DST HOMEWORK ASSIGNMENT 2: MYCIELSKI'S THEOREM; THE SPACE OF SUBSETS; BOREL CODES

Homework: This is the second of three mandatory homework assignments. You must hand it in at the beginning of lecture on Friday, December 21, 2012.

Exercise 1

Prove the following theorem due to Mycielski:

Theorem. *Let E be an equivalence relation*[1] *on a Polish space X. Suppose that E is meagre as a subset of $X \times X$, when the latter is given the product topology. Then there is a continuous injection $f : 2^\omega \to X$ such that*

$$(\forall x, y \in 2^\omega) x \neq y \implies f(x) \not\!E f(y),$$

i.e., $f(2^\omega)$ meets each E equivalence class in at most *one point.*

Hint: Fix a decreasing sequence $U_n \subseteq X^2$ of dense open sets such that

$$E \cap \bigcap_{n \in \omega} U_n = \emptyset.$$

Recursively (on $\ell\text{h}(s)$, say) define a Cantor scheme $(V_s)_{s \in 2^{<\omega}}$ of open sets in X such that

(1) $\overline{V_{s^\frown i}} \subseteq V_s$ for all $s \in 2^{<\omega}$, $i \in \{0, 1\}$;
(2) $\text{diam}(V_s) \leq 2^{-\ell\text{h}(s)}$;
(3) For all $n \in \omega$ and $s, t \in 2^{<\omega}$ with $\ell\text{h}(s) = \ell\text{h}(t) = n$, if $t \neq s$ then $V_s \times V_t \subseteq U_n$.

Exercise 2

(a) Let $\mathcal{P}(A)$ denote the powerset of A (i.e., the set of all subsets of A), and for $x, y \in \mathcal{P}(A)$, let $x \triangle y = x \setminus y \cup y \setminus x$ be the symmetric difference. Show that if $A = \{a_n : n \in \omega\}$ is a countable set, then

$$d(x, y) = 2^{-\min\{n \in \omega : a_n \in x \triangle y\}}$$

(where as per our usual convention $\min(\emptyset) = \infty$) defines a complete metric on $\mathcal{P}(A)$. Show that the set

$$\text{FIN}(A) = \{x \in \mathcal{P}(A) : x \text{ is finite}\}$$

is dense in $\mathcal{P}(A)$, and conclude that $(\mathcal{P}(A), d)$ is a Polish metric space.

(b) Show that $x_n \to x$ in $\mathcal{P}(A)$ if and only if for all $m \in \omega$ we have:

$$a_m \in x \implies (\exists N)(\forall n \geq N) a_m \in x_n$$

and

$$a_m \notin x \implies (\exists N)(\forall n \geq N) a_m \notin x_n.$$

(c) Consider now $\mathcal{P}(\omega^{<\omega})$. Show that $\text{Tree}(\omega) \subseteq \mathcal{P}(\omega^{<\omega})$ is a closed subset of $\mathcal{P}(\omega^{<\omega})$. Is the set of finite branching trees closed? What about the set of pruned trees? Or the set of wellfounded trees?

[1] i.e., a reflexive, symmetric and transitive binary relation

Exercise 3

This exercise requires some preparatory definitions.

Definition. A *Borel code* is a pair (T, f), where $T \in \text{Tree}(\omega)$ is a well-founded, $T \supsetneq \{\emptyset\}$, and where

$$f : \{t \in T : t \text{ is terminal}\} \to \omega$$

is a function. (Thus a Borel code can be thought of as a well-founded tree where each terminal node has been labeled by some n.) The *rank* of the Borel code (T, f), denoted $\text{rk}(T, f)$, is the rank of $\emptyset \in T$, i.e., $\text{rk}(T, f) = \rho_T(\emptyset)$. (See Kechris 2.E.)

Let X be a Polish space, and fix a basis $(U_n)_{n\in\omega}$ for the topology. Let (T, f) be a Borel code. We define recursively for $t \in T$ the sets $B^t(T, f)$ as follows: If $t \in T$ is terminal in T, then $B^t(T, f) = U_{f(t)}$. If $t \in T$ is not terminal, we define

$$B^t(T, f) = \bigcap \{X \setminus B^{t^\frown n}(T, f) : n \in \omega \wedge t^\frown n \in T\}.$$

Finally, we define $B(T, f) = B^{\emptyset}(T, f)$.

(a) Prove that $B(T, f) \in \mathbf{\Pi}^0_{\text{rk}(T,f)}$.
(b) Prove that if $A \subseteq X$ is $\mathbf{\Pi}^0_\alpha$ then $A = B(T, f)$ for some Borel code (T, f) of rank α.

Hint: In both cases, use induction on ordinals.