DAVID AYALA — BORDISM HYPOTHESIS I

GOAL:
- DEFINE ⊗ (∞,n)-CAT $Bord_n$ [INSPIRED BY LURIE AND TILLMANN
- DEFINE NOTION OF DUALIZABILITY
- → STATE BORDISM HYPOTHESIS*
- SAY SOMETHING ~~ABOUT~~ THE PROOF.

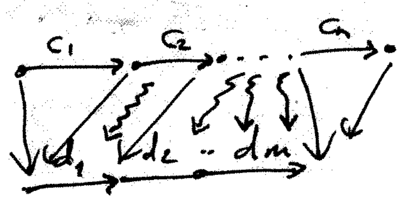
A SYMM MONOIDAL (⊗) (∞,n)-CAT IS A FUNCTOR

$$\Gamma^{op} \xrightarrow{\ \mathcal{C}\ } Cat_{(\infty,n)} \xhookrightarrow[\text{FIBRANT}]{} sPsh(\Theta_n)^{inj}_{\mathcal{C}_{n,\infty}}$$

'' CAT OF BASED FINITE SETS, BASED MAPS

$$\Theta_n = \Delta \wr \cdots \wr \Delta$$

$\Delta \wr C =$



SUCH THAT

- $\mathcal{C}(+) \simeq *$, $+ = [0_+]$
- $\mathcal{C}(A_+) \xrightarrow{\ \simeq\ } \prod_{i=1}^{n} \mathcal{C}(\{a\}_+)$

ACTUALLY, ∃ CARTESIAN MODEL STRUCTURE ON $\left(sPsh(\Theta_n)^{inj}_{\mathcal{C}_{n,\infty}}\right)^{\Gamma^{op}}$ FOR WHICH ⊗-(∞,n)-CATS ARE FIBRANT OBJECTS, BECAUSE THE CONDITIONS ARE LOCALIZATIONS (AS IN DUGGER)

EX: $n=1$, $\mathcal{D}$ AN HONEST ⊗-CAT ⇝ ⊗-$(\infty,1)$-CAT:

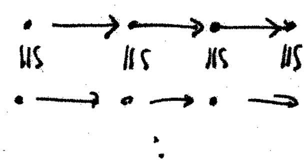
$$\mathcal{D}^{(n)} \simeq \underbrace{\mathcal{D}\times\cdots\times\mathcal{D}}_{n}$$

$$\Gamma^{op} \longrightarrow Cat \xrightarrow{\ N.\ } Psh(\Delta) \longrightarrow sPsh($$

$$n_+ \longmapsto \mathcal{D}^{(n)}$$

SETS → SPACES

$N = $ Nerve(iso —) ⇝ COMPLETE SEGAL SPACE



$$\Psi_n^{\delta} : \mathcal{O}(V)^{op} \longrightarrow Set$$

V fin dim Vect/$\mathbb{R}$

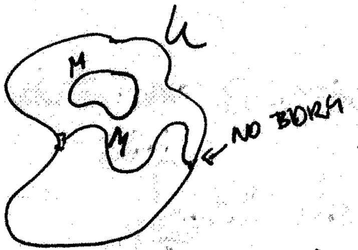
POSET OF OPEN SUBSETS

V = fdim vect $/\mathbb{R}$

$\mathcal{O}(V)$ = POSET OF OPEN SUBSETS OF V

$$\Psi_n^\delta : \mathcal{O}(V)^{op} \longrightarrow \mathrm{Set} \qquad \text{SHEAF}$$

$$U \longmapsto \Psi_n^\delta(U) := \left\{ \underset{\substack{\text{SUBMFD} \\ \dim n \\ \partial M = \emptyset}}{M \subset U} \;\middle|\; \begin{array}{l} M \text{ CLOSED AS} \\ \text{A SUBSET} \end{array} \right\}$$

EX: 

Ψ IS A SHEAF SINCE

$\{U_\alpha\}_{\alpha \in A}$ OPEN COVER OF U AND

$$M_\alpha \in \Psi_n^\delta(U_\alpha) \text{ with } M_\alpha|_{U_\alpha \cap U_\beta} = M_\beta|_{U_\alpha \cap U_\beta}$$

$$\rightsquigarrow M = \bigcup_\alpha M_\alpha \subset U .$$

$$\Psi_n : \mathcal{O}(V)^{op} \longrightarrow \mathrm{Spaces} \qquad \text{SHEAF}$$

$$\Psi_n(U) \overset{?}{=} \coprod_{\substack{[M^n \text{ with} \\ \partial M = \emptyset]}} \mathrm{Emb}^{proper}(M, U) / \mathrm{Diff}(M)$$

USE THE TOPOLOGY GENERATED BY

$C(K, U)$ WHERE $K \subset U$ COMPACT

$(M \hookrightarrow U) \in W \subset \mathrm{Emb}^{prop}(M, U)$

$$C(K, W) := \left\{ M' \subset U \;\middle|\; \begin{array}{l} \exists j \in W \text{ with} \\ j(M) \cap K = M' \cap K \end{array} \right\}$$

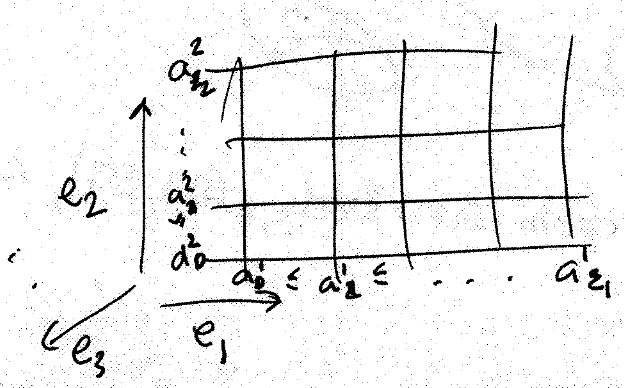
($\rightarrow$ CAN PUSH THINGS TO INFINITY)

$\mathbb{R}^\delta$ ($\mathbb{R}$ AS POSET, DISCRETE TOP) $\rightsquigarrow$ $\Delta^{op} \xrightarrow{R_1} \mathrm{Set}$

$$[n] \longmapsto \mathbb{R}^{n+1}_{\leq} = \{(a_0 \leq \dots \leq a_n)\}$$

$$\rightsquigarrow \quad \underbrace{(\Delta\times\dots\times\Delta)^{op} \xrightarrow{R_1\times\dots\times R_1} \underbrace{Set\times\dots\times Set}_{n} \longrightarrow Set}_{R_n}$$

$$([k_1],\dots,[k_n]) \longmapsto \left\{ (\underline{a}^1,\dots,\underline{a}^n) \;\middle|\; \begin{array}{l} \underline{a}^i \in \mathbb{R}_{\le}^{k_i+1} \\ \forall\, 1\le i\le n \end{array} \right\}$$

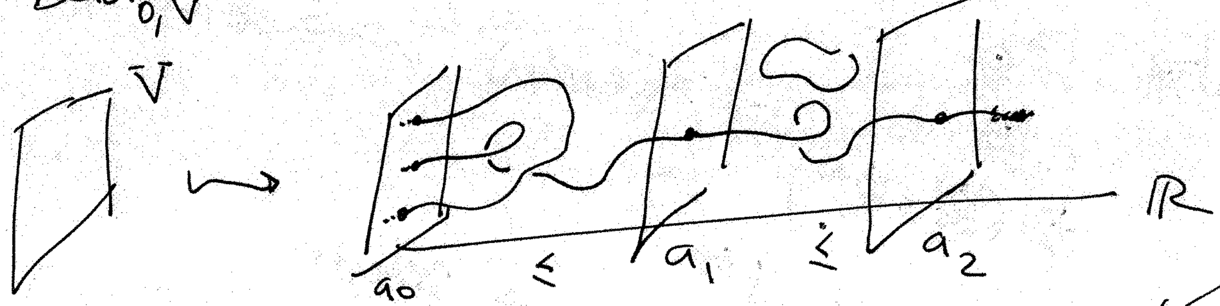


DEFINE $\mathrm{Bord}_{n,V} : (\Delta\times\dots\times\Delta)^{op} \longrightarrow$ SPACES

$$([k_1],\dots,[k_n]) \longmapsto \left\{ (\underline{a}^1,\dots,\underline{a}^n; M^n) \;\middle|\; \begin{array}{l} M \in \Psi_n\left(\prod_{i=1}^n [a^i_0, a^i_{k_i}] \times V\right) \\ \text{s.t. } (1), (2), (3) \end{array} \right\}$$

$\underline{n=0}$: $\mathrm{Bord}_{0,V}$ = THE SPACE $\Psi_0(V)$ CONFIGURATIONS IN V

$\underline{n=1}$:

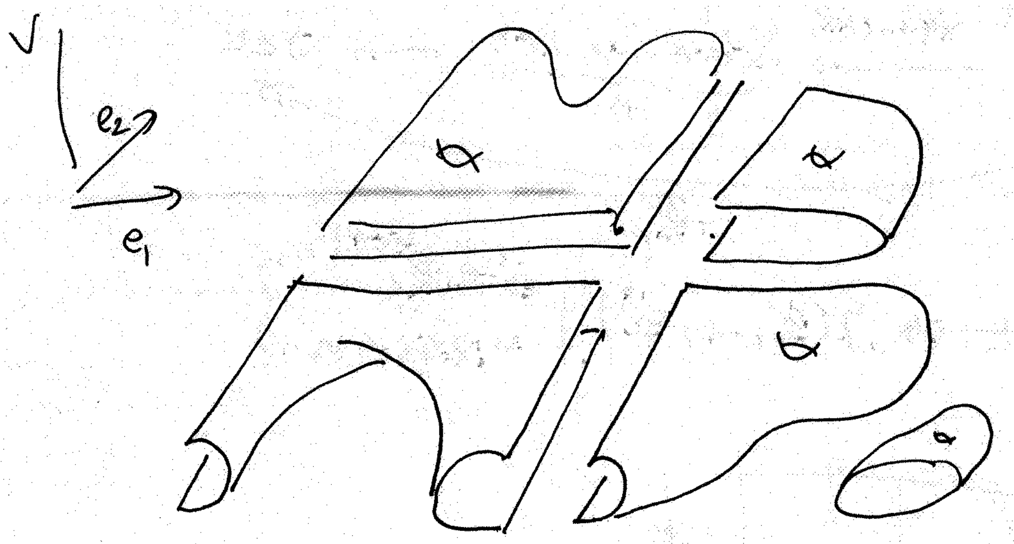


(1) "COMPACT": $M \hookrightarrow \mathbb{R}^n \times V \xrightarrow{pr} V$ PROPER

(2) "TRANSVERSE": $\forall\, S \subset \{1,\dots,n\}$ AND $(1 \le j_i \le k_i)_{i\in S}$

$$M \xrightarrow{pr} \mathbb{R}^S \text{ IS TRANSVERSE TO } (a_{j_i})_{i\in S} \in \mathbb{R}^S$$

(3) "UNIT": $k_i = 0$ FOR SOME $1 \le i \le n \implies M \xrightarrow{pr} \mathbb{R}^{\{i+1,\dots,n\}}$ IS A SUBMERSION



$\in \mathrm{Bord}_{2,V}([2],[2])$

CONDITION (3): IF WE RESTRICT TO SAY $\mathrm{Bord}_{2,V}([2],[0])$, GET A PRODUCT MFD.

<u>VARIANTS</u>: EACH $M \in \Psi_n(\mathcal{U})$, $\mathcal{U} \subset \mathbb{R}^n \times V$ DETERMINES A WELL-DEFINED MAP

$$\begin{aligned} M &\xrightarrow{\tau_M} Gr_n(\mathbb{R}^n \times V) \\ x &\longmapsto T_x M \end{aligned}$$

SUPPOSE $V \subset \mathbb{R}^\infty$.

FOR
$$\begin{array}{c} B\theta \\ \downarrow \theta \\ Gr(\mathbb{R}^n \times \mathbb{R}^\infty) \end{array}$$
A FIBRATION $\rightsquigarrow$ DEFINE A NEW SHEAF

$$\Psi_n^\theta : \mathcal{O}(\mathbb{R}^n \times V)^{op} \longrightarrow \text{SPACES}$$

$$\mathcal{U} \longmapsto \left\{ (M, \ell) \;\middle|\; M \in \Psi_n(\mathcal{U}),\; \begin{array}{ccc} & & B\theta|_V \\ & \overset{\ell}{\nearrow} & \downarrow \theta \\ M & \longrightarrow & Gr_n(\mathbb{R}^n \times V) \end{array} \right\}$$

WITH NATURAL TOPOLOGY AS BEFORE.

<u>EX</u>: θ = ORIENTATION, FRAMING, ...

DEFINE $\mathrm{Bord}^{\vee}_{n,V}$ AS BEFORE AND

$$\mathrm{Bord}^{\theta}_{n} := \operatorname*{colim}_{\substack{V \in \text{fin dim SUBSP} \\ \text{OF } \mathbb{R}^{\infty}, \text{ INCLUSION}}} \mathrm{Bord}^{\theta}_{n,V}$$

$$\mathrm{Bord}^{\theta}_{n} \in s\mathrm{Psh}(\underbrace{\Delta \times \dots \times \Delta}_{n})$$

s ← TAKING SINGULAR CPX TO GET SIMPLICIAL SET.

NOTE: DIMENSION OF GRID = DIM OF MFD SO THAT OBJECTS WILL CORRESPOND TO POINTS, MORPH TO 1-MFDS ETC.

FACT: $\mathrm{Bord}_{n}([1], \dots, [1]) \simeq \coprod_{[M^n, (\pm)^n]} BDiff(M, (\pm)^n)$

M^n: WITH CORNERS

$BDiff(M, (\pm)^n)$: DIFFEO PRESERVE DIRECTIONS, BDRIE ALLOWED TO MOVE INSIDE SUBSPACE

$$\underbrace{(\mathrm{SPsh}(\Theta_n);\ \mathrm{inj},\ J_{n,\infty})}^{\text{REZK}}$$

- PRESENTED (EASY TO VERIFY FIBRANT OBJECTS)
- CARTESIAN CLOSED!

VIOLENT: CHOPPING AND h-COBORDISM COMPLETING

$$\mathrm{SPsh}(\Delta \times \dots \times \Delta)$$

DISCRETIZING THE LOWER MORPHISMS — TOO RIGID TO PROVE THE BH?

VIOLENT! "GROUP COMPLETE WRT TO h-COBORDISMS"

$$\underbrace{(\mathrm{SPsh}(\Delta \times \dots \times \Delta),\ \text{SEGAL } n\text{-CAT})}_{\text{SIMPSON-HIRSCHOWITZ}}$$

- NOT NECESSARILY PRESENTED
- CARTESIAN CLOSED

$$\underbrace{(\mathrm{SPsh}(\Delta \times \dots \times \Delta);\ n\text{-FOLD COMPLETE SEGAL SPA}}_{\text{LURIE}}$$

[REZK: PRESENTATION FOR THIS MODEL CATEGORY NOT CARTESIAN CLOSE!

AND WON'T PURSUE IT.

COULD ALSO TRY TO GO TO CHAIN CPXS OR SOMETHING ...

NOTE: "BORDISM HYPOTHESIS" MEANS THAT IF IT DOESN'T HOLD IN YOUR MODEL, YOUR MODEL IS BAD!

$$\Delta \times \mathcal{C} \longrightarrow \Delta \wr \mathcal{C}$$
$$([n], c) \longmapsto [n](c, \dots, c)$$

$$\Rightarrow \Delta \times \dots \times \Delta \longrightarrow \Delta \wr \dots \wr \Delta = \Theta_n$$

$$\begin{tikzcd} & \Theta_n \arrow[dr, dotted] & \\ \Delta \times \dots \times \Delta \arrow[ur] \arrow[rr, "\mathrm{Bord}_n"] & & \text{Spaces} \end{tikzcd}$$

Left Kan extension
" $\mathrm{Lan\,Bord}_n \in s\mathrm{PSh}(\Theta_n)$

DEFINE $\overline{\mathrm{Lan\,Bord}_n} :=$ FIBRANT REPLACEMENT OF $\mathrm{Lan\,Bord}_n$

IT'S ~~AN~~ (∞, n)-CATEGORY BY CONSTRUCTION.

$\wedge$ CAN WRITE DOWN AN EXPLICIT MODEL OF $\overline{\mathrm{Lan\,Bord}_n}$
(DAVID) (USING THE TOPOLOGY WHERE THINGS DISAPPEAR AT ∞ AND CHOPPING THINGS UP --)

WANT TO CONSIDER

$\mathrm{Hom}_{s\mathrm{PSh}(\Theta_n)}(\overline{\mathrm{Lan\,Bord}_n}, \mathcal{C})$ — (∞, n)-CAT

((∞, n)-CAT AS CARTESIAN CLOSED)

$\downarrow$ IS ($\mathcal{C}$ FIBRANT, $\mathrm{Lan\,Bord}_n \xrightarrow{\sim} \overline{\mathrm{Lan\,Bord}_n}$)

$\mathrm{Hom}_{s\mathrm{PSh}(\Theta_n)}(\mathrm{Lan\,Bord}_n, \mathcal{C})$ ISO OF SETS .... (HOMOTOPY VERSION?)

$\downarrow$ IIS (KAN EXTENSION)

$\mathrm{Hom}_{s\mathrm{PSh}(\Delta \times \dots \times \Delta)}(\mathrm{Bord}_n, \mathcal{C}|_{\Delta \times \dots \times \Delta})$

MIGHT BE AN n-FOLD COMPLETE SEGAL SPACE?

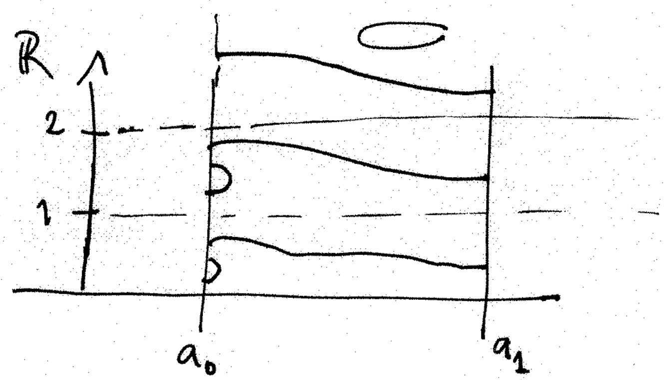
Bord_n AS A Γ-OBJECT

$$\otimes\mathrm{Bord}^{\otimes}_{n,V\times\mathbb{R}} : \Gamma^{op} \longrightarrow \mathrm{SPSh}(\Delta\times\dots\times\Delta)$$

$$\begin{pmatrix} \ell_+ \\ [k_1],\dots,[k_n]\end{pmatrix} \longmapsto \left\{(\underline{a}^1,\dots,\underline{a}^n; M) \in \mathrm{Bord}^{\otimes}_{n,V\times\mathbb{R}}([k_1],\dots,[k_n]) \;\middle|\; (*)\right\}$$

$$(*)\quad M\cap V\times \overbrace{\{1,\dots,\ell-1\}}^{\subset\mathbb{R}} = \emptyset$$

$V=0$
$\ell=3$
$n=1$
$k_1=1$



<u>QUESTION</u>: How does Γ^{op} fit in the previous diagram?