DST HOMEWORK ASSIGNMENT 1:
A GUIDED TOUR OF ORDINALS

Homework: This is the first of three mandatory homework assignments. You must hand it in at the beginning of lecture on Friday, November 30, 2012.

The theme of this homework assignment is *ordinals*. First we recall some basic notions related to relations and orderings, which you've probably seen previously in other courses.

Definition 1. Let X be a set.

(A) A subset $R \subseteq X \times X$ is called a *binary relation* on X. We write xRy whenever $(x, y) \in R$, and we write $x \not R\, y$ whenever $(x, y) \notin R$.

(B) An *ordering* (in the strict sense) of a set X is a binary relation $R \subseteq X \times X$ which satisfies:

- *Irreflexivity*: $(\forall x) x \not R\, x$.
- *Transitivity*: $(\forall x, y, z)(xRy \wedge yRz) \implies xRz$.

(C) A *linear* ordering of a set X is an ordering R on X which additionally satisfies:

- *Trichotomy*: $(\forall x, y \in X) xRy \vee x = y \vee yRx$.

A linear order is sometimes also called a *total* order. We will not use this word.

(D) A linear ordering R of X is said to be a *wellordering* of X if every non-empty subset of X contains a least element.

Some examples to think about are the usual ordering of $\mathbb{R}$, or the usual ordering of $\mathbb{N}$. The ordering of $\mathbb{R}$ is not a wellordering (why?), but the ordering of $\mathbb{N}$ is.

Exercise 1. Show that the following are equivalent for a linear ordering R on a set X:

(1) R is a wellordering of X.
(2) There is no sequence $(x_n)_{n \in \mathbb{N}}$ in X such that

$$(\forall n \in \mathbb{N}) x_{n+1} R x_n.$$

Definition 2. A set x is called *transitive* if

$$(\forall y)(\forall z)(y \in x \wedge z \in y) \implies z \in x$$

Exercise 2. Prove that x is transitive if and only if

$$\bigcup_{y \in x} y \subseteq x.$$

Definition 3. (A) A set x is called an *ordinal* if x is transitive and wellordered by $\in$.

(B) We define the *successor function* $\mathcal{S}$ by

$$\mathcal{S}(x) = x \cup \{x\}$$

for any set x.

Exercise 3. (A) Show that for no ordinal x do we have $x \in x$.[1]

(B) Show that if x is transitive, then so is $\mathcal{S}(x)$, and that if x is an ordinal then so is $\mathcal{S}(x)$.

[1]This is a bit silly because the usual formulation of set theory does not allow *any* set x with the property that $x \in x$.

You should quickly convince yourself now that $\emptyset$ is an ordinal, and so $\mathcal{S}^n(\emptyset)$ is an ordinal for all $n \in \mathbb{N}$ by the previous exercise. We define $0 = \emptyset = \mathcal{S}^0(0)$.

Exercise 4. Prove that
$$\omega \stackrel{\text{def.}}{=} \{0\} \cup \{\mathcal{S}^n(0) : n \in \mathbb{N}\}$$
is an ordinal.

Hint: Prove by induction on $n \in \mathbb{N}$ that $\mathcal{S}^n(0) = \{0, \dots, \mathcal{S}^{n-1}(0)\}$.

Announcement: For the rest of this course, we will identify n with $\mathcal{S}^n(0)$, thus $n = \{0, \dots, n-1\}$.

We usually denote ordinals by lowercase greek letters, α, β, γ, etc. We let
$$\mathbb{ON} = \{\alpha : \alpha \text{ is an ordinal}\},$$
in other words, $\mathbb{ON}$ is the class[2] of all ordinals.

Definition 4. For $\alpha, \beta \in \mathbb{ON}$, write $\alpha < \beta$ iff $\alpha \in \beta$.

At this point, you quickly check for yourself that $\alpha < \mathcal{S}(\alpha)$, in fact, if $\beta < \mathcal{S}(\alpha)$, then either $\beta < \alpha$, or $\alpha = \beta$.

Exercise 5. Prove that for all $\alpha, \beta \in \mathbb{ON}$, either $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$. Conclude that the class $\mathbb{ON}$ is linearly ordered by $<$. Why is transitivity satisfied?

Hint: This is the only exercise with any teeth here. Start by arguing that $z = \alpha \cap \beta$ is an ordinal, and then show that either $z \in \alpha$ or $\alpha = z$, and also that either $z \in \beta$ or $z = \beta$. For this, assume that $z \neq \alpha$. Argue that $\alpha \setminus z$ is non-empty, and so has a $\in$-least member $\gamma \in \alpha$. Prove that $z = \gamma$ by showing that z and γ have the same members.

As a bonus exercise, you may prove for yourself that the class of ordinals is wellordered by $<$, but please don't give us bonus grading to do by handing in your solution.

Definition 5. An ordinal α is called a *successor ordinal* if $\alpha = \mathcal{S}(\beta)$ for some $\beta \in \mathbb{ON}$. An ordinal α is called a *limit ordinal* if
$$\alpha = \bigcup_{\beta < \alpha} \beta.$$

By this definition, 0 is a limit (!), while $\mathcal{S}^n(0)$ is a successor for all $n \in \mathbb{N}$.

Exercise 6. (A) Show that ω is a limit ordinal.
(B) Show that every ordinal is either a successor or a limit, but never both.

Hint: Suppose that α is not a limit. Show that $\gamma = \bigcup_{\beta<\alpha} \beta$ is an ordinal, and that $\mathcal{S}(\gamma) = \alpha$.

This ends the exercises. However, it is worthwhile pointing out that since ordinals are wellordered by $<$, one can make definitions by recursion and proofs by induction. Here, the distinction between successor and limit ordinals turns out to be very useful. One can also show that every wellordered set is order-isomorphic to an ordinal. You can try to prove this yourself if you have nothing better to do. The idea is to define the order-isomorphism by recursion.

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[2]Some of you may now be tempted to show off your knowledge by pointing out that in standard formulations of set theory, $\mathbb{ON}$ is not a set, but a proper class. You're right. Give yourself a pat on the back.