HOSSEIN ABBASPOUR: - $(\infty,0)$ - CATEGORIES
- $(\infty,1)$ - CATEGORIES

STRICT HIGHER CATEGORY:

DEF: AN ENRICHED CATEGORY $\mathcal{C}$ OVER A MONOIDAL CATEGORY $(M, \otimes, I)$ IS A CATEGORY WITH THE FOLLOWING DATA:

$\mathrm{OBJECTS}(\mathcal{C}) = X, Y, Z, \ldots$ (MAY BE NOT A SET)

$X \longmapsto Id_X \in \mathrm{Hom}_M(I, \mathrm{Hom}_{\mathcal{C}}(X,X))$

$\mathrm{Hom}_{\mathcal{C}}(X,X) \in \mathrm{Obj}(M)$

$(X,Y,Z) \longmapsto \alpha_{XYZ}: \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ IN M

+ SOME COMPATIBILITY CONDITIONS:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(W,X) \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \to & \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Z) \\ \downarrow & \circlearrowright & \downarrow \end{array}$$

COMMUTES UP TO ISO WHICH IS PART OF THE STRUCTURE

(NEED TO ASSUME WE HAVE A FUNCTOR $\overrightarrow{M \to \mathrm{Sets}}$?)

$0-\mathrm{Cat}$ = SET WITH CARTESIAN PRODUCT, MONOIDAL CATEGORY

$n-\mathrm{Cat}$ IS THE CATEGORY OF CATEGORIES ENRICHED OVER $(n-1)-\mathrm{CAT}$.

DEF: AN n-CATEGORY IS AN OBJECT OF $n-\mathrm{Cat}$.

NOTATION: AN n-CATEGORY IS GIVEN BY A DIAGRAM

$$C_n \rightrightarrows C_{n-1} \rightrightarrows \cdots \quad C_1 \rightrightarrows C_0 .$$

$n = \infty$: $\cdots \rightrightarrows C_n \rightrightarrows \cdots \quad C_0$

Def: $\cdots \rightrightarrows C_n \rightrightarrows C_{n-1} \rightrightarrows \cdots \rightrightarrows C_1 \rightrightarrows C_0$

IS AN ∞-CATEGORY IF

i) $\forall k > l$, $(C_k \rightrightarrows C_l)$ IS A CATEGORY

ii) $\forall k > l > m$, $(C_k \rightrightarrows C_l \rightrightarrows C_m)$ IS A 2-CATEGORY

EQUIVALENT TO $\forall n$, $[C_n \rightrightarrows \cdots \rightrightarrows C_0]$ IS AN n-CATEGORY ?

[BAEZ]

EX:
$$\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta"] & z \end{tikzcd}$$

WEAK HIGHER CATEGORY:

TWO IDEAS: 1) WEAK n-CATEGORY: ENRICH THEN RELAX THE CONDITIONS

NOT GOOD ENOUGH

2) WEAKLY ENRICHED OVER WEAK CATEGORIES

NEED TO SOMEHOW DO THAT...

$(\infty, 0)$-CATEGORY:

KAN COMPLEX: LET $S = \{S_n\}$ BE SIMPLICIAL SET

i-HORNS $\Lambda^i_n(S) = \left\{ \{\sigma_j\}_{\substack{0 \le j \le n \\ j \ne i}} \;\middle|\; \sigma_j \in S_{n-1},\ d_j \sigma_k = d_{k-1} \sigma_j \in S_{n-2},\ j < k \right\}$

EX: $\Lambda^1_2(\Delta[2]) = $
$$\begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 0 & & 2 \end{tikzcd}$$

RESTRICTION MAP: $\forall i$, $0 \le i \le n$:
$$S_n \longrightarrow \Lambda^i_n(S)$$
$$z \longmapsto (d_0 z, d_1 z, \ldots, \widehat{d_i z}, \ldots, d_i z)$$

Def: S IS A KAN COMPLEX IF ALL RESTRICTION MAPS ARE SURJECTIVE $\forall i$

~~EX~~ERCISE! $S = \{S_n\}$ IS ISOMORPHIC TO THE NERVE OF A CATEGORY IF ALL RESTRICTION MAPS ARE BIJECTIONS $\forall\ 0<i<n$ (GROUPOID IF ALSO FOR $i=0,n$)

CONSTRUCTION:

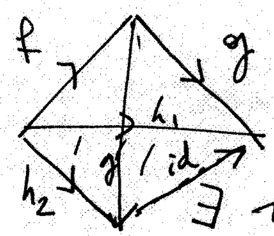
$$S = \{S_n\} \text{ KAN COMPLEX } \xrightarrow[\text{ASSOCIATE}]{} \mathcal{C} \quad \underset{(\infty,0)}{\overset{\shortparallel}{\infty\text{-CATEGORY}}}$$

$$\mathrm{Obj}(\mathcal{C}) = S_0$$

$$1\text{-}\mathrm{Mor}(\mathcal{C}) = S_1 \ni f, g$$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, dashed, bend right] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

$\exists\, h$ BY KAN CONDITION, NOT UNIQUE BUT WELL-DEFINED UP TO HOMOTOPY:



$\exists$ HOMOTOPY BY KAN CONDITION

(CHOICE OF h IS A CONTRACTIBLE SIMPLICIAL SET.)

$f \in 1\text{-}\mathrm{Mor}$ IS WEAKLY INVERTIBLE:

$$\begin{tikzcd} & \bullet \arrow[dr, dotted, "\exists f^{-1}"] & \\ \bullet \arrow[ur, "f"] & & \bullet \arrow[ll, "id"] \end{tikzcd}$$

A <u>MODEL CATEGORY FOR $(\infty,0)$-CATEGORIES</u> IS GIVEN BY $\mathcal{C}$ WHERE

$\mathrm{Top} \overset{\langle\sim\rangle}{\simeq}$
- $\mathrm{Obj}(\mathcal{C}) = S\mathrm{Sets}$
- w.e. = WEAK HOMOTOPY OF GEOM REALIZ.
- Cof = LEVELWISE INJECTION

KAN COMPLEXES = FIBRANT OBJECTS IN $\mathcal{C}$

(WANT TO WORK WITH KAN COMPLEXES $\rightsquigarrow$ SOME CONSTRUCTIONS LEAVE KAN COMPLEXES $\rightsquigarrow$ WANT A MODEL CAT WITH FIBRANT REPLACEMENT TO GET BACK TO KAN COMPLEXES
$\shortparallel$
FIBRANT OBJECTS)

AN $(\infty,0)$-CATEGORY IS A FIBRANT OBJECT IN $\mathcal{C}$.

CHANGING THE MODEL CATEGORY (FOR TOP) CHANGES THE MODEL FOR $(\infty,0)$-CATEGORIES.

$(\infty,1)$-CATEGORY

SUPPOSE WE HAVE DEFINED SUCH A THING, AND $\mathcal{C}$ ← HAVE ONE
(k-MORPHISMS ARE INVERTIBLE FOR $k \geq 2$)

$\mathcal{C} \rightsquigarrow \mathcal{C}_0$ A $(\infty,0)$-CATEGORY

$$\mathrm{Obj}(\mathcal{C}_0) = \mathrm{Obj}(\mathcal{C})$$

1-$\mathrm{Mor}_{\mathcal{C}_0}$ = INVERTIBLE 1-$\mathrm{Mor}_{\mathcal{C}}$

2-$\mathrm{Mor}_{\mathcal{C}_0}$ = 2-$\mathrm{Mor}_{\mathcal{C}}$ BETWEEN INVERTIBLE 1-$\mathrm{Mor}_{\mathcal{C}}$

$\mathcal{C}_0 \longmapsto X_0$ TOP SPACE

$\forall n$, $\mathrm{Fun}([n],\mathcal{C})$ $\quad [n] = \{0 < 1 < \dots < n\}$
is $(\infty,1)$-CAT ??
$\rightsquigarrow$ AS BEFORE
$(\infty,0)$-CAT
$\rightsquigarrow$
TOP SPACE X_n

~~REM~~: $X = \{X_n\}$ IS A SIMPLICIAL SPACE.

$$X_{n+m} \ni A_0 \xrightarrow{f_1} A_1 \longrightarrow \dots \xrightarrow{f_{n+m}} A_{n+m}$$

$$\swarrow \qquad \searrow$$

$$(A_0 \to A_1 \to \dots \to A_n) \text{ AND } (A_n \to \dots \to A_{n+m})$$

WANT THE TOP TO BE A HOMOTOPY FIBER PRODUCT OF THE BOTTOM TWO.

$\mathcal{C}$ IN $(\infty,1)$-CAT $\longrightarrow$

$$(**)\quad \begin{array}{ccc} X_{n+m} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & X_0 \end{array}$$

HOMOTOPY PULL-BAC[K]
(i.e. $X_{n+m} \xrightarrow{\text{w.e.}}$ HPull-[...]

$C(V,1)$ ... $\supset X = \{X_n\}$ ...

VERIFYING (**)

LOOSES SOME DATA

CSS = COMPLETE SEGAL SPACES

MODEL CATEGORY WITH ~~OBJECTS~~

Obj = SIMPLICIAL SPACES

SEGAL MAP: $\forall i \quad \alpha_i : [1] \longrightarrow [k]$, $0 \leq i < k$

$$\begin{aligned} 0 &\longmapsto i \\ 1 &\longmapsto i+1 \end{aligned}$$

$X = \{X_k\}$ S.SPACE

HTPY FIBER PRODUCT

$$\varphi_k : X_k \longrightarrow X_1 \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \cdots X_1 \quad \text{INDUCED BY } \{\alpha_i\}$$

DEF: X IS A <u>SEGAL SPACE</u> IF ALL φ_k'S ARE w.e.

<u>HOMOTOPY CATEGORY hX OF A SEGAL SPACE X</u>:

$\mathrm{Obj}(hX)$ = POINTS OF X_0

$$\mathrm{Hom}_{hX}(x,y) = \pi_0\left(\{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\}\right)$$

LET $f \in X_1$, $\quad x \xrightarrow{f} y \longmapsto [f] \in \mathrm{Hom}_{hX}(x,y)$

IN PARTICULAR, FOR $x \in X_0 \longmapsto \delta(x) \in X \qquad \delta : X_0 \to X$ DEG.

$$x \longmapsto [\delta(x)] \in \mathrm{Hom}_{hX}(x,x)$$

<u>DEF</u>: LET $Z \subseteq X_1$ BE THE SET OF ALL INVERTIBLE ELEMENTS (WHEN PASSING TO hX). WE SAY THAT $X = \{X_n\}$ IS A <u>COMPLETE SEGAL SPACE</u> IF $\delta : X_0 \to Z$ IS A w-e. (WEAK HTPY EQUIV)

THM [REZK] THERE IS A MODEL STRUCTURE

OBJ = SIMPLICIAL SPACES, MORPH OF SIMPL SPA

FIBRANT OBJ = COMPLETE SEGAL SPACE

(1) WE BETWEEN SEGAL SPACE ARE DK - EQUIV

(2) COFIBRATIONS = MONOMORPHISMS

(3)

WHERE DK-EQUIV $f : U \to V$ s.t. U, V SEGAL SPACES

i) $\forall x, y \in U_0$, $\mathrm{Map}_U(x,y) \to \mathrm{Map}_V(fx, fy)$ IS A W.E. OF SPACES

$\mathrm{Map}_U(x,y) = \{x\} \times_{U_0} U_1 \times_{U_1} \{y\}$

ii) INDUCED MAP $hf : hU \to hV$ IS AN EQUIV. OF CATEGORIES.