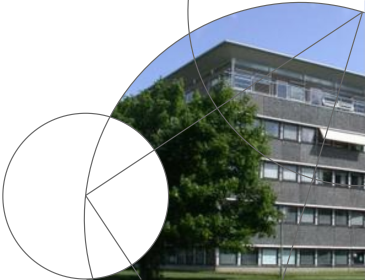
Do phantom Cuntz-Krieger algebras exist?

Sara Arklint
Department of Mathematical Sciences



What is a phantom Cuntz-Krieger algebra?

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Properties of Cuntz-Krieger algebras

A Cuntz-Krieger algebra A has the following properties:

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Properties of Cuntz-Krieger algebras

A Cuntz-Krieger algebra A has the following properties:

- A is unital, purely infinite, nuclear, separable, and of real rank zero,

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Properties of Cuntz-Krieger algebras

A Cuntz-Krieger algebra A has the following properties:

- A is unital, purely infinite, nuclear, separable, and of real rank zero,
- A has finitely many ideals,

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Properties of Cuntz-Krieger algebras

A Cuntz-Krieger algebra A has the following properties:

- A is unital, purely infinite, nuclear, separable, and of real rank zero,
- A has finitely many ideals,
- for all $I \trianglelefteq J \trianglelefteq A$, the group $\mathrm{K}_*(J/I)$ is finitely generated, the group $\mathrm{K}_1(J/I)$ is free, and $\operatorname{rank} \mathrm{K}_0(J/I) = \operatorname{rank} \mathrm{K}_1(J/I)$,

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Properties of Cuntz-Krieger algebras

A Cuntz-Krieger algebra A has the following properties:

- A is unital, purely infinite, nuclear, separable, and of real rank zero,
- A has finitely many ideals,
- for all $I \trianglelefteq J \trianglelefteq A$, the group $\mathrm{K}_*(J/I)$ is finitely generated, the group $\mathrm{K}_1(J/I)$ is free, and $\operatorname{rank}\mathrm{K}_0(J/I) = \operatorname{rank}\mathrm{K}_1(J/I)$,
- the simple subquotients of A are in the bootstrap class.

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Definition

A C^*-algebra A *looks like a Cuntz-Krieger algebra* if

- A is unital, purely infinite, nuclear, separable, and of real rank zero,
- A has finitely many ideals,
- for all $I \trianglelefteq J \trianglelefteq A$, the group $\mathrm{K}_*(J/I)$ is finitely generated, the group $\mathrm{K}_1(J/I)$ is free, and $\operatorname{rank} \mathrm{K}_0(J/I) = \operatorname{rank} \mathrm{K}_1(J/I)$,
- the simple subquotients of A are in the bootstrap class.

What is a phantom Cuntz-Krieger algebra?

Definition

A *Cuntz-Krieger algebra* is a graph algebra $C^*(E)$ arising from a finite graph E that has no sinks and sources, and satisfies condition (K).

Definition

A C^*-algebra A *looks like a Cuntz-Krieger algebra* if

- A is unital, purely infinite, nuclear, separable, and of real rank zero,
- A has finitely many ideals,
- for all $I \trianglelefteq J \trianglelefteq A$, the group $\mathrm{K}_*(J/I)$ is finitely generated, the group $\mathrm{K}_1(J/I)$ is free, and $\operatorname{rank} \mathrm{K}_0(J/I) = \operatorname{rank} \mathrm{K}_1(J/I)$,
- the simple subquotients of A are in the bootstrap class.

Definition

A *phantom Cuntz-Krieger algebra* is a C^*-algebra that looks like a Cuntz-Krieger algebra but is not isomorphic to a Cuntz-Krieger algebra.

We really don't know if phantom Cuntz-Krieger algebras exist

We really don't know if phantom Cuntz-Krieger algebras exist

- Can $M_n \otimes A$ be a phantom Cuntz-Krieger algebra if A is a Cuntz-Krieger algebra?

We really don't know if phantom Cuntz-Krieger algebras exist

- Can $M_n \otimes A$ be a phantom Cuntz-Krieger algebra if A is a Cuntz-Krieger algebra?
- Can a graph algebra be a phantom Cuntz-Krieger algebra?

Can a phantom Cuntz-Krieger algebra be simple?

Can a phantom Cuntz-Krieger algebra be simple?

Theorem (Szymański, Eilers-Katsura-Tomforde-West)

Let G and F be finitely generated groups, let $g \in G$, and assume that F is free and that $\operatorname{rank} G = \operatorname{rank} F$. Then there exists a simple Cuntz-Krieger algebra A realising $(G \oplus F, g)$ as $(\mathsf{K}_*(A), [1_A]_0)$.

Can a phantom Cuntz-Krieger algebra be simple?

Theorem (Szymański, Eilers-Katsura-Tomforde-West)

Let G and F be finitely generated groups, let $g \in G$, and assume that F is free and that $\operatorname{rank} G = \operatorname{rank} F$. Then there exists a simple Cuntz-Krieger algebra A realising $(G \oplus F, g)$ as $(\mathsf{K}_*(A), [1_A]_0)$.

Theorem (Kirchberg-Phillips)

Let A and B be unital, simple, purely infinite, nuclear, separable C^*-algebras in the bootstrap class.
Then $(\mathsf{K}_*(A), [1_A]_0) \cong (\mathsf{K}_*(B), [1_B]_0)$ implies $A \cong B$.

Can a phantom Cuntz-Krieger algebra be simple?

Theorem (Szymański, Eilers-Katsura-Tomforde-West)

Let G and F be finitely generated groups, let $g \in G$, and assume that F is free and that $\operatorname{rank} G = \operatorname{rank} F$. Then there exists a simple Cuntz-Krieger algebra A realising $(G \oplus F, g)$ as $(\mathrm{K}_*(A), [1_A]_0)$.

Theorem (Kirchberg-Phillips)

Let A and B be unital, simple, purely infinite, nuclear, separable C^*-algebras in the bootstrap class.
Then $(\mathrm{K}_*(A), [1_A]_0) \cong (\mathrm{K}_*(B), [1_B]_0)$ implies $A \cong B$.

Corollary

Simple phantom Cuntz-Krieger algebras do not exist.

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Definition

Let A be a C^*-algebra with exactly one ideal I, and define $\mathrm{K}_{\mathrm{six}}(A)$ as

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A/I) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A/I) & \longleftarrow & \mathrm{K}_1(A) & \longleftarrow & \mathrm{K}_1(I). \end{array}$$

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Definition

Let A be a C^*-algebra with exactly one ideal I, and define $\mathrm{K}_{\mathrm{six}}(A)$ as

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A/I) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A/I) & \longleftarrow & \mathrm{K}_1(A) & \longleftarrow & \mathrm{K}_1(I). \end{array}$$

Theorem (Eilers-Katsura-Tomforde-West)

Let a six-term exact sequence $\mathcal{E}$

$$\begin{array}{ccccc} G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 \\ \uparrow & & & & \downarrow 0 \\ F_3 & \longleftarrow & F_2 & \longleftarrow & F_1. \end{array}$$

be given with G_1, G_2, G_3 and F_1, F_2, F_3 finitely generated groups, let $g \in G_2$, assume that the groups F_1, F_2, F_3 are free, that $\operatorname{rank} G_i = \operatorname{rank} F_i$ for all $i = 1, 2, 3$. Then there exists a Cuntz-Krieger algebra A with exactly one ideal realising $(\mathcal{E}, g)$ as $(\mathrm{K}_{\mathrm{six}}(A), [1_A]_0)$.

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Definition

Let A be a C^*-algebra with exactly one ideal I, and define $\mathrm{K}_{\mathrm{six}}(A)$ as

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A/I) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A/I) & \longleftarrow & \mathrm{K}_1(A) & \longleftarrow & \mathrm{K}_1(I). \end{array}$$

Theorem (Eilers-Katsura-Tomforde-West)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra and has exactly one ideal. Then there exists a Cuntz-Krieger algebra A with exactly one ideal for which $(\mathrm{K}_{\mathrm{six}}(A), [1_A]_0)$ *is isomorphic to* $(\mathrm{K}_{\mathrm{six}}(B), [1_B]_0)$.

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Theorem (Eilers-Katsura-Tomforde-West)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra and has exactly one ideal. Then there exists a Cuntz-Krieger algebra A with exactly one ideal for which $(\mathsf{K}_{\mathrm{six}}(A), [1_A]_0)$ is isomorphic to $(\mathsf{K}_{\mathrm{six}}(B), [1_B]_0)$.

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Theorem (Eilers-Katsura-Tomforde-West)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra and has exactly one ideal. Then there exists a Cuntz-Krieger algebra A with exactly one ideal for which $(\mathsf{K}_{\mathsf{six}}(A), [1_A]_0)$ is isomorphic to $(\mathsf{K}_{\mathsf{six}}(B), [1_B]_0)$.

Theorem (Bonkat)

Let A and B be unital, purely infinite, nuclear, separable C^*-algebras that have exactly one ideal and whose simple subquotients are in the bootstrap class. Then $(\mathsf{K}_{\mathsf{six}}(A), [1_A]_0) \cong (\mathsf{K}_{\mathsf{six}}(B), [1_B]_0)$ implies $A \cong B$.

Can a phantom Cuntz-Krieger algebra have exactly one ideal?

Theorem (Eilers-Katsura-Tomforde-West)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra and has exactly one ideal. Then there exists a Cuntz-Krieger algebra A with exactly one ideal for which $(\mathsf{K}_{\mathsf{six}}(A), [1_A]_0)$ is isomorphic to $(\mathsf{K}_{\mathsf{six}}(B), [1_B]_0)$.

Theorem (Bonkat)

Let A and B be unital, purely infinite, nuclear, separable C^*-algebras that have exactly one ideal and whose simple subquotients are in the bootstrap class. Then $(\mathsf{K}_{\mathsf{six}}(A), [1_A]_0) \cong (\mathsf{K}_{\mathsf{six}}(B), [1_B]_0)$ implies $A \cong B$.

Corollary

Phantom Cuntz-Krieger algebras with exactly one ideal do not exist.

What about general phantom Cuntz-Krieger algebras?

What about general phantom Cuntz-Krieger algebras?

Definition

For a C^*-algebra A with finitely many ideals, its *reduced filtered K-theory* $\mathrm{FK}_{\mathcal{R}}(A)$ consists of

$$\mathrm{K}_1(J/I) \longrightarrow \mathrm{K}_0(I) \longrightarrow \mathrm{K}_0(J)$$

for all $I \trianglelefteq J \trianglelefteq A$ for which J/I is simple, and $J = J_1 + J_2$ implies $J = J_i$.

What about general phantom Cuntz-Krieger algebras?

Definition

For a C^*-algebra A with finitely many ideals, its *reduced filtered K-theory* $\mathrm{FK}_{\mathcal{R}}(A)$ consists of

$$\mathrm{K}_1(J/I) \longrightarrow \mathrm{K}_0(I) \longrightarrow \mathrm{K}_0(J)$$

for all $I \trianglelefteq J \trianglelefteq A$ for which J/I is simple, and $J = J_1 + J_2$ implies $J = J_i$.

Theorem (A-Bentmann-Katsura)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra. Then there exists a Cuntz-Krieger algebra A with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ for which $\mathrm{FK}_{\mathcal{R}}(A)$ is isomorphic to $\mathrm{FK}_{\mathcal{R}}(B)$.

What about general phantom Cuntz-Krieger algebras?

Definition

For a C^*-algebra A with finitely many ideals, its *reduced filtered K-theory* $\mathsf{FK}_{\mathcal{R}}(A)$ consists of

$$\mathsf{K}_1(J/I) \longrightarrow \mathsf{K}_0(I) \longrightarrow \mathsf{K}_0(J)$$

for all $I \trianglelefteq J \trianglelefteq A$ for which J/I is simple, and $J = J_1 + J_2$ implies $J = J_i$.

Theorem (A-Bentmann-Katsura)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra. Then there exists a Cuntz-Krieger algebra A with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ for which $\mathsf{FK}_{\mathcal{R}}(A)$ is isomorphic to $\mathsf{FK}_{\mathcal{R}}(B)$.

Theorem (Meyer-Nest, A-Restorff-Ruiz, A-Bentmann-Katsura)

Let A and B be C^*-algebras that look like Cuntz-Krieger algebras and have the same suitably nice primitive ideal space.
Then $\mathsf{FK}_{\mathcal{R}}(A) \cong \mathsf{FK}_{\mathcal{R}}(B)$ implies $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

What about general phantom Cuntz-Krieger algebras?

Definition

For a C^*-algebra A with finitely many ideals, its *reduced filtered K-theory* $\mathrm{FK}_{\mathcal{R}}(A)$ consists of

$$\mathrm{K}_1(J/I) \longrightarrow \mathrm{K}_0(I) \longrightarrow \mathrm{K}_0(J)$$

for all $I \trianglelefteq J \trianglelefteq A$ for which J/I is simple, and $J = J_1 + J_2$ implies $J = J_i$.

Theorem (A-Bentmann-Katsura)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra. Then there exists a Cuntz-Krieger algebra A with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ for which $\mathrm{FK}_{\mathcal{R}}(A)$ is isomorphic to $\mathrm{FK}_{\mathcal{R}}(B)$.

Theorem (Meyer-Nest, A-Restorff-Ruiz, A-Bentmann-Katsura)

Let A and B be C^*-algebras that look like Cuntz-Krieger algebras and have the same suitably nice primitive ideal space.
Then $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(B)$ implies $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

What about general phantom Cuntz-Krieger algebras?

Definition

For a C^*-algebra A with finitely many ideals, its *reduced filtered K-theory* $\mathrm{FK}_{\mathcal{R}}(A)$ consists of

$$\mathrm{K}_1(J/I) \longrightarrow \mathrm{K}_0(I) \longrightarrow \mathrm{K}_0(J)$$

for all $I \trianglelefteq J \trianglelefteq A$ for which J/I is simple, and $J = J_1 + J_2$ implies $J = J_i$.

Theorem (A-Bentmann-Katsura)

Let B be a C^*-algebra that looks like a Cuntz-Krieger algebra. Then there exists a Cuntz-Krieger algebra A with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ for which $\mathrm{FK}_{\mathcal{R}}(A)$ is isomorphic to $\mathrm{FK}_{\mathcal{R}}(B)$.

Theorem (Meyer-Nest, A-Restorff-Ruiz, A-Bentmann-Katsura)

Let A and B be C^*-algebras that look like Cuntz-Krieger algebras and have the same suitably nice primitive ideal space.
Then $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(B)$ implies $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.