Trivial Fusion Systems

Intro: ~~Fs~~ Origins of work, Frobenius p-nilpotency theorem, $\mathcal{F}$ saturated f.s. on S convention

$\boxed{\text{BOARD 1A}}$

<u>Theorem</u>: <u>Glauberman-Thompson</u> (K-L) If p is odd, then if $\mathcal{F} \in \underline{\mathcal{F}us}(S)$

$$\mathcal{F} = \mathcal{F}_S(S) \iff N_{\mathcal{F}}(Z(J(S))) = \mathcal{F}_S(S)$$

<u>Pf</u>: "$\Rightarrow$": clear

"$\Leftarrow$" Let $\mathcal{F}$ be a minimal counterexample and let $\mathcal{G}$ be a proper subfusion system of $\mathcal{F}$ on S. As $\mathcal{F}_S(S) \le N_{\mathcal{G}}(Z(J(S))) \le N_{\mathcal{F}}(Z(J(S))) = \mathcal{F}_S(S)$, the minimality of $\mathcal{F}$ implies $\mathcal{G} = \mathcal{F}_S(S)$.

$\boxed{\text{BOARD 1B}}$

<u>Theorem</u>: (Navarro) Let $S \in Syl_p(G)$. If $S = N_G(S)$, then $N_G(S')$ is p-nilpotent.

<u>Theorem</u> (Glesser) Let $\mathcal{F} \in \underline{\mathcal{F}us}(S)$. If $N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$, then

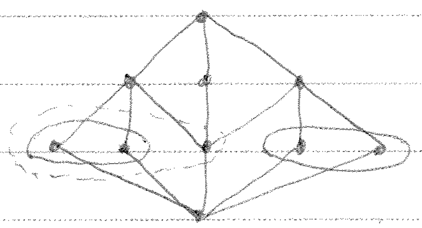
$$N_{\mathcal{F}}(Q) = \mathcal{F}_S(S) \quad \text{for all } S' \le Q \le \Phi(S).$$

<u>Pf</u>: For $S' \le Q \le \Phi(S)$, $N_{N_{\mathcal{F}}(Q)}(S) = \mathcal{F}_S(S)$ and $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(Q)}(Q)$, so we may assume $Q \trianglelefteq \mathcal{F}$, i.e., we want to show that $\mathcal{F} = \mathcal{F}_S(S)$.
Let $\mathcal{F}$ be a minimal counterexample and let $\mathcal{G}$ be a proper fusion system of $\mathcal{F}$ on S. As $N_{\mathcal{G}}(S) \le N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$, the minimality of $\mathcal{F}$ implies $\mathcal{G} = \mathcal{F}_S(S)$.

$\boxed{\text{BOARD 2A}}$

<u>Def'n</u>: A nontrivial $\mathcal{F} \in \underline{\mathcal{F}us}(S)$ is <u>sparse</u> if $\mathcal{F}_S(S)$ is the only proper subfusion system of $\mathcal{F}$ on S.

<u>Ex</u>: $p=2$, $S = D_8$, $G = S_4$, $\mathcal{F} = \mathcal{F}_S(G)$



$D_8 = \langle (1324), (12) \rangle$

$V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$

$V_4 \trianglelefteq \mathcal{F}$

$C_S(V_4) = V_4$

$\Rightarrow \mathcal{F}$ is constrained.

<u>Conjecture</u>: Every sparse system on S is constrained.

BOARD 2B

Theorem (Thompson, DGMP): Let p be odd or $\mathcal{F}$ S_4-free for $\mathcal{F} \in \underline{\mathrm{Fus}}(S)$

$$\mathcal{F} = \mathcal{F}_S(S) \iff C_{\mathcal{F}}(Z(S)) = \mathcal{F}_S(S) = N_{\mathcal{F}}(J(S)).$$

Theorem (Glesser) Let $\mathcal{F} \in \underline{\mathrm{Fus}}(S)$ be sparse.

(1) If $Q \trianglelefteq \mathcal{F}$ and $QC_S(Q) \ntrianglelefteq \mathcal{F}$, then $\mathcal{F} = SC_{\mathcal{F}}(Q)$

(where $SC_{\mathcal{F}}(Q)$ is the fusion system on S where ~~$\mathrm{Hom}_{SC_{\mathcal{F}}(Q)}(P,R) = \{$~~

$$\mathrm{Hom}_{SC_{\mathcal{F}}(Q)}(P, S) = \left\{ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S) \;\middle|\; \begin{array}{l} \exists \text{ extension } \tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QP,S) \text{ of } \varphi \text{ s.t.} \\ \tilde{\varphi}|_Q \in \mathrm{Aut}_S(Q) \end{array} \right\})$$

(2) If p is odd or $\mathcal{F}$ is S_4-free, then $\mathcal{F}$ is constrained.

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Pf: (1) Stancu: $\mathcal{F} = \langle SC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_S(Q)) \rangle = SC_{\mathcal{F}}(Q)$

(since $Q \trianglelefteq \mathcal{F}$, $QC_S(Q) \ntrianglelefteq \mathcal{F}$ and $\mathcal{F}$ is sparse)

(2) By Thompson, $J(S) \trianglelefteq \mathcal{F}$ or $Z(S) \trianglelefteq \mathcal{F} \Rightarrow Q := O_p(\mathcal{F}) \neq 1$.

If $Q \in \mathcal{F}^c$, $\mathcal{F}$ is constrained. Otherwise, $\mathcal{F} = SC_{\mathcal{F}}(Q)$. From here, the proof uses results of Kessar-Linckelmann on quotient fusion systems to obtain a contradiction. □

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As $\mathcal{F}$ is sparse and p is odd, $\mathcal{F}$ is constrained $\Rightarrow \mathcal{F} = \mathcal{F}_S(G)$.

$\Rightarrow \mathcal{F}_S(N_G(Z(J(S)))) = N_{\mathcal{F}}(Z(J(S))) = \mathcal{F}_S(S)$

$\Rightarrow \mathcal{F}_S(G) = \mathcal{F}_S(S)$ by the original [GT] result. □

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$\mathcal{F}$ is sparse. If $Q \in \mathcal{F}^c$, then $\mathcal{F}$ is constrained and the result will follow from Navarro. Otherwise, $\mathcal{F} = SC_{\mathcal{F}}(Q)$

Lemma: If $\mathcal{F} \in \mathrm{Fus}(S)$, $N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$, $S' \leq Q \leq S$ and $\mathcal{F} = SC_{\mathcal{F}}(Q)$, then $\mathcal{F} = \mathcal{F}_S(S)$.

This gives a contradiction, completing the proof.

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Corollary: Let $b \in Bl(G|S)$ with inertial index 1 (i.e., $|N_G(S,e)/SC_G(S)| = 1$ where $(S,e) \in Syl_b(G)$)

If $\exists S' \in Q \leq \mathcal{F}(S)$ s.t. $Q \trianglelefteq G$, then b is nilpotent. In particular, if S is abelian, then b is nilpotent.

BOARD 2A

Def'n A nontrivial $\mathcal{F} \in \underline{Fus}(S)$ is extremely sparse if $\mathcal{F}_Q(Q)$ is the only proper subfusion system of $\mathcal{F}$ on Q for every $Q \leq S$.

Rem: Extremely sparse $\Rightarrow$ sparse.

Theorem: Let $\mathcal{F} \in \underline{Fus}(S)$ be extremely sparse.

(1) $O_p(\mathcal{F}) \neq 1$

(2) $\mathcal{F}$ is constrained or $\mathcal{F} = SC_{\mathcal{F}}(O_p(\mathcal{F}))$

Pf: (1) Frobenius' Theorem.

(2) If $O_p(\mathcal{F}) \in \mathcal{F}^c$, then $\mathcal{F}$ is constrained. Otherwise, the result follows since $\mathcal{F}$ is sparse. $\square$

BOARD 3A ~~Recall~~ $\mathcal{F} \in \underline{Fus}(S)$

Recall $[Q, \mathcal{F}] = \langle u^{-1}\varphi(u) \mid u \in Q, \varphi \in Hom_{\mathcal{F}}(\langle u \rangle, S) \rangle$

Set $[Q, \mathcal{F}; 0] = Q$, $[Q; \mathcal{F}; i] = [[Q, \mathcal{F}; i-1], \mathcal{F}]$, $i > 0$.

As $[Q, \mathcal{F}; i] \leq [Q, \mathcal{F}; i-1]$ $\forall i \geq 0$, we may define

$$[Q, \mathcal{F}; \infty] = \bigcap_{i=0}^{\infty} [Q, \mathcal{F}; i]$$

Theorem: Let $\mathcal{F} \in \underline{Fus}(S)$. If $[S, \mathcal{F}; \infty] = 1$, then $\mathcal{F} = \mathcal{F}_S(S)$.

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Pf: Let $\mathcal{F}$ be a minimal counterexample and let $\mathcal{G}$ be a proper subfusion system of $\mathcal{F}$ on $Q \leq S$. As $[Q, \mathcal{F}; \infty] \leq [S, \mathcal{F}; \infty] = 1$, the minimality of $\mathcal{F}$ implies $\mathcal{G} = \mathcal{F}_Q(Q) \Rightarrow \mathcal{F}$ is extremely sparse.
If $\mathcal{F}$ is constrained, we are done by a result of Passman.
If $\mathcal{F} = C_{\mathcal{F}}(O_p(\mathcal{F}))$, then another argument involving quotient fusion systems completes the proof. $\blacksquare$