AST & FORCING HOMEWORK ASSIGNMENT 1

Homework: This is the first of three mandatory homework assignments. You must hand it in at the beginning of lecture on Friday, February 14, 2014.

The purpose of this homework is to remind you of things related to *cardinality*: Countability, uncountability, equinumerousity, etc. As always in set theory and logic, $\omega = \{0, 1, 2, \ldots\}$ denotes the set of all natural numbers (including zero).

Exercise 1 (4 points)**.** Recall that two sets X and Y are *equinumerous* (also called *equipotent*) if there is a bijection $f : X \to Y$. (So, for instance, the set of fingers on my left hand is equinumerous with the set of fingers on my right hand because I can pair them up: My left thumb to my right thumb, my left index finger to my right index finger, etc.)

Prove the *Schröder-Bernstein Theorem*: If X and Y are sets and there is an injection $g : X \to Y$ and an injection $h : Y \to X$, then X and Y are equinumerous.

Hint: We need to define a bijection $f : X \to Y$. If g is already a bijection, there is nothing to show. So assume not. Then $Y \setminus g(X) \neq \emptyset$. We might try to define $f(x) = h^{-1}(x)$ for $x \in h(Y \setminus g(X))$ and $f(x) = g(x)$ otherwise, but this is not onto, since $g(h(Y \setminus g(X)))$ is not contained in the range of this f. So we try to remedy this by (re-)defining $f(x) = h^{-1}(x)$ for $x \in h(g(h(Y \setminus g(X))))$, only to realize that then $g(h(g(h(Y \setminus g(X)))))$ is not contined in the range, so we need to make another correction, etc. At any rate, you should by now realize that the set

$$\bigcup_{n \in \omega} (h \circ g)^n(h(Y \setminus g(X))),$$

where $(h \circ g)^0 = I$ is by definition the identity function, might be a good set to look at.

Exercise 2 (6 points)**.** (1) Prove the *Baire Category Theorem*: If (X, d) is a complete metric space (i.e., a metric space where every Cauchy sequence is convergent), and if $(U_n)_{n\in\omega}$ is a collection of dense open sets, then $\bigcap_{n\in\omega} U_n$ is dense.

Hint: We need to show that $\bigcap_{n\in\omega} U_n$ meets every open ball in (X, d). So fix some ball open ball B. Find a closed (!) ball B_0 of positive radius at most 1 contained in $B \cap U_0$; then find a closed ball B_1 of positive radius at most $\frac{1}{2^1}$ contained in $B_0 \cap U_0 \cap U_1$; then find a closed ball B_2 of positive radius at most $\frac{1}{2^2}$ contained in $B_1 \cap U_0 \cap U_1 \cap U_2$. Continue defining B_n recursively in this manner. Now argue that the set $\cap_{n\in\omega} B_n$ contains a single point which is in $B \cap U_n$ for all n.

(2) Recall that a set is *countable* if there is a surjection from ω onto it; otherwise it is called *uncountable*. Use the Baire Category Theorem to prove that $\mathbb{R}$ is uncountable.

Hint: For any $x \in \mathbb{R}$, the set $\mathbb{R} \setminus \{x\}$ is open dense.

Exercise 3 (2 points)**.** Let X be a set, and let $\mathcal{P}(X)$ denote the *power set*, i.e., the set of all subsets of X.

(A) Show that there is an injection $X \to \mathcal{P}(X)$.

(B) Prove that there is no surjection $X \to \mathcal{P}(X)$.

Hint: In (B), suppose there were such a surjection, call it f. Consider

$$\{x \in X : x \notin f(x)\}.$$

Exercise 4 (6 points)**.** Let in general ${}^X Y$ denote the set of functions $f : X \to Y$. (This is also sometimes denoted X^Y, though we will reserve this for a different use.) Recall that $n = \{0, 1, \ldots, n-1\}$, and let

$${}^{<\omega}2 = \bigcup_{n \in \omega} {}^n 2,$$

that is, ${}^{<\omega}2$ is the set of finite binary sequences. Graphically, ${}^{<\omega}2$ can be represented as a complete binary tree. If $p, q \in {}^{<\omega}2$, then we write $p \leq q$ if p extends q (this is no typo! Think of a complete binary tree that grows *down*), that is $p \leq q$ just in case $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and $p(n) = q(n)$ whenever both are defined. (The relation $p \leq q$ is read "p extends q", or sometimes "p is stronger than q".)

A set $D \subseteq {}^{<\omega}2$ is called *dense* if for every $q \in {}^{<\omega}2$ there is $p \in D$ such that $p \leq q$. A non-empty set $F \subseteq {}^{<\omega}2$ is called a *filter* if (1) whenever $p \leq q$ and $p \in F$, then $q \in F$, and (2) whenever $p, q \in F$ there is $r \in F$ such that $r \leq p$ and $r \leq q$.

(a) What does a filter "look like" if we represent ${}^{<\omega}2$ is a complete binary tree? Can a filter be infinite? Explain with your own words, and make a drawing to illustrate.

(b) Prove that if $(D_n)_{n \in \omega}$ is a sequence of dense sets, $p \in {}^{<\omega}2$, then there is a filter F such that $p \in F$ and $F \cap D_n \neq \emptyset$ for all $n \in \omega$.

(c) Can you see see any connection between (b) and the Baire Category Theorem? Explain your answer.

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