Functorial Dimensions

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Dear reader,

The present manuscript comprises my Ph.D. thesis as submitted to the Faculty of Science at the University of Copenhagen on October 1st 1999.

The thesis is composed of a survey article and four preprints of accepted papers and papers in preparation.

Lars Winther Christensen
Copenhagen, fall 1999

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This thesis is divided into four self contained parts:

Part I

Functorial Dimensions

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Contents

Introduction

This manuscript, dated October 1999, forms part of my thesis for the Ph.D. degree; like the rest of the thesis it was written at the University of Copenhagen under supervision of professor Hans–Bjørn Foxby. The aim of the present manuscript is to describe the general ideas behind my work as presented in the papers listed below.

[TD] *A Closer Look on Tor–dimensions*, in preparation.

[AC] *Semi-dualizing Complexes and their Auslander Categories*, to appear in Trans. Amer. Math. Soc.

[SC] *Sequences for Complexes*, to appear in Math. Scand.

[PC] *Parameters for Complexes*, in preparation.

The bulk of my work is contained in [**AC**] and [**SC**], while [**PC**] grew from an example given in [**SC**]. The paper [**TD**] in preparation is, actually, a spin-off to another manuscript, *Gorenstein Dimensions* [**9**], to be completed this fall.

I want to emphasize that **this is neither a résumé nor a quick guide** to the highlights of my work; rather, it is an attempt to give the grounds for my work and review parts of it in the proper perspective.

The Ingredients

The key words are 'Gorenstein dimensions', 'functorial dimensions', and 'ultimate formulas'. This entire manuscript is devoted to defining, explaining, and exemplifying these concepts and showing how they play together in my work. But to give a flavor of what is to come, I want to dwell for a second on, what I call, 'the ultimate addition formula for G–dimensions'. I want to advertise this result — Theorem (6.5) in [**AC**] — because it "has it all" in the sense that it combines the three ingredients in a simple and beautiful way.

An Appetizer

We consider a finite local homomorphism of rings, $\varphi\colon R \to S$; this means that R and S are (commutative and Noetherian) local rings, and that φ maps the maximal ideal of R into that of S and makes S a finitely generated R–module. The complex C is semi-dualizing for R; this means that C has finite homology and the homothety morphism $\chi^R_C\colon R \to \mathbf{R}\mathrm{Hom}_R(C,C)$ is invertible in the derived category. For example, C could be a dualizing complex for R or just the ring itself; the latter is a most important special case.

In this setting the ultimate addition formula holds:

Theorem. *If* $\text{G–dim}_C S$ *is finite, then the complex* $S^{\dagger_C} = \mathbf{R}\text{Hom}_R(S,C)$ *is semi-dualizing for* S*, and for every* S*–complex* Z *with finite homology there is an equality:*

$$\text{G–dim}_C Z = \text{G–dim}_C S + \text{G–dim}_{S^{\dagger_C}} Z.$$

In particular, $\text{G–dim}_C Z$ *and* $\text{G–dim}_{S^{\dagger_C}} Z$ *are simultaneously finite.*

The generalized *Gorenstein dimensions* $\text{G–dim}_C Z$, $\text{G–dim}_C S$, and $\text{G–dim}_{S^{\dagger_C}} Z$ are *functorial dimensions*; and by this I mean that they are defined solely in terms of derived functors and without reference to resolutions. Sometimes, however, these dimensions can be interpreted, or defined, in terms of resolutions, and then they are known to obey certain addition formulas. All these formulas — due to Auslander and Bridger, Peskine and Szpiro, Golod, and Avramov and Foxby — are special cases of the equality $\text{G–dim}_C Z = \text{G–dim}_C S + \text{G–dim}_{S^{\dagger_C}} Z$ which, hereby, qualifies as an *ultimate formula*.

The Menu

Most of my published work is inspired by my efforts to understand Gorenstein dimensions, so I have found it appropriate to start with an introduction to the basics of this theory. The first chapter, therefore, deals with the work of other people.

I have tried hard to keep this manuscript low on notation and terminology; nevertheless, the first chapter opens with a short section on these matters, just to make sure we speak the same language.

The paper [**TD**] in preparation deals exclusively with functorial dimensions, so, given the scope of this manuscript, it requires little explanation. In the course of the second chapter I will show where the motivation came from, and I will point out a few results of importance to the theory of Gorenstein dimensions.

One of the original results in my, otherwise, largely expository [**9**] can be motivated, stated, and proved using only the fundamentals of the theory of Gorenstein dimensions; it will make its first public appearance in chapter two.

In [**AC**] several themes are played at the same time. The all-embracing main theme is 'studying semi-dualizing complexes via their Auslander categories'; subsidiary themes include 'Golod's question', 'relative dualizing complexes', and 'G–dimensions'. In the third chapter we focus on the latter theme.

The fourth chapter covers [**SC**] and [**PC**]. Here we look at a well-known functorial dimension: the depth. For finite modules it has a classical interpretation in terms of regular sequences, and now we search for similar interpretations for complexes.

Toasts

Almost four years ago Hans–Bjørn Foxby gave me a copy of [**1**] and suggested Gorenstein dimensions as the subject for my Master's thesis; I consider this a great success and have, already, expressed my gratitude on several occasions.

Here I, first and foremost, want to thank Foxby for his never ending inspiration and support and for guiding me tactfully while letting me enjoy the feeling of having a free hand.

The joint work of Avramov and Foxby, [**5**] and [**6**] in particular, has been a great source for inspiration, and I want to thank Luchezar Avramov for teaching me a lot during our discussions at Purdue University. Special thanks for fruitful discussions also go to Srikanth Iyengar, Anders Frankild, and Peter Jørgensen.

CHAPTER 1

Background

This chapter gives a brief introduction to Auslander's Gorenstein dimension, the extensions due to Enochs, Jenda, Torrecillas, and Xu, and the related categories introduced by Foxby. Conventions for the entire manuscript are laid down in the first section. No original results are presented in this chapter.

1. Notation and Terminology

In this manuscript all rings are commutative and Noetherian; in particular, R always denotes such a ring. We will often assume that R is *local*, meaning that it has a unique maximal ideal $\mathfrak{m}$. Finitely generated modules are, for short, called *finite* modules.

(1.1.1) ***Complexes***. An *R–complex* X is a sequence of R–modules X_ℓ and R–linear maps $\partial_\ell^X : X_\ell \to X_{\ell-1}$, with $\partial_\ell^X \partial_{\ell+1}^X = 0$ for $\ell \in \mathbb{Z}$. The *degree* of an element x is denoted by $|x|$; that is $|x| = \ell \Leftrightarrow x \in X_\ell$. If $X_\ell = 0$ for $\ell \neq 0$, then we identify X with the module in degree 0; and an R–module M is considered as a complex $0 \to M \to 0$ with M in degree 0.

A *morphism* $\alpha : X \to Y$ of R–complexes is a sequence of R–linear maps $\alpha_\ell : X_\ell \to Y_\ell$, satisfying $\partial_\ell^Y \alpha_\ell - \alpha_{\ell-1}\partial_\ell^X = 0$ for $\ell \in \mathbb{Z}$. A morphism that induces an isomorphism in homology is called a *quasi-isomorphism*.

The numbers *supremum*, $\sup X = \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}$, *infimum*, $\inf X = \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}$, and *amplitude*, $\operatorname{amp} X = \sup X - \inf X$, capture the homological position and size of X. A complex X is *homologically trivial* if $\mathrm{H}(X) = 0$; by convention, $\sup X = -\infty$ and $\inf X = \infty$ for such complexes.

The *support* of X is the set

$$\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X_\mathfrak{p}) \neq 0\} = \bigcup_\ell \operatorname{Supp}_R \mathrm{H}_\ell(X).$$

(1.1.2) ***Derived Category***. We often work in $\mathcal{D}(R)$: the *derived category* of the category of R–modules. This is the category of R–complexes localized at the class of all quasi-isomorphisms, see [**24**], [**27**], or [**28**, Chapter 10] for details. The symbol $\simeq$ is used for isomorphisms in $\mathcal{D}(R)$, while $\cong$ denotes isomorphisms in the category of complexes (and, thereby, in the category of modules).

The full subcategories of $\mathcal{D}(R)$: $\mathcal{D}_+(R)$, $\mathcal{D}_-(R)$, $\mathcal{D}_\mathrm{b}(R)$, and $\mathcal{D}_0(R)$ consist of complexes X with $\mathrm{H}_\ell(X) = 0$ for, respectively, $\ell \ll 0$, $\ell \gg 0$, $|\ell| \gg 0$, and $\ell \neq 0$. By $\mathcal{D}^\mathrm{f}(R)$ we denote the full subcategory of complexes with finite homology modules. We also use combined notations: $\mathcal{D}_-^\mathrm{f}(R) = \mathcal{D}_-(R) \cap \mathcal{D}^\mathrm{f}(R)$ etc.

The right derived functor of the homomorphism functor for R–complexes is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$, and $-\otimes^{\mathbf{L}}_R-$ is the left derived functor of the tensor product functor for R–complexes.

(1.1.3) ***Invariants***. The standard homological dimensions (projective, flat, and injective dimension of modules) extend to complexes, see [**4**] for details. We use two-letter abbreviations (pd, fd, and id) for these dimensions; and for convenience we denote by $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{I}(R)$ the full subcategories of $\mathcal{D}_\mathrm{b}(R)$ consisting of complexes of, respectively, finite projective, flat, and injective dimension. That is, $X \in \mathcal{P}(R) \Leftrightarrow \mathrm{pd}_R X < \infty$ etc. We freely use combined notations so, e.g., $\mathcal{F}_0(R)$ and $\mathcal{P}_0^\mathrm{f}(R)$ are (equivalent to) the full subcategories of, respectively, modules of finite flat dimension and finite modules of finite projective dimension.

The (*Krull*) *dimension* of a complex is given by:

$$\dim_R X = \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R X\};$$

and when R is local with residue field $k = R/\mathfrak{m}$, the *depth* of a complex $X \in \mathcal{D}_-(R)$ is defined as:

$$\mathrm{depth}_R X = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X)).$$

This definition goes back to [**20**], where it is also proved that the *Cohen–Macaulay defect*,

$$\mathrm{cmd}_R X = \dim_R X - \mathrm{depth}_R X,$$

is non-negative for complexes $X \in \mathcal{D}_\mathrm{b}(R)$ with $\mathrm{depth}_R X < \infty$. In particular,

$$0 \leq \mathrm{cmd}_R X$$

for $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$, and X is said to be *Cohen–Macaulay* if and only if equality holds.

(1.1.4) ***Dualizing Complexes***. When R is local, a complex $D \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ is said to be *dualizing* for R if and only if it has finite injective dimension and the homothety morphism $\chi_D^R \colon R \to \mathbf{R}\mathrm{Hom}_R(D,D)$ is invertible. This definition goes back to [**24**].

(1.1.5) ***Koszul Complexes***. The *Koszul complex* on an element $x \in R$ is the complex $\mathrm{K}(x) = 0 \to R \xrightarrow{x} R \to 0$ concentrated in degrees 1 and 0. For a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of elements in R the Koszul complex $\mathrm{K}(\boldsymbol{x}) = \mathrm{K}(x_1,\ldots,x_n)$ is the tensor product $\mathrm{K}(x_1) \otimes_R \cdots \otimes_R \mathrm{K}(x_n)$. The Koszul complex on the empty sequence is R.

2. Fundamentals of Gorenstein Dimensions

The *G–dimension*, short for *Gorenstein dimension*, for finite modules was introduced by Auslander in [**1**] and developed in [**2**]. For a finite R–module M the G–dimension, $\mathrm{G\text{–}dim}_R M$, is defined as the minimal length of a resolution of M by modules from the, so-called, G–class. This class contains all finite projective modules, and the G–dimension is a *finer invariant* than the projective dimension in the (strict) sense that there is always an inequality:

$$\text{G–dim}_R\, M \leq \text{pd}_R\, M,$$

and equality holds if $\text{pd}_R\, M < \infty$.

If the Gorenstein dimension of a finite R–module M is finite, then it can be computed by non-vanishing of Ext modules of M against the ring:

(1.2.1) $$\text{G–dim}_R\, M = \sup\{m \in \mathbb{N}_0 \mid \text{Ext}^m_R(M, R) \neq 0\};$$

and if R is local, then

(1.2.2) $$\text{G–dim}_R\, M = \text{depth}\, R - \text{depth}_R\, M.$$

This extension [**2**, Theorem (4.13)(b)] of the classical Auslander–Buchsbaum formula is known as the *Auslander–Bridger formula*.

The name 'Gorenstein dimension' is justified by the next result [**1**, Théorème 3, p. 64].

(1.2.3) **Theorem.** *If R is a local ring with residue field k, then the following are equivalent:*

(*i*) *R is Gorenstein.*

(*ii*) $\text{G–dim}_R\, k < \infty$.

(*iii*) $\text{G–dim}_R\, M < \infty$ *for all finite R–modules M.*

This is parallel to the Auslander–Buchsbaum–Serre characterization of regular local rings, but to make the analogy complete a fourth condition dealing with non-finite modules is needed. So far, the most successful approach to Gorenstein dimensions for non-finite modules is the one taken by Enochs et al. in [**12, 13, 15**]. In combination with some powerful ideas due to Foxby, the work of Enochs' group has resulted in a beautiful theory for Gorenstein projective and flat dimensions (extensions of the original G–dimension) and Gorenstein injective dimension (dual to the Gorenstein projective one) over Cohen–Macaulay local rings with dualizing module, see [**22**] and [**16**]. A full account of this theory is given in [**9**], here we only give a survey of the basic ideas.

The G–class can be defined as follows (see [**1**, Proposition 8, p. 67] and [**10**, Proposition 3.(3.3)]):

(1.2.4) **Definition.** A finite R–module M belongs to the *G–class*, if and only if there exists a homologically trivial complex of finite free R–modules,

$$L \ = \ \cdots \to L_2 \to L_1 \to L_0 \to L_{-1} \to L_{-2} \to \cdots \ ,$$

which stays trivial under dualization (that is, also $\text{Hom}_R(L, R)$ is homologically trivial) and has $\text{Coker}(L_1 \to L_0) \cong M$.

This is not Auslander's original definition [**1**, p. 55], but it is far better suited for generalization and dualization. It led Enochs et al. to introduce Gorenstein injective modules as follows [**12**, Definition 2.1]:

(1.2.5) **Definitions.** A homologically trivial complex,

$$I \;=\; \cdots \to I_2 \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots\,,$$

of injective R–modules is said to be a *complete injective resolution* if and only if the complex $\mathrm{Hom}_R(J,I)$ is homologically trivial for every injective R–module J.

A module N is said to be *Gorenstein injective* if and only if there exists a complete injective resolution I with $\mathrm{Ker}(I_0 \to I_{-1}) \cong N$.

Dualizing this definition leads to a notion of Gorenstein projective modules [**16**, p. 3224]:

(1.2.6) **Definitions.** A homologically trivial complex,

$$P \;=\; \cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots\,,$$

of projective R–modules is said to be a *complete projective resolution* if and only if the complex $\mathrm{Hom}_R(P,Q)$ is homologically trivial for every projective R–module Q.

A module M is said to be *Gorenstein projective* if and only if there exists a complete projective resolution P with $\mathrm{Coker}(P_1 \to P_0) \cong M$.

This is an extension of the G–class: a finite module belongs to the G–class if and only if it is Gorenstein projective in the sense of (1.2.6); a proof due to Avramov et al. is given [**9**, Section 4.2].

Enochs et al. have also introduced a notion of Gorenstein flat modules [**15**, Definition 2.2]:

(1.2.7) **Definitions.** A homologically trivial complex,

$$F \;=\; \cdots \to F_2 \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots\,,$$

of flat R–modules is said to be a *complete flat resolution* if and only if the complex $J \otimes_R F$ is homologically trivial for every injective R–module J.

A module M is said to be *Gorenstein flat* if and only if there exists a complete flat resolution F with $\mathrm{Coker}(F_1 \to F_0) \cong M$.

Gorenstein projective, injective, and flat dimensions can now be defined as usual, that is, by way of resolutions. The existence of appropriate resolutions is immediate: if, for example, I' is an injective module, then the complex $I = 0 \to I' \xrightarrow{=} I' \to 0$, concentrated in degrees 1 and 0, is a complete injective resolution with $\mathrm{Ker}(I_0 \to I_{-1}) = I'$, so every injective module is Gorenstein injective. It is equally easy to see that every projective module is Gorenstein projective, and every flat module is Gorenstein flat.

One of the fundamental problems in the theory of Gorenstein dimensions is to characterize the modules and complexes of finite Gorenstein projective, flat, or injective dimension. Once finiteness of a Gorenstein dimension is established, it is usually easy to give formulas for computing it by certain derived functors, cf. (1.2.1).

3. Finiteness of Gorenstein Dimensions

For a local ring R, admitting a dualizing complex D, Foxby has introduced two full subcategories of $\mathcal{D}_{\mathrm{b}}(R)$, the so-called *Auslander categories* [**5**, (3.1)]. In the notation of [**AC**] these are the *Auslander class*, ${}_D\mathcal{A}(R)$, and the *Bass class* ${}_D\mathcal{B}(R)$:

(1.3.1) **Definitions.** A complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ belongs to ${}_D\mathcal{A}(R)$ if and only if $D \otimes^{\mathbf{L}}_R X$ is bounded and the canonical morphism $\gamma^D_X \colon X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ is invertible.

A complex $Y \in \mathcal{D}_{\mathrm{b}}(R)$ belongs to ${}_D\mathcal{B}(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded and the canonical morphism $\xi^D_Y \colon D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y) \to Y$ is invertible.

The Auslander and Bass classes are equivalent categories and extensions of the full subcategories of complexes of, respectively, finite flat dimension and finite injective dimension [**5**, Theorem (3.2)]:

(1.3.2) **Theorem.** *Let R be a local ring. If D is a dualizing complex for R, then there is a commutative diagram*

$$\begin{array}{ccc}
\mathcal{D}(R) & \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} & \mathcal{D}(R) \\
\cup\!\mathsf{I} & & \cup\!\mathsf{I} \\
{}_D\mathcal{A}(R) & \rightleftarrows & {}_D\mathcal{B}(R) \\
\cup\!\mathsf{I} & & \cup\!\mathsf{I} \\
\mathcal{F}(R) & \rightleftarrows & \mathcal{I}(R)
\end{array}$$

in which the vertical inclusions are full embeddings, and the unlabeled horizontal arrows are quasi-inverse equivalences of categories.

Through [**16**] and [**29**] this has become known as *Foxby duality*, and [**AC**] follows this convention; but the involved functors are covariant, so *Foxby equivalence* would be more appropriate.

If R is a Cohen–Macaulay local ring with a dualizing module D, then the complexes in the Auslander class ${}_D\mathcal{A}(R)$ are exactly those of finite Gorenstein projective or, equivalently, finite Gorenstein flat dimension, see [**22, 16, 17, 10**], and R is Gorenstein if and only if all bounded complexes have finite Gorenstein projective/flat dimension, i.e., ${}_D\mathcal{A}(R) = \mathcal{D}_{\mathrm{b}}(R)$. The complexes in ${}_D\mathcal{B}(R)$ are those of finite Gorenstein injective dimension. The Gorenstein flat dimension, like the usual flat one, cf. [**8**, Corollary 1.2], satisfies a formula of the Auslander–Buchsbaum type, cf. [**22**, (4.2)]. That is, for $X \in {}_D\mathcal{A}(R)$ there is an equality:

(1.3.3) $$\mathrm{Gfd}_R\, X = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\};$$

needless to say, 'Gfd' stands for 'Gorenstein flat dimension'.

The Gorenstein projective and flat dimensions are extensions of Auslander's G–dimension for finite modules so, in particular, the finite modules in ${}_D\mathcal{A}(R)$ are those of finite G–dimension in the sense of [**1**]. By the next result, due to Foxby, this is actually true over any local ring admitting a dualizing complex.

(1.3.4) **Theorem.** *If R is local and admits a dualizing complex D, then*

$$ {}_D\mathcal{A}^{\mathrm{f}}(R) = {}_R\mathcal{R}(R). $$

In particular, the next biconditional holds for finite R–modules.

$$ M \in {}_D\mathcal{A}_0^{\mathrm{f}}(R) \quad \Longleftrightarrow \quad \text{G–dim}_R\, M < \infty. $$

Here ${}_R\mathcal{R}(R)$ denotes the full subcategory of R–reflexive complexes (see below); by [**30**, Theorem (2.7)] the modules in ${}_R\mathcal{R}(R)$ are exactly those of finite G–dimension, and this holds true over any Noetherian ring.

In [**30**] Yassemi studied G–dimension of complexes (notation has been adjusted to agree with [**AC**]):

(1.3.5) **Definitions.** A complex $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *R–reflexive* if and only if $\mathbf{R}\mathrm{Hom}_R(Z,R)$ is bounded and the biduality morphism

$$ \delta_Z^R :\ Z \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Z,R),R) $$

is invertible.

The *G–dimension*, $\text{G–dim}_R\, Z$, of $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is defined as follows:

$$ \text{G–dim}_R\, Z = \begin{cases} -\inf\left(\mathbf{R}\mathrm{Hom}_R(Z,R)\right) & \text{if } Z \in {}_R\mathcal{R}(R); \quad \text{and} \\ \infty & \text{if } Z \notin {}_R\mathcal{R}(R). \end{cases} $$

By (1.2.1) and [**30**, Theorem (2.7)] this is an extension of Auslander's concept. This definition makes no mention of resolutions (it looks like a functorial dimension), but in [**10**, Chapter 2] it is proved that the R–reflexive complexes are exactly those that allow a bounded resolution by modules from the G–class.

Yassemi also extended the Auslander–Bridger formula (1.2.2) to complexes [**30**, Theorem 2.9]:

(1.3.6) **Theorem.** *If R is local and Z is an R–reflexive complex, then*

$$ \text{G–dim}_R\, Z = \operatorname{depth} R - \operatorname{depth}_R Z. $$

So much for the introduction. Some of my own contributions to the theory of Gorenstein dimensions proper are explained in the next chapter, and others will appear in [**9**]. Generalizations of Auslander's G–dimension are dealt with in chapter three.

CHAPTER 2

Contributions to the Theory of Gorenstein Dimensions

While working on [**9**] I have been pondering the open questions related to Gorenstein dimensions, and in a few cases I have been able to give, at least, partial answers. The purpose of this chapter is to advertise some of these results from [**9**] and [**TD**].

1. Gorenstein Flat and Injective Modules

A classical result, due to Ishikawa [**25**], says that an R–module M is flat if and only if $\mathrm{Hom}_R(M,E)$ is injective for every injective R–module E. It is natural to ask if something similar holds for Gorenstein flat and Gorenstein injective modules, and this section gives an affirmative answer. We start with a straightforward lemma.

(2.1.1) **Lemma.** *Let E be an injective R–module. If F is a complete flat resolution, then* $\mathrm{Hom}_R(F,E)$ *is a complete injective resolution; and the converse holds if E is faithfully injective.*

Proof. If F is a complex of flat modules and E is injective, then $\mathrm{Hom}_R(F,E)$ is a complex of injective modules. Furthermore, if F is homologically trivial, then so is $\mathrm{Hom}_R(F,E)$; and the converse holds if E is faithfully injective. For every (injective) module J we have

$$\mathrm{Hom}_R(J \otimes_R F, E) \cong \mathrm{Hom}_R(J, \mathrm{Hom}_R(F,E)),$$

by adjointness, so if $J \otimes_R F$ is homologically trivial, then so is $\mathrm{Hom}_R(J, \mathrm{Hom}_R(F,E))$; and, again, the converse holds if E is faithful. $\square$

It is almost immediate from this lemma that $\mathrm{Hom}_R(M,E)$ is Gorenstein injective if E is injective and M is Gorenstein flat; the converse, however, requires another ingredient also developed by Enochs [**11**]:

(2.1.2) **Definition.** Let M be an R–module. A homomorphism $\phi\colon M \to F$, where F is a flat R–module, is said to be a *flat preenvelope* of M if and only if the sequence

$$\mathrm{Hom}_R(F,F') \xrightarrow{\mathrm{Hom}_R(\phi,F')} \mathrm{Hom}_R(M,F') \to 0$$

is exact for every flat R–module F'. That is, if F' is flat and $\nu : M \to F'$ is a homomorphism, then there exists a $\nu' \in \mathrm{Hom}_R(F,F')$ such that $\nu = \nu'\phi$.

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\nu"'] & F \arrow[d, "\nu'"] \\ & F' \end{tikzcd}$$

By [**11**, Proposition 5.1] every module over a Noetherian ring has a flat preenvelope.

(2.1.3) **Theorem.** *An R–module M is Gorenstein flat if and only if* $\mathrm{Hom}_R(M,E)$ *is Gorenstein injective for every injective R–module E.*

PROOF. “Only if”: Let F be a complete flat resolution with $\mathrm{Coker}(F_1 \to F_0) \cong M$, and let E be injective. Then, by the lemma above, $\mathrm{Hom}_R(F,E)$ is a complete injective resolution, and it follows by exactness of the functor $\mathrm{Hom}_R(-,E)$ that

$$\begin{aligned}\mathrm{Ker}(\mathrm{Hom}_R(F,E)_0 \to \mathrm{Hom}_R(F,E)_{-1}) &\cong \mathrm{Hom}_R(\mathrm{Coker}(F_1 \to F_0),E)\\ &\cong \mathrm{Hom}_R(M,E),\end{aligned}$$

so $\mathrm{Hom}_R(M,E)$ is Gorenstein injective as wanted.

“If”: We assume that $\mathrm{Hom}_R(M,E)$ is Gorenstein injective for every injective module E, and we set out to construct a complete flat resolution F with $\mathrm{Coker}(F_1 \to F_0) \cong M$. If we can construct a short exact sequence

$$(*) \qquad 0 \to M \to F_{-1} \to C_{-1} \to 0,$$

where F_{-1} is flat and C_{-1} is a module with the same properties as M (that is, $\mathrm{Hom}_R(C_{-1},E)$ is Gorenstein injective for every injective module E), then the right half of a complex F of flat modules can be constructed recursively: the n-th step supplies a flat module F_{-n} (and an obvious differential) and a module C_{-n} with the same properties as M. The left half of F we get for free by taking a flat resolution of M, and a complex F established this way is homologically trivial with $\mathrm{Coker}(F_1 \to F_0) \cong M$. Let E be a faithfully injective R–module, and consider the homologically trivial complex $\mathrm{Hom}_R(F,E)$ of injective modules. For brevity we will use the notation

$$K_\ell = \mathrm{Ker}(\mathrm{Hom}_R(F,E)_\ell \to \mathrm{Hom}_R(F,E)_{\ell-1}).$$

We want to prove that $\mathrm{Hom}_R(J,\mathrm{Hom}_R(F,E))$ is homologically trivial for every injective module J; this is the case when $\mathrm{Hom}_R(J,-)$ leaves all the short exact sequences

$$(**) \qquad 0 \to K_\ell \to \mathrm{Hom}_R(F,E)_\ell \to K_{\ell-1} \to 0$$

exact, so it is sufficient to show that $\mathrm{Ext}^1_R(J,K_\ell)=0$ for all $\ell \in \mathbb{Z}$ and every injective module J. On the other hand, if N is Gorenstein injective, then

$$(\dagger) \qquad \mathrm{Ext}^m_R(J,N)=0$$

for all $m>0$ and all injective modules J, cf. [**12**, Proposition 2.4]. As above it follows by exactness of $\mathrm{Hom}_R(-,E)$ that

$$(\dagger\dagger) \qquad K_\ell \cong \mathrm{Hom}_R(\mathrm{Coker}(F_{-\ell+1} \to F_{-\ell}),E),$$

so for $\ell>0$ the kernel K_ℓ is a Gorenstein injective module, because $\mathrm{Coker}(F_{-\ell+1} \to F_{-\ell}) = C_{-\ell}$ is a module with the same properties as M. Let J be an injective module, for $\ell>0$ we then have $\mathrm{Ext}^1_R(J,K_\ell)=0$, cf. ($\dagger$); and using standard identities derived from $(**)$, it follows by $(\dagger\dagger)$, $(\dagger)$, and the assumptions on M that

$$\begin{aligned}\operatorname{Ext}^1_R(J,K_\ell) &= \operatorname{Ext}^{1-\ell}_R(J,K_0)\\ &= \operatorname{Ext}^{1-\ell}_R(J,\operatorname{Hom}_R(M,E)) = 0\end{aligned}$$

for $\ell \leq 0$. Thus, $\operatorname{Hom}_R(J,\operatorname{Hom}_R(F,E))$ is homologically trivial for every injective module J; that is, $\operatorname{Hom}_R(F,E)$ is a complete injective resolution and, therefore, F is a complete flat resolution, cf. Lemma (2.1.1). To complete the proof it is now sufficient to construct the short exact sequence $(*)$.

Let $\mathrm{E}_R(M)$ denote the injective hull of M (or any other injective module containing M), then $M^\vee = \operatorname{Hom}_R(M,\mathrm{E}_R(M))$ is a Gorenstein injective module, so by definition, cf. (1.2.5), we have a short exact sequence

$$0 \to Z \to I \xrightarrow{\partial} M^\vee \to 0,$$

where I is injective. Applying the exact functor $-^\vee = \operatorname{Hom}_R(-,\mathrm{E}_R(M))$ we get another short exact sequence

$$0 \to M^{\vee\vee} \xrightarrow{\partial^\vee} I^\vee \to Z^\vee \to 0.$$

The biduality map $\delta\colon M \to M^{\vee\vee}$, given by $m \longmapsto [\mu \mapsto \mu(m)]$, is injective, so we have an injective map $\nu = \partial^\vee\delta$ from M into the flat module $I^\vee$. Let $\phi\colon M \to F_{-1}$ be a flat preenvelope of M, then there exists a homomorphism $\nu' \in \operatorname{Hom}_R(F_{-1},I^\vee)$ such that $\nu = \nu'\phi$, and since ν is injective, so is ϕ. With $C_{-1} = \operatorname{Coker}\phi$ we now have an exact sequence

$$(\ddagger) \qquad 0 \to M \xrightarrow{\phi} F_{-1} \to C_{-1} \to 0.$$

Let E be an injective R–module, we now want to prove that $\operatorname{Hom}_R(C_{-1},E)$ is Gorenstein injective. From $(\ddagger)$ we get a short exact sequence

$$0 \to \operatorname{Hom}_R(C_{-1},E) \to \operatorname{Hom}_R(F_{-1},E) \xrightarrow{\operatorname{Hom}_R(\phi,E)} \operatorname{Hom}_R(M,E) \to 0,$$

where the module $\operatorname{Hom}_R(F_{-1},E)$ is injective and $\operatorname{Hom}_R(M,E)$ is Gorenstein injective by assumption. To prove that also $\operatorname{Hom}_R(C_{-1},E)$ is Gorenstein injective it is, by [**12**, Theorem 2.13], sufficient to see that $\operatorname{Ext}^1_R(J,\operatorname{Hom}_R(C_{-1},E)) = 0$ for all injective modules J. Let J be injective, $\operatorname{Ext}^1_R(J,\operatorname{Hom}_R(C_{-1},E))$ vanishes if and only if the map

$$\operatorname{Hom}_R(J,\operatorname{Hom}_R(\phi,E)):\ \operatorname{Hom}_R(J,\operatorname{Hom}_R(F_{-1},E)) \longrightarrow \operatorname{Hom}_R(J,\operatorname{Hom}_R(M,E))$$

is surjective, so we consider the commutative diagram

$$\begin{array}{ccc}\operatorname{Hom}_R(J,\operatorname{Hom}_R(F_{-1},E)) & \xrightarrow{\operatorname{Hom}_R(J,\operatorname{Hom}_R(\phi,E))} & \operatorname{Hom}_R(J,\operatorname{Hom}_R(M,E))\\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow\\ \operatorname{Hom}_R(F_{-1},\operatorname{Hom}_R(J,E)) & \xrightarrow[\operatorname{Hom}_R(\phi,\operatorname{Hom}_R(J,E))]{} & \operatorname{Hom}_R(M,\operatorname{Hom}_R(J,E))\end{array}$$

The module $\operatorname{Hom}_R(J,E)$ is flat and ϕ is a flat preenvelope of M, so the induced map $\operatorname{Hom}_R(\phi,\operatorname{Hom}_R(J,E))$ is surjective, cf. (2.1.2), and hence so is $\operatorname{Hom}_R(J,\operatorname{Hom}_R(\phi,E))$. This concludes the proof. $\square$

I should mention that a special case of Theorem (2.1.3) follows from [**13**, Lemma 3.4]: a module M over a Gorenstein ring is Gorenstein flat if and only if its Pontryagin dual, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, is Gorenstein injective.

2. Tor–dimensions

The *restricted Tor–dimension*, $\mathrm{Td}_R\, X$, of $X \in \mathcal{D}_+(R)$ has been introduced by Foxby as:

$$\mathrm{Td}_R\, X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{F}_0(R)\}.$$

This functorial dimension is always finite (provided that R has finite Krull dimension), and it satisfies a formula of the Auslander–Buchsbaum type:

(2.2.1) $$\mathrm{Td}_R\, X = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

for $X \in \mathcal{D}_{\mathrm{b}}(R)$, see [**10**, 4.(1.7)].

The restricted Tor–dimension is a finer invariant than the flat dimension, so Chouinard's formula [**8**, Corollary 1.2]:

(2.2.2) $$\mathrm{fd}_R\, M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\},$$

for modules of finite flat dimension, is a special case of (2.2.1). If R is a Cohen–Macaulay local ring with a dualizing module, then the restricted Tor–dimension is a finer invariant than the Gorenstein flat dimension, so also the formula (1.3.3) is a special case of (2.2.1); this makes the latter the *ultimate Auslander–Buchsbaum formula.*

In [**TD**] we take a closer look on the restricted Tor–dimension, and because it is a finer invariant than the Gorenstein flat dimension (as described above), the study also sheds new light on this dimension.

For a non-local ring the Cohen–Macaulay defect is the supremum over defects at all prime or, equivalently, all maximal ideals:

$$\operatorname{cmd} R = \sup\{\dim R_{\mathfrak{p}} - \operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Rings with $\operatorname{cmd} R \leq 1$ are said to be "almost" Cohen–Macaulay, and over such rings one can compute the restricted Tor–dimension using only cyclic test modules:

(2.2.3) **Theorem.** [**TD**, (4.3)] *If* $\operatorname{cmd} R \leq 1$ *and* $X \in \mathcal{D}_+(R)$*, then*

$$\mathrm{Td}_R\, X = \sup\{\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} = x_1, \dots, x_n \text{ is an } R\text{–sequence}\}.$$

In fact, this characterizes local rings of Cohen–Macaulay defect at most one [**TD**, (4.2)]; and local Cohen–Macaulay rings are characterized by:

(2.2.4) **Theorem.** [**TD**, (4.4)] *A local ring* R *is Cohen–Macaulay if and only if*

$$\mathrm{Td}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X$$

for all complexes $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

In view of (2.2.1) the theorem tells us that a local ring R is Cohen–Macaulay if and only if

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \operatorname{depth} R - \operatorname{depth}_R X$$

for all complexes $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$. This is a special case of the next result.

(2.2.5) **Theorem.** [**TD**, (4.7)] *When* $\mathfrak{p}$, $\mathfrak{q}$, *and* $\mathfrak{m}$ *denote ideals* $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ *and* $\mathfrak{m} \in \operatorname{Max} R$, *the following equalities hold:*

$$\begin{aligned}
\operatorname{cmd} R = \sup\{&(\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}) \\
&\qquad - (\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} X_{\mathfrak{q}}) \mid X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R),\ \mathfrak{p} \subseteq \mathfrak{q}\} \\
= \sup\{&(\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) \\
&\qquad - (\operatorname{depth} R_{\mathfrak{m}} - \operatorname{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}) \mid M \in \mathcal{D}_{0}^{\mathrm{f}}(R),\ \mathfrak{p} \subseteq \mathfrak{m}\}.
\end{aligned}$$

3. The Bass Formula

Both the original Auslander–Buchsbaum formula, $\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ [**3**, Theorem 3.7], for finite modules of finite projective dimension over local rings, and Chouinard's formula for flat dimension (2.2.2) have been extended to Gorenstein dimensions. The first one by Auslander and Bridger, cf. (1.2.2), and the latter by Foxby cf. (1.3.3).

The celebrated *Bass Formula*, $\operatorname{id}_R N = \operatorname{depth} R$ [**7**, Lemma (3.3)], for finite modules $N \neq 0$ of finite injective dimension over local rings, can also be extended:

(2.3.1) **Theorem.** *If R is a Cohen–Macaulay local ring with a dualizing module, and $N \neq 0$ is a finite R–module of finite Gorenstein injective dimension, then*

$$\operatorname{Gid}_R N = \operatorname{depth} R.$$

Up to now this formula was only known to hold over Gorenstein local rings, cf. [**14**, Corollary 4.11]. I will not try to describe the new proof on the basis of last chapter's short introduction to Gorenstein dimensions, but it will appear in [**9**].

CHAPTER 3

Auslander Categories

This chapter deals with generalizations of Auslander's Gorenstein dimension. The first work in this direction was done by Golod [**23**] and Foxby [**18**].

For simplicity we will assume that all rings are local. In line with the conventions for rings, a homomorphism, $\varphi\colon R \to S$, of rings is said to be *finite* if it imposes on S the structure of a finite R–module. If $\mathfrak{m}$ and $\mathfrak{n}$ are the maximal ideals of R and S, and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, then φ is said to be *local*.

1. Gorenstein Dimension and Semi-dualizing Complexes

We start by demonstrating how the notion of semi-dualizing complexes is closely related to Auslander's concept of Gorenstein dimension for finite modules.

(3.1.1) **Definition.** [**AC**, (2.1)] An R–complex C is said to be *semi-dualizing* for R if and only if $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and the homothety morphism

$$\chi^R_C :\; R \longrightarrow \mathbf{R}\mathrm{Hom}_R(C, C)$$

is an isomorphism.

The immediate examples are dualizing complexes for R (these are semi-dualizing complexes of finite injective dimension, cf. (1.1.4)) and the ring itself.

A local ring R is Gorenstein if and only if the R–module R is dualizing; this is a classic characterization cf. [**24**, Proposition V.3.4 and Section V.10]. If S is a homomorphic image of such a ring, then $\mathbf{R}\mathrm{Hom}_R(S, R)$ is a dualizing complex for S; and, actually, the converse also holds:

(3.1.2) **Theorem.** [**AC**, (6.2) and (6.3)] *If $\varphi : R \to S$ is a finite local homomorphism, then $\mathbf{R}\mathrm{Hom}_R(S, R)$ is dualizing for S if and only if R is Gorenstein.*

Gorenstein or not, the ring R is always semi-dualizing, so it is natural to ask when $\mathbf{R}\mathrm{Hom}_R(S, R)$ will be semi-dualizing for S. The answer is provided by:

(3.1.3) **Theorem.** [**AC**, (6.1)] *If $\varphi\colon R \to S$ is a finite local homomorphism, then $\mathbf{R}\mathrm{Hom}_R(S, R)$ is semi-dualizing for S if and only if S has finite Gorenstein dimension over R, i.e.,* $\mathrm{G\text{–}dim}_R\, S < \infty$.

Thus, under finite local homomorphisms, semi-dualizing complexes relate to finiteness of Gorenstein dimensions the same way dualizing complexes relate to Gorensteinness of rings.

2. Generalized G–dimension

Inspired by Yassemi [**30**], cf. (1.3.5), we make the following:

(3.2.1) **Definitions.** [**AC**, (2.7) and (3.11)] Let C be a semi-dualizing complex for R. A complex $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *C–reflexive* if and only if $\mathbf{R}\mathrm{Hom}_R(Z, C)$ is bounded and the biduality morphism

$$\delta_Z^C : \; Z \;\longrightarrow\; \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Z, C), C)$$

generated by

$$z \;\longmapsto\; [\mu \mapsto (-1)^{|\mu||z|}\mu(z)]$$

is invertible.

By ${}_C\mathcal{R}(R)$ we denote the full subcategory of $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ whose objects are the C–reflexive complexes.

The *G–dimension* of $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ with respect to C is defined as follows:

$$\text{G–dim}_C\, Z = \begin{cases} \inf C - \inf\left(\mathbf{R}\mathrm{Hom}_R(Z, C)\right) & \text{if } Z \in {}_C\mathcal{R}(R); \quad \text{and} \\ \infty & \text{if } Z \notin {}_C\mathcal{R}(R). \end{cases}$$

The G–dimension with respect to a semi-dualizing complex C is a functorial dimension. For $C = R$ the definition agrees with Yassemi's [**30**, Definition 2.8], cf. (1.3.5), and in this case the dimension can be interpreted in terms of resolutions by modules from the G–class, see [**10**, Chapter 2]. More generally, if K is a semi-dualizing module (that is, a semi-dualizing complex concentrated in one degree), then the G–dimension with respect to K can be interpreted in terms of resolutions by modules from a generalized G–class. In [**23**] Golod showed how to do this for finite modules. On the other hand, let C be a semi-dualizing complex with $\operatorname{amp} C = s > 0$. We are free to assume that $\inf C = 0$, and it is then immediate from the definition that $\text{G–dim}_C\, C = 0$; but a resolution of C must have length at least s, so the G–dimension with respect to C can not be interpreted in terms of resolutions.

The next results show that the generalized G–dimension shares many of the nice properties of Auslander's original Gorenstein dimension.

First of all, it is a finer invariant than the projective dimension:

(3.2.2) **Proposition.** [**AC**, (3.15)] *Let C be a semi-dualizing complex for R. For $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ there is an inequality:*

$$\text{G–dim}_C\, Z \leq \operatorname{pd}_R Z,$$

and equality holds if $\operatorname{pd}_R Z < \infty$.

There is a formula of the Auslander–Buchsbaum type:

(3.2.3) **Theorem.** [**AC**, (3.14)] *Let C be a semi-dualizing complex for R. If Z is C–reflexive, then*

$$\text{G–dim}_C\, Z = \operatorname{depth} R - \operatorname{depth}_R Z.$$

The next result is the general version of Theorem (3.1.3).

(3.2.4) **Theorem.** [**AC**, (6.1)] *Let $\varphi\colon R \to S$ be a finite local homomorphism, and let C be a semi-dualizing complex for R. Then $\mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S if and only if S has finite G–dimension with respect to C, i.e.,* $\mathrm{G\text{–}dim}_C\, S < \infty$.

And last but not least we have the *ultimate addition formula*:

(3.2.5) **Theorem.** [**AC**, (6.5)] *Let $\varphi\colon R \to S$ be a finite local homomorphism, and let C be a semi-dualizing complex for R. If* $\mathrm{G\text{–}dim}_C\, S$ *is finite, then the complex $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S, and for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ there is an equality:*

$$\mathrm{G\text{–}dim}_C\, Z = \mathrm{G\text{–}dim}_C\, S + \mathrm{G\text{–}dim}_{S^{\dagger_C}}\, Z.$$

In particular, $\mathrm{G\text{–}dim}_C\, Z$ *and* $\mathrm{G\text{–}dim}_{S^{\dagger_C}}\, Z$ *are simultaneously finite; that is,*

$$Z \in {}_C\mathcal{R}(R) \quad\Longleftrightarrow\quad Z \in {}_{S^{\dagger_C}}\mathcal{R}(S).$$

To substantiate the alleged importance of this result, let's work out a couple of special cases to recover the addition formulas found in the literature.

If S is R modulo a regular sequence, $S = R/(x_1,\dots,x_n)$, and φ is the canonical surjection, then $\mathrm{G\text{–}dim}_R\, S = \mathrm{pd}_R\, S = n$, cf. Proposition (3.2.2). Thus, $\mathbf{R}\mathrm{Hom}_R(S,R)$ is semi-dualizing for S, and it is easy to see that

$$\mathbf{R}\mathrm{Hom}_R(S,R) \simeq \mathrm{Hom}_R(\mathrm{K}(x_1,\dots,x_n),R) \simeq \pm^{-n} S.$$

($\pm^{-n}S$ is the complex with S in degree $-n$ and 0 elsewhere.) By the definition, (3.2.1), the G–dimension is independent of the position of the semi-dualizing complex [**AC**, (3.12)], so for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ the addition formula reads:

$$\mathrm{G\text{–}dim}_R\, Z = n + \mathrm{G\text{–}dim}_S\, Z.$$

This particular case of Theorem (3.2.5) was proved in part by Auslander and Bridger [**2**, Corollary (4.33)], and the full strength can be derived from a result due to Peskine and Szpiro [**2**, Proposition (4.35)]. That such a formula holds is a remarkable feature of the Gorenstein dimension and not shared by the projective dimension. This reflects the fact that Gorensteinness of a ring is preserved when we "mod out" a regular element, but regularity is not.

A finite local homomorphism $\varphi : R \to S$ is, so-called, *quasi-Gorenstein* if and only if $\mathrm{G\text{–}dim}_R\, S < \infty$ and $\mathbf{R}\mathrm{Hom}_R(S,R) \sim S$, see [**5**]. In this more general case the addition formula takes the form of [**5**, Theorem (7.11)]:

$$\mathrm{G\text{–}dim}_R\, Z = \mathrm{G\text{–}dim}_R\, S + \mathrm{G\text{–}dim}_S\, Z.$$

Let K be a semi-dualizing module for R, and assume that S is a homomorphic image of R such that $\mathrm{G\text{–}dim}_K\, S < \infty$ and $\mathbf{R}\mathrm{Hom}_R(S,K)$ is concentrated in one degree; that is, $\mathbf{R}\mathrm{Hom}_R(S,K)$ is a semi-dualizing module for S, and we will call it K'. The addition formula now reads:

$$\mathrm{G\text{–}dim}_K\, Z = \mathrm{G\text{–}dim}_K\, S + \mathrm{G\text{–}dim}_{K'}\, Z.$$

This was originally proved by Golod [**23**, Proposition 5], and in the terminology of [**23**], S is R modulo a G_K–perfect ideal.

A dualizing complex D for R is Cohen–Macaulay [**AC**, (3.5)], and all complexes $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ are D–reflexive, cf. [**24**, Proposition V.2.1]. The next result, therefore, includes the well-known formula, $\operatorname{cmd}_R Z = \operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(Z, D))$ [**20**, Proposition 3.14], for Cohen–Macaulay defect of complexes with finite homology.

(3.2.6) **Theorem.** [**AC**, (3.8)] *Let C be a Cohen–Macaulay semi-dualizing complex for R. If $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is C–reflexive, i.e., $\mathrm{G}\text{–}\dim_C Z < \infty$, then*

$$\operatorname{cmd}_R Z = \operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(Z, C)).$$

Also a number of other results from the literature can be read off directly from this new "CMD formula". E.g., if R is Cohen–Macaulay and M is a finite module with $\mathrm{G}\text{–}\dim_R M < \infty$, then M is Cohen–Macaulay if and only if

$$\operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(M, R)) = \mathrm{G}\text{–}\dim_R M - \operatorname{grade}_R M = 0;$$

that is, if and only if M is *quasi-perfect* in the sense of [**19**]. In particular, a finite R–module of finite projective dimension is *perfect* ($\operatorname{pd}_R M = \operatorname{grade}_R M$) if and only if it is Cohen–Macaulay.

3. Generalized Foxby Equivalence

Inspired by Foxby [**5**, (3.1)], cf. (1.3.1), we make the following:

(3.3.1) **Definitions.** [**AC**, (4.1)] Let C be a semi-dualizing complex for R. The *C–Auslander class* of R, ${}_C\mathcal{A}(R)$, and the *C–Bass class* of R, ${}_C\mathcal{B}(R)$, are the full subcategories of $\mathcal{D}_{\mathrm{b}}(R)$ defined by specifying their objects as follows:

- A complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ belongs to ${}_C\mathcal{A}(R)$ if and only if $C \otimes_R^{\mathbf{L}} X$ is bounded and the canonical map
$$\gamma_X^C : \; X \longrightarrow \mathbf{R}\mathrm{Hom}_R(C, C \otimes_R^{\mathbf{L}} X)$$
generated by
$$x \longmapsto [c \mapsto (-1)^{|c||x|} c \otimes x]$$
is an isomorphism.
- A complex $Y \in \mathcal{D}_{\mathrm{b}}(R)$ belongs to ${}_C\mathcal{B}(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(C, Y)$ is bounded and the canonical map
$$\xi_Y^C : \; C \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(C, Y) \longrightarrow Y$$
generated by
$$c \otimes \mu \longmapsto (-1)^{|c||\mu|} \mu(c)$$
is an isomorphism.

We want to see how far the analogy with Theorem (1.3.2) goes. The first step is to ascertain that:

(3.3.2) **Proposition.** [**AC**, (4.4)] *If C is a semi-dualizing complex for R, then there are two full embeddings:*

$$\mathcal{F}(R) \subseteq {}_C\mathcal{A}(R) \qquad \textit{and} \qquad \mathcal{I}(R) \subseteq {}_C\mathcal{B}(R).$$

The next step is to show that the Auslander and Bass classes are equivalent categories.

(3.3.3) **Theorem.** [**AC**, (4.6)] *Let C be a semi-dualizing complex for R. The two functors $C \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(C, -)$ give quasi-inverse equivalences of ${}_C\mathcal{A}(R)$ and ${}_C\mathcal{B}(R)$, and they take semi-dualizing complexes to semi-dualizing complexes.*

And the analogy goes no further than this: only when the complex C is truly dualizing do the equivalences restrict to the subcategories $\mathcal{F}(R)$ and $\mathcal{I}(R)$ [**AC**, (8.2)].

A connection to the generalized G–dimension is provided by the next generalization of Theorem (1.3.4).

(3.3.4) **Theorem.** [**AC**, (4.7) and (2.12)] *Let R be local, and let C be a semi-dualizing complex for R. If D is dualizing for R, then $C^{\dagger_D} = \mathbf{R}\mathrm{Hom}_R(C, D)$ is semi-dualizing for R, and there is an equality of full subcategories:*

$$ {}_C\mathcal{A}^{\mathrm{f}}(R) = {}_{C^{\dagger_D}}\mathcal{R}(R). $$

An attempt to establish a theory of generalized G–dimensions for non-finite modules might, therefore, take the Auslander and Bass classes ${}_C\mathcal{A}(R)$ and ${}_C\mathcal{B}(R)$ as its starting point.

CHAPTER 4

Sequences and Parameters

The depth, as defined in (1.1.3), is without doubt the most successful functorial dimension to date. Classical proofs of results like the Auslander–Buchsbaum formula [**3**, Theorem 3.7] use the connection between regular sequences and depth for finite modules. Modern proofs, like that of Chouinard's formula [**8**, Corollary 1.2], cf. (2.2.2), draw on properties of depth without referring to sequences; and they do so with good reason, because the sequences may not be there!

Now, with its importance well-established — and expanding into the realm of non-commutative algebra through the work of Jørgensen [**26**] and others — it is tempting to ask how and when the depth of a complex can be interpreted in terms of sequences. This chapter reports on some of the answers to that question given in [**SC**] and [**PC**].

Throughout the chapter R is a local ring with maximal ideal $\mathfrak{m}$.

1. Depth of Complexes and Length of Sequences

Our approach to sequences for complexes goes via extensions to complexes of the notions of zero-divisors and associated prime ideals.

(4.1.1) **Definitions.** [**SC**, (2.1) and (3.3)] For a complex $Y \not\simeq 0$ in $\mathcal{D}_-(R)$ we denote by $\mathrm{z}_R\, Y$ the set of zero-divisors for the top homology module; that is

$$\mathrm{z}_R\, Y = \mathrm{z}_R(\mathrm{H}_{\sup Y}(Y)).$$

An element $x \in R$ is said to be *regular* for Y if and only if $x \notin \mathrm{z}_R\, Y$. If $Y \not\simeq 0$, then a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in R is said to be a *Y–sequence* if and only if $\mathrm{K}(\boldsymbol{x}) \otimes_R Y \not\simeq 0$ and x_j is regular for $\mathrm{K}(x_1, \ldots, x_{j-1}) \otimes_R Y$ for each $j \in \{1, \ldots, n\}$.

For an R–module M these definitions agree with the classical notions of zero-divisors, M–regular elements, and M–sequences [**SC**, (2.5) and (3.4)].

A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ is said to be a *maximal Y–sequence* if and only if it is a Y–sequence and not the first part of a longer one [**SC**, (4.1)]. Because R is Noetherian, every Y–sequence can be extended to a maximal one [**SC**, (4.3)]. The next result is an extension of the classical interpretation of depth for finite modules.

(4.1.2) **Theorem.** [**SC**, (5.1) and (5.5)] *If $Y \not\simeq 0$ belongs to $\mathcal{D}^{\mathrm{f}}_-(R)$, then any permutation of a Y–sequence is, again, a Y–sequence. The number*

$$\operatorname{depth}_R Y + \sup Y$$

is the maximal length of a Y–sequence; and any maximal Y–sequence is of this *length.*

Furthermore, the following inequalities hold:

$$\operatorname{depth}_R Y + \sup Y \leq \dim_R(\mathrm{H}_{\sup Y}(Y)) \leq \dim R.$$

As the next result shows, finite depth of a bounded complex Y always implies an upper bound for the length of a Y–sequence, and this bound is, itself, bounded by the Krull dimension of the ring and the amplitude of the complex.

(4.1.3) **Proposition.** [**SC**, (5.9)] *Let* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *and assume that* $\operatorname{depth}_R Y < \infty$. *If* $\boldsymbol{x} = x_1, \ldots, x_n$ *is a* Y*–sequence, then*

$$n \leq \operatorname{depth}_R Y + \sup Y \leq \dim_R Y + \sup Y \leq \dim R + \operatorname{amp} Y.$$

From the above, these sequences for complexes seem to provide a reasonable extension of the classical notion; there is, however, one snag: they may fail to localize properly [**SC**, (3.13)]. This leads us to make the following:

(4.1.4) **Definitions.** [**SC**, (2.3) and (3.3)] For a complex $Y \in \mathcal{D}_{-}(R)$ we set

$$\mathrm{Z}_R Y = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R Y} \mathfrak{p};$$

where

$$\operatorname{Ass}_R Y = \{\mathfrak{p} \in \operatorname{Supp}_R Y \mid \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p} = 0\}.$$

An element $x \in R$ is said to be *strongly regular* for Y if and only if $x \notin \mathrm{Z}_R Y$. If $Y \not\simeq 0$, then a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in R is said to be a *strong* Y*–sequence* if and only if $\mathrm{K}(\boldsymbol{x}) \otimes_R Y \not\simeq 0$ and x_j is strongly regular for $\mathrm{K}(x_1, \ldots, x_{j-1}) \otimes_R Y$ for each $j \in \{1, \ldots, n\}$.

There is always an inclusion [**SC**, (2.4)]:

$$\mathrm{z}_R Y \subseteq \mathrm{Z}_R Y,$$

so every strong Y–sequence is a Y–sequence [**SC**, (3.4)]. For an R–module M the set $\operatorname{Ass}_R M$ is just the usual set of associated prime ideals [**SC**, (2.5)], so $\mathrm{z}_R M = \mathrm{Z}_R M$, and M–sequences and strong M–sequences are one and the same thing [**SC**, (3.8)].

Thus, Y–sequences and strong Y–sequences both extend the classical notion of M–sequences; for (real) complexes, however, the two notions are different, and the strong sequences have a desirable property:

(4.1.5) **Theorem.** [**SC**, (3.11)] *Let* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *and* $\mathfrak{p} \in \operatorname{supp}_R Y$; *if* $\boldsymbol{x} = x_1, \ldots, x_n$ *is a strong* Y*–sequence in* $\mathfrak{p}$, *then* ${}^{x_1}/_1, \ldots, {}^{x_n}/_1$ *in the maximal ideal of* $R_\mathfrak{p}$ *is a strong* $Y_\mathfrak{p}$*–sequence.*

That $\mathfrak{p}$ is in the *homological support* $\operatorname{supp}_R Y$ means that $Y_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} \not\simeq 0$. For $Y \in \mathcal{D}_{+}(R)$ there is always an inclusion $\operatorname{supp}_R Y \subseteq \operatorname{Supp}_R Y$, and equality holds if $Y \in \mathcal{D}^{\mathrm{f}}_{+}(R)$, cf. [**20**, Section 3].

For a bounded complex of finite depth an upper bound for the length of a strong sequence is provided by Proposition (4.1.3). Even for complexes with finite homology there may be more sequences than strong sequences [**SC**, (3.13)], but the maximal ones have the same length:

(4.1.6) **Theorem.** [**SC**, (5.7)] *Let $Y \not\simeq 0$ belong to $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$. A maximal strong Y–sequence is a maximal Y–sequence; in particular, the maximal length of a strong Y–sequence is a well-determined number n:*

$$n = \operatorname{depth}_R Y + \sup Y \leq \dim_R(\mathrm{H}_{\sup Y}(Y)) \leq \dim R;$$

and any maximal strong Y–sequence is of this length.

As a matter of course, a strong Y–sequence is deemed maximal if and only if it is not the first part of a longer one, and every strong Y–sequence can be extended to a maximal one [**SC**, (4.1) and (4.3)].

2. Dimension of Complexes and Systems of Parameters

To study parameters for complexes it is natural to go via a generalized notion of minimal prime ideals:

(4.2.1) **Definition.** [**PC**, (2.1)] Let $Y \in \mathcal{D}_+(R)$; we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an *anchor prime ideal* for Y if and only if $\dim_{R_\mathfrak{p}} Y_\mathfrak{p} = -\inf Y_\mathfrak{p} > -\infty$. The set of anchor prime ideals for Y is denoted by $\operatorname{Anc}_R Y$; that is,

$$\operatorname{Anc}_R Y = \{\mathfrak{p} \in \operatorname{Supp}_R Y \mid \dim_{R_\mathfrak{p}} Y_\mathfrak{p} + \inf Y_\mathfrak{p} = 0\}.$$

For an R–module M the anchor prime ideals are exactly the minimal ones in the support [**PC**, (2.3)], and the next result extends the classical connection between parameters and Krull dimension for finite modules.

(4.2.2) **Theorem.** [**PC**, (2.9)] *If Y belongs to $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$, then the next two numbers are equal.*

$$\begin{aligned} \mathrm{d}(Y) &= \dim_R Y + \inf Y; \quad \textit{and} \\ \mathrm{s}(Y) &= \inf\{s \in \mathbb{N}_0 \mid \exists\, x_1, \ldots, x_s : \mathfrak{m} \in \operatorname{Anc}_R(\mathrm{K}(x_1, \ldots, x_s) \otimes_R Y)\}. \end{aligned}$$

This justifies the following:

(4.2.3) **Definitions.** [**PC**, (3.1)] Let Y belong to $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ and set $d = \dim_R Y + \inf Y$. A set of elements $x_1, \ldots, x_d \in \mathfrak{m}$ are said to be a *system of parameters* for Y if and only if $\mathfrak{m} \in \operatorname{Anc}_R(\mathrm{K}(x_1, \ldots, x_d) \otimes_R Y)$.

A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ is said to be a *Y–parameter sequence* if and only if it is part of a system of parameters for Y.

The amplitude and the Cohen–Macaulay defect are often considered to be dual measures of size of complexes; this view is warranted further by:

(4.2.4) **Theorem.** [**PC**, (3.7)] *The following hold for $Y \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$:*

(a) *If $\operatorname{amp} Y = 0$, then any Y–sequence is a Y–parameter sequence.*
(b) *If $\operatorname{cmd}_R Y = 0$, then any Y–parameter sequence is a strong Y–sequence.*

Finally, sequences and parameters for complexes are linked together by the next result.

(4.2.5) **Theorem.** [**PC**, (3.9)] *Let C be a Cohen–Macaulay semi-dualizing complex for R, and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $\mathfrak{m}$. If Y is C–reflexive, then $\boldsymbol{x}$ is a Y–parameter sequence if and only if it is a $\mathbf{R}\mathrm{Hom}_R(Y, C)$–sequence; that is*

$$\boldsymbol{x} \text{ is a } Y\text{–parameter sequence} \iff \boldsymbol{x} \text{ is a } \mathbf{R}\mathrm{Hom}_R(Y, C)\text{–sequence.}$$

A dualizing complex D for R is Cohen–Macaulay [**AC**, (3.5)] and every complex $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is D–reflexive, cf. [**24**, Proposition V.2.1], so in particular we have the following:

(4.2.6) **Corollary.** *If D is a dualizing complex for R and $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{m}$ is a Y–parameter sequence if and only if it is a $\mathbf{R}\mathrm{Hom}_R(Y, D)$–sequence.*

My motivation for studying parameters for complexes actually came from an early version of this corollary [**SC**, (5.10)] going back to Foxby's [**21**, Remark 12.13].

Closing Remarks

I am now left with only one duty which I perform with joy. I record my gratitude to everyone with whom I have had the opportunity to discuss mathematics: teachers, colleagues, and students alike.

Bibliography

1. Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles.
2. Maurice Auslander and Mark Bridger, *Stable module theory*, American Mathematical Society, Providence, R.I., 1969, Memoirs of the American Mathematical Society, No. 94.
3. Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405.
4. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155.
5. ______, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270.
6. ______, *Cohen-Macaulay properties of ring homomorphisms*, Adv. Math. **133** (1998), no. 1, 54–95.
7. Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
8. Leo G. Chouinard, II, *On finite weak and injective dimension*, Proc. Amer. Math. Soc. **60** (1976), 57–60 (1977).
9. Lars Winther Christensen, *Gorenstein Dimensions*, notes in preparation.
10. ______, *Gorenstein dimensions*, Master's thesis, Matematisk Institut, Københavns Universitet, 1996.
11. Edgar E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), no. 3, 189–209.
12. Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633.
13. ______, *Gorenstein injective and flat dimensions*, Math. Japon. **44** (1996), no. 2, 261–268.
14. ______, *Gorenstein injective dimension and Tor-depth of modules*, Arch. Math. (Basel) **72** (1999), no. 2, 107–117.
15. Edgar E. Enochs, Overtoun M. G. Jenda, and Blas Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9.
16. Edgar E. Enochs, Overtoun M. G. Jenda, and Jinzhong Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3223–3234.
17. Hans-Bjørn Foxby, *Auslander categories*, in preparation.
18. ______, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973).
19. ______, *Quasi-perfect modules over Cohen-Macaulay rings*, Math. Nachr. **66** (1975), 103–110.
20. ______, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
21. ______, *A homological theory for complexes of modules*, Preprint Series 1981, nos. 19a&b, Matematisk Institut, Københavns Universitet, 1981.
22. ______, *Gorenstein dimension over Cohen-Macaulay rings*, Proceedings of international conference on commutative algebra (W. Bruns, ed.), Universität Onsabrück, 1994.
23. Evgeniy S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66, Algebraic geometry and its applications.
24. Robin Hartshorne, *Residues and duality*, Springer-Verlag, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20.

25. Takeshi Ishikawa, *On injective modules and flat modules*, J. Math. Soc. Japan **17** (1965), 291–296.
26. Peter Jørgensen, *Non-commutative graded homological identities*, J. London Math. Soc. (2) **57** (1998), no. 2, 336–350.
27. Jean-Louis Verdier, *Catégories dérivées. quelques résultats* (*état 0*), SGA $4\frac{1}{2}$, Springer, Berlin Heidelberg New York, 1977, pp. 262–311.
28. Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.
29. Jinzhong Xu, *Flat covers of modules*, Springer-Verlag, Berlin, 1996.
30. Siamak Yassemi, *G-dimension*, Math. Scand. **77** (1995), no. 2, 161–174.

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Part II

A CLOSER LOOK ON TOR–DIMENSIONS

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Contents

Introduction

Let R be a commutative Noetherian ring. The flat dimension of an R–module M can be computed by non-vanishing of Tor modules,

$$\text{(I.1)} \qquad \mathrm{fd}_R\, M = \sup\,\{m \in \mathbb{N}_0 \mid \mathrm{Tor}_m^R(T,M) \neq 0 \text{ for some module } T\},$$

and, accordingly, it is sometimes referred to as the *Tor–dimension*. If the flat dimension of M is finite, then it can be computed by Chouinard's formula [6, Cor. 1.2]:

$$\text{(I.2)} \qquad \mathrm{fd}_R\, M = \sup\,\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}_{R_\mathfrak{p}}\, M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

($\mathrm{depth}_{R_\mathfrak{p}}\, M_\mathfrak{p}$ denotes the number of the first non-vanishing $\mathrm{Ext}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}, M_\mathfrak{p})$ module.)

Foxby has studied the *restricted* Tor–dimension:

$$\mathrm{Td}_R\, M = \sup\,\{m \in \mathbb{N}_0 \mid \mathrm{Tor}_m^R(T,M) \neq 0 \text{ for some module } T \text{ with } \mathrm{fd}_R\, T < \infty\}.$$

Over a ring of finite Krull dimension this restricted Tor–dimension is finite for every R–module M, and it can always be computed by the formula:

$$\text{(I.3)} \qquad \mathrm{Td}_R\, M = \sup\,\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}_{R_\mathfrak{p}}\, M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

Chouinard's formula generalizes the classical Auslander–Buchsbaum formula, $\mathrm{pd}_R\, M = \mathrm{depth}\, R - \mathrm{depth}_R\, M$ [1, Thm. 3.7], for finite modules of finite projective dimension over a local ring, and (I.2) is often called the *Auslander–Buchsbaum formula for flat dimension.* The restricted Tor–dimension is a *finer invariant* than the flat dimension: by this we mean that $\mathrm{Td}_R\, M \leq \mathrm{fd}_R\, M$ for all R–modules M, and equality holds if $\mathrm{fd}_R\, M < \infty$. In particular, Chouinard's formula (I.2) is a special case of (I.3), and the latter is, therefore, called the *ultimate Auslander–Buchsbaum formula.*

When testing flat dimension by non-vanishing of Tor modules, cf. (I.1), it is sufficient to use finite, even cyclic, test modules. It is natural to ask if something similar holds for the restricted Tor–dimension, and it is easily verified — see (3.6) — that the answer, in general, is negative. However, testing by only finite modules of finite flat dimension gives rise to a new invariant, the *small* restricted Tor–dimension,

$$\operatorname{td}_R M = \sup\{m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(T, M) \neq 0 \text{ for some f.g. module } T \text{ with } \operatorname{pd}_R T < \infty\},$$

with interesting properties of its own. E.g., this new dimension always satisfies the formula:

$$\operatorname{td}_R M = \sup\{\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

($\operatorname{depth}_R(\mathfrak{p}, M)$ is sometimes called the grade of $\mathfrak{p}$ on M.)

In this paper we investigate the small restricted Tor–dimension, and we ask when it coincides with the large one. The latter question leads to a short study of "almost" Cohen–Macaulay rings (these are the rings with $\dim R_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \leq 1$ at every prime ideal $\mathfrak{p}$), and we show how (almost) Cohen–Macaulay rings can be characterized by properties of the restricted Tor–dimensions.

1. Homological Algebra for Complexes

Throughout this paper R is a non-trivial, commutative, and Noetherian ring.

We say that R is *local* if it has a unique maximal ideal $\mathfrak{m}$, and the residue field $R/\mathfrak{m}$ is then denoted by k. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ the residue field of the local ring $R_\mathfrak{p}$ is denoted by $k(\mathfrak{p})$, i.e., $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. As usual, the set of prime ideals containing an ideal $\mathfrak{a}$ is written $\mathrm{V}(\mathfrak{a})$.

Finitely generated modules are often, for brevity, called *finite* modules.

(1.1) ***Complexes***. An *R–complex* X is a sequence of R–modules X_ℓ and R–linear maps $\partial_\ell^X : X_\ell \to X_{\ell-1}$, the so-called *differentials*, satisfying $\partial_\ell^X \partial_{\ell+1}^X = 0$ for all $\ell \in \mathbb{Z}$. If $X_\ell = 0$ for $\ell \neq 0$, then we identify X with the module in degree 0; and an R–module M is considered as a complex $0 \to M \to 0$ with M in degree 0.

For $m \in \mathbb{Z}$ we denote by $\Sigma^m X$ the complex X *shifted* m degrees (to the left); it is given by $(\Sigma^m X)_\ell = X_{\ell-m}$ and $\partial_\ell^{\Sigma^m X} = (-1)^m \partial_{\ell-m}^X$.

A *morphism* $\alpha : X \to Y$ of R–complexes is a sequence of R–linear maps $\alpha_\ell : X_\ell \to Y_\ell$ with $\partial_\ell^Y \alpha_\ell - \alpha_{\ell-1}\partial_\ell^X = 0$ for all $\ell \in \mathbb{Z}$. We say that a morphism is a *quasi-isomorphism* if it induces an isomorphism in homology. The symbol $\simeq$ indicates quasi-isomorphisms, and $\cong$ indicates isomorphisms of complexes and, thereby, modules.

The homological position and size of a complex X are measured by the *supremum*, *infimum*, and *amplitude*:

$$\begin{aligned}
\sup X &= \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \\
\inf X &= \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \quad \text{and} \\
\operatorname{amp} X &= \sup X - \inf X.
\end{aligned}$$

If $X \simeq 0$, that is, if all the homology modules $\mathrm{H}_\ell(X)$ vanish, then $\sup X = -\infty$ and $\inf X = \infty$ by convention.

The *support* of a complex X is the set

$$\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \not\simeq 0\} = \cup_\ell \operatorname{Supp}_R \mathrm{H}_\ell(X).$$

As usual $\operatorname{Min}_R X$ and $\operatorname{Max}_R X$ are the subsets of, respectively, minimal and maximal elements in the support

(1.2) ***Derived Category***. The *derived category* of the category of R–modules is the category of R–complexes localized at the class of all quasi-isomorphisms (see [15], [19], or [20, Chp. 10] for details), we denote it by $\mathcal{D}(R)$. The symbol $\simeq$ is used for isomorphisms in $\mathcal{D}(R)$, and this is in agreement with the notation introduced above, because a morphism of complexes is a quasi-isomorphism if and only if it represents an isomorphism in the derived category.

The full subcategories $\mathcal{D}_+(R)$, $\mathcal{D}_-(R)$, $\mathcal{D}_\mathrm{b}(R)$, and $\mathcal{D}_0(R)$ consist of complexes X with $\mathrm{H}_\ell(X) = 0$ for, respectively, $\ell \ll 0$, $\ell \gg 0$, $|\ell| \gg 0$, and $\ell \neq 0$. By $\mathcal{D}^\mathrm{f}(R)$ we denote the full subcategory consisting of complexes X with all homology modules $\mathrm{H}_\ell(X)$ finitely generated over R. The category of R–modules, respectively, finite R–modules, is naturally identified with $\mathcal{D}_0(R)$, respectively, $\mathcal{D}_0^\mathrm{f}(R)$.

We also use combined notations, e.g., $\mathcal{D}_\mathrm{b}^\mathrm{f}(R) = \mathcal{D}_\mathrm{b}(R) \cap \mathcal{D}^\mathrm{f}(R)$. More generally: for a subcategory $\mathcal{S}(R) \subseteq \mathcal{D}(R)$ we set $\mathcal{S}^\mathrm{f}(R) = \mathcal{S}(R) \cap \mathcal{D}^\mathrm{f}(R)$, $\mathcal{S}_0(R) = \mathcal{S}(R) \cap \mathcal{D}_0(R)$, etc.

(1.3) ***Derived Functors***. The right derived functor of the homomorphism functor for R–complexes is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$, and $- \otimes_R^{\mathbf{L}} -$ is the left derived functor of the tensor product functor for R–complexes. By [3] and [18] no boundedness conditions are needed on the arguments, so for $X, Y \in \mathcal{D}(R)$ the complexes $\mathbf{R}\mathrm{Hom}_R(X,Y)$ and $X \otimes_R^{\mathbf{L}} Y$ are uniquely determined up to isomorphisms in $\mathcal{D}(R)$, and they have the expected functorial properties. Note that $\mathrm{Tor}_m^R(M,N) = \mathrm{H}_m(M \otimes_R^{\mathbf{L}} N)$ and $\mathrm{Ext}_R^m(M,N) = \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(M,N))$ for R–modules M and N and $m \in \mathbb{N}_0$.

Let $\mathfrak{p}$ be a prime ideal in R. There are isomorphisms $(X \otimes_R^{\mathbf{L}} Y)_{\mathfrak{p}} \simeq X_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} Y_{\mathfrak{p}}$ and $\mathbf{R}\mathrm{Hom}_R(Z,Y)_{\mathfrak{p}} \simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(Z_{\mathfrak{p}}, Y_{\mathfrak{p}})$ in $\mathcal{D}(R_{\mathfrak{p}})$; the first one always holds, and the second holds when $Y \in \mathcal{D}_-(R)$ and $Z \in \mathcal{D}_+^\mathrm{f}(R)$, cf. [3, Lem. 5.2].

The next results are standard, cf. [12, Lem. 2.1]. Let $X \in \mathcal{D}_+(R)$ and $Y \in \mathcal{D}_-(R)$, then $\mathbf{R}\mathrm{Hom}_R(X,Y) \in \mathcal{D}_-(R)$ and there is an inequality:

$$\sup\left(\mathbf{R}\mathrm{Hom}_R(X,Y)\right) \leq \sup Y - \inf X. \tag{1.3.1}$$

Moreover, with $i = \inf X$ and $s = \sup Y$ (both finite) we have $\mathrm{H}_{s-i}(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y))$; in particular,

$$\sup\left(\mathbf{R}\mathrm{Hom}_R(X,Y)\right) = \sup Y - \inf X \iff \mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y)) \neq 0. \tag{1.3.2}$$

Let $X, Y \in \mathcal{D}_+(R)$, then $X \otimes_R^{\mathbf{L}} Y \in \mathcal{D}_+(R)$ and there is an inequality:

$$\inf\left(X \otimes_R^{\mathbf{L}} Y\right) \geq \inf X + \inf Y. \tag{1.3.3}$$

With $i = \inf X$ and $j = \inf Y$ we have $\mathrm{H}_{i+j}(X \otimes_R^{\mathbf{L}} Y) = \mathrm{H}_i(X) \otimes_R \mathrm{H}_j(Y)$; in particular,

$$\inf\left(X \otimes_R^{\mathbf{L}} Y\right) = \inf X + \inf Y \iff \mathrm{H}_i(X) \otimes_R \mathrm{H}_j(Y) \neq 0. \tag{1.3.4}$$

(1.4) ***Depth***. Over a local ring R the *depth* of a complex $Y \in \mathcal{D}_-(R)$ is defined as:

$$\operatorname{depth}_R Y = -\sup\left(\mathbf{R}\mathrm{Hom}_R(k, Y)\right),$$

cf. [13, Sec. 3]. For finite modules this agrees with the classical definition.

For $Y \in \mathcal{D}_-^{\mathrm{f}}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$ there is an inequality (R is still local):

$$\operatorname{depth}_R X \leq \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p}; \tag{1.4.1}$$

this follows by a complex version of [5, Lem. (3.1)], cf. [10, Chp. 13].

In the following R is any (commutative Noetherian) ring. For $Y \in \mathcal{D}_-(R)$ the first inequality below is immediate by (1.3.1):

$$\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \geq -\sup Y_\mathfrak{p} \geq -\sup Y; \tag{1.4.2}$$

and for $Y \not\simeq 0$ it follows by (1.3.2) that

$$\mathfrak{p} \in \operatorname{Ass}_R(\mathrm{H}_{\sup Y}(Y)) \quad\Longleftrightarrow\quad \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} = -\sup Y. \tag{1.4.3}$$

Let $\mathfrak{a}$ be an ideal in R and let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite set of generators for $\mathfrak{a}$. By definition, cf. [16, Sec. 2], the $\mathfrak{a}$–*depth* of $Y \in \mathcal{D}(R)$ is the number:

$$\operatorname{depth}_R(\mathfrak{a}, Y) = t - \sup\left(\mathrm{K}(\boldsymbol{a}) \otimes_R Y\right). \tag{1.4.4}$$

($\mathrm{K}(\boldsymbol{a})$ is the Koszul complex on the generators for $\mathfrak{a}$.) For ideals $\mathfrak{b} \supseteq \mathfrak{a}$ there are always inequalities [8, Thm. 4.7(b)]:

$$\operatorname{depth}_R(\mathfrak{b}, Y) \geq \operatorname{depth}_R(\mathfrak{a}, Y) \geq -\sup Y; \tag{1.4.5}$$

and for $Y \in \mathcal{D}_-(R)$ the relation between non-local and local depths is given by [8, Prop. 4.5]:

$$\operatorname{depth}_R(\mathfrak{a}, Y) = \inf\{\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}. \tag{1.4.6}$$

In particular, we have

$$\operatorname{depth}_R(\mathfrak{p}, Y) \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \tag{1.4.7}$$

for $Y \in \mathcal{D}_-(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$; and if R is local, then

$$\operatorname{depth}_R(\mathfrak{m}, Y) = \operatorname{depth}_R Y. \tag{1.4.8}$$

The $\mathfrak{a}$–depth is an extension to complexes of a well-known invariant, the grade, for (finite) modules. In particular, $\operatorname{depth}_R(\mathfrak{a}, R)$ is the maximal length of an R–sequence in $\mathfrak{a}$ (and the length of every maximal R–sequence in $\mathfrak{a}$).

The *Cohen–Macaulay defect* of a local ring is the (always non-negative) difference between the Krull dimension and the depth:

$$\operatorname{cmd} R = \dim R - \operatorname{depth} R.$$

For a non-local ring R the Cohen–Macaulay defect is the supremum over defects at all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$, and by (1.4.11) below it is sufficient to look at the maximal ideals:

$$\operatorname{cmd} R = \sup\{\operatorname{cmd} R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{cmd} R_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max} R\}. \tag{1.4.9}$$

By (1.4.1) and the trivial inequality

$$\dim R_\mathfrak{p} + \dim R/\mathfrak{p} \leq \dim R \tag{1.4.10}$$

the Cohen–Macaulay defect cannot grow under localization. That is,

(1.4.11) $$\operatorname{cmd} R_{\mathfrak{p}} \leq \operatorname{cmd} R_{\mathfrak{q}}$$

for prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{Spec} R$.

2. Background in Homological Dimensions

The *projective*, *injective*, and *flat dimensions* are the standard homological dimensions; we use two-letter abbreviations (pd, id, and fd) for these dimensions.

(2.1) ***Computability***. In [3] the standard homological dimensions for modules, defined in terms of resolutions, are extended to complexes $X \in \mathcal{D}(R)$ and shown to be computable in terms of derived functors [3, Thm. 2.4 and Cor. 2.5]:

(2.1.1) $$\operatorname{pd}_R X = \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{D}_0(R)\};$$

(2.1.2) $$\operatorname{id}_R X = \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(T,X)) \mid T \in \mathcal{D}_0(R)\}; \quad \text{and}$$

(2.1.3) $$\operatorname{fd}_R X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{D}_0(R)\}.$$

These are extensions of the well-known formulas for computing homological dimensions of modules.

(2.2) ***Finiteness***. By $\mathcal{P}(R)$, $\mathcal{I}(R)$, and $\mathcal{F}(R)$ we denote the full subcategories of $\mathcal{D}_{\mathrm{b}}(R)$ consisting of complexes of, respectively, finite projective, injective, and flat dimension. That is, X belongs to $\mathcal{P}(R)$ if and only if $X \simeq P$, where P is a bounded complex of projective modules etc. Note that $\mathcal{F}_0(R)$ and $\mathcal{P}^{\mathrm{f}}_0(R)$ are equivalent to the full subcategories of, respectively, modules of finite flat dimension and finite modules of finite projective dimension.

If X is bounded, then so are the complexes $\mathbf{R}\mathrm{Hom}_R(P,X)$, $\mathbf{R}\mathrm{Hom}_R(X,I)$, and $F \otimes^{\mathbf{L}}_R X$ when $P \in \mathcal{P}(R)$, $I \in \mathcal{I}(R)$, and $F \in \mathcal{F}(R)$; there is, for example, an inequality [3, Thm. 2.4.P]:

(2.2.1) $$\inf(\mathbf{R}\mathrm{Hom}_R(P,X)) \geq \inf X - \operatorname{pd}_R P.$$

If R is local and $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then the next equalities hold by [3, Cor. 2.10.F and Prop. 5.5].

(2.2.2) $$\operatorname{pd}_R X = \operatorname{fd}_R X = \sup(X \otimes^{\mathbf{L}}_R k);$$

in particular, $\mathcal{P}^{\mathrm{f}}(R) = \mathcal{F}^{\mathrm{f}}(R)$.

If $M \in \mathcal{P}_0(R)$, i.e., M is a module of finite projective dimension, then

(2.2.3) $$\operatorname{pd}_R M \leq \dim R,$$

cf. [17, Thm. (3.2.6)].

The classical Auslander–Buchsbaum formula [1, Thm. 3.7], cited in the introduction, also extends to complexes [14, (0.1)]:

(2.3) **Auslander–Buchsbaum formula.** *If R is local and $X \in \mathcal{P}^{\mathrm{f}}(R)$, then*

$$\operatorname{pd}_R X = \operatorname{depth} R - \operatorname{depth}_R X. \quad \square$$

Actually, it is a special case of the following [14, Lem. 2.1]:

(2.4) **Theorem.** *Let R be local. If $X \in \mathcal{F}(R)$ and $Y \in \mathcal{D}_{\mathrm{b}}(R)$, then the next three equalities hold.*

(a) $$\operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = -\sup\,(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth}_R Y.$$

(b) $$\operatorname{depth}_R X = -\sup\,(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth} R.$$

(c) $$\operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{depth}_R X + \operatorname{depth}_R Y - \operatorname{depth} R.$$

In [16, Thm. 4.1] it was demonstrated that it is sufficient to take Y bounded on the left, i.e., $Y \in \mathcal{D}_-(R)$.

Finally, we recall a shrewd observation due to Auslander and Buchsbaum [2]; it often comes in handy.

(2.5) **Lemma.** *If R is local and $\dim R > 0$, then there exists a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ such that $\operatorname{depth} R_{\mathfrak{p}} = \dim R - 1$.*

In particular, for any local ring we have

$$\sup\{\operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \begin{cases} \dim R & \textit{if } R \textit{ is Cohen–Macaulay; and} \\ \dim R - 1 & \textit{if } R \textit{ is not Cohen–Macaulay.} \end{cases} \qquad \square$$

3. Tor–dimensions

We start by reviewing Foxby's Tor–dimension.

(3.1) ***Restricted Tor–dimension.*** Foxby has studied the (large) *restricted Tor–dimension*, $\operatorname{Td}_R X$, defined for $X \in \mathcal{D}_+(R)$ as:

(3.1.1) $$\operatorname{Td}_R X = \sup\{\sup\,(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{F}_0(R)\}.$$

For an R–module M the definition reads:

$$\operatorname{Td}_R M = \sup\{m \in \mathbb{N}_0 \mid \exists\, T \in \mathcal{F}_0(R) : \operatorname{Tor}^R_m(T, M) \neq 0\},$$

and this explains the name. The principal properties of this dimension are cataloged below, see [9] and [7] for proofs.

The following inequalities always hold:

(3.1.2) $$\sup X \leq \operatorname{Td}_R X \leq \sup X + \dim R;$$

in particular,

$$\operatorname{Td}_R X = -\infty \quad \Longleftrightarrow \quad X \simeq 0,$$

and if $\dim R < \infty$, then

$$\operatorname{Td}_R X < \infty \quad \Longleftrightarrow \quad X \in \mathcal{D}_{\mathrm{b}}(R).$$

The restricted Tor–dimension behaves as expected under localization; for every $\mathfrak{p} \in \operatorname{Spec} R$ there is an inequality:

(3.1.3) $$\operatorname{Td}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \operatorname{Td}_R X.$$

The restricted Tor–dimension is a finer invariant than the flat (Tor–) dimension, i.e., there is always an inequality:

$$\mathrm{Td}_R\, X \leq \mathrm{fd}_R\, X, \tag{3.1.4}$$

and equality holds if $\mathrm{fd}_R\, X < \infty$.

Finally, the restricted Tor–dimension satisfies the *ultimate Auslander–Buchsbaum formula*: If $X \in \mathcal{D}_{\mathrm{b}}(R)$, then

$$\mathrm{Td}_R\, X = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}. \tag{3.1.5}$$

One consequence of this formula is that

$$\mathrm{Td}_R\, X = \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{F}(R) \ \wedge\ U \not\simeq 0\}.$$

(3.2) **Definition.** We define the *small restricted Tor–dimension*, $\mathrm{td}_R\, X$, of $X \in \mathcal{D}_+(R)$ as:

$$\mathrm{td}_R\, X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{P}^{\mathrm{f}}_0(R)\}.$$

(3.3) **Observation.** Let $X \in \mathcal{D}_+(R)$. It is immediate from the definition that

$$\sup X = \sup(R \otimes^{\mathbf{L}}_R X) \leq \mathrm{td}_R\, X \leq \mathrm{Td}_R\, X \leq \sup X + \dim R, \tag{3.3.1}$$

cf. (3.1.2). In particular,

$$\mathrm{td}_R\, X = -\infty \quad \Longleftrightarrow \quad X \simeq 0,$$

and if $\dim R < \infty$, then

$$\mathrm{td}_R\, X < \infty \quad \Longleftrightarrow \quad X \in \mathcal{D}_{\mathrm{b}}(R).$$

By (3.1.5) the large restricted Tor–dimension is a supremum of differences in local depths; the next result shows that the small one is a supremum of differences in non-local depths.

(3.4) **Theorem.** *If $X \in \mathcal{D}_{\mathrm{b}}(R)$, then*

$$\begin{aligned}\mathrm{td}_R\, X &= \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{P}^{\mathrm{f}}(R) \ \wedge\ U \not\simeq 0\}\\ &= \sup\{\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R\}.\end{aligned}$$

Proof. It is clear from the definition that

$$\mathrm{td}_R\, X \leq \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{P}^{\mathrm{f}}(R) \ \wedge\ U \not\simeq 0\};$$

this leaves us two inequalities to prove.

Let a homologically non-trivial complex $U \in \mathcal{P}^{\mathrm{f}}(R)$ be given; we then want to prove the existence of a prime ideal $\mathfrak{p}$ such that

$$\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \leq \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X). \tag{$*$}$$

We can assume that $U \otimes^{\mathbf{L}}_R X \not\simeq 0$, otherwise $(*)$ holds for every $\mathfrak{p}$. Set $s = \sup(U \otimes^{\mathbf{L}}_R X)$, choose $\mathfrak{p}$ in $\mathrm{Ass}_R(\mathrm{H}_s(U \otimes^{\mathbf{L}}_R X))$, and choose by (1.4.6) a prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$, such that $\operatorname{depth}_R(\mathfrak{p}, R) = \operatorname{depth} R_{\mathfrak{q}}$. The first equality in the computation below follows by (1.4.3), the second by (2.4)(a) and (2.2.2), and the third by (2.3); the last inequality is by (1.4.7) and (1.4.2).

$$\begin{aligned}\sup(U \otimes_R^{\mathbf{L}} X) - \sup U &= -\operatorname{depth}_{R_{\mathfrak{p}}}(U_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} X_{\mathfrak{p}}) - \sup U\\ &= \operatorname{pd}_{R_{\mathfrak{p}}} U_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \sup U\\ &\leq \operatorname{pd}_{R_{\mathfrak{q}}} U_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \sup U\\ &= \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} U_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \sup U\\ &= \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{q}}} U_{\mathfrak{q}} - \sup U\\ &\leq \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X).\end{aligned}$$

Now, let $\mathfrak{p} \in \operatorname{Spec} R$ be given; the task is then to find a finite module T of finite projective dimension with

$$\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \leq \sup(T \otimes_R^{\mathbf{L}} X).$$

Set $d = \operatorname{depth}_R(\mathfrak{p}, R)$, choose a maximal R–sequence $\boldsymbol{x} = x_1, \dots, x_d$ in $\mathfrak{p}$, and set $T = R/(\boldsymbol{x})$. Then T belongs to $\mathcal{P}_0^{\mathrm{f}}(R)$ and $\mathrm{K}(\boldsymbol{x})$ is its minimal free resolution. By (1.4.4) and (1.4.5) we now have

$$\begin{aligned}\sup(T \otimes_R^{\mathbf{L}} X) &= \sup(\mathrm{K}(\boldsymbol{x}) \otimes_R X)\\ &= d - \operatorname{depth}_R((\boldsymbol{x}), X)\\ &\geq d - \operatorname{depth}_R(\mathfrak{p}, X)\\ &= \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X)\end{aligned}$$

as desired. □

(3.5) **Observation.** Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. In view of (1.4.8) it follows from the second equality in (3.4) that

$$\sup\{\operatorname{depth} R_{\mathfrak{m}} - \operatorname{depth}_{R_{\mathfrak{m}}} X_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max} R\} \leq \operatorname{td}_R X;$$

and in view of (1.4.6) and (1.4.5) we also have

$$\operatorname{td}_R X \leq \sup\{\operatorname{depth} R_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max}_R X\} + \sup X.$$

In particular: if R is local, then

(3.5.1) $$\operatorname{depth} R - \operatorname{depth}_R X \leq \operatorname{td}_R X \leq \operatorname{depth} R + \sup X.$$

The example below shows that the two restricted Tor–dimensions may differ, even for finite modules over local rings, and it shows that the small Tor–dimension can grow under localization. The latter, unfortunate, property is reflected in the non-local nature of the formula given in (3.4).

(3.6) **Example.** Let R be a local ring with $\dim R = 2$ and $\operatorname{depth} R = 0$. By (2.5) choose a prime ideal $\mathfrak{q}$ in R, such that $\operatorname{depth} R_{\mathfrak{q}} = 1$, choose an element $x \in \mathfrak{q}$ such that the fraction ${}^{x}/_{1}$ is $R_{\mathfrak{q}}$–regular, and set $M = R/(x)$. It follows by (3.5.1) that

$\operatorname{td}_R M = 0$, but

$$\begin{aligned}\operatorname{Td}_R M \geq \operatorname{Td}_{R_\mathfrak{q}} M_\mathfrak{q} \geq \operatorname{td}_{R_\mathfrak{q}} M_\mathfrak{q} &\geq \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} \\ &= 1 - 0 \\ &> \operatorname{td}_R M\end{aligned}$$

by (3.1.3), (3.3.1), and (3.5.1).

The ring considered above is of Cohen–Macaulay defect two, so in a sense — to be made clear by (4.2) — the example is a minimal one.

4. Almost Cohen–Macaulay Rings

If $\mathfrak{p}$ is a prime ideal in R and $x_1, \ldots, x_n$ is a maximal R–sequence in $\mathfrak{p}$, then the sequence of fractions ${}^{x_1}/_1, \ldots, {}^{x_n}/_1$ in $\mathfrak{p}_\mathfrak{p}$ is an $R_\mathfrak{p}$–sequence, but it need not be a maximal one. The present section will show that this basic fact, captured by (1.4.7), is what obstructs equality between the small and the large restricted Tor–dimension.

The first lemma gives a number of convenient characterizations of the rings where maximality of an R–sequence is preserved under localization; these are the "almost" Cohen–Macaulay rings.

(4.1) **Lemma.** *The following are equivalent:*

(i) $\operatorname{cmd} R \leq 1$.

(ii) $\operatorname{depth} R_\mathfrak{p} \leq \operatorname{depth} R_\mathfrak{q}$ *for all prime ideals* $\mathfrak{p} \subseteq \mathfrak{q}$.

(iii) $\operatorname{depth}_R(\mathfrak{p}, R) = \operatorname{depth} R_\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.

(iv) *For every* $\mathfrak{p} \in \operatorname{Spec} R$ *and every maximal R–sequence* $\boldsymbol{x} = x_1, \ldots, x_n$ *in* $\mathfrak{p}$ *the ideal* $\mathfrak{p}$ *is associated to* $R/(\boldsymbol{x})$*, that is,* $\mathfrak{p} \in \operatorname{Ass}_R R/(\boldsymbol{x})$.

(v) *For every* $\mathfrak{p} \in \operatorname{Spec} R$ *there exists a module* $M \in \mathcal{P}_0^{\mathrm{f}}(R)$ *with* $\mathfrak{p} \in \operatorname{Ass}_R M$.

(vi) *For every* $\mathfrak{p} \in \operatorname{Spec} R$ *there exists a complex* $X \in \mathcal{P}^{\mathrm{f}}(R)$ *such that* $\mathfrak{p}$ *is associated to the top homology module of* X*, that is,* $\mathfrak{p} \in \operatorname{Ass}_R(\mathrm{H}_{\sup X}(X))$.

Proof. Conditions (i) through (iv) are the equivalent conditions (3), (2), (4), and (5) in [11, Prop. 3.3].

(iv) ⇒ (v): Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a maximal R–sequence in $\mathfrak{p}$ and set $M = R/(\boldsymbol{x})$, then M is a finite R–module of finite projective dimension and $\mathfrak{p} \in \operatorname{Ass}_R M$.

(v) ⇒ (vi): Immediate because M is the top (and only) homology module of M viewed as a complex.

(vi) ⇒ (i): By (1.4.9) it is sufficient to prove that $\operatorname{cmd} R_\mathfrak{m} \leq 1$ for all maximal ideals $\mathfrak{m}$ in R, so we can assume that R is local. If $\dim R = 0$ there is nothing to prove, so we assume that $\dim R > 0$ and choose by (2.5) a prime ideal $\mathfrak{p}$, such that $\operatorname{depth} R_\mathfrak{p} = \dim R - 1$. Let $X \in \mathcal{P}^{\mathrm{f}}(R)$ be a complex with $\mathfrak{p} \in \operatorname{Ass}_R(\mathrm{H}_{\sup X}(X))$, then

$$\begin{aligned}\operatorname{depth} R &= \operatorname{pd}_R X + \operatorname{depth}_R X \\ &\geq \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup X \\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup X \\ &= \dim R - 1\end{aligned}$$

by (2.3), (1.4.2), and (1.4.3). □

(4.2) **Theorem.** *If R is local, then the following are equivalent:*

(i) $\operatorname{cmd} R \leq 1$.

(ii) $\operatorname{td}_R X = \operatorname{Td}_R X$ *for all complexes* $X \in \mathcal{D}_+(R)$.

(iii) $\operatorname{td}_R M = \operatorname{Td}_R M$ *for all finite R–modules M.*

Proof. The second condition is, clearly, stronger than the third, so there are two implications to prove.

(i) $\Rightarrow$ (ii): If X is not bounded, then $\infty = \operatorname{td}_R X = \operatorname{Td}_R X$, cf. (3.3.1). Suppose $X \in \mathcal{D}_\mathrm{b}(R)$; the (in)equalities in the computation below follow by, respectively, (3.4), (4.1), (1.4.6), and (3.1.5).

$$\begin{aligned}\operatorname{td}_R X &= \sup\{\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &\geq \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \operatorname{Td}_R X.\end{aligned}$$

The opposite inequality always holds, cf. (3.3.1), whence equality holds.

(iii) $\Rightarrow$ (i): We can assume that $\dim R > 0$ and choose a prime ideal $\mathfrak{q}$ such that $\operatorname{depth} R_\mathfrak{q} = \dim R - 1$, cf. (2.5). Set $M = R/\mathfrak{q}$, then

$$\operatorname{depth} R \geq \operatorname{td}_R M = \operatorname{Td}_R M \geq \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} = \dim R - 1$$

by (3.5.1) and (3.1.5). □

Over almost Cohen–Macaulay rings it is, actually, sufficient to use cyclic modules for testing the restricted Tor–dimensions:

(4.3) **Corollary.** *If* $\operatorname{cmd} R \leq 1$ *and* $X \in \mathcal{D}_+(R)$, *then*

$$\operatorname{td}_R X = \operatorname{Td}_R X = \sup\{\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} = x_1, \dots, x_n \text{ is an } R\text{–sequence}\}.$$

Proof. If X is not bounded, then

$$\operatorname{td}_R X = \operatorname{Td}_R X = \infty = \sup(R \otimes^{\mathbf{L}}_R X).$$

For $X \in \mathcal{D}_\mathrm{b}(R)$ the proof of (i) $\Rightarrow$ (ii) in (4.2) shows that $\operatorname{td}_R X = \operatorname{Td}_R X$, and it is obvious from the definition that

$$\sup\{\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} = x_1, \dots, x_n \text{ is an } R\text{–sequence}\} \leq \operatorname{td}_R X.$$

By (3.1.5) it is now sufficient to prove that for each $\mathfrak{p} \in \operatorname{Spec} R$ there is an R–sequence $\boldsymbol{x} = x_1, \dots, x_n$ such that

$$\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \leq \sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X).$$

But this is easy: let $\boldsymbol{x} = x_1, \dots, x_n$ be any maximal R–sequence in $\mathfrak{p}$, then, by (4.1), $\mathfrak{p}$ is associated to $R/(\boldsymbol{x})$, in particular, $\operatorname{depth}_{R_\mathfrak{p}}(R/(\boldsymbol{x}))_\mathfrak{p} = 0$, so by (1.4.2) and (2.4)(c) we have

$$\begin{aligned}\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) &\geq -\operatorname{depth}_{R_\mathfrak{p}}(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X)_\mathfrak{p} \\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}}(R/(\boldsymbol{x}))_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}. \quad \square\end{aligned}$$

For finite modules the large restricted Tor–dimension is a finer invariant than the projective dimension, and over a local ring we, therefore, have $\operatorname{Td}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ for $M \in \mathcal{P}_0^{\mathrm{f}}(R)$. Now we ask when such a formula holds for all finite modules:

(4.4) **Theorem.** *If R is local, then the following are equivalent:*

(i) R is Cohen–Macaulay.

(ii) $\operatorname{Td}_R X = \operatorname{depth} R - \operatorname{depth}_R X$ for all complexes $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.

(iii) $\operatorname{td}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ for all finite R–modules M.

Proof. (i) $\Rightarrow$ (ii): Let $\mathfrak{p} \in \operatorname{Spec} R$, by (1.4.1) we have

$$\begin{aligned}\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} &\leq \operatorname{depth} R_{\mathfrak{p}} - (\operatorname{depth}_R X - \dim R/\mathfrak{p})\\ &\leq \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} - \operatorname{depth}_R X\\ &\leq \dim R - \operatorname{depth}_R X\\ &= \operatorname{depth} R - \operatorname{depth}_R X.\end{aligned}$$

The desired equality now follows by (3.1.5).

(ii) $\Rightarrow$ (iii): Immediate as

$$\operatorname{depth} R - \operatorname{depth}_R M \leq \operatorname{td}_R M \leq \operatorname{Td}_R M$$

by (3.5.1) and (3.3.1).

(iii) $\Rightarrow$ (i): We assume that R is not Cohen–Macaulay and seek a contradiction. Set $d = \operatorname{depth} R$ and let $\boldsymbol{x} = x_1, \ldots, x_d$ be a maximal R–sequence. Since R is not Cohen–Macaulay, the ideal generated by the sequence is not $\mathfrak{m}$–primary; that is, there exists a prime ideal $\mathfrak{p}$ such that $(\boldsymbol{x}) \subseteq \mathfrak{p} \subset \mathfrak{m}$. Set $M = R/\mathfrak{p}$, then $\operatorname{depth}_R M > 0$, but $\operatorname{depth}_R(\mathfrak{p}, M) = 0$ and $\operatorname{depth}_R(\mathfrak{p}, R) = d$, so by (3.4) we have

$$\operatorname{depth} R - \operatorname{depth}_R M < d = \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, M) \leq \operatorname{td}_R M,$$

and the desired contradiction has been obtained. □

(4.5) **Corollary.** *If R is a Cohen–Macaulay local ring and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\operatorname{td}_R X = \operatorname{Td}_R X = \operatorname{depth} R - \operatorname{depth}_R X.$$

Proof. Immediate by (4.4) and (4.2). □

(4.6) **Observation.** For convenience we set

$$\operatorname{dd}_R(\mathfrak{p}, X) = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$$

for $X \in \mathcal{D}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$. If R is Cohen–Macaulay and $\mathfrak{q} \in \operatorname{Spec} R$, then $R_{\mathfrak{q}}$ is a local Cohen–Macaulay ring, cf. (1.4.9), and view of (3.1.5) it follows from (4.5) that

$$\operatorname{dd}_R(\mathfrak{p}, X) \leq \operatorname{dd}_R(\mathfrak{q}, X)$$

for all complexes $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and all prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$. This reflects next result.

(4.7) **Theorem.** *When* $\mathfrak{p}$, $\mathfrak{q}$, *and* $\mathfrak{m}$ *denote ideals* $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ *and* $\mathfrak{m} \in \operatorname{Max} R$, *the following equalities hold:*

$$\begin{aligned}\operatorname{cmd} R &= \sup\{\operatorname{dd}_R(\mathfrak{p}, X) - \operatorname{dd}_R(\mathfrak{q}, X) \mid X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R),\ \mathfrak{p} \subseteq \mathfrak{q}\}\\ &= \sup\{\operatorname{dd}_R(\mathfrak{p}, M) - \operatorname{dd}_R(\mathfrak{m}, M) \mid M \in \mathcal{D}^{\mathrm{f}}_{0}(R),\ \mathfrak{p} \subseteq \mathfrak{m}\}.\end{aligned}$$

Proof. We start by proving that the Cohen–Macaulay defect of R is greater than or equal to the first supremum: Let $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ be given; we may assume that R is local with maximal ideal $\mathfrak{q}$. The inequality in demand now follows by (1.4.1) and (1.4.10):

$$\begin{aligned}\operatorname{dd}_R(\mathfrak{p}, X) - \operatorname{dd}_R(\mathfrak{q}, X) &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - (\operatorname{depth} R - \operatorname{depth}_R X)\\ &\leq \dim R_{\mathfrak{p}} + \operatorname{depth}_R X - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \operatorname{depth} R\\ &\leq \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} - \operatorname{depth} R\\ &\leq \operatorname{cmd} R.\end{aligned}$$

It it obvious that the second supremum is no larger than the first. To see that the second supremum is larger than or equal to $\operatorname{cmd} R$, we are free to assume that R is local. If $\dim R = 0$ there is nothing to prove because there are no prime ideals $\mathfrak{p} \subset \mathfrak{m}$. If $\dim R > 0$ we choose by (2.5) a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ such that $\operatorname{depth} R_{\mathfrak{p}} = \dim R - 1$. Set $M = R/\mathfrak{p}$ and observe that $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$ and $\operatorname{depth}_R M > 0$, then

$$\begin{aligned}\operatorname{dd}_R(\mathfrak{p}, M) - \operatorname{dd}_R(\mathfrak{m}, M) &= \dim R - 1 - (\operatorname{depth} R - \operatorname{depth}_R M)\\ &= \operatorname{cmd} R - 1 + \operatorname{depth}_R M\\ &\geq \operatorname{cmd} R\end{aligned}$$

as desired. □

5. Comments on an Ext–dimension

Chouinard has also proved the following dual of (I.2) [6, Cor. 3.1]:

$$\operatorname{id}_R N = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

for modules of finite injective dimension. ($\operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ denotes the number of the first non-vanishing $\operatorname{Tor}^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, k(\mathfrak{p}))$ module, other words for this are 'codepth' and 'Tor–depth'.)

It is natural to look for a generalization of this formula, parallel to Foxby's generalization (I.3) of (I.2). This calls for a dual notion of the large restricted Tor–dimension, and the immediate dual is the (large) *restricted Ext–dimension* defined for R–modules N as:

$$\operatorname{Ed}_R N = \sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}^m_R(T, N) \neq 0 \text{ for some module } T \text{ with } \operatorname{pd}_R T < \infty\}.$$

It is shown below that this Ext–dimension is a finer invariant than the injective dimension, at least over almost Cohen–Macaulay rings. For a module N of finite injective dimension over such a ring we, therefore, have

$$\operatorname{Ed}_R N = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\};$$

but it is easy to see that this formula fails in general. Let R be local and not Cohen–Macaulay, and let T be a module with $\mathrm{pd}_R\, T = \dim R$, cf. [4, Prop. 5.4]. By (2.1.1) there is then an R–module N with $\mathrm{Ext}_R^{\dim R}(T, N) \neq 0$, so $\mathrm{Ed}_R\, N \geq \dim R$, but

$$\begin{aligned}\sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} &\leq \sup\{\operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \dim R - 1\end{aligned}$$

by Nakayama's lemma and (2.5).

(5.1) **Proposition.** *For every R–module N there is an inequality:*

$$\mathrm{Ed}_R\, N \leq \mathrm{id}_R\, N,$$

and equality holds if $\mathrm{id}_R\, N < \infty$ *and* $\mathrm{cmd}\, R \leq 1$.

Proof. The inequality is immediate by the definition and (2.1.2), and equality holds if N is trivial. If $\mathrm{id}_R\, N = n \in \mathbb{N}_0$, then

$$\mathrm{Ext}_R^n(R/\mathfrak{p}, N) \neq 0 \tag{$*$}$$

for some $\mathfrak{p} \in \operatorname{Spec} R$. This is because the Bass numbers,

$$\mu_R^m(\mathfrak{p}, N) = \operatorname{rank}_{k(\mathfrak{p})}(\mathrm{Ext}_R^m(R/\mathfrak{p}, N)_{\mathfrak{p}}),$$

are determined by the minimal injective resolution of N, which has length n, and, therefore, $\mu_R^n(\mathfrak{p}, N) \neq 0$ for some prime ideal $\mathfrak{p}$. Choose, by (4.1), a module $T \in \mathcal{P}_0^{\mathrm{f}}(R)$ with $\mathfrak{p} \in \mathrm{Ass}_R\, T$; then $R/\mathfrak{p}$ is isomorphic to a submodule of T, so we have a short exact sequence of R–modules

$$0 \to R/\mathfrak{p} \to T \to C \to 0,$$

which induces a long exact sequence:

$$\cdots \to \mathrm{Ext}_R^n(T, N) \to \mathrm{Ext}_R^n(R/\mathfrak{p}, N) \to \mathrm{Ext}_R^{n+1}(C, N) \to \cdots.$$

Since $\mathrm{Ext}_R^{n+1}(C, N) = 0$, cf. (2.1.2), while $\mathrm{Ext}_R^n(R/\mathfrak{p}, N) \neq 0$ by $(*)$, we conclude that also $\mathrm{Ext}_R^n(T, N)$ is non-vanishing. By the definition we now have

$$\mathrm{ed}_R\, N \geq n,$$

and, with that, the desired equality holds. □

References

1. Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405.
2. ______, *Homological dimension in Noetherian rings. II*, Trans. Amer. Math. Soc. **88** (1958), 194–206.
3. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155.
4. Hyman Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 18–29.
5. ______, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
6. Leo G. Chouinard, II, *On finite weak and injective dimension*, Proc. Amer. Math. Soc. **60** (1976), 57–60 (1977).
7. Lars Winther Christensen, *Gorenstein Dimensions*, notes in preparation.
8. ______, *Sequences for complexes*, to appear in Math. Scand.

9. ———, *Gorenstein dimensions*, Master's thesis, Matematisk Institut, Københavns Universitet, 1996.
10. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.
11. ———, *n-Gorenstein rings*, Proc. Amer. Math. Soc. **42** (1974), 67–72.
12. ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19.
13. ———, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
14. ———, *Homological dimensions of complexes of modules*, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Springer, Berlin, 1980, pp. 360–368.
15. Robin Hartshorne, *Residues and duality*, Springer-Verlag, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20.
16. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567.
17. Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.
18. Nicolas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.
19. Jean-Louis Verdier, *Catégories dérivées. quelques résultats (état 0)*, SGA $4\frac{1}{2}$, Springer, Berlin Heidelberg New York, 1977, pp. 262–311.
20. Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.

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Part III

SEMI-DUALIZING COMPLEXES AND THEIR AUSLANDER CATEGORIES

LARS WINTHER CHRISTENSEN

Abstract. Let R be a commutative Noetherian ring. We study R–modules, and complexes of such, with excellent duality properties. While their common properties are strong enough to admit a rich theory, we count among them such, potentially, diverse objects as *dualizing complexes* for R on one side, and on the other, the *ring* itself. In several ways, these two examples constitute the extremes, and their well-understood properties serve as guidelines for our study; however, also the employment, in recent studies of ring homomorphisms, of complexes "lying between" these extremes is incentive.

Contents

Introduction

In this paper all rings are commutative and Noetherian, in particular, R always denotes such a ring. We study R–complexes (that is, complexes of R–modules) with certain excellent duality properties. The canonical example is the *ring* itself, considered as a complex concentrated in degree zero. Another example is *dualizing complexes* for R; these were introduced by A. Grothendieck in [21] and have proved to be a powerful tool, as demonstrated by P. Roberts in [26], and by C. Peskine and L. Szpiro in [25]. Modules with excellent duality properties have been studied by H.–B. Foxby in [13], and in [19] by E.S. Golod, who used the name *suitable*[1] for these modules. Other complexes of the kind considered here were used tacitly by L.L. Avramov and H.–B. Foxby in their study of homological properties of ring homomorphisms [6], and consistently in the ensuing paper [5]; in the latter the name *relative dualizing complexes* was used. This paper offers a generalized and unified

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[1]The paper is in Russian and uses the word удобный, which allows several translations.

treatment of some of the notions and techniques studied and applied in [6], [5], and [19]; it provides a common language for stating and proving not only generalizations of results from these papers, but also new results.

The common language developed here is that of *semi-dualizing complexes*. To illustrate the idea, we start by looking at the modules among these complexes:

A *semi-dualizing module* for R is a finite (that is, finitely generated) R–module C with $\mathrm{Hom}_R(C,C)$ canonically isomorphic to R and $\mathrm{Ext}^i_R(C,C)=0$ for $i>0$. It is not obvious that a local ring may posses semi-dualizing modules other than itself and, possibly, a dualizing module. This was put forward as a question in 1985 by E.S. Golod, see [20], and in 1987 H.–B. Foxby gave examples of rings with three different semi-dualizing modules. As a spin-off to the results established here for semi-dualizing complexes, we will be able to describe a procedure for constructing Cohen–Macaulay local rings with any finite number of semi-dualizing modules; this is done in section 7.

Semi-dualizing complexes are defined — in section 2 — by generalizing the above definition to the derived category of R–modules.

Now, which are the generalized notions and techniques?

The *Gorenstein dimension*, or G–dimension, for finite modules was introduced by M. Auslander in [1] and developed in [2]; two of its main features are that it is a finer invariant than the projective dimension and that satisfies an equality of the Auslander–Buchsbaum type. In [31] S. Yassemi studied Gorenstein dimension for complexes through a consistent use of the functor $\mathbf{R}\mathrm{Hom}_R(-,R)$ and the related category $\mathcal{R}(R)$.

In section 3 we study the functor $\mathbf{R}\mathrm{Hom}_R(-,C)$ and the related category ${}_C\mathcal{R}(R)$ for a semi-dualizing complex C, and we show that it is possible to define a G–dimension with respect to C, sharing the nice properties of Auslander's classical G–dimension. That is, there is an inequality:

$$\text{G–dim}_C\, M \le \mathrm{pd}_R\, M$$

for all finite R–modules M, and equality holds if $\mathrm{pd}_R\, M<\infty$. Furthermore,

$$\text{G–dim}_C\, M = \operatorname{depth} R - \operatorname{depth}_R M$$

for finite modules with $\text{G–dim}_C\, M<\infty$.

Recall that a finite module is said to be *perfect* if its projective dimension equals its grade, and *Cohen–Macaulay* if its Krull dimension equals its depth. Perfectness and Cohen–Macaulayness capture qualitative properties of the module, and perfect modules and Cohen–Macaulay modules have always been considered close kin; indeed, a module of finite projective dimension over a Cohen–Macaulay ring is perfect if and only if it is Cohen–Macaulay.

As it is usual in modern algebra, numerical invariants are introduced to measure 'how far' a module is from possessing such qualitative properties. Our study will make it clear that the invariants *imperfection*, $\mathrm{imp}_R\, M = \mathrm{pd}_R\, M - \mathrm{grade}_R\, M$, and *Cohen–Macaulay defect*, $\mathrm{cmd}_R\, M = \dim_R M - \operatorname{depth}_R M$, for finite R–modules M, are close kin and in fact, by ($\sharp$) below, coincide for modules of finite projective dimension over a Cohen–Macaulay ring. Actually, the truly related invariants are the Cohen–Macaulay defect and the *quasi-imperfection*, cf. [14], and they coincide for modules of finite Gorenstein dimension (in the sense of [1]).

The study of ${}_C\mathcal{R}(R)$ also provides us with valuable information about the semi-dualizing complex C. It turns out that the Cohen–Macaulay defect of R is an upper bound for the Cohen–Macaulay defect as well as the amplitude (homological size) of C, and a lower bound for the sum of these invariants. These results are special cases of inequalities like the following:

$$(\sharp)\qquad \operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(M,C)) \leq \operatorname{cmd}_R M \leq \operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(M,C)) + \operatorname{cmd}_R C;$$

it holds for modules with $\text{G–dim}_C\, M < \infty$. The homological formula $\operatorname{cmd}_R M = \operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(M,D))$, which holds for finite modules M when D is a dualizing complex for R, is also contained in $(\sharp)$.

It was discovered in [6] that important properties of a local ring homomorphism $\varphi\colon R \to S$ of finite flat (Tor–) dimension are reflected in the way φ base changes a possible dualizing complex for the source ring R. And in [5] the larger class of homomorphisms of finite Gorenstein dimension was studied via associated semi-dualizing complexes for the target ring S. We pick up this track in sections 5 and 6 and ask the question, 'When does a semi-dualizing complex for the source ring, in a natural way, induce a semi-dualizing complex for the target ring?'

For finite local homomorphisms an answer can be neatly phrased in terms of the generalized G–dimension:

> **Theorem.** *Let C be a semi-dualizing complex for R. The complex $A = \mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S if and only if $\text{G–dim}_C\, S < \infty$; and when this is the case, the equality*
>
> $$\text{G–dim}_C\, N = \text{G–dim}_C\, S + \text{G–dim}_A\, N$$
>
> *holds for all finite S–modules N.*

This result represents a vast extension and generalization of [19, Proposition 5]. Also for non-finite homomorphisms the answer is related to the concept of G–dimension. To see this we introduce — in section 4 — two categories, the so-called *Auslander* and *Bass classes* ${}_C\mathcal{A}(R)$ and ${}_C\mathcal{B}(R)$, for a semi-dualizing complex C, and we show that the functors $C \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(C,-)$ provide quasi-inverse equivalences of these categories. For dualizing complexes these categories were introduced in [5], and the described equivalences were named *Foxby duality* in [10]. We establish a link to G–dimension by proving (essentially) that finite modules in an Auslander class will have finite G–dimension with respect to some semi-dualizing complex.

In general, by base changing a semi-dualizing complex C for the source ring R we obtain a semi-dualizing complex for the target ring S, if and only if S belongs to the Auslander class ${}_C\mathcal{A}(R)$. Every Auslander class will contain all R–modules of finite flat dimension, and we can prove more detailed results for homomorphisms of finite flat dimension, along with a variety of ascent and descent results. The general base change result also establishes a converse to the key result in [5] on existence of relative dualizing complexes.

Together, $\mathcal{R}$, $\mathcal{A}$, and $\mathcal{B}$ are known as *Auslander categories*, and the underlying idea of this paper is to study semi-dualizing complexes via their Auslander categories — hence the title. This idea is distinctly present in section 8; there we show how a local ring R and a dualizing complex for R can be distinguished from other semi-dualizing complexes for R by special properties of the functors and categories studied in the previous sections. We also prove that (up to isomorphism and shift) R is the only semi-dualizing complex of finite Gorenstein dimension (in the sense of [1]); and the well-known result that a semi-dualizing complex of finite injective dimension (a dualizing complex) is unique, is sharpened as we show that a semi-dualizing complex of finite Gorenstein injective dimension (in the sense of [11]) is unique, and in fact dualizing.

I avail myself of this opportunity to thank my supervisor[2] professor H.–B. Foxby for many valuable discussions concerning the material presented here.

1. Homological Algebra for Complexes

In this paper results are stated and proved in the derived category of the category of R–modules. In this first section we recall the vocabulary and some basic, but important, results.

First, a few conventions: All rings are commutative, Noetherian and non-trivial. A ring R is said to be *local* if it has a unique maximal ideal $\mathfrak{m}$, and we denote the $\mathfrak{m}$–adic completion by $\widehat{R}$. Applied to modules the word *finite* means finitely generated.

By the *flat dimension* of a homomorphism of rings $\varphi\colon R \to S$ we understand the flat (Tor-) dimension of S (with the imposed structure) over R; in particular, we say that φ is (*faithfully*) *flat* if it makes S a (faithfully) flat R–module. We call φ *finite* if it makes S a finite R–module, and we say that φ is *local* if R and S are local rings and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the maximal ideals of R and S.

(1.1) ***Complexes***. An *R–complex* X is a sequence of R–modules X_i and R–linear maps $\partial_i^X : X_i \to X_{i-1}$, $i \in \mathbb{Z}$. The module X_i is called the *module in degree* i, and ∂_i^X is the *i-th differential*; composition of two consecutive differentials always yields the zero map, i.e., $\partial_i^X \partial_{i+1}^X = 0$. If $X_i = 0$ for $i \neq 0$ we identify X with the module in degree 0, and an R–module M is thought of as a complex $0 \to M \to 0$, with M in degree 0. When m is an integer we denote by $\Sigma^m X$ the complex X *shifted* m degrees (to the left); it is given by $(\Sigma^m X)_i = X_{i-m}$ and $\partial_i^{\Sigma^m X} = (-1)^m \partial_{i-m}^X$.

The homological position and size of a complex is captured by the numbers *supremum*, *infimum*, and *amplitude* defined by

$$\begin{aligned} \sup X &= \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}; \\ \inf X &= \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}; \quad \text{and} \\ \operatorname{amp} X &= \sup X - \inf X. \end{aligned}$$

By convention $\sup X = -\infty$ and $\inf X = \infty$ if $X \simeq 0$.

[2]The author is a Ph.D.-student at the University of Copenhagen.

A *morphism* $\alpha : X \to Y$ of R–complexes is a sequence of R–linear maps $\alpha_i : X_i \to Y_i$ satisfying $\partial_i^Y \alpha_i - \alpha_{i-1}\partial_i^X = 0$ for $i \in \mathbb{Z}$. We say that a morphism is a *quasi-isomorphism* if it induces an isomorphism in homology. The symbol $\simeq$ is used to indicate quasi-isomorphisms while $\cong$ indicates isomorphisms of complexes (and hence modules).

(1.2) ***Derived Category***. The *derived category* of the category of R–modules is the category of R–complexes localized at the class of all quasi-isomorphisms (see [29] and [21]), it is denoted by $\mathcal{D}(R)$. We use the symbol $\simeq$ for isomorphisms in $\mathcal{D}(R)$; this is in line with the notation introduced above, as a morphism of complexs is a quasi-isomorphism exactly if it represents an isomorphism in the derived category. The symbol $\sim$ indicates isomorphism up to shift.

The full subcategories $\mathcal{D}_+(R)$, $\mathcal{D}_-(R)$, $\mathcal{D}_\mathrm{b}(R)$, and $\mathcal{D}_0(R)$ consist of complexes X with $\mathrm{H}_i(X) = 0$ for, respectively, $i \ll 0$, $i \gg 0$, $|i| \gg 0$, and $i \neq 0$. By $\mathcal{D}^\mathrm{f}(R)$ we denote the full subcategory of $\mathcal{D}(R)$ consisting of complexes X with $\mathrm{H}_i(X)$ a finite R–module for all $i \in \mathbb{Z}$. For a subcategory $\mathcal{S}(R) \subseteq \mathcal{D}(R)$ we set $\mathcal{S}^\mathrm{f}(R) = \mathcal{S}(R) \cap \mathcal{D}^\mathrm{f}(R)$, $\mathcal{S}_0(R) = \mathcal{S}(R) \cap \mathcal{D}_0(R)$, etc. The category of R–modules, respectively, finite R–modules, is naturally identified with $\mathcal{D}_0(R)$, respectively, $\mathcal{D}_0^\mathrm{f}(R)$.

(1.3) ***Derived Functors***. The left derived functor of the tensor product functor of R–complexes is denoted by $- \otimes_R^\mathbf{L} -$, and $\mathbf{R}\mathrm{Hom}_R(-,-)$ denotes the right derived functor of the homomorphism functor of complexes; by [28] and [3] no boundedness conditions are needed on the arguments. That is, for $X, Y \in \mathcal{D}(R)$ the complexes $X \otimes_R^\mathbf{L} Y$, and $\mathbf{R}\mathrm{Hom}_R(X, Y)$ are uniquely determined up to isomorphism in $\mathcal{D}(R)$, and they have the usual functorial properties. For $i \in \mathbb{Z}$ we set $\mathrm{Tor}_i^R(X, Y) = \mathrm{H}_i(X \otimes_R^\mathbf{L} Y)$ and $\mathrm{Ext}_R^i(X, Y) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X, Y))$. For modules X and Y this agrees with the notation of classical homological algebra, so no confusion arises.

Let $\mathfrak{p} \in \operatorname{Spec} R$; by [3, 5.2] there are isomorphisms $(X \otimes_R^\mathbf{L} Y)_\mathfrak{p} \simeq X_\mathfrak{p} \otimes_{R_\mathfrak{p}}^\mathbf{L} Y_\mathfrak{p}$ and $\mathbf{R}\mathrm{Hom}_R(Z, Y)_\mathfrak{p} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(Z_\mathfrak{p}, Y_\mathfrak{p})$ in $\mathcal{D}(R_\mathfrak{p})$. The first one always holds, and the second holds when $Y \in \mathcal{D}_-(R)$ and $Z \in \mathcal{D}_+^\mathrm{f}(R)$. We use these isomorphisms without further comment.

When $R \to R'$ is a homomorphism of rings the following hold [5, (1.2.1) and (1.2.2)]:

(1.3.1) If $X \in \mathcal{D}_+^\mathrm{f}(R)$ and $X' \in \mathcal{D}_+^\mathrm{f}(R')$ then $X \otimes_R^\mathbf{L} X' \in \mathcal{D}_+^\mathrm{f}(R')$.

(1.3.2) If $X \in \mathcal{D}_+^\mathrm{f}(R)$ and $Y' \in \mathcal{D}_-^\mathrm{f}(R')$ then $\mathbf{R}\mathrm{Hom}_R(X, Y') \in \mathcal{D}_-^\mathrm{f}(R')$.

The next two inequalities hold for $X, Z \in \mathcal{D}_+(R)$ and $Y \in \mathcal{D}_-(R)$, cf. [15, (2.1)].

$$\sup\left(\mathbf{R}\mathrm{Hom}_R(X, Y)\right) \leq \sup Y - \inf X; \quad \text{and} \tag{1.3.3}$$

$$\inf\left(X \otimes_R^\mathbf{L} Z\right) \geq \inf X + \inf Z. \tag{1.3.4}$$

If R is local, equality holds in the latter when $X, Z \in \mathcal{D}_+^\mathrm{f}(R)$. This is Nakayama's Lemma for complexes.

(1.4) ***Homological Dimensions***. A complex $X \in \mathcal{D}_\mathrm{b}(R)$ is said to be of *finite projective* (respectively, *injective* or *flat*) *dimension* if $X \simeq U$, where U is a complex of projective (respectively, injective or flat) modules and $U_i = 0$ for $|i| \gg 0$.

By $\mathcal{P}(R)$, $\mathcal{I}(R)$, and $\mathcal{F}(R)$ we denote the full subcategories of $\mathcal{D}_{\mathrm{b}}(R)$ consisting of complexes of, respectively, finite projective, injective, or flat dimension. Note that $\mathcal{P}_0(R)$, $\mathcal{I}_0(R)$, and $\mathcal{F}_0(R)$ are equivalent, respectively, to the full subcategories of modules of finite projective, injective or flat dimension. We use two-letter abbreviations (pd, id, fd) for the homological dimensions.

If X belongs to $\mathcal{D}_{\mathrm{b}}(R)$, then so do the complexes $F \otimes^{\mathbf{L}}_R X$, $\mathbf{R}\mathrm{Hom}_R(X,I)$, and $\mathbf{R}\mathrm{Hom}_R(P,X)$ when $F \in \mathcal{F}(R)$, $I \in \mathcal{I}(R)$, and $P \in \mathcal{P}(R)$. To be specific, there are inequalities [3, 2.4.F,I, and P]:

(1.4.1) $$\sup\,(F \otimes^{\mathbf{L}}_R X) \leq \mathrm{fd}_R\, F + \sup X;$$

(1.4.2) $$\inf\,(\mathbf{R}\mathrm{Hom}_R(X,I)) \geq -\sup X - \mathrm{id}_R\, I; \quad \text{and}$$

(1.4.3) $$\inf\,(\mathbf{R}\mathrm{Hom}_R(P,X)) \geq \inf X - \mathrm{pd}_R\, P.$$

Let R be a local ring with residue field k. For $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ the next equalities hold by [3, 2.10.F and 5.5]; note that $\mathcal{P}^{\mathrm{f}}(R) = \mathcal{F}^{\mathrm{f}}(R)$.

(1.4.4) $$\mathrm{pd}_R\, Z = \mathrm{fd}_R\, Z = \sup\,(Z \otimes^{\mathbf{L}}_R k); \quad \text{and}$$

(1.4.5) $$\mathrm{id}_R\, Z = -\inf\,(\mathbf{R}\mathrm{Hom}_R(k,Z)).$$

(1.5) ***Canonical Morphisms***. When $R \to R'$ is a homomorphism of rings, a number of canonical homomorphisms in $\mathcal{D}(R')$ are associated with complexes $X', Y', Z' \in \mathcal{D}(R')$ and $Y, Z \in \mathcal{D}(R)$. We consider the *associativity* and *adjointness* isomorphisms:

(1.5.1) $$(Z \otimes^{\mathbf{L}}_R Y') \otimes^{\mathbf{L}}_{R'} X' \simeq Z \otimes^{\mathbf{L}}_R (Y' \otimes^{\mathbf{L}}_{R'} X');$$

(1.5.2) $$\mathbf{R}\mathrm{Hom}_{R'}(Z \otimes^{\mathbf{L}}_R X', Y') \simeq \mathbf{R}\mathrm{Hom}_R(Z, \mathbf{R}\mathrm{Hom}_{R'}(X',Y')); \quad \text{and}$$

(1.5.3) $$\mathbf{R}\mathrm{Hom}_R(Z' \otimes^{\mathbf{L}}_{R'} X', Y) \simeq \mathbf{R}\mathrm{Hom}_{R'}(Z', \mathbf{R}\mathrm{Hom}_R(X',Y)).$$

These *standard isomorphisms* are usually used without comment, and so is *commutativity* of the derived tensor product: $X \otimes^{\mathbf{L}}_R Y \simeq Y \otimes^{\mathbf{L}}_R X$. The following special cases of the standard isomorphisms are often very useful:

(1.5.4) $$(Z \otimes^{\mathbf{L}}_R R') \otimes^{\mathbf{L}}_{R'} X' \simeq Z \otimes^{\mathbf{L}}_R X';$$

(1.5.5) $$\mathbf{R}\mathrm{Hom}_{R'}(Z \otimes^{\mathbf{L}}_R R', Y') \simeq \mathbf{R}\mathrm{Hom}_R(Z,Y'); \quad \text{and}$$

(1.5.6) $$\mathbf{R}\mathrm{Hom}_{R'}(Z', \mathbf{R}\mathrm{Hom}_R(R',Y)) \simeq \mathbf{R}\mathrm{Hom}_R(Z',Y).$$

We also consider the *evaluation* morphisms

$$\omega_{ZY'X'} : \mathbf{R}\mathrm{Hom}_R(Z,Y') \otimes^{\mathbf{L}}_{R'} X' \to \mathbf{R}\mathrm{Hom}_R(Z, Y' \otimes^{\mathbf{L}}_{R'} X'); \quad \text{and}$$
$$\theta_{ZX'Y'} : Z \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_{R'}(X',Y') \to \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(Z,X'),Y').$$

In general, these are not invertible, but by [3, 4.4] the following hold when $Z \in \mathcal{D}^{\mathrm{f}}_{+}(R)$:

(1.5.7) $\omega_{ZY'X'}$ is an isomorphism if $Y' \in \mathcal{D}_{-}(R')$, and $X' \in \mathcal{F}(R')$ or $Z \in \mathcal{P}(R)$.

(1.5.8) $\theta_{ZX'Y'}$ is an isomorphism if $X' \in \mathcal{D}_{\mathrm{b}}(R')$, and $Y' \in \mathcal{I}(R')$ or $Z \in \mathcal{P}(R)$.

We also use the *homothety* morphism

$$\chi^R_X \colon R \to \mathbf{R}\mathrm{Hom}_R(X,X);$$

and the *biduality* morphism

$$\delta_X^Y \colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,Y),Y).$$

(1.6) ***Numerical Invariants***. Let R be local with residue field k. The *depth* of an R–complex X is defined by

$$\operatorname{depth}_R X = -\sup(\mathbf{R}\mathrm{Hom}_R(k,X)); \tag{1.6.1}$$

and the (*Krull*) *dimension* of X is defined as follows:

$$\begin{aligned}\dim_R X &= \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\},\end{aligned} \tag{1.6.2}$$

where $\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \not\simeq 0\} = \bigcup_{i\in\mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_i(X)$. Note that for modules these notions agree with the standard ones.

For $X \not\simeq 0$ in $\mathcal{D}_+(R)$ the inequalities in (1.6.3) and (1.6.5) below follow by the definition of dimension. For $Y \not\simeq 0$ in $\mathcal{D}_-(R)$ the inequality in (1.6.4) follows by (1.3.3).

$$-\infty < -\inf X \le \dim_R X \le \dim R - \inf X < \infty. \tag{1.6.3}$$

$$-\infty < -\sup Y \le \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \quad \text{for } \mathfrak{p} \in \operatorname{Spec} R. \tag{1.6.4}$$

$$\dim_R X \ge \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p} \quad \text{for } \mathfrak{p} \in \operatorname{Spec} R. \tag{1.6.5}$$

Equality holds in (1.6.4) if and only if $\mathfrak{p}$ is associated to the top homology module in Y; that is,

$$\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} = -\sup Y \iff \mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_{\sup Y}(Y). \tag{1.6.6}$$

By [16, 3.9 and 2.8] the next inequality holds for $X \in \mathcal{D}_\mathrm{b}(R)$ with $\operatorname{depth}_R X < \infty$:

$$\operatorname{depth}_R X \le \dim_R X. \tag{1.6.7}$$

The *Cohen–Macaulay defect* of $X \in \mathcal{D}(R)$ is as usual the difference $\operatorname{cmd}_R X = \dim_R X - \operatorname{depth}_R X$; by (1.6.7) it is non-negative for $X \in \mathcal{D}_\mathrm{b}(R)$ with $\operatorname{depth}_R X < \infty$, in particular, for $X \not\simeq 0$ in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$.

(1.7) ***Formal Invariants***. Let R be local with residue field k. For $X \in \mathcal{D}_+^\mathrm{f}(R)$ the *Betti numbers* $\beta_i^R(X) = \operatorname{rank}_k \operatorname{Tor}_i^R(X,k)$ are finite and vanish for $0 \gg i$, cf. (1.3.1). The formal Laurant series $\mathrm{P}_X^R(t) = \sum_{i\in\mathbb{Z}} \beta_i^R(X)t^i$ is the so-called *Poincaré series* of X; it has non-negative integer coefficients, and by Nakayama's Lemma and (1.4.4) there are equalities:

$$\operatorname{ord} \mathrm{P}_X^R(t) = \inf X \quad \text{and} \quad \deg \mathrm{P}_X^R(t) = \operatorname{pd}_R X. \tag{1.7.1}$$

Furthermore, the following hold:

$$X \sim R \iff \mathrm{P}_X^R(t) = t^d \text{ for some } d \in \mathbb{Z}. \tag{1.7.2}$$

For $Y \in \mathcal{D}_-^\mathrm{f}(R)$ the *Bass numbers* $\mu_R^i(Y) = \operatorname{rank}_k \operatorname{Ext}_R^i(k,Y)$ are finite and vanish for $i \gg 0$, cf. (1.3.2). The formal Laurant series $\mathrm{I}_R^Y(t) = \sum_{i\in\mathbb{Z}} \mu_R^i(Y)t^i$ is called the

Bass series of Y; it has non-negative integer coefficients, and by the definition of depth (1.6.1) and (1.4.5) there are equalities:

$$\text{(1.7.3)} \qquad \operatorname{ord} \mathrm{I}_R^Y(t) = \operatorname{depth}_R Y \quad \text{and} \quad \deg \mathrm{I}_R^Y(t) = \operatorname{id}_R Y.$$

For convenience we set $\mu_R^i = \mu_R^i(R)$ and $\mathrm{I}_R^R(t) = \mathrm{I}_R(t)$.

Let $Y \in \mathcal{D}_-^{\mathrm{f}}(R)$, let $\mathfrak{p} \in \operatorname{Spec} R$, and set $n = \dim R/\mathfrak{p}$. The next implication holds for all $i \in \mathbb{Z}$ (of course, $\mu_R^i(\mathfrak{p}, Y)$ is the rank of the $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$–vector space $\operatorname{Ext}_{R_\mathfrak{p}}^i(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}, Y_\mathfrak{p})$):

$$\text{(1.7.4)} \qquad \mu_R^i(\mathfrak{p}, Y) \neq 0 \quad \Longrightarrow \quad \mu_R^{i+n}(Y) \neq 0.$$

This is a complex version of [8, (3.1)], cf. [12, Chapter 13], and as for modules the next inequality follows, cf. (1.7.3).

$$\text{(1.7.5)} \qquad \operatorname{depth}_R Y \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \dim R/\mathfrak{p}.$$

Let $\varphi : R \to R'$ be a local homomorphism; the next two equalities of formal Laurant series are established in [5, (1.5.3)].

$$\text{(1.7.6)} \qquad \mathrm{P}_{X \otimes_R^{\mathbf{L}} X'}^{R'}(t) = \mathrm{P}_X^R(t)\, \mathrm{P}_{X'}^{R'}(t)$$

holds for $X \in \mathcal{D}_+^{\mathrm{f}}(R)$ and $X' \in \mathcal{D}_+^{\mathrm{f}}(R')$; and

$$\text{(1.7.7)} \qquad \mathrm{I}_{R'}^{\mathbf{R}\mathrm{Hom}_R(X,Y')}(t) = \mathrm{P}_X^R(t)\, \mathrm{I}_{R'}^{Y'}(t)$$

holds for $X \in \mathcal{D}_+^{\mathrm{f}}(R)$ and $Y' \in \mathcal{D}_-^{\mathrm{f}}(R')$. We shall also need the following:

(1.7.8) **Lemma.** *Let $\varphi : R \to R'$ be a finite local homomorphism. If $Y \in \mathcal{D}_-^{\mathrm{f}}(R)$ and $X' \in \mathcal{D}_+^{\mathrm{f}}(R')$, then $\mathbf{R}\mathrm{Hom}_R(X', Y)$ belongs to $\mathcal{D}_-^{\mathrm{f}}(R')$, and there is an equality of formal Laurant series:*

$$\text{(1.7.9)} \qquad \mathrm{I}_{R'}^{\mathbf{R}\mathrm{Hom}_R(X',Y)}(t) = \mathrm{P}_{X'}^{R'}(t)\, \mathrm{I}_R^Y(t).$$

Proof. The R'–structure, and thereby the R–structure, of $\mathbf{R}\mathrm{Hom}_R(X', Y)$ is determined by X'. Since φ is finite X' belongs to $\mathcal{D}_+^{\mathrm{f}}(R)$, so $\mathbf{R}\mathrm{Hom}_R(X', Y) \in \mathcal{D}_-^{\mathrm{f}}(R)$ by (1.3.2) and hence $\mathbf{R}\mathrm{Hom}_R(X', Y) \in \mathcal{D}_-^{\mathrm{f}}(R')$ as desired. We denote the residue fields of R and R' by, respectively, k and k'; using that φ is local we find that

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{R'}(k', \mathbf{R}\mathrm{Hom}_R(X', Y)) &\simeq \mathbf{R}\mathrm{Hom}_R(k' \otimes_{R'}^{\mathbf{L}} X', Y) \\
&\simeq \mathbf{R}\mathrm{Hom}_R((X' \otimes_{R'}^{\mathbf{L}} k') \otimes_k k, Y) \\
&\simeq \mathrm{Hom}_k(X' \otimes_{R'}^{\mathbf{L}} k', \mathbf{R}\mathrm{Hom}_R(k, Y)).
\end{aligned}$$

Hence we have $\operatorname{Ext}_{R'}^*(k', \mathbf{R}\mathrm{Hom}_R(X', Y)) \cong \mathrm{Hom}_k(\operatorname{Tor}_*^{R'}(X', k'), \operatorname{Ext}_R^*(k, Y))$, and the equality of Laurant series follows. $\square$

(1.8) ***Dualizing Complexes***. Let R be a local ring. Recall that an R–complex D is said to be *dualizing* for R if and only if $D \in \mathcal{I}^{\mathrm{f}}(R)$ and the homothety morphism $\chi_D^R : R \to \mathbf{R}\mathrm{Hom}_R(D, D)$ is invertible. The following hold [21, V.3.4]:

$$\text{(1.8.1)} \qquad D \text{ is dualizing for } R \quad \Longleftrightarrow \quad \mathrm{I}_R^D(t) = t^d \text{ for some } d \in \mathbb{Z}.$$

We say that D is a *normalized dualizing complex* for R if $\mathrm{I}_R^D(t) = 1$. This is in keeping with the convention used in [5].

Not all rings have a dualizing complex, but an ample supply of rings that do is ensured by the following:

(1.8.2) R is Gorenstein if and only if the R–module R is a dualizing complex for R [21, V.3.4 and V.10].

(1.8.3) Every homomorphic image of a Gorenstein local ring has a dualizing complex. In particular, every complete local ring has a dualizing complex [21, V.10].

2. Dagger Duality

In this section we define semi-dualizing complexes and collect some technical results for later reference. We also introduce the first Auslander category — the category of reflexive complexes — and its related duality functor.

(2.1) **Definition.** An R–complex C is said to be *semi-dualizing* for R if and only if $C \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and the homothety morphism $\chi_C^R \colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism.

(2.2) **Remark.** If R is local, then a complex $D \in \mathcal{D}(R)$ is dualizing for R if and only if it is semi-dualizing and of finite injective dimension, cf. (1.8).

(2.3) **Examples.** The canonical example of a semi-dualizing complex for R is the ring itself; other examples are dualizing complexes, when these exist, and relative dualizing complexes, cf. [5]. More examples of semi-dualizing complexes are given in section 7.

(2.4) **Observation.** If C is a semi-dualizing complex for R, then so are all the shifted complexes $\Sigma^n C$, $n \in \mathbb{Z}$.

The next two results on localization and completion are easily derived from, respectively, (5.1) and (5.6). Since we draw heavily on the first one in section 3, we have included the straightforward proof.

(2.5) **Lemma.** *If C is a semi-dualizing complex for R, then* $\mathrm{Supp}_R\, C = \mathrm{Spec}\, R$, *and for each* $\mathfrak{p} \in \mathrm{Spec}\, R$ *the complex $C_\mathfrak{p}$ is semi-dualizing for $R_\mathfrak{p}$.*

Proof. Let $\mathfrak{p} \in \mathrm{Spec}\, R$. The commutative diagram

$$\begin{tikzcd} \mathbf{R}\mathrm{Hom}_R(C,C)_\mathfrak{p} \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p}) \\ R_\mathfrak{p} \arrow[u, "(\chi_C^R)_\mathfrak{p}", "\simeq"'] \arrow[r, equal] & R_\mathfrak{p} \arrow[u, "\chi_{C_\mathfrak{p}}^{R_\mathfrak{p}}"] \end{tikzcd}$$

shows that $C_\mathfrak{p}$ is semi-dualizing for $R_\mathfrak{p}$. In particular, it follows that $\mathrm{Supp}_R\, C = \mathrm{Spec}\, R$ as claimed. □

(2.6) **Lemma.** *Let R be a local ring. A complex $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is semi-dualizing for R if and only if the complex $C \otimes_R \widehat{R} \in \mathcal{D}(\widehat{R})$ is semi-dualizing for $\widehat{R}$.* $\square$

(2.7) **Definitions.** Let C be a semi-dualizing complex for R. For $Z \in \mathcal{D}(R)$ the *dagger dual* with respect to C is the complex $Z^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(Z, C)$, and $-^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(-, C)$ is the corresponding *dagger duality functor*. For convenience we set $Z^{\dagger_C\dagger_C} = (Z^{\dagger_C})^{\dagger_C}$.

An R–complex Z is said to be *C–reflexive* if and only if Z and the dagger dual $Z^{\dagger_C}$ belong to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, and the biduality morphism $\delta^C_Z \colon Z \to Z^{\dagger_C\dagger_C}$ is invertible. By ${}_C\mathcal{R}(R)$ we denote the full subcategory of $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ whose objects are the C–reflexive complexes.

(2.8) **Remarks.** It is straightforward to check that R and C belong to ${}_C\mathcal{R}(R)$ when C is semi-dualizing for R. By [31, 2.7 and 2.8] the complexes in ${}_R\mathcal{R}(R)$ are exactly those of finite Gorenstein dimension; and it is well-known that ${}_D\mathcal{R}(R) = \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ if R is local and D is a dualizing complex for R, cf. [21, V.2.1].

A word of caution: An R–module M with the property that the canonical map $\varepsilon_M \colon M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism is sometimes called reflexive. However, such a module need not be R–reflexive in the sense defined above, and vice versa: an R–reflexive module, in the sense of (2.7), need not have the property. See [31, 2.6] for examples.

(2.9) **Proposition.** *If C is a semi-dualizing complex for R, then there is a full embedding:*

$$\mathcal{P}^{\mathrm{f}}(R) \subseteq {}_C\mathcal{R}(R).$$

Proof. Let $P \in \mathcal{P}^{\mathrm{f}}(R)$, then the complex $P^{\dagger_C}$ belongs to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and the evaluation morphism θ_{PCC} is invertible, cf. (1.4.3) and (1.5.8). The commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\delta^C_P"] \arrow[d, "\simeq"] & P^{\dagger_C\dagger_C} \\
P \otimes^{\mathbf{L}}_R R \arrow[r, "\simeq", "P \otimes^{\mathbf{L}}_R \chi^R_C"'] & P \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, C) \arrow[u, "\theta_{PCC}"', "\simeq"]
\end{tikzcd}$$

shows that δ^C_P is an isomorphism, and hence P is C–reflexive. $\square$

The behavior of C–reflexive complexes under completion and localization is explained by (5.10). Since we shall use it repeatedly in the next section, we spell out the result on localization:

(2.10) **Lemma.** *Let C be a semi-dualizing complex for R. The next implication holds for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\mathfrak{p} \in \mathrm{Spec}\, R$:*

$$Z \in {}_C\mathcal{R}(R) \quad \Longrightarrow \quad Z_{\mathfrak{p}} \in {}_{C_{\mathfrak{p}}}\mathcal{R}(R_{\mathfrak{p}}). \quad \square$$

(2.11) **Dagger Duality Theorem.** *Let C be a semi-dualizing complex for R. Dagger duality with respect to C is an endofunctor of ${}_C\mathcal{R}(R)$, and it takes semi-dualizing complexes to semi-dualizing complexes.*

Proof. Suppose $Z \in {}_C\mathcal{R}(R)$, then $Z^{\dagger_C}$ belongs to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and so does $(Z^{\dagger_C})^{\dagger_C} \simeq Z$. The commutative diagram

$$\begin{tikzcd}
(Z^{\dagger_C})^{\dagger_C\dagger_C} \arrow[r, equal] & (Z^{\dagger_C\dagger_C})^{\dagger_C} \arrow[d, "\simeq"', "(\delta^C_Z)^{\dagger_C}"] \\
Z^{\dagger_C} \arrow[u, "\delta^C_{Z^{\dagger_C}}"] \arrow[r, equal] & Z^{\dagger_C}
\end{tikzcd}$$

shows that the biduality map $\delta^C_{Z^{\dagger_C}}$ is an isomorphism, and hence $Z^{\dagger_C} \in {}_C\mathcal{R}(R)$. Since ${}_C\mathcal{R}(R)$ is a full subcategory, it follows that $-^{\dagger_C}$ is an endofunctor of ${}_C\mathcal{R}(R)$.

Suppose $Z \in {}_C\mathcal{R}(R)$ is semi-dualizing for R. The dagger dual $Z^{\dagger_C}$ belongs to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, and the diagram

$$\begin{tikzcd}
R \arrow[r, "\chi^R_Z", "\simeq"'] \arrow[d, "\chi^R_{Z^{\dagger_C}}"'] & \mathbf{R}\mathrm{Hom}_R(Z,Z) \arrow[d, "\mathbf{R}\mathrm{Hom}_R(Z,\delta^C_Z)"', "\simeq"] \\
\mathbf{R}\mathrm{Hom}_R(Z^{\dagger_C}, Z^{\dagger_C}) \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_R(Z, Z^{\dagger_C\dagger_C})
\end{tikzcd}$$

where the bottom row isomorphism involves only standard isomorphisms, is commutative. The homothety morphism $\chi^R_{Z^{\dagger_C}}$ is, therefore, invertible and $Z^{\dagger_C}$, indeed, a semi-dualizing complex for R. $\square$

As mentioned in (2.8) every complex in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is D–reflexive when R is local and D is dualizing for R. Thus, the dagger duality functor $-^{\dagger_D}$ is an endofunctor of $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, and we have the following:

(2.12) **Corollary.** *If R is local and D is a dualizing complex for R, then $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is semi-dualizing for R if and only if $C^{\dagger_D}$ is so.* $\square$

For every semi-dualizing complex C the complexes with finite homology and finite projective dimension form a full subcategory of ${}_C\mathcal{R}(R)$, cf. (2.9). The next theorem shows that for $C = R$ dagger duality is stable on this subcategory, and in section 8 we show that this property distinguishes a local ring R from its other semi-dualizing complexes.

(2.13) **Theorem.** *Dagger duality with respect to R is an endofunctor of $\mathcal{P}^{\mathrm{f}}(R)$, and for $P \in \mathcal{P}^{\mathrm{f}}(R)$ the next two equalities hold.*

(a) $$\mathrm{pd}_R\, P^{\dagger_R} = -\inf P \quad \textit{and} \quad \inf P^{\dagger_R} = \mathrm{pd}_R\, P.$$

Furthermore, if R is local and $P \in \mathcal{P}^{\mathrm{f}}(R)$, then there is an equality of Poincaré series:

(b) $$\mathrm{P}^R_{P^{\dagger_R}}(t) = \mathrm{P}^R_P(t^{-1}).$$

Proof. Let $P \in \mathcal{P}^{\mathrm{f}}(R)$. By [3, 2.5.P], (1.5.8), and (1.3.4) we have

$$\begin{aligned}
\operatorname{pd}_R P^{\dagger_R} &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(P^{\dagger_R}, N)) \mid N \in \mathcal{D}_0(R)\} \\
&= \sup\{-\inf(P \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R, N)) \mid N \in \mathcal{D}_0(R)\} \\
&= \sup\{-\inf(P \otimes^{\mathbf{L}}_R N) \mid N \in \mathcal{D}_0(R)\} \\
&= -\inf P.
\end{aligned}$$

This proves the first equality in (a), and it follows that $-^{\dagger_R}$ is an endofunctor of $\mathcal{P}^{\mathrm{f}}(R)$. The second equality in (a) follows by the first one, as $P = P^{\dagger_R\dagger_R}$ by (2.9).

Now, let R be local with residue field k, and let $P \in \mathcal{P}^{\mathrm{f}}(R)$. The equality of Poincaré series, (b), follows by the calculation below, where $\mathrm{E}_R(k)$ denotes the injective hull of the residue field. This part of the proof is taken from [12, Chapter 11].

$$\begin{aligned}
\beta^R_i(P^{\dagger_R}) &= \operatorname{rank}_k \operatorname{Tor}^R_i(P^{\dagger_R}, k) \\
&= \operatorname{rank}_k \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R k) \\
&= \operatorname{rank}_k \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(P, R \otimes^{\mathbf{L}}_R k)) && \text{by (1.5.7)} \\
&= \operatorname{rank}_k \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(P, k), \mathrm{E}_R(k))) \\
&= \operatorname{rank}_k \mathrm{H}_{-i}(P \otimes^{\mathbf{L}}_R \mathrm{Hom}_R(k, \mathrm{E}_R(k))) && \text{by (1.5.8)} \\
&= \operatorname{rank}_k \operatorname{Tor}^R_{-i}(P, k) \\
&= \beta^R_{-i}(P). \quad \square
\end{aligned}$$

(2.14) **Corollary (from** [12]**).** *Let R be local. For $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $P \in \mathcal{P}^{\mathrm{f}}(R)$ there is an equality of formal Laurant series:*

$$\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(P,Z)}(t) = \mathrm{P}^R_P(t^{-1})\,\mathrm{P}^R_Z(t).$$

Proof. If $P \in \mathcal{P}^{\mathrm{f}}(R)$, then $P \simeq P^{\dagger_R\dagger_R}$ and $P^{\dagger_R} \in \mathcal{P}^{\mathrm{f}}(R)$ by (2.9) and (2.13). Let $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$; by (1.5.8) we have

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(P, Z) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(P^{\dagger_R}, R), Z) \\
&\simeq P^{\dagger_R} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R, Z) \\
&\simeq P^{\dagger_R} \otimes^{\mathbf{L}}_R Z.
\end{aligned}$$

The desired equality now follows by (1.7.6) and (2.13)(b). $\square$

3. Reflexive Complexes and G–dimension

Throughout this section R is a local ring.

We make a detailed study of reflexive complexes, focusing on the interplay between dagger duality and the invariants dimension and depth. The essential results are described in (3.3) and (3.8). The study also reveals strong relations between invariants of a ring and those of its semi-dualizing complexes. The essentials in this direction are (3.4) and (3.7).

The opening lemma is central for the study of reflexive complexes; the proof is deferred to the end of the section.

(3.1) **Lemma.** *Let C be a semi-dualizing complex for R. If Z is C–reflexive, then the following hold:*

(a) $\operatorname{depth}_R Z - \inf Z^{\dagger_C} = \operatorname{depth}_R C$;

(b) $\operatorname{depth}_R C \leq \dim_R Z - \sup Z^{\dagger_C} \leq \dim_R C$;

(c) $\dim_R Z - \sup Z^{\dagger_C} \leq \dim_R Z^{\dagger_C} - \inf Z$; *and*

(d) $\operatorname{Supp}_R Z^{\dagger_C} = \operatorname{Supp}_R Z$.

(3.2) **Corollary.** *If C is a semi-dualizing complex for R, then the following hold:*

(a) $\operatorname{depth}_R C = \operatorname{depth} R - \inf C$; *and*

(b) $\dim R - \sup C \leq \dim_R C \leq \dim R - \inf C$.

Proof. (a) is immediate by applying (3.1)(a) to $Z = R$. The first inequality in (b) similarly follows by applying (3.1)(b) to $Z = R$ while the second holds in general, cf. (1.6.3). □

(3.3) **Theorem.** *Let C be a semi-dualizing complex for R. If Z is C–reflexive, then the following hold:*

(a) $\operatorname{amp} Z^{\dagger_C} \leq \operatorname{cmd}_R Z \leq \operatorname{amp} Z^{\dagger_C} + \operatorname{cmd}_R C$; *and*

(b) $\operatorname{cmd}_R Z \leq \operatorname{amp} Z^{\dagger_C} + \operatorname{cmd}_R Z^{\dagger_C}$.

Proof. (a) follows immediately by (a) and (b) in (3.1). To establish (b) we use that also $Z^{\dagger_C} \in {}_C\mathcal{R}(R)$:

$$\begin{aligned}
\operatorname{cmd}_R Z &= \dim_R Z - \operatorname{depth}_R Z \\
&\leq \dim_R Z^{\dagger_C} + \sup Z^{\dagger_C} - \inf Z - \operatorname{depth}_R Z && \text{by (3.1)(c)} \\
&= \dim_R Z^{\dagger_C} + \sup Z^{\dagger_C} - \inf Z - \operatorname{depth}_R C - \inf Z^{\dagger_C} && \text{by (3.1)(a)} \\
&= \dim_R Z^{\dagger_C} + \operatorname{amp} Z^{\dagger_C} - \operatorname{depth}_R Z^{\dagger_C} && \text{by (3.1)(a)} \\
&= \operatorname{amp} Z^{\dagger_C} + \operatorname{cmd}_R Z^{\dagger_C}. \quad \square
\end{aligned}$$

(3.4) **Corollary.** *If C is a semi-dualizing complex for R, then the next three inequalities hold.*

(a) $\operatorname{amp} C \leq \operatorname{cmd} R$;

(b) $\operatorname{cmd}_R C \leq \operatorname{cmd} R$; *and*

(c) $\operatorname{cmd} R \leq \operatorname{amp} C + \operatorname{cmd}_R C$.

Equality holds in (a) *if C is Cohen–Macaulay and in* (b) *if* $\operatorname{amp} C = 0$, *i.e., if C is a module up to a shift.*

Proof. Applying (3.3)(a) to $Z = R$ yields (a) and (c) while (3.3)(b) applied to $Z = C$ accounts for (b). It follows from (a) and (c) that $\operatorname{amp} C = \operatorname{cmd} R$ if $\operatorname{cmd}_R C = 0$, and similarly it follows from (b) and (c) that $\operatorname{cmd}_R C = \operatorname{cmd} R$ if $\operatorname{amp} C = 0$. □

In the appendix we show how defects in chains of prime ideals obstruct equality in (3.4)(c).

(3.5) **Remarks.** A dualizing complex D is Cohen–Macaulay: The inequality below follows by (1.7.4) (as for modules, cf. [8, (3.2)]), and the equalities are, respectively, the Bass formula for complexes, cf. [15, 4.1(b) and 4.3(1)], and (3.2)(a).

$$\dim_R D \leq \operatorname{id}_R D = \operatorname{depth} R - \inf D = \operatorname{depth}_R D.$$

Thus, $\dim_R D = \operatorname{depth}_R D$ by (1.6.7), and (3.4) contains the fact that $\operatorname{amp} D = \operatorname{cmd} R$.

For $Z \in {}_R\mathcal{R}(R)$ the two inequalities in (3.3), applied to $Z^{\dagger_R} \in {}_R\mathcal{R}(R)$, read:

$$\operatorname{cmd}_R Z^{\dagger_R} \leq \operatorname{amp} Z + \operatorname{cmd} R \qquad \text{and} \qquad \operatorname{cmd}_R Z^{\dagger_R} \leq \operatorname{amp} Z + \operatorname{cmd}_R Z.$$

When Z is a module, the second inequality is known from [31, 2.18]. We note that for $Z \in \mathcal{P}^{\mathrm{f}}(R)$ the first inequality is stronger than the second by the New Intersection Theorem, cf. (3.6.1) below.

(3.6) ***New Intersection Theorem***. Let R be a local ring. The following inequalities hold for $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and $P \not\simeq 0$ in $\mathcal{P}^{\mathrm{f}}(R)$:

(3.6.1) $$\operatorname{cmd}_R(Z \otimes_R^{\mathbf{L}} P) \geq \operatorname{cmd}_R Z; \quad \text{and}$$

(3.6.2) $$\operatorname{amp}(Z \otimes_R^{\mathbf{L}} P) \geq \operatorname{amp} Z.$$

The first inequality can, cf. [12, Chapter 18], be derived from the New Intersection Theorem due to Peskine and Szpiro [25], Hochster [22], and Roberts [26, 27]. The second inequality is Iversen's Amplitude Inequality [23, (3.2)]. Its proof uses the New Intersection Theorem; note that it holds without restrictions on R since, with the appearance of [27], the extra requirement of [23] that R be an algebra over a field is no longer needed.

Before we carry on with another important corollary to (3.3), we note that (3.4) offers the following characterization of Cohen–Macaulay rings:

(3.7) **Corollary.** *The next three conditions are equivalent.*

(*i*) *R is Cohen–Macaulay.*

(*ii*) *R has a Cohen–Macaulay semi-dualizing module.*

(*iii*) *Every semi-dualizing complex for R is a maximal Cohen–Macaulay module* (*up to a shift*).

Proof. (*i*) $\Rightarrow$ (*iii*): Suppose R is Cohen–Macaulay and let C be a semi-dualizing complex for R. It follows by (a) and (b) in (3.4) that C is Cohen–Macaulay and $\operatorname{amp} C = 0$. We may assume that C is concentrated in degree 0, and then it follows by (3.2)(a) that $\dim_R C = \operatorname{depth}_R C = \operatorname{depth} R = \dim R$.

The implication (*iii*) $\Rightarrow$ (*ii*) is obvious, and (*ii*) $\Rightarrow$ (*i*) follows by (3.4)(c). $\square$

The next result is immediate by (3.3)(a). It generalizes the homological formula $\mathrm{cmd}_R\, Z = \mathrm{amp}\, Z^{\dagger_D}$, which holds for complexes Z in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R) = {}_D\mathcal{R}(R)$ when D is dualizing for R, cf. [16, 3.14].

(3.8) **Corollary (CMD Formula).** *Let C be a Cohen–Macaulay semi-dualizing complex for R. If Z is C–reflexive, then*

$$\mathrm{cmd}_R\, Z = \mathrm{amp}\, Z^{\dagger_C}. \quad \square$$

(3.9) ***(Quasi-)Imperfection***. An R–module M is said to be *perfect* if its projective dimension equals its grade, and the integer $\mathrm{imp}_R\, M = \mathrm{pd}_R\, M - \mathrm{grade}_R\, M$ is referred to as the *imperfection* of M. For $M \in \mathcal{P}_0^\mathrm{f}(R)$ there is an equality $\mathrm{imp}_R\, M = \mathrm{amp}\, M^{\dagger_R}$. With the convention that $\mathrm{imp}_R\, Z = \mathrm{pd}_R\, Z - \mathrm{grade}_R\, Z$ and $\mathrm{grade}_R\, Z = -\sup Z^{\dagger_R}$ for Z in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$, cf. [16, 5.3], the equality $\mathrm{imp}_R\, Z = \mathrm{amp}\, Z^{\dagger_R}$ is valid for $Z \in \mathcal{P}^\mathrm{f}(R)$.

Replacing the projective dimension by Auslander's G–dimension [1, 2] we obtain the notion of *quasi-perfect* modules, cf. [14]; and the *quasi-imperfection*, $\text{q–imp}_R\, M = \text{G–dim}_R\, M - \mathrm{grade}_R\, M$, satisfies the formula $\text{q–imp}_R\, M = \mathrm{amp}\, M^{\dagger_R}$, when $\text{G–dim}_R\, M$ is finite. By a result due to Foxby, cf. [31, 2.7], a finite R–module has finite G–dimension if and only if it belongs to ${}_R\mathcal{R}(R)$; and with the extension of the G–dimension to complexes [31, 2.8] the R–reflexive complexes are exactly those of finite G–dimension. Moreover, $\text{G–dim}_R\, Z = -\inf Z^{\dagger_R}$ for these complexes, and hence

$$\text{q–imp}_R\, Z = \mathrm{amp}\, Z^{\dagger_R} \quad \text{for} \;\; Z \in {}_R\mathcal{R}(R).$$

Thus, at least on the formal level of homological formulae, the quasi-imperfection and the Cohen–Macaulay defect are invariants of the same kind. By (3.8) they coincide for complexes of finite G–dimension over a Cohen–Macaulay ring; in particular, they agree for all complex in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ when R is Gorenstein, see also (8.5).

The next proposition is an extension of [6, (2.6.2)] to complexes.

(3.10) **Proposition.** *For $Z \in \mathcal{P}^\mathrm{f}(R)$ the following inequalities hold:*

$$0 \leq \mathrm{cmd}_R\, Z - \mathrm{cmd}\, R \leq \mathrm{imp}_R\, Z \leq \mathrm{cmd}_R\, Z.$$

In particular, the next two conditions are equivalent.

(*i*) *R is Cohen–Macaulay and* $\mathrm{imp}_R\, Z = 0$.

(*ii*) *Z is Cohen–Macaulay.*

Proof. The first inequality is a consequence of the New Intersection Theorem, cf. (3.6.1), while the second and third follow by (3.3)(a). The equivalence of conditions (*i*) and (*ii*) is evident from the inequalities. $\square$

Closing this section we show that it is possible, for any semi-dualizing complex C, to define a G–dimension that mimics the nice properties of the original. To be exact, (3.14) and (3.17) below are covered in the case $C = R$ by Theorem 2.9 and Lemmas 2.14 and 2.15 in [31].

(3.11) **Definition.** Let C be a semi-dualizing complex for R. For $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ we define the *G–dimension* of Z with respect to C as follows:

$$\text{G–dim}_C Z = \begin{cases} \inf C - \inf Z^{\dagger_C} & \text{if } Z \in {}_C\mathcal{R}(R); \quad \text{and} \\ \infty & \text{if } Z \notin {}_C\mathcal{R}(R). \end{cases}$$

For $C = R$ this definition agrees with [31, 2.8].

(3.12) **Lemma.** *Let C be a semi-dualizing complex for R. If $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\sup Z \leq \operatorname{amp} C + \text{G–dim}_C Z;$$

and for $m \in \mathbb{Z}$ there are equalities:

$$\begin{aligned} \text{G–dim}_C \Sigma^m Z &= \text{G–dim}_C Z + m; \quad \textit{and} \\ \text{G–dim}_{\Sigma^m C} Z &= \text{G–dim}_C Z. \end{aligned}$$

Proof. First, note that the inequality as well as the two equalities trivially hold if $Z \notin {}_C\mathcal{R}(R)$. For $Z \in {}_C\mathcal{R}(R)$ the two equalities follow by inspection and the inequality follows by (1.3.3):

$$\sup Z = \sup Z^{\dagger_C \dagger_C} \leq \sup C - \inf Z^{\dagger_C} = \operatorname{amp} C + \text{G–dim}_C Z. \quad \square$$

We can now compile some basic properties of the G–dimension.

(3.13) **Observation.** Let C be a semi-dualizing complex for R. The following hold for $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$:

$$\begin{aligned} &\text{G–dim}_C Z \in \{\infty\} \cup \mathbb{Z} \cup \{-\infty\}; \\ &\text{G–dim}_C Z \geq \sup Z - \operatorname{amp} C; \\ &\text{G–dim}_C Z = -\infty \iff Z \simeq 0; \quad \text{and} \\ &\text{G–dim}_C Z < \infty \iff Z \in {}_C\mathcal{R}(R). \end{aligned}$$

The next theorem generalizes Auslander and Bridger's equality, $\text{G–dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ [2, (4.13)(b)], for finite modules of finite Gorenstein dimension. This Auslander–Bridger Equality was, in turn, a generalization of the Auslander–Buchsbaum Equality for projective dimension, so it seems appropriate that an equality of this type should be called, simply, an ABE.

(3.14) **Theorem (ABE for Reflexive Complexes).** *Let C be a semi-dualizing complex for R. If Z is C–reflexive, then*

$$\text{G–dim}_C Z = \operatorname{depth} R - \operatorname{depth}_R Z.$$

Proof. By (3.1)(a) and (3.2)(a) we have

$$\begin{aligned} \text{G–dim}_C Z &= \inf C - \inf Z^{\dagger_C} \\ &= \inf C + \operatorname{depth}_R C - \operatorname{depth}_R Z \\ &= \operatorname{depth} R - \operatorname{depth}_R Z. \quad \square \end{aligned}$$

The next proposition shows that G–dimension with respect to a semi-dualizing complex is a finer invariant than the projective dimension.

(3.15) **Proposition.** *Let C be a semi-dualizing complex for R. For $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ there is an inequality:*

$$\text{G–dim}_C\, Z \le \operatorname{pd}_R Z,$$

and equality holds if $\operatorname{pd}_R Z < \infty$.

Proof. The inequality trivially holds if $\operatorname{pd}_R Z = \infty$. If $\operatorname{pd}_R Z < \infty$, then also $\text{G–dim}_C\, Z$ is finite, cf. (2.9), and we have $\operatorname{pd}_R Z = \operatorname{depth} R - \operatorname{depth}_R Z$ by the Auslander–Buchsbaum Equality, cf. [17, (0.1)], so equality holds by (3.14). $\square$

Let C be a semi-dualizing complex for R, and let $\mathfrak{p} \in \operatorname{Spec} R$. For $Z \in \mathcal{P}^{\mathrm{f}}(R)$ there is an inequality $\text{G–dim}_{C_\mathfrak{p}}\, Z_\mathfrak{p} \le \text{G–dim}_C\, Z$, as the G–dimension equals the projective dimension. However, such an inequality need not hold in general for $Z \in {}_C\mathcal{R}(R)$; for example, we have $\text{G–dim}_{C_\mathfrak{p}}\, C_\mathfrak{p} = \inf C_\mathfrak{p} \ge \inf C = \text{G–dim}_C\, C$. If C is a semi-dualizing module (that is, $\operatorname{amp} C{=}0$), then the desired inequality holds for all C–reflexive complexes; but in general, the best we will get is the following:

(3.16) **Proposition.** *Let C be a semi-dualizing complex for R. For $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$ there is an inequality:*

$$\text{G–dim}_{C_\mathfrak{p}}\, Z_\mathfrak{p} \le \text{G–dim}_C\, Z + \inf C_\mathfrak{p} - \inf C.$$

Proof. Suppose $\text{G–dim}_C\, Z < \infty$, i.e., $Z \in {}_C\mathcal{R}(R)$, then $Z_\mathfrak{p} \in {}_{C_\mathfrak{p}}\mathcal{R}(R)$ by (2.10), and a straightforward computation establishes the desired inequality:

$$\begin{aligned}\text{G–dim}_{C_\mathfrak{p}}\, Z_\mathfrak{p} &= \inf C_\mathfrak{p} - \inf\,(Z^{\dagger_C})_\mathfrak{p}\\ &\le \inf C_\mathfrak{p} - \inf Z^{\dagger_C} = \text{G–dim}_C\, Z + \inf C_\mathfrak{p} - \inf C. \quad \square\end{aligned}$$

The behavior of G–dimension under completion is accounted for in (5.11).

Auslander categories have some remarkable stability properties. For the Auslander and Bass classes (to be introduced in the next section) these are discussed in [11]; here we only deal with those pertaining to the class ${}_C\mathcal{R}(R)$ of reflexive complexes.

(3.17) **Theorem.** *Let C be a semi-dualizing complex for R. If $Z \in {}_C\mathcal{R}(R)$ and $P \in \mathcal{P}^{\mathrm{f}}(R)$, then also the complexes $\mathbf{R}\mathrm{Hom}_R(P,Z)$ and $P \otimes_R^{\mathbf{L}} Z$ belong to ${}_C\mathcal{R}(R)$, and there are equalities:*

(a) $\text{G–dim}_C\, \mathbf{R}\mathrm{Hom}_R(P,Z) = \text{G–dim}_C\, Z - \inf P$; *and*

(b) $\text{G–dim}_C(P \otimes_R^{\mathbf{L}} Z) = \text{G–dim}_C\, Z + \operatorname{pd}_R P$.

Proof. Let $Z \in {}_C\mathcal{R}(R)$ and $P \in \mathcal{P}^{\mathrm{f}}(R)$. The complexes $\mathbf{R}\mathrm{Hom}_R(P,Z)$ and $P \otimes_R^{\mathbf{L}} Z$ belong to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, cf. (1.4), and so do their dagger duals as we have

$$\begin{aligned}(P \otimes_R^{\mathbf{L}} Z)^{\dagger_C} &\simeq \mathbf{R}\mathrm{Hom}_R(P, Z^{\dagger_C}) && \text{by (1.5.2); and}\\ \mathbf{R}\mathrm{Hom}_R(P,Z)^{\dagger_C} &\simeq P \otimes_R^{\mathbf{L}} Z^{\dagger_C} && \text{by (1.5.8).}\end{aligned}$$

Moreover, the commutative diagrams below show that the relevant biduality maps are invertible.

$$\begin{tikzcd}
P \otimes^{\mathbf{L}}_R Z \arrow[r, "\delta^C_{P\otimes^{\mathbf{L}}_R Z}"] \arrow[d, "P\otimes^{\mathbf{L}}_R \delta^C_Z"', "\simeq"] & (P \otimes^{\mathbf{L}}_R Z)^{\dagger_C\dagger_C} \\
P \otimes^{\mathbf{L}}_R Z^{\dagger_C\dagger_C} \arrow[r, "\simeq", "\theta_{PZ^{\dagger_C}C}"'] & \mathbf{R}\mathrm{Hom}_R(P, Z^{\dagger_C})^{\dagger_C} \arrow[u, "\simeq"]
\end{tikzcd}$$

The evaluation morphisms $\theta_{PZ^{\dagger_C}C}$ and θ_{PZC} are invertible as $P \in \mathcal{P}^{\mathrm{f}}(R)$, cf. (1.5.8).

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_R(P, Z) \arrow[r, "\delta^C_{\mathbf{R}\mathrm{Hom}_R(P,Z)}"] \arrow[d, "\mathbf{R}\mathrm{Hom}_R(P,\delta^C_Z)"', "\simeq"] & \mathbf{R}\mathrm{Hom}_R(P, Z)^{\dagger_C\dagger_C} \arrow[d, "\simeq"', "(\theta_{PZC})^{\dagger_C}"] \\
\mathbf{R}\mathrm{Hom}_R(P, Z^{\dagger_C\dagger_C}) \arrow[r, "\simeq"] & (P \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Z, C))^{\dagger_C}
\end{tikzcd}$$

This proves that $\mathbf{R}\mathrm{Hom}_R(P, Z)$ and $P \otimes^{\mathbf{L}}_R Z$ belong to ${}_C\mathcal{R}(R)$, i.e., they have finite G–dimension, and the next step is to compute these dimensions.

(a): The desired equality follows by the definition of G–dimension and the next computation:

$$\begin{aligned}
-\inf(\mathbf{R}\mathrm{Hom}_R(P, Z)^{\dagger_C}) &= -\inf(P \otimes^{\mathbf{L}}_R Z^{\dagger_C}) \quad \text{by (1.5.8)} \\
&= -\inf Z^{\dagger_C} - \inf P \quad \text{by (1.3.4)}.
\end{aligned}$$

(b): We have $(P \otimes^{\mathbf{L}}_R Z)^{\dagger_C} \simeq \mathbf{R}\mathrm{Hom}_R(P, Z^{\dagger_C})$ by (1.5.2), and hence

$$\begin{aligned}
-\inf(P \otimes^{\mathbf{L}}_R Z)^{\dagger_C} &= -\operatorname{ord} \mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(P,Z^{\dagger_C})}(t) \\
&= -\operatorname{ord}(\mathrm{P}^R_P(t^{-1})\, \mathrm{P}^R_{Z^{\dagger_C}}(t)) \quad \text{by (2.14)} \\
&= \deg \mathrm{P}^R_P(t) - \operatorname{ord} \mathrm{P}^R_{Z^{\dagger_C}}(t) \\
&= \mathrm{pd}_R P - \inf Z^{\dagger_C}.
\end{aligned}$$

Again the desired equality follows by the definition of G–dimension. □

(3.18) **Observation.** Let C be a semi-dualizing complex for R. If Z is C–reflexive, then $Z \simeq \mathbf{R}\mathrm{Hom}_R(Z^{\dagger_C}, C)$, so by (1.7.7) there is an equality of formal Laurant series:

(3.18.1) $$\mathrm{I}^Z_R(t) = \mathrm{P}^R_{Z^{\dagger_C}}(t)\, \mathrm{I}^C_R(t).$$

In particular,

(3.18.2) $$\mathrm{I}_R(t) = \mathrm{P}^R_C(t)\, \mathrm{I}^C_R(t).$$

If D is dualizing for R, then $\mathrm{I}^D_R(t) = t^d$ for some $d \in \mathbb{Z}$, cf. (1.8.1), and for every complex Z in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ we have

(3.18.3) $$\mathrm{I}^Z_R(t) = \mathrm{P}^R_{Z^{\dagger_D}}(t) t^d \quad \text{and} \quad \mathrm{P}^R_Z(t) = \mathrm{I}^{Z^{\dagger_D}}_R(t) t^{-d}.$$

Proof of (3.1). (a): By (1.7.1), (3.18.1), and (1.7.3) we have

$$\operatorname{depth}_R Z = \operatorname{ord} \mathrm{I}_R^Z(t) = \operatorname{ord} \mathrm{P}^R_{Z^{\dagger_C}}(t) + \operatorname{ord} \mathrm{I}_R^C(t) = \inf Z^{\dagger_C} + \operatorname{depth}_R C.$$

(d): The equality is evident by the next chain of inclusions:

$$\operatorname{Supp}_R Z = \operatorname{Supp}_R Z^{\dagger_C\dagger_C} \subseteq \operatorname{Supp}_R Z^{\dagger_C} \subseteq \operatorname{Supp}_R Z.$$

(b): Using (d) and applying (a) to $(Z^{\dagger_C})_{\mathfrak{p}} \in {}_{C_{\mathfrak{p}}}\mathcal{R}(R_{\mathfrak{p}})$, cf. (2.10), we find that

$$\begin{aligned}\dim_R Z &= \sup\{\dim R/\mathfrak{p} - \inf Z_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z\}\\ &= \sup\{\dim R/\mathfrak{p} + \operatorname{depth}_{R_{\mathfrak{p}}} C_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z^{\dagger_C}\}.\end{aligned}$$

Thus, it follows by (1.7.5), (1.6.4), and (1.6.6) that

$$\begin{aligned}\dim_R Z &\geq \sup\{\operatorname{depth}_R C - \operatorname{depth}_{R_{\mathfrak{p}}} (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z^{\dagger_C}\}\\ &= \operatorname{depth}_R C + \sup Z^{\dagger_C};\end{aligned}$$

and by (1.6.7) and (1.6.5) we get

$$\begin{aligned}\dim_R Z &\leq \sup\{\dim R/\mathfrak{p} + \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z^{\dagger_C}\}\\ &\leq \sup\{\dim_R C - \operatorname{depth}_{R_{\mathfrak{p}}} (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z^{\dagger_C}\}\\ &= \dim_R C + \sup Z^{\dagger_C}.\end{aligned}$$

(c): Again we use (d), and the desired inequality follows by a straightforward computation:

$$\begin{aligned}\dim_R Z - \sup Z^{\dagger_C} &= \sup\{\dim R/\mathfrak{p} - \inf Z_{\mathfrak{p}} - \sup Z^{\dagger_C} \mid \mathfrak{p} \in \operatorname{Supp}_R Z\}\\ &\leq \sup\{\dim R/\mathfrak{p} - \inf Z_{\mathfrak{p}} - \inf (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z^{\dagger_C}\}\\ &\leq \sup\{\dim R/\mathfrak{p} - \inf (Z^{\dagger_C})_{\mathfrak{p}} - \inf Z \mid \mathfrak{p} \in \operatorname{Supp}_R Z^{\dagger_C}\}\\ &= \dim_R Z^{\dagger_C} - \inf Z. \quad \square\end{aligned}$$

4. Foxby Duality

When C is semi-dualizing for R, canonical maps $\gamma_X^C \colon X \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$ and $\xi_Y^C : C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y) \to Y$, for $X, Y \in \mathcal{D}_{\mathrm{b}}(R)$, are defined by requiring commutativity of the diagrams

(4.0.1)
$$\begin{array}{ccc} X & \xrightarrow{\gamma_X^C} & \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \\ \big\downarrow{\simeq} & & \big\uparrow{\omega_{CCX}} \\ R \otimes^{\mathbf{L}}_R X & \xrightarrow[\chi_C^R \otimes^{\mathbf{L}}_R X]{\simeq} & \mathbf{R}\mathrm{Hom}_R(C,C) \otimes^{\mathbf{L}}_R X \end{array}$$

and

(4.0.2)
$$\begin{array}{ccc} C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y) & \xrightarrow{\xi_Y^C} & Y \\ {\theta_{CCY}}\big\downarrow & & \big\uparrow{\simeq} \\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,C),Y) & \xrightarrow[\mathbf{R}\mathrm{Hom}_R(\chi_C^R,Y)]{\simeq} & \mathbf{R}\mathrm{Hom}_R(R,Y) \end{array}$$

(4.1) **Definitions.** Let C be a semi-dualizing complex for R. The *C–Auslander class* of R, ${}_C\mathcal{A}(R)$, and the *C–Bass class* of R, ${}_C\mathcal{B}(R)$, are the full subcategories of $\mathcal{D}_\mathrm{b}(R)$ defined by specifying their objects as follows:

(A) X belongs to ${}_C\mathcal{A}(R)$ if and only if $C \otimes^{\mathbf{L}}_R X \in \mathcal{D}_\mathrm{b}(R)$ and the canonical map $\gamma^C_X \colon X \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$ is an isomorphism.

(B) Y belongs to ${}_C\mathcal{B}(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(C,Y) \in \mathcal{D}_\mathrm{b}(R)$ and the canonical map $\xi^C_Y \colon C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y) \to Y$ is an isomorphism.

(4.2) ***Genuine Foxby Duality***. Let R be a local ring. If D is a dualizing complex for R, then there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}(R) & \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} & \mathcal{D}(R) \\
\cup\!\mid & & \cup\!\mid \\
{}_D\mathcal{A}(R) & \rightleftarrows & {}_D\mathcal{B}(R) \\
\cup\!\mid & & \cup\!\mid \\
\mathcal{F}(R) & \rightleftarrows & \mathcal{I}(R)
\end{array}$$

where the vertical inclusions are full embeddings, and the unlabeled horizontal arrows are quasi-inverse equivalences of categories. This has become known as *Foxby duality*, cf. [10] and [30]; a proof is given in [5, (3.2)].

The purpose of this section is to establish a theory of Foxby duality for semi-dualizing complexes. We seek to take the analogy with (4.2) as far as possible, and it does go quite far: the full embeddings are established in (4.4) and the equivalence of Auslander and Bass classes in (4.6). On the other hand, it is out of the question that every semi-dualizing complex C will give equivalences of $\mathcal{F}(R)$ and $\mathcal{I}(R)$ the way a dualizing complex does. For $C = R$, e.g., this would imply that $\mathcal{F}(R) = \mathcal{I}(R)$, which means that R is Gorenstein. This problem is treated carefully in section 8.

(4.3) **Remarks.** It is straightforward to check that $R \in {}_C\mathcal{A}(R)$ and $C \in {}_C\mathcal{B}(R)$ when C is a semi-dualizing complex for R; and, obviously, ${}_R\mathcal{A}(R) = {}_R\mathcal{B}(R) = \mathcal{D}_\mathrm{b}(R)$. When R is local and Cohen–Macaulay with a dualizing module D, the complexes in the Auslander class ${}_D\mathcal{A}(R)$ are exactly those of finite Gorenstein projective dimension or, equivalently, finite Gorenstein flat dimension, while the complexes in the Bass class ${}_D\mathcal{B}(R)$ are those of finite Gorenstein injective dimension, cf. [5], [10], [11], and [9].

Auslander and Bass classes behave as expected under completion and localization. That is, $X_\mathfrak{p} \in {}_{C_\mathfrak{p}}\mathcal{A}(R_\mathfrak{p})$ if $X \in {}_C\mathcal{A}(R)$; and if R is local, then X belongs to ${}_C\mathcal{A}(R)$ if and only if $X \otimes_R \widehat{R} \in {}_{C\otimes_R\widehat{R}}\mathcal{A}(\widehat{R})$. Similar results hold for Bass classes; they are all special cases of (5.8) and (5.9).

(4.4) **Proposition.** *If C is a semi-dualizing complex for R, then there are two full embeddings:*

$$\mathcal{F}(R) \subseteq {}_C\mathcal{A}(R) \qquad \textit{and} \qquad \mathcal{I}(R) \subseteq {}_C\mathcal{B}(R).$$

Proof. The first embedding is immediate by (1.4.1) and (4.0.1) and the second by (1.4.2) and (4.0.2). $\square$

(4.5) **Lemma.** *Let C be a semi-dualizing complex for R. The following hold for $X \in {}_C\mathcal{A}(R)$, $Y \in {}_C\mathcal{B}(R)$, and $U \in \mathcal{D}_\mathrm{b}(R)$:*

(a) $$\mathbf{R}\mathrm{Hom}_R(U,X) \simeq \mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R U, C \otimes^{\mathbf{L}}_R X);$$

(b) $$\mathbf{R}\mathrm{Hom}_R(Y,U) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,Y), \mathbf{R}\mathrm{Hom}_R(C,U)); \quad \textit{and}$$

(c) $$Y \otimes^{\mathbf{L}}_R U \simeq (C \otimes^{\mathbf{L}}_R U) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y).$$

Proof. The proof of (a) is straightforward:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(U,X) &\simeq \mathbf{R}\mathrm{Hom}_R(U, \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X))\\ &\simeq \mathbf{R}\mathrm{Hom}_R(U \otimes^{\mathbf{L}}_R C, C \otimes^{\mathbf{L}}_R X)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R U, C \otimes^{\mathbf{L}}_R X);\end{aligned}$$

and the proofs of (b) and (c) are similar. $\square$

(4.6) **Foxby Duality Theorem.** *Let C be a semi-dualizing complex for R. The two functors $C \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(C,-)$ give quasi-inverse equivalences of ${}_C\mathcal{A}(R)$ and ${}_C\mathcal{B}(R)$, and they take semi-dualizing complexes to semi-dualizing complexes.*

Furthermore, the following hold for $X, Y \in \mathcal{D}_\mathrm{b}(R)$:

(a) *If $C \otimes^{\mathbf{L}}_R X \in {}_C\mathcal{B}(R)$, then $X \in {}_C\mathcal{A}(R)$; and*

(b) *if $\mathbf{R}\mathrm{Hom}_R(C,Y) \in {}_C\mathcal{A}(R)$, then $Y \in {}_C\mathcal{B}(R)$.*

Proof. The proof of [5, (3.2)] applies verbatim to show that $\mathbf{R}\mathrm{Hom}_R(C,-)$ and $C \otimes^{\mathbf{L}}_R -$ give quasi-inverse equivalences of ${}_C\mathcal{A}(R)$ and ${}_C\mathcal{B}(R)$ with properties (a) and (b).

Suppose $X \in {}_C\mathcal{A}(R)$ is semi-dualizing for R, then $C \otimes^{\mathbf{L}}_R X \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R)$, cf. (1.3.1), and we have

$$R \xrightarrow[\simeq]{\chi^R_X} \mathbf{R}\mathrm{Hom}_R(X,X) \simeq \mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R X, C \otimes^{\mathbf{L}}_R X).$$

The last isomorphism is (4.5)(a), it only involves natural maps, and it is straightforward to check that also $\chi^R_{C \otimes^{\mathbf{L}}_R X}$ is an isomorphism. A similar argument, using (4.5)(b), shows that $\mathbf{R}\mathrm{Hom}_R(C,Y)$ is semi-dualizing for R if $Y \in {}_C\mathcal{B}(R)$ is so. $\square$

The next result provides a connection to the concept of (generalized) G–dimension introduced in previous section. As a corollary we recover [5, (4.1.7)], the fact that: if R is local with a dualizing complex D, then the finite modules in ${}_D\mathcal{A}(R)$ are exactly those with $\text{G–dim}_R\, M < \infty$.

(4.7) **Theorem.** *Let R be local, and let C be a semi-dualizing complex for R. If D is dualizing for R, then there is an equality of full subcategories:*

$$_C\mathcal{A}^{\mathrm{f}}(R) = {}_{C^{\dagger_D}}\mathcal{R}(R).$$

Proof. Set $B = C^{\dagger_D}$, B is semi-dualizing for R by (2.12), and we want to prove that $_C\mathcal{A}^{\mathrm{f}}(R) = {}_B\mathcal{R}(R)$.

First, note that for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ we have two chains of isomorphisms:

$$(*)\qquad \begin{aligned} Z^{\dagger_B} &= \mathbf{R}\mathrm{Hom}_R(Z, \mathbf{R}\mathrm{Hom}_R(C, D)) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(Z \otimes^{\mathbf{L}}_R C, D) \\ &\simeq (C \otimes^{\mathbf{L}}_R Z)^{\dagger_D} \\ &\simeq \mathbf{R}\mathrm{Hom}_R(C, Z^{\dagger_D}); \quad \text{and} \end{aligned}$$

$$(**)\qquad \begin{aligned} C \otimes^{\mathbf{L}}_R Z &\simeq C^{\dagger_D\dagger_D} \otimes^{\mathbf{L}}_R Z \\ &\simeq Z \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B, D) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Z, B), D) \quad \text{by (1.5.8)} \\ &= (Z^{\dagger_B})^{\dagger_D}. \end{aligned}$$

Thus, if Z and $Z^{\dagger_B}$ belong to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ we have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\gamma^C_Z} & \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R Z) \\ {\scriptstyle \delta^B_Z}\big\downarrow & & \big\downarrow{\scriptstyle \simeq} \\ Z^{\dagger_B\dagger_B} & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_R(C, (Z^{\dagger_B})^{\dagger_D}) \end{array}$$

where the unlabeled maps are (induced by) $(*)$ and $(**)$.

"$\subseteq$": Suppose $Z \in {}_C\mathcal{A}^{\mathrm{f}}(R)$, then $C \otimes^{\mathbf{L}}_R Z$, and thereby $(C \otimes^{\mathbf{L}}_R Z)^{\dagger_D}$, belongs to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, cf. (1.4.2). By $(*)$ we conclude that $Z^{\dagger_B} \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The canonical map γ^C_Z is an isomorphism, and the diagram above shows that so is δ^B_Z.

"$\supseteq$": Let $Z \in {}_B\mathcal{R}(R)$, then $Z^{\dagger_B} \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and hence also $(Z^{\dagger_B})^{\dagger_D} \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, so $(**)$ shows that $C \otimes^{\mathbf{L}}_R Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The biduality map δ^B_Z is an isomorphism, and the diagram shows that so is γ^C_Z. $\square$

The last results of this section answer the question of 'how much the size of a complex can change under Foxby duality', and it paves the way for a description of Foxby duality for semi-dualizing modules in terms of classical homological algebra. The proof of the next inequalities is found at the end of the section.

(4.8) **Proposition.** *Let C be a semi-dualizing complex for R. For $X \in {}_C\mathcal{A}(R)$ and $Y \in {}_C\mathcal{B}(R)$ there are inequalities:*

(a) $\qquad \sup X + \inf C \leq \sup (C \otimes^{\mathbf{L}}_R X) \leq \sup X + \sup C;$

(b) $\qquad \operatorname{amp} X - \operatorname{amp} C \leq \operatorname{amp}(C \otimes^{\mathbf{L}}_R X) \leq \operatorname{amp} X + \operatorname{amp} C;$

(c) $\qquad \inf Y - \sup C \leq \inf (\mathbf{R}\mathrm{Hom}_R(C, Y)) \leq \inf Y - \inf C;$ *and*

(d) $\qquad \operatorname{amp} Y - \operatorname{amp} C \leq \operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(C, Y)) \leq \operatorname{amp} Y + \operatorname{amp} C.$

(4.9) **Corollary.** *Let C be a semi-dualizing complex for R. The next three conditions are equivalent.*

(i) $\operatorname{amp} C = 0$.

(ii) $\operatorname{amp}(C \otimes^{\mathbf{L}}_R X) = \operatorname{amp} X$ *for all* $X \in {}_C\mathcal{A}(R)$.

(iii) $\operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(C, Y)) = \operatorname{amp} Y$ *for all* $Y \in {}_C\mathcal{B}(R)$.

And if R is local and D is dualizing for R, then they are equivalent with:

(iv) $C^{\dagger_D}$ *is Cohen–Macaulay.*

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow by (b) and (d) in (4.8), and the reverse implications follow by setting $X = R$ and $Y = C$. If R is local and D is dualizing for R, then $\operatorname{cmd}_R C^{\dagger_D} = \operatorname{amp} C^{\dagger_D \dagger_D} = \operatorname{amp} C$, cf. (3.8), and hence ($iv$) is equivalent to ($i$). ☐

(4.10) **Observation.** Corollary (4.9) generalizes [5, (3.3)]. A semi-dualizing complex C satisfying the equivalent conditions in (4.9) has only one non-vanishing homology module, and we may, after a shift, assume that it is located in degree 0 and hence identify C with the module $K = \mathrm{H}_0(C)$. Thus, K is a *semi-dualizing module*: The endomorphism ring $\mathrm{Hom}_R(K, K)$ is canonically isomorphic to R, and $\mathrm{Ext}^i_R(K, K) = 0$ for $i > 0$.

Modules in the Auslander and Bass classes ${}_K\mathcal{A}(R)$ and ${}_K\mathcal{B}(R)$ can be easily described in terms of classical homological algebra:

$$M \in {}_K\mathcal{A}_0(R) \quad \iff \quad \begin{cases} \mathrm{Tor}^R_i(K, M) = 0 & \text{for } i > 0; \\ \mathrm{Ext}^i_R(K, K \otimes_R M) = 0 & \text{for } i > 0; \quad \text{and} \\ M \cong \mathrm{Hom}_R(K, K \otimes_R M) & \text{canonically.} \end{cases}$$

Similarly, N belongs to ${}_K\mathcal{B}_0(R)$ if and only if

$$\mathrm{Ext}^i_R(K, N) = 0 = \mathrm{Tor}^R_i(K, \mathrm{Hom}_R(K, N))$$

for $i > 0$ and $K \otimes_R \mathrm{Hom}_R(K, N)$ is canonically isomorphic to N. The functors $K \otimes_R -$ and $\mathrm{Hom}_R(K, -)$ give quasi-inverse equivalences of the categories ${}_K\mathcal{A}_0(R)$ and ${}_K\mathcal{B}_0(R)$. These claims are proved in detail in [5, (3.4) and (3.6)] for the case where K is dualizing (and R hence local Cohen–Macaulay), and it is again easily verified that the proofs apply verbatim in our setting. Semi-dualizing modules are *PG–modules* (of rank 1) and partial proofs can be found in [13].

(4.11) **Lemma.** *Let C be a semi-dualizing complex for R. For $X \in \mathcal{D}_\mathrm{b}(R)$ the next two inequalities hold.*

(a) $$\sup X - \sup C \leq \sup(\mathbf{R}\mathrm{Hom}_R(C, X)) \leq \sup X - \inf C; \quad \textit{and}$$

(b) $$\inf X + \inf C \leq \inf(C \otimes^{\mathbf{L}}_R X) \leq \inf X + \sup C.$$

Proof. The first inequality in (a) follows by [15, 2.2] as $\mathrm{Supp}_R\, C = \mathrm{Spec}\, R$, cf. (2.5), and the second is (1.3.3).

Let J be a faithfully injective R–module (e.g., $J = \prod_{\mathfrak{m}\in\operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$, where $\mathrm{E}_R(M)$ denotes the injective hull of an R–module M); by (a) we have

$$\begin{aligned}\inf(C \otimes^{\mathbf{L}}_R X) &= -\sup(\operatorname{Hom}_R(C \otimes^{\mathbf{L}}_R X, J))\\ &= -\sup(\mathbf{R}\operatorname{Hom}_R(C, \operatorname{Hom}_R(X, J)))\\ &\leq \sup C - \sup(\operatorname{Hom}_R(X, J))\\ &= \sup C + \inf X.\end{aligned}$$

This proves the second inequality in (b), and the first one is (1.3.4). □

Proof of (4.8). (a): Since $X \simeq \mathbf{R}\operatorname{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$ it follows by (4.11)(a) that

$$\sup(C \otimes^{\mathbf{L}}_R X) - \sup C \leq \sup X \leq \sup(C \otimes^{\mathbf{L}}_R X) - \inf C$$

and, therefore,

$$-\sup X - \sup C \leq -\sup(C \otimes^{\mathbf{L}}_R X) \leq -\sup X - \inf C.$$

(b): Using the inequalities in (a) and (4.11)(b) we find:

$$\begin{aligned}\operatorname{amp}(C \otimes^{\mathbf{L}}_R X) &= \sup(C \otimes^{\mathbf{L}}_R X) - \inf(C \otimes^{\mathbf{L}}_R X)\\ &\leq \sup X + \sup C - \inf(C \otimes^{\mathbf{L}}_R X)\\ &\leq \sup X + \sup C - (\inf X + \inf C)\\ &= \operatorname{amp} X + \operatorname{amp} C; \quad \text{and}\end{aligned}$$

$$\begin{aligned}\operatorname{amp}(C \otimes^{\mathbf{L}}_R X) &= \sup(C \otimes^{\mathbf{L}}_R X) - \inf(C \otimes^{\mathbf{L}}_R X)\\ &\geq \sup X + \inf C - \inf(C \otimes^{\mathbf{L}}_R X)\\ &\geq \sup X + \inf C - (\inf X + \sup C)\\ &= \operatorname{amp} X - \operatorname{amp} C.\end{aligned}$$

The proof of (c) is similar to that of (a), only it uses (4.11)(b). The proof of (d) uses (c) and (4.11)(a), otherwise it is analogous to the proof of (b). □

5. Base Change

In this section $\varphi\colon R \to S$ is a homomorphism of rings.

We study the behavior of semi-dualizing complexes and Auslander categories under base change; the focus is on (local) homomorphisms of finite flat dimension.

The proofs of the principal results (5.1), (5.3), (5.8), (5.9), and (5.10) require some technical auxiliary results, and they have, therefore, been gathered at the end of the section.

(5.1) **Theorem.** *Let C be a semi-dualizing complex for R. The next two conditions are equivalent.*

(*i*) $S \in {}_C\mathcal{A}(R)$.

(*ii*) $C \otimes^{\mathbf{L}}_R S$ *is semi-dualizing for* S.

When they are satisfied, there is an inequality of amplitudes:

(a) $$\operatorname{amp}(C \otimes^{\mathbf{L}}_R S) \leq \operatorname{amp} C;$$

and, provided that φ is local, an equality of formal Laurant series:

(b) $$\mathrm{I}_S(t) = \mathrm{P}^R_C(t)\, \mathrm{I}_S^{C \otimes^{\mathbf{L}}_R S}(t).$$

Furthermore, if S is local and E is a dualizing complex for S, then

(c) $$(C \otimes^{\mathbf{L}}_R S)^{\dagger_E} \simeq \mathbf{R}\mathrm{Hom}_R(C, E),$$

and the next two conditions are equivalent and equivalent to (*i*) *and* (*ii*).

(*iii*) $E \in {}_C\mathcal{B}(R)$.

(*iv*) $\mathbf{R}\mathrm{Hom}_R(C, E)$ *is semi-dualizing for* S.

(5.2) **Remarks.** Theorem (5.1) is is the general result on base change for semi-dualizing complexes, special cases will be considered below; among its consequences we note the following:

Applying it to a dualizing complex C for R we obtain a converse to the key result in [5] on existence of relative dualizing complexes, see also (7.1).

Suppose φ is local and S belongs to ${}_C\mathcal{A}(R)$; the equality (5.1)(b) generalizes [5, (7.1)] and shows that if S is Gorenstein, then $\mathrm{P}^R_C(t)$ is a monomial and hence $C \sim R$ as will be proved in (8.3); this should be compared to [5, (7.7.2)].

The following descent result generalizes [5, (4.6)(a) and (7.9)].

(5.3) **Proposition.** *Let C be a semi-dualizing complex for R, and assume that $S \in {}_C\mathcal{A}(R)$. The base changed complex $C \otimes^{\mathbf{L}}_R S$ is then semi-dualizing for S, and the following hold for S–complexes X and Y:*

(a) $$X \in {}_{C \otimes^{\mathbf{L}}_R S}\mathcal{A}(S) \iff X \in {}_C\mathcal{A}(R); \quad \textit{and}$$

(b) $$Y \in {}_{C \otimes^{\mathbf{L}}_R S}\mathcal{B}(S) \iff Y \in {}_C\mathcal{B}(R).$$

In particular, there are full embeddings:

(c) $$\mathcal{F}(S) \subseteq {}_{C \otimes^{\mathbf{L}}_R S}\mathcal{A}(S) \subseteq {}_C\mathcal{A}(R); \quad \textit{and}$$

(d) $$\mathcal{I}(S) \subseteq {}_{C \otimes^{\mathbf{L}}_R S}\mathcal{B}(S) \subseteq {}_C\mathcal{B}(R).$$

We now turn our attention to homomorphisms of finite flat dimension; we start by reviewing a few results from [4] and [6].

(5.4) ***Flat Base Change***. If φ is flat, then the functors $- \otimes^{\mathbf{L}}_R S$ and $- \otimes_R S$ are naturally isomorphic, and we shall not distinguish between them. If φ is also local, then it makes S a faithfully flat R–module, so the amplitude of an R–complex does not change under base change. In this case, behavior of the Cohen–Macaulay defect is governed by the closed fiber $S/\mathfrak{m}S$: for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ there is an equality [6, (1.2)]:

$$\operatorname{cmd}_S(Z \otimes_R S) = \operatorname{cmd}_R Z + \operatorname{cmd} S/\mathfrak{m}S. \tag{5.4.1}$$

(5.5) ***Bass Series***. For a local homomorphism φ of finite flat dimension a Bass series $\mathrm{I}_\varphi(t)$ can be defined (see [4, (2.1)] but also [5, (7.2)]). It is a formal Laurant series with non-negative integer coefficients, and for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ there is an equality of formal Laurant series [4, (2.2)]:

$$\mathrm{I}_S^{Z \otimes^{\mathbf{L}}_R S}(t) = \mathrm{I}_R^Z(t)\,\mathrm{I}_\varphi(t). \tag{5.5.1}$$

Applying the equality to $Z = R$ we see that $\operatorname{ord} \mathrm{I}_\varphi(t) = \operatorname{depth} S - \operatorname{depth} R$, cf. (1.7.3).

If φ is flat, then $\mathrm{I}_\varphi(t)$ is the Bass series of the closed fiber $S/\mathfrak{m}S$, cf. [18].

(5.6) **Theorem.** *Let $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. If φ is local and flat, then the next two conditions are equivalent.*

(*i*) *C is semi-dualizing for R.*

(*ii*) *$C \otimes_R S$ is semi-dualizing for S.*

Furthermore, the following hold:

(a) $\operatorname{amp}(C \otimes_R S) = \operatorname{amp} C$; *and*

(b) $\mathrm{I}_S^{C \otimes_R S}(t) = \mathrm{I}_R^C(t)\,\mathrm{I}_{S/\mathfrak{m}S}(t)$.

Proof. Since S is flat we have $C \otimes_R S \simeq C \otimes^{\mathbf{L}}_R S$, and this complex belongs to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ by (1.3.1). By (1.5.7) the evaluation morphism ω_{CCS} is invertible, and the equivalence of (*i*) and (*ii*) follows from the commutative diagram below, as S is faithfully flat over R.

$$\begin{array}{ccccc}
S & \xrightarrow[\simeq]{} & R \otimes_R S & \xrightarrow{\chi_C^R \otimes_R S} & \mathbf{R}\mathrm{Hom}_R(C,C) \otimes_R S \\
{\scriptstyle \chi^S_{C \otimes_R S}}\Big\downarrow & & & & \simeq\Big\downarrow{\scriptstyle \omega_{CCS}} \\
\mathbf{R}\mathrm{Hom}_S(C \otimes_R S, C \otimes_R S) & & \xrightarrow{\simeq} & & \mathbf{R}\mathrm{Hom}_R(C, C \otimes_R S)
\end{array}$$

The bottom row isomorphism is (1.5.5).

Also (a) follows by faithful flatness, and (b) is (5.5.1). □

(5.7) **Proposition.** *Let C be a semi-dualizing complex for R. If φ is local and of finite flat dimension, then the base changed complex $C \otimes^{\mathbf{L}}_R S$ is semi-dualizing for S, and the following hold:*

(a) $\operatorname{amp}(C \otimes^{\mathbf{L}}_R S) = \operatorname{amp} C$; *and*

(b) $\mathrm{I}_S^{C \otimes^{\mathbf{L}}_R S}(t) = \mathrm{I}_R^C(t)\,\mathrm{I}_\varphi(t)$.

Proof. It follows from (4.4) and (5.1) that $C \otimes^{\mathbf{L}}_R S$ is a semi-dualizing complex for S, and the equality in (b) is (5.5.1). By (5.1) we have $\operatorname{amp}(C \otimes^{\mathbf{L}}_R S) \leq \operatorname{amp} C$, and the opposite inequality is (5.13)(a); this proves (a). □

The next three results explore the ascent and descent properties of Auslander categories under homomorphisms of finite flat dimension.

(5.8) **Proposition.** *Let C be a semi-dualizing complex for R. If φ is of finite flat dimension, then $C \otimes^{\mathbf{L}}_R S$ is semi-dualizing for S, and for $X \in \mathcal{D}(R)$ we can consider the two statements:*

$$(i)\ X \in {}_C\mathcal{A}(R); \qquad \text{and} \qquad (ii)\ X \otimes^{\mathbf{L}}_R S \in {}_{C\otimes^{\mathbf{L}}_R S}\mathcal{A}(S).$$

The following hold:

(a) *(i) implies (ii); and*

(b) *if φ is faithfully flat, then the two conditions are equivalent.*

(5.9) **Proposition.** *Let C be a semi-dualizing complex for R. If φ is of finite flat dimension, then $C \otimes^{\mathbf{L}}_R S$ is semi-dualizing for S, and for $Y \in \mathcal{D}(R)$ we can consider the two statements:*

$$(i)\ Y \in {}_C\mathcal{B}(R); \qquad \text{and} \qquad (ii)\ Y \otimes^{\mathbf{L}}_R S \in {}_{C\otimes^{\mathbf{L}}_R S}\mathcal{B}(S).$$

The following hold:

(a) *(i) implies (ii); and*

(b) *if φ is faithfully flat, then the two conditions are equivalent.*

(5.10) **Theorem.** *Let C be a semi-dualizing complex for R. If φ is of finite flat dimension, then $C \otimes^{\mathbf{L}}_R S$ is semi-dualizing for S, and for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ we can consider the two statements:*

$$(i)\ Z \in {}_C\mathcal{R}(R); \qquad \text{and} \qquad (ii)\ Z \otimes^{\mathbf{L}}_R S \in {}_{C\otimes^{\mathbf{L}}_R S}\mathcal{R}(S).$$

The following hold:

(a) *(i) implies (ii);*

(b) *if φ is faithfully flat, then (ii) and (i) are equivalent; and*

(c) *if φ is local, then the next equality holds for $Z \in {}_C\mathcal{R}(R)$:*

$$\text{G–dim}_{C\otimes^{\mathbf{L}}_R S}(Z \otimes^{\mathbf{L}}_R S) = \text{G–dim}_C Z.$$

The next result describes the behavior of G–dimension under flat extensions, it generalizes [5, (4.1.4)].

(5.11) **Corollary.** *Let C be a semi-dualizing complex for R. If φ is local and flat, then $C \otimes_R S$ is semi-dualizing for S, and for $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ there is an equality:*

$$\text{G–dim}_{C\otimes_R S}(Z \otimes_R S) = \text{G–dim}_C Z.$$

In particular, the two dimensions are simultaneously finite.

Proof. The complex $C \otimes_R S$ is semi-dualizing for S by (5.6). It follows by the definition (3.11) and (5.10)(b) that $\text{G–dim}_{C\otimes_R S}(Z \otimes_R S)$ is finite if and only if $\text{G–dim}_C Z$ is so, and the equality follows by (5.10)(c). $\square$

Proof of (5.10). We assume that $S \in \mathcal{F}(R)$ and $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$. It follows by (5.7) that $C \otimes_R^{\mathbf{L}} S$ is a semi-dualizing complex for S; and $Z \otimes_R^{\mathbf{L}} S$ belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$ by (1.3.1) and (1.4.1). The isomorphisms in the next chain follow by (1.5.5) and (1.5.7).

$$(*)\qquad \begin{aligned} (Z \otimes_R^{\mathbf{L}} S)^{\dagger_{C \otimes_R^{\mathbf{L}} S}} &= \mathbf{R}\mathrm{Hom}_S(Z \otimes_R^{\mathbf{L}} S, C \otimes_R^{\mathbf{L}} S) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(Z, C \otimes_R^{\mathbf{L}} S) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(Z, C) \otimes_R^{\mathbf{L}} S \\ &= Z^{\dagger_C} \otimes_R^{\mathbf{L}} S. \end{aligned}$$

(a): Suppose $Z \in {}_C\mathcal{R}(R)$, then $Z^{\dagger_C} \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and by (1.4.1) we conclude from $(*)$ that $(Z \otimes_R^{\mathbf{L}} S)^{\dagger_{C \otimes_R^{\mathbf{L}} S}} \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$. In the commutative diagram

$$\begin{array}{ccc} Z \otimes_R^{\mathbf{L}} S & \xrightarrow{\delta^{C \otimes_R^{\mathbf{L}} S}_{Z \otimes_R^{\mathbf{L}} S}} & (Z \otimes_R^{\mathbf{L}} S)^{\dagger_{C \otimes_R^{\mathbf{L}} S} \dagger_{C \otimes_R^{\mathbf{L}} S}} \\ {\scriptstyle \delta_Z^C \otimes_R^{\mathbf{L}} S}\big\downarrow & & \big\uparrow{\scriptstyle \simeq} \\ Z^{\dagger_C \dagger_C} \otimes_R^{\mathbf{L}} S & \xrightarrow{\simeq} & (Z^{\dagger_C} \otimes_R^{\mathbf{L}} S)^{\dagger_{C \otimes_R^{\mathbf{L}} S}} \end{array}$$

the unlabeled isomorphisms are (induced by) $(*)$; it shows that $\delta^{C \otimes_R^{\mathbf{L}} S}_{Z \otimes_R^{\mathbf{L}} S}$ is invertible if δ_Z^C is so. We have now proved that (i) implies (ii).

(b): Suppose S is a faithfully flat R–module. Then it follows by $(*)$ that $Z^{\dagger_C}$ is bounded if and only if $(Z \otimes_R^{\mathbf{L}} S)^{\dagger_{C \otimes_R^{\mathbf{L}} S}}$ is so, and we see from the diagram that δ_Z^C is invertible if and only if $\delta^{C \otimes_R^{\mathbf{L}} S}_{Z \otimes_R^{\mathbf{L}} S}$ is so.

(c): Suppose φ is local and $Z \in {}_C\mathcal{R}(R)$. It follows by (a) that $\mathrm{G\text{–}dim}_{C \otimes_R^{\mathbf{L}} S}(Z \otimes_R^{\mathbf{L}} S)$ is finite, and the equality of G–dimensions follows by (3.14) and (5.5.1):

$$\begin{aligned} \mathrm{G\text{–}dim}_{C \otimes_R^{\mathbf{L}} S}(Z \otimes_R^{\mathbf{L}} S) &= \operatorname{depth} S - \operatorname{depth}_S(Z \otimes_R^{\mathbf{L}} S) \\ &= \operatorname{depth} S - (\operatorname{depth}_R Z + (\operatorname{depth} S - \operatorname{depth} R)) \\ &= \operatorname{depth} R - \operatorname{depth}_R Z \\ &= \mathrm{G\text{–}dim}_C Z. \quad \square \end{aligned}$$

The next proposition extends the results in (5.4) to homomorphisms of finite flat dimension; part (a) was used in the proof of (5.7)(a). Note that the inequality $\operatorname{cmd} R \le \operatorname{cmd} S$, [6, (4.3)], for local homomorphisms of finite flat dimension, is a corollary to part (b). For the proof we need the following:

(5.12) **Lemma.** *Let φ be local and finite. If $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$, then Z belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, and there is an equality of Cohen–Macaulay defects:*

$$\operatorname{cmd}_S Z = \operatorname{cmd}_R Z.$$

Proof. If φ is local and finite, then so is the completion $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$. By (5.4.1) the Cohen–Macaulay defect is not affected by completion, so we may assume that R is complete and hence possesses a dualizing complex D, cf. (1.8.3). Let $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$, it follows by finiteness of φ that $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$. The complex $\mathbf{R}\mathrm{Hom}_R(S,D)$ is dualizing for S, cf. [12, Chapter 15] or (6.2), and

$$\begin{aligned} \mathrm{cmd}_S\, Z &= \mathrm{amp}(\mathbf{R}\mathrm{Hom}_S(Z, \mathbf{R}\mathrm{Hom}_R(S,D))) && \text{by (3.8)} \\ &= \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(Z,D)) && \text{by (1.5.6)} \\ &= \mathrm{cmd}_R\, Z && \text{by (3.8).} \quad \square \end{aligned}$$

(5.13) **Proposition.** *If φ is local and of finite flat dimension, then the next two inequalities hold for $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.*

$$\text{(a)} \qquad \mathrm{amp}(Z \otimes_R^{\mathbf{L}} S) \geq \mathrm{amp}\, Z; \quad \textit{and}$$

$$\text{(b)} \qquad \mathrm{cmd}_S(Z \otimes_R^{\mathbf{L}} S) \geq \mathrm{cmd}_R\, Z.$$

Proof. Let $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$. First we deal with the special case of a finite homomorphism, and then we deal with the general case.

If φ is finite, then $S \in \mathcal{P}_0^{\mathrm{f}}(R)$, so (a) is Iversen's Amplitude Inequality (3.6.2). The inequality in (b) is a consequence of the New Intersection Theorem (3.6.1):

$$\mathrm{cmd}_S(Z \otimes_R^{\mathbf{L}} S) = \mathrm{cmd}_R(Z \otimes_R^{\mathbf{L}} S) \geq \mathrm{cmd}_R\, Z;$$

the equality is (5.12), it applies as $Z \otimes_R^{\mathbf{L}} S \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$ by (1.3.1) and (1.4.1).

We denote the maximal ideals of R and S by, respectively, $\mathfrak{m}$ and $\mathfrak{n}$. If φ is not finite, we consider a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of the semi-completion $\acute{\varphi}\colon R \to \widehat{S}$ (the composition of φ and the the canonical map from S to the $\mathfrak{n}$–adic completion $\widehat{S}$). That is, R' is complete, $R'/\mathfrak{m}R'$ is regular, and we have $\acute{\varphi} = \varphi'\dot{\varphi}$, where $\dot{\varphi}$ is a flat local homomorphism and φ' is a finite local homomorphism with $\mathrm{fd}\,\varphi' < \infty$; cf. [7, (1.1) and (3.3)]. In particular, there is an isomorphism of $\widehat{S}$–complexes:

$$(*) \qquad Z \otimes_R^{\mathbf{L}} \widehat{S} \simeq (Z \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} \widehat{S}.$$

Now, we have

$$\begin{aligned} \mathrm{cmd}_S(Z \otimes_R^{\mathbf{L}} S) &= \mathrm{cmd}_{\widehat{S}}((Z \otimes_R^{\mathbf{L}} S) \otimes_S \widehat{S}) && \text{by (5.4.1)} \\ &= \mathrm{cmd}_{\widehat{S}}(Z \otimes_R^{\mathbf{L}} \widehat{S}) && \text{by (1.5.4)} \\ &= \mathrm{cmd}_{\widehat{S}}((Z \otimes_R^{\mathbf{L}} R') \otimes_{R'}^{\mathbf{L}} \widehat{S}) && \text{by } (*) \\ &\geq \mathrm{cmd}_{R'}(Z \otimes_R^{\mathbf{L}} R') && \text{as } \varphi' \text{ is finite} \\ &= \mathrm{cmd}_R\, Z && \text{by (5.4.1).} \end{aligned}$$

This proves (b), and the proof of (a) is similar, only easier. $\square$

Proof of (5.1). *(i)* $\Rightarrow$ *(ii)*: Suppose $S \in {}_C\mathcal{A}(R)$, then $C \otimes_R^{\mathbf{L}} S \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$, cf. (1.3.1), and the commutative diagram below shows that the homothety morphism $\chi^S_{C \otimes_R^{\mathbf{L}} S}$ is invertible.

$$\begin{array}{ccc} S & \xrightarrow{\chi^S_{C\otimes^{\mathbf{L}}_R S}} & \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S, C\otimes^{\mathbf{L}}_R S) \\ {\scriptstyle\gamma^C_S}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} \\ \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R S) & = & \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R S) \end{array}$$

The unlabeled isomorphism is (1.5.5).

$(ii) \Rightarrow (i)$: If $C\otimes^{\mathbf{L}}_R S$ is semi-dualizing for S, then, in particular, $C\otimes^{\mathbf{L}}_R S \in \mathcal{D}_{\mathrm{b}}(R)$ and the diagram above shows that γ^C_S is an isomorphism.

The inequality of amplitudes, (a), follows from (4.8)(b) and the equality of formal Laurant series, (b), from (3.18.2) and (1.7.6):

$$\mathrm{I}_S(t) = \mathrm{P}^S_{C\otimes^{\mathbf{L}}_R S}(t)\,\mathrm{I}^{C\otimes^{\mathbf{L}}_R S}_S(t) = \mathrm{P}^R_C(t)\,\mathrm{I}^{C\otimes^{\mathbf{L}}_R S}_S(t).$$

In the following E is a dualizing complex for S. We have

$$(C\otimes^{\mathbf{L}}_R S)^{\dagger_E} = \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S, E) \simeq \mathbf{R}\mathrm{Hom}_R(C, E),$$

by (1.5.5), and this isomorphism shows that the equivalence of (ii) and (iv) is simply (2.12). It follows by (5.3)(d) (which only uses the implication $(i) \Rightarrow (ii)$ of (5.1)) that (i) implies (iii); this leaves us only one implication to prove.

$(iii) \Rightarrow (iv)$: We assume that E belongs to ${}_C\mathcal{B}(R)$; the complex $\mathbf{R}\mathrm{Hom}_R(C,E)$ is then bounded, i.e., it belongs to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ as desired, cf. (1.3.2). The unlabeled isomorphism in the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\chi^S_{\mathbf{R}\mathrm{Hom}_R(C,E)}} & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(C,E), \mathbf{R}\mathrm{Hom}_R(C,E)) \\ {\scriptstyle\chi^S_E}\big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\simeq} \\ \mathbf{R}\mathrm{Hom}_S(E,E) & \xrightarrow[\mathbf{R}\mathrm{Hom}_S(\xi^C_E, E)]{\simeq} & \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,E), E) \end{array}$$

follows by (1.5.5), (1.5.2), and (1.5.4):

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(C,E), \mathbf{R}\mathrm{Hom}_R(C,E)) \\ &\qquad\simeq \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(C,E), \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S, E)) \\ &\qquad\simeq \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(C,E)\otimes^{\mathbf{L}}_S (C\otimes^{\mathbf{L}}_R S), E) \\ &\qquad\simeq \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,E), E). \end{aligned}$$

It follows that the homothety morphism $\chi^S_{\mathbf{R}\mathrm{Hom}_R(C,E)}$ is invertible, and the proof is complete. ☐

Proof of (5.3). It follows by (5.1) that $C\otimes^{\mathbf{L}}_R S$ is semi-dualizing for S. By (1.5.4) and (1.5.5) we have

$$\begin{aligned} (C\otimes^{\mathbf{L}}_R S)\otimes^{\mathbf{L}}_S X \in \mathcal{D}_{\mathrm{b}}(S) \quad &\Longleftrightarrow \quad C\otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R); \quad \text{and} \\ \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S, Y) \in \mathcal{D}_{\mathrm{b}}(S) \quad &\Longleftrightarrow \quad \mathbf{R}\mathrm{Hom}_R(C, Y) \in \mathcal{D}_{\mathrm{b}}(R). \end{aligned}$$

These standard isomorphisms also account for the unlabeled isomorphisms in the diagrams below, and since they only involve natural maps, commutativity it is easily checked.

$$
\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_S(C \otimes^{\mathbf{L}}_R S, (C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S X) \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_R(C, (C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S X) \\
X \arrow[u, "\gamma_X^{C \otimes^{\mathbf{L}}_R S}"] \arrow[r, "\gamma^C_X"'] & \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \arrow[u, "\simeq"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_S(C \otimes^{\mathbf{L}}_R S, Y) \arrow[r, "\xi_Y^{C \otimes^{\mathbf{L}}_R S}"] & Y \\
(C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(C, Y) \arrow[u, "\simeq"'] \arrow[r, "\simeq"] & C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, Y) \arrow[u, "\xi^C_Y"']
\end{tikzcd}
$$

This establishes (a) and (b), and the full embeddings follow in view of (4.4). $\square$

Proof of (5.8) and (5.9). We assume that $S \in \mathcal{F}(R)$; then $C \otimes^{\mathbf{L}}_R S$ is a semi-dualizing complex for S by (5.7). Note that for $X \in \mathcal{D}(R)$ we have

$$
(*) \qquad (C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S (X \otimes^{\mathbf{L}}_R S) \simeq C \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R S) \simeq (C \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R S.
$$

Also note that, when $C \otimes^{\mathbf{L}}_R X$ is bounded, we have the following commutative diagram:

$$
\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_S(C \otimes^{\mathbf{L}}_R S, (C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S (X \otimes^{\mathbf{L}}_R S)) \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_R(C, (C \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R S) \\
X \otimes^{\mathbf{L}}_R S \arrow[u, "\gamma_{X \otimes^{\mathbf{L}}_R S}^{C \otimes^{\mathbf{L}}_R S}"] \arrow[r, "\gamma^C_X \otimes^{\mathbf{L}}_R S"'] & \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R S \arrow[u, "\simeq", "\omega_{C(C \otimes^{\mathbf{L}}_R X)S}"']
\end{tikzcd}
$$

The unlabeled isomorphism is induced by (1.5.4) and $(*)$, and the evaluation morphism is invertible by (1.5.7).

(a): If $X \in {}_C\mathcal{A}(R)$, then X is bounded and hence so is $X \otimes^{\mathbf{L}}_R S$, cf. (1.4.1). Also $C \otimes^{\mathbf{L}}_R X$ is bounded, and from $(*)$ we conclude that $(C \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S (X \otimes^{\mathbf{L}}_R S) \in \mathcal{D}_{\mathrm{b}}(S)$. The diagram shows that $\gamma_{X \otimes^{\mathbf{L}}_R S}^{C \otimes^{\mathbf{L}}_R S}$ is invertible since γ^C_X is so, and hence $X \otimes^{\mathbf{L}}_R S \in {}_{C \otimes^{\mathbf{L}}_R S}\mathcal{A}(S)$ as desired.

(b): If $X \otimes^{\mathbf{L}}_R S \in {}_{C \otimes^{\mathbf{L}}_R S}\mathcal{A}(S)$ we conclude by faithful flatness of S over R that both X and $C \otimes^{\mathbf{L}}_R X$ (in view of $(*)$) must belong to $\mathcal{D}_{\mathrm{b}}(R)$. The diagram now shows that the induced map $\gamma^C_X \otimes^{\mathbf{L}}_R S$, and hence also γ^C_X, is invertible. This concludes the proof of (5.8), and the proof of (5.9) is analogous. $\square$

6. Finite Local Homomorphisms

In this section $\varphi \colon R \to S$ is a finite local homomorphism of rings.

We study induced semi-dualizing complexes for S of the form $\mathbf{R}\mathrm{Hom}_R(S, C)$, where C is semi-dualizing for R. The motivation for doing so, of course, comes from [21].

The proofs of (6.1) and (6.4) are deferred to the end of the section.

(6.1) **Theorem.** *Let C be a semi-dualizing complex for R. The next two conditions are equivalent.*

(*i*) $S \in {}_{C}\mathcal{R}(R)$.

(*ii*) $\mathbf{R}\mathrm{Hom}_R(S,C)$ *is semi-dualizing for* S.

And there is an equality of Bass series:

$$\text{(a)} \qquad \mathrm{I}_S^{\mathbf{R}\mathrm{Hom}_R(S,C)}(t) = \mathrm{I}_R^C(t).$$

When the equivalent conditions (*i*) *and* (*ii*) *are satisfied, there is also an inequality:*

$$\text{(b)} \qquad \operatorname{cmd}_S \mathbf{R}\mathrm{Hom}_R(S,C) \leq \operatorname{cmd}_R C.$$

We start by spelling out some easy consequences of the theorem. The corollary below is a well-known result, cf. [12, Chapter 15], and it follows immediately from the theorem by (1.8.1).

(6.2) **Corollary.** *If D is a (normalized) dualizing complex for R, then $\mathbf{R}\mathrm{Hom}_R(S,D)$ is a (normalized) dualizing complex for S.* □

(6.3) **Observation.** If the complex $\mathbf{R}\mathrm{Hom}_R(S,R)$ is semi-dualizing for S, then $S \in {}_{R}\mathcal{R}(R)$ by (6.1), that is, φ makes S an R–module of finite Gorenstein dimension (in the sense of [1]). If $\mathbf{R}\mathrm{Hom}_R(S,R)$ is dualizing for S, then $1 = \mathrm{I}_S^{\mathbf{R}\mathrm{Hom}_R(S,R)}(t) = \mathrm{I}_R(t)$ by (1.8.1) and (6.1)(b), so R is Gorenstein, cf. (1.8.2), and by [2, (4.20)] we have $\mathrm{G\text{–}dim}_R M < \infty$ for all finite R–modules M (see also (8.5)).

(6.4) **Proposition.** *Let C be a semi-dualizing complex for R. If φ is of finite flat dimension, then the complex $\mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S, and the following hold:*

(a) $\operatorname{cmd}_S \mathbf{R}\mathrm{Hom}_R(S,C) = \operatorname{cmd}_R C$; *and*

(b) $\operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(S,C)) \geq \operatorname{amp} C$.

The next result generalizes and extends [19, Proposition 5]; applying it to $C = R$ we recover [5, (7.11)].

(6.5) **Theorem.** *Let C be a semi-dualizing complex for R, and assume that $\mathrm{G\text{–}dim}_C S$ is finite. Then the complex $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S, and for $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$ there is an equality:*

$$\mathrm{G\text{–}dim}_C Z = \mathrm{G\text{–}dim}_C S + \mathrm{G\text{–}dim}_{S^{\dagger_C}} Z.$$

In particular, $\mathrm{G\text{–}dim}_{S^{\dagger_C}} Z$ and $\mathrm{G\text{–}dim}_C Z$ are simultaneously finite, that is,

$$Z \in {}_{S^{\dagger_C}}\mathcal{R}(S) \quad \Longleftrightarrow \quad Z \in {}_{C}\mathcal{R}(R).$$

Proof. By definition, (3.11), finiteness of $\mathrm{G\text{–}dim}_C S$ is tantamount to S being C–reflexive. It, therefore, follows by (6.1) that $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ is a semi-dualizing complex for S. We first prove that the two G–dimensions are simultaneously finite. Let $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$; since φ is finite Z belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, and by (1.5.6) we have

$$(*)\qquad \begin{aligned} Z^{\dagger_{(S^{\dagger_C})}} &= \mathbf{R}\mathrm{Hom}_S(Z, S^{\dagger_C}) \\ &= \mathbf{R}\mathrm{Hom}_S(Z, \mathbf{R}\mathrm{Hom}_R(S,C)) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(Z,C) \\ &= Z^{\dagger_C}, \end{aligned}$$

so $Z^{\dagger_{(S^{\dagger_C})}}$ belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$ if and only if $Z^{\dagger_C} \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$. It now follows by the commutative diagram below that the complex Z belongs to ${}_{S^{\dagger_C}}\mathcal{R}(S)$ if and only if it belongs to ${}_C\mathcal{R}(R)$.

$$\begin{array}{ccc} Z & \xrightarrow{\delta_Z^C} & Z^{\dagger_C\dagger_C} \\ {\scriptstyle \delta_Z^{S^{\dagger_C}}}\big\downarrow & & \big\uparrow{\scriptstyle\simeq} \\ Z^{\dagger_{(S^{\dagger_C})}\dagger_{(S^{\dagger_C})}} & \xrightarrow{\simeq} & (Z^{\dagger_{(S^{\dagger_C})}})^{\dagger_C} \end{array}$$

The unlabeled isomorphisms are induced by $(*)$.

Finally, for $Z \in {}_{S^{\dagger_C}}\mathcal{R}(S)$ we have

$$\begin{aligned} \text{G–dim}_{S^{\dagger_C}} Z &= \inf S^{\dagger_C} - \inf Z^{\dagger_{(S^{\dagger_C})}} \\ &= \inf C - \inf Z^{\dagger_C} - (\inf C - \inf S^{\dagger_C}) \quad \text{by } (*) \\ &= \text{G–dim}_C Z - \text{G–dim}_C S. \quad \square \end{aligned}$$

There is also a descent result for complexes with non-finite homology modules:

(6.6) **Proposition.** *Let C be a semi-dualizing complex for R. If φ is of finite flat dimension, then $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ is a semi-dualizing complex for S, and the next implication holds for $X \in \mathcal{D}(R)$:*

$$X \in {}_C\mathcal{A}(R) \quad \Longrightarrow \quad X \otimes_R^{\mathbf{L}} S \in {}_{S^{\dagger_C}}\mathcal{A}(S).$$

Proof. We assume that φ has finite flat dimension, i.e., $S \in \mathcal{P}_0^{\mathrm{f}}(R)$, then it follows by (6.4) that $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S. Since X is bounded so is $X \otimes_R^{\mathbf{L}} S$, cf. (1.4.1). Also $C \otimes_R^{\mathbf{L}} X$ is bounded, so it follows by (1.4.3) and the next chain of isomorphisms that $S^{\dagger_C} \otimes_S^{\mathbf{L}} (X \otimes_R^{\mathbf{L}} S) \in \mathcal{D}_{\mathrm{b}}(S)$.

$$(*)\qquad \begin{aligned} S^{\dagger_C} \otimes_S^{\mathbf{L}} (X \otimes_R^{\mathbf{L}} S) &\simeq S^{\dagger_C} \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} X) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(S,C) \otimes_R^{\mathbf{L}} X \\ &\simeq \mathbf{R}\mathrm{Hom}_R(S, C \otimes_R^{\mathbf{L}} X) \quad \text{by (1.5.7).} \end{aligned}$$

Finally, the commutative diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_S(S^{\dagger_C}, (S^{\dagger_C} \otimes^{\mathbf{L}}_S (X \otimes^{\mathbf{L}}_R S))) \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_S(S^{\dagger_C}, \mathbf{R}\mathrm{Hom}_R(S, C \otimes^{\mathbf{L}}_R X)) \arrow[d, "\simeq"'] \\
X \otimes^{\mathbf{L}}_R S \arrow[u, "\gamma^{S^{\dagger_C}}_{X\otimes^{\mathbf{L}}_R S}"] \arrow[d, "\gamma^{C}_{X}\otimes^{\mathbf{L}}_R S"', "\simeq"] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(S, C), C \otimes^{\mathbf{L}}_R X) \\
\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R S \arrow[r, "\simeq"] & S \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \arrow[u, "\simeq"', "\theta_{SC(C\otimes^{\mathbf{L}}_R X)}"]
\end{tikzcd}$$

shows that $\gamma^{S^{\dagger_C}}_{X\otimes^{\mathbf{L}}_R S}$ is an isomorphism. The top row isomorphism is induced by $(*)$, the unlabeled vertical isomorphism follows by (1.5.6), and the evaluation morphism $\theta_{SC(C\otimes^{\mathbf{L}}_R X)}$ is invertible by (1.5.8). $\square$

(6.7) **Observation.** Let C be a semi-dualizing complex for R, and assume that R has a dualizing complex D. If S belongs to ${}_C\mathcal{R}(R)$ it also belongs to ${}_{C^{\dagger_D}}\mathcal{A}(R)$, cf. (4.7), so by (6.1) and (5.1) the complexes $\mathbf{R}\mathrm{Hom}_R(S,C)$ and $C^{\dagger_D} \otimes^{\mathbf{L}}_R S$ are both semi-dualizing for S. By the next chain of isomorphisms, they are the dagger duals of each other with respect to the dualizing complex $E = \mathbf{R}\mathrm{Hom}_R(S,D)$ for S, cf. (6.2).

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(S,C)^{\dagger_E} &= \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S,C), \mathbf{R}\mathrm{Hom}_R(S,D)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(S,C), D) && \text{by (1.5.6)} \\
&\simeq S \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,D) && \text{by (1.5.8)} \\
&\simeq C^{\dagger_D} \otimes^{\mathbf{L}}_R S.
\end{aligned}$$

If S belongs to ${}_C\mathcal{R}(R)$ and to ${}_C\mathcal{A}(R)$ (e.g., $S \in \mathcal{P}^{\mathrm{f}}_0(R)$), then, by (2.12), the pair $(C, C^{\dagger_D})$ of semi-dualizing complexes for R gives rise to two pairs of semi-dualizing complexes for S, namely

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_R(S,C) \quad \text{and its dagger dual} \quad \mathbf{R}\mathrm{Hom}_R(S,C)^{\dagger_E} \simeq C^{\dagger_D} \otimes^{\mathbf{L}}_R S; \quad \text{and} \\
&\mathbf{R}\mathrm{Hom}_R(S,C^{\dagger_D}) \text{ and its dagger dual } \mathbf{R}\mathrm{Hom}_R(S,C^{\dagger_D})^{\dagger_E} \simeq C \otimes^{\mathbf{L}}_R S.
\end{aligned}$$

Proof of (*6.1*). The equality of Bass series, (a), follows by (1.7.9).

$(i) \Rightarrow (ii)$: If $S \in {}_C\mathcal{R}(R)$, then the dagger dual $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ is bounded, so $\mathbf{R}\mathrm{Hom}_R(S,C) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ as required, cf. (1.7.8). The unlabeled isomorphism in the commutative diagram

$$\begin{tikzcd}
S \arrow[r, "\chi^{S}_{\mathbf{R}\mathrm{Hom}_R(S,C)}"] \arrow[d, "\delta^C_S"'] & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S,C), \mathbf{R}\mathrm{Hom}_R(S,C)) \arrow[d, "\simeq"'] \\
S^{\dagger_C\dagger_C} \arrow[r, equal] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(S,C), C)
\end{tikzcd}$$

is (1.5.6); the diagram shows that the homothety morphism $\chi^{S}_{\mathbf{R}\mathrm{Hom}_R(S,C)}$ is invertible since δ^C_S is so.

$(ii) \Rightarrow (i)$: If the complex $\mathbf{R}\mathrm{Hom}_R(S,C) = S^{\dagger_C}$ is semi-dualizing for S, then it belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$ and hence to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, as φ is finite. The diagram above now shows that the biduality map δ_S^C is invertible, so $S \in {}_C\mathcal{R}(R)$.

When $S \in {}_C\mathcal{R}(R)$ also the complex $S^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(S,C)$ belongs to ${}_C\mathcal{R}(R)$, cf. (2.11), and the inequality of Cohen–Macaulay defects, (b), follows by applying (3.3)(a) to $S^{\dagger_C}$ and using (5.12). □

Proof of (6.4). If $\mathrm{fd}\,\varphi < \infty$, then S belongs to $\mathcal{P}_0^{\mathrm{f}}(R)$, so it follows by (2.9) and (6.1) that $\mathbf{R}\mathrm{Hom}_R(S,C)$ is semi-dualizing for S. By (6.1)(b) we have

$$\mathrm{cmd}_S\,\mathbf{R}\mathrm{Hom}_R(S,C) \leq \mathrm{cmd}_R\,C,$$

and by (5.12) we have $\mathrm{cmd}_S\,\mathbf{R}\mathrm{Hom}_R(S,C) = \mathrm{cmd}_R\,\mathbf{R}\mathrm{Hom}_R(S,C)$, so (a) proved by establishing the inequality $\mathrm{cmd}_R\,\mathbf{R}\mathrm{Hom}_R(S,C) \geq \mathrm{cmd}_R\,C$. By (5.4.1) we are free to assume that R is complete and, thereby, has a dualizing complex D, cf. (1.8.3). The desired inequality follows by the next calculation:

$$\begin{aligned}
\mathrm{cmd}_R\,\mathbf{R}\mathrm{Hom}_R(S,C) &= \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(S,C)^{\dagger_D}) && \text{by (3.8)}\\
&= \mathrm{amp}(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(S,C),D)) &&\\
&= \mathrm{amp}(S \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(C,D)) && \text{by (1.5.8)}\\
&= \mathrm{amp}(C^{\dagger_D} \otimes_R^{\mathbf{L}} S) &&\\
&\geq \mathrm{amp}\,C^{\dagger_D} && \text{by (3.6.2)}\\
&= \mathrm{cmd}_R\,C && \text{by (3.8).}
\end{aligned}$$

The inequality in (b) follows by a similar calculation, only it uses (3.6.1) instead of (3.6.2). □

7. Examples

In this section $\varphi\colon R \to S$ is a local homomorphism of rings.

We shall apply the results from the previous sections to give some examples of semi-dualizing complexes. First, we review the special semi-dualizing complexes used by L.L. Avramov and H.–B. Foxby in [5], and we apply the main results from sections 5 and 6 in their setting to obtain converses to some key results in that paper. Next, in search of an appropriate invariant for telling semi-dualizing complexes apart, we make a short study of *Gorenstein homomorphisms*, and the so-called *type* emerges a suitable choice. The section closes with an answer to Golod's question: a recipe for constructing rings with any desired (finite) number of semi-dualizing complexes, or modules.

(7.1) ***Relative Dualizing Complexes***. The concept of *relative dualizing complexes*, that is, dualizing complexes for local homomorphisms, was introduced and studied in [5]; we catalogue a few facts about these complexes:

A complex $A \in \mathcal{D}(S)$ is said to be *dualizing* for φ if and only if

- A is semi-dualizing for S; and
- $D' \otimes_{\widehat{R}}^{\mathbf{L}} (A \otimes_S \widehat{S}) \in \mathcal{I}(\widehat{S})$ when D' is a dualizing complex for $\widehat{R}$.

If R has a dualizing complex D, then this is equivalent to requiring that A be semi-dualizing for S and $D \otimes_R^{\mathbf{L}} A \in \mathcal{I}(S)$ [5, (5.2)(b)].

We say that φ is of *finite Gorenstein dimension* [5, (4.3)] if $\widehat{S} \in {}_{D'}\mathcal{A}(\widehat{R})$, where D' is a dualizing complex for $\widehat{R}$; this is certainly the case if $\operatorname{fd}\varphi < \infty$ [5, (4.4.2)]. For homomorphisms of finite Gorenstein dimension relative dualizing complexes often exist, and when they do, they are uniquely determined up to isomorphism and shift [5, (5.4)]. Examples:

- If φ is finite of finite Gorenstein dimension, i.e., $S \in {}_R\mathcal{R}(R)$ cf. (4.7) and (5.11), then $\mathbf{R}\mathrm{Hom}_R(S,R)$ is dualizing for φ [5, (6.5)]. Conversely, by (6.1) φ is of finite Gorenstein dimension if $\mathbf{R}\mathrm{Hom}_R(S,R)$ is semi-dualizing for S, cf. (6.3)
- If D and E are dualizing complexes for R and S, respectively, and $S \in {}_D\mathcal{A}(R)$, then $\mathbf{R}\mathrm{Hom}_R(D,E)$ is a dualizing complex for φ [5, (6.1)]. Conversely, by (5.1) S belongs to ${}_D\mathcal{A}(R)$ if $\mathbf{R}\mathrm{Hom}_R(D,E)$ is semi-dualizing for S.

Let φ be of finite flat dimension; recall that φ is said to be *Gorenstein* [4] if the Bass series $\mathrm{I}_\varphi(t)$ is a monomial. Suppose R and S have dualizing complexes D and E, it was proved in [5] that φ is Gorenstein if and only if $E \sim D \otimes_R^{\mathbf{L}} S$. This condition can be rewritten, clumsily, as $(R \otimes_R^{\mathbf{L}} S)^{\dagger_E} \sim R^{\dagger_D} \otimes_R^{\mathbf{L}} S$ to suggest that base change commutes with dagger duality; the next result makes this idea explicit.

(7.2) **Theorem.** *Let φ be local and of finite flat dimension. If R has a dualizing complex D, then the next two conditions are equivalent.*

(*i*) *φ is Gorenstein.*
(*ii*) *$D \otimes_R^{\mathbf{L}} S$ is dualizing for S.*

And when S has a dualizing complex E, they are equivalent with the following:

(*iii*) *$(Z \otimes_R^{\mathbf{L}} S)^{\dagger_E} \sim Z^{\dagger_D} \otimes_R^{\mathbf{L}} S$ for all $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.*
(*iv*) *$(C \otimes_R^{\mathbf{L}} S)^{\dagger_E} \sim C^{\dagger_D} \otimes_R^{\mathbf{L}} S$ for some semi-dualizing complex C for R.*

Proof. The equivalence of (*i*) and (*ii*) is immediate by (5.5.1) and (1.8.1), and known from [5, (7.8) and (7.7.1)]. In the following E is a dualizing complex for S.

(*ii*) $\Rightarrow$ (*iii*): Suppose $E \sim D \otimes_R^{\mathbf{L}} S$. For $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ we have

$$\begin{aligned}
(Z \otimes_R^{\mathbf{L}} S)^{\dagger_E} &= \mathbf{R}\mathrm{Hom}_S(Z \otimes_R^{\mathbf{L}} S, E) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(Z, E) && \text{by (1.5.5)} \\
&\sim \mathbf{R}\mathrm{Hom}_R(Z, D \otimes_R^{\mathbf{L}} S) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(Z, D) \otimes_R^{\mathbf{L}} S && \text{by (1.5.7)} \\
&= Z^{\dagger_D} \otimes_R^{\mathbf{L}} S.
\end{aligned}$$

Condition (*iii*) is, obviously, stronger than (*iv*), so we need only prove (*iv*) $\Rightarrow$ (*ii*): Let C be a semi-dualizing complex for R, and assume that $(C \otimes_R^{\mathbf{L}} S)^{\dagger_E} \sim C^{\dagger_D} \otimes_R^{\mathbf{L}} S$.

Since, by (5.3)(d) and (4.4), both E and D belong to ${}_C\mathcal{B}(R)$, we have

$$\begin{aligned} E &\simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, E) \\ &\simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_S(C \otimes^{\mathbf{L}}_R S, E) \qquad \text{by (1.5.5)} \\ &= C \otimes^{\mathbf{L}}_R (C \otimes^{\mathbf{L}}_R S)^{\dagger_E} \\ &\sim C \otimes^{\mathbf{L}}_R (C^{\dagger_D} \otimes^{\mathbf{L}}_R S) \\ &\simeq (C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, D)) \otimes^{\mathbf{L}}_R S \\ &\simeq D \otimes^{\mathbf{L}}_R S. \quad \square \end{aligned}$$

(7.3) ***Type***. For a complex $Z \in \mathcal{D}^{\mathrm{f}}_{-}(R)$ we refer to its first non-vanishing Bass number as its *type*, that is, $\operatorname{type}_R Z = \mu_R^{\operatorname{depth}_R Z}(Z)$, cf. (1.7). For the ring we write $\operatorname{type} R = \operatorname{type}_R R = \mu_R^{\operatorname{depth} R}$. When $\operatorname{fd}\varphi < \infty$ we also write $\operatorname{type}\varphi$ for the first non-vanishing coefficient of the Bass series $\mathrm{I}_\varphi(t)$, cf. (5.5.1); no confusion arises, as this is in agreement with the more general definition of $\operatorname{type}\varphi$ given in [6, (7.1)], cf. [6, (7.8)].

(7.4) **Observation.** Let φ be of finite flat dimension, and assume that R and S have dualizing complexes D and E, respectively. Let C be a semi-dualizing complex for R. Since S belongs to ${}_C\mathcal{A}(R)$ as well as to ${}_{C^{\dagger_D}}\mathcal{A}(R)$, the pair $(C, C^{\dagger_D})$ of semi-dualizing complexes for R gives rise to four semi-dualizing complexes for S: namely, the base changed complexes $C \otimes^{\mathbf{L}}_R S$ and $C^{\dagger_D} \otimes^{\mathbf{L}}_R S$, cf. (5.7), and their dagger duals $(C \otimes^{\mathbf{L}}_R S)^{\dagger_E}$ and $(C^{\dagger_D} \otimes^{\mathbf{L}}_R S)^{\dagger_E}$, cf. (2.12). It follows by (5.7)(b) that

$$\begin{aligned} \operatorname{type}_S(C \otimes^{\mathbf{L}}_R S) &= \operatorname{type}_R C \cdot \operatorname{type}\varphi \quad \text{and} \\ \operatorname{type}_S(C^{\dagger_D} \otimes^{\mathbf{L}}_R S) &= \operatorname{type}_R C^{\dagger_D} \cdot \operatorname{type}\varphi. \end{aligned}$$

By (3.18.3) and (1.7.6) we have

$$\mathrm{I}_S^{(C \otimes^{\mathbf{L}}_R S)^{\dagger_E}}(t) = \mathrm{P}^S_{C \otimes^{\mathbf{L}}_R S}(t) t^e = \mathrm{P}^R_C(t) t^e = \mathrm{I}_R^{C^{\dagger_D}}(t) t^{e-d},$$

for suitable $d, e \in \mathbb{Z}$ and, therefore,

$$\operatorname{type}_S(C \otimes^{\mathbf{L}}_R S)^{\dagger_E} = \operatorname{type}_R C^{\dagger_D} \quad \text{and} \quad \operatorname{type}_S(C^{\dagger_D} \otimes^{\mathbf{L}}_R S)^{\dagger_E} = \operatorname{type}_R C.$$

We know from (7.2) that unless φ is Gorenstein, i.e., $\operatorname{type}\varphi = 1$, the semi-dualizing complexes $(C \otimes^{\mathbf{L}}_R S)^{\dagger_E}$ and $C^{\dagger_D} \otimes^{\mathbf{L}}_R S$ must be different, and the equations above show that the type is a convenient numerical invariant for telling semi-dualizing complexes apart.

(7.5) **Example.** Let φ be of finite flat dimension, and assume that D and E are dualizing complexes for, respectively, R and S. Then we have

$$\begin{aligned} \operatorname{type} S &= \operatorname{type} R \cdot \operatorname{type}\varphi, & \operatorname{type}_S E &= 1, \\ \operatorname{type}_S(D \otimes^{\mathbf{L}}_R S) &= \operatorname{type}\varphi, \quad \text{and} & \operatorname{type}_S A &= \operatorname{type} R, \end{aligned}$$

where $A = (D \otimes^{\mathbf{L}}_R S)^{\dagger_E} \simeq \mathbf{R}\mathrm{Hom}_R(D, E)$ is dualizing for φ, cf. (7.1) and (5.1)(c). Thus, if R and φ are not Gorenstein, i.e., $\operatorname{type} R > 1$ and $\operatorname{type}\varphi > 1$, then S has at least three semi-dualizing complexes of different type, and if type $R \neq$ type φ, then all the four semi-dualizing complexes above are of different type.

(7.6) **Observation.** Let φ be finite of finite flat dimension, i.e., $S \in \mathcal{P}_0^{\mathrm{f}}(R)$, and assume that φ is not Gorenstein, that is, type $\varphi > 1$. (An example follows below.) Let $C_1, \dots, C_n$ be $n \geq 1$ semi-dualizing complexes for R of different type. For each $\ell \in \{1, \dots, n\}$ the complexs $\mathbf{R}\mathrm{Hom}_R(S, C_\ell)$ and $C_\ell \otimes_R^{\mathbf{L}} S$ are semi-dualizing for S with $\operatorname{type}_S \mathbf{R}\mathrm{Hom}_R(S, C_\ell) = \operatorname{type}_R C_\ell$ and $\operatorname{type}_S(C_\ell \otimes_R^{\mathbf{L}} S) = \operatorname{type}_R C_\ell \cdot \operatorname{type} \varphi$, cf. (6.1)(a) and (5.7)(b). Thus, S will have at least $n+1$ semi-dualizing complexes of different type, namely (assuming that C_1 has minimal type among $C_1, \dots, C_n$) the complexes

$$\mathbf{R}\mathrm{Hom}_R(S, C_1), C_1 \otimes_R^{\mathbf{L}} S, \dots, C_n \otimes_R^{\mathbf{L}} S.$$

We note that if type φ does not divide $\operatorname{type}_R C_\ell$ for any $\ell \in \{1, \dots, n\}$, then the $2n$ semi-dualizing complexes

$$\mathbf{R}\mathrm{Hom}_R(S, C_1), \dots, \mathbf{R}\mathrm{Hom}_R(S, C_n), C_1 \otimes_R^{\mathbf{L}} S, \dots, C_n \otimes_R^{\mathbf{L}} S$$

will all have different type.

(7.7) **Example.** Consider the trivial extension $S = R \ltimes L$ of R by a free R–module L. It is a Noetherian local ring with maximal ideal $\mathfrak{m} \times L$, $\dim S = \dim R$, and $\operatorname{depth} S = \operatorname{depth} R$. Viewed as an R–module S is isomorphic to the direct sum $R \oplus L$, so the inclusion map $\varphi \colon R \to S$ is a finite flat local homomorphism, and its type is determined by the closed fiber: type φ = type $S/\mathfrak{m}S$, cf. (5.5). There is an isomorphism of R–modules $S/\mathfrak{m}S \cong k \ltimes k^r$, where r is the rank of L, and it is straightforward to check that it is also an isomorphism of rings. The ring $k \ltimes k^r$ is Artinian (because k is so) with maximal ideal $\mathfrak{M} = 0 \times k^r$. The type of $k \ltimes k^r$, and hence of φ, is the rank of the socle $\operatorname{Soc} k \ltimes k^r$, which is r as $\mathfrak{M}^2 = 0$. Thus, we have type $\varphi = \operatorname{rank} L$.

Let C be a semi-dualizing complex for R. It follows by flatness (freeness) of S over R that the induced semi-dualizing complexes are $C \otimes_R S$ and $\mathrm{Hom}_R(S, C)$.

(7.8) ***Golod's Question***. In [20] E.S. Golod asked for examples of semi-dualizing modules for local rings in addition to the canonical ones, that is, rings and dualizing modules. This question is answered by the example and the observation above:

If R is Cohen–Macaulay, then so are the trivial extensions $S = R \ltimes L$ considered above. It follows from (7.6) that if $\operatorname{rank} L > 1$, then the extension ring $S = R \ltimes L$ will have at least two different semi-dualizing complexes, and by (3.7) these will both be modules. For any finite number n we can, by iterating this procedure, construct a Cohen–Macaulay local ring possessing at least n different semi-dualizing modules.

By (3.18.2) the type of a semi-dualizing complex must divide the type of the ring. Let R be Gorenstein, then type $R = 1$, cf. (1.8), and R has exactly one semi-dualizing module (this will be proved in (8.6)). Setting $S_{(1)} = R \ltimes R^2$ and recursively defining $S_{(\ell+1)} = S_{(\ell)} \ltimes S_{(\ell)}^{2^{2^\ell}}$, we establish a sequence of Cohen–Macaulay rings, where the ℓ-th ring has type $S_{(\ell)} = 2^{2^\ell - 1}$ and possesses semi-dualizing modules of each of the possible 2^ℓ different types.

Golod's question has previously (in 1987) been addressed by H.–B. Foxby. Starting with a non-Gorenstein ring R, and setting $S = R \ltimes M$ for some finite module M (e.g., $M = R^2$) with $\mathrm{G\text{–}dim}_R\, M = 0$ and $\mathrm{Hom}_R(M,R)$ non-cyclic, he proved that $\mathrm{Hom}_R(S,R)$ is semi-dualizing for S and neither isomorphic to S nor dualizing for S.

8. Uniqueness Results

In this section R is a local ring with residue field k.

Foxby duality and dagger duality has already been studied for rings and dualizing complexes, see for example [5], [12], [11], and [31]. We start by summing up some results from these studies:

Let D be a dualizing complex for R. The next implications are part of the original Foxby duality theorem [5, (3.2)], cf. (4.2).

$$X \in \mathcal{F}(R) \implies D \otimes^{\mathbf{L}}_R X \in \mathcal{I}(R); \quad \text{and} \tag{8.0.1}$$

$$Y \in \mathcal{I}(R) \implies \mathbf{R}\mathrm{Hom}_R(D,Y) \in \mathcal{F}(R). \tag{8.0.2}$$

For complexes Z in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R) = {}_D\mathcal{R}(R)$ it follows by (3.18.3) that

$$Z \in \mathcal{P}^{\mathrm{f}}(R) \iff Z^{\dagger_D} \in \mathcal{I}^{\mathrm{f}}(R). \tag{8.0.3}$$

Foxby duality with respect to R is trivial. As proved in (2.13) dagger duality with respect to R is an endofunctor of $\mathcal{P}^{\mathrm{f}}(R)$, in particular,

$$Z \in \mathcal{P}^{\mathrm{f}}(R) \implies Z^{\dagger_R} \in \mathcal{P}^{\mathrm{f}}(R). \tag{8.0.4}$$

The two main results of this section characterize R and a dualizing complex for R in terms of special properties of their associated dagger and Foxby duality functors. Theorem (8.1) shows that only for $C \sim R$ is the dagger duality functor $-^{\dagger_C}$ stable on the subcategory $\mathcal{P}^{\mathrm{f}}(R)$ of ${}_C\mathcal{R}(R)$. Theorem (8.2) shows that only if C is dualizing for R do we get genuine Foxby duality, cf. (4.2), meaning that the functors $C \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(C,-)$ also provide quasi-inverse equivalences of the subcategories $\mathcal{F}(R)$ and $\mathcal{I}(R)$ of ${}_C\mathcal{A}(R)$ and ${}_C\mathcal{B}(R)$.

The proofs of the two theorems are given at the end of the section.

(8.1) **Theorem.** *If C is a semi-dualizing complex for R, then the following are equivalent:*

(*i*) $C \sim R$.

(*ii*) $Z^{\dagger_C} \in \mathcal{P}^{\mathrm{f}}(R)$ *for some* $Z \not\simeq 0$ *in* $\mathcal{P}^{\mathrm{f}}(R)$.

(*iii*) $C \otimes^{\mathbf{L}}_R X \in \mathcal{F}(R)$ *for some* $X \in {}_C\mathcal{A}(R)$ *with* $\mathrm{depth}_R\, X < \infty$.

(*iii'*) $\mathrm{depth}_R\, Y < \infty$ *for some* $Y \in {}_C\mathcal{B}(R) \cap \mathcal{F}(R)$.

(*iv*) $\mathbf{R}\mathrm{Hom}_R(C,Y) \in \mathcal{I}(R)$ *for some* $Y \in {}_C\mathcal{B}(R)$ *with* $\mathrm{depth}_R\, Y < \infty$.

(*iv'*) $\mathrm{depth}_R\, X < \infty$ *for some* $X \in {}_C\mathcal{A}(R) \cap \mathcal{I}(R)$.

In (8.1) the second condition should be compared to (8.0.4), and (*iii*) and (*iv*) should be compared to (8.0.1) and (8.0.2), respectively.

Conditions (*iii*) and (*iv*) in the next theorem show that a dualizing complex can be unveiled by 'a single instance of genuine Foxby duality', cf. (8.0.1) and (8.0.2). The second condition should be compared to (8.0.3).

(8.2) **Theorem.** *If C is a semi-dualizing complex for R, then the following are equivalent:*

(*i*) *C is dualizing for R.*
(*ii*) *$Z^{\dagger_C} \in \mathcal{I}^{\mathrm{f}}(R)$ for some $Z \not\simeq 0$ in $\mathcal{D}^{\mathrm{f}}_{+}(R)$.*
(*ii'*) *There is a complex $Y \not\simeq 0$ in ${}_C\mathcal{R}(R) \cap \mathcal{I}(R)$.*
(*iii*) *$C \otimes^{\mathbf{L}}_R X \in \mathcal{I}(R)$ for some $X \in \mathcal{F}(R)$ with $\operatorname{depth}_R X < \infty$.*
(*iv*) *$\mathbf{R}\mathrm{Hom}_R(C, Y) \in \mathcal{F}(R)$ for some $Y \in \mathcal{I}(R)$ with $\operatorname{depth}_R Y < \infty$.*

As a first step towards a proof of these theorems we establish two results characterizing semi-dualizing complexes in terms of their Poincaré and Bass series and special elements in their Auslander categories.

In [24] it was proved that a PG–module (see [13]) of finite G–dimension is projective, so over a local ring all PG–modules are free, and R is the unique rank 1 PG–module. This result is extended to complexes by (8.3): up to isomorphism and shift, R is the unique semi-dualizing complex in ${}_R\mathcal{R}(R)$, in particular, it is the only semi-dualizing complex of finite projective dimension.

From [21, V.3.1] we know that, up to isomorphism and shift, a dualizing complex D is the only semi-dualizing complex of finite injective dimension. From (8.4) it follows that D is also the unique semi-dualizing complex in ${}_D\mathcal{B}(R)$.

(8.3) **Proposition.** *If C is a semi-dualizing complex for R, then the following are equivalent:*

(*i*) *$C \sim R$.*
(*ii*) *$\mathrm{P}^R_C(t)$ is a monomial.*
(*iii*) *$C \in {}_R\mathcal{R}(R)$.*
(*iv*) *$k \in {}_C\mathcal{A}(R)$.*
(*iv'*) *$k \in {}_C\mathcal{B}(R)$.*
(*v*) *${}_C\mathcal{A}(R) = \mathcal{D}_{\mathrm{b}}(R)$.*
(*v'*) *${}_C\mathcal{B}(R) = \mathcal{D}_{\mathrm{b}}(R)$.*

(8.4) **Proposition.** *If C is a semi-dualizing complex for R, then the following are equivalent:*

(*i*) *C is dualizing for R.*
(*ii*) *$\mathrm{I}^C_R(t)$ is a monomial.*
(*iii*) *R has a dualizing complex D, and $C \in {}_D\mathcal{B}(R)$.*
(*iv*) *$k \in {}_C\mathcal{R}(R)$.*
(*v*) *${}_C\mathcal{R}(R) = \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.*

Proof of (8.3). It is well-known that conditions (i) and (ii) are equivalent, cf. (1.7.2), and they certainly imply (iii), cf. (2.8). Also the implications (i) $\Rightarrow$ (v) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v') $\Rightarrow$ (iv') are clear, cf. (4.3).

(iii) $\Rightarrow$ (ii): Assume that $C \in {}_R\mathcal{R}(R)$, then we have

$$\begin{aligned} \mathrm{I}_R^C(t) &= \mathrm{P}_{C^{\dagger_R}}^R(t)\,\mathrm{I}_R(t) && \text{by (3.18.1)} \\ &= \mathrm{P}_{C^{\dagger_R}}^R(t)\,\mathrm{P}_C^R(t)\,\mathrm{I}_R^C(t) && \text{by (3.18.2).} \end{aligned}$$

This implies that $\mathrm{P}_{C^{\dagger_R}}^R(t)\,\mathrm{P}_C^R(t) = 1$, and since $\mathrm{P}_{C^{\dagger_R}}^R(t)$ and $\mathrm{P}_C^R(t)$ are formal Laurant series with non-negative coefficients, we can conclude that they are monomials.

(iv) $\Rightarrow$ (iii): If $k \in {}_C\mathcal{A}(R)$ then, in particular, $C \otimes_R^{\mathbf{L}} k \in \mathcal{D}_{\mathrm{b}}(R)$ so $\mathrm{pd}_R\, C = \sup(C \otimes_R^{\mathbf{L}} k) < \infty$, cf. (1.4.4), and hence C is R–reflexive.

(iv') $\Rightarrow$ (iii): As we just saw, it is sufficient to prove that $\sup(C \otimes_R^{\mathbf{L}} k) < \infty$; we do so by using Matlis duality ($\mathrm{E}_R(k)$ is the injective hull of the residue field):

$$\begin{aligned} \sup(C \otimes_R^{\mathbf{L}} k) &= -\inf(\mathrm{Hom}_R(C \otimes_R^{\mathbf{L}} k, \mathrm{E}_R(k))) \\ &= -\inf(\mathbf{R}\mathrm{Hom}_R(C, \mathrm{Hom}_R(k, \mathrm{E}_R(k)))) \\ &= -\inf(\mathbf{R}\mathrm{Hom}_R(C, k)) \\ &< \infty, \end{aligned}$$

as $\mathrm{Hom}_R(k, \mathrm{E}_R(k)) \cong k$ and $\mathbf{R}\mathrm{Hom}_R(C, k) \in \mathcal{D}_{\mathrm{b}}(R)$. This concludes the proof. □

Proof of (8.4). The equivalence of conditions (i) and (ii) is well-known, cf. (1.8.1), and they obviously, cf. (4.4) and (2.8), imply (iii) and (v), the latter of which, in turn, implies (iv).

(iii) $\Rightarrow$ (ii): Suppose D is a dualizing complex for R and $C \in {}_D\mathcal{B}(R)$. We may assume that D is normalized, and we find:

$$\begin{aligned} \mathrm{P}_C^R(t) &= \mathrm{P}^R_{D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D,C)}(t) \\ &= \mathrm{P}_D^R(t)\,\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(D,C)}(t) && \text{by (1.7.6)} \\ &= \mathrm{I}_R(t)\,\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(D,C)}(t) && \text{by (3.18.3)} \\ &= \mathrm{P}_C^R(t)\,\mathrm{I}_R^C(t)\,\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(D,C)}(t) && \text{by (3.18.2).} \end{aligned}$$

This implies that $\mathrm{I}_R^C(t)\,\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(D,C)}(t) = 1$, and since these Laurant series have non-negative coefficients, we conclude that $\mathrm{I}_R^C(t)$ is a monomial.

(iv) $\Rightarrow$ (i): If k belongs to ${}_C\mathcal{R}(R)$ then, in particular, $\mathbf{R}\mathrm{Hom}_R(k, C) \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ so $\mathrm{id}_R\, C = -\inf(\mathbf{R}\mathrm{Hom}_R(k, C)) < \infty$, cf. (1.4.5), and C is, indeed, dualizing for R. □

(8.5) **Remarks.** Recall that R is said to be *Gorenstein* if and only if $\mathrm{id}_R\, R < \infty$ or, equivalently, if and only if R is a dualizing complex for R, cf. (1.8.2). We note that (8.3) and (8.4) contain the fact that the conditions

(i) R is Gorenstein; (ii) $\text{G–dim}_R\, k < \infty$; and

(iii) $\text{G–dim}_R\, M < \infty$ for all finite R–modules M;

are equivalent. This result [1, Théorème 3, p. 64] explains the name 'Gorenstein dimension'.

The reader is invited to apply (8.1) to a dualizing complex for R and (8.2) to R and, thereby, obtain a series (with some redundancy) of conditions equivalent with R being Gorenstein. The equivalence of conditions (*iii'*) and (*iv'*) in (8.1) applied to a dualizing complex were originally discovered by Foxby, cf. [11]. Further conditions can be extracted from (8.3) and (8.4); we only spell out the following:

(8.6) **Corollary.** *If D is dualizing complex for R, then the following are equivalent:*

(*i*) *R is Gorenstein.*
(*ii*) *R has only one semi-dualizing complex* (*up to isomorphism and shift*).
(*iii*) $D \sim R$.
(*iv*) ${}_D\mathcal{A}(R) = {}_D\mathcal{B}(R) = \mathcal{D}_{\mathrm{b}}(R)$.

Proof. The equivalence (*i*) $\Leftrightarrow$ (*iii*) is (1.8.2), and (*iii*) $\Leftrightarrow$ (*iv*) follows by (8.3). It follows by (8.4) that (*iv*) implies (*ii*), as any semi-dualizing complex will belong to ${}_D\mathcal{B}(R)$, and the implication (*ii*) $\Rightarrow$ (*iii*) is obvious. $\square$

(8.7) **Lemma.** *Let C be a semi-dualizing complex for R. For $U \in \mathcal{D}_{\mathrm{b}}(R)$ the following are equivalent:*

(*i*) $\mathbf{R}\mathrm{Hom}_R(k, U) \not\simeq 0$.
(*i'*) $\operatorname{depth}_R U < \infty$.
(*ii*) $k \otimes^{\mathbf{L}}_R U \not\simeq 0$.
(*iii*) $\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(C, U)) \not\simeq 0$.
(*iv*) $k \otimes^{\mathbf{L}}_R (C \otimes^{\mathbf{L}}_R U) \not\simeq 0$.

Proof. The equivalence of (*i*) and (*i'*) is evident by the definition of depth, (1.6.1), and (*i*) $\Leftrightarrow$ (*ii*) by [16, 2.8]. The complex C has finite homology modules and, therefore, finite depth, so it follows from what we have already proved that $\mathbf{R}\mathrm{Hom}_R(k, C) \not\simeq 0$ and $k \otimes^{\mathbf{L}}_R C \not\simeq 0$. The equivalences (*i*) $\Leftrightarrow$ (*iii*) and (*ii*) $\Leftrightarrow$ (*iv*) now follow straight from the next two chains of isomorphisms.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(C, U)) &\simeq \mathbf{R}\mathrm{Hom}_R(k \otimes^{\mathbf{L}}_R C, U) \\
&\simeq \mathbf{R}\mathrm{Hom}_R((k \otimes^{\mathbf{L}}_R C) \otimes_k k, U) \\
&\simeq \mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R C, \mathbf{R}\mathrm{Hom}_R(k, U)); \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
k \otimes^{\mathbf{L}}_R (C \otimes^{\mathbf{L}}_R U) &\simeq (k \otimes^{\mathbf{L}}_R C) \otimes^{\mathbf{L}}_R U \\
&\simeq (k \otimes^{\mathbf{L}}_R C) \otimes_k (k \otimes^{\mathbf{L}}_R U). \quad \square
\end{aligned}$$

Proof of (*8.1*). Clearly, (*i*) implies (*ii*), (*iii*), and (*iv*).

(*ii*) $\Rightarrow$ (*i*): When $Z \in \mathcal{P}^{\mathrm{f}}(R)$ we have $\mathrm{P}^R_{Z^{\dagger_C}}(t) = \mathrm{P}^R_Z(t^{-1})\,\mathrm{P}^R_C(t)$ by (2.14), and hence $\operatorname{pd}_R Z^{\dagger_C} = \operatorname{pd}_R C - \inf Z$, cf. (1.7.3). This shows that if $Z^{\dagger_C}$ belongs to $\mathcal{P}^{\mathrm{f}}(R)$, then so does C and hence $C \sim R$ by (8.3).

(*iii*) $\Rightarrow$ (*iii'*): Set $Y = C \otimes^{\mathbf{L}}_R X$, then $Y \in {}_C\mathcal{B}(R) \cap \mathcal{F}(R)$ by (4.6). Furthermore, $k \otimes^{\mathbf{L}}_R Y \not\simeq 0$, by (8.7), and hence $\operatorname{depth}_R Y < \infty$, as $Y \in \mathcal{D}_{\mathrm{b}}(R)$.

(*iii'*) $\Rightarrow$ (*i*): Suppose $Y \in {}_C\mathcal{B}(R) \cap \mathcal{F}(R)$ has $\operatorname{depth}_R Y < \infty$. By (8.7) we have $\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(C,Y)) \not\simeq 0$ and, again by (8.7), $k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y) \not\simeq 0$; in particular, $-\sup(k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y)) < \infty$. Now, by (4.5)(c) we have

$$Y \otimes^{\mathbf{L}}_R k \simeq (C \otimes^{\mathbf{L}}_R k) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y) \simeq (C \otimes^{\mathbf{L}}_R k) \otimes_k (k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y)),$$

and hence

$$\sup(Y \otimes^{\mathbf{L}}_R k) = \sup(C \otimes^{\mathbf{L}}_R k) + \sup(k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y)).$$

By (1.4.4) and (1.4.1) we have

$$\begin{aligned}\operatorname{pd}_R C = \sup(C \otimes^{\mathbf{L}}_R k) &= \sup(Y \otimes^{\mathbf{L}}_R k) - \sup(k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y))\\ &\le \operatorname{fd}_R Y - \sup(k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y))\\ &< \infty;\end{aligned}$$

and it follows by (8.3) that $C \sim R$.

(*iv*) $\Rightarrow$ (*iv'*): Set $X = \mathbf{R}\mathrm{Hom}_R(C,Y)$, then $\operatorname{depth}_R X < \infty$ by (8.7), and $X \in {}_C\mathcal{A}(R) \cap \mathcal{I}(R)$ by (4.6).

(*iv'*) $\Rightarrow$ (*i*): If $X \in {}_C\mathcal{A}(R) \cap \mathcal{I}(R)$ has $\operatorname{depth}_R X < \infty$, then, still applying (8.7), we see that $\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X) \not\simeq 0$; in particular, $\inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X)) < \infty$. By (4.5)(a) we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(k,X) &\simeq \mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R k, C \otimes^{\mathbf{L}}_R X)\\ &\simeq \mathbf{R}\mathrm{Hom}_R((C \otimes^{\mathbf{L}}_R k) \otimes_k k, C \otimes^{\mathbf{L}}_R X)\\ &\simeq \mathrm{Hom}_k(C \otimes^{\mathbf{L}}_R k, \mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X)),\end{aligned}$$

and it follows that

$$\inf(\mathbf{R}\mathrm{Hom}_R(k,X)) = \inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X)) - \sup(C \otimes^{\mathbf{L}}_R k).$$

Hence, by (1.4.4) and (1.4.2),

$$\begin{aligned}\operatorname{pd}_R C = \sup(C \otimes^{\mathbf{L}}_R k) &= \inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X)) - \inf(\mathbf{R}\mathrm{Hom}_R(k,X))\\ &\le \inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X)) + \operatorname{id}_R X\\ &< \infty;\end{aligned}$$

and the desired result again follows by (8.3). $\square$

Proof of (8.2). Certainly, (*i*) implies (*ii'*), and also (*iii*) and (*iv*), cf. (8.0.1) and (8.0.2).

(*ii'*) $\Rightarrow$ (*ii*): Set $Z = Y^{\dagger_C}$.

(*ii*) $\Rightarrow$ (*i*): By (1.7.7) we have $\mathrm{I}_R^{Z^{\dagger_C}}(t) = \mathrm{P}_Z^R(t)\,\mathrm{I}_R^C(t)$ and therefore $\operatorname{id}_R Z^{\dagger_C} = \operatorname{pd}_R Z + \operatorname{id}_R C$; cf. (1.7.1) and (1.7.3). Thus, $Z^{\dagger_C} \in \mathcal{I}^{\mathrm{f}}(R)$ implies that $C \in \mathcal{I}^{\mathrm{f}}(R)$ (and $Z \in \mathcal{P}^{\mathrm{f}}(R)$), and it follows that C is, in fact, dualizing for R.

$(iii) \Rightarrow (i)$: It is sufficient to prove that $\mathrm{id}_R C < \infty$. It follows by (1.5.7) that

$$\begin{aligned}\inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X)) &= \inf(\mathbf{R}\mathrm{Hom}_R(k, C) \otimes^{\mathbf{L}}_R X)\\ &= \inf(\mathbf{R}\mathrm{Hom}_R(k, C) \otimes_k (k \otimes^{\mathbf{L}}_R X))\\ &= \inf(\mathbf{R}\mathrm{Hom}_R(k, C)) + \inf(k \otimes^{\mathbf{L}}_R X).\end{aligned}$$

Hence,

$$\begin{aligned}-\inf(\mathbf{R}\mathrm{Hom}_R(k, C)) &= \inf(k \otimes^{\mathbf{L}}_R X) - \inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes^{\mathbf{L}}_R X))\\ &\le \inf(k \otimes^{\mathbf{L}}_R X) + \mathrm{id}_R(C \otimes^{\mathbf{L}}_R X)\\ &< \infty,\end{aligned}$$

where the inequalities follow by (1.4.2), (8.7), and the assumptions on X. Since $\mathrm{id}_R C = -\inf(\mathbf{R}\mathrm{Hom}_R(k, C))$ by (1.4.5), we are done.

$(iv) \Rightarrow (i)$: As above it is sufficient to prove that $-\inf(\mathbf{R}\mathrm{Hom}_R(k, C)) < \infty$. It follows by (1.5.8) that

$$\begin{aligned}\sup(k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, Y)) &= \sup(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, C), Y))\\ &= \sup(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, C) \otimes_k k, Y))\\ &= \sup(\mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_R(k, C), \mathbf{R}\mathrm{Hom}_R(k, Y)))\\ &= \sup(\mathbf{R}\mathrm{Hom}_R(k, Y)) - \inf(\mathbf{R}\mathrm{Hom}_R(k, C)).\end{aligned}$$

Hence, by (1.4.1) and the assumptions on Y, we have

$$\begin{aligned}-\inf(\mathbf{R}\mathrm{Hom}_R(k, C)) &= \sup(\mathbf{R}\mathrm{Hom}_R(C, Y) \otimes^{\mathbf{L}}_R k) - \sup(\mathbf{R}\mathrm{Hom}_R(k, Y))\\ &\le \mathrm{fd}_R \mathbf{R}\mathrm{Hom}_R(C, Y) + \mathrm{depth}_R Y\\ &< \infty. \quad \square\end{aligned}$$

Appendix. Chain Defects

Throughout the appendix R is a local ring.

The inequality

$$(\ddagger) \qquad \mathrm{cmd}\, R \le \mathrm{amp}\, C + \mathrm{cmd}_R C$$

holds for every semi-dualizing complex for R; this was established in (3.4)(c). Obviously, equality holds if C is dualizing for R or $C \sim R$, and it was established ibid. that equality holds if C is Cohen–Macaulay or $\mathrm{amp}\, C = 0$. The purpose of this appendix is to show that the difference between the two sides in (‡) is governed by (differences in) so-called *catenary defects* of C and R. Thus, the main result is (A.5), but our approach is more general, cf. (A.4).

(A.1) **Definitions.** Let $Z \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. For $\mathfrak{p} \in \mathrm{Supp}_R Z$ we consider the two integers:

$$\mathrm{dimd}_R(\mathfrak{p}, Z) = \dim_R Z + \inf Z_{\mathfrak{p}} - \dim R/\mathfrak{p}; \quad \text{and}$$
$$\mathrm{catd}_R(\mathfrak{p}, Z) = \dim_R Z - \dim_{R_{\mathfrak{p}}} Z_{\mathfrak{p}} - \dim R/\mathfrak{p}.$$

Both numbers belong to $\mathbb{N}_0$ by, respectively, the definition of dimension, (1.6.2), and (1.6.5). The number $\mathrm{catd}_R(\mathfrak{p}, Z)$ is the *catenary defect* of Z at $\mathfrak{p}$. See also the appendix in [6], where these numbers are related to certain conjectures of M. Auslander.

(A.2) **Observation.** Let $Z \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and $\mathfrak{p} \in \operatorname{Supp}_R Z$; by (1.6.3) and the definitions above we have

$$0 \leq \operatorname{dimd}_R(\mathfrak{p}, Z) - \operatorname{catd}_R(\mathfrak{p}, Z) = \inf Z_{\mathfrak{p}} + \dim_{R_{\mathfrak{p}}} Z_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}}.$$

(A.3) **Definition.** Let C be a semi-dualizing complex for R. For $Z \in {}_C\mathcal{R}(R)$ we consider the integer

$$\Delta_C(Z) = \sup \{\operatorname{catd}_R(\mathfrak{p}, C) - \operatorname{catd}_R(\mathfrak{p}, Z) \mid \mathfrak{p} \in \operatorname{Supp}_R Z\}.$$

Note from (A.2) that $\operatorname{dimd}_R(\mathfrak{p}, Z) = 0$ implies $\operatorname{catd}_R(\mathfrak{p}, Z) = 0$, and hence $\Delta_C(Z) \geq 0$.

(A.4) **Theorem.** *Let C be a semi-dualizing complex for R. For $Z \in {}_C\mathcal{R}(R)$ there is an inequality:*

(a) $$\operatorname{cmd}_R Z \leq \operatorname{amp} Z^{\dagger_C} + \operatorname{cmd}_R C - \Delta_C(Z);$$

and for $Z \in \mathcal{P}^{\mathrm{f}}(R)$ also the next inequality holds:

(b) $$\operatorname{amp} Z^{\dagger_C} + \operatorname{cmd}_R C - \Delta_C(Z) \leq \operatorname{cmd}_R Z + \operatorname{amp} Z.$$

In particular, for $M \in \mathcal{P}_0^{\mathrm{f}}(R)$ there is an equality:

(c) $$\operatorname{cmd}_R M = \operatorname{amp} M^{\dagger_C} + \operatorname{cmd}_R C - \Delta_C(M).$$

The proof of the theorem is found at the end of the section; applying part (c) to $R \in {}_C\mathcal{R}(R)$ we get the following:

(A.5) **Corollary.** *If C is a semi-dualizing complex for R, then*

$$\operatorname{cmd} R = \operatorname{amp} C + \operatorname{cmd}_R C - \Delta_C(R). \quad \square$$

(A.6) **Remark.** Let C be a semi-dualizing complex for R. It follows by (3.3)(a) and (A.4)(a) that

$$0 \leq \Delta_C(Z) \leq \operatorname{cmd}_R C \quad \text{for } Z \in {}_C\mathcal{R}(R).$$

The proof of (A.4) requires two lemmas and a proposition; the first lemma has the following interesting consequence:

(A.7) **Corollary (to A.8).** *Let C be a semi-dualizing complex for R, and set $s = \sup C$. For $\mathfrak{p} \in \operatorname{Spec} R$ the next biconditional holds:*

$$\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(C) \quad \Longleftrightarrow \quad \operatorname{depth} R_{\mathfrak{p}} = 0 \ \wedge\ \sup C = \inf C_{\mathfrak{p}};$$

and there is an equality:

$$\operatorname{dimd}_R(\mathfrak{p}, C) - \operatorname{catd}_R(\mathfrak{p}, C) = \operatorname{depth} R_{\mathfrak{p}} + \operatorname{cmd}_{R_{\mathfrak{p}}} C_{\mathfrak{p}}.$$

(A.8) **Lemma.** *Let C be a semi-dualizing complex for R, and let Z be C–reflexive. The following hold for $\mathfrak{p} \in \operatorname{Supp}_R Z$:*

$$\text{(a)} \qquad \begin{aligned} \inf Z_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} Z_\mathfrak{p} &= \operatorname{dimd}_R(\mathfrak{p}, Z) - \operatorname{catd}_R(\mathfrak{p}, Z) - \operatorname{cmd}_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &= \inf\,(Z^{\dagger_C})_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} (Z^{\dagger_C})_\mathfrak{p}; \quad \textit{and} \end{aligned}$$

$$\text{(b)} \qquad \inf Z_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} Z_\mathfrak{p} \geq \inf\,(Z^{\dagger_C})_\mathfrak{p} - \sup Z^{\dagger_C}.$$

Equality holds in (b) *if and only if* $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(Z^{\dagger_C})$ *for* $s = \sup Z^{\dagger_C}$.

Proof. (a) follows by the next computation:

$$\begin{aligned} &\operatorname{dimd}_R(\mathfrak{p}, Z) - \operatorname{catd}_R(\mathfrak{p}, Z) - \operatorname{cmd}_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &\qquad = \inf Z_\mathfrak{p} + \dim_{R_\mathfrak{p}} Z_\mathfrak{p} - \operatorname{cmd}_{R_\mathfrak{p}} Z_\mathfrak{p} && \text{by (A.2)} \\ &\qquad = \inf Z_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &\qquad = \inf Z_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} + \inf\,(Z^{\dagger_C})_\mathfrak{p} && \text{by (3.1)(a)} \\ &\qquad = \operatorname{depth}_{R_\mathfrak{p}} (Z^{\dagger_C})_\mathfrak{p} + \inf\,(Z^{\dagger_C})_\mathfrak{p} && \text{by (3.1)(a).} \end{aligned}$$

(b) follows from (a) as $\operatorname{depth}_{R_\mathfrak{p}} (Z^{\dagger_C})_\mathfrak{p} \geq -\sup Z^{\dagger_C}$ with equality if and only if $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(Z^{\dagger_C})$ for $s = \sup Z^{\dagger_C}$, cf. (1.6.6). □

Proof of (A.7). Let $\mathfrak{p} \in \operatorname{Spec} R$ and recall that by (3.2)(a) we have $\operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} + \inf C_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p}$. The equality now follows by applying (A.8)(a) to $Z = C$ and the biconditional by applying (A.8)(b) to $Z = R$. □

(A.9) **Lemma.** *Let C be a semi-dualizing complex for R, and let Z be C–reflexive. For $\mathfrak{p} \in \operatorname{Supp}_R Z$ there is an inequality:*

$$\text{(a)} \qquad \operatorname{catd}_R(\mathfrak{p}, C) - \operatorname{catd}_R(\mathfrak{p}, Z) \leq \dim_R C - \dim_R Z + \sup\,(Z^{\dagger_C})_\mathfrak{p};$$

and if $\operatorname{cmd}_{R_\mathfrak{p}} Z_\mathfrak{p} \geq \operatorname{cmd}_{R_\mathfrak{p}} C_\mathfrak{p}$ *also the next inequality holds:*

$$\text{(b)} \qquad \dim_R C - \dim_R Z + \inf\,(Z^{\dagger_C})_\mathfrak{p} \leq \operatorname{catd}_R(\mathfrak{p}, C) - \operatorname{catd}_R(\mathfrak{p}, Z).$$

Proof. Both inequalities follow by straightforward computations.

$$\text{(a):} \qquad \begin{aligned} \operatorname{catd}_R(\mathfrak{p}, C) - \operatorname{catd}_R(\mathfrak{p}, Z) &= \dim_R C - \dim_R Z - \dim_{R_\mathfrak{p}} C_\mathfrak{p} + \dim_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &\leq \dim_R C - \dim_R Z + \sup\,(Z^{\dagger_C})_\mathfrak{p}; \end{aligned}$$

the inequality follows by (3.1)(b).

$$\text{(b):} \qquad \begin{aligned} \operatorname{catd}_R(\mathfrak{p}, C) - \operatorname{catd}_R(\mathfrak{p}, Z) &\geq \dim_R C - \dim_R Z - \dim_{R_\mathfrak{p}} C_\mathfrak{p} + \dim_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &\quad + \operatorname{cmd}_{R_\mathfrak{p}} C_\mathfrak{p} - \operatorname{cmd}_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &= \dim_R C - \dim_R Z - \operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} Z_\mathfrak{p} \\ &= \dim_R C - \dim_R Z + \inf\,(Z^{\dagger_C})_\mathfrak{p}; \end{aligned}$$

the last equality is (3.1)(a). □

The next proposition extends and generalizes [6, (A.4)]

(A.10) **Proposition.** *Let C be a semi-dualizing complex for R. For $Z \in {}_C\mathcal{R}(R)$ there is an inequality:*

(a) $$\Delta_C(Z) \leq \dim_R C - \dim_R Z + \sup Z^{\dagger_C};$$

and for $Z \in \mathcal{P}^{\mathrm{f}}(R)$ there are also inequalities:

(b) $$\begin{aligned}\Delta_C(Z) &\geq \dim_R C - \dim_R Z + \sup\{\inf (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z\}\\ &\geq \dim_R C - \dim_R Z + \sup Z^{\dagger_C} - \operatorname{amp} Z.\end{aligned}$$

In particular, the next equality holds for $M \in \mathcal{P}_0^{\mathrm{f}}(R)$:

(c) $$\Delta_C(M) = \dim_R C - \dim_R M + \sup M^{\dagger_C}.$$

Proof. (a) is an immediate consequence of (A.9)(a).

(b): Suppose $Z \in \mathcal{P}^{\mathrm{f}}(R) \subseteq {}_C\mathcal{R}(R)$. For each $\mathfrak{p} \in \operatorname{Spec} R$ the complex $C_{\mathfrak{p}}$ is semi-dualizing for $R_{\mathfrak{p}}$ and $Z_{\mathfrak{p}} \in \mathcal{P}^{\mathrm{f}}(R_{\mathfrak{p}}) \subseteq {}_{C_{\mathfrak{p}}}\mathcal{R}(R_{\mathfrak{p}})$, so by (3.4)(b) and (3.6.1) we have inequalities $\operatorname{cmd}_{R_{\mathfrak{p}}} C_{\mathfrak{p}} \leq \operatorname{cmd} R_{\mathfrak{p}} \leq \operatorname{cmd}_{R_{\mathfrak{p}}} Z_{\mathfrak{p}}$. The first inequality now follows by (A.9)(b). To prove the second set $s = \sup Z^{\dagger_C}$, choose $\mathfrak{q}$ in $\operatorname{Ass}_R \mathrm{H}_s(Z^{\dagger_C})$ and recall that $\mathfrak{q} \in \operatorname{Supp}_R Z^{\dagger_C} = \operatorname{Supp}_R Z$ by (3.1)(d). We now have

$$\begin{aligned}\sup\{\inf (Z^{\dagger_C})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Z\} &\geq \inf (Z^{\dagger_C})_{\mathfrak{q}}\\ &= \sup Z^{\dagger_C} + \inf Z_{\mathfrak{q}} + \operatorname{depth}_{R_{\mathfrak{q}}} Z_{\mathfrak{q}} && \text{by (A.8)(b)}\\ &\geq \sup Z^{\dagger_C} + \inf Z_{\mathfrak{q}} - \sup Z && \text{by (1.6.4)}\\ &\geq \sup Z^{\dagger_C} - \operatorname{amp} Z;\end{aligned}$$

and this proves the desired inequality.

Finally, (c) follows from (a) and (b) as $\operatorname{amp} M = 0$. □

Proof of (A.4). By (A.10) we have inequalities:

$$\dim_R Z \leq \dim_R C + \sup Z^{\dagger_C} - \Delta_C(Z)$$

for $Z \in {}_C\mathcal{R}(R)$; and

$$\dim_R C + \sup Z^{\dagger_C} - \Delta_C(Z) \leq \dim_R Z + \operatorname{amp} Z$$

for $Z \in \mathcal{P}^{\mathrm{f}}(R)$. The inequalities (a) and (b) now follow by (3.1)(a), and (c) is a consequence of (a) and (b). □

References

1. Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles.
2. Maurice Auslander and Mark Bridger, *Stable module theory*, American Mathematical Society, Providence, R.I., 1969, Memoirs of the American Mathematical Society, No. 94.
3. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155.
4. ______, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), no. 5, 1007–1047.
5. ______, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270.

6. ______, *Cohen-Macaulay properties of ring homomorphisms*, Adv. Math. **133** (1998), no. 1, 54–95.
7. Luchezar L. Avramov, Hans-Bjørn Foxby, and Bernd Herzog, *Structure of local homomorphisms*, J. Algebra **164** (1994), no. 1, 124–145.
8. Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
9. Lars Winther Christensen, *Gorenstein dimensions*, Master's thesis, Matematisk Institut, Københavns Universitet, 1996.
10. Edgar E. Enochs, Overtoun M. G. Jenda, and Jinzhong Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3223–3234.
11. Hans-Bjørn Foxby, *Auslander categories*, in preparation.
12. ______, *Hyperhomological algebra & commutative rings*, notes in preparation.
13. ______, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973).
14. ______, *Quasi-perfect modules over Cohen-Macaulay rings*, Math. Nachr. **66** (1975), 103–110.
15. ______, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19.
16. ______, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
17. ______, *Homological dimensions of complexes of modules*, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Springer, Berlin, 1980, pp. 360–368.
18. Hans-Bjørn Foxby and Anders Thorup, *Minimal injective resolutions under flat base change*, Proc. Amer. Math. Soc. **67** (1977), no. 1, 27–31.
19. Evgeniy S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66, Algebraic geometry and its applications.
20. ______, Uspekhi Mat. Nauk **40** (1985), no. 1(241), 234, (Question, in Russian).
21. Robin Hartshorne, *Residues and duality*, Springer-Verlag, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20.
22. Melvin Hochster, *Topics in the homological theory of modules over commutative rings*, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975, Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24.
23. Birger Iversen, *Amplitude inequalities for complexes*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 547–558.
24. M. I. Jinnah, *A note on PG-modules*, Math. Scand. **37** (1975), no. 1, 27–28.
25. Christian Peskine and Lucien Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119.
26. Paul Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 103–106.
27. ______, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 7, 177–180.
28. Nicolas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.
29. Jean-Louis Verdier, *Catégories dérivées. quelques résultats (état 0)*, SGA $4\frac{1}{2}$, Springer, Berlin Heidelberg New York, 1977, pp. 262–311.
30. Jinzhong Xu, *Flat covers of modules*, Springer-Verlag, Berlin, 1996.
31. Siamak Yassemi, *G-dimension*, Math. Scand. **77** (1995), no. 2, 161–174.

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Part IV

Contents

SEQUENCES FOR COMPLEXES

LARS WINTHER CHRISTENSEN

Introduction

Let R be a commutative Noetherian ring and let $M \neq 0$ be a finite (that is, finitely generated) R–module. The concept of *M–sequences* is central for the study of R–modules by methods of homological algebra. Largely, the usefulness of these sequences is based on the following properties:

1° When $\mathfrak{a}$ is an ideal in R and $M/\mathfrak{a}M \neq 0$, the number
$$\inf\{\ell \in \mathbb{Z} \mid \operatorname{Ext}_R^\ell(R/\mathfrak{a}, M) \neq 0\},$$
the so-called $\mathfrak{a}$–depth of M, is the maximal length of an M–sequence in $\mathfrak{a}$, and any maximal M–sequence in $\mathfrak{a}$ is of this finite length.

2° If $x_1, \dots, x_n$ is an M–sequence contained in $\mathfrak{p} \in \operatorname{Supp}_R M$, then the sequence of fractions ${}^{x_1}/_1, \dots, {}^{x_n}/_1$, in the maximal ideal of $R_\mathfrak{p}$, is an $M_\mathfrak{p}$–sequence.

In commutative algebra, a wave of work dealing with complexes of modules was started by A. Grothendieck, see [9]. The underlying idea is the following: Complexes (that is, complexes of modules) are tacitly involved whenever homological methods are applied, and since hyperhomological algebra, that is, homological algebra for complexes, is a very powerful tool, it is better to work consistently with complexes. Modules are also complexes, concentrated in degree zero, so results for complexes yield results for modules as special cases.

Like most concepts for modules that of M–sequences can be extended to complexes in several non-equivalent ways; this short paper explores two such possible extensions: (ordinary) *sequences* and *strong sequences* for complexes. Ordinary sequences have a property corresponding to 1°, at least over local rings where they coincide with the *regular sequences* suggested by H.–B. Foxby in [8, Sec. 12]. But ordinary sequences may fail to localize properly, whereas strong sequences not only enjoy the correspondent property of 2°, but also that of 1° in the special case where R is local and $\mathfrak{a}$ the maximal ideal.

As a rule, the hyperhomological approach not only reproduces known results for modules, but also strengthens some of them. In this case we show, among other things, that also for a non-finite module M is the $\mathfrak{a}$–depth an upper bound for the maximal length of an M–sequence in $\mathfrak{a}$, and the $\mathfrak{a}$–depth of such a module may be finite even if $M/\mathfrak{a}M = 0$.

1. Conventions, Notation, and Background

Throughout this paper R is a non-trivial, commutative, Noetherian ring. We work in the derived category of the category of R–modules; this first section fixes the notation and sums up a few basic results.

(1.1) ***Notation***. As usual, the set of prime ideals containing an ideal $\mathfrak{a}$ in R is written $\mathrm{V}(\mathfrak{a})$; when $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence in R we write $\mathrm{V}(\boldsymbol{x})$ for the set of prime ideals containing $\boldsymbol{x}$. The set of *zero-divisors* for an R–module M is denoted by $\mathrm{z}_R\, M$.

The ring R is said to be *local* if it has a unique maximal ideal $\mathfrak{m}$, the residue field $R/\mathfrak{m}$ is then denoted by k. In general, for $\mathfrak{p} \in \operatorname{Spec} R$ the residue field of the local ring $R_\mathfrak{p}$ is denoted by $k(\mathfrak{p})$, that is, $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.

(1.2) ***Complexes***. An *R–complex* X is a sequence of R–modules X_ℓ and R–linear maps, so-called *differentials*, $\partial_\ell^X : X_\ell \to X_{\ell-1}$, $\ell \in \mathbb{Z}$. Composition of two consecutive differentials always yields the zero map, i.e. $\partial_\ell^X \partial_{\ell+1}^X = 0$. If $X_\ell = 0$ for $\ell \neq 0$, we identify X with the module in degree 0, and an R–module M is considered as a complex $0 \to M \to 0$ with M in degree 0.

A *morphism* $\alpha : X \to Y$ of R–complexes is a sequence of R–linear maps $\alpha_\ell : X_\ell \to Y_\ell$ satisfying $\partial_\ell^Y \alpha_\ell - \alpha_{\ell-1}\partial_\ell^X = 0$ for $\ell \in \mathbb{Z}$. We say that a morphism is a *quasi-isomorphism* if it induces an isomorphism in homology. The symbol $\simeq$ is used to indicate quasi-isomorphisms while $\cong$ indicates isomorphisms of complexes (and hence modules). For an element $r \in R$ the morphism $r_X : X \to X$ is given by multiplication by r.

The numbers *supremum*, *infimum*, and *amplitude*:

$$\begin{aligned} \sup X &= \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \\ \inf X &= \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \quad \text{and} \\ \operatorname{amp} X &= \sup X - \inf X \end{aligned}$$

capture the homological position and size of X. By convention, $\sup X = -\infty$ and $\inf X = \infty$ if $X \simeq 0$.

(1.3) ***Derived Functors***. The *derived category* of the category of R–modules is the category of R–complexes localized at the class of all quasi-isomorphisms (see [9] and [13]), we denote it by $\mathcal{D}(R)$. The symbol $\simeq$ is used for isomorphisms in $\mathcal{D}(R)$; a morphism of complexs is a quasi-isomorphism exactly if it represents an isomorphism in the derived category, so this is in agreement with the notation introduced above.

The full subcategories $\mathcal{D}_+(R)$, $\mathcal{D}_-(R)$, $\mathcal{D}_\mathrm{b}(R)$, and $\mathcal{D}_0(R)$ consist of complexes X with $\mathrm{H}_\ell(X) = 0$ for, respectively, $\ell \ll 0$, $\ell \gg 0$, $|\ell| \gg 0$, and $\ell \neq 0$. By $\mathcal{D}^\mathrm{f}(R)$ we denote the full subcategory of $\mathcal{D}(R)$ consisting of complexes X with $\mathrm{H}_\ell(X)$ a finite R–module for all $\ell \in \mathbb{Z}$. We also use combined notations: $\mathcal{D}_-^\mathrm{f}(R) = \mathcal{D}_-(R) \cap \mathcal{D}^\mathrm{f}(R)$, etc. The category of R–modules, respectively, finite R–modules, is naturally identified with $\mathcal{D}_0(R)$, respectively, $\mathcal{D}_0^\mathrm{f}(R)$.

The right derived functor of the homomorphism functor for R–complexes is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$, and $- \otimes_R^\mathbf{L} -$ is the left derived functor of the tensor product functor for R–complexes; by [2] and [12] no boundedness conditions are needed on the arguments. That is, for $X, Y \in \mathcal{D}(R)$ the complexes $\mathbf{R}\mathrm{Hom}_R(X,Y)$ and $X \otimes_R^\mathbf{L} Y$ are uniquely determined up to isomorphism in $\mathcal{D}(R)$, and they have the expected functorial properties. Note that $\mathrm{Tor}_\ell^R(M,N) = \mathrm{H}_\ell(M \otimes_R^\mathbf{L} N)$ and $\mathrm{Ext}_R^\ell(M,N) = \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(M,N))$ for $M, N \in \mathcal{D}_0(R)$ and $\ell \in \mathbb{Z}$.

Let $\mathfrak{p} \in \operatorname{Spec} R$; by [2, 5.2] there are isomorphisms: $(X \otimes^{\mathbf{L}}_R Y)_{\mathfrak{p}} \simeq X_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}$ and $\mathbf{R}\mathrm{Hom}_R(Z,Y)_{\mathfrak{p}} \simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(Z_{\mathfrak{p}}, Y_{\mathfrak{p}})$ in $\mathcal{D}(R_{\mathfrak{p}})$. The first one always holds, and the second holds when $Y \in \mathcal{D}_-(R)$ and $Z \in \mathcal{D}^{\mathrm{f}}_+(R)$.

The next results are standard, cf. [6, (2.1)]. Let $X \in \mathcal{D}_+(R)$ and $Y \in \mathcal{D}_-(R)$, then $\mathbf{R}\mathrm{Hom}_R(X,Y) \in \mathcal{D}_-(R)$ and there is an inequality:

$$\sup \mathbf{R}\mathrm{Hom}_R(X,Y) \leq \sup Y - \inf X. \tag{1.3.1}$$

Setting $i = \inf X$ and $s = \sup Y$ we have

$$\mathrm{H}_{s-i}(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y));$$

in particular,

$$\sup \mathbf{R}\mathrm{Hom}_R(X,Y) = \sup Y - \inf X \iff \mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y)) \neq 0. \tag{1.3.2}$$

Let $X, Y \in \mathcal{D}_+(R)$, then $X \otimes^{\mathbf{L}}_R Y \in \mathcal{D}_+(R)$ and there is an inequality

$$\inf (X \otimes^{\mathbf{L}}_R Y) \geq \inf X + \inf Y; \tag{1.3.3}$$

furthermore, with $i = \inf X$ and $j = \inf Y$ we have

$$\mathrm{H}_{i+j}(X \otimes^{\mathbf{L}}_R Y) = \mathrm{H}_i(X) \otimes_R \mathrm{H}_j(Y). \tag{1.3.4}$$

(1.4) ***Depth over Local Rings***. Let R be local; in [7, Sec. 3] the *depth* and (*Krull*) *dimension* of an R–complex X are defined as follows:

$$\operatorname{depth}_R X = -\sup \mathbf{R}\mathrm{Hom}_R(k, X), \text{ for } X \in \mathcal{D}_-(R); \quad \text{and}$$
$$\dim_R X = \sup \{\dim R/\mathfrak{p} - \inf X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Note that for modules these notions agree with the usual ones.

It follows immediately by (1.3.1) that $-\sup X \leq \operatorname{depth}_R X$ for $X \in \mathcal{D}_-(R)$, and if $s = \sup X > -\infty$ the next biconditional holds, cf. (1.3.2).

$$\operatorname{depth}_R X = -\sup X \iff \mathfrak{m} \in \operatorname{Ass}_R \mathrm{H}_s(X). \tag{1.4.1}$$

For $X \in \mathcal{D}_-(R)$ and $M \in \mathcal{D}^{\mathrm{f}}_0(R)$ the next equality holds, cf. [7, 3.4].

$$-\sup \mathbf{R}\mathrm{Hom}_R(M, X) = \inf \{\operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}. \tag{1.4.2}$$

Let $X \in \mathcal{D}^{\mathrm{f}}_-(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$; a complex version of [3, (3.1)], cf. [5, (13.13)], accounts for the inequality

$$\operatorname{depth}_R X \leq \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} + \dim R/\mathfrak{p}. \tag{1.4.3}$$

Finally, let $X \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_-(R)$ and set $s = \sup X$; applying (1.4.3) to $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(X)$ with $\dim R/\mathfrak{p} = \dim_R \mathrm{H}_s(X)$ and using (1.4.1) we obtain the next inequalities.

$$\operatorname{depth}_R X + \sup X \leq \dim_R \mathrm{H}_s(X) \leq \dim R. \tag{1.4.4}$$

2. Ann, Supp, and Ass for Complexes

As for modules, regular elements for complexes are linked to concepts of *zero-divisors* and *associated prime ideals*. These are introduced below within the relevant setting of support and annihilators.

(2.1) ***Weak Notions***. Weak notions of *support* and *annihilators* for $X \in \mathcal{D}(R)$ are defined by uniting/intersecting the corresponding sets for the homology modules $\mathrm{H}_\ell(X)$, cf. [7, Sec. 2] and [1, Sec. 2]:

$$\begin{aligned} \operatorname{Supp}_R X &= \bigcup_{\ell\in\mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_\ell(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \not\simeq 0\}; \quad \text{and} \\ \operatorname{Ann}_R X &= \bigcap_{\ell\in\mathbb{Z}} \operatorname{Ann}_R \mathrm{H}_\ell(X) = \{r \in R \mid \mathrm{H}(r_X) = 0\}. \end{aligned}$$

These are complemented by the next definitions. For $X \not\simeq 0$ in $\mathcal{D}_-(R)$ we set

$$\operatorname{ass}_R X = \operatorname{Ass}_R \mathrm{H}_{\sup X}(X) \quad \text{and} \quad \mathrm{z}_R X = \mathrm{z}_R \mathrm{H}_{\sup X}(X),$$

cf. [8, Sec. 12], and for $X \simeq 0$ we set $\operatorname{ass}_R X = \emptyset$ and $\mathrm{z}_R X = \emptyset$.

(2.2) ***The Small Support***. The *small*, or *homological*, *support* for $X \in \mathcal{D}_+(R)$ was introduced in [7, Sec. 2]:

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p}) \not\simeq 0\}.$$

Its principal properties developed ibid. are as follows:
Let $X \in \mathcal{D}_+(R)$. Then

$$X \not\simeq 0 \iff \operatorname{supp}_R X \neq \emptyset; \tag{2.2.1}$$

there is an inclusion

$$\operatorname{supp}_R X \subseteq \operatorname{Supp}_R X, \tag{2.2.2}$$

and equality holds when $X \in \mathcal{D}^{\mathrm{f}}_+(R)$. For $X, Y \in \mathcal{D}_+(R)$ the next equality holds.

$$\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{supp}_R X \cap \operatorname{supp}_R Y. \tag{2.2.3}$$

If R is local, the next biconditional holds for $X \in \mathcal{D}_\mathrm{b}(R)$.

$$\mathfrak{m} \in \operatorname{supp}_R X \quad \iff \quad \operatorname{depth}_R X < \infty. \tag{2.2.4}$$

(2.3) **Definitions.** Let $X \in \mathcal{D}_-(R)$; we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an *associated prime ideal* for X if and only if $\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} = -\sup X_\mathfrak{p} < \infty$, that is,

$$\begin{aligned} \operatorname{Ass}_R X &= \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \sup X_\mathfrak{p} = 0\} \\ &= \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \mathfrak{p}_\mathfrak{p} \in \operatorname{ass}_{R_\mathfrak{p}} X_\mathfrak{p}\}, \end{aligned}$$

cf. (1.4.1). The union of the associated prime ideals forms the set of *zero-divisors* for X:

$$\mathrm{Z}_R X = \bigcup_{\mathfrak{p}\in\operatorname{Ass}_R X} \mathfrak{p}.$$

(2.4) **Observations.** Let $X \in \mathcal{D}_-(R)$, $\mathfrak{p} \in \operatorname{Supp}_R X$, and set $s = \sup X_\mathfrak{p}\,(\in \mathbb{Z})$; then

$$\mathfrak{p} \in \operatorname{Ass}_R X \iff \mathfrak{p}_\mathfrak{p} \in \operatorname{ass}_{R_\mathfrak{p}} X_\mathfrak{p} \iff \mathfrak{p} \in \operatorname{Ass}_R \operatorname{H}_s(X).$$

That is, $\mathfrak{p} \in \operatorname{Ass}_R X$ if and only if there exists an $m \in \mathbb{Z}$ such that $\mathfrak{p} \in \operatorname{Ass}_R \operatorname{H}_m(X)$ and $\mathfrak{p} \notin \operatorname{Supp}_R \operatorname{H}_\ell(X)$ for $\ell > m$. In particular there is an inclusion

$$\operatorname{ass}_R X \subseteq \operatorname{Ass}_R X; \tag{2.4.1}$$

and since $\operatorname{z}_R X = \cup_{\mathfrak{p} \in \operatorname{ass}_R X}\, \mathfrak{p}$, also the next inclusion holds.

$$\operatorname{z}_R X \subseteq \operatorname{Z}_R X. \tag{2.4.2}$$

We also note that $\operatorname{Ass}_R X$ is a finite set for X in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

(2.5) **Modules.** For $M \in \mathcal{D}_0(R)$ the weak notions in (2.1) agree with the classical notions for modules; furthermore, $\operatorname{ass}_R M = \operatorname{Ass}_R M$ and $\operatorname{z}_R M = \operatorname{Z}_R M$, but $\operatorname{supp}_R M$ and $\operatorname{Supp}_R M$ may differ if M is not finite.

(2.6) **Proposition.** *Let $X \in \mathcal{D}_-(R)$; every minimal prime ideal in $\operatorname{Supp}_R X$ belongs to $\operatorname{Ass}_R X$, that is,*

$$\operatorname{Min}_R X \subseteq \operatorname{Ass}_R X;$$

and for $X \in \mathcal{D}_{\mathrm{b}}(R)$ also the next inclusion holds.

$$\operatorname{Ass}_R X \subseteq \operatorname{supp}_R X.$$

Proof. Let $X \in \mathcal{D}_-(R)$ and assume that $\mathfrak{p}$ is minimal in $\operatorname{Supp}_R X$. As $\operatorname{Supp}_{R_\mathfrak{p}} X_\mathfrak{p} = \{\mathfrak{p}_\mathfrak{p}\}$ it follows that $\mathfrak{p}_\mathfrak{p} \in \operatorname{ass}_{R_\mathfrak{p}} X_\mathfrak{p}$ and hence $\mathfrak{p} \in \operatorname{Ass}_R X$.

Let $X \in \mathcal{D}_{\mathrm{b}}(R)$; the first biconditional in the next chain is (2.2.4).

$$\begin{aligned} \mathfrak{p} \in \operatorname{Ass}_R X \quad &\Longrightarrow \quad \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty \\ &\iff \quad \mathfrak{p}_\mathfrak{p} \in \operatorname{supp}_{R_\mathfrak{p}} X_\mathfrak{p} \\ &\iff \quad X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p}) \not\simeq 0 \quad \iff \quad \mathfrak{p} \in \operatorname{supp}_R X. \quad \square \end{aligned}$$

(2.7) **Lemma.** *Let S be a multiplicative system in R; the following hold for $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$:*

(a) $\mathfrak{p} \in \operatorname{supp}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{supp}_{S^{-1}R} S^{-1}X$, *provided that $X \in \mathcal{D}_+(R)$; and*

(b) $\mathfrak{p} \in \operatorname{Ass}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{Ass}_{S^{-1}R} S^{-1}X$, *provided that $X \in \mathcal{D}_-(R)$.*

Proof. $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$ and $k(S^{-1}\mathfrak{p}) = (S^{-1}R/S^{-1}\mathfrak{p})_{S^{-1}\mathfrak{p}} \cong k(\mathfrak{p})$, so

$$(S^{-1}X)_{S^{-1}\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^{-1}R)_{S^{-1}\mathfrak{p}}} k(S^{-1}\mathfrak{p}) \simeq X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p}); \quad \text{and}$$
$$\mathbf{R}\mathrm{Hom}_{(S^{-1}R)_{S^{-1}\mathfrak{p}}}(k(S^{-1}\mathfrak{p}), (S^{-1}X)_{S^{-1}\mathfrak{p}}) \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), X_\mathfrak{p}).$$

(a) follows directly from the first isomorphism, and (b) follows from the second by the definition of depth. $\square$

3. Three Types of Sequences

We are now ready to define sequences — and strong and weak ones — for complexes $Y \in \mathcal{D}_-(R)$. The main results of this section are that strong Y–sequences localize properly, and that for $M \in \mathcal{D}_0(R)$ the notions of M–sequences and strong M–sequences both agree with the classical notion for modules.

(3.1) ***Koszul Complexes***. For $x \in R$ the complex $\mathrm{K}(x) = 0 \to R \xrightarrow{x} R \to 0$, concentrated in degrees 1 and 0, is called the *Koszul complex* of x. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in R, the Koszul complex $\mathrm{K}(\boldsymbol{x}) = \mathrm{K}(x_1, \ldots, x_n)$ of $\boldsymbol{x}$ is the tensor product $\mathrm{K}(x_1) \otimes_R \cdots \otimes_R \mathrm{K}(x_n)$. The Koszul complex of the empty sequence is R.

For $Y \in \mathcal{D}(R)$ we set $\mathrm{K}(\boldsymbol{x}; Y) = Y \otimes_R \mathrm{K}(\boldsymbol{x})$, and for $m \in \{1, \ldots, n\}$ we write $\mathrm{K}(\boldsymbol{x}_m; Y)$ for the complex $\mathrm{K}(x_1, \ldots, x_m; Y)$. We also set $\mathrm{K}(\boldsymbol{x}_0; Y) = Y$, corresponding to the empty sequence.

(3.2) **Observations.** In the following $\boldsymbol{x} = x_1, \ldots, x_n$ is a sequence in R and $Y \in \mathcal{D}(R)$.

For $m \in \{0, \ldots, n-1\}$ we have

(3.2.1) $$\mathrm{K}(\boldsymbol{x}; Y) = \mathrm{K}(x_{m+1}, \ldots, x_n; \mathrm{K}(\boldsymbol{x}_m; Y)),$$

by associativity of the tensor product. Let $\mathfrak{p} \in \operatorname{Spec} R$ and denote by ${}^{x_1}/_1, \ldots, {}^{x_n}/_1$ the sequence of fractions in $R_\mathfrak{p}$ corresponding to $\boldsymbol{x}$. There is an isomorphism:

(3.2.2) $$\mathrm{K}(x_1, \ldots, x_n; Y)_\mathfrak{p} \cong \mathrm{K}({}^{x_1}/_1, \ldots, {}^{x_n}/_1; Y_\mathfrak{p}).$$

For each j the Koszul complex $\mathrm{K}(x_j)$ is a complex of finite free, in particular flat, modules, and hence so is $\mathrm{K}(\boldsymbol{x})$. We can, therefore, identify $\mathrm{K}(\boldsymbol{x})$ with $\mathrm{K}(x_1) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathrm{K}(x_n)$ and $\mathrm{K}(\boldsymbol{x}; Y)$ with $Y \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x})$. It follows by (1.3.3) and (1.3.4) that

(3.2.3) $$\inf \mathrm{K}(\boldsymbol{x}) \geq 0 \quad \text{and} \quad \mathrm{H}_0(\mathrm{K}(\boldsymbol{x})) = R/(\boldsymbol{x}).$$

It is well-known (see [1, Sec. 2] or [11, 16.4]) that

(3.2.4) $$(x_1, \ldots, x_n) \subseteq \operatorname{Ann}_R \mathrm{K}(\boldsymbol{x}; Y).$$

It is easy to see that $\operatorname{Supp}_R \mathrm{K}(x_j) = \mathrm{V}(x_j)$, and it follows by (2.2.2) and (2.2.3) that $\operatorname{Supp}_R \mathrm{K}(\boldsymbol{x}) = \operatorname{supp}_R \mathrm{K}(\boldsymbol{x}) = \mathrm{V}(\boldsymbol{x})$. If $Y \in \mathcal{D}_+(R)$ it follows, also by (2.2.3), that

(3.2.5) $$\operatorname{supp}_R \mathrm{K}(\boldsymbol{x}; Y) = \operatorname{supp}_R Y \cap \mathrm{V}(\boldsymbol{x}).$$

Finally, it follows by the definition of tensor product complexes that

(3.2.6) if Y belongs $\mathcal{D}_-(R)$, respectively, $\mathcal{D}^{\mathrm{f}}_-(R)$ then also $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}_-(R)$, respectively, $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}^{\mathrm{f}}_-(R)$; and

(3.2.7) if Y belongs $\mathcal{D}_\mathrm{b}(R)$, respectively, $\mathcal{D}^{\mathrm{f}}_\mathrm{b}(R)$ then also $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}_\mathrm{b}(R)$, respectively, $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}^{\mathrm{f}}_\mathrm{b}(R)$.

In view of (3.2.6) the next definitions make sense.

(3.3) **Definitions.** Let $Y \in \mathcal{D}_-(R)$. An element $x \in R$ is said to be *regular* for Y if and only if $x \notin \mathrm{z}_R Y$ and *strongly regular* for Y if and only if $x \notin \mathrm{Z}_R Y$.

Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R. We say that

- $\boldsymbol{x}$ is a *weak Y–sequence* if and only if x_j is regular for $\mathrm{K}(\boldsymbol{x}_{j-1}; Y)$ for each $j \in \{1, \dots, n\}$;
- $\boldsymbol{x}$ is a *Y–sequence* if and only if $\boldsymbol{x}$ is a weak Y–sequence, and $\mathrm{K}(\boldsymbol{x}; Y) \not\simeq 0$ or $Y \simeq 0$; and
- $\boldsymbol{x}$ is a *strong Y–sequence* if and only if x_j is strongly regular for $\mathrm{K}(\boldsymbol{x}_{j-1}; Y)$ for each $j \in \{1, \dots, n\}$, and $\mathrm{K}(\boldsymbol{x}; Y) \not\simeq 0$ or $Y \simeq 0$.

(3.4) **Remarks.** For $M \in \mathcal{D}_0(R)$ regular and strongly regular elements are the same, cf. (2.5), and the definition agrees with the usual definition of *M–regular elements*, cf. [11, Sec. 16]. In (3.8) we prove that also the definition of M–sequences agrees with the classical one.

Let $Y \in \mathcal{D}_-(R)$. By (2.4.2) a strongly regular element for Y is also regular for Y; hence any strong Y–sequence is a Y–sequence and, thereby, a weak one.

The empty sequence is a strong Y–sequence for any complex $Y \in \mathcal{D}_-(R)$. A unit $u \in R$ is a strongly regular element for any complex $Y \in \mathcal{D}_-(R)$ and constitutes a weak Y–sequence, u can, however, not be part of a Y–sequence if $Y \not\simeq 0$. On the other hand, if $Y \simeq 0$, then any sequence is a strong Y–sequence. Later we supply an example — (3.13) — to show that a Y–sequence need not be a strong one.

(3.5) **Observation.** Let $Y \in \mathcal{D}_-(R)$, let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R, and let $m \in \{1, \dots, n-1\}$. It follows by (3.2.1) that $\boldsymbol{x}$ is a Y–sequence, respectively, a weak or a strong one, if and only if $x_1, \dots, x_m$ is a Y–sequence, respectively, a weak or strong one, and $x_{m+1}, \dots, x_n$ is a $\mathrm{K}(\boldsymbol{x}_m; Y)$–sequence, respectively, a weak or a strong one.

(3.6) **Lemma.** *The following hold for $x \in R$ and $Y \not\simeq 0$ in $\mathcal{D}_-(R)$:*

(a) $\sup \mathrm{K}(x; Y) \leq \sup Y + 1$*;*

(b) $\sup \mathrm{K}(x; Y) = \sup Y + 1$ *if and only if* $x \in \mathrm{z}_R Y$*; and*

(c) $\sup \mathrm{K}(x; Y) \geq \sup Y$ *if* $x \mathrm{H}_{\sup Y}(Y) \neq \mathrm{H}_{\sup Y}(Y)$.

Proof. It is easy to see that $\mathrm{K}(x; Y)$ is the mapping cone for the morphism x_Y, multiplication by x on Y. Thus, $\mathrm{K}(x; Y)$ fits in the exact sequence of complexes

$$0 \to Y \to \mathrm{K}(x; Y) \to Y[1] \to 0,$$

where $Y[1]$ is a shift of Y: $Y[1]_\ell = Y_{\ell-1}$ and $\partial_\ell^{Y[1]} = -\partial_{\ell-1}^Y$. Now, set $s = \sup Y$ and examine the corresponding long exact sequence of homology modules:

$$0 \to \mathrm{H}_{s+1}(\mathrm{K}(x; Y)) \to \mathrm{H}_s(Y) \xrightarrow{x_{\mathrm{H}_s(Y)}} \mathrm{H}_s(Y) \to \mathrm{H}_s(\mathrm{K}(x; Y)) \to \cdots . \quad \square$$

Parts (a) and (b) have the following immediate consequence:

(3.7) **Corollary.** *Let $Y \in \mathcal{D}_-(R)$; a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in R is a weak Y–sequence if and only if* $\sup \mathrm{K}(\boldsymbol{x}_j; Y) \leq \sup \mathrm{K}(\boldsymbol{x}_{j-1}; Y)$ *for each* $j \in \{1, \dots, n\}$. $\square$

(3.8) **Sequences for Modules.** *Let M be an R–module; the following hold for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in R:*

(a) $\mathrm{H}_0(\mathrm{K}(\boldsymbol{x}_j; M)) = M/(x_1, \dots, x_j)M$ *for* $j \in \{1, \dots, n\}$.

(b) *The next three conditions are equivalent.*

(i) $\boldsymbol{x}$ *is a weak M–sequence.*

(ii) $\mathrm{K}(\boldsymbol{x}_j; M) \simeq M/(x_1, \dots, x_j)M$ *for each* $j \in \{1, \dots, n\}$.

(iii) $x_j \notin \mathrm{z}_R\, M/(x_1, \dots, x_{j-1})M$ *for each* $j \in \{1, \dots, n\}$[1].

(c) *The next three conditions are equivalent.*

(i) $\boldsymbol{x}$ *is a weak M–sequence, and* $M/(x_1, \dots, x_n)M \neq 0$ *or* $M = 0$.

(ii) $\boldsymbol{x}$ *is an M–sequence.*

(iii) $\boldsymbol{x}$ *is a strong M–sequence.*

Proof. All three assertions are trivial if $M = 0$, so we assume that M is non-zero and let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R.

(a): Considering, as always, M as a complex concentrated in degree 0, we see that

$$(*) \qquad \inf \mathrm{K}(\boldsymbol{x}_j; M) \geq 0 \quad \text{for } \ j \in \{1, \dots, n\},$$

cf. (3.2.3) and (1.3.3), and $\mathrm{H}_0(\mathrm{K}(\boldsymbol{x}_j; M)) = M \otimes_R R/(x_1, \dots, x_j)$, cf. (1.3.4).

(b): For each $j \in \{1, \dots, n\}$ we have $\inf \mathrm{K}(\boldsymbol{x}_j; M) \geq 0$, cf. ($*$), so by (3.7) it follows that $\boldsymbol{x}$ is a weak Y–sequence if and only if $\mathrm{K}(\boldsymbol{x}_j; M) \in \mathcal{D}_0(R)$ for each j, that is (by (a)), if and only if $\mathrm{K}(\boldsymbol{x}_j; M) \simeq M/(x_1, \dots, x_j)M$ for each j. This proves the equivalence of (i) and (ii); that of (ii) and (iii) follows from (a), (3.2.1), and (3.6) by induction on n.

(c): First note that (i) $\Rightarrow$ (ii) by (a); it is then sufficient to prove that (ii) implies (iii): Suppose $\boldsymbol{x}$ is an M–sequence; for $j \in \{1, \dots, n\}$ we have $x_j \notin \mathrm{z}_R\, \mathrm{K}(\boldsymbol{x}_{j-1}; M)$, and $\mathrm{K}(\boldsymbol{x}_{j-1}; M) \in \mathcal{D}_0(R)$ by (b), so $\mathrm{z}_R\, \mathrm{K}(\boldsymbol{x}_{j-1}; M) = \mathrm{Z}_R\, \mathrm{K}(\boldsymbol{x}_{j-1}; M)$, cf. (2.5), whence $\boldsymbol{x}$ is a strong M–sequence. $\square$

(3.9) **Remark.** Let M be a non-zero R–module and let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R. Classically, cf. [11, Sec. 16], $\boldsymbol{x}$ is said to be an M–sequence if and only if (1) $x_j \notin \mathrm{z}_R\, M/(x_1, \dots, x_{j-1})M$ for $j \in \{1, \dots, n\}$, and (2) $M/(x_1, \dots, x_n)M \neq 0$. A sequence satisfying only the first condition is called a weak M–sequence, cf. [4, 1.1.1]. It follows by (b) and (c) in (3.8) that the notions of (weak) M–sequences defined in (3.3) agree with the classical ones.

(3.10) **Observation.** Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$; it follows by (3.6) that a sufficient condition for $x \in R$ to be a Y–sequence is that x is a $\mathrm{H}_{\sup Y}(Y)$–sequence. This condition is, of course, not necessary, see (5.3) for an example.

(3.11) **Theorem.** *Let $Y \in \mathcal{D}_\mathrm{b}(R)$ and $\mathfrak{p} \in \mathrm{supp}_R\, Y$; if $\boldsymbol{x} = x_1, \dots, x_n$ is a strong Y–sequence in $\mathfrak{p}$, then $x_1/1, \dots, x_n/1$ in the maximal ideal of $R_\mathfrak{p}$ is a strong $Y_\mathfrak{p}$–sequence.*

Proof. Let $\boldsymbol{x}/1 = x_1/1, \dots, x_n/1$ denote the sequence of fractions in $R_\mathfrak{p}$ corresponding to $\boldsymbol{x}$. Since $\mathfrak{p} \in \mathrm{supp}_R\, \mathrm{K}(\boldsymbol{x}; Y)$ by (3.2.5), it follows by (2.7)(a) and (3.2.2)

[1]For $j = 1$ this means $x_1 \notin \mathrm{z}_R\, M$.

that $\mathfrak{p}_\mathfrak{p} \in \operatorname{supp}_{R_\mathfrak{p}} \mathrm{K}({}^{\boldsymbol{x}}/_1; Y_\mathfrak{p})$; in particular, $\mathrm{K}({}^{\boldsymbol{x}}/_1; Y_\mathfrak{p}) \not\simeq 0$. We are now required to prove that ${}^{x_j}/_1 \notin \mathrm{Z}_{R_\mathfrak{p}} \mathrm{K}({}^{x_1}/_1, \dots, {}^{x_{j-1}}/_1; Y_\mathfrak{p})$ for $j \in \{1, \dots, n\}$. This follows by the lemma below as $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{K}(\boldsymbol{x}_{j-1}; Y)$, $x_j \notin \mathrm{Z}_R \mathrm{K}(\boldsymbol{x}_{j-1}; Y)$, and $\mathrm{Z}_{R_\mathfrak{p}} \mathrm{K}(\boldsymbol{x}_{j-1}; Y)_\mathfrak{p} = \mathrm{Z}_{R_\mathfrak{p}} \mathrm{K}({}^{x_1}/_1, \dots, {}^{x_{j-1}}/_1; Y_\mathfrak{p})$. □

(3.12) **Lemma.** *Let Y belong to $\mathcal{D}_-(R)$ and $\mathfrak{p} \in \operatorname{Supp}_R Y$; if $x \in \mathfrak{p}$ and $x \notin \mathrm{Z}_R Y$, then ${}^{x}/_1 \notin \mathrm{Z}_{R_\mathfrak{p}} Y_\mathfrak{p}$.*

Proof. We assume that ${}^{x}/_1 \in \mathrm{Z}_{R_\mathfrak{p}} Y_\mathfrak{p}$ and want to prove that x belongs to $\mathrm{Z}_R Y$. By assumption ${}^{x}/_1$ belongs to a prime ideal in $\operatorname{Ass}_{R_\mathfrak{p}} Y_\mathfrak{p}$, that is, ${}^{x}/_1 \in \mathfrak{q}_\mathfrak{p}$ for some $\mathfrak{q} \in \operatorname{Spec} R$ contained in $\mathfrak{p}$. Then $x \in \mathfrak{q}$, and $\mathfrak{q} \in \operatorname{Ass}_R Y$ by (2.7)(b), so $x \in \mathrm{Z}_R Y$ as wanted. □

As the next example demonstrates, a Y–sequence does not necessarily localize properly, not even if R is local and $Y \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$.

(3.13) **Example.** Let R be a local ring, assume that there exist $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{p} \not\subseteq \mathfrak{q}$ and $\mathfrak{q} \not\subseteq \mathfrak{p}$, and consider the complex $Y = 0 \to R/\mathfrak{q} \xrightarrow{0} R/\mathfrak{p} \to 0$. Let x be an element in $\mathfrak{p}$ not in $\mathfrak{q}$; it follows by (3.6) that x is a Y–sequence, but the localization of Y at $\mathfrak{p}$ is the field $k(\mathfrak{p})$, and ${}^{x}/_1 \in R_\mathfrak{p}$ is certainly not a $k(\mathfrak{p})$–sequence.

Note that if $\mathfrak{p} \cap \mathfrak{q} = 0$, then there is no non-empty strong Y–sequence in $\mathfrak{p}$.

4. Length of Sequences and Depth of Complexes

In this section we prove that any (strong) sequence can be extended to a maximal (strong) sequence, and we discuss various upper bounds for the length of such sequences.

(4.1) ***Maximal Sequences***. Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$ and let $\mathfrak{a}$ be an ideal in R. A sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$ is said to be a *maximal* (*strong*) *Y–sequence* in $\mathfrak{a}$ if and only if it is a (strong) Y–sequence and not the first part of a longer (strong) Y–sequence in $\mathfrak{a}$.

(4.2) **Lemma.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$; if $\boldsymbol{x} = x_1, \dots, x_n$ is a Y–sequence then $x_n \notin (x_1, \dots, x_{n-1})$.*

Proof. By (3.2.4) we have $(x_1, \dots, x_{n-1}) \subseteq \operatorname{Ann}_R \mathrm{K}(\boldsymbol{x}_{n-1}; Y)$, hence $(x_1, \dots, x_{n-1}) \subseteq \mathrm{z}_R \mathrm{K}(\boldsymbol{x}_{n-1}; Y)$ as $\mathrm{K}(\boldsymbol{x}_{n-1}; Y) \not\simeq 0$, and it follows that $x_n \notin (x_1, \dots, x_{n-1})$ as desired.

(4.3) **Corollary.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$ and let $\mathfrak{a}$ be an ideal in R. Any Y–sequence, respectively, strong Y–sequence in $\mathfrak{a}$ can be extended to a maximal Y–sequence, respectively, a maximal strong Y–sequence in $\mathfrak{a}$.*

Proof. The assertions follow immediately by (4.2) as R is Noetherian. □

(4.4) ***Depth***. Let $\mathfrak{a}$ be an ideal in R and let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite set of generators for $\mathfrak{a}$. By definition, cf. [10, Sec. 2], the $\mathfrak{a}$–*depth* of $Y \in \mathcal{D}(R)$ is the number

$$\operatorname{depth}_R(\mathfrak{a}, Y) = t - \sup \mathrm{K}(\boldsymbol{a}; Y);$$

it is, of course, independent of the choice of generating set $\boldsymbol{a}$.

We note that $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ if and only if $\mathrm{K}(\boldsymbol{a}; Y) \not\simeq 0$ for some, equivalently any, finite set of generators for $\mathfrak{a}$. Thus, by (2.2.1) and (3.2.5) we have

$$\text{(4.4.1)} \qquad \operatorname{depth}_R(\mathfrak{a}, Y) < \infty \iff \operatorname{supp}_R Y \cap \mathrm{V}(\mathfrak{a}) \neq \emptyset, \quad \text{for } Y \in \mathcal{D}_{\mathrm{b}}(R).$$

(4.5) **Proposition.** *Let $Y \in \mathcal{D}_-(R)$, let $\mathfrak{a}$ be a proper ideal in R, and let M belong to $\mathcal{D}_0^{\mathrm{f}}(R)$ with* $\operatorname{Supp}_R M = \mathrm{V}(\mathfrak{a})$*. The following equalities hold:*

$$\begin{aligned} \operatorname{depth}_R(\mathfrak{a}, Y) &= -\sup \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y) \\ &= \inf \{\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \\ &= -\sup \mathbf{R}\mathrm{Hom}_R(M, Y). \end{aligned}$$

Proof. The first equality is [10, 6.1], the second and third both follow by (1.4.2). □

(4.6) **Remark.** It follows from the first equality in (4.5) that the $\mathfrak{a}$–depth for complexes extends the usual concept of $\mathfrak{a}$–depth for modules, cf. [11, 16.7]; furthermore, it generalizes the concept of depth over local rings, that is, $\operatorname{depth}_R Y = \operatorname{depth}_R(\mathfrak{m}, Y)$ for $Y \in \mathcal{D}_-(R)$, when R is local with maximal ideal $\mathfrak{m}$. By the second equality in (4.5) the next inequality holds for all $Y \in \mathcal{D}_-(R)$ and all $\mathfrak{p} \in \operatorname{Spec} R$.

$$\operatorname{depth}_R(\mathfrak{p}, Y) \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p}.$$

Part (a) of the next theorem is often referred to as the 'depth sensitivity of the Koszul complex'.

(4.7) **Theorem.** *Let $Y \in \mathcal{D}(R)$ and let $\mathfrak{a}$ be an ideal in R. The following hold:*

(a) *For any sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$ there is an equality:*

$$\operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x}; Y)) = \operatorname{depth}_R(\mathfrak{a}, Y) - n.$$

(b) *For any ideal $\mathfrak{b} \subseteq \mathfrak{a}$ there is an inequality:*

$$\operatorname{depth}_R(\mathfrak{b}, Y) \leq \operatorname{depth}_R(\mathfrak{a}, Y).$$

Proof. Let $\boldsymbol{a} = a_1, \ldots, a_t$ be a set of generators for $\mathfrak{a}$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $\mathfrak{a}$. Also $\boldsymbol{x}, \boldsymbol{a} = x_1, \ldots, x_n, a_1, \ldots, a_t$ is a generating set for $\mathfrak{a}$, and by (3.2.1) we have $\mathrm{K}(\boldsymbol{x}, \boldsymbol{a}; Y) = \mathrm{K}(\boldsymbol{a}; \mathrm{K}(\boldsymbol{x}; Y))$. Hence,

$$\begin{aligned} \operatorname{depth}_R(\mathfrak{a}, Y) &= n + t - \sup \mathrm{K}(\boldsymbol{x}, \boldsymbol{a}; Y) \\ &= n + t - \sup \mathrm{K}(\boldsymbol{a}; \mathrm{K}(\boldsymbol{x}; Y)) \\ &= n + \operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x}; Y)); \end{aligned}$$

and this proves (a).

To prove (b), let $\boldsymbol{b} = b_1, \ldots, b_u$ be a generating set for $\mathfrak{b}$, then $\boldsymbol{b}, \boldsymbol{a} = b_1, \ldots, b_u, a_1, \ldots, a_t$ is a generating set for $\mathfrak{a}$. If $\sup \mathrm{K}(\boldsymbol{b}; Y) = \infty$ the inequality is trivial, so we assume that $\mathrm{K}(\boldsymbol{b}; Y) \in \mathcal{D}_-(R)$. As above we have $\mathrm{K}(\boldsymbol{b}, \boldsymbol{a}; Y) = \mathrm{K}(\boldsymbol{a}; \mathrm{K}(\boldsymbol{b}; Y))$, so it follows by (3.6)(a) that $\sup \mathrm{K}(\boldsymbol{b}, \boldsymbol{a}; Y) \leq \sup \mathrm{K}(\boldsymbol{b}; Y) + t$, whence

$$\begin{aligned} \operatorname{depth}_R(\mathfrak{a}, Y) &= u + t - \sup \mathrm{K}(\boldsymbol{b}, \boldsymbol{a}; Y) \\ &\geq u + t - (\sup \mathrm{K}(\boldsymbol{b}; Y) + t) \\ &= \operatorname{depth}_R(\mathfrak{b}, Y), \end{aligned}$$

as desired. □

(4.8) **Corollary.** *Let $Y \in \mathcal{D}_-(R)$ and let $\mathfrak{a}$ be a proper ideal in R. If $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ then the following hold for a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$.*

(a) *If $\boldsymbol{x}$ is a weak Y–sequence then $\boldsymbol{x}$ is a Y–sequence.*

(b) *If $\boldsymbol{x}$ is a Y–sequence then $\boldsymbol{x}$ is maximal in $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathrm{z}_R \mathrm{K}(\boldsymbol{x}; Y)$.*

(c) *If $\boldsymbol{x}$ is a strong Y–sequence then $\boldsymbol{x}$ is maximal in $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathrm{Z}_R \mathrm{K}(\boldsymbol{x}; Y)$.*

Proof. Denote by $\mathfrak{b}$ the ideal generated by $\boldsymbol{x}$. It follows by 4.7(b) that $\operatorname{depth}_R(\mathfrak{b}, Y) < \infty$, in particular, $\mathrm{K}(\boldsymbol{x}; Y) \not\simeq 0$, cf. 4.4. The three assertions are now immediate by the definitions in 3.3. □

(4.9) **Proposition.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a weak Y–sequence. The next inequality holds for any ideal $\mathfrak{a}$ containing $\boldsymbol{x}$.*

$$n \leq \operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y.$$

Proof. Let $\mathfrak{b}$ be the ideal generated by the sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$. By 4.4, 4.7(b), and 3.7 we have

$$\begin{aligned} n &= \operatorname{depth}_R(\mathfrak{b}, Y) + \sup \mathrm{K}(\boldsymbol{x}; Y) \\ &\leq \operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y. \end{aligned}$$

(4.10) **Corollary.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_\mathrm{b}(R)$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a strong Y–sequence. The following inequality holds:*

(a) $$n \leq \inf \{\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R Y \cap \mathrm{V}(\boldsymbol{x})\};$$

and if $Y \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R)$, also the next inequality holds.

(b) $$n \leq \inf \{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(\boldsymbol{x})\}.$$

Proof. Let $Y \in \mathcal{D}_\mathrm{b}(R)$ and assume that $\boldsymbol{x} = x_1, \ldots, x_n$ is a strong Y–sequence in $\mathfrak{p} \in \operatorname{supp}_R Y$. By (3.11) the sequence ${}^{x_1}/_1, \ldots, {}^{x_n}/_1$ in the maximal ideal of $R_\mathfrak{p}$ is a strong $Y_\mathfrak{p}$–sequence, so by (4.9) we have $n \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p}$, and this proves (a). If $Y \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R)$ then $\operatorname{supp}_R Y = \operatorname{Supp}_R Y$, cf. (2.2.2), and $Y_\mathfrak{p} \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R_\mathfrak{p})$, so (b) follows from (a) as $\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p} \leq \dim R_\mathfrak{p}$ by (1.4.4). □

(4.11) **Theorem.** *Let $Y \in \mathcal{D}^\mathrm{f}_-(R)$ and let $\mathfrak{a}$ be a proper ideal in R. If $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$, then the following conditions are equivalent for a Y–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$:*

(*i*) *$\boldsymbol{x}$ is a maximal Y–sequence in $\mathfrak{a}$.*

(*ii*) *$\mathfrak{a} \subseteq \mathrm{z}_R \mathrm{K}(\boldsymbol{x}; Y)$.*

(*iii*) *$\operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x}; Y)) + \sup \mathrm{K}(\boldsymbol{x}; Y) = 0$.*

(*iv*) *$\operatorname{depth}_R(\mathfrak{a}, Y) + \sup \mathrm{K}(\boldsymbol{x}; Y) = n$.*

Proof. We assume that $Y \in \mathcal{D}^\mathrm{f}_-(R)$ with $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$; the equivalence of (*i*) and (*ii*) is (4.8)(b). From (4.7)(a) it follows that (*iii*) $\Leftrightarrow$ (*iv*); this leaves us with one equivalence to prove:

Set $K = \mathrm{K}(\boldsymbol{x};Y)$ and $s = \sup K$ $(\in \mathbb{Z})$; by (4.5) and (1.3.1) we have

$$-\operatorname{depth}_R(\mathfrak{a}, K) = \sup \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, K) \leq s,$$

and equality holds if and only if $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s(K)) \neq 0$, cf. (1.3.2). Since $\mathrm{H}_s(K)$ is a finite module, cf. (3.2.6), it is well-known that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s(K)) \neq 0$ if and only if $\mathfrak{a} \subseteq \mathrm{z}_R\, K$, and this proves the equivalence of (*ii*) and (*iii*). □

(4.12) **Remarks.** Let Y, $\mathfrak{a}$, and $\boldsymbol{x}$ be as in (4.11). Since $\mathrm{K}(\boldsymbol{x};Y) \in \mathcal{D}_{-}^{\mathrm{f}}(R)$, cf. (3.2.6), it follows that

$$\mathfrak{a} \subseteq \mathrm{z}_R\, \mathrm{K}(\boldsymbol{x};Y) \quad \Longleftrightarrow \quad \mathfrak{a} \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{ass}_R\, \mathrm{K}(\boldsymbol{x};Y).$$

This should be compared to (*ii*) and (*iii*) in (4.15).

For a finite R–module M and an ideal $\mathfrak{a}$ in R it follows by (4.4.1) and (2.2.2) that

$$\operatorname{depth}_R(\mathfrak{a}, M) < \infty \quad \Longleftrightarrow \quad M/\mathfrak{a}M \neq 0.$$

Spelling out (4.11) for modules — as done in (4.14) — we recover the property 1° advertised in the introduction. Thus, in a sense, (4.11) describes the corresponding property for complexes $Y \in \mathcal{D}_{-}^{\mathrm{f}}(R)$; but unless R is local (see (5.4)) the length of a maximal Y–sequence need not be a well-determined integer:

Let Y and $\mathfrak{a}$ be as in (4.11). If $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y = 0$ then $\mathfrak{a} \subseteq \mathrm{z}_R\, Y$, so the empty sequence is the only Y–sequence in $\mathfrak{a}$. If $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y = 1$ then all maximal Y–sequences in $\mathfrak{a}$ are of length 1, but if $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y > 1$ there can be maximal Y–sequences in $\mathfrak{a}$ of different length. This is illustrated by the example below.

(4.13) **Example.** Let k be a field, set $R = k[U,V]$, and consider the R–complex $Y = 0 \to R/(U-1) \to 0 \to k \to 0$ concentrated in degrees 2, 1, and 0. Let $\mathfrak{a}$ be the maximal ideal $\mathfrak{a} = (U,V)$, then $\operatorname{depth}_R(\mathfrak{a}, Y) = 2 - 2 = 0$ and it is straightforward to check that U as well as V, U is a maximal Y–sequence in $\mathfrak{a}$.

(4.14) **Corollary.** *Let M be a finite R–module and let $\mathfrak{a}$ be a proper ideal in R. If $\operatorname{depth}_R(\mathfrak{a}, M) < \infty$, then the next four conditions are equivalent for an M–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$.*

(*i*) *$\boldsymbol{x}$ is a maximal M–sequence in $\mathfrak{a}$.*

(*ii*) *$\mathfrak{a} \subseteq \mathrm{z}_R\, M/(x_1, \ldots, x_n)M$.*

(*iii*) *$\operatorname{depth}_R(\mathfrak{a}, M/(x_1, \ldots, x_n)M) = 0$.*

(*iv*) *$\operatorname{depth}_R(\mathfrak{a}, M) = n$.*

In particular, the maximal length of an M–sequence in $\mathfrak{a}$ is a well-determined integer: $\operatorname{depth}_R(\mathfrak{a}, M) = \inf\{\ell \in \mathbb{Z} \mid \mathrm{Ext}_R^{\ell}(R/\mathfrak{a}, M) \neq 0\}$, and all maximal M–sequences in $\mathfrak{a}$ have this length.

Proof. By (3.8) we have

$$\mathrm{K}(\boldsymbol{x}; M) \simeq M/(x_1, \ldots, x_n)M \neq 0,$$

in particular, $\sup \mathrm{K}(\boldsymbol{x}; M) = 0$. The equivalence of the four conditions now follows from (4.11), and the last assertions are immediate, cf. (4.5). □

Other well-known characterizations of maximal sequences for finite modules are recovered by reading $M/(x_1,\dots,x_n)M$ for $\mathrm{K}(\boldsymbol{x};M)$ in the next theorem.

(4.15) **Theorem.** *Let $Y \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ and let $\mathfrak{a}$ be a proper ideal in R. If $\operatorname{depth}_R(\mathfrak{a},Y) < \infty$, then the next four conditions are equivalent for a strong Y–sequence $\boldsymbol{x} = x_1,\dots,x_n$ in $\mathfrak{a}$.*

- (i) *$\boldsymbol{x}$ is a maximal strong Y–sequence in $\mathfrak{a}$.*
- (ii) *$\mathfrak{a} \subseteq \mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x};Y)$.*
- (iii) *$\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)$.*
- (iv) *There is a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R Y$ containing $\mathfrak{a}$ such that the strong $Y_\mathfrak{p}$–sequence ${}^{x_1}/_1,\dots,{}^{x_n}/_1$ in $R_\mathfrak{p}$ is a maximal $Y_\mathfrak{p}$–sequence.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is immediate as $\operatorname{depth}_R(\mathfrak{a},Y) < \infty$, cf. (4.8)(c).

(ii) $\Leftrightarrow$ (iii): Clearly, (iii) implies (ii). On the other hand, $\mathrm{K}(\boldsymbol{x};Y) \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ by (3.2.7), so $\mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x};Y) = \cup_{\mathfrak{p}\in\operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)}\,\mathfrak{p}$ is a finite union, cf. (2.4). Thus, if $\mathfrak{a} \subseteq \mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x};Y)$ then $\mathfrak{a}$ must be contained in one of the prime ideals $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)$.

(iii)$\Leftrightarrow$(iv): Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Supp}_R Y$ containing $\mathfrak{a}$, then $\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} < \infty$, cf. (2.2.2) and (2.2.4), and by (3.11) the sequence of fractions ${}^{\boldsymbol{x}}/_1 = {}^{x_1}/_1,\dots,{}^{x_n}/_1$ in $R_\mathfrak{p}$ is a strong $Y_\mathfrak{p}$–sequence. By (3.2.2) there is an equality:

$$\operatorname{depth}_{R_\mathfrak{p}} \mathrm{K}(\boldsymbol{x};Y)_\mathfrak{p} + \sup \mathrm{K}(\boldsymbol{x};Y)_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} \mathrm{K}({}^{\boldsymbol{x}}/_1;Y_\mathfrak{p}) + \sup \mathrm{K}({}^{\boldsymbol{x}}/_1;Y_\mathfrak{p}).$$

By (4.11) and the definition of associated prime ideals it now follows that ${}^{\boldsymbol{x}}/_1$ is a maximal $Y_\mathfrak{p}$–sequence if and only if $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)$. □

5. Local Rings

In this section R is local with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We focus on (strong) sequences for complexes in $\mathcal{D}_{-}^\mathrm{f}(R)$ and strengthen some of the results from the previous section. The results established here are essentially those lined out by H.–B. Foxby in [8, Sec. 12], exceptions are (5.7) and (5.9).

(5.1) **Proposition.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_{-}^\mathrm{f}(R)$; the following hold for a sequence $\boldsymbol{x} = x_1,\dots,x_n$ in $\mathfrak{m}$:*

(a) *There are inequalities*

$$\sup \mathrm{K}(\boldsymbol{x};Y) \geq \cdots \geq \sup \mathrm{K}(\boldsymbol{x}_j;Y) \geq \sup \mathrm{K}(\boldsymbol{x}_{j-1};Y) \geq \cdots \geq \sup Y;$$

in particular, $\mathrm{K}(\boldsymbol{x};Y) \not\simeq 0$.

(b) *The next three conditions are equivalent.*

- (i) *$\boldsymbol{x}$ is a weak Y–sequence.*
- (ii) *$\boldsymbol{x}$ is a Y–sequence.*
- (iii) *$\sup \mathrm{K}(\boldsymbol{x};Y) = \sup Y$.*

(c) *If $\boldsymbol{x}$ is a Y–sequence then so is any permutation of $\boldsymbol{x}$.*

Proof. (a): The inequalities hold by Nakayama's lemma and (3.6)(c); in particular we have $\sup \mathrm{K}(\boldsymbol{x};Y) \geq \sup Y > -\infty$, so $\mathrm{K}(\boldsymbol{x};Y) \not\simeq 0$.

(b): It follows by (3.7) that $\boldsymbol{x}$ is a weak Y–sequence if and only if equality holds in each of the inequalities in (a). This proves the equivalence of (i) and (iii); also $(i) \Leftrightarrow (ii)$ is immediate by (a).

(c): By commutativity of the tensor product the number $\sup \mathrm{K}(\boldsymbol{x};Y)$ is unaffected by permutations of $\boldsymbol{x}$, so the last assertion follows by (b). $\square$

The next corollary is an immediate consequence of (5.1)(a). The example below shows that the equality $\sup \mathrm{K}(\boldsymbol{x};Y) = \sup Y$ need not hold, not even for strong Y–sequence, if Y does not have finite homology modules.

(5.2) **Corollary.** *Let $\mathfrak{a}$ be a proper ideal in R. If $Y \in \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, then*

$$\operatorname{depth}_R(\mathfrak{a}, Y) < \infty \quad \Longleftrightarrow \quad Y \not\simeq 0. \quad \square$$

(5.3) **Example.** Let R be a local integral domain, not a field, and let $B = R_{(0)} \neq R$ be the field of fractions. Consider the complex $Y = 0 \to B \xrightarrow{0} R \to 0$. For any $\mathfrak{p} \in \operatorname{Spec} R$ we have $\sup Y_{\mathfrak{p}} = \sup Y$, so $\operatorname{Ass}_R Y = \operatorname{ass}_R Y = \operatorname{Ass}_R B = \{0\}$. Let $x \neq 0$ be an element in the maximal ideal of R, it follows that $x \notin \mathrm{Z}_R\, Y$ and $\mathrm{K}(x;Y) \simeq R/(x) \neq 0$, so x is a strong Y–sequence, but $\sup \mathrm{K}(x;Y) < \sup Y$.

(5.4) **Theorem.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ and let $\mathfrak{a}$ be a proper ideal in R. The next four conditions are equivalent for a Y–sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$.*

- (i) *$\boldsymbol{x}$ is a maximal Y–sequence in $\mathfrak{a}$.*
- (ii) $\mathfrak{a} \subseteq \mathrm{z}_R\, \mathrm{K}(\boldsymbol{x};Y)$.
- (iii) $\operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x};Y)) + \sup Y = 0$.
- (iv) $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y = n$.

In particular, the maximal length of a Y–sequence in $\mathfrak{a}$ is a well-determined integer: $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y$, and all maximal Y–sequences in $\mathfrak{a}$ have this length.

Proof. By (5.2) and (5.1) we have $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ and $\sup \mathrm{K}(\boldsymbol{x};Y) = \sup Y$, so the equivalence is a special case of (4.11), and the last assertions follow. $\square$

(5.5) **Corollary.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$; the integer*

$$\operatorname{depth}_R Y + \sup Y$$

is the maximal length of a Y–sequence, and any maximal Y–sequence is of this length. Furthermore, the following inequalities hold:

$$\operatorname{depth}_R Y + \sup Y \leq \dim_R \mathrm{H}_{\sup Y}(Y) \leq \dim R.$$

Proof. A Y–sequence must be contained in $\mathfrak{m}$, and the first part is (5.4) applied to $\mathfrak{a} = \mathfrak{m}$. The inequalities are (1.4.4). $\square$

(5.6) **Corollary.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_{-}(R)$ and $M \in \mathcal{D}^{\mathrm{f}}_{0}(R)$. The maximal length of a Y–sequence in $\operatorname{Ann}_R M$ is a well-determined integer n:*

$$\begin{aligned} n &= -\sup \mathbf{R}\mathrm{Hom}_R(M,Y) + \sup Y \\ &= \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{V}(\operatorname{Ann}_R M)\} + \sup Y; \end{aligned}$$

and any maximal Y–sequence in $\operatorname{Ann}_R M$ is of this length.

Proof. It follows by (5.4) that a Y–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\operatorname{Ann}_R M$ is maximal if and only if $n = \operatorname{depth}_R(\operatorname{Ann}_R M, Y) + \sup Y$. As M is finite $\operatorname{Supp}_R M = \mathrm{V}(\operatorname{Ann}_R M)$, and the desired equalities follow by (4.5). $\square$

It follows from the last remark in (3.13) that (5.6) has no counterpart for strong sequences, but (5.5) does have one:

(5.7) **Corollary (to 4.15).** *Let $Y \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. A maximal strong Y–sequence is a maximal Y–sequence; in particular, the maximal length of a strong Y–sequence is a well-determined integer n:*

$$n = \operatorname{depth}_R Y + \sup Y \leq \dim_R \mathrm{H}_{\sup Y}(Y) \leq \dim R;$$

and any maximal strong Y–sequence is of this length.

Proof. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a maximal strong Y–sequence, that is, maximal in $\mathfrak{m}$. Since $\operatorname{depth}_R Y < \infty$ by (5.2) it follows by (4.15) that $\boldsymbol{x}$ is a maximal Y–sequence, and the desired equality and inequalities follow from (5.5). $\square$

The number $\operatorname{depth}_R Y + \sup Y$ provides an upper bound for the length of a Y–sequence, even if Y does not have finite homology modules, cf. (4.9). In view of (5.5) it is natural to ask if also $\dim R$ is a bound. If $\dim R = 0$ it obviously is, cf. (1.4.1), and so it is if $\dim R = 1$ and $\operatorname{depth}_R \mathrm{H}_{\sup Y}(Y) < \infty$ (this follows by [10, 2.3]); but the next example shows that the answer is negative. For bounded complexes, however, a bound involving $\dim R$ is available, see (5.9).

(5.8) **Example.** Let k be a field and consider the local ring $R = k[\![U,V]\!]/(UV)$ with $\dim R = 1$. The residue classes u and v of, respectively, U and V generate prime ideals in R; we set $Y = 0 \to R_{(v)} \xrightarrow{0} R/(u) \to 0$. Multiplication by u on $R_{(v)}$ is an isomorphism, v is a $R/(u)$–sequence, and it follows that u, v is a Y–sequence.

(5.9) **Corollary (to 4.9).** *Let $Y \in \mathcal{D}_{\mathrm{b}}(R)$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a weak Y–sequence in $\mathfrak{m}$. If $\mathfrak{m} \in \operatorname{supp}_R Y$, then $\boldsymbol{x}$ is a Y–sequence, and*

$$n \leq \operatorname{depth}_R Y + \sup Y \leq \dim_R Y + \sup Y \leq \dim R + \operatorname{amp} Y.$$

Proof. It follows by (2.2.4) that $\operatorname{depth}_R Y < \infty$, so $\boldsymbol{x}$ is a Y–sequence by (4.8)(a). The first inequality is a special case of (4.9). The inequality $\operatorname{depth}_R Y \leq \dim_R Y$ holds by [7, 3.9]; this gives the second inequality, and the third one follows as $\dim_R Y \leq \dim R - \inf Y$ by the definition of dimension. $\square$

We close with an example, illustrating an application of sequences for complexes.

(5.10) **Example (Parameter Sequences).** In the following we assume that R admits a dualizing complex D, cf. [9], and let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R. For $Y \not\simeq 0$ in $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ it follows by (3.2.7), (3.6), and well-known properties of dualizing complexes that

$$\dim_R \mathrm{K}(\boldsymbol{x}; Y) = \dim_R Y - n \quad \Longleftrightarrow \quad \boldsymbol{x} \text{ is a } \mathbf{R}\mathrm{Hom}_R(Y, D)\text{–sequence;}$$

and by [7, 3.12] there is an equality:

$$\dim_R \mathrm{K}(\boldsymbol{x}; Y) = \sup \{\dim_R(Y \otimes_R^{\mathbf{L}} \mathrm{H}_\ell(\mathrm{K}(\boldsymbol{x}))) - \ell \mid \ell \in \mathbb{Z}\}.$$

Let M be a finite R–module; we say that $\boldsymbol{x}$ is an *M–parameter sequence* if and only if $\dim_R M/(x_1, \dots, x_n)M = \dim_R M - n$, that is, if and only if $\boldsymbol{x}$ is part of a system of parameters for M. It follows by the definition of Krull dimension, Nakayama's lemma, and (3.2.3) that

$$\begin{aligned} \dim_R \mathrm{K}(\boldsymbol{x}; M) &= \sup \{\dim_R(M \otimes_R^{\mathbf{L}} \mathrm{H}_\ell(\mathrm{K}(\boldsymbol{x}))) - \ell \mid \ell \in \mathbb{Z}\} \\ &= \sup \{\dim_R(M \otimes_R \mathrm{H}_\ell(\mathrm{K}(\boldsymbol{x}))) - \ell \mid \ell \in \mathbb{Z}\} \\ &= \dim_R M/(x_1, \dots, x_n)M. \end{aligned}$$

Thus, $\boldsymbol{x}$ is an M–parameter sequence if and only if $\boldsymbol{x}$ is a $\mathbf{R}\mathrm{Hom}_R(M, D)$–sequence. In particular, any M–sequence is a $\mathbf{R}\mathrm{Hom}_R(M, D)$–sequence. Only if M is Cohen–Macaulay will $\mathbf{R}\mathrm{Hom}_R(M, D)$ have homology concentrated in one degree, that is, be equivalent to a module up to a shift.

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References

1. Dmitri Apassov, *Annihilating complexes of modules*, to appear in Math. Scand.
2. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155.
3. Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
4. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge University Press, Cambridge, 1998.
5. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.
6. ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19.
7. ———, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
8. ———, *A homological theory for complexes of modules*, Preprint Series 1981, nos. 19a&b, Matematisk Institut, Københavns Universitet, 1981.
9. Robin Hartshorne, *Residues and duality*, Springer-Verlag, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20.
10. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567.
11. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

12. Nicolas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.
13. Jean-Louis Verdier, *Catégories dérivées. quelques résultats (état 0)*, SGA $4\frac{1}{2}$, Springer, Berlin Heidelberg New York, 1977, pp. 262–311.

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PARAMETERS FOR COMPLEXES

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1. Introduction and Notation

This note elaborates on an example given in [4] to illustrate an application of sequences for complexes:

> Let R be a local ring with a dualizing complex D, and let M be a finitely generated R–module; then a sequence $x_1, \ldots, x_n$ is part of a system of parameters for M if and only if it is a $\mathbf{R}\mathrm{Hom}_R(M, D)$–sequence [4, 5.10].

The final Theorem (3.9) of this note generalizes the result above in two directions: the dualizing complex is replaced by a Cohen–Macaulay semi–dualizing complex (see [3, Sec. 2] or (3.8) below for definitions), and the finite module is replaced by a complex with finite homology.

Before we can even state, let alone prove, this generalization of [4, 5.10] we must introduce and study *parameters* for complexes. For a finite R–module M every M–sequence is part of a system of parameters for M, so, loosely speaking, regular elements are just special parameters. For a complex X, however, parameters and regular elements are two different things, and kinship between them implies strong relations between two measures of the size of X: the *amplitude* and the *Cohen–Macaulay defect* (both defined below). This is described in (3.5), (3.6), and (3.7).

Throughout R denotes a commutative, Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We use the same notation as in [4], but for convenience we recall a few basic facts.

The homological position and size of a complex X is captured by the *supremum*, *infimum*, and *amplitude*:

$$\begin{aligned} \sup X &= \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \\ \inf X &= \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \quad \text{and} \\ \operatorname{amp} X &= \sup X - \inf X. \end{aligned}$$

By convention, $\sup X = -\infty$ and $\inf X = \infty$ if $\mathrm{H}(X) = 0$.

The *support* of a complex X is the set

$$\mathrm{Supp}_R\, X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \not\simeq 0\} = \bigcup_\ell \mathrm{Supp}_R\, \mathrm{H}_\ell(X).$$

As usual $\mathrm{Min}_R\, X$ is the subset of minimal elements in the support.

The *depth* and the (*Krull*) *dimension* of an R–complex X are defined as follows:

$$\begin{aligned} \operatorname{depth}_R X &= -\sup \mathbf{R}\mathrm{Hom}_R(k, X), \text{ for } X \in \mathcal{D}_-(R), \quad \text{and} \\ \dim_R X &= \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R\, X\}, \end{aligned}$$

cf. [6, Sec. 3]. For modules these notions agree with the usual ones. It follows from the definition that

$$\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p} \tag{1.0.1}$$

for $X \in \mathcal{D}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$; and there are always inequalities:

$$-\inf X \leq \dim_R X \quad \text{for} \quad X \in \mathcal{D}_+(R); \quad \text{and} \tag{1.0.2}$$

$$-\sup X \leq \operatorname{depth}_R X \quad \text{for} \quad X \in \mathcal{D}_-(R). \tag{1.0.3}$$

A complex $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ is *Cohen–Macaulay* if and only if $\dim_R X = \operatorname{depth}_R X$, that is, if an only if the *Cohen–Macaulay defect*,

$$\operatorname{cmd}_R X = \dim_R X - \operatorname{depth}_R X,$$

is zero. For complexes in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ the Cohen–Macaulay defect is always non-negative, cf. [6, Cor. 3.9].

2. Anchor Prime Ideals

(2.1) **Definitions.** Let $X \in \mathcal{D}_+(R)$; we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an *anchor prime ideal* for X if and only if $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p} > -\infty$. The set of anchor prime ideals for X is denoted by $\operatorname{Anc}_R X$; that is,

$$\operatorname{Anc}_R X = \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \inf X_\mathfrak{p} = 0\}.$$

For $n \in \mathbb{N}_0$ we set

$$\mathrm{W}_n(X) = \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \dim_R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} \leq n\}.$$

(2.2) **Observation.** Let S be a multiplicative system in R, and let $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p} \cap S = \emptyset$ then $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$, and for $X \in \mathcal{D}(R)$ there is an isomorphism $S^{-1}X_{S^{-1}\mathfrak{p}} \simeq X_\mathfrak{p}$ in $\mathcal{D}(R_\mathfrak{p})$. In particular, $\inf S^{-1}X_{S^{-1}\mathfrak{p}} = \inf X_\mathfrak{p}$ and $\dim_{S^{-1}R_{S^{-1}\mathfrak{p}}} S^{-1}X_{S^{-1}\mathfrak{p}} = \dim_{R_\mathfrak{p}} X_\mathfrak{p}$. Thus, the next biconditional holds for $X \in \mathcal{D}_+(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$.

$$\mathfrak{p} \in \operatorname{Anc}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{Anc}_{S^{-1}R} S^{-1}X. \tag{2.2.1}$$

(2.3) **Theorem.** *For $X \in \mathcal{D}_+(R)$ there are inclusions:*

(a) $$\operatorname{Min}_R X \subseteq \operatorname{Anc}_R X; \quad \textit{and}$$

(b) $$\mathrm{W}_0(X) \subseteq \operatorname{Anc}_R X.$$

Furthermore, if $\operatorname{amp} X = 0$, that is, if X is equivalent to a module up to a shift, then

(c) $$\operatorname{Anc}_R X = \operatorname{Min}_R X \subseteq \operatorname{Ass}_R X;$$

and if X is Cohen–Macaulay, that is, $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ and $\dim_R X = \operatorname{depth}_R X$, then

(d) $$\operatorname{Ass}_R X \subseteq \operatorname{Anc}_R X = \mathrm{W}_0(X).$$

Proof. In the following X belongs to $\mathcal{D}_+(R)$.

(a): If $\mathfrak{p}$ belongs to $\operatorname{Min}_R X$ then $\operatorname{Supp}_{R_\mathfrak{p}} X_\mathfrak{p} = \{\mathfrak{p}_\mathfrak{p}\}$, so $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, that is, $\mathfrak{p}_\mathfrak{p} \in \operatorname{Anc}_{R_\mathfrak{p}} X_\mathfrak{p}$ and hence $\mathfrak{p} \in \operatorname{Anc}_R X$ by (2.2.1).

(b): Assume that $\mathfrak{p}$ belongs to $\mathrm{W}_0(X)$, then $\dim_R X = \dim R/\mathfrak{p} - \inf X_\mathfrak{p}$, and since $\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p}$ and $\dim_{R_\mathfrak{p}} X_\mathfrak{p} \geq -\inf X_\mathfrak{p}$, cf. (1.0.1) and (1.0.2), it follows that $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, as desired.

(c): For $M \in \mathcal{D}_0(R)$ we have

$$\operatorname{Anc}_R M = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \dim_{R_\mathfrak{p}} M_\mathfrak{p} = 0\} = \operatorname{Min}_R M,$$

and the inclusion $\operatorname{Min}_R M \subseteq \operatorname{Ass}_R M$ is well-known.

(d): Assume that $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\dim_R X = \operatorname{depth}_R X$, then $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp}_R X$, cf. [5, (16.17)]. If $\mathfrak{p} \in \operatorname{Ass}_R X$ we have

$$\dim_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} = -\sup X_\mathfrak{p} \leq -\inf X_\mathfrak{p},$$

cf. [4, Def. 2.3], and it follows by (1.0.2) that equality must hold, so $\mathfrak{p}$ belongs to $\operatorname{Anc}_R X$.

For each $\mathfrak{p} \in \operatorname{Supp}_R X$ there is an equality

$$\dim_R X = \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p},$$

cf. [5, (17.4)(b)], so $\dim_R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} = 0$ for $\mathfrak{p}$ with $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$. This proves the inclusion $\operatorname{Anc}_R X \subseteq \mathrm{W}_0(X)$. □

(2.4) **Corollary.** *For $X \in \mathcal{D}_{\mathrm{b}}(R)$ there is an inclusion:*

(a) $$\operatorname{Min}_R X \subseteq \operatorname{Ass}_R X \cap \operatorname{Anc}_R X;$$

and for $\mathfrak{p} \in \operatorname{Ass}_R X \cap \operatorname{Anc}_R X$ there is an equality:

(b) $$\operatorname{cmd}_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{amp} X_\mathfrak{p}.$$

Proof. Part (a) follows by (2.3)(a) and [4, Prop. 2.6]; part (b) is immediate by the definitions of associated and anchor prime ideals, cf. [4, Def. 2.3]. □

(2.5) **Corollary.** *If $X \in \mathcal{D}^{\mathrm{f}}_+(R)$, then*

$$\dim_R X = \sup\{\dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Anc}_R X\}.$$

Proof. It is immediate by the definitions that

$$\begin{aligned}\dim_R X &= \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}\\ &\geq \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Anc}_R X\}\\ &= \sup\{\dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Anc}_R X\};\end{aligned}$$

and the opposite inequality follows by (2.3)(b). □

(2.6) **Proposition.** *The following hold:*

(a) *If $X \in \mathcal{D}_+(R)$ and $\mathfrak{p}$ belongs to $\operatorname{Anc}_R X$, then $\dim_{R_\mathfrak{p}} \mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p}) = 0$.*

(b) *If $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then $\operatorname{Anc}_R X$ is a finite set.*

Proof. (a): Assume that $\mathfrak{p} \in \operatorname{Anc}_R X$; by [6, Prop. 3.5] we have

$$-\inf X_{\mathfrak{p}} = \dim_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \geq \dim_{R_{\mathfrak{p}}} \mathrm{H}_{\inf X_{\mathfrak{p}}}(X_{\mathfrak{p}}) - \inf X_{\mathfrak{p}},$$

and hence $\dim_{R_{\mathfrak{p}}} \mathrm{H}_{\inf X_{\mathfrak{p}}}(X_{\mathfrak{p}}) = 0$.

(b): By (a) every anchor prime ideal for X is minimal for one of the homology modules of X, and when $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ each of the finitely many homology modules has a finite number of minimal prime ideals. □

(2.7) **Observation.** By Nakayama's lemma it follows that

$$\inf \mathrm{K}(x_1, \dots, x_n; Y) = \inf Y,$$

for $Y \in \mathcal{D}_{+}^{\mathrm{f}}(R)$ and elements $x_1, \dots, x_n \in \mathfrak{m}$.

(2.8) **Proposition (Dimension of Koszul Complexes).** *The following hold for a complex $Y \in \mathcal{D}_{+}^{\mathrm{f}}(R)$ and elements $x_1, \dots, x_n \in \mathfrak{m}$:*

(a)
$$\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \sup\{\dim R/\mathfrak{p} - \inf Y_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(x_1, \dots, x_n)\}; \quad \textit{and}$$

(b)
$$\dim_R Y - n \leq \dim_R \mathrm{K}(x_1, \dots, x_n; Y) \leq \dim_R Y.$$

Furthermore:

(c) *The elements $x_1, \dots, x_n$ are contained in a prime ideal $\mathfrak{p} \in \mathrm{W}_n(Y)$; and*

(d) $\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y$ *if and only if* $x_1, \dots, x_n \in \mathfrak{p}$ *for some* $\mathfrak{p} \in \mathrm{W}_0(Y)$.

Proof. Since $\operatorname{Supp}_R \mathrm{K}(x_1, \dots, x_n; Y) = \operatorname{Supp}_R Y \cap \mathrm{V}(x_1, \dots, x_n)$ (see [6, p. 157] and [4, 3.2]) (a) follows by the definition of Krull dimension and (2.7). In (b) the second inequality follows from (a); the first one is established through four steps:

1° $Y = R$: The second equality below follows from the definition of Krull dimension as $\operatorname{Supp}_R \mathrm{K}(x_1, \dots, x_n) = \operatorname{Supp}_R \mathrm{H}_0(\mathrm{K}(x_1, \dots, x_n)) = \mathrm{V}(x_1, \dots, x_n)$, cf. [4, 3.2]; the inequality is a consequence of Krull's Principal Ideal Theorem, see for example [8, Thm. 13.6].

$$\begin{aligned}
\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_1, \dots, x_n) \\
&= \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(x_1, \dots, x_n)\} \\
&= \dim R/(x_1, \dots, x_n) \\
&\geq \dim R - n \\
&= \dim_R Y - n.
\end{aligned}$$

2° $Y = B$, a cyclic module: By $\bar{x}_1, \dots, \bar{x}_n$ we denote the residue classes in B of the elements $x_1, \dots, x_n$; the inequality below is by 1°.

$$\begin{aligned}
\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(\bar{x}_1, \dots, \bar{x}_n) \\
&= \dim_B \mathrm{K}(\bar{x}_1, \dots, \bar{x}_n) \\
&\geq \dim B - n \\
&= \dim_R Y - n.
\end{aligned}$$

3° $Y = H \in \mathcal{D}_0^{\mathrm{f}}(R)$: We set $B = R/\operatorname{Ann}_R H$; the first equality below follows by [6, Prop. 3.11] and the inequality by 2°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1,\dots,x_n;Y) &= \dim_R \mathrm{K}(x_1,\dots,x_n;B)\\ &\geq \dim_R B - n\\ &= \dim_R Y - n.\end{aligned}$$

4° $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$: The first equality below follows by [6, Prop. 3.12] and the last by [6, Prop. 3.5]; the inequality is by 3°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1,\dots,x_n;Y) &= \sup\{\dim_R \mathrm{K}(x_1,\dots,x_n;\mathrm{H}_\ell(Y)) - \ell \mid \ell \in \mathbb{Z}\}\\ &\geq \sup\{\dim_R \mathrm{H}_\ell(Y) - n - \ell \mid \ell \in \mathbb{Z}\}\\ &= \dim_R Y - n.\end{aligned}$$

This proves (b).

In view of (a) it now follows that

$$\dim_R Y - n \leq \dim R/\mathfrak{p} - \inf Y_{\mathfrak{p}}$$

for some $\mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(x_1,\dots,x_n)$. That is, the elements $x_1,\dots,x_n$ are contained in a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R Y$ with

$$\dim_R Y - \dim R/\mathfrak{p} + \inf Y_{\mathfrak{p}} \leq n,$$

and this proves (c).

Finally, it is immediate by the definitions that

$$\dim_R Y = \sup\{\dim R/\mathfrak{p} - \inf Y_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(x_1,\dots,x_n)\}$$

if and only if $\mathrm{W}_0(Y) \cap \mathrm{V}(x_1,\dots,x_n) \neq \emptyset$. This proves (d). □

(2.9) **Theorem.** *If $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then the next two numbers are equal.*

$$\begin{aligned}\mathrm{d}(Y) &= \dim_R Y + \inf Y; \quad \textit{and}\\ \mathrm{s}(Y) &= \inf\{s \in \mathbb{N}_0 \mid \exists\, x_1,\dots,x_s : \mathfrak{m} \in \operatorname{Anc}_R \mathrm{K}(x_1,\dots,x_s;Y)\}.\end{aligned}$$

Proof. There are two inequalities to prove.

$\mathrm{d}(Y) \leq \mathrm{s}(Y)$: Let $x_1,\dots,x_s \in \mathfrak{m}$ be such that $\mathfrak{m} \in \operatorname{Anc}_R \mathrm{K}(x_1,\dots,x_s;Y)$; by (2.8)(b) and (2.7) we then have

$$\dim_R Y - s \leq \dim_R \mathrm{K}(x_1,\dots,x_s;Y) = -\inf \mathrm{K}(x_1,\dots,x_s;Y) = -\inf Y,$$

so $\mathrm{d}(Y) \leq s$, and the desired inequality follows.

$\mathrm{s}(Y) \leq \mathrm{d}(Y)$: We proceed by induction on $\mathrm{d}(Y)$. If $\mathrm{d}(Y) = 0$ then $\mathfrak{m} \in \operatorname{Anc}_R Y$ so $\mathrm{s}(Y) = 0$. If $\mathrm{d}(Y) > 0$ then $\mathfrak{m} \notin \operatorname{Anc}_R Y$, and since $\operatorname{Anc}_R Y$ is a finite set, by (2.6)(b), we can choose an element $x \in \mathfrak{m} - \cup_{\mathfrak{p}\in\operatorname{Anc}_R Y}\mathfrak{p}$. We set $K = \mathrm{K}(x;Y)$; it is cleat that $\mathrm{s}(Y) \leq \mathrm{s}(K) + 1$. Furthermore, it follows by (2.8)(a) and (2.3)(b) that $\dim_R K < \dim_R Y$ and thereby $\mathrm{d}(K) < \mathrm{d}(Y)$, cf. (2.7). Thus, by the induction hypothesis we have

$$\mathrm{s}(Y) \leq \mathrm{s}(K) + 1 \leq \mathrm{d}(K) + 1 \leq \mathrm{d}(Y);$$

as desired. □

3. Parameters

By (2.9) the next definitions extend the classical notions of systems and sequences of parameters for finite modules (e.g., see [8, § 14] and the appendix in [2]).

(3.1) **Definitions.** Let Y belong to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and set $d = \dim_R Y + \inf Y$. A set of elements $x_1, \ldots, x_d \in \mathfrak{m}$ are said to be a *system of parameters* for Y if and only if $\mathfrak{m} \in \operatorname{Anc}_R \mathrm{K}(x_1, \ldots, x_d; Y)$.

A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ is said to be a *Y–parameter sequence* if and only if it is part of a system of parameters for Y.

(3.2) **Lemma.** *Let Y belong to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and set $d = \dim_R Y + \inf Y$. The next two conditions are equivalent for elements $x_1, \ldots, x_d \in \mathfrak{m}$ and $j \in \{0, \ldots, d\}$.*

(*i*) *$x_1, \ldots, x_d$ is a system of parameters for Y.*

(*ii*) *There is an equality:*

$$\dim_R \mathrm{K}(x_1, \ldots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \ldots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \ldots, x_j; Y)$.

Proof. (*i*) $\Rightarrow$ (*ii*): Assume that $x_1, \ldots, x_d$ is a system of parameters for Y, then

$$\begin{aligned} -\inf \mathrm{K}(x_1, \ldots, x_n; Y) &= \dim_R \mathrm{K}(x_1, \ldots, x_n; Y) \\ &= \dim_R \mathrm{K}(x_{j+1}, \ldots, x_d; \mathrm{K}(x_1, \ldots, x_j; Y)) \\ &\geq \dim_R \mathrm{K}(x_1, \ldots, x_j; Y) - (d - j) && \text{by (2.8)(b)} \\ &\geq \dim_R Y - j - (d - j) && \text{by (2.8)(b)} \\ &= \dim_R Y - d \\ &= -\inf Y. \end{aligned}$$

By (2.7) it now follows that $-\inf Y = \dim_R \mathrm{K}(x_1, \ldots, x_j; Y) - (d - j)$, so

$$\dim_R \mathrm{K}(x_1, \ldots, x_j; Y) = d - j - \inf Y = \dim_R Y - j,$$

as desired. It also follows that $\mathrm{d}(\mathrm{K}(x_1, \ldots, x_j; Y)) = d - j$, and since

$$\mathfrak{m} \in \operatorname{Anc}_R \mathrm{K}(x_1, \ldots, x_d; Y) = \operatorname{Anc}_R \mathrm{K}(x_{j+1}, \ldots, x_d; \mathrm{K}(x_1, \ldots, x_j; Y)),$$

we conclude that $x_{j+1}, \ldots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \ldots, x_j; Y)$.

(*ii*)$\Rightarrow$(*i*): If $\dim_R \mathrm{K}(x_1, \ldots, x_j; Y) = \dim_R Y - j$ then $\mathrm{d}(\mathrm{K}(x_1, \ldots, x_j; Y)) = d - j$; and if $x_{j+1}, \ldots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \ldots, x_j; Y)$ then $\mathfrak{m}$ belongs to $\operatorname{Anc}_R \mathrm{K}(x_{j+1}, \ldots, x_d; \mathrm{K}(x_1, \ldots, x_j; Y)) = \operatorname{Anc}_R \mathrm{K}(x_1, \ldots, x_d; Y)$, so $x_1, \ldots, x_d$ must be a system of parameters for Y. $\square$

(3.3) **Proposition.** *Let $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$. The following conditions are equivalent for a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{m}$.*

(*i*) *$\boldsymbol{x}$ is a Y–parameter sequence.*

(*ii*) *For each $j \in \{0, \ldots, n\}$ there is an equality:*

$$\dim_R \mathrm{K}(x_1, \ldots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \ldots, x_n$ is a $\mathrm{K}(x_1, \ldots, x_j; Y)$–parameter sequence.

(*iii*) *There is an equality:*

$$\dim_R \mathrm{K}(x_1,\ldots,x_n;Y) = \dim_R Y - n.$$

Proof. It follows by (3.2) that (*i*) implies (*ii*), and (*iii*) follows from (*ii*). Now, set $K = \mathrm{K}(\boldsymbol{x};Y)$ and assume that $\dim_R K = \dim_R Y - n$. Choose, by (2.9), $s = \mathrm{s}(K) = \dim_R K + \inf K$ elements $w_1,\ldots,w_s$ in $\mathfrak{m}$ such that $\mathfrak{m}$ belongs to $\mathrm{Anc}_R \mathrm{K}(w_1,\ldots,w_s;K) = \mathrm{Anc}_R \mathrm{K}(x_1,\ldots,x_n,w_1,\ldots,w_s;Y)$. Then, by (2.7), we have

$$n + s = (\dim_R Y - \dim_R K) + (\dim_R K + \inf K) = \dim_R Y + \inf Y = d,$$

so $x_1,\ldots,x_n,w_1,\ldots,w_s$ is a system of parameters for Y, whence $x_1,\ldots,x_n$ is a Y–parameter sequence. □

We now recover a classical result (e.g., see [2, Prop. A.4]):

(3.4) **Corollary.** *Let M be an R–module. The following conditions are equivalent for a sequence $\boldsymbol{x} = x_1,\ldots,x_n$ in $\mathfrak{m}$.*

(*i*) *$\boldsymbol{x}$ is an M–parameter sequence.*

(*ii*) *For each $j \in \{0,\ldots,n\}$ there is an equality:*

$$\dim_R M/(x_1,\ldots,x_j)M = \dim_R M - j;$$

and $x_{j+1},\ldots,x_n$ is an $M/(x_1,\ldots,x_j)M$–parameter sequence.

(*iii*) *There is an equality:*

$$\dim_R M/(x_1,\ldots,x_n)M = \dim_R M - n.$$

Proof. By [6, Prop. 3.12] and [5, (16.22)] we have

$$\begin{aligned}\dim_R \mathrm{K}(x_1,\ldots,x_j;M) &= \sup\{\dim_R(M \otimes^{\mathbf{L}}_R \mathrm{H}_\ell(\mathrm{K}(x_1,\ldots,x_j))) - \ell \mid \ell \in \mathbb{Z}\}\\ &= \sup\{\dim_R(M \otimes_R \mathrm{H}_\ell(\mathrm{K}(x_1,\ldots,x_j))) - \ell \mid \ell \in \mathbb{Z}\}\\ &= \dim_R(M \otimes_R R/(x_1,\ldots,x_j)). \quad \square\end{aligned}$$

(3.5) **Theorem.** *Let $Y \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The following hold for a sequence $\boldsymbol{x} = x_1,\ldots,x_n$ in $\mathfrak{m}$.*

(a) *There is an inequality:*

$$\operatorname{amp} \mathrm{K}(\boldsymbol{x};Y) \geq \operatorname{amp} Y;$$

and equality holds if and only if $\boldsymbol{x}$ is a Y–sequence.

(b) *There is an inequality:*

$$\operatorname{cmd}_R \mathrm{K}(\boldsymbol{x};Y) \geq \operatorname{cmd}_R Y;$$

and equality holds if and only if $\boldsymbol{x}$ is a Y–parameter sequence.

(c) *If $\boldsymbol{x}$ is a maximal Y–sequence, then*

$$\operatorname{amp} Y \leq \operatorname{cmd}_R \mathrm{K}(\boldsymbol{x};Y).$$

(d) *If $\boldsymbol{x}$ is a system of parameters for Y, then*

$$\operatorname{cmd}_R Y \leq \operatorname{amp} \mathrm{K}(\boldsymbol{x};Y).$$

Proof. In the following K denotes the Koszul complex $\mathrm{K}(\boldsymbol{x};Y)$.

(a): Immediate by (2.7) and [4, Prop. 5.1].

(b): By [4, Thm. 4.7(a)] and (2.8)(b) we have

$$\operatorname{cmd}_R K = \dim_R K - \operatorname{depth}_R K = \dim_R K + n - \operatorname{depth}_R Y \geq \operatorname{cmd}_R Y,$$

and by (3.3) equality holds if and only if $\boldsymbol{x}$ is a Y–parameter sequence.

(c): Suppose $\boldsymbol{x}$ is a maximal Y–sequence, then

$$\begin{aligned} \operatorname{amp} Y &= \sup Y - \inf K && \text{by (2.7)} \\ &= -\operatorname{depth}_R K - \inf K && \text{by [4, Thm. 5.4]} \\ &\leq \operatorname{cmd}_R K && \text{by (1.0.2).} \end{aligned}$$

(d): Suppose $\boldsymbol{x}$ is system of parameters for Y, then

$$\begin{aligned} \operatorname{amp} K &= \sup K + \dim_R K \\ &\geq \dim_R K - \operatorname{depth}_R K && \text{by (1.0.3)} \\ &= \operatorname{cmd}_R Y && \text{by (b).} \quad \square \end{aligned}$$

(3.6) **Theorem.** *The following hold for* $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.

(a) *The next four conditions are equivalent.*

- (i) *There is a maximal Y–sequence which is also a Y–parameter sequence.*
- (ii) $\operatorname{depth}_R Y + \sup Y \leq \dim_R Y + \inf Y$.
- (ii') $\operatorname{amp} Y \leq \operatorname{cmd}_R Y$.
- (iii) *There is a maximal strong Y–sequence which is also a Y–parameter sequence.*

(b) *The next four conditions are equivalent.*

- (i) *There is a system of parameters for Y which is also a Y–sequence.*
- (ii) $\dim_R Y + \inf Y \leq \operatorname{depth}_R Y + \sup Y$.
- (ii') $\operatorname{cmd}_R Y \leq \operatorname{amp} Y$.
- (iii) *There is a system of parameters for Y which is also a strong Y–sequence.*

(c) *The next four conditions are equivalent.*

- (i) *There is a system of parameters for Y which is also a maximal Y–sequence.*
- (ii) $\dim_R Y + \inf Y = \operatorname{depth}_R Y + \sup Y$.
- (ii') $\operatorname{cmd}_R Y = \operatorname{amp} Y$.
- (iii) *There is a system of parameters for Y which is also a maximal strong Y–sequence.*

Proof. Let $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, set $\mathrm{n}(Y) = \operatorname{depth}_R Y + \sup Y$ and $\mathrm{d}(Y) = \dim_R Y + \inf Y$.

(a): A maximal Y–sequence is of length $\mathrm{n}(Y)$, cf. [4, Cor. 5.5], and the length of a Y–parameter sequence is at most $\mathrm{d}(Y)$. Thus, (i) implies (ii) which in turn is equivalent to (ii'). Furthermore, a maximal strong Y–sequence is, in particular, a maximal Y–sequence, cf. [4, Cor. 5.7], so (iii) is stronger than (i). It is now sufficient to prove the implication $(ii) \Rightarrow (iii)$: We proceed by induction. If $\mathrm{n}(Y) = 0$ then the empty sequence is a maximal strong Y–sequence and a Y–parameter sequence. Let $\mathrm{n}(Y) > 0$; the two sets $\mathrm{Ass}_R Y$ and $\mathrm{W}_0(Y)$ are both finite, and since $0 < \mathrm{n}(Y) \leq \mathrm{d}(Y)$ none of them contain $\mathfrak{m}$. We can, therefore, choose an element $x \in \mathfrak{m} - \cup_{\mathrm{Ass}_R Y \cup \mathrm{W}_0(Y)} \mathfrak{p}$, and x is then a strong Y–sequence, cf. [4, Def. 3.3], and a Y–parameter sequence, cf. (3.3) and (2.8). Set $K = \mathrm{K}(x;Y)$, by [4, Thm. 4.7 and Prop. 5.1], respectively, (2.8) and (2.7) we have

$$\begin{aligned} \operatorname{depth}_R K + \sup K &= \mathrm{n}(Y) - 1 \\ &\leq \mathrm{d}(Y) - 1 = \dim_R K + \inf K. \end{aligned}$$

By the induction hypothesis there exists a maximal strong K–sequence $w_1, \dots, w_{n-1}$ which is also a K–parameter sequence, and it follows by [4, 3.5] and (3.3) that $x, w_1, \dots, w_{n-1}$ is a strong Y–sequence and a Y–parameter sequence, as wanted.

The proof of (b) i similar to the proof of (a), and (c) follows immediately by (a) and (b). □

(3.7) **Theorem.** *The following hold for $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$:*

(a) *If* $\operatorname{amp} Y = 0$, *then any Y–sequence is a Y–parameter sequence.*

(b) *If* $\operatorname{cmd}_R Y = 0$, *then any Y–parameter sequence is a strong Y–sequence.*

Proof. The empty sequence is a Y–parameter sequence as well as a strong Y–sequence, this founds the base for a proof by induction on the length n of the sequence $\boldsymbol{x} = x_1, \dots, x_n$. Let $n > 0$ and set $K = \mathrm{K}(x_1, \dots, x_{n-1}; Y)$; by (2.8)(a) we have

$$(*) \qquad \begin{aligned} \dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_n; K) \\ &= \sup\{\dim R/\mathfrak{p} - \inf K_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R K \cap \mathrm{V}(x_n)\}. \end{aligned}$$

Assume that $\operatorname{amp} Y = 0$. If $\boldsymbol{x}$ is a Y–sequence, then $\operatorname{amp} K = 0$ by (3.5)(a) and $x_n \notin \mathrm{z}_R K$, cf. [4, Def. 3.3]. As $\mathrm{z}_R K = \cup_{\mathfrak{p} \in \mathrm{Ass}_R K} \mathfrak{p}$, cf. [4, 2.5], it follows by (b) and (c) in (2.3) that x_n is not contained in any prime ideal $\mathfrak{p} \in \mathrm{W}_0(K)$; so from $(*)$ we conclude that $\dim_R \mathrm{K}(x_n; K) < \dim_R K$, and it follows by (2.8)(b) that $\dim_R \mathrm{K}(x_n; K) = \dim_R K - 1$. By the induction hypothesis $\dim_R K = \dim_R Y - (n-1)$, so $\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y - n$ and it follows by (3.3) that $\boldsymbol{x}$ is a Y–parameter sequence. This proves (a).

We now assume that $\operatorname{cmd}_R Y = 0$. If $\boldsymbol{x}$ is a Y–parameter sequence then, by the induction hypothesis, $x_1, \dots, x_{n-1}$ is a strong Y–sequence, so it is sufficient to prove that $x_n \notin \mathrm{Z}_R K$, cf. [4, 3.5]. By (3.3) it follows that x_n is a K–parameter sequence, so $\dim_R \mathrm{K}(x_n; K) = \dim_R K - 1$ and we conclude from $(*)$ that $x_n \notin \cup_{\mathfrak{p} \in \mathrm{W}_0(K)} \mathfrak{p}$. Now, by (3.5)(b) we have $\operatorname{cmd}_R K = 0$, so it follows from (2.3)(d) that $x_n \notin \cup_{\mathfrak{p} \in \mathrm{Ass}_R K} \mathfrak{p} = \mathrm{Z}_R K$. This proves (b). □

(3.8) ***Semi-dualizing Complexes***. We recall two basic definitions from [3]:

A complex $C \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *semi-dualizing* for R if and only if the homothety morphism $\chi_C^R \colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism [3, (2.1)].

Let C be a semi–dualizing complex for R. A complex $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *C–reflexive* if and only if the *dagger dual* $Y^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(Y, C)$ belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$

and the biduality morphism $\delta_Y^C : Y \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Y, C), C)$ is invertible in $\mathcal{D}(R)$ [3, (2.7)].

A *dualizing complex*, cf. [7], is a semi–dualizing complex of finite injective dimension, in particular, it is Cohen–Macaulay, cf. [3, (3.5)]. If D is a dualizing complex for R, then, by [7, Prop. V.2.1], all complexes $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ are D–reflexive; in particular, all finite R–modules are D–reflexive and, therefore, [4, 5.10] is a special case of the following:

(3.9) **Theorem.** *Let C be a Cohen–Macaulay semi–dualizing complex for R, and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $\mathfrak{m}$. If Y is C–reflexive, then $\boldsymbol{x}$ is a Y–parameter sequence if and only if it is a $\mathbf{R}\mathrm{Hom}_R(Y, C)$–sequence; that is*

$$\boldsymbol{x} \text{ is a } Y\text{–parameter sequence} \iff \boldsymbol{x} \text{ is a } \mathbf{R}\mathrm{Hom}_R(Y, C)\text{–sequence.}$$

Proof. We assume that C is a Cohen–Macaulay semi–dualizing complex for R and that Y is C–reflexive, cf. (3.8). The desired biconditional follows by the next chain, and each step is explained below (we use the notation $-^{\dagger_C}$ introduced in (3.8)).

$$\begin{aligned}
\boldsymbol{x} \text{ is a } Y\text{–parameter sequence} &\iff \operatorname{cmd}_R \mathrm{K}(\boldsymbol{x}; Y) = \operatorname{cmd}_R Y \\
&\iff \operatorname{amp} \mathrm{K}(\boldsymbol{x}; Y)^{\dagger_C} = \operatorname{amp} Y^{\dagger_C} \\
&\iff \operatorname{amp} \mathrm{K}(\boldsymbol{x}; Y^{\dagger_C}) = \operatorname{amp} Y^{\dagger_C} \\
&\iff \boldsymbol{x} \text{ is a } Y^{\dagger_C}\text{–sequence.}
\end{aligned}$$

The first biconditional follows by (3.5)(b) and the last by (3.5)(a). Since $\mathrm{K}(\boldsymbol{x})$ is a bounded complex of free modules (hence of finite projective dimension), it follows from [3, Thm. (3.17)] that also $\mathrm{K}(\boldsymbol{x}; Y)$ is C–reflexive, and the second biconditional is then immediate by the CMD–formula [3, Cor. (3.8)]. The third one is established as follows:

$$\begin{aligned}
\mathrm{K}(\boldsymbol{x}; Y)^{\dagger_C} &\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}) \otimes_R^{\mathbf{L}} Y, C) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), Y^{\dagger_C}) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R \otimes_R^{\mathbf{L}} Y^{\dagger_C}) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R) \otimes_R^{\mathbf{L}} Y^{\dagger_C} \\
&\sim \mathrm{K}(\boldsymbol{x}) \otimes_R^{\mathbf{L}} Y^{\dagger_C} \\
&\simeq \mathrm{K}(\boldsymbol{x}; Y^{\dagger_C}),
\end{aligned}$$

where the second isomorphism is by adjointness and the fourth by, so-called, tensor-evaluation, cf. [1, (1.4.2)]. It is straightforward to check that $\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R)$ is isomorphic to the Koszul complex $\mathrm{K}(\boldsymbol{x})$ shifted n degrees to the right, and the symbol $\sim$ denotes isomorphism up to shift. □

References

1. Luchezar L. Avramov and Hans-Bjørn Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270.
2. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge University Press, Cambridge, 1998.

3. Lars Winther Christensen, *Semi–dualizing complexes and their Auslander categories*, to appear in Trans. Amer. Math. Soc.
4. ______, *Sequences for complexes*, to appear in Math. Scand.
5. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.
6. ______, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
7. Robin Hartshorne, *Residues and duality*, Springer-Verlag, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20.
8. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

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