16/2
2010

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Strings in Copenhagen

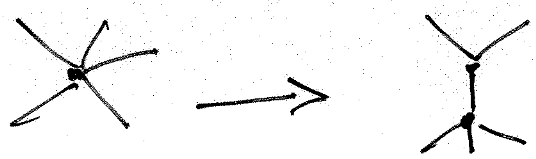
Fat Graphs.

Idea:

| Categories of fat graphs
Morphisms = Edge collapses | $\xrightarrow{B}$ | Space of metric graphs
edge collapsing corresponds to length $\to 0$ |

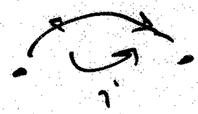
$\downarrow$

Chain complex
Generators = graphs
degree of graph = $\sum |v| - 3$
Boundary maps = "blowing up vertices"



$+ \dots$

A <u>graph</u> G is (V, H, i, s)



V = set of vertices

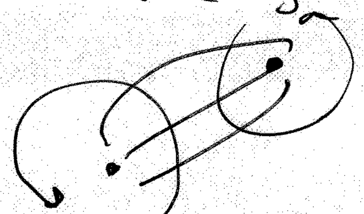
H = set of half-edges

$i : H \longrightarrow H$ fixed point free involution

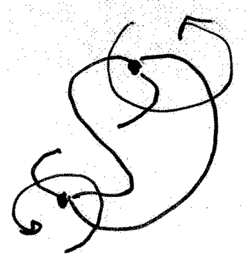
$s : H \longrightarrow V$ source map

$E = H/i$ = set of edges.

A <u>fat graph</u> is a graph + a cyclic ordering of the sets $s^{-1}(v)$ for each vertex $v \in V$.
Can see this as the cycles of a perm. σ of H

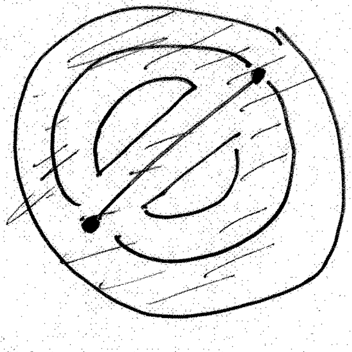


cyclic ordering at each vertex



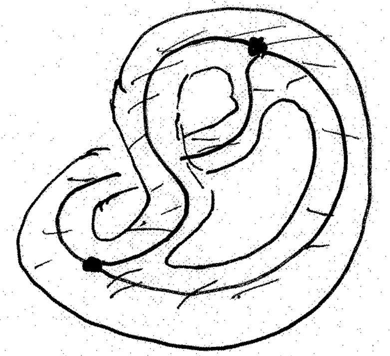
A fat graph has <u>boundary cycles</u> which are the cycles of the permutation $\sigma \circ i$, of H.

(1)



3 boundary components

(2)



1 boundary component

The surface associated to a fat graph by thickening has the same # of boundary cycles / components and the same Euler characteristic.

Ex. (1) $\chi = V - E = -1$
$= 2 - 2g - 3 \Rightarrow g = 0$

(2) $\chi = 2 - 2g - 1 = -1 \Rightarrow g = 1$

Could do open-closed cobordisms

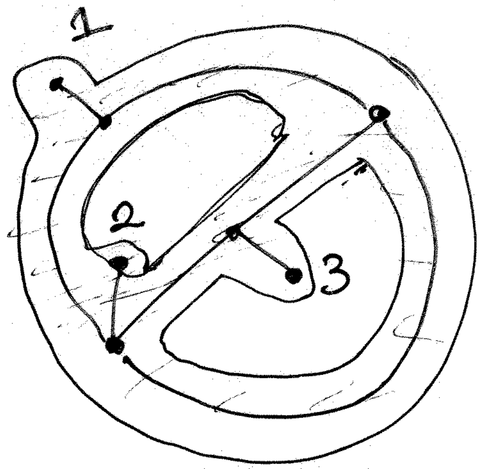
but this becomes cumbersome...

restrict to the "closed part"

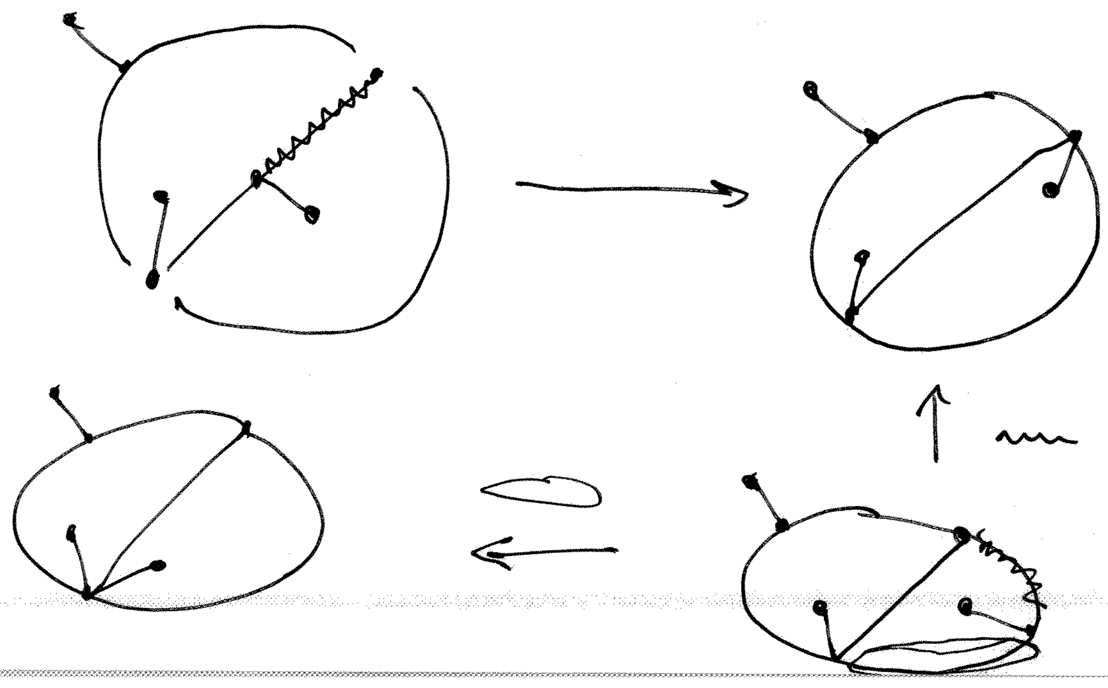
Model the cobordisms of circles.

<u>Definition</u>: $\mathcal{F}at_{g,k}$ = Category with
($k>0$)

<u>Objects</u>: fat graphs of type $S_{g,k}$
(genus g, k boundary components)
with a labelled valence 1
vertex in each cycle.



<u>Morphisms</u>: Induced by edge collapses
+ identifying the leaves.



Theorem [Godin / Penner - Strebel-Harer / Igusa]

For $k > 0$,

$$N\,\mathcal{F}at_{g,k} \simeq B\Gamma_{g,k} \simeq B\mathrm{Diff}(S_{g,k} \text{ rel } \partial)$$

$$\Gamma_{g,k} = \pi_0 \mathrm{Diff}(S_{g,k} \text{ rel } \partial).$$

Plan of proof:

(1). Define $E\mathcal{F}at$ ~ Teichmüller space

$\circlearrowleft \Gamma_{g,k}$ free action

with $E\mathcal{F}at/\Gamma \cong \mathcal{F}at$

(2). $E\mathcal{F}at \cong$ complex of arcs $\simeq *$.

[Recent proof by Hatcher]

(1) Definition: $E\mathcal{F}at_{g,k}$ = category with

Objects: $(G, [\varphi])$

$G \in \mathcal{F}at_{g,k}$, and

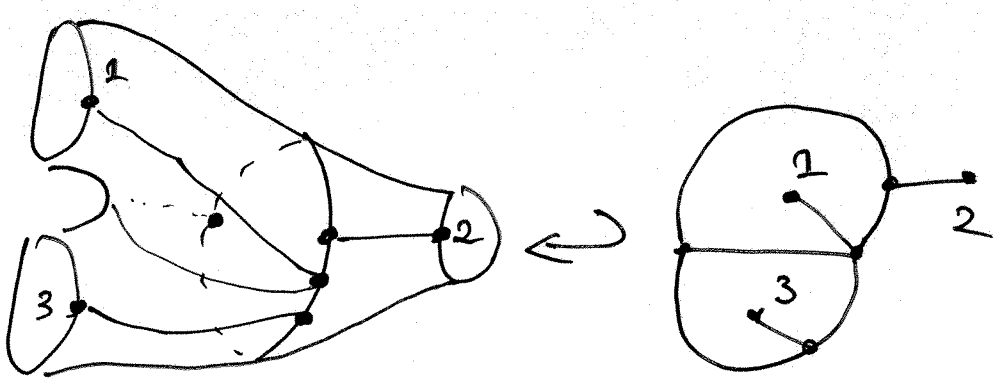
$\varphi : G \hookrightarrow S_{g,k}$ embedding such that

- i^{th} leaf $\overset{\varphi}{\longmapsto}$ i^{th} boundary component (to a chosen point p_i)
- cyclic orderings of G are to agree with those induced by S on $\varphi(G)$.

$(\Rightarrow G \overset{\sim}{\hookrightarrow} S_{g,k})$

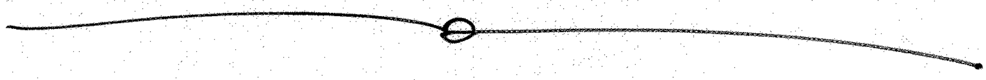
$[\varphi]$ = isotopy class of φ, rel ∂

Ex:

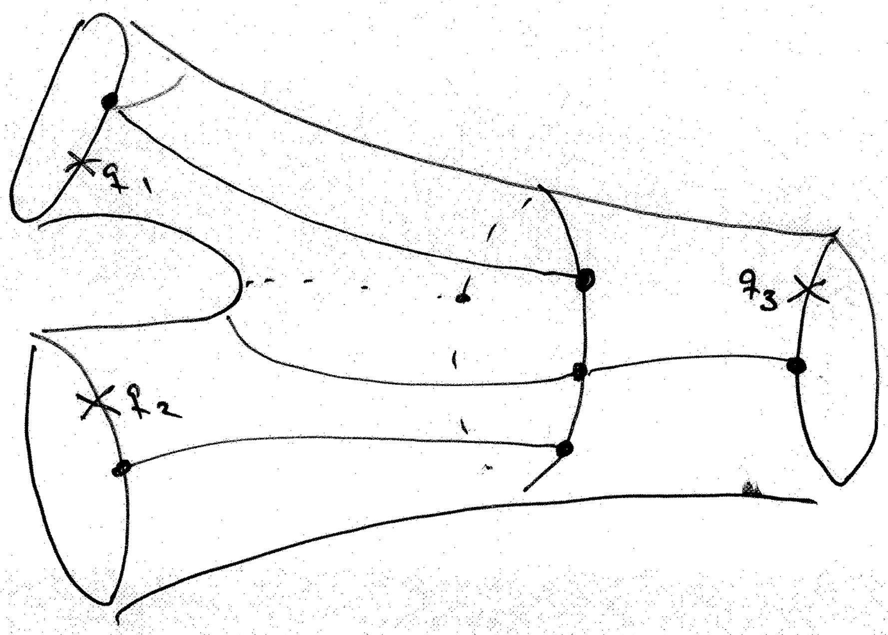


Note: $\Gamma_{g,k}$ acts freely on $E\mathcal{F}at$ because the complement of $\varphi(G)$ is a union of strips (along ∂S) and $\mathrm{Diff}(D^2 \text{ rel } \partial) \simeq *$

<u>Morphisms</u>: Induced by edge collapses



(2) Taking dual graphs



gives an arc system on (S, Δ), where $\Delta = \{q_1, \ldots, q_k\}$, whose ~~[illegible]~~ complement is a union of polygons, each with at least 3 sides. "Filling arc systems" (cf. ideal triangulations)

[Harer-Hatcher] This is a contractible space.

Admissible fat graphs

$\mathcal{F}at^{a_p}_{g,k} \hookrightarrow \mathcal{F}at_{g,k}$ subcategory of graphs $[G]$ such that the first $p<k$ boundary cycles are disjointly embedded in G

Theorem [Godin]
For all $p<k$ $\quad \mathcal{F}at^{a_p}_{g,k} \hookrightarrow \mathcal{F}at_{g,k}$ induces a weak equivalence upon taking classifying spaces.

Proof: Induction on p.

$p=0$ ✓

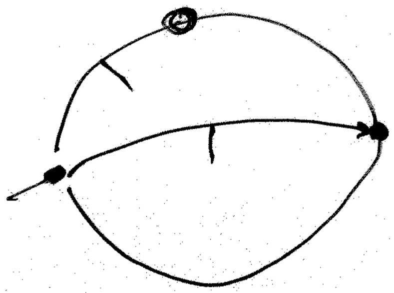
Want to show that

$$\mathcal{F}at^{a_{p+1}}_{g,k+1} \hookrightarrow \mathcal{F}at^{a_p}_{g,k+1}$$

is a weak equivalence.

Define $\mathcal{F}at_{\underline{1}}^{ap}(S_{g,q}) =$

p-admissible fat graphs with a puncture = a special vertex, allowed to be bivalent



$$\begin{array}{c} \mathcal{F}at_{\underline{1}}^{ap}(S_{g,q}) \\ \Big\downarrow \pi \\ \mathcal{F}at^{ap}(S_{g,p}) \end{array}$$

The fiber $\pi/\Gamma = \left\{ \begin{array}{c} \pi(\Gamma_1) \xrightarrow{\alpha_1} \Gamma \\ \downarrow \pi(\varphi) \nearrow \\ \pi(\Gamma_2) \end{array} \right\}$

$$\begin{array}{c} \pi/\Gamma \\ \uparrow \Big\downarrow m \\ \pi^{-1}(\Gamma) \end{array}$$

The map m is $(\Gamma_1, \alpha_1) \rightsquigarrow (\overline{\Gamma}, id)$

collapse Γ_1 according to α_1

There are natural transformations between the composites and identities

Thus, π/Γ and $\pi^{-1}(\Gamma)$ have homotopy equivalent nerves, and we get a fibration sequence

$$
\begin{tikzcd}
N(\pi^{-1}(\Gamma)) \arrow[r] \arrow[d, "\cong"] & N\mathcal{F}at_1^{a_p}(S_{g,q}) \arrow[r] \arrow[dd, dashed, "\simeq"] & N\mathcal{F}at^{a_p}(S_{g,q}) \arrow[dd, "\simeq \text{ by induction}"] \\
\Gamma \arrow[d, "\simeq"] & & \\
S_{g,p} \arrow[r] & B\Gamma^1_{g,q} \arrow[r] & B\Gamma_{g,q}
\end{tikzcd}
$$

This gives a model for $\mathcal{F}at_1^{a_p}(S_{g,p})$.

We have a functor

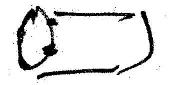
$$\mathcal{F}at^{a_{p+1}}(S_{g,q+1}) \xrightarrow{\Psi} \mathcal{F}at_1^{a_p}(S_{g,p})$$

which collapses the $(p+1)$-st cycle to a puncture.

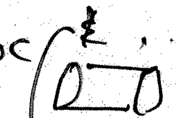
$$
\begin{tikzcd}
\Psi/\Gamma = \left\{ \begin{tikzcd}[ampersand replacement=\&] \Psi(\Gamma_1) \arrow[d] \arrow[dr, "\alpha_1"] \& \\ \Psi(\Gamma_2) \arrow[r, "\alpha_2"'] \& \Gamma \end{tikzcd} \right\} \\
\Psi^{-1}(\Gamma) \arrow[u, hook]
\end{tikzcd}
$$

<u>Case 1</u>: The puncture in Γ is not in one of the first p cycles.

Then $\Psi/\Gamma \xrightarrow{\simeq} \Psi^{-1}(\Gamma)$.

and $\Psi^{-1}(\Gamma) \cong \mathcal{F}at^{oc}($  $)$

whose classifying space is $\simeq S^1$.

<u>Case 2</u>: $\Psi/\Gamma \xrightarrow{\simeq} \Psi^{-1}(\Gamma) \simeq \mathcal{F}at^{oc}($  $)$

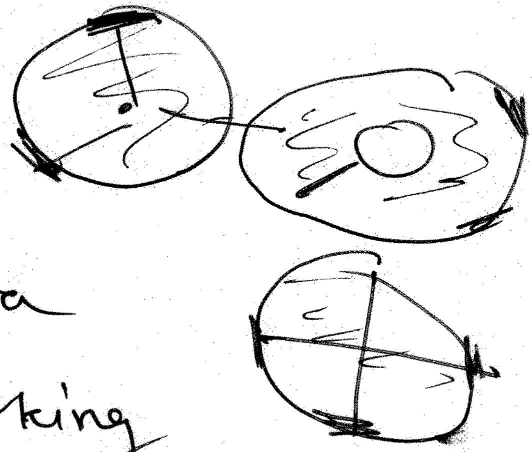
$\simeq S^1$

$\rightsquigarrow$ get fibration

$$
\begin{tikzcd}
S^1 \arrow[r] \arrow[d, equal] & \mathcal{F}at^{a_{p+1}}(S_{g,q+1}) \arrow[r] \arrow[d, "\simeq\,!"] & \mathcal{F}at_1^{a_p}(S_{g,p}) \arrow[d, "\simeq"] \\
S^1 \arrow[r] & B\Gamma_{g,p+1} \arrow[r] & B\Gamma^1_{g,q}
\end{tikzcd}
$$

Relation to Costello

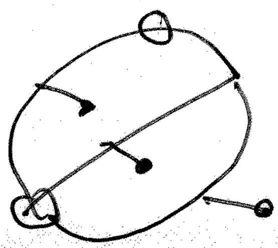
Looks like



$\mathcal{F}at_p(S_{g,q})$ + extra circle marking

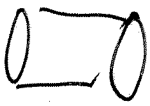
$\Longleftrightarrow$

Fat graphs with p punctures



Gluing?

$$\begin{array}{ccc} G_1 \; (8) & \cdot & 0 \; G_2 \\ \downarrow \sim & & \downarrow \sim \\ S_1 & & S_2 \end{array}$$



$$\begin{array}{c} \mathcal{G}_{S_1,S_2} \longrightarrow \mathcal{F}at(S_1 * S_2) \\ \downarrow \simeq \\ \mathcal{F}at(S_1) \\ \times \\ \mathcal{F}at(S_2) \end{array}$$

subcategory of "gluable graphs"