Nadia Mazza

Oliver's Conjecture (joint w/ D. Green & L. Héthelyi)

1. Oliver subgroup + Bob's conjecture

2. Reformulation of Oliver's conjecture (Green - Héthelyi -

3. Little improvements (Green - Héthelyi - Mazza)

Notation: · $S \in p\text{-gr}$, $p > 2$

· $[x,y] = x^{-1}y^{-1}xy$ $\quad [x,y;i] = [[x,y;i-1],y]$

· $\Omega_n(S) = \langle x \in S \mid x^{p^n} = 1 \rangle$

Def'n: Let $\mathfrak{X}(S)$ be the largest subgroup of S s.t. $\exists\ 1 = Q_0 \leq Q_1 \leq \cdots \leq Q_n = \mathfrak{X}(S)$ with $Q_i \trianglelefteq S$ $\forall i$ and $[\Omega_1(C_S(Q_{i-1})), Q_i; p-1] = 1$ $\forall i$.

$\mathfrak{X}(S)$ is the Oliver subgroup of S.

Rem: 1) $\mathfrak{X}(S)$ is well-defined: $\left.\begin{array}{l} 1 = Q_0 \leq \cdots \leq Q_n \\ 1 = R_0 \leq \cdots \leq R_m \end{array}\right\} \Rightarrow 1 \leq Q_0 \leq \cdots \leq Q_n \leq Q_n R_1 \leq \cdots \leq Q_n R_m$

2) $p = 2$: $\mathfrak{X}(S) = C_S(\Omega_1(S))$

Properties: 1) $\mathfrak{X}(S) \geq A$ $\forall A \in \mathcal{J}_{ab}^{\trianglelefteq}(S)$. In particular, $\mathfrak{X}(S)$ is centric.

2) If $Q \trianglelefteq S$ and $[\Omega_1(Z(\mathfrak{X}(S))), Q; p-1] = 1 \Rightarrow Q \leq \mathfrak{X}(S)$.

3) $\mathfrak{X}(S) = S$ or $p\text{-rank}(Z(\mathfrak{X}(S))) \geq p$.

Exercise: $S = C_p \wr C_p \Rightarrow \mathfrak{X}(S) = J(S) \cong C_p^p$

OC = Conjecture (Oliver): $J(S) \leq \mathfrak{X}(S)$ $\quad p > 2$

(Here $J(S) = \langle$ elem. abelian p-subgps of S of maximal order $\rangle$)

Conjecture $\Rightarrow$ $\mathcal{Z}$-functor on fusion systems that is acyclic.

Reformulation of (OC)

(1) Define the property

(PS) Let $G \in p\text{-}gr$ $(p>2)$ and V a faithful f.g. $\mathbb{F}_p G$-module. For all $z \in \Omega_1(Z(G))^{\#}$, $\mathrm{Res}^G_{\langle z \rangle}(V)$ has a nonzero projective summand.

(2) (OC) $\Longleftrightarrow$ Any nontrivial finite p-group G has no "F-module" satisfying (PS).

Def: V is an F-module if $\exists$ $A \leq G$ non-trivial el. abel. subgp s.t. $|A| \cdot |C_V(A)| \geq |V|$. and A is called an offender.

(F stands for "failure of Thompson factorization")

Thm [GHL]: (OC) holds $\forall S$ s.t. $S/\mathfrak{X}(S)$ has nilpotence class ≤ 2.

$p>2$ Little Improvements by [GHM] (here "module" means "f.g. faithful right module")

Thm: (OC) holds for any S s.t. $G = S/\mathfrak{X}(S)$ satisfies one of:

(1) G has nilpotence class ≤ 4

(2) G is metabelian (G' is abelian)

(3) G has maximal nilpotence class. ($\Rightarrow$ (4))

(4) G has p-rank at most p.

Sketchy proof Let $1 \neq G \in p\text{-}gr$ $p>2$ (think $G = S/\mathfrak{X}(S)$) and let V be a faithful f.g. $\mathbb{F}_p$-module.

Thm A: If $\Omega_1(Z(G))$ has no quadratic element and G satisfies (1) or (2), above, then V cannot be an F-module.

Thm B: If $\forall x \in \Omega_1(Z(G))^{\#}$, x acts on V with min. polynomial $x^p - 1$ (i.e., V satis (PS)) and G satisfies (3) or (4), then V cannot be an F-module.

$\Gamma = G \ltimes V$ with multiplication.

$(g,v)\cdot(h,w) = (gh,\ v * h + w$

Commutators in Γ : $[v,g] = [(1,v),(g,0)] = \dots = v * \overbrace{(g-1)}^{\in \mathbb{F}_p G}$

<u>Def'n</u> : $x \in G^{\#}$ is <u>quadratic</u> if $[V,x,x]=0$

$A \leq G$ acts quadratically on V if $[V,A,A]=0$

<u>Rem</u> : (1) $x \in G$ is quadratic $\Rightarrow$ $o(x)=p$.

$[V,x;2]=0 \Rightarrow \underbrace{[V,x;p]}_{=[V,x^p]}=0$ $\quad$ V faithful, p odd $\Rightarrow x^p=1$

(2) $H \leq G \Rightarrow C_V(H) = V^H$

Timmesfeld : There are offenders $\Leftrightarrow$ there are quadratic offenders

<u>Notation</u> : $\mathcal{E}(G) = \{$el. abelian p-subgroups of $G\}$

<u>Lemma 1</u> : $1 \neq E \in \mathcal{E}(G)$ and E is an offender. If $\Omega_1(Z(G))$ has no quadratic elements, then $E \cap Z(G) = 1$. In particular, $J^{\triangleleft}_{\leq E}(G) = 1$.

<u>Thm</u> : Assume $\Omega_1(Z(G))$ has no quadratic elements :

(1) $A \in \mathcal{J}^{\triangleleft}_{ab}(G) \Rightarrow A$ does not contain any offender.

(2) $E \in \mathcal{E}(G)$ offender $\Rightarrow [G', E] \neq 1$

<u>Lemma 2</u> Set n = nilpotence class of G and $G = G_1 \geq G_2 \geq \dots \geq G_n > 1$.

If $r \in \mathbb{N}$ s.t. $2r \geq n$ and if G_{r+1} has no quadratic elements, then $[G_i, E] = 1$ $\forall$ quadratic offenders E. In particular, if $n \leq 4$ and $\Omega_1(Z(G))$ has no quadratic elements, then $[G_{n-2}, E] = 1$.

$\Rightarrow$ Thm (A) (1)

For (2) $\rightarrow$ Huppert's exercise : if $A \in \mathcal{J}^{\triangleleft}_{ab}(G)$, G/A cyclic, then $G' = \{[a,g] \mid a \in A, \langle gA \rangle =$