# MODELS FOR $(\infty, n)$-CATEGORIES

PHILOSOPHY

ALEXANDER

" $(\infty,n)$-CATEGORY = $\infty$-CATEGORY WHERE $k$-MORPHISMS ARE INVERTIBLE FOR $k>n$. "

" $(\infty,0)$-CATEGORY = $\omega$-GROUPOID = KAN COMPLEX "

" COBORDISM HYPOTHESIS: $\mathcal{C}$ SYMM. MONOIDAL $n$-CAT
THE $(\infty,n)$-CAT $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \cong \ldots$ "

<u>NEED</u>: GOOD THEORY OF $(\infty,n)$-CATEGORY, IN PARTICULAR CARTESIAN CLOSED $\mathcal{C}\times\mathcal{D}$

$\mathrm{Fun}(\mathcal{C},\mathcal{D})$: $(\infty,n)$-CAT OF FUNCTORS

$$\mathcal{C}\times\mathcal{D}\to\mathcal{E} \quad \leftrightarrow \quad \mathcal{C}\to\mathrm{Fun}(\mathcal{D},\mathcal{E})$$

" <u>HOMOTOPY HYPOTHESIS</u>: (GROTHENDIECK'S DREAM) "

THE FUNDAMENTAL $n$-GROUPOID $X \longmapsto \pi_{\leq n} X$

ESTABLISHES A BIJECTION

0 points
1 paths
2 homotopies
⋮
$n$ homotopy classes of homotopies

$n$-TYPES UP TO W.E. $\longleftrightarrow$ $n$-GROUPOIDS $= (n,0)$-CAT UP TO EQUIV.

$(\pi_i(X)=0 \; i>n)$

(WOULD LIKE ANY DEFINITION OF WEAK $n$-CAT TO SATISFY THIS)

LET $n\to\infty$

HOMOTOPY TYPES $\longleftrightarrow$ $\omega$-GROUPOIDS

YET ANOTHER MOTIVATION FOR WHY $(\infty,0)$-CAT = SPACES.

A WEAK $n$-CATEGORY SHOULD BE A CATEGORY "WEAKLY ENRICHED" IN $(n-1)$-CATEGORIES.

AN $(\infty,n)$-CATEGORY SHOULD BE A CATEGORY "WEAKLY ENRICHED" IN $(\infty,n-1)$-CATEGORIES.

STRICT ENRICHMENTS — NO GOOD BECAUSE STRICT n-CATEGORIES DO NOT SATISFY THE HOMOTOPY HYPOTHESIS.

TWO APPROACHES:

REZK: $\Theta_n$-SPACES = ITERATIVE VERSION OF COMPLETE SEGAL SPACES

SIMPSON: n-FOLD SEGAL CATEGORIES : ITERATION OF SEGAL CATEGORIES.

BOTH GIVE CARTESIAN MODEL STRUCTURES WHERE FIBRANT OBJECTS ARE MODELS FOR $(\infty, n)$-CATEGORIES

WILL CONCENTRATE ON REZK'S APPROACH.

PLAN:
- REVIEW COMPLETE SEGAL SPACES [REZK '00]
- CARTESIAN PRESENTATION OF MODEL CATEGORIES [DUGGER '01]
- $\Theta_n$-SPACES [REZK '09]

COMPLETE SEGAL SPACES – $(\infty, 1)$-CATEGORIES

NOTATION: $Sp = sSets$ = "SPACES"

$\forall \mathcal{C}$ small cat, $sPSh(\mathcal{C}) = Sp^{\mathcal{C}^{op}}$ = "SIMPLICIAL PRESHEAVES"

INJECTIVE MODEL STRUCTURE ON $sPSh(\mathcal{C})$:
- ~~LEVELWISE~~ POINTWISE w.e. AND COFIBRATIONS

START WITH $\mathcal{C} = \Delta$

$sPSh(\Delta)$ = SIMPLICIAL SPACES.

LET $\mathcal{D}$ BE A SMALL CATEGORY. LET $N\mathcal{D}$ = BISIMPLICIAL NERVE OF CLASSIFYING DIAGRAM

THE BICATEGORY WITH
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\simeq"'] & \bullet \arrow[d, "\simeq"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
VERTICAL MORPH = ISO'S
HORIZONTAL MORPH = ANY

$N\mathcal{D}_m = \mathrm{nerve}(\mathrm{iso}\, \mathcal{D}^{[m]})$ $\qquad [m] = 0 \to 1 \to \dots \to m$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\cong"'] & \bullet \arrow[r] \arrow[d, "\cong"] & \dots \arrow[r] & \bullet \arrow[d, "\cong"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \dots \arrow[r] & \bullet \end{tikzcd}$$

(BY WALDHAUSEN'S SWALLOWING LEMMA,
$|N\mathcal{D}| \simeq B\mathcal{D}$ )

<u>THM</u> (REZK '00) $f : \mathcal{D} \to \mathcal{E}$ IS AN EQUIV OF CATEGORIES
$\iff N\mathcal{D} \xrightarrow[Nf]{} N\mathcal{E}$ WEAK EQUIV. IN $sPSh(\Delta)$, INJECTIVE MODEL STRUCT
(i.e. LEVELWISE EQUIV. )

- $N\mathcal{D}$ IS ALWAYS FIBRANT IN THE INJECTIVE MODEL STRUCT
(~ NERVE OF A GROUPOID IS FIBRANT )
- $N\mathcal{C}_m \xrightarrow[\text{SEGAL MAP}]{\cong} N\mathcal{C}_1 \times_{N\mathcal{C}_0} \dots \times_{N\mathcal{C}_0} N\mathcal{C}_1 = \lim \left( N\mathcal{C}_1 \xrightarrow{d_0} N\mathcal{C}_0 \xleftarrow{d_1} N\mathcal{C}_1 \dots \mathcal{C}_1 \right)$

THERE IS AN ADJUNCTION
$$sPSh(\Delta) \underset{N}{\overset{L}{\rightleftarrows}} Cat$$

$N$ IS FULLY FAITHFUL.

<u>DEFINITION</u>: $X \in sPSh(\Delta)$ IS A SEGAL SPACE IF
- $X$ FIBRANT IN $sPSh(\Delta)^{inj}$ (IMPLIES $d_0, d_1$ FIBRATIONS → CAN TAKE PULL-BACKS INSTEAD OF HTPY PB)
- THE SEGAL MAP $X_\ell \to X_1 \times_{X_0} \dots \times_{X_0} X_1$ IS W.E. OF SIMPLICIAL SETS

$X \mapsto X_\ell$ REPRESENTABLE : $\quad \mathrm{Map}(F(\ell), X) \xrightarrow[\alpha^*]{} \mathrm{Map}(G(\ell), X)$
$\mapsto X_1 \times_{X_0} \dots \times_{X_0} X_1$

AND $\alpha : G(\ell) \to F(\ell)$ MAP IN $sPSh(\Delta)$.

<u>EXAMPLE</u>: $N\mathcal{C}$ IS ALWAYS A SEGAL SPACE.

LET $X$ BE A SEGAL SPACE.

$Obj(X) = X_{0,0}$ = VERTICES OF $X_0$

$x_0, \dots, x_m \in Obj(X)$, DEFINE

$$\begin{tikzcd}
map_X(x_0,\dots,x_m) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[r, "X({[0]} \xrightarrow{i} {[m]})"] \arrow[d] & X_0 \\
* \arrow[r, "(x_0,\dots,x_m)"] & \underbrace{X_0 \times \dots \times X_0}_{m+1} \arrow[ur, "pr_i"'] &
\end{tikzcd}$$

- IF $X = N\mathcal{D}$, THEN $map_X(x,y) \cong Hom_{\mathcal{D}}(x,y)$.
- $$\begin{tikzcd}
map_X(x_0,x_1,x_2) \arrow[r, "d_1"] \arrow[d, "s"', "(d_0{,}d_2)" , "\simeq"] & map_X(x_0;x_2) \\
map_X(x_1,x_2) \times map_X(x_0,x_1) &
\end{tikzcd}$$

$(g,f)$ $\qquad g \circ f = d_1(\xi)$

$f \sim g \in map_X(x,y)$ IF THEY LIE IN THE SAME PATH COMPONENT.

COMPOSITION IS WELL-DEFINED AND ASSOCIATIVE UP TO ~~HOMOT~~.

SO WE GET A CATEGORY $hX = \begin{cases} Obj(hX) = Obj(X) \\ Hom_{hX}(x,y) = \pi_0 \, Map_X(x,y) \end{cases}$

<u>FACT</u>: $X$ SEGAL SPACE $\Rightarrow LX \cong hX$

LEFT ADJOINT TO $N$

$map_X(x,y)$ IS A KAN COMPLEX $= (\infty, 0)$-CAT.

$g \in map_X(x,y)$ is a HOMOTOPY EQUIVALENCE if $[g]$ $\in \mathrm{Hom}_{hX}(x,y)$ is an isomorphism.

FACT: If $g \in X_{1,0}$ can be connected by a path in $X_1$ to a homotopy equiv, then $g$ is also a htpy equiv.

Let $X_{hoequ} \subseteq X_1$ union of all components containing htpy equiv.

$$s_0 : X_0 \longrightarrow X_{hoequ} \subseteq X_1$$
$$s_0 x = id_x$$

Def: A Segal space $X$ is complete $\overset{DEF}{\Leftrightarrow}$ $X \xrightarrow{s_0} X_{hoequ}$ is a weak equiv.

- $N\mathcal{D}$ is always complete

TFAE:
- $X$ complete
- $x, y \in Obj(X)$ the space $hoequ(x,y)$ is weakly equiv to the space of paths in $X_0$ from $x$ to $y$.
- Let $E = \underbrace{N^\delta}_{\text{discrete nerve}}(\bullet \underset{\cong}{\overset{\cong}{\rightleftarrows}} \bullet)$

discrete nerve: $N^\delta(\mathcal{D})_m = \text{NERVE}(id\, \mathcal{D}^{[m]})$

$$\bullet \rightleftarrows \bullet \longrightarrow \bullet$$

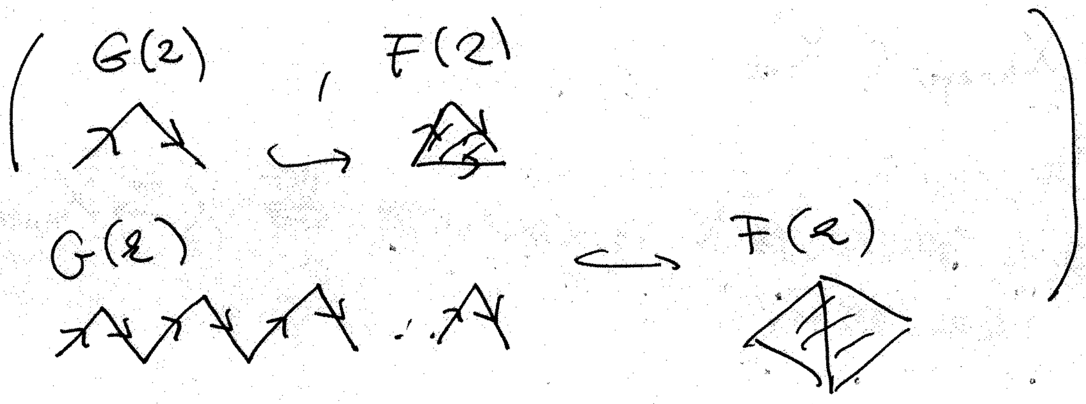
$$\rightsquigarrow E \xrightarrow[\varepsilon]{} (F(0) =) N^\delta([0])$$

$$Map(F(0), X) \xrightarrow{\varepsilon^*} Map(E, X) \text{ is a w.e.}$$

UPSHOT: COMPLETE ...

OBJECTS $X \in sPSh(\Delta)$ WITH

- $X$ FIBRANT IN $sPSh(\Delta)^{inj}$
- $Map(F(\ell),X) \xrightarrow[\alpha^*]{\simeq} Map(G(\ell),X)$
- $Map(F(0),X) \xrightarrow[\varepsilon^*]{\simeq} Map(E,X)$




## PRESENTATION OF MODEL CATEGORIES

(DUGGER, UNIVERSAL HOMOTOPY THEORIES)

A PRESENTATION IS A PAIR $(\mathcal{C}, \mathcal{S})$ WHERE

$\mathcal{C}$ = SMALL CATEGORY

$\mathcal{S}$ = SET OF MORPHISMS IN $sPSh(\mathcal{C})$

$X \in sPSh(\mathcal{C})$ IS $\mathcal{S}$-LOCAL IF FIBRANT IN $sPSh(\mathcal{C})^{inj}$ AND

$s^*: Map(S',X) \xrightarrow[\text{(in sSets)}]{\simeq} Map(S,X) \quad \forall \underset{s: S \to S'}{s \in \mathcal{S}}$

$f: A \to B$ IN $sPSh(\mathcal{C})$ IS AN $\mathcal{S}$-EQUIV IF

$f^*: Map(B,X) \xrightarrow{\simeq} Map(A,X) \quad \forall\ \mathcal{S}\text{-LOCAL } X$

$\bar{\mathcal{S}} = \{\mathcal{S}\text{-EQUIVALENCES}\} \supseteq \mathcal{S}.$

PROPOSITION: GIVEN A PRESENTATION $(\mathcal{C}, \mathcal{J})$, THERE IS A COFIBRANTLY GENERATED, LEFT PROPER, SIMPLICIAL MODEL STRUCTURE ON $sPSh(\mathcal{C})$ CHARACTERIZED BY

- WE = $\mathcal{J}$-EQUIV
- COFIBRATIONS = MONOMORPHISMS
- FIBRANT OBJECTS = $\mathcal{J}$-LOCAL OBJECTS

FURTHERMORE,

- $X \to Y$ LEVELWISE EQUIV $\Rightarrow$ $\mathcal{J}$-EQUIV.
  CONVERSE TRUE IF $X, Y$ LOCAL.
- AN $\mathcal{J}$-FIBRATION $X \to Y$ IS AN INJECTIVE FIBRATION
  CONVERSE HOLDS IF $X, Y$ $\mathcal{J}$-LOCAL.

LET $sPSh(\mathcal{C})_{\mathcal{J}}$ DENOTE THIS MODEL CATEGORY.

A PRESENTATION IS CARTESIAN IF ANY OF THE FOLLOWING EQUIVALENT CONDITIONS HOLD:

- $f, g \in \bar{\mathcal{J}} \to f \times g \in \bar{\mathcal{J}}$
- THE MODEL CATEGORY $sPSh(\mathcal{C})_{\mathcal{J}}$ IS CARTESIAN:
  - PRODUCT BEHAVES WELL WRT COF
  + INNER HOM'S $Y^X$

$X \times Y \to Z \quad \Leftrightarrow \quad X \to Z^{\cancel{X}}$

CSS = MODEL CAT OF COMPLETE SEGAL SPACES

$= sPSh(\Delta)_{\mathcal{J}}$ FOR $\mathcal{J} = Se \cup Cpt$

$Se = \{\alpha : G(k) \to F(k)\}_{k \geq 2}$

$Cpt = \{F(0) \to E\}$

"COMPLETE"

## $\Theta_n$-SPACES AND $(\infty,n)$-CATEGORIES [REZK '09]

$(\infty,1)$-CATEGORIES = COMPLETE SEGAL SPACES
= FIBRANT OBJECTS IN $sPSh(\Delta)_{\mathcal{J}}$

$(\infty,n)$-CATEGORIES = FIBRANT OBJECTS IN $sPSh(\Theta_n)_{\mathcal{J}_{n,\infty}}$

FOR A CARTESIAN PRESENTATION $(\Theta_n, \mathcal{J}_{n,\infty})$.

$\Theta_1 = \Delta$, $\mathcal{J}_{1,\infty} = Se \cup Cpt$

$\Theta_{n+1} = \Delta \wr \Theta_n = \underbrace{\Delta \wr \cdots \wr \Delta}_{n+1}$ ITERATED WREATH PRODUCT

<u>INDUCTIVE</u> STEP: GIVEN A CARTESIAN PRESENTATION $(\mathcal{C},\mathcal{J})$,
DEFINE A NEW CARTESIAN PRESENTATION $(\Delta \wr \mathcal{C}, \mathcal{J}_\Theta)$:

$\mathcal{C}$ SMALL CATEGORY, $\Theta\mathcal{C} = \Delta \wr \mathcal{C} = \Delta_{\mathcal{C}}$ IS "$\mathcal{C}$-COLORED VERSION OF $\Delta$":

$\mathrm{Obj} := \{[n](c_1,\dots,c_n) \mid n \geq 0,\ c_i \in \mathrm{Obj}(\mathcal{C})\}$

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} 2 \to \cdots \xrightarrow{c_n} n$$

$\mathrm{Mor}([m](c_1,\dots,c_m),[n](d_1,\dots,d_n)) = \{(\delta, \{f_{ij}\})\}$

$\delta : [m] \to [n]$

$f_{ij} : c_i \to d_j$ in $\mathcal{C}$ if $\delta(i-1) < j$

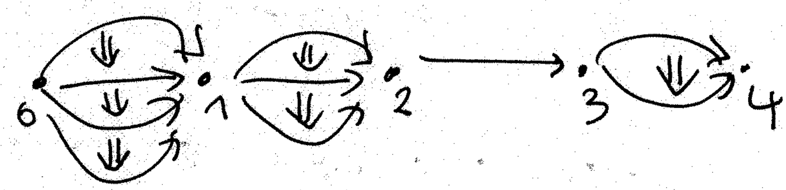
EX: $[3](c_1,c_2,c_3) \to [4](d_1,d_2,d_3,d_4)$

$$\begin{tikzcd}
0 \arrow[r, "c_1"] \arrow[d] \arrow[dr, phantom, "\rightsquigarrow f_{11}"] & 1 \arrow[r, "c_2"] \arrow[drr] \arrow[dr, phantom, "\rightsquigarrow f_{12}"] & 2 \arrow[r, "c_3"] \arrow[d] & 3 \arrow[d] \arrow[dl, phantom, "\rightsquigarrow f_{33}"] & \\
0 \arrow[r, "d_1"] & 1 \arrow[r, "d_2"] & 2 \arrow[r, "d_3"] & 3 \arrow[r, "d_4"] & 4
\end{tikzcd}$$

<u>NOTE</u>: $\Delta \wr * = \Delta$.

$\Theta_2 = \Delta \wr \Delta$ $\qquad [4]([3],[2],[0],[1]) \in \Theta_2$

FREE STRICT 2-CAT ON DIAGRAM




MORE GENERALLY, $\Theta_n \hookrightarrow$ STRICT n-CATEGORIES

EXTENDING $\Delta \hookrightarrow$ CATEGORIES

TAKING PRESHEAVES, WE ADD HOMOTOPY COLIMITS ...

~~[illegible]~~

$J_\Theta = Se_e \cup Cpt_e \cup V(Se \cup Cpt)$

SEGAL CONDITIONS ON THE TOP THINGS

"PROPAGATES THE CONDITIONS TO LOWER CELLS" (?)