HOMOTOPY INVARIANTS OF DAVIS-JANUSZKIEWICZ SPACES AND MOMENT-ANGLE COMPLEXES

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Abstract. We give calculations of certain homotopy invariants of Davis-Januszkiewicz spaces $DJ(\Delta)$ and moment-angle complexes $\mathcal{Z}_\Delta$ associated to finite simplicial complexes Δ. For any Δ, we compute the Hilbert series of the loop space homology $\mathrm{H}_*(\Omega DJ(\Delta);k)$ with coefficients in any field k, we characterize when $\mathcal{Z}_\Delta$ is rationally homotopy equivalent to a wedge of spheres and we calculate the rational homotopy Lie algebra $\pi_*(\Omega\mathcal{Z}_\Delta)\otimes\mathbb{Q}$.

1. Introduction

Davis-Januszkiewicz spaces $DJ(\Delta)$ and moment-angle complexes $\mathcal{Z}_\Delta$ associated to finite simplicial complexes Δ, first introduced in [12], are spaces that have been studied intensely recently from the point of view of toric topology, see [4, 11, 14, 13, 17, 19, 20, 21, 22]. The purpose of this paper is to show how the results of [6, 5, 10, 9, 8] can be used to give calculations of certain homotopy invariants of these spaces. The main results are Theorem 2, which gives a computation of the Hilbert series of the loop space homology $\mathrm{H}_*(\Omega DJ(\Delta);k)$ for any field k, Theorem 6, which gives a characterization of when $\mathcal{Z}_\Delta$ is rationally homotopy equivalent to a wedge of spheres, and Theorem 13, which gives a description of the rational homotopy groups $\pi_*(\mathcal{Z}_\Delta)\otimes\mathbb{Q}$ and the Whitehead product on these.

Both spaces $DJ(\Delta)$ and $\mathcal{Z}_\Delta$ are examples of 'generalized moment-angle complexes' or 'polyhedral products' (cf. [4]): Suppose that the simplicial complex Δ has vertex set $[n]=\{1,2,\ldots,n\}$. For a pair of topological spaces $A\subseteq X$, the polyhedral product is defined as

$$(X,A)^\Delta=\bigcup_{\sigma\in\Delta}(X,A)^\sigma$$

where

$$(X,A)^\sigma=\{(x_1,\ldots,x_n)\in X^n \mid x_i\in A \text{ if } i\notin\sigma\}.$$

In this notation we have that $DJ(\Delta)=(\mathbb{C}\mathrm{P}^\infty,*)^\Delta$ and $\mathcal{Z}_\Delta=(D^2,S^1)^\Delta$.

2. Arrangements associated to simplicial complexes

To state the main theorems, we will need to introduce certain complex linear subspace arrangements associated to a simplicial complex, and their intersection lattices. Let $M(\Delta)$ denote the set of minimal non-faces of Δ, i.e., the set of subsets $F\subseteq[n]$ such that $F\notin\Delta$, but $F'\in\Delta$ for any proper subset $F'\subset F$.

Definition 1. The *coordinate arrangement* associated to Δ is the collection

$$\mathcal{C}(\Delta)=\{H_\sigma \mid \sigma\in M(\Delta)\},$$

and the *diagonal arrangement* associated to Δ is the collection

$$\mathcal{D}(\Delta)=\{D_\sigma \mid \sigma\in M(\Delta)\},$$

where

$$H_\sigma=\{z\in\mathbb{C}^n \mid z_{i_1}=\ldots=z_{i_r}=0\},$$

and

$$D_\sigma = \{z \in \mathbb{C}^n \mid z_{i_1} = \ldots = z_{i_r}\}$$

if $\sigma = \{i_1, \ldots, i_r\}$.

The *intersection lattice* $L_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the set of all intersections of subspaces in $\mathcal{A}$ partially ordered by reverse inclusion. The empty intersection is interpreted as the ambient space $\mathbb{C}^n$. If $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint subsets of $[n]$ then we let $D_{\sigma_1|\ldots|\sigma_r}$ denote the intersection $D_{\sigma_1} \cap \ldots \cap D_{\sigma_r}$, and we say that the number of *blocks* of $D = D_{\sigma_1|\ldots|\sigma_r}$ is $c(D) = r$. The *complement* of an arrangement $\mathcal{A}$ of subspaces in $\mathbb{C}^n$ is defined to be the space

$$X(\mathcal{A}) = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} H \subseteq \mathbb{C}^n$$

with the subspace topology.

3. Hilbert series

If $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a graded vector space over a field k then we let $V(z)$ denote the formal power series

$$V(z) = \sum_{i \in \mathbb{Z}} \dim_k V_i z^i.$$

Theorem 2. *For any field k, the Hilbert series of the loop space homology of the Davis-Januszkiewicz space is given by*

$$\mathrm{H}_*(\Omega DJ(\Delta); k)(z) = \frac{(1+z)^n}{b_\Delta(z)}, \tag{1}$$

where $b_\Delta(z)$ is the polynomial

$$b_\Delta(z) = 1 + \sum_{\mathbb{C}^n \neq S \in L_{\mathcal{D}(\Delta)}} (-z)^{c(S) + 2\operatorname{codim}_{\mathbb{C}}(S) - 2} \widetilde{\mathrm{H}}_*((\mathbb{C}^n, S)_{L_{\mathcal{D}(\Delta)}}; k)(z^{-1}). \tag{2}$$

Proof. For any coefficient field k, the singular cochain algebra $C^*(DJ(\Delta); k)$ is quasi-isomorphic to the cohomology algebra $\mathrm{H}^*(DJ(\Delta); k)$, see [21]. Furthermore, this cohomology algebra is isomorphic to the Stanley-Reisner algebra $k[\Delta]$, where the variables $x_1, \ldots, x_n$ are assigned cohomological degree 2, see [11]. Therefore, by Adams' theorem there is an isomorphism of graded algebras

$$\mathrm{H}_*(\Omega DJ(\Delta); k) \cong \mathrm{Ext}^*_{C^*(DJ(\Delta);k)}(k, k) \cong \mathrm{Ext}^*_{k[\Delta]}(k, k).$$

The Hilbert series of the Ext-algebra of $k[\Delta]$ was computed in [6]. Using this computation, and keeping track of the additional cohomological grading of $k[\Delta]$, one obtains the desired description. □

Remark 3. Note the similarity between the above formula for the denominator polynomial $b_\Delta(z)$ and the Goresky-MacPherson formula [16] for the Hilbert series of the cohomology of the complement of the complex diagonal subspace arrangement $\mathcal{D}(\Delta)$ associated to Δ:

$$\mathrm{H}^*(X(\mathcal{D}(\Delta)); k)(z) = 1 + \sum_{\mathbb{C}^n \neq S \in L_{\mathcal{D}(\Delta)}} z^{2\operatorname{codim}_{\mathbb{C}}(S) - 2} \widetilde{\mathrm{H}}_*((\mathbb{C}^n, S)_{L_{\mathcal{D}(\Delta)}}; k)(z^{-1}). \tag{3}$$

It would be interesting to compare the two using Dobrinskaya's decomposition [14, Theorem 1.1].

4. Koszul complexes

Let k be any commutative ring and let $K^{k[\Delta]}$ denote the Koszul complex of the Stanley-Reisner ring $k[\Delta]$. This is the exterior algebra over $k[\Delta]$ with generators $T_1, \ldots, T_n$ of degree 1 and differential determined by $dT_i = x_i$ and the Leibniz rule $d(xy) = d(x)y + (-1)^{|x|}xd(y)$. The cohomology algebra of $K^{k[\Delta]}$ is isomorphic to $\mathrm{Tor}_*^{k[x_1,\ldots,x_n]}(k[\Delta], k)$.

Proposition 4. *If k is a field of characteristic zero, then the Sullivan cochain algebra $A^*_{PL}(\mathcal{Z}_\Delta; k)$ of polynomial differential forms on the moment-angle complex $\mathcal{Z}_\Delta$ (see* [15]*) is quasi-isomorphic to the Koszul complex $K^{k[\Delta]}$.*

Proof. From the proof of [21, Theorem 5.5] one sees that the morphism of commutative differential graded algebras

$$A^*_{PL}(BT^n; k) \to A^*_{PL}(DJ(\Delta); k)$$

is weakly equivalent to the quotient morphism

$$k[x_1, \ldots, x_n] \to k[\Delta].$$

On one hand, the homotopy cofiber of the first morphism is quasi-isomorphic to $A^*_{PL}(\mathcal{Z}_\Delta; k)$, since $\mathcal{Z}_\Delta$ is homotopy equivalent to the homotopy fiber of the inclusion $DJ(\Delta) \to BT^n$, and by [15, Theorem II.15.3]. On the other hand, the Koszul complex $K^{k[\Delta]}$ is one realization of the homotopy cofiber of the second morphism. Since the homotopy cofiber of a morphism of commutative dg-algebras is an invariant, up to quasi-isomorphism, of the quasi-isomorphism class of the morphism, it follows that $A^*_{PL}(\mathcal{Z}_\Delta; k)$ is quasi-isomorphic to $K^{k[\Delta]}$. □

In particular, it follows there is an isomorphism of cohomology algebras

$$\mathrm{H}^*(\mathcal{Z}_\Delta; k) \cong \mathrm{Tor}_*^{k[x_1,\ldots,x_n]}(k[\Delta], k).$$

Buchstaber and Panov have proved that one has such an isomorphism also for $k = \mathbb{Z}$, and not only for fields of characteristic zero. Therefore, one might ask the following question.

Question 5. Let k be any commutative ring. Is the Koszul complex $K^{k[\Delta]}$ quasi-isomorphic to the singular cochain algebra $C^*(\mathcal{Z}_\Delta; k)$ in the category of cochain algebras over k?

5. When $\mathcal{Z}_\Delta$ is rationally a wedge of spheres

Theorem 6. *The following are equivalent:*

1. *The moment-angle complex $\mathcal{Z}_\Delta$ is rationally homotopy equivalent to a wedge of spheres.*
2. *The Stanley-Reisner ring $\mathbb{Q}[\Delta]$ is a Golod ring.*
3. *The cohomology algebra $\mathrm{H}^*(\mathcal{Z}_\Delta; \mathbb{Q})$ is trivial, in the sense that any product of two elements of positive degree is zero.*
4. *For every $H \in L_{\mathcal{C}(\Delta)}$ there is an equality of polynomials*
$$\widetilde{\mathrm{H}}_*((\mathbb{C}^n, H)_{L_{\mathcal{C}(\Delta)}}; \mathbb{Q})(z) = \sum_{D \in \epsilon^{-1}(H)} (-z)^{c(D)-1} \widetilde{\mathrm{H}}_*((\mathbb{C}^n, D)_{L_{\mathcal{D}(\Delta)}}; \mathbb{Q})(z),$$
where $\epsilon\colon L_{\mathcal{D}(\Delta)} \to L_{\mathcal{C}(\Delta)}$ is the surjective morphism of join-semilattices that maps $D_{\sigma_1|\ldots|\sigma_r}$ to H_σ, where $\sigma = \sigma_1 \cup \ldots \cup \sigma_r$.

Proof. It follows from the characterization in [2] that R is a Golod ring if and only if the Koszul complex K^R is quasi-isomorphic to its homology $\mathrm{H}_*(K^R)$ and the product on $\mathrm{H}_*(K^R)$ is trivial. On the other hand, the same property of $A^*_{PL}(X; \mathbb{Q})$ characterizes when X is rationally homotopy equivalent to a wedge of spheres [15, Theorem 24.5]. In view of Proposition 4, one therefore sees that $\mathcal{Z}_\Delta$ is rationally homotopy equivalent to a wedge of spheres if and only if $\mathbb{Q}[\Delta]$ is a Golod ring. This proves the equivalence of the first two statements. The other equivalences follow from the characterizations in [10] and [6] of when $\mathbb{Q}[\Delta]$ is a Golod ring. □

Question 7. Is it possible to interpret the equality of polynomials in characterization (4) geometrically, somehow relating the complements $X(\mathcal{C}(\Delta))$ and $X(\mathcal{D}(\Delta))$?

Problem 8. Characterize simplicial complexes Δ for which $\mathcal{Z}_\Delta$ is homotopy equivalent to a wedge of spheres. The class of such complexes include 'shifted complexes', see [17].

Problem 9. (See [8]) Find a simplicial complex Δ which is rationally Golod but for which $\mathbb{F}_p[\Delta]$ is not Golod, where p is some prime number.

Find a simplicial complex Δ which is rationally Golod but for which $\mathcal{Z}_\Delta$ is not homotopy equivalent to a wedge of speheres.

6. Homotopy Lie algebras

The rational homotopy Lie algebra of the moment-angle complex $\pi_*(\Omega\mathcal{Z}_\Delta)\otimes\mathbb{Q}$ is, after regrading, isomorphic to the homotopy Lie algebra $\pi^*(K^{\mathbb{Q}[\Delta]})$ of the Koszul complex of the Stanley-Reisner ring. This follows from the fact that the commutative cochain algebras $K^{\mathbb{Q}[\Delta]}$ and $A^*_{PL}(\mathcal{Z}_\Delta)$ are quasi-isomorphic. Here the homotopy Lie algebra of the Koszul complex is taken in the sense of Avramov [1]. The homotopy Lie algebra $\pi^*(K^{k[\Delta]})$ was computed in [5], for any field k, and we will now regrade the description of $\pi^*(K^{\mathbb{Q}[\Delta]})$ given in [5] to obtain a description of the homotopy Lie algebra $\pi_*(\Omega\mathcal{Z}_\Delta)\otimes\mathbb{Q}$.

Definition 10. The *graph of missing faces* of Δ is the undirected graph $G(\Delta)$ whose set of vertices is the set of missing faces $M(\Delta)$ and and whose edges are the pairs of missing faces $\{\sigma,\tau\}$ such that $\sigma\cap\tau\neq\emptyset$.

By a subgraph we will mean an induced subgraph, i.e., a subgraph is determined by its set of vertices. In order to deal with signs, fix a total order on $M(\Delta)$ and let $\{v_\sigma\}_{\sigma\in M(\Delta)}$ be anti-commuting variables. For a subset $S=\{\sigma_1,\ldots,\sigma_m\}$ of $M(\Delta)$, where $\sigma_1<\ldots<\sigma_m$, let $v_S=v_{\sigma_1}\wedge\ldots\wedge v_{\sigma_m}$. If $S=S_1\cup\ldots\cup S_r$ is a partition of S, then define the sign $\operatorname{sgn}(S_1,\ldots,S_r)\in\{-1,1\}$ by

$$v_S=\operatorname{sgn}(S_1,\ldots,S_r)v_{S_1}\wedge\ldots\wedge v_{S_r}.$$

Set $\operatorname{sgn}(S_1,\ldots,S_r)=0$ if $S_i\cap S_j\neq\emptyset$ for some $i\neq j$.

Definition 11. The L_∞-algebra (see [18]) associated to Δ is the graded vector space $\mathfrak{L}_\infty(\Delta)$ over $\mathbb{Q}$ with basis all non-empty connected subgraphs G of the graph of missing faces $G(\Delta)$. The homological degree of G is defined to be $2|\cup G|-|G|-1$. For each $r\geq 1$ we have an r-ary bracket

$$[\cdot,\ldots,\cdot]\colon \mathfrak{L}_\infty(\Delta)^{\otimes r}\to\mathfrak{L}_\infty(\Delta)$$

of degree $r-2$ defined as follows. If $G_1,\ldots,G_r$ are *separated*, meaning that $(\cup G_i)\cap(\cup G_j)=\emptyset$ if $i\neq j$, then

$$[G_1,\ldots,G_r]=(-1)^\epsilon\sum_{\substack{\sigma\in M(\Delta)-G\\ \sigma\subseteq\cup G\\ G\cup\{\sigma\}\text{connected}}}\operatorname{sgn}(\{\sigma\},G_r,G_{r-1},\ldots,G_1)\;G\cup\{\sigma\}$$

where $G=G_1\cup\ldots\cup G_r$. If $G_1,\ldots,G_r$ are not separated, then $[G_1,\ldots,G_r]=0$. The sign is given by

$$\epsilon=1+\sum_{i=1}^r(r-i+1)(|G_i|+1).$$

We will denote the unary bracket by d, that is, $dx = [x]$.

Proposition 12. *The above definitions make $\mathfrak{L}_\infty(\Delta)$ into an L_∞-algebra. In particular, for any $x, y, z \in \mathfrak{L}_\infty(\Delta)$*

$$\begin{aligned} d^2x &= 0, \\ [x, y] &= -(-1)^{|x||y|}[y, x], \\ d[x, y] &= [dx, y] + (-1)^{|x|}[x, dy], \\ J(x, y, z) &= d[x, y, z] + [dx, y, z] + (-1)^{|x|}[x, dy, z] + (-1)^{|x|+|y|}[x, y, dz], \end{aligned}$$

where $J(x, y, z) = [[x, y], z] - [x, [y, z]] - (-1)^{|y||z|}[[x, z], y]$.

It follows that the homology $\mathrm{H}_* \mathfrak{L}_\infty(\Delta)$ inherits the structure of a graded Lie algebra.

In addition to the homological grading, this L_∞-algebra has an $\mathbb{N}^n$-grading. If G is a basis element, i.e., a non-empty connected subgraph of $G(\Delta)$, then G has multidegree $\mathrm{mdeg}(G) = \chi_{\cup G}$, where $\chi_{\cup G}$ is the characteristic function of the subset $\cup G \subseteq [n]$. In other words,

$$\mathrm{mdeg}(G) = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n,$$

where $\alpha_i = 1$ if $i \in \cup G$ and $\alpha_i = 0$ if $i \notin \cup G$. It is easily seen that all brackets respect the multigrading. In particular the homology Lie algebra $\mathrm{H}_* \mathfrak{L}_\infty(\Delta)$ inherits a multigrading. If L is any multigraded Lie algebra, then define

$$FL = \frac{\mathbb{L}(L)}{\langle [\![x, y]\!] - [x, y] \mid x \perp y \in L \rangle}$$

Here $[\![x, y]\!]$ denotes the Lie bracket in the free graded Lie algebra $\mathbb{L}(L)$ and $[x, y]$ denotes the bracket in the Lie algebra L. For multihomogeneous elements x, y of a multigraded vector space, we write $x \perp y$ if the multidegrees of x and y have disjoint supports, i.e., if $\alpha_i\beta_i = 0$ for all i, where $\alpha = \mathrm{mdeg}(x)$ and $\beta = \mathrm{mdeg}(y)$.

Theorem 13. *The rational homotopy Lie algebra $\pi_*(\Omega\mathcal{Z}_\Delta) \otimes \mathbb{Q}$ is isomorphic to the graded Lie algebra*

$$F\, \mathrm{H}_* \mathfrak{L}_\infty(\Delta).$$

Definition 14. A multigraded vector space $V = \bigoplus_{\alpha \in \mathbb{N}^n} V_\alpha$ is called *truncated* if $V_\alpha = 0$ for $\alpha \notin \{0, 1\}^n$. The *truncation* of a multigraded vector space V is the quotient

$$V_\tau = V / \bigoplus_{\alpha \notin \{0,1\}^n} V_\alpha.$$

Note that $\mathfrak{L}_\infty(\Delta)$ is a truncated vector space. Hence, so is the homology Lie algebra $H_*\mathfrak{L}_\infty(\Delta)$.

Remark 15. The functor F is left adjoint of the truncation functor from the category of multigraded Lie algebras to the category of truncated Lie algebras. If L is a truncated Lie algebra with generators V and truncated relations R, then FL has the same generators and relations. For instance, if L is the truncated Lie algebra with generators two odd elements x and y with $x \perp y$, and relations $[x, y] = 0$, $[x, x] = [y, y] = 0$ then FL has generators x and y and the single relation $[x, y] = 0$.

Problem 16. Relate this to Dobrinskaya's method [14], using higher order Whitehead products, of producing generators for the rational loop space homology algebra $\mathrm{H}_*(\Omega DJ(\Delta); \mathbb{Q})$.

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