

## NOTES ON CATEGORIES AND FUNCTORS

These notes collect basic definitions and facts about categories and functors that have been mentioned in the Homological Algebra course. For further reading about category theory, consult [4].

### 1. CATEGORIES AND FUNCTORS

**Definition 1.** A *category*  $\mathcal{C}$  consists of

- A collection  $\text{Ob}\mathcal{C}$  of *objects*.
- For every two objects  $X$  and  $Y$  in  $\mathcal{C}$  a set of *morphisms*  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- For any triple of objects  $X, Y, Z$  a *composition* function

$$\begin{aligned} \circ: \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{C}}(X, Z) \\ g, f &\mapsto g \circ f, \end{aligned}$$

subject to the following conditions:

(C1) The composition is associative: Given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

we have that  $h \circ (g \circ f) = (h \circ g) \circ f$ .

(C2) For every object  $X$  in  $\mathcal{C}$  there is an *identity morphism*  $1_X: X \rightarrow X$  with the property that

$$1_X \circ f = f, \quad g \circ 1_X = g$$

for any morphisms  $f: Y \rightarrow X$  and  $g: X \rightarrow Z$ .

(C3) The sets  $\text{Hom}_{\mathcal{C}}(X, Y)$  and  $\text{Hom}_{\mathcal{C}}(X', Y')$  are disjoint unless  $X = X'$  and  $Y = Y'$ .

**Remark 2.** • Condition (C3) simply means that the domain and codomain of a morphism is determined by the morphism.

- If  $e_1$  and  $e_2$  are morphisms from  $X$  to itself that satisfy the conditions for an identity morphism, i.e.,  $e_i \circ f = f$  and  $g \circ e_i = g$  for any morphisms  $f$  and  $g$ , then  $e_1 = e_1 \circ e_2 = e_2$ , so we may speak of *the* identity morphism  $1_X$  of the object  $X$ .

For objects  $X$  and  $Y$  in a category  $\mathcal{C}$  we will write, and we have already written,  $f: X \rightarrow Y$  or  $X \xrightarrow{f} Y$  to indicate that  $f$  is an element of the set  $\text{Hom}_{\mathcal{C}}(X, Y)$ .

**Example 3.** •  $\mathcal{S}$  is the category of sets, whose objects are sets and whose morphisms are functions between sets.

- ${}_R\text{Mod}$  is the category of left modules over an associative unital ring  $R$ , whose objects are all left  $R$ -modules and whose morphisms are homomorphisms of left  $R$ -modules. Similarly,  $\text{Mod}_R$  denotes the category of right modules over  $R$ .
- $\text{Top}$  is the category of topological spaces, whose objects are all topological spaces and whose morphisms are continuous maps.
- A category  $\mathcal{C}$  with only one object  $*$  is the same thing as a monoid  $M = \text{Hom}_{\mathcal{C}}(*, *)$  with the composition of  $\mathcal{C}$  as multiplication. A category could therefore be thought of as a ‘monoid with several objects’.

A category  $\mathcal{C}$  is called *small* if the collection of objects  $\text{Ob}\mathcal{C}$  forms a set. The category of sets is not a small category (Russel's paradox), but the category of subsets of a given set would be an example of a small category.

**Definition 4.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A (*covariant*) *functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  consists of

- For every object  $X$  in  $\mathcal{C}$  an object  $F(X)$  in  $\mathcal{D}$ .
- For every morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  a morphism  $F(f): F(X) \rightarrow F(Y)$  in  $\mathcal{D}$ .

subject to the following conditions:

- (F1) Given morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z$ , we have that  $F(g) \circ F(f) = F(g \circ f)$ .  
 (F2) For any object  $X$  in  $\mathcal{C}$ , we have that  $F(1_X) = 1_{F(X)}$ .

**Definition 5.** Let  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be functors. A *natural transformation*  $\eta: F \Rightarrow G$  consists of

- For every object  $X$  in  $\mathcal{C}$  a morphism  $\eta_X: F(X) \rightarrow G(X)$  in  $\mathcal{D}$ .

subject to the following condition:

- (N1) For every morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

is commutative.

That  $\eta$  is a natural transformation from  $F$  to  $G$  may be pictorially indicated as:

$$\begin{array}{ccc} & F & \\ & \curvearrowright & \\ \mathcal{C} & & \mathcal{D} \\ & \curvearrowleft & \\ & G & \\ & \Downarrow \eta & \end{array}$$

## 2. SPECIAL MORPHISMS

A morphism  $f: X \rightarrow Y$  is called an *isomorphism* if there is a morphism  $g: Y \rightarrow X$  such that  $f \circ g = 1_Y$  and  $g \circ f = 1_X$ .

A morphism  $f$  in a category  $\mathcal{C}$  is called a *monomorphism* if  $f \circ g = f \circ h$  implies  $g = h$ . It is called an *epimorphism* if  $g \circ f = h \circ f$  implies  $g = h$ .

Any isomorphism is necessarily both a monomorphism and an epimorphism, but the converse need not be true. A category is called *balanced* if any morphism which is both a monomorphism and an epimorphism is an isomorphism.

## 3. CONSTRUCTIONS ON CATEGORIES

**Opposite category.** The *opposite category*, or *dual category*, of  $\mathcal{C}$  is the category  $\mathcal{C}^{op}$  whose objects are the same as those of  $\mathcal{C}$  but where morphisms are reversed in the sense that

$$\text{Hom}_{\mathcal{C}^{op}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$$

for any objects  $X$  and  $Y$ . The composition

$$\circ_{op}: \text{Hom}_{\mathcal{C}^{op}}(Y, Z) \times \text{Hom}_{\mathcal{C}^{op}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}^{op}}(X, Z)$$

is defined using the composition in  $\mathcal{C}$ : Given composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in  $\mathcal{C}^{op}$ , this is by definition the same thing as morphisms

$$X \xleftarrow{f} Y \xleftarrow{g} Z$$

in  $\mathcal{C}$ , and we define

$$g \circ_{op} f = f \circ g.$$

This is a morphism from  $Z$  to  $X$  in  $\mathcal{C}$ , in other words a morphism from  $X$  to  $Z$  in  $\mathcal{C}^{op}$ .

**Definition 6.** A *contravariant functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a functor  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$ . A contravariant functor from  $\mathcal{C}$  to  $\mathcal{D}$  can also be thought of as a functor from  $\mathcal{C}$  to  $\mathcal{D}^{op}$ .

**Product category.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. The *product category*  $\mathcal{C} \times \mathcal{D}$  has objects

$$\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob} \mathcal{C} \times \text{Ob} \mathcal{D}$$

and morphisms

$$\text{Hom}_{\mathcal{C} \times \mathcal{D}}((X, X'), (Y, Y')) = \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{D}}(X', Y').$$

Composition is defined componentwise: Given morphisms

$$(X, X') \xrightarrow{(f, f')} (Y, Y') \xrightarrow{(g, g')} (Z, Z')$$

we set  $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$ .

**The category of categories.** Ignoring set theoretical issues, we can define the category  $\text{Cat}$  whose objects are all categories, and where morphisms are given by

$$\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) = \{\text{functors } F: \mathcal{C} \rightarrow \mathcal{D}\}.$$

The composition of functors is defined by  $(G \circ F)(X) = G(F(X))$  on objects  $X$  of  $\mathcal{C}$  and by  $(G \circ F)(f) = G(F(f))$  on morphisms  $f$  in  $\mathcal{C}$ . The identity functor  $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$  is given by  $1_{\mathcal{C}}(X) = X$  and  $1_{\mathcal{C}}(f) = f$  for all objects  $X$  and all morphisms  $f$  in  $\mathcal{C}$ .

**The category of functors between two categories.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Given natural transformations

$$F \xrightarrow{\eta} G \xrightarrow{\theta} H$$

between functors  $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$ , we can define the composite natural transformation  $\theta \circ \eta: F \Rightarrow H$  by

$$(\theta \circ \eta)_X = \theta_X \circ \eta_X$$

for objects  $X$  in  $\mathcal{C}$ . For every functor  $F$  there is an identity natural transformation  $1_F: F \Rightarrow F$  defined by  $(1_F)_X = 1_{F(X)}$  for all objects  $X$  in  $\mathcal{C}$ . That the composition of natural transformations is associative follows from the associativity of the composition in  $\mathcal{D}$ . The *functor category*  $\mathcal{D}^{\mathcal{C}}$  has objects all functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  and morphisms

$$\text{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) = \{\text{natural transformations } \eta: F \Rightarrow G\}.$$

#### 4. EQUIVALENCES OF CATEGORIES AND ADJOINT FUNCTORS

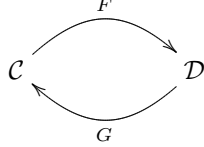
**Definition 7.** Let  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be functors. A natural transformation  $\eta: F \Rightarrow G$  is called a *natural isomorphism*, or an *isomorphism of functors*, if  $\eta_X: F(X) \rightarrow G(X)$  is an isomorphism in  $\mathcal{D}$  for every object  $X$  in  $\mathcal{C}$ . If  $F$  and  $G$  are functors such that there exists a natural isomorphism  $\eta: F \Rightarrow G$ , then we say that  $F$  and  $G$  are isomorphic, and we write  $F \cong G$ .

**Definition 8.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is called an *equivalence of categories* if there is a functor  $G: \mathcal{D} \rightarrow \mathcal{C}$  such that there are isomorphisms of functors

$$F \circ G \cong 1_{\mathcal{D}}, \quad G \circ F \cong 1_{\mathcal{C}}.$$

In this situation,  $F$  is called an *isomorphism of categories* if  $G$  can be chosen so that  $F \circ G = 1_{\mathcal{D}}$  and  $G \circ F = 1_{\mathcal{C}}$ .

**Definition 9.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. An *adjunction* consists of two functors



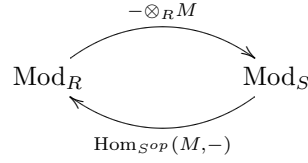
such that there is a natural isomorphism of functors from  $\mathcal{C}^{op} \times \mathcal{D}$  to  $\mathcal{S}$

$$\eta_{X,Y}: \text{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(X, G(Y)).$$

In this situation, we say that  $F$  is left adjoint to  $G$  and that  $G$  is right adjoint to  $F$ . Sometimes, this is written  $F \dashv G$ .

**Example 10.** Let  $R$  and  $S$  be rings, and let  $M$  be an  $R$ - $S$ -bimodule. Then there are two important adjunctions:

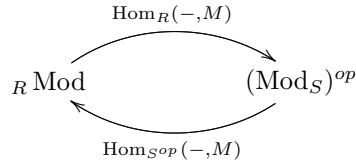
- (1) If  $L$  is a right  $R$ -module then  $L \otimes_R M$  is a right  $S$ -module via  $(l \otimes m)s = l \otimes (ms)$ , and if  $N$  is a right  $S$ -module then  $\text{Hom}_{S^{op}}(M, N)$  is a right  $R$ -module via  $(\phi r)(m) = \phi(rm)$ . There is an adjunction



$$\eta_{L,N}: \text{Hom}_{S^{op}}(L \otimes_R M, N) \xrightarrow{\cong} \text{Hom}_{R^{op}}(L, \text{Hom}_{S^{op}}(M, N))$$

$$\eta_{L,N}(\phi)(l)(m) = \phi(l \otimes m).$$

- (2) If  $L$  is a left  $R$ -module then  $\text{Hom}_R(L, M)$  is a right  $S$ -module via  $(\phi s)(l) = \phi(l)s$ , and if  $N$  is a right  $S$ -module then  $\text{Hom}_{S^{op}}(N, M)$  is a left  $R$ -module via  $(r\phi)(n) = r\phi(n)$ . There is an adjunction



$$\theta_{L,N}: \text{Hom}_{S^{op}}(N, \text{Hom}_R(L, M)) \xrightarrow{\cong} \text{Hom}_R(L, \text{Hom}_{S^{op}}(N, M))$$

$$\theta_{L,N}(\phi)(l)(n) = \phi(n)(l).$$

**Definition 11.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is called *faithful*, or *full* respectively, if the function

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ f &\mapsto F(f) \end{aligned}$$

is injective, or surjective respectively. Sometimes, a functor which is both full and faithful is called *fully faithful*.

**Definition 12.** An object  $X$  in a category  $\mathcal{C}$  is called a *generator* if the functor

$$\text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow \mathcal{S}$$

is faithful. Dually,  $X$  is called a *cogenerator* if the functor

$$\text{Hom}_{\mathcal{C}}(-, X): \mathcal{C}^{op} \rightarrow \mathcal{S}$$

is faithful.

### 5. CONSTRUCTIONS WITHIN CATEGORIES

Constructions within a category are often defined by universal properties. A construction is characterized by how it maps out of, or into, other objects in the category. Pinning down a construction by means of a universal property ensures that the construction is determined in the strongest possible way, namely up to unique isomorphism. However, existence of a certain construction must be proved separately in each case. This is usually done by writing down an explicit formula.

**Definition 13.** A *direct product* of a family of objects  $\{X_i\}_{i \in I}$  is an object  $X$  together with morphisms  $\pi_i: X \rightarrow X_i$ , called *projections*, that satisfy the following universal property:

Given an object  $Y$  and morphisms  $\phi_i: Y \rightarrow X_i$  for every  $i \in I$ , there is a unique morphism  $\psi: Y \rightarrow X$  such that the diagram

$$\begin{array}{ccc} & X & \\ \psi \nearrow & & \searrow \pi_i \\ Y & \xrightarrow{\phi_i} & X_i \end{array}$$

commutes for every  $i \in I$ .

As usual when something is defined by a universal property, it is unique up to unique isomorphism. Therefore, one can allow oneself to use the notation  $X = \prod_{i \in I} X_i$  if  $X$  is a product of the family of objects  $\{X_i\}$ . Note however that the direct product is more than the object  $\prod_{i \in I} X_i$  — the projections  $\pi_i: \prod_{i \in I} X_i \rightarrow X_i$  are part of the data.

**Definition 14.** A *coproduct*, or *direct sum*, of a family of objects  $\{X_i\}_{i \in I}$  is an object  $X$  together with morphisms  $\iota_i: X_i \rightarrow X$ , called *injections*, that satisfy the following universal property:

Given an object  $Y$  and morphisms  $\mu_i: X_i \rightarrow Y$ , there is a unique morphism  $\theta: X \rightarrow Y$  such that the diagram

$$\begin{array}{ccc} & X & \\ \iota_i \nearrow & & \searrow \theta \\ X_i & \xrightarrow{\mu_i} & Y \end{array}$$

commutes for every  $i \in I$ .

If  $X$  is a direct sum of the family  $\{X_i\}$ , then we use the coproduct notation  $X = \coprod_{i \in I} X_i$  or the direct sum notation  $X = \bigoplus_{i \in I} X_i$ . The direct sum notation is commonly used in additive categories, and the coproduct notation is used in non-additive categories.

**Definition 15.** A *terminal object* is an object  $*$  that satisfies the following universal property:

For every object  $Y$  there is a unique morphism  $t_Y: Y \rightarrow *$ .

An *initial object* is an object  $\emptyset$  that satisfies the following universal property:

For every object  $Y$  there is a unique morphism  $i_Y: \emptyset \rightarrow Y$ .

A *zero object* is an object  $0$  which is both a terminal and an initial object.

Note that a terminal object is a product and an initial object a coproduct of the empty family of objects. A category is called *pointed* if it possesses a zero object. A category is pointed if and only if it has a terminal and an initial object and the unique morphism  $\emptyset \rightarrow *$  is an isomorphism; in this case both  $\emptyset$  and  $*$  are zero objects. For any two objects  $X$  and  $Y$  in a pointed category, the *zero morphism*  $0: X \rightarrow Y$  is defined to be the composite  $X \xrightarrow{t_X} 0 \xrightarrow{i_Y} Y$ .

**Definition 16.** Let  $f: X \rightarrow Y$  be a morphism in a pointed category  $\mathcal{C}$ . A *kernel* of  $f$  is an object  $K$  together with a morphism  $\kappa: K \rightarrow X$  such that  $f \circ \kappa = 0$  and which satisfy the following universal property:

Given an object  $Z$  and a morphism  $\zeta: Z \rightarrow X$  such that  $f \circ \zeta = 0$ , there is a unique morphism  $\lambda: Z \rightarrow K$  such that the diagram

$$\begin{array}{ccc} & Z & \\ \lambda \swarrow & \downarrow \zeta & \\ K & \xrightarrow{\kappa} X & \xrightarrow{f} Y \end{array}$$

commutes.

**Definition 17.** Let  $f: X \rightarrow Y$  be a morphism in a pointed category  $\mathcal{C}$ . A *cokernel* of  $f$  is an object  $C$  together with a morphism  $\gamma: Y \rightarrow C$  such that  $\gamma \circ f = 0$  which satisfy the following universal property:

Given an object  $Z$  and a morphism  $\zeta: Y \rightarrow Z$  such that  $\zeta \circ f = 0$ , there is a unique morphism  $\lambda: C \rightarrow Z$  such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} Y & \xrightarrow{\gamma} C \\ & \downarrow \zeta & \swarrow \lambda \\ & Z & \end{array}$$

commutes.

## 6. ADDITIVE AND ABELIAN CATEGORIES

**Definition 18.** A *pre-additive category* is a category  $\mathcal{A}$  where the set of morphisms  $\text{Hom}_{\mathcal{A}}(X, Y)$  between any two objects  $X$  and  $Y$  has the structure of an abelian group, and moreover the composition

$$\text{Hom}_{\mathcal{A}}(Y, Z) \times \text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear, i.e.,  $(g + g') \circ f = g \circ f + g' \circ f$  and  $g \circ (f + f') = g \circ f + g \circ f'$  for any morphisms  $f, f': X \rightarrow Y$  and  $g, g': Y \rightarrow Z$ .

A pre-additive category  $\mathcal{A}$  with one object  $*$  is the same thing as an associative unital ring  $R = \text{Hom}_{\mathcal{A}}(*, *)$ . The ring multiplication is the composition in  $\mathcal{A}$ . Therefore, a pre-additive category could be thought of as a 'ring with many objects'. The opposite category  $\mathcal{A}^{op}$  corresponds to the opposite ring  $R^{op}$ .

**Definition 19.** An *additive functor* between pre-additive categories  $\mathcal{A}$  and  $\mathcal{B}$  is a functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  such that for every two objects  $X$  and  $Y$  in  $\mathcal{A}$ , the function

$$\text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is a homomorphism of abelian groups, i.e.,  $F(f + g) = F(f) + F(g)$  for any morphisms  $f, g: X \rightarrow Y$ .

If  $\mathcal{A}$  is a pre-additive category with one object, thought of as a ring  $R$ , then additive functors  $\mathcal{A} \rightarrow \mathbb{Z}\text{Mod}$  correspond to left  $R$ -modules, and additive contravariant functors  $\mathcal{A} \rightarrow \mathbb{Z}\text{Mod}$  correspond to right  $R$ -modules.

**Definition 20.** An *additive category* is a pre-additive category  $\mathcal{A}$  in which any finite family of objects has a product and a coproduct.

**Proposition 21.** *Let  $\mathcal{A}$  be an additive category. Then for any finite family of objects  $\{X_i\}_{i \in I}$  the natural morphism*

$$\alpha: \bigoplus_{i \in I} X_i \rightarrow \prod_{i \in I} X_i$$

is an isomorphism.

*Proof.* We can use the abelian group structure on the set of homomorphisms to define a morphism  $\beta: \prod_{i \in I} X_i \rightarrow \bigoplus_{i \in I} X_i$  by

$$\beta = \sum_{i \in I} \iota_i \pi_i.$$

This makes sense because  $I$  is a finite set. We claim that  $\beta$  is an inverse to  $\alpha$ . Indeed,  $\alpha$  is characterized by  $\pi_i \alpha \iota_i = 1_{X_i}$  and  $\pi_j \alpha \iota_i = 0$  if  $i \neq j$ . Hence, for every  $j \in I$  we have that

$$\pi_j \alpha \beta = \sum_{i \in I} \pi_j \alpha \iota_i \pi_i = \pi_j = \pi_j \circ 1.$$

By the uniqueness part of the universal property defining products, we must have  $\alpha \beta = 1$ , the identity on  $\prod_{i \in I} X_i$ . A dual argument shows that  $\beta \alpha = 1$ , the identity on  $\bigoplus_{i \in I} X_i$ .  $\square$

In particular, by specializing the above proposition to the empty family of objects, the morphism  $\emptyset \rightarrow *$  from the initial to the terminal object in  $\mathcal{A}$  is an isomorphism. Hence,  $\mathcal{A}$  is pointed, i.e.,  $\mathcal{A}$  has a zero object. Note that the zero morphism  $0: X \rightarrow Y$ , that is defined in any pointed category, coincides with the zero element in the abelian group  $\text{Hom}_{\mathcal{A}}(X, Y)$ .

**Definition 22.** An *abelian category* is an additive category  $\mathcal{A}$  where

- Every morphism has a kernel and a cokernel.
- Given a diagram

$$K \xrightarrow{\kappa} X \xrightarrow{\gamma} C$$

where  $\kappa$  is a monomorphism and  $\gamma$  is an epimorphism,  $(K, \kappa)$  is a kernel of  $\gamma$  if and only if  $(C, \gamma)$  is a cokernel of  $\kappa$ .

A category is said to have *enough projectives* if for every object  $X$  there is an epimorphism  $\epsilon: P \rightarrow X$  where  $P$  is a projective object. It is said to have *enough injectives* if the dual statement is true, namely that for every object  $X$  there is a monomorphism  $\mu: X \rightarrow I$  where  $I$  is an injective object.

If an abelian category  $\mathcal{A}$  has enough projectives, then every object admits a projective resolution, and hence the left derived functors  $L_n T$  of any additive functor  $T: \mathcal{A} \rightarrow \mathcal{B}$  can be defined. Dually, if  $\mathcal{A}$  has enough injectives then every object admits an injective resolution, and hence the right derived functors  $R^n T$  of any additive functor  $T: \mathcal{A} \rightarrow \mathcal{B}$  can be defined.

The theory of homological algebra in abelian categories owes much to Grothendieck's paper [3] and also to Gabriel's work [2].

**Example 23.** • The category  ${}_R\text{Mod}$  of left modules over a ring  $R$  is an abelian category. It has enough projectives and enough injectives.

- If  $\mathcal{A}$  is an abelian category, then so is the opposite category  $\mathcal{A}^{op}$ . In particular, the opposite category  $({}_R\text{Mod})^{op}$  of left modules over a ring  $R$  is abelian. This is a simple example of an abelian category which is not equivalent to a module category.
- If  $\mathcal{A}$  is abelian and if  $I$  is any small category, then the functor category  $\mathcal{A}^I$  is abelian. Sums, products, kernels and cokernels, etc, are defined pointwise.
- For any abelian category  $\mathcal{A}$  one can define the category  $\text{Com}(\mathcal{A})$  of complexes of objects in  $\mathcal{A}$ . The category  $\text{Com}(\mathcal{A})$  is again abelian, by defining sums, products, etc, levelwise.
- The category  $\text{Sh}(X)$  of sheaves on a topological space  $X$  is abelian. It has enough injectives, but it does not have enough projectives.

Abelian categories are the proper setting in which to ‘do homological algebra’. However, in proving diagram lemmas like the Five Lemma, or when deriving the long exact homology sequence from a short exact sequence of complexes, one selects elements from the modules involved and then one chases them around the diagram at hand. The objects in an abstract abelian category are not necessarily sets with extra structure, so it does not make sense to talk about elements. Since there are examples of abelian categories that are not equivalent to module categories, this is a real problem. It is possible to develop techniques for performing diagram chases in arbitrary abelian categories (ask AB — he will be happy to tell you about it), and these techniques suffice to develop the elements of homological algebra. There is also another, more sophisticated, approach. Even though not every abelian category is equivalent to a module category, one can find an exact embedding of every *small* abelian category into a module category.

**Theorem 24** (Freyd-Mitchell Embedding Theorem). *If  $\mathcal{A}$  is a small abelian category, then there is a ring  $R$  and a fully faithful exact functor  $\mathcal{A} \rightarrow {}_R\text{Mod}$ .*

In particular, when proving a diagram lemma, one can restrict attention to a suitable small abelian subcategory. A consequence of the embedding theorem is the following ‘metatheorem’.

**Theorem 25** (Metatheorem). *Any diagram lemma that is true in all categories of  $R$ -modules is true in all abelian categories.*

These theorems are proved in the book [1]. For instance, the metatheorem applies to the Five Lemma, the  $(3 \times 3)$ -Lemma, the Snake Lemma, the Horseshoe Lemma, etc.

This situation is in a sense somewhat similar to the theory of finite dimensional vector spaces over a field  $k$ . In a course on linear algebra, you first deal with  $k^n$  and linear maps, aka matrices, between these. Then you develop an axiomatic theory of abstract vector spaces, and you encounter some exotic examples that do not look very much like  $k^n$ . But then you learn that every finite dimensional vector space is actually isomorphic to  $k^n$  for some  $n$ , so in proving statements about finite dimensional vector spaces, you might as well assume that they are of this form.

Similarly, there are definitely quite exotic examples of abelian categories that bear little or no resemblance to the category of modules over a ring. However, the Freyd-Mitchell Embedding Theorem tells you that when working within an abelian category, you might as well assume that you are in a module category.

## REFERENCES

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