Dynamic portfolio optimization with stochastic investment opportunities

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This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen.

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This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen on the 23rd of December 2022.

ISBN:978-87-7125-067-1.

Abstract

This thesis is comprised of nine independent research projects pertaining primarily to dynamic portfolio optimization with time-varying investment opportunities, model ambiguity, and relative performance concerns. We start with the optimal mean-variance portfolio selection problem with a 3/2 stochastic volatility in a complete market setting, where we derive, in closed form, both the static and dynamic optimality using a backward stochastic differential equation approach. Then, in incomplete market settings, we study more complicated cases within the framework of the mean-variance criteria under a hybrid model of stochastic volatility and stochastic interest rates, the family of state-of-the-art 4/2 stochastic volatility models with derivatives trading and uncontrollable random liabilities, and in the presence of mispricing, respectively. Next, three portfolio optimization problems under the expected utility maximization paradigm are investigated, where we consider the presence of stochastic volatility and affine short rates, stochastic income and stochastic inflation, and random liabilities under the hyperbolic absolute risk aversion preferences, respectively. The last two projects revolve around optimal asset-liability management problems with stochastic volatility in the non-Markovian cases, demonstrating the impact of model ambiguity and relative performance concerns on the behavior of the optimal investment strategies by means of the backward stochastic differential equations, in which the former one is modeled as a zero-sum stochastic differential game between the manager and the adverse market whereas the latter one is described by a non-zero-sum stochastic differential game between two competitive managers.

Preface

This thesis has been submitted in partial fulfillment of the requirements for the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen. The work was carried out between January 2020 and December 2022 under the supervision of Associate Professor Jesper Lund Pedersen.

The thesis is comprised of an introductory chapter and nine manuscripts, of which two are submitted for review and seven have been published in international peer-reviewed journals at the time of writing. The introduction, Chapter 1, provides an overview and contextualization of the contributions and outlines the interconnections of the manuscripts. Each manuscript is independent, constituting a chapter, and some minor notational discrepancies may exist between the contents of a chapter and the corresponding manuscript. The author takes full responsibility for any typographical and mathematical errors.

Acknowledgments

Pursuing a PhD degree at the Department of Mathematical Sciences at the University of Copenhagen is a valuable and unforgettable experience for me. First and foremost, I would like to express my deepest gratitude to my supervisor Prof. Jesper Lund Pedersen for giving me the opportunity to conduct my doctoral studies under his supervision. I am grateful for his precious advice and consistent support in challenging times throughout the course, and needless to say, for keeping faith with me and giving me enough academic freedom to walk my own way.

I would like to thank my good friend Peter Pommergård Lind for providing me with life and emotional support during my research visit to Aalborg University Business School from April to June 2022. I much appreciate our discussions on the applications of backward stochastic differential equations to finance and many other interesting themes beyond mathematics. I also would like to express my gratitude to Prof. Frederik Steen Lundtofte, Douglas Eduardo Turatti, Luca Branco, and Michael Simonsen for their hospitality, and I enjoy our entertaining conversations on a broad range of topics.

Many thanks to everyone in administration at the Department of Mathematical

Sciences, in particular, Nina Weisse, Mette Fulling, and Lone Baltzersen, for their always kind and patient assistance with bureaucratic matters and ordering textbooks for me. I am thankful to Christian Furrer, Prof. Mogens Bladt, and Prof. Mogens Steffen for organizing the very instructive Seminar on Insurance and Economics. Especially, I wish to thank Christian Furrer for providing me with the template for this thesis and prompt feedback when I have difficulty running it. I also kindly thank Prof. Thomas Mikosch for lending his textbooks to me and allowing me to keep them for a long period of time, and Prof. Rolf Poulsen for his career-related advice and help.

I want to thank my high school classmate and good friend, Hanqing Xiang, for patiently listening to my complaints and confusion when I receive either rejections or a variety of revision requests from editors and referees. Most of all, I am exceptionally grateful to my parents, Xinwen Zhang and Wenting Shen, for their unconditional love and support, as well as to my girlfriend Gina Ge Gao for always accompanying, encouraging, and loving me. This thesis is dedicated to them.

Yumo Zhang
Copenhagen, December 2022

A few minor errors have been corrected and an ISBN has been provided in the final version of the thesis compared to the previous version submitted to the PhD School on 23rd of December 2022.

Yumo Zhang
Copenhagen, March 2023

List of papers

Besides Chapter 1, which has been prepared specially for this thesis, each of the nine remaining chapters presents a self-contained and submitted or already published manuscript according to the following scheme:

Chapter 2:
Zhang, Y. (2021). Dynamic optimal mean-variance portfolio selection with a 3/2 stochastic volatility. *Risks* **9**, p. 61. DOI: 10.3390/risks9040061.

Chapter 3:
Zhang, Y. (2022). Dynamic optimal mean-variance portfolio selection with stochastic volatility and stochastic interest rate. *Annals of Finance* **18**, pp. 511-544. DOI: 10.1007/s10436-022-00414-x.

Chapter 4:
Zhang, Y. (2023). Mean-variance asset-liability management under CIR interest rate and the family of 4/2 stochastic volatility models with derivative trading. *Journal of Industrial and Management Optimization* **19**, pp. 4022-4063. DOI: 10.3934/jimo.2022121.

Chapter 5:
Zhang, Y. (2021). Dynamic optimal mean-variance investment with mispricing in the family of 4/2 stochastic volatility models. *Mathematics* **9**, p. 2293. DOI: 10.3390/math9182293.

Chapter 6:
Zhang, Y. (2022). Utility maximization in a stochastic affine interest rate and CIR risk premium framework: a BSDE approach. *Decisions in Economics and Finance*. DOI: 10.1007/s10203-022-00374-x.

Chapter 7:
Zhang, Y. (2022). Optimal DC pension investment with square-root factor processes under stochastic income and inflation risks. *Optimization: A Journal of Mathematical Programming and Operations Research*. DOI: 10.1080/02331934.2022.2081083.

Chapter 8:
Zhang, Y. (2022). Optimal investment strategies for asset-liability management with affine diffusion factor processes and HARA preferences. *Journal of Industrial and Management Optimization*. DOI: 10.3934/jimo.2022194.

Chapter 9:
Zhang, Y. (2022). Robust optimal asset-liability management under square-root factor processes and model ambiguity: a BSDE approach. *Under review*. DOI: 10.2139/ssrn.4182575.

Chapter 10:
Zhang, Y. (2022). Non-zero-sum stochastic differential games for asset-liability management with stochastic inflation and stochastic volatility. *Under review*. DOI: 10.2139/ssrn.4262759.

Summary

This thesis is comprised of an introductory chapter and nine research papers written from January 2020 to December 2022. Each paper is self-contained and constitutes a chapter, investigating problems and techniques around dynamic portfolio optimization problems with time-varying investment opportunities under different market scenarios. The introduction, Chapter 1, provides the scientific background and an overview of the main contributions of the papers and their interconnections. The abstracts of the subsequent Chapters 2–10 are listed below:

- **Dynamic optimal mean-variance portfolio selection with a 3/2 stochastic volatility**. This paper considers a mean-variance portfolio selection problem when the stock price has a 3/2 stochastic volatility in a complete market. Specifically, we assume that the stock price and the volatility are perfectly negatively correlated. By applying a backward stochastic differential equation (BSDE) approach, closed-form expressions for the statically optimal (time-inconsistent) strategy and the value function are derived. Due to the time inconsistency of the mean-variance criterion, a dynamic formulation of the problem is presented. We obtain the dynamically optimal (time-consistent) strategy explicitly which is shown to keep the wealth process strictly below the target (expected terminal wealth) before the terminal time. Finally, we provide numerical studies to show the impact of main model parameters on the efficient frontier and illustrate the differences between the two optimal wealth processes.

- **Dynamic optimal mean-variance portfolio selection with stochastic volatility and stochastic interest rate**. This paper studies optimal portfolio selection problems in the presence of stochastic volatility and stochastic interest rate under the mean-variance criterion. The financial market consists of a risk-free asset (cash), a zero-coupon bond (roll-over bond), and a risky asset (stock). Specifically, we assume that the interest rate follows the Vasicek model, and the risky asset's return rate not only depends on a Cox-Ingersoll-Ross (CIR) process but also has stochastic covariance with the interest rate, which embraces the family of state-of-the-art 4/2 stochastic

volatility models as an exceptional case. By adopting a backward stochastic differential equation (BSDE) approach and solving two related BSDEs, we derive, in closed form, the static optimal (time-inconsistent) strategy and optimal value function. Given the time inconsistency of the mean-variance criterion, a dynamic formulation of the problem is further investigated and the explicit expression for the dynamic optimal (time-consistent) strategy is derived. In addition, analytical solutions to some special cases of our model are provided. Finally, the impact of the model parameters on the efficient frontier and the behavior of the static and dynamic optimal asset allocations is illustrated with numerical examples.

- **Mean-variance asset-liability management under CIR interest rate and the family of 4/2 stochastic volatility models with derivative trading.** This paper investigates the effects of derivative trading on the performance of asset-liability management in the presence of stochastic interest rate and stochastic volatility under the mean-variance criterion. Specifically, the asset-liability manager can invest not only in a money market account, a zero-coupon (rollover) bond, and a stock index but also in stock derivatives. It is assumed that the interest rate follows a Cox-Ingersoll-Ross (CIR) process, and the instantaneous variance of the stock index is governed by the family of 4/2 stochastic volatility models, which embraces the Heston model and 3/2 model, as particular cases. By solving a system of three backward stochastic differential equations, closed-form expressions for the optimal strategies and optimal value functions are derived in two cases: with and without the stock derivatives. Moreover, we consider the special cases without random liabilities. Numerical examples are provided to illustrate theoretical results and explore the effects of derivative trading on efficient frontiers.

- **Dynamic optimal mean-variance investment with mispricing in the family of 4/2 stochastic volatility models.** This paper considers an optimal investment problem with mispricing in the family of 4/2 stochastic volatility models under the mean-variance criterion. The financial market consists of a risk-free asset, a market index, and a pair of mispriced stocks. By applying the linear–quadratic stochastic control theory and solving the corresponding Hamilton–Jacobi–Bellman equation, explicit expressions for the statically optimal (pre-commitment) strategy and the corresponding optimal value function are derived. Moreover, a necessary verification theorem is provided based on an assumption of the model parameters with the investment horizon. Due to the time inconsistency under the mean-variance criterion, we give a dynamic formulation of the problem and obtain the closed-form expression of the dynamically optimal (time-consistent) strategy. This strategy is shown to keep the wealth process strictly below the target (expected terminal wealth) before the terminal time. Results on the special case without mispricing are included. Finally, some numerical examples are given to illustrate the effects of model parameters on the efficient frontier and the difference between static and dynamic optimality.

- **Utility maximization in a stochastic affine interest rate and CIR risk premium framework: a BSDE approach.** This paper investigates optimal investment problems in the presence of stochastic interest rates and stochastic volatility under the expected utility maximization criterion. The financial market consists of three assets: a risk-free asset, a risky asset, and zero-coupon bonds (rolling bonds). The short interest rate is assumed to follow an affine diffusion process, which includes the Vasicek and the Cox-Ingersoll-Ross (CIR) models, as special cases. The risk premium of the risky asset depends on a square-root diffusion (CIR) process, while the return rate and volatility coefficient are unspecified and possibly given by non-Markovian processes. This framework embraces the family of state-of-the-art 4/2 stochastic volatility models and some non-Markovian models, as exceptional examples. The investor aims to maximize the expected utility of the terminal wealth for two types of utility functions, power utility, and logarithmic utility. By adopting a backward stochastic differential equation (BSDE) approach to overcome the potentially non-Markovian framework and solving two BSDEs explicitly, we derive, in closed form, the optimal investment strategies and optimal value functions. Furthermore, explicit solutions to some special cases of our model are provided. Finally, numerical examples illustrate our results under one specific case, the hybrid Vasicek-4/2 model.

- **Optimal DC pension investment with square-root factor processes under stochastic income and inflation risks.** This paper studies optimal defined contribution (DC) pension investment problems under the expected utility maximization framework with stochastic income and inflation risks. The member has access to a financial market consisting of a risk-free asset (money account), an inflation-indexed bond, and a stock. The market price of volatility risk is assumed to depend on an affine-form, Markovian, square-root factor process, while the return rate and the volatility of the stock are possibly given by general non-Markovian, unbounded stochastic processes. This financial framework recovers the Black-Scholes model, constant elasticity of variance (CEV) model, Heston model, 3/2 model, 4/2 model, and some non-Markovian models as exceptional cases. To tackle the potentially non-Markovian structures, we adopt a backward stochastic differential equation (BSDE) approach. By solving the associated BSDEs explicitly, closed-form expressions for the optimal investment strategies and optimal value functions are obtained for the power, logarithmic, and exponential utility functions. Moreover, explicit solutions to some special cases of our portfolio model are provided. Finally, numerical examples are provided to illustrate the effects of model parameters on the optimal investment strategies under the 4/2 model.

- **Optimal investment strategies for asset-liability management with affine diffusion factor processes and HARA preferences.** This paper investigates an optimal asset-liability management problem within the expected utility maximization framework. The general hyperbolic absolute risk aversion (HARA) utility is adopted to describe the risk preference of the asset-liability manager. The financial market comprises a risk-free asset and a risky asset. The market price of risk depends on an affine diffusion factor process, which includes, but is not limited to, the constant elasticity of variance (CEV), Stein-Stein, Schöbel and Zhu, Heston, 3/2, 4/2 models, and some non-Markovian models, as exceptional examples. The accumulative liability process is featured by a generalized drifted Brownian motion with possibly unbounded and non-Markovian drift and diffusion coefficients. Due to the sophisticated structure of HARA utility and the non-Markovian framework of the incomplete financial market, a backward stochastic differential equation (BSDE) approach is adopted. By solving a recursively coupled BSDE system, closed-form expressions for both the optimal investment strategy and optimal value function are derived. Moreover, explicit solutions to some particular cases of our model are provided. Finally, numerical examples are presented to illustrate the effect of model parameters on the optimal investment strategies in several particular cases.

- **Robust optimal asset-liability management under square-root factor processes and model ambiguity: a BSDE approach.** This paper studies robust optimal asset-liability management problems for an ambiguity-averse manager in a possibly non-Markovian environment with stochastic investment opportunities. The manager has access to one risk-free asset and one risky asset in a financial market. The market price of risk relies on a stochastic factor process satisfying an affine-form, square-root, Markovian model, whereas the risky asset's return rate and volatility are potentially given by general non-Markovian, unbounded stochastic processes. This financial framework includes, but is not limited to, the constant elasticity of variance (CEV) model, the family of 4/2 stochastic volatility models, and some path-dependent non-Markovian models, as exceptional cases. As opposed to most of the papers using the Hamilton-Jacobi-Bellman-Issacs (HJBI) equation to deal with model ambiguity in the Markovian cases, we address the non-Markovian case by proposing a backward stochastic differential equation (BSDE) approach. By solving the associated BSDEs explicitly, we derive, in closed form, the robust optimal controls and robust optimal value functions for power and exponential utility, respectively. In addition, analytical solutions to some particular cases of our model are provided. Finally, the effects of model ambiguity and market parameters on the robust optimal investment strategies are illustrated under the CEV model and 4/2 model with numerical examples.

- **Non-zero-sum stochastic differential games for asset-liability management with stochastic inflation and stochastic volatility.** This paper investigates the optimal asset-liability management problems for two managers subject to relative performance concerns in the presence of stochastic inflation and stochastic volatility. The objective of the two managers is to maximize the expected utility of their relative terminal surplus with respect to that of their competitor. The problem of finding the optimal investment strategies for both managers is modeled as a non-zero-sum stochastic differential game. Both managers have access to a financial market consisting of a risk-free asset, a risky asset, and an inflation-linked index bond. The risky asset's price process and uncontrollable random liabilities are not only affected by the inflation risk but also driven by a general class of stochastic volatility models including the constant elasticity of variance model, the family of state-of-the-art 4/2 models, and some path-dependent models as particular cases. By adopting a backward stochastic differential equation (BSDE) approach to overcome the possibly non-Markovian setting, closed-form expressions for the equilibrium investment strategies and corresponding value functions are derived under power and exponential utility preferences. Moreover, explicit solutions to some special cases of our model are provided. Finally, we perform numerical studies to illustrate the impact of model parameters on the equilibrium strategies and draw some economic interpretations.

Resumé

Dette speciale består af et indledende kapitel og ni forskningsartikler som er skrevet fra januar 2020 til december 2022. Hvert papir er selvstændigt og udgør et kapitel, der undersøger problemer og teknikker omkring dynamiske porteføljeoptimeringsproblemer med tidsvarierende investeringsmuligheder under forskellige markedsscenarier. Indledningen, kapitel 1, giver den videnskabelige baggrund og et overblik over de vigtigste bidrag fra artiklerne og deres sammenhænge. Resuméerne af de efterfølgende kapitler 2–10 er anført nedenfor:

- **Dynamisk optimal middel-varians porteføljevalg med en 3/2 stokastisk volatilitet**. Denne artikel betragter et problem med porteføljeudvælgelse med middel varians, når aktiekursen har en 3/2 stokastisk volatilitet på et komplet marked. Mere konkret antager vi, at aktiekursen og volatiliteten er perfekt negativt korreleret. Ved at anvende en baglæns stokastisk differentialligningstilgang (BSDE) udledes udtryk i lukket form for den statisk optimale (tids-inkonsistente) strategi og værdifunktionen. På grund af tidsinkonsistensen af middelvarianskriteriet præsenteres en dynamisk formulering af problemet. Vi opnår eksplicit den dynamisk optimale (tidskonsistente) strategi, som er vist at holde rigdomsprocessen strengt under målet (forventet terminal rigdom) før terminaltidspunktet. Endelig giver vi numeriske undersøgelser for at vise indvirkningen af hovedmodelparametre på den effektive grænse og illustrere forskellene mellem de to optimale velstandsprocesser.

- **Dynamisk optimal middel-varians porteføljevalg med stokastisk volatilitet og stokastisk rente**. Denne artikel studerer optimale porteføljeudvælgelsesproblemer i nærvær af stokastisk volatilitet og stokastisk rente under middelvarianskriteriet. Det finansielle marked består af et risikofrit aktiv (kontanter), en nulkuponobligation (roll-over-obligation) og et risikabelt aktiv (aktie). Mere konkret antager vi, at renten følger Vasicek-modellen, og det risikable aktivs afkastsats afhænger ikke kun af en Cox-Ingersoll-Ross (CIR) proces, men har også stokastisk kovarians med renten, som favner familien af state-of-the-art 4/2 stokastiske volatilitetsmodeller som et ekstraordinært tilfælde. Ved at anvende en baglæns stokastisk differentialligningstilgang (BS-

DE) og løse to relaterede BSDE'er, udleder vi i lukket form den statiske optimale (tids-inkonsistente) strategi og optimale værdifunktion. Givet tidsinkonsistensen af middelvarianskriteriet, undersøges en dynamisk formulering af problemet yderligere, og det eksplicitte udtryk for den dynamiske optimale (tidskonsistente) strategi udledes. Derudover tilbydes der analytiske løsninger til nogle særlige tilfælde af vores model. Til sidst illustreres modelparametrenes indvirkning på den effektive grænse og adfærden af de statiske og dynamiske optimale aktivallokeringer med numeriske eksempler.

- **Middel-varians aktiv-passivstyring under CIR-rente og familien af 4/2 stokastiske volatilitetsmodeller med derivathandel**. Denne artikel undersøger virkningerne af derivathandel på ydeevnen af aktiv-passivstyring i tilstedeværelsen af stokastisk rente og stokastisk volatilitet under middelvarianskriteriet. Konkret kan aktiv-passivforvalteren investere ikke kun i en pengemarkedskonto, en nulkupon (roll-over) obligation og et aktieindeks, men også i aktiederivater. Det antages, at renten følger en Cox-Ingersoll-Ross (CIR) proces, og aktieindeksets øjeblikkelige varians er styret af familien af 4/2 stokastiske volatilitetsmodeller, som omfatter Heston modellen og 3/2 modellen, som særlige tilfælde. Ved at løse et system af tre bagudrettede stokastiske differentialligninger udledes udtryk i lukket form for de optimale strategier og optimale værdifunktioner i to tilfælde: med og uden aktiederivaterne. Desuden betragter vi de særlige tilfælde uden tilfældige forpligtelser. Der gives numeriske eksempler for at illustrere teoretiske resultater og udforske virkningerne af derivathandel på effektive grænser.

- **Dynamisk optimal middel-variance-investering med fejlpriser i familien af 4/2 stokastiske volatilitetsmodeller**. Denne artikel betragter et optimalt investeringsproblem med fejlprissætning i familien af 4/2 stokastiske volatilitetsmodeller under middelvarianskriteriet. Det finansielle marked består af et risikofrit aktiv, et markedsindeks og et par forkert prissatte aktier. Ved at anvende den lineære–kvadratiske stokastiske kontrolteori og løse den tilsvarende Hamilton–Jacobi–Bellman-ligning, udledes eksplicitte udtryk for den statisk optimale (pre-commitment) strategi og den tilsvarende optimale værdifunktion. Desuden er der givet et nødvendigt verifikationsteorem baseret på en antagelse af modelparametrene med investeringshorisonten. På grund af tidsinkonsistensen under middelvarianskriteriet giver vi en dynamisk formulering af problemet og opnår det lukkede udtryk for den dynamisk optimale (tidskonsistente) strategi. Denne strategi er vist for at holde rigdomsprocessen strengt under målet (forventet terminal rigdom) før terminaltidspunktet. Resultater på den særlige sag uden fejlpriser er inkluderet. Til sidst gives nogle numeriske eksempler for at illustrere effekterne af modelparametre på den effektive grænse og forskellen mellem statisk og dynamisk optimalitet.

- **Maksimering af nytte i en stokastisk affin rente og CIR-risikopræmie-ramme: en BSDE-tilgang**. Denne artikel undersøger optimale investeringsproblemer i nærvær af stokastiske renter og stokastisk volatilitet under det forventede nyttemaksimeringskriterium. Det finansielle marked består af tre aktiver: et risikofrit aktiv, et risikabelt aktiv og nulkuponobligationer (rullende obligationer). Den korte rente antages at følge en affin diffusionsproces, som inkluderer Vasicek og Cox-Ingersoll-Ross (CIR) modellerne, som særlige tilfælde. Risikopræmien for det risikable aktiv afhænger af en kvadratrodsdiffusionsproces (CIR), mens afkastraten og volatilitetskoefficienten er uspecificerede og muligvis givet af ikke-markovske processer. Denne ramme omfatter familien af state-of-the-art 4/2 stokastiske volatilitetsmodeller og nogle ikke-markovske modeller, som ekstraordinære eksempler. Investoren sigter mod at maksimere den forventede nytte af terminalformuen for to typer hjælpefunktioner, elforsyning og logaritmisk nytte. Ved at anvende en baglæns stokastisk differentialligning (BSDE) tilgang til at overvinde den potentielt ikke-markovske ramme og eksplicit løse to BSDE'er, udleder vi, i lukket form, de optimale investeringsstrategier og optimale værdifunktioner. Desuden er der eksplicitte løsninger på nogle særlige tilfælde af vores model. Endelig illustrerer numeriske eksempler vores resultater under ét specifikt tilfælde, hybrid Vasicek-4/2-modellen.
- **Optimal DC pensionsinvestering med kvadratrodsfaktorprocesser under stokastiske indkomst- og inflationsrisici**. Denne artikel studerer optimale pensionsinvesteringsproblemer (DC) under den forventede nyttemaksimeringsramme med stokastiske indkomst- og inflationsrisici. Medlemmet har adgang til et finansielt marked bestående af et risikofrit aktiv (pengekonto), en inflationsindekseret obligation og en aktie. Markedsprisen på volatilitetsrisiko antages at afhænge af en affin-form, markovsk, kvadratrodsfaktorproces, mens afkastraten og volatiliteten af aktien muligvis er givet af generelle ikke-markovske, ubegrænsede stokastiske processer. Denne økonomiske ramme genskaber Black-Scholes-modellen, konstant varianselasticitet (CEV)-modellen, Heston-modellen, 3/2-modellen, 4/2-modellen og nogle ikke-markovske modeller som ekstraordinære tilfælde. For at tackle de potentielt ikke-markovske strukturer anvender vi en baglæns stokastisk differentialligning (BSDE) tilgang. Ved eksplicit at løse de tilknyttede BSDE'er opnås lukkede udtryk for de optimale investeringsstrategier og optimale værdifunktioner for effekt-, logaritmiske og eksponentielle nyttefunktioner. Desuden er der eksplicitte løsninger på nogle særlige tilfælde af vores porteføljemodel. Afslutningsvis gives numeriske eksempler for at illustrere effekterne af modelparametre på de optimale investeringsstrategier under 4/2-modellen.
- **Optimale investeringsstrategier til aktiv-passivstyring med affine diffusionsfaktorprocesser og HARA-præferencer**. Denne artikel undersøger

et optimalt aktiv-passivstyringsproblem inden for den forventede nyttemaksimeringsramme. Den generelle hyperbolske absolutte risikoaversion (HARA) bruges til at beskrive risikopræferencen for aktiv-passiver manager. Det finansielle marked består af et risikofrit aktiv og et risikofyldt aktiv. Markedsprisen for risiko afhænger af en affin diffusionsfaktorproces, som inkluderer, men ikke er begrænset til, den konstante varianselasticitet (CEV), Stein-Stein, Schöbel og Zhu, Heston, 3/2, 4/2-modeller og nogle ikke-markovske modeller, som ekstraordinære eksempler. Den akkumulerede ansvarsproces er karakteriseret ved en generaliseret drift af Brownsk bevægelse med muligvis ubegrænsede og ikke-markovske drift- og diffusionskoefficienter. På grund af den sofistikerede struktur af HARA-nytte- og den ikke-markovske ramme for det ufuldstændige finansielle marked, anvendes en baglæns stokastisk differentialligning (BSDE). Ved at løse et rekursivt koblet BSDE-system udledes udtryk i lukket form for både den optimale investeringsstrategi og optimal værdifunktion. Der gives eksplicitte løsninger på nogle særlige tilfælde af vores model. Til sidst præsenteres numeriske eksempler for at illustrere effekten af modelparametre på de optimale investeringsstrategier i flere særlige tilfælde.

- **Robust optimal aktiv-passivstyring under kvadratrodsfaktorprocesser og modeluklarhed: en BSDE-tilgang**. Denne artikel studerer robuste, optimale aktiv-passivstyringsproblemer for en tvetydighedsvillig forvalter i et muligvis ikke-markovsk miljø med stokastiske investeringsmuligheder. Forvalteren har adgang til ét risikofrit aktiv og ét risikabelt aktiv på et finansielt marked. Markedsprisen for risiko er afhængig af en stokastisk faktorproces, der opfylder en affin-form, kvadratrods, Markovian model, hvorimod det risikable aktivs afkastrate og volatilitet potentielt er givet af generelle ikke-markovske, ubegrænsede stokastiske processer. Denne økonomiske ramme inkluderer, men er ikke begrænset til, modellen for konstant varianselasticitet (CEV), familien af 4/2 stokastiske volatilitetsmodeller og nogle sti-afhængige ikke-Markovske modeller, som undtagelsestilfælde. I modsætning til de fleste artikler, der bruger Hamilton-Jacobi-Bellman-Issacs (HJBI)-ligningen til at håndtere modeluklarhed i de Markovske tilfælde, adresserer vi det ikke-Markovianske tilfælde ved at foreslå en baglæns stokastisk differentialligning (BSDE) tilgang. Ved at løse de tilknyttede BSDE'er eksplicit, udleder vi i lukket form de robuste optimale kontroller og robuste optimalværdifunktioner for henholdsvis kraft og eksponentiel nytte. Derudover tilbydes der analytiske løsninger til nogle særlige tilfælde af vores model. Endelig er effekterne af modeluklarhed og markedsparametre på de robuste optimale investeringsstrategier illustreret under CEV-modellen og 4/2-modellen med numeriske eksempler.

- **Ikke-nul-sum stokastiske differentielle spil til aktiv-passivstyring med stokastisk inflation og stokastisk volatilitet**. Denne artikel undersøger de optimale problemer med aktiv-passivstyring for to forvaltere, der er underlagt relative præstationsbekymringer i nærvær af stokastisk inflation og stokastisk volatilitet. Målet for de to forvaltere er at maksimere den forventede nytte af deres relative terminaloverskud i forhold til deres konkurrents. Problemet med at finde de optimale investeringsstrategier for begge forvaltere er modelleret som et stokastisk differentielle spil, der ikke er nul sum. Begge forvaltere har adgang til et finansielt marked bestående af et risikofrit aktiv, et risikabelt aktiv og en inflationsindekseret indeksobligation. Det risikable aktivs prisproces og ukontrollerbare tilfældige forpligtelser er ikke kun påvirket af inflationsrisikoen, men også drevet af en generel klasse af stokastiske volatilitetsmodeller, herunder modellen med konstant varianselasticitet, familien af avancerede 4/2-modeller, og nogle sti-afhængige modeller som særlige tilfælde. Ved at anvende en baglæns stokastisk differentialligning (BSDE) tilgang til at overvinde den muligvis ikke-markovske indstilling, udledes udtryk i lukket form for ligevægtsinvesteringsstrategier og tilsvarende værdifunktioner under magt og eksponentielle nyttepræferencer. Desuden er der eksplicitte løsninger på nogle særlige tilfælde af vores model. Til sidst udfører vi numeriske undersøgelser for at illustrere effekten af modelparametre på ligevægtsstrategierne og drage nogle økonomiske fortolkninger.

Contents

Chapter 1

Introduction

This introductory chapter serves as a prelude to a series of investigations that are mainly concerned with the problems and techniques within the mathematics of dynamic portfolio optimization. Section 1.1 briefly reviews the background and existing studies related to the classical single-agent portfolio optimization problems under both the expected utility maximization and mean-variance criteria. In Section 1.2, the seminal studies and results in the field of robust portfolio optimization under model ambiguity are reviewed. Section 1.3 presents the most relevant works on the multi-agent portfolio optimization problems taking into account relative performance concerns. In Section 1.4, some essential methodologies that are commonly adopted to address dynamic portfolio optimization problems are summarised, including the dynamic programming approach, martingale approach, and backward stochastic differential equation (BSDE) approach. Finally, the introduction provides an outline of the thesis and the interconnections among the reminder chapters.

1.1 Portfolio optimization problems

This section introduces a brief background on the classical single-agent portfolio optimization problems over a finite horizon to either maximize the expected integrated utility of running consumption and terminal wealth or minimize the variance of terminal wealth given a specific expected return.

The earliest attempt to analyze portfolio selection problems in a quantitative way appears to be Markowitz (1952), in which the agent was concerned with the trade-off between the profit (expected return) maximization and risk (variance) minimization in a single period. Although the pioneering work of Markowitz laid the foundation of modern portfolio optimization theory, the results of Markowitz's work, limited by the single-period setting, lead to myopic investment strategies and fail to account for the optimal investment behavior with time-varying opportunities within long-term investment problems. Around two decades later, Samuelson (1969) and

Hakansson (1975) formulated and solved the multi-period investment-consumption problems under the framework of expected utility maximization, where the agents' risk preferences were described by constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) utilities. Due to the variance operator within the objective function under the mean-variance criterion precluding the use of Bellman's principle of optimality, Markowitz's results were not generalized to the multi-period case until the seminal work of Li and Ng (2000), where an embedding technique was first proposed to reduce the mean-variance problem to the resolution of an auxiliary control problem.

To tackle the constantly dynamic financial markets and to react to new information immediately, portfolio optimization problems in a continuous-time setting have also been extensively investigated by quite a few scholars, among which Merton (1969, 1971) initiated research on continuous-time portfolio optimization under the framework of expected utility maximization using the dynamic programming approach and solving the associated Hamilton-Jacobi-Bellman (HJB) equations. The dynamic programming approach, however, entails the Markovian structures of state variable processes. Pliska (1986) and Karatzas, Lehoczky, and Shreve (1987) solved continuous-time consumption/portfolio optimization problems in a more generally complete but non-Markovian market setting by proposing the martingale approach. In short, this approach decomposes a dynamic portfolio optimization problem into a more tractable static optimization problem and a financial replication problem. From a mathematical point of view, the martingale approach essentially hinges on the uniqueness of the risk-neutral measure (market completeness) and the martingale representation theorem for determining the attainable optimal terminal wealth and the associated replication strategy, respectively. In other words, the martingale approach falls apart in generally incomplete markets where there exist infinitely many risk-neutral measures and some contingent claims cannot be hedged against by using the underlying assets solely. To extend the usage of the martingale approach to incomplete market settings, Karatzas et al. (1991) proposed the fictitious completion method by introducing additional fictitious assets into the original incomplete market and making them unfavorable to the agent. Nevertheless, finding such fictitious assets is not straightforward and might be computationally intensive. El Karoui, Peng, and Quenez (1997) opted for the BSDE (Pardoux and Peng (1990)) approach to portfolio optimization problems in an incomplete and non-Markovian market setting, where the solution to a BSDE with the concave generator was represented as the objective function of the portfolio optimization problem and the comparison theorem for BSDEs played a key role in the determinization of the optimal control and value function. Hu, Imkeller, and Müller (2005) adopted an alternative BSDE approach to El Karoui, Peng, and Quenez (1997) to solve portfolio optimization problems, where the idea of this BSDE approach for determining the value function and the associated optimal control is to construct a family of stochastic processes depending on the admissible strategies in a way that their values at time zero do not depend on any admissible control and their terminal values coincide with the utility of the agent's terminal wealth. For all the admissible strategies, the stochastic processes are (local) super-martingale, while there exists one particular strategy such that it is a (local) martingale.

In the context of mean-variance theory, following a similar embedding technique pioneered by Li and Ng (2000) for discrete-time models, Zhou and Li (2000) transferred the continuous-time mean-variance portfolio optimization problem in a complete market setting into a tractable stochastic linear-quadratic control problem. The latter was solved explicitly premised on deterministic coefficients. Besides the embedding technique, Li, Zhou, and Lim (2002) proposed the Lagrange dual method to investigate a continuous-time mean-variance problem with no-shorting constraint upon noticing that the mean-variance problem is a convex optimization problem with a linear constraint which can be dealt with by introducing a Lagrange multiplier. To extend the results of Zhou and Li (2000) to a more general model with random market coefficients, Lim and Zhou (2002) used the Lagrange dual method and the standard results of existence and uniqueness of linear BSDEs with uniformly Lipschitz continuity (El Karoui, Peng, and Quenez (1997) and Yong and Zhou (1998)) to derive the efficient frontier and optimal investment strategy which are expressed in terms of the solutions to a backward stochastic Riccati equation (BSRE) and a standard linear BSDE. Owing to the complete market setting and uniformly bounded market coefficients, Lim and Zhou (2002) showed that the global solvability of the BSRE can be addressed via an auxiliary linear BSDE. Lim (2004) adopted a similar approach to Lim and Zhou (2002) to study a continuous-time mean-variance problem in a more complicated incomplete market. The proof of the solvability of the BSRE used the results of BSDEs with quadratic growth (Lepeltier and Martin (1998) and Kobylanski (2000)). However, since the coefficients of the linear BSDE in Lim (2004) were only square integrable rather than uniformly bounded, the existence and uniqueness results were not evident and an intricate variance optimal martingale measure was adopted to finish the proof. Recently, Shen (2015) generalized the results of Lim and Zhou (2002) to the case with unbounded market coefficients in a complete market. The global solvability of the associated BSRE and linear BSDE were proved by imposing an exponential integrability assumption on the market price of risk, which was a sufficient condition to define a class of equivalent probability measures and ensure that the solutions to the associated BSDEs belong to proper spaces. Lv, Wu, and Yu (2016) investigated a continuous-time mean-variance portfolio selection problem in an incomplete market with uniformly bounded market coefficients and an uncertain investment horizon. By virtue of the martingales of bounded mean oscillation (Kazamaki (1994)) and Girsanov's measure change techniques, the existence results of the associated BSRE and linear BSDE were determined. Moreover, the proof of the uniqueness of the solution to the linear BSDE was completed with the help of the results on BSDEs with stochastic Lipschitz condition (Briand and Confortola (2008)). Another development for continuous-time mean-variance portfolio selection problems that has been receiving much attention in recent years is the time-inconsistent control (Strotz (1956)), which means that the optimal control derived at the initial time might not be optimal at a future time point due to the non-separability of the variance operator under the mean-variance criterion in the sense of Bellman's optimality principle. In other words, once the agent arrives at any new position at a future time, the optimal strategy determined at the new position is inconsistent with the initial one unless the agent commits himself/herself to the initial strategy over the whole investment period. As such, the time-inconsistent strategy is also referred to as the pre-committed strategy in the literature. To tackle the time inconsistency, Basak and Chabakauri (2010) derived a time-consistent strategy that is determined by applying a backward recursion starting from the terminal date. Within a reasonably general Markovian framework, Björk, Khapko, and Murgoci (2017) developed a game theoretical approach that essentially studies the subgame-perfect Nash equilibrium, and they derived the equilibrium strategy and equilibrium value function by solving an extended HJB equation. Lately, Pedersen and Peskir (2017) introduced the dynamically optimal approach to investigate the time inconsistency of mean-variance problems. They overcame the time inconsistency by recomputing the statically optimal (pre-committed) strategy during the investment period, and they can, therefore, obtain dynamically optimal (time-consistent) strategies by solving infinitely many optimization problems.

It is widely accepted that the volatility of stock returns displays a stochastic fashion rather than a constant or deterministic one. Empirical studies on equity market data reveal many stylized facts including fat tails, the leverage effect, volatility clustering, and volatility smile/skew, which cannot be explained by the Black-Scholes model (Black and Scholes (1973)). Over the last few decades, various stochastic (local) volatility models have been developed to interpret the phenomena observed in the market. See, for example, French, Schwert, and Stambaugh (1987), Wiggins (1987), Hull and White (1987), Stein and Stein (1991), Heston (1993), Schöbel and Zhu (1999), and Grasselli (2017). Besides volatility risk, changing interest rates also constitute one of the major risk sources, and modeling the term-structure movements of interest rates is deemed a challenging but crucial task. The rigorous treatment of the term-structure models stemmed from the seminal work of Vasicek (1977), where the short rates of interest were characterized by an Ornstein-Uhlenbeck (OU) model. Cox, Ingersoll, and Ross (1985) (hereafter called CIR) adopted an affine-form, square-root process to describe the evolution of the short rates of interest. Contrarily to the Vasicek model, the CIR specification precludes negative interest rates while preserving tractability. Duffie and Kan (1996) stepped further by introducing a multi-factor affine-form diffusion equation to track the evolution of bond prices. Duffee (2002) proposed a broader class of "essentially affine" models which not only retained the analytical tractability of completely affine models but also allowed compensation for interest rate risk to vary independently of interest rate volatility.

The early research of stochastic volatility and stochastic interest rate models focused on derivative pricing problems. In recent years, there has been emerging interest in portfolio optimization problems under various stochastic environments. Under the framework of expected utility maximization, Kraft (2005) provided an explicit solution for a CRRA utility maximizer under the Heston model (Heston (1993)) by imposing a specific condition on the model parameters and solving the associated HJB equation. Chacko and Viceira (2005) investigated an investment-consumption problem under the 3/2 model (Lewis (2000)), where an explicit solution and an approximation solution were derived for CRRA utility and generally recursive utility, respectively. Liu (2007) solved a CRRA utility maximization problem when the asset returns were (non-affine) quadratic, up to the solution to an ordinary differential equation (ODE). With the help of a martingale criterion, Kallsen and Muhle-Karbe (2010) derived explicit solutions for a CRRA utility maximization problem in a number of affine-form stochastic volatility models. Jung and Kim (2012) considered an optimal investment problem under the constant elasticity of variance (CEV) model and hyperbolic absolute risk aversion (HARA) utility. Zeng and Taksar (2013) studied an optimal investment problem under CRRA utility and a general stochastic volatility model in a Markovian setting, and closed-form solutions for the Heston model were derived by solving the associated HJB equation under a more relaxed assumption than that in Kraft (2005). Cheng and Escobar (2021a) investigated an optimal investment problem for a CRRA agent under the family of state-of-the-art 4/2 stochastic volatility models (Grasselli (2017)) in both complete and incomplete markets and obtained closed-form solutions by solving the corresponding HJB equations. By using the martingale approach, Bajeux-Besnainou, Jordan, and Portait (2003) and Deelstra, Grasselli, and Koehl (2000) considered CRRA utility maximization problems under the Vasicek model and CIR model, respectively. Grasselli (2003) studied a more complicated HARA utility maximization problem under the CIR interest rate and derived closed-form solutions by applying the dynamic programming approach. Assuming that the stock price and its volatility were perfectly correlated, Li and Wu (2009) investigated CRRA utility maximization problems in a hybrid CIR-Heston framework. Escobar, Neykova, and Zagst (2017) studied a HARA utility maximization problem in a Markov-switching bond-stock market. Chang et al. (2020) considered an optimal asset allocation problem for a defined contribution plan member with a stochastic affine interest rate and mean-reverting returns under the HARA preference. Recently, Zhang (2022f) solved a utility maximization problem with a stochastic interest rate model and volatility risk using the BSDE approach, where the short rate of interest followed an affine diffusion process established by Duffie and Kan (1996) and the market price of risk was described by an affine-form, square-root factor process. In a similar modeling framework, Zhang (2022c) investigated a defined contribution (DC) pension investment problem with stochastic volatility and stochastic inflation and obtained explicit solutions for CRRA and CARA utility. Zhang (2022d) considered an asset-liability management (ALM) problem for a HARA utility maximizer in a non-Markovian market setting, where the return rate and volatility of the risky asset were potentially non-Markovian path-dependent processes, while the market price of risk was governed by an affine diffusion process. As the literature on utility maximization problems with various stochastic investment opportunities is abundant, the above review is not exclusive. For more literature on continuous-time utility maximization problems, one may refer to Zariphipoulou (2001), Pham (2002), Benth and Karlsen (2005), Liang, Yuen, and Guo (2011), Shen and Siu (2012), Zhao and Rong (2012), Kraft, Seifried, and Steffensen (2013), Guan and Liang (2014), Pan and Xiao (2017a,b), Xing (2017), Pan, Hu, and Zhou (2019), Ma, Zhao, and Rong (2020), and references therein.

Under Markowitz's mean-variance paradigm, Černý and Kallsen (2008) studied an optimal investment and hedging problem under the Heston model by using the martingale approach. Ferland and Watier (2010) considered a portfolio selection problem in a complete market under an extended CIR interest rate model. By means of BSDEs, Shen, Zhang, and Siu (2014) investigated a portfolio selection problem under the CEV model premised on a sufficient condition on the market price of risk, where the generator of the associated BSDE satisfied the stochastic Lipschitz condition established by Bender and Kohlmann (2000). Shen and Zeng (2015) further studied an optimal investment-reinsurance problem for a mean-variance insurer in an incomplete market, where the market price of risk is proportional to a Markovian, affine-form, and square-root factor process. The modeling framework recovered the CEV model, Heston model, and some non-Markovian path-dependent models, as particular cases. Zhang and Chen (2016) considered an ALM problem with multiple risky assets under the CEV model, and the solutions were expressed in terms of the solutions to two BSDEs. Li, Shen, and Zeng (2018) stepped further by incorporating derivative trading into an ALM problem under the Heston model. Sun, Zhang, and Yuen (2020) investigated an ALM problem in a complete market setting with multiple risky assets and reinsurance options, where the variance processes of the risky assets followed an affine diffusion equation. Using a similar technique, Tian, Guo, and Sun (2021) considered an optimal investment-reinsurance problem, where the return rate of the risky asset was described by an OU process. Zhang (2023) investigated a derivative-based ALM problem in the presence of both stochastic interest rates and stochastic volatility, where the interest rate and risky asset's volatility were driven by the CIR model and 4/2 stochastic volatility model, respectively. It is worth mentioning that the optimal strategies derived in the above-mentioned literature are pre-committed not time-consistent. Other previous works along this line include Chiu and Wong (2014a), Chang (2015), Pan and Xiao (2017c), Sun and Guo (2018), Pan, Zhang, and Zhou (2018), Shen, Wei, and Zhao (2020), to name but a few. Following the game theoretical approach pioneered by Björk, Khapko, and Murgoci (2017), Li, Rong, and Zhao (2015) considered a time-consistent reinsurance-investment problem under stochastic interest rate and stochastic inflation. Li, Zeng, and Lai (2012) and Lin and Qian (2016) investigated time-consistent reinsurance-investment problems under the Heston model and CEV model, respectively. Zhu and Li (2020) studied a time-consistent reinsurance-investment problem with stochastic interest rates and stochastic volatility. Recently, within the framework developed by Pedersen and Peskir (2017), Zhang (2021b,a, 2022a) derived the dynamically optimal strategies explicitly under the 3/2 model, 4/2 model with mispricing phenomenon, and a hybrid model with Vasicek interest rates and general stochastic volatility, respectively. For more literature about continuous-time mean-variance portfolio selection under different types of constraints and scenarios, readers may refer to the review paper by Zhang, Li, and Guo (2018).

1.2 Robust portfolio optimization problems with model ambiguity

In the preceding literature, the risk-averse agents are assumed to know the probability distributions of all relevant random quantities and fully trust the formulated models. As pointed out by Merton (1980) and Cochrane (1997), however, economic agents are indeed not confident in the formulated models due to the lack of complete information or the difficulty to estimate model parameters with precision. Moreover, following the early ideas originated from Knight (1921), the empirical studies of Ellsberg (1961) and Bossaerts et al. (2010) demonstrated that economic agents display aversion not only to risk but also to ambiguity (unknown probability distribution). In this sense, incorporating model ambiguity into traditional portfolio optimization problems is plausible. Great developments have been made in dealing with model ambiguity in recent years. Among them, Andersen, Hansen, and Sargent (2000) proposed the penalty-based robust control approach, behind which the fundamental idea is that the ambiguity-averse agent takes the formulated reference model as an approximation to the unknown true model and believes that the true model belongs to a family of adverse models which do not deviate from the reference model too much, and a penalty term, penalizing the situation where the agent accepts an improper alternative model far away from the reference model, is embedded into the objective function. Within this framework, the economic agent seeks the robust optimal investment strategy to optimize the penalized objective function measured in the worst-case scenario. Andersen, Hansen, and Sargent (2003) further formulated the optimal investment problems under continuous-time Markovian models with jump and diffusion components, provided the associated HJB equation, and solved it explicitly. Maenhout (2004) refined the penalty-based robust control approach by introducing the notion of homothetic robustness and investigated the impact of model ambiguity on the optimal investment-consumption problems in the setting of constant investment opportunity. Uppal and Wang (2003) extended the work of Maenhout (2004) to the case with different levels of ambiguity aversion about state variables. Maenhout (2006) defined a utility loss function measuring the influence of model ambiguity with stochastic investment opportunities, where a mean-reverting process described the expected return rate of the risky asset.

The most common treatment for robust control problems with penalization is to reformulate the original problems in terms of zero-sum, stochastic differential games between the risk- and ambiguity-averse agent and the market, where the economic agent aims at maximizing (minimizing) the value function by opting for an investment strategy, while the market acts adversely by choosing a real-world probability measure to minimize (maximize) the value function in the meantime. For more details on the game-theoretic formulation, readers may refer to Mataramvura and Øksendal (2008). Owing to the mathematical tractability and the consistency with economic intuition under the penalty-based robust control approach, there has been emerging literature on robust portfolio optimization problems under various market settings. For example, Liu (2010) studied a robust investment-investment problem under recursive utility. Yi et al. (2013) considered a robust optimal reinsurance-investment problem under the Heston model. Flor and Larsen (2014) investigated a robust investment problem with stochastic interest rates described by the Vasicek model. Munk and Rubtsov (2014) extended the work of Flor and Larsen (2014) to the case with both stochastic interest rates and stochastic inflation. Escobar, Ferrando, and Rubtsov (2015) discussed the effect of derivative trading on the robust portfolio optimization problem with stochastic volatility and jump risk. Zheng, Zhou, and Sun (2016) and Gu, Viens, and Yi (2017) investigated robust reinsurance-investment problems under the CEV model and with mispricing, respectively. Zeng et al. (2018) considered a robust derivative-based pension investment problem with stochastic income and Heston's stochastic volatility. Wang and Li (2018) discussed a robust DC pension investment problem with affine interest rates and Heston's stochastic volatility. Yang et al. (2020) investigated a robust portfolio optimization problem with multi-factor stochastic volatility. Chang, Li, and Zhao (2022) studied a robust mean-variance DC pension investment problem under the Heston model. By disentangling a BSDE approach, Zhang (2022e) investigated robust ALM problems for CRRA and CARA utility in a potentially non-Markovian market featured by a general stochastic volatility model including the CEV model, the family of state-of-the-art 4/2 model, and some path-dependent models, as exceptional cases. The above review is not exhaustive as the literature on the robust control problems under the penalty-based robust control framework is plentiful. For more recent work along this line, one may refer to Cheng and Escobar (2021b), Yuan and Mi (2022b), Chen, Huang, and Li (2022), Baltas et al. (2022), Wei, Yang, and Zhuang (2023), to name but a few.

1.3 Multi-agent portfolio optimization problems with relative performance concerns

The majority of the above-mentioned studies do not consider the strategic interaction, i.e., competition, among economic agents. However, as documented by a large literature, such as Abel (1990), Gali (1994), DeMarzo, Kaniel, and Kremer (2008), and Gomez (2009), relative performance concerns play a key role in explaining various financial and economic phenomena in a competitive market. In recent years, great advances have been achieved in dealing with optimal investment under relative performance concerns within the framework with multi-agents. Among them, two strands of literature, Basak and Makarov (2014) and Espinosa and Touzi (2015), are noteworthy, for they not only pioneered the studies on dynamic portfolio selection with relative performance concerns in continuous time and analyzed the problems under a non-zero-sum stochastic differential game formulation in the sense of Issacs (1965), among others but also proposed approaches to incorporating relative performance concerns into optimal investment from different angles. More specifically, Basak and Makarov (2014) focused primarily on a stochastic differential game among money managers in a canonical Merton's setting, i.e., the return rate and volatility of risky assets were constants, and the utility function for each manager was an average over his/her own terminal wealth and his/her relative wealth aggregated via a constant elasticity Cobb-Douglas function. The Nash equilibrium optimal position was derived by employing a martingale approach owing to the complete market setting. Alternatively, Espinosa and Touzi (2015) investigated a complete market situation where the economic agents were heterogeneous in terms of utility functions as well as liquidity constraints sets, and instead of considering only his/her absolute wealth, each agent cared about a convex combination of his/her wealth and the difference between his/her wealth and the average wealth of their peers via an interaction coefficient. By using the BSDE technique developed by Hu, Imkeller, and Müller (2005), they provided general conditions for the existence and uniqueness of the Nash equilibrium for the cases of unconstrained and constrained agents with exponential utilities within a Black-Scholes market setting. The results of both Basak and Makarov (2014) and Espinosa and Touzi (2015) show that the competitive agents have tendencies to increase the weight of risky assets in their portfolios as they are more concerned about their peers' performances and quantify the impact of interaction coefficients on the investment decisions of agents.

Recently, there is growing interest in continuous-time portfolio optimization problems with relative performance concerns under various market settings. For instance, following the framework of Basak and Makarov (2014), Guan and Liang (2016) investigated a non-zero-sum stochastic differential game between two DC pension funds with constant investment opportunities and inflation risk. Kraft, Meyer-Wehmann, and Seifried (2020) went beyond constant investment opportunities and considered a dynamic asset allocation problem with relative wealth concerns in incomplete markets with unhedgeable stochastic volatility. In the case of heterogeneous agents, i.e., agents have different levels of risk aversion, solutions were derived up to solving a system of ODEs. In a setting with homogeneous agents, explicit solutions were obtained to the problem. Different from the conclusion drawn by Basak and Makarov (2014) that relative wealth concerns only gave rise to additional myopic demand for risky assets, the work of Kraft, Meyer-Wehmann, and Seifried (2020) showed that relative wealth concerns lead to new hedge terms beyond the usual Merton-Breeden terms as well. Following the modeling framework of Espinosa and Touzi (2015), Bensoussan et al. (2014) studied a class of non-zero-sum stochastic differential reinsurance and investment games between two insurance companies whose surplus processes were modulated by continuous-time Markov chains. Kwok, Chiu, and Wong (2016) explored the impact of relative performance concerns on the longevity risk transfer market in the presence of stochastic interest and mortality rates. Deng, Zeng, and Zhu (2018) considered a non-zero-sum stochastic differential investment and reinsurance game with default risk and Heston's stochastic volatility. Under Markowitz's mean-variance criterion, Hu and Wang (2018) derived the optimal time-consistent investment and reinsurance strategies for two mean-variance insurance managers with relative performance concerns in a Black-Scholes market. Zhu, Cao, and Zhang (2019) and Zhu, Cao, and Zhu (2021) extended the results of Hu and Wang (2018) to the cases with the Heston model and CEV model, respectively. For more previous works along this line, readers may refer to Meng, Li, and Jin (2015), Pun and Wong (2016), Dong, Rong, and Zhao (2022), Savku and Weber (2022), among others.

It is worth mentioning that the majority of the preceding literature on portfolio optimization problems with relative performance concerns was studied under Markovian market settings. Hence, the Nash equilibrium can be essentially constructed as a solution to a system of HJB equations thanks to the dynamic programming principle (Mataramvura and Øksendal (2008)). In the recent work of Zhang (2022b), non-zero-sum stochastic differential games for ALM in incomplete markets with inflation and volatility risks for CARA and CRRA utility were considered, where the two heterogeneous asset-liability managers aimed at maximizing the expected utility of their relative terminal surplus with respect to that of their competitor in the sense of Espinosa and Touzi (2015). More importantly, inspired by the BSDE technique proposed by Hu, Imkeller, and Müller (2005), the author overcame the potentially non-Markovian environment induced by the path dependence of the return rate and volatility of the risky asset. Closed-form solutions to the Nash equilibrium and optimal value functions were derived for a general class of stochastic (local) volatility models including the CEV model, Heston model, 3/2 model, 4/2 model, and some path-dependent stochastic volatility models, as exceptional cases.

1.4 Methodologies

This subsection summarizes the key methodologies adopted to solving dynamic portfolio optimization problems. We convey the standard ideas of the dynamic programming approach, martingale approach, and BSDE approach, provide their basic procedures, and present a concise comparison among the three approaches. As the optimal investment strategy under the mean-variance criterion is ingeniously related to that under Merton's expected utility maximization paradigm via Lagrange multipliers, in what follows we only consider the context of continuous-time portfolio optimization within the framework of expected utility maximization.

Let $T \in \mathbb{R}^+$ be a fixed and finite constant describing the decision-making horizon for an economic agent. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a one-dimensional Brownian motion $\{W_t\}_{t\in[0,T]}$. The filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is the completion of the filtration generated by W_t. The economic agent has access to a financial market consisting of a risk-free asset (money account) and a risky asset (stock). The dynamics of the risk-free asset B_t are given by

$$dB_t = rB_t\,dt,\ B_0 = 1,$$

where the constant $r \in \mathbb{R}$ stands for the risk-free short rate of interest. The price process of the risky asset S_t is governed by the following stochastic differential equation (SDE):

$$dS_t = S_t\left(\mu_t\,dt + \sigma_t\,dW_t\right),\ S_0 = s_0 \in \mathbb{R},$$

where μ_t and σ_t are two $\mathbb{R}$-valued progressively measurable processes such that the above SDE of S_t is well-defined. Denote by π_t and X_t^π the proportion of the agents' wealth invested in the risky asset and the controlled wealth associated with π_t, respectively. Suppose that the financial market is frictionless and infinite short-selling and leverage are allowed. Under a self-financing condition, the dynamics of the agent's wealth X_t^π with initial endowment $x_0 \in \mathbb{R}$ read

$$dX_t^\pi = X_t^\pi\left[\left(r + (\mu_t - r)\pi_t\right)\,dt + \sigma_t\pi_t dW_t\right],\ t \in [0,T]. \tag{1.4.1}$$

The task at hand is to choose an admissible strategy $\pi \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of admissible strategies, that maximizes the agent's expected utility with respect to some utility function U from his/her total wealth X_T^π, i.e.,

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U(X_T^\pi)\right]. \tag{1.4.2}$$

1.4.1 Dynamic programming approach

Within a Markovian market setting when the two stochastic processes μ_t and σ_t can be described in a way of some measurable functions on the state variable S_t and t, i.e.,

$$\mu_t = \bar{\mu}(t, S_t), \;\; \sigma_t = \bar{\sigma}(t, S_t). \tag{1.4.3}$$

Then, the classical approach for solving (1.4.2) is that of dynamic programming. As the name suggests, this approach hinges on Bellman's dynamic programming principle and ties the problem (1.4.2) to a special kind of second-order and nonlinear partial differential equation (PDE), called the HJB equation, which is indeed the infinitesimal version of the dynamic programming principle as the result of Itô's formula.

More precisely, denote by V_t the value function to the following problem associated with (1.4.2):

$$V_t = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^{\pi}) \middle| \, \mathcal{F}_t \right].$$

Thanks to the Markov property of the risky asset price process when (1.4.3) holds true, the value function V_t is can be equivalently written as

$$V_t = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^{\pi}) \middle| \, \mathcal{F}_t \right] = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^{\pi}) \middle| \, X_t, S_t \right] = \bar{V}(t, X_t, S_t), \tag{1.4.4}$$

where $\bar{V}$ is some unknown deterministic function, and more importantly, the dynamic programming principle is viable in this context. In other words, for any $0 \le t \le t' \le T$, we have

$$\bar{V}(t, X_t, S_t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\bar{V}(t', X_{t'}^{\pi}, S_{t'}) \right].$$

By further assuming that $\bar{V}$ is a smooth function, an application of Itô's formula leads to the local behavior of $\bar{V}$ which is governed by the following HJB equation:

$$\frac{\partial \bar{V}}{\partial t}(t,x,s) + \frac{\partial \bar{V}}{\partial s}(t,x,s)\bar{\mu}(t,s)s + \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial s^2}(t,x,s)\bar{\sigma}^2(t,s)s^2 + \sup_{\pi}\left\{ \frac{\partial \bar{V}}{\partial x}(t,x,s)(r + (\bar{u}(t,s) \right.$$
$$\left. -r)\pi)x + \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial x^2}(t,x,s)x^2\bar{\sigma}^2(t,s)\pi^2 + \frac{\partial^2 \bar{V}}{\partial x \partial s}(t,x,s)xs\bar{\sigma}^2(t,s)\pi \right\} = 0,$$

with terminal condition $\bar{V}(T, x, s) = U(x)$. By solving the above HJB equation explicitly or showing the existence of a smooth solution by PDE techniques, a candidate solution to the HJB equation denoted by $\tilde{V}(t, x, s)$, along with an optimal feedback control denoted by $\pi^*(t, x, s)$, is derived. Finally, it is indispensable to check the concavity of the candidate solution with respect to x and the admissibility condition of the optimal control so that the candidate solution $\tilde{V}(t, x, s)$ coincides with the value function $\bar{V}(t, x, s)$, which is referred to as the verification theorem in the literature; see, for example, Yong and Zhou (1998) or Pham (2009).

To summarize, the steps for finding the optimal controls and value functions via the dynamic programming approach are as follows:

1. Make sure the control problem is in the Markovian context. If that is the case, derive the HJB equation formally.

2. Assume the value function is concave with respect to the wealth and use the static optimization method to derive the expression of the optimal control from the HJB equation.

3. Substitute the optimal control into the HJB equation and solve the resultant PDE governing the value function explicitly by providing a candidate solution.

4. Verify all the technical conditions so that the candidate solution coincides with the value function.

1.4.2 Martingale approach

The martingale approach stemmed from the seminal works of Pliska (1986) and Karatzas, Lehoczky, and Shreve (1987). In the complete market case, it provides an alternative methodology to the dynamic programming approach. Compared with the dynamic programming approach, the biggest advantage of the martingale approach is that it does not require the Markovian structure of the model. The basic idea is to decompose the dynamic portfolio optimization problem into a static optimization problem and a financial replication problem, from which the optimal terminal wealth and the optimal investment strategy are derived, respectively. Following the setup (1.4.1) and (1.4.2), by imposing the Novikov's condition on the following Radon-Nikodym derivative process L_t:

$$L_t := \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{\mu_s - r}{\sigma_s}\right)^2 ds - \int_0^t \frac{\mu_s - r}{\sigma_s}\, dW_s\right\},$$

the risk-neutral measure $\mathbb{Q}$ is well-defined and equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = L_T.$$

Since the market is complete, the equivalent risk-neutral measure $\mathbb{Q}$ is unique. Moreover, all the attainable terminal wealth X_T^π can be regarded as the payoff of some contingent claims which then must have a price equal to the agent's initial wealth x_0, i.e.,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T^\pi\right] = x_0. \tag{1.4.5}$$

Combining (1.4.1) and (1.4.5) indicates that the agent faces a constrained optimization problem which can be conveniently expressed in terms of the Lagrangian:

$$\mathcal{L} := \mathbb{E}\left[U(X_T^\pi) - \lambda(e^{-rT}L_T X_T^\pi - x_0)\right], \tag{1.4.6}$$

where λ is the Lagrange multiplier. A point-wise maximization within the expectation in (1.4.6) leads to the optimal attainable terminal wealth $X_T^* = (U')^{-1}(\lambda e^{-rT}L_T)$, where $(U')^{-1}$ denotes the inverse of the utility function's first-order derivative and the Lagrange multiplier λ is obtained such that the linear constraint (1.4.5) holds for X_T^*. To find the optimal strategy π^* such that the associated wealth process $X_t^{\pi^*}$ coincides with X_T^*, $\mathbb{P}$ almost surely at the terminal date T, applying Itô's formula to $e^{r(T-t)}X_t^{\pi^*}$ under $\mathbb{Q}$ measure shows that

$$de^{r(T-t)}X_t^{\pi^*} = e^{r(T-t)}X_t^{\pi^*}\sigma_t\pi_t^*\,dW_t^{\mathbb{Q}},$$

where $W_t^{\mathbb{Q}} = \int_0^t \frac{\mu_s - r}{\sigma_s}\,ds + W_t$ is the Brownian motion under $\mathbb{Q}$ due to Girsanov's theorem. Under some integrability condition, we know that $e^{r(T-t)}X_t^{\pi^*}$ is an $(\mathbb{F},\mathbb{Q})$-martingale, and upon considering the terminal condition that $X_T^{\pi^*} = X_T^*$, we find the optimal wealth process is given by

$$X_t^{\pi^*} = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}X_T^* \middle| \mathcal{F}_t \right],$$

from which we further observe that the process $e^{-rt}L_tX_t^{\pi^*} = \mathbb{E}\left[e^{-rT}L_TX_T^* \middle| \mathcal{F}_t \right]$ is an $(\mathbb{F},\mathbb{P})$-martingale. Due to the martingale representation theorem, on one hand, there exists a progressively measurable process ψ_t such that

$$de^{-rt}L_tX_t^{\pi^*} = \psi_t\,dW_t. \tag{1.4.7}$$

On the other hand, an application of Itô's formula yields the dynamics of $e^{-rt}L_tX_t^{\pi^*}$ as follows:

$$de^{-rt}L_tX_t^{\pi^*} = e^{-rt}L_tX_t^{\pi^*}\left(\pi_t^*\sigma_t - \frac{\mu_t - r}{\sigma_t}\right)dW_t. \tag{1.4.8}$$

Comparing the diffusion coefficients of (1.4.7) and (1.4.8) leads to the following expression of the optimal investment strategy π_t^*:

$$\pi_t^* = \left(\frac{\psi_t}{e^{-rt}L_tX_t^{\pi^*}} + \frac{\mu_t - r}{\sigma_t}\right)\frac{1}{\sigma_t}. \tag{1.4.9}$$

In summary, the martingale approach adopted here for determining the value function and the optimal control in the complete market case is based on the following procedure:

1. Compute the optimal attainable terminal wealth $X_T^* = (U')^{-1}(\lambda e^{-rT}L_T)$.
2. Determine the Lagrange multiplier λ from the budget equation $\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T^*\right] = x_0$.
3. Derive the optimal control π_t^* from (1.4.9) and the value function $V_t = \mathbb{E}\left[U(X_T^*)\middle| \mathcal{F}_t\right]$.

It is worth mentioning that the validity of the martingale approach relies on market completeness ensuring the uniqueness of the equivalent risk-neutral measure $\mathbb{Q}$ and that any contingent claims with maturity T which have a price equal to the initial wealth x_0 can be replicated. When the market is incomplete, Karatzas et al. (1991) proposed the fictitious completion method to complete the incomplete market by introducing additional fictitious assets. Hence, in addition to following the above procedures for deriving the optimal control and value function in the fictitious complete market, it is indispensable to determine the market price of risk ultimately so that the fictitious assets are unfavorable to the agent and the solution to the fictitious complete market case coincides with that to the original incomplete market case.

1.4.3 BSDE approach

BSDE is a type of SDEs prescribed by the terminal condition. The theory of linear BSDE was initiated by Bismut (1976) in the context of stochastic linear quadratic control. The problem of the existence and uniqueness of the solutions to nonlinear BSDEs was solved by Pardoux and Peng (1990) under the uniformly Lipschitz condition on the generator, i.e., for the following general form of nonlinear BSDE of (Y_t, Z_t):

$$\begin{cases} dY_t = - f(t, Y_t, Z_t)\, dt + Z_t\, dW_t, \\ Y_T = \xi, \end{cases} \tag{1.4.10}$$

where ξ is a given $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable satisfying $\mathbb{E}|\xi|^2 < +\infty$, the generator $f : \Omega \otimes [0, T] \otimes \mathbb{R} \otimes \mathbb{R} \mapsto \mathbb{R}$ is a progressively measurable function, and there exists a positive constant $k \in \mathbb{R}^+$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq k(|y_1 - y_2| + |z_1 - z_2|), \tag{1.4.11}$$

for all $t \in [0, T]$ and $y_1, y_2, z_1, z_2 \in \mathbb{R}$. To weaken the assumption of uniformly Lipschitz condition (1.4.11), Bender and Kohlmann (2000), among others, proposed a type of nonlinear BSDE with stochastic Lipschitz continuity: there exists two non-negative $\mathbb{F}$-measurable processes $c_{1,t}$ and $c_{2,t}$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c_{1,t}|y_1 - y_2| + c_{2,t}|z_1 - z_2|.$$

By strengthening the integrability conditions on the generator and the terminal value ξ, Bender and Kohlmann (2000) derived the existence and uniqueness results of the solutions in a proper space. Wang, Ran, and Hong (2006) further established the well-posedness of the results of Bender and Kohlmann (2000) in some larger spaces.

Another important weakening of the uniformly Lipschitz continuity (1.4.11) on the generator was given in Kobylanski (2000), which pioneered the studies on the BSDEs whose generator has quadratic growth in the variable z. More precisely, the author assumed that there exists a positive constant $k \in \mathbb{R}^+$ such that

$$|f(t, y, z)| \leq k(1 + |y| + |z^2|),$$

and that there exists a positive constant $k \in \mathbb{R}^+$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq k \left(|y_1 - y_2| + (1 + |y_1| + |y_2| + |z_1| + |z_2|)|z_1 - z_2| \right),$$

for all $t \in [0, T]$ and $y_1, y_2, z_1, z_2 \in \mathbb{R}$. By imposing a uniform boundedness condition on the terminal value ξ, the existence and uniqueness result of the solution to quadratic BSDE was demonstrated in Kobylanski (2000). We should point out that quadratic BSDEs have found wide applicability in the fields of stochastic control and mathematical finance; see, for example, Hu, Imkeller, and Müller (2005), Yu (2013), Lv, Wu, and Yu (2016), to name but a few. Briand and Hu (2006) proved the existence result of the solution to quadratic BSDEs with unbounded terminal value, where the uniform boundedness condition on the terminal value in Kobylanski (2000) was replaced by an exponential integrability condition. However, no uniqueness result was stated in that work since the comparison theorem for this kind of BSDE was not presented. By further assuming the generator is convex in the variable z, i.e., for all $t \in [0, T]$ and $y \in \mathbb{R}$,

$$z \mapsto f(t, y, z) \text{ is convex},$$

Briand and Hu (2008) filled this gap. In recent years, there has been growing interest in developing and studying various types of BSDEs; see, for example, Briand and Carmona (2000), Briand and Confortola (2008), Delbaen, Hu, and Adrien (2011), Fan, Hu, and Tang (2020), and references therein. For a comprehensive textbook reading on the theory of BSDEs, see, for instance, Pham (2009) and Zhang (2017).

In the context of dynamic portfolio optimization, compared with the above-mentioned dynamic programming approach and martingale approach, the BSDE approach does not entail the Markovian structures of state variables as well as the complete market setting. There are numerous papers considering optimal investment using the tools of BSDEs, among which the seminal works of Lim and Zhou (2002) and Hu, Imkeller, and Müller (2005) are the milestones viewed from our perspective.

To put it short, the primary ideas of both Lim and Zhou (2002) and Hu, Imkeller, and Müller (2005) are to reformulate the term within the objective function under either the mean-variance criterion or the expected utility maximization paradigm as the terminal value of an unknown stochastic process. The determination of such stochastic process then leads to several BSDEs with various and specific forms, and usually, the resultant BSDEs would be in the types of backward stochastic Riccati equations (BSREs), quadratic BSDEs, and linear BSDEs. Thus, the optimality of portfolio optimization problems boils down to the solvability of the associated BSDEs. Once the existence and uniqueness results can be established, the optimal investment strategy along with the value function can be obtained via the martingale optimality principle. We follow the above setup (1.4.1) and (1.4.2) to illustrate this idea. Construct a family of stochastic processes denoted by M_t^π, $\pi \in \mathcal{A}$, such that

- the terminal value $M_T^\pi = U(X_T^\pi)$, for all $\pi \in \mathcal{A}$;
- the initial value at time zero M_0^π is independent of all $\pi \in \mathcal{A}$;
- M_t^π is an $(\mathbb{F}, \mathbb{P})$-supermartingale for all $\pi \in \mathcal{A}$, and there exists a $\pi^* \in \mathcal{A}$ such that $M_t^{\pi^*}$ is an $(\mathbb{F}, \mathbb{P})$-martingale.

The above argument immediately shows that π^* is the optimal investment strategy and $M_0^{\pi^*}$ is the value function of problem (1.4.2) since

$$\mathbb{E}\left[U(X_T^\pi)\right] = \mathbb{E}\left[M_T^\pi\right] \leq M_0^\pi = M_0^{\pi^*} = \mathbb{E}\left[M_T^{\pi^*}\right] = \mathbb{E}\left[U(X_T^{\pi^*})\right].$$

The determination of M_t^π normally hinges on the specific form of the utility function and we can construct M_t^π in the form

$$M_t^\pi = U(Y_{1,t} X_t^\pi + Y_{2,t}), \tag{1.4.12}$$

where $Y_{1,t}$ and $Y_{2,t}$ are the first components of the solutions to the following two uncontrollable BSDEs of $(Y_{1,t}, Z_{1,t})$ and $(Y_{2,t}, Z_{2,t})$, respectively:

$$\begin{cases} dY_{1,t} = - f_1(t, Y_{1,t}, Z_{1,t})\, dt + Z_{1,t}\, dW_t, \\ Y_{1,T} = 1, \end{cases} \tag{1.4.13}$$

and

$$\begin{cases} dY_{2,t} = - f_2(t, Y_{2,t}, Z_{2,t})\, dt + Z_{2,t}\, dW_t, \\ Y_{2,T} = 0. \end{cases} \tag{1.4.14}$$

The expressions of generators f_1 and f_2 can be obtained by using Itô's formula to M_t^π. By solving BSDEs (1.4.13) and (1.4.14) explicitly or at least establishing their existence and uniqueness results, the value function of problem (1.4.2) turns out to be $M_0^\pi = M_0^{\pi^*} = U(Y_{1,0} x_0 + Y_{2,0})$ and the expression of the optimal strategy can be found from the dynamics of M_t^π. To summarize, the basic procedure for finding the optimal controls and value functions via the BSDE approach is as follows:

1. Define an auxiliary process M_t^π for all $\pi \in \mathcal{A}$ by (1.4.12), where the dynamics of $Y_{1,t}$ and $Y_{2,t}$ are given by (1.4.13) and (1.4.14) and the expressions for generators f_1 and f_2 are not given at this step.
2. Apply Itô's formula to M_t^π and reformulate the drift terms of dM_t^π in a way that f_1 and f_2 are independent of all $\pi \in \mathcal{A}$.
3. Substitute the specific forms of f_1 and f_2 determined from the last step into (1.4.13) and (1.4.14) and solve the resultant BSDEs (1.4.13) and (1.4.14) explicitly or prove their existence and uniqueness results.
4. Represent the value function in terms of the solutions to BSDEs (1.4.13) and (1.4.14), i.e., $V_0 = U(Y_{1,0} x_0 + Y_{2,0})$, and derive the optimal investment strategy π^* from the drift coefficients of dM_t^π.

We conclude this subsection by providing the following table which exhibits the principles and applicable scenarios of the above-mentioned three approaches to solving dynamic portfolio optimization problems:

		Market scenarios			
Approaches	**Principles**	Complete	Incomplete	Markovian	Non-Markovian
Dynamic programming approach	Bellman's dynamic programming principle	✓	✓	✓	✗
Martingale approach (Pliska (1986) and Karatzas, Lehoczky, and Shreve (1987))	Uniqueness of the risk-neutral measure and martingale representation theorem	✓	✗ (✓ with fictitious completion by Karatzas et al. (1991))	✓	✓
BSDE approach (Lim and Zhou (2002) and Hu, Imkeller, and Müller (2005))	Martingale optimality principle	✓	✓	✓	✓

Table 1.1: *Methodologies for solving dynamic portfolio optimization problems under different market scenarios*

1.4.4 Outline of thesis

This thesis contains an introduction and nine self-contained research papers. Chapters 2-5 are concerned with the single-agent portfolio optimization problems under Markowitz's mean-variance criteria in the presence of 3/2 stochastic volatility in a complete market, a hybrid model of stochastic interest rates and stochastic volatility in an incomplete market, uncontrollable random liabilities and the opportunities of derivative trading, and the family of 4/2 stochastic volatility models with mispricing phenomenon, respectively. Chapters 6-8 focus on dynamic portfolio optimization for a single agent within the framework of Merton's expected utility maximization under various market scenarios. In Chapter 6, we consider utility maximization with stochastic affine interest rates and stochastic volatility. Chapter 7 deals with optimal DC pension investment problems with stochastic income, stochastic inflation, and stochastic volatility taken into consideration in the meantime. Chapter 8 contains a study on optimal ALM problems with affine diffusion factor processes and HARA utility preferences in a non-Markovian market setting. Chapter 9 and 10 are on the subject of stochastic differential games between two agents in non-Markovian market economies. Particularly, in Chapter 9, we consider a robust optimal ALM problem with stochastic volatility with model ambiguity. Chapter 10 investigates an optimal ALM problem for two asset-liability managers subject to relative performance concerns in the presence of stochastic inflation and stochastic volatility, and the problem is modeled as a non-zero-sum stochastic differential game. The following hierarchical graph exhibits the interactions among the remaining chapters.

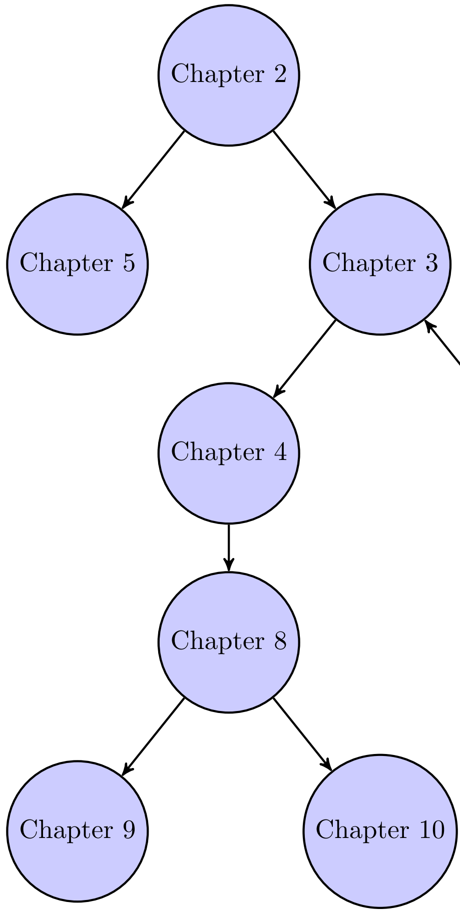


Figure 1.1: *Structure of the remaining chapters*

Chapter 2

Dynamic optimal mean-variance portfolio selection with a 3/2 stochastic volatility

Abstract

This paper considers a mean-variance portfolio selection problem when the stock price has a 3/2 stochastic volatility in a complete market. Specifically, we assume that the stock price and the volatility are perfectly negatively correlated. By applying a backward stochastic differential equation (BSDE) approach, closed-form expressions for the statically optimal (time-inconsistent) strategy and the value function are derived. Due to the time inconsistency of the mean-variance criterion, a dynamic formulation of the problem is presented. We obtain the dynamically optimal (time-consistent) strategy explicitly which is shown to keep the wealth process strictly below the target (expected terminal wealth) before the terminal time. Finally, we provide numerical studies to show the impact of main model parameters on the efficient frontier and illustrate the differences between the two optimal wealth processes.

Keywords: Mean-variance portfolio selection; 3/2 stochastic volatility; Backward stochastic differential equation; Dynamic optimality; Complete market

2.1 Introduction

In the last several decades, various stochastic volatility models have been developed in the literature to explain the volatility smile and heavy tails of return distribution as widely observed in the financial market. See, for example, Heston (1993), Hull and White (1987), Lewis (2000), and Stein and Stein (1991). Among them, a non-affine model with a mean reverting structure called the 3/2 stochastic volatility model (Lewis (2000)) enjoys empirical support in the bond and the stock market by previous works, such as Ahn and Cao (1999), Bakshi, Ju, and Ou-Yang (2006), and Jones (2003). Efforts have been made under the 3/2 stochastic volatility in derivative pricing problems such as Carr and Sun (2007), Drimus (2012), and Yuen, Zheng, and Kwok (2015). It seems, however, that little attention has been paid to portfolio selection problems under Markowitz (1952)'s mean-variance criterion.

The single-period asset allocation theory under the mean-variance criterion is first introduced by the seminal paper Markowitz (1952). Thereafter there has been increasing attention on extensions and applications of Markowitz's work. Two milestones are the work of Li and Ng (2000) and Zhou and Li (2000) which generalize Markowitz's work to a multi-period and a continuous-time setting by using embedding techniques. In Zhou and Li (2000), they assume that all the market parameters are deterministic functions or constants. To extend the results to more realistic models with random parameters, on the assumption that the return rate, the volatility, and the risk premium are all bounded stochastic processes, the backward stochastic differential equation (BSDE) approach is introduced by Lim and Zhou (2002) to solve a mean-variance problem in a complete market. From then on, many papers work on the mean-variance portfolio selection problem under various financial models by using the BSDE approach. Chiu and Wong (2011) consider the problem where asset prices are cointegrated. Shen, Zhang, and Siu (2014) investigate the same problem under a constant elasticity of variance model by assuming that the risk premium process satisfies exponential integrability. Zhang and Chen (2016) extend the results in Shen, Zhang, and Siu (2014) by further incorporating a liability process. Shen and Zeng (2015) study the optimal investment-reinsurance problem for a mean-variance insurer in an incomplete market where the risk premium process is proportional to a Markovian, affine-form and square-root process, and a modified locally square-integrable optimal strategy is derived by imposing an exponential integrability of order 2 on the risk premium process. Under similar conditions considered in Shen and Zeng (2015), a mean-variance problem under the Heston model with a liability process and a financial derivative is considered in Li, Shen, and Zeng (2018). Other relevant works on mean-variance portfolio selection problems by applying not only the BSDE approach but also other approaches (for example, the dynamic programming approach and the martingale approach (Pliska (1986))) include, such as Bielecki et al. (2005), Chang (2015), Ferland and Watier (2010), Han and Wong (2021), Lv, Wu, and Yu (2016), Pan and Xiao (2017c), Pan, Zhang, and Zhou (2018), Peng and Chen (2021), Peng and Chen (2022), Shen (2015), Shen (2020), Shen, Wei, and Zhao (2020), Tian, Guo, and Sun (2021), and Yu (2013).

The literature mentioned above under Markowitz's paradigm, however, shares one characteristic, that is, all deals with pre-committed strategies (Strotz (1956)). The resulting optimal strategy always depends on the initial wealth level and thus is called time-inconsistent. Recently, the time-consistent mean-variance portfolio selection problem has received considerable attention. To tackle the time inconsistency, Basak and Chabakauri (2010) derive a time-consistent strategy which is determined by applying a backward recursion starting from the terminal date. Björk, Khapko, and Murgoci (2017) develop a game theoretical approach under Markovian settings which essentially studies the subgame-perfect Nash equilibrium, and they derive the equilibrium strategy and the equilibrium value function by solving an extended Hamilton-Jacobi-Bellman (HJB) equation. Along this approach, previous works include Li, Zeng, and Lai (2012), Li, Rong, and Zhao (2015), Lin and Qian (2016), and Zhu and Li (2020), to name but only a few. Alternatively, Pedersen and Peskir (2017) introduces the dynamically optimal approach to investigate the time inconsistency of mean-variance problems. They overcome the time inconsistency by recomputing the statically optimal (pre-committed) strategy during the investment period, and they can therefore obtain dynamically optimal (time-consistent) strategies by solving infinitely many optimization problems.

Motivated by these aspects, we consider a dynamic mean-variance portfolio selection problem within the framework developed in Pedersen and Peskir (2017) in a complete market with two primitive assets, a risk-free asset and a stock with 3/2 stochastic volatility. In particular, the market is completed by fixing a perfectly negative correlation between the stock price and the volatility. To make the problem analytically tractable, the return rate of the stock is constant so that the risk premium process is linear in the reciprocal of the volatility process. We adopt the BSDE approach to solve this problem. The Lagrange multiplier is first applied to transform the mean-variance problem into an unconstrained optimization problem. By making an assumption on model parameters, the uniqueness and existence of the solution to a special type BSDE (Bender and Kohlmann (2000)) are established. We then solve the BSDE explicitly and obtain the optimal strategy in a closed form for the unconstrained optimization problem. Furthermore, we derive the analytic expression of the statically optimal strategy of the mean-variance portfolio selection problem by the Lagrange duality theorem. Finally, by solving the statically optimal strategy at each time, we obtain the dynamically optimal strategy which is shown to keep the corresponding wealth process strictly below the target (expected terminal wealth) before the terminal time. To summarize, this paper has main contributions in three aspects: (1) We make an assumption on the model parameters instead of on the risk premium process. This assumption guarantees the existence and uniqueness of solutions to the BSDEs. (2) We manage to derive the square-integrable optimal strategy instead of the locally square-integrable optimal strategy and verify the admissibility. (3) We provide both static and dynamic optimality.

The rest of this paper is organized as follows. Section 2.2 formulates the financial market and the portfolio selection problem. In Section 2.3, we derive the explicit solutions to the BSDEs as well as the closed-from expression of the optimal investment strategy of the unconstrained problem. Section 2.4 presents the static and dynamic optimality of the mean-variance portfolio selection problem. In Section 2.5, we provide numerical examples to present the efficient frontier under the statically optimal strategy and illustrate the differences between the two optimal controlled wealth processes. Section 2.6 concludes the paper.

2.2 Formulation of the problem

Let $[0,T]$ be a finite horizon and $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space that carries a one-dimensional standard Brownian motion $W = (W_t)_{t\in[0,T]}$. The right-continuous, P-complete filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by the Brownian motion W.

We consider a market where two primitive assets, one risk-free asset and one stock, are available to the investor. The price of the risk-free asset B solves

$$dB_t = rB_t\,dt,$$

with $B_{t_0} = b_0 \in \mathbb{R}^+$ at time $t_0 \in [0,T)$ fixed and given, where $r > 0$ stands for the interest rate. The price of the stock follows

$$dS_t = \mu S_t\,dt + \sqrt{V_t} S_t\,dW_t, \tag{2.2.1}$$

with $S_{t_0} = s_0 \in \mathbb{R}^+$ at time t_0. The return rate of the stock price $\mu > r$ is a constant and $V = (V_t)_{t\in[t_0,T]}$ is the stochastic variance of the stock price described by a 3/2 model: (see, for example, Lewis (2000))

$$dV_t = \kappa V_t(\theta - V_t)\,dt - \sigma V_t^{3/2}\,dW_t, \tag{2.2.2}$$

with initial value $V_{t_0} = v_0 \in \mathbb{R}^+$ at time t_0, where three parameters κ, θ and σ are all assumed to be positive. We hereby put the minus sign in front of σ in (2.2.2) to emphasize the assumption that the dynamics of the stock price S_t and the volatility V_t are perfectly negatively correlated.

We shall consider Markov controls $u(t, V_t, X_t^u)$ denoting the wealth invested in the stock at time $t \in [t_0, T]$ and such a deterministic function $u(\cdot,\cdot,\cdot)$ is called a feedback control law. We assume that there are no transaction costs in the trading as well as other restrictions. The investor wishes to create a self-financing portfolio of the risk-free asset B and the stock S dynamically. Thus, the controlled wealth process $(X_t^u)_{t\in[t_0,T]}$ of the investor can be described by the system of SDEs below

$$\begin{cases} dX_t^u = [rX_t^u + (\mu - r)u(t, V_t, X_t^u)]\,dt + u(t, V_t, X_t^u)\sqrt{V_t}\,dW_t, \\ \;dV_t = \kappa V_t(\theta - V_t)\,dt - \sigma V_t^{3/2}\,dW_t, \end{cases} \tag{2.2.3}$$

with $X_{t_0}^u = x_0$ at time $t_0 \in [0,T)$. We let P_{t_0,v_0,x_0} denote the probability measure with initial value $(V_{t_0}, X_{t_0}^u) = (v_0, x_0)$ at time $t_0 \in [0,T)$.

Definition 2.2.1. *Given any fixed $t_0 \in [0,T)$, if for any $(v_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$, it holds that*

1. $\mathrm{E}_{t_0,v_0,x_0}\left[\int_{t_0}^{T} u^2(t, V_t, X_t^u) V_t \, dt\right] < \infty$,
2. $\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^u|^2\right] < \infty$,

then the (Markovian) strategy u is called admissible. We denote by $\mathcal{U}$ the set of admissible portfolio strategies.

We are first interested in determining an admissible strategy $u \in \mathcal{U}$ that solves the following portfolio problem:

Definition 2.2.2. *The mean-variance portfolio problem is an optimization problem denoted by*

$$\begin{cases} \min_{u\in\mathcal{U}} \mathrm{Var}_{t_0,v_0,x_0}(X_T^u) \\ \text{subject to } \mathrm{E}_{t_0,v_0,x_0}[X_T^u] = \xi, \end{cases} \tag{2.2.4}$$

where ξ is a fixed and given constant playing the financial role of a target. The corresponding value function is denoted by $V_{MV}(t_0, v_0, x_0)$.

Remark 2.2.3. Here we impose $\xi > x_0 e^{r(T-t_0)}$, in line with previous studies such as Lim and Zhou (2002), Shen and Zeng (2015), and Shen, Wei, and Zhao (2020). Otherwise, the investor can simply take the risk-free strategy $u \equiv 0$ over $[t_0, T]$ which dominates any other admissible strategy.

As a result of the quadratic non-linearity of the variance operator, problem (2.2.4) falls outside of Bellman's principle. Denote by u^* the optimal strategy in problem (2.2.4) which refers to the *static optimality* (refer to Definition 1 in Pedersen and Peskir (2017)) and is relative to the initial position (t_0, v_0, x_0). The investor might not be committed to the statically optimal strategy u^* chosen at the very initial position (t_0, v_0, x_0) during the following investment period $(t_0, T]$. Therefore, we shall also consider a dynamic formulation of problem (2.2.4). Here, we opt for the framework developed in Pedersen and Peskir (2017). We now review the definition of *dynamic optimality* in problem (2.2.4) for the readers' convenience.

Definition 2.2.4. *For a triple $(t_0, v_0, x_0) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$ fixed and given, we call a Makrov strategy u^{d*} dynamically optimal in problem (2.2.4), if for every $(t, v, x) \in [t_0, T) \times \mathbb{R}^+ \times \mathbb{R}$ and every strategy $\pi \in \mathcal{U}$ with $\pi(t,v,x) \neq u^{d*}(t,v,x)$ and $\mathrm{E}_{t,v,x}[X_T^\pi] = \xi$, there is a Markov strategy w satisfying $w(t,v,x) = u^{d*}(t,v,x)$ and $\mathrm{E}_{t,v,x}[X_T^w] = \xi$ such that*

$$\mathrm{Var}_{t,v,x}(X_T^w) < \mathrm{Var}_{t,v,x}(X_T^\pi).$$

The dynamically optimal strategy u^{d*} is essentially derived by solving the statically optimal strategy u^* at each time and implementing it in an infinitesimally small period of time, which in turn implies that we shall first address problem (2.2.4) in the sense of static optimality so as to derive the dynamic optimality.

We observe that problem (2.2.4) is, in fact, a convex optimization problem with linear constraint $\mathrm{E}_{t_0,v_0,x_0}[X_T^u] = \xi$. Thus, we can handle the constraint by introducing a Lagrange multiplier $\theta \in \mathbb{R}$, and define the following Lagrangian:

$$
\begin{aligned}
L(x_0, v_0; u, \theta) &= \mathrm{E}_{t_0,v_0,x_0}[(X_T^u - \xi)^2] + 2\theta \mathrm{E}_{t_0,v_0,x_0}[X_T^u - \xi] \\
&= \mathrm{E}_{t_0,v_0,x_0}\left[(X_T^u - (\xi - \theta))^2\right] - \theta^2.
\end{aligned}
\tag{2.2.5}
$$

Then the Lagrangian duality theorem (see, for example, Luenberger (1968)) indicates that we can derive the static optimality u^* in problem (2.2.4) by solving the following equivalent min-max stochastic control problem

$$
\max_{\theta \in \mathbb{R}} \min_{u \in \mathcal{U}} L(x_0, v_0; u, \theta). \tag{2.2.6}
$$

This shows that we can solve problem (2.2.6) with two steps, of which the first step is to solve the unconstrained stochastic optimization problem with respect to $u \in \mathcal{U}$ given a fixed $\theta \in \mathbb{R}$ and the second step is to solve the static optimization problem with respect to the Lagrange multiplier $\theta \in \mathbb{R}$. Hence, we can first address the following unconstrained quadratic-loss minimization problem:

$$
\min_{u \in \mathcal{U}} J(x_0, v_0; u, \gamma) = \mathrm{E}_{t_0,v_0,x_0}\left[(X_T^u - \gamma)^2\right], \tag{2.2.7}
$$

with $\gamma = \xi - \theta$ fixed and given.

2.3 Solution to the unconstrained problem

In this section, we opt for the BSDE approach so as to solve problem (2.2.7) above. Before formulating the main results in this section, we make the following notations to facilitate the discussions below. For any $\mathbb{R}^+$-valued, $\mathcal{F}_t$-adapted stochastic process $\eta := (\eta_t)_{t\in[0,T]}$, a continuous process $A := (A_t)_{t\in[0,T]}$ associated with η is defined by $A_t = \int_0^t \eta_s^2 ds$. Let $\beta \geq 0$ be a generic constant; we denote by

- $\mathcal{L}^2_{\mathrm{P}}(\beta, \eta, [0,T]; \mathbb{R})$: the space of $\mathcal{F}_t$-adapted, $\mathbb{R}$-valued stochastic processes f satisfying
$$
\|f\|_\beta^2 := \mathrm{E}\left[\int_0^T e^{\beta A_t}|f_t|^2 dt\right] < \infty;
$$
- $\mathcal{L}^{2,\eta}_{\mathrm{P}}(\beta, \eta, [0,T]; \mathbb{R})$: the space of $\mathcal{F}_t$-adapted, $\mathbb{R}$-valued stochastic processes f satisfying
$$
\|\eta f\|_\beta^2 := \mathrm{E}\left[\int_0^T \eta_t^2 e^{\beta A_t}|f_t|^2 dt\right] < \infty;
$$

- $\mathcal{L}_{\mathrm{P}}^{2,c}(\beta,\eta,[0,T];\mathbb{R})$ the space of $\mathcal{F}_t$-adapted, $\mathbb{R}$-valued stochastic processes f satisfying

$$\|f\|_{\beta,c}^2 := \mathrm{E}\left[\sup_{0\le t\le T} e^{\beta A_t}|f_t|^2\right] < \infty.$$

Hence, we have the following Banach space:

$$\mathcal{M}(\beta,\eta,[0,T];\mathbb{R}^2) := \Big(\mathcal{L}_{\mathrm{P}}^{2,\eta}(\beta,\eta,[0,T];\mathbb{R})) \cap (\mathcal{L}_{\mathrm{P}}^{2,c}(\beta,\eta,[0,T];\mathbb{R})\Big) \times \mathcal{L}_{\mathrm{P}}^{2}(\beta,\eta,[0,T];\mathbb{R})$$

with the norm $\|(Y,Z)\|_\beta^2 = \|\eta Y\|_\beta^2 + \|Y\|_{\beta,c}^2 + \|Z\|_\beta^2$.

In addition, we introduce

$$\begin{cases} \Delta = [\kappa\theta + 2(\mu-r)\sigma]^2 - 2\sigma^2(\mu-r)^2, \\ n_1 = \dfrac{-[\kappa\theta+2(\mu-r)\sigma]+\sqrt{\Delta}}{-\sigma^2}, \ n_2 = \dfrac{-[\kappa\theta+2(\mu-r)\sigma]-\sqrt{\Delta}}{-\sigma^2}, \\ C_b = \max\left\{(60+16\sqrt{14})\left((\mu-r)^2 + \dfrac{\sigma^2 n_1^2 n_2^2 (1-e^{\sqrt{\Delta}T})^2}{(n_1 - n_2 e^{\sqrt{\Delta}T})^2}\right),\right. \\ \qquad \left. 8(\mu-r)^2 + 8(\mu-r)\sigma\dfrac{n_1 n_2(1-e^{\sqrt{\Delta}T})}{n_1 - n_2 e^{\sqrt{\Delta}T}}\right\}. \end{cases} \tag{2.3.1}$$

It can be easily checked that $\Delta > 0$ due to $\mu > r$. The following standing assumption is imposed on the model parameters throughout the paper.

Assumption 2.3.1. *$C_b \le \kappa^2\theta^2/2\sigma^2$.*

Remark 2.3.2. It follows from Lemma 2.3.8 below that C_b is strictly increasing in T. In particular, when $T \to 0$, $C_b \to (60+16\sqrt{14})(\mu-r)^2$. This indicates the feasibility of Assumption 2.3.1. Moreover, Assumption 2.3.1 is crucial to guarantee that three BSDEs (2.3.2), (2.3.5), and (2.3.10) admit unique solutions and the statically optimal strategy (2.4.4) is admissible.

The following linear BSDE of (P,Γ) is considered so as to solve problem (2.2.7)

$$\begin{cases} dP_t = \left\{\left[2r - \dfrac{(\mu-r)^2}{V_t}\right]P_t + \dfrac{2(\mu-r)}{\sqrt{V_t}}\Gamma_t\right\}dt + \Gamma_t\,dW_t, \\ P_T = 1. \end{cases} \tag{2.3.2}$$

Clearly, due to the randomness and unboundedness of the driver of (2.3.2), this linear BSDE is without the uniform Lipschitz continuity with respect to both P_t and Γ_t. Thus, BSDE (2.3.2) is out of the scope of El Karoui, Peng, and Quenez (1997). Nevertheless, we observe that BSDE (2.3.2) follows a stochastic Lipschitz continuity which is first proposed in Bender and Kohlmann (2000). To proceed, some useful results on the BSDE with stochastic Lipschitz continuity adapted from Definition 2 and Theorem 3 in Bender and Kohlmann (2000) are presented below.

Definition 2.3.3. *We call a pair* (ζ, f) *standard data for the BSDE of* (Y, Z)*:*

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)\,dt - Z_t\,dW_t, \\ \quad Y_T = \zeta,\ \ t \in [0, T], \end{cases}$$

if the following four conditions hold:

1. *There exist two* $\mathbb{R}^+$*-valued,* $\mathcal{F}_t$*-adapted stochastic processes* $(\eta_{1,t})_{t\in[0,T]}$ *and* $(\eta_{2,t})_{t\in[0,T]}$ *such that* $\forall t \in [0, T], \forall (y_1, z_1), (y_2, z_2) \in \mathbb{R}^2$

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \le \eta_{1,t}|y_1 - y_2| + \eta_{2,t}|z_1 - z_2|.$$

   We refer to this inequality as the stochastic Lipschitz continuity.

2. *There exists a positive constant* $\varepsilon > 0$ *satisfying* $\eta_t^2 := \eta_{1,t} + \eta_{2,t}^2 \ge \varepsilon$*.*

3. *The terminal condition* ζ *satisfies* $\mathrm{E}\left[\exp\left(\beta \int_0^T \eta_t^2\,dt\right)|\zeta|^2\right] < \infty$ *in which* β *is a positive constant.*

4. $\frac{f(\cdot,0,0)}{\eta} \in \mathcal{L}^2_{\mathrm{P}}(\beta, \eta, [0, T]; \mathbb{R})$*.*

Lemma 2.3.4. *The BSDE of* (Y, Z)

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)\,dt - Z_t\,dW_t, \\ \quad Y_T = \zeta,\ \ t \in [0, T], \end{cases}$$

admits a unique solution $(Y, Z) \in \mathcal{M}(\beta, \eta, [0, T]; \mathbb{R}^2)$ *if* (ζ, f) *is standard data for a sufficiently large* β*, in particular, for* $\beta > 3 + \sqrt{21}$*.*

Before adapting the above results to establish the uniqueness and existence of the solution to BSDE (2.3.2), we recall the following useful result from Theorem 5.1 in Zeng and Taksar (2013).

Lemma 2.3.5. *Suppose the process* $(r_t)_{t\in[0,T]}$ *follows the Cox-Ingersoll-Ross (CIR) model:*

$$dr_t = (\kappa\theta - \kappa r_t)\,dt + \sigma\sqrt{r_t}\,dW_t,$$

where κ, θ *and* σ *are positive constants. Then we have*

$$\mathrm{E}\left[\exp\left(\beta \int_0^T r_t\,dt\right)\right] < \infty \text{ if and only if } \beta \le \kappa^2/2\sigma^2.$$

Lemma 2.3.6. *Assume Assumption 2.3.1 holds true, then there is a constant* $3 + \sqrt{21} < \beta \le \frac{\kappa^2\theta^2}{6(\mu-r)^2\sigma^2}$ *such that the unique solution* $(P, \Gamma) \in \mathcal{M}(\beta, \eta, [t_0, T]; \mathbb{R}^2)$ *with* $\eta_t = \left(2r + \frac{3(\mu-r)^2}{V_t}\right)^{1/2}$ *to BSDE (2.3.2) exists.*

Proof. Let $\eta_{1,t} = 2r - (\mu - r)^2/V_t$ and $\eta_{2,t} = 2(\mu - r)/\sqrt{V_t}$. Denote in this case the non-negative $\mathcal{F}_t$-adapted process η_t by

$$\eta_t^2 := \eta_{1,t} + \eta_{2,t}^2,$$

and accordingly, define the increasing process A_t by

$$A_t := \int_{t_0}^{t} \eta_s^2 \, ds = \int_{t_0}^{t} \left(2r + \frac{3(\mu - r)^2}{V_s} \right) ds.$$

We then have

$$\mathrm{E}_{t_0,v_0,x_0} \left[\exp\left(\beta A_T \right) \right] \leq C \mathrm{E}_{t_0,v_0,x_0} \left[\exp\left(3(\mu - r)^2 \beta \int_{t_0}^{T} \frac{1}{V_t} dt \right) \right],$$

where the positive constant $C > 0$ is independent of β. By Itô's lemma, we then have the following dynamics of the reciprocal of the variance process (2.2.2):

$$d\left(\frac{1}{V_t} \right) = \kappa\theta \left(\frac{\kappa + \sigma^2}{\kappa\theta} - \frac{1}{V_t} \right) dt + \sigma \sqrt{\frac{1}{V_t}} \, dW_t,$$

which is a CIR process. It follows from Lemma 2.3.5 that if

$$3(\mu - r)^2 \beta \leq \frac{\kappa^2\theta^2}{2\sigma^2},$$

then we have

$$\mathrm{E}_{t_0,v_0,x_0} \left[\exp\left(3(\mu - r)^2 \beta \int_{t_0}^{T} \frac{1}{V_t} dt \right) \right] < \infty.$$

Indeed, when Assumption 2.3.1 holds, there exists a constant β such that $3 + \sqrt{21} < \beta \leq \frac{\kappa^2\theta^2}{6(\mu - r)^2\sigma^2}$, and the driver and the terminal condition of BSDE (2.3.2) then constitute standard data. Finally, by Lemma 2.3.4 above, we see that a unique solution $(P, \Gamma) \in \mathcal{M}(\beta, \eta, [t_0, T]; \mathbb{R}^2)$ to BSDE (2.3.2) with $3 + \sqrt{21} < \beta \leq \frac{\kappa^2\theta^2}{6(\mu - r)^2\sigma^2}$ and $\eta_t = \left(2r + \frac{3(\mu - r)^2}{V_t} \right)^{1/2}$ exists. □

In what follows, we shall give the explicit expression of the unique solution (P, Γ) of BSDE (2.3.2).

Lemma 2.3.7. *Assume Assumption 2.3.1 holds, then the unique solution (P, Γ) of BSDE (2.3.2) has the following explicit expression:*

$$\begin{cases} P_t = \exp\left(-2r(T - t) \right) g(t, V_t), \\ \Gamma_t = \sigma a(t) \dfrac{P_t}{\sqrt{V_t}}, \end{cases} \tag{2.3.3}$$

for $t \in [t_0, T]$, where $g(t, v) = \exp\left\{ a(t)\frac{1}{v} + b(t) \right\}$, and $a(t)$ and $b(t)$ are solutions to the following system of ODEs:

$$\begin{cases} \dfrac{da(t)}{dt} - (\kappa\theta + 2(\mu - r)\sigma) a(t) + \dfrac{1}{2}\sigma^2 a^2(t) + (\mu - r)^2 = 0, & a(T) = 0, \\ \dfrac{db(t)}{dt} + (\kappa + \sigma^2) a(t) = 0, & b(T) = 0. \end{cases} \tag{2.3.4}$$

Proof. We first introduce the likelihood process $(L_t)_{t\in[t_0,T]}$ from the dynamics

$$dL_t = -\frac{2(\mu-r)}{\sqrt{V_t}} L_t\, dW_t,\ \ L_{t_0} = 1.$$

Similar to the reasoning in Lemma 2.3.6, it can be easily verified from Assumption 2.3.1 above that

$$\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left(\int_{t_0}^{T} \frac{2(\mu-r)^2}{V_t}\, dt\right)\right] < \infty.$$

That is, the Novikov's condition is satisfied for $(L_t)_{t\in[t_0,T]}$. Thus, $(L_t)_{t\in[t_0,T]}$ is a uniformly integrable martingale under P_{t_0,v_0,x_0} measure and we can define an equivalent probability measure $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$ on $\mathcal{F}_T$ through the Radon-Nikodym derivative

$$d\tilde{\mathrm{P}}_{t_0,v_0,x_0} = L_T\, d\mathrm{P}_{t_0,v_0,x_0}.$$

From the Girsanov's theorem, Brownian motions under $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$ and P_{t_0,v_0,x_0} are related to each other through

$$dW_t^{\tilde{\mathrm{P}}} = \frac{2(\mu-r)}{\sqrt{V_t}}\, dt + dW_t,$$

and we can rewrite (2.3.2) under $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$-measure as follows

$$\begin{cases} dP_t = \left\{\left[2r - \dfrac{(\mu-r)^2}{V_t}\right] P_t\right\} dt + \Gamma_t\, dW_t^{\tilde{\mathrm{P}}}, \\ P_T = 1. \end{cases} \tag{2.3.5}$$

We see that the driver of BSDE (2.3.5) again satisfies the stochastic Lipschitz continuity with in this case $\eta_t^2 = |2r - \frac{(\mu-r)^2}{V_t}| + \varepsilon$ for any $\varepsilon > 0$ fixed and given and $A_t = \int_{t_0}^t \eta_s^2\, ds$ such that using Hölder's inequality we have for some $\beta > 3 + \sqrt{21}$

$$\begin{aligned}
\tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\exp\left(\beta A_T\right)\right] \leq& K\tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\exp\left((\mu-r)^2\beta\int_{t_0}^{T}\frac{1}{V_t}\, dt\right)\right] \\
=& K\mathrm{E}_{t_0,v_0,x_0}\left[L_T\exp\left((\mu-r)^2\beta\int_{t_0}^{T}\frac{1}{V_t}\, dt\right)\right] \\
\leq& K\left(\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left(-\int_{t_0}^{T}\frac{4(\mu-r)}{\sqrt{V_t}}\, dW_t - \int_{t_0}^{T}\frac{8(\mu-r)^2}{V_t}\, dt\right)\right]\right)^{\frac{1}{2}} \\
&\cdot\left(\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left((4+2\beta)(\mu-r)^2\int_{t_0}^{T}\frac{1}{V_t}\, dt\right)\right]\right)^{\frac{1}{2}} \\
=& K\left(\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left((4+2\beta)(\mu-r)^2\int_{t_0}^{T}\frac{1}{V_t}\, dt\right)\right]\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

where $K > 0$ is constant independent of β, the second equality follows from the fact that $\left(\exp\left(-\int_{t_0}^t \frac{4(\mu-r)}{\sqrt{V_u}}\, dW_u - \int_{t_0}^t \frac{8(\mu-r)^2}{V_u}\, du\right)\right)_{t\in[t_0,T]}$ is a P_{t_0,v_0,x_0} martingale due to Assumption 2.3.1, and the last strict inequality is due to Assumption 2.3.1. This shows that the terminal condition and the driver of BSDE (2.3.5) constitute standard data. Then by Lemma 2.3.4 above, the BSDE (2.3.5) admits a unique solution (P,Γ) satisfying $\Gamma \in \mathcal{L}^2_{\tilde{\mathrm{P}}_{t_0,v_0,x_0}}(\beta,\eta,[t_0,T];\mathbb{R})$ with some $\beta > 3+\sqrt{21}$ and $\eta_t = \sqrt{|2r - \frac{(\mu-r)^2}{V_t}| + \epsilon}$. Moreover, we see that under $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$ measure

$$d\left[P_t \exp\left(\int_{t_0}^t \left(\frac{(\mu-r)^2}{V_u} - 2r\right) du\right)\right] = \exp\left(\int_{t_0}^t \left(\frac{(\mu-r)^2}{V_u} - 2r\right) du\right) \Gamma_t \, dW_t^{\tilde{\mathrm{P}}}.$$

This shows that $\left(P_t \exp\left(\int_{t_0}^t \left(\frac{(\mu-r)^2}{V_u} - 2r\right) du\right)\right)_{t\in[t_0,T]}$ is a local martingale under measure $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$. By Burkholder-Davis-Gundy inequality and Hölder's inequality, we then find that

$$\begin{aligned}
&\tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\sup_{t_0\le t\le T}\left|\int_{t_0}^t \exp\left(\int_{t_0}^s \left(\frac{(\mu-r)^2}{V_u} - 2r\right) du\right)\Gamma_s \, dW_s^{\tilde{\mathrm{P}}}\right|\right] \\
\le & K\tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\left(\int_{t_0}^T \exp\left(2\int_{t_0}^t \left(\frac{(\mu-r)^2}{V_u} - 2r\right) du\right)\Gamma_t^2 \, dt\right)^{\frac{1}{2}}\right] \\
\le & K\left(\tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\exp\left(2\int_{t_0}^T \left|\frac{(\mu-r)^2}{V_t} - 2r\right| dt\right)\right] + \tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\int_{t_0}^T \Gamma_t^2 dt\right]\right) \\
\le & K\left(\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left(-\int_{t_0}^T \frac{4(\mu-r)}{\sqrt{V_t}} \, dW_t - \int_{t_0}^T \frac{8(\mu-r)^2}{V_t} \, dt\right)\right]\right)^{\frac{1}{2}} \\
& \cdot \left(\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left(8(\mu-r)^2 \int_{t_0}^T \frac{1}{V_t} \, dt\right)\right]\right)^{\frac{1}{2}} + K\tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\int_{t_0}^T \Gamma_t^2 dt\right] \\
= & K\left(\left(\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left(8(\mu-r)^2 \int_{t_0}^T \frac{1}{V_t} \, dt\right)\right]\right)^{\frac{1}{2}} + \tilde{\mathrm{E}}_{t_0,v_0,x_0}\left[\int_{t_0}^T \Gamma_t^2 dt\right]\right) < \infty,
\end{aligned}$$

where the positive constant $K > 0$ might vary between lines; the equality follows from the fact that $\left(\exp\left(-\int_{t_0}^t \frac{4(\mu-r)}{\sqrt{V_u}} \, dW_u - \int_{t_0}^u \frac{8(\mu-r)^2}{V_u} \, du\right)\right)_{t\in[t_0,T]}$ is a P_{t_0,v_0,x_0} martingale due to Assumption 2.3.1, and the last strict inequality is due to Assumption 2.3.1 and $\Gamma \in \mathcal{L}^2_{\tilde{\mathrm{P}}_{t_0,v_0,x_0}}(\beta,\eta,[t_0,T];\mathbb{R})$. This shows that

$$P_t \exp\left(\int_{t_0}^t \left(\frac{(\mu-r)^2}{V_u} - 2r\right) du\right)$$

is, in fact, a uniformly integrable martingale under $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$ measure (refer to Corollary 5.17 in Le Gall (2016)). Upon noticing the boundary condition that $P_T = 1$, we have the expectational form for $(P_t)_{t\in[t_0,T]}$ below

$$P_t = \exp(-2r(T-t))\mathrm{E}^{\tilde{\mathrm{P}}}_{t_0,v_0,x_0}\left[\exp\left(\int_t^T \frac{(\mu-r)^2}{V_u} \, du\right)\middle|\mathcal{F}_t\right].$$

Denote by

$$g(t,v) = \mathrm{E}^{\tilde{\mathrm{P}}}_{t,v}\left[\exp\left(\int_t^T \frac{(\mu-r)^2}{V_u} \, du\right)\right],$$

where $\mathrm{E}^{\tilde{\mathrm{P}}}_{t,v}[\cdot]$ is the expectation at time $t \in [0,T)$ such that $V_t = v$ under $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$-measure. Due to the Markovian structure of the variance process $(V_t)_{t\in[t_0,T]}$ with respect to $(\mathcal{F}_t)_{t\in[t_0,T]}$, we can obviously rewrite $(P_t)_{t\in[t_0,T]}$ as follows

$$P_t = \exp\left(-2r(T-t)\right) g(t,V_t).$$

Note that the variance process V_t has $\tilde{\mathrm{P}}_{t_0,v_0,x_0}$-dynamics

$$dV_t = \left\{[\kappa\theta + 2(\mu - r)\sigma]V_t - \kappa V_t^2\right\} dt - \sigma V_t^{3/2}\, dW_t^{\tilde{\mathrm{P}}}.$$

Suppose the deterministic function $g(\cdot,\cdot) \in C^{1,2}([t_0,T] \times \mathbb{R}^+)$, then applying the Feynman-Kac theorem yields the following PDE governing function g:

$$\begin{cases} \dfrac{\partial g}{\partial t} + [(\kappa\theta + 2(\mu - r)\sigma)v - \kappa v^2]\dfrac{\partial g}{\partial v} + \dfrac{1}{2}\sigma^2 v^3 \dfrac{\partial^2 g}{\partial v^2} + \dfrac{(\mu - r)^2}{v} g = 0, \\ \hfill g(T,v) = 1. \end{cases} \tag{2.3.6}$$

We conjecture that g admits the following exponential expression:

$$g(t,v) = \exp\left(a(t)\frac{1}{v} + b(t)\right),$$

with boundary condition $a(T) = b(T) = 0$. Its derivatives are given by

$$\begin{cases} \dfrac{\partial g}{\partial t} = g\left(\dfrac{1}{v}\dfrac{da(t)}{dt} + \dfrac{db(t)}{dt}\right), \\ \dfrac{\partial g}{\partial v} = -g\dfrac{a(t)}{v^2}, \\ \dfrac{\partial^2 g}{\partial v^2} = g\left(a^2(t)\dfrac{1}{v^4} + a(t)\dfrac{2}{v^3}\right). \end{cases} \tag{2.3.7}$$

Substituting (2.3.7) into (2.3.6) yields

$$\left[\frac{da(t)}{dt} - (\kappa\theta + 2(\mu - r)\sigma)a(t) + \frac{1}{2}\sigma^2 a^2(t) + (\mu - r)^2\right]\frac{1}{v} + \frac{db(t)}{dt} + (\sigma^2 + \kappa)a(t) = 0.$$

The arbitrariness of $v \in \mathbb{R}^+$ in turn leads to the system of ODEs (2.3.4) as claimed above. Applying Itô's lemma to P_t, we obtain

$$\Gamma_t = \sigma a(t)\frac{P_t}{\sqrt{V_t}}$$

by the uniqueness result of BSDE (2.3.2). □

Lemma 2.3.8. *Assume Assumption 2.3.1 holds true, then the explicit solutions of the ODE system (2.3.4) are*

$$a(t) = \frac{n_1 n_2 (1 - e^{\sqrt{\Delta}(T-t)})}{n_1 - n_2 e^{\sqrt{\Delta}(T-t)}}, \tag{2.3.8}$$

$$b(t) = \int_t^T \left(\kappa + \sigma^2\right) a(s)\, ds, \tag{2.3.9}$$

where n_1, n_2 and Δ are given in (2.3.1). Moreover, function $a(t)$ is strictly decreasing in t.

Proof. By reformulating the Riccati ODE of $a(t)$, we have

$$\frac{da(t)}{dt} = -\frac{1}{2}\sigma^2(a(t) - n_1)(a(t) - n_2),$$

where n_1 and n_2 are given in (2.3.1) above. After some tedious calculations upon considering $a(T) = 0$, we obtain (2.3.8). Integrating both sides of ODE of $b(t)$ from t to T upon considering the boundary condition $b(T) = 0$ gives (2.3.9). Furthermore, differentiating (2.3.8) with respect to t yields

$$\frac{da(t)}{dt} = \frac{-4(\mu - r)^2\sqrt{\Delta}e^{\sqrt{\Delta}(T-t)}}{\sigma^4(n_1 - n_2 e^{\sqrt{\Delta}(T-t)})^2} < 0.$$

□

Denote by $Y_t := 1/P_t$ the reciprocal process of $(P_t)_{t\in[t_0,T]}$. Then a direct application of Itô's lemma to Y_t yields the backward stochastic Riccati equation (BSRE) of (Y, Λ) below

$$\begin{cases} dY_t = \left\{\left[-2r + \dfrac{(\mu - r)^2}{V_t}\right] Y_t + \dfrac{2(\mu - r)}{\sqrt{V_t}}\Lambda_t + \dfrac{\Lambda_t^2}{Y_t}\right\} dt + \Lambda_t\, dW_t, \\ Y_T = 1, \end{cases} \tag{2.3.10}$$

where $\Lambda_t = -Y_t^2\Gamma_t$. Since (P, Γ) given in (2.3.3) is the unique solution of BSDE (2.3.2), from the relationship of (P, Γ) and (Y, Λ), we see that BSRE (2.3.10) admits a unique solution as well.

Lemma 2.3.9. *Assume Assumption 2.3.1 holds true, then the unique solution (Y, Λ) of BSRE (2.3.10) is*

$$\begin{cases} Y_t = \exp\left(2r(T - t) - a(t)\dfrac{1}{V_t} - b(t)\right), \\ \Lambda_t = -\sigma a(t)\dfrac{1}{\sqrt{V_t}}Y_t, \end{cases} \tag{2.3.11}$$

with $a(t)$ and $b(t)$ given in (2.3.8) and (2.3.9), respectively.

Proof. The equations (2.3.11) can be directly derived from the relationship of (P, Γ) and (Y, Λ) above. □

We now define a Doléans-Dade exponential $(\Pi_t)_{t\in[t_0,T]}$ of $\left(\frac{\mu-r-\sigma a(t)}{\sqrt{V_t}}\right)_{t\in[t_0,T]}$ by

$$\Pi_t = \exp\left(\int_{t_0}^t -\frac{\mu - r - \sigma a(u)}{\sqrt{V_u}}\, dW_u - \int_{t_0}^t \frac{1}{2}\frac{(\mu - r - \sigma a(u))^2}{V_u}\, du\right). \tag{2.3.12}$$

In the next lemma, we shall study the integrablity of Π_t which will be useful when we verify the admissibility of optimal strategy (2.3.13) below.

Lemma 2.3.10. *Assume Assumption 2.3.1 holds true, then the Doléans-Dade exponential* Π_t *(2.3.12) satisfies*

$$\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]}|\Pi_t|^8\right]<\infty.$$

Proof. We know that the following equation of k

$$p=\frac{k}{2\sqrt{k}-1}$$

admits two positive solutions

$$k_1=2p\sqrt{p(p-1)}+p(2p-1),\ \ k_2=-2p\sqrt{p(p-1)}+p(2p-1),$$

for any given constant $p>1$, where the first solution satisfies $k_1>1$. In particular, when $p=8$, we have $k_1=120+32\sqrt{14}$. Using Assumption 2.3.1, Lemma 2.3.8, and the reasoning given in the proof of Lemma 2.3.6 above, we see that

$$\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left((60+16\sqrt{14})\int_{t_0}^{T}\frac{(\mu-r-\sigma a(t))^2}{V_t}\,dt\right)\right]<\infty.$$

Then Theorem 15.4.6 in Cohen and Elliott (2015) yields

$$\begin{aligned}&\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]}|\Pi_t|^8\right]\\ \le&\frac{8}{7}\left\{\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left((60+16\sqrt{14})\int_{t_0}^{T}\frac{(\mu-r-\sigma a(t))^2}{V_t}\,dt\right)\right]\right\}^{\frac{\sqrt{120+32\sqrt{14}}-1}{120+32\sqrt{14}}}<\infty.\end{aligned}$$

This completes the proof. ☐

To end this section, we shall relate the optimal Markovian strategy and the corresponding value function of problem (2.2.7) to the solution (Y,Λ) of BSRE (2.3.10).

Proposition 2.3.11. *Assume Assumption 2.3.1 holds true, then for* $(t_0,v_0,x_0)\in[0,T)\times\mathbb{R}^+\times\mathbb{R}$ *fixed and given, the optimal (Markovian) strategy of problem (2.2.7) is*

$$u^*(t,v,x)=-\left(x-\gamma e^{-r(T-t)}\right)\frac{\mu-r-\sigma a(t)}{v},\tag{2.3.13}$$

for $t\in[t_0,T]$. *The corresponding value function is*

$$J(x_0,v_0;u^*(\cdot),\gamma)=\exp\left(2r(T-t_0)-a(t_0)\frac{1}{v_0}-b(t_0)\right)\left(x_0-\gamma e^{-r(T-t_0)}\right)^2,\tag{2.3.14}$$

The controlled wealth process X_t^* evolves as

$$X_t^* = \left(x_0 e^{r(t-t_0)} - \gamma e^{-r(T-t)}\right) \Pi_t \exp\left\{-\int_{t_0}^{t} (\mu - r)\frac{\mu - r - \sigma a(u)}{V_u}\, du\right\} + \gamma e^{-r(T-t)}, \tag{2.3.15}$$

where Π_t is given in (2.3.12). Moreover, the optimal strategy u^* belongs to $\mathcal{U}$.

Proof. Using Itô's lemma to $G_t = X_t^u - \gamma e^{-r(T-t)}$, we obtain

$$dG_t = [rG_t + (\mu - r)u(t, S_t, X_t^u)]\, dt + u(t, S_t, X_t^u)\sqrt{V_t}\, dW_t,\ G_0 = x_0 - \gamma e^{-r(T-t_0)}.$$

Furthermore, applying Itô's lemma to $Y_t G_t^2$ yields

$$\begin{aligned} dY_t G_t^2 =& Y_t \left\{u(t, V_t, X_t^u)\sqrt{V_t} + \left(\frac{\mu - r}{\sqrt{V_t}} + \frac{\Lambda_t}{Y_t}\right) G_t\right\}^2 dt \\ &+ \left[\Lambda_t G_t^2 + 2Y_t G_t u(t, V_t, X_t^u)\sqrt{V_t}\right] dW_t. \end{aligned} \tag{2.3.16}$$

We observe that the stochastic integral on the right-hand side of (2.3.16) is a local martingale, and thus, we can define stopping times $(\tau_n)_{n\geq 1}$ as follows

$$\tau_n = \inf\left\{t \geq t_0 : \int_{t_0}^{t} \left|\Lambda_{t'} G_{t'}^2 + 2Y_{t'} G_{t'} u(t', V_{t'}, X_{t'}^u)\sqrt{V_{t'}}\right|^2 dt' \geq n\right\},$$

such that $\tau_n \to \infty$, P_{t_0,v_0,x_0} almost surely as $n \to \infty$. We integrate (2.3.16) from t_0 to $T \wedge \tau_n$ and take expectations on both sides of (2.3.16)

$$\begin{aligned} &\mathrm{E}_{t_0,v_0,x_0}\left[Y_{T\wedge\tau_n} G_{T\wedge\tau_n}^2\right] \\ =&\mathrm{E}_{t_0,v_0,x_0}\left[\int_{t_0}^{T\wedge\tau_n} Y_t \left\{u(t, V_t, X_t^u)\sqrt{V_t} + \left(\frac{\mu - r}{\sqrt{V_t}} + \frac{\Lambda_t}{Y_t}\right) G_t\right\}^2 dt\right] \\ &+ y_0 \left(x_0 - \gamma e^{-r(T-t_0)}\right)^2, \end{aligned} \tag{2.3.17}$$

where $y_0 = \exp\left(2r(T - t_0) - a(t_0)\frac{1}{v_0} - b(t_0)\right)$. From the definition of function $g(t, v)$ in Lemma 2.3.7 above, we see that $0 < Y_t < e^{2rT}$ for any $t \in [t_0, T]$, P_{t_0,v_0,x_0}-a.s. Moreover, in view of Definition 2.2.1, we have $\mathrm{E}_{t_0,v_0,x_0}[\sup_{t\in[t_0,T]} |G_t|^2] < \infty$ for $u \in \mathcal{U}$. As a result of the Lebesgue's dominated convergence theorem and the monotone convergence theorem working on (2.3.17), then we have

$$\begin{aligned} \mathrm{E}_{t_0,v_0,x_0}[(X_T^u - \gamma)^2] =& \mathrm{E}_{t_0,v_0,x_0}\left[\int_{t_0}^{T} Y_t \left\{u(t, V_t, X_t^u)\sqrt{V_t} + \left(\frac{\mu - r}{\sqrt{V_t}} + \frac{\Lambda_t}{Y_t}\right) G_t\right\}^2 dt\right] \\ &+ y_0 \left(x_0 - \gamma e^{-r(T-t_0)}\right)^2. \end{aligned} \tag{2.3.18}$$

Upon considering explicit expressions of Y_t and Λ_t (2.3.11), we obtain the optimal Markov strategy (2.3.13) and the value function (2.3.14) for problem (2.2.7).

Substituting u^* (2.3.13) into the wealth process (2.2.3), we obtain

$$dX_t^* = \left[rX_t^* + (\mu - r)\frac{\mu - r - \sigma a(t)}{\sqrt{V_t}} \left(\gamma e^{-r(T-t)} - X_t^* \right) \right] dt \\ + \frac{\mu - r - \sigma a(t)}{\sqrt{V_t}} \left(\gamma e^{-r(T-t)} - X_t^* \right) dW_t.$$

A direction application of Itô's lemma to $e^{r(T-t)}X_t^* - \gamma$ then yields the controlled wealth process X_t^* (2.3.15).

In the following, we show that the optimal strategy u^* (2.3.13) is admissible. For this, we first show that

$$\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^*|^4 \right] < \infty. \tag{2.3.19}$$

Indeed, from Assumption 2.3.1 and Lemma 2.3.10 above we find that

$$\begin{aligned}
&\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^*|^4 \right] \\
\le & K\mathrm{E}_{t_0,v_0,x_0}\left[1 + \sup_{t\in[t_0,T]} \left| \exp\left\{ -\int_{t_0}^t (\mu - r)\frac{\mu - r - \sigma a(u)}{V_u}\, du \right\} \Pi_t \right|^4 \right] \\
\le & K + K\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} \exp\left\{ -8\int_{t_0}^t (\mu - r)\frac{\mu - r - \sigma a(u)}{V_u}\, du \right\} \right] \\
& + K\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} |\Pi_t|^8 \right] \\
\le & K + K\mathrm{E}_{t_0,v_0,x_0}\left[\exp\left(C\int_{t_0}^T \frac{1}{V_t}\, dt \right) \right] + K\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} |\Pi_t|^8 \right] < \infty,
\end{aligned}$$

where $K > 0$ is a constant that differs between lines and $C = 8(\mu - r)(\mu - r + \sigma a(t_0)) > 0$. This shows that the second condition in Definition 2.2.1 that $\mathrm{E}_{t_0,v_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^*|^2 \right] < \infty$ is satisfied by Jensen's inequality. In view of (2.3.19), we further find that the first condition in Definition 2.2.1 holds as well since

$$\begin{aligned}
&\mathrm{E}_{t_0,v_0,x_0}\left[\int_{t_0}^T (u^*(t, V_t, X_t^*))^2 V_t\, dt \right] \\
= & \int_{t_0}^T \mathrm{E}_{t_0,v_0,x_0}\left[\frac{(X_t^* - \gamma e^{-r(T-t)})^2(\mu - r - \sigma a(t))^2}{V_t} \right] dt \\
\le & K\int_{t_0}^T \mathrm{E}_{t_0,v_0,x_0}\left[|X_t^* - \gamma e^{-r(T-t)}|^4 + \frac{1}{V_t^2} \right] dt \\
< & K\left\{ \mathrm{E}_{t_0,v_0,x_0}[\sup_{t\in[t_0,T]} |X_t^*|^4] + \int_{t_0}^T \mathrm{E}_{t_0,v_0,x_0}\left[\frac{1}{V_t^2} \right] dt \right\} < \infty,
\end{aligned}$$

where $K > 0$ is a constant that differs between lines and last strict inequality comes from (2.3.19) and the fact that $1/V_t$ is a CIR process (see the proof of Lemma

2.3.6 above) with finite second moment $\mathrm{E}_{t_0,v_0,x_0}\left[\frac{1}{V_t^2}\right]$ at time $t \in [t_0, T]$ which is continuous in t (see, for example, Cox, Ingersoll, and Ross (1985)). These results show that the optimal strategy u^* (2.3.13) is admissible. □

2.4 Static and dynamic optimality of the problem

In this section, we devote to deriving the static and dynamic optimality of problem (2.2.4) by exploiting the results above. In regard to the static optimality of problem (2.2.4), it now suffices to maximize the following optimization problem with respect to the Lagrange multiplier $\theta \in \mathbb{R}$ in view of (2.2.5) and (2.2.6) above

$$\max_{\theta \in \mathbb{R}} J(x_0, v_0; u^*, \xi - \theta) - \theta^2. \tag{2.4.1}$$

Reformulating (2.4.1) in terms of a quadratic functional over $\theta \in \mathbb{R}$, we find that the value function of problem (2.2.4) can be obtained from

$$\begin{aligned} V_{MV}(t_0, v_0, x_0) = \max_{\theta \in \mathbb{R}} \Big\{ &\left(e^{-a(t_0)\frac{1}{v_0} - b(t_0)} - 1\right)\theta^2 + 2e^{-a(t_0)\frac{1}{v_0} - b(t_0)}\left(x_0 e^{r(T-t_0)} - \xi\right)\theta \\ &+ e^{-a(t_0)\frac{1}{v_0} - b(t_0)}\left(x_0 e^{r(T-t_0)} - \xi\right)^2 \Big\}. \end{aligned} \tag{2.4.2}$$

Upon considering the exponential expression of function $g(t, v)$ given in Lemma 2.3.7 above, the right-hand side of (2.4.2) is then a quadratic function of $\theta \in \mathbb{R}$ with strictly negative leading coefficient. Therefore, to the right-hand side of (2.4.2) the maximum is uniquely attained at

$$\theta^* = \frac{x_0 e^{r(T-t_0)} - \xi}{e^{a(t_0)\frac{1}{v_0} + b(t_0)} - 1}. \tag{2.4.3}$$

Theorem 2.4.1. *Assume Assumption 2.3.1 holds true, then for $(t_0, s_0, x_0) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R}$ given and fixed such that $x_0 e^{r(T-t_0)} < \xi$, the statically optimal (Markovian) strategy of problem (2.2.4) is*

$$u^*(t, v, x) = -\left(x - \frac{\xi e^{-r(T-t) + a(t_0)\frac{1}{v_0} + b(t_0)} - x_0 e^{r(t-t_0)}}{e^{a(t_0)\frac{1}{v_0} + b(t_0)} - 1}\right)\frac{\mu - r - \sigma a(t)}{v} \tag{2.4.4}$$

for $t \in [t_0, T)$, where functions $a(t)$ and $b(t)$ are given in (2.3.8) and (2.3.9), respectively. The corresponding value function is

$$V_{MV}(t_0, v_0, x_0) = \frac{1}{e^{a(t_0)\frac{1}{v_0} + b(t_0)} - 1}\left(x_0 e^{r(T-t_0)} - \xi\right)^2. \tag{2.4.5}$$

The controlled wealth process $X_t^{u^*}$ is given by

$$\begin{aligned} X_t^* = &\frac{x_0 e^{r(t-t_0) + a(t_0)\frac{1}{v_0} + b(t_0)} - \xi e^{-r(T-t) + a(t_0)\frac{1}{v_0} + b(t_0)}}{e^{a(t_0)\frac{1}{v_0} + b(t_0)} - 1}\Pi_t \\ &\cdot \exp\left\{-\int_{t_0}^t (\mu - r)\frac{\mu - r - \sigma a(u)}{V_u}\,du\right\} + \frac{\xi e^{-r(T-t) + a(t_0)\frac{1}{v_0} + b(t_0)} - x_0 e^{r(t-t_0)}}{e^{a(t_0)\frac{1}{v_0} + b(t_0)} - 1}, \end{aligned} \tag{2.4.6}$$

where Π_t is given in (2.3.12). Moreover, the statically optimal strategy u^* belongs to $\mathcal{U}$.

Proof. Substituting (2.4.3) into (2.4.2) gives the value function (2.4.5). Replacing the constant γ in (2.3.13) and (2.3.15) with $\xi - \theta^*$ yields the statically optimal strategy (2.4.4) and the wealth process (2.4.6), respectively. In view of the proof in Proposition 2.3.11 above, it is obvious that the statically optimal strategy u^* (2.4.4) is admissible. □

As discussed in Section 2.2, the statically optimal strategy u^* (2.4.4) derived in Theorem 2.4.1 relies on the initial value (t_0, v_0, x_0). This implies that once the investor arrives at a new position (t, v, x) at later times, the statically optimal strategy u^* determined at the initial position would be sub-optimal. Now we give the dynamically optimal strategy u^{d*} of the problem (2.2.4) within the framework developed in Pedersen and Peskir (2017).

Theorem 2.4.2. *Assume Assumption 2.3.1 holds, then for $(t_0, v_0, x_0) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R}$ given and fixed such that $x_0 e^{r(T-t_0)} < \xi$, the dynamically optimal (Markovian) strategy of problem (2.2.4) for $t \in [t_0, T)$ is*

$$u^{d*}(t, v, x) = -\frac{x e^{a(t)\frac{1}{v}+b(t)} - \xi e^{-r(T-t)+a(t)\frac{1}{v}+b(t)}}{e^{a(t)\frac{1}{v}+b(t)} - 1} \frac{\mu - r - \sigma a(t)}{v}. \tag{2.4.7}$$

The corresponding controlled wealth process X_t^{d*} is

$$\begin{aligned} X_t^{d*} = &\left(x_0 e^{r(t-t_0)} - \xi e^{-r(T-t)}\right) \exp\left\{ \int_{t_0}^t -(\mu - r) \frac{e^{a(u)\frac{1}{V_u}+b(u)}}{e^{a(u)\frac{1}{V_u}+b(u)} - 1} \frac{\mu - r - \sigma a(u)}{V_u} \, du \right. \\ &\left. - \frac{1}{2} \frac{e^{2a(u)\frac{1}{V_u}+2b(u)}}{\left(e^{a(u)\frac{1}{V_u}+b(u)} - 1\right)^2} \frac{(\mu - r - \sigma a(u))^2}{V_u} \, du \right\} \\ &\cdot \exp\left\{ - \int_{t_0}^t \frac{e^{a(u)\frac{1}{V_u}+b(u)}}{e^{a(u)\frac{1}{V_u}+b(u)} - 1} \frac{\mu - r - \sigma a(u)}{\sqrt{V_u}} \, dW_u \right\} + \xi e^{-r(T-t)} \end{aligned} \tag{2.4.8}$$

satisfying $X_t^{d*} e^{r(T-t)} < \xi$ for $t \in [t_0, T)$.

Proof. To derive a candidate for the dynamic optimality u^{d*} over $t \in [t_0, T)$, we identify t_0 with t, x_0 with x and v_0 with v in the statically optimal strategy given in (2.4.4). We then immediately find a candidate for the dynamically optimal strategy

$$u^{d*}(t, v, x) = -\frac{x e^{a(t)\frac{1}{v}+b(t)} - \xi e^{-r(T-t)+a(t)\frac{1}{v}+b(t)}}{e^{a(t)\frac{1}{v}+b(t)} - 1} \frac{\mu - r - \sigma a(t)}{v}. \tag{2.4.9}$$

In what follows, we show that this candidate (2.4.9) is indeed dynamically optimal in problem (2.2.4). To see this, we take any other admissible control $\pi \in \mathcal{U}$ such that $\mathrm{E}_{t,v,x}[X_T^\pi] = \xi$ and $\pi(t,v,x) \neq u^{d*}(t,v,x)$, and we set $w = u^*$ under the measure $\mathrm{P}_{t,v,x}$. We note from (2.4.4) with (t_0, v_0, x_0) replaced by (t,v,x) that $u^*(t,v,x) = u^{d*}(t,v,x)$, and thus, we have $w(t,v,x) = u^*(t,v,x) = u^{d*}(t,v,x) \neq \pi(t,v,x)$ for any $t \in [0,T)$. Then by continuity of π and w, there exists a ball $B_\varepsilon := [t, t+\varepsilon] \times [v-\varepsilon, v+\varepsilon] \times [x-\varepsilon, x+\varepsilon]$ such that $w(\tilde{t}, \tilde{v}, \tilde{x}) \neq \pi(\tilde{t}, \tilde{v}, \tilde{x})$ for any $(\tilde{t}, \tilde{v}, \tilde{x}) \in B_\varepsilon$ when $\varepsilon > 0$ is small enough and satisfies $t + \varepsilon \leq T$. We observe from (2.3.18) that $w = u^*$ is, in fact, the unique continuous function such that the minimum within the expectation on the right-hand side of (2.3.18) (with $\xi - \theta^*$ and (t,v,x) in place of γ and (t_0, v_0, x_0), respectively) is attained up to probability one. Therefore, we can set exiting time $\tau_\varepsilon = \inf\{t \wedge T | \ (t, V_t, X_t^\pi) \notin B_\varepsilon\}$, and we see that for $\tilde{t} \leq \tau_\varepsilon$

$$Y_{\tilde{t}} \left\{ \pi(\tilde{t}, V_{\tilde{t}}, X_{\tilde{t}}^\pi)\sqrt{V_{\tilde{t}}} + \left(\frac{\mu - r}{\sqrt{V_{\tilde{t}}}} + \frac{\Lambda_{\tilde{t}}}{Y_{\tilde{t}}} \right) G_{\tilde{t}} \right\}^2 \geq \zeta > 0, \ \mathrm{P}_{t,v,x}\text{-}a.s.$$

where ζ is a fixed positive constant. Now, from (2.3.18) with $\xi - \theta^*$ and (t,v,x) in place of γ and (t_0, v_0, x_0) respectively, we find that

$$\begin{aligned}
&\mathrm{E}_{t,v,x}[(X_T^\pi - (\xi - \theta^*))^2] \\
=&\mathrm{E}_{t,v,x}\left[\int_t^{\tau_\varepsilon} Y_{\tilde{t}} \left\{ \pi(\tilde{t}, V_{\tilde{t}}, X_{\tilde{t}}^\pi)\sqrt{V_{\tilde{t}}} + \left(\frac{\mu - r}{\sqrt{V_{\tilde{t}}}} + \frac{\Lambda_{\tilde{t}}}{Y_{\tilde{t}}} \right) G_{\tilde{t}} \right\}^2 d\tilde{t} \right] \\
&+ \mathrm{E}_{t,v,x}\left[\int_{\tau_\varepsilon}^{T} Y_{t'} \left\{ \pi(t', V_{t'}, X_{t'}^\pi)\sqrt{V_{t'}} + \left(\frac{\mu - r}{\sqrt{V_{t'}}} + \frac{\Lambda_{t'}}{Y_{t'}} \right) G_{t'} \right\}^2 dt' \right] \\
&+ c\left(x - (\xi - \theta^*)e^{-r(T-t)} \right)^2 \\
\geq&\zeta \mathrm{E}_{t,v,x}[\tau_\varepsilon - t] + c\left(x - (\xi - \theta^*)e^{-r(T-t)} \right)^2 \\
>&c\left(x - (\xi - \theta^*)e^{-r(T-t)} \right)^2 \\
=&\mathrm{E}_{t,v,x}[(X_T^w - (\xi - \theta^*))^2],
\end{aligned} \tag{2.4.10}$$

where $c = \exp\left(2r(T-t) - a(t)\frac{1}{v} - b(t)\right)$ is a constant at position (t,v,x), and the strict inequality makes use of the fact that $\tau_\varepsilon > t$ since the pair (V, X^π) has continuous sample paths with probability one under $\mathrm{P}_{t,v,x}$ measure. From (2.4.10) we see that

$$\begin{aligned}
\mathrm{Var}_{t,v,x}(X_T^\pi) &= \mathrm{E}_{t,v,x}[(X_T^\pi)^2] - \xi^2 \\
&= \mathrm{E}_{t,v,x}[(X_T^\pi - (\xi - \theta^*))^2] - (\theta^*)^2 \\
&> \mathrm{E}_{t,v,x}[(X_T^w - (\xi - \theta^*))^2] - (\theta^*)^2 \\
&= \mathrm{Var}_{t,v,x}(X_T^w).
\end{aligned}$$

This shows that for any $(t,v,x) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$, the candidate u^{d*} (2.4.9) is the dynamically optimal (Markovian) strategy for problem (2.2.4).

We substitute u^{d*} (2.4.9) into the controlled wealth process (2.2.3) and denote the corresponding wealth process by X_t^{d*}. Using Itô's lemma to $Z_t := \xi - e^{r(T-t)}X_t^{d*}$ yields

$$\begin{aligned} dZ_t =& -(\mu-r)\frac{e^{a(t)\frac{1}{V_t}+b(t)}}{e^{a(t)\frac{1}{V_t}+b(t)}-1}\frac{\mu-r-\sigma a(t)}{V_t}Z_t\,dt \\ & -\frac{e^{a(t)\frac{1}{V_t}+b(t)}}{e^{a(t)\frac{1}{V_t}+b(t)}-1}\frac{\mu-r-\sigma a(t)}{\sqrt{V_t}}Z_t\,dW_t. \end{aligned} \tag{2.4.11}$$

We then obtain the closed-form expression of Z_t by solving the linear SDE (2.4.11)

$$\begin{aligned} Z_t =& z_0 \exp\left\{\int_{t_0}^t -(\mu-r)\frac{e^{a(u)\frac{1}{V_u}+b(u)}}{e^{a(u)\frac{1}{V_u}+b(u)}-1}\frac{\mu-r-\sigma a(u)}{V_u}\,du \right. \\ & \left. -\frac{1}{2}\frac{e^{2a(u)\frac{1}{V_u}+2b(u)}}{\left(e^{a(u)\frac{1}{V_u}+b(u)}-1\right)^2}\frac{(\mu-r-\sigma a(u))^2}{V_u}\,du\right\} \\ & \cdot \exp\left\{-\int_{t_0}^t \frac{e^{a(u)\frac{1}{V_u}+b(u)}}{e^{a(u)\frac{1}{V_u}+b(u)}-1}\frac{\mu-r-\sigma a(u)}{\sqrt{V_u}}\,dW_u\right\}, \end{aligned} \tag{2.4.12}$$

where $z_0 = \xi - x_0 e^{r(T-t_0)} > 0$. From the definition of Z_t and (2.4.8) we conclude that $X_t^{d*}e^{r(T-t)} < \xi$ for $t \in [t_0, T)$. Finally, the corresponding wealth process X_t^{d*} (2.4.8) follows from (2.4.12). □

2.5 Numerical examples

In this section, numerical examples are provided to analyze the impact of different parameters on the efficient frontier when the wealth process is controlled by the statically optimal strategy as well as to illustrate the differences between the dynamically optimal wealth and the statically optimal wealth derived in Section 4. Unless otherwise stated, we consider the following model parameters adapted from previous empirical studies (see, for example, Drimus (2012)): $r = 0.04,\ \mu = 0.2,\ \kappa = 22.84,\ \theta = 0.4689,\ \sigma = 8.56,\ x_0 = 1,\ v_0 = 0.245,\ t_0 = 0,\ T = 1,\ \xi = 4$.

Figure 2.1 shows us how the interest rate r affects the efficient frontier. We find that higher interest rate r results in larger $\mathrm{Var}_{t_0,v_0,x_0}(X_T^*)$ with the same $\mathrm{E}_{t_0,v_0,x_0}[X_T^*]$. One of the possible reasons is that although the investor can get a higher return by investing in the risk-free asset, the risk premium $(\mu-r)/\sqrt{V_t}$ decreases as r increases so that the investor indeed derives less expected return from the stock, and thus undertakes more risk. In summary, the impact of r on the stock is more significant than that on the risk-free asset.

Figure 2.2 shows how the return rate of the stock μ influences the efficient frontier. A higher level of the return rate of the stock price μ lowers the variance of terminal wealth $\mathrm{Var}_{t_0,v_0,x_0}(X_T^*)$ with the same $\mathrm{E}_{t_0,v_0,x_0}[X_T^*]$, which is quite clear due to the

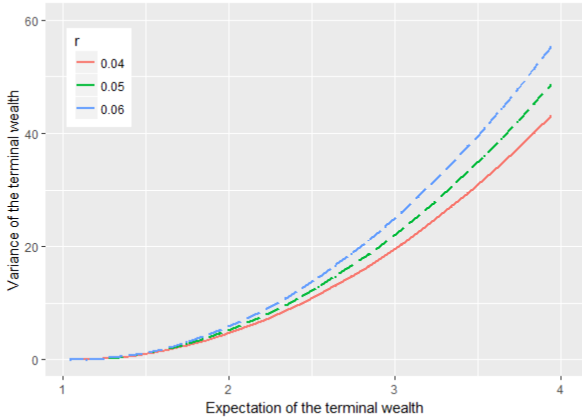


Figure 2.1: *Impact of r on the efficient frontier*

fact that the investor receives more risk premium as μ increases and the investor can therefore undertake less risk by investing less into the stock and more into the risk-free asset so as to have the same expected terminal wealth.

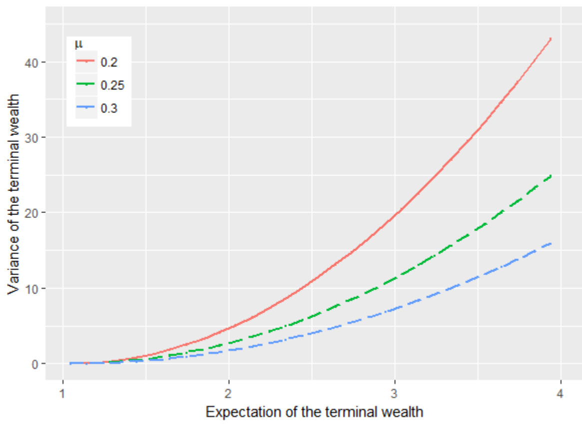


Figure 2.2: *Impact of μ on the efficient frontier*

The impact of the parameter κ on the efficient frontier is presented in Figure 2.3 below. We see that larger κ results in larger $\mathrm{Var}_{t_0,v_0,x_0}(X_T^*)$ with the same $\mathrm{E}_{t_0,v_0,x_0}[X_T^*]$. One possible reason is that as κ partly stands for the mean-reversion speed of the reciprocal of the stochastic volatility $1/V_t$ (recall the proof of Lemma 2.3.7 above), a larger κ results in a faster speed of $1/V_t$ towards the long-term level $(\kappa+\sigma^2)/\kappa\theta$. Meanwhile, we see that the long-term level is, in fact, decreasing in κ. These two aspects in turn make the volatility of the stock V_t stay longer around the relatively higher level $\kappa\theta/(\kappa+\sigma^2)$. Hence, the investor has to undertake more risk.

The effect of the parameter σ on the efficient frontier is given in Figure 2.4 which

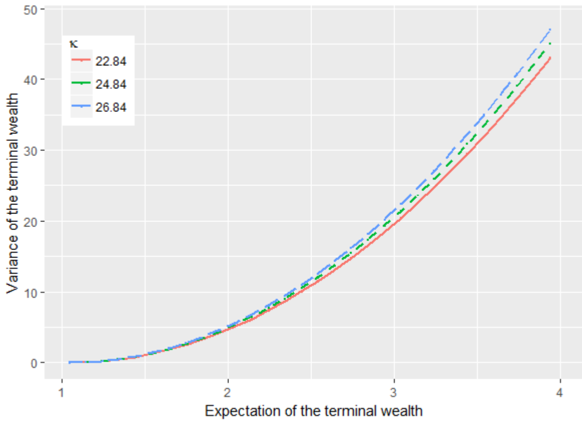


Figure 2.3: *Impact of κ on the efficient frontier*

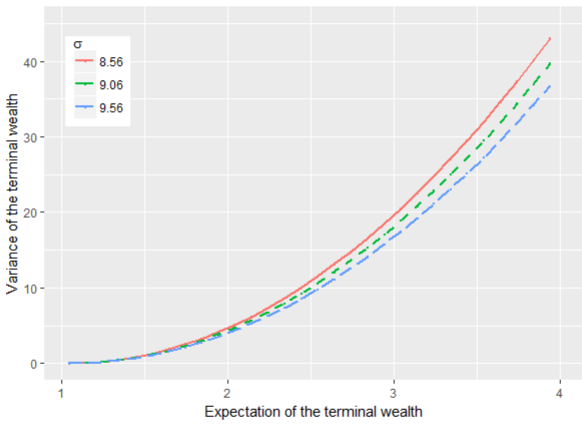


Figure 2.4: *Impact of σ on the efficient frontier*

shows that $\mathrm{Var}_{t_0,v_0,x_0}(X_T^*)$ decreases with the same $\mathrm{E}_{t_0,v_0,x_0}[X_T^*]$ as σ increases. Again, from the proof of Lemma 2.3.7 above, we see that σ plays a role as the volatility of the reciprocal of volatility process $1/V_t$, and a larger σ results in milder movements of the volatility process V_t. In addition, we see that the long-run level of volatility $\kappa\theta/(\kappa+\sigma^2)$ decreases as σ increases. Therefore, these two factors help the investor bear less risk.

To end this section, we show the dynamics of wealth processes controlled by the statically optimal strategy u^* (2.4.4) and the dynamically optimal strategy u^{d*} (2.4.7), respectively. By setting 500 equidistant time points over $[0,1]$, we simulate two paths of optimal wealth processes X_t^* and X_t^{d*}. Figure 2.5 illustrates the significant difference between the dynamically optimal wealth process X_t^{d*} and the statically optimal wealth process X_t^*. In particular, we see that the result supports the conclusion of Theorem 2.4.2 above that the dynamically optimal wealth X_t^{d*} is strictly smaller than the expected terminal wealth $\xi = 4$ when $t < 1$.

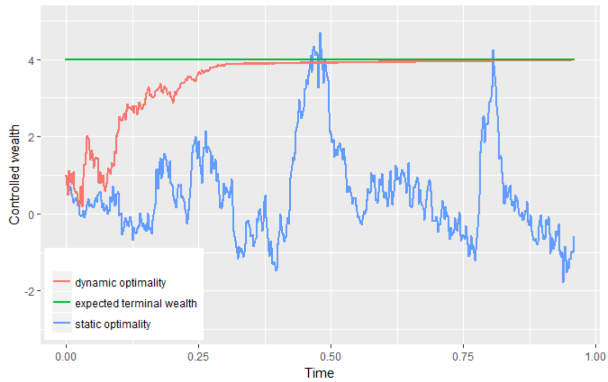


Figure 2.5: *Statically optimal wealth X_t^* and dynamically optimal wealth X_t^{d*}*

2.6 Conclusions

In this paper, a dynamically optimal mean-variance portfolio selection problem within the framework developed in Pedersen and Peskir (2017) in a stochastic environment has been investigated. A 3/2 stochastic volatility model is used to characterize the stochastic volatility of the stock. Considering the methodology in Pedersen and Peskir (2017) to tackle the time inconsistency of the optimality under the mean-variance criterion, we first address the static optimality and solve it by using a general BSDE approach. Under an assumption on model parameters, we obtain the static optimality and the corresponding value function explicitly. By solving the static optimality in an infinitesimally small period of time, the closed-form expression of the dynamic optimality is derived. Considering some technical difficulties, however, we have only studied the case without any state constraint. One branch of research topics in the future is to impose path-wise constraints on the wealth process; see, for example, Pedersen and Peskir (2018).

Acknowledgments and declarations of interest

We would like to thank Prof. Jesper Lund Pedersen for his general comments and discussions during this work.

Chapter 3

Dynamic optimal mean-variance portfolio selection with stochastic volatility and stochastic interest rate

Abstract

This paper studies optimal portfolio selection problems in the presence of stochastic volatility and stochastic interest rate under the mean-variance criterion. The financial market consists of a risk-free asset (cash), a zero-coupon bond (roll-over bond), and a risky asset (stock). Specifically, we assume that the interest rate follows the Vasicek model, and the risky asset's return rate not only depends on a Cox-Ingersoll-Ross (CIR) process but also has stochastic covariance with the interest rate, which embraces the family of state-of-the-art 4/2 stochastic volatility models as an exceptional case. By adopting a backward stochastic differential equation (BSDE) approach and solving two related BSDEs, we derive, in closed form, the static optimal (time-inconsistent) strategy and optimal value function. Given the time inconsistency of the mean-variance criterion, a dynamic formulation of the problem is further investigated and the explicit expression for the dynamic optimal (time-consistent) strategy is derived. In addition, analytical solutions to some special cases of our model are provided. Finally, the impact of the model parameters on the efficient frontier and the behavior of the static and dynamic optimal asset allocations is illustrated with numerical examples.

Keywords: Mean-variance portfolio selection; Vasicek interest rate; CIR process; Dynamic optimality; Backward stochastic differential equation

3.1 Introduction

Mean-variance portfolio selection problem is concerned with the trade-off between profit (expected return) maximization and risk (variance) minimization. The pioneering work of Markowitz (1952) laid the foundation for portfolio selection under the mean-variance criterion in a single-period setting. By applying an embedding technique and taking advantage of the stochastic linear-quadratic control theory, Li and Ng (2000) and Zhou and Li (2000) extended Markowitz's work to a multi-period and continuous-time setting, respectively. A notable feature of Zhou and Li (2000) is that the exogenous parameter processes are assumed to be only constants or deterministic functions. To generalize Zhou and Li (2000)'s results to more realistic environments, Lim and Zhou (2002) considered a complete market where the model coefficients are assumed to be uniformly bounded stochastic processes. By exploiting the backward stochastic differential equation (BSDE) theory (El Karoui, Peng, and Quenez (1997)), they solved the mean-variance problem by relating the optimal strategy to the solution to the associated BSDEs. Lim (2004) went a step forward by extending the results and methods of Lim and Zhou (2002) to an incomplete market setting under similar model assumptions. The uniform boundedness hypothesis, however, precludes the applications of local volatility and stochastic volatility models to the mean-variance portfolio selection problem, such as the constant elasticity of variance (CEV) model, Heston model (Heston (1993)), 3/2 model (Lewis (2000)), and the state-of-the-art 4/2 model (Grasselli (2017)). For this reason, many researchers drew on a more general market by relaxing the uniform boundedness hypothesis in recent years. For example, Shen, Zhang, and Siu (2014) investigated a mean-variance portfolio selection problem under the CEV model, and explicit solutions were obtained by using a BSDE approach and assuming that the market price of volatility risk satisfies exponential integrability of infinitely large order. Shen and Zeng (2015) further considered the optimal investment-reinsurance problem for a mean-variance insurer in an incomplete market, where the market price of risk is proportional to a Markovian, affine-form, and square-root factor process. By using similar techniques, Tian, Guo, and Sun (2021) studied a mean-variance investment-reinsurance problem when the return rate of the stock follows an Ornstein-Uhlenbeck (OU) process. As the literature on the mean-variance portfolio selection problems is abundant, the above review is not exhaustive. Other relevant works include Chiu and Wong (2011), Yu (2013), Lv, Wu, and Yu (2016), Sun and Guo (2018), Sun, Zhang, and Yuen (2020), to name but only a few.

Although the mean-variance portfolio selection problems have been extensively investigated in the last decade, two aspects deserve further exploration. First, most of the preceding literature assumes that the interest rates are constants or deterministic functions, which violates the well-documented evidence that the short rates are stochastic, mainly referred to Vasicek (1977), Cox, Ingersoll, and Ross (1985), and Duffie and Kan (1996). It is noteworthy that, in the last few years, some research results on portfolio optimization problems with stochastic interest rates have been achieved. For example, Ferland and Watier (2010) considered a portfolio selection problem under the mean-variance criterion in a complete market with an extended CIR interest rate and obtained the optimal strategy by using a BSDE approach. Assuming that the stochastic interest rate follows the Vasicek model, Shen and Siu (2012) studied an asset allocation problem with regime switching in an exponential utility maximization framework by using the dynamic programming approach. Chang (2015) concerned a mean-variance problem with random liabilities and Vasicek's stochastic interest rate and solved the problem explicitly for two special cases by using the dynamic programming approach. Guan and Liang (2014) investigated a defined contribution pension management problem under power utility in the presence of stochastic interest rates and stochastic volatility. By using similar methods to Guan and Liang (2014), Guan and Liang (2015) considered a similar problem under the mean-variance criterion with an affine-form stochastic interest rate and a stochastic return rate driven by an OU process. Recent works on the portfolio selection problems with stochastic interest rates include Yao, Li, and Lai (2016), Pan and Xiao (2017c), Escobar, Neykova, and Zagst (2017) and Escobar, Ferrando, and Rubtsov (2018), Chang et al. (2020), and references therein.

Second, the optimal investment strategies derived in most of the aforementioned literature on the mean-variance portfolio selection problems are time-inconsistent (Strotz (1956)), in the sense that the optimal strategies determined at the initial time might not be optimal at a future time point since the nonlinear operator within the objective function under the mean-variance criterion precludes the use of Bellman's principle of optimality. In recent years, there has been a growing interest in developing time-consistent mean-variance approaches. To deal with the time inconsistency under the mean-variance criterion, Basak and Chabakauri (2010) applied a backward recursion approach starting from the terminal date to determine a time-consistent optimal strategy. Alternatively, Björk, Khapko, and Murgoci (2017) proposed the Nash equilibrium approach by imposing a time-consistent constraint on the optimal strategy and derived the equilibrium optimal strategy and equilibrium value function by essentially solving an HJB equation under the Markovian market settings. Along this approach, readers may refer to Li, Zeng, and Lai (2012), Wei and Wang (2017), and Zhang, Li, and Lai (2020). Different from the equilibrium approach, the dynamic optimal approach championed by Pedersen and Peskir (2017) tackled the time inconsistency of the static optimal (time-inconsistent) strategy by performing an infinite number of the static optimality over the investment period, and they, therefore, derived a dynamic optimal (time-consistent) strategy. For other previous works along this line, one can refer to Pedersen and Peskir (2018), Zhang (2021b,a), and references therein.

Motivated by the above aspects, in this paper, we study a mean-variance portfolio selection problem that takes into consideration interest rate and volatility risks within the framework developed by Pedersen and Peskir (2017). Three primitive assets, one risk-free asset, one risky asset, and one zero-coupon bond, can be freely traded in the market. We assume that the stochastic interest rate is described by the Vasicek model. Inspired by Escobar, Ferrando, and Rubtsov (2018), the risky asset price exhibits not only stochastic volatility but also stochastic covariance with the interest rate. As opposed to most of the above-mentioned literature on the mean-variance portfolio selection problems, the risky asset's return rate and volatility are not specifically given. We only assume that the market price of volatility risk relies on a Cox-Ingersoll-Ross (CIR) process, which embraces the family of state-of-the-art 4/2 stochastic volatility models (Cheng and Escobar (2021a)) as a particular case. By applying a BSDE approach and solving the associated BSDE explicitly, closed-form expressions for the static optimal (time-inconsistent) strategy and optimal value function (efficient frontier) are derived. Following the methodology of Pedersen and Peskir (2017), we further consider a dynamic formulation of the mean-variance problem, and the explicit expression for the dynamic optimal (time-consistent) strategy is obtained by solving an infinite number of the static optimality over the investment period. Moreover, analytical solutions to some special cases of our model are provided. Finally, the economic impact of some model parameters on the efficient frontier as well as on the static and dynamic optimal asset allocations is illustrated with numerical examples. To sum up, the main contributions of this paper are as follows: (1) we consider a mean-variance portfolio selection problem in an incomplete market with interest rate and volatility risks, where the stochastic interest rate follows the Vasicek model while the market price of volatility risk is driven by a CIR process recovering the Heston model, 3/2 model, and 4/2 model as special cases. (2) Explicit expressions for the static optimal (time-inconsistent) and dynamic optimal (time-consistent) strategies are obtained by applying a BSDE approach. (3) The impact of some model parameters on the efficient frontier and the static and dynamic optimal asset allocations is shown.

The remainder of this paper is structured as follows. Section 3.2 introduces the financial market and formulates the mean-variance portfolio selection problems. In Section 3.3, we explore the solvability of a BSRE and a linear BSDE and obtain the explicit solutions. Section 3.4 presents both the static and dynamic optimality of the mean-variance problem, and closed-form solutions to some special cases are recovered. In Section 3.5, some numerical experiments are implemented to illustrate our theoretical results. Section 3.6 concludes the paper.

3.2 Formulation of the problem

Let $[0,T]$ be a fixed and finite horizon of decision making and $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ be a filtered complete probability space satisfying the usual conditions on which are defined three one-dimensional, mutually independent Brownian motions $\left\{W_t^0\right\}_{t\in[0,T]}, \left\{W_t^1\right\}_{t\in[0,T]}$, and $\left\{W_t^2\right\}_{t\in[0,T]}$. where filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by the above three Brownian motions, and P is a real-world probability measure.

3.2.1 Financial market

We consider a financial market with interest rate and volatility risks, where a risk-free asset (cash), a zero-coupon bond, and a risky asset (stock) can be continuously traded. Assume that the price of the risk-free asset, denoted by S_t^0, satisfies the following dynamics:

$$dS_t^0 = r_t S_t^0 \, dt$$

with initial value $S_{t_0}^0 = s_0$ at time $t_0 \in [0,T)$ fixed and given, and that the instantaneous interest rate r_t is governed by the Vasicek model:

$$dr_t = (a - br_t)\, dt - \sigma_r \, dW_t^0, \tag{3.2.1}$$

with initial value $r_{t_0} = r_0$, where $b \in \mathbb{R}^+$ is the mean-reversion speed, $a/b \in \mathbb{R}^+$ is the long-run level, and σ_r is the volatility of the interest rate. Suppose that the market price of interest rate risk is $\lambda_r \in \mathbb{R}^+$. From Vasicek (1977), the price process $B_t(u)$ of the zero-coupon bond with bond maturity u satisfies the following stochastic differential equation (SDE):

$$dB_t(u) = r_t B_t(u)\, dt + h_0(u-t) B_t(u) \sigma_r \left(\lambda_r \, dt + dW_t^0\right), \; t \le u \tag{3.2.2}$$

with boundary condition $B_u(u) = 1$, where the deterministic function $h_0(t)$ is given by

$$h_0(t) = \frac{1}{b}\left(1 - e^{-bt}\right).$$

We notice that the maturity of the zero-coupon bond $B_t(u)$, $u-t$, varies continuously as time t evolves. However, as it is stated in Boulier, Huang, and Taillard (2001), there may not exist zero-coupon bonds with any maturity in the market. We, therefore, introduce a rollover bond with a fixed time-to-maturity $K \in \mathbb{R}^+$ into the market. Denote by $B_t(K)$ the price of the rollover bond at time t. Then, the rollover bond $B_t(K)$ is of the form:

$$dB_t(K) = r_t B_t(K)\, dt + h_0(K)\sigma_r B_t(K) \left(\lambda_r \, dt + dW_t^0\right). \tag{3.2.3}$$

The risky asset price process S_t^1 is related to the risk of interest rate and governed by the following general stochastic volatility model:

$$dS_t^1 = S_t^1 \left[\mu\left(t, r_t, \alpha_t\right) dt + \eta_r \sigma_r \, dW_t^0 + \sigma\left(t, \alpha_t\right) dW_t^1\right], \; S_{t_0}^1 = s_0^1 \in \mathbb{R}^+, \tag{3.2.4}$$

where μ and $\sigma \neq 0$ are two possibly unbounded and continuous functions and related to each other via:

$$\mu(t, r_t, \alpha_t) - r_t = \lambda\sqrt{\alpha_t}\sigma(t, \alpha_t) + \lambda_r\eta_r\sigma_r$$

with $\lambda, \eta_r \in \mathbb{R}$, and α_t is an observable stochastic factor process following the CIR model:

$$d\alpha_t = \kappa(\theta - \alpha_t)\, dt + \sigma_\alpha\sqrt{\alpha_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right), \ \alpha_{t_0} = \alpha_0 \in \mathbb{R}^+, \quad (3.2.5)$$

where $\kappa \in \mathbb{R}^+$ is the speed of mean reversion, $\theta \in \mathbb{R}^+$ is the long-run mean, σ_α is the volatility of the factor process α_t, and $\rho \in [-1, 1]$ is the correlation coefficient between the risky asset price and factor process. In particular, we posit that the Feller condition is satisfied, i.e. $2\kappa\theta \geq \sigma_\alpha^2$, so that the factor process α_t driving the volatility of the risky asset price is strictly positive $\mathbb{P}$ almost surely, for $t \in [t_0, T]$.

Remark 3.2.1. Notice that the risky asset price process (3.2.4) exhibits not only stochastic volatility via the function σ and factor process α_t (3.2.5), but also stochastic instantaneous correlation $\frac{\eta_r\sigma_r}{\sqrt{\eta_r^2\sigma_r^2+\sigma^2(t,\alpha_t)}} \in [-1, 1]$ with the interest rate process r_t (3.2.1), in which the parameter η_r measures the impact of the interest rate dynamics on the risky asset price, and the specification $\eta_r = 0$ corresponds to the case when the interest rate and risky asset price are uncorrelated. It is also noteworthy that functions μ and σ allow for more flexibility in modeling the risky asset price. In what follows, we shall see that the modeling framework includes the family of the state-of-the-art 4/2 stochastic volatility models, as an exceptional case.

Example 3.2.2 (The 4/2 model)**.** If $\sigma(t, \alpha_t) = c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$, $\mu(t, \alpha_t, r_t) = r_t + \lambda(c_1\alpha_t + c_2) + \lambda_r\eta_r\sigma_r$ with constants $c_1 \geq 0$ and $c_2 \geq 0$, and $\alpha_t = V_t$, then the risky asset price process is given by the 4/2 model (Grasselli (2017)):

$$\begin{cases} dS_t^1 = S_t^1\left[(r_t + \lambda(c_1V_t + c_2) + \lambda_r\eta_r\sigma_r)\, dt + \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}\right) dW_t^1 + \eta_r\sigma_r\, dW_t^0\right], \\ dV_t = \kappa(\theta - V_t)\, dt + \sigma_\alpha\sqrt{V_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right), \end{cases} \quad (3.2.6)$$

where V_t is the instantaneous variance driver process, and parameters c_1 and c_2 characterize the superposition of the two embedded parsimonious models, the Heston model (Heston (1993)) and 3/2 model (Lewis (2000)). More specifically, the case $(c_1, c_2) = (1, 0)$ stands for the Heston model, while $(c_1, c_2) = (0, 1)$ corresponds to the 3/2 model.

Suppose that the investor has an initial wealth $x_0 \in \mathbb{R}^+$ at time t_0. Denote by two Markovian controls $\pi_B(t, \alpha_t, r_t, X_t^\pi)$ and $\pi_{S^1}(t, \alpha_t, r_t, X_t^\pi)$ the market value of wealth invested in the rollover bond $B_t(K)$ and risky asset S_t^1, respectively, where

$\pi := \left(\{\pi_B(\cdot)\}_{t\in[t_0,T]}, \{\pi_{S^1}(\cdot)\}_{t\in[t_0,T]} \right)$ represents the investment strategy and X_t^π is the associated wealth process. Under a self-financing condition, the wealth of the investor evolves according to

$$\begin{aligned} dX_t^\pi = &\Big[r_t X_t^\pi + \pi_B(t,\alpha_t,r_t,X_t^\pi) h_0(K)\sigma_r\lambda_r + \pi_{S^1}(t,\alpha_t,r_t,X_t^\pi) \Big(\sigma(t,\alpha_t)\lambda\sqrt{\alpha_t} \\ &+ \lambda_r\eta_r\sigma_r \Big) \Big] \, dt + \left(\pi_B(t,\alpha_t,r_t,X_t^\pi) h_0(K)\sigma_r + \pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)\eta_r\sigma_r \right) dW_t^0 \\ &+ \pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)\sigma(t,\alpha_t)\, dW_t^1. \end{aligned} \tag{3.2.7}$$

Throughout the rest of the paper, we denote by P_{t_0} the probability measure with initial data $(\alpha_{t_0}, r_{t_0}, X_{t_0}^\pi) = (\alpha_0, r_0, x_0)$ at time $t_0 \in [0,T]$, and $\mathrm{E}_{t_0}[\cdot]$ and $\mathrm{Var}_{t_0}(\cdot)$ denote the associated expectation and variance, respectively.

Definition 3.2.3 (Admissible strategy)). *Given any fixed initial time $t_0 \in [0,T)$, a Markovian investment strategy π is said to be admissible if the following conditions are met:*

1. *SDE (3.2.7) associated with π has a pathwise unique solution;*
2. $\mathrm{E}_{t_0}\left[\int_0^T \pi_B^2(t,\alpha_t,r_t,X_t^\pi) + \pi_{S_1}^2(t,\alpha_t,r_t,X_t^\pi)\, dt\right] < +\infty$;
3. $\mathrm{E}_{t_0}\left[\int_0^T \pi_{S_1}^2(t,\alpha_t,r_t,X_t^\pi)\sigma^2(t,\alpha_t)\, dt\right] < +\infty$;
4. $\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]} |X_t^\pi|^4\right] < +\infty$.

The set of admissible strategies is denoted by $\mathcal{A}$.

Remark 3.2.4. Due to the unboundedness of interest rate process r_t and factor process α_t in the meantime, the square integrability condition for the associated wealth process adopted by some preceding literature, such as Tian, Guo, and Sun (2021), Sun, Zhang, and Yuen (2020), and Zhang (2021b,a), is not sufficient to apply the dominated convergence theorem on the left-hand side of (3.G.2) to exchange the order of limit and expectation. We, therefore, opt for the fourth-order integrability condition for the wealth process X_t^π, i.e. condition 4 in Definition 3.2.3.

3.2.2 Optimization problems

We consider the investor who wants to trade over the time interval $[t_0, T]$ to minimize the variance of the terminal wealth, while the expected value is exogenously determined, i.e., under the mean-variance criterion. Formally, the mean-variance portfolio selection problem is defined as follows.

Definition 3.2.5. *The mean-variance portfolio problem is a constrained stochastic optimization problem:*

$$\begin{cases} \min\limits_{\pi \in \mathcal{A}} \ \mathrm{Var}_{t_0}(X_T^{\pi}) \\ \text{subject to } \mathrm{E}_{t_0}[X_T^{\pi}] = \xi, \end{cases} \tag{3.2.8}$$

where ξ is a fixed and given constant. We denoted by $V_{MV}(t_0, \alpha_0, r_0, x_0)$ and π^* the optimal value function and optimal investment strategy, respectively.

Considering the time inconsistency of the mean-variance criterion as discussed in the introduction, it is expected to see that the resulting optimal strategy relies on the initial value of state variables $(t_0, \alpha_0, r_0, x_0)$ and might not be guaranteed to be optimal at a future time point. To address this problem, we opt for the dynamic optimal approach introduced by Pedersen and Peskir (2017). For the readers' convenience, we now present the definition of dynamic optimality, which is slightly modified from Definition 2 in Pedersen and Peskir (2017), to adapt to the current context.

Definition 3.2.6 (Dynamic optimality)**.** *Given any fixed initial time $t_0 \in [0, T)$, a Markovian investment strategy $\pi^{d*} =: \left(\left\{ \pi_B^{d*}(\cdot) \right\}_{t \in [t_0, T]}, \left\{ \pi_{S^1}^{d*}(\cdot) \right\}_{t \in [t_0, T]} \right)$ is referred to as the dynamic optimality for the mean-variance problem (3.2.8) if for every $(t, \alpha, r, x) \in [t_0, T) \otimes \mathbb{R}^+ \otimes \mathbb{R} \otimes \mathbb{R}$ and every admissible strategy $u \in \mathcal{A}$ with $u(t, \alpha, r, x) \neq \pi^{d*}(t, \alpha, r, x)$ and $\mathrm{E}_{t,\alpha,r,x}[X_T^u] = \xi$, there is a Markovian strategy w satisfying $w(t, \alpha, r, x) = \pi^{d*}(t, \alpha, r, x)$ and $\mathrm{E}_{t,\alpha,r,x,}[X_T^w] = \xi$ such that*

$$\mathrm{Var}_{t,\alpha,r,x}(X_T^w) < \mathrm{Var}_{t,\alpha,r,x}(X_T^u),$$

where $\mathrm{E}_{t,\alpha,r,x}[X_T^{\pi}] = \mathrm{E}[X_T^{\pi} | \ \alpha_t = \alpha, r_t = r, X_t^{\pi} = x]$ and $\mathrm{Var}_{t,\alpha,r,x}(X_T^{\pi}) = \mathrm{E}_{t,\alpha,r,x}[(X_T^{\pi})^2] - (\mathrm{E}_{t,\alpha,r,x}[X_T^{\pi}])^2$.

Remark 3.2.7. According to Pedersen and Peskir (2017), the dynamic optimality π^{d*} is essentially derived by solving the static optimal strategy, i.e., π^* at each time and implementing it in an infinitesimally small period of time. In other words, the static optimality shall be considered in the first place.

Since the mean-variance problem (3.2.8) involves a convex objective functional, the associated linear constraint $\mathrm{E}_{t_0}[X_T^{\pi}] = \xi$ can be eliminated by introducing the following auxiliary Lagrange dual function:

$$\begin{aligned} \mathcal{L}(\alpha_0, r_0, x_0; \pi, \theta) :&= \mathrm{E}_{t_0}[(X_T^{\pi} - \xi)^2] + 2\theta \mathrm{E}_{t_0}[X_T^{\pi} - \xi] \\ &= \mathrm{E}_{t_0}\left[(X_T^{\pi} - (\xi - \theta))^2 \right] - \theta^2, \end{aligned} \tag{3.2.9}$$

where $\theta \in \mathbb{R}$ is the Lagrange multiplier. According to the Lagrangian duality theorem (see, for example, Luenberger (1968)), problem (3.2.8) is equivalent to the following min-max problem:

$$\max_{\theta\in\mathbb{R}}\min_{\pi\in\mathcal{A}}\mathcal{L}(\alpha_0,r_0,x_0;\pi,\theta), \tag{3.2.10}$$

which implies that it remains to first consider the following benchmark problem:

$$\min_{\pi\in\mathcal{A}} J(\alpha_0,r_0,x_0;\pi,\gamma) := \min_{\pi\in\mathcal{A}} \mathrm{E}_{t_0}\left[(X_T^\pi-\gamma)^2\right], \tag{3.2.11}$$

where $\gamma=\xi-\theta\in\mathbb{R}$.

3.3 Solution to the benchmark problem

In this section, we mainly focus on the benchmark problem (3.2.11) using a BSDE approach. Before introducing the BSDEs associated with the benchmark problem (3.2.11), we present the following auxiliary results on the Vasicek model (3.2.1) and CIR model (3.2.5), which are modified from Lemma 4.3 in Benth and Karlsen (2005), Lemma 4.1 in Wei and Wang (2017), and Theorem 5.1 in Zeng and Taksar (2013), respectively.

Lemma 3.3.1. *For the Vasicek model (3.2.1), when c is a constant such that $c<\frac{b}{2\sigma_r^2(T-t_0)}$, the Laplace transform of r_t^2 is well-defined, i.e.,*

$$\mathrm{E}_{t_0}\left[\exp\left\{c\int_{t_0}^T r_t^2\,dt\right\}\right]<+\infty.$$

Lemma 3.3.2. *For the Vasicek model (3.2.1), $|r_t|$ has exponential moment of all order, i.e.,*

$$\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]} e^{p|r_t|}\right]<+\infty,\ \ \forall p\geq 0.$$

Lemma 3.3.3. *For the CIR model (3.2.5), when c is a constant such that $c\leq\kappa^2/2\sigma_\alpha^2$, the Laplace transform of α_t is well-defined, i.e.,*

$$\mathrm{E}_{t_0}\left[\exp\left\{c\int_{t_0}^T \alpha_t\,dt\right\}\right]<+\infty.$$

Having reviewed the above preliminary results, we now impose the following assumption to facilitate further discussions:

Assumption 3.3.4. *$48\sigma_r^2T<b$ and $\max\Big\{24\lambda(\lambda+\sigma_\alpha|\rho b(t_0)|),(276+48\sqrt{33})(\lambda^2+\sigma_\alpha^2 b^2(t_0))\Big\}\leq\kappa^2/2\sigma_\alpha^2$, where function $b(t)$ is given by (3.3.9) below.*

Remark 3.3.5. The monotonicity of function $b(t)$ shown in Proposition 3.3.9 implies that $|b(t)|$ decreases to 0 as T approaches 0, which indicates the mathematical feasibility of the assumption above when the investment horizon is small enough. From an economic point of view, Assumption 3.3.4 presents an upper bound for the slope λ of the market price of volatility risk. As stated in Korn and Kraft (2004), when λ is too large, undertaking volatility risk is rewarded too much by the market, and the optimal investment strategy might not be uniquely determined. Mathematically speaking, if the above technical condition is violated, the uniqueness result to the following BSRE (3.3.1) and linear BSDE (3.3.2) might not be ensured.

Considering the following BSRE and linear BSDE:

$$\begin{cases} dP_t = \left\{ \left[-2r_t + \lambda_r^2 + \lambda^2\alpha_t\right] P_t + 2\lambda_r\Gamma_{0,t} + 2\lambda\sqrt{\alpha_t}\Gamma_{1,t} + \dfrac{\Gamma_{0,t}^2}{P_t} + \dfrac{\Gamma_{1,t}^2}{P_t} \right\} dt \\ \qquad\quad + \Gamma_{0,t}\, dW_t^0 + \Gamma_{1,t}\, dW_t^1 + \Gamma_{2,t}\, dW_t^2, \\ P_T = 1, \\ P_t > 0, \text{ for all } t \in [t_0, T], \end{cases} \tag{3.3.1}$$

and

$$\begin{cases} dY_t = (r_t Y_t + Z_t\lambda_r)\, dt + Z_t\, dW_t^0, \\ Y_T = -\gamma. \end{cases} \tag{3.3.2}$$

Here, a solution to (3.3.1) is a triplet of $\mathbb{F}$-adapted processes $(P_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$; a solution to (3.3.2) is a pair of $\mathbb{F}$-adapted processes (Y_t, Z_t). It is noteworthy that these two kinds of BSDEs are with unbounded coefficients due to the unboundedness of the interest rate process r_t and factor process α_t, and thus, the results in Lim (2004) and El Karoui, Peng, and Quenez (1997) cannot be used in our case. Nevertheless, by observing that the driver of linear BSDE (3.3.2) follows a stochastic Lipschitz continuity (Bender and Kohlmann (2000)), we derive the unique solution to BSDE (3.3.2) in the next lemma.

Lemma 3.3.6. *Suppose that Assumption 3.3.4 holds true. The unique solution (Y_t, Z_t) to linear BSDE (3.3.2) is given by*

$$\begin{cases} Y_t = -\gamma \exp\left\{g(t) + h(t) r_t\right\}, \\ Z_t = -\sigma_r h(t) Y_t, \end{cases} \tag{3.3.3}$$

where functions $g(t)$ and $h(t)$ are given by

$$\begin{cases} g(t) = \left(\frac{\sigma_r^2}{2b^2} - \frac{a+\sigma_r\lambda_r}{b}\right)(T-t) + \left(\frac{a+\lambda_r\sigma_r}{b^2} - \frac{\sigma_r^2}{b^3}\right)\left(1-e^{-b(T-t)}\right) \\ \qquad + \frac{\sigma_r^2}{4b^3}\left(1-e^{-2b(T-t)}\right), \\ h(t) = \frac{1}{b}\left(e^{-b(T-t)}-1\right). \end{cases} \tag{3.3.4}$$

Proof. See Appendix 3.A. □

By using the Markovian structures of the interest rate process r_t and factor process α_t, we next manage to derive one explicit solution to BSRE (3.3.1) and show its uniqueness.

Lemma 3.3.7. *One solution $(P_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ to BSRE (3.3.1) is given by*

$$P_t = \exp\left\{a(t)r_t + b(t)\alpha_t\right\}\phi(t), \tag{3.3.5}$$

and

$$(\Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t}) = (-\sigma_r a(t)P_t, \sigma_\alpha\rho b(t)\sqrt{\alpha_t}P_t, \sigma_\alpha\sqrt{1-\rho}b(t)\sqrt{\alpha_t}P_t), \tag{3.3.6}$$

where functions $a(t), b(t)$, and $\phi(t)$ are solutions to the following ordinary differential equations (ODEs):

$$\begin{cases} \frac{da(t)}{dt} - ba(t) + 2 = 0, \ a(T) = 0, \\ \frac{db(t)}{dt} - (\kappa + 2\rho\sigma_\alpha\lambda)b(t) + \left(\frac{1}{2}-\rho^2\right)\sigma_\alpha^2 b^2(t) - \lambda^2 = 0, \ b(T) = 0, \\ \frac{d\phi(t)}{dt} + \left[(a+2\sigma_r\lambda_r)a(t) + \kappa\theta b(t) - \frac{1}{2}a^2(t)\sigma_r^2 - \lambda_r^2\right]\phi(t) = 0, \ \phi(T) = 1. \end{cases} \tag{3.3.7}$$

Proof. See Appendix 3.B. □

In the next proposition, we derive explicit solutions to ODEs (3.3.7), which provides the closed-form solution to BSRE (3.3.1).

Proposition 3.3.8. *The explicit solutions of $a(t), b(t)$, and $\phi(t)$ to ODEs (3.3.7) are given as follows:*

$$a(t) = \frac{2}{b}\left(1-e^{-b(T-t)}\right), \tag{3.3.8}$$

and

$$b(t)=\begin{cases}\dfrac{\lambda^2}{k+2\lambda\rho\sigma_\alpha}\left(e^{(k+2\lambda\rho\sigma_\alpha)(t-T)}-1\right), & \rho^2=\dfrac{1}{2},\ k+2\lambda\rho\sigma_\alpha\neq 0;\\ \lambda^2(t-T), & \rho^2=\dfrac{1}{2},\ k+2\lambda\rho\sigma_\alpha=0;\\ \dfrac{n_1n_2(1-e^{\sqrt{\Delta}(T-t)})}{n_1-n_2e^{\sqrt{\Delta}(T-t)}}, & \rho^2\neq\dfrac{1}{2},\ \Delta>0;\\ \dfrac{\sigma_\alpha^2(\rho^2-\frac{1}{2})(T-t)n_0^2}{\sigma_\alpha^2(\rho^2-\frac{1}{2})(T-t)n_0-1}, & \rho^2\neq\dfrac{1}{2},\ \Delta=0;\\ \dfrac{\sqrt{-\Delta}}{\sigma_\alpha^2(2\rho^2-1)}\tan\left(\arctan\left(\dfrac{k+2\lambda\rho\sigma_\alpha}{\sqrt{-\Delta}}\right)-\dfrac{\sqrt{-\Delta}}{2}(T-t)\right)+n_0, & \rho^2\neq\dfrac{1}{2},\ \Delta<0,\end{cases}\tag{3.3.9}$$

and

$$\phi(t)=\exp\left\{\int_t^T(a+2\sigma_r\lambda_r)a(s)+\kappa\theta b(s)-\frac{1}{2}a^2(s)\sigma_r^2-\lambda_r^2\,ds\right\},\tag{3.3.10}$$

where $\Delta, n_0, n_1,$ *and* n_2 *are given by*

$$\begin{cases}\Delta=(k+2\lambda\rho\sigma_\alpha)^2-(4\rho^2-2)\sigma_\alpha^2\lambda^2,\ n_0=\dfrac{-(k+2\lambda\rho\sigma_\alpha)}{\sigma_\alpha^2(2\rho^2-1)},\\ n_1=\dfrac{-(k+2\lambda\rho\sigma_\alpha)+\sqrt{\Delta}}{\sigma_\alpha^2(2\rho^2-1)},\ n_2=\dfrac{-(k+2\lambda\rho\sigma_\alpha)-\sqrt{\Delta}}{\sigma_\alpha^2(2\rho^2-1)}.\end{cases}\tag{3.3.11}$$

Proof. See Appendix 3.C. □

The next proposition shows that $b(t)$ is a strictly increasing function over $[t_0,T]$. In other words, the maximum value of $|b(t)|$ is attained at the initial time t_0.

Proposition 3.3.9. *Function* $b(t)$ *is monotonically increasing over* $[t_0,T]$.

Proof. See Appendix 3.D. □

Lemma 3.3.10. *Suppose that Assumption 3.3.4 holds true. The solution given in (3.3.5) and (3.3.6) is the unique solution to BSRE (3.3.1).*

Proof. See Appendix 3.E. □

Having derived the uniqueness results of BSRE (3.3.1) and linear BSDE (3.3.2), we now define the following two stochastic exponential processes $\Pi_{0,t}$ and $\Pi_{1,t}$, for $t\in[t_0,T]$,

$$\begin{cases}\Pi_{0,t}=\exp\left\{-\displaystyle\int_{t_0}^t(\lambda_r-\sigma_ra(s))\,dW_s^0-\int_{t_0}^t\frac{1}{2}\left(\lambda_r-\sigma_ra(s)\right)^2\,ds\right\},\\ \Pi_{1,t}=\exp\left\{-\displaystyle\int_{t_0}^t(\lambda+\sigma_\alpha\rho b(s))\sqrt{\alpha_s}\,dW_s^1-\int_{t_0}^t\frac{1}{2}(\lambda+\sigma_\alpha\rho b(s))^2\alpha_s\,ds\right\}.\end{cases}\tag{3.3.12}$$

In the next lemma, we investigate the integrability of $\Pi_{0,t}$ and $\Pi_{1,t}$, which shall be used in the proof of Proposition 3.3.12 below.

Lemma 3.3.11. *Suppose that Assumption 3.3.4 holds true. The stochastic exponential processes $\Pi_{0,t}$ and $\Pi_{1,t}$ defined in (3.3.12) satisfy*

$$\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]}|\Pi_{0,t}|^{12}+|\Pi_{1,t}|^{12}\right]<+\infty.$$

Proof. See Appendix 3.F. □

Based on the preceding results, we are ready to present the first main result of this paper, which relates the optimal strategy and optimal value function of the benchmark problem (3.2.11) to the solutions to BSRE (3.3.1) and linear BSDE (3.3.2).

Proposition 3.3.12. *Suppose that Assumption 3.3.4 holds true. For any initial data $(t_0,\alpha_0,r_0,x_0)\in[0,T)\otimes\mathbb{R}^+\otimes\mathbb{R}\otimes\mathbb{R}$ fixed and given, the optimal investment strategy, denoted by π^*, of the benchmark problem (3.2.11) is given by*

$$\begin{cases}\pi^*_{S^1}(t,\alpha_t,r_t,X^*_t)=-\dfrac{(X^*_t+Y_t)\left(\frac{\Gamma_{1,t}}{P_t}+\lambda\sqrt{\alpha_t}\right)}{\sigma(t,\alpha_t)},\\[2ex] \pi^*_B(t,\alpha_t,r_t,X^*_t)=-\dfrac{(X^*_t+Y_t)\left(\frac{\Gamma_{0,t}}{P_t}+\lambda_r\right)+Z_t+\pi^*_{S^1}(t,\alpha_t,r_t,X^*_t)\eta_r\sigma_r}{h_0(K)\sigma_r},\end{cases}\tag{3.3.13}$$

where $Y_t,Z_t,P_t,\Gamma_{0,t}$, and $\Gamma_{1,t}$ are given by (3.3.3), (3.3.5), and (3.3.6), respectively. The optimal value function is given by

$$J(\alpha_0,r_0,x_0;\pi^*,\gamma)=P_{t_0}\left(x_0+Y_{t_0}\right)^2,\tag{3.3.14}$$

and the wealth process X^*_t associated with the optimal strategy (3.3.13) evolves as

$$\begin{aligned}X^*_t=&\left(x_0-\gamma e^{g(t_0)+h(t_0)r_0}\right)\exp\left\{\int_{t_0}^t r_s-(\lambda_r^2-\lambda_r\sigma_r a(s))-(\lambda^2+\lambda\sigma_\alpha\rho b(s))\alpha_s\,ds\right\}\\&\times\Pi_{0,t}\Pi_{1,t}+\gamma e^{g(t)+h(t)r_t},\end{aligned}\tag{3.3.15}$$

where $g(t),h(t),a(t),b(t),\Pi_{0,t}$, and $\Pi_{1,t}$ are given in (3.3.4), (3.3.8), (3.3.9), and (3.3.12), respectively. Moreover, the optimal strategy given in (3.3.13) is admissible.

Proof. See Appendix 3.G.

□

3.4 Static and dynamic optimality of the problem

In this section, we derive the static and dynamic optimality of the mean-variance problem (3.2.8). The static optimal investment strategy and optimal value function (efficient frontier) of problem (3.2.8) are obtained by solving (3.2.9) and (3.2.11) in a backward sequence.

Specifically, based on the relationship between the mean-variance problem (3.2.8) and benchmark problem (3.2.11) as shown in (3.2.10), we have

$$
\begin{aligned}
&V_{MV}(t_0,\alpha_0,r_0,x_0)\\
=&\max_{\theta\in\mathbb{R}} J(\alpha_0,r_0,x_0;\pi^*,\xi-\theta)-\theta^2\\
=&\max_{\theta\in\mathbb{R}}\Big\{\left[\exp\{b(t_0)\alpha_0+2g(t_0)\}\phi(t_0)-1\right]\theta^2\\
&+2\exp\{(a(t_0)+h(t_0))r_0+b(t_0)\alpha_0+g(t_0)\}\phi(t_0)\left(x_0-\xi e^{g(t_0)+h(t_0)r_0}\right)\theta\\
&+\exp\{a(t_0)r_0+b(t_0)\alpha_0\}\phi(t_0)\left(x_0-\xi e^{g(t_0)+h(t_0)r_0}\right)^2\Big\}.
\end{aligned}
\tag{3.4.1}
$$

It can be easily checked that the leading coefficient of the above quadratic function of θ is negative, i.e,

$$
\exp\{b(t_0)\alpha_0+2g(t_0)\}\phi(t_0)<\exp\left\{-\int_{t_0}^{T}\left(\frac{a(t)\sigma_r}{2}-\lambda_r\right)^2dt\right\}\leq 1,
$$

where the strict inequality follows from the negativeness of function $b(t)$ implied by Proposition 3.3.9. As such, the maximum of the right-hand side of (3.4.1) is attained at

$$
\theta^*=\frac{\exp\{(a(t_0)+h(t_0))r_0+b(t_0)\alpha_0+g(t_0)\}\phi(t_0)\left(\xi e^{g(t_0)+h(t_0)r_0}-x_0\right)}{\exp\{b(t_0)\alpha_0+2g(t_0)\}\phi(t_0)-1}. \tag{3.4.2}
$$

Now we are ready to state our second main result.

Theorem 3.4.1. *Suppose that Assumption 3.3.4 holds true. For any initial data $(t_0,\alpha_0,r_0,x_0)\in[0,T)\otimes\mathbb{R}^+\otimes\mathbb{R}\otimes\mathbb{R}$ fixed and given, the static optimal investment strategy of the mean-variance problem (3.2.8) is given by*

$$
\begin{cases}
\pi^*_{S^1}(t,\alpha_t,r_t,X^*_t)=-\dfrac{\left(X^*_t-(\xi-\theta^*)e^{g(t)+h(t)r_t}\right)(\lambda+\sigma_\alpha\rho b(t))\sqrt{\alpha_t}}{\sigma(t,\alpha_t)},\\
\pi^*_B(t,\alpha_t,r_t,X^*_t)=-\dfrac{(X^*_t-(\xi-\theta^*)e^{g(t)+h(t)r_t})(\lambda_r-\sigma_r a(t))+(\xi-\theta^*)\sigma_r h(t)e^{g(t)+h(t)r_t}}{h_0(K)\sigma_r}\\
\qquad\qquad\qquad\qquad -\dfrac{\pi^*_{S^1}(t,\alpha_t,r_t,X^*_t)\eta_r}{h_0(K)},
\end{cases}
\tag{3.4.3}
$$

where θ^* is given by (3.4.2), and $g(t),h(t),a(t)$, and $b(t)$ are given by (3.3.4), (3.3.8), and (3.3.9), respectively. The optimal value function (efficient frontier) is *given by*

$$V_{MV}(t_0,\alpha_0,r_0,x_0) = \frac{\exp\left\{a(t_0)r_0 + b(t_0)\alpha_0\right\}\phi(t_0)\left(x_0 - \xi e^{g(t_0)+h(t_0)r_0}\right)^2}{1-\exp\left\{b(t_0)\alpha_0 + 2g(t_0)\right\}\phi(t_0)}, \quad (3.4.4)$$

and the wealth process X_t^* associated with (3.4.3) evolves according to

$$\begin{aligned} X_t^* = &\left(x_0 - (\xi-\theta^*)e^{g(t_0)+h(t_0)r_0}\right)\exp\left\{\int_{t_0}^t r_s - (\lambda_r^2 - \lambda_r\sigma_r a(s)) - (\lambda^2 + \lambda\sigma_\alpha\rho b(s))\alpha_s\, ds\right\} \\ &\times \Pi_{0,t}\Pi_{1,t} + (\xi-\theta^*)e^{g(t)+h(t)r_t}, \end{aligned} \quad (3.4.5)$$

where $\Pi_{0,t}$ and $\Pi_{1,t}$ are given by (3.3.12). Moreover, the static optimality (3.4.3) is admissible.

Proof. Replacing the constant γ in (3.3.13) and (3.3.15) by $\xi-\theta^*$ leads to the static optimal strategy (3.4.3) and the associated wealth process (3.4.5), respectively. Plugging θ^* given in (3.4.2) back into the right-hand side of (3.4.1) yields the optimal value function (3.4.4). Moreover, following the proof of Proposition 3.3.12, it is evident that the static optimal strategy (3.4.3) is admissible, i.e., $\pi^* \in \mathcal{A}$. □

The next corollary provides the explicit results for one special case of our model, the 4/2 stochastic volatility model.

Corollary 3.4.2 (The 4/2 model)**.** *Suppose that Assumption 3.3.4 holds true. If the risky asset price S_t^1 follows the 4/2 model (3.2.6), then the static optimal investment strategy and optimal value function of the mean-variance problem (3.2.8) are, respectively, given by*

$$\begin{cases} \pi_B^*(t,V_t,r_t,X_t^*) = -\dfrac{(X_t^* - (\xi-\theta^*)e^{g(t)+h(t)r_t})(\lambda_r - \sigma_r a(t)) + (\xi-\theta^*)\sigma_r h(t)e^{g(t)+h(t)r_t}}{h_0(K)\sigma_r} \\ \qquad\qquad\qquad\qquad - \dfrac{\pi_{S^1}^*(t,V_t,r_t,X_t^*)\eta_r}{h_0(K)}, \\ \pi_{S^1}^*(t,V_t,r_t,X_t^*) = -\dfrac{\left(X_t^* - (\xi-\theta^*)e^{g(t)+h(t)r_t}\right)(\lambda + \sigma_\alpha\rho b(t))\sqrt{V_t}}{c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}}, \end{cases}$$

and

$$V_{MV}(t_0,v_0,r_0,x_0) = \frac{\exp\left\{a(t_0)r_0 + b(t_0)v_0\right\}\phi(t_0)\left(x_0 - \xi e^{g(t_0)+h(t_0)r_0}\right)^2}{1-\exp\left\{b(t_0)v_0 + 2g(t_0)\right\}\phi(t_0)}.$$

Proof. Plugging the specified parameters of the 4/2 model (3.2.6) given in Example 3.2.2 into (3.4.3)-(3.4.4) leads to the above results. □

Remark 3.4.3. If we further specify $(c_1,c_2) = (1,0)$ and $(c_1,c_2) = (0,1)$ in Corollary 3.4.2, explicit solutions to the embedded Heston model and 3/2 stochastic volatility model are derived, respectively. To the best of our knowledge, there is no existing literature on the portfolio selection problems reporting the above results for the hybrid Vasicek-4/2 model under the mean-variance criterion.

As discussed in Section 3.2, the static optimal investment strategy given in Theorem 3.4.1 is time-inconsistent because it depends on the initial value of the state variables via θ^*, and thus, the mean-variance investor might deviate from it whenever any new position at a future time is arrived at. Now, we proceed to derive the dynamic optimality of the mean-variance problem (3.2.8) within the framework championed by Pedersen and Peskir (2017), which is the third main result of this paper.

Theorem 3.4.4. *Suppose that Assumption 3.3.4 holds true. For any initial data* $(t_0, \alpha_0, r_0, x_0) \in [0,T) \otimes \mathbb{R}^+ \otimes \mathbb{R} \otimes \mathbb{R}$ *fixed and given, the dynamic optimal investment strategy* π^{d*} *of the mean-variance problem (3.2.8) is given by*

$$\begin{cases} \pi^{d*}_{S^1}(t,\alpha_t,r_t,X^{d*}_t) = \dfrac{\left(X^{d*}_t - \xi e^{g(t)+h(t)r_t}\right)(\lambda+\sigma_\alpha\rho b(t))\sqrt{\alpha_t}}{(\exp\{b(t)\alpha_t+2g(t)\}\phi(t)-1)\sigma(t,\alpha_t)}, \\ \pi^{d*}_{B}(t,\alpha_t,r_t,X^{d*}_t) = \dfrac{X^{d*}_t\left(\lambda_r - \sigma_r a(t) - \exp\{b(t)\alpha_t+2g(t)\}\phi(t)h(t)\sigma_r\right)}{(\exp\{b(t)\alpha_t+2g(t)\}\phi(t)-1)h_0(K)\sigma_r} \\ \qquad - \dfrac{\xi e^{g(t)+h(t)r_t}(\lambda_r + h(t)\sigma_r)}{(\exp\{b(t)\alpha_t+2g(t)\}\phi(t)-1)h_0(K)\sigma_r} - \dfrac{\pi^{d*}_{S^1}(t,\alpha_t,r_t,X^{d*}_t)\eta_r}{h_0(K)}, \end{cases} \tag{3.4.6}$$

where X^{d*}_t *is the wealth process associated with* π^{d*} *and evolves according to*

$$\begin{aligned} X^{d*}_t =& \xi e^{g(t)+h(t)r_t} + \exp\left\{\int_{t_0}^t \left[\frac{\lambda_r\left(\lambda_r - \exp\{b(u)\alpha_u+2g(u)\}\phi(u)h(u)\sigma_r - \sigma_r a(u)\right)}{\exp\{b(u)\alpha_u+2g(u)\}\phi(u)-1}\right.\right. \\ &\left.\left.+\frac{(\lambda+\sigma_\alpha\rho b(u))\lambda\alpha_u}{\exp\{b(u)\alpha_u+2g(u)\}\phi(u)-1} + r_u\right]du\right\}\Pi_{2,t}\Pi_{3,t}\left(x_0 - \xi e^{g(t_0)+h(t_0)r_0}\right), \end{aligned} \tag{3.4.7}$$

with $\Pi_{2,t}$ *and* $\Pi_{3,t}$ *given by*

$$\begin{cases} \Pi_{2,t} = \exp\left\{\displaystyle\int_{t_0}^t \frac{(\lambda+\sigma_\alpha\rho b(u))\sqrt{\alpha_u}}{\exp\{b(u)\alpha_u+2g(u)\}\phi(u)-1}\,dW^1_u \right. \\ \qquad \left. -\dfrac{1}{2}\displaystyle\int_{t_0}^t \frac{(\lambda+\sigma_\alpha\rho b(u))^2\alpha_u}{(\exp\{b(u)\alpha_u+2g(u)\}\phi(u)-1)^2}\,du\right\}, \\ \Pi_{3,t} = \exp\left\{\displaystyle\int_{t_0}^t \frac{\lambda_r - \sigma_r a(u) - \exp\{b(u)\alpha_u+2g(u)\}\phi(u)h(u)\sigma_r}{\exp\{b(u)\alpha_u+2g(u)\}\phi(u)-1}\,dW^0_u \right. \\ \qquad \left. -\dfrac{1}{2}\displaystyle\int_{t_0}^t \frac{(\lambda_r - \sigma_r a(u) - \exp\{b(u)\alpha_u+2g(u)\}\phi(u)h(u)\sigma_r)^2}{(\exp\{b(u)\alpha_u+2g(u)\}\phi(u)-1)^2}\,du\right\}. \end{cases}$$

Furthermore, if the initial data satisfies $x_0 \leq \xi e^{g(t_0)+h(t_0)r_0}$*, then it holds that* $X^{d*}_t \leq \xi e^{g(t)+h(t)r_t}$*,* P_{t_0} *almost surely.*

Proof. See Appendix 3.H. □

Corollary 3.4.5 (The 4/2 model)**.** *Suppose that Assumption 3.3.4 holds true. If the risky asset price S_t^1 follows the 4/2 model (3.2.6), then the dynamic optimal investment strategy π^{d*} of the mean-variance problem (3.2.8) is given by*

$$\begin{cases} \pi_{S^1}^{d*}(t, V_t, r_t, X_t^{d*}) = \dfrac{\left(X_t^{d*} - \xi e^{g(t)+h(t)r_t}\right)(\lambda + \sigma_\alpha \rho b(t))V_t}{(\exp\{b(t)V_t + 2g(t)\}\phi(t) - 1)(c_1 V_t + c_2)}, \\ \pi_B^{d*}(t, V_t, r_t, X_t^{d*}) = \dfrac{X_t^{d*}(\lambda_r - \sigma_r a(t) - \exp\{b(t)V_t + 2g(t)\}\phi(t)h(t)\sigma_r)}{(\exp\{b(t)V_t + 2g(t)\}\phi(t) - 1)h_0(K)\sigma_r} \\ \qquad\qquad - \dfrac{\xi e^{g(t)+h(t)r_t}(\lambda_r + h(t)\sigma_r)}{(\exp\{b(t)V_t + 2g(t)\}\phi(t) - 1)h_0(K)\sigma_r} - \dfrac{\pi_{S^1}^{d*}(t, V_t, r_t, X_t^{d*})\eta_r}{h_0(K)}, \end{cases}$$

where the wealth process X_t^{d*} satisfies that $X_t^{d*} \leq \xi e^{g(t)+h(t)r_t}$, for $t \in [t_0, T]$, P_{t_0} almost surely.

Proof. Substituting the specified parameters of the 4/2 model (3.2.6) given in Example 3.2.2 into (3.4.6) yields the results immediately. □

Remark 3.4.6. Setting either $(c_1, c_2) = (1, 0)$ or $(c_1, c_2) = (0, 1)$ in Corollary 3.4.5, we provide the closed-form expressions for the dynamic optimal strategies under the Heston model and 3/2 model, respectively.

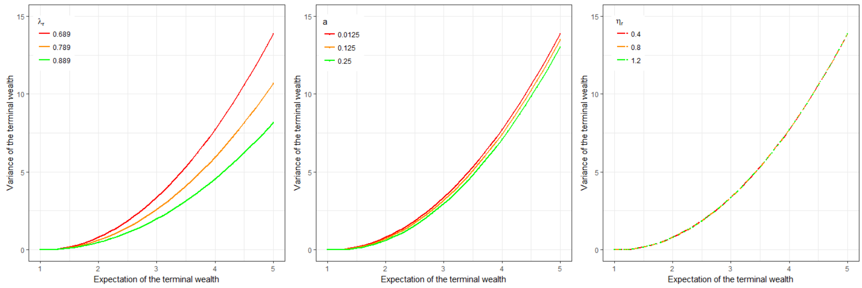
3.5 Numerical analysis

This section investigates the impact of the model parameters on the efficient frontier and the static and dynamic optimal investment strategies. The formula of efficient frontier is given by (3.4.4) and the closed-form expressions for the static and dynamic optimality are presented in (3.4.3) and (3.4.6), respectively. We show the case when the market model is characterized by the hybrid Vasicek-Heston model. Throughout this section, unless otherwise stated, the values of the parameters modified from Escobar, Neykova, and Zagst (2017) are listed below: $a = 0.0125, b = 0.266, \sigma_r = 0.013, \lambda_r = 0.689, \eta_r = 0.4, \lambda = 2.234, \kappa = 2.115, \theta = 0.051, \sigma_\alpha = 0.505, \rho = -0.514, x_0 = 1, r_0 = 0.05, v_0 = 0.03, T = 1, \xi = 3, K = 20$.

3.5.1 Efficient frontier

In this subsection, we present how the model parameters affect the efficient frontier. In the following numerical experiments, we vary the value of one parameter with others fixed and given.

Figure 3.1 contributes to the impact of parameters λ_r, a, and η_r on the efficient frontier. We observe from Figure 3.1(a) that given the fixed expected value of terminal wealth, the efficient frontier moves downwards as λ_r increases from 0.689 to 0.889. Since λ_r characterizes the market price of interest rate risk, a greater value of λ_r implies that the investor can obtain higher returns by investing in the

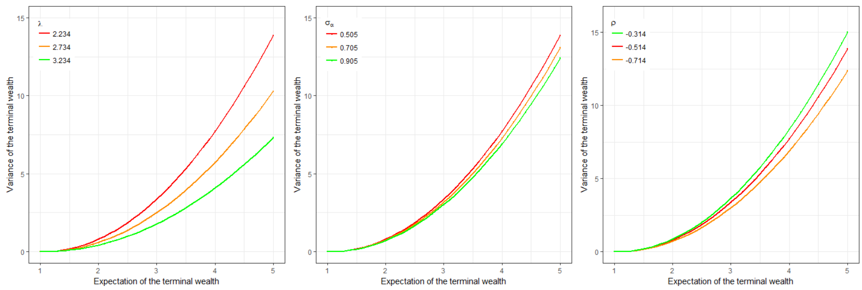


(a) **(b)** **(c)**

Figure 3.1: *Impact of parameters $\lambda_r, a,$ and η_r on the efficient frontier*

rollover bond. As such, the investor can take fewer risks from the market if he wants to gain the same expected wealth at the terminal date. Figure 3.1(b) shows the relationship between the efficient frontier and parameter a. We find that along with the growth of a, the variance of terminal wealth decreases. As revealed by (3.2.1), parameter a partially depicts the long-run mean of the short interest rate. As a increases, the return rate of the risk-free asset becomes higher, while the risk premiums of investing in both the roll-over bond and the risky asset are not influenced. In such a case, the investor can bear fewer risks if the same expected terminal wealth is acquired. From Figure 3.1(c), we find that the scale parameter η_r has no impact on the efficient frontier. This is consistent with our intuition that although η_r changes the optimal allocations on the roll-over bond and risky asset, the optimal risk exposures to the interest rate and volatility risks remain unchanged. Therefore, the investor undertakes the same investment risks, and he has the same variance of terminal wealth if he acquires the same expected terminal wealth.

Figure 3.2 reveals the relationship between the efficient frontier and parameters λ, σ_α, and ρ. From Figure 3.2(a), we find that the efficient frontier moves down as λ increases from 2.234 to 3.234. λ characterizes the slope of the market price of volatility risk. So, along with the growth of λ, the investor can obtain a higher volatility risk premium. In such a case, the investor can invest less in the risky asset to obtain the same expected terminal wealth. In Figure 3.2(b), we vary σ_α from 0.505 to 0.705, and find that the efficient frontier moves down. In other words, as σ_α increases, to derive a fixed expected terminal wealth, the investor will undertake fewer risks. One explanation is that as the volatility of volatility σ_α increases, the fluctuation of the stochastic volatility becomes larger, and thus, the return rate of the risky asset is more likely to increase, which helps the investor derive the same expected return by investing less in the risky asset and hence bearing fewer risks. In Figure 3.2(c), we vary ρ from -0.314 to -0.714, and find that the efficient frontier moves down. This can be explained by the fact that as the correlation parameter ρ approaches -1, the risky asset price and its instantaneous variance become more negatively correlated. Therefore, the offset between the risk caused by fluctuations in the risky asset price and its volatility becomes more. Consequently, investing the same amount in the risky asset reduces the investor's exposure to volatility risk.



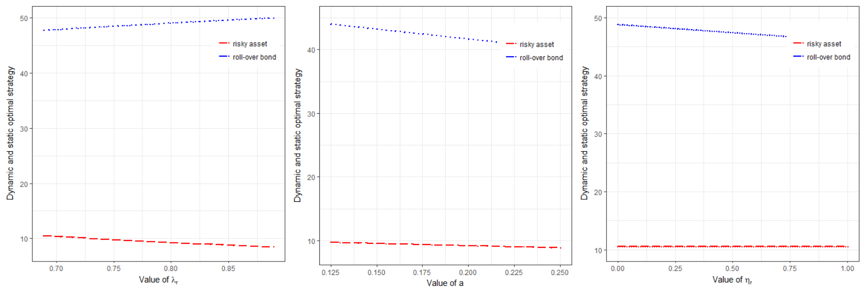
(a) **(b)** **(c)**

Figure 3.2: *Impact of parameters* λ, σ_α, *and* ρ *on the efficient frontier*

3.5.2 Static and dynamic optimal strategies

In this subsection, we investigate the impact of some model parameters and the fixed expected terminal wealth ξ on the behavior of the static and dynamic optimal investment strategies. For simplicity, we pay attention to the results at time $t_0 = 0$ in the following numerical experiments. From the definition of dynamic optimality above (see Definition 3.2.6), we know that $\pi^* = \pi^{d*}$ at the initial time t_0.

Figure 3.3 illustrates the relationship between the parameters λ_r, a, and η_r on the dynamic and static optimal investment strategies. In Figure 3.3(a), we vary λ_r from 0.689 to 0.889 and find that the market value of wealth invested in the roll-over bond is positively correlated with λ_r, while the investment in the risky asset is negatively correlated with λ_r. As the previous section explains, as the market price of interest rate risk λ_r increases, the investor can obtain a higher risk premium from investing in the roll-over bond. It is thus better to allocate more in the roll-over bond to reduce the overall risks when the same expected terminal wealth is acquired. Figure 3.3(b) shows that the amount of wealth invested in both the roll-over bond and the risky asset decreases as a increases from 0.125 to 0.25. Indeed, as a becomes larger, the long-run level of the short rate a/b increases, such that the return rate of the risk-free asset is amplified. Hence, the investor can undertake fewer risks by investing more in the risk-free asset. It is shown from Figure 3.3(c) that as η_r increases from 0 to 1, the amount of wealth invested in the roll-bond is reduced, while the investment in the risky asset remains unchanged. As a matter of fact, the overall interest rate and volatility risks are not changed when η_r varies since it only measures the impact of the interest rate dynamics on the risky asset price. Namely, when η_r becomes larger, the investor faces the same amount of volatility and interest rate risks, but the interest rate risk can be more easily hedged against by investing in the risky asset.

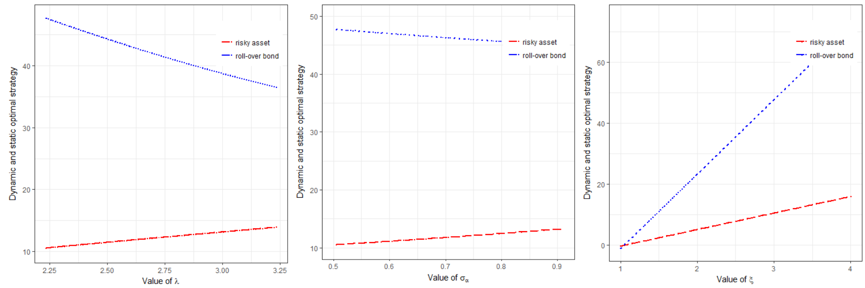


(a) **(b)** **(c)**

Figure 3.3: *Impact of parameters λ_r, a, and η_r on the dynamic and static optimality*

Figure 3.4 shows how the dynamic and static optimal investment strategies change with respect to the parameters λ, σ_α, and ξ. From Figure 3.4(a), we find that as λ increases, the investor is willing to invest more in the risky asset and less in the rollover bond. This can be explained by the fact that λ characterizes the slope of the market price of volatility risk, and the investor can derive a higher risk premium from the risky asset as λ becomes larger. In Figure 3.4(b), we vary σ_α from 0.505 to 0.905 and find that the amount of wealth invested in the risky asset becomes larger as σ_α increases. One of the possible explanations is that as the volatility of volatility σ_α increases, it is more likely for the investor to derive a higher volatility risk premium from the risky asset, i.e., $\lambda\sqrt{V_t}$. In such a case, the investor tends to adopt a more aggressive investment strategy. In the meantime, since the optimal risk exposure to the interest rate risk is not affected by the change of σ_α, the investor can invest less in the roll-over bond to hedge against the interest rate risk due to the stochastic correlation between interest rate and risky asset price. Finally, we see from Figure 3.4(c) that the amount of wealth invested in the roll-over bond and the risky asset has a positive relationship with the expected terminal value ξ. This is consistent with our intuition that to obtain a greater value of the expected terminal wealth, the investor has to invest more in both the roll-over bond and the risky asset such that the overall interest rate and volatility risks can be hedged against.

To end this subsection, we highlight the difference between static and dynamic optimality, i.e., π^* and π^{d*}. By setting 500 equidistant time points over the investment horizon $[0, 1]$ and using some Monte Carlo techniques, we simulate two paths of X_t^* and X_t^{d*}, as well as one path of the stochastic process $\xi e^{g(t)+h(t)r_t}$, which is referred to as the bound in Figure 3.5. As shown in Figure 3.5, the trajectories of two optimal wealth processes X_t^* and X_t^{d*} are significantly different even though the same random numbers are used. In particular, we observe that



(a) **(b)** **(c)**

Figure 3.4: *Impact of parameters* λ, σ_α, *and* ξ *on the dynamic and static optimality*

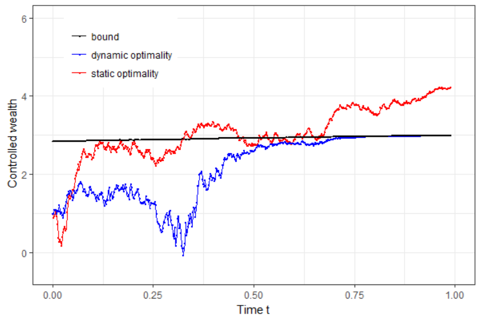


Figure 3.5: *Two trajectories of static and dynamic optimality*

the dynamic optimal wealth process X_t^{d*} is strictly below the process $\xi e^{g(t)+h(t)r_t}$, which is consistent with the theoretical results derived in Theorem 3.4.4 above.

3.6 Conclusion

In this paper, we consider dynamic mean-variance portfolio selection problems in a stochastic environment. The risks in the market come from the interest rate and the risky asset. The interest rate follows the Vasicek model while the risky asset's return rate not only relies on a CIR process but also exhibits stochastic covariance with the interest rate. The modeling framework embraces the family of the state-of-the-art 4/2 stochastic volatility models, as an exceptional case. Given the time inconsistency of the mean-variance criterion, the problems are investigated in line with the dynamic optimal approach. For this, we first address the static optimal (time-inconsistent) strategy by using a BSDE approach. Under the assumption of some model parameters, the associated BSDEs are solved explicitly. Analytical expressions for the static optimality and optimal value function (efficient frontier) are derived via the explicit solutions to the BSDEs. By recomputing the static optimality in an infinitesimally small period of time, we derive, in closed form, the dynamic optimal (time-consistent) strategy. Moreover, results on the Vasicek-

Heston, Vasicek-3/2, and Vasicek-4/2 models are provided, as particular cases. Finally, the economic impact of some model parameters on the efficient frontier as well as on the static and dynamic optimal asset allocations is illustrated with numerical examples. As far as we know, there is no existing literature on the mean-variance portfolio selection problems considering the time-inconsistent and time-consistent solutions in the presence of stochastic volatility and interest rate. So, this study is meaningful from both theoretical and practical perspectives.

Built on the present paper, several potential topics in the future may be followed; for instance, one may extend the current framework with a single risky asset to that with multiple risky assets. In addition, since it is difficult to estimate the return rate of the risky asset and interest rate with precision in practice, the investor might be ambiguous about the financial market. It is thus of interest to explore the mean-variance portfolio selection problems with stochastic interest rate and volatility under model ambiguity.

Acknowledgement(s)

The authors are very grateful to Prof. Jesper Lund Pedersen, the editor, and an anonymous reviewer for their constructive comments that led to a significant improvement of this work.

3.A Proof of Lemma 3.3.6

Proof. We start by introducing the likelihood process $L_{1,t}$, for $t \in [t_0, T]$ from the following dynamic:

$$dL_{1,t} = -\lambda_r L_{1,t}\, dW_t^0,$$

for which Novikov's condition is satisfied. Thus, $L_{1,t}$ is an $(\mathbb{F}, \mathrm{P}_{t_0})$-uniformly integrable martingale, and the equivalent probability measure, denoted by $\tilde{\mathrm{P}}_{t_0}$, is well-defined on $\mathcal{F}_T$ via the Radon-Nikodym derivative:

$$\frac{d\tilde{\mathrm{P}}_{t_0}}{d\mathrm{P}_{t_0}}\Big|_{\mathcal{F}_T} = L_{1,T}.$$

Let $\tilde{\mathrm{E}}_{t_0}[\cdot]$ denote the corresponding expectation under measure $\tilde{\mathrm{P}}_{t_0}$. From Girsanov's theorem, three processes $\tilde{W}_t^0, \tilde{W}_t^1$, and $\tilde{W}_t^2$ given by

$$d\tilde{W}_t^0 = \lambda_r\, dt + dW_t^0,\ d\tilde{W}_t^1 = dW_t^1,\ d\tilde{W}_t^2 = dW_t^2$$

are three standard $(\mathbb{F}, \tilde{\mathrm{P}}_{t_0})$ Brownian motions. Then, linear BSDE (3.3.2) can be reformulated as follows:

$$\begin{cases} dY_t = r_t Y_t\, dt + Z_t\, d\tilde{W}_t^0, \\ Y_T = -\gamma. \end{cases} \tag{3.A.1}$$

Notice that the driver of BSDE (3.A.1) satisfies the stochastic Lipschitz continuity (refer to Definition 2 (H2) in Bender and Kohlmann (2000)) with $\eta_t^2 := r_t + \varepsilon$ as its coefficient for any $\varepsilon \in \mathbb{R}^+$ fixed and given. Setting $A_t := \int_{t_0}^t \eta_s^2 \, ds$ and using Hölder's inequality, from Assumption 3.3.4 and Lemma 3.3.1, we have for some constant $\beta > 3 + \sqrt{21}$

$$
\begin{aligned}
&\tilde{\mathrm{E}}_{t_0}\left[|-\gamma|^2 \exp\{\beta A_T\}\right] \\
\leq& c\left\{\mathrm{E}_{t_0}\left[\exp\left\{-2\int_{t_0}^T \lambda_r \, dW_t^0 - 2\int_{t_0}^T \lambda_r^2 \, dt\right\}\right]\right\}^{\frac{1}{2}}\left\{\mathrm{E}_{t_0}\left[\exp\left\{\int_{t_0}^T (\lambda_r^2 + 2\beta|r_t|)\, dt\right\}\right]\right\}^{\frac{1}{2}} \\
=& c\left\{\mathrm{E}_{t_0}\left[\exp\left\{\int_{t_0}^T (\lambda_r^2 + 2\beta|r_t|)\, dt\right\}\right]\right\}^{\frac{1}{2}} \\
\leq& c\left\{\mathrm{E}_{t_0}\left[\exp\left\{2\beta\int_{t_0}^T |r_t|^2 \, dt\right\}\right]\right\}^{\frac{1}{2}} < +\infty,
\end{aligned}
$$

where the constant c might differ between lines, and the second inequality follows from the basic result that $x^2 + \frac{1}{4} \geq x$ for $x \in \mathbb{R}^+$. This shows that the driver and terminal condition of BSDE (3.A.1) constitute standard data (Definition 2 in Bender and Kohlmann (2000)). According to Theorem 3 in Bender and Kohlmann (2000), BSDE (3.A.1) admits a unique solution (Y_t, Z_t) such that

$$
\tilde{\mathrm{E}}_{t_0,}\left[\int_{t_0}^T e^{\beta A_t}|Z_t|^2 \, dt\right] < +\infty.
$$

Applying Itô's formula to $Y_t \exp\left\{-\int_{t_0}^t r_u \, du\right\}$ under measure $\tilde{\mathrm{P}}_{t_0}$ yields

$$
d\left[Y_t \exp\left\{-\int_{t_0}^t r_u \, du\right\}\right] = Z_t \exp\left\{\int_{t_0}^t -r_u \, du\right\} d\tilde{W}_t^0, \tag{3.A.2}
$$

which means $Y_t \exp\left\{-\int_{t_0}^t r_u \, du\right\}$ is a $(\mathbb{F}, \tilde{\mathrm{P}}_{t_0})$-local martingale. Moreover, by Lemma 3.3.1, Burkholder-Davis-Gundy inequality, and Hölder's inequality, we find that $Y_t \exp\left\{-\int_{t_0}^t r_u \, du\right\}$ is, in fact, an $(\mathbb{F}, \tilde{\mathrm{P}}_{t_0})$-uniformly integrable martingale under Assumption 3.3.4, since

$$
\begin{aligned}
&\tilde{\mathrm{E}}_{t_0}\left[\sup_{t\in[t_0,T]}\left|\int_{t_0}^{t}\exp\left\{-\int_{t_0}^{s} r_u\,du\right\} Z_s\,d\tilde{W}_s^0\right|\right]\\
\leq & c\tilde{\mathrm{E}}_{t_0}\left[\left(\int_{t_0}^{T}\exp\left\{-\int_{t_0}^{t} 2r_u\,du\right\} Z_t^2\,dt\right)^{\frac{1}{2}}\right]\\
\leq & c\left(\tilde{\mathrm{E}}_{t_0}\left[\exp\left\{2\int_{t_0}^{T}|r_t|\,dt\right\}\right]+\tilde{\mathrm{E}}_{t_0}\left[\int_{t_0}^{T}Z_t^2\,dt\right]\right)\\
\leq & c\left\{\mathrm{E}_{t_0}\left[\exp\left\{\int_{t_0}^{T}(\lambda_r^2+4|r_t|)\,dt\right\}\right]\right\}^{\frac{1}{2}}+c\tilde{\mathrm{E}}_{t_0}\left[\int_{t_0}^{T}Z_t^2\,dt\right]\\
\leq & c\left\{\mathrm{E}_{t_0}\left[\exp\left(\int_{t_0}^{T}4|r_t|^2\,dt\right)\right]\right\}^{\frac{1}{2}}+c\tilde{\mathrm{E}}_{t_0}\left[\int_{t_0}^{T}Z_t^2\,dt\right]<+\infty,
\end{aligned}
$$

where the constant c might differ between lines. Therefore, from (3.A.2) and the Markovian structure of interest rate process r_t, we have the following expectation formulation for Y_t:

$$
Y_t=-\gamma\tilde{\mathrm{E}}_{t_0}\left[\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{F}_t\right]=-\gamma\tilde{\mathrm{E}}_{t_0}\left[\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,r_t\right]=-\gamma f(t,r_t),
$$

where the deterministic function $f(t,r)=\tilde{\mathrm{E}}_{t,r}\left[\exp\left\{-\int_t^T r_s\,ds\right\}\right]$. Observe that the interest rate process r_t has the following $\tilde{\mathrm{P}}_{t_0}$ dynamic:

$$
dr_t=(a+\sigma_r\lambda_r-br_t)\,dt-\sigma_r\,d\tilde{W}_t^0.
$$

Then, from the Feynman-Kac theorem, we find the following partial differential equation (PDE) governing function $f(t,r)$:

$$
\begin{cases}
\dfrac{\partial f}{\partial t}+(a+\sigma_r\lambda_r-br)\dfrac{\partial f}{\partial r}+\dfrac{1}{2}\sigma_r^2\dfrac{\partial^2 f}{\partial r^2}-rf=0,\\
f(T,r)=1.
\end{cases}
$$

Conjecture that $f(t,r)$ admits the following exponential-affine form, i.e.,

$$
f(t,r)=\exp\{g(t)+h(t)r\}
$$

with boundary conditions $g(T)=h(T)=0$. Then, we can decompose the above PDE into the following two ODEs of $g(t)$ and $h(t)$:

$$
\begin{cases}
\dfrac{dg(t)}{dt}+(a+\sigma_r\lambda_r)h(t)+\dfrac{1}{2}\sigma_r^2h^2(t)=0,\ g(T)=0,\\
\dfrac{dh(t)}{dt}-bh(t)-1=0,\ h(T)=0.
\end{cases}
$$

After some tedious calculations, the closed-form expressions for $g(t)$ and $h(t)$ are given by (3.3.4), and Y_t is given by (3.3.3). Finally, applying Itô's formula to Y_t under measure $\tilde{\mathrm{P}}_{t_0}$ and comparing with the diffusive part of (3.A.1) lead to the explicit expression for Z_t given in (3.3.3). □

3.B Proof of Lemma 3.3.7

Proof. Applying Itô's lemma to P_t given by (3.3.5) and making use of (3.3.6)-(3.3.7), we have

$$\begin{aligned} dP_t =& P_t\Bigg[a(t)a - 2r_t + \alpha_t\left(2\rho\sigma_\alpha\lambda b(t) + \left(\rho^2 - \frac{1}{2}\right)\sigma_\alpha^2 b^2(t) + \lambda^2\right) \\ &+ \kappa\theta b(t) + \frac{1}{2}\left(\sigma_\alpha^2 b^2(t)\alpha_t + \sigma_r^2 a^2(t)\right)\Bigg]\,dt - P_t\sigma_r a(t)\,dW_t^0 \\ &+ P_t\sigma_\alpha\sqrt{\alpha_t}\rho b(t)\,dW_t^1 + P_t\sigma_\alpha\sqrt{\alpha_t}\sqrt{1-\rho^2}b(t)\,dW_t^2 - P_t\Bigg((a + 2\sigma_r\lambda_r)a(t) \\ &+ \kappa\theta b(t) - \frac{1}{2}a^2(t)\sigma_r^2 - \lambda_r^2\Bigg)\,dt \\ =& \Big[\left(-2r_t + \lambda_r^2 + \lambda^2\alpha_t\right) - 2\sigma_r\lambda_r a(t) + 2\rho\sigma_\alpha\lambda\alpha_t b(t) + \rho^2\sigma_\alpha^2 b^2(t)\alpha_t \\ &+ a^2(t)\sigma_r^2\Big]P_t\,dt - P_t\sigma_r a(t)\,dW_t^0 + P_t\sigma_\alpha\sqrt{\alpha_t}\rho b(t)\,dW_t^1 \\ &+ P_t\sigma_\alpha\sqrt{\alpha_t}\sqrt{1-\rho^2}b(t)\,dW_t^2 \\ =& \left[\left(-2r_t + \lambda_r^2 + \lambda^2\alpha_t\right)P_t + 2\lambda_r\Gamma_{0,t} + 2\lambda\sqrt{\alpha_t}\Gamma_{1,t} + \frac{\Gamma_{1,t}^2}{P_t} + \frac{\Gamma_{0,t}^2}{P_t}\right]dt \\ &+ \Gamma_{0,t}\,dW_t^0 + \Gamma_{1,t}\,dW_t^1 + \Gamma_{2,t}\,dW_t^2. \end{aligned}$$

This shows that P_t given in (3.3.5) satisfies the first equation of BSRE (3.3.1). The terminal condition $P_T = 1$ follows from the boundary conditions $a(T) = b(T) = 0$ and $\phi(T) = 1$ given by (3.3.7). Finally, considering the canonical exponential expression of the solution to the first-order homogeneous linear equation, we know that $\phi(t)$ given in (3.3.7) must have an exponential formulation, which implies from (3.3.5) that $P_t > 0$ over $[t_0, T]$. □

3.C Proof of Proposition 3.3.8

Proof. Since the ODE of $a(t)$ in (3.3.7) is a first-order linear equation, we can reformulate it as follows:

$$\frac{da(t)}{ba(t) - 2} = dt.$$

By integrating both sides from t to T upon considering the boundary condition $a(T) = 0$, we obtain

$$a(t) = \frac{2}{b}\left(1 - e^{-b(T-t)}\right).$$

We next consider the ODE of $b(t)$. Reshuffling the terms in (3.3.7) yields

$$\frac{db(t)}{dt} = \left(\rho^2 - \frac{1}{2}\right)\sigma_\alpha^2 b^2(t) + (\kappa + 2\sigma_\alpha\rho\lambda)b(t) + \lambda^2, \; b(T) = 0. \tag{3.C.1}$$

When $\rho^2 = \frac{1}{2}$ and $\kappa + 2\sigma_\alpha\rho\lambda = 0$, it follows from (3.C.1) that $b(t) = \lambda^2(t - T)$. When $\rho^2 = \frac{1}{2}$ and $\kappa + 2\sigma_\alpha\rho\lambda \neq 0$, we have the following linear equation:

$$db(t) = (\kappa + 2\sigma_\alpha\rho\lambda)b(t)\,dt + \lambda^2\,dt,$$

from which we obtain

$$b(t) = \frac{\lambda^2}{k + 2\lambda\rho\sigma_\alpha}\left(e^{(k+2\lambda\rho\sigma_\alpha)(t-T)} - 1\right).$$

When $\rho^2 \neq \frac{1}{2}$, we denote by $\Delta = (k + 2\lambda\rho\sigma_\alpha)^2 - (4\rho^2 - 2)\sigma_\alpha^2\lambda^2$. It follows from (3.C.1) that

$$\frac{db(t)}{dt} = \begin{cases} \sigma_\alpha^2\left(\rho^2 - \frac{1}{2}\right)(b(t) - n_1)(b(t) - n_2), \; \Delta > 0; \\ \sigma_\alpha^2\left(\rho^2 - \frac{1}{2}\right)(b(t) - n_0)^2\,dt, \; \Delta = 0; \\ \sigma_\alpha^2\left(\rho^2 - \frac{1}{2}\right)\left[\left(b(t) + \frac{k + 2\lambda\rho\sigma_\alpha}{\sigma_\alpha^2(2\rho^2 - 1)}\right)^2 + \frac{-\Delta}{\sigma_\alpha^4(2\rho^2 - 1)^2}\right], \; \Delta < 0, \end{cases}$$

where n_0, n_1, and n_2 are given by (3.3.11). After some tedious calculations, we derive the explicit expressions for $b(t)$ presented in (3.3.9). Finally, noticing the boundary condition that $\phi(T) = 1$ and substituting $a(t)$ and $b(t)$ back into the ODE of $\phi(t)$, we have the closed-form solution of $\phi(t)$ given in (3.3.10). □

3.D Proof of Proposition 3.3.9

Proof. Differentiating (3.3.10) with respect to t yields

$$\frac{db(t)}{dt} = \begin{cases} \lambda^2 e^{(k+2\lambda\rho\sigma_\alpha)(t-T)}, & \rho^2 = \frac{1}{2}, \; k + 2\lambda\rho\sigma_\alpha \neq 0; \\ \lambda^2, & \rho^2 = \frac{1}{2}, \; k + 2\lambda\rho\sigma_\alpha = 0; \\ \frac{4\lambda^2\Delta e^{\sqrt{\Delta}(T-t)}}{\sigma_\alpha^4(2\rho^2-1)^2}\frac{1}{(n_1 - n_2 e^{\sqrt{\Delta}(T-t)})^2}, & \rho^2 \neq \frac{1}{2}, \; \Delta > 0; \\ \frac{\sigma_\alpha^2\left(\rho^2 - \frac{1}{2}\right)n_0^2}{\left(\sigma_\alpha^2(\rho^2 - \frac{1}{2})(T-t)n_0 - 1\right)^2}, & \rho^2 \neq \frac{1}{2}, \; \Delta = 0; \\ \frac{-\Delta}{2\sigma_\alpha^2(2\rho^2-1)}\sec^2\left(\arctan\left(\frac{k + 2\lambda\rho\sigma_\alpha}{\sqrt{-\Delta}}\right) - \frac{\sqrt{-\Delta}}{2}(T-t)\right), & \rho^2 \neq \frac{1}{2}, \; \Delta < 0; \end{cases}$$

It is obvious that $\frac{db(t)}{dt} > 0$ holds for the first three cases. As for the last two cases, we see that $\rho^2 > \frac{1}{2}$ must hold when $\Delta \leq 0$. □

3.E Proof of Lemma 3.3.10

Proof. Observing that P_t given in (3.3.5) is positive, we can apply Itô's lemma to $\log(P_t)$ and find that

$$d\log(P_t) = \left[-2r_t + \lambda_r^2 + \lambda^2\alpha_t + 2\lambda_r\frac{\Gamma_{0,t}}{P_t} + 2\lambda\sqrt{\alpha_t}\frac{\Gamma_{1,t}}{P_t} + \frac{1}{2}\frac{\Gamma_{0,t}^2}{P_t^2} + \frac{1}{2}\frac{\Gamma_{1,t}^2}{P_t^2} - \frac{1}{2}\frac{\Gamma_{2,t}^2}{P_t^2}\right] dt + \frac{\Gamma_{0,t}}{P_t}\, dW_t^0 + \frac{\Gamma_{1,t}}{P_t}\, dW_t^1 + \frac{\Gamma_{2,t}}{P_t}\, dW_t^2. \tag{3.E.1}$$

Now, we introduce the likelihood process $L_{2,t}$ from the following dynamic:

$$dL_{2,t} = -2\lambda_r L_{2,t}\, dW_t^0 - 2\lambda\sqrt{\alpha_t} L_{2,t}\, dW_t^1,$$

which can be easily shown to be an $(\mathbb{F}, \mathrm{P}_{t_0})$-uniformly integrable martingale by the Novikov's condition and Assumption 3.3.4. Thus, we can define an equivalent probability measure $\hat{\mathrm{P}}_{t_0}$ on $\mathcal{F}_T$ via the following Radon-Nikodym derivative:

$$\left.\frac{d\hat{\mathrm{P}}_{t_0}}{d\mathrm{P}_{t_0}}\right|_{\mathcal{F}_T} = L_{2,T}.$$

From Girsanov's theorem, the following three processes $\hat{W}_t^0, \hat{W}_t^1$, and $\hat{W}_t^2$:

$$d\hat{W}_t^0 = 2\lambda_r\, dt + dW_t^0,\ \ d\hat{W}_t^1 = 2\lambda\sqrt{\alpha_t}\, dt + dW_t^1,\ \ d\hat{W}_t^2 = dW_t^2$$

are standard Brownian motions under measure $\hat{\mathrm{P}}_{t_0}$. Then, BSDE (3.E.1) of $\left(\log(P_t), \frac{\Gamma_{0,t}}{P_t}, \frac{\Gamma_{1,t}}{P_t}, \frac{\Gamma_{2,t}}{P_t}\right)$ can be rewritten as follows:

$$\begin{cases} d\log(P_t) = \left[-2r_t + \lambda_r^2 + \lambda^2\alpha_t + \frac{1}{2}\frac{\Gamma_{0,t}^2}{P_t^2} + \frac{1}{2}\frac{\Gamma_{1,t}^2}{P_t^2} - \frac{1}{2}\frac{\Gamma_{2,t}^2}{P_t^2}\right] dt + \frac{\Gamma_{0,t}}{P_t}\, d\hat{W}_t^0 \\ \qquad\qquad + \frac{\Gamma_{1,t}}{P_t}\, d\hat{W}_t^1 + \frac{\Gamma_{2,t}}{P_t}\, d\hat{W}_t^2, \\ \log(P_T) = 0. \end{cases} \tag{3.E.2}$$

Suppose that there exists another solution, denoted by $(\hat{P}_t, \hat{\Gamma}_{0,t}, \hat{\Gamma}_{1,t}, \hat{\Gamma}_{2,t})$, to BSRE (3.3.1). It follows from (3.E.2) that the following difference process

$$(\Delta\log(P_t), \Delta\Gamma_{0,t}, \Delta\Gamma_{1,t}, \Delta\Gamma_{2,t}) := \left(\log(P_t) - \log(\hat{P}_t), \frac{\Gamma_{0,t}}{P_t} - \frac{\hat{\Gamma}_{0,t}}{\hat{P}_t}, \frac{\Gamma_{1,t}}{P_t} - \frac{\hat{\Gamma}_{1,t}}{\hat{P}_t}, \frac{\Gamma_{2,t}}{P_t} - \frac{\hat{\Gamma}_{2,t}}{\hat{P}_t}\right)$$

must solve the following BSDE:

$$\begin{cases} d\Delta\log(P_t) = \left[\frac{1}{2}\left(\frac{\Gamma_{0,t}^2}{P_t^2} - \frac{\hat{\Gamma}_{0,t}^2}{\hat{P}_t^2}\right) + \frac{1}{2}\left(\frac{\Gamma_{1,t}^2}{P_t^2} - \frac{\hat{\Gamma}_{1,t}^2}{\hat{P}_t^2}\right) - \frac{1}{2}\left(\frac{\Gamma_{2,t}^2}{P_t^2} - \frac{\hat{\Gamma}_{2,t}^2}{\hat{P}_t^2}\right)\right] dt \\ \qquad\qquad + \Delta\Gamma_{0,t}\, d\hat{W}_t^0 + \Delta\Gamma_{1,t}\, d\hat{W}_t^1 + \Delta\Gamma_{2,t}\, d\hat{W}_t^2, \\ \Delta\log(P_T) = 0. \end{cases} \tag{3.E.3}$$

We now introduce another likelihood process $L_{3,t}$ from the dynamic:

$$dL_{3,t} = -\frac{\Gamma_{0,t}}{P_t} L_{3,t}\, d\hat{W}_t^0 - \frac{\Gamma_{1,t}}{P_t} L_{3,t}\, d\hat{W}_t^1 + \frac{\Gamma_{2,t}}{P_t} L_{3,t}\, d\hat{W}_t^2.$$

By using the explicit expressions for P_t and $(\Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ given in (3.3.5) and (3.3.6), Hölder's inequality, Proposition 3.3.9, and Assumption 3.3.4, we find that Novikov's condition is satisfied for $L_{3,t}$, i.e.,

$$\begin{aligned}
&\hat{\mathrm{E}}_{t_0}\left[\exp\left\{\frac{1}{2}\int_{t_0}^{T} \frac{\Gamma_{0,t}^2}{P_t^2} + \frac{\Gamma_{1,t}^2}{P_t^2} + \frac{\Gamma_{2,t}^2}{P_t^2}\, dt\right\}\right] \\
=&\mathrm{E}_{t_0}\left[L_{2,T}\exp\left\{\frac{1}{2}\int_{t_0}^{T} \sigma_r^2 a^2(t) + \sigma_\alpha^2 b^2(t)\alpha_t\, dt\right\}\right] \\
\le&\left\{\mathrm{E}_{t_0}\left[\exp\left\{\int_{t_0}^{T} -4\lambda_r\, dW_t^0 - \int_{t_0}^{T} 4\lambda\sqrt{\alpha_t}\, dW_t^1 - \int_{t_0}^{T} \left(8\lambda_r^2 + 8\lambda^2\alpha_t\right)\, dt\right\}\right]\right\}^{\frac{1}{2}} \\
&\times\left\{\mathrm{E}_{t_0}\left[\exp\left\{\int_{t_0}^{T} 4\lambda_r^2 + \sigma_r^2 a^2(t) + (4\lambda^2 + \sigma_\alpha^2 b^2(t))\alpha_t\, dt\right\}\right]\right\}^{\frac{1}{2}} \\
=&c\left\{\mathrm{E}_{t_0}\left[\exp\left\{\int_{t_0}^{T} (4\lambda^2 + \sigma_\alpha^2 b^2(t))\alpha_t\, dt\right\}\right]\right\}^{\frac{1}{2}} < +\infty,
\end{aligned}$$

where c is a positive constant. Thus, the equivalent probability measure $\bar{\mathrm{P}}_{t_0}$ is well-defined on $\mathcal{F}_T$ via

$$\left.\frac{d\bar{\mathrm{P}}_{t_0}}{d\hat{\mathrm{P}}_{t_0}}\right|_{\mathcal{F}_T} = L_{3,T}.$$

Accordingly, the standard Brownian motions $\bar{W}_t^0, \bar{W}_t^1, \bar{W}_t^2$ under $\bar{\mathrm{P}}_{t_0}$ are given as follows due to the Girsanov's theorem:

$$d\bar{W}_t^0 = d\hat{W}_t^0 + \frac{\Gamma_{0,t}}{P_t}\, dt,\ d\bar{W}_t^1 = d\hat{W}_t^1 + \frac{\Gamma_{1,t}}{P_t}\, dt.\ d\bar{W}_t^2 = d\hat{W}_t^2 - \frac{\Gamma_{2,t}}{P_t}\, dt. \tag{3.E.4}$$

Plugging (3.E.4) into (3.E.3) yields the BSDE of $(\Delta\log(P_t), \Delta\Gamma_{0,t}, \Delta\Gamma_{1,t}, \Delta\Gamma_{2,t})$

$$\begin{cases}
d\Delta\log(P_t) = \left[-\frac{1}{2}(\Delta\Gamma_{0,t})^2 - \frac{1}{2}(\Delta\Gamma_{1,t})^2 + \frac{1}{2}(\Delta\Gamma_{2,t})^2\right] dt \\
\qquad\qquad\qquad + \Delta\Gamma_{0,t}\, d\bar{W}_t^0 + \Delta\Gamma_{1,t}\, d\bar{W}_t^1 + \Delta\Gamma_{2,t}\, d\bar{W}_t^2, \\
\Delta\log(P_T) = 0.
\end{cases} \tag{3.E.5}$$

It is easy to check that quadratic BSDE (3.E.5) satisfies all regularity conditions in Kobylanski (2000). Then according to Theorem 2.3 and Theorem 2.6 in Kobylanski (2000), we know that BSDE (3.E.5) admits unique solution $(0, 0, 0, 0)$, which, in turn, reveals

$$(P_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t}) = (\hat{P}_t, \hat{\Gamma}_{0,t}, \hat{\Gamma}_{1,t}, \hat{\Gamma}_{2,t}).$$

Hence, we can conclude that $(P_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ given in Lemma 3.3.6 is the unique solution to BSRE (3.3.1). □

3.F Proof of Lemma 3.3.11

Proof. For any given constant $p > 1$, it is straightforward to see that the following equation of k

$$p = \frac{k}{2\sqrt{k} - 1}$$

admits two positive solutions:

$$k_1 = 2p\sqrt{p(p-1)} + p(2p-1),\;\; k_2 = -2p\sqrt{p(p-1)} + p(2p-1),$$

where the first solution satisfies $k_1 > 1$. In particular, we have $k_1 = 276 + 48\sqrt{33}$ when $p = 12$. By Assumption 3.3.4 and Proposition 3.3.9, we have

$$\mathrm{E}_{t_0}\left[\exp\left\{(138 + 24\sqrt{33})\int_{t_0}^{T}(\lambda + \sigma_\alpha \rho b(t))^2 \alpha_t\, dt\right\}\right] < +\infty.$$

Then, according to Theorem 15.4.6 in Cohen and Elliott (2015), we have

$$\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]}|\Pi_{1,t}|^{12}\right] \le \frac{12}{11}\left\{\mathrm{E}_{t_0,\alpha_0,r_0,x_0}\left[\exp\left\{(138+24\sqrt{33})\int_{t_0}^{T}(\lambda+\sigma_\alpha\rho b(t))^2\alpha_t\, dt\right\}\right]\right\}^{\frac{\sqrt{276+48\sqrt{33}}-1}{276+48\sqrt{33}}} < +\infty.$$

By using the same technique, we also have $\mathrm{E}_{t_0}\left[\sup_{t\in t_0,T}|\Pi_{0,t}|^{12}\right] < \infty$. □

3.G Proof of Proposition 3.3.12

Proof. For any admissible strategy $\pi \in \mathcal{A}$, applying Itô's formula to $P_t(X_t^\pi + Y_t)^2$ and completing of squares, we have

$$\begin{aligned}
& dP_t(X_t^\pi + Y_t)^2 \\
=& P_t\left\{\left[\pi_B(t,\alpha_t,r_t,X_t^\pi)h_0(K)\sigma_r + \pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)\eta_r\sigma_r + Z_t + (X_t^\pi+Y_t)\left(\frac{\Gamma_{0,t}}{P_t}+\lambda_r\right)\right]^2\right. \\
&+\left.\left[\pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)\sigma(t,\alpha_t) + (X_t^\pi+Y_t)\left(\frac{\Gamma_{1,t}}{P_t}+\lambda\sqrt{\alpha_t}\right)\right]^2\right\}dt \\
&+\left[(X_t^\pi+Y_t)^2\Gamma_{0,t} + 2(X_t^\pi+Y_t)P_t\left(\pi_B(t,\alpha_t,r_t,X_t^\pi)h_0(K)\sigma_r + \pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)\eta_r\sigma_r + Z_t\right)\right]dW_t^0 \\
&+\left[(X_t^\pi+Y_t)^2\Gamma_{1,t} + 2(X_t^\pi+Y_t)P_t\pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)\sigma(t,\alpha_t)\right]dW_t^1 \\
&+(X_t^\pi+Y_t)^2\Gamma_{2,t}\,dW_t^2.
\end{aligned} \tag{3.G.1}$$

Due to the continuity of $Y_t, Z_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t}, P_t, \pi_B(t,\alpha_t,r_t,X_t^\pi), \pi_{S^1}(t,\alpha_t,r_t,X_t^\pi)$, X_t^π and $\sigma(t,\alpha_t,r_t,X_t^\pi)$, the stochastic integrals on the right-hand side of (3.G.1) are $(\mathbb{F},\mathrm{P}_{t_0})$-local martingales. Hence, there exists a localizing sequence, denoted by $\{\tau_n\}_{n\in\mathbb{N}}$, such that $\tau_n \uparrow +\infty$, P_{t_0} almost surely as $n \to +\infty$, and when stopped by such a sequence, the aforementioned local martingales are $(\mathbb{F},\mathrm{P}_{t_0})$-martingales. Then, integrating both sides of (3.G.1) from t_0 to $\tau_n \wedge T$ and taking expectations, we obtain

$$\begin{aligned}
&\mathrm{E}_{t_0}\left[P_{T\wedge\theta_n}(X^{\pi}_{T\wedge\theta_n}+Y_{T\wedge\theta_n})^2\right]\\
=&\mathrm{E}_{t_0}\left[\int_{t_0}^{T\wedge\theta_n} P_t\left(\pi_{S^1}(t,\alpha_t,r_t,X^{\pi}_t)\sigma(t,\alpha_t)+(X^{\pi}_t+Y_t)\left(\frac{\Gamma_{1,t}}{P_t}+\lambda\sqrt{\alpha_t}\right)\right)^2 dt\right]\\
&+\mathrm{E}_{t_0}\left[\int_{t_0}^{T\wedge\theta_n} P_t\left(\pi_B(t,\alpha_t,r_t,X^{\pi}_t)h_0(K)\sigma_r+\pi_{S^1}(t,\alpha_t,r_t,X^{\pi}_t)\eta_r\sigma_r+Z_t\right.\right.\\
&\left.\left.+(X^{\pi}_t+Y_t)\left(\frac{\Gamma_{0,t}}{P_t}+\lambda_r\right)\right)^2 dt\right]+P_{t_0}(x_0+Y_{t_0})^2.
\end{aligned} \tag{3.G.2}$$

For the term within the expectation on the left-hand side of (3.G.2), we have from (3.3.3), (3.3.5), and Proposition 3.3.9 that

$$P_{T\wedge\theta_n}(X^{\pi}_{T\wedge\theta_n}+Y_{T\wedge\theta_n})^2\leq c\left(\phi_b^2\sup_{t\in[t_0,T]}e^{2a(t_0)|r_t|}+\sup_{t\in[t_0,T]}|X^{\pi}_t|^4+\gamma^2e^{2g_b}\sup_{t\in[t_0,T]}e^{2h_b|r_t|}\right), \tag{3.G.3}$$

where c is a positive constant, and ϕ_b, g_b, and h_b denote the bound of continuous functions $\phi(t), g(t)$, and $h(t)$ over $[t_0,T]$, respectively. Then, from Definition 3.2.3 and Lemma 3.3.2, we know the family $\left\{P_{T\wedge\theta_n}(X^{\pi}_{T\wedge\theta_n}+Y_{T\wedge\theta_n})^2\right\}_{n\in\mathbb{N}}$ is integrable, and thus, sending n to infinity and applying the dominated convergence theorem and monotone convergence theorem to the left-hand and right-hand side of (3.G.2), respectively, we have

$$\mathrm{E}_{t_0}\left[(X^{\pi}_T-\gamma)^2\right]\geq P_{t_0}\left(x_0+Y_{t_0}\right)^2. \tag{3.G.4}$$

In particular, the right-hand side of (3.G.4) is attained when we opt for the investment strategy given in (3.3.13). In other words, the strategy (3.3.13) is the optimal strategy for the benchmark problem (3.2.11).

In the remaining part of this proof, we aim to show that the optimal strategy (3.3.13) is admissible. Denote the wealth process (3.2.7) associated with the strategy (3.3.13) by X^*_t. Then, we find that

$$\begin{aligned}
\frac{d(X^*_t+Y_t)}{X^*_t+Y_t}=&\left[r_t-\lambda_r\left(\frac{\Gamma_{0,t}}{P_t}+\lambda_r\right)-\lambda\sqrt{\alpha_t}\left(\frac{\Gamma_{1,t}}{P_t}+\lambda\sqrt{\alpha_t}\right)\right]dt\\
&-\left(\frac{\Gamma_{0,t}}{P_t}+\lambda_r\right)dW^0_t-\left(\frac{\Gamma_{1,t}}{P_t}+\lambda\sqrt{\alpha_t}\right)dW^1_t.
\end{aligned} \tag{3.G.5}$$

Solving the linear SDE (3.G.5) and using the explicit expressions for $Y_t, \Gamma_{0,t}, \Gamma_{1,t}$, and P_t, we obtain (3.3.15). Moreover, by (3.3.15), Lemma 3.3.1-3.3.3, Assumption

3.3.4, Lemma 3.3.11, and Hölder's inequality, we have

$$\begin{aligned}
&\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]}|X_t^*|^4\right]\\
\leq& c\left\{\mathrm{E}_{t_0}\left[\exp\left\{24(\lambda^2+\lambda\sigma_\alpha|\rho b(t_0)|)\int_{t_0}^{T}\alpha_t\,dt\right\}\right]+\mathrm{E}_{t_0}\left[\exp\left\{24\int_{t_0}^{T}r_t^2\,dt\right\}\right]\right.\\
&\left.+\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]}|\Pi_{0,t}|^{12}\right]+\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]}|\Pi_{1,t}|^{12}\right]+\mathrm{E}_{t_0}\left[\sup_{t\in[t_0,T]}e^{4h_b|r_t|}\right]\right\}<+\infty,
\end{aligned} \tag{3.G.6}$$

where c is a positive constant and h_b denotes the bound of $h(t)$ over $[t_0, T]$. Finally, from (3.G.6) and the explicit expressions for the optimal investment strategy given in (3.3.13), it is easy to verify that

$$\mathrm{E}_{t_0}\left[\int_{t_0}^{T}(\pi_{S^1}^*(t,\alpha_t,r_t,X_t^*))^2(\sigma^2(t,\alpha_t)+1)+(\pi_B^*(t,\alpha_t,r_t,X_t^*))^2\,dt\right]<+\infty. \tag{3.G.7}$$

Hence, we can conclude from (3.G.5)-(3.G.7) that the optimal strategy (3.3.13) is admissible. □

3.H Proof of Theorem 3.4.4

Proof. Identifying t_0, α_0, r_0, and x_0 with t, α_t, r_t, and X_t^* in (3.4.3), respectively, we can write a candidate for the dynamic optimality of problem (3.2.8), which is given by (3.4.6). We claim that the candidate solution (3.4.6) is dynamic optimal for problem (3.2.8). Indeed, for any initial data $(t, \alpha, r, x) \in [t_0, T) \otimes \mathbb{R}^+ \otimes \mathbb{R} \otimes \mathbb{R}$, we can take any other admissible strategy $u \in \mathcal{A}$ such that $\mathrm{E}_{t,\alpha,r,x}[X_T^u] = \xi$ and $u(t, \alpha, r, x) \neq \pi^{d*}(t, \alpha, r, x)$. Additionally, we set $w = \pi^*$, in which the initial data $(t_0, \alpha_0, r_0, x_0)$ is replaced by (t, α, r, x). In other words, it holds that $w(t, \alpha, r, x) = \pi^*(t, \alpha, r, x) = \pi^{d*}(t, \alpha, r, x) \neq u(t, \alpha, r, x)$ when the initial data is (t, α, r, x). Then, by the continuity of the feedback controls u and w, there exists a ball $B_\varepsilon := [t, t+\varepsilon]\otimes[\alpha-\varepsilon, \alpha+\varepsilon]\otimes[r-\varepsilon, r+\varepsilon]\otimes[x-\varepsilon, x+\varepsilon]$ such that $w(\tilde{t}, \tilde{\alpha}, \tilde{r}, \tilde{x}) \neq u(\tilde{t}, \tilde{\alpha}, \tilde{r}, \tilde{x})$ for any $(\tilde{t}, \tilde{\alpha}, \tilde{r}, \tilde{x}) \in B_\varepsilon$ when ε is small enough such that $t + \varepsilon \leq T$. Replacing $(t_0, \alpha_0, r_0, x_0)$ and γ in (3.G.2) by (t, α, r, x) and $\xi - \tilde{\theta}^*$, where $\tilde{\theta}^*$ is given by

$$\tilde{\theta}^*=\frac{\exp\{(a(t)+h(t))r+b(t)\alpha+g(t)\}\,\phi(t)\left(\xi e^{g(t)+h(t)r}-x\right)}{\exp\{b(t)\alpha+2g(t)\}\,\phi(t)-1},$$

we observe that $w = \pi^*$ is the unique continuous function such that the minimum within the expectations on the right-hand side of (3.G.2) is attained, $\mathrm{P}_{t,\alpha,r,x}$ almost surely, which indicates that by setting $\tau_\varepsilon = \inf\{t\wedge T|\ (t, \alpha_t, r_t, X_t^u) \notin B_\varepsilon\}$, it holds that for $\tilde{t} \leq \tau_\varepsilon$

$$P_{\tilde{t}}\Bigg[\bigg(u_{S^1}(\tilde{t},\alpha_{\tilde{t}},r_{\tilde{t}},X^u_{\tilde{t}})\sigma(\tilde{t},\alpha_{\tilde{t}}) + (X^u_{\tilde{t}}+Y_{\tilde{t}})\bigg(\frac{\Gamma_{1,\tilde{t}}}{P_{\tilde{t}}}+\lambda\sqrt{\alpha_{\tilde{t}}}\bigg)\bigg)^2 + \bigg(u_B(\tilde{t},\alpha_{\tilde{t}},r_{\tilde{t}},X^u_{\tilde{t}})h_0(K)\sigma_r$$
$$+ u_{S^1}(\tilde{t},\alpha_{\tilde{t}},r_{\tilde{t}},X^u_{\tilde{t}})\eta_r\sigma_r + Z_{\tilde{t}} + (X^u_{\tilde{t}}+Y_{\tilde{t}})\bigg(\frac{\Gamma_{0,\tilde{t}}}{P_{\tilde{t}}}+\lambda_r\bigg)\bigg)^2\Bigg] \geq \zeta,\ \mathrm{P}_{t,\alpha,r,x}-a.s.,$$

where $\zeta \in \mathbb{R}^+$. In other words, replacing (t_0,α_0,r_0,x_0) and γ in (3.G.2) by (t,α,r,x) and $\xi-\tilde{\theta}^*$, respectively, we find that

$$\begin{aligned}
\mathrm{E}_{t,\alpha,r,x}\left[(X^u_t-(\xi-\tilde{\theta}^*))^2\right] \geq & \zeta\mathrm{E}_{t,\alpha,r,x}[\tau_\varepsilon - t] + e^{a(t)r+b(t)\alpha}\phi(t)\left(x-(\xi-\tilde{\theta}^*)e^{g(t)+h(t)r}\right)\\
> & e^{a(t)r+b(t)\alpha}\phi(t)\left(x-(\xi-\tilde{\theta}^*)e^{g(t)+h(t)r}\right)\\
= & \mathrm{E}_{t,\alpha,r,x}[(X^w_T-(\xi-\tilde{\theta}^*))^2],
\end{aligned}$$

where the strict inequality follows from the fact that $\tau_\varepsilon > t$ holds $\mathrm{P}_{t,\alpha,r,x}$ almost surely due to the path-wise continuity of the state variables. This result, in turn, leads to

$$\begin{aligned}
\mathrm{Var}_{t,\alpha,r,x}(X^u_T) &= \mathrm{E}_{t,\alpha,r,x}\left[(X^u_T)^2\right] - \xi^2\\
&= \mathrm{E}_{t,\alpha,r,x}[(X^u_T-(\xi-\tilde{\theta}^*))^2] - (\tilde{\theta}^*)^2\\
&> \mathrm{E}_{t,\alpha,r,x}[(X^w_T-(\xi-\tilde{\theta}^*))^2] - (\tilde{\theta}^*)^2\\
&= \mathrm{Var}_{t,\alpha,r,x}(X^w_T),
\end{aligned}$$

by which we can conclude that the candidate solution presented in (3.4.6) is the dynamic optimality of the mean-variance problem (3.2.8).

Substitute the dynamic optimal strategy (3.4.6) into the wealth process (3.2.7) and denote by $K_t = X^{d*}_t + \frac{\xi}{\gamma}Y_t$. Then, we have the following linear SDE of K_t:

$$\begin{aligned}
dK_t = &\Bigg[r_t + \frac{\lambda_r\left(\lambda_r - \sigma_r a(t) - \exp\left\{b(t)\alpha_t + 2g(t)\right\}\phi(t)h(t)\sigma_r\right)}{\exp\left\{b(t)\alpha_t+2g(t)\right\}\phi(t)-1}\\
&+\frac{(\lambda+\sigma_\alpha\rho b(t))\lambda\alpha_t}{\exp\left\{b(t)\alpha_t+2g(t)\right\}\phi(t)-1}\Bigg]K_t\,dt\\
&+\frac{(\lambda+\sigma_\alpha\rho b(t))\sqrt{\alpha_t}}{\exp\left\{b(t)\alpha_t+2g(t)\right\}\phi(t)-1}K_t\,dW^1_t\\
&+\frac{\lambda_r - \sigma_r a(t) - \exp\left\{b(t)\alpha_t+2g(t)\right\}\phi(t)h(t)\sigma_r}{\exp\left\{b(t)\alpha_t+2g(t)\right\}\phi(t)-1}K_t\,dW^0_t,
\end{aligned}$$

with $K_{t_0} = x_0 - \xi e^{g(t_0)+h(t_0)r_0}$. Solving the above SDE explicitly, we obtain (3.4.7). Particularly, when the initial data satisfies $x_0 \leq \xi e^{g(t_0)+h(t_0)r_0}$, from (3.4.7), we see that $X^{d*}_t \leq \xi e^{g(t)+h(t)r_t}$, for $t \in [t_0,T]$, P_{t_0} almost surely.

□

Chapter 4

Mean-variance asset-liability management under CIR interest rate and the family of 4/2 stochastic volatility models with derivative trading

Abstract

This paper investigates the effects of derivative trading on the performance of asset-liability management in the presence of stochastic interest rate and stochastic volatility under the mean-variance criterion. Specifically, the asset-liability manager can invest not only in a money market account, a zero-coupon (rollover) bond, and a stock index but also in stock derivatives. It is assumed that the interest rate follows a Cox-Ingersoll-Ross (CIR) process, and the instantaneous variance of the stock index is governed by the family of 4/2 stochastic volatility models, which embraces the Heston model and 3/2 model, as particular cases. By solving a system of three backward stochastic differential equations, closed-form expressions for the optimal strategies and optimal value functions are derived in two cases: with and without the stock derivatives. Moreover, we consider the special cases without random liabilities. Numerical examples are provided to illustrate theoretical results and explore the effects of derivative trading on efficient frontiers.

Keywords: Asset-liability management; CIR interest rate; 4/2 stochastic volatility; Derivative trading; Backward stochastic differential equation.

4.1 Introduction

The asset-liability management (ALM) problem is a topic of concern to financial institutions such as banks, pension funds, and insurance companies. The main purpose of ALM is to pursue investment return at an adequate level with the presence of liability. The ALM problem in a single-period setting under Markowitz (1952)'s mean-variance criterion can be traced back to the pioneering work of Sharpe and Tint (1990). In recent years, many attempts were made to extend the work of Sharpe and Tint (1990) to different scenarios. Leippold, Trojani, and Vanini (2004) investigated the multi-period case and derived explicit expressions for the optimal investment strategy and the efficient frontier. By utilizing the stochastic linear-quadratic theory and solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation, Chiu and Li (2006) solved a mean-variance ALM problem in a continuous-time setting with the assumption that both the risky asset price and random liability followed geometric Brownian motions. Different from Chiu and Li (2006), Xie, Li, and Wang (2008) considered the case when the random liability was governed by a Brownian motion with drift. Chen, Yang, and Yin (2008) and Chen and Yang (2011) extended the work of Chiu and Li (2006) and Leippold, Trojani, and Vanini (2004) to the cases with Markovian regime-switching markets, respectively. By applying the backward stochastic differential equation (BSDE) approach, Chiu and Wong (2012, 2013) studied a mean-variance ALM problem with cointegrated risky assets. Peng and Chen (2021) considered a mean-variance ALM problem with partial information and an uncertain time horizon by using the BSDE approach. For other previous works, one can refer to Chiu and Wong (2014a), Shen, Wei, and Zhao (2020), A, Shen, and Zeng (2022), and references therein.

Although ALM problems under the mean-variance criterion have been extensively studied, two aspects are worthy of being further explored. Firstly, most of the aforementioned literature was studied in the context of constant or deterministic volatility, which is, however, not consistent with many stylized facts observed in the financial market, such as volatility smiles and volatility clustering. Therefore, as natural extensions of constant volatility models, varieties of local volatility and stochastic volatility models have been proposed in recent years, such as the constant elasticity of variance (CEV) model, Stein-Stein model (Stein and Stein (1991)), Heston model (Heston (1993)), and 3/2 model (Lewis (2000)). This has led many researchers onto this path. For example, Černý and Kallsen (2008) applied a martingale approach to study a mean-variance optimal investment and hedging problem under the Heston model. Zhang and Chen (2016) considered a mean-variance ALM problem under a CEV model with multiple risky assets, and the optimal strategy and efficient frontier were expressed via the solutions to two BSDEs. Li, Shen, and Zeng (2018) considered a derivative-based mean-variance

ALM problem under the Heston model, and explicit solutions were derived for two cases: with and without the derivative asset. By alternatively characterizing the liability process by a generalized Brownian motion, Pan, Zhang, and Zhou (2018) studied a mean-variance ALM problem under the Heston model, and they obtained explicit solutions for the special cases when two fundamental risk factors were perfectly correlated or anti-correlated. By using the BSDE approach, Sun, Zhang, and Yuen (2020) further studied a mean-variance ALM problem in a complete market setting with multiple risky assets, where the variance processes of the risky assets were described by an affine diffusion equation. More recently, Han and Wong (2021) incorporated rough volatility into a mean-variance portfolio selection problem and derived the closed-form solution under the rough Heston model. Secondly, most of the literature mentioned above assumes that interest rates are constant or deterministic, which excludes the applications of some specific models for modeling interest rates in practice, such as the Vasicek model (Vasicek (1977)) and Cox-Ingersoll-Ross (CIR) model (Cox, Ingersoll, and Ross (1985)). Recently, some literature has focused on the ALM problem with interest rate risk. For example, Pan and Xiao (2017c) studied an optimal mean-variance ALM problem in the presence of interest rate and inflation risks, and they obtained explicit solutions by using the dynamic programming approach. Besides Markowitz's mean-variance criterion, Pan and Xiao (2017a) considered an optimal ALM problem under the expected utility maximization framework with stochastic interest rates, and closed-form expressions for the optimal strategies were derived for the power and exponential utility functions. Other preceding research outputs on the ALM problem with stochastic interest rates include Pan and Xiao (2017b), Zhu, Zhang, and Jin (2020), to name but only a few. It is also worth mentioning that the optimal strategies derived in the above-mentioned literature are pre-committed but not time-consistent in the sense that, the optimal strategies determined at the initial time might not be optimal at a future time point. This is because the nonlinear operator within the objective function under the mean-variance criterion violates the Bellman optimality principle. In the last few years, time-consistent mean-variance portfolio selection problems within the game theoretical framework (Strotz (1956), Björk, Murgoci, and Zhou (2014) and Björk, Khapko, and Murgoci (2017)) and open-loop control framework (Hu, Jin, and Zhou (2017)) have been extensively studied under different specific scenarios, such as Li, Zeng, and Lai (2012), Li, Rong, and Zhao (2015), Lin and Qian (2016), Yan and Wong (2019), Zhu and Li (2020), and references therein.

In 2017, a new stochastic volatility model, known as the 4/2 model, was proposed by Grasselli (2017). The state-of-the-art model recovers two parsimonious models, the Heston model and 3/2 model, as particular cases. By combining the advantages of the Heston model and 3/2 model and canceling their limitations, the 4/2 model can better predict the dynamics of the implied volatility surface (Grasselli (2017), Cui, Kirkby, and Nguyen (2018) and Zhu and Wang (2019)). Recently, the potential of applications of the 4/2 model to dynamic portfolio optimization problems has been realized. For example, Cheng and Escobar (2021a) considered a utility maximization problem under the 4/2 model for power utility, and closed-form solutions for the optimal strategy were obtained by applying the dynamic programming approach. Alternatively, Zhang (2021a) investigated a mean-variance portfolio selection problem with mispricing in the family of 4/2 models, and the optimal strategy and efficient frontier were derived explicitly by solving the corresponding HJB equation.

To the best of our knowledge, there is no existing literature discussing the pre-committed strategies for the mean-variance ALM problem in the presence of stochastic volatility as well as stochastic interest rates. This paper aims to fill the gap. Specifically, we assume that the stochastic interest rate follows the CIR model (Cox, Ingersoll, and Ross (1985)), and the stock index process exhibits not only the 4/2 stochastic volatility (Grasselli (2017)) but also a stochastic correlation with the interest rate. The uncontrollable random liability is described by a generalized geometric Brownian motion with the return rate and volatility dependent on the stochastic interest rate. In addition to a money market account, a zero-coupon (rollover) bond, and a stock index, we follow Escobar, Ferrando, and Rubtsov (2018) to introduce stock derivatives to complete the market. Moreover, to investigate the effects of derivatives trading on the mean-variance ALM problem, we also consider the incomplete market case where the stock derivatives are not available to the asset-liability manager. We employ a BSDE approach to solve the two problems. By considering the canonical decomposition of continuous semi-martingales, we introduce two systems of three BSDEs including a backward stochastic Riccati equation (BSRE) and two linear BSDEs into the complete and incomplete markets, respectively. Particularly, in the incomplete market case, the drivers of two linear BSDEs are not only with unbounded coefficients but also dependent on the solution to the BSRE. This makes the problem more technically challenging. By making an assumption on the model parameters and using measure change techniques, we prove the existence and uniqueness results to the two BSDE systems and derive the corresponding solutions in closed form. Explicit expressions for both the optimal strategies and efficient frontiers are then obtained. Furthermore, we provide the results of special cases without random liabilities in both the complete and incomplete market cases. Finally, some numerical experiments are given to illustrate the effects of derivatives trading on efficient frontiers. For clarity, we list the main contributions of the paper below:

1. We incorporate derivatives trading into an ALM problem under the mean-variance criterion in the presence of stochastic volatility and stochastic interest rates, where the interest rate is modeled by the CIR process and the stochastic volatility is described by the 4/2 model (Grasselli (2017)).

2. Explicit expressions for the optimal pre-committed (efficient) strategy and optimal value function (efficient frontier) are derived by applying a BSDE approach, which reduces a matter of conjecturing a candidate solution to a sophisticated HJB equation and proving a necessary verification theorem along the dynamic programming approach to solving three BSDEs and extends the recent results of Cheng and Escobar (2021a) and Zhang (2021a) on portfolio optimization problems under the 4/2 model to the cases with random liabilities and stochastic interest rates.

3. We solve the induced linear BSDEs with unbounded coefficients by using a comparison method combined with Girsanov's measure transformation and the linear BSDEs with uniformly Lipschitz continuity (El Karoui, Peng, and Quenez (1997)) rather than the results of linear BSDEs with stochastic Lipschitz continuity (Bender and Kohlmann (2000) and Wang, Ran, and Hong (2006)), in which strong assumptions on the model parameters are needed to ensure the existence and uniqueness results of the solution. This differentiates the present paper from Shen, Zhang, and Siu (2014), Zhang and Chen (2016), and Zhang (2021b) from a technical point of view.

The remainder of this paper is organized as follows. Section 4.2 introduces the market model and formulates the problem in the complete market case. Section 4.3 explores the solvability of three BSDEs and derives the optimal strategy and efficient frontier explicitly for the complete market case. In Section 4.4, we formulate the problem in the incomplete market case and obtain the closed-form solutions without derivatives trading by solving three BSDEs. Section 4.5 presents some numerical experiments to illustrate our results. Section 4.6 concludes the paper.

4.2 Problem formulation

Let $T > 0$ be a fixed and finite time of decision making and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions and carrying three one-dimensional, mutually independent Brownian motions $\{W_t^S\}_{t\in[0,T]}$, $\{W_t^V\}_{t\in[0,T]}$, and $\{W_t^r\}_{t\in[0,T]}$, where filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ is generated by the above Brownian motions, and $\mathbb{P}$ is a real-world probability measure. The expectation under $\mathbb{P}$ is denoted by $\mathbb{E}[\cdot]$.

4.2.1 Financial markets

We consider a financial market consisting of a money market account (cash), a stock index, zero-coupon bonds, and stock derivatives. The money market account follows the dynamics:

$$dA_t = r_t A_t \, dt, A_0 = 1,$$

where the short-term interest rate r_t is governed by the Cox-Ingersoll-Ross (CIR) model:

$$dr_t = (\varphi_r - \kappa_r r_t)\, dt + \sigma_r \sqrt{r_t}\, dW_t^r \tag{4.2.1}$$

with initial value $r_0 \in \mathbb{R}^+$, where $\kappa_r \in \mathbb{R}^+$ is the mean-reversion speed, $\varphi_r/\kappa_r \in \mathbb{R}^+$ is the long-run mean, and $\sigma_r \in \mathbb{R}^+$ is the volatility of the interest rate. Moreover, we require the Feller condition $2\varphi_r \geq \sigma_r^2$ to ensure that r_t is strictly positive $\mathbb{P}$ almost surely, for $t \in [0, T]$.

Suppose that the market price of interest rate risk is $\frac{\lambda_r}{\sigma_r}\sqrt{r_t}$, where $\lambda_r \in \mathbb{R}^+$. According to Cox, Ingersoll, and Ross (1985), the price process $B_t^{\bar{T}}$ of zero-coupon bond with bond maturity $\bar{T}$ evolves according to

$$dB_t^{\bar{T}} = \left(r_t - \lambda_r b(\bar{T} - t) r_t\right) B_t^{\bar{T}}\, dt - b(\bar{T} - t)\sigma_r \sqrt{r_t} B_t^{\bar{T}}\, dW_t^r, \tag{4.2.2}$$

with boundary condition $B_{\bar{T}}^{\bar{T}} = 1$, where the deterministic function $b(t)$ is given by

$$b(t) = \frac{2\left(e^{\zeta t} - 1\right)}{2\zeta + (\zeta + \kappa_r + \lambda_r)\left(e^{\zeta t} - 1\right)},$$

with $\zeta = \sqrt{(\kappa_r + \lambda_r)^2 + 2\sigma_r^2}$. As discussed in Boulier, Huang, and Taillard (2001), it is quite unlikely to find all zero-coupon bonds in the market. We, therefore, introduce a rollover bond with constant maturity $K \in \mathbb{R}^+$ into the market. Denote the price of the rollover bond by B_t^K. Then, the price process B_t^K is described by the following SDE (Deelstra, Grasselli, and Koehl (2003)):

$$dB_t^K = (r_t - \lambda_r b(K) r_t)\, B_t^K\, dt - b(K)\sigma_r \sqrt{r_t} B_t^K\, dW_t^r. \tag{4.2.3}$$

The stock index S_t is described by the following dynamics:

$$\begin{aligned} dS_t = &\left(r_t + \lambda_v \rho(c_1 V_t + c_2) + \lambda_s \sqrt{1-\rho^2}(c_1 V_t + c_2) + \eta \lambda_r r_t\right) S_t\, dt + \eta \sigma_r \sqrt{r_t} S_t\, dW_t^r \\ &+ S_t \left(c_1 \sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}\right)\left(\rho\, dW_t^V + \sqrt{1-\rho^2}\, dW_t^S\right),\ S_0 = s_0 \in \mathbb{R}^+, \end{aligned} \tag{4.2.4}$$

where parameters $\eta \in \mathbb{R}$, $c_1 \geq 0$, $c_2 \geq 0$, and $\rho \in (-1, 1)$, and the variance driver V_t follows a CIR process:

$$dV_t = (\varphi_v - \kappa_v V_t)\, dt + \sigma_v \sqrt{V_t}\, dW_t^V,\ V_0 = v_0 \in \mathbb{R}^+, \tag{4.2.5}$$

where parameters $\kappa_v, \varphi_v/\kappa_v, \sigma_v \in \mathbb{R}^+$ have similar economic meaning to that of the parameters for the interest rate process (4.2.1). Furthermore, we assume that the Feller condition $2\varphi_v \geq \sigma_v^2$ is satisfied, so that V_t is strictly positive $\mathbb{P}$ almost surely, for $t \in [0, T]$.

Remark 4.2.1. Note that the stock index process (4.2.4) exhibits not only the 4/2 stochastic volatility (refer to Grasselli (2017)) but also a stochastic instantaneous correlation with interest rate process r_t (4.2.1). Specifically, the correlation between the stock index return and interest rate is $\frac{\eta\sigma_r\sqrt{r_t}}{\sqrt{(c_1\sqrt{V_t}+\frac{c_2}{\sqrt{V_t}})^2+\eta^2\sigma_r^2 r_t}} \in (-1,1)$ and the parameter η characterizes how large the effect of the interest rate on the stock index dynamic is. In particular, the specification $\eta = 0$ means that the stock index dynamic is not affected by interest rate shocks. Moreover, it is worth mentioning that the case $(c_1, c_2) = (1, 0)$ corresponds to the Heston model (Heston (1993)), while the specification $(c_1, c_2) = (0, 1)$ is known as the 3/2 model (Lewis (2000)).

Apart from investing in the money market account, rollover bond, and stock index, we posit that the manager has access to stock derivatives in the financial market. Following Escobar, Ferrando, and Rubtsov (2018), we assume that the value of the stock derivative D_t relies on the underlying stock index S_t, stock index instantaneous variance V_t, and interest rate r_t via some twice continuously differentiable function $g(\cdot,\cdot,\cdot,\cdot)$, i.e., $D_t = g(S_t, V_t, r_t, t)$. Then applying Itô's lemma to D_t and making use of the fundamental pricing equation that function $g(\cdot,\cdot,\cdot\cdot)$ should satisfy, the dynamic for the derivative price D_t is given by

$$\begin{aligned}\frac{dD_t}{D_t} =& r_t\,dt + C_1\left(\lambda_s V_t\,dt + \sqrt{V_t}\,dW_t^S\right) + C_2\left(\lambda_r r_t\,dt + \sigma_r\sqrt{r_t}\,dW_t^r\right) \\ &+ C_3\left(\lambda_v V_t\,dt + \sqrt{V_t}\,dW_t^V\right),\end{aligned} \tag{4.2.6}$$

where

$$\begin{cases} C_1 = D_t^{-1} g_s S_t\sqrt{1-\rho^2}\left(c_1 + \dfrac{c_2}{V_t}\right), \\ C_2 = D_t^{-1}(g_s S_t\eta + g_r), \\ C_3 = D_t^{-1}\left(g_s S_t\rho\left(c_1 + \dfrac{c_2}{V_t}\right) + \sigma_v g_v\right). \end{cases}$$

Here, notations g_s, g_v, and g_r represent the partial derivatives of D_t with S_t, V_t, and r_t, respectively, i.e., we write

$$g_s = \left.\frac{\partial g(s,v,r,t)}{\partial s}\right|_{(S_t,V_t,r_t,t)},\ g_v = \left.\frac{\partial g(s,v,r,t)}{\partial v}\right|_{(S_t,V_t,r_t,t)},\ g_r = \left.\frac{\partial g(s,v,r,t)}{\partial r}\right|_{(S_t,V_t,r_t,t)}.$$

4.2.2 Asset and liability processes

Denote by π_t^B, π_t^S, and π_t^D the proportions of asset invested in the rollover bond, stock index, and stock derivative, respectively. Let $\pi := \left(\{\pi_t^B\}_{t\in[0,T]}, \{\pi_t^S\}_{t\in[0,T]}, \{\pi_t^D\}_{t\in[0,T]}\right)$ denote the investment strategy, and X_t^π denote the asset process associated with π. Under a self-financing condition, the asset process X_t^π is given by

$$
\begin{aligned}
dX_t^\pi =&\pi_t^B X_t^\pi \frac{dB_t^K}{B_t^K} + \pi_t^S X_t^\pi \frac{dS_t}{S_t} + \pi_t^D X_t^\pi \frac{dD_t}{D_t} + (1-\pi_t^B-\pi_t^S-\pi_t^D) X_t^\pi \frac{dA_t}{A_t} \\
=& \left(r_t + \theta_t^S \lambda_s V_t + \theta_t^r \lambda_r r_t + \theta_t^V \lambda_v V_t\right) X_t^\pi \, dt + \theta_t^S \sqrt{V_t} X_t^\pi \, dW_t^S \\
&+ \theta_t^V \sqrt{V_t} X_t^\pi \, dW_t^V + \theta_t^r \sigma_r \sqrt{r_t} X_t^\pi \, dW_t^r ,
\end{aligned}
\tag{4.2.7}
$$

with initial asset value $X_0^\pi = x_0 \in \mathbb{R}^+$, where θ_t^S, θ_t^r, and θ_t^V represent the risk exposures to the fundamental risk factors W_t^S, W_t^r, and W_t^V, respectively, and are related to investment strategy π_t^S, π_t^B, and π_t^D via

$$
\begin{bmatrix} \theta_t^S \\ \theta_t^r \\ \theta_t^V \end{bmatrix} = \begin{bmatrix} \sqrt{1-\rho^2}\,(c_1 + c_2/V_t) & 0 & C_1 \\ \eta & -b(K) & C_2 \\ \rho\,(c_1 + c_2/V_t) & 0 & C_3 \end{bmatrix} \begin{bmatrix} \pi_t^S \\ \pi_t^B \\ \pi_t^D \end{bmatrix} . \tag{4.2.8}
$$

Notice from SDE (4.2.7) that we work with the risk exposures rather than the investment strategies to make our analysis independent of the stock derivative specifically chosen in Section 4.3, and we replace X_t^π by X_t^θ in the following discussions.

Apart from making investments in the above financial market, the asset manager is also subject to an exogenous liability commitment. The uncontrollable liability process L_t follows the dynamics:

$$
\frac{dL_t}{L_t} = \mu_r r_t \, dt + \beta_r \sqrt{r_t} \, dW_t^r , \;\; L_0 = l_0 \in \mathbb{R}^+, \tag{4.2.9}
$$

where constant $\mu_r \in \mathbb{R}$ is the drift coefficient, and $\beta_r \in \mathbb{R}^+$ controls the volatility of the liability process.

Definition 4.2.2 (Admissible strategy). *A risk exposure strategy $\theta = (\{\theta_t^S\}_{t\in[0,T]}, \{\theta_t^V\}_{t\in[0,T]}, \{\theta_t^r\}_{t\in[0,T]})$ is said to be admissible if the following conditions are satisfied:*

1. *θ_t^S, θ_t^V, and θ_t^r are $\mathbb{F}$-adapted processes such that*

$$
\int_0^T |\theta_t^S \sqrt{V_t}|^2 \, dt < \infty, \;\; \int_0^T |\theta_t^r \sqrt{r_t}|^2 \, dt < \infty, \;\; \int_0^T |\theta_t^V \sqrt{V_t}|^2 \, dt < \infty, \;\; \mathbb{P}-a.s.;
$$

2. *SDE (4.2.7) associated with θ has a unique strong solution X_t^θ;*

3. *The family of random variables $Y_{\tau_n \wedge T}\left(X_{\tau_n\wedge T}^\theta - G_{1,\tau_n\wedge T} L_{\tau_n \wedge T} - \gamma G_{2,\tau_n\wedge T}\right)^2$, $n \in \mathbb{N}$, is uniformly integrable, for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, where $\gamma \in \mathbb{R}$, and $Y_t, G_{1,t}, G_{2,t}$ are respectively given by (4.3.9), (4.3.17), and (4.3.20).*

The set of all admissible risk exposure strategies is denoted by Θ.

4.2.3 Optimization problems

We now introduce the mean-variance ALM problem. Under the mean-variance criterion, the asset-liability manager aims to find an admissible risk exposure strategy such that the variance of the terminal surplus is minimized, while the expected terminal surplus has been exogenously determined. Formally, the mean-variance ALM problem is defined below.

Definition 4.2.3. *The mean-variance ALM problem is the following constrained stochastic optimization problem:*

$$\begin{cases} \min\limits_{\theta \in \Theta} \ J_{MV}(\theta) := \mathbb{E}\left[(X_T^\theta - L_T - \xi)^2\right] \\ \text{subject to } \mathbb{E}\left[X_T^\theta - L_T\right] = \xi \end{cases} \tag{4.2.10}$$

with $\xi \in \mathbb{R}$ fixed and given. Denote by J_{MV}^* the optimal value function corresponding to the optimal strategy $\theta^* = \left(\left\{\theta_t^{S*}\right\}_{t\in[0,T]}, \left\{\theta_t^{r*}\right\}_{t\in[0,T]}, \left\{\theta_t^{V*}\right\}_{t\in[0,T]}\right)$.

To obtain the optimal risk exposure strategy for problem (4.2.10), we introduce an auxiliary Lagrangian dual functional to eliminate the constraint $\mathbb{E}\left[X_T^\theta - L_T\right] = \xi$ in problem (4.2.10)

$$\begin{aligned} L(\theta, \lambda) :&= \mathbb{E}\left[(X_T^\theta - L_T - \xi)^2\right] + 2\lambda\mathbb{E}\left[X_T^\theta - L_T - \xi\right] \\ &= \mathbb{E}\left[(X_T^\theta - L_T - (\xi - \lambda))^2\right] - \lambda^2, \end{aligned} \tag{4.2.11}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. According to the Lagrangian duality theorem (refer to Luenberger (1968)), problem (4.2.10) is equivalent to the following min-max problem:

$$\max_{\lambda \in \mathbb{R}} \min_{\theta \in \Theta} L(\theta, \lambda). \tag{4.2.12}$$

This means that we shall first address the following benchmark problem:

$$\min_{\theta \in \Theta} J_{BM}(\theta; \gamma) := \mathbb{E}\left[\left(X_T^\theta - L_T - \gamma\right)^2\right], \tag{4.2.13}$$

where $\gamma := \xi - \lambda \in \mathbb{R}$ is an exogenous constant. We denote the optimal strategy of problem (4.2.13) by $\theta_{BM}^* = \left(\left\{\theta_{BM,t}^{S*}\right\}_{t\in[0,T]}, \left\{\theta_{BM,t}^{V*}\right\}_{t\in[0,T]}, \left\{\theta_{BM,t}^{r*}\right\}_{t\in[0,T]}\right)$.

4.3 Solution to the complete market case

In this section, we first use the BSDE approach to solve the above benchmark problem (4.2.13) and then obtain the optimal risk exposure strategy and optimal value function of mean-variance ALM problem (4.2.10), when derivative trading is available to the asset-liability manager.

4.3.1 Solution to the benchmark problem

To find the BSDEs associated with benchmark problem (4.2.13), we consider two continuous $(\mathbb{F},\mathbb{P})$-semi-martingales, Y_t and G_t, with the following canonical decomposition:

$$dY_t = H_t\,dt + Z_t^r\,dW_t^r + Z_t^S\,dW_t^S + Z_t^V\,dW_t^V, \tag{4.3.1}$$

and

$$dG_t = \Psi_t\,dt + P_t^r\,dW_t^r + P_t^S\,dW_t^S + P_t^V\,dW_t^V, \tag{4.3.2}$$

where $H_t, Z_t^r, Z_t^S, Z_t^V, \Psi_t, P_t^r, P_t^S, P_t^V$ are some $\mathbb{F}$-adapted processes that will be determined later. For any admissible strategy $\theta \in \Theta$, applying Itô's formula to $Y_t\left(X_t^\theta - G_t\right)^2$, we have

$$\begin{aligned}
&Y_t\left(X_t^\theta - G_t\right)^2\\
&=\int_0^t Y_s\left[\left(\theta_s^S\sqrt{V_s}X_s^\theta - P_s^S\right) + (X_s^\theta - G_s)\left(\frac{Z_s^S}{Y_s} + \lambda_s\sqrt{V_s}\right)\right]^2 ds\\
&\quad+\int_0^t Y_s\left[\left(\theta_s^V\sqrt{V_s}X_s^\theta - P_s^V\right) + (X_s^\theta - G_s)\left(\frac{Z_s^V}{Y_s} + \lambda_v\sqrt{V_s}\right)\right]^2 ds\\
&\quad+\int_0^t Y_s\left[\left(\theta_s^r\sigma_r\sqrt{r_s}X_s^\theta - P_s^r\right) + (X_s^\theta - G_s)\left(\frac{Z_s^r}{Y_s} + \frac{\lambda_r}{\sigma_r}\sqrt{r_s}\right)\right]^2 ds\\
&\quad+\int_0^t 2(X_s^\theta - G_s)Y_s\left(r_sG_s - \Psi_s + \lambda_sP_s^S\sqrt{V_s} + \lambda_vP_s^V\sqrt{V_s} + \frac{\lambda_r}{\sigma_r}P_s^r\sqrt{r_s}\right) ds\\
&\quad+\int_0^t (X_s^\theta - G_s)^2\left(H_s + 2r_sY_s - Y_s\left(\frac{Z_s^S}{Y_s} + \lambda_s\sqrt{V_s}\right)^2 - Y_s\left(\frac{Z_s^V}{Y_s} + \lambda_v\sqrt{V_s}\right)^2\right.\\
&\quad\left.-Y_s\left(\frac{Z_s^r}{Y_s} + \frac{\lambda_r}{\sigma_r}\sqrt{r_s}\right)^2\right) ds + Y_0\left(x_0 - G_0\right)^2\\
&\quad+\int_0^t \left[\left(X_s^\theta - G_s\right)^2 Z_s^r + 2Y_s\left(X_s^\theta - G_s\right)\left(\theta_s^r\sigma_r\sqrt{r_s}X_s^\theta - P_s^r\right)\right] dW_s^r\\
&\quad+\int_0^t \left[\left(X_s^\theta - G_s\right)^2 Z_s^S + 2Y_s(X_s^\theta - G_s)\left(\theta_s^S\sqrt{V_s}X_s^\theta - P_s^S\right)\right] dW_s^S\\
&\quad+\int_0^t \left[\left(X_s^\theta - G_s\right)^2 Z_s^V + 2Y_s(X_s^\theta - G_s)\left(\theta_s^V\sqrt{V_s}X_s^\theta - P_s^V\right)\right] dW_s^V,
\end{aligned} \tag{4.3.3}$$

for any $t \in [0,T]$. Inspired by (4.3.3), we introduce the BSRE of $\left(Y_t, Z_t^r, Z_t^S, Z_t^V\right)$:

$$\begin{cases}
dY_t = \left[\left(\lambda_s^2V_t + \lambda_v^2V_t + \dfrac{\lambda_r^2}{\sigma_r^2}r_t - 2r_t\right)Y_t + 2\lambda_s\sqrt{V_t}Z_t^S + 2\lambda_v\sqrt{V_t}Z_t^V + 2\dfrac{\lambda_r}{\sigma_r}\sqrt{r_t}Z_t^r\right.\\
\qquad\qquad\left.+\dfrac{(Z_t^S)^2}{Y_t} + \dfrac{(Z_t^V)^2}{Y_t} + \dfrac{(Z_t^r)^2}{Y_t}\right] dt + Z_t^S\,dW_t^S + Z_t^V\,dW_t^V + Z_t^r\,dW_t^r,\\
Y_T = 1,\\
Y_t > 0,\ \text{for all } t \in [0,T],
\end{cases} \tag{4.3.4}$$

and the linear BSDE of $\left(G_t, P_t^r, P_t^S, P_t^V\right)$:

$$\begin{cases} dG_t = \left(r_t G_t + \lambda_s \sqrt{V_t} P_t^S + \lambda_v \sqrt{V_t} P_t^V + \dfrac{\lambda_r}{\sigma_r}\sqrt{r_t} P_t^r\right) dt + P_t^S\, dW_t^S + P_t^V\, dW_t^V \\ \qquad\quad + P_t^r\, dW_t^r, \\ G_T = L_T + \gamma. \end{cases} \tag{4.3.5}$$

Furthermore, by separating the dependence of BSDE (4.3.5) on the liability value L_T and applying Itô's lemma, we can decompose BSDE (4.3.5) of $(G_t, P_t^r, P_t^S, P_t^V)$ into the following two linear BSDEs of $(G_{1,t}, \Lambda_t^S, \Lambda_t^V, \Lambda_t^r)$ and $(G_{2,t}, \Gamma_t^S, \Gamma_t^V, \Gamma_t^r)$, respectively:

$$\begin{cases} dG_{1,t} = \Bigg[\left(1 + \dfrac{\lambda_r \beta_r}{\sigma_r} - \mu_r\right) r_t G_{1,t} + \lambda_s \sqrt{V_t}\Lambda_t^S + \lambda_v \sqrt{V_t}\Lambda_t^V \\ \qquad\qquad + \left(\dfrac{\lambda_r}{\sigma_r} - \beta_r\right)\sqrt{r_t}\Lambda_t^r\Bigg] dt + \Lambda_t^S\, dW_t^S + \Lambda_t^V\, dW_t^V + \Lambda_t^r\, dW_t^r, \\ G_{1,T} = 1, \end{cases} \tag{4.3.6}$$

and

$$\begin{cases} dG_{2,t} = \left(r_t G_{2,t} + \lambda_s \sqrt{V_t}\Gamma_t^S + \lambda_v \sqrt{V_t}\Gamma_t^V + \dfrac{\lambda_r}{\sigma_r}\sqrt{r_t}\Gamma_t^r\right) dt \\ \qquad\qquad + \Gamma_t^S\, dW_t^S + \Gamma_t^V\, dW_t^V + \Gamma_t^r\, dW_t^r, \\ G_{2,T} = 1. \end{cases} \tag{4.3.7}$$

Specifically, $(G_t, P_t^r, P_t^S, P_t^V), (G_{1,t}, \Lambda_t^S, \Lambda_t^V, \Lambda_t^r)$, and $(G_{2,t}, \Gamma_t^S, \Gamma_t^V, \Gamma_t^r)$ are related via the following linear formulation:

$$\begin{cases} G_t = L_t G_{1,t} + \gamma G_{2,t}, \\ P_t^r = L_t\left(\Lambda_t^r + G_{1,t}\beta_r\sqrt{r_t}\right) + \gamma\Gamma_t^r, \\ P_t^S = L_t\Lambda_t^S + \gamma\Gamma_t^S, \\ P_t^V = L_t\Lambda_t^V + \gamma\Gamma_t^V. \end{cases} \tag{4.3.8}$$

Throughout the rest of this paper, we make the following assumption on the market parameters.

Assumption 4.3.1. $\lambda_r^2 \geq 2\sigma_r^2$.

Remark 4.3.2. From an economic perspective, Assumption 4.3.1 is closely related to the slope of the market price of interest rate risk λ_r/σ_r and implies that taking interest rate risk should be highly rewarded, so that the asset-liability manager is willing to invest in the rollover (zero-coupon) bond to hedge against the interest rate risk. Otherwise, investing in the bond is too risky compared with the money market account; in the extreme scenario when $\lambda_r = 0$, the return rates of the money market account and bond are the same, but the bond dynamic involves an additional diffusion term (interest rate shocks). In other words, large enough λ_r/σ_r assures that the interest rate risk can be fully hedged, and thus, leading to the finite variance of terminal surplus. Mathematically speaking, Assumption 4.3.1 guarantees the non-positiveness and boundedness of functions $A_3(t)$ and $\bar{A}_3(t)$ given by (4.3.15) and (4.4.16) over $[0, T]$. This is, in turn, essential to ensure that BSREs (4.3.4) and (4.4.8) admit unique solutions, and the first-order conditions work in the proof of Theorem 4.3.14 and 4.4.8.

To derive the optimal risk exposure strategy and optimal value function of benchmark problem (4.2.13), we investigate the solvability of BSDEs (4.3.4), (4.3.6), and (4.3.7).

Proposition 4.3.3. *One solution $\left(Y_t, Z_t^r, Z_t^S, Z_t^V\right)$ to BSRE (4.3.4) is given by*

$$Y_t = \exp\left\{A_1(t) + A_2(t)V_t + A_3(t)r_t\right\}, \tag{4.3.9}$$

and

$$\begin{cases} Z_t^r = Y_t A_3(t)\sigma_r\sqrt{r_t}, \\ Z_t^S = 0, \\ Z_t^V = Y_t A_2(t)\sigma_v\sqrt{V_t}, \end{cases} \tag{4.3.10}$$

where functions $A_1(t), A_2(t)$, and $A_3(t)$ solve the following ordinary differential equations (ODEs):

$$\frac{dA_2(t)}{dt} = \left(\lambda_s^2 + \lambda_v^2\right) + \left(\kappa_v + 2\lambda_v\sigma_v\right)A_2(t) + \frac{1}{2}\sigma_v^2 A_2^2(t), \;\; A_2(T) = 0, \tag{4.3.11}$$

$$\frac{dA_3(t)}{dt} = \left(\frac{\lambda_r^2}{\sigma_r^2} - 2\right) + (\kappa_r + 2\lambda_r)A_3(t) + \frac{1}{2}\sigma_r^2 A_3^2(t), \;\; A_3(T) = 0, \tag{4.3.12}$$

$$\frac{dA_1(t)}{dt} = -\,\varphi_v A_2(t) - \varphi_r A_3(t), \;\; A_1(T) = 0. \tag{4.3.13}$$

Proof. See Appendix 4.A. □

Proposition 4.3.4. *Closed-form solutions to (4.3.11), (4.3.12), and (4.3.13) are, respectively, given by*

$$A_2(t) = \begin{cases} \dfrac{n_{A_2}^+ n_{A_2}^- \left(1 - e^{\sqrt{\Delta_{A_2}}(T-t)}\right)}{n_{A_2}^+ - n_{A_2}^- e^{\sqrt{\Delta_{A_2}}(T-t)}}, \;\; \Delta_{A_2} > 0; \\ \dfrac{n_{A_2}^2\sigma_v^2(T-t)}{n_{A_2}\sigma_v^2(T-t) - 2}, \;\; \Delta_{A_2} = 0; \\ \dfrac{\sqrt{-\Delta_{A_2}}}{\sigma_v^2}\tan\left(\arctan\left(\dfrac{\kappa_v + 2\lambda_v\sigma_v}{\sqrt{-\Delta_{A_2}}}\right) - \dfrac{\sqrt{-\Delta_{A_2}}(T-t)}{2}\right) + n_{A_2}, \;\; \Delta_{A_2} < 0, \end{cases} \tag{4.3.14}$$

with

$$\begin{cases} \Delta_{A_2} = (\kappa_v + 2\lambda_v\sigma_v)^2 - 2(\lambda_s^2 + \lambda_v^2)\sigma_v^2, \; n_{A_2} = -\dfrac{\kappa_v + 2\lambda_v\sigma_v}{\sigma_v^2}, \\ n_{A_2}^+ = \dfrac{-(\kappa_v + 2\lambda_v\sigma_v) + \sqrt{\Delta_{A_2}}}{\sigma_v^2}, \; n_{A_2}^- = \dfrac{-(\kappa_v + 2\lambda_v\sigma_v) - \sqrt{\Delta_{A_2}}}{\sigma_v^2}, \end{cases}$$

$$A_3(t) = \begin{cases} \dfrac{n_{A_3}^+ n_{A_3}^- \left(1 - e^{\sqrt{\Delta_{A_3}}(T-t)}\right)}{n_{A_3}^+ - n_{A_3}^- e^{\sqrt{\Delta_{A_3}}(T-t)}}, \; \Delta_{A_3} > 0; \\ \dfrac{n_{A_3}^2 \sigma_r^2 (T-t)}{n_{A_3}\sigma_r^2(T-t) - 2}, \; \Delta_{A_3} = 0; \\ \dfrac{\sqrt{-\Delta_{A_3}}}{\sigma_r^2} \tan\left(\arctan\left(\dfrac{\kappa_r + 2\lambda_r}{\sqrt{-\Delta_{A_3}}}\right) - \dfrac{\sqrt{-\Delta_{A_3}}(T-t)}{2}\right) + n_{A_3}, \; \Delta_{A_3} < 0, \end{cases} \tag{4.3.15}$$

with

$$\begin{cases} \Delta_{A_3} = (\kappa_r + 2\lambda_r)^2 - 2\lambda_r^2 + 4\sigma_r^2, \; n_{A_3} = -\dfrac{\kappa_r + 2\lambda_r}{\sigma_r^2}, \\ n_{A_3}^+ = \dfrac{-(\kappa_r + 2\lambda_r) + \sqrt{\Delta_{A_3}}}{\sigma_r^2}, \; n_{A_3}^- = \dfrac{-(\kappa_r + 2\lambda_r) - \sqrt{\Delta_{A_3}}}{\sigma_r^2}, \end{cases}$$

and

$$A_1(t) = \int_t^T \varphi_v A_2(s) + \varphi_r A_3(s) \, ds. \tag{4.3.16}$$

Proof. See Appendix 4.B. □

Proposition 4.3.5. *Function $A_2(t)$ given in (4.3.14) is non-positive and bounded over $[0,T]$. Furthermore, suppose that Assumption 4.3.1 holds, then $A_3(t)$ given in (4.3.15) is also non-positive and bounded over $[0,T]$.*

Proof. See Appendix 4.C. □

We now provide an auxiliary result before showing that the solution presented in Proposition 4.3.3 above is the unique solution to BSRE (4.3.4). The following lemma (Lemma 4.3.6) is adapted from Lemma A1 in Shen and Zeng (2015).

Lemma 4.3.6. *For the CIR processes r_t and V_t given in (4.2.1) and (4.2.5), if functions $m_1(t), m_2(t)$, and $m_3(t)$ are bounded on $[0,T]$, then the stochastic exponential processes:*

$$\exp\left\{-\frac{1}{2}\int_0^t m_1^2(s) r_s \, ds + \int_0^t m_1(s)\sqrt{r_s}\, dW_s^r\right\}$$

and

$$\exp\left\{-\frac{1}{2}\int_0^t \left(m_2^2(s) + m_3^2(s)\right) V_s \, ds + \int_0^t m_2(s)\sqrt{V_s}\, dW_s^V + \int_0^t m_3(s)\sqrt{V_s}\, dW_s^S\right\}$$

are $(\mathbb{F}, \mathbb{P})$-martingales.

Remark 4.3.7. Lemma 4.3.6 essentially states the bonafide martingale properties of the stochastic exponential processes of $\int_0^t m_1(s)\sqrt{r_s}\,dW_s^r$ and $\big(\int_0^t m_2(s)\sqrt{V_s}\,dW_s^V,\ \int_0^t m_3(s)\sqrt{V_s}\,dW_s^S\big)$ under $\mathbb{P}$ measure. Hence, without Novikov's condition imposed, the above stochastic exponential processes are true $(\mathbb{F},\mathbb{P})$-martingales.

Corollary 4.3.8. *Suppose that Assumption 4.3.1 holds true. The stochastic exponential process*

$$\exp\left\{-\frac{1}{2}\int_0^t A_3^2(s)\sigma_r^2 r_s + A_2^2(s)\sigma_v^2 V_s\,ds - \int_0^t A_3(s)\sigma_r\sqrt{r_s}\,d\tilde{W}_s^r - \int_0^t A_2(s)\sigma_v\sqrt{V_s}\,d\tilde{W}_s^V\right\}$$

is an $(\mathbb{F},\tilde{\mathbb{P}})$-martingale, where the probability measure $\tilde{\mathbb{P}}$ is defined by

$$\begin{aligned}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = \exp\bigg\{ & -2\int_0^T \lambda_s\sqrt{V_t}\,dW_t^S - 2\int_0^T \lambda_v\sqrt{V_t}\,dW_t^V - 2\int_0^T \frac{\lambda_r}{\sigma_r}\sqrt{r_t}\,dW_t^r \\ & - 2\int_0^T (\lambda_s^2+\lambda_v^2)V_t + \frac{\lambda_r^2}{\sigma_r^2}r_t\,dt\bigg\},\end{aligned}$$

and $\tilde{W}_t^r$ and $\tilde{W}_t^V$ are two independent Brownian motions under $\tilde{\mathbb{P}}$.

Proof. See Appendix 4.D. □

Lemma 4.3.9. *Suppose that Assumption 4.3.1 holds true. The solution $\big(Y_t, Z_t^r, Z_t^S, Z_t^V\big)$ given in Proposition 4.3.3 is the unique solution to BSRE (4.3.4).*

Proof. See Appendix 4.E. □

In the next two propositions (Proposition 4.3.10 and 4.3.11), we derive closed-form expressions for the unique solutions to linear BSDEs (4.3.6) and (4.3.7), respectively.

Proposition 4.3.10. *The unique solution $(G_{1,t},\Lambda_t^S,\Lambda_t^V,\Lambda_t^r)$ to linear BSDE (4.3.6) is given by*

$$\begin{cases} G_{1,t} = \exp\left\{f_1(t)+f_2(t)r_t\right\}, \\ \Lambda_t^S = 0, \\ \Lambda_t^V = 0, \\ \Lambda_t^r = \sigma_r\sqrt{r_t}f_2(t)G_{1,t}, \end{cases} \tag{4.3.17}$$

where $f_1(t)$ and $f_2(t)$ are given by

$$f_2(t) = \begin{cases} \dfrac{n^+_{f_2} n^-_{f_2}\left(1 - e^{\sqrt{\Delta_{f_2}}(T-t)}\right)}{n^+_{f_2} - n^-_{f_2} e^{\sqrt{\Delta_{f_2}}(T-t)}}, \ \Delta_{f_2} > 0; \\ \dfrac{n^2_{f_2}\sigma^2_r(T-t)}{n_{f_2}\sigma^2_r(T-t)+2}, \ \Delta_{f_2} = 0; \\ -\dfrac{\sqrt{-\Delta_{f_2}}}{\sigma^2_r} \tan\left(\arctan\left(\dfrac{\kappa_r + \lambda_r - \beta_r\sigma_r}{\sqrt{-\Delta_{f_2}}}\right) - \dfrac{\sqrt{-\Delta_{f_2}}(T-t)}{2}\right) \\ + n_{f_2}, \ \Delta_{f_2} < 0, \end{cases} \tag{4.3.18}$$

where

$$\begin{cases} \Delta_{f_2} = (\kappa_r + \lambda_r - \beta_r\sigma_r)^2 + 2\sigma^2_r\left(1 + \dfrac{\lambda_r\beta_r}{\sigma_r} - \mu_r\right), \ n_{f_2} = \dfrac{\kappa_r + \lambda_r - \beta_r\sigma_r}{\sigma^2_r}, \\ n^+_{f_2} = \dfrac{-(\kappa_r + \lambda_r - \beta_r\sigma_r) + \sqrt{\Delta_{f_2}}}{-\sigma^2_r}, \ n^-_{f_2} = \dfrac{-(\kappa_r + \lambda_r - \beta_r\sigma_r) - \sqrt{\Delta_{f_2}}}{-\sigma^2_r} \end{cases}$$

and

$$f_1(t) = \int_t^T \varphi_r f_2(s)\, ds. \tag{4.3.19}$$

Proof. See Appendix 4.F. □

Proposition 4.3.11. *The unique solution $(G_{2,t}, \Gamma^S_t, \Gamma^V_t, \Gamma^r_t)$ to linear BSDE (4.3.7) is given by*

$$\begin{cases} G_{2,t} = \exp\{g_1(t) + g_2(t)r_t\}, \\ \Gamma^S_t = 0, \\ \Gamma^V_t = 0, \\ \Gamma^r_t = g_2(t)\sigma_r\sqrt{r_t}G_{2,t}, \end{cases} \tag{4.3.20}$$

where $g_1(t)$ and $g_2(t)$ are given by

$$g_2(t) = \frac{n^+_{g_2} n^-_{g_2}\left(1 - e^{\sqrt{\Delta_{g_2}}(T-t)}\right)}{n^+_{g_2} - n^-_{g_2} e^{\sqrt{\Delta_{g_2}}(T-t)}}, \tag{4.3.21}$$

with $\Delta_{g_2} = (\kappa_r + \lambda_r)^2 + 2\sigma^2_r, n^+_{g_2} = \frac{-(\kappa_r+\lambda_r)+\sqrt{\Delta_{g_2}}}{-\sigma^2_r}, n^-_{g_2} = \frac{-(\kappa_r+\lambda_r)-\sqrt{\Delta_{g_2}}}{-\sigma^2_r}$, and

$$g_1(t) = \int_t^T \varphi_r g_2(s)\, ds. \tag{4.3.22}$$

Moreover, function $g_2(t)$ is non-positive and bounded over $[0, T]$.

The proof of Proposition 4.3.11 is similar to that of Proposition 4.3.5 and 4.3.10, and so we omit it here.

Remark 4.3.12. It is worth mentioning that the drivers of linear BSDEs (4.3.6) and (4.3.7) are, in fact, with stochastic Lipschitz conditions (refer to Bender and Kohlmann (2000) and Wang, Ran, and Hong (2006)). As shown either in Theorem 4 of Bender and Kohlmann (2000) or in Theorem 4.1 of Wang, Ran, and Hong (2006), however, strict assumptions on coefficients are required, so that a contraction mapping is constructed to prove the uniqueness and existence of an adapted solution to the associated BSDEs. Unlike most of the related literature (see, for example, Shen, Zhang, and Siu (2014), Zhang and Chen (2016), and Zhang (2021b)), we verify the existence and uniqueness of the solutions to BSDEs (4.3.6) and (4.3.7) by adopting Girsanov's measure transformation techniques and the results of linear BSDEs with uniformly Lipschitz condition (El Karoui, Peng, and Quenez (1997)) in the proof of Proposition 4.3.10 and 4.3.11.

Theorem 4.3.13. *Suppose that Assumption 4.3.1 holds true. For any initial data* $(r_0, s_0, v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *fixed and given, the optimal risk exposure strategy and optimal value function of benchmark problem (4.2.13) are, respectively, given by*

$$\begin{cases} \theta^{S*}_{BM,t} = -\dfrac{1}{X_t^*}\lambda_s\left(X_t^* - \gamma e^{g_1(t)+g_2(t)r_t} - e^{f_1(t)+f_2(t)r_t}L_t\right), \\ \theta^{r*}_{BM,t} = -\dfrac{1}{X_t^*}\left(\left(A_3(t)+\dfrac{\lambda_r}{\sigma_r^2}\right)X_t^* - \gamma e^{g_1(t)+g_2(t)r_t}\left(A_3(t)+g_2(t)+\dfrac{\lambda_r}{\sigma_r^2}\right)\right. \\ \qquad\qquad \left. - \left(A_3(t)+f_2(t)+\dfrac{\beta_r}{\sigma_r}+\dfrac{\lambda_r}{\sigma_r^2}\right)e^{f_1(t)+f_2(t)r_t}L_t\right), \\ \theta^{V*}_{BM,t} = -\dfrac{1}{X_t^*}(\sigma_v A_2(t)+\lambda_v)\left(X_t^* - \gamma e^{g_1(t)+g_2(t)r_t} - e^{f_1(t)+f_2(t)r_t}L_t\right), \end{cases} \tag{4.3.23}$$

and

$$J_{BM}(\theta^*_{BM};\gamma) = e^{A_1(0)+A_2(0)v_0+A_3(0)r_0}\left(x_0 - l_0 e^{f_1(0)+f_2(0)r_0} - \gamma e^{g_1(0)+g_2(0)r_0}\right)^2, \tag{4.3.24}$$

where functions $A_1(t), A_2(t), A_3(t), f_1(t), f_2(t), g_1(t)$, *and* $g_2(t)$ *are given by (4.3.16), (4.3.14), (4.3.15), (4.3.19), (4.3.18), (4.3.22), and (4.3.21), respectively. Furthermore, the optimal strategy* θ^*_{BM} *given in (4.3.23) is admissible, i.e.,* $\theta^*_{BM} \in \Theta$.

Proof. See Appendix 4.G. □

4.3.2 Solution to the mean-variance problem

In the next theorem, we obtain closed-form expressions for the optimal risk exposure strategy and optimal value function of the mean-variance ALM problem (4.2.10). In addition, we provide the results of the special case without random liabilities.

Theorem 4.3.14. *Suppose that Assumption 4.3.1 holds true. For any initial data $(r_0, s_0, v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ fixed and given, the optimal strategy and optimal value function of mean-variance ALM problem (4.2.10) are, respectively, given by*

$$
\begin{cases}
\theta_t^{S*} = -\dfrac{1}{X_t^*}\lambda_s \left(X_t^* - (\xi - \lambda^*)e^{g_1(t)+g_2(t)r_t} - e^{f_1(t)+f_2(t)r_t}L_t\right), \\
\theta_t^{r*} = -\dfrac{1}{X_t^*}\left[\left(A_3(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)X_t^* - (\xi - \lambda^*)e^{g_1(t)+g_2(t)r_t}\left(A_3(t) + g_2(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)\right. \\
\qquad \left. - \left(A_3(t) + f_2(t) + \dfrac{\beta_r}{\sigma_r} + \dfrac{\lambda_r}{\sigma_r^2}\right)e^{f_1(t)+f_2(t)r_t}L_t\right], \\
\theta_t^{V*} = -\dfrac{1}{X_t^*}\left[(\sigma_v A_2(t) + \lambda_v)\left(X_t^* - (\xi - \lambda^*)e^{g_1(t)+g_2(t)r_t}\right)\right. \\
\qquad \left. - (\sigma_v A_2(t) + \lambda_v)\, e^{f_1(t)+f_2(t)r_t}L_t\right],
\end{cases}
\tag{4.3.25}
$$

and

$$
J_{MV}^* = \frac{Y_0\left(x_0 - l_0 G_{1,0} - \xi G_{2,0}\right)^2}{1 - Y_0 G_{2,0}^2}, \tag{4.3.26}
$$

with λ^* given by

$$
\lambda^* = \frac{Y_0 G_{2,0}\left(x_0 - l_0 G_{1,0} - \xi G_{2,0}\right)}{1 - Y_0 G_{2,0}^2}, \tag{4.3.27}
$$

where $Y_t, G_{1,t}, G_{2,t}, A_1(t), A_2(t), A_3(t), f_1(t), f_2(t), g_1(t), g_2(t)$ are given by (4.3.9), (4.3.17), (4.3.20), (4.3.16), (4.3.14), (4.3.15), (4.3.19), (4.3.18), (4.3.22), and (4.3.21), respectively. In addition, the optimal risk exposure strategy (4.3.25) is admissible, i.e., $\theta^* \in \Theta$. Moreover, from the relationship between π and θ given in (4.2.8), we have the optimal investment strategy $\pi^* = (\pi_t^{S*}, \pi_t^{D*}, \pi_t^{B*})$ of mean-variance ALM problem (4.2.10) as follows:

$$
\begin{cases}
\pi_t^{S*} = \dfrac{C_1\theta_t^{V*} - C_3\theta_t^{S*}}{C_1\rho\left(c_1 + \frac{c_2}{V_t}\right) - C_3\sqrt{1-\rho^2}\left(c_1 + \frac{c_2}{V_t}\right)}, \\
\pi_t^{D*} = \dfrac{\rho\theta_t^{S*} - \sqrt{1-\rho^2}\theta_t^{V*}}{\rho C_1 - \sqrt{1-\rho^2}C_3}. \\
\pi_t^{B*} = \dfrac{\eta\pi_t^{S*} + C_2\pi_t^{D*} - \theta_t^{r*}}{b(K)}.
\end{cases}
\tag{4.3.28}
$$

Proof. See Appendix 4.H. □

Remark 4.3.15. The above results for the mean-variance ALM problem with derivatives trading under the hybrid CIR-4/2 model are not considered in the existing literature. In this sense, the present paper extends the results on the mean-variance ALM problems, such as Zhang and Chen (2016), Li, Shen, and Zeng (2018), and Sun, Zhang, and Yuen (2020), to the case that simultaneously takes into consideration derivatives trading, stochastic volatility as well as stochastic interest rates.

Remark 4.3.16. Note that the optimal value function J^*_{MV} and optimal exposures θ^* to the stock index risk, volatility risk, and interest rate risk are independent of the specific type of derivative asset as said in Section 4.2, whereas the optimal demand of stock index, rollover bond, and derivative asset depends on C_1, C_2, and C_3, which reflects the deltas of the derivative asset. In particular, for a volatility/variance derivative asset such as VIX and variance swap, it follows from (4.2.6) that $C_1 = 0$, $C_2 = D_t^{-1} g_r$, and $C_3 = D_t^{-1}\sigma_v g_v$. Moreover, we observe that the optimal value function and optimal exposures are also irrelevant to the specification of two non-negative constants c_1 and c_2 in the dynamics of stock index (4.2.4). If we further set $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$, we obtain the corresponding optimal investment strategies of mean-variance ALM problem (4.2.10) under the CIR-Heston model and CIR-3/2 model, respectively. These two findings reveal that the solvability of problem (4.2.10) essentially hinges upon the specifications of the market prices of stock index risk $\lambda_s\sqrt{V_t}$, volatility risk $\lambda_v\sqrt{V_t}$, and interest risk $\frac{\lambda_r}{\sigma_r}\sqrt{r_t}$.

Corollary 4.3.17. *(Without liability). Suppose that Assumption 4.3.1 holds true. If there is no liability, then for any initial data $(r_0, s_0, v_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ fixed and given, the optimal strategy and optimal value function of mean-variance problem (4.2.10) are, respectively, given by*

$$\begin{cases} \theta_t^{S*} = -\dfrac{1}{X_t^*}\lambda_s\left(X_t^* - (\xi - \tilde{\lambda}^*)e^{g_1(t)+g_2(t)r_t}\right), \\ \theta_t^{r*} = -\dfrac{1}{X_t^*}\left[\left(A_3(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)X_t^* - (\xi - \tilde{\lambda}^*)e^{g_1(t)+g_2(t)r_t}\left(A_3(t) + g_2(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)\right], \\ \theta_t^{V*} = -\dfrac{1}{X_t^*}(\sigma_v A_2(t) + \lambda_v)\left(X_t^* - (\xi - \tilde{\lambda}^*)e^{g_1(t)+g_2(t)r_t}\right), \end{cases} \tag{4.3.29}$$

and

$$\tilde{J}^*_{MV} = \frac{Y_0\left(x_0 - \xi G_{2,0}\right)^2}{1 - Y_0 G_{2,0}^2}, \tag{4.3.30}$$

where

$$\tilde{\lambda}^* = \frac{Y_0 G_{2,0}\left(x_0 - \xi G_{2,0}\right)}{1 - Y_0 G_{2,0}^2}. \tag{4.3.31}$$

Proof. Substituting $l_0 = \mu_r = \beta_r = 0$ into (4.3.25)–(4.3.27) yields (4.3.29)–(4.3.31), respectively. □

4.4 Solution to the incomplete market case

In this section, we study the alternative scenario where stock derivatives are not available, and we devote to deriving closed-form expressions for the optimal investment strategy and optimal value function of the mean-variance ALM problem in the incomplete market case.

To facilitate further discussions in this section, we work with the following three one-dimensional, mutually independent Brownian motions (W_t^0, W_t^1, W_t^2) under $\mathbb{P}$ measure defined by

$$\begin{bmatrix} dW_t^0 \\ dW_t^1 \\ dW_t^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \rho & \sqrt{1-\rho^2} & 0 \\ \sqrt{1-\rho^2} & -\rho & 0 \end{bmatrix} \begin{bmatrix} dW_t^V \\ dW_t^S \\ dW_t^r \end{bmatrix},$$

which are clearly equivalent to (W_t^S, W_t^V, W_t^r) due to Levy's characterization of Brownian motions. It then follows from (4.2.1), (4.2.3), (4.2.4), (4.2.5), and (4.2.9) that the dynamics of the market model and random liability can be reformulated as follows:

$$\begin{cases} dr_t = (\varphi_r - \kappa_r r_t)\, dt + \sigma_r \sqrt{r_t}\, dW_t^0, \\ dB_t^K = (r_t - \lambda_r b(K) r_t)\, B_t^K\, dt - b(K)\sigma_r\sqrt{r_t} B_t^K\, dW_t^0, \\ dS_t = \left[r_t + \left(\lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right)(c_1 V_t + c_2) + \eta\lambda_r r_t \right] S_t\, dt + \eta\sigma_r\sqrt{r_t}\, dW_t^0 \\ \qquad + S_t \left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}} \right) dW_t^1, \\ dV_t = (\varphi_v - \kappa_v V_t)\, dt + \sigma_v\sqrt{V_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right), \\ dL_t = L_t \left(\mu_r r_t\, dt + \beta_r\sqrt{r_t}\, dW_t^0\right). \end{cases} \tag{4.4.1}$$

With the additional constraint that the investor cannot trade stock derivatives, it follows from (4.2.7) and (4.4.1) that the asset process X_t^π evolves as:

$$\begin{aligned} dX_t^\pi = & \left[r_t + \left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)(c_1 V_t + c_2)\,\pi_t^S + \left(\eta\pi_t^S - b(K)\pi_t^B\right)\lambda_r r_t \right] X_t^\pi\, dt \\ & + \pi_t^S X_t^\pi \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}} \right) dW_t^1 + \left(\eta\pi_t^S - b(K)\pi_t^B\right)\sigma_r\sqrt{r_t} X_t^\pi\, dW_t^0, \end{aligned} \tag{4.4.2}$$

with initial asset value $x_0 \in \mathbb{R}^+$.

Definition 4.4.1 (Admissible strategy)**.** *In the market without derivatives trading, an investment strategy π is said to be admissible if the following conditions are satisfied:*

1. *π_t^S and π_t^B are $\mathbb{F}$-adapted processes such that*
$$\int_0^T \left| \pi_t^S \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}} \right) \right|^2 dt < \infty, \ \int_0^T |(\eta\pi_t^S - b(K)\pi_t^B)\sqrt{r_t}|^2\, dt < \infty, \ \mathbb{P} - a.s.;$$
2. *SDE (4.4.2) associated with π has a unique strong solution X_t^π;*
3. *The family of random variables $Y_{\tau_n\wedge T}^{ic}\left(X_{\tau_n\wedge T}^\pi - G_{1,\tau_n\wedge T}^{ic} L_{\tau_n\wedge T} - \gamma G_{2,\tau_n\wedge T}^{ic}\right)^2$, $n \in \mathbb{N}$ is uniformly integrable, for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$*

such that $\tau_n \uparrow \infty$, where $Y_t^{ic}, G_{1,t}^{ic}$, and $G_{2,t}^{ic}$ are respectively given by (4.4.13), (4.4.18), and (4.4.21), and $\gamma \in \mathbb{R}$.

The set of all admissible investment strategies is denoted as $\mathcal{A}$.

Definition 4.4.2. *In the market without derivatives trading, the mean-variance ALM problem is the following constrained stochastic optimization problem:*

$$\begin{cases} \min_{\pi \in \mathcal{A}} \; J_{MV}^{ic}(\pi) := \mathbb{E}\left[(X_T^\pi - L_T - \xi)^2 \right] \\ \text{subject to } \mathbb{E}\left[X_T^\pi - L_T \right] = \xi \end{cases} \tag{4.4.3}$$

with $\xi \in \mathbb{R}$ fixed and given. Denote by J_{MV}^{ic*} the optimal value function corresponding to the optimal investment strategy $\pi^{ic,*} := \left(\left\{ \pi_t^{ic,S*} \right\}_{t\in[0,T]}, \left\{ \pi_t^{ic,B*} \right\}_{t\in[0,T]} \right)$.

Similar to the previous sections, we shall first address the following benchmark problem (4.4.4) before solving the mean-variance ALM problem (4.4.3) in the incomplete market case:

$$\min_{\pi \in \mathcal{A}} J_{BM}(\pi; \gamma) = \mathbb{E}\left[(X_T^\pi - L_T - \gamma)^2 \right], \tag{4.4.4}$$

where $\gamma := \xi - \lambda \in \mathbb{R}$ is an exogenous constant, and we denote the corresponding optimal strategy of problem (4.4.4) by π_{BM}^{ic*}.

4.4.1 Solution to the benchmark problem

To apply the BSDE approach to solve the above benchmark problem (4.4.4), we introduce two continuous $(\mathbb{F}, \mathbb{P})$-semi-martingales, Y_t^{ic} and G_t^{ic}, with the following decomposition:

$$dY_t^{ic} = H_t^{ic}\, dt + Z_{0,t}\, dW_t^0 + Z_{1,t}\, dW_t^1 + Z_{2,t}\, dW_t^2, \tag{4.4.5}$$

and

$$dG_t^{ic} = \Psi_t^{ic}\, dt + P_{0,t}\, dW_t^0 + P_{1,t}\, dW_t^1 + P_{2,t}\, dW_t^2, \tag{4.4.6}$$

where $H_t^{ic}, Z_{0,t}, Z_{1,t}, Z_{2,t}, \Psi_t^{ic}, P_{0,t}, P_{1,t}, P_{2,t}$ are some undetermined $\mathbb{F}$-adapted processes. For any admissibile strategy $\pi \in \mathcal{A}$, using Itô's formula to $Y_t^{ic}\left(X_t^\pi - G_t^{ic} \right)^2$

and completing the square yield

$$
\begin{aligned}
&Y_t^{ic}(X_t^\pi - G_t^{ic})^2 \\
&= \int_0^t \left[(X_s^\pi - G_s^{ic})^2 Z_{0,s} + 2(X_s^\pi - G_s^{ic}) Y_s^{ic} \left(\left(\eta\pi_s^S - b(K)\pi_s^B \right) \sigma_r \sqrt{r_s} X_s^\pi - P_{0,s} \right) \right] dW_s^0 \\
&\quad + \int_0^t \left[(X_s^\pi - G_s^{ic})^2 Z_{1,s} + 2(X_s^\pi - G_s^{ic}) Y_s^{ic} \left(\pi_s^S X_s^\pi \left(c_1\sqrt{V_s} + \frac{c_2}{\sqrt{V_s}} \right) - P_{1,s} \right) \right] dW_s^1 \\
&\quad + \int_0^t \left[(X_s^\pi - G_s^{ic})^2 Z_{2,s} - 2(X_s^\pi - G_s^{ic}) Y_s^{ic} P_{2,s} \right] dW_s^2 + Y_0^{ic} \left(x_0 - G_0^{ic} \right)^2 \\
&\quad + \int_0^t Y_s^{ic} \left[\left(\eta\pi_s^S - b(K)\pi_s^B \right) \sigma_r X_s^\pi \sqrt{r_s} - P_{0,s} + (X_s^\pi - G_s^{ic}) \left(\frac{Z_{0,s}}{Y_s^{ic}} + \frac{\lambda_r}{\sigma_r}\sqrt{r_s} \right) \right]^2 ds \\
&\quad + \int_0^t Y_s^{ic} \Bigg[\pi_s^S X_s^\pi \left(c_1\sqrt{V_s} + \frac{c_2}{\sqrt{V_s}} \right) - P_{1,s} + (X_s^\pi - G_s^{ic}) \Bigg(\frac{Z_{1,s}}{Y_s^{ic}} + \Big(\lambda_v \rho \\
&\quad + \lambda_s\sqrt{1-\rho^2} \Big) \sqrt{V_s} \Bigg) \Bigg]^2 ds + \int_0^t Y_s^{ic} (P_{2,s})^2 \, ds + \int_0^t (X_s^\pi - G_s^{ic})^2 \Bigg[H_s^{ic} + 2r_s Y_s^{ic} \\
&\quad - \left(\frac{Z_{0,s}}{Y_s^{ic}} + \frac{\lambda_r}{\sigma_r}\sqrt{r_s} \right)^2 Y_s^{ic} - \left(\frac{Z_{1,s}}{Y_s^{ic}} + \left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2} \right) \sqrt{V_s} \right)^2 Y_s^{ic} \Bigg] ds \\
&\quad + \int_0^t 2(X_s^\pi - G_s^{ic}) Y_s^{ic} \Bigg(r_s G_s^{ic} + \frac{\lambda_r}{\sigma_r}\sqrt{r_s} P_{0,s} + \left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2} \right) \sqrt{V_s} P_{1,s} \\
&\quad - \frac{Z_s^2}{Y_s^{ic}} P_{2,s} - \Psi_s^{ic} \Bigg) ds.
\end{aligned}
\tag{4.4.7}
$$

In view of the right-hand side of (4.4.7), we propose the BSRE of $(Y_t^{ic}, Z_{0,t}, Z_{1,t}, Z_{2,t})$:

$$
\begin{cases}
dY_t^{ic} = \Bigg[\Bigg(\left(\dfrac{\lambda_r^2}{\sigma_r^2} - 2 \right) r_t + \left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2} \right)^2 V_t \Bigg) Y_t^{ic} + 2\dfrac{\lambda_r}{\sigma_r}\sqrt{r_t} Z_{0,t} + 2\Big(\lambda_v\rho \\
\qquad + \lambda_s\sqrt{1-\rho^2} \Big) \sqrt{V_t} Z_{1,t} + \dfrac{(Z_{0,t})^2}{Y_t^{ic}} + \dfrac{(Z_{1,t})^2}{Y_t^{ic}} \Bigg] dt + Z_{0,t}\, dW_t^0 + Z_{1,t}\, dW_t^1 \\
\qquad + Z_{2,t}\, dW_t^2, \\
Y_T^{ic} = 1, \\
Y_t^{ic} > 0, \ \text{for all } t \in [0, T],
\end{cases}
\tag{4.4.8}
$$

and the linear BSDE of $(G_t^{ic}, P_{0,t}, P_{1,t}, P_{2,t})$:

$$
\begin{cases}
dG_t^{ic} = \left(r_t G_t^{ic} + \dfrac{\lambda_r}{\sigma_r}\sqrt{r_t} P_{0,t} + \left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2} \right) \sqrt{V_t} P_{1,t} - \dfrac{Z_{2,t}}{Y_t^{ic}} P_{2,t} \right) dt \\
\qquad + P_{0,t}\, dW_t^0 + P_{1,t}\, dW_t^1 + P_{2,t}\, dW_t^2, \\
G_T^{ic} = L_T + \gamma.
\end{cases}
\tag{4.4.9}
$$

It can be shown that linear BSDE (4.4.9) is related to the following two linear BSDEs of $(G_{1,t}^{ic}, \Lambda_{0,t}, \Lambda_{1,t}, \Lambda_{2,t})$ and $(G_{2,t}^{ic}, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$:

$$
\begin{cases}
dG^{ic}_{1,t} = & \Bigg[\left(1 + \dfrac{\lambda_r \beta_r}{\sigma_r} - \mu_r\right) r_t G^{ic}_{1,t} + \left(\dfrac{\lambda_r}{\sigma_r} - \beta_r\right) \sqrt{r_t}\Lambda_{0,t} + (\lambda_v \rho + \\
& \lambda_s \sqrt{1-\rho^2}\Big) \sqrt{V_t}\Lambda_{1,t} - \dfrac{Z_{2,t}}{Y^{ic}_t}\Lambda_{2,t}\Bigg] dt + \Lambda_{0,t}\, dW^0_t + \Lambda_{1,t}\, dW^1_t + \Lambda_{2,t}\, dW^2_t, \\
G^{ic}_{1,T} = & 1,
\end{cases}
\tag{4.4.10}
$$

and

$$
\begin{cases}
dG^{ic}_{2,t} = & \left[r_t G^{ic}_{2,t} + \dfrac{\lambda_r}{\sigma_r}\sqrt{r_t}\Gamma_{0,t} + \left(\lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right)\sqrt{V_t}\Gamma_{1,t} - \dfrac{Z_{2,t}}{Y^{ic}_t}\Gamma_{2,t}\right] dt \\
& + \Gamma_{0,t}\, dW^0_t + \Gamma_{1,t}\, dW^1_t + \Gamma_{2,t}\, dW^2_t, \\
G^{ic}_{2,T} = & 1,
\end{cases}
\tag{4.4.11}
$$

via the following linear formulation:

$$
\begin{cases}
G^{ic}_t = & L_t G^{ic}_{1,t} + \gamma G^{ic}_{2,t}, \\
P_{0,t} = & L_t \left(\Lambda_{0,t} + G^{ic}_{1,t}\beta_r \sqrt{r_t}\right) + \gamma \Gamma_{0,t}, \\
P_{1,t} = & L_t \Lambda_{1,t} + \gamma \Gamma_{1,t}, \\
P_{2,t} = & L_t \Lambda_{2,t} + \gamma \Gamma_{2,t}.
\end{cases}
\tag{4.4.12}
$$

The next lemma presents a closed-form expression for the unique solution to BSRE (4.4.8).

Lemma 4.4.3. *Suppose that Assumption 4.3.1 holds true. The unique solution to BSRE (4.4.8) is given by*

$$
\begin{cases}
Y^{ic}_t = & \exp\left\{\bar{A}_1(t) + \bar{A}_2(t)V_t + \bar{A}_3(t) r_t\right\}, \\
Z_{0,t} = & Y^{ic}_t \bar{A}_3(t)\sigma_r \sqrt{r_t}, \\
Z_{1,t} = & Y^{ic}_t \bar{A}_2(t)\sigma_v \rho \sqrt{V_t}, \\
Z_{2,t} = & Y^{ic}_t \bar{A}_2(t)\sigma_v \sqrt{1-\rho^2}\sqrt{V_t},
\end{cases}
\tag{4.4.13}
$$

where functions $\bar{A}_1(t), \bar{A}_2(t)$, and $\bar{A}_3(t)$ solve

$$
\begin{cases}
\dfrac{d\bar{A}_2(t)}{dt} = & \left(\lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right)^2 + \left(\kappa_v + 2\left(\lambda_v \rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v \rho\right)\bar{A}_2(t) \\
& + \sigma_v^2 \left(\rho^2 - \dfrac{1}{2}\right)\bar{A}_2^2(t), \;\; \bar{A}_2(T) = 0, \\
\dfrac{d\bar{A}_3(t)}{dt} = & \left(\dfrac{\lambda_r^2}{\sigma_r^2} - 2\right) + (\kappa_r + 2\lambda_r)\bar{A}_3(t) + \dfrac{1}{2}\sigma_r^2 \bar{A}_3^2(t), \;\; \bar{A}_3(T) = 0, \\
\dfrac{d\bar{A}_1(t)}{dt} = & -\varphi_v \bar{A}_2(t) - \varphi_r \bar{A}_3(t), \;\; \bar{A}_1(T) = 0.
\end{cases}
\tag{4.4.14}
$$

Moreover, closed-form solutions to ODEs (4.4.14) are given by

$$\bar{A}_2(t) = \begin{cases} \dfrac{\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)^2 \left(e^{\left(\kappa_v + 2\left(\lambda_v\rho+\lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho\right)(t-T)} - 1\right)}{\kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho}, \ \rho^2 = \dfrac{1}{2} \\ \text{and } \kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho \neq 0; \\ \left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)^2 (t-T), \ \rho^2 = \dfrac{1}{2} \text{ and } \kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho = 0; \\ \dfrac{n^+_{\bar{A}_2} n^-_{\bar{A}_2}\left(1 - e^{\sqrt{\Delta_{\bar{A}_2}}(T-t)}\right)}{n^+_{\bar{A}_2} - n^-_{\bar{A}_2} e^{\sqrt{\Delta_{\bar{A}_2}}(T-t)}}, \ \rho^2 \neq \dfrac{1}{2} \text{ and } \Delta_{\bar{A}_2} > 0; \\ \dfrac{\sigma_v^2(\rho^2 - \frac{1}{2})(T-t) n^2_{\bar{A}_2}}{\sigma_v^2(\rho^2 - \frac{1}{2})(T-t) n_{\bar{A}_2} - 1}, \ \rho^2 \neq \dfrac{1}{2} \text{ and } \Delta_{\bar{A}_2} = 0; \\ \dfrac{\sqrt{-\Delta_{\bar{A}_2}}}{\sigma_v^2(2\rho^2-1)} \tan\left(\arctan\left(\dfrac{\kappa_v + 2\sigma_v\rho\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)}{\sqrt{-\Delta_{\bar{A}_2}}}\right)\right. \\ \left. - \dfrac{\sqrt{-\Delta_{\bar{A}_2}}}{2}(T-t)\right) + n_{\bar{A}_2}, \ \rho^2 \neq \dfrac{1}{2} \text{ and } \Delta_{\bar{A}_2} < 0, \end{cases} \tag{4.4.15}$$

where

$$\begin{cases} \Delta_{\bar{A}_2} = \left(\kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho\right)^2 - (4\rho^2 - 2)\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)^2 \sigma_v^2, \\ n_{\bar{A}_2} = \dfrac{-\left(\kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho\right)}{\sigma_v^2(2\rho^2-1)}, \\ n^+_{\bar{A}_2} = \dfrac{-\left(\kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho\right) + \sqrt{\Delta_{\bar{A}_2}}}{\sigma_v^2(2\rho^2-1)}, \\ n^-_{\bar{A}_2} = \dfrac{-\left(\kappa_v + 2\left(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)\sigma_v\rho\right) - \sqrt{\Delta_{\bar{A}_2}}}{\sigma_v^2(2\rho^2-1)}, \end{cases}$$

$$\bar{A}_3(t) = \begin{cases} \dfrac{n^+_{A_3} n^-_{A_3}\left(1 - e^{\sqrt{\Delta_{A_3}}(T-t)}\right)}{n^+_{A_3} - n^-_{A_3} e^{\sqrt{\Delta_{A_3}}(T-t)}}, \ \Delta_{A_3} > 0; \\ \dfrac{n^2_{A_3}\sigma_r^2(T-t)}{n_{A_3}\sigma_r^2(T-t) - 2}, \ \Delta_{A_3} = 0; \\ \dfrac{\sqrt{-\Delta_{A_3}}}{\sigma_r^2} \tan\left(\arctan\left(\dfrac{\kappa_r + 2\lambda_r}{\sqrt{-\Delta_{A_3}}}\right) - \dfrac{\sqrt{-\Delta_{A_3}}(T-t)}{2}\right) + n_{A_3}, \ \Delta_{A_3} < 0, \end{cases} \tag{4.4.16}$$

where

$$\begin{cases} \Delta_{A_3} = (\kappa_r + 2\lambda_r)^2 - 2\lambda_r^2 + 4\sigma_r^2, \ n_{A_3} = -\dfrac{\kappa_r + 2\lambda_r}{\sigma_r^2}, \\ n^+_{A_3} = \dfrac{-(\kappa_r + 2\lambda_r) + \sqrt{\Delta_{A_3}}}{\sigma_r^2}, \ n^-_{A_3} = \dfrac{-(\kappa_r + 2\lambda_r) - \sqrt{\Delta_{A_3}}}{\sigma_r^2}, \end{cases}$$

and

$$\bar{A}_1(t) = \int_t^T \varphi_v \bar{A}_2(s) + \varphi_r \bar{A}_3(s)\, ds. \tag{4.4.17}$$

Furthermore, functions $\bar{A}_2(t)$ and $\bar{A}_3(t)$ are non-positive and bounded over $[0, T]$.

Proof. See Appendix 4.I. □

Based on the unique solution $(Y_t^{ic}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ derived in Lemma 4.4.3, we next obtain the unique solutions to BSDEs (4.4.10) and (4.4.11), respectively.

Lemma 4.4.4. *Suppose that Assumption 4.3.1 holds true. The unique solution to linear BSDE (4.4.10) is given by*

$$\begin{cases} G_{1,t}^{ic} = \exp\left\{\bar{f}_1(t) + \bar{f}_2(t) r_t\right\}, \\ \Lambda_{0,t} = G_{1,t}^{ic}\sigma_r \bar{f}_2(t)\sqrt{r_t}, \\ \Lambda_{1,t} = 0, \\ \Lambda_{2,t} = 0, \end{cases} \tag{4.4.18}$$

where closed-form expressions for $\bar{f}_1(t)$ and $\bar{f}_2(t)$ are given by

$$\bar{f}_2(t) = \begin{cases} \dfrac{n_{f_2}^+ n_{f_2}^- \left(1 - e^{\sqrt{\Delta_{f_2}}(T-t)}\right)}{n_{f_2}^+ - n_{f_2}^- e^{\sqrt{\Delta_{f_2}}(T-t)}}, \ \Delta_{f_2} > 0; \\ \dfrac{n_{f_2}^2 \sigma_r^2 (T-t)}{n_{f_2}\sigma_r^2 (T-t) + 2}, \ \Delta_{f_2} = 0; \\ -\dfrac{\sqrt{-\Delta_{f_2}}}{\sigma_r^2} \tan\left(\arctan\left(\dfrac{\kappa_r + \lambda_r - \beta_r \sigma_r}{\sqrt{-\Delta_{f_2}}}\right) - \dfrac{\sqrt{-\Delta_{f_2}}(T-t)}{2}\right) \\ + n_{f_2}, \ \Delta_{f_2} < 0, \end{cases} \tag{4.4.19}$$

where

$$\begin{cases} \Delta_{f_2} = (\kappa_r + \lambda_r - \beta_r\sigma_r)^2 + 2\sigma_r^2\left(1 + \dfrac{\lambda_r \beta_r}{\sigma_r} - \mu_r\right), \ n_{f_2} = \dfrac{\kappa_r + \lambda_r - \beta_r \sigma_r}{\sigma_r^2}, \\ n_{f_2}^+ = \dfrac{-(\kappa_r + \lambda_r - \beta_r\sigma_r) + \sqrt{\Delta_{f_2}}}{-\sigma_r^2}, \ n_{f_2}^- = \dfrac{-(\kappa_r + \lambda_r - \beta_r\sigma_r) - \sqrt{\Delta_{f_2}}}{-\sigma_r^2} \end{cases}$$

and

$$\bar{f}_1(t) = \int_t^T \varphi_r \bar{f}_2(s)\, ds. \tag{4.4.20}$$

Proof. See Appendix 4.J. □

Lemma 4.4.5. *Suppose that Assumption 4.3.1 holds true. The unique solution to linear BSDE (4.4.11) is given by*

$$\begin{cases} G_{2,t}^{ic} = \exp\left\{\bar{g}_1(t) + \bar{g}_2(t) r_t\right\}, \\ \Gamma_{0,t} = G_{2,t}^{ic}\sigma_r \bar{g}_2(t)\sqrt{r_t}, \\ \Gamma_{1,t} = 0, \\ \Gamma_{2,t} = 0, \end{cases} \tag{4.4.21}$$

where functions $\bar{g}_1(t)$ and $\bar{g}_2(t)$ are given by

$$\bar{g}_2(t) = \frac{n_{g_2}^{+} n_{g_2}^{-} \left(1 - e^{\sqrt{\Delta_{g_2}}(T-t)}\right)}{n_{g_2}^{+} - n_{g_2}^{-} e^{\sqrt{\Delta_{g_2}}(T-t)}}, \tag{4.4.22}$$

with $\Delta_{g_2} = (\kappa_r + \lambda_r)^2 + 2\sigma_r^2, n_{g_2}^{+} = \frac{-(\kappa_r+\lambda_r)+\sqrt{\Delta_{g_2}}}{-\sigma_r^2}, n_{g_2}^{-} = \frac{-(\kappa_r+\lambda_r)-\sqrt{\Delta_{g_2}}}{-\sigma_r^2}$, *and*

$$\bar{g}_1(t) = \int_t^T \varphi_r \bar{g}_2(s)\, ds. \tag{4.4.23}$$

Furthermore, function $\bar{g}_2(t)$ is non-positive and bounded over $[0, T]$.

The proof of Lemma 4.4.5 is similar to that of Lemma 4.4.4, so we omit it here. It is worth noting that the non-positiveness and boundedness of $\bar{g}_2(t)$ over $[0, T]$ follow from the fact that $\bar{g}_2(t) = g_2(t)$ with $g_2(t)$ given in Proposition 4.3.11 above.

Remark 4.4.6. It should be noted that the results obtained in Lemma 4.4.4 and 4.4.5 do not rely on the unique solution $(Y_t^{ic}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ to BSRE (4.4.8), although the term $Z_{2,t}/Y_t^{ic}$ appears in the drivers of linear BSDEs (4.4.10) and (4.4.11). Due to the boundedness of function $\bar{A}_2(t)$ as shown in Lemma 4.4.3, we can change the original probability measure $\mathbb{P}$ to some equivalent probability measures, which substantially simplifies the forms of the drivers of BSDEs (4.4.10) and (4.4.11), so that the only state variable process involved in their respective drivers is the interest rate r_t. This, in turn, allows us to obtain the explicit solutions to these two BSDEs upon utilizing the Markovian structure of r_t.

Theorem 4.4.7. *Suppose that Assumption 4.3.1 holds true. For any initial data $(r_0, s_0, v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given, the optimal strategy and optimal value function of benchmark problem (4.4.4) are, respectively, given by*

$$\begin{cases} \pi_{BM,t}^{ic,S*} = -\dfrac{1}{X_t^*(c_1 V_t + c_2)} \left(X_t^* - \gamma e^{\bar{g}_1(t)+\bar{g}_2(t) r_t} - e^{\bar{f}_1(t)+\bar{f}_2(t) r_t} L_t\right) \\ \qquad \times \left(\bar{A}_2(t)\sigma_v \rho + \lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right) V_t, \\ \pi_{BM,t}^{ic,B*} = \dfrac{1}{b(K) X_t^*} \left[X_t^* \left(\bar{A}_3(t) + \dfrac{\lambda_r}{\sigma_r^2}\right) - \gamma e^{\bar{g}_1(t)+\bar{g}_2(t) r_t} \left(\bar{A}_3(t) + \bar{g}_2(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)\right. \\ \qquad \left. - L_t e^{\bar{f}_1(t)+\bar{f}_2(t) r_t} \left(\bar{A}_3(t) + \bar{f}_2(t) + \dfrac{\beta_r}{\sigma_r} + \dfrac{\lambda_r}{\sigma_r^2}\right)\right] + \dfrac{\eta}{b(K)} \pi_{BM,t}^{ic,S*}, \end{cases} \tag{4.4.24}$$

and

$$J_{BM}(\pi_{BM}^{ic*}; \gamma) = e^{\bar{A}_1(0)+\bar{A}_2(0) v_0 + \bar{A}_3(0) r_0} \left(x_0 - l_0 e^{\bar{f}_1(0)+\bar{f}_2(0) r_0} - \gamma e^{\bar{g}_1(0)+\bar{g}_2(0) r_0}\right)^2, \tag{4.4.25}$$

where functions $\bar{A}_1(t), \bar{A}_2(t), \bar{A}_3(t), \bar{f}_1(t), \bar{f}_2(t), \bar{g}_1(t)$, and $\bar{g}_2(t)$ are given by (4.4.7), (4.4.15), (4.4.16), (4.4.20), (4.4.19), (4.4.23), and (4.4.22), respectively. Furthermore, optimal strategy (4.4.24) is admissible, i.e., $\pi_{BM}^{ic*} \in \mathcal{A}$.

Proof. See Appendix 4.K. ☐

4.4.2 Solution to the mean-variance problem

In the next theorem, we obtain the explicit solutions to the optimal investment strategy and optimal value function of mean-variance ALM problem (4.4.3) in the incomplete market without derivatives trading.

Theorem 4.4.8. *Suppose that Assumption 4.3.1 holds true. For any initial data $(r_0, s_0, v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ fixed and given, the optimal investment strategy and optimal value function of mean-variance ALM problem (4.4.3) are, respectively, given by*

$$\begin{cases} \pi_t^{ic,S*} = -\dfrac{1}{X_t^*(c_1 V_t + c_2)} \left(X_t^* - (\xi - \lambda^{ic*}) e^{\bar{g}_1(t)+\bar{g}_2(t) r_t} - e^{\bar{f}_1(t)+\bar{f}_2(t) r_t} L_t \right) \\ \qquad \times \left(\bar{A}_2(t) \sigma_v \rho + \lambda_v \rho + \lambda_s \sqrt{1-\rho^2} \right) V_t, \\ \pi_t^{ic,B*} = \dfrac{1}{b(K) X_t^*} \left[X_t^* \left(\bar{A}_3(t) + \dfrac{\lambda_r}{\sigma_r^2} \right) - \gamma e^{\bar{g}_1(t)+\bar{g}_2(t) r_t} \left(\bar{A}_3(t) + \bar{g}_2(t) + \dfrac{\lambda_r}{\sigma_r^2} \right) \right. \\ \qquad \left. - L_t e^{\bar{f}_1(t)+\bar{f}_2(t) r_t} \left(\bar{A}_3(t) + \bar{f}_2(t) + \dfrac{\beta_r}{\sigma_r} + \dfrac{\lambda_r}{\sigma_r^2} \right) \right] + \dfrac{\eta}{b(K)} \pi_t^{ic,S*}, \end{cases} \tag{4.4.26}$$

and

$$J_{MV}^{ic*} = \frac{Y_0^{ic} \left(x_0 - l_0 G_{1,0}^{ic} - \xi G_{2,0}^{ic} \right)^2}{1 - Y_0^{ic} (G_{2,0}^{ic})^2}, \tag{4.4.27}$$

with λ^{ic*} given by

$$\lambda^{ic*} = \frac{Y_0^{ic} G_{2,0}^{ic} \left(x_0 - l_0 G_{1,0}^{ic} - \xi G_{2,0}^{ic} \right)}{1 - Y_0^{ic} (G_{2,0}^{ic})^2}, \tag{4.4.28}$$

where $Y_t^{ic}, G_{1,t}^{ic}, G_{2,t}^{ic}, \bar{A}_1(t), \bar{A}_2(t), \bar{A}_3(t), \bar{f}_1(t), \bar{f}_2(t), \bar{g}_1(t), \bar{g}_2(t)$ are given by (4.4.13), (4.4.18), (4.4.21), (4.4.17), (4.4.15), (4.4.16), (4.4.20), (4.4.19), (4.4.23), and (4.4.22), respectively. Furthermore, optimal investment strategy (4.4.26) is admissible, i.e., $\pi^{ic*} \in \mathcal{A}$.

The proof of Theorem 4.4.8 is similar to that of Theorem 4.3.14, and so we omit it here.

Remark 4.4.9. Note that, when specifying $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$ in Theorem 4.4.8, we obtain the optimal investment strategies of mean-variance ALM problem (4.4.3) under the Heston model and 3/2 model, respectively. In addition, it is not surprising to see that the optimal exposures to interest rate risk in the non-derivative and derivative markets are the same, namely, $\eta \pi_t^{ic,S*} - b(K) \pi_t^{ic,B*} = \theta_t^{r*}$. This finding corresponds to the fact that the interest rate risk can be perfectly hedged by the zero-coupon (rollover) bond regardless of the market completeness.

Corollary 4.4.10. *(Without liability). Suppose that Assumption 4.3.1 holds true. If there is no liability, then for any initial data* $(r_0, s_0, v_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *fixed and given, the optimal investment strategy and optimal value function of mean-variance ALM problem (4.4.3) are, respectively, given by*

$$\begin{cases} \pi_t^{ic,S*} = -\dfrac{\left(X_t^* - (\xi - \tilde{\lambda}^{ic*})e^{\bar{g}_1(t)+\bar{g}_2(t)r_t}\right)\left(\bar{A}_2(t)\sigma_v\rho + \lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)V_t}{X_t^*(c_1V_t + c_2)}, \\ \pi_t^{ic,B*} = \dfrac{1}{b(K)X_t^*}\left[X_t^*\left(\bar{A}_3(t) + \dfrac{\lambda_r}{\sigma_r^2}\right) - \gamma e^{\bar{g}_1(t)+\bar{g}_2(t)r_t}\left(\bar{A}_3(t) + \bar{g}_2(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)\right] \\ \qquad + \dfrac{\eta}{b(K)}\pi_t^{ic,S*}, \end{cases} \tag{4.4.29}$$

and

$$J_{MV}^{ic*} = \frac{Y_0^{ic}\left(x_0 - \xi G_{2,0}^{ic}\right)^2}{1 - Y_0^{ic}(G_{2,0}^{ic})^2} \tag{4.4.30}$$

with $\tilde{\lambda}^{ic*}$ *given by*

$$\tilde{\lambda}^{ic*} = \frac{Y_0^{ic}G_{2,0}^{ic}\left(x_0 - \xi G_{2,0}^{ic}\right)}{1 - Y_0^{ic}(G_{2,0}^{ic})^2}. \tag{4.4.31}$$

Proof. Substituting $l_0 = \mu_r = \beta_r = 0$ into (4.4.26)–(4.4.28) returns (4.4.29)–(4.4.31), respectively. □

Remark 4.4.11. The results provided in Theorem 4.4.8 and Corollary 4.4.10 above for the mean-variance ALM problem under the CIR-4/2 model without derivatives trading are not reported in the existing literature. In this sense, the present paper extends some papers on portfolio optimization problems under the 4/2 model, such as Cheng and Escobar (2021a) and Zhang (2021a), to the case with stochastic interest rates and random liabilities.

To end this section, we verify that derivatives trading can improve the efficacy of portfolio optimization under certain conditions.

Proposition 4.4.12. *Suppose that Assumption 4.3.1 holds true. For any initial data* $(r_0, s_0, v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *fixed and given, we have*

$$J_{MV}^* \leq J_{MV}^{ic*},$$

if the following condition is satisfied:

$$\lambda_v\sqrt{1-\rho^2} \geq \lambda_s\rho. \tag{4.4.32}$$

Proof. See Appendix 4.L. □

4.5 Numerical illustration

In this section, we provide some numerical experiments to illustrate the effects of derivatives trading on the economic behavior of efficient frontiers, when condition (4.4.32) is not satisfied. Throughout this section, unless otherwise stated, the values of model parameters are given as follows: $\lambda_s = 0.22472$, $\lambda_r = -0.1132$, $\lambda_v = -0.66932$, $\eta = -0.5973$, $\rho = -0.2292$, $\kappa_r = 1.3$, $\varphi_r = 0.0025$, $\sigma_r = 0.0566$, $\kappa_v = 2.8278$, $\varphi_v = 0.0563$, $\sigma_v = 0.2941$, $\mu_r = 0.05$, $\beta_r = 0.2$, $x_0 = 1$, $l_0 = 1$, $v_0 = 0.0225$, $r_0 = 0.03$, $T = 1$. Most of the model parameters are adapted from Escobar, Ferrando, and Rubtsov (2018). It is easy to verify that Assumption 4.3.1 is satisfied, while condition (4.4.32) does not hold in this case. In the following numerical experiments, we vary the value of one parameter with others fixed and given.

Figure 4.1 shows the effects of parameter λ_s on the efficient frontiers. We observe that given the expectation of the terminal surplus, both the efficient frontiers, J^*_{MV} and J^{ic*}_{MV}, move downwards concerning λ_s. One possible reason is that larger λ_s indicates that the stock index exhibits a higher return rate, which, in turn, allows the asset-liability manager to undertake less risk for deriving the same value of the expected return. We also notice that $J^{ic*}_{MV} - J^*_{MV}$ decreases concerning λ_s. This can be explained by the economic implication of parameter λ_v. When λ_s becomes larger, $|\lambda_v|$ gets relatively smaller. In this case, the volatility risk induced by Brownian motion W^V_t is lower. Given that this risk can be hedged by the stock derivatives but not by the stock index only, the effectiveness of derivatives trading is reduced when λ_s increases.

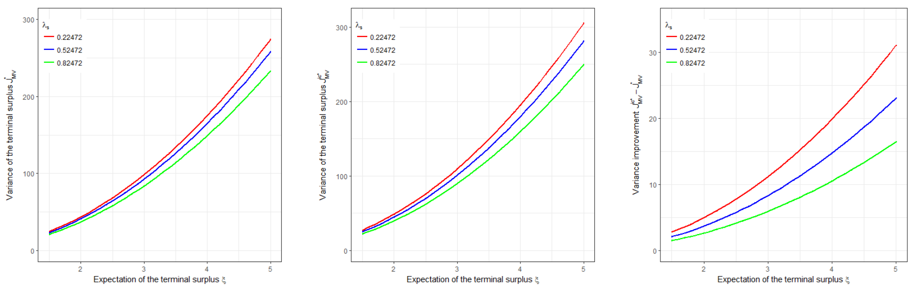


Figure 4.1: *The effects of λ_s on the efficient frontiers*

Figure 4.2 depicts the effects of λ_v on the efficient frontiers. From Figure 4.2(a)-(b) we find that when $|\lambda_v|$ increases, the efficient frontiers move downwards substantially when derivatives trading is available whereas the efficient frontiers move downwards subtly with no investment in the stock derivatives. As a matter of fact, the larger $|\lambda_v|$ becomes, the larger the volatility risk induced by Brownian motion W^V_t is. In this case, the stock derivative is more useful as a hedging tool. This is consistent with the result shown in Figure 4.2(c) that as $|\lambda_v|$ increases, the value of $J^{ic*}_{MV} - J^*_{MV}$ becomes larger.

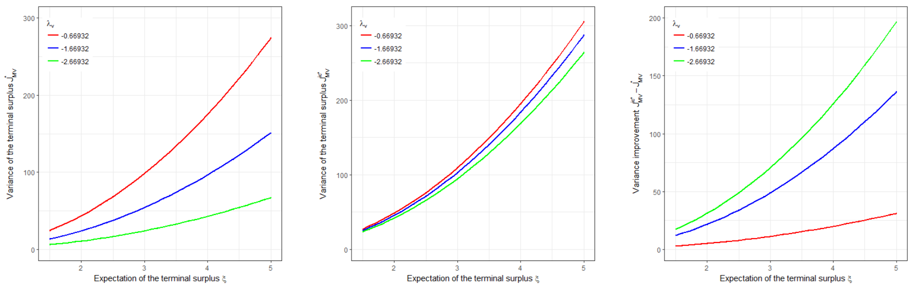


Figure 4.2: *The effects of λ_v on the efficient frontiers*

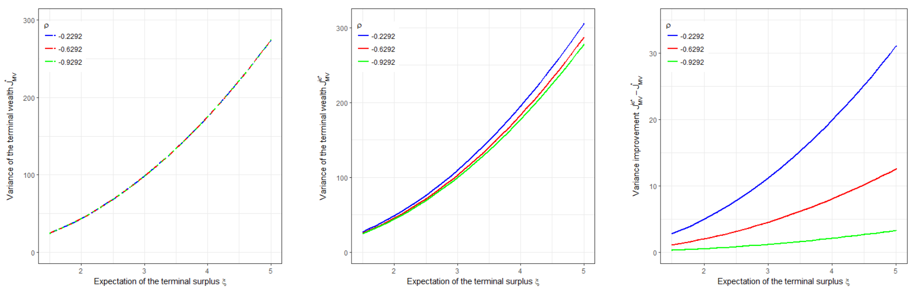


Figure 4.3: *The effects of ρ on the efficient frontiers*

Figure 4.3 contributes to the evolution of the efficient frontiers with respect to ρ. From Figure 4.3(a) we find that the value of ρ has no impact on the efficient frontier J^*_{MV} in the complete market case. This corresponds to the results given in Theorem 4.3.14 above. Furthermore, Figure 4.3(c) shows that when ρ decreases from -0.2292 to -0.9292, the value of $J^{ic*}_{MV} - J^*_{MV}$ becomes smaller, that is, derivatives trading becomes less useful. This is because when $|\rho|$ approaches 1, the dynamic of the stock index is less affected by Brownian motion W^S_t but more affected by W^V_t. Consequently, fewer derivatives trading is needed to hedge the stock index risk. In the extreme scenario when $\rho = -1$, both the dynamics of the stock index and its instantaneous variance are driven by the same Brownian motion W^V_t, and the stock derivative becomes a redundant asset in this case. This also explains the results shown in Figure 4.3(b) that the efficient frontiers J^{ic*}_{MV} move upwards when the value of $|\rho|$ decreases.

4.6 Conclusion

In this paper, we investigated a mean-variance ALM problem with derivatives trading in the presence of the state-of-the-art 4/2 stochastic volatility (Grasselli (2017)) and CIR stochastic interest rate. The asset-liability manager is allowed to invest in not only a money market account, a stock index, and zero-coupon bonds, but also a stock derivative, the price dynamic of which depends on the interest rate, the stock index, and the instantaneous variance of the stock index.

By adopting a BSDE approach and solving a system of three BSDEs, we obtained closed-form expressions for the optimal investment strategies and optimal value functions for both the complete and incomplete market cases: with and without derivatives trading. Furthermore, results for the CIR-4/2 model, CIR-Heston model, and CIR-3/2 model without random liabilities were also provided explicitly, as exceptional cases. Finally, some numerical experiments were given to illustrate the effects of derivatives trading on the efficient frontiers, and we found that derivatives trading can reduce investment risk under the mean-variance criterion. To the best of our knowledge, there is no existing literature on the mean-variance ALM problem with the hybrid CIR-4/2 model and derivatives trading taken into consideration simultaneously.

Built on our current work, several potential topics deserve further investigation. For example, one may consider the case with model ambiguity. One may also introduce rough volatility into the market.

Acknowledgment(s)

We are grateful to two anonymous referees and the editors for their valuable comments and suggestions to improve on the earlier version of this paper.

4.A Proof of Proposition 4.3.3

Proof. We conjecture that the first component Y_t of the solution to BSRE (4.3.4) has the following exponential form:

$$Y_t = \exp\left\{A_1(t) + A_2(t)V_t + A_3(t)r_t\right\}, \tag{4.A.1}$$

where $A_1(t), A_2(t)$, and $A_3(t)$ are undetermined differentiable functions of t with boundary conditions $A_1(T) = A_2(T) = A_3(T) = 0$. Applying Itô's formula to Y_t, we have

$$\begin{aligned} &dY_t \\ =&Y_t\left(\frac{dA_1(t)}{dt} + \frac{dA_2(t)}{dt}V_t + (\varphi_v - \kappa_v V_t)A_2(t) + \frac{dA_3(t)}{dt}r_t + (\varphi_r - \kappa_r r_t)A_3(t)\right. \\ &\left. + \frac{1}{2}A_2^2(t)\sigma_v^2 V_t + \frac{1}{2}A_3^2(t)\sigma_r^2 r_t\right)dt + Y_t\left(A_2(t)\sigma_v\sqrt{V_t}\,dW_t^V + A_3(t)\sigma_r\sqrt{r_t}\,dW_t^r\right). \end{aligned} \tag{4.A.2}$$

By matching the diffusion coefficients of SDEs (4.3.4) and (4.A.2):

$$Z_t^S = 0,\ Z_t^V = Y_t A_2(t)\sigma_v\sqrt{V_t},\ Z_t^r = Y_t A_3(t)\sigma_r\sqrt{r_t},$$

the driver of BSRE (4.3.4) turns out to be

$$Y_t \left\{ \left[(\lambda_s^2 + \lambda_v^2) + 2\lambda_v \sigma_v A_2(t) + A_2^2(t)\sigma_v^2 \right] V_t + \left[\left(\frac{\lambda_r^2}{\sigma_r^2} - 2 \right) + 2\lambda_r A_3(t) + \sigma_r^2 A_3^2(t) \right] r_t \right\}. \tag{4.A.3}$$

By comparing (4.A.3) and the drift coefficient of SDE (4.A.2), and separating the dependence on r_t and V_t, we find that functions $A_1(t), A_2(t)$, and $A_3(t)$ must solve the following ODEs:

$$\begin{cases} \dfrac{dA_2(t)}{dt} = (\lambda_s^2 + \lambda_v^2) + (\kappa_v + 2\lambda_v\sigma_v) A_2(t) + \dfrac{1}{2}\sigma_v^2 A_2^2(t), \; A_2(T) = 0, \\ \dfrac{dA_3(t)}{dt} = \left(\dfrac{\lambda_r^2}{\sigma_r^2} - 2 \right) + (\kappa_r + 2\lambda_r) A_3(t) + \dfrac{1}{2}\sigma_r^2 A_3^2(t), \; A_3(T) = 0, \\ \dfrac{dA_1(t)}{dt} = -\varphi_v A_2(t) - \varphi_r A_3(t), \; A_1(T) = 0. \end{cases}$$

This completes the proof. □

4.B Proof of Proposition 4.3.4

Proof. Denote $\Delta_{A_2} := (\kappa_v + 2\lambda_v\sigma_v)^2 - 2(\lambda_s^2 + \lambda_v^2)\sigma_v^2$. When $\Delta_{A_2} > 0$, Riccati ODE (4.3.11) is equivalent to

$$\frac{dA_2(t)}{dt} = \frac{1}{2}\sigma_v^2 \left(A_2(t) - n_{A_2}^+ \right) \left(A_2(t) - n_{A_2}^- \right). \tag{4.B.1}$$

where $n_{A_2}^+ = \frac{-(\kappa_v + 2\lambda_v\sigma_v) + \sqrt{\Delta_{A_2}}}{\sigma_v^2}$ and $n_{A_2}^- = \frac{-(\kappa_v + 2\lambda_v\sigma_v) - \sqrt{\Delta_{A_2}}}{\sigma_v^2}$. Moreover, we can rewrite (4.B.1) as follows:

$$\frac{dA_2(t)}{A_2(t) - n_{A_2}^+} - \frac{dA_2(t)}{A_2(t) - n_{A_2}^-} = \sqrt{\Delta_{A_2}}\, dt. \tag{4.B.2}$$

Integrating both sides of (4.B.2) with respect to t, and using the boundary condition $A_2(T) = 0$, we have

$$A_2(t) = \frac{n_{A_2}^+ n_{A_2}^- \left(1 - e^{\sqrt{\Delta_{A_2}}(T-t)} \right)}{n_{A_2}^+ - n_{A_2}^- e^{\sqrt{\Delta_{A_2}}(T-t)}}.$$

When $\Delta_{A_2} = 0$, then Riccati ODE (4.3.11) can be reformed as follows:

$$\frac{dA_2(t)}{\left(A_2(t) - n_{A_2} \right)^2} = \frac{1}{2}\sigma_v^2\, dt, \tag{4.B.3}$$

where $n_{A_2} = -\frac{\kappa_v + 2\lambda_v\sigma_v}{\sigma_v^2}$. Integrating both sides of (4.B.3) and using the boundary condition $A_2(T) = 0$, we have

$$A_2(t) = \frac{n_{A_2}^2 \sigma_v^2 (T-t)}{n_{A_2}\sigma_v^2 (T-t) - 2}.$$

When $\Delta_{A_2} < 0$, we can reformulate ODE (4.3.11) as follows:

$$\frac{dA_2(t)}{\left(A_2(t) - n_{A_2}\right)^2 + \frac{-\Delta_{A_2}}{\sigma_v^4}} = \frac{1}{2}\sigma_v^2\, dt \tag{4.B.4}$$

Using the separation variable method to (4.B.4), we have

$$A_2(t) = \frac{\sqrt{-\Delta_{A_2}}}{\sigma_v^2} \tan\left(\arctan\left(\frac{\kappa + 2\lambda_v\sigma_v}{\sqrt{-\Delta_{A_2}}}\right) - \frac{\sqrt{-\Delta_{A_2}}(T-t)}{2}\right) + n_{A_2}.$$

Similarly, denote $\Delta_{A_3} := (\kappa_r + 2\lambda_r)^2 - 2\lambda_r^2 + 4\sigma_r^2$. When $\Delta_{A_3} > 0$, Riccati ODE (4.3.12) can be rewritten as follows:

$$\frac{dA_3(t)}{dt} = \frac{1}{2}\sigma_r^2\left(A_3(t) - n_{A_3}^+\right)\left(A_3(t) - n_{A_3}^-\right).$$

where $n_{A_3}^+ = \frac{-(\kappa_r+2\lambda_r)+\sqrt{\Delta_{A_3}}}{\sigma_r^2}$ and $n_{A_3}^- = \frac{-(\kappa_r+2\lambda_r)-\sqrt{\Delta_{A_3}}}{\sigma_r^2}$. By the boundary condition $A_3(T) = 0$, a direct integral calculation yields

$$A_3(t) = \frac{n_{A_3}^+ n_{A_3}^-\left(1 - e^{\sqrt{\Delta_{A_3}}(T-t)}\right)}{n_{A_3}^+ - n_{A_3}^- e^{\sqrt{\Delta_{A_3}}(T-t)}}.$$

When $\Delta_{A_3} = 0$, similar to (4.B.3), we derive

$$A_3(t) = \frac{n_{A_3^2}\sigma_r^2(T-t)}{n_{A_3}\sigma_r^2(T-t) - 2},$$

where $n_{A_3} = -\frac{\kappa_r+2\lambda_r}{\sigma_r^2}$. When $\Delta_{A_3} < 0$, we have

$$A_3(t) = \frac{\sqrt{-\Delta_{A_3}}}{\sigma_r^2} \tan\left(\arctan\left(\frac{\kappa_r + 2\lambda_r}{\sqrt{-\Delta_{A_3}}}\right) - \frac{\sqrt{-\Delta_{A_3}}(T-t)}{2}\right) + n_{A_3}.$$

Finally, a direct integral calculation gives the solution $A_1(t)$ to ODE (4.3.13) below

$$A_1(t) = \int_t^T \varphi_v A_2(s) + \varphi_r A_3(s)\, ds.$$

This completes the proof. □

4.C Proof of Proposition 4.3.5

Proof. A direct differentiation of $A_2(t)$ given in (4.3.14) with respect to t leads to

$$\frac{dA_2(t)}{dt} = \begin{cases} \dfrac{4(\lambda_s^2 + \lambda_v^2)\Delta_v e^{\sqrt{\Delta_{A_2}}(T-t)}}{\left((\kappa_v + 2\lambda_v\sigma_v) - \sqrt{\Delta_{A_2}} - \left(\kappa_v + 2\lambda_v\sigma_v + \sqrt{\Delta_{A_2}}\right)e^{\sqrt{\Delta_{A_2}}(T-t)}\right)^2}, & \Delta_{A_2} > 0; \\[2ex] \dfrac{2(\kappa_v + 2\lambda_v\sigma_v)^2}{\sigma_v^2\left((\kappa_v + 2\lambda_v\sigma_v)(T-t) + 2\right)^2}, & \Delta_{A_2} = 0; \\[2ex] \dfrac{-\Delta_{A_2}}{2\sigma_v^2}\sec^2\left(\arctan\left(\dfrac{\kappa_v + 2\lambda_v\sigma_v}{\sqrt{-\Delta_{A_2}}}\right) - \dfrac{\sqrt{-\Delta_{A_2}}(T-t)}{2}\right), & \Delta_{A_2} < 0. \end{cases}$$

This clearly shows that $dA_2(t)/dt > 0$ over $[0, T]$. Thus, by the boundary condition $A_2(T) = 0$, we see that $A_2(t)$ is non-positive over $[0, T]$, and more precisely, $A_2(t)$ is within the interval $[A_2(0), 0]$.

Similarly, for the function $A_3(t)$ given in (4.3.15), we have

$$\frac{dA_3(t)}{dt} = \begin{cases} \dfrac{4\left(\frac{\lambda_r^2}{\sigma_r^2} - 2\right)\Delta_{A_3} e^{\sqrt{\Delta_{A_3}}(T-t)}}{\left(-(\kappa_r + 2\lambda_r) + \sqrt{\Delta_{A_3}} + (\kappa_r + 2\lambda_r + \sqrt{\Delta_{A_3}})e^{\sqrt{\Delta_{A_3}}(T-t)}\right)^2}, & \Delta_{A_3} > 0; \\ \dfrac{2(\kappa_r + 2\lambda_r)^2}{\sigma_r^2\left((\kappa_r + 2\lambda_r)(T-t) + 2\right)^2}, & \Delta_{A_3} = 0; \\ \dfrac{-\Delta_{A_3}}{2\sigma_r^2}\sec^2\left(\arctan\left(\dfrac{\kappa_r + 2\lambda_r}{\sqrt{-\Delta_{A_3}}}\right) - \dfrac{\sqrt{-\Delta_{A_3}}(T-t)}{2}\right), & \Delta_{A_3} < 0. \end{cases}$$

Under Assumption 4.3.1, the above equalities reveal that $dA_3(t)/dt > 0$ over $[0, T]$. Thus, $A_3(t)$ is non-positive and bounded by $[A_3(0), 0]$ over $[0, T]$. □

4.D Proof of Corollary 4.3.8

Proof. By Lemma 4.3.6, the following two stochastic exponential processes:

$$\exp\left\{-2\int_0^t \lambda_s\sqrt{V_s}\,dW_s^S - 2\int_0^t \lambda_v\sqrt{V_s}\,dW_s^V - 2\int_0^t (\lambda_s^2 + \lambda_v^2)V_s\,ds\right\} \tag{4.D.1}$$

and

$$\exp\left\{-2\int_0^t \frac{\lambda_r}{\sigma_r}\sqrt{r_s}\,dW_s^r - 2\int_0^t \frac{\lambda_r^2}{\sigma_r^2} r_s\,ds\right\} \tag{4.D.2}$$

are $(\mathbb{F}, \mathbb{P})$-martingales. Since $(\mathbb{F}, \mathbb{P})$-martingales (4.D.1) and (4.D.2) are mutually independent with continuous sample paths under measure $\mathbb{P}$, by Theorem 2.4 in Cherny (2006), the product of (4.D.1) and (4.D.2)

$$\exp\left\{-2\int_0^t \lambda_s\sqrt{V_t}\,dW_s^S - 2\int_0^t \lambda_v\sqrt{V_t}\,dW_s^V - 2\int_0^t \frac{\lambda_r}{\sigma_r}\sqrt{r_s}\,dW_s^r \right.$$
$$\left. -2\int_0^t \left[(\lambda_s^2 + \lambda_v^2)V_s + \frac{\lambda_r^2}{\sigma_r^2} r_s\right] ds\right\}$$

is also an $(\mathbb{F}, \mathbb{P})$-martingale. Hence, the probability measure $\tilde{\mathbb{P}}$ defined by

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{ -2\int_0^T \lambda_s\sqrt{V_t}\,dW_t^S - 2\int_0^T \lambda_v\sqrt{V_t}\,dW_t^V - 2\int_0^T \frac{\lambda_r}{\sigma_r}\sqrt{r_t}\,dW_t^r \right.$$
$$\left. -2\int_0^T \left[(\lambda_s^2 + \lambda_v^2)V_t + \frac{\lambda_r^2}{\sigma_r^2} r_t\right] dt\right\}$$

is equivalent to $\mathbb{P}$ measure on $\mathcal{F}_T$. As a result of Girsanov's theorem, the dynamics of Brownian motions under $\tilde{\mathbb{P}}$ are related to the dynamics of Brownian motions under $\mathbb{P}$ via

$$d\tilde{W}_t^S = 2\lambda_s\sqrt{V_t}\,dt + dW_t^S,\ d\tilde{W}_t^V = 2\lambda_v\sqrt{V_t}\,dt + dW_t^V,\ d\tilde{W}_t^r = 2\frac{\lambda_r}{\sigma_r}\sqrt{r_t}\,dt + dW_t^r.$$

Under measure $\tilde{\mathbb{P}}$, we observe that V_t and r_t:

$$dV_t = (\varphi_v - (\kappa_v + 2\lambda_v\sigma_v)V_t)\ dt + \sigma_v\sqrt{V_t}\, d\tilde{W}_t^V$$

and

$$dr_t = (\varphi_r - (\kappa_r + 2\lambda_r)r_t)\ dt + \sigma_r\sqrt{r_t}\, d\tilde{W}_t^r$$

retain CIR structures. Therefore, by Proposition 4.3.5 and Lemma 4.3.6 again, the following stochastic exponential processes:

$$\exp\left\{-\frac{1}{2}\int_0^t A_3^2(s)\sigma_r^2 r_s\, ds - \int_0^t A_3(s)\sigma_r\sqrt{r_s}\, d\tilde{W}_s^r\right\} \tag{4.D.3}$$

and

$$\exp\left\{-\frac{1}{2}\int_0^t A_2^2(s)\sigma_v^2 V_s\, ds - \int_0^t A_2(s)\sigma_v\sqrt{V_s}\, d\tilde{W}_s^V\right\} \tag{4.D.4}$$

are $(\mathbb{F}, \tilde{\mathbb{P}})$-martingales. Finally, given that martingales (4.D.3) and (4.D.4) are mutually independent with continuous sample paths under $\tilde{\mathbb{P}}$, by Theorem 2.4 in Cherny (2006), we know that

$$\exp\left\{-\frac{1}{2}\int_0^t \left(A_3^2(s)\sigma_r^2 r_s + A_2^2(s)\sigma_v^2 V_s\right)\, ds - \int_0^t A_3(s)\sigma_r\sqrt{r_s}\, d\tilde{W}_s^r \right.$$
$$\left. - \int_0^t A_2(s)\sigma_v\sqrt{V_s}\, d\tilde{W}_s^V\right\}$$

is an $(\mathbb{F}, \tilde{\mathbb{P}})$-martingale. This completes the proof. □

4.E Proof of Lemma 4.3.9

Proof. By applying Itô's formula to $\log(Y_t)$ with Y_t given in (4.3.9) and changing from measure $\mathbb{P}$ to $\tilde{\mathbb{P}}$, $\left(\log(Y_t), \frac{Z_t^S}{Y_t}, \frac{Z_t^V}{Y_t}, \frac{Z_t^r}{Y_t}\right)$ is a solution to the following quadratic BSDE:

$$\begin{cases} d\log(Y_t) = & \left[\left(-2r_t + \lambda_s^2 V_t + \lambda_v^2 V_t + \dfrac{\lambda_r^2}{\sigma_r^2} r_t\right) + \dfrac{1}{2}\left(\dfrac{Z_t^S}{Y_t}\right)^2 + \dfrac{1}{2}\left(\dfrac{Z_t^V}{Y_t}\right)^2 \right. \\ & \left. + \dfrac{1}{2}\left(\dfrac{Z_t^r}{Y_t}\right)^2\right] dt + \dfrac{Z_t^S}{Y_t}\, d\tilde{W}_t^S + \dfrac{Z_t^V}{Y_t}\, d\tilde{W}_t^V + \dfrac{Z_t^r}{Y_t}\, d\tilde{W}_t^r, \\ \log(Y_T) = & 0. \end{cases} \tag{4.E.1}$$

Suppose there exists another solution to BSRE (4.3.4) denoted by $(\tilde{Y}_t, \tilde{Z}_t^S, \tilde{Z}_t^V, \tilde{Z}_t^r)$, which is different from the solution given in Proposition 4.3.3. By the above transformation, $\left(\log(\tilde{Y}_t), \frac{\tilde{Z}_t^S}{\tilde{Y}_t}, \frac{\tilde{Z}_t^V}{\tilde{Y}_t}, \frac{\tilde{Z}_t^r}{\tilde{Y}_t}\right)$ also solves BSDE (4.E.1). Therefore, the difference process $(\Delta\log(Y_t), \Delta Z_t^S, \Delta Z_t^V, \Delta Z_t^r)$ defined by

$$\left(\Delta\log(Y_t), \Delta Z_t^S, \Delta Z_t^V, \Delta Z_t^r\right) = \left(\log(Y_t) - \log(\tilde{Y}_t), \frac{Z_t^S}{Y_t} - \frac{\tilde{Z}_t^S}{\tilde{Y}_t}, \frac{Z_t^V}{Y_t} - \frac{\tilde{Z}_t^V}{\tilde{Y}_t}, \frac{Z_t^r}{Y_t} - \frac{\tilde{Z}_t^r}{\tilde{Y}_t}\right)$$

solves the following quadratic BSDE:

$$\begin{cases} d\Delta\log(Y_t) = & \frac{1}{2}\Bigg[\left(\frac{(Z_t^S)^2}{Y_t^2} - \frac{(\tilde{Z}_t^S)^2}{\tilde{Y}_t^2}\right) + \left(\frac{(Z_t^V)^2}{Y_t^2} - \frac{(\tilde{Z}_t^V)^2}{\tilde{Y}_t^2}\right) \\ & + \left(\frac{(Z_t^r)^2}{Y_t^2} - \frac{(\tilde{Z}_t^r)^2}{\tilde{Y}_t^2}\right)\Bigg] dt + \Delta Z_t^S\, d\tilde{W}_t^S + \Delta Z_t^V\, d\tilde{W}_t^V + \Delta Z_t^r\, d\tilde{W}_t^r, \\ \Delta\log(Y_T) = & 0. \end{cases}$$

From Corollary 4.3.8, we know that the probability measure $\hat{\mathbb{P}}$ defined by

$$\begin{aligned} \frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}}\Big|_{\mathcal{F}_T} &= \exp\Bigg\{ -\frac{1}{2}\int_0^T \left(A_3^2(t)\sigma_r^2 r_t + A_2^2(t)\sigma_v^2 V_t\right)\, dt - \int_0^T A_3(t)\sigma_r\sqrt{r_t}\, d\tilde{W}_t^r \\ &\quad - \int_0^T A_2(t)\sigma_v\sqrt{V_t}\, d\tilde{W}_t^V \Bigg\} \\ &= \exp\Bigg\{ -\frac{1}{2}\int_0^T \left(\frac{Z_t^S}{Y_t}\right)^2 + \left(\frac{Z_t^V}{Y_t}\right)^2 + \left(\frac{Z_t^r}{Y_t}\right)^2 dt - \int_0^T \frac{Z_t^S}{Y_t}\, d\tilde{W}_t^S \\ &\quad - \int_0^T \frac{Z_t^V}{Y_t}\, d\tilde{W}_t^V - \int_0^T \frac{Z_t^r}{Y_t}\, d\tilde{W}_t^r \Bigg\} \end{aligned}$$

is equivalent to $\tilde{\mathbb{P}}$ on $\mathcal{F}_T$, where the second equality makes use of the result in Proposition 4.3.3 that $\left(Z_t^r, Z_t^S, Z_t^V\right) = \left(Y_t A_3(t)\sigma_r\sqrt{r_t}, 0, Y_t A_2(t)\sigma_v\sqrt{V_t}\right)$. Therefore, from Girsanov's theorem, the Brownian motions $\hat{W}_t^r, \hat{W}_t^S, \hat{W}_t^V$ under measure $\hat{\mathbb{P}}$ are given by

$$d\hat{W}_t^r = d\tilde{W}_t^r + \frac{Z_t^r}{Y_t}\, dt,\ d\hat{W}_t^S = d\tilde{W}_t^S + \frac{Z_t^S}{Y_t}\, dt,\ d\hat{W}_t^V = d\tilde{W}_t^V + \frac{Z_t^V}{Y_t}\, dt.$$

Then, it can be shown that the difference process $(\Delta\log(Y_t), \Delta Z_t^S, \Delta Z_t^V, \Delta Z_t^r)$ solves the following quadratic BSDE under $\hat{\mathbb{P}}$ measure:

$$\begin{cases} d\Delta\log(Y_t) = & -\frac{1}{2}\left[(\Delta Z_t^V)^2 + (\Delta Z_t^r)^2 + (\Delta Z_t^S)^2\right]\, dt + \Delta Z_t^S\, d\hat{W}_t^S + \Delta Z_t^V\, d\hat{W}_t^V \\ & + \Delta Z_t^r\, d\hat{W}_t^r, \\ \Delta\log(Y_T) = & 0, \end{cases} \tag{4.E.2}$$

which satisfies all the regularity conditions in Kobylanski (2000). Hence, it follows from Theorem 2.3 and 2.6 in Kobylanski (2000) that quadratic BSDE (4.E.2) admits a unique solution $(0, 0, 0, 0)$. Finally, we can conclude that the solution $(Y_t, Z_t^r, Z_t^S, Z_t^V)$ given in Proposition 4.3.3 is the unique solution to BSRE (4.3.4).
□

4.F Proof of Proposition 4.3.10

Proof. We conjecture that the first component $G_{1,t}$ of the solution to linear BSDE (4.3.6) admits the following exponential form:

$$G_{1,t} = \exp\left\{f_1(t) + f_2(t)r_t\right\}, \tag{4.F.1}$$

where $f_1(t)$ and $f_2(t)$ are undetermined differentiable functions of t with boundary conditions $f_1(T) = f_2(T) = 0$. Applying Itô' formula to $G_{1,t}$ given in (4.F.1) yields

$$\begin{aligned} dG_{1,t} =& G_{1,t}\left(\frac{df_1(t)}{dt} + \frac{df_2(t)}{dt}r_t + (\varphi_r - \kappa_r r_t)f_2(t) + \frac{1}{2}\sigma_r^2 r_t f_2^2(t)\right) dt \\ &+ G_{1,t}\sigma_r\sqrt{r_t}f_2(t)\, dW_t^r. \end{aligned} \tag{4.F.2}$$

Let $\Lambda_t^S = 0, \Lambda_t^V = 0$, and $\Lambda_t^r = \sigma_r\sqrt{r_t}f_2(t)G_{1,t}$. The driver of linear BSDE (4.3.6) can be reformulated as follows:

$$G_{1,t}\left(1 + \frac{\lambda_r\beta_r}{\sigma_r} - \mu_r + \left(\frac{\lambda_r}{\sigma_r} - \beta_r\right)\sigma_r f_2(t)\right) r_t. \tag{4.F.3}$$

A direct comparison between (4.F.3) and the drift coefficient of SDE (4.F.2) shows that functions $f_1(t)$ and $f_2(t)$ must solve the following two ODEs:

$$\begin{cases} \dfrac{df_2(t)}{dt} = \left(1 + \dfrac{\lambda_r\beta_r}{\sigma_r} - \mu_r\right) + (\kappa_r + \lambda_r - \beta_r\sigma_r)\, f_2(t) - \dfrac{1}{2}\sigma_r^2 f_2^2(t), \;\; f_2(T) = 0, \\ \dfrac{df_1(t)}{dt} = -\varphi_r f_2(t), \;\; f_1(T) = 0. \end{cases} \tag{4.F.4}$$

Notice that ODEs (4.F.4) have similar structures to the ones given in Proposition 4.3.3. By repeating the calculations in Appendix 4.B, closed-form expressions of $f_1(t)$ and $f_2(t)$ are then given by (4.3.18) and (4.3.18), respectively.

In the following, we verify that the solution given in (4.3.17) is the unique solution to BSDE (4.3.6) by using a comparison method. By Lemma 4.3.6, we first notice the following two stochastic exponential processes:

$$\exp\left\{-\int_0^t \left(\frac{\lambda_r}{\sigma_r} - \beta_r\right)\sqrt{r_s}\, dW_s^r - \frac{1}{2}\int_0^t \left(\frac{\lambda_r}{\sigma_r} - \beta_r\right)^2 r_s\, ds\right\}$$

and

$$\exp\left\{-\int_0^t \lambda_s\sqrt{V_s}\, dW_s^S - \int_0^t \lambda_v\sqrt{V_s}\, dW_s^V - \frac{1}{2}\int_0^t \left(\lambda_s^2 + \lambda_v^2\right) V_s\, ds\right\}$$

are $(\mathbb{F}, \mathbb{P})$-martingales. Clearly, the above two martingales are with continuous sample paths and are mutually independent. It then follows from Theorem 2.4 in Cherny (2006) that the stochastic exponential in the following Radon-Nikodym derivative is an $(\mathbb{F},\mathbb{P})$-martingale:

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{ -\frac{1}{2}\int_0^T (\lambda_s^2+\lambda_v^2)\,V_t + \left(\frac{\lambda_r}{\sigma_r}-\beta_r\right)^2 r_t\,dt \right.$$
$$\left. - \int_0^T \left(\frac{\lambda_r}{\sigma_r}-\beta_r\right)\sqrt{r_t}\,dW_t^r - \int_0^T \lambda_s\sqrt{V_t}\,dW_t^S - \int_0^T \lambda_v\sqrt{V_t}\,dW_t^V \right\},$$

and thus, $\bar{\mathbb{P}}$ measure is equivalent to $\mathbb{P}$ measure on $\mathcal{F}_T$. By Girsanov's theorem, we have the dynamics of Brownian motions $\bar{W}_t^r, \bar{W}_t^S, \bar{W}_t^V$ under $\bar{\mathbb{P}}$ as follows:

$$d\bar{W}_t^r = dW_t^r + \left(\frac{\lambda_r}{\sigma_r}-\beta_r\right)\sqrt{r_t}\,dt,\;\; d\bar{W}_t^S = dW_t^S + \lambda_s\sqrt{V_t}\,dt,\;\; d\bar{W}_t^V = dW_t^V + \lambda_v\sqrt{V_t}\,dt.$$

Therefore, the solution presented in (4.3.17) must be a solution to the following linear BSDE (4.F.5) under $\bar{\mathbb{P}}$ measure as well:

$$\begin{cases} dG_{1,t} = \left(1+\dfrac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right) r_t G_{1,t}\,dt + \Lambda_t^S\,d\bar{W}_t^S + \Lambda_t^V\,d\bar{W}_t^V + \Lambda_t^r\,d\bar{W}_t^r, \\ G_{1,T} = 1. \end{cases} \tag{4.F.5}$$

Suppose that besides $\left(G_{1,t}, \Lambda_t^S, \Lambda_t^V, \Lambda_t^r\right)$ given in (4.3.17), there exists another solution to BSDE (4.3.6), which is denoted as $(\bar{G}_{1,t}, \bar{\Lambda}_t^S, \bar{\Lambda}_t^V, \bar{\Lambda}_t^r)$. Then the difference process defined by

$$\left(\Delta G_{1,t}, \Delta\Lambda_t^S, \Delta\Lambda_t^V, \Delta\Lambda_t^r\right) := \left(G_{1,t}-\bar{G}_{1,t}, \Lambda_t^S-\bar{\Lambda}_t^S, \Lambda_t^V-\bar{\Lambda}_t^V, \Lambda_t^r-\bar{\Lambda}_t^r\right) \tag{4.F.6}$$

solves the following BSDE (4.F.7):

$$\begin{cases} d\Delta G_{1,t} = \left(1+\dfrac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right) r_t \Delta G_{1,t}\,dt + \Delta\Lambda_t^S\,d\bar{W}_t^S + \Delta\Lambda_t^V\,d\bar{W}_t^V + \Delta\Lambda_t^r\,d\bar{W}_t^r, \\ \Delta G_{1,T} = 0. \end{cases} \tag{4.F.7}$$

We now introduce the following BSDE of $\left(\Delta G_{1,t}^{'}, \Delta\Lambda_t^{S'}, \Delta\Lambda_t^{V'}, \Delta\Lambda_t^{r'}\right)$ with uniformly Lipschitz continuity:

$$\begin{cases} d\Delta G_{1,t}^{'} = \Delta\Lambda_t^{S'}\,d\bar{W}_t^S + \Delta\Lambda_t^{V'}\,d\bar{W}_t^V + \Delta\Lambda_t^{r'}\,d\bar{W}_t^r, \\ \Delta G_{1,T}^{'} = 0. \end{cases} \tag{4.F.8}$$

It is clear that linear BSDE (4.F.8) is with standard data (refer to El Karoui, Peng, and Quenez (1997)). Then by Theorem 2.1 and Proposition 2.2 in El Karoui, Peng, and Quenez (1997), BSDE (4.F.8) has the unique solution $(0,0,0,0)$. Let

$$\begin{cases} \Delta G_{1,t} = \exp\left\{\int_0^t \left(1+\frac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right) r_s\,ds\right\}\Delta G_{1,t}^{'},\;\; \Delta\Lambda_t^S = \exp\left\{\int_0^t \left(1+\frac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right) r_s\,ds\right\}\Delta\Lambda_t^{S'} \\ \Delta\Lambda_t^V = \exp\left\{\int_0^t \left(1+\frac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right) r_s\,ds\right\}\Delta\Lambda_t^{V'},\;\; \Delta\Lambda_t^r = \exp\left\{\int_0^t \left(1+\frac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right) r_s\,ds\right\}\Delta\Lambda_t^{r'}. \end{cases} \tag{4.F.9}$$

Applying Itô's formula to $\Delta G_{1,t}$ defined in (4.F.9) yields

$$d\Delta G_{1,t} = \left(1 + \frac{\lambda_r \beta_r}{\sigma_r} - \mu_r\right) r_t \Delta G_{1,t}\, dt + \Delta\Lambda_t^S\, d\bar{W}_t^S + \Delta\Lambda_t^V\, d\bar{W}_t^V + \Delta\Lambda_t^r\, d\bar{W}_t^r,$$

with terminal condition $\Delta G_{1,T} = 0$, which is exactly BSDE (4.F.7). Hence, we can conclude that BSDE (4.F.7) has the unique solution $(0,0,0,0)$ from the one-to-one correspondence between $\left(\Delta G_{1,t}, \Delta\Lambda_t^S, \Delta\Lambda_t^V, \Delta\Lambda_t^r\right)$ and $\left(\Delta G'_{1,t}, \Delta\Lambda_t^{S'}, \Delta\Lambda_t^{V'}, \Delta\Lambda_t^{r'}\right)$ given in (4.F.9). This result, in turn, means that

$$\left(G_{1,t}, \Lambda_t^S, \Lambda_t^V, \Lambda_t^r\right) = (\bar{G}_{1,t}, \bar{\Lambda}_t^S, \bar{\Lambda}_t^V, \bar{\Lambda}_t^r),$$

that is, the solution given in (4.3.17) is the unique solution to BSDE (4.3.6). This completes the proof. □

4.G Proof of Theorem 4.3.13

Proof. Applying Itô's formula to $Y_t\left(X_t^\theta - G_t\right)^2$ and completing of squares yield

$$\begin{aligned} dY_t(X_t^\theta - G_t)^2 = &\left[\left(X_t^\theta - G_t\right)^2 Z_t^r + 2Y_t\left(X_t^\theta - G_t\right)\left(\theta_t^r \sigma_r \sqrt{r_t} X_t^\theta - P_t^r\right)\right] dW_t^r \\ &+ \left[\left(X_t^\theta - G_t\right)^2 Z_t^S + 2Y_t(X_t^\theta - G_t)\left(\theta_t^S \sqrt{V_t} X_t^\theta - P_t^S\right)\right] dW_t^S \\ &+ \left[\left(X_t^\theta - G_t\right)^2 Z_t^V + 2Y_t(X_t^\theta - G_t)\left(\theta_t^V \sqrt{V_t} X_t^\theta - P_t^V\right)\right] dW_t^V \\ &+ Y_t\left[\left(\theta_t^S \sqrt{V_t} X_t^\theta - P_t^S\right) + (X_t^\theta - G_t)\left(\frac{Z_t^S}{Y_t} + \lambda_s \sqrt{V_t}\right)\right]^2 dt \\ &+ Y_t\left[\left(\theta_t^V \sqrt{V_t} X_t^\theta - P_t^V\right) + (X_t^\theta - G_t)\left(\frac{Z_t^V}{Y_t} + \lambda_v \sqrt{V_t}\right)\right]^2 dt \\ &+ Y_t\left[\left(\theta_t^r \sigma_r \sqrt{r_t} X_t^\theta - P_t^r\right) + (X_t^\theta - G_t)\left(\frac{Z_t^r}{Y_t} + \frac{\lambda_r}{\sigma_r}\sqrt{r_t}\right)\right]^2 dt. \end{aligned} \tag{4.G.1}$$

Due to the path-wise continuity of $G_t, Y_t, r_t, V_t, P_t^r, P_t^S, P_t^V, Z_t^r, Z_t^S, Z_t^V, \theta_t^r, \theta_t^S, \theta_t^V$, and X_t^θ for any admissible strategy $\theta \in \Theta$, stochastic integrals on the right-hand side of (4.G.1) are $(\mathbb{F}, \mathbb{P})$-local martingales. Therefore, there exists a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \uparrow +\infty$ $\mathbb{P}$ almost surely as $n \to +\infty$, and when stopped by such a sequence, the aforementioned local martingales are $(\mathbb{F}, \mathbb{P})$-martingales.

By integrating both sides of above equality from 0 to $\tau_n \wedge T$ and taking expectations, we have

$$
\begin{aligned}
&\mathbb{E}\left[Y_{\tau_n\wedge T}\left(X^{\theta}_{T\wedge\tau_n}-G_{T\wedge\tau_n}\right)^2\right]\\
=&Y_0\left(x_0-G_0\right)^2+\mathbb{E}\left[\int_0^{T\wedge\tau_n}Y_t\left[\left(\theta^S_t\sqrt{V_t}X^{\theta}_t-P^S_t\right)+(X^{\theta}_t-G_t)\left(\frac{Z^S_t}{Y_t}+\lambda_s\sqrt{V_t}\right)\right]^2dt\right]\\
&+\mathbb{E}\left[\int_0^{T\wedge\tau_n}Y_t\left[\left(\theta^V_t\sqrt{V_t}X^{\theta}_t-P^V_t\right)+(X^{\theta}_t-G_t)\left(\frac{Z^V_t}{Y_t}+\lambda_v\sqrt{V_t}\right)\right]^2dt\right]\\
&+\mathbb{E}\left[\int_0^{T\wedge\tau_n}Y_t\left[\left(\theta^r_t\sigma_r\sqrt{r_t}X^{\theta}_t-P^r_t\right)+(X^{\theta}_t-G_t)\left(\frac{Z^r_t}{Y_t}+\frac{\lambda_r}{\sigma_r}\sqrt{r_t}\right)\right]^2dt\right].
\end{aligned}
\tag{4.G.2}
$$

Notice from the correspondence between $G_t, G_{1,t}$, and $G_{2,t}$ given in (4.3.8) that the term in the expectation on the left-hand side of (4.G.2) is uniformly integrable for any admissible strategy $\theta \in \Theta$, and the terms in the expectations on the right-hand side of (4.G.2) are all non-negative and increasing with respect to $n \in \mathbb{N}$. Hence, by applying the monotone convergence theorem and the equivalence between the uniform integrability and $\mathcal{L}^1$ convergence, sending n to infinity in (4.G.2) leads to

$$
\begin{aligned}
&\mathbb{E}\left[\left(X^{\theta}_T-L_T-\gamma\right)^2\right]\\
=&Y_0\left(x_0-G_0\right)^2+\mathbb{E}\left[\int_0^{T}Y_t\left[\left(\theta^S_t\sqrt{V_t}X^{\theta}_t-P^S_t\right)+(X^{\theta}_t-G_t)\left(\frac{Z^S_t}{Y_t}+\lambda_s\sqrt{V_t}\right)\right]^2dt\right]\\
&+\mathbb{E}\left[\int_0^{T}Y_t\left[\left(\theta^V_t\sqrt{V_t}X^{\theta}_t-P^V_t\right)+(X^{\theta}_t-G_t)\left(\frac{Z^V_t}{Y_t}+\lambda_v\sqrt{V_t}\right)\right]^2dt\right]\\
&+\mathbb{E}\left[\int_0^{T}Y_t\left[\left(\theta^r_t\sigma_r\sqrt{r_t}X^{\theta}_t-P^r_t\right)+(X^{\theta}_t-G_t)\left(\frac{Z^r_t}{Y_t}+\frac{\lambda_r}{\sigma_r}\sqrt{r_t}\right)\right]^2dt\right].
\end{aligned}
\tag{4.G.3}
$$

Recalling the preceding results given in Proposition 4.3.3, 4.3.10, 4.3.11 and (4.3.8), we find from (4.G.3) that the optimal strategy and optimal value function of benchmark problem (4.2.13) are, respectively, given by

$$
\begin{cases}
\theta^{S*}_{BM,t}=-\dfrac{1}{X^*_t}\lambda_s\left(X^*_t-\gamma e^{g_1(t)+g_2(t)r_t}-e^{f_1(t)+f_2(t)r_t}L_t\right),\\
\theta^{r*}_{BM,t}=-\dfrac{1}{X^*_t}\left(\left(A_3(t)+\dfrac{\lambda_r}{\sigma_r^2}\right)X^*_t-\gamma e^{g_1(t)+g_2(t)r_t}\left(A_3(t)+g_2(t)+\dfrac{\lambda_r}{\sigma_r^2}\right)\right.\\
\qquad\qquad\left.-\left(A_3(t)+f_2(t)+\dfrac{\beta_r}{\sigma_r}+\dfrac{\lambda_r}{\sigma_r^2}\right)e^{f_1(t)+f_2(t)r_t}L_t\right),\\
\theta^{V*}_{BM,t}=-\dfrac{1}{X^*_t}(\sigma_vA_2(t)+\lambda_v)\left(X^*_t-\gamma e^{g_1(t)+g_2(t)r_t}-e^{f_1(t)+f_2(t)r_t}L_t\right),
\end{cases}
\tag{4.G.4}
$$

and

$$
J_{BM}(\theta^*_{BM};\gamma)=e^{A_1(0)+A_2(0)v_0+A_3(0)r_0}\left(x_0-l_0e^{f_1(0)+f_2(0)r_0}-\gamma e^{g_1(0)+g_2(0)r_0}\right)^2.
$$

In the following, we devote ourselves to verifying that optimal strategy (4.G.4) is admissible, that is, $\theta^*_{BM} \in \Theta$.

Denote by X^*_t the asset process associated with the optimal strategy θ^*_{BM} given by (4.G.4). Then combining (4.2.7), (4.3.5), and (4.G.4), we observe that

$$\begin{aligned} & d(X^*_t - G_t) \\ =& \left[r_t - \lambda_s \sqrt{V_t} \left(\frac{Z^S_t}{Y_t} + \lambda_s \sqrt{V_t} \right) - \lambda_v \sqrt{V_t} \left(\frac{Z^V_t}{Y_t} + \lambda_v \sqrt{V_t} \right) \right. \\ & \left. - \frac{\lambda_r}{\sigma_r} \sqrt{r_t} \left(\frac{Z^r_t}{Y_t} + \frac{\lambda_r}{\sigma_r} \sqrt{r_t} \right) \right] (X^*_t - G_t)\, dt - \left(\frac{Z^S_t}{Y_t} + \lambda_s \sqrt{V_t} \right) (X^*_t - G_t)\, dW^S_t \\ & - \left(\frac{Z^V_t}{Y_t} + \lambda_v \sqrt{V_t} \right) (X^*_t - G_t)\, dW^V_t - \left(\frac{Z^r_t}{Y_t} + \frac{\lambda_r}{\sigma_r} \sqrt{r_t} \right) (X^*_t - G_t)\, dW^r_t. \end{aligned}$$

By solving this linear SDE explicitly and using the preceding results given in Proposition 4.3.3, 4.3.10, and 4.3.11, the asset process X^*_t is given by

$$\begin{aligned} & X^*_t \\ =& \left(x_0 - l_0 e^{f_1(0) + f_2(0) r_0} - \gamma e^{g_1(0) + g_2(0) r_0} \right) \exp\left\{ - \int_0^t \lambda_s \sqrt{V_s}\, dW^S_s - \frac{1}{2} \int_0^t \lambda_s^2 V_s\, ds \right\} \\ & \times \exp\left\{ - \int_0^t (A_2(s)\sigma_v + \lambda_v) \sqrt{V_s}\, dW^V_s - \frac{1}{2} \int_0^t (A_2(s)\sigma_v + \lambda_v)^2 V_s\, ds \right\} \\ & \times \exp\left\{ - \int_0^t \left(A_3(s)\sigma_r + \frac{\lambda_r}{\sigma_r} \right) \sqrt{r_s}\, dW^r_s - \frac{1}{2} \int_0^t \left(A_3(s)\sigma_r + \frac{\lambda_r}{\sigma_r} \right)^2 r_s\, ds \right\} \\ & \times \exp\left\{ \int_0^t \left(1 - A_3(s)\lambda_r - \frac{\lambda_r^2}{\sigma_r^2} \right) r_s - (A_2(s)\lambda_v\sigma_v + \lambda_s^2 + \lambda_v^2) V_s\, ds \right\} \\ & + l_0 \exp\left\{ \int_0^t \left(\mu_r - \frac{1}{2}\beta_r^2 \right) r_s\, ds + \int_0^t \beta_r \sqrt{r_s}\, dW^r_s + f_1(t) + f_2(t) r_t \right\} \\ & + \gamma \exp\left\{ g_1(t) + g_2(t) r_t \right\}, \ \ X^*_t = x_0. \end{aligned} \tag{4.G.5}$$

Since X^*_t given in (4.G.5) is $\mathbb{F}$-adapted and has continuous sample paths $\mathbb{P}$ almost surely, it follows from (4.G.4) above that $\theta^{S*}_{BM,t}, \theta^{V*}_{BM,t}$, and $\theta^{r*}_{BM,t}$ are also $\mathbb{F}$-adapted processes with continuous sample paths $\mathbb{P}$ almost surely. Thus, we must have

$$\mathbb{P}\left(\int_0^T |\theta^{S*}_{BM,t}|^2 V_t\, dt < \infty \right) = \mathbb{P}\left(\int_0^T |\theta^{V*}_{BM,t}|^2 V_t\, dt < \infty \right) = \mathbb{P}\left(\int_0^T |\theta^{r*}_{BM,t}|^2 r_t\, dt < \infty \right) = 1.$$

Now, it remains to show that $\left\{ Y_{\tau_n \wedge T} \left(X^*_{\tau_n \wedge T} - G_{1,\tau_n \wedge T} L_{\tau_n \wedge T} - \gamma G_{2,\tau_n \wedge T} \right)^2 \right\}_{n \in \mathbb{N}}$ is a uniformly integrable family for any stopping time sequences $\{\tau_n\}_{n \in \mathbb{N}}$ such that $\tau_n \uparrow \infty$, $\mathbb{P}$ almost surely as $n \to +\infty$. Denote

$$M_t = Y_t (X^*_t - G_t)^2 = Y_t (X^*_t - L_t G_{1,t} - \gamma G_{2,t})^2.$$

From (4.G.1) above, we observe

$$
\begin{aligned}
dM_t = &-\left(A_3(t)\sigma_r + 2\frac{\lambda_r}{\sigma_r}\right)\sqrt{r_t}M_t\,dW_t^r - 2\lambda_s\sqrt{V_t}M_t\,dW_t^S \\
&-(A_2(t)\sigma_v + 2\lambda_v)\sqrt{V_t}M_t\,dW_t^V.
\end{aligned}
$$

Solving the above equation explicitly, we have

$$
\begin{aligned}
M_t =& M_0\exp\left\{-\int_0^t \left(A_3(s)\sigma_r + 2\frac{\lambda_r}{\sigma_r}\right)\sqrt{r_s}\,dW_s^r - \frac{1}{2}\int_0^t \left(A_3(s)\sigma_r + 2\frac{\lambda_r}{\sigma_r}\right)^2 r_s\,ds\right\} \\
&\times\exp\left\{-\int_0^t 2\lambda_s\sqrt{V_s}\,dW_s^S - \int_0^t (A_2(s)\sigma_v + 2\lambda_v)\sqrt{V_s}\,dW_s^V\right. \\
&\left.-\frac{1}{2}\int_0^t (4\lambda_s^2 + (A_2(s)\sigma_v + 2\lambda_v)^2)V_s\,ds\right\} > 0,
\end{aligned}
\tag{4.G.6}
$$

where $M_0 = J_{BM}(\theta^*_{BM})$. It follows from Lemma 4.3.6 that two stochastic exponential processes on the right-hand side of (4.G.6) are $(\mathbb{F},\mathbb{P})$-martingales. Furthermore, due to the independence of the above two stochastic exponential processes with continuous sample paths, it follows from Theorem 2.4 in Cherny (2006) that M_t is an $(\mathbb{F},\mathbb{P})$-martingale.

For any sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, it is clear that $\tau_n \wedge T$ and T are two bounded stopping times. Thus, by Doob's optional sampling theorem (for bounded stopping times), we have

$$
M_{\tau_n\wedge T} = \mathbb{E}\left[M_T|\mathcal{F}_{\tau_n\wedge T}\right].
$$

Since $\{\mathcal{F}_{\tau_n\wedge T}\}_{n\in\mathbb{N}}$ is a family of sub $\sigma-$algebra of $\mathcal{F}_T$ and $\mathbb{E}\left[|M_T|\right] = J_{BM}(\theta^*_{BM}) < \infty$, it follows from Theorem 4.6.1 in Durrett (2019) that the family $\{M_{\tau_n\wedge T}\}_{n\in\mathbb{N}}$ is uniformly integrable. This completes the proof. □

4.H Proof of Theorem 4.3.14

Proof. The solution is obtained via the relationship between the mean-variance problem (4.2.10) and benchmark problem (4.2.13) shown in Section 4.2. Specifically, we have

$$
\begin{aligned}
J^*_{MV} =& \max_{\lambda\in\mathbb{R}} J_{BM}(\theta^*_{BM};\xi-\lambda) - \lambda^2 \\
=& \max_{\lambda\in\mathbb{R}}\left\{e^{A_1(0)+A_2(0)v_0+A_3(0)r_0}\left(x_0 - l_0 e^{f_1(0)+f_2(0)r_0}\right.\right. \\
&\left.\left.-(\xi-\lambda)e^{g_1(0)+g_2(0)r_0}\right)^2 - \lambda^2\right\}.
\end{aligned}
\tag{4.H.1}
$$

Note that

$$
\begin{aligned}
&\frac{d^2 J_{BM}(\theta^*_{BM}; \xi - \lambda) - \lambda^2}{d\lambda^2} \\
=&2\exp\left\{A_1(0) + 2g_1(0) + A_2(0)v_0 + (A_3(0) + 2g_2(0))r_0\right\} - 2 \\
=&2\exp\left\{\int_0^T (\varphi_v A_2(t) + \varphi_r A_3(t))\, dt + 2\int_0^T \varphi_r g_2(t)\, dt + A_2(0)v_0\right. \\
&\left. + (A_3(0) + 2g_2(0))r_0\right\} - 2 < 0,
\end{aligned}
$$

where the last strict inequality follows from the non-positiveness of $A_2(t), A_3(t)$, and $g_2(t)$ as shown in Proposition 4.3.5 and 4.3.11 above. Thus, the maximum of the right-hand side of (4.H.1) is attained at

$$
\lambda^* = \frac{Y_0 G_{2,0}\left(x_0 - l_0 G_{1,0} - \xi G_{2,0}\right)}{1 - Y_0 G_{2,0}^2}, \tag{4.H.2}
$$

where $Y_t, G_{1,t}$, and $G_{2,t}$ are given by (4.3.9), (4.3.17), and (4.3.20), respectively. Substituting (4.H.2) into the right-hand side of (4.H.1) yields the optimal value function of mean-variance ALM problem (4.2.10):

$$
J^*_{MV} = \frac{Y_0\left(x_0 - l_0 G_{1,0} - \xi G_{2,0}\right)^2}{1 - Y_0 G_{2,0}^2}.
$$

Replacing γ in (4.3.23) with $\xi - \lambda^*$ gives the optimal risk exposure strategy as shown in (4.3.25). Finally, the admissibility of optimal strategy (4.3.25) can be shown as Theorem 4.3.13. This completes the proof. □

4.I Proof of Lemma 4.4.3

Proof. We conjecture that the first component Y_t^{ic} of the solution to BSRE (4.4.8) admits the following exponential form:

$$
Y_t^{ic} = \exp\left\{\bar{A}_1(t) + \bar{A}_2(t)V_t + \bar{A}_3(t)r_t\right\},
$$

where $\bar{A}_1(t), \bar{A}_2(t)$, and $\bar{A}_3(t)$ are undetermined differentiable functions with boundary conditions $\bar{A}_1(T) = \bar{A}_2(T) = \bar{A}_3(T) = 0$. A direct application of Itô's formula to Y_t^{ic} returns

$$
\begin{aligned}
dY_t^{ic} =&Y_t^{ic}\left(\frac{d\bar{A}_1(t)}{dt} + \frac{d\bar{A}_2(t)}{dt}V_t + \frac{d\bar{A}_3(t)}{dt}r_t + (\varphi_v - \kappa_v V_t)\bar{A}_2(t) + \frac{1}{2}\sigma_v^2 \bar{A}_2^2(t)V_t\right. \\
&\left. + (\varphi_r - \kappa_r r_t)\bar{A}_3(t) + \frac{1}{2}\sigma_r^2 r_t \bar{A}_3^2(t)\right) dt + \sigma_r\sqrt{r_t}\bar{A}_3(t)Y_t^{ic}\, dW_t^0 \\
&+ \sigma_v\sqrt{V_t}\rho\bar{A}_2(t)Y_t^{ic}\, dW_t^1 + \sigma_v\sqrt{V_t}\sqrt{1-\rho^2}\bar{A}_2(t)Y_t^{ic}\, dW_t^2.
\end{aligned} \tag{4.I.1}
$$

Let

$$\begin{cases} Z_{0,t} = & \sigma_r \sqrt{r_t} \bar{A}_3(t) Y_t^{ic}, \\ Z_{1,t} = & \sigma_v \sqrt{V_t} \rho \bar{A}_2(t) Y_t^{ic}, \\ Z_{2,t} = & \sigma_v \sqrt{V_t} \sqrt{1-\rho^2} \bar{A}_2(t) Y_t^{ic}. \end{cases}$$

By matching the driver of BSRE (4.4.8) and the drift coefficient of SDE (4.I.1), we find that functions $\bar{A}_1(t), \bar{A}_2(t)$, and $\bar{A}_3(t)$ solve the following ODEs:

$$\begin{cases} \frac{d\bar{A}_2(t)}{dt} = \left(\lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right)^2 + \left(\kappa_v + 2\left(\lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right) \sigma_v \rho\right) \bar{A}_2(t) \\ \qquad\qquad + \sigma_v^2 \left(\rho^2 - \frac{1}{2}\right) \bar{A}_2^2(t), \;\; \bar{A}_2(T) = 0, \\ \frac{d\bar{A}_3(t)}{dt} = \left(\frac{\lambda_r^2}{\sigma_r^2} - 2\right) + (\kappa_r + 2\lambda_r) \bar{A}_3(t) + \frac{1}{2}\sigma_r^2 \bar{A}_3^2(t), \;\; \bar{A}_3(T) = 0, \\ \frac{d\bar{A}_1(t)}{dt} = -\varphi_v \bar{A}_2(t) - \varphi_r \bar{A}_3(t), \;\; \bar{A}_1(T) = 0. \end{cases} \tag{4.I.2}$$

Notice that ODEs (4.I.2) have similar structures to the ones given in Proposition 4.3.3 above. By repeating the same calculations as shown in Proposition 4.3.4, we obtain closed-form expressions for $\bar{A}_1(t), \bar{A}_2(t)$, and $\bar{A}_3(t)$ given by (4.4.17), (4.4.15), and (4.4.16), respectively. Moreover, it follows from direct differentiation that functions $\bar{A}_2(t)$ and $\bar{A}_3(t)$ are strictly increasing over $[0,T]$ and are bounded over $[0,T]$. Therefore, by the similar arguments of Lemma 4.3.9, it can be shown that $(Y_t^{ic}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ given by (4.4.13) is the unique solution to BSRE (4.4.8). □

4.J Proof of Lemma 4.4.4

Proof. Based on the unique solution to BSRE (4.4.8) given in Lemma 4.4.3, we can rewrite the linear BSDE (4.4.10) of $(G_{1,t}^{ic}, \Lambda_{0,t}, \Lambda_{1,t}, \Lambda_{2,t})$ as follows:

$$\begin{cases} dG_{1,t}^{ic} = \Bigg[\left(1 + \frac{\lambda_r \beta_r}{\sigma_r} - \mu_r\right) r_t G_{1,t}^{ic} + \left(\frac{\lambda_r}{\sigma_r} - \beta_r\right) \sqrt{r_t} \Lambda_{0,t} \\ \qquad + \left(\lambda_v \rho + \lambda_s \sqrt{1-\rho^2}\right) \sqrt{V_t} \Lambda_{1,t} - \sigma_v \sqrt{1-\rho^2} \bar{A}_2(t) \sqrt{V_t} \Lambda_{2,t}\Bigg] \, dt \\ \qquad + \Lambda_{0,t} \, dW_t^0 + \Lambda_{1,t} \, dW_t^1 + \Lambda_{2,t} \, dW_t^2, \\ G_{1,T}^{ic} = 1. \end{cases} \tag{4.J.1}$$

We make a conjecture that $G_{1,t}^{ic}$ has the following exponential form:

$$G_{1,t}^{ic} = \exp\left\{\bar{f}_1(t) + \bar{f}_2(t) r_t\right\}. \tag{4.J.2}$$

Applying Itô's formula to G_t^{ic} given in (4.J.2) gives

$$\begin{aligned} dG_{1,t}^{ic} = & G_{1,t}^{ic} \left(\frac{d\bar{f}_1(t)}{dt} + \frac{d\bar{f}_2(t)}{dt} r_t + (\varphi_r - \kappa_r r_t) \bar{f}_2(t) + \frac{1}{2}\sigma_r^2 \bar{f}_2^2(t) r_t\right) dt \\ & + \sigma_r \sqrt{r_t} \bar{f}_2(t) G_{1,t}^{ic} \, dW_t^0 \end{aligned} \tag{4.J.3}$$

Let

$$\Lambda_{0,t} = G^{ic}_{1,t}\sigma_r\sqrt{r_t}\bar{f}_2(t),\ \ \Lambda_{1,t} = 0,\ \ \Lambda_{2,t} = 0.$$

Then the driver of BSDE (4.J.1) is given by

$$r_t G^{ic}_{1,t}\left[\left(1+\frac{\lambda_r}{\sigma_r}\beta_r-\mu_r\right)+(\lambda_r-\beta_r\sigma_r)\bar{f}_2(t)\right]. \tag{4.J.4}$$

It follows from (4.J.4) and the drift coefficient of SDE (4.J.3) that $\bar{f}_1(t)$ and $\bar{f}_2(t)$ must solve the ODEs below

$$\begin{cases}\dfrac{d\bar{f}_2(t)}{dt} = \left(1+\dfrac{\lambda_r\beta_r}{\sigma_r}-\mu_r\right)+(\kappa_r+\lambda_r-\beta_r\sigma_r)\bar{f}_2(t)-\dfrac{1}{2}\sigma_r^2\bar{f}_2^2(t),\ \ \bar{f}_2(T)=0,\\ \dfrac{d\bar{f}_1(t)}{dt} = -\ \varphi_r\bar{f}_2(t),\ \ \bar{f}_1(T)=0.\end{cases} \tag{4.J.5}$$

Notice that ODEs (4.J.5) are the same as ODEs (4.F.4). Hence, closed-form solutions to ODEs (4.J.5) can be immediately given in (4.4.19) and (4.4.20). Due to the boundedness of function $\bar{A}_2(t)$ as shown in Lemma 4.4.3, by the similar arguments of Proposition 4.3.10 above, it can be shown that $(G^{ic}_{1,t},\Lambda_{0,t},\Lambda_{1,t},\Lambda_{2,t})$ given by (4.4.18) is the unique solution to linear BSDE (4.4.10). □

4.K Proof of Theorem 4.4.7

Proof. The proof is similar to that of Theorem 4.3.13, so we only provide the modifications here. It follows from (4.4.7) that

$$\begin{aligned}
& dY^{ic}_t\left(X^\pi_t-G^{ic}_t\right)^2\\
=& \left[(X^\pi_t-G^{ic}_t)^2Z_{0,t}+2(X^\pi_t-G^{ic}_t)Y^{ic}_t\left(\left(\eta\pi^S_t-b(K)\pi^B_t\right)\sigma_r\sqrt{r_t}X^\pi_t-P_{0,t}\right)\right]dW^0_t\\
&+\left[(X^\pi_t-G^{ic}_t)^2Z_{1,t}+2(X^\pi_t-G^{ic}_t)Y^{ic}_t\left(\pi^S_tX^\pi_t\left(c_1\sqrt{V_t}+\frac{c_2}{\sqrt{V_t}}\right)-P_{1,t}\right)\right]dW^1_t\\
&+\left[(X^\pi_t-G^{ic}_t)^2Z_{2,t}-2(X^\pi_t-G^{ic}_t)Y^{ic}_tP_{2,t}\right]dW^2_t+Y^{ic}_t(P_{2,t})^2\,dt\\
&+Y^{ic}_t\left[\left(\eta\pi^S_t-b(K)\pi^B_t\right)\sigma_rX^\pi_t\sqrt{r_t}-P_{0,t}+(X^\pi_t-G^{ic}_t)\left(\frac{Z_{0,t}}{Y^{ic}_t}+\frac{\lambda_r}{\sigma_r}\sqrt{r_t}\right)\right]^2dt\\
&+Y^{ic}_t\left[\pi^S_tX^\pi_t\left(c_1\sqrt{V_t}+\frac{c_2}{\sqrt{V_t}}\right)-P_{1,t}+(X^\pi_t-G^{ic}_t)\left(\frac{Z_{1,t}}{Y^{ic}_t}+\left(\lambda_v\rho\right.\right.\right.\\
&\left.\left.\left.+\lambda_s\sqrt{1-\rho^2}\right)\sqrt{V_t}\right)\right]^2dt.
\end{aligned} \tag{4.K.1}$$

By applying localization techniques and taking expectation and integration on both sides of (4.K.1) from 0 to T, we obtain

$$\begin{aligned}
&\mathbb{E}\left[(X_t^\pi - L_T - \gamma)^2\right] - Y_0^{ic}(x_0 - l_0 G_{1,0}^{ic} - \gamma G_{2,0}^{ic})^2 \\
=&\mathbb{E}\left[\int_0^T Y_t^{ic}\left[\pi_t^S X_t^\pi \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}\right) - P_{1,t} + (X_t^\pi - G_t^{ic})\left(\frac{Z_{1,t}}{Y_t^{ic}} + \left(\lambda_v \rho\right.\right.\right.\right. \\
&\left.\left.\left.\left. +\lambda_s\sqrt{1-\rho^2}\right)\sqrt{V_t}\right)\right]^2 dt\right] + \mathbb{E}\left[\int_0^T Y_t^{ic}\left[\left(\eta\pi_t^S - b(K)\pi_t^B\right)\sigma_r X_t^\pi \sqrt{r_t} - P_{0,t}\right.\right. \\
&\left.\left. +(X_t^\pi - G_t^{ic})\left(\frac{Z_{0,t}}{Y_t^{ic}} + \frac{\lambda_r}{\sigma_r}\sqrt{r_t}\right)\right]^2 dt\right] + \mathbb{E}\left[\int_0^T Y_t^{ic}(P_{2,t})^2\, dt\right].
\end{aligned} \tag{4.K.2}$$

We observe from (4.4.12), (4.4.18), and (4.4.21) above that $P_{2,t} = 0$. Therefore, it follows from (4.K.2) that the optimal investment strategy and optimal value function of benchmark problem (4.4.4) are given by

$$\begin{cases}
\pi_{BM,t}^{ic,S*} = -\dfrac{\left(X_t^* - \gamma e^{\bar{g}_1(t)+\bar{g}_2(t)r_t} - e^{\bar{f}_1(t)+\bar{f}_2(t)r_t}L_t\right)\left(\bar{A}_2(t)\sigma_v\rho + \lambda_v\rho + \lambda_s\sqrt{1-\rho^2}\right)V_t}{X_t^*(c_1 V_t + c_2)}, \\
\pi_{BM,t}^{ic,B*} = \dfrac{1}{b(K)X_t^*}\left[X_t^*\left(\bar{A}_3(t) + \dfrac{\lambda_r}{\sigma_r^2}\right) - \gamma e^{\bar{g}_1(t)+\bar{g}_2(t)r_t}\left(\bar{A}_3(t) + \bar{g}_2(t) + \dfrac{\lambda_r}{\sigma_r^2}\right)\right. \\
\qquad\left. -L_t e^{\bar{f}_1(t)+\bar{f}_2(t)r_t}\left(\bar{A}_3(t) + \bar{f}_2(t) + \dfrac{\beta_r}{\sigma_r} + \dfrac{\lambda_r}{\sigma_r^2}\right)\right] + \dfrac{\eta}{b(K)}\pi_{BM,t}^{ic,S*}.
\end{cases}$$

□

4.L Proof of Proposition 4.4.12

Proof. Considering functions $A_2(t)$ and $\bar{A}_2(t)$ determined by Riccati ODEs (4.3.11) and (4.4.14), we observe that

$$\begin{aligned}
&\begin{pmatrix} -(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2})^2 & -(\kappa_v + 2(\lambda_v\rho + \lambda_s\sqrt{1-\rho^2})\sigma_v\rho) \\ 0 & -\left(\rho^2 - \frac{1}{2}\right)\sigma_v^2 \end{pmatrix} \\
\geq &\begin{pmatrix} -(\lambda_v^2 + \lambda_s^2) & -(\kappa_v + 2\lambda_v\sigma_v) \\ 0 & -\frac{1}{2}\sigma_v^2 \end{pmatrix}
\end{aligned} \tag{4.L.1}$$

holds, when $\lambda_v\sqrt{1-\rho^2} \geq \lambda_s\rho$ is satisfied. According to the comparison theorem for Riccati ODEs (refer to Theorem 2.1 in Freiling, Jank, and Abou-Kandil (1996)), we have $A_2(t) \leq \bar{A}_2(t) \leq 0$, for all $t \in [0,T]$. On the other hand, it is straightforward to see from Proposition 4.3.4–4.3.11 and Lemma 4.4.3-4.4.5 that

$$\bar{A}_3(t) = A_3(t) \leq 0,\ \bar{g}_1(t) = g_1(t) \leq 0,\ \bar{g}_2(t) = g_2(t) \leq 0,\ \bar{f}_1(t) = f_1(t),\ \bar{f}_2(t) = f_2(t).$$

These results indicate that $G_{1,t}^{ic} = G_{1,t},\ 0 < G_{2,t}^{ic} = G_{2,t} < 1$, and $0 < Y_{1,t} \leq Y_{1,t}^{ic} < 1$ for $t \in [0,T]$. Finally, since function $f(x) = \frac{x}{1-ax}$ is monotonically increasing for

$x \in (0,1)$ with an exogenous constant $a \in (0,1)$, we can conclude that

$$J_{MV}^{*} \leq J_{MV}^{ic*}.$$

This completes the proof. □

Chapter 5

Dynamic optimal mean-variance investment with mispricing in the family of 4/2 stochastic volatility models

ABSTRACT

This paper considers an optimal investment problem with mispricing in the family of 4/2 stochastic volatility models under the mean-variance criterion. The financial market consists of a risk-free asset, a market index, and a pair of mispriced stocks. By applying the linear–quadratic stochastic control theory and solving the corresponding Hamilton–Jacobi–Bellman equation, explicit expressions for the statically optimal (pre-commitment) strategy and the corresponding optimal value function are derived. Moreover, a necessary verification theorem is provided based on an assumption of the model parameters with the investment horizon. Due to the time inconsistency under the mean-variance criterion, we give a dynamic formulation of the problem and obtain the closed-form expression of the dynamically optimal (time-consistent) strategy. This strategy is shown to keep the wealth process strictly below the target (expected terminal wealth) before the terminal time. Results on the special case without mispricing are included. Finally, some numerical examples are given to illustrate the effects of model parameters on the efficient frontier and the difference between static and dynamic optimality.

Keywords: Mean-variance investment; 4/2 stochastic volatility model; Mispricing; Hamilton–Jacobi–Bellman equation; Dynamic optimality

5.1 Introduction

The development of continuous-time stochastic volatility models is deemed crucial in the field of modern finance. The attraction of stochastic volatility models mainly resides in their capacity to explain many stylized facts observed in the financial market such as fat tails, the leverage effect, and the volatility smile/skew on implied volatility surfaces. See, for example, Hull and White (1987), Stein and Stein (1991), Heston (1993) and Lewis (2000). In 2017, Grasselli (2017) proposed a new model called the 4/2 stochastic volatility model which embraces the celebrated Heston model and the 3/2 model (Lewis (2000)) as special cases. The superposition of these two parsimonious models makes it possible for the new 4/2 model to better predict the evolution of the implied volatility surface. This leads to emerging interests in applications of Grasselli's work to derivative pricing problems, such as Cui, Kirkby, and Nguyen (2017, 2018) and Zhu, Zhang, and Jin (2020). In view of the success of the 4/2 model in terms of option pricing, Cheng and Escobar (2021a) recently investigated a utility maximization problem under the 4/2 model. It seems, however, that little attention has been paid to portfolio optimization problems with the 4/2 model under Markowitz (1952)'s mean–variance criterion.

The single-period portfolio selection problem under the mean-variance criterion can be traced back to the seminal work of Markowitz (1952). Li and Ng (2000) and Zhou and Li (2000) generalized Markowitz's work to multi-period and continuous settings, respectively. In particular, Zhou and Li (2000) applied the standard results on the linear–quadratic stochastic control theory combined with an embedding technique to solve the problem in a financial market where all the market coefficients are deterministic. Many researchers then realized the potential of diversification. For example, Shen, Zhang, and Siu (2014) solved the problem under the constant elasticity of the variance model by imposing an exponential integrability condition on the market price of risk. Shen and Zeng (2015) went a step forward by considering the optimal investment–reinsurance problem for a mean-variance insurer in an incomplete market where the market price of risk depends on an affine-form and square-root process, and they derived the modified locally square-integrable optimal strategy. Sun, Zhang, and Yuen (2020) further extended Shen and Zeng (2015)'s results to the case with multiple risky assets and random liabilities. For other previous works, one can refer to Chiu and Wong (2011), Yu (2013), Lv, Wu, and Yu (2016), Tian, Guo, and Sun (2021), Sun, Zhang, and Yuen (2020) and the references therein.

In the aforementioned literature, however, the optimal strategies depend on the initial position of state variables, which is due to the non-separability of the variance operator under the mean-variance criterion in the sense of Bellman's optimality principle. In other words, once the investor arrives at any new position at a future time, the optimal strategy determined at the new position is inconsistent with the initial one unless the investor commits to the initial strategy over the whole investment period. This optimal strategy is therefore time-inconsistent, and is referred to as the pre-commitment strategy in the literature. The notion of time-inconsistency under the mean-variance paradigm stemmed from the work of Strotz (1956). In recent years, the time inconsistency of the mean-variance portfolio selection problem has received considerable attention. For example, Basak and Chabakauri (2010) determined a time-consistent strategy by using a backward recursion approach starting from the terminal time. Alternatively, Björk, Khapko, and Murgoci (2017) proposed the game theoretical approach and studied the subgame-perfect Nash equilibrium for the mean-variance problem. The equilibrium value function and the equilibrium strategy can be explicitly derived under Markovian settings by essentially solving an extended Hamilton–Jacobi–Bellman (HJB) equation. Rather than searching for the time-consistent equilibrium strategy, Pedersen and Peskir (2017) pioneered the dynamically optimal approach to deal with the time inconsistency of the statically optimal (pre-commitment) strategy. Along with this approach, previous works include Pedersen and Peskir (2017), Zhang (2021b), and the references therein.

According to the law of one price, identical assets must have an identical price. There is, however, ample evidence of violations in the law of one price and of the prevalence of a mispricing phenomenon in the financial market. See, for example, Lamont and Thaler (2003), Liu and Longstaff (2004), and Liu and Timmermann (2013). This leads to growing interest in portfolio optimization problems with mispricing in recent years. Yi et al. (2015) studied a utility maximization problem with model ambiguity and mispricing in a financial market consisting of a risk-free asset, a market index, and a pair of mispricing stocks with a constant return rate and volatility. Ma, Zhao, and Rong (2020) considered a problem for a defined contribution plan with mispricing under the Heston model. Considering the methodology developed by Björk, Khapko, and Murgoci (2017) to deal with the time inconsistency under the mean-variance paradigm, Wang et al. (2022) investigated a mean-variance investment–reinsurance problem with mispricing in the context of constant volatility. Other preceding research outputs on the portfolio optimization problems with mispricing include Gu, Viens, and Yi (2017), Gu, Viens, and Yao (2018), Wang et al. (2021), to name but only a few.

Motivated by the above aspects, within the framework introduced by Pedersen and Peskir (2017) to overcome the time inconsistency under the mean-variance criterion, in this paper we study a mean-variance portfolio selection problem that takes into consideration the family of 4/2 stochastic volatility models and mispricing simultaneously. The financial market consists of a risk-free asset, a market index, and a pair of mispriced stocks. To solve this problem, we first apply the Lagrange multiplier method to relate the original problem to an unconstrained optimization problem. To solve the latter by using the dynamic programming approach, we establish the corresponding HJB equation. By solving the HJB equation explicitly, closed-form expressions of the statically optimal strategy and the corresponding optimal value function are derived. Based on an assumption on the model parameters combined with the investment horizon, we prove the necessary verification theorem from scratch and verify the admissibility of the optimal strategy. By solving the statically optimal strategy each time, the dynamically optimal strategy is explicitly derived. This time-consistent strategy keeps the wealth process strictly below the target (expected terminal wealth) before the terminal time. Moreover, we provide the results without mispricing and consider the special cases under the Heston and the 3/2 models. Finally, we present some numerical examples to illustrate the effects of some model parameters on the efficient frontier and the difference between static and dynamic optimality. In summary, compared with some related current research studies, the main contributions of this paper are as follows:

- The market model incorporates the 4/2 model and mispricing simultaneously;
- By making an assumption on the model parameters, a verification theorem is provided to guarantee that the candidate solution to the HJB equation is the optimal value function, and the admissibility of the optimal strategy is verified;
- We derive both the statically optimal (pre-commitment) and the dynamically optimal (time-consistent) strategies explicitly for the mean-variance problem.

The remainder of this paper is structured as follows. In Section 5.2, we formulate the market model and the mean-variance portfolio problem. Section 5.3 is devoted to solving the HJB equation and deriving the closed-form expression of the optimal investment strategy of the unconstrained problem. In Section 5.4, we present the statically optimal strategy and the dynamically optimal strategy for the mean-variance problem, and provide the results on some special cases. In Section 5.5, some numerical examples are given to illustrate our theoretical results. Section 5.6 concludes the paper.

5.2 Formulation of the problem

Let $T > 0$ be a fixed terminal time of decision making and $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space carrying five one-dimensional, mutually independent standard Brownian motions W^1, W^2, Z, Z^1, Z^2. The probability space is further equipped with a right-continuous, P-complete filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian motions.

We consider a financial market setting where a risk-free asset, a market index, and a pair of stocks with mispricing can be continuously traded. The risk-free asset price $B = (B_t)_{t\in[t_0,T]}$ evolves over time as:

$$dB_t = rB_t\,dt,$$

with the initial value $B_{t_0} = b_0 \in \mathbb{R}^+$ at time $t_0 \in [0,T)$, where the positive constant $r > 0$ is the risk-free interest rate. Let the price dynamic of the market index $S_m = (S_{m,t})_{t\in[t_0,T]}$ be governed by the 4/2 stochastic volatility model (Grasselli (2017)):

$$\begin{cases} dS_{m,t} = (r + \lambda(c_1V_t + c_2))\,S_{m,t}\,dt + \left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}}\right) S_{m,t}\,dW_t^1, \\ dV_t = \kappa(\theta_v - V_t)\,dt + \sigma_v\sqrt{V_t}\left(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^2\right), \end{cases} \tag{5.2.1}$$

with $S_{m,t_0} = s_{m,0} \in \mathbb{R}^+$ and $V_{t_0} = v_0 \in \mathbb{R}^+$ at time $t_0 \in [0,T)$, where the constant $\lambda > 0$ stands for a controller of the excess return, and the variance process V_t follows a Cox–Ingersoll–Ross (CIR) process with mean-reversion speed $\kappa > 0$, long-term mean $\theta_v > 0$ and volatility of volatility $\sigma_v > 0$. The Feller condition $2\kappa\theta_v > \sigma_v^2$ is required such that V_t is strictly positive. We assume that two parameters c_1 and c_2 are non-negative constants and $\rho \in [-1,1]$.

Remark 5.2.1. It should be noted that the two non-negative constants $c_1 \geq 0$ and $c_2 \geq 0$ are critical in the 4/2 model (5.2.1), which makes the 4/2 model a superposition of the Heston model (Heston (1993)) and the 3/2 model (Lewis (2000)). Specifically, the case $(c_1, c_2) = (1, 0)$ is known as the Heston model, while the case $(c_1, c_2) = (0, 1)$ corresponds to the 3/2 model.

The two mispriced processes are modeled as a pair of stocks $S_1 = (S_{1,t})_{t\in[t_0,T]}$ and $S_2 = (S_{2,t})_{t\in[t_0,T]}$ which are coupled via the pricing error:

$$M_t = \ln\frac{S_{1,t}}{S_{2,t}},$$

where $S_{1,t}$ and $S_{2,t}$ evolve according to the following system of stochastic differential equations (SDEs):

$$\begin{cases} dS_{1,t} =(r + \beta\lambda(c_1V_t + c_2))S_{1,t}\,dt + \beta\left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}}\right) S_{1,t}\,dW_t^1 \\ \qquad\qquad + \sigma S_{1,t}\,dZ_t + bS_{1,t}\,dZ_t^1 - l_1M_tS_{1,t}\,dt, \\ dS_{2,t} =(r + \beta\lambda(c_1V_t + c_2))S_{2,t}\,dt + \beta\left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}}\right) S_{2,t}\,dW_t^1 \\ \qquad\qquad + \sigma S_{2,t}\,dZ_t + bS_{2,t}\,dZ_t^2 + l_2M_tS_{2,t}\,dt, \end{cases} \tag{5.2.2}$$

with initial values $S_{1,t_0} = s_{1,0}$ and $S_{2,t_0} = s_{2,0}$ at time $t_0 \in [0,T)$, where l_1, l_2, β, σ and b are constant parameters. The term $\beta\left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}\right) dW_t^1$ characterizes the systematic risk of the market, while $\sigma\,dZ_t + b\,dZ_t^i$ stands for the idiosyncratic risk of stock i, $i = 1, 2$. In particular, $\sigma\, dZ_t$ describes the common risk whereas $b\, dZ_t^i$ represents the individual risk generated by the stock i, $i = 1, 2$, respectively. The term $l_i M_t$ reveals the effect of mispricing on ith stock's price via the pricing error M_t defined above. Moreover, it can be shown that the pricing error M_t follows an Ornstein–Uhlenbeck (OU) process as a result of Itô's formula:

$$dM_t = -(l_1 + l_2)M_t\, dt + b\, dZ_t^1 - b\, dZ_t^2, \tag{5.2.3}$$

with $M_{t_0} = m_0 = \ln(s_{1,0}/s_{2,0}) \in \mathbb{R}$, where two constant parameters l_1 and l_2 can be explained as liquidity terms which control the mean-reversion rate of the pricing error. To be specific, the lower liquidity decreases the velocity of reversion of the pricing error towards the long-term mean of zero. Following some previous studies, such as Liu and Timmermann (2013), Ma, Zhao, and Rong (2020) and Wang et al. (2022), we hereby assume that $l_1 + l_2 > 0$, which ensures the stability of the financial market.

Let $\pi_m(t, V_t, M_t, X_t^\pi)$, $\pi_1(t, V_t, M_t, X_t^\pi), \pi_2(t, V_t, M_t, X_t^\pi)$ be three Markov controls denoting the proportions of wealth invested in the market index S_m, and the pair of stocks S_1 and S_2 at time t, respectively. We write $\pi := (\pi_m, \pi_1, \pi_2)$ and such deterministic functions π_m, π_1, π_2 are referred to as feedback control laws in the literature. Suppose that the market is frictionless and no restrictions on leverage and short-selling are enforced, the investor decides to construct a self-financing portfolio of B, S_m, S_1 and S_2 over the investment period $[t_0, T]$. So the controlled wealth process $X^\pi = (X_t^\pi)_{t\in[t_0,T]}$ is described by the following system of SDEs:

$$\begin{cases} dX_t^\pi = X_t^\pi \left[r + u_m(t, V_t, M_t, X_t^\pi)\lambda(c_1 V_t + c_2) - (\pi_1(t, V_t, M_t, X_t^\pi) l_1 \right. \\ \qquad \left. -\pi_2(t, V_t, M_t, X_t^\pi) l_2) M_t\right] dt + X_t^\pi \left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}}\right) u_m(t, V_t, M_t, X_t^\pi)\, dW_t^1 \\ \qquad + X_t^\pi \sigma(\pi_1(t, V_t, M_t, X_t^\pi) + \pi_2(t, V_t, M_t, X_t^\pi))\, dZ_t \\ \qquad + X_t^\pi b\left(\pi_1(t, V_t, M_t, X_t^\pi)\, dZ_t^1 + \pi_2(t, V_t, M_t, X_t^\pi)\, dZ_t^2\right), \\ dV_t = \kappa(\theta_v - V_t)\, dt + \sigma_v\sqrt{V_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right), \\ dM_t = -(l_1 + l_2)M_t\, dt + b\, dZ_t^1 - b\, dZ_t^2, \end{cases} \tag{5.2.4}$$

with $X_{t_0}^\pi = x_0$, where we write $u_m := \pi_m + \beta(\pi_1 + \pi_2)$ to simplify our notation. Let $\mathrm{P}_{t_0,v_0,m_0,x_0}$ denote the probability measure with the initial value $(V_{t_0}, M_{t_0}, X_{t_0}^\pi) = (v_0, m_0, x_0)$ at time $t_0 \in [0, T)$. Accordingly, $\mathrm{E}_{t_0,v_0,m_0,x_0}[\cdot]$ and $\mathrm{Var}_{t_0,v_0,m_0,x_0}(\cdot)$ denote the associated expectation and variance under the probability measure $\mathrm{P}_{t_0,v_0,m_0,x_0}$, respectively.

Definition 5.2.2 (Admissible strategy). *Given any fixed $t_0 \in [0, T)$, a strategy π is said to be admissible if for any $(v_0, m_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, it holds that:*

1. $\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^T (X_t^\pi)^2 \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}\right)^2 u_m^2(t, V_t, M_t, X_t^\pi)\, dt\right] < \infty,$

2. $\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^{T}(X_t^\pi)^2\left(\pi_1^2(t,V_t,M_t,X_t^\pi)+\pi_2^2(t,V_t,M_t,X_t^\pi)\right)\,dt\right]<\infty,$

3. $\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]}|X_t^\pi|^2\right]<\infty.$

The set of all admissible strategies is denoted by $\mathcal{A}$.

The investor wishes to determine an admissible strategy $\pi \in \mathcal{A}$ solving the following mean-variance portfolio problem.

Definition 5.2.3. *The mean-variance portfolio problem is a stochastic optimization problem denoted by*

$$\begin{cases} \min_{\pi\in\mathcal{A}} \mathrm{Var}_{t_0,v_0,m_0,x_0}(X_T^\pi) \\ \text{subject to } \mathrm{E}_{t_0,v_0,m_0,x_0}[X_T^\pi]=\xi, \end{cases} \tag{5.2.5}$$

where ξ is a fixed and given constant serving as a target. We denote the corresponding optimal value function by $V_{MV}(t_0,v_0,m_0,x_0)$.

Remark 5.2.4. Here, we impose $\xi > x_0e^{r(T-t_0)}$, which precludes the trivial case when the investor simply takes the risk-free strategy $\pi \equiv 0$ over the investment period $[t_0,T]$. This condition is consistent with some previous studies, such as Shen, Zhang, and Siu (2014), Sun and Guo (2018) and Sun, Zhang, and Yuen (2020).

As discussed in the Introduction, the mean–variance problem (5.2.5) is time-inconsistent due to the presence of the variance operator in the mean-variance objective. We take the dynamically optimal approach as championed by Pedersen and Peskir (2017) to address the problem of time inconsistency. For readers' convenience, we adapted the definition of dynamic optimality (Definition 2 in Pedersen and Peskir (2017)) into the current context.

Definition 5.2.5 (Dynamic optimality)**.** *A control π^{d*} is said to be dynamically optimal in mean–variance portfolio problem (5.2.5) for (t_0,v_0,m_0,x_0) given and fixed, if for every given and fixed $(t,v,m,x)\in[t_0,T)\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$ and every strategy $u\in\mathcal{A}$ such that $u(t,v,m,x)\neq\pi^{d*}(t,v,m,x)$ with $\mathrm{E}_{t,v,m,x}[X_T^u]=\xi$, there exists a control w satisfying $w(t,v,m,x)=\pi^{d*}(t,v,m,\,x)$ with $\mathrm{E}_{t,v,m,x}[X_T^w]=\xi$ such that*

$$\mathrm{Var}_{t,v,m,x}(X_T^w)<\mathrm{Var}_{t,v,m,x}(X_T^\pi).$$

Upon considering the nature of the dynamically optimal approach, as discussed in the Introduction, we shall first pay attention to the static optimality (pre-commitment) for the mean-variance problem (5.2.5).

Due to the convexity of the objective function in the problem (5.2.5), we can deal with the linear constraint $\mathrm{E}_{t_0,v_0,m_0,x_0}[X_T^\pi]=\xi$ by introducing a Lagrange multiplier

$\theta \in \mathbb{R}$. The associated (dual) Lagrangian is formulated as follows:

$$\begin{aligned} L(x_0, v_0, m_0; \pi, \theta) =& \mathrm{E}_{t_0,v_0,m_0,x_0}[(X_T^\pi - \xi)^2] + 2\theta \mathrm{E}_{t_0,v_0,m_0,x_0}[X_T^\pi - \xi] \\ =& \mathrm{E}_{t_0,v_0,m_0,x_0}\left[(X_T^\pi - (\xi - \theta))^2\right] - \theta^2. \end{aligned} \tag{5.2.6}$$

According to the Lagrangian duality theorem (Luenberger (1968)), mean-variance problem (5.2.5) is, in fact, equivalent to the following min–max stochastic optimization problem:

$$\max_{\theta \in \mathbb{R}} \min_{\pi \in \mathcal{A}} L(x_0, v_0, m_0; \pi, \theta). \tag{5.2.7}$$

This suggests that two steps are involved to obtain the static optimality of the mean-variance problem (5.2.5). First of all, we should solve the internal unconstrained stochastic optimization problem with regard to $\pi \in \mathcal{A}$ with $\theta \in \mathbb{R}$ fixed and given. Subsequently, we turn to optimize Lagrange multiplier $\theta \in \mathbb{R}$ in the external static problem. Hence, we are supposed to determine the optimal strategy of the following quadratic-loss minimization problem in the first place:

$$\min_{\pi \in \mathcal{A}} J(x_0, v_0, m_0; \pi, \gamma) = \mathrm{E}_{t_0,v_0,m_0,x_0}\left[(X_T^\pi - \gamma)^2\right], \tag{5.2.8}$$

with $\gamma = \xi - \theta$ fixed and given.

5.3 Solution to the unconstrained problem

In this section, we devote ourselves to solving the unconstrained quadratic-loss minimization problem (5.2.8) by using the dynamic programming approach. For this, we first define the optimal value function as

$$H(t, x, v, m) = \inf_{\pi \in \mathcal{A}} \mathrm{E}_{t,v,m,x}\left[(X_T^\pi - \gamma)^2\right], \; t_0 \le t \le T, \tag{5.3.1}$$

where $\mathrm{E}_{t,v,m,x}[\cdot]$ is short for $\mathrm{E}[\cdot|\, X_t^\pi = x, V_t = v, M_t = m]$ at time $t \in [t_0, T]$. For the function $H(t, x, v, m) \in C^{1,2,2,2}([t_0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, it must satisfy the following HJB equation due to dynamic programming principle:

$$\inf_{\pi \in \mathcal{A}} \mathcal{D}^{\pi \in \mathcal{A}} H(t, x, v, m) = 0, \tag{5.3.2}$$

where we denote $\mathcal{D}^{\pi \in \mathcal{A}} H(t, x, v, m)$ as the following differential operator:

$$\begin{aligned} \mathcal{D}^{\pi \in \mathcal{A}} H(t, x, v, m) =& H_t + H_x x[u_m \lambda(c_1 v + c_2) - (\pi_1 l_1 - \pi_2 l_2)m + r] + \kappa(\theta_v - v)H_v \\ &+ \frac{1}{2} H_{xx} x^2 \left[\left(c_1\sqrt{v} + \frac{c_2}{\sqrt{v}}\right)^2 u_m^2 + \sigma^2(\pi_1 + \pi_2)^2 + b^2(\pi_1^2 + \pi_2^2)\right] \\ &+ H_{xv} u_m \rho \sigma_v x(c_1 v + c_2) + \frac{1}{2}\sigma_v^2 v H_{vv} - (l_1 + l_2)m H_m \\ &+ H_{xm} x b^2(\pi_1 - \pi_2) + b^2 H_{mm}, \end{aligned}$$

for $t \in [t_0, T)$, with the boundary condition $H(T, x, v, m) = (x-\gamma)^2$. Then, the first-order minimization condition yields the optimal control:

$$\begin{cases} u_m^* = -\dfrac{(H_x\lambda + H_{xv}\rho\sigma_v)v}{H_{xx}x(c_1v+c_2)}, \\ \pi_1^* = -\dfrac{H_{xm}}{H_{xx}x} + \dfrac{H_xm[(\sigma^2+b^2)l_1+\sigma^2 l_2]}{H_{xx}x(2\sigma^2+b^2)b^2}, \\ \pi_2^* = \dfrac{H_{xm}}{H_{xx}x} - \dfrac{H_xm[(\sigma^2+b^2)l_2+\sigma^2 l_1]}{H_{xx}x(2\sigma^2+b^2)b^2}. \end{cases} \tag{5.3.3}$$

Inserting (5.3.3) into the HJB Equation (5.3.2) and simplifying the expression, we obtain the following second-order partial differential Equation (PDE) for function H:

$$\begin{aligned} H_t + \frac{1}{2H_{xx}}\left[2H_xH_{xm}m(l_1+l_2) - H_x^2\lambda^2 v - 2H_xH_{xv}\rho\sigma_v\lambda vs. - H_{xv}^2\rho^2\sigma_v^2 v - 2H_{xm}^2 b^2\right] \\ +\kappa(\theta_v - v)H_v - \frac{H_x^2m^2}{2H_{xx}(2\sigma^2+b^2)b^2}\left[(\sigma^2+b^2)(l_1^2+l_2^2)+2\sigma^2 l_1l_2\right] \\ +rxH_x + \frac{1}{2}\sigma_v^2 vH_{vv} - (l_1+l_2)mH_m + b^2H_{mm} =0. \end{aligned} \tag{5.3.4}$$

In the next proposition, we shall construct an explicit solution denoted by $G(t, x, v, m)$ to PDE (5.3.4).

Proposition 5.3.1. *One solution to second-order PDE (5.3.4) is*

$$G(t, x, v, m) = e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}\left(x - \gamma e^{-r(T-t)}\right)^2, \tag{5.3.5}$$

and the optimal feedback control is given by

$$\begin{cases} u_m^*(t, v, m, x) = -\dfrac{(\lambda + \rho\sigma_v\beta(t))\left(x-\gamma e^{-r(T-t)}\right)v}{x(c_1v+c_2)}, \\ \pi_1^*(t, v, m, x) = \left[-2\gamma(t) + \dfrac{(\sigma^2+b^2)l_1+\sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right]\dfrac{m\left(x-\gamma e^{-r(T-t)}\right)}{x}, \\ \pi_2^*(t, v, m, x) = \left[2\gamma(t) - \dfrac{(\sigma^2+b^2)l_2+\sigma^2 l_1}{(2\sigma^2+b^2)b^2}\right]\dfrac{m\left(x-\gamma e^{-r(T-t)}\right)}{x}, \end{cases} \tag{5.3.6}$$

where

$$\alpha(t) = \int_t^T \kappa\theta_v\beta(s) + 2b^2\gamma(s) + 2r\,ds, \tag{5.3.7}$$

$$\beta(t) = \begin{cases} \lambda^2(t-T), \ \rho^2 = \frac{1}{2}, \ k + 2\lambda\rho\sigma_v = 0; \\ \frac{\lambda^2}{k+2\lambda\rho\sigma_v}\left(e^{(k+2\lambda\rho\sigma_v)(t-T)} - 1\right), \ \rho^2 = \frac{1}{2}, \ k+2\lambda\rho\sigma_v \neq 0; \\ \frac{n_1 n_2(1-e^{\sqrt{\Delta}(T-t)})}{n_1 - n_2 e^{\sqrt{\Delta}(T-t)}}, \ \rho^2 \neq \frac{1}{2}, \ \Delta > 0; \\ \frac{\sigma_v^2(\rho^2-\frac{1}{2})(T-t)n_0^2}{\sigma_v^2(\rho^2-\frac{1}{2})(T-t)n_0 - 1}, \ \rho^2 \neq \frac{1}{2}, \ \Delta = 0; \\ \frac{\sqrt{-\Delta}}{\sigma_v^2(2\rho^2-1)}\tan\left(\arctan\left(\frac{k+2\lambda\rho\sigma_v}{\sqrt{-\Delta}}\right) - \frac{\sqrt{-\Delta}}{2}(T-t)\right) + n_0, \ \rho^2 \neq \frac{1}{2}, \ \Delta < 0 \end{cases} \tag{5.3.8}$$

with

$$\begin{cases} \Delta = (k+2\lambda\rho\sigma_v)^2 - (4\rho^2-2)\sigma_v^2\lambda^2, \\ n_1 = \frac{-(k+2\lambda\rho\sigma_v)+\sqrt{\Delta}}{\sigma^2(2\rho^2-1)}, \\ n_2 = \frac{-(k+2\lambda\rho\sigma_v)-\sqrt{\Delta}}{\sigma^2(2\rho^2-1)}, \\ n_0 = \frac{-(k+2\lambda\rho\sigma_v)}{\sigma^2(2\rho^2-1)}, \end{cases} \tag{5.3.9}$$

and

$$\gamma(t) = \frac{\sqrt{-\Delta_\gamma}}{8b^2}\tan\left(\arctan\left(-\frac{2(l_1+l_2)}{\sqrt{-\Delta_\gamma}}\right) - \frac{\sqrt{-\Delta_\gamma}}{2}(T-t)\right) + \frac{l_1+l_2}{4b^2}, \tag{5.3.10}$$

with $\Delta_\gamma = 4(l_1+l_2)^2 - \frac{16(\sigma^2+b^2)(l_1^2+l_2^2)+32\sigma^2 l_1 l_2}{(2\sigma^2+b^2)} < 0.$

Proof. We propose a candidate solution to the second-order PDE (5.3.4) in the following form:

$$G(t,x,v,m) = e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}[x-a(t)]^2,$$

with $\alpha(T) = \beta(T) = \gamma(T) = 0$ and $a(T) = \gamma$. Then, we have the following partial derivatives:

$$\begin{cases} G_t = \left[\dfrac{d\alpha(t)}{dt} + \dfrac{d\beta(t)}{dt}v + \dfrac{d\gamma(t)}{dt}m^2\right] G - 2e^{\alpha(t)+\beta(t)v+\gamma(t)m^2} \\ \qquad \cdot (x - a(t))\dfrac{da(t)}{dt}, \\ G_x = 2e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}(x - a(t)),\ G_m = 2\gamma(t)mG,\ G_v = \beta(t)G, \\ G_{xx} = 2e^{\alpha(t)+\beta(t)v+\gamma(t)m^2},\ G_{mm} = 2\gamma(t)G + 4\gamma^2(t)m^2G,\ G_{vv} = \beta^2(t)G, \\ G_{xm} = 4\gamma(t)me^{\alpha(t)+\beta(t)v+\gamma(t)m^2}(x - a(t)), \\ G_{xv} = 2\beta(t)e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}(x - a(t)). \end{cases} \tag{5.3.11}$$

Substituting (5.3.11) into (5.3.4) and reshuffling terms yield

$$\begin{aligned} &G\Bigg[\frac{d\alpha(t)}{dt} + \frac{d\beta(t)}{dt}v + \frac{d\gamma(t)}{dt}m^2 + 4\gamma(t)m^2(l_1 + l_2 - \lambda^2 v) - \lambda^2 v - 2\rho\sigma_v\lambda v\beta(t) - \rho^2\sigma_v^2 v\beta^2(t) \\ &- 8b^2m^2\gamma^2(t) + \kappa(\theta_v - v)\beta(t) - \frac{(\sigma^2 + b^2)(l_1^2 + l_2^2) + 2\sigma^2 l_1 l_2}{(2\sigma^2 + b^2)b^2}m^2 + \frac{1}{2}\sigma_v^2 v\beta^2(t) + 4b^2\gamma^2(t)m^2 \\ &- 2\,(l_1 + l_2)\,\gamma(t)m^2 + 2b^2\gamma(t) + 2r\Bigg] + 2(x - a(t))e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}\left[ra(t) - \frac{da(t)}{dt}\right] = 0. \end{aligned}$$

This indicates that we have the following two identities:

$$ra(t) - \frac{da(t)}{dt} = 0, \tag{5.3.12}$$

and

$$\begin{aligned} &\frac{d\alpha(t)}{dt} + \frac{d\beta(t)}{dt}v + \frac{d\gamma(t)}{dt}m^2 + 4\gamma(t)m^2(l_1 + l_2 - \lambda^2 v) - \lambda^2 v - 2\rho\sigma_v\lambda v\beta(t) - \rho^2\sigma_v^2 v\beta^2(t) \\ &- 8b^2m^2\gamma^2(t) + \kappa(\theta_v - v)\beta(t) - \frac{(\sigma^2 + b^2)(l_1^2 + l_2^2) + 2\sigma^2 l_1 l_2}{(2\sigma^2 + b^2)b^2}m^2 + \frac{1}{2}\sigma_v^2 v\beta^2(t) + 4b^2\gamma^2(t)m^2 \\ &- 2(l_1 + l_2)\gamma(t)m^2 + 2b^2\gamma(t) + 2r = 0. \end{aligned} \tag{5.3.13}$$

Upon considering the boundary condition $a(T) = \gamma$, we obtain the following expression of $a(t)$ by solving (5.3.12):

$$a(t) = \gamma e^{-r(T-t)}.$$

As for (5.3.13), we can separate it with respect to variables v and m^2 as follows:

$$\begin{aligned} &\left[\frac{d\beta(t)}{dt} - (2\rho\sigma_v + \kappa)\lambda\beta(t) + \left(\frac{1}{2} - \rho^2\right)\sigma_v^2\beta^2(t) - \lambda^2\right]v + \left[\frac{d\gamma(t)}{dt} - 4b^2\gamma^2(t)\right. \\ &\left.+2(l_1 + l_2)\gamma(t) - \frac{(\sigma^2 + b^2)(l_1^2 + l_2^2) + 2\sigma^2 l_1 l_2}{(2\sigma^2 + b^2)b^2}\right]m^2 + \frac{d\alpha(t)}{dt} + \kappa\theta_v\beta(t) + 2b^2\gamma(t) + 2r = 0. \end{aligned} \tag{5.3.14}$$

Thus, we have the following system of ordinary differential equations (ODEs) from (5.3.14) due to the arbitrariness of $v \in \mathbb{R}^+$ and $m \in \mathbb{R}$:

$$\frac{d\beta(t)}{dt} = \left(\rho^2 - \frac{1}{2}\right)\sigma_v^2\beta^2(t) + (\kappa + 2\lambda\rho\sigma_v)\beta(t) + \lambda^2,\ \beta(T) = 0, \tag{5.3.15}$$

$$\frac{d\gamma(t)}{dt} = 4b^2\gamma^2(t) - 2(l_1 + l_2)\gamma(t) + \frac{(\sigma^2 + b^2)(l_1^2 + l_2^2) + 2\sigma^2 l_1 l_2}{(2\sigma^2 + b^2)b^2},\ \gamma(T) = 0, \tag{5.3.16}$$

$$\frac{d\alpha(t)}{dt} = -\kappa\theta_v\beta(t) - 2b^2\gamma(t) - 2r,\ \alpha(T) = 0. \tag{5.3.17}$$

We see that both (5.3.15) and (5.3.16) are Riccati ODEs, and once these two equations are solved, the explicit expression of solution $\alpha(t)$ to (5.3.17) can be immediately derived.

In the following, we first solve Equation (5.3.15) of $\beta(t)$. When $\rho^2 = \frac{1}{2}$ and $\kappa + 2\lambda\rho\sigma_v = 0$, we have

$$\beta(t) = \lambda^2(t - T).$$

When $\rho^2 = \frac{1}{2}$ and $\kappa + 2\lambda\rho\sigma_v \neq 0$, Riccati ODE (5.3.15) is reduced to the following linear ODE:

$$\frac{d\beta(t)}{dt} = (\kappa + 2\lambda\rho\sigma_v)\beta(t) + \lambda^2. \tag{5.3.18}$$

Integrating both sides of (5.3.18) with respect to time t yields

$$\beta(t) = \frac{\lambda^2}{k + 2\lambda\rho\sigma_v}\left(e^{(k+2\lambda\rho\sigma_v)(t-T)} - 1\right).$$

When $\rho^2 \neq \frac{1}{2}$, we set $\Delta := (k + 2\lambda\rho\sigma_v)^2 - (4\rho^2 - 2)\sigma_v^2\lambda^2$ as given in (5.3.9) above. If $\Delta > 0$, we can rewrite (5.3.15) as follows:

$$\frac{d\beta(t)}{dt} = \sigma_v^2\left(\rho^2 - \frac{1}{2}\right)(\beta(t) - n_1)(\beta(t) - n_2), \tag{5.3.19}$$

where n_1 and n_2 are given by (5.3.9). Upon considering the boundary condition $\beta(T) = 0$, we find

$$\beta(t) = \frac{n_1 n_2\left(1 - e^{\sqrt{\Delta}(T-t)}\right)}{n_1 - n_2 e^{\sqrt{\Delta}(T-t)}}.$$

If $\Delta = 0$, then (5.3.19) can be simplified to

$$\frac{1}{(\beta(t) - n_0)^2}\, d\beta(t) = \sigma_v^2\left(\rho^2 - \frac{1}{2}\right) dt, \tag{5.3.20}$$

where n_0 is given in (5.3.9) above. Integrating both sides of (5.3.20) with respect to time t upon considering the boundary condition $\beta(T) = 0$, we obtain

$$\beta(t) = \frac{\sigma_v^2(\rho^2 - \frac{1}{2})(T - t)n_0^2}{\sigma_v^2(\rho^2 - \frac{1}{2})(T - t)n_0 - 1}.$$

If $\Delta < 0$, then (5.3.15) can reformulated as follows:

$$\frac{d\beta(t)}{\left(\beta(t) + \frac{k+2\lambda\rho\sigma_v}{\sigma_v^2(2\rho^2-1)}\right)^2 + \frac{-\Delta}{\sigma_v^4(2\rho^2-1)^2}} = \sigma_v^2\left(\rho^2 - \frac{1}{2}\right) dt.$$

After calculations upon considering the boundary condition $\beta(T) = 0$, we find

$$\beta(t) = \frac{\sqrt{-\Delta}}{\sigma_v^2(2\rho^2-1)} \tan\left(\arctan\left(\frac{k+2\lambda\rho\sigma_v}{\sqrt{-\Delta}}\right) - \frac{\sqrt{-\Delta}}{2}(T-t)\right) + n_0.$$

Then, we pay attention to the ODE (5.3.16) of $\gamma(t)$. Considering

$$\begin{aligned}\Delta_\gamma :=& 4(l_1+l_2)^2 - \frac{16(\sigma^2+b^2)(l_1^2+l_2^2) + 32\sigma^2 l_1 l_2}{(2\sigma^2+b^2)} \\ =& \frac{-8\sigma^2(l_1+l_2)^2 - 4b^2(l_1-l_2)^2 - 8b^2(l_1^2+l_2^2)}{2\sigma^2+b^2} < 0,\end{aligned}$$

we can rearrange the terms in (5.3.16) to have the following formulation:

$$\frac{d\gamma(t)}{\left(\gamma(t) - \frac{1}{4b^2}(l_1+l_2)\right)^2 + \frac{-\Delta_\gamma}{64b^4}} = 4b^2\, dt. \tag{5.3.21}$$

After some calculations, upon considering the boundary condition $\gamma(T) = 0$, we have

$$\gamma(t) = \frac{\sqrt{-\Delta_\gamma}}{8b^2} \tan\left(\arctan\left(-\frac{2(l_1+l_2)}{\sqrt{-\Delta_\gamma}}\right) - \frac{\sqrt{-\Delta_\gamma}}{2}(T-t)\right) + \frac{l_1+l_2}{4b^2}.$$

Finally, a direct integral calculation on both sides of (5.3.17) upon considering the boundary condition $\alpha(T) = 0$ yields (5.3.7). □

The following proposition presents strict monotonicity results of $\beta(t)$ and $\gamma(t)$ with respect to time t, which in turn leads to the non-positiveness of $\beta(t)$ and $\gamma(t)$ over $[t_0, T]$.

Proposition 5.3.2. *Functions $\beta(t)$ and $\gamma(t)$ given by (5.3.8) and (5.3.10), respectively, are strictly increasing with respect to time t, and thus non-positive over $[t_0, T]$.*

Proof. By differentiating $\beta(t)$ given in (5.3.8) with respect to t, we obtain

$$\frac{d\beta(t)}{dt} = \begin{cases} \lambda^2, \ \rho^2 = \frac{1}{2}, \ k+2\lambda\rho\sigma_v = 0; \\ \lambda^2 e^{(k+2\lambda\rho\sigma_v)(t-T)}, \ \rho^2 = \frac{1}{2}, \ k+2\lambda\rho\sigma_v \neq 0; \\ \frac{4\lambda^2\Delta e^{\sqrt{\Delta}(T-t)}}{\sigma_v^4(2\rho^2-1)^2}\frac{1}{(n_1 - n_2 e^{\sqrt{\Delta}(T-t)})^2}, \ \rho^2 \neq \frac{1}{2}, \ \Delta > 0; \\ \frac{\sigma_v^2(\rho^2-\frac{1}{2})n_0^2}{(\sigma_v^2(\rho^2-\frac{1}{2})(T-t)n_0 - 1)^2}, \ \rho^2 \neq \frac{1}{2}, \ \Delta = 0; \\ \frac{-\Delta}{2\sigma_v^2(2\rho^2-1)} \sec^2\left(\arctan\left(\frac{k+2\lambda\rho\sigma_v}{\sqrt{-\Delta}}\right) - \frac{\sqrt{-\Delta}}{2}(T-t)\right), \ \rho^2 \neq \frac{1}{2}, \ \Delta < 0. \end{cases}$$

It is obvious that $\frac{d\beta(t)}{dt} > 0$ holds for the first three cases. As for the fourth and the fifth cases, note that when $\Delta \leq 0$, we must have $\rho^2 > \frac{1}{2}$.

Similarly, a direct differentiation of $\gamma(t)$ given in (5.3.10) leads to

$$\frac{d\gamma(t)}{dt} = \frac{-\Delta_\gamma}{16b^2} \sec^2\left(\arctan\left(-\frac{2(l_1+l_2)}{\sqrt{-\Delta_\gamma}}\right) - \frac{\sqrt{-\Delta_\gamma}}{2}(T-t)\right) > 0.$$

Finally, upon considering the boundary condition $\beta(T) = \gamma(T) = 0$, we can conclude that $\beta(t)$ and $\gamma(t)$ are non-positive over $[t_0, T]$. □

To facilitate further discussions, we now present some auxiliary results on the OU process and the CIR process in the literature. The first lemma (Lemma 5.3.3) is adapted from Lemma 4.3 in Benth and Karlsen (2005).

Lemma 5.3.3. *Consider the OU process M_t in (5.2.3). If ε is a constant such that*

$$\varepsilon < \frac{l_1+l_2}{4b^2(T-t_0)},$$

then we have

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\exp\left(\varepsilon\int_{t_0}^{T} M_u^2\,du\right)\right] < \infty.$$

The second lemma (lemma 5.3.4) follows from Theorem 5.1 in Zeng and Taksar (2013).

Lemma 5.3.4. *Consider the CIR process V_t in (5.2.1). We have*

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\exp\left(\varepsilon\int_{t_0}^{T} V_t\,dt\right)\right] < \infty \text{ if and only if } \varepsilon \leq \frac{\kappa^2}{2\sigma_v^2}.$$

Inspired by the above results, throughout the rest of paper, we impose the following assumption on the model parameters and the investment horizon $[t_0, T]$:

Assumption 5.3.5. *The model parameters and the investment horizon $[t_0, T]$ satisfy:*

$$C_b \leq \frac{\kappa^2}{2\sigma_v^2} \text{ and } C_\gamma < \frac{l_1+l_2}{4b^2(T-t_0)},$$

where

$$C_b = \max\left\{24\lambda(\lambda - \sigma_v|\rho|\beta(t_0)), (1128+96\sqrt{138})\left(\lambda^2 + \rho^2\sigma_v^2\beta^2(t_0)\right)\right\},$$

and

$$C_\gamma = \max\Bigg\{(564+48\sqrt{138})\frac{(l_1-l_2)^2\sigma^2}{(2\sigma^2+b^2)^2},\\ (1128+96\sqrt{138})\left(4b^2\gamma^2(t_0)+\frac{((\sigma^2+b^2)l_1+\sigma^2 l_2)^2}{(2\sigma^2+b^2)^2b^2}\right),\\ (1128+96\sqrt{138})\left(4b^2\gamma^2(t_0)+\frac{((\sigma^2+b^2)l_2+\sigma^2 l_1)^2}{(2\sigma^2+b^2)^2b^2}\right)\Bigg\}.$$

Remark 5.3.6. It follows from Proposition 5.3.2 above that as $t_0 \to T$, we have $C_b \to (1128+96\sqrt{138})\lambda^2$, which indicates the feasibility of the assumption on C_b. As for the assumption on C_γ, it is straightforward to have $(l_1+l_2)/4b^2(T-t_0)$ and C_γ are decreasing and increasing with respect to T, respectively. This means when the investment horizon $T-t_0$ is small enough, the assumption on C_γ is well established as well.

We next define four Doléans–Dade exponential processes $\Pi_{0,t}, \Pi_{1,t}, \Pi_{2,t}$ and $\Pi_{3,t}$ as follows:

$$\begin{cases} \Pi_{0,t} = \exp\left(\int_{t_0}^t -(\lambda+\rho\sigma_v\beta(s))\sqrt{V_s}\,dW_s^1 - \frac{1}{2}\int_{t_0}^t (\lambda+\rho\sigma_v\beta(s))^2 V_s\,ds\right), \\ \Pi_{1,t} = \exp\left(\int_{t_0}^t \frac{(l_1-l_2)\sigma}{2\sigma^2+b^2} M_s\,dZ_s - \frac{1}{2}\int_{t_0}^t \frac{(l_1-l_2)^2\sigma^2}{(2\sigma^2+b^2)^2} M_s^2\,ds\right), \\ \Pi_{2,t} = \exp\left(\int_{t_0}^t \left(-2\gamma(s)+\frac{(\sigma^2+b^2)l_1+\sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right) bM_s\,dZ_s^1 - \frac{1}{2}\int_{t_0}^t (-2\gamma(s)\right. \\ \qquad \left. + \frac{(\sigma^2+b^2)l_1+\sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right)^2 b^2 M_s^2\,ds\Bigg), \\ \Pi_{3,t} = \exp\left(\int_{t_0}^t \left(2\gamma(s)-\frac{(\sigma^2+b^2)l_2+\sigma^2 l_1}{(2\sigma^2+b^2)b^2}\right) bM_s\,dZ_s^2 - \frac{1}{2}\int_{t_0}^t (2\gamma(s)\right. \\ \qquad \left. - \frac{(\sigma^2+b^2)l_2+\sigma^2 l_1}{(2\sigma^2+b^2)b^2}\right)^2 b^2 M_s^2\,ds\Bigg). \end{cases} \tag{5.3.22}$$

We shall study the integrability of $\Pi_{0,t}, \Pi_{1,t}, \Pi_{2,t}$ and $\Pi_{3,t}$ which will be used in the proof of Theorem 5.3.8 below.

Lemma 5.3.7. *Suppose that Assumption 5.3.5 holds. Then,* $\Pi_{0,t}, \Pi_{1,t}, \Pi_{2,t}$ *and* $\Pi_{3,t}$ *satisfy*

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]}|\Pi_{0,t}|^{24} + \sup_{t\in[t_0,T]}|\Pi_{1,t}|^{24} + \sup_{t\in[t_0,T]}|\Pi_{2,t}|^{24} + \sup_{t\in[t_0,T]}|\Pi_{3,t}|^{24}\right] < \infty. \tag{5.3.23}$$

Proof. Let $p>1$ be any given constant. Then, the following equation of k

$$p = \frac{k}{2\sqrt{k}-1}$$

admits two positive roots:

$$k_1 = p(2p-1) + 2p\sqrt{p(p-1)} \text{ and } k_2 = p(2p-1) - 2p\sqrt{p(2p-1)},$$

with the first root satisfying $k_1 > 1$. In particular, when $p = 24$, we have $k_1 = 1128 + 96\sqrt{138}$. From Assumption 5.3.5, we have

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\exp\left((564 + 48\sqrt{138})\int_{t_0}^{T}(\lambda + \rho\sigma_v\beta(s))^2 V_t\, dt\right)\right] < \infty.$$

According to Theorem 15.4.6 in Cohen and Elliott (2015), we then find that Π_0 satisfies

$$\begin{aligned}
&\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]}|\Pi_{0,t}|^{24}\right] \\
\leq&\frac{24}{23}\left\{\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\exp\left((564 + 48\sqrt{138})\int_{t_0}^{T}(\lambda + \rho\sigma_v\beta(s))^2 V_t\, dt\right)\right]\right\}^{\frac{\sqrt{1128+96\sqrt{138}}-1}{1128+96\sqrt{138}}} \\
<&\infty.
\end{aligned}$$

By applying the same technique to Π_1, Π_2 and Π_3, it is straightforward to obtain (5.3.23) due to Assumption 5.3.5. So we omit the details here. □

To end this section, we shall prove a verification theorem from scratch which guarantees that the candidate solution $G(t, x, v, m)$ derived in (5.3.5) coincides with the optimal value function $H(t, x, v, m)$ defined in (5.3.1) to the quadratic-loss minimization problem (5.2.8). Furthermore, we will also prove the admissibility of the optimal strategy obtained in (5.3.6) in the sense of Definition 5.2.2.

Theorem 5.3.8 (Verification theorem)**.** *Suppose that Assumption 5.3.5 holds. Then, the optimal strategy given in (5.3.6) for the problem (5.2.8) is admissible, and the optimal controlled wealth process X_t^* evolves as*

$$\begin{aligned}
X_t^* =&\Pi_{0,t}\Pi_{1,t}\Pi_{2,t}\Pi_{3,t}\exp\left\{\int_{t_0}^{t}\left[2(l_1+l_2)\gamma(u) - \frac{\sigma^2(l_1+l_2)^2 + b^2\left(l_1^2+l_2^2\right)}{(2\sigma^2+b^2)b^2}\right]M_u^2\right. \\
&\left.-\lambda(\lambda+\rho\sigma_v\beta(u))V_u\, du\right\}\left(x_0 e^{r(t-t_0)} - \gamma e^{-r(T-t)}\right),
\end{aligned} \tag{5.3.24}$$

for $t \in [t_0, T]$, with $(t_0, v_0, m_0, x_0) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ given and fixed such that $x_0 e^{r(T-t_0)} < \xi$, where processes $\Pi_{0,t}, \Pi_{1,t}, \Pi_{2,t}$, and $\Pi_{3,t}$ are given in (5.3.22). Moreover, we have

$$G(t, x, v, m) = H(t, x, v, m)$$

for any $(t, x, v, m) \in [t_0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$*. In particular, the optimal value function of problem (5.2.8) is given by*

$$G(t_0, x_0, v_0, m_0) = e^{\alpha(t_0)+\beta(t_0)v_0+\gamma(t_0)m_0^2}\left(x_0 - \gamma e^{-r(T-t_0)}\right)^2, \tag{5.3.25}$$

with $\alpha(t), \beta(t)$*, and* $\gamma(t)$ *given by (5.3.7), (5.3.8), and (5.3.10), respectively.*

Proof. In the following, we will finish the proof with two steps. At step 1, we show that the optimal strategy $\pi^* = (u_m^*, \pi_1^*, \pi_2^*)$ given in (5.3.6) is admissible. At step 2, we verify that the candidate solution G given in (5.3.5) is indeed the optimal value function H defined in (5.3.1).

Step 1. Substituting the optimal strategy (5.3.6) into the controlled wealth process (5.2.4) leads to

$$\begin{aligned} dX_t^* =& \Bigg\{ \left[2(l_1+l_2)\gamma(t) - \frac{\sigma^2(l_1+l_2)^2 + b^2(l_1^2+l_2^2)}{(2\sigma^2+b^2)b^2}\right]\left(X_t^* - \gamma e^{-r(T-t)}\right)M_t^2 \\ &- \lambda(\lambda+\rho\sigma_v\beta(t))\left(X_t^* - \gamma e^{-r(T-t)}\right)V_t + rX_t^*\Bigg\}\,dt - (\lambda+\rho\sigma_v\beta(t)) \\ &\cdot \left(X_t^* - \gamma e^{-r(T-t)}\right)\sqrt{V_t}\,dW_t^1 + \left[-2\gamma(t) + \frac{(\sigma^2+b^2)l_1 + \sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right]b \\ &\cdot \left(X_t^* - \gamma e^{-r(T-t)}\right)M_t\,dZ_t^1 + \left[2\gamma(t) - \frac{(\sigma^2+b^2)l_2 + \sigma^2 l_1}{(2\sigma^2+b^2)b^2}\right]b \\ &\cdot \left(X_t^* - \gamma e^{-r(T-t)}\right)M_t\,dZ_t^2 + \frac{(l_1-l_2)\sigma}{2\sigma^2+b^2}\left(X_t^* - \gamma e^{-r(T-t)}\right)M_t\,dZ_t, \end{aligned}$$

with $X_{t_0}^* = x_0$. Applying Itô's lemma to $Y_t := e^{r(T-t)}X_t^* - \gamma$, we have

$$\begin{aligned} dY_t =& \left\{\left[2(l_1+l_2)\gamma(t) - \frac{\sigma^2(l_1+l_2)^2 + b^2(l_1^2+l_2^2)}{(2\sigma^2+b^2)b^2}\right]Y_tM_t^2 - \lambda(\lambda+\rho\sigma_v\beta(t))Y_tV_t\right\}dt \\ &- (\lambda+\rho\sigma_v\beta(t))Y_t\sqrt{V_t}\,dW_t^1 + \left[-2\gamma(t) + \frac{(\sigma^2+b^2)l_1+\sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right]bY_tM_t\,dZ_t^1 \\ &+ \left[2\gamma(t) - \frac{(\sigma^2+b^2)l_2+\sigma^2 l_1}{(2\sigma^2+b^2)b^2}\right]bY_tM_t\,dZ_t^2 + \frac{(l_1-l_2)\sigma}{2\sigma^2+b^2}Y_tM_t\,dZ_t, \end{aligned}$$

with $Y_{t_0} = x_0e^{r(T-t)} - \gamma$. By explicitly solving the linear SDE of Y_t, we then have the following closed-form expression:

$$\begin{aligned} Y_t = \left(x_0e^{r(T-t_0)} - \gamma\right)\Pi_{0,t}\Pi_{1,t}\Pi_{2,t}\Pi_{3,t}\exp\Bigg\{\int_{t_0}^{t}\Bigg[&2(l_1+l_2)\gamma(u) \\ &- \frac{\sigma^2(l_1+l_2)^2 + b^2(l_1^2+l_2^2)}{(2\sigma^2+b^2)b^2}\Bigg]M_u^2 - \lambda(\lambda+\rho\sigma_v\beta(u))V_u\,du\Bigg\}, \end{aligned}$$

where $\Pi_{0,t}, \Pi_{1,t}, \Pi_{2,t}$ and $\Pi_{3,t}$ are defined in (5.3.22) above. This in turn shows the optimal controlled wealth process X_t^* given by (5.3.24). We now proceed to show that the optimal strategy $\pi^* = (u_m^*, \pi_1^*, \pi_2^*)$ given in (5.3.6) is admissible. To this end, we first show that

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^*|^4\right] < \infty. \tag{5.3.26}$$

Indeed, from the expression of X_t^* given in (5.3.24), we have

$$\begin{aligned}
&\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^*|^4\right] \\
\leq &K\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]}\left|\Pi_{0,t}\Pi_{1,t}\Pi_{2,t}\Pi_{3,t}\exp\left(\int_{t_0}^t -\lambda(\lambda+\rho\sigma_v\beta(u))V_u\,du\right)\right.\right. \\
&\left.\left.\cdot\exp\left(\int_{t_0}^t\left(2(l_1+l_2)\gamma(u)-\frac{\sigma^2(l_1+l_2)^2+b^2(l_1^2+l_2^2)}{(2\sigma^2+b^2)b^2}\right)M_u^2\,du\right)\right|^4+1\right] \\
\leq &K\Bigg\{\mathrm{E}_{t_0,v_0,m_0,x_0}\left[1+\sup_{t\in[t_0,T]}\Pi_{0,t}^{24}+\Pi_{1,t}^{24}+\Pi_{2,t}^{24}+\Pi_{3,t}^{24}\right] \\
&+\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\exp\left(24\lambda(\lambda-\sigma_v|\rho|\beta(t_0))\int_{t_0}^T V_t\,dt\right)\right] \\
&+\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\exp\left(-\frac{24\sigma^2(l_1+l_2)^2+24b^2(l_1^2+l_2^2)}{(2\sigma^2+b^2)b^2}\int_{t_0}^T M_t^2\,dt\right)\right]\Bigg\} \\
<&\infty,
\end{aligned}$$

where the positive constant K might differ between lines, the second inequality makes use of Jensen's inequality and the non-positiveness of functions $\beta(t)$ and $\gamma(t)$ from Proposition 5.3.2, and the last strictly inequality is due to Assumption 5.3.5 on C_b and Lemma 5.3.3. This in turn leads to the establishment of Condition 3 in Definition 5.2.2 by Jensen's inequality. Then, we show that Condition 1 in Definition 5.2.2 is satisfied:

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^T (X_t^*)^2\left(u_m^*(t,V_t,M_t,X_t^*)\right)^2\left(c_1\sqrt{V_t}+\frac{c_2}{\sqrt{V_t}}\right)^2 dt\right] < \infty.$$

Indeed, in view of the expression of u_m^* given in (5.3.6), we obtain

$$\begin{aligned}
&\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^T (X_t^*)^2\left(u_m^*(t,V_t,M_t,X_t^*)\right)^2\left(c_1\sqrt{V_t}+\frac{c_2}{\sqrt{V_t}}\right)^2 dt\right] \\
=&\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^T (\lambda+\rho\sigma_v\beta(t))^2 V_t\left(X_t^*-\gamma e^{-r(T-t)}\right)^2 dt\right] \\
\leq &K\left\{\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]}|X_t^*|^4\right]+\int_{t_0}^T\mathrm{E}_{t_0,v_0,m_0,x_0}\left[V_t^2\right]dt\right\}<\infty,
\end{aligned}$$

where K is a positive constant, and the last strict inequality follows from (5.3.26) as well as the fact that the CIR process V_t has a finite second moment at time $t \in [t_0, T]$, which is continuous in time t (see, for example, Cox, Ingersoll, and Ross (1985)). Recalling that M_t given in (5.2.3) is an OU process, we can write the solution explicitly:

$$M_t = m_0 e^{-(l_1+l_2)t} + \sqrt{2} b \int_{t_0}^{t} e^{-(l_1+l_2)(t-s)}\, dZ_s^3,$$

where $Z_t^3 = Z_t^1/\sqrt{2} - Z_t^2/\sqrt{2}$ is $\mathrm{P}_{t_0,v_0,m_0,x_0}$ Brownian motion due to Lévy's characterization of Brownian motion. Then, upon noticing that $\int_{t_0}^{t} e^{-(l_1+l_2)(t-s)}\, dZ_s^3$ is normally distributed with mean zero and variance $\int_{t_0}^{t} e^{-2(l_1+l_2)(t-s)}\, ds$, we find that

$$\begin{aligned} \mathrm{E}_{t_0,v_0,m_0,x_0}[M_t^4] &\le K\left[1 + \mathrm{E}_{t_0,v_0,m_0,x_0}\left[\left(\int_{t_0}^{t} e^{-(l_1+l_2)(t-s)}\, dZ_s^3\right)^4\right]\right] \\ &= K\left[1 + 3\left(\int_{t_0}^{t} e^{-2(l_1+l_2)(t-s)}\, ds\right)^2\right] \\ &\le K\left(1 + 3(t-t_0)^2\right), \end{aligned}$$

where $K > 0$ is a positive constant. Therefore, in view of the expressions of π_1^* and π_2^* given in (5.3.6), we find that Condition 2 in Definition 5.2.2 holds as well:

$$\begin{aligned} &\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^{T} (X_t^*)^2 \left(\pi_1^*(t, V_t, M_t, X_t^*)\right)^2 dt\right] \\ =&\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^{T} \left(-2\gamma(t) + \frac{(\sigma^2+b^2)l_1 + \sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right) M_t^2 \left(X_t^* - \gamma e^{-r(T-t)}\right)^2\right] \\ \le& K\left\{\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\sup_{t\in[t_0,T]} |X_t^*|^4\right] + \int_{t_0}^{T} \mathrm{E}_{t_0,v_0,m_0,x_0}\left[M_t^4\right] dt\right\} < \infty, \end{aligned}$$

where K is a positive constant. Using the same technique, we also have

$$\mathrm{E}_{t_0,v_0,m_0,x_0}\left[\int_{t_0}^{T} (X_t^*)^2 \left(\pi_2^*(t, V_t, M_t, X_t^*)\right)^2 dt\right] < \infty.$$

The above results show that the optimal strategy (5.3.6) $\pi^* \in \mathcal{A}$ and completes the first part of the proof.

Step 2. Applying Itô's lemma to the candidate solution G given in (5.3.5) of the

HJB Equation (5.3.2) for any admissible strategy $\pi \in \mathcal{A}$, we have

$$
\begin{aligned}
& dG(t, X_t^\pi, V_t, M_t) \\
=& \mathcal{D}^{\pi \in \mathcal{A}} G(t, X_t^\pi, V_t, M_t)\, dt + G_v(t, X_t^\pi, V_t, M_t) \left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right) \\
& \cdot \sigma_v \sqrt{V_t} + G_x(t, X_t^\pi, V_t, M_t) \Bigg[X_t^\pi \left(c_1 \sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}\right) u_m(t, V_t, M_t, X_t^\pi)\, dW_t^1 \\
& + X_t^\pi \sigma(\pi_1(t, V_t, M_t, X_t^\pi) + \pi_2(t, V_t, M_t, X_t^\pi))\, dZ_t + \Bigg(\pi_1(t, V_t, M_t, X_t^\pi)\, dZ_t^1 \\
& + \pi_2(t, V_t, M_t, X_t^\pi)\, dZ_t^2 \Bigg) X_t^\pi b \Bigg] + G_m(t, X_t^\pi, V_t, M_t) \left(b\, dZ_t^1 - b\, dZ_t^2\right).
\end{aligned}
\tag{5.3.27}
$$

Due to the pathwise continuity of $X^\pi, \pi_1, \pi_2, u_m, V, G_x, G_m$, all the stochastic integrals on the right-hand side of (5.3.27) are clearly continuous local martingales under measure $\mathrm{P}_{t_0,v_0,m_0,x_0}$. Then, there exists a sequence of stopping times localizing all the local martingales (see, for example, page 76 in Le Gall (2016)). We therefore denote the associated localizing sequence by $(\tau_n)_{n\geq 1}$ such that $\tau_n \to \infty$ $\mathrm{P}_{t_0,v_0,m_0,x_0}$ almost surely as $n \to \infty$. Similar to the preceding definition of the probability measure $\mathrm{P}_{t_0,v_0,m_0,x_0}$, we let $\mathrm{P}_{t,v,m,x}$ denote the probability measure with initial data $(V_t, M_t, X_t^\pi) = (v, m, x)$ given and fixed at time $t \in [t_0, T)$. Thus, integrating both sides of (5.3.27) from t to $T \wedge \tau_n$ and taking expectation lead to

$$
\begin{aligned}
& \mathrm{E}_{t,v,m,x}\left[G(T\wedge\tau_n, X_{T\wedge\tau_n}^\pi, V_{T\wedge\tau_n}, M_{T\wedge\tau_n})\right] \\
=& \mathrm{E}_{t,v,m,x}\left[\int_t^{T\wedge\tau_n} \mathcal{D}^{\pi\in\mathcal{A}} G(t', X_{t'}^\pi, V_{t'}, M_{t'})\, dt'\right] + G(t, x, v, m).
\end{aligned}
\tag{5.3.28}
$$

From the expression of candidate function G given in (5.3.5), we find

$$
\begin{aligned}
& G(T\wedge\tau_n, X_{T\wedge\tau_n}^\pi, V_{T\wedge\tau_n}, M_{T\wedge\tau_n}) \\
=& \left(X_{T\wedge\tau_n}^\pi - \gamma e^{-r(T-T\wedge\tau_n)}\right)^2 \exp\left(\alpha(T\wedge\tau_n) + \beta(T\wedge\tau_n) V_{T\wedge\tau_n}\right. \\
& \left. + \gamma(T\wedge\tau_n) M_{T\wedge\tau_n}^2\right) \\
\leq& K \left(X_{T\wedge\tau_n}^\pi - \gamma e^{-r(T-T\wedge\tau_n)}\right)^2,
\end{aligned}
\tag{5.3.29}
$$

where K is a positive constant independent of V and M^2, and the inequality makes use of the non-positiveness of functions $\beta(t)$ and $\gamma(t)$ over $[t_0, T]$ from Proposition 5.3.2. On the one hand, we notice that $\left(X_{T\wedge\tau_n}^\pi - \gamma e^{-r(T-T\wedge\tau_n)}\right)^2$ is $\mathrm{P}_{t,v,m,x}$ integrable for any admissible strategy $\pi \in \mathcal{A}$. On the other hand, since candidate function G given in (5.3.5) satisfies the HJB Equation (5.3.2), then we must have $\mathcal{D}^{\pi\in\mathcal{A}} G(t', X_{t'}^\pi, V_{t'}, M_{t'}) \geq 0$, $\mathrm{P}_{t,v,m,x}$ almost surely for all $t' \in [t, T]$. Hence, passing to the limit in (5.3.28) and applying Lebesgue's dominated convergence theorem to the left-hand side and the monotone convergence theorem to the right-hand side of (5.3.28), respectively, we obtain

$$\begin{aligned}&\mathrm{E}_{t,v,m,x}\left[(X_T^\pi-\gamma)^2\right]\\=&\mathrm{E}_{t,v,m,x}\left[\int_t^T \mathcal{D}^{\pi\in\mathcal{A}}G(t',X_{t'}^\pi,V_{t'},M_{t'})\,dt'\right]+G(t,x,v,m)\\\geq& G(t,x,v,m),\end{aligned} \tag{5.3.30}$$

which implies that, for any admissible strategy $\pi\in\mathcal{A}$, we have

$$H(t,x,v,m)=\inf_{\pi\in\mathcal{A}}\mathrm{E}_{t,v,m,x}\left[(X_T^\pi-\gamma)^2\right]\geq G(t,x,v,m)$$

with any $(t,x,v,m)\in[t_0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ fixed and given. Meanwhile, from Proposition 5.3.1 above, we know:

$$G(t,x,v,m)=\mathrm{E}_{t,v,m,x}\left[(X_T^*-\gamma)^2\right]$$

with admissible strategy $\pi^*=(u_m^*,\pi_1^*,\pi_2^*)\in\mathcal{A}$ given by (5.3.6), which means

$$H(t,x,v,m)=\inf_{\pi\in\mathcal{A}}\mathrm{E}_{t,v,m,x}\left[(X_T^\pi-\gamma)^2\right]\leq \mathrm{E}_{t,v,m,x}\left[(X_T^*-\gamma)^2\right]=G(t,x,v,m).$$

Combining these two results, we can finally conclude that the candidate solution G coincides with the optimal value function H, i.e.

$$G(t,x,v,m)=H(t,x,v,m),$$

for any $(t,x,v,m)\in[t_0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ fixed and given. In particular, the optimal value function of the quadratic-loss minimization problem (5.2.8) is given by (5.3.25). □

5.4 Static and dynamic optimality of the problem

In this section, we derive the statically optimal strategy and the dynamically optimal strategy of the mean-variance portfolio problem (5.2.5) by utilizing the preceding results. As a matter of fact, in view of (5.2.6) and (5.2.7) above, we now only need to solve the following static optimization problem with respect to the Lagrange multiplier $\theta\in\mathbb{R}$ to obtain the static optimality and the corresponding optimal value function for the mean-variance problem (5.2.5)

$$\max_{\theta\in\mathbb{R}} J(x_0,v_0,m_0;\pi^*,\xi-\theta)-\theta. \tag{5.4.1}$$

Reformulating (5.4.1) as a quadratic functional over $\theta \in \mathbb{R}$, we find that the optimal value function of the mean-variance problem (5.2.5) can be obtained from

$$\begin{aligned} & V_{MV}(t_0, v_0, m_0, x_0) \\ =& \max_{\theta\in\mathbb{R}} \left\{ \left[e^{\alpha(t_0)+\beta(t_0)v_0+\gamma(t_0)m_0^2-2r(T-t_0)} - 1 \right] \theta^2 + 2e^{\alpha(t_0)+\beta(t_0)v_0+\gamma(t_0)m_0^2-r(T-t_0)} \right. \\ & \left. \cdot \left(x_0 - \xi e^{-r(T-t_0)} \right) \theta + e^{\alpha(t_0)+\beta(t_0)v_0+\gamma(t_0)m_0^2} \left(x_0 - \xi e^{-r(T-t_0)} \right)^2 \right\}, \end{aligned} \tag{5.4.2}$$

if the coefficient of the quadratic term is strictly negative. Indeed, upon noticing that π^* given in (5.3.6) is the unique optimal strategy for the quadratic loss minimization problem (5.2.8), we must have

$$\begin{aligned} H(t_0, x_0, v_0, m_0) &= e^{\alpha(t_0)+\beta(t_0)v_0+\gamma(t_0)m_0^2} \left(x_0 - \gamma e^{-r(T-t_0)} \right)^2 \\ &< \left(x_0 e^{r(T-t_0)} - \gamma \right)^2 \\ &= \mathrm{E}_{t_0,v_0,m_0,x_0} \left[(X_T^{\bar{\pi}} - \gamma)^2 \right], \end{aligned}$$

where: $\bar{\pi} := (\bar{\pi}_m, \bar{\pi}_1, \bar{\pi}_2) = (0,0,0)$ stands for the risk-free strategy over the period $[t_0, T]$. This implies that the quadratic coefficient of θ in (5.4.2) is strictly negative as desired. Therefore, the maximum to the right-hand side of (5.4.2) is uniquely attained at

$$\theta^* = \frac{x_0 e^{r(T-t_0)} - \xi}{e^{-\alpha(t_0)-\beta(t_0)v_0-\gamma(t_0)m_0^2+2r(T-t_0)} - 1}. \tag{5.4.3}$$

Theorem 5.4.1. *Suppose that Assumption 5.3.5 holds. For any initial data* $(t_0, v_0, m_0, x_0) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ *given and fixed such that* $x_0 < e^{-r(T-t_0)}\xi$, *the statically optimal strategy of the mean–variance portfolio problem (5.2.5) is given by*

$$\begin{cases} \pi_m^*(t,v,m,x) = -\left[\dfrac{(\lambda + \rho\sigma_v\beta(t))v}{c_1 v + c_2} + \dfrac{\beta m(l_1 - l_2)}{2\sigma^2 + b^2} \right] \dfrac{x - (\xi - \theta^*)e^{-r(T-t)}}{x}, \\ \pi_1^*(t,v,m,x) = \left[-2\gamma(t) + \dfrac{(\sigma^2 + b^2)l_1 + \sigma^2 l_2}{(2\sigma^2 + b^2)b^2} \right] m \dfrac{x - (\xi - \theta^*)e^{-r(T-t)}}{x}, \\ \pi_2^*(t,v,m,x) = \left[2\gamma(t) - \dfrac{(\sigma^2 + b^2)l_2 + \sigma^2 l_1}{(2\sigma^2 + b^2)b^2} \right] m \dfrac{x - (\xi - \theta^*)e^{-r(T-t)}}{x}, \end{cases} \tag{5.4.4}$$

for $t \in [t_0, T]$, *and the corresponding optimal value function is*

$$V_{MV}(t_0, v_0, m_0, x_0) = \frac{1}{e^{-\alpha(t_0)-\beta(t_0)v_0-\gamma(t_0)m_0^2+2r(T-t_0)} - 1} (x_0 e^{r(T-t_0)} - \xi)^2, \tag{5.4.5}$$

where $\alpha(t), \beta(t)$, and $\gamma(t)$ are given in (5.3.7), (5.3.8), and (5.3.10), respectively, and θ^* is given by (5.4.3). The controlled wealth process X_t^* is given by

$$
\begin{aligned}
X_t^* =& \Pi_{0,t}\Pi_{1,t}\Pi_{2,t}\Pi_{3,t} \exp\left(\int_{t_0}^{t} \left[2(l_1+l_2)\gamma(u) - \frac{\sigma^2(l_1+l_2)^2 + b^2\left(l_1^2+l_2^2\right)}{(2\sigma^2+b^2)b^2} \right] M_u^2 \right. \\
& \left. - \lambda(\lambda + \rho\sigma_v\beta(u))V_u \, du \right) \left(x_0 e^{r(t-t_0)} - (\xi - \theta^*)e^{-r(T-t)} \right),
\end{aligned} \tag{5.4.6}
$$

where processes $\Pi_{0,t}, \Pi_{1,t}, \Pi_{2,t}$, and $\Pi_{3,t}$ are given in (5.3.22). Moreover, the statically optimal strategy given by (5.4.4) is admissible, i.e., $\pi^* = (\pi_m^*, \pi_1^*, \pi_2^*) \in \mathcal{A}$.

Proof. Substituting θ^* given by (5.4.3) into (5.4.2) leads to the optimal value function (5.4.5). Replacing γ in (5.3.6) and (5.3.24) with $\xi - \theta^*$ gives the statically optimal strategy (5.4.4) and the statically optimal controlled wealth process (5.4.6), respectively. Following the proof in Theorem 5.3.8 above, it is obvious to see that the statically optimal strategy $\pi^* \in \mathcal{A}$. □

Remark 5.4.2. If we set either $(c_1, c_2) = (1, 0)$ or $(c_1, c_2) = (0, 1)$ in (5.4.4), then we obtain explicit solutions to the mean-variance problem with mispricing under the Heston model and the 3/2 model, respectively.

Corollary 5.4.3. *(No mispricing under the 4/2 model). Suppose that Assumption 5.3.5 holds. For any initial data $(t_0, v_0, x_0) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R}$ given and fixed such that $x_0 < e^{-r(T-t_0)}\xi$, the statically optimal strategy of the mean-variance portfolio problem (5.2.5) without mispricing is given by*

$$
\pi_m^*(t, v, x) = -\frac{(\lambda + \rho\sigma_v\beta(t))\, v}{c_1 v + c_2} \frac{x - (\xi - \bar{\theta}^*)e^{-r(T-t)}}{x}, \tag{5.4.7}
$$

for $t \in [t_0, T]$. The corresponding optimal value function is

$$
V_{MV}(t_0, v_0, x_0) = \frac{1}{e^{-\bar{\alpha}(t_0) - \beta(t_0)v_0 + 2r(T-t_0)} - 1} (x_0 e^{r(T-t_0)} - \xi)^2, \tag{5.4.8}
$$

where $\beta(t)$ is given in (5.3.8) and $\bar{\alpha}(t)$ is given by

$$
\bar{\alpha}(t) = \int_t^T \kappa\theta_v\beta(s) + 2r \, ds, \tag{5.4.9}
$$

and $\bar{\theta}^*$ is given by

$$
\bar{\theta}^* = \frac{x_0 e^{r(T-t_0)} - \xi}{e^{-\bar{\alpha}(t_0) - \beta(t_0)v_0 + 2r(T-t_0)} - 1}. \tag{5.4.10}
$$

The controlled wealth process X_t^* is given by

$$
X_t^* = \exp\left\{ \int_{t_0}^{t} -\lambda(\lambda + \rho\sigma_v\beta(u))V_u \, du \right\} \Pi_{0,t} \left(x_0 e^{r(t-t_0)} - (\xi - \bar{\theta}^*)e^{-r(T-t)} \right), \tag{5.4.11}
$$

with $\Pi_{0,t}$ *given in (5.3.22). Moreover, the optimal strategy given in (10.4.7) is admissible, i.e.,* $\pi_m^* \in \mathcal{A}$.

Proof. If there is no mispricing in the market, then $M_t \equiv 0$, which reveals that $\pi_1^* = \pi_2^* = 0$ and $\pi_m^* = u_m^*$ due to (5.3.3). Moreover, since m vanishes from the HJB Equation (5.3.2) in this case, then $\gamma(t)$ disappears as well. This in turn leads to (5.4.7)–(5.4.11) following from (5.3.7), (5.4.3)–(5.4.6), respectively. □

As discussed in Section 5.2, the statically optimal strategy $\pi^* = (\pi_m^*, \pi_1^*, \pi_2^*)$ in Theorem 5.4.1 relies on the initial position of state variables (t_0, v_0, m_0, x_0). We will now proceed to derive the dynamically optimal strategy under the framework developed by Pedersen and Peskir (2017).

Theorem 5.4.4. *Suppose that Assumption 5.3.5 holds. For any initial data* $(t_0, v_0, m_0, x_0) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ *given and fixed such that* $x_0 < e^{-r(T-t_0)}\xi$*, the dynamically optimal strategy of the mean-variance portfolio problem (5.2.5) is given by*

$$\begin{cases} \pi_m^{d*}(t,v,m,x) = -\left[\dfrac{(\lambda+\rho\sigma_v\beta(t))v}{c_1 v + c_2} + \dfrac{\beta m(l_1-l_2)}{2\sigma^2+b^2}\right] \dfrac{x-\xi e^{-r(T-t)}}{\left(1-e^{\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t)}\right)x}, \\ \pi_1^{d*}(t,v,m,x) = \left[-2\gamma(t) + \dfrac{(\sigma^2+b^2)l_1+\sigma^2 l_2}{(2\sigma^2+b^2)b^2}\right] \dfrac{m\left(x-\xi e^{-r(T-t)}\right)}{\left(1-e^{\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t)}\right)x}, \\ \pi_2^{d*}(t,v,m,x) = \left[2\gamma(t) - \dfrac{(\sigma^2+b^2)l_2+\sigma^2 l_1}{(2\sigma^2+b^2)b^2}\right] \dfrac{m\left(x-\xi e^{-r(T-t)}\right)}{\left(1-e^{\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t)}\right)x}, \end{cases} \tag{5.4.12}$$

for $t \in [t_0, T)$*. The controlled wealth process* X_t^{d*} *evolves over time as*

$$\begin{aligned} X_t^{d^*} =& \Gamma_{0,t}\Gamma_{1,t}\Gamma_{2,t}\Gamma_{3,t} \exp\left(\int_{t_0}^t \left[\left(2(l_1+l_2)\gamma(s) - \frac{\sigma^2(l_1+l_2)^2 + b^2\left(l_1^2+l_2^2\right)}{(2\sigma^2+b^2)b^2}\right)M_s^2 \right.\right. \\ & \left.\left. - \lambda(\lambda+\rho\sigma_v\beta(s))V_s\right] f(s,V_s,M_s)\,ds\right)\left(x_0 e^{r(t-t_0)} - \xi e^{-r(T-t)}\right) + \xi e^{-r(T-t)}, \end{aligned} \tag{5.4.13}$$

with $X_t^{d*}e^{r(T-t)} < \xi$ for $t \in [t_0, T)$, where processes $\Gamma_0, \Gamma_1, \Gamma_2$, and Γ_3 are given by

$$
\begin{cases}
\Gamma_{0,t} = \exp\left(\int_{t_0}^t -(\lambda+\rho\sigma_v\beta(s))\sqrt{V_s}f(s,V_s,M_s)\,dW_s^1 - \frac{1}{2}\int_{t_0}^t(\lambda+\rho\sigma_v\beta(s))^2V_sf^2(s,V_s,M_s)\,ds\right),\\
\Gamma_{1,t} = \exp\left(\int_{t_0}^t \frac{(l_1-l_2)\sigma}{2\sigma^2+b^2}M_sf(s,V_s,M_s)\,dZ_s - \frac{1}{2}\int_{t_0}^t\frac{(l_1-l_2)^2\sigma^2}{(2\sigma^2+b^2)^2}M_s^2f^2(s,V_s,M_s)\,ds\right),\\
\Gamma_{2,t} = \exp\left(\int_{t_0}^t\left(-2\gamma(s)+\frac{(\sigma^2+b^2)l_1+\sigma^2l_2}{(2\sigma^2+b^2)b^2}\right)bM_sf(s,V_s,M_s)\,dZ_s^1\right.\\
\qquad\left.-\frac{1}{2}\int_{t_0}^t\left(-2\gamma(s)+\frac{(\sigma^2+b^2)l_1+\sigma^2l_2}{(2\sigma^2+b^2)b^2}\right)^2b^2M_s^2f^2(s,V_s,M_s)\,ds\right),\\
\Gamma_{3,t} = \exp\left(\int_{t_0}^t\left(2\gamma(s)-\frac{(\sigma^2+b^2)l_2+\sigma^2l_1}{(2\sigma^2+b^2)b^2}\right)bM_sf(s,V_s,M_s)\,dZ_s^2\right.\\
\qquad\left.-\frac{1}{2}\int_{t_0}^t\left(2\gamma(s)-\frac{(\sigma^2+b^2)l_2+\sigma^2l_1}{(2\sigma^2+b^2)b^2}\right)^2b^2M_s^2f^2(s,V_s,M_s)\,ds\right),
\end{cases}
\tag{5.4.14}
$$

and the help function $f(t,v,m) : [t_0,T)\times\mathbb{R}^+\times\mathbb{R}\mapsto\mathbb{R}$ is given by

$$
f(t,v,m) = \frac{1}{1-\exp(\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t))},
\tag{5.4.15}
$$

with $\alpha(t), \beta(t)$, and $\gamma(t)$ given in (5.3.7), (5.3.8), and (5.3.10), respectively.

Proof. We start with identifying t_0 with t, x_0 with x, v_0 with v and m_0 with m in (5.4.4). This leads to the following candidate:

$$
\begin{cases}
\pi_m^{d*}(t,v,m,x) = -\left[\frac{(\lambda+\rho\sigma_v\beta(t))v}{c_1v+c_2}+\frac{\beta m(l_1-l_2)}{2\sigma^2+b^2}\right]\frac{x-\xi e^{-r(T-t)}}{\left(1-e^{\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t)}\right)x},\\
\pi_1^{d*}(t,v,m,x) = \left[-2\gamma(t)+\frac{(\sigma^2+b^2)l_1+\sigma^2l_2}{(2\sigma^2+b^2)b^2}\right]\frac{m\left(x-\xi e^{-r(T-t)}\right)}{\left(1-e^{\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t)}\right)x},\\
\pi_2^{d*}(t,v,m,x) = \left[2\gamma(t)-\frac{(\sigma^2+b^2)l_2+\sigma^2l_1}{(2\sigma^2+b^2)b^2}\right]\frac{m\left(x-\xi e^{-r(T-t)}\right)}{\left(1-e^{\alpha(t)+\beta(t)v+\gamma(t)m^2-2r(T-t)}\right)x}.
\end{cases}
\tag{5.4.16}
$$

We next show that this candidate $\pi^{d*} = (\pi_m^{d*}, \pi_1^{d*}, \pi_2^{d*})$ (5.4.16) is dynamically optimal in the mean-variance portfolio problem (5.2.5). To this end, we first take any other admissible strategy $u \in \mathcal{A}$ such that $u(t,v,m,x) \neq \pi^{d*}(t,v,m,x)$ with $\mathrm{E}_{t,v,m,x}[X_T^u] = \xi$. Then, set $w = \pi^*$ under the measure $\mathrm{P}_{t,v,m,x}$. Replacing (t_0, v_0, m_0, x_0) with (t,v,m,x) in (5.4.4), we see from (5.4.4) that $\pi^*(t,v,m,x) = \pi^{d*}(t,v,m,x)$, and thus $w(t,v,m,x) = \pi^*(t,v,m,x) = \pi^{d*}(t,v,m,x) \neq u(t,v,m,x)$ for any $t \in [0,T)$. Due to the continuity of functions u and w, there exists a ball $B_\epsilon := [t, t+\epsilon]\times[v-\epsilon, v+\epsilon]\times[m-\epsilon, m+\epsilon]\times[x-\epsilon, x+\epsilon]$ such that $w(\tilde{t},\tilde{v},\tilde{m},\tilde{x}) \neq u(\tilde{t},\tilde{v},\tilde{m},\tilde{x})$ for $(\tilde{t},\tilde{v},\tilde{m},\tilde{x}) \in B_\epsilon$ when $\epsilon > 0$ is small enough such that $t+\epsilon \leq T$. Therefore, since $w = \pi^*$ is the unique continuous function such that the infimum within the HJB Equation (5.3.2) is attained for any (t,v,m,x), then we can set the exiting time $\tau_\epsilon = \inf\{t \wedge T |\ (t, V_t, M_t, X_t^u) \notin B_\epsilon\}$ such that for $\tilde{t} \leq \tau_\epsilon$, it holds that

$$\mathcal{D}^{u\in\mathcal{A}} H(\tilde{t}, X_{\tilde{t}}^u, V_{\tilde{t}}, M_{\tilde{t}}) \geq \zeta > 0,\ \mathrm{P}_{t,v,m,x} - a.s.$$

where ζ is a fixed positive constant. Replacing γ by $\xi - \tilde{\theta}^*$ in the boundary condition of the HJB Equation (5.3.2) with $\tilde{\theta}^*$ given by

$$\tilde{\theta}^* = \frac{xe^{r(T-t)} - \xi}{e^{-\alpha(t)-\beta(t)v-\gamma(t)m^2+2r(T-t)} - 1},$$

it follows from (5.3.30) with $\xi - \tilde{\theta}^*$ in place of γ that

$$\begin{aligned}
&\mathrm{E}_{t,v,m,x}\left[(X_T^\pi - (\xi - \tilde{\theta}^*))^2\right] \\
=&\mathrm{E}_{t,v,m,x}\left[\int_t^{\tau_\epsilon} \mathcal{D}^{u\in\mathcal{A}} H(\tilde{t}, X_{\tilde{t}}^u, V_{\tilde{t}}, M_{\tilde{t}})\, d\tilde{t}\right] + \mathrm{E}_{t,v,m,x}\left[\int_{\tau_\epsilon}^T \mathcal{D}^{u\in\mathcal{A}} H(t', X_{t'}^u, V_{t'}, M_{t'})\, dt'\right] \\
&+ e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}\left(x - (\xi - \tilde{\theta}^*)e^{-r(T-t)}\right)^2 \\
\geq&\zeta\mathrm{E}_{t,v,m,x}[\tau_\epsilon - t] + e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}\left(x - (\xi - \tilde{\theta}^*)e^{-r(T-t)}\right)^2 \\
>&e^{\alpha(t)+\beta(t)v+\gamma(t)m^2}\left(x - (\xi - \tilde{\theta}^*)e^{-r(T-t)}\right)^2 \\
=&\mathrm{E}_{t,x,v,m}\left[(X_T^w - (\xi - \tilde{\theta}^*))^2\right],
\end{aligned} \tag{5.4.17}$$

where the strict inequality follows from the fact that $\tau_\varepsilon > t$, since the triple (V, M, X^u) has continuous sample paths with probability one under $\mathrm{P}_{t,v,m,x}$ measure. From (5.4.17), we then have

$$\begin{aligned}
\mathrm{Var}_{t,v,m,x}(X_T^u) &= \mathrm{E}_{t,v,m,x}[(X_T^u)^2] - \xi^2 \\
&= \mathrm{E}_{t,v,m,x}[(X_T^u - (\xi - \tilde{\theta}^*))^2] - (\tilde{\theta}^*)^2 \\
&> \mathrm{E}_{t,v,m,x}[(X_T^w - (\xi - \tilde{\theta}^*))^2] - (\tilde{\theta}^*)^2 \\
&= \mathrm{Var}_{t,v,m,x}(X_T^w).
\end{aligned}$$

This shows that the candidate $\pi^{d*} = (\pi_m^{d*}, \pi_1^{d*}, \pi_2^{d*})$ proposed in (5.4.16) is the dynamically optimal strategy for mean–variance portfolio problem (5.2.5).

Substitute π^{d*} into (5.2.4) and denote the corresponding wealth process by X_t^{d*}.

Applying Itô's lemma to $Y_t := e^{r(T-t)} X_t^{d*} - \xi$ yields

$$
\begin{aligned}
dY_t = & \Bigg\{ \left[2\gamma(t)(l_1 + l_2) - \frac{b^2(l_1^2 + l_2^2) + \sigma^2(l_1 + l_2)^2}{(2\sigma^2 + b^2)b^2} \right] M_t^2 f(t, V_t, M_t) \\
& - \lambda(\lambda + \rho\sigma_v\beta(t)) f(t, V_t, M_t) V_t \Bigg\} Y_t \, dt + \frac{(l_1 - l_2)\sigma}{2\sigma^2 + b^2} M_t f(t, V_t, M_t) Y_t \, dZ_t \\
& - (\lambda + \rho\sigma_v\beta(t)) \sqrt{V_t} f(t, V_t, M_t) Y_t \, dW_t^1 + \left[-2\gamma(t) + \frac{(\sigma^2 + b^2)l_1 + \sigma^2 l_2}{(2\sigma^2 + b^2)b^2} \right] bM_t \\
& \cdot f(t, V_t, M_t) Y_t \, dZ_t^1 + \left[2\gamma(t) - \frac{(\sigma^2 + b^2)l_2 + \sigma^2 l_1}{(2\sigma^2 + b^2)b^2} \right] bM_t f(t, V_t, M_t) Y_t \, dZ_t^2,
\end{aligned}
\tag{5.4.18}
$$

where the help function f is defined in (5.4.15). Solving this linear SDE (5.4.18) of Y_t explicitly, we obtain the closed-form expression of X_t^{d*} given in (5.4.13). Moreover, it is easy to see that the initial value $Y_{t_0} = x_0 e^{r(T-t_0)} - \xi < 0$ leads to $X_t^{d*} e^{r(T-t)} < \xi$ for $t \in [t_0, T)$. □

Corollary 5.4.5. *(No mispricing under the 4/2 model). Suppose that the Assumption 5.3.5 holds. For any initial data $(t_0, v_0, x_0) \in [0, T) \times \mathbb{R}^+ \times \mathbb{R}$ given and fixed such that $x_0 < e^{-r(T-t_0)}\xi$, the dynamically optimal strategy of the mean-variance portfolio problem (5.2.5) without mispricing is given by*

$$
\pi_m^{d*}(t, v, x) = -\frac{(\lambda + \rho\sigma_v\beta(t))\, v}{c_1 v + c_2} \frac{x - \xi e^{-r(T-t)}}{\left(1 - e^{\bar{\alpha}(t) + \beta(t)v - 2r(T-t)}\right) x},
$$

for $t \in [t_0, T)$. The controlled wealth process X_t^{d*} evolves as:

$$
\begin{aligned}
X_t^{d*} = & \left(x_0 e^{r(t-t_0)} - \xi e^{-r(T-t)} \right) \exp\left\{ -\int_{t_0}^t (\lambda + \rho\sigma_v\beta(u)) \sqrt{V_u} \bar{f}(u, V_u) \, dW_u^1 \right\} \\
& \exp\left\{ \int_{t_0}^t -\lambda(\lambda + \rho\sigma_v\beta(u)) \bar{f}(u, V_u) V_u - \frac{1}{2}(\lambda + \rho\sigma_v\beta(u))^2 V_u \bar{f}^2(u, V_u) \, du \right\} \\
& + \xi e^{-r(T-t)},
\end{aligned}
$$

with $X_t^{d*} e^{r(T-t)} < \xi$ for $t \in [t_0, T)$, where the help function $\bar{f}(t, v) : [t_0, T) \times \mathbb{R}^+ \mapsto \mathbb{R}$ is given by

$$
\bar{f}(t, v) = \frac{1}{1 - e^{\bar{\alpha}(t) + \beta(t)v - 2r(T-t)}},
$$

with $\bar{\alpha}(t)$ and $\beta(t)$ are given in (5.4.9) and (5.3.8), respectively.

Proof. The results follow from Corollary 5.4.3 and Theorem 5.4.4 directly. □

Remark 5.4.6. If we specify $(c_1, c_2) = (1, 0)$ in Corollary 5.4.5, then we have the dynamically optimal strategy under the Heston model without mispricing; if we choose $(c_1, c_2) = (0, 1)$ instead, then the results in Corollary 5.4.5 correspond to the ones under the 3/2 model without mispricing.

5.5 Numerical examples

This section presents some numerical results to illustrate the theoretical results derived in the previous section. Throughout this section, unless stated otherwise, we consider the following market parameter setting adapted from Cheng and Escobar (2021a) and Ma, Zhao, and Rong (2020): $\kappa = 7.3479, \theta_v = 0.0328, \sigma_v = 0.6612, c_1 = 0.9051, c_2 = 0.0023, \lambda = 2.9428, \rho = -0.7689, r = 0.05, \sigma = 0.3, b = 0.3, l_1 = 0.1, l_2 = 0.2, \beta = 1.1, x_0 = 1, \xi = 3, v_0 = 0.02, m_0 = 0.04, T = 1$.

Figures 5.1 and 5.2 below display the effects of r and λ on the efficient frontier, respectively. As a matter of fact, when the interest rate r increases, the investor can obtain more expected return by investing in the risk-free asset, and thus undertaking less risk. Meanwhile, from the economic implications of λ, the investor can obtain a higher risk premium of W^1 as λ increases. This leads to a lower value of $\mathrm{Var}_{t_0,v_0,m_0,x_0}(X_T^*)$ when the same $\mathrm{E}_{t_0,v_0,m_0,x_0}[X_T^*]$ is asked for.

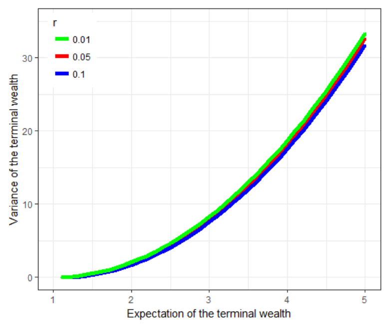


Figure 5.1: *Effects of r on the efficient frontier.*

Figure 5.3 contributes to the evolution of the efficient frontier with respect to l_1. When we vary l_1 from 0.1 to 0.5, the efficient frontier moves downwards. One possible explanation is that since l_1 partially characterizes the liquidity term, then as l_1 increases, the pricing error M_t in (5.2.3) has a faster mean-reversion rate towards the long-term zero such that the investor can bear less risk coming out of the pricing error between S_1 and S_2.

We finally give a simulation experiment to illustrate the difference between the dynamics of X^* and X^{d*}. As shown in Figure 5.4, two optimal wealth processes have significantly different trajectories while using the same random numbers. Particularly, we observe that the dynamically optimal wealth process X^{d*} is strictly below the expected terminal wealth $\xi = 3$ when $t < T = 1$ in this case, which is consistent with the conclusion derived in Theorem 5.4.4 above.

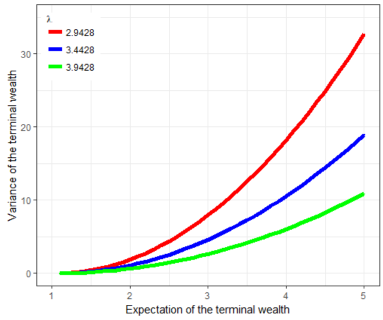


Figure 5.2: *Effects of* λ *on the efficient frontier.*

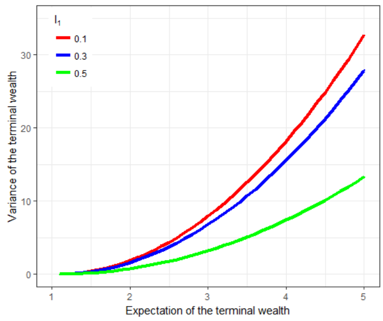


Figure 5.3: *Effects of* l_1 *on the efficient frontier.*

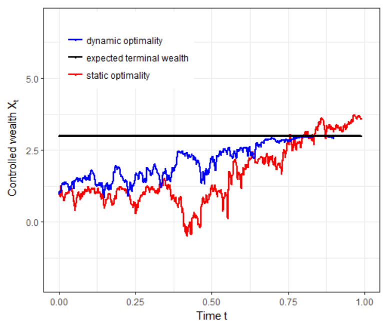


Figure 5.4: *Trajectories of static and dynamic optimality.*

5.6 Conclusions

In this paper, we consider an optimal investment problem with mispricing in the family of 4/2 stochastic volatility models (Grasselli (2017)) which embraces the 3/2 and the Heston models as special cases under Markowitz's mean–variance criterion.

By applying the dynamic programming approach and establishing the corresponding HJB equation, we derive the closed-form expressions of the statically optimal (pre-commitment) strategy and the optimal value function. A verification theorem is further provided from scratch to ensure that the candidate solution to the HJB equation coincides with the optimal value function and that the optimal strategy is admissible. By recomputing the statically optimal strategy in an infinitesimally small period of time, we explicitly obtain the dynamically optimal (time-consistent) strategy (Pedersen and Peskir (2017)). Moreover, some results on special cases, such as that without mispricing and that under the 3/2 and Heston models, are included. Finally, some numerical examples are presented to illustrate our results. To the best of our knowledge, there is no existing literature on the mean-variance problem with the new influential 4/2 stochastic volatility model and mispricing taken into consideration simultaneously.

Based on our current work, several potential topics in the future may be followed; for example, one may incorporate the stochastic interest rate into the model. One may also introduce random liabilities into the mean–variance problem.

Acknowledgment(s)

We are grateful to Prof. Jesper Lund Pedersen for his valuable suggestions.

Chapter 6

Utility maximization in a stochastic affine interest rate and CIR risk premium framework: a BSDE approach

Abstract

This paper investigates optimal investment problems in the presence of stochastic interest rates and stochastic volatility under the expected utility maximization criterion. The financial market consists of three assets: a risk-free asset, a risky asset, and zero-coupon bonds (rolling bonds). The short interest rate is assumed to follow an affine diffusion process, which includes the Vasicek and the Cox-Ingersoll-Ross (CIR) models, as special cases. The risk premium of the risky asset depends on a square-root diffusion (CIR) process, while the return rate and volatility coefficient are unspecified and possibly given by non-Markovian processes. This framework embraces the family of state-of-the-art 4/2 stochastic volatility models and some non-Markovian models, as exceptional examples. The investor aims to maximize the expected utility of the terminal wealth for two types of utility functions, power utility, and logarithmic utility. By adopting a backward stochastic differential equation (BSDE) approach to overcome the potentially non-Markovian framework and solving two BSDEs explicitly, we derive, in closed form, the optimal investment strategies and optimal value functions. Furthermore, explicit solutions to some special cases of our model are provided. Finally, numerical examples illustrate our results under one specific case, the hybrid Vasicek-4/2 model.

Keywords: Affine diffusion process; CIR risk premium; Power utility; Logarithmic utility; Backward stochastic differential equation

6.1 Introduction

Continuous-time portfolio optimization under Merton (1969)'s utility-maximization criterion is one of the central topics in mathematical finance. In the classical Merton's model, all the market coefficients are assumed to be deterministic or constants. However, this oversimplified assumption is not consistent with many phenomena observed in the financial market, such as fat tails of return distribution and the volatility smile on implied volatility surfaces. To better predict the dynamics of implied volatility surfaces, different types of stochastic (local) volatility models have been proposed in the last several decades; see, for example, the constant variance of elasticity model, Heston model (Heston (1993)), 3/2 model (Lewis (2000)). This leads to growing interest in extending Merton's seminal work to cases in a stochastic volatility environment. For instance, under a specific condition on the model parameters, Kraft (2005) provided an explicit solution to a power utility maximization problem under the Heston model. Chacko and Viceira (2005) investigated a consumption and investment problem with an infinite horizon under the 3/2 model. To extend Kraft's work (Kraft (2005)), Zeng and Taksar (2013) further studied an optimal portfolio selection problem for a broad class of stochastic volatility models, and closed-form solutions for the Heston model were obtained under more relaxed assumptions. Recently, a state-of-the-art 4/2 stochastic volatility model was introduced in Grasselli (2017), which recovers two parsimonious models, the Heston model and 3/2 model, as particular cases. This new influential model draws the attention of many scholars in the field of derivatives pricing; see, for example, Cui, Kirkby, and Nguyen (2017), Lin et al. (2017), and Zhu, Cao, and Zhang (2019). By using the dynamic programming approach and solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation, Cheng and Escobar (2021a) derived analytical expressions for the optimal solutions for a power utility maximization problem under the 4/2 model. For other related work concerning utility maximization problems with stochastic volatility, one can refer to Liu (2007), Kallsen and Muhle-Karbe (2010), Pan, Hu, and Zhou (2019), and references therein.

Although stochastic volatility has been considered in the aforementioned literature, most were studied on the preconditions that interest rates are either constant or deterministic functions. However, it is generally accepted in the literature that interest rates are stochastic (see, for example, Duffie and Kan (1996)) and can be described by some specific Markovian models, such as the Vasicek model (Vasicek (1977)) and Cox-Ingersoll-Ross (CIR) model (Cox, Ingersoll, and Ross (1985)). Some research outputs on portfolio optimization problems under Merton's utility-maximization criterion with stochastic interest rates have been achieved in recent years. For example, by assuming that the stock price and volatility are perfectly correlated, Li and Wu (2009) investigated power utility maximization problems in a CIR interest rate and Heston's stochastic volatility framework. Chang and Rong

(2013) extended the results of Li and Wu (2009) by further considering the optimal consumption. Assuming that the interest rate is driven by an affine process and the stock price is characterized by the Heston model, Guan and Liang (2014) studied DC pension management problems for the power utility. Considering the same market model as Guan and Liang (2014), Chang and Li (2016) studied optimal investment-consumption problems for the power and logarithmic utility functions. Chang et al. (2020) investigated DC pension management problems for a hyperbolic absolute risk averse (HARA) type utility function when an affine model drives the interest rate, and a mean-reverting process governs the stock's return rate. As the literature on utility maximization problems with stochastic interest rates is abundant, the above review is not exhaustive. For other relevant works, one can refer to Korn and Kraft (2002), Deelstra, Grasselli, and Koehl (2003), Shen and Siu (2012), Escobar, Neykova, and Zagst (2017), to name but only a few.

In the literature concerning utility maximization problems, three approaches have been extensively applied, the dynamic programming approach, the martingale approach, and the backward stochastic differential equation (BSDE) approach. The dynamic programming approach characterizes the optimal value function by the induced HJB equation. Once the HJB equation can be solved in the viscosity sense along with a necessary verification theorem, the candidate solution to the HJB equation then coincides with the optimal value function. This approach, however, requires a Markovian framework. One may refer to the monographs Fleming and Soner (2006) and Pham (2009) for a detailed exposition of this approach. Compared with the dynamic programming approach, the martingale approach (refer to Pliska (1986) and Karatzas, Lehoczky, and Shreve (1987)) does not entail the Markovian structures of state variable processes. This approach, however, falls apart in an incomplete market setting. Mathematically speaking, the martingale approach essentially hinges on the uniqueness of the risk-neutral measure (market completeness) and the martingale representation theorem for determining the attainable optimal terminal wealth and the associated replication strategy, respectively. To overcome the problem of incompleteness, Karatzas et al. (1991) proposed the fictitious completion method by introducing additional fictitious assets into the original incomplete market and making them unfavorable to the investor. Nevertheless, finding such fictitious assets is not straightforward and might be computationally intensive. Alternatively, Hu, Imkeller, and Müller (2005) introduced a BSDE approach to utility maximization problems in an incomplete and non-Markovian market setting. In contrast to the martingale approach, the BSDE approach addresses the primal problem directly rather than the dual problem. The idea of the BSDE approach is to construct a stochastic process depending on the investment strategy such that it coincides with the utility of the investor's wealth at the terminal date. For every admissible strategy, this stochastic process is a (local) super-martingale, while there exists one particular strategy such that it is a (local) martingale. It is worth noting that, however, most of the literature adopting the BSDE approach to address utility maximization problems (see, for example, Hu, Imkeller, and Müller (2005), Cheridito and Hu (2011), and Huang, Wang, and Wu (2020)) assumed that the short interest rates, the risky asset's return rates, and volatility coefficients are all uniformly bounded processes and only concentrated on the existence and uniqueness of solutions to the corresponding BSDEs rather than the explicit solutions.

In this paper, we proceed to study utility maximization problems in a stochastic interest rate and stochastic volatility environment. We consider a risk-averse investor with two types of utility functions, power utility and logarithmic utility. Apart from a risk-free asset (money account) and a risky asset (stock), zero-coupon bonds are available in the financial market to hedge against interest rate risk. The stochastic interest rate is assumed to follow an affine diffusion process, which recovers the Vasicek model and CIR model, as exceptional cases. Unlike most of the preceding literature on utility maximization problems, the risky asset's return rate and volatility coefficient are not specified and might be unbounded, non-Markovian processes in our model. We only assume that the risk premium of the volatility risk relies on a CIR process, which embraces the family of 4/2 stochastic volatility models and some non-Markovian models (refer to Siu (2012)), as particular cases. Because of the potentially non-Markovian and incomplete market setting, we solve the problem by adopting a BSDE approach. For more details on the theory and applications of BSDEs, one may refer to El Karoui, Peng, and Quenez (1997), Kobylanski (2000), Briand and Hu (2008), Zhang (2017), Shen and Zeng (2015), Sun, Zhang, and Yuen (2020), and references therein. To find the associated BSDEs, we consider the canonical decomposition of semi-martingales with continuous sample paths and use the completion method of squares. Explicit expressions for both the optimal investment strategies and optimal value functions are derived from the unique solutions to the BSDEs. Moreover, we provide the analytical results for several special cases of our model. Finally, we concentrate on the effects of model parameters on the behavior of the optimal investment strategy by numerical analysis. In summary, the main contribution of this paper are: (1) We incorporate stochastic volatility and stochastic interest rates into utility maximization problems simultaneously, which extends the results of Kraft (2005), Zeng and Taksar (2013), and Cheng and Escobar (2021a) to a more general non-Markovian framework. (2) We derive explicit expressions for the optimal strategies and optimal value functions by presenting closed-form solutions to the corresponding BSDEs instead of only considering the existence and unique results. (3) We give some numerical studies with the hybrid Vasicek-4/2 model as the working example to illustrate our results.

The remainder of this paper is structured as follows. In Section 6.2, we introduce the financial market and formulate two utility maximization problems. In Section 6.3, we consider the power utility case, and closed-form expressions for the optimal strategy and optimal value function are obtained by solving a non-linear BSDE. Section 6.4 contributes to addressing the logarithmic utility case. Section 6.5 provides numerical studies to illustrate our results under a specific case. Section 6.6 concludes the paper.

6.2 Model formulation

This section will formulate the market model and two portfolio optimization problems.

Let $[0,T]$ be a fixed and finite horizon for decision-making. We fix a complete probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where three one-dimensional, mutually independent standard Brownian motions $\left\{W_t^0\right\}_{t\in[0,T]}, \left\{W_t^1\right\}_{t\in[0,T]}, \left\{W_t^2\right\}_{t\in[0,T]}$ are carried, $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ is the $\mathbb{P}$-augmentation of the filtration generated by the above three Brownian motions, and $\mathbb{P}$ is a real-world probability measure. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}[\cdot]$. To facilitate the discussions throughout the rest of the paper, we make use of the following notations:

- $\mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all $\mathbb{R}$-valued, $\mathbb{F}$-adapted processes with $\mathbb{P}$-a.s. continuous sample paths;
- $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all $\mathbb{R}$-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{P}\left(\int_0^T |f_t|^2\,dt<\infty\right)=1$;
- $\mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all $\mathbb{R}$-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}\left[\sup_{0\le t\le T}|f_t|^2\right]<\infty$;
- $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all $\mathbb{R}$-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}\left[\left(\int_0^T |f_t|^2\,dt\right)\right]<\infty$.

We consider a financial market that comprises a risk-free asset (money account), a risky asset (stock), and a zero-coupon bond. The price process of the risk-free asset S_t^0 evolves according to

$$dS_t^0=r_tS_t^0\,dt,\;\; S_0^0=1,$$

where the short interest rate r_t is assumed to follow an affine diffusion process (refer to Duffie and Kan (1996)):

$$dr_t=(a-br_t)\,dt-\sqrt{\eta_1 r_t+\eta_2}\,dW_t^0, \tag{6.2.1}$$

with initial value r_0 at time zero, where $a\in\mathbb{R}^+, b\in\mathbb{R}^+$, and η_1,η_2 are two non-negative constants.

Remark 6.2.1. When $\eta_1 \neq 0$, a direct application of Itô's formula shows that $\eta_1 r_t + \eta_2$ follows a Cox-Ingersoll-Ross (CIR) process:

$$d(\eta_1 r_t + \eta_2) = [\eta_1 a + \eta_2 b - b(\eta_1 r_t + \eta_2)]\, dt - \eta_1 \sqrt{\eta_1 r_t + \eta_2}\, dW_t^0, \tag{6.2.2}$$

with mean-reverting rate b, long-run mean $(\eta_1 a + \eta_2 b)/b$, and volatility η_1. It follows from Cox, Ingersoll, and Ross (1985) that $\eta_1 r_t + \eta_2 > 0$, $\mathbb{P}$ almost surely for $t \in [0, T]$ when the Feller condition $\frac{\eta_2}{\eta_1} b + a > \frac{\eta_1}{2}$ is satisfied. For the special case when $\eta_1 \neq 0$ and $\eta_2 = 0$, (6.2.1) is reduced to the CIR model (Cox, Ingersoll, and Ross (1985)), and the Feller condition $2a > \eta_1$ is required such that $r_t > 0$, $\mathbb{P}$ almost surely for $t \in [0, T]$. The specification $\eta_1 = 0$ is known as the Vasicek model (Vasicek (1977)).

Denote by $B_t(u)$ the price of the zero-coupon bond with maturity u at time t, and assume that the market price of interest rate risk is $\lambda_r \sqrt{\eta_1 r_t + \eta_2}$, where $\lambda_r \in \mathbb{R} \setminus \{0\}$. It follows from (6.2.2) and the exponentially affine term structure of the CIR model that the bond price $B_t(u)$ is given by (see also Proposition 1 and Lemma 2 in Deelstra, Grasselli, and Koehl (2003))

$$B_t(u) = h_1(u-t) e^{-h_0(u-t) r_t}, \text{ for } t \leq u,$$

with functions $h_0(t)$ and $h_1(t)$ given by

$$\begin{cases} h_0(t) = \dfrac{2(e^{mt} - 1)}{m - (b - \eta_1 \lambda_r) + e^{mt}(m + b - \eta_1 \lambda_r)}, \\ h_1(t) = \exp\left\{ \dfrac{\eta_2}{\eta_1} t - \dfrac{2(\eta_1 a + \eta_2 b)}{\eta_1^2} \log\left(\dfrac{2m e^{\frac{m + b - \eta_1 \lambda_r}{2} t}}{m - (b - \eta_1 \lambda_r) + e^{mt}(m + b - \eta_1 \lambda_r)} \right) \right. \\ \qquad\qquad \left. - \dfrac{\eta_2}{\eta_1} h_0(t) \right\}, \end{cases}$$

where $m = \sqrt{(b - \eta_1 \lambda_r)^2 + 2\eta_1}$. Then, a direct application of Itô's formula to $B_t(u)$ leads to the following stochastic differential equation (SDE):

$$dB_t(u) = r_t B_t(u)\, dt + h_0(u-t) B_t(u) \sqrt{\eta_1 r_t + \eta_2} \left(\lambda_r \sqrt{\eta_1 r_t + \eta_2}\, dt + dW_t^0 \right), \tag{6.2.3}$$

with the terminal condition $B_u(u) = 1$. It is noteworthy that the maturity $u - t$ of the zero-coupon bond $B_t(u)$ varies continuously over time. As suggested by Boulier, Huang, and Taillard (2001), it is quite unlikely to find all of the zero-coupon bonds in the market. To amend the drawback, we follow Boulier, Huang, and Taillard (2001) to introduce a rolling bond with constant maturity K into the market. Denote the price of the rolling bond at time t by $B_t(K)$, then $B_t(K)$ is supposed to satisfy the following SDE (refer to Boulier, Huang, and Taillard (2001)):

$$dB_t(K) = r_t B_t(K)\, dt + h_0(K) B_t(K) \sqrt{\eta_1 r_t + \eta_2} \left(\lambda_r \sqrt{\eta_1 r_t + \eta_2}\, dt + dW_t^0 \right). \tag{6.2.4}$$

The price process S_t^1 of the risky asset is assumed to evolve according to

$$\begin{cases} dS_t^1 = \mu_t S_t^1 \, dt + \sigma_t S_t^1 \, dW_t^1, \\ d\alpha_t = \kappa(\theta - \alpha_t) \, dt + \sigma_\alpha \sqrt{\alpha_t} \left(\rho \, dW_t^1 + \sqrt{1-\rho^2} \, dW_t^2 \right), \end{cases} \tag{6.2.5}$$

with initial value $s_0^1 \in \mathbb{R}^+$ and $\alpha_0 \in \mathbb{R}^+$ at time zero, where μ_t and $\sigma_t \neq 0$ are two $\mathbb{F}$-adapted processes standing for the risky asset's return rate and volatility coefficient at time t, respectively, and the affine form, square-root factor (CIR) process α_t is related to the price process of the risky asset S_t^1 via the following specification of the market price of volatility risk:

$$\frac{\mu_t - r_t}{\sigma_t} = \lambda \sqrt{\alpha_t}, \ \ \lambda \in \mathbb{R} \setminus \{0\} .$$

In (6.2.5), $\kappa \in \mathbb{R}^+$ is the mean-reverting rate, $\theta \in \mathbb{R}^+$ is the long-run mean, and $\sigma_\alpha \in \mathbb{R}^+$ is the volatility of the CIR process, respectively. Particularly, it is required that the Feller condition $2\kappa\theta > \sigma_\alpha^2$ holds such that the CIR process $\alpha_t > 0$, $\mathbb{P}$ almost surely over $[0, T]$. The correlation coefficient ρ between the risky asset price and the stochastic factor lies in $[-1, 1]$.

Remark 6.2.2. It should be noted that the risky asset's return rate μ_t and volatility coefficient α_t given in (6.2.5) are not specified and may be unbounded and non-Markovian processes, which reflects the generalization of the above modeling framework. We shall see below that this modeling framework includes, but is not limited to, the family of 4/2 stochastic volatility models (Grasselli (2017)) and some non-Markovian models as special examples.

Example 6.2.3. (The 4/2 model). If we set $\mu_t = r_t + \lambda(c_1\alpha_t + c_2)$ and $\sigma_t = c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$, where $c_1 \geq 0$ and $c_2 \geq 0$, then the risky asset S_t^1 corresponds to the 4/2 model:

$$\begin{cases} dS_t^1 = S_t^1 \left[(r_t + \lambda(c_1\alpha_t + c_2)) \, dt + \left(c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}} \right) dW_t^1 \right], \\ d\alpha_t = \kappa(\theta - \alpha_t) \, dt + \sigma_\alpha \sqrt{\alpha_t} \left(\rho \, dW_t^1 + \sqrt{1-\rho^2} \, dW_t^2 \right). \end{cases} \tag{6.2.6}$$

In this case, α_t is the variance driver of the instantaneous volatility $c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$ of the risky asset price; κ, θ, and σ_α are the mean-reverting speed, the long-run average level, and the volatility of the variance driver, respectively.

Remark 6.2.4. The specification $(c_1, c_2) = (1, 0)$ in (6.2.6) corresponds to the Heston model (Heston (1993)), and the case $(c_1, c_2) = (0, 1)$ is known as the 3/2 model (Lewis (2000)).

Example 6.2.5. (A path-dependent model). If we set $\mu_t = r_t + \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]})$ and $\sigma_t = \hat{\sigma}(\alpha_{[0,t]})$ for some functional $\hat{\sigma} : C(0, t; \ \mathbb{R}) \to \mathbb{R}^+$, where $\alpha_{[0,t]} := (\alpha_s)_{s\in[0,t]}$ is the restriction of $\alpha \in C(0, T; \ \mathbb{R})$ to $C(0, t; \ \mathbb{R})$, i.e. the space of $\mathbb{R}$-valued, continuous functions defined on $[0,t]$. In this case, the risky asset is governed by a path-dependent model:

$$\begin{cases} dS_t^1 = S_t^1 \left[\left(r_t + \lambda\sqrt{\alpha_t}\hat{\sigma}\left(\alpha_{[0,t]}\right)\right) dt + \hat{\sigma}\left(\alpha_{[0,t]}\right) dW_t^1 \right], \\ d\alpha_t = \kappa(\theta-\alpha_t)\, dt + \sigma_\alpha\sqrt{\alpha_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right). \end{cases} \tag{6.2.7}$$

For more details on model (6.2.7), readers may refer to Siu (2012).

Let π_t^B and $\pi_t^{S^1}$ denote the proportions of wealth invested in the rolling bond $B_t(K)$ and the risky asset S_t^1 at time t, respectively. The two-dimensional process $\pi := \left(\left\{\pi_t^{S^1}\right\}_{t\in[0,T]}, \left\{\pi_t^B\right\}_{t\in[0,T]}\right)$ represents the investment strategy. Let X_t^π be the wealth process associated with π. Suppose that the financial market is frictionless and infinite short-selling and leverage are allowed. Under a self-financing condition, the controlled wealth process X_t^π is described by the following SDE:

$$\begin{aligned} dX_t^\pi = &\left(\pi_t^B h_0(K)\lambda_r(\eta_1 r_t+\eta_2) + \pi_t^{S^1}\sigma_t\lambda\sqrt{\alpha_t} + r_t\right) X_t^\pi\, dt + X_t^\pi \pi_t^{S^1}\sigma_t\, dW_t^1 \\ &+ X_t^\pi\pi_t^B h_0(K)\sqrt{\eta_1 r_t+\eta_2}\, dW_t^0,\ X_0^\pi = x_0\in\mathbb{R}^+. \end{aligned} \tag{6.2.8}$$

In this paper, we consider two utility maximization problems when the risk preferences of the investor are characterized by a power utility function $U_1(x)=\gamma^{-1}x^\gamma$, where $\gamma\in(0,1)$ and $x\in\mathbb{R}^+$, and a logarithmic utility function $U_2(x)=\log(x),\ x\in\mathbb{R}^+$, respectively. To this end, we shall present the formal definitions of admissible strategies for these two problems.

Definition 6.2.6. *(Admissible strategy). Consider the power utility function $U_1(\cdot)$. An investment strategy π is said to be admissible if*

1. *$\pi_t^B \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ and $\sigma_t\pi_t^{S^1}\in\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$;*
2. *for any initial data $(\alpha_0,r_0,x_0)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ fixed and given, the SDE (6.2.8) admits a unique solution $X_t^\pi\in\mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ satisfying $X_t^\pi>0$, $\mathbb{P}$-a.s. for all $t\in[0,T]$.*

The set of admissible strategies is denoted as $\mathcal{A}_p$.

In this case, the control problem corresponds to finding an admissible strategy $\pi\in\mathcal{A}_p$ such that the expected utility derived from the terminal wealth X_T^π is maximized, i.e.,

$$\sup_{\pi\in\mathcal{A}_p}\mathbb{E}\left[U_1\left(X_T^\pi\right)\right]. \tag{6.2.9}$$

We denote the corresponding optimal value function by $V_p(\alpha_0,r_0,x_0)$.

Definition 6.2.7. *(Admissible strategy). Consider the logarithmic utility function $U_2(\cdot)$. An investment strategy π is said to be admissible if*

1. $\pi_t^B \sqrt{\eta_1 r_t + \eta_2} \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ *and* $\pi_t^{S^1} \sigma_t \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$;
2. *for any initial data* $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ *fixed and given, the SDE (6.2.8) admits a unique solution* $X_t^\pi \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ *such that* $X_t^\pi > 0$, $\mathbb{P}$*-a.s. for all* $t \in [0,T]$.

We denote the set of admissible strategies by $\mathcal{A}_l$.

In this case, the investor aims to solve the following optimization problem by opting for an admissible strategy $\pi \in \mathcal{A}_l$:

$$\sup_{\pi \in \mathcal{A}_l} \mathbb{E}\left[U_2\left(X_T^\pi\right)\right]. \tag{6.2.10}$$

The corresponding optimal value function is denoted by $V_l(\alpha_0, r_0, x_0)$.

6.3 Solution to the power utility case

In this section, we solve the power utility maximization problem (6.2.9) and derive the corresponding optimal investment strategy by applying a BSDE approach. To find the BSDE associated with problem (6.2.9), we introduce a continuous $(\mathbb{F},\mathbb{P})$-semi-martingale Y_t with canonical decomposition:

$$dY_t = \Psi_t\, dt + \Gamma_{0,t}\, dW_t^0 + \Gamma_{1,t}\, dW_t^1 + \Gamma_{2,t}\, dW_t^2,$$

where Ψ_t is an $\mathbb{F}$-adapted process that shall be determined in the sequel, and $\Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t} \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. For any admissible strategy $\pi \in \mathcal{A}_p$, applying Itô's formula to $\frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t}$ and using the method of completion of squares, we have

$$\begin{aligned}
& d\left(\frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t}\right) \\
=& \frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t} \left(\gamma h_0(K)\sqrt{\eta_1 r_t + \eta_2}\pi_t^B + \Gamma_{0,t}\right) dW_t^0 + \frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t}\left(\gamma\sigma_t\pi_t^{S^1} + \Gamma_{1,t}\right) dW_t^1 \\
& + \frac{(X_t^\pi)^\gamma e^{Y_t}}{2(\gamma-1)} \left\{\left[\Gamma_{1,t} + \lambda\sqrt{\alpha_t} + (\gamma-1)\sigma_t\pi_t^{S^1}\right]^2 + \left[\Gamma_{0,t} + \lambda_r\sqrt{\eta_1 r_t + \eta_2}\right.\right. \\
& \left.\left. + (\gamma-1)h_0(K)\sqrt{\eta_1 r_t + \eta_2}\pi_t^B\right]^2\right\} dt + \frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t}\left[\Psi_t + \frac{\Gamma_{0,t}^2}{2} + \frac{\Gamma_{1,t}^2}{2} + \frac{\Gamma_{2,t}^2}{2} + \gamma r_t\right. \\
& \left. - \frac{\gamma(\lambda\sqrt{\alpha_t} + \Gamma_{1,t})^2}{2(\gamma-1)} - \frac{\gamma(\lambda_r\sqrt{\eta_1 r_t + \eta_2} + \Gamma_{0,t})^2}{2(\gamma-1)}\right] dt.
\end{aligned} \tag{6.3.1}$$

Inspired by the above equation, we can choose Ψ_t such that the last term on the right-hand side of (6.3.1) turns out to be zero, which, in turn, leads to the following nonlinear BSDE of $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ we shall investigate:

$$\begin{cases} dY_t = \Big[-\gamma r_t + \dfrac{\Gamma_{0,t}^2 + 2\gamma\lambda_r\sqrt{\eta_1 r_t + \eta_2}\Gamma_{0,t} + \gamma\lambda_r^2(\eta_1 r_t + \eta_2)}{2(\gamma-1)} \\ \qquad + \dfrac{\Gamma_{1,t}^2 + 2\gamma\lambda\sqrt{\alpha_t}\Gamma_{1,t} + \gamma\lambda^2\alpha_t}{2(\gamma-1)} - \dfrac{\Gamma_{2,t}^2}{2} \Big] dt + \Gamma_{0,t}\, dW_t^0 + \Gamma_{1,t}\, dW_t^1 + \Gamma_{2,t}\, dW_t^2, \\ Y_T = 0. \end{cases} \tag{6.3.2}$$

Remark 6.3.1. The construction for the stochastic process $e^{Y_t}(X_t^\pi)^\gamma/\gamma$ is enlightened by some existing results on portfolio optimization problems with power utility functions in Markovian settings (see, for example, Kraft (2005), Zeng and Taksar (2013), Guan and Liang (2014), and Cheng and Escobar (2021a)).

Remark 6.3.2. It is worth mentioning that the driver of BSDE (6.3.2) depends on the market prices of interest rate and volatility risks, i.e., $\lambda_r\sqrt{\eta_1 r_t + \eta_2}$ and $\lambda\sqrt{\alpha_t}$ rather than the risky asset's return rate μ_t and volatility σ_t. This means that the solvability of BSDE (6.3.2) is completely determined by these two risk premium processes, and it is, therefore, irrelevant exactly how to specify μ_t and σ_t per se. However, due to the randomness and unboundedness of r_t and α_t within the driver of BSDE (6.3.2), condition (H1) in Kobylanski (2000) and Assumption (A.2) in Briand and Hu (2008) are violated so that the standard existence and uniqueness results of quadratic BSDEs are not applicable directly in this case. Nevertheless, upon considering the Markovian structures of two risk premium processes, we manage to derive an explicit solution in the next lemma.

Lemma 6.3.3. *One solution $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ to BSDE (6.3.2) is given by*

$$\begin{cases} Y_t = f_1(t) + f_2(t) r_t + f_3(t)\alpha_t, \\ \Gamma_{0,t} = -\sqrt{\eta_1 r_t + \eta_2} f_2(t), \\ \Gamma_{1,t} = \rho\sigma_\alpha\sqrt{\alpha_t} f_3(t), \\ \Gamma_{2,t} = \sqrt{1-\rho^2}\sigma_\alpha\sqrt{\alpha_t} f_3(t), \end{cases} \tag{6.3.3}$$

where functions $f_1(t), f_2(t)$, and $f_3(t)$ are solutions to the following ordinary differential equation (ODE) system:

$$\begin{cases} \dfrac{df_1(t)}{dt} + \left(a + \lambda_r\eta_2\dfrac{\gamma}{\gamma-1}\right) f_2(t) + \kappa\theta f_3(t) - \dfrac{\eta_2}{2(\gamma-1)} f_2^2(t) - \dfrac{\lambda_r^2\gamma\eta_2}{2(\gamma-1)} = 0, \\ \dfrac{df_2(t)}{dt} - \dfrac{\eta_1}{2(\gamma-1)} f_2^2(t) - \left(b - \lambda_r\eta_1\dfrac{\gamma}{\gamma-1}\right) f_2(t) + \gamma - \dfrac{\lambda_r^2\gamma\eta_1}{2(\gamma-1)} = 0, \\ \dfrac{df_3(t)}{dt} + \dfrac{\sigma_\alpha^2}{2}\left(1 - \rho^2\dfrac{\gamma}{\gamma-1}\right) f_3^2(t) - \left(\kappa + \rho\lambda\sigma_\alpha\dfrac{\gamma}{\gamma-1}\right) f_3(t) - \dfrac{\lambda^2\gamma}{2(\gamma-1)} = 0, \end{cases} \tag{6.3.4}$$

with boundary condition $f_1(T) = f_2(T) = f_3(T) = 0$.

Proof. We conjecture that the first component Y_t of one solution to BSDE (6.3.2) admits an affine form:

$$Y_t = f_1(t) + f_2(t)r_t + f_3(t)\alpha_t,$$

where $f_1(t), f_2(t)$, and $f_3(t)$ are three undetermined functions with boundary conditions $f_1(T) = f_2(T) = f_3(T) = 0$. Applying Itô's formula to Y_t, we have

$$\begin{aligned} dY_t =& d(f_1(t) + f_2(t)r_t + f_3(t)\alpha_t) \\ =& \left(\frac{df_1(t)}{dt} + \frac{df_2(t)}{dt}r_t + \frac{df_3(t)}{dt}\alpha_t + (a - br_t)f_2(t) + \kappa(\theta - \alpha_t)f_3(t)\right) dt \\ & - \sqrt{\eta_1 r_t + \eta_2} f_2(t)\, dW_t^0 + \rho\sigma_\alpha\sqrt{\alpha_t} f_3(t)\, dW_t^1 + \sqrt{1-\rho^2}\sigma_\alpha\sqrt{\alpha_t} f_3(t)\, dW_t^2. \end{aligned} \tag{6.3.5}$$

By comparing the diffusion coefficients of BSDE (6.3.2) and (6.3.5), we observe that $\Gamma_{0,t} = -\sqrt{\eta_1 r_t + \eta_2} f_2(t), \Gamma_{1,t} = \rho\sigma_\alpha\sqrt{\alpha_t} f_3(t)$, and $\Gamma_{2,t} = \sqrt{1-\rho^2}\sigma_\alpha\sqrt{\alpha_t} f_3(t)$. Moreover, the generator of BSDE (6.3.2) turns out to be

$$\begin{aligned} &\left[\frac{\eta_1 f_2^2(t) - 2\gamma\lambda_r\eta_1 f_2(t) + \lambda_r^2\gamma\eta_1}{2(\gamma-1)} - \gamma\right] r_t + \left[\frac{\rho^2\sigma_\alpha^2 f_3^2(t) + 2\gamma\lambda\rho\sigma_\alpha f_3(t) + \lambda^2\gamma}{2(\gamma-1)}\right. \\ &\left. - \frac{(1-\rho^2)\sigma_\alpha^2 f_3^2(t)}{2}\right] \alpha_t + \frac{\eta_2 f_2^2(t) - 2\gamma\lambda_r\eta_2 f_2(t) + \lambda_r^2\gamma\eta_2}{2(\gamma-1)}. \end{aligned} \tag{6.3.6}$$

Then comparing the drift coefficient of (6.3.5) and (6.3.6) and separating the dependence on r_t and α_t, we see that $f_1(t), f_2(t)$, and $f_3(t)$ must be governed by the ODE system (6.3.4). □

The next proposition provides the explicit expressions for functions $f_1(t), f_2(t)$, and $f_3(t)$.

Proposition 6.3.4. *Explicit solutions to ODE system (6.3.4) are given by*

$$f_1(t) = \int_t^T \left\{ \left(a + \lambda_r\eta_2\frac{\gamma}{\gamma-1}\right) f_2(s) + \kappa\theta f_3(s) - \frac{\eta_2}{2(\gamma-1)} f_2^2(s) - \frac{\lambda_r^2\gamma\eta_2}{2(\gamma-1)} \right\} ds, \tag{6.3.7}$$

$$
f_2(t) = \begin{cases} \dfrac{\gamma}{b}\left(1 - e^{b(t-T)}\right), \ \textit{if } \eta_1 = 0; \\ \dfrac{n_1 n_2 \left(1 - e^{\sqrt{\Delta_r}(T-t)}\right)}{n_1 - n_2 e^{\sqrt{\Delta_r}(T-t)}}, \ \textit{if } \eta_1 \neq 0 \ \textit{and } \Delta_r > 0; \\ \dfrac{\eta_1 (T-t) n_0^2}{\eta_1 (T-t) n_0 - 2(\gamma - 1)}, \ \textit{if } \eta_1 \neq 0 \ \textit{and } \Delta_r = 0; \\ \dfrac{\sqrt{-\Delta_r}(\gamma-1)}{\eta_1} \tan\left(\arctan\left(\dfrac{b(\gamma-1) - \lambda_r \eta_1 \gamma}{\sqrt{-\Delta_r}(\gamma-1)}\right) - \dfrac{\sqrt{-\Delta_r}}{2}(T-t)\right) \\ - \dfrac{b(\gamma-1) - \lambda_r \eta_1 \gamma}{\eta_1}, \ \textit{if } \eta_1 \neq 0 \ \textit{and } \Delta_r < 0, \end{cases} \tag{6.3.8}
$$

and

$$
f_3(t) = \begin{cases} \dfrac{l_1 l_2 \left(1 - e^{\sqrt{\Delta_\alpha}(T-t)}\right)}{l_1 - l_2 e^{\sqrt{\Delta_\alpha}(T-t)}}, \ \textit{if } \Delta_\alpha > 0; \\ \dfrac{\sigma_\alpha^2 \left(\rho^2 \frac{\gamma}{\gamma-1} - 1\right)(T-t) l_0^2}{\sigma_\alpha^2 \left(\rho^2 \frac{\gamma}{\gamma-1} - 1\right)(T-t) l_0 - 2}, \ \textit{if } \Delta_\alpha = 0; \\ \dfrac{\sqrt{-\Delta_\alpha}}{\sigma_\alpha^2 \left(\rho^2 \frac{\gamma}{\gamma-1} - 1\right)} \tan\left(\arctan\left(\dfrac{\kappa + \rho\lambda\sigma_\alpha \frac{\gamma}{\gamma-1}}{\sqrt{-\Delta_\alpha}}\right) - \dfrac{\sqrt{-\Delta_\alpha}}{2}(T-t)\right) \\ + l_0, \ \textit{if } \Delta_\alpha < 0, \end{cases} \tag{6.3.9}
$$

where $\Delta_r, n_0, n_1,$ *and* n_2 *are given by*

$$
\begin{cases} \Delta_r = b^2 + \dfrac{\gamma}{\gamma-1}(\lambda_r^2 \eta_1^2 + 2\eta_1 - 2\eta_1 \lambda_r b), \ n_0 = \dfrac{-(b(\gamma-1) - \lambda_r \eta_1 \gamma)}{\eta_1}, \\ n_1 = \dfrac{-(b(\gamma-1) - \lambda_r \eta_1 \gamma) + (\gamma-1)\sqrt{\Delta_r}}{\eta_1}, \ n_2 = \dfrac{-(b(\gamma-1) - \lambda_r \eta_1 \gamma) - (\gamma-1)\sqrt{\Delta_r}}{\eta_1}, \end{cases} \tag{6.3.10}
$$

and $\Delta_\alpha, l_0, l_1,$ *and* l_2 *are given by*

$$
\begin{cases} \Delta_\alpha = \left(\kappa + \rho\lambda\sigma_\alpha \dfrac{\gamma}{\gamma-1}\right)^2 + \left(1 - \rho^2 \dfrac{\gamma}{\gamma-1}\right)\dfrac{\lambda^2 \sigma_\alpha^2 \gamma}{\gamma-1}, \ l_0 = \dfrac{-\left(\kappa + \rho\lambda\sigma_\alpha \frac{\gamma}{\gamma-1}\right)}{\sigma_\alpha^2 \left(\rho^2 \frac{\gamma}{\gamma-1} - 1\right)}, \\ l_1 = \dfrac{-\left(\kappa + \rho\lambda\sigma_\alpha \frac{\gamma}{\gamma-1}\right) + \sqrt{\Delta_\alpha}}{\sigma_\alpha^2 \left(\rho^2 \frac{\gamma}{\gamma-1} - 1\right)}, \ l_2 = \dfrac{-\left(\kappa + \rho\lambda\sigma_\alpha \frac{\gamma}{\gamma-1}\right) - \sqrt{\Delta_\alpha}}{\sigma_\alpha^2 \left(\rho^2 \frac{\gamma}{\gamma-1} - 1\right)}. \end{cases} \tag{6.3.11}
$$

Proof. We first solve the equation of $f_2(t)$. When $\eta_1 = 0$, the Riccati equation of $f_2(t)$ is reduced to the following first-order linear equation:

$$
\frac{df_2(t)}{bf_2(t) - \gamma} = dt.
$$

Integrating both sides from t to T and noticing the boundary condition $f_2(T) = 0$ lead to

$$f_2(t) = \frac{\gamma}{b}\left(1 - e^{b(t-T)}\right).$$

When $\eta_1 \neq 0$, we set $\Delta_r = b^2 + \frac{\gamma}{\gamma-1}(\lambda_r^2\eta_1^2 + 2\eta_1 - 2\eta_1\lambda_r b)$. If $\Delta_r > 0$, we can reformulate the Riccati equation of $f_2(t)$ in (6.3.4) as follows:

$$\frac{df_2(t)}{dt} = \frac{\eta_1}{2(\gamma-1)}\left(f_2(t) - n_1\right)\left(f_2(t) - n_2\right), \tag{6.3.12}$$

where n_1 and n_2 are given in (6.3.10) above. After taking integration on both sides from t to T, we obtain

$$f_2(t) = \frac{n_1 n_2\left(1 - e^{\sqrt{\Delta_r}(T-t)}\right)}{n_1 - n_2 e^{\sqrt{\Delta_r}(T-t)}}.$$

If $\Delta_r = 0$, then (6.3.12) can be rewritten as follows

$$\frac{1}{(f_2(t) - n_0)^2}\,df_2(t) = \frac{\eta_1}{2(\gamma-1)}\,dt, \tag{6.3.13}$$

where n_0 is given in (6.3.10). Thus, implementing an integral calculation yields

$$f_2(t) = \frac{\eta_1(T-t)n_0^2}{\eta_1(T-t)n_0 - 2(\gamma-1)}.$$

If $\Delta_r < 0$, then the Riccati equation of $f_2(t)$ in (6.3.4) can be reformulated by

$$\frac{df_2(t)}{\left[\left(f_2(t) + \frac{b(\gamma-1)-\lambda_r\eta_1\gamma}{\eta_1}\right)^2 + \frac{(-\Delta_r)(\gamma-1)^2}{\eta_1^2}\right]} = \frac{\eta_1}{2(\gamma-1)}\,dt.$$

Doing an integral calculation with respect to t upon noticing the boundary condition $f_2(T) = 0$, we obtain

$$\begin{aligned} f_2(t) = &\frac{\sqrt{-\Delta_r}(\gamma-1)}{\eta_1}\tan\left(\arctan\left(\frac{b(\gamma-1)-\lambda_r\eta_1\gamma}{\sqrt{-\Delta_r}(\gamma-1)}\right) - \frac{\sqrt{-\Delta_r}}{2}(T-t)\right) \\ &- \frac{b(\gamma-1)-\lambda_r\eta_1\gamma}{\eta_1}. \end{aligned}$$

The derivation of $f_3(t)$ is similar to that of $f_2(t)$ above, so we omit it here. Finally, upon obtaining the explicit expressions for $f_2(t)$ and $f_3(t)$, a direct integral calculation to the first-order linear equation of $f_1(t)$ leads to (6.3.7). □

The next proposition shows that $f_2(t)$ and $f_3(t)$ are strictly decreasing functions over $[0, T]$. In other words, the maximum values of $f_2(t)$ and $f_3(t)$ are attained at time zero.

Proposition 6.3.5. *Functions $f_2(t)$ and $f_3(t)$ are monotonically decreasing over $[0,T]$.*

Proof. Differentiating $f_2(t)$ given in (6.3.8) with respect to time t leads to

$$\frac{df_2(t)}{dt} = \begin{cases} -\gamma e^{b(t-T)}, \text{ if } \eta_1 = 0; \\ \frac{2\gamma(\gamma-1)(\lambda_r^2 - 2(\gamma-1))\Delta_r e^{\sqrt{\Delta_r}(T-t)}}{\eta_1^2\left(n_1 - n_2 e^{\sqrt{\Delta_r}(T-t)}\right)^2}, \text{ if } \eta_1 \neq 0 \text{ and } \Delta_r > 0; \\ \frac{2(\gamma-1)\eta_1 n_0^2}{(\eta_1(T-t)n_0 - 2(\gamma-1))^2}, \text{ if } \eta_1 \neq 0 \text{ and } \Delta_r = 0; \\ \frac{(-\Delta_r)(\gamma-1)}{2\eta_1}\sec^2\left(\arctan\left(\frac{b(\gamma-1) - \lambda_r\eta_1\gamma}{\sqrt{-\Delta_r}(\gamma-1)}\right) - \frac{\sqrt{-\Delta_r}(T-t)}{2}\right), \text{ if } \eta_1 \neq 0 \text{ and } \Delta_r < 0. \end{cases}$$

Given that $\gamma \in (0,1)$, this result shows $\frac{df_2(t)}{dt} < 0$. Similarly, by doing a differentiation to $f_3(t)$ given in (6.3.9), we obtain

$$\frac{df_3(t)}{dt} = \begin{cases} \frac{2\gamma\Delta_\alpha e^{\sqrt{\Delta_\alpha}(T-t)}}{(\gamma-1)\sigma_\alpha^4\left(\rho^2\frac{\gamma}{\gamma-1} - 1\right)^2\left(l_1 - l_2 e^{\sqrt{\Delta_\alpha}(T-t)}\right)^2}, \text{ if } \Delta_\alpha > 0; \\ \frac{2\sigma_\alpha^2 l_0^2\left(\rho^2\frac{\gamma}{\gamma-1} - 1\right)}{\left(\sigma_\alpha^2\left(\rho^2\frac{\gamma}{\gamma-1} - 1\right)(T-t)l_0 - 2\right)^2}, \text{ if } \Delta_\alpha = 0; \\ \frac{-\Delta_\alpha}{2\sigma_\alpha^2\left(\rho^2\frac{\gamma}{\gamma-1} - 1\right)}\sec\left(\arctan\left(\frac{\kappa + \rho\lambda\sigma_\alpha\frac{\gamma}{\gamma-1}}{\sqrt{-\Delta_\alpha}}\right) - \frac{\sqrt{-\Delta_\alpha}}{2}\right), \text{ if } \Delta_\alpha < 0, \end{cases}$$

which implies that $\frac{df_3(t)}{dt} < 0$ by $\gamma \in (0,1)$. □

Lemma 6.3.6. *The solution $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ given by (6.3.3) lies in $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.*

Proof. For the non-trivial case when $\eta_1 \neq 0$, from Lemma 6.3.3, Proposition 6.3.5 and Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^T |\Gamma_{0,t}|^2\, dt\right] &\leq f_2^2(0)\int_0^T \mathbb{E}[\eta_1 r_t + \eta_2]\, dt \\ &= f_2^2(0)\int_0^T \left[(\eta_1 r_0 + \eta_2)e^{-bt} + \frac{\eta_1 a + \eta_2 b}{b}\left(1 - e^{-bt}\right)\right] dt < \infty. \end{aligned}$$

This means $\Gamma_{0,t} \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. By applying the same procedure to $\Gamma_{1,t}$ and $\Gamma_{2,t}$, it can be checked that $\Gamma_{1,t}$ and $\Gamma_{2,t}$ also lie in $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. When $\eta_1 \neq 0$, $\eta_1 r_t + \eta_2$ and α_t are both CIR processes with

$$\begin{aligned} \mathbb{E}\left[(\eta_1 r_t + \eta_2)^2\right] = &\left[(\eta_1 r_0 + \eta_2)e^{-bt} + \frac{\eta_1 a + \eta_2 b}{b}\left(1 - e^{-bt}\right)\right]^2 \\ &+ r_0\frac{\eta_1^2\left(e^{-bt} - e^{-2bt}\right)}{b} + \frac{(\eta_1 a + \eta_2 b)\eta_1^2\left(1 - e^{-bt}\right)^2}{2b^2} \end{aligned}$$

and

$$\mathbb{E}\left[\alpha_t^2\right]=\left[\alpha_0 e^{-\kappa t}+\theta\left(1-e^{-\kappa t}\right)\right]^2+\alpha_0\frac{\sigma_\alpha^2\left(e^{-\kappa t}-e^{-2\kappa t}\right)}{\kappa}+\frac{\theta\sigma_\alpha^2\left(1-e^{-\kappa t}\right)^2}{2\kappa}.$$

Then, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T |Y_t|^2\,dt\right] =&\mathbb{E}\left[\int_0^T \left|f_1(t)-\frac{\eta_2}{\eta_1}f_2(t)+\frac{f_2(t)}{\eta_1}(\eta_1 r_t+\eta_2)+f_3(t)\alpha_t\right|^2 dt\right]\\ \leq& c\left(1+\frac{f_2^2(0)}{\eta_1^2}\int_0^T \mathbb{E}\left[(\eta_1 r_t+\eta_2)^2\right]\,dt+f_3^2(0)\int_0^T \mathbb{E}[\alpha_t^2]\,dt\right)<\infty,\end{aligned}$$

where $c\in\mathbb{R}^+$. As for when $\eta_1=0$, r_t follows the Vasicek model and

$$\mathbb{E}[r_t^2]=\left[r_0 e^{-bt}+\frac{a}{b}\left(1-e^{-bt}\right)\right]^2+\frac{\eta_2}{2b}\left(1-e^{-2bt}\right).$$

Thus, we obtain

$$\mathbb{E}\left[\int_0^T |Y_t|^2\,dt\right]\leq c\left(1+f_2^2(0)\int_0^T \mathbb{E}\left[r_t^2\right]\,dt+f_3^2(0)\int_0^T \mathbb{E}[\alpha_t^2]\,dt\right)<\infty,$$

where $c\in\mathbb{R}^+$. These results reveal that $Y_t\in\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. □

Before showing that $(Y_t,\Gamma_{0,t},\Gamma_{1,t},\Gamma_{2,t})$ obtained in Lemma 6.3.3 is the unique solution to BSDE (6.3.2) in $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\times\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\times\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\times\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, we present an auxiliary result on the CIR process which is adapted from Theorem 5.1 in Zeng and Taksar (2013) provides an equivalent condition for the exponential integrability of the integrated CIR process.

Lemma 6.3.7. *For the CIR process α_t given in (6.2.5), we have*

$$\mathbb{E}\left[\exp\left\{\beta\int_0^T \alpha_t\,dt\right\}\right]<\infty \text{ if and only if } \beta\leq\frac{\kappa^2}{2\sigma_\alpha^2}.$$

If $\eta_1\neq 0$, the CIR process $\eta_1 r_t+\eta_2$ given in (6.2.2) satisfies

$$\mathbb{E}\left[\exp\left\{\beta\int_0^T (\eta_1 r_t+\eta_2)\,dt\right\}\right]<\infty \text{ if and only if } \beta\leq\frac{b^2}{2\eta_1^2}.$$

In view of the above results, throughout the rest of this section, we impose the following assumption on the model parameters:

Assumption 6.3.8. $\frac{2\gamma^2}{(1-\gamma)^2}(\lambda_r^2+f_2^2(0))\leq\frac{b^2}{2\eta_1^2}$ *and* $\frac{2\gamma^2}{(1-\gamma)^2}\left(\lambda^2+\sigma_\alpha^2 f_3^2(0)\right)\leq\frac{\kappa^2}{2\sigma_\alpha^2}$.

Remark 6.3.9. The monotonicity of functions $f_2(t)$ and $f_3(t)$ shown in Proposition 6.3.5 implies that $f_2(0)$ and $f_3(0)$ decrease to 0 as T approaches 0, which indicates the mathematical feasibility of the assumption above when the investment horizon is small enough. From an economic point of view, Assumption 6.3.8 gives upper bounds for the slope λ_r and λ of the market prices of interest and volatility risks. As stated in Korn and Kraft (2004), when λ_r or λ is too large, undertaking risk is rewarded too much by the market, and the optimal investment strategy might not be uniquely determined. Mathematically speaking, if the above technical condition is violated, the uniqueness result to BSDE (6.3.2) might not be ensured.

The uniqueness of the solution to BSDE (6.3.2) follows from the comparison arguments below.

Lemma 6.3.10. *Under Assumption 6.3.8, the solution* $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ *derived in Lemma 6.3.3 is the unique solution to nonlinear BSDE (6.3.2) in* $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.

Proof. We start by introducing the likelihood process $L_{1,t}$ for $t \in [0,T]$ from the following dynamics:

$$dL_{1,t} = -\frac{\gamma}{\gamma-1}\lambda_r\sqrt{\eta_1 r_t + \eta_2}L_{1,t}\,dW_t^0 - \frac{\gamma}{\gamma-1}\lambda\sqrt{\alpha_t}L_{1,t}\,dW_t^1.$$

It is straightforward to see that Novikov's condition is satisfied by Assumption 6.3.8 and Cauchy-Schwarz inequality, i.e.,

$$\mathbb{E}\left[\exp\left\{\int_0^T \frac{\gamma^2}{2(\gamma-1)^2}\left(\lambda_r^2(\eta_1 r_t + \eta_2) + \lambda^2\alpha_t\right)\,dt\right\}\right] < \infty.$$

Thus, the likelihood process $L_{1,t}$ is an $(\mathbb{F},\mathbb{P})$-uniformly integrable martingale, and the equivalent probability measure denoted by $\hat{\mathbb{P}}$ on $\mathcal{F}_T$ is well-defined via the Radon-Nikodym derivative:

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = L_{1,T}.$$

Let $\hat{\mathbb{E}}[\cdot]$ denote the corresponding expectation under measure $\hat{\mathbb{P}}$. From Girsanov's theorem, three processes $\hat{W}_t^0, \hat{W}_t^1$, and $\hat{W}_t^2$ given by

$$\hat{W}_t^0 = \int_0^t \frac{\gamma}{\gamma-1}\lambda_r\sqrt{\eta_1 r_s + \eta_2}\,ds + W_t^0,\ \hat{W}_t^1 = \int_0^t \frac{\gamma}{\gamma-1}\lambda\sqrt{\alpha_s}\,ds + W_t^1,\ \hat{W}_t^2 = W_t^2$$

are three standard $(\mathbb{F},\hat{\mathbb{P}})$ Brownian motions. And the solution $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}\,\Gamma_{2,t})$ given by (6.3.3) forms a solution to the following BSDE:

$$\begin{cases} dY_t = \left[\dfrac{\Gamma_{0,t}^2 + \gamma\lambda_r^2(\eta_1 r_t + \eta_2) + \Gamma_{1,t}^2 + \gamma\lambda^2\alpha_t}{2(\gamma-1)} - \dfrac{\Gamma_{2,t}^2}{2} - \gamma r_t\right] dt + \Gamma_{0,t}\,d\hat{W}_t^0 \\ \qquad\quad + \Gamma_{1,t}\,d\hat{W}_t^1 + \Gamma_{2,t}\,d\hat{W}_t^2, \\ Y_T = 0. \end{cases}$$

Denote by $(\hat{Y}_t, \hat{\Gamma}_{0,t}, \hat{\Gamma}_{1,t}, \hat{\Gamma}_{2,t})$ another solution to BSDE (6.3.2). Then, the following difference process

$$(\Delta Y_t, \Delta\Gamma_{0,t}, \Delta\Gamma_{1,t}, \Delta\Gamma_{2,t}) = (Y_t - \hat{Y}_t, \Gamma_{0,t} - \hat{\Gamma}_{0,t}, \Gamma_{1,t} - \hat{\Gamma}_{1,t}, \Gamma_{2,t} - \hat{\Gamma}_{2,t})$$

must solve the following BSDE under $\hat{\mathbb{P}}$ measure:

$$\begin{aligned} d\Delta Y_t = &\left[\frac{(\Gamma_{0,t}^2 - \hat{\Gamma}_{0,t}^2) + (\Gamma_{1,t}^2 - \hat{\Gamma}_{1,t}^2)}{2(\gamma-1)} - \frac{\Gamma_{2,t}^2 - \hat{\Gamma}_{2,t}^2}{2}\right] dt + \Delta\Gamma_{0,t}\, d\hat{W}_t^0 \\ &+ \Delta\Gamma_{1,t}\, d\hat{W}_t^1 + \Delta\Gamma_{2,t}\, d\hat{W}_t^2. \end{aligned} \tag{6.3.14}$$

Now, we introduce the second likelihood process $L_{2,t}$ for which the dynamics are given by

$$dL_{2,t} = -\frac{\Gamma_{0,t}}{\gamma-1} L_{2,t}\, d\hat{W}_t^0 - \frac{\Gamma_{1,t}}{\gamma-1} L_{2,t}\, d\hat{W}_t^1 + \Gamma_{2,t} L_{2,t}\, d\hat{W}_t^2,$$

with $\Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t}$ given in (6.3.3) above. Then, by Cauchy-Schwarz inequality and Assumption 6.3.8, we see that Novikov's condition holds for likelihood process $L_{2,t}$

$$\begin{aligned} &\hat{\mathbb{E}}\left[\exp\left\{\frac{1}{2}\int_0^T \frac{1}{\gamma-1}\Gamma_{0,t}^2 + \frac{1}{\gamma-1}\Gamma_{1,t}^2 + \Gamma_{2,t}^2\, dt\right\}\right] \\ =&\mathbb{E}\left[L_{1,T}\exp\left\{\frac{1}{2}\int_0^T \frac{1}{\gamma-1}\Gamma_{0,t}^2 + \frac{1}{\gamma-1}\Gamma_{1,t}^2 + \Gamma_{2,t}^2\, dt\right\}\right] \\ \le&\left\{\mathbb{E}\left[\exp\left\{\int_0^T -\frac{2\gamma}{\gamma-1}\lambda_r\sqrt{\eta_1 r_t + \eta_2}\, dW_t^0 - \int_0^T \frac{2\gamma}{\gamma-1}\lambda\sqrt{\alpha_t}\, dW_t^1\right.\right.\right. \\ &\left.\left.\left.- \int_0^T \frac{2\gamma^2}{(\gamma-1)^2}\left(\lambda_r^2(\eta_1 r_t + \eta_2) + \lambda^2\alpha_t\right)\, dt\right\}\right]\right\}^{\frac{1}{2}} \\ &\times\left\{\mathbb{E}\left[\exp\left\{\int_0^T \left(\frac{\lambda^2\gamma^2}{(\gamma-1)^2} + \frac{1}{\gamma-1}\rho^2\sigma_\alpha^2 f_3^2(t) + (1-\rho^2)\sigma_\alpha^2 f_3^2(t)\right)\alpha_t\right.\right.\right. \\ &\left.\left.\left.+ \int_0^T \left(\frac{\lambda_r^2\gamma^2}{(\gamma-1)^2} + \frac{f_2^2(t)}{\gamma-1}\right)(\eta_1 r_t + \eta_2)\, dt\right\}\right]\right\}^{\frac{1}{2}} \\ \le&\frac{\sqrt{2}}{2}\left\{\mathbb{E}\left[\exp\left\{\frac{2\lambda_r^2\gamma^2}{(\gamma-1)^2}\int_0^T (\eta_1 r_t + \eta_2)\, dt\right\}\right]\right. \\ &\left.+\mathbb{E}\left[\exp\left\{\left(2(1-\rho^2)\sigma_\alpha^2 f_3^2(0) + \frac{2\lambda^2\gamma^2}{(\gamma-1)^2}\right)\int_0^T \alpha_t\, dt\right\}\right]\right\}^{\frac{1}{2}} < \infty. \end{aligned}$$

This shows that probability measure $\tilde{\mathbb{P}}$ is well-defined on $\mathcal{F}_T$ via the Radon-Nikodym derivative:

$$\frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}}\bigg|_{\mathcal{F}_T} = L_{2,T}.$$

Accordingly, from Girsanov's theorem, three processes $\tilde{W}_t^0, \tilde{W}_t^1$, and $\tilde{W}_t^2$ defined by

$$\tilde{W}_t^0 = \int_0^t \frac{\Gamma_{0,s}}{\gamma - 1}\, ds + \hat{W}_t^0,\ \ \tilde{W}_t^1 = \int_0^t \frac{\Gamma_{1,s}}{\gamma - 1}\, ds + \hat{W}_t^1,\ \ \tilde{W}_t^2 = \int_0^t -\Gamma_{2,s}\, ds + \hat{W}_t^2, \tag{6.3.15}$$

are three standard Brownian motions under measure $\tilde{\mathbb{P}}$. Therefore, substituting (6.3.15) into (6.3.14) shows that $(\Delta Y_t, \Delta\Gamma_{0,t}, \Delta\Gamma_{1,t}, \Delta\Gamma_{2,t})$ solves the following quadratic BSDE under measure $\tilde{\mathbb{P}}$:

$$\begin{cases} d\Delta Y_t = \left(-\frac{1}{2(\gamma-1)} \Delta\Gamma_{0,t}^2 - \frac{1}{2(\gamma-1)} \Delta\Gamma_{1,t}^2 + \frac{1}{2} \Delta\Gamma_{2,t}^2 \right) dt + \Delta\Gamma_{0,t}\, d\tilde{W}_t^0 \\ \qquad\qquad + \Delta\Gamma_{1,t}\, d\tilde{W}_t^1 + \Delta\Gamma_{2,t}\, d\tilde{W}_t^2, \\ \Delta Y_T = 0. \end{cases} \tag{6.3.16}$$

It can be checked that this quadratic BSDE (6.3.16) satisfies all regularity conditions in Kobylanski (2000). Hence, according to Theorem 2.3 and Theorem 2.6 in Kobylanski (2000), BSDE (6.3.16) admits a unique solution $(0, 0, 0, 0)$, which, in turn, indicates

$$Y_t = \hat{Y}_t,\ \ \Gamma_{0,t} = \hat{\Gamma}_{0,t},\ \ \Gamma_{1,t} = \hat{\Gamma}_{1,t},\ \ \Gamma_{2,t} = \hat{\Gamma}_{2,t}.$$

We can, therefore, conclude that the solution $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ given by (6.3.3) is the unique solution to BSDE (6.3.2) in $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. □

Remark 6.3.11. Girsanov's measure change techniques are applied within the proof of Lemma 6.3.10, of which the validity critically hinges on the specifications of the market prices of interest rate and volatility risks. By assuming the market prices of risks to be linear in the square root of two CIR processes $\eta_1 r_t + \eta_2$ and α_t and making use of the exponential integrability condition with respect to CIR processes (Assumption 6.3.8), the likelihood processes $L_{1,t}$ and $L_{2,t}$ presented in the proof of Lemma 6.3.10 are indeed uniformly integrable martingales rather than strictly (positive) local martingales under measures $\mathbb{P}$ and $\hat{\mathbb{P}}$, respectively. Consequently, before implementing the optimal strategy presented in Theorem 6.3.12 below, one must cautiously specify the form of the market prices of risks. Some parametric settings might lead to local martingales and, therefore, a lack of likely changes of measure (see, for example, Platen and Heath (2006), Grasselli (2017), and Gnoatto, Grasselli, and Platen (2022)).

To end this section, we give the following theorem which relates the optimal investment strategy and optimal value function for the power utility maximization problem (6.2.9) to the unique solution $(Y_t, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ to BSDE (6.3.2).

Theorem 6.3.12. *Under Assumption 6.3.8, for any initial data* $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ *fixed and given, the optimal strategy and optimal value function for problem (6.2.9) are given by*

$$\begin{cases} \pi_t^{B*} = \dfrac{1}{h_0(K)} \left(\dfrac{\lambda_r}{1-\gamma} - \dfrac{f_2(t)}{1-\gamma} \right), \\ \pi_t^{S^1*} = \dfrac{\sqrt{\alpha_t}}{\sigma_t} \left(\dfrac{\rho \sigma_\alpha f_3(t)}{1-\gamma} + \dfrac{\lambda}{1-\gamma} \right), \end{cases} \tag{6.3.17}$$

and

$$V_p(\alpha_0, r_0, x_0) = \frac{(x_0)^\gamma}{\gamma} \exp\left\{ f_1(0) + f_2(0) r_0 + f_3(0) \alpha_0 \right\}, \tag{6.3.18}$$

where functions $f_1(t), f_2(t)$, *and* $f_3(t)$ *are explicitly given by (6.3.7), (6.3.8), and (6.3.9). Moreover, the optimal strategy is admissible, i.e.,* $\pi^* \in \mathcal{A}_p$.

Proof. For any admissible strategy $\pi \in \mathcal{A}_p$, it follows from (6.3.1) that

$$\begin{aligned} & d\left(\frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t} \right) \\ =& \frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t} \left(\gamma h_0(K) \sqrt{\eta_1 r_t + \eta_2} \pi_t^B + \Gamma_{0,t} \right) dW_t^0 + \frac{(X_t^\pi)^\gamma}{\gamma} e^{Y_t} \left(\gamma \sigma_t \pi_t^{S^1} + \Gamma_{1,t} \right) dW_t^1 \\ & + \frac{(X_t^\pi)^\gamma e^{Y_t}}{2(\gamma-1)} \left\{ \left[\Gamma_{1,t} + \lambda \sqrt{\alpha_t} + (\gamma-1) \sigma_t \pi_t^{S^1} \right]^2 + \left[\Gamma_{0,t} + \lambda_r \sqrt{\eta_1 r_t + \eta_2} \right. \right. \\ & \left. \left. + (\gamma-1) h_0(K) \sqrt{\eta_1 r_t + \eta_2} \pi_t^B \right]^2 \right\} dt. \end{aligned} \tag{6.3.19}$$

Since $X_t^\pi \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, $Y_t, \Gamma_{0,t}, \Gamma_{1,t} \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, $\pi_t^{S^1}\sigma_t, \pi_t^B \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, two stochastic integrals on the right-hand side of (6.3.19) are $(\mathbb{F},\mathbb{P})$-local martingales. Thus, there exists a sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, $\mathbb{P}$ almost surely as $n \to \infty$, and the aforementioned local martingales are indeed $(\mathbb{F},\mathbb{P})$-martingales when stopped by $\{\tau_n\}_{n\in\mathbb{N}}$. In other words, we have the following equation by integrating both sides of (6.3.19) from 0 to $T \wedge \tau_n$ and taking expectations:

$$\begin{aligned} \mathbb{E}\left[\frac{(X^\pi_{T\wedge\tau_n})}{\gamma} e^{Y_{T\wedge\tau_n}} \right] =& \mathbb{E}\left[\int_0^{T\wedge\tau_n} \frac{(X_t^\pi)^\gamma e^{Y_t}}{2(\gamma-1)} \left(\left[(\gamma-1)\sigma_t \pi_t^{S^1} + \Gamma_{1,t} + \lambda\sqrt{\alpha_t} \right]^2 \right. \right. \\ & \left. \left. + \left[(\gamma-1) h_0(K) \sqrt{\eta_1 r_t + \eta_2} \pi_t^B + \Gamma_{0,t} + \lambda_r \sqrt{\eta_1 r_t + \eta_2} \right]^2 \right) dt \right] \\ & + \frac{(x_0)^\gamma}{\gamma} e^{Y_0}. \end{aligned} \tag{6.3.20}$$

Recall from Definition 6.2.6 that for any admissible $\pi \in \mathcal{A}_p$, the corresponding wealth process $X_t^\pi > 0$, $\mathbb{P}$ almost surely. Then the term within the expectation on the right-hand side of (6.3.20) is non-positive due to $\gamma \in (0,1)$, which means

$$\mathbb{E}\left[\frac{(X^{\pi}_{T\wedge\tau_n})}{\gamma}e^{Y_{T\wedge\tau_n}}\right] \leq \frac{(x_0)^{\gamma}}{\gamma}e^{Y_0}. \tag{6.3.21}$$

Then passing to the limit in (6.3.21) and making use of Fatou's lemma, we find

$$\mathbb{E}\left[\frac{(X^{\pi}_{T})^{\gamma}}{\gamma}\right] \leq \liminf_{n\to\infty}\mathbb{E}\left[\frac{(X^{\pi}_{T\wedge\tau_n})}{\gamma}e^{Y_{T\wedge\tau_n}}\right] \leq \frac{(x_0)^{\gamma}}{\gamma}e^{Y_0},$$

and thus, upon considering the explicit expression of Y_0 from (6.3.3), we obtain

$$\sup_{\pi\in\mathcal{A}_p}\mathbb{E}\left[\frac{(X^{\pi}_{T})^{\gamma}}{\gamma}\right] \leq \frac{(x_0)^{\gamma}}{\gamma}\exp\left\{f_1(0)+f_2(0)r_0+f_3(0)\alpha_0\right\}. \tag{6.3.22}$$

In particular, when we opt for the strategy π^{B*}_t and $\pi^{S^1*}_t$ given in (6.3.17) and denote by X^*_t the corresponding wealth process, (6.3.19) yields

$$\begin{aligned} d\left(\frac{(X^*_t)^{\gamma}}{\gamma}e^{Y_t}\right) =&\frac{(X^*_t)^{\gamma}}{\gamma}e^{Y_t}\left[\left(\frac{1}{1-\gamma}\Gamma_{0,t}+\frac{\gamma}{1-\gamma}\lambda_r\sqrt{\eta_1 r_t+\eta_2}\right)dW^0_t\right.\\ &\left.+\left(\frac{1}{1-\gamma}\Gamma_{1,t}+\frac{\gamma}{1-\gamma}\lambda\sqrt{\alpha_t}\right)dW^1_t\right]. \end{aligned} \tag{6.3.23}$$

Under Assumption 6.3.8, we find that process $\left\{\frac{(X^*_t)^{\gamma}}{\gamma}e^{Y_t}\right\}_{t\in[0,T]}$ is an $(\mathbb{F},\mathbb{P})$-uniformly integrable martingale because the following Novikov's condition is satisfied

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left(\frac{1}{1-\gamma}\Gamma_{0,t}+\frac{\gamma}{1-\gamma}\lambda_r\sqrt{\eta_1 r_t+\eta_2}\right)^2+\left(\frac{1}{1-\gamma}\Gamma_{1,t}+\frac{\gamma}{1-\gamma}\lambda\sqrt{\alpha_t}\right)^2dt\right\}\right]\\ =&\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left(\frac{\gamma}{1-\gamma}\lambda_r-\frac{1}{1-\gamma}f_2(t)\right)^2(\eta_1 r_t+\eta_2)+\left(\frac{\gamma}{1-\gamma}\lambda+\frac{1}{1-\gamma}\rho\sigma_\alpha f_3(t)\right)^2\alpha_t\,dt\right\}\right]\\ \leq&\frac{1}{2}\mathbb{E}\left[\exp\left\{\frac{2\gamma^2\lambda_r^2+2f_2^2(0)}{(1-\gamma)^2}\int_0^T(\eta_1 r_t+\eta_2)\,dt\right\}+\exp\left\{\frac{2\lambda^2\gamma^2+2\rho^2\sigma_\alpha^2 f_3^2(0)}{(1-\gamma)^2}\int_0^T\alpha_t\,dt\right\}\right]<\infty. \end{aligned}$$

Hence, we have

$$\mathbb{E}\left[\frac{(X^*_T)^{\gamma}}{\gamma}\right] = \frac{(x_0)^{\gamma}}{\gamma}\exp\left\{f_1(0)+f_2(0)r_0+f_3(0)\alpha_0\right\}. \tag{6.3.24}$$

Moreover, solving linear SDE (6.3.23) of $\frac{(X^*_t)^{\gamma}}{\gamma}e^{Y_t}$ explicitly gives the following dynamics of wealth process X^*_t:

$$\begin{aligned} X^*_t =&x_0\exp\left\{\frac{f_1(0)+f_2(0)r_0+f_3(0)\alpha_0-f_1(t)-f_2(t)r_t-f_3(t)\alpha_t}{\gamma}\right\}\\ &\times\exp\left\{\int_0^t\frac{\gamma\lambda_r-f_2(s)}{\gamma(1-\gamma)}\sqrt{\eta_1 r_s+\eta_2}\,dW^0_s-\frac{1}{2}\int_0^t\frac{(\gamma\lambda_r-f_2(s))^2}{\gamma(1-\gamma)^2}(\eta_1 r_s+\eta_2)\,ds\right\}\\ &\times\exp\left\{\int_0^t\frac{\gamma\lambda+\rho\sigma_\alpha f_3(s)}{\gamma(1-\gamma)}\sqrt{\alpha_s}\,dW^1_s-\frac{1}{2}\int_0^t\frac{(\gamma\lambda+\rho\sigma_\alpha f_3(s))^2}{\gamma(1-\gamma)^2}\alpha_s\,ds\right\}>0. \end{aligned} \tag{6.3.25}$$

Given (6.3.22) and (6.3.24), we know that π^* given in (6.3.17) is the optimal investment strategy, and the optimal value function is given by (6.3.18). Finally, it follows from (6.3.17) and (6.3.25) that π_t^{B*} and $\sigma_t \pi_t^{S^1*} \in \mathcal{L}_{\mathbb{F},\mathbb{P}}^{2,loc}(0,T;\mathbb{R})$, $X_t^* \in \mathcal{L}_{\mathbb{F},\mathbb{P}}^{0}(0,T;\mathbb{R})$ and $X_t^* > 0$, i.e., the optimal strategy π^* is admissible. □

Remark 6.3.13. Notice from (6.3.17) that the optimal allocation in the risky asset $\pi_t^{S^1*}$ comprises three terms, a multiplier $\sqrt{\alpha_t}/\sigma_t$, a myopic (time-independent) component $\lambda/(1-\gamma)$, and an inter-temporal (time-dependent) hedging component $\rho\sigma_\alpha f_3(t)/(1-\gamma)$ (refer to Merton (1973)). The myopic component is increasing in both λ and γ; λ partially characterizes the market price of volatility risk, and γ depicts the risk-averse preference of the investor. This is intuition-consistent because a more aggressive investment strategy will be adopted when the investor is less risk-averse or realizes a greater volatility risk premium. As for the effect of the hedging demand, recall from Proposition 6.3.5 that $f_3(t)$ approaches zero as the investment horizon shrinks. In other words, the hedging demand decreases in absolute value as the investment horizon approaches its terminal. In the extreme case of one-period investments, it is noticeable that no inter-temporal hedging is needed. It is also important to point out from (6.3.17) that $\pi_t^{S^1*}$ is affected by the ratio $\sqrt{\alpha_t}/\sigma_t$ rather than σ_t only. This can be explained by our choice of the market price of volatility risk as well as the fact that not only the volatility σ_t of the risky asset price but also the volatility risk premium $\lambda\sqrt{\alpha_t}$ influences asset allocation strategies. It is far more interesting to realize from the wealth equation (6.2.8) and (6.3.17) that $\sigma_t\pi_t^{S^1*}$ is exactly the optimal risk exposure to the volatility risk and it is not relevant to the specifications of the return rate μ_t and volatility σ_t of the risky asset. This finding, combined with the expression for the optimal value function given in (6.3.18), reveals that the optimal investment strategies essentially hinge on the market price specification of volatility risk rather than the dynamics of the risky asset price as a whole.

Remark 6.3.14. The optimal allocation of wealth in the rolling bond π_t^{B*} is a deterministic and continuous function. Similar to the form of $\pi_t^{S^1*}$ except for the stochastic multiplier component, π_t^{B*} includes a constant multiplier $1/h_0(K)$. This is due to our specific choice of the market price of interest rate risk $\lambda_r\sqrt{\eta_1 r_t + \eta_2}$ and the volatility of the rolling bond $h_0(K)\sqrt{\eta_1 r_t + \eta_2}$.

The next two corollaries provide the results for two special cases of our model, the 4/2 model and Siu's non-Markovian model, respectively.

Corollary 6.3.15. *(The 4/2 model). Under Assumption 6.3.8, when the risky asset S_t^1 follows the 4/2 model (6.2.6), then for any initial data* $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$

fixed and given, the optimal strategy for problem (6.2.9) is given by

$$\begin{cases} \pi_t^{B*} = \dfrac{1}{h_0(K)} \left(\dfrac{\lambda_r}{1-\gamma} - \dfrac{f_2(t)}{1-\gamma} \right), \\ \pi_t^{S^1*} = \dfrac{\alpha_t}{c_1 \alpha_t + c_2} \left(\dfrac{\rho \sigma_\alpha f_3(t)}{1-\gamma} + \dfrac{\lambda}{1-\gamma} \right), \end{cases}$$

and the optimal value function is given by

$$V_p(\alpha_0, r_0, x_0) = \frac{(x_0)^\gamma}{\gamma} \exp \left\{ f_1(0) + f_2(0) r_0 + f_3(0) \alpha_0 \right\},$$

where functions $f_1(t), f_2(t)$, and $f_3(t)$ are given by (6.3.7), (6.3.8), and (6.3.9), respectively.

Proof. Plugging the specified parameters in Example 6.2.3 back into Theorem 6.3.12 leads to the results in Corollary 6.3.15. □

Remark 6.3.16. The optimal asset allocation in the risky asset $\pi_t^{S_1*}$ given in Corollary 6.3.15 is in line with the results derived in Cheng and Escobar (2021a), where constant interest rates are considered. This means that our results generalize the results of Cheng and Escobar (2021a) to the case with stochastic affine interest rates. If we further specify $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$, then Corollary 6.3.15 recovers the optimal strategies under the Heston model and 3/2 model with affine interest rates, respectively. It can be checked that the resulting Heston solution is consistent with the results presented in Kraft (2005) and Zeng and Taksar (2013).

Corollary 6.3.17. *(Siu's non-Markovian model). Under Assumption 6.3.8, when the risky asset S_t^1 follows the non-Markovian model (6.2.7), then for any initial data $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given, the optimal strategy for problem (6.2.9) is given by*

$$\begin{cases} \pi_t^{B*} = \dfrac{1}{h_0(K)} \left(\dfrac{\lambda_r}{1-\gamma} - \dfrac{f_2(t)}{1-\gamma} \right), \\ \pi_t^{S^1*} = \dfrac{\sqrt{\alpha_t}}{\hat{\sigma}\left(\alpha_{[0,t]}\right)} \left(\dfrac{\rho \sigma_\alpha f_3(t)}{1-\gamma} + \dfrac{\lambda}{1-\gamma} \right), \end{cases}$$

and the optimal value function is given by

$$V_p(\alpha_0, r_0, x_0) = \frac{(x_0)^\gamma}{\gamma} \exp \left\{ f_1(0) + f_2(0) r_0 + f_3(0) \alpha_0 \right\},$$

where functions $f_1(t), f_2(t)$, and $f_3(t)$ are given by (6.3.7), (6.3.8), and (6.3.9), respectively.

Proof. Substituting the specified parameters in Example 6.2.5 into Theorem 6.3.12 leads to the results in Corollary 6.3.17. □

6.4 Solution to the logarithmic utility case

In this section, we consider the logarithmic utility maximization problem (6.2.10) by means of BSDE. Similar to the last section, we introduce a continuous $(\mathbb{F},\mathbb{P})$-semi-martingale P_t with canonical decomposition:

$$dP_t = H_t\,dt + Z_{0,t}\,dW_t^0 + Z_{1,t}\,dW_t^1 + Z_{2,t}\,dW_t^2,$$

where H_t is an undetermined $\mathbb{F}$-adapted process, and $Z_{0,t}, Z_{1,t}, Z_{2,t} \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. Applying Itô's formula to $\log(X_t^\pi) + P_t$ and using the method of completion of squares for any admissible strategy $\pi \in \mathcal{A}_l$ yield

$$\begin{aligned} d\left(\log(X_t^\pi) + P_t\right) = {} & \left(\pi_t^B h_0(K)\sqrt{\eta_1 r_t + \eta_2} + Z_{0,t}\right)\,dW_t^0 + \left(\pi_t^{S^1}\sigma_t + Z_{1,t}\right)\,dW_t^1 \\ & + Z_{2,t}\,dW_t^2 - \frac{1}{2}\left(\pi_t^B h_0(K) - \lambda_r\right)^2(\eta_1 r_t + \eta_2)\,dt - \frac{1}{2}\left(\pi_t^{S^1}\sigma_t\right. \\ & \left.-\lambda\sqrt{\alpha_t}\right)^2\,dt + \left(H_t + r_t + \frac{1}{2}\lambda^2\alpha_t + \frac{1}{2}\lambda_r^2(\eta_1 r_t + \eta_2)\right)\,dt. \end{aligned} \tag{6.4.1}$$

Intuitively, we can choose H_t such that the last term on the right-hand side of (6.4.1) is zero. This leads to the following linear BSDE of $(P_t, Z_{0,t}, Z_{1,t}, Z_{2,t})$ we shall consider:

$$\begin{cases} dP_t = \left[-\frac{1}{2}\lambda^2\alpha_t - \frac{1}{2}\lambda_r^2(\eta_1 r_t + \eta_2) - r_t\right]\,dt + Z_{0,t}\,dW_t^0 + Z_{1,t}\,dW_t^1 + Z_{2,t}\,dW_t^2, \\ P_T = 0. \end{cases} \tag{6.4.2}$$

Remark 6.4.1. Despite of the unboundedness of r_t and α_t, it is clear that the driver of linear BSDE (6.4.2) is uniformly Lipschitz continuous with respect to $P_t, Z_{0,t}, Z_{1,t}$, and $Z_{2,t}$. In addition, it can be checked $\mathbb{E}\left[\int_0^T(\frac{1}{2}\lambda^2\alpha_t + \frac{1}{2}\lambda_r^2(\eta_1 r_t + \eta_2) + r_t)^2\,dt\right] < \infty$, linear BSDE (6.4.2) is therefore with standard data (see El Karoui, Peng, and Quenez (1997)) and the uniqueness and existence result of BSDE (6.4.2) is ensured by Theorem 2.1 in El Karoui, Peng, and Quenez (1997).

In the following proposition, we explicitly present the unique solution to BSDE (6.4.2).

Proposition 6.4.2. *The unique solution $(P_t, Z_{0,t}, Z_{1,t}, Z_{2,t})$ to linear BSDE (6.4.2) is given by*

$$\begin{cases} P_t = g_1(t) + g_2(t)r_t + g_3(t)\alpha_t, \\ Z_{0,t} = -g_2(t)\sqrt{\eta_1 r_t + \eta_2}, \\ Z_{1,t} = g_3(t)\rho\sigma_\alpha\sqrt{\alpha_t}, \\ Z_{2,t} = g_3(t)\sqrt{1-\rho^2}\sigma_\alpha\sqrt{\alpha_t}, \end{cases} \tag{6.4.3}$$

where functions $g_1(t), g_2(t), g_3(t)$ *are solutions to the following ODE system:*

$$\begin{cases} \dfrac{dg_1(t)}{dt} + a g_2(t) + \kappa\theta g_3(t) + \dfrac{1}{2}\lambda_r^2\eta_2 = 0, \;\; g_1(T) = 0, \\ \qquad \dfrac{dg_2(t)}{dt} - b g_2(t) + \dfrac{1}{2}\lambda_r^2\eta_1 + 1 = 0, \;\; g_2(T) = 0, \\ \qquad\qquad \dfrac{dg_3(t)}{dt} - \kappa g_3(t) + \dfrac{1}{2}\lambda^2 = 0, \;\; g_3(T) = 0. \end{cases} \tag{6.4.4}$$

Moreover, $(P_t, Z_{0,t}, Z_{1,t}, Z_{2,t})$ *lies in* $\mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.

Proof. Since linear BSDE (6.4.2) satisfies all the regularity condition in El Karoui, Peng, and Quenez (1997), it follows from Proposition 2.2 in El Karoui, Peng, and Quenez (1997) and Proposition 4.1.1 in Zhang (2017) that the unique solution $(P_t, Z_{0,t}, Z_{1,t}, Z_{2,t}) \in \mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, and moreover, P_t can be represented by the following conditional expectation form:

$$P_t = \mathbb{E}\left[\int_t^T \frac{1}{2}\lambda^2\alpha_s + \frac{1}{2}\lambda_r^2(\eta_1 r_s + \eta_2) + r_s \, ds \,\middle|\, \mathcal{F}_t\right].$$

Denote by $g(t,\alpha,r) = \mathbb{E}_{t,\alpha,r}\left[\int_t^T \frac{1}{2}\lambda^2\alpha_s + \frac{1}{2}\lambda_r^2(\eta_1 r_s + \eta_2) + r_s \, ds\right]$,where $\mathbb{E}_{t,\alpha,r}[\cdot]$ denotes the conditional expectation under $\mathbb{P}$ measure given that $\alpha_t = \alpha$ and $r_t = r$ at time $t \in [0,T]$. Due to the Markovian structures of interest rate r_t and factor process α_t with respect to $\mathcal{F}_t$, we then have

$$P_t = g(t, \alpha_t, r_t).$$

Suppose that function $g(\cdot,\cdot,\cdot) \in C^{1,2,2}([0,T] \times \mathbb{R}^+ \times \mathbb{R})$. Then, applying the Feynman-Kac formula leads to the following partial differential equation:

$$\begin{cases} \dfrac{\partial g}{\partial t} + (a - br)\dfrac{\partial g}{\partial r} + \kappa(\theta - \alpha)\dfrac{\partial g}{\partial \alpha} + \dfrac{1}{2}(\eta_1 r + \eta_2)\dfrac{\partial^2 g}{\partial r^2} \\ \qquad + \dfrac{1}{2}\sigma_\alpha^2\alpha\dfrac{\partial^2 g}{\partial \alpha^2} + \dfrac{1}{2}\lambda^2\alpha + \dfrac{1}{2}\lambda_r^2(\eta_1 r + \eta_2) + r = 0, \\ \qquad\qquad g(T,\alpha,r) = 0. \end{cases} \tag{6.4.5}$$

We conjecture that solution of $g(t,\alpha,r)$ admits an affine form:

$$g(t,\alpha,r) = g_1(t) + g_2(t)r + g_3(t)\alpha, \tag{6.4.6}$$

with boundary conditions that $g_1(T) = g_2(T) = g_3(T) = 0$. Substituting (6.4.6) into (6.4.5) and separating the dependence on r and α result in the ODE system (6.4.4). Finally, applying Itô's lemma to P_t and matching the coefficients show us that

$$Z_{0,t} = -g_2(t)\sqrt{\eta_1 r_t + \eta_2}, \;\; Z_{1,t} = g_3(t)\rho\sigma_\alpha\sqrt{\alpha_t}, \;\; Z_{2,t} = g_3(t)\sqrt{1-\rho^2}\sigma_\alpha\sqrt{\alpha_t},$$

by the uniqueness of the solution to linear BSDE (6.4.2). □

Proposition 6.4.3. *Explicit solutions to ODE system (6.4.4) are given by*

$$\begin{cases} g_1(t) = \dfrac{(\lambda_r^2\eta_1+2)a}{2b^2}\left(e^{b(t-T)}-1\right)+\dfrac{\lambda^2\theta}{2\kappa}\left(e^{\kappa(t-T)}-1\right) \\ \qquad +\dfrac{1}{2}(T-t)\left(\dfrac{(\lambda_r^2\eta_1+2)a}{b}+\lambda^2\theta+\lambda_r^2\eta_2\right), \\ g_2(t) = -\dfrac{\lambda_r^2\eta_1+2}{2b}\left(e^{b(t-T)}-1\right), \\ g_3(t) = -\dfrac{\lambda^2}{2\kappa}\left(e^{\kappa(t-T)}-1\right). \end{cases} \tag{6.4.7}$$

Proof. Since equations for $g_2(t)$ and $g_3(t)$ are both first-order linear ODEs, a direct integral calculation leads to the analytical solution of $g_2(t)$ and $g_3(t)$ given by (6.4.7). Inserting the explicit representations of $g_2(t)$ and $g_3(t)$ into the equation of $g_1(t)$ and integrating both sides from t to T yield the analytical representation of $g_1(t)$. □

We now proceed to derive the optimal investment strategy and optimal value function for the logarithmic utility maximization problem (6.2.10) via the unique solution $(P_t, Z_{0,t}, Z_{1,t}, Z_{2,t})$ to linear BSDE (6.4.2).

Theorem 6.4.4. *For any initial data $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given, the optimal strategy and optimal value function for problem (6.2.10) are respectively given by*

$$\begin{cases} \pi_t^{B*} = \dfrac{\lambda_r}{h_0(K)}, \\ \pi_t^{S^1*} = \dfrac{\lambda\sqrt{\alpha_t}}{\sigma_t}, \end{cases} \tag{6.4.8}$$

and

$$V_l(\alpha_0, r_0, x_0) = \log(x_0) + (g_1(0) + g_2(0)r_0 + g_3(0)\alpha_0), \tag{6.4.9}$$

where functions $g_1(t), g_2(t)$, and $g_3(t)$ are explicitly given by (6.4.7) above. Moreover, the optimal investment strategy is admissible, i.e., $\pi^* \in \mathcal{A}_l$.

Proof. For any admissible strategy $\pi \in \mathcal{A}_l$, it follows from (6.4.1) that

$$\begin{aligned} d(\log(X_t^\pi) + P_t) = &\left(\pi_t^B h_0(K)\sqrt{\eta_1 r_t + \eta_2} + Z_{0,t}\right) dW_t^0 + \left(\pi_t^{S^1}\sigma_t + Z_{1,t}\right) dW_t^1 \\ &+ Z_{2,t}\, dW_t^2 - \frac{1}{2}\left(\pi_t^B h_0(K) - \lambda_r\right)^2(\eta_1 r_t + \eta_2)\, dt - \frac{1}{2}\left(\pi_t^{S^1}\sigma_t\right. \\ &\left.-\lambda\sqrt{\alpha_t}\right)^2 dt. \end{aligned} \tag{6.4.10}$$

Furthermore, for any admissible strategy $\pi \in \mathcal{A}_l$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(\pi_t^B h_0(K)\sqrt{\eta_1 r_t + \eta_2} + Z_{0,t}\right)^2 + \left(\pi_t^{S^1}\sigma_t + Z_{1,t}\right)^2 + Z_{2,t}^2\, dt\right] \\
\leq & 2h_0^2(K)\mathbb{E}\left[\int_0^T \left(\pi_t^B\right)^2(\eta_1 r_t + \eta_2)\, dt\right] + k_1 \int_0^T \mathbb{E}\left[\eta_1 r_t + \eta_2\right]\, dt \\
& + 2\mathbb{E}\left[\int_0^T \left(\pi_t^{S^1}\right)^2 \sigma_t^2\, dt\right] + k_2 \int_0^T \mathbb{E}\left[\alpha_t\right]\, dt < \infty,
\end{aligned}
$$

where constants $k_1 = 2\sup_{t\in[0,T]} g_2^2(t)$ and $k_2 = (1+\rho^2)\sigma_\alpha^2 \sup_{t\in[0,T]} g_3^2(t)$. This means that stochastic integrals on the right-hand side of (6.4.10) are $(\mathbb{F},\mathbb{P})$-martingales. Hence, integrating both sides of (6.4.10) from 0 to T and taking expectation yield

$$
\begin{aligned}
\mathbb{E}\left[\log(X_T^\pi)\right] =& \log(x_0) + P_0 - \frac{1}{2}\mathbb{E}\left[\int_0^T \left(\pi_t^B h_0(K) - \lambda_r\right)^2(\eta_1 r_t + \eta_2)\, dt\right] \\
& - \frac{1}{2}\mathbb{E}\left[\int_0^T \left(\pi_t^{S^1}\sigma_t - \lambda\sqrt{\alpha_t}\right)^2\, dt\right] \\
\leq & \log(x_0) + P_0,
\end{aligned}
$$

and thus, by making use of the explicit expression for P_t given in (6.4.3), we have

$$
\sup_{\pi\in\mathcal{A}_l} \mathbb{E}\left[\log(X_T^\pi)\right] \leq \log(x_0) + g_1(0) + g_2(0)r_0 + g_3(0)\alpha_0. \tag{6.4.11}
$$

In particular, when we opt for the strategy π_t^{B*} and $\pi_t^{S^1*}$ given by (6.4.8), we find that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(\pi_t^{B*} h_0(K)\sqrt{\eta_1 r_t + \eta_2} + Z_{0,t}\right)^2 + \left(\pi_t^{S^1*}\sigma_t + Z_{1,t}\right)^2 + Z_{2,t}^2\, dt\right] \\
\leq & k_1 \int_0^T \mathbb{E}\left[\eta_1 r_t + \eta_2\right]\, dt + k_2 \int_0^T \mathbb{E}[\alpha_t]\, dt < \infty,
\end{aligned}
\tag{6.4.12}
$$

where constants $k_1 = \sup_{t\in[0,T]}(\lambda_r - g_2(t))^2$ and $k_2 = \sup_{t\in[0,T]}(\lambda + g_3(t)\rho\sigma_\alpha)^2 + (1-\rho^2)g_3^2(t)\sigma_\alpha^2$. This implies that by replacing π_t^B and $\pi_t^{S^1}$ in (6.4.10) with π_t^{B*} and $\pi_t^{S^1*}$, we arrive at

$$
\mathbb{E}\left[\log(X_T^*)\right] = \log(x_0) + g_1(0) + g_2(0)r_0 + g_3(0)\alpha_0, \tag{6.4.13}
$$

where X_t^* denotes the wealth process associated with π_t^{B*} and $\pi_t^{S^1*}$ given in (6.4.8). Therefore, combining (6.4.11) and (6.4.13), we can conclude that $(\pi_t^{B*}, \pi_t^{S^{1*}})$ is the optimal strategy, and the optimal value function is given by (6.4.9). Finally, by solving the following linear SDE of X_t^* explicitly:

$$
\frac{dX_t^*}{X_t^*} = \left[\lambda_r^2(\eta_1 r_t + \eta_2) + \lambda^2\alpha_t + r_t\right]\, dt + \lambda_r\sqrt{\eta_1 r_t + \eta_2}\, dW_t^0 + \lambda\sqrt{\alpha_t}\, dW_t^1,
$$

we obtain the dynamic of wealth process X_t^* as follows:

$$\begin{aligned} X_t^* =& x_0 \exp\left\{ \int_0^t \left[\frac{\lambda_r^2}{2}(\eta_1 r_s + \eta_2) + \frac{\lambda^2}{2}\alpha_s + r_s \right] ds + \int_0^t \lambda_r \sqrt{\eta_1 r_s + \eta_2}\, dW_s^0 \right. \\ &\left. + \int_0^t \lambda \sqrt{\alpha_s}\, dW_s^1 \right\} > 0. \end{aligned} \tag{6.4.14}$$

From (6.4.9), (6.4.12), and (6.4.14), we know that $\pi^* \in \mathcal{A}_l$. This completes the proof. □

Remark 6.4.5. It is straightforward to see that the optimal investment strategy (6.4.8) for the logarithmic utility function can be obtained by letting $\gamma = 0$ in (6.3.17). This is not surprising since the case of $U_2(x) = \log(x)$ is a limiting case of $U_1(x) - 1/\gamma = (x^\gamma - 1)/\gamma$ as $\gamma \to 0$.

Corollary 6.4.6. *(The 4/2 model). If the risky asset S_t^1 follows the 4/2 model (6.2.6), then for any initial data $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given, the optimal strategy and optimal value function for problem (6.2.10) are, respectively, given by*

$$\begin{cases} \pi_t^{B*} = \dfrac{\lambda_r}{h_0(K)}, \\ \pi_t^{S^1*} = \dfrac{\lambda \alpha_t}{c_1 \alpha_t + c_2}, \end{cases}$$

and

$$V_l(\alpha_0, r_0, x_0) = \log(x_0) + (g_1(0) + g_2(0) r_0 + g_3(0)\alpha_0).$$

Proof. Plugging the specified parameters in Example 6.2.3 into Theorem 6.4.4 leads to the results in Corollary 6.4.6. □

Remark 6.4.7. By specifying $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$, Corollary 6.4.6 provides the optimal strategies under the Heston model and 3/2 model, respectively.

Corollary 6.4.8. *(Siu's non-Markovian model). If the risky asset price S_t^1 follows the non-Markovian model (6.2.7), then for any initial data $(\alpha_0, r_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given, the optimal strategy and optimal value function for the problem (6.2.10) are, respectively, given by*

$$\begin{cases} \pi_t^{B*} = \dfrac{\lambda_r}{h_0(K)}, \\ \pi_t^{S^1*} = \dfrac{\lambda \sqrt{\alpha_t}}{\hat{\sigma}\left(\alpha_{[0,t]}\right)}, \end{cases}$$

and

$$V_l(\alpha_0, r_0, x_0) = \log(x_0) + (g_1(0) + g_2(0) r_0 + g_3(0)\alpha_0).$$

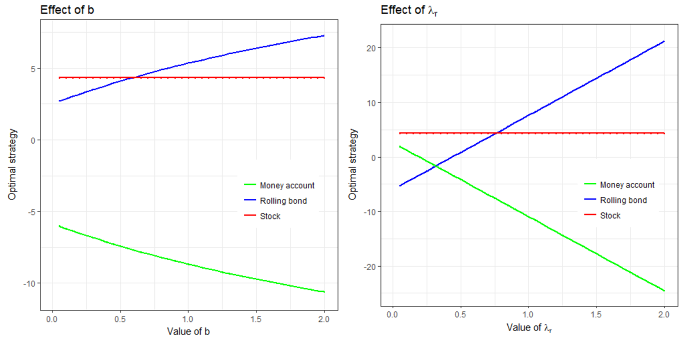
Proof. Plugging the specified parameters in Example 6.2.5 into Theorem 6.4.4 leads to the results in Corollary 6.4.8. □

6.5 Numerical studies

This section presents a sensitivity analysis of the optimal investment strategies with respect to some model parameters. We mainly focus on the hybrid Vasicek-4/2 model in the following numerical experiments because, as stated above, the 4/2 model not only recovers two parsimonious models, the Heston model and 3/2 model, as special cases but also shows practical significance in the context of derivatives pricing in the past few years. In addition, we shall concentrate on the power utility case since the logarithmic utility case can be seen as a limiting case of the former. Unless otherwise stated, we consider the following model parameters, of which the values are modified from some previous studies (see, for example, Escobar, Neykova, and Zagst (2017) and Cheng and Escobar (2021a)): $a = 0.0125,\ b = 0.266,\ \eta_2 = 0.00169,\ \lambda_r = 0.689,\ \lambda = 2.234,\ \kappa = 2.115,\ \theta = 0.051,\ \sigma_\alpha = 0.505,\ \rho = -0.514,\ T = 1,\ \alpha_0 = 0.03,\ r_0 = 0.05,\ K = 0.15,\ \gamma = 0.5,\ c_1 = 0.9051,\ c_2 = 0.0023$. And without loss of generality, we shall only investigate the impact of model parameters on the optimal investment strategy at time zero.

The relationship between optimal strategies $\left(\pi_0^{B*}, \pi_0^{S^1*}\right)$ and parameter b is presented in Figure 6.1(a). It can be seen that under the Vasicek-4/2 model, the proportion of wealth invested in the rolling bond π_0^{B*} at time zero increases with parameter b, while $\pi_0^{S^1*}$ remains unchanged. As revealed by (6.2.1), the parameter b characterizes both the mean-reversion rate and long-term mean of the interest rate process r_t. As b becomes larger, interest rate r_t moves faster towards a relatively lower level of long-term mean a/b. Therefore, the expected return rate of the risk-free asset (money account) decreases, and the investor is willing to put more wealth into the rolling bond in this case. As for the relationship between the optimal asset allocation and parameter λ_r, Figure 6.1(b) shows that π_0^{B*} increases in λ_r, while $\pi_0^{S^1*}$ is not affected. This is intuition-consistent because λ_r partially depicts the market price of interest rate risk, and a greater value of λ_r allows the investor to realize a larger interest rate risk premium by undertaking the same amount of risk.

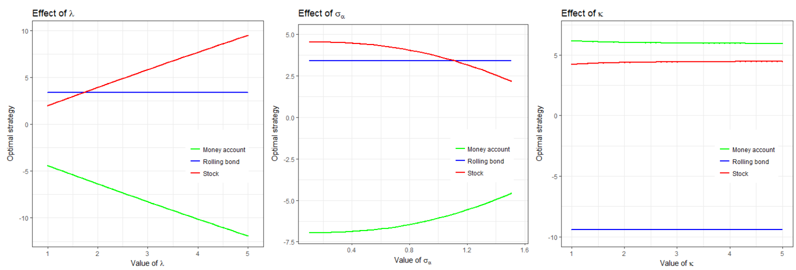
The relationships between the optimal asset allocation and parameters λ, σ_α, and κ are shown in Figures 6.2(a), 6.2(b), and 6.2(c), respectively. Specifically, Figure 6.2(a) shows that $\pi_0^{S^1*}$ increases in λ, while π_0^{B*} is not affected when λ varies. This is in line with the economic implication of parameter λ which describes the market price of volatility risk. As λ increases, the return per unit risk by investing in the risky asset (stock) increases accordingly. As such the investor tends to put more wealth into the risky asset and reduces the proportion of wealth in the risk-free asset to derive a greater expected utility at the terminal date. Conversely, as shown in Figure 6.2(b), the investor is less willing to invest in the risky asset as the volatility of variance driver σ_α under the 4/2 model becomes larger. One of the possible explanations is that by specifying $(c_1, c_2) = (0.9051, 0.0023) \approx (1, 0)$ in the above



(a) **(b)**

Figure 6.1: *Impact of parameters b and λ_r on the optimal investment strategy*

model parameter setting, the 4/2 model resembles the embedded Heston model. Thus, the state variable α_t to some extent represents the instantaneous variance of the risky asset price in this case. Consequently, as the volatility of volatility σ_α increases, the investor faces more volatility risk and is, therefore, reluctant to invest in the risky asset. Figure 6.2(c) shows that $\pi_0^{S^1*}$ has a positive relationship with κ, of which the reason is fairly similar to that of parameter σ_α. Notice that κ stands for the mean-reverting speed of the instantaneous variance of the risky asset price. As κ increases, α_t moves faster towards the long-run mean θ. Hence, the risky asset price is less volatile, and the investor tends to invest more in the risky asset.

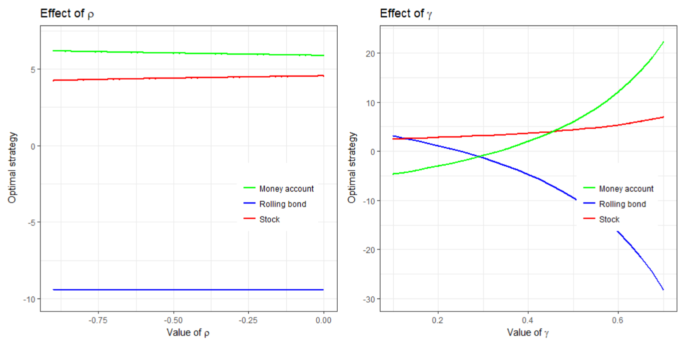


(a) **(b)** **(c)**

Figure 6.2: *Impact of parameters λ, σ_α and κ on the optimal investment strategy*

Figures 6.3(a) and 6.3(b) contribute to the evolution of the optimal asset allocation with respect to ρ and γ, respectively. Figure 6.3(a) reveals that $\pi_0^{S^1*}$ moves down as ρ varies from 0 to -0.9. The risky asset price S_t^1 and the variance driver process α_t become more negatively correlated as ρ approaches -1, which leads to more offset between the risks caused by fluctuations of the risky asset price and its volatility. Accordingly, the same amount of volatility risk can be hedged against with less investment in the risky asset. Lastly, it can be seen from Figure 6.3(b) that as the risk-aversion parameter γ becomes more positive, $\pi_0^{S^1*}$ moves upwards, whereas π_0^{B*}

moves downwards. Indeed, the investor becomes less risk-averse as γ increases, and myopic allocation increases for the asset with a higher risk premium. As revealed in our parameter setting above, the volatility risk premium is larger than the interest rate risk premium at time zero, i.e., $\lambda\sqrt{\alpha_0} > \lambda_r\sqrt{\eta_2}$. Comparatively speaking, the investor tends to increase the proportion of risky asset and, meanwhile, decrease that of rolling bonds in asset allocation.



(a) **(b)**

Figure 6.3: *Impact of parameters ρ and γ on the optimal investment strategy*

6.6 Conclusions

This paper investigates utility maximization problems in a stochastic interest rate and stochastic volatility environment. The dynamics of interest rates are described by an affine diffusion process, including the Vasicek and CIR models, as special cases. The risky asset's return rate and volatility are not specified except that the stochastic volatility risk premium is assumed to depend on a square-root diffusion (CIR) process. This general modeling framework recovers some celebrated Markovian models, such as the Heston, 3/2, 4/2 stochastic volatility models, and some non-Markovian models, as exceptional cases. The investor's objective is to maximize the expected utility of the terminal wealth for power and logarithmic utility. Given the potentially non-Markovian and incomplete market setting, we adopt a BSDE approach. To find the BSDEs, we consider the canonical decomposition of semimartingales. By exploring the uniqueness and existence results of the associated BSDEs and solving the BSDEs completely, explicit expressions for the optimal strategies and optimal value functions are derived. Furthermore, we provide analytical solutions to some special cases of our model. Finally, a sensitivity analysis on the optimal strategies to some model parameters under the hybrid Vasicek-4/2 model is presented with numerical experiments. To the best of our knowledge, there is no existing literature on utility maximization problems that simultaneously considers an affine stochastic interest rate and a general class of (non-Markovian) stochastic volatility models.

Built on our current work, several directions in the future may be followed: for instance, instead of considering the power and logarithmic utility functions, one may investigate the HARA utility maximization problems with affine stochastic interest rates and affine diffusion factor processes. One may also extend the current framework to that with multiple risky assets. We hope to address these problems in future research.

Acknowledgement(s)

The authors are very grateful to Prof. Jesper Lund Pedersen, the editor, and two anonymous reviewers for their constructive comments and suggestions, which greatly improve the quality of this paper.

Chapter 7

Optimal DC pension investment with square-root factor processes under stochastic income and inflation risks

Abstract

This paper studies optimal defined contribution (DC) pension investment problems under the expected utility maximization framework with stochastic income and inflation risks. The member has access to a financial market consisting of a risk-free asset (money account), an inflation-indexed bond, and a stock. The market price of volatility risk is assumed to depend on an affine-form, Markovian, square-root factor process, while the return rate and the volatility of the stock are possibly given by general non-Markovian, unbounded stochastic processes. This financial framework recovers the Black-Scholes model, constant elasticity of variance (CEV) model, Heston model, 3/2 model, 4/2 model, and some non-Markovian models as exceptional cases. To tackle the potentially non-Markovian structures, we adopt a backward stochastic differential equation (BSDE) approach. By solving the associated BSDEs explicitly, closed-form expressions for the optimal investment strategies and optimal value functions are obtained for the power, logarithmic, and exponential utility functions. Moreover, explicit solutions to some special cases of our portfolio model are provided. Finally, numerical examples are provided to illustrate the effects of model parameters on the optimal investment strategies under the 4/2 model.

Keywords: Expected utility maximization; DC pension; Stochastic income; Inflation risk; Square-root factor process; Backward stochastic differential equation

7.1 Introduction

Owing to the prevalence of lifespans increasing and fertility declining in recent decades, the economic role of pension management is more and more prominent. There are two different ways of pension fund management: defined benefit (DB) and defined contribution (DC) pension plans. The benefits in a DB pension plan are predetermined by the sponsor, while the contributions are initially set and subsequently adjusted to maintain a balance of the pension fund. On the contrary, in a DC pension plan, the contributions are fixed in advance by the sponsor and the benefits depend on the investment performance of the pension fund during the period up to retirement. Compared with DB pension plans, DC pension plans have the advantage of transferring the financial, longevity, and inflation risks from the sponsor to the member, and therefore relieve the pressure on social security programs (Poterba et al. (2007)). Since the benefits of retirees depend on the investment return of DC pension plans, a topic of concern to pension funds is searching for the optimal investment strategies of DC pension plans. In recent years, many attempts were made to study the optimal investment problems for DC pension plans before retirement under the expected utility maximization framework (see, for example, Boulier, Huang, and Taillard (2001), Deelstra, Grasselli, and Koehl (2003), Cairns, Blake, and Dowd (2006), Korn, Siu, and Zhang (2011), and Giacinto, Federico, and Gozzi (2011)). Since the investment of a pension plan usually lasts for a long period, it is of interest to take the risk of inflation rate into account. Battocchio and Menoncin (2004) introduced inflation risk into a DC pension management problem and derived the optimal investment strategy for an exponential utility function. By using the martingale method, Zhang and Ewald (2010) considered an optimal DC pension management problem with inflation risk for a power utility function. Han and Hung (2012) investigated the case with downside protection under stochastic inflation. For other related work concerning DC pension management with inflation risk, one may refer to Yao, Yang, and Chen (2013), Wu, Zhang, and Chen (2015), Chen et al. (2017), Wang, Li, and Sun (2021), and references therein.

In the aforementioned literature, however, the stock price dynamics were generally assumed to follow a geometric Brownian motion, that is, the volatility of the stock price was described by a constant or a deterministic function. This is not consistent with many empirical studies supporting the existence of local volatility and stochastic volatility models (see, for example, Heston (1993), Cox (1996), and Lewis (2000)). Recently, some research outputs on DC pension management problems with stochastic (local) volatility were achieved. For example, Xiao, Hong, and Qin (2007) studied the constant elasticity of variance (CEV) model for a DC pension plan problem, and they obtained the explicit solutions to the logarithmic utility function by using the Legendre transform and dual theory. Gao (2009) extended the results of Xiao, Hong, and Qin (2007) to the cases for the power and exponential utility functions. Apart from the CEV model, many attempts were made to study Heston's model (Heston (1993)) for portfolio optimization problems under utility maximization and mean-variance criteria. See, for example, Kraft (2005), Zeng and Taksar (2013) and Li, Shen, and Zeng (2018). In the field of DC pension plans, Guan and Liang (2014) considered Heston's model and an affine stochastic interest rate simultaneously. Wang and Li (2018) stepped forward by incorporating probability measure ambiguity into the framework of Guan and Liang (2014). Zeng et al. (2018) further introduced derivatives trading into DC pension plans under Heston's model. Chang, Li, and Zhao (2022) alternatively studied a robust optimal DC pension management problem under mean-variance criteria. In 2017, a state-of-the-art stochastic volatility model named the 4/2 model was proposed in Grasselli (2017) and a strong rationale for introducing this model into the field of option pricing was reported by the author. This new influential model recovers the Heston model and 3/2 model (Lewis (2000)) as special cases, and due to the superposition of these two embedded parsimonious models, the 4/2 model can accurately capture the evolution of the implied volatility surface. Considering the recent success of the 4/2 model in the context of derivatives pricing (see, for example, Cui, Kirkby, and Nguyen (2017), Lin et al. (2017), and Zhu and Wang (2019)), Cheng and Escobar (2021a) and Zhang (2021a) studied the 4/2 model on portfolio optimization problems under utility maximization and mean-variance criteria, respectively. It seems that DC pension management problems under the 4/2 model may have not yet been well-explored.

In this paper, we study a DC pension investment problem with inflation and volatility risk taken into consideration simultaneously under a more general model. Specifically, the inflation level and the member's stochastic income are modeled by two different geometric Brownian motions. The DC pension plan member has access to a financial market consisting of a risk-free asset (money account), an inflation-indexed bond, and a stock. Unlike most of the previous literature on the optimal DC pension management problems in which the price process of stock is usually assumed to satisfy some specific Markovian structures, such as the CEV model and Heston model, it is not a prerequisite to specify the structures of return rate and volatility of the stock price in our paper as they may be general unbounded, non-Markovian stochastic processes. On the contrary, we only assume that the market price of volatility risk depends on an affine-form, square-root, Markovian model, which includes the Black-Scholes model, CEV model, Heston model, 3/2 model, 4/2 model, and some non-Markovian models as particular cases (see Example 7.2.2-7.2.5). Based on the above settings, we subsequently formulate optimal DC pension investment problems for the power, logarithmic, and exponential utility functions. The potentially non-Markovian structures of the state variable processes in the modeling framework lead to the failure of Bellman's optimality principle, and thus, exclude the application of the dynamic programming approach which is used in most of the aforementioned literature on DC pension investment problems. We, therefore, opt for a backward stochastic differential equation (BSDE) approach. To be specific, by considering the canonical decomposition of semi-martingales with continuous sample paths and using the method of completion of squares, we establish the BSDEs associated with the above utility maximization problems. By discussing the uniqueness and existence results of the induced BSDEs and solving the BSDEs explicitly, we derive the analytical expressions of the optimal investment strategies and optimal value functions for power, logarithmic, and exponential utility. Furthermore, we provide the results for two special cases of our model: the CEV model and 4/2 model. Finally, we present some numerical examples to illustrate our results and analyze the effects of model parameters on the behavior of the optimal strategies under the 4/2 model. To sum up, compared with the existing literature, the contributions of this paper are as follows:

1. We incorporate stochastic volatility, stochastic income, and stochastic inflation simultaneously into an optimal DC pension investment problem in a general non-Markovian modeling framework, which can reduce to some special cases in the existing literature.

2. We introduce a BSDE approach to DC pension investment problems for the power, logarithmic, and exponential utility functions by utilizing the canonical decomposition of semi-martingales. Analytical representations of the optimal investment strategies, optimal wealth processes, and optimal value functions are obtained. Moreover, explicit solutions to some special cases are recovered.

3. We use Girsanov's measure change techniques, the uniqueness theorem for linear BSDEs with uniformly Lipschitz continuity (El Karoui, Peng, and Quenez (1997)), and a contradiction method to solve the induced linear BSDEs with stochastic Lipschitz continuity (Bender and Kohlmann (2000) and Wang, Ran, and Hong (2006)), which differs from the technical methods presented in Shen, Zhang, and Siu (2014), Sun and Guo (2018) and Zhang (2021b).

The remainder of this paper is organized as follows. Section 7.2 formulates the model and establishes the DC pension investment problems for the power, logarithmic and exponential utility functions. Section 7.3 discusses the solvability of associated BSDEs; the explicit expressions of the optimal strategies and optimal value functions are derived, and two special cases are provided. Section 7.4 concentrates on the effects of model parameters on the optimal strategy with numerical analysis. Section 7.5 concludes the paper.

7.2 Model formulation

In this section, we introduce the financial market and formulate the optimal DC pension investment problems under the expected utility maximization framework.

Let $T > 0$ be a fixed constant describing the retirement time of the DC pension plan member and $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ be a filtered complete probability space satisfying the usual conditions on which are defined three one-dimensional, mutually independent Brownian motions $\left\{W_t^0\right\}_{t\in[0,T]}$, $\left\{W_t^1\right\}_{t\in[0,T]}$, and $\left\{W_t^2\right\}_{t\in[0,T]}$. The filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be generated by the three Brownian motions, and P is a real-world probability measure.

7.2.1 Financial market and income

The financial market consists of a risk-free asset (money account), an inflation-indexed bond, and a stock. The price process of the money account B_t evolves as

$$dB_t = RB_t\,dt,\ B_0 = 1,$$

where the constant $R \in \mathbb{R}$ stands for the nominal risk-free interest rate. The price process of the stock S_t is governed by the following system of stochastic differential equations (SDEs):

$$\begin{cases} dS_t = \mu_t S_t\,dt + \sigma_t S_t\,dW_t^1, \\ d\alpha_t = \kappa(\theta - \alpha_t)\,dt + \sqrt{\alpha_t}\left(\rho_1\,dW_t^1 + \rho_2\,dW_t^2\right), \end{cases} \tag{7.2.1}$$

with initial value $S_0 = s_0 \in \mathbb{R}^+$ and $\alpha_0 \in \mathbb{R}^+$ at time zero, where μ_t and $\sigma_t > 0$ are two $\mathbb{F}$-progressively measurable processes representing the appreciation rate and the volatility of the stock at time t, respectively; α_t is an affine-form, square-root process with the speed of mean reversion κ, long-run level θ and volatility $\sqrt{\rho_1^2 + \rho_2^2}$ and is related to the market price of risk $(\mu_t - R)/\sigma_t$ by

$$\frac{\mu_t - R}{\sigma_t} = \lambda\sqrt{\alpha_t},\ \lambda \in \mathbb{R}\setminus\{0\}.$$

In line with Chapter 6.3 in Jeanblanc, Chesney, and Yor (2009), we assume that the constants $\kappa, \theta \in \mathbb{R}$ satisfy $\kappa\theta \in \mathbb{R}^+$ to ensure the process α_t is non-negative for all $t \in [0, T]$, P almost surely, while we do not impose any specific conditions on the parameters $\rho_1, \rho_2 \in \mathbb{R}$. Notice that the Feller condition, i.e. $2\kappa\theta \geq \rho_1^2 + \rho_2^2$, is not imposed on the parameters $\kappa, \theta, \rho_1, \rho_2 \in \mathbb{R}$ in our case, which makes the modeling framework more general.

Remark 7.2.1. We shall see below that the modeling framework includes, but is not limited to, some classical Markovian models in finance, such as the Black-Scholes model, CEV model (Cox (1996)), Heston model (Heston (1993)), 3/2 model (Lewis (2000)) and 4/2 model (Grasselli (2017)), as well as some non-Markovian models (Siu (2012)) as exceptional cases.

Example 7.2.2. (CEV model). If $\mu_t = \mu \neq R$, $\sigma_t = \sigma S_t^{\beta}$, where parameters $\mu \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$ and $\beta \leq -\frac{1}{2}$, then the stock price is governed by the CEV model:

$$dS_t = S_t \left(\mu \, dt + \sigma S_t^{\beta} \, dW_t^1\right), \; S_0 = s_0 > 0, \tag{7.2.2}$$

where β is called the elasticity parameter. By setting $\alpha_t = S_t^{-2\beta}, \kappa = 2\beta\mu, \theta = (\beta + \frac{1}{2})\frac{\sigma^2}{\mu}, \rho_1 = -2\beta\sigma, \rho_2 = 0$ and $\lambda = \frac{\mu - R}{\sigma}$, we see

$$\begin{aligned} d\alpha_t =& 2\beta\mu \left[\left(\beta + \frac{1}{2}\right)\frac{\sigma^2}{\mu} - S_t^{-2\beta}\right] dt - 2\beta\sigma S_t^{-\beta} \, dW_t^1 \\ =& \kappa(\theta - \alpha_t) \, dt + \sqrt{\alpha_t} \left(\rho_1 \, dW_t^1 + \rho_2 \, dW_t^2\right). \end{aligned}$$

In particular, if we set $\beta = 0$, then the condition $\kappa\theta \geq 0$ still holds, and the CEV model (7.2.2) is reduced to the Black-Scholes model in this case.

Example 7.2.3. (The 4/2 model). If $\mu_t = R + \lambda(c_1\alpha_t + c_2)$, $\sigma_t = c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$, $V_t = \alpha_t$, $\kappa \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$, $\rho_1 = \sigma_v \rho$ and $\rho_2 = \sigma_v\sqrt{1 - \rho^2}$, where constants $c_1 \geq 0, c_2 \geq 0, \sigma_v \in \mathbb{R}^+$, and $\rho \in (-1, 1)$, then the stock price process corresponds to the 4/2 model (Grasselli (2017)):

$$\begin{cases} dS_t =& S_t \left[(R + \lambda(c_1 V_t + c_2)) \; dt + \left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}}\right) \, dW_t^1\right], \; S_0 = s_0 \in \mathbb{R}^+, \\ dV_t =& \kappa(\theta - V_t) \, dt + \sigma_v\sqrt{V_t}\left(\rho \, dW_t^1 + \sqrt{1 - \rho^2} \, dW_t^2\right), \; V_0 = v_0 = \alpha_0 \in \mathbb{R}^+. \end{cases} \tag{7.2.3}$$

Here $\kappa \in \mathbb{R}^+$ is the mean-reversion rate, $\theta \in \mathbb{R}^+$ is the long-run mean, $\sigma_v \in \mathbb{R}^+$ is the volatility of the variance driver process, and $\rho \in (-1, 1)$ is the correlation coefficient between the stock price and its variance driver. For the 4/2 model (7.2.3), we posit that the Feller condition holds, i.e. $2\kappa\theta \geq \sigma_v^2$ so that the process V_t which drives the volatility of the stock price is strictly positive for all $t \in [0, T]$, P almost surely.

Remark 7.2.4. The parameters c_1 and c_2 are crucial in the 4/2 model characterizing the superposition of two embedded parsimonious models, the Heston model (Heston (1993)) and 3/2 model (Lewis (2000)), and predicting a flattening and steepening of the implied volatility skew, respectively (see, for example, Grasselli (2017)). Particularly, by specifying $(c_1, c_2) = (1, 0)$ in (7.2.3), the 4/2 model degenerates to the Heston model, while the case $(c_1, c_2) = (0, 1)$ corresponds to the 3/2 model.

Example 7.2.5. (A non-Markovian model). If $\alpha_t = V_t$, $\mu_t = R + \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]})$ and $\sigma_t = \hat{\sigma}(\alpha_{[0,t]})$ for some functional $\hat{\sigma} : \mathcal{C}([0, t]; \mathbb{R}) \mapsto \mathbb{R}^+$, where $\alpha_{[0,t]} := (\alpha_s)_{s\in[0,t]}$ is the restriction of $\alpha \in \mathcal{C}([0, T]; \mathbb{R})$ to $\mathcal{C}([0, t]; \mathbb{R})$, i.e. the space of real-valued, continuous functions defined on $[0, t]$. The stock price process is then given by the following path-dependent model:

$$\begin{cases} dS_t =& S_t \left[\left(R + \lambda\sqrt{V_t}\hat{\sigma}\left(V_{[0,t]}\right)\right) \, dt + \hat{\sigma}\left(V_{[0,t]}\right) \, dW_t^1\right], \; S_0 = s_0 \in \mathbb{R}^+, \\ dV_t =& \kappa(\theta - V_t) \, dt + \sqrt{V_t}\left(\rho_1 \, dW_t^1 + \rho_2 \, dW_t^2\right), \; V_0 = \alpha_0 \in \mathbb{R}^+. \end{cases} \tag{7.2.4}$$

Since the appreciation rate and the volatility of the stock price are path-dependent, the model (7.2.4) is a specific case of non-Markovian models. For more details on model (7.2.4), readers may refer to Siu (2012).

As mentioned in the introduction, since the investment of pension funds usually involves a long period, we shall consider the impact of inflation risk on the DC pension member's wealth. To describe the inflation risk, we follow Wang et al. (2021) to assume that the price index P_t is given by the geometric Brownian motion below:

$$dP_t = P_t \left(\mu_p \, dt + \sigma_p \, dW_t^0 \right), \; P_0 = p_0 \in \mathbb{R}^+, \tag{7.2.5}$$

where constants $\mu_p \in \mathbb{R}$ and $\sigma_p \in \mathbb{R}^+$ stand for the appreciation rate and the volatility of inflation, respectively. To hedge against the inflation risk, we introduce an inflation-indexed bond into the market and assume the price process of the inflation-indexed bond I_t follows the following SDE:

$$\frac{dI_t}{I_t} = r \, dt + \frac{dP_t}{P_t} = (r + \mu_p) \, dt + \sigma_p \, dW_t^0, \tag{7.2.6}$$

where the constant $r \in \mathbb{R}$ represents the real interest rate. Similar to Wang, Li, and Sun (2021), we further posit that $r + \mu_p > R$ holds true, because of which the inflation-indexed bond admits a positive risk premium.

Apart from making investments in the above financial market, in the DC pension plan, the member continuously contributes a fixed percentage $\eta \in [0, 1]$ of her nominal income to her wealth during the period up to retirement time T. As Zhang and Ewald (2010) and Wang, Li, and Sun (2021), we assume that the nominal income $L_{N,t}$ is stochastic and is driven by

$$\frac{dL_{N,t}}{L_{N,t}} = \mu_l \, dt + \sigma_l \, dW_t^0, \; L_{N,0} = l_{N,0} \in \mathbb{R}^+, \tag{7.2.7}$$

where the constants $\mu_l \in \mathbb{R}$ and $\sigma_l \in \mathbb{R}^+$ stand for the expected growth rate and the volatility of the nominal income, respectively.

Let π_t^S and π_t^I denote the proportions of nominal wealth invested in the stock and the inflation-indexed bond at time t, respectively. The two-dimensional process $\pi := \left(\{\pi_t^S\}_{t \in [0,T]}, \{\pi_t^I\}_{t \in [0,T]} \right)$ represents the investment strategy. Let $X_{N,t}^\pi$ be the nominal wealth process associated with π. Suppose that the financial market is friction-less and infinite short-selling and leverage are allowed, under a self-financing condition, the dynamics of the nominal wealth process $X_{N,t}^\pi$ is described by the following SDE:

$$\begin{aligned} dX_{N,t}^\pi = & \left[X_{N,t}^\pi R + X_{N,t}^\pi \pi_t^S (\mu_t - R) + X_{N,t}^\pi \pi_t^I (r + \mu_p - R) + \eta L_{N,t} \right] \, dt \\ & + X_{N,t}^\pi \pi_t^I \sigma_p \, dW_t^0 + X_{N,t}^\pi \pi_t^S \sigma_t \, dW_t^1, \; X_{N,0}^\pi = x_{N,0} \in \mathbb{R}. \end{aligned} \tag{7.2.8}$$

Denote by $X_t^\pi := X_{N,t}^\pi / P_t$ and $L_t := L_{N,t}/P_t$ the real wealth and the real income at time t after stripping out inflation, respectively. Applying Itô's formula to X_t^π yields the following dynamics of inflation-adjusted wealth process:

$$\begin{aligned} dX_t^\pi = &\left[rX_t^\pi + X_t^\pi \pi_t^S (\mu_t - R) + X_t^\pi (\pi_t^I - 1)\left(r + \mu_p - R - \sigma_p^2\right) + \eta L_t \right] dt \\ &+ X_t^\pi (\pi_t^I - 1)\sigma_p \, dW_t^0 + X_t^\pi \pi_t^S \sigma_t \, dW_t^1, \;\; X_0^\pi = x_0 = x_{N,0}/p_0 \in \mathbb{R}. \end{aligned} \tag{7.2.9}$$

Similarly, the dynamics of the inflation-adjusted income L_t is driven by

$$\frac{dL_t}{L_t} = \left(\mu_l - \mu_p + \sigma_p^2 - \sigma_l \sigma_p\right) dt + (\sigma_l - \sigma_p) \, dW_t^0, \;\; L_0 = l_0 = l_{N,0}/p_0 \in \mathbb{R}^+. \tag{7.2.10}$$

Throughout the rest of the paper, we denote by P_0 the real-word probability measure conditioned on the initial data (α_0, x_0, l_0) at time zero, and $\mathrm{E}_0[\cdot]$ represents the associated expectation.

7.2.2 Portfolio optimization problems

In this paper, we will subsequently consider three utility maximization problems when the risk preferences of the member are characterized by a power utility function $U_1(x) = \gamma^{-1} x^\gamma$ with parameter $\gamma \in (0,1)$, a logarithmic utility function $U_2(x) = \log(x)$, and an exponential utility function $U_3(x) = -e^{-qx}$ with parameter $q \in \mathbb{R}^+$. To this end, we give below the formal definitions of the admissible strategies for these three optimization problems, respectively.

Definition 7.2.6. *(Admissible strategy for power utility). For the power utility function $U_1(\cdot)$, an investment strategy π is called admissible if*

1. *π is $\mathbb{F}$-progressively measurable;*
2. *for any initial data $(\alpha_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ such that $x_0 + G_{1,0} \in \mathbb{R}^+$, the associated inflation-adjusted wealth process (7.2.9) admits a pathwise unique solution such that $X_t^\pi + G_{1,t} > 0$ holds for all $t \in [0,T]$, where $G_{1,t}$ is given by (7.3.13) below.*

The set of admissible strategies is denoted by Π_p.

In this case, the control problem corresponds to determining an admissible strategy $\pi \in \Pi_p$ such that the following expected utility of the terminal real wealth X_T^π is maximized:

$$\sup_{\pi \in \Pi_p} \mathrm{E}_0 \left[U_1(X_T^\pi) \right]. \tag{7.2.11}$$

We denote by $V_1(\alpha_0, x_0, l_0)$ the corresponding optimal value function.

Definition 7.2.7. *(Admissible strategy for logarithmic utility). For the logarithmic utility function $U_2(\cdot)$, an investment strategy π is called admissible if*

1. *the family of random variables* $\left\{\log(X^{\pi}_{\tau_n\wedge T}+G_{2,\tau_n\wedge T})+Y_{2,\tau_n\wedge T}\right\}_{n\in\mathbb{N}}$ *is uniformly integrable, for any sequence of* $\mathbb{F}$*-stopping times* $\{\tau_n\}_{n\in\mathbb{N}}$ *such that* $\tau_n\uparrow\infty$, P_0 *almost surely as* $n\to\infty$*, where* $Y_{2,t}$ *and* $G_{2,t}$ *are given by (7.3.20) and (7.3.23), respectively;*

2. *for any initial data* $(\alpha_0,x_0,l_0)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ *such that* $x_0+G_{2,0}\in\mathbb{R}^+$*, the associated inflation-adjusted wealth process (7.2.9) admits a pathwise unique solution such that* $X^{\pi}_t+G_{2,t}>0$ *holds for all* $t\in[0,T]$*, where* $G_{2,t}$ *is given by (7.3.23); in particular,* $X^{\pi}_T>0$ *holds.*

3. π *is* $\mathbb{F}$*-progressively measurable.*

The set of admissible strategies is denoted by Π_l*.*

For the logarithmic utility function $U_2(\cdot)$, the control problem becomes

$$\sup_{\pi\in\Pi_l}\mathrm{E}_0\left[U_2(X^{\pi}_T)\right], \tag{7.2.12}$$

and the corresponding optimal value function is denoted as $V_2(\alpha_0,x_0,l_0)$.

Definition 7.2.8. *(Admissible strategy for exponential utility). For the exponential utility function* $U_3(\cdot)$*, an investment strategy* π *is called admissible if*

1. *the family of random variables* $\left\{-e^{-\tilde{q}(\tilde{X}^{\pi}_{T\wedge\tau_n}-Y_{3,T\wedge\tau_n})}\right\}_{n\in\mathbb{N}}$ *is uniformly integrable, for any sequence of* $\mathbb{F}$*-stopping times* $\{\tau_n\}_{n\in\mathbb{N}}$ *such that* $\tau_n\uparrow\infty$, P_0 *almost surely as* $n\to\infty$*, where* $\tilde{q}=qe^{rT}\in\mathbb{R}^+$*, and* $\tilde{X}^{\pi}_t$ *and* $Y_{3,t}$ *are given by (7.3.27) and (7.3.32), respectively.*

2. *for any initial data* $(\alpha_0,x_0,l_0)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$*, the associated inflation-adjusted wealth process (7.2.9) admits a pathwise unique solution;*

3. π *is* $\mathbb{F}$*-progressively measurable.*

The set of admissible strategies is denoted by Π_e*.*

For the exponential utility function $U_3(\cdot)$, the control problem is given by

$$\sup_{\pi\in\Pi_e}\mathrm{E}_0\left[U_3(X^{\pi}_T)\right]. \tag{7.2.13}$$

In this case, the corresponding optimal value function is denoted by $V_3(\alpha_0,x_0,l_0)$.

Remark 7.2.9. As discussed in the introduction, the non-Markovian structures of the stock price process (7.2.1) and inflation-adjusted wealth process (7.2.9) lead to the failure of Bellman's optimality principle, so the dynamic programming approach used in most of the related literature (see, for example, Xiao, Hong, and Qin (2007), Gao (2009), Kraft (2005), Zeng and Taksar (2013), Wang and Li (2018), Wang, Li, and Sun (2021), and Cheng and Escobar (2021a)) cannot be applied in our paper. We, therefore, opt for a general BSDE approach to solve the above three expected utility maximization problems in Section 7.3.

7.3 Solution to the optimization problem

In this section, we devote ourselves to solving the utility maximization problems (7.2.11)-(7.2.13) explicitly by using a BSDE approach.

7.3.1 Optimal investment strategies for the power utility function

To find the BSDEs associated with problem (7.2.11), we introduce the following two continuous $(\mathbb{F}, \mathrm{P}_0)$-semi-martingales $Y_{1,t}$ and $G_{1,t}$ with canonical decomposition:

$$dY_{1,t} = \Psi_{1,t}\, dt + Z_{0,t}\, dW_t^0 + Z_{1,t}\, dW_t^1 + Z_{2,t}\, dW_t^2,$$

and

$$dG_{1,t} = H_{1,t}\, dt + \Lambda_{0,t}\, dW_t^0 + \Lambda_{1,t}\, dW_t^1 + \Lambda_{2,t}\, dW_t^2,$$

where $\Psi_{1,t}, Z_{0,t}, Z_{1,t}, H_{1,t}, \Lambda_{0,t}, \Lambda_{1,t}, \Lambda_{2,t}$ are undetermined $\mathbb{F}$-progressively measurable processes. For any admissible strategy $\pi \in \Pi_p$, applying Itô's formula to $Y_{1,t}\frac{(X_t^\pi + G_{1,t})^\gamma}{\gamma}$ and using the method of completion of squares lead to

$$\begin{aligned}
& d\left(Y_{1,t}\frac{(X_t^\pi + G_{1,t})^\gamma}{\gamma}\right) \\
=& \frac{1}{2}(\gamma-1)Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-2}\left[(X_t^\pi\left(\pi_t^I - 1\right)\sigma_p + \Lambda_{0,t}) + \frac{1}{\gamma-1}(X_t^\pi + G_{1,t})\Bigg(\frac{Z_{0,t}}{Y_{1,t}} + \right. \\
& \left.\left.\frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\right)\right]^2 dt + \frac{1}{2}(\gamma-1)Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-2}\left[(X_t^\pi \pi_t^S \sigma_t + \Lambda_{1,t}) + \frac{1}{\gamma-1}\right. \\
& \left.\times(X_t^\pi + G_{1,t})\left(\frac{Z_{1,t}}{Y_{1,t}} + \lambda\sqrt{\alpha_t}\right)\right]^2 dt + (X_t^\pi + G_{1,t})^\gamma\left[\frac{\Psi_{1,t}}{\gamma} + rY_{1,t} - \frac{Y_{1,t}}{2(\gamma-1)}\Bigg(\frac{Z_{0,t}}{Y_{1,t}}\right. \\
& \left. + \frac{r+\mu_p - R-\sigma_p^2}{\sigma_p}\Bigg)^2 - \frac{Y_{1,t}}{2(\gamma-1)}\left(\frac{Z_{1,t}}{Y_{1,t}} + \lambda\sqrt{\alpha_t}\right)^2\right] dt + Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-1}\Bigg[H_{1,t} \\
& \left. + \eta L_t - rG_{1,t} - \frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\Lambda_{0,t} - \lambda\sqrt{\alpha_t}\Lambda_{1,t} + \frac{Z_{2,t}}{Y_{1,t}}\Lambda_{2,t}\right] dt + \frac{1}{2}(\gamma-1)Y_{1,t} \\
& \times (X_t^\pi + G_{1,t})^{\gamma-2}\Lambda_{2,t}^2\, dt + \Bigg[\frac{(X_t^\pi + G_{1,t})^\gamma}{\gamma}Z_{0,t} + Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-1}(X_t^\pi(\pi_t^I - 1)\sigma_p \\
& \left. + \Lambda_{0,t})\right] dW_t^0 + \left[\frac{(X_t^\pi + G_{1,t})^\gamma}{\gamma}Z_{1,t} + Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-1}(X_t^\pi \pi_t^S \sigma_t + \Lambda_{1,t})\right] dW_t^1 \\
& + \left[\frac{(X_t^\pi + G_{1,t})^\gamma}{\gamma}Z_{2,t} + Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-1}\Lambda_{2,t}\right] dW_t^2.
\end{aligned} \tag{7.3.1}$$

Inspired by the right-hand side of (7.3.1), we shall consider the following nonlinear BSDE of $(Y_{1,t}, Z_{0,t}, Z_{1,t}, Z_{2,t})$:

$$\begin{cases} dY_{1,t} = \Bigg[\left(\dfrac{\gamma}{2(\gamma-1)}\dfrac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2}+\dfrac{\gamma}{2(\gamma-1)}\lambda^2\alpha_t-r\gamma\right)Y_{1,t}+\dfrac{\lambda\gamma}{\gamma-1}\sqrt{\alpha_t}Z_{1,t} \\ \qquad +\dfrac{\gamma}{\gamma-1}\dfrac{r+\mu_p-R-\sigma_p^2}{\sigma_p}Z_{0,t}+\dfrac{\gamma}{2(\gamma-1)}\dfrac{Z_{0,t}^2}{Y_{1,t}}+\dfrac{\gamma}{2(\gamma-1)}\dfrac{Z_{1,t}^2}{Y_{1,t}}\Bigg]\,dt+Z_{0,t}\,dW_t^0 \\ \qquad + Z_{1,t}\,dW_t^1+Z_{2,t}\,dW_t^2, \\ Y_{1,T} = 1, \\ Y_{1,t} > 0, \text{ for all } t\in[0,T), \end{cases} \tag{7.3.2}$$

and the linear BSDE of $(G_{1,t}, \Lambda_{0,t}, \Lambda_{1,t}, \Lambda_{2,t})$:

$$\begin{cases} dG_{1,t} = \left[rG_{1,t}+\dfrac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\Lambda_{0,t}+\lambda\sqrt{\alpha_t}\Lambda_{1,t}-\dfrac{Z_{2,t}}{Y_{1,t}}\Lambda_{2,t}-\eta L_t\right]dt+\Lambda_{0,t}\,dW_t^0 \\ \qquad + \Lambda_{1,t}\,dW_t^1+\Lambda_{2,t}\,dW_t^2, \\ G_{1,T} = 0. \end{cases} \tag{7.3.3}$$

Remark 7.3.1. The generators of two BSDEs (7.3.2) and (7.3.3) are unbounded due to the unboundedness of processes α_t and L_t. Hence, although there are some established existence and uniqueness results of BSDEs theory (see, for example, El Karoui, Peng, and Quenez (1997), Bender and Kohlmann (2000), Wang, Ran, and Hong (2006), Kobylanski (2000) and Briand and Hu (2008)), none of them can be applied to (7.3.2) and (7.3.3) immediately. It is also worth mentioning that the presence of the term $Z_{2,t}/Y_{1,t}$ in the generator of linear BSDE (7.3.3) makes the problem more technically challenging. Nevertheless, by utilizing the Markovian structures of processes α_t and L_t, we prove the uniqueness of solutions to the above two BSDEs.

In this subsection, we make the following assumption on the model parameters:

Assumption 7.3.2. $\kappa+\rho_1\lambda\frac{\gamma}{\gamma-1}\neq 0$.

Remark 7.3.3. Assumption 7.3.2 assures that the process α_t given in (7.2.1) maintains an affine-form, square-root structure under another probability measure $\tilde{\mathrm{P}}_0$ which is equivalent to P_0 and defined in the proof of Corollary 7.3.7 below.

Proposition 7.3.4. *One solution $(Y_{1,t}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ to BSDE (7.3.2) is given by*

$$Y_{1,t} = \exp\{f_1(t)+g_1(t)\alpha_t\}, \tag{7.3.4}$$

and

$$(Z_{0,t}, Z_{1,t}, Z_{2,t}) = (0, \rho_1 g_1(t)\sqrt{\alpha_t}Y_{1,t}, \rho_2 g_1(t)\sqrt{\alpha_t}Y_{1,t}), \tag{7.3.5}$$

where functions $f_1(t)$ and $g_1(t)$ solve the following ordinary differential equations (ODEs):

$$\frac{dg_1(t)}{dt} = \left(\frac{1}{2(\gamma-1)}\rho_1^2 - \frac{1}{2}\rho_2^2\right) g_1^2(t) + \left(\kappa + \frac{\gamma}{\gamma-1}\lambda\rho_1\right) g_1(t) + \frac{\gamma}{2(\gamma-1)}\lambda^2, \tag{7.3.6}$$

$$\frac{df_1(t)}{dt} = -\kappa\theta g_1(t) + \frac{\gamma}{2(\gamma-1)}\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2} - r\gamma, \tag{7.3.7}$$

with boundary conditions $f_1(T) = g_1(T) = 0$. Moreover, the closed-form solutions to ODEs (7.3.6) and (7.3.7) are given by

$$g_1(t) = \begin{cases} \dfrac{n_{g_1^+} n_{g_1^-}\left(1 - e^{\sqrt{\Delta_{g_1}}(T-t)}\right)}{n_{g_1^+} - n_{g_1^-} e^{\sqrt{\Delta_{g_1}}(T-t)}}, \ \textit{if } \Delta_{g_1} > 0; \\ \dfrac{\left(\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2\right)(T-t) n_{g_1}^2}{\left(\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2\right)(T-t) n_{g_1} - 2}, \ \textit{if } \Delta_{g_1} = 0; \\ \dfrac{\sqrt{-\Delta_{g_1}}}{\left(\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2\right)} \tan\left(\arctan\left(\dfrac{\kappa + \frac{\gamma}{\gamma-1}\lambda\rho_1}{\sqrt{-\Delta_{g_1}}}\right) - \dfrac{\sqrt{-\Delta_{g_1}}}{2}(T-t)\right) + n_{g_1}, \ \textit{if } \Delta_{g_1} < 0, \end{cases} \tag{7.3.8}$$

and

$$f_1(t) = \int_t^T \left[\kappa\theta g_1(s) - \frac{\gamma}{2(\gamma-1)}\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2} + r\gamma\right] ds, \tag{7.3.9}$$

where

$$\begin{cases} \Delta_{g_1} = \left(\kappa + \lambda\rho_1\dfrac{\gamma}{\gamma-1}\right)^2 - \left(\dfrac{1}{\gamma-1}\rho_1^2 - \rho_2^2\right)\dfrac{\gamma}{\gamma-1}\lambda^2, \ n_{g_1} = \dfrac{-\left(\kappa + \frac{\gamma}{\gamma-1}\lambda\rho_1\right)}{\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2}, \\ n_{g_1^+} = \dfrac{-\left(\kappa + \frac{\gamma}{\gamma-1}\lambda\rho_1\right) + \sqrt{\Delta_{g_1}}}{\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2}, \ n_{g_1^-} = \dfrac{-\left(\kappa + \frac{\gamma}{\gamma-1}\lambda\rho_1\right) - \sqrt{\Delta_{g_1}}}{\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2}. \end{cases} \tag{7.3.10}$$

Proof. See Appendix 7.A. □

The next proposition reveals that function $g_1(t)$ is non-negative and bounded over $[0, T]$.

Proposition 7.3.5. *Function $g_1(t)$ is strictly decreasing in t, and $0 \le g_1(t) \le g_1(0)$ for $t \in [0, T]$.*

Proof. See Appendix 7.B. □

To facilitate further discussions, we now provide an auxiliary result that follows from Lemma A1 in Shen and Zeng (2015).

Lemma 7.3.6. *(Bona-fide martingale property). If $m_1(t)$ and $m_2(t)$ are bounded functions on $[0,T]$, then the stochastic exponential processes defined by*

$$\exp\left\{-\frac{1}{2}\int_0^t (m_1^2(s)+m_2^2(s))\alpha_s\,ds + \int_0^t m_1(s)\sqrt{\alpha_s}\,dW_s^1 + \int_0^t m_2(s)\sqrt{\alpha_s}\,dW_s^2\right\}$$

is an $(\mathbb{F}, \mathrm{P}_0)$-martingale.

Corollary 7.3.7. *Under Assumption 7.3.2, the stochastic exponential process*

$$\begin{aligned}&\exp\left\{\int_0^t \frac{1}{1-\gamma}\rho_1 g_1(s)\sqrt{\alpha_s}\,d\tilde{W}_s^1 + \int_0^t \rho_2 g_1(s)\sqrt{\alpha_s}\,d\tilde{W}_s^2\right.\\ &\left.-\frac{1}{2}\int_0^t\left(\frac{\rho_1^2}{(1-\gamma)^2}+\rho_2^2\right)g_1^2(s)\alpha_s\,ds\right\}\end{aligned} \tag{7.3.11}$$

is an $(\mathbb{F}, \tilde{\mathrm{P}}_0)$-martingale, where probability measure $\tilde{\mathrm{P}}_0$ is defined by

$$\begin{aligned}\left.\frac{d\tilde{\mathrm{P}}_0}{d\mathrm{P}_0}\right|_{\mathcal{F}_T} = \exp\Bigg\{&-\int_0^T \frac{\gamma}{\gamma-1}\frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\,dW_t^0 - \int_0^T \frac{\gamma}{\gamma-1}\lambda\sqrt{\alpha_t}\,dW_t^1\\ &-\frac{1}{2}\int_0^T \frac{\gamma^2}{(\gamma-1)^2}\left[\frac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2}+\lambda^2\alpha_t\right]dt\Bigg\},\end{aligned}$$

and $\tilde{W}_t^1$ and $\tilde{W}_t^2$ are two mutually independent Brownian motions under $\tilde{\mathrm{P}}_0$ measure.

Proof. See Appendix 7.C. □

In the next theorem, we show that the solution to nonlinear BSDE (7.3.2) given in (7.3.4) and (7.3.5) must be unique.

Theorem 7.3.8. *Under Assumption 7.3.2, the solution $(Y_{1,t}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ given in (7.3.4) and (7.3.5) is the unique solution to BSDE (7.3.2).*

Proof. See Appendix 7.D. □

After solving nonlinear BSDE (7.3.2) explicitly, we can now simplify linear BSDE (7.3.3) as follows:

$$\begin{cases} dG_{1,t} = \left[rG_{1,t} + \dfrac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\Lambda_{0,t} + \lambda\sqrt{\alpha_t}\Lambda_{1,t} - \rho_2 g_2(t)\sqrt{\alpha_t}\Lambda_{2,t} - \eta L_t\right]dt \\ \qquad\qquad + \Lambda_{0,t}\,dW_t^0 + \Lambda_{1,t}\,dW_t^1 + \Lambda_{2,t}\,dW_t^2, \\ G_{1,T} = 0. \end{cases} \tag{7.3.12}$$

Remark 7.3.9. Instead of opting for the uniqueness and existence results of linear BSDEs with stochastic Lipscthiz continuity which entail strong assumptions on the coefficients within the generator (see Theorem 4 in Bender and Kohlmann (2000) or Theorem 4.1 in Wang, Ran, and Hong (2006)), we observe that a linear BSDE with uniform Lipschitz continuity (7.E.2) constructed by the difference of two possible solutions to BSDE (7.3.12) always admits a zero solution. This result allows us to conclude that linear BSDE (7.3.12) admits a unique solution and so does BSDE (7.3.3).

The next proposition presents the explicit expression of the unique solution to linear BSDE (7.3.12).

Proposition 7.3.10. *The unique solution to linear BSDE (7.3.12) is given by*

$$(G_{1,t}, \Lambda_{0,t}, \Lambda_{1,t}, \Lambda_{2,t}) = (b_1(t)L_t, b_1(t)(\sigma_l - \sigma_p)L_t, 0, 0)\,, \tag{7.3.13}$$

where function $b_1(t)$ is given by

$$b_1(t) = \begin{cases} \eta(T-t), & \text{if } m = 0; \\ \dfrac{\eta\left(e^{m(T-t)} - 1\right)}{m}, & \text{if } m \neq 0, \end{cases} \tag{7.3.14}$$

and $m = \mu_l - R - \sigma_l\sigma_p - \frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\sigma_l$.

Proof. See Appendix 7.E. □

Theorem 7.3.11. *Under Assumption 7.3.2, for any initial data $(\alpha_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given at time zero such that $x_0 + b_1(0)l_0 > 0$ holds, the optimal investment strategy and optimal value function of problem (7.2.11) are respectively given by*

$$\begin{cases} \pi_t^{S*} = \dfrac{\lambda\sqrt{\alpha_t}(X_t^* + G_{1,t})}{(1-\gamma)\sigma_t X_t^*} + \dfrac{Z_{1,t}(X_t^* + G_{1,t})}{(1-\gamma)Y_{1,t}\sigma_t X_t^*}, \\ \pi_t^{I*} = \dfrac{(r+\mu_p - R - \sigma_p^2)}{(1-\gamma)\sigma_p^2}\dfrac{X_t^* + G_{1,t}}{X_t^*} + 1 - \dfrac{\Lambda_{0,t}}{\sigma_p X_t^*}, \end{cases} \tag{7.3.15}$$

and

$$V_1(\alpha_0, x_0, l_0) = \frac{(x_0 + b_1(0)l_0)^\gamma}{\gamma}\exp\left(f_1(0) + g_1(0)\alpha_0\right), \tag{7.3.16}$$

where X_t^* is the inflation-adjusted wealth process associated with π_t^{S*} and π_t^{I*}, and $Y_{1,t}, Z_{1,t}, G_{1,t}, \Lambda_{0,t}$ are explicitly given by (7.3.4), (7.3.5), and (7.3.13); functions $f_1(t), g_1(t)$, and $b_1(t)$ are given by (7.3.9), (7.3.8), and (7.3.14), respectively. Moreover, the optimal strategy $\pi^* = \left(\{\pi_t^{S*}\}_{t\in[0,T]}, \{\pi_t^{I*}\}_{t\in[0,T]}\right)$ is admissible, i.e. $\pi^* \in \Pi_p$.

Proof. See Appendix 7.F. ☐

The next two corollaries provide the results for two special cases of our model: the CEV model (7.2.2) and the 4/2 model (7.2.3).

Corollary 7.3.12. *(CEV model). Suppose that $\mu - \gamma R \neq 0$ holds true. For any initial data $(s_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given at time zero such that $x_0 + b_1(0)l_0 > 0$ holds, if the stock price S_t follows the CEV model (7.2.2) in Example 7.2.2, then the optimal strategy and optimal value function of problem (7.2.11) are, respectively, given by*

$$\begin{cases} \pi_t^{S*} = & \dfrac{(\mu - R)(X_t^* + b_1(t)L_t)}{(1-\gamma)\sigma^2 S_t^{2\beta} X_t^*} - \dfrac{2\beta\tilde{g}_1(t)(X_t^* + b_1(t)L_t)}{(1-\gamma)S_t^{2\beta}X_t^*}, \\ \pi_t^{I*} = & \dfrac{r + \mu_p - R - \sigma_p^2}{(1-\gamma)\sigma_p^2}\dfrac{X_t^* + b_1(t)L_t}{X_t^*} + \dfrac{(\sigma_p - \sigma_l)b_1(t)L_t}{\sigma_p X_t^*} + 1, \end{cases}$$

and

$$V_1(s_0, x_0, l_0) = \frac{(x_0 + b_1(0)l_0)^\gamma}{\gamma} \exp\left(\tilde{f}_1(0) + \tilde{g}_1(0)s_0^{-2\beta}\right),$$

where functions $\tilde{f}_1(t)$ and $\tilde{g}_1(t)$ are given by

$$\tilde{f}_1(t) = \int_t^T \left[(2\beta+1)\beta\sigma^2\tilde{g}_1(s) - \frac{\gamma}{2(\gamma-1)}\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2} + r\gamma\right] ds,$$

and

$$\tilde{g}_1(t) = \begin{cases} \dfrac{n_{\tilde{g}_1^+} n_{\tilde{g}_1^-}\left(1 - e^{\sqrt{\Delta_{\tilde{g}_1}}(T-t)}\right)}{n_{\tilde{g}_1^+} - n_{\tilde{g}_1^-} e^{\sqrt{\Delta_{\tilde{g}_1}}(T-t)}}, & \text{if } \Delta_{\tilde{g}_1} > 0; \\ \dfrac{\frac{1}{\gamma-1}2\beta^2\sigma^2(T-t)n_{\tilde{g}_1}^2}{\frac{1}{\gamma-1}2\beta^2\sigma^2(T-t)n_{\tilde{g}_1} - 1}, & \text{if } \Delta_{\tilde{g}_1} = 0; \\ \dfrac{\sqrt{-\Delta_{\tilde{g}_1}}}{\frac{1}{\gamma-1}4\beta^2\sigma^2}\tan\left(\arctan\left(\dfrac{2\beta\left(\mu - \frac{\lambda\sigma\gamma}{\gamma-1}\right)}{\sqrt{-\Delta_{\tilde{g}_1}}}\right) - \dfrac{\sqrt{-\Delta_{\tilde{g}_1}}}{2}(T-t)\right) + n_{\tilde{g}_1}, & \text{if } \Delta_{\tilde{g}_1} < 0, \end{cases}$$

where

$$\begin{cases} \Delta_{\tilde{g}_1} = 4\beta^2\left[\left(\mu + \dfrac{\gamma(\mu - R)}{1-\gamma}\right)^2 - \dfrac{\gamma}{(1-\gamma)^2}(\mu - R)^2\right], \; n_{\tilde{g}_1} = \dfrac{(\mu - R)\gamma - 2\beta\mu(\gamma - 1)}{2\beta\sigma^2}, \\ n_{\tilde{g}_1^+} = \dfrac{2\beta\left((\mu - R)\frac{\gamma}{\gamma-1} - 2\beta\mu\right) + \sqrt{\Delta_{\tilde{g}_1}}}{\frac{1}{\gamma-1}4\beta^2\sigma^2}, \; n_{\tilde{g}_1^-} = \dfrac{2\beta\left((\mu - R)\frac{\gamma}{\gamma-1} - 2\beta\mu\right) - \sqrt{\Delta_{\tilde{g}_1}}}{\frac{1}{\gamma-1}4\beta^2\sigma^2}. \end{cases}$$

Proof. Substituting the parameters specified in Example 7.2.2 into Theorem 7.3.11 yields the results. ☐

Remark 7.3.13. The specifications $\mu_l - \mu_p + \sigma_p^2 - \sigma_l\sigma_p = \sigma_l - \sigma_p = 0$ and $\eta = 1$ in Corollary 7.3.12 recover the results of Proposition 1 in Gao (2009).

Corollary 7.3.14. *(The 4/2 model). Suppose that $\kappa+\lambda\sigma_v\rho\frac{\gamma}{\gamma-1}\neq 0$ holds true. For any initial data $(v_0,x_0,l_0)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ fixed and given at time zero such that $x_0+b_1(0)l_0>0$ holds, if the stock price S_t follows the 4/2 model (7.2.3) in Example 7.2.3, then the optimal strategy and optimal value function of problem (7.2.11) are given by*

$$\begin{cases}\pi_t^{S*}=\dfrac{\lambda V_t(X_t^*+b_1(t)L_t)}{(1-\gamma)(c_1V_t+c_2)X_t^*}+\dfrac{\sigma_v\rho\hat{g}_1(t)V_t(X_t^*+b_1(t)L_t)}{(1-\gamma)(c_1V_t+c_2)X_t^*},\\ \pi_t^{I*}=\dfrac{r+\mu_p-R-\sigma_p^2}{(1-\gamma)\sigma_p^2}\dfrac{X_t^*+b_1(t)L_t}{X_t^*}+\dfrac{(\sigma_p-\sigma_l)b_1(t)L_t}{\sigma_pX_t^*}+1,\end{cases}$$

and

$$V_1(v_0,x_0,l_0)=\frac{(x_0+b_1(0)l_0)^\gamma}{\gamma}\exp\left(\hat{f}_1(0)+\hat{g}_1(0)v_0\right),$$

where functions $\hat{f}_1(t)$ and $\hat{g}_1(t)$ are given by

$$\hat{f}_1(t)=\int_t^T\left[\kappa\theta\hat{g}_1(s)-\frac{\gamma}{2(\gamma-1)}\frac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2}+r\gamma\right]ds,$$

and

$$\hat{g}_1(t)=\begin{cases}\dfrac{n_{\hat{g}_1^+}n_{\hat{g}_1^-}\left(1-e^{\sqrt{\Delta_{\hat{g}_1}}(T-t)}\right)}{n_{\hat{g}_1^+}-n_{\hat{g}_1^-}e^{\sqrt{\Delta_{\hat{g}_1}}(T-t)}}, & \text{if } \Delta_{\hat{g}_1}>0;\\ \dfrac{\sigma_v^2\left(\frac{\gamma}{\gamma-1}\rho^2-1\right)(T-t)n_{\hat{g}_1}^2}{\sigma_v^2\left(\frac{\gamma}{\gamma-1}\rho^2-1\right)(T-t)n_{\hat{g}_1}-2}, & \text{if } \Delta_{\hat{g}_1}=0;\\ \dfrac{\sqrt{-\Delta_{\hat{g}_1}}}{\sigma_v^2\left(\frac{\gamma}{\gamma-1}\rho^2-1\right)}\tan\left(\arctan\left(\dfrac{\kappa+\frac{\gamma\lambda\sigma_v\rho}{\gamma-1}}{\sqrt{-\Delta_{\hat{g}_1}}}\right)-\dfrac{\sqrt{-\Delta_{\hat{g}_1}}}{2}(T-t)\right)+n_{\hat{g}_1}, & \text{if } \Delta_{\hat{g}_1}<0,\end{cases}$$

where

$$\begin{cases}\Delta_{\hat{g}_1}=\left(\kappa+\lambda\sigma_v\rho\dfrac{\gamma}{\gamma-1}\right)^2-\sigma_v^2\left(\dfrac{\gamma}{\gamma-1}\rho^2-1\right)\dfrac{\gamma}{\gamma-1}\lambda^2,\ n_{\hat{g}_1}=\dfrac{-\left(\kappa+\lambda\sigma_v\rho\frac{\gamma}{\gamma-1}\right)}{\sigma_v^2\left(\frac{\gamma}{\gamma-1}\rho^2-1\right)},\\ n_{\hat{g}_1^+}=\dfrac{-\left(\kappa+\lambda\sigma_v\rho\frac{\gamma}{\gamma-1}\right)+\sqrt{\Delta_{\hat{g}_1}}}{\sigma_v^2\left(\frac{\gamma}{\gamma-1}\rho^2-1\right)},\ n_{\hat{g}_1^-}=\dfrac{-\left(\kappa+\lambda\sigma_v\rho\frac{\gamma}{\gamma-1}\right)-\sqrt{\Delta_{\hat{g}_1}}}{\sigma_v^2\left(\frac{\gamma}{\gamma-1}\rho^2-1\right)}.\end{cases}$$

Proof. Substituting the specified parameters of the 4/2 model (7.2.3) in Example 7.2.3 into Theorem 7.3.11 yields the results. □

Remark 7.3.15. Note that if we further specify either $(c_1,c_2)=(1,0)$ or $(c_1,c_2)=(0,1)$ in the 4/2 model (7.2.3), then Corollary 7.3.14 provides the explicit expressions for the optimal strategy and optimal value function of problem (7.2.11) under the Heston model and the 3/2 model, respectively.

Remark 7.3.16. By imposing $\mu_l - \mu_p + \sigma_p^2 - \sigma_l\sigma_p = \sigma_l - \sigma_p$=0 and $\eta = 1$, then $L_t = c \in \mathbb{R}$ over $[0, T]$. In this case, Corollary 7.3.14 provides the results of the DC pension management problems under the 4/2 model with inflation risk and constant income and therefore, generalizes the results presented in Ma, Zhao, and Rong (2020) (for more details, see Remark 3.1 in Ma, Zhao, and Rong (2020)).

Remark 7.3.17. By setting $\eta = 0$ in Corollary 7.3.14, the pension investment problem is reduced to a pure investment problem under the 4/2 model, and it can be verified that the optimal allocation in the stock π_t^{S*} provided in Corollary 7.3.14 is then identical to the result of Proposition 3.1 in Cheng and Escobar (2021a) in this special case. If we further consider the Heston model by specifying $(c_1, c_2) = (1, 0)$, then the optimal strategy π_t^{S*} corresponds to the result in Kraft (2005) (refer to Eq. (28) in Kraft (2005)).

7.3.2 Optimal investment strategies for the logarithmic utility function

In this subsection, we consider the logarithmic utility maximization problem (7.2.12) by applying a BSDE approach. For this, we introduce the following two $(\mathbb{F}, \mathrm{P}_0)$-semi-martingales, $Y_{2,t}$ and $G_{2,t}$, to find the BSDEs associated with problem (7.2.12):

$$dY_{2,t} = \Psi_{2,t}\,dt + P_{0,t}\,dW_t^0 + P_{1,t}\,dW_t^1 + P_{2,t}\,dW_t^2,$$

and

$$dG_{2,t} = H_{2,t}\,dt + \Gamma_{0,t}\,dW_t^0 + \Gamma_{1,t}\,dW_t^1 + \Gamma_{2,t}\,dW_t^2,$$

where $\Psi_{2,t}, P_{0,t}, P_{1,t}, P_{2,t}, H_{2,t}, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t}$ are some $\mathbb{F}$-progressively measurable processes which will be determined in the sequel. For any admissible strategy $\pi \in \Pi_l$, applying Itô's formula to $\log(X_t^\pi + G_{2,t}) + Y_{2,t}$ yields

$$\begin{aligned}
&d\left(\log(X_t^\pi + G_{2,t}) + Y_{2,t}\right)\\
&= -\frac{1}{2(X_t^\pi + G_{2,t})^2}\left[\left(X_t^\pi\left(\pi_t^I - 1\right)\sigma_p + \Gamma_{0,t}\right) - \frac{(X_t^\pi + G_{2,t})(r + \mu_p - R - \sigma_p^2)}{\sigma_p}\right]^2 dt\\
&\quad - \frac{1}{2(X_t^\pi + G_{2,t})^2}\left[\left(X_t^\pi \pi_t^S \sigma_t + \Gamma_{1,t}\right) - (X_t^\pi + G_{2,t})\lambda\sqrt{\alpha_t}\right]^2 dt\\
&\quad - \frac{1}{X_t^\pi + G_{2,t}}\left[\frac{r + \mu_p - R - \sigma_p^2}{\sigma_p}\Gamma_{0,t} + \lambda\sqrt{\alpha_t}\Gamma_{1,t} + rG_{2,t} - H_{2,t} - \eta L_t\right] dt\\
&\quad + \left[\Psi_{2,t} + r + \frac{(r + \mu_p - R - \sigma_p^2)^2}{2\sigma_p^2} + \frac{\lambda^2\alpha_t}{2}\right] dt\\
&\quad + \left[\frac{1}{X_t^\pi + G_{2,t}}\left(X_t^\pi\left(\pi_t^I - 1\right)\sigma_p + \Gamma_{0,t}\right) + P_{0,t}\right] dW_t^0 + \left[\frac{1}{X_t^\pi + G_{2,t}}\left(X_t^\pi \pi_t^S \sigma_t + \Gamma_{1,t}\right)\right.\\
&\quad \left. + P_{1,t}\right] dW_t^1 + \left(\frac{1}{X_t^\pi + G_{2,t}}\Gamma_{2,t} + P_{2,t}\right) dW_t^2.
\end{aligned} \tag{7.3.17}$$

Therefore, inspired by the right-hand side of (7.3.17), we shall consider the linear BSDE of $(Y_{2,t}, P_{0,t}, P_{1,t}, P_{2,t})$:

$$\begin{cases} dY_{2,t} = -\left(r + \frac{1}{2}\lambda^2\alpha_t + \frac{1}{2}\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2}\right) dt + P_{0,t}\, dW_t^0 \\ \qquad\qquad + P_{1,t}\, dW_t^1 + P_{2,t}\, dW_t^2, \\ Y_{2,T} = 0, \end{cases} \tag{7.3.18}$$

and the linear BSDE of $(G_{2,t}, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$:

$$\begin{cases} dG_{2,t} = \left(rG_{2,t} + \frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\Gamma_{0,t} + \lambda\sqrt{\alpha_t}\Gamma_{1,t} - \eta L_t\right) dt + \Gamma_{0,t}\, dW_t^0 \\ \qquad\qquad + \Gamma_{1,t}\, dW_t^1 + \Gamma_{2,t}\, dW_t^2, \\ G_{2,T} = 0. \end{cases} \tag{7.3.19}$$

The following Proposition 7.3.18 and 7.3.19 provide explicit solutions to linear BSDEs (7.3.18) and (7.3.19), respectively.

Proposition 7.3.18. *The unique solution $(Y_{2,t}, P_{0,t}, P_{1,t}, P_{2,t})$ to linear BSDE (7.3.18) is given by*

$$(Y_{2,t}, P_{0,t}, P_{1,t}, P_{2,t}) = \left(f_2(t) + g_2(t)\alpha_t, 0, \rho_1 g_2(t)\sqrt{\alpha_t}, \rho_2 g_2(t)\sqrt{\alpha_t}\right), \tag{7.3.20}$$

where the closed-form expressions of functions $g_2(t)$ and $f_2(t)$ are given by

$$g_2(t) = \frac{-\lambda^2\left(e^{\kappa(t-T)} - 1\right)}{2\kappa}, \tag{7.3.21}$$

and

$$f_2(t) = \left(r + \frac{1}{2}\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2} + \frac{1}{2}\lambda^2\theta\right)(T-t) - \frac{\lambda^2\theta}{2\kappa}\left(1 - e^{\kappa(t-T)}\right). \tag{7.3.22}$$

Proposition 7.3.19. *The unique solution $(G_{2,t}, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t})$ to linear BSDE (7.3.19) is given by*

$$(G_{2,t}, \Gamma_{0,t}, \Gamma_{1,t}, \Gamma_{2,t}) = \left(b_2(t)L_t, b_2(t)(\sigma_l - \sigma_p)L_t, 0, 0\right), \tag{7.3.23}$$

where the closed-form expression of function $b_2(t)$ is given by

$$b_2(t) = \begin{cases} \eta(T-t), & \text{if } m = 0; \\ \frac{\eta\left(e^{m(T-t)} - 1\right)}{m}, & \text{if } m \neq 0, \end{cases} \tag{7.3.24}$$

and $m = \mu_l - R - \sigma_l\sigma_p - \frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\sigma_l$.

The proofs of Proposition 7.3.18 and 7.3.19 are similar to that of Proposition 7.3.10, and so we omit it here.

Based on the explicit solutions to BSDE (7.3.18) and (7.3.19), we can derive the optimal strategy and the optimal value function of problem (7.2.12).

Theorem 7.3.20. *For any initial data $(\alpha_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given at time zero such that $x_0 + b_2(0)l_0 > 0$ holds, the optimal investment strategy and optimal value function of problem (7.2.12) are respectively given by*

$$\begin{cases} \pi_t^{S*} = \dfrac{(X_t^* + G_{2,t})\lambda\sqrt{\alpha_t}}{X_t^*\sigma_t}, \\ \pi_t^{I*} = \dfrac{1}{X_t^*\sigma_p}\left(\dfrac{(X_t^* + G_{2,t})(r + \mu_p - R - \sigma_p^2)}{\sigma_p} - \Gamma_{0,t}\right) + 1, \end{cases} \tag{7.3.25}$$

and

$$V_2(\alpha_0, x_0, l_0) = \log\left(x_0 + b_2(0)l_0\right) + f_2(0) + g_2(0)\alpha_0, \tag{7.3.26}$$

where X_t^* is the inflation-adjusted wealth process associated with π_t^{S*} and π_t^{I*}, and $G_{2,t}$ and $\Gamma_{0,t}$ are explicitly given by (7.3.23); functions $f_2(t), g_2(t)$, and $b_2(t)$ are respectively given by (7.3.22), (7.3.21), and (7.3.24). Moreover, the optimal strategy $\pi^* = \left(\{\pi_t^{S*}\}_{t\in[0,T]}, \{\pi_t^{I*}\}_{t\in[0,T]}\right)$ is admissible, i.e. $\pi^* \in \Pi_l$.

Proof. See Appendix 7.G. □

The following two corollaries provide the results for the CEV model and 4/2 model, respectively.

Corollary 7.3.21. *(CEV model). For any initial data $(s_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given at time zero such that $x_0 + b_2(0)l_0 > 0$ holds, if the risky asset S_t follows the CEV model (7.2.2) in Example 7.2.2, then the optimal strategy and optimal value function of problem (7.2.12) are respectively given by*

$$\begin{cases} \pi_t^{S*} = \dfrac{(X_t^* + b_2(t)L_t)(\mu - R)}{X_t^*\sigma^2 S_t^{2\beta}}, \\ \pi_t^{I*} = \dfrac{1}{X_t^*\sigma_p}\left(\dfrac{(X_t^* + b_2(t)L_t)(r + \mu_p - R - \sigma_p^2)}{\sigma_p} - (\sigma_l - \sigma_p)b_2(t)L_t\right) + 1, \end{cases}$$

and

$$V_2(s_0, x_0, l_0) = \log\left(x_0 + b_2(0)l_0\right) + \tilde{f}_2(0) + \tilde{g}_2(0)s_0^{-2\beta},$$

where functions $\tilde{f}_2(t)$ and $\tilde{g}_2(t)$ are given by

$$\begin{cases} \tilde{f}_2(t) = \left(r + \dfrac{(r + \mu_p - R - \sigma_p^2)}{2\sigma_p^2} + \dfrac{(\beta + \frac{1}{2})(\mu - R)^2}{2\mu}\right) - \dfrac{(\mu - R)^2(\beta + \frac{1}{2})\left(1 - e^{2\beta\mu(t-T)}\right)}{4\beta\mu^2}, \\ \tilde{g}_2(t) = \dfrac{(\mu - R)^2\left(1 - e^{2\beta\mu(t-T)}\right)}{4\beta\mu\sigma^2}. \end{cases}$$

Proof. Substituting the specified parameters of the CEV model (7.2.2) in Example 7.2.2 into Theorem 7.3.20 gives the results. □

Remark 7.3.22. By specifying $\mu_l - \mu_p + \sigma_p^2 - \sigma_l\sigma_p = \sigma_l - \sigma_p = 0$ and $\eta = 1$, then $L_t = c \in \mathbb{R}$ over $[0, T]$, and the optimal strategy π_t^{S*} derived in Corollary 7.3.21 is identical to the results in Xiao, Hong, and Qin (2007).

Corollary 7.3.23. *(The 4/2 model). For any initial data $(v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given at time zero such that $x_0 + b_2(0)l_0 > 0$ holds, if the risky asset S_t follows the 4/2 model (7.2.3) in Example 7.2.3, then the optimal investment strategy and optimal value function of problem (7.2.12) are respectively given by*

$$\begin{cases} \pi_t^{S*} = \dfrac{(X_t^* + b_2(t)L_t)\lambda V_t}{X_t^*\,(c_1 V_t + c_2)}, \\ \pi_t^{I*} = \dfrac{1}{X_t^*\sigma_p}\left(\dfrac{(X_t^* + b_2(t)L_t)(r + \mu_p - R - \sigma_p^2)}{\sigma_p} - (\sigma_l - \sigma_p)b_2(t)L_t\right) + 1, \end{cases}$$

and

$$V_2(v_0, x_0, l_0) = \log(x_0 + b_2(0)l_0) + \hat{f}_2(0) + \hat{g}_2(0)v_0,$$

where functions $\hat{f}_2(t)$ and $\hat{g}_2(t)$ are given by

$$\begin{cases} \hat{f}_2(t) = \left(r + \dfrac{(r + \mu_p - R - \sigma_p^2)^2}{2\sigma_p^2} + \dfrac{\lambda^2\theta}{2}\right)(T - t) - \dfrac{\lambda^2\theta}{2\kappa}\left(1 - e^{\kappa(t-T)}\right), \\ \hat{g}_2(t) = \dfrac{-\lambda^2\left(e^{\kappa(t-T)} - 1\right)}{2\kappa}. \end{cases}$$

Proof. Plugging the specified parameters of the 4/2 model (7.2.3) in Example 7.2.3 into Theorem 7.3.20 yields the results. □

Remark 7.3.24. If we specify $(c_1, c_2) = (1, 0)$ in Corollary 7.3.23, we obtain the results under Heston's model. Instead, setting $(c_1, c_2) = (0, 1)$ in Corollary 7.3.23 provides the results under the 3/2 model.

7.3.3 Optimal investment strategies for the exponential utility function

In this subsection, we investigate the exponential utility maximization problem (7.2.13). To facilitate further discussions in this subsection, we denote by $\tilde{X}_t^\pi := e^{-rt}X_t^\pi$ the discounted inflation-adjusted wealth process which is discounted by the real interest rate r. Using Itô's formula leads to

$$\begin{aligned} d\tilde{X}_t^\pi =& \left[\tilde{X}_t^\pi \pi_t^S(\mu_t - R) + \tilde{X}_t^\pi\left(\pi_t^I - 1\right)\left(r + \mu_p - R - \sigma_p^2\right) + \eta\tilde{L}_t\right] dt \\ &+ \tilde{X}_t^\pi\left(\pi_t^I - 1\right)\sigma_p\, dW_t^0 + \tilde{X}_t^\pi \pi_t^S \sigma_t\, dW_t^1,\ \tilde{X}_0^\pi = X_0^\pi = x_0, \end{aligned} \tag{7.3.27}$$

where $\tilde{L}_t := e^{-rt}L_t$ represents the discounted value of the inflation-adjusted income, and it has the following dynamics:

$$d\tilde{L}_t = \tilde{L}_t \left[(\mu_l - \mu_p + \sigma_p^2 - \sigma_l\sigma_p - r)\, dt + (\sigma_l - \sigma_p)\, dW_t^0 \right], \ \tilde{L}_0 = L_0 = l_0. \tag{7.3.28}$$

Thus, the exponential utility maximization problem (7.2.13) can be reformulated as

$$\sup_{\pi\in\Pi_e} \mathrm{E}_0 \left[-e^{-\tilde{q}\tilde{X}_T^\pi} \right], \tag{7.3.29}$$

where $\tilde{q} = qe^{rT} \in \mathbb{R}^+$. To obtain the BSDEs associated with problem (7.3.29), we introduce an $(\mathbb{F}, \mathrm{P}_0)$-semi-martingale $Y_{3,t}$ with the following decomposition:

$$dY_{3,t} = \Psi_{3,t}\, dt + M_{0,t}\, dW_t^0 + M_{1,t}\, dW_t^1 + M_{2,t}\, dW_t^2,$$

where $\Psi_{3,t}, M_{0,t}, M_{1,t}, M_{2,t}$ are some $\mathbb{F}$-progressively measurable processes which shall be determined later. For any admissible strategy $\pi \in \Pi_e$, by applying Itô's formula to $-e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}$ and using the method of completion of squares, we have

$$\begin{aligned}
& d\left(-e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}\right) \\
= & -\frac{1}{2}\tilde{q}^2 e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})} \left(\tilde{X}_t^\pi \left(\pi_t^I - 1\right)\sigma_p - M_{0,t} - \frac{r + \mu_p - R - \sigma_p^2}{\tilde{q}\sigma_p} \right)^2 dt \\
& - \frac{1}{2}\tilde{q}^2 e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})} \left(\tilde{X}_t^\pi \pi_t^S \sigma_t - M_{1,t} - \frac{\lambda\sqrt{\alpha_t}}{\tilde{q}} \right)^2 dt \\
& + \tilde{q}e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})} \Bigg[\eta\tilde{L}_t + \frac{\lambda^2\alpha_t}{2\tilde{q}} + \frac{(r+\mu_p - R - \sigma_p^2)^2}{2\tilde{q}\sigma_p^2} + \frac{r + \mu_r - R - \sigma_p^2}{\sigma_p} M_{0,t} \\
& + \lambda\sqrt{\alpha_t}M_{1,t} - \frac{\tilde{q}M_{2,t}^2}{2} - \Psi_{3,t} \Bigg] dt + \tilde{q}e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})} \left(\tilde{X}_t^\pi \left(\pi_t^I - 1\right)\sigma_p - M_{0,t} \right) dW_t^0 \\
& + \tilde{q}e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})} \left(\tilde{X}_t^\pi \pi_t^S \sigma_t - M_{1,t} \right) dW_t^1 - \tilde{q}e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})} M_{2,t}\, dW_t^2.
\end{aligned} \tag{7.3.30}$$

The right-hand side of (7.3.30) induces the following BSDE of $(Y_{3,t}, M_{0,t}, M_{1,t}, M_{2,t})$ we shall investigate in the sequel:

$$\begin{cases}
dY_{3,t} = \Bigg[\dfrac{\lambda^2\alpha_t}{2\tilde{q}} + \dfrac{(r+\mu_p - R - \sigma_p^2)^2}{2\tilde{q}\sigma_p^2} + \dfrac{r+\mu_p - R - \sigma_p^2}{\sigma_p} M_{0,t} + \lambda\sqrt{\alpha_t}M_{1,t} \\
\qquad\qquad - \dfrac{\tilde{q}M_{2,t}^2}{2} + \eta\tilde{L}_t \Bigg] dt + M_{0,t}\, dW_t^0 + M_{1,t}\, dW_t^1 + M_{2,t}\, dW_t^2, \\
Y_{3,T} = 0.
\end{cases} \tag{7.3.31}$$

Throughout the rest of this subsection, we impose the following assumption:

Assumption 7.3.25. $\kappa + \lambda\rho_1 \neq 0$.

Remark 7.3.26. Similar to the role of Assumption 7.3.2 above, Assumption 7.3.25 is essential to assure that BSDE (7.3.31) admits a unique solution.

Proposition 7.3.27. *One solution* $(Y_{3,t}, M_{0,t}, M_{1,t}, M_{2,t})$ *to BSDE (7.3.31) is given by*

$$Y_{3,t} = f_3(t) + g_3(t)\alpha_t + b_3(t)\tilde{L}_t, \tag{7.3.32}$$

and

$$(M_{0,t}, M_{1,t}, M_{2,t}) = \left((\sigma_l - \sigma_p)b_3(t)\tilde{L}_t, \rho_1 g_3(t)\sqrt{\alpha_t}, \rho_2 g_3(t)\sqrt{\alpha_t}\right), \tag{7.3.33}$$

where the closed-form expressions of functions $f_3(t), g_3(t)$, *and* $b_3(t)$ *are respectively given by*

$$g_3(t) = \begin{cases} \dfrac{n_{g_3^+}n_{g_3^-}\left(1 - e^{\sqrt{\Delta_{g_3}}(T-t)}\right)}{n_{g_3^+} - n_{g_3^-}e^{\sqrt{\Delta_{g_3}}(T-t)}}, & \text{if } \rho_2 \neq 0 \text{ and } \Delta_{g_3} > 0; \\ \dfrac{-\tilde{q}\rho_2^2(T-t)n_{g_3}^2}{-\tilde{q}\rho_2^2(T-t)n_{g_3} - 2}, & \text{if } \rho_2 \neq 0 \text{ and } \Delta_{g_3} = 0; \\ \dfrac{\lambda^2\left(e^{(\kappa+\lambda\rho_1)(t-T)} - 1\right)}{2\tilde{q}(\kappa + \lambda\rho_1)}, & \text{if } \rho_2 = 0, \end{cases} \tag{7.3.34}$$

$$b_3(t) = \begin{cases} \eta(t-T), & \text{if } m = 0; \\ \dfrac{\eta\left(1 - e^{m(T-t)}\right)}{m}, & \text{if } m \neq 0, \end{cases} \tag{7.3.35}$$

and

$$f_3(t) = \int_t^T \left(\kappa\theta g_3(s) - \frac{(r + \mu_p - R - \sigma_p^2)^2}{2\tilde{q}\sigma_p^2}\right) ds, \tag{7.3.36}$$

where $m = \mu_l - R - \sigma_l\sigma_p - \frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\sigma_l$ *and*

$$\begin{cases} \Delta_{g_3} = (\kappa + \rho_1\lambda)^2 + \rho_2^2\lambda^2, \; n_{g_3} = \dfrac{\kappa + \lambda\rho_1}{\tilde{q}\rho_2^2}, \\ n_{g_3^-} = \dfrac{-(\kappa + \lambda\rho_1) - \sqrt{\Delta_{g_3}}}{-\tilde{q}\rho_2^2}, \; n_{g_3^+} = \dfrac{-(\kappa + \lambda\rho_1) + \sqrt{\Delta_{g_3}}}{-\tilde{q}\rho_2^2}. \end{cases}$$

Proof. See Appendix 7.H. □

Proposition 7.3.28. *Function* $g_3(t)$ *is bounded over* $[0, T]$.

Proof. See Appendix 7.I. □

Theorem 7.3.29. *Under Assumption 7.3.25, the solution* $(Y_{3,t}, M_{0,t}, M_{1,t}, M_{2,t})$ *given in (7.3.32) and (7.3.33) is the unique solution to BSDE (7.3.31).*

The proof of Theorem 7.3.29 is almost identical to that of Theorem 7.3.8 above, so we omit it here. The following theorem relates the optimal strategy and optimal value function to the explicit solution to BSDE (7.3.31).

Theorem 7.3.30. *Under Assumption 7.3.25, for any initial data* $(\alpha_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ *fixed and given at time zero, the optimal investment strategy and optimal value function of problem (7.2.13) are respectively given by*

$$\begin{cases} \pi_t^{S*} = \dfrac{1}{X_t^* \sigma_t} \left(e^{rt} M_{1,t} + e^{-r(T-t)} \dfrac{\lambda}{q} \sqrt{\alpha_t} \right), \\ \pi_t^{I*} = \dfrac{1}{X_t^* \sigma_p} \left(e^{rt} M_{0,t} + e^{-r(T-t)} \dfrac{r + \mu_p - R - \sigma_p^2}{q \sigma_p} \right) + 1, \end{cases} \tag{7.3.37}$$

and

$$V_3(\alpha_0, x_0, l_0) = -\exp\left\{ -q e^{rT} \left(x_0 - f_3(0) - g_3(0)\alpha_0 - b_3(0) l_0 \right) \right\}, \tag{7.3.38}$$

where X_t^* *is the inflation-adjusted wealth process associated with* π_t^{S*} *and* π_t^{I*}*, and* $M_{0,t}$ *and* $M_{1,t}$ *are explicitly given by (7.3.33); functions* $f_3(t), g_3(t)$*, and* $b_3(t)$ *are given by (7.3.36), (7.3.34), and (7.3.35), respectively. Moreover, the optimal strategy* $\pi^* = \left(\{\pi_t^{S*}\}_{t\in[0,T]}, \{\pi_t^{I*}\}_{t\in[0,T]} \right)$ *is admissible, i.e.* $\pi^* \in \Pi_e$.

Proof. See Appendix 7.J. □

Remark 7.3.31. It should be noted that different from the power utility maximization problem (7.2.11) and logarithmic utility maximization problem (7.2.12), Theorem 7.3.30 shows that neither the inflation-adjusted wealth X_t^* nor the stochastic income L_t affects the optimal market value of inflation-adjusted wealth invested in the stock, i.e. $X_t^* \pi_t^{S*}$, which can be possibly explained by the constant absolute risk aversion q under the exponential utility function.

Corollary 7.3.32. *(CEV model). For any initial data* $(s_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ *fixed and given at time zero, if the risky asset* S_t *follows the CEV model (7.2.2) in Example 7.2.2, then the optimal strategy and optimal value function of problem (7.2.13) are given by*

$$\begin{cases} \pi_t^{S*} = \dfrac{\mu - R}{X_t^* q \sigma^2 S_t^{2\beta}} e^{r(t-T)} \left(1 + \left(1 - e^{2\beta R(t-T)} \right) \dfrac{\mu - R}{2R} \right), \\ \pi_t^{I*} = \dfrac{1}{X_t^* \sigma_p} \left((\sigma_l - \sigma_p) b_3(t) L_t + e^{-r(T-t)} \dfrac{r + \mu_p - R - \sigma_p^2}{\sigma_p q} \right) + 1, \end{cases}$$

and

$$V_3(s_0, x_0, l_0) = -\exp\left\{ -q e^{rT} \left(x_0 - \tilde{f}_3(0) - \tilde{g}_3(0) s_0^{-2\beta} - b_3(0) l_0 \right) \right\},$$

where functions $\tilde{f}_3(t)$ *and* $\tilde{g}_3(t)$ *are given by*

$$\begin{cases} \tilde{f}_3(t) = \int_t^T \left(\beta(2\beta+1)\sigma^2 - \frac{e^{-rT}(r+\mu_p-R-\sigma_p^2)^2}{2q\sigma_p^2} \right) ds, \\ \tilde{g}_3(t) = \frac{(\mu-R)^2\left(e^{2\beta R(t-T)}-1\right)}{4\beta R q \sigma^2 e^{rT}}. \end{cases}$$

Proof. Substituting the specified parameters of the CEV model (7.2.2) in Example 7.2.2 into Theorem 7.3.30 gives the results. Moreover, it is straightforward to see that Assumption 7.3.25 always holds for the CEV model due to $2\beta R \neq 0$. □

Remark 7.3.33. If we further impose that $r = R$ in Corollary 7.3.32, the optimal allocation in the stock π_t^{S*} recovers the results presented in Sun, Yong, and Gao (2020) and Proposition 2 in Gao (2009) in which stochastic income and inflation risks are not considered. In other words, Corollary 7.3.32 generalizes the existing work of Gao (2009) and Sun, Yong, and Gao (2020).

Corollary 7.3.34. *(The 4/2 model). Suppose that* $\kappa + \lambda\sigma_v\rho \neq 0$ *holds true. For any initial data* $(v_0, x_0, l_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ *fixed and given at time zero, if the risky asset* S_t *follows the 4/2 model (7.2.3) in Example 7.2.3, then the optimal strategy and optimal value function of problem (7.2.13) are respectively given by*

$$\begin{cases} \pi_t^{S*} = \frac{V_t}{X_t^*(c_1V_t+c_2)} \left(\sigma_v\rho\hat{g}_3(t)e^{rt} + e^{-r(T-t)}\frac{\lambda}{q} \right), \\ \pi_t^{I*} = \frac{1}{X_t^*\sigma_p} \left((\sigma_l-\sigma_p)b_3(t)L_t + e^{-r(T-t)}\frac{r+\mu_p-R-\sigma_p^2}{\sigma_p q} \right) + 1, \end{cases}$$

and

$$V_3(v_0, x_0, l_0) = -\exp\left\{ -qe^{rT}\left(x_0 - \hat{f}_3(0) - \hat{g}_3(0)v_0 - b_3(0)l_0 \right) \right\},$$

where functions $\hat{f}_3(t)$ *and* $\hat{g}_3(t)$ *are given by*

$$\begin{cases} \hat{f}_3(t) = \int_t^T \left(\kappa\theta\hat{g}_3(s) - \frac{(r+\mu_p-R-\sigma_p^2)^2}{2\tilde{q}\sigma_p^2} \right) ds, \\ \hat{g}_3(t) = \frac{n_{\hat{g}_3^+}n_{\hat{g}_3^-}\left(1-e^{\sqrt{\Delta_{\hat{g}_3}}(T-t)}\right)}{n_{\hat{g}_3^+} - n_{\hat{g}_3^-}e^{\sqrt{\Delta_{\hat{g}_3}}(T-t)}}, \end{cases}$$

where

$$\Delta_{\hat{g}_3} = (\kappa+\lambda\sigma_v\rho)^2 + (1-\rho^2)\sigma_v^2\lambda^2, \; n_{\hat{g}_3^-} = \frac{-(\kappa+\lambda\sigma_v\rho) - \sqrt{\Delta_{\hat{g}_3}}}{-\tilde{q}\sigma_v^2(1-\rho^2)}, \; n_{\hat{g}_3^+} = \frac{-(\kappa+\lambda\sigma_v\rho) + \sqrt{\Delta_{\hat{g}_3}}}{-\tilde{q}\sigma_v^2(1-\rho^2)}.$$

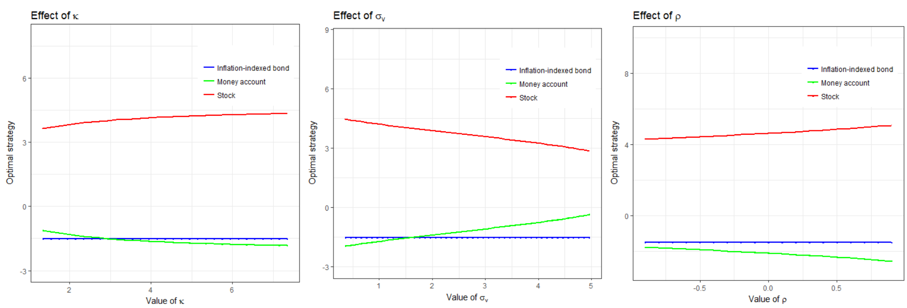
Proof. Substituting the specified parameters of the 4/2 model (7.2.3) in Example 7.2.3 into Theorem 7.3.30 yields the results. □

Remark 7.3.35. The specifications $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$ in Corollary 7.3.34 give the results under the Heston model and the 3/2 model, respectively. Moreover, to the best of our knowledge, the results provided in Corollary 7.3.34 are not reported in the existing literature. In this sense, this paper extends the results of Cheng and Escobar (2021a) and Zhang (2021a) to the case with stochastic income and inflation risks under the exponential utility framework.

7.4 Numerical analysis

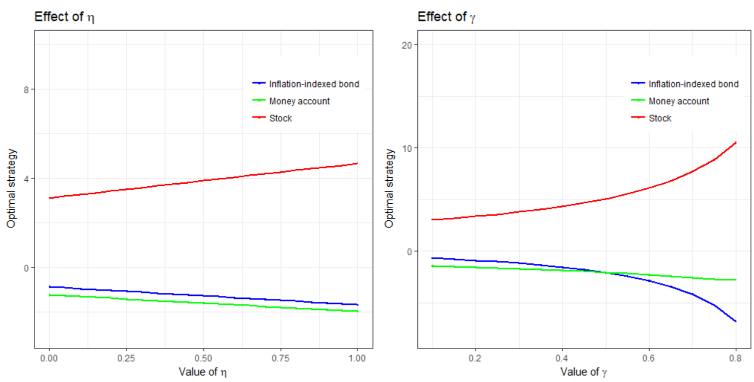
This section presents some numerical examples to illustrate the effect of some model parameters on the behavior of the optimal investment strategy. We pay attention to the sensitivity of the optimal investment strategy to some parameters within the 4/2 model (7.2.3) in the following numerical illustrations because this model includes two parsimonious models, the Heston model and the 3/2 model, as particular cases, and it has shown practical significance in the last few years as discussed in the introduction. Additionally, we concentrate on the power utility framework because the numerical experiments of the logarithmic and exponential functions can be conducted similarly. Unless otherwise stated, the hypothetical values of model parameters are given as follows: $R = 0.05$, $r = 0.02$, $\mu_l = 0.02$, $\sigma_l = 0.3$, $\mu_p = 0.01$, $\sigma_p = 0.4$, $\eta = 0.8$, $T = 1$, $\gamma = 0.4$, $x_0 = 1$, $l_0 = 0.5$, $\lambda = 2.9428$, $\kappa = 7.3479$, $\theta = 0.0328$, $\sigma_v = 0.6612$, $\rho = -0.7689$, $c_1 = 0.9051$, $c_2 = 0.023$ and $v_0 = 0.04$. Most of the parameters are taken from the recent paper of Cheng and Escobar (2021a). In the following numerical examples, we vary the value of one parameter with others fixed at each time. The range allowed for the parameters is the possibility that the conditions in Corollary 7.3.14 are satisfied, i.e., $\kappa + \lambda\sigma_v\rho\frac{\gamma}{\gamma-1} \neq 0$ and $x_0 + b_1(0)l_0 > 0$.

Figure 7.1 contributes to the evolution of the optimal investment strategy with respect to parameters κ, σ_v and ρ. In Figure 7.1(a), we vary κ from 1.3479 to 7.3479. It shows that π_0^{S*} is positively correlated with κ, while π_0^{I*} remains unchanged. Indeed, since κ depicts the mean-reversion speed of the state variable V_t towards its long-run mean θ in the 4/2 model (7.2.3), the state variable V_t moves faster towards $\theta = 0.0328 < 0.04 = v_0$ as κ increases in our case. Furthermore, we notice from the model parameter setting above that by specifying $(c_1, c_2) = (0.9051, 0.023) \approx (1, 0)$, the 4/2 model (7.2.3) resembles the 3/2 model less than the embedded Heston model of which V_t stands for the instantaneous variance. In other words, when κ becomes larger, the member faces less volatility risk. Therefore, the member is willing to adopt a more aggressive strategy in the stock as κ increases from 1.3479 to 7.3479. In Figure 7.1(b), we find that the π_0^{S*} decreases as σ_v increases. Namely, as the volatility of the state variable V_t increases, the investor has to undertake more volatility risk. Indeed, the explanation is similar to that of parameter κ above in the sense that the 4/2 model in our case has a resemblance to the Heston model such that state variable V_t plays the role as the instantaneous variance of the stock price. Thus, as the volatility of volatility increases, the member faces more volatility risk and is less willing to invest in the stock. In Figure 7.1(c), we vary ρ from 0.9 to -0.9 and find that π_0^{S*} moves downwards as ρ decreases, while π_0^{I*} remains unchanged. This is consistent with intuition because when ρ varies from 0.9 to 0, the stock price and its volatility become less positively related, while when ρ changes from 0 to -0.9, the stock price and its volatility become more negatively correlated. In such a case, the offset between the risks caused by fluctuations of the stock price and its volatility becomes more. Accordingly, the member can hedge against the same amount of volatility risk with less investment in the stock.



(a) *The effect of* κ **(b)** *The effect of* σ_v **(c)** *The effect of* ρ

Figure 7.1: *The effects of parameters* κ, σ_v, *and* ρ *on the optimal investment strategy*



(a) *The effects of* η **(b)** *The effect of* γ

Figure 7.2: *The effects of parameters* η *and* γ *on the optimal investment strategy*

The relationship between optimal strategy and parameters η and γ is presented in Figure 7.2(a) and 7.2(b), respectively. It can be seen that the optimal allocation in the stock π_0^{S*} increases with contribution rate η and risk-aversion γ whereas the optimal allocation in the inflation-indexed bond π_0^{I*} decreases with these two parameters. As η or γ increases, the member becomes less risk-averse. Therefore, myopic allocation increases for the asset with a positive market price of risk, and it decreases if the asset admits a negative market price of risk. In the 4/2 model (7.2.3), the market price of volatility risk is positive, i.e., $\lambda\sqrt{V_t} > 0$ by setting $\lambda > 0$, while the market price of inflation risk is negative, i.e., $(r + \mu_p - R - \sigma_p^2)/\sigma_p < 0$, given the values of model parameters above, we can therefore see from Figure 7.2 that π_0^{S*} and π_0^{I*} move in opposite directions as η and γ change.

7.5 Conclusion

In this paper, we investigate optimal investment problems for a DC pension member with volatility and inflation risks taken into account simultaneously. We focus on an affine-form, Markovian, square-root process for describing the market price of volatility risk, whereas the return rate and the volatility of stock price are not specified and possibly unbounded, non-Markovian processes. This modeling framework embraces the Black-Scholes model, CEV model, Heston model, 3/2 model, 4/2 model, and some non-Markovian models, as exceptional cases. Moreover, the member faces the risk of a stochastic income stream.

Due to the failure of Bellman's optimality principle in this context, we introduce a BSDE approach. To find the BSDEs, we consider the canonical decomposition of semi-martingales. By exploring the uniqueness and existence results of the induced BSDEs and solving the BSDEs explicitly, analytical expressions for the optimal strategies, optimal wealth processes, and optimal value functions are derived for the power, logarithmic, and exponential utility functions, respectively. Particularly, explicit solutions to some special cases of our portfolio model are provided: the CEV, Heston, 3/2, 4/2 models. Finally, we present some numerical examples to analyze the effects of model parameters on the behavior of the optimal strategies under the 4/2 model. To the best of our knowledge, there is no existing literature on the optimal DC pension investment problems that incorporate stochastic income, stochastic inflation, and a general class of potentially non-Markovian, stochastic volatility models simultaneously.

Built on the current work, various directions might be followed in future research on DC pension investment problems. For example, (1) this paper assumes that the income risk faced by the member can be completely hedged. One may consider a more general model for describing the stochastic income process. (2) In addition to inflation, income, and volatility risks, the member might face model misspecification (Andersen, Hansen, and Sargent (2003)). (3) It may also be of interest to extend the framework of this paper to the case with multiple risky assets.

Acknowledgement(s)

The authors are grateful to Prof. Jesper Lund Pedersen, two anonymous reviewers, and the editors for their insightful comments and suggestions, which are very helpful for revising and improving our manuscript.

7.A Proof of Proposition 7.3.4

Proof. We conjecture that the first component of one solution to nonlinear BSDE (7.3.2), $Y_{1,t}$, admits an exponential form:

$$Y_{1,t} = \exp\{f_1(t) + g_1(t)\alpha_t\},$$

where $f_1(t)$ and $g_1(t)$ are two undetermined differentiable functions with boundary conditions $f_1(T) = g_1(T) = 0$. Using Itô's formula to $Y_{1,t}$ yields

$$\begin{aligned} dY_{1,t} =& Y_{1,t}\left[\frac{df_1(t)}{dt} + \frac{dg_1(t)}{dt}\alpha_t + \kappa(\theta - \alpha_t)g_1(t) + \frac{1}{2}\left(\rho_1^2 + \rho_2^2\right)g_1^2(t)\alpha_t\right]dt \\ &+ \rho_1 g_1(t)\sqrt{\alpha_t}Y_{1,t}\,dW_t^1 + \rho_2 g_1(t)\sqrt{\alpha_t}Y_{1,t}\,dW_t^2. \end{aligned} \tag{7.A.1}$$

Let $Z_{0,t} = 0$, $Z_{1,t} = \rho_1 g_1(t)\sqrt{\alpha_t}Y_{1,t}$, and $Z_{2,t} = \rho_2 g_1(t)\sqrt{\alpha_t}Y_{1,t}$, then the generator of BSDE (7.3.2) can be reformulated as follows:

$$Y_{1,t}\left[\left(\frac{\gamma\lambda\rho_1 g_1(t)}{\gamma-1} + \frac{\gamma g_1^2(t)\rho_1^2}{2(\gamma-1)} + \frac{\gamma\lambda^2}{2(\gamma-1)}\right)\alpha_t - r\gamma + \frac{\gamma}{2(\gamma-1)}\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2}\right]. \tag{7.A.2}$$

It then follows from a direct comparison between (7.A.2) and the drift of (7.A.1) that functions $f_1(t)$ and $g_1(t)$ are governed by ODEs (7.3.6) and (7.3.7), respectively.

Moreover, denote by $\Delta_{g_1} = \left(\kappa + \frac{\gamma}{\gamma-1}\lambda\rho_1\right)^2 - \left(\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2\right)\frac{\gamma}{\gamma-1}\lambda^2$. The Riccati equation governing $g_1(t)$ can be reformulated as follows:

$$\frac{dg_1(t)}{dt} = \begin{cases} \left(\frac{1}{2(\gamma-1)}\rho_1^2 - \frac{1}{2}\rho_2^2\right)\left(g_1(t) - n_{g_1^-}\right)\left(g_1(t) - n_{g_1^+}\right), & \text{if } \Delta_{g_1} > 0; \\ \left(\frac{1}{2(\gamma-1)}\rho_1^2 - \frac{1}{2}\rho_2^2\right)\left(g_1(t) - n_{g_1}\right)^2, & \text{if } \Delta_{g_1} = 0; \\ \left(\frac{1}{2(\gamma-1)}\rho_1^2 - \frac{1}{2}\rho_2^2\right)\left[\left(g_1(t) - n_{g_1}\right)^2 + \frac{-\Delta_{g_1}}{\left(\frac{1}{\gamma-1}\rho_1^2 - \rho_2^2\right)^2}\right], & \text{if } \Delta_{g_1} < 0, \end{cases} \tag{7.A.3}$$

where $n_{g_1}, n_{g_1^-}$ and $n_{g_1^+}$ are given by (7.3.10). By applying the separation variable method to the ODE (7.A.3) and combining the boundary condition that $g_1(T) = 0$, we have the closed-form expressions of $g_1(t)$ given in (7.3.8). Plugging $g_1(t)$ back into the ODE (7.3.7) yields the closed-form expression of $f_1(t)$ given by (7.3.9). □

7.B Proof of Proposition 7.3.5

Proof. Differentiating $g_1(t)$ given in (7.3.8) with respect to t, we obtain

$$\frac{dg_1(t)}{dt} = \begin{cases} \dfrac{2\gamma\lambda^2\Delta_{g_1} e^{\sqrt{\Delta_{g_1}}(T-t)}}{(\gamma-1)\left(\frac{1}{\gamma-1}\rho_1^2-\rho_2^2\right)^2\left(n_{g_1^+}-n_{g_1^-}e^{\sqrt{\Delta_{g_1}}(T-t)}\right)^2}, & \text{if } \Delta_{g_1}>0; \\ \dfrac{2n_{g_1}^2\left(\frac{1}{\gamma-1}\rho_1^2-\rho_2^2\right)}{\left[\left(\frac{1}{\gamma-1}\rho_1^2-\rho_2^2\right)(T-t)n_{g_1}-2\right]^2}, & \text{if } \Delta_{g_1}=0; \\ \dfrac{-\Delta_{g_1}}{2\left(\frac{1}{\gamma-1}\rho_1^2-\rho_2^2\right)}\sec\left(\tan\left(\dfrac{\kappa+\rho_1\lambda\frac{\gamma}{\gamma-1}}{\sqrt{-\Delta_{g_1}}}\right)-\dfrac{\sqrt{-\Delta_{g_1}}}{2}\right), & \text{if } \Delta_{g_1}<0. \end{cases}$$

This result reveals that $dg_1(t)/dt<0$ holds for $\gamma\in(0,1)$. Therefore, we can deduce that $0\le g_1(t)\le g_1(0)$ for $t\in[0,T]$. □

7.C Proof of Corollary 7.3.7

Proof. In the first place, it follows from Lemma 7.3.6 that

$$M_{1,t} := \exp\left\{-\int_0^t \frac{\gamma}{\gamma-1}\lambda\sqrt{\alpha_s}\,dW_s^1 - \frac{1}{2}\int_0^t \frac{\gamma^2}{(\gamma-1)^2}\lambda^2\alpha_s\,ds\right\} \tag{7.C.1}$$

is an $(\mathbb{F},\mathrm{P}_0)$-martingale. In addition, we know that

$$M_{2,t} := \exp\left\{-\int_0^t \frac{\gamma}{\gamma-1}\frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\,dW_s^0 - \frac{1}{2}\int_0^t \frac{\gamma^2}{(\gamma-1)^2}\frac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2}\,ds\right\} \tag{7.C.2}$$

is also an $(\mathbb{F},\mathrm{P}_0)$-martingale since Novikov's condition holds. Due to the path-wise continuity of both (7.C.1) and (7.C.2) and the independence between Brownian motions W_t^0 and W_t^1, it follows from Theorem 2.4 in Cherny (2006) that the stochastic exponential process $M_t := M_{1,t}M_{2,t}$ is an $(\mathbb{F},\mathrm{P}_0)$-martingale as well, which in turn means that the probability measure $\tilde{\mathrm{P}}_0$ defined by $\frac{d\tilde{\mathrm{P}}_0}{d\mathrm{P}_0}|_{\mathcal{F}_T}=M_T$ is equivalent to P_0 on $\mathcal{F}_T$. By Girsanov's theorem, then we have

$$d\tilde{W}_t^0 = \frac{\gamma}{\gamma-1}\frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\,dt + dW_t^0,\ d\tilde{W}_t^1 = \frac{\gamma}{\gamma-1}\lambda\sqrt{\alpha_t}\,dt + dW_t^1,\ d\tilde{W}_t^2 = dW_t^2,$$

where $\tilde{W}_t^0, \tilde{W}_t^1$ and $\tilde{W}_t^2$ are mutually independent Brownian motions under $\tilde{\mathrm{P}}_0$ measure. Moreover, we find that the dynamics of α_t under $\tilde{\mathrm{P}}_0$ measure reserves the affine-form, square-root structure

$$d\alpha_t = \left(\kappa+\rho_1\lambda\frac{\gamma}{\gamma-1}\right)\left(\frac{\kappa\theta}{\kappa+\rho_1\lambda\frac{\gamma}{\gamma-1}}-\alpha_t\right)dt + \sqrt{\alpha_t}\left(\rho_1\,d\tilde{W}_t^1+\rho_2\,d\tilde{W}_t^2\right),$$

under Assumption 7.3.2. Due to the boundedness of function $g_1(t)$ as shown in Proposition 7.3.5 above, it therefore follows from Lemma 7.3.6 that the stochastic exponential process given by (7.3.11) is an $(\mathbb{F},\tilde{\mathrm{P}}_0)$-martingale. This completes the proof. □

7.D Proof of Theorem 7.3.8

Proof. In view of the proof of Corollary 7.3.7, we can reformulate the nonlinear BSDE of $(Y_{1,t}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ under $\tilde{\mathrm{P}}_0$ measure as follows:

$$\begin{cases} dY_{1,t} = \left[\left(\frac{\gamma}{2(\gamma-1)}\frac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2} + \frac{\gamma}{2(\gamma-1)}\lambda^2\alpha_t - r\gamma\right)Y_{1,t} + \frac{\gamma}{2(\gamma-1)}\frac{Z_{0,t}^2}{Y_{1,t}} \right.\\ \qquad\qquad \left. + \frac{\gamma}{2(\gamma-1)}\frac{Z_{1,t}^2}{Y_{1,t}}\right] dt + Z_{0,t}\, d\tilde{W}_t^0 + Z_{1,t}\, d\tilde{W}_t^1 + Z_{2,t}\, d\tilde{W}_t^2, \\ Y_{1,T} = 1, \\ Y_{1,t} > 0, \text{ for } t \in [0,T). \end{cases}$$

Suppose that there exists another solution $(\tilde{Y}_{1,t}, \tilde{Z}_{0,t}, \tilde{Z}_{1,t}, \tilde{Z}_{2,t})$ to nonlinear BSDE (7.3.2), which is different from the one obtained in Proposition 7.3.4 above. Then by defining the following difference process:

$$(\Delta\log(Y_{1,t}), \Delta Z_{0,t}, \Delta Z_{1,t}, \Delta Z_{2,t}) := \left(\log(Y_{1,t}) - \log(\tilde{Y}_{1,t}), \frac{Z_{0,t}}{Y_{1,t}} - \frac{\tilde{Z}_{0,t}}{\tilde{Y}_{1,t}}, \right.$$
$$\left. \frac{Z_{1,t}}{Y_{1,t}} - \frac{\tilde{Z}_{1,t}}{\tilde{Y}_{1,t}}, \frac{Z_{2,t}}{Y_{1,t}} - \frac{\tilde{Z}_{2,t}}{\tilde{Y}_{1,t}}\right)$$

and applying Itô's formula to $\Delta\log(Y_1)$, we find $(\Delta\log(Y_{1,t}), \Delta Z_{0,t}, \Delta Z_{1,t}, \Delta Z_{2,t})$ solves the following BSDE under $\tilde{\mathrm{P}}_0$ measure:

$$\begin{cases} d\Delta\log(Y_{1,t}) = \frac{1}{2}\left[\frac{1}{\gamma-1}\left(\frac{Z_{0,t}^2}{Y_{1,t}^2} - \frac{\tilde{Z}_{0,t}^2}{\tilde{Y}_{1,t}^2}\right) + \frac{1}{\gamma-1}\left(\frac{Z_{1,t}^2}{Y_{1,t}^2} - \frac{\tilde{Z}_{1,t}^2}{\tilde{Y}_{1,t}^2}\right) - \left(\frac{Z_{2,t}^2}{Y_{1,t}^2} - \frac{\tilde{Z}_{2,t}^2}{\tilde{Y}_{1,t}^2}\right)\right] dt \\ \qquad\qquad + \Delta Z_{0,t}\, d\tilde{W}_t^0 + \Delta Z_{1,t}\, d\tilde{W}_t^1 + \Delta Z_{2,t}\, d\tilde{W}_t^2, \\ \Delta\log(Y_{1,T}) = 0. \end{cases} \tag{7.D.1}$$

By Corollary 7.3.7, we can define another equivalent probability measure $\hat{\mathrm{P}}_0$ on $\mathcal{F}_T$ via the following Radon-Nikodym derivative:

$$\begin{aligned} \left.\frac{d\hat{\mathrm{P}}_0}{d\tilde{\mathrm{P}}_0}\right|_{\mathcal{F}_T} &= \exp\left\{-\int_0^T \frac{1}{\gamma-1}\rho_1 g_1(t)\sqrt{\alpha_t}\, d\tilde{W}_t^1 + \int_0^T \rho_2 g_1(t)\sqrt{\alpha_t}\, d\tilde{W}_t^2 \right. \\ &\qquad \left. -\frac{1}{2}\int_0^T \left(\frac{1}{(\gamma-1)^2}\rho_1^2 + \rho_2^2\right) g_1^2(t)\alpha_t\, dt\right\} \\ &= \exp\left\{-\int_0^T \frac{1}{\gamma-1}\frac{Z_{0,t}}{Y_{1,t}}\, d\tilde{W}_t^0 - \int_0^T \frac{1}{\gamma-1}\frac{Z_{1,t}}{Y_{1,t}}\, d\tilde{W}_t^1 + \int_0^T \frac{Z_{2,t}}{Y_{1,t}}\, d\tilde{W}_t^2 \right. \\ &\qquad \left. -\frac{1}{2}\int_0^T \left[\frac{1}{(\gamma-1)^2}\frac{Z_{0,t}^2}{Y_{1,t}^2} + \frac{1}{(\gamma-1)^2}\frac{Z_{1,t}^2}{Y_{1,t}^2} + \frac{Z_{2,t}^2}{Y_{1,t}^2}\right] dt\right\}, \end{aligned}$$

where the second equality follows from the explicit expressions of $(Y_{1,t}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ obtained in Proposition 7.3.4 above. By Girsanov's theorem, we have the following dynamics of standard Brownian motions $\hat{W}_t^0, \hat{W}_t^1$ and $\hat{W}_t^2$ under $\hat{\mathrm{P}}_0$ measure:

$$d\hat{W}_t^0 = \frac{1}{\gamma-1}\frac{Z_{0,t}}{Y_{1,t}}\,dt + d\tilde{W}_t^0,\ d\hat{W}_t^1 = \frac{1}{\gamma-1}\frac{Z_{1,t}}{Y_{1,t}}\,dt + d\tilde{W}_t^1,\ d\hat{W}_t^2 = -\frac{Z_{2,t}}{Y_{1,t}}\,dt + d\tilde{W}_t^2. \tag{7.D.2}$$

Finally, combining (7.D.1) and (7.D.2) shows that $(\Delta \log(Y_{1,t}), \Delta Z_{0,t}, \Delta Z_{1,t}, \Delta Z_{2,t})$ solves the following quadratic BSDE under $\hat{\mathrm{P}}_0$ measure:

$$\begin{cases} d\Delta \log(Y_{1,t}) = -\dfrac{1}{2}\left[\dfrac{1}{\gamma-1}\Delta Z_{0,t}^2 + \dfrac{1}{\gamma-1}\Delta Z_{1,t}^2 - \Delta Z_{2,t}^2\right] dt + \Delta Z_{0,t}\, d\hat{W}_t^0 \\ \qquad\qquad\qquad\qquad \Delta Z_{1,t}\, d\hat{W}_t^1 + \Delta Z_{2,t}\, d\hat{W}_t^2, \\ \Delta \log(Y_{1,T}) = 0. \end{cases} \tag{7.D.3}$$

It is straightforward to verify that quadratic BSDE (7.D.3) satisfies all the regularity conditions in Kobylanski (2000). Hence, we can conclude that quadratic BSDE (7.D.3) admits a unique solution by Theorem 2.3 and 2.6 in Kobylanski (2000), and it is easy to see that $(\Delta \log(Y_{1,t}), \Delta Z_{0,t}, \Delta Z_{1,t}, \Delta Z_{2,t}) = (0,0,0,0)$ in our case, which indicates that the solution $(Y_{1,t}, Z_{0,t}, Z_{1,t}, Z_{2,t})$ given in (7.3.4) and (7.3.5) forms the unique solution to BSDE (7.3.2). □

7.E Proof of Proposition 7.3.10

Proof. By the similar arguments given in the proof of Corollary 7.3.7, we can define an equivalent probability measure $\bar{\mathrm{P}}_0$ to P_0 on $\mathcal{F}_T$ via the following Radon-Nikodym derivative:

$$\left.\frac{d\bar{\mathrm{P}}_0}{d\mathrm{P}_0}\right|_{\mathcal{F}_T} = \exp\left\{-\int_0^T \frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\, dW_t^0 - \int_0^T \lambda\sqrt{\alpha_t}\, dW_t^1 + \int_0^T \rho_2 g_1(t)\sqrt{\alpha_t}\, dW_t^2 \right.$$
$$\left. -\frac{1}{2}\int_0^T \left(\frac{(r+\mu_p - R - \sigma_p^2}{\sigma_p^2} + \left(\lambda^2 + \rho_2^2 g_1^2(t)\right)\alpha_t\right) dt\right\}.$$

By Girsanov's theorem, we have the following dynamics of standard Brownian motions $\bar{W}_t^0, \bar{W}_t^1$ and $\bar{W}_t^2$ under $\bar{\mathrm{P}}_0$ measure:

$$\begin{cases} d\bar{W}_t^0 = \dfrac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\, dt + dW_t^0, \\ d\bar{W}_t^1 = \lambda\sqrt{\alpha_t}\, dt + dW_t^1, \\ d\bar{W}_t^2 = -\rho_2 g_1(t)\sqrt{\alpha_t}\, dt + dW_t^2. \end{cases}$$

Thus, we can reformulate linear BSDE (7.3.12) of $(G_{1,t}, \Lambda_{0,t}, \Lambda_{1,t}, \Lambda_{2,t})$ under $\bar{\mathrm{P}}_0$ measure:

$$\begin{cases} dG_{1,t} = (rG_{1,t} - \eta L_t)\ dt + \Lambda_{0,t}\, d\bar{W}_t^0 + \Lambda_{1,t}\, d\bar{W}_t^1 + \Lambda_{2,t}\, d\bar{W}_t^2. \\ G_{1,T} = 0. \end{cases} \tag{7.E.1}$$

To solve BSDE (7.E.1) explicitly, we conjecture that the first component of solution, $G_{1,t}$, has an affine form:

$$G_{1,t} = a_1(t) + b_1(t)L_t,$$

where $a_1(t)$ and $b_1(t)$ are undetermined differentiable functions with boundary conditions $a_1(T) = b_1(T) = 0$. Apply Itô's formula to $G_{1,t}$ under $\bar{\mathrm{P}}_0$ measure and let $\Lambda_{0,t} = b_1(t)(\sigma_l - \sigma_p)L_t,\ \Lambda_{1,t} = 0$ and $\Lambda_{2,t} = 0$. It can be shown that $a_1(t) = 0$ and the explicit expression of $b_1(t)$ is given by (7.3.14).

In the sequel, we will verify the solution given in (7.3.13) is the unique solution to BSDE (7.3.12). To this end, we denote by $(\bar{G}_{1,t}, \bar{\Lambda}_{0,t}, \bar{\Lambda}_{1,t}, \bar{\Lambda}_{2,t})$ any other solution to BSDE (7.3.12) which is different from the one given in (7.3.13). Then the following difference process

$$(\Delta G_{1,t}, \Delta\Lambda_{0,t}, \Delta\Lambda_{1,t}, \Delta\Lambda_{2,t}) := \big(G_{1,t} - \bar{G}_{1,t}, \Lambda_{0,t} - \bar{\Lambda}_{0,t}, \Lambda_{1,t} - \bar{\Lambda}_{1,t}, \Lambda_{2,t} - \bar{\Lambda}_{2,t}\big)$$

must solve the following linear BSDE under $\bar{\mathrm{P}}_0$ measure:

$$\begin{cases} d\Delta G_{1,t} = r\Delta G_{1,t}\, dt + \Delta\Lambda_{0,t}\, d\bar{W}_t^0 + \Delta\Lambda_{1,t}\, d\bar{W}_t^1 + \Delta\Lambda_{2,t}\, d\bar{W}_t^2, \\ \Delta G_{1,T} = 0. \end{cases} \tag{7.E.2}$$

The generator of linear BSDE (7.E.2) is clearly uniformly Lipschitz continuous with respect to $\Delta G_{1,t}, \Delta\Lambda_{0,t}, \Delta\Lambda_{1,t}$ and $\Delta\Lambda_{2,t}$ and satisfies all the regularity conditions in El Karoui, Peng, and Quenez (1997). Hence, by Theorem 2.1 in El Karoui, Peng, and Quenez (1997), we can conclude that BSDE (7.E.2) admits a unique solution which is $(0, 0, 0, 0)$ in our case. This, in turn, indicates that the solution given in (7.3.13) is the unique solution to BSDE (7.3.12). □

7.F Proof of Theorem 7.3.11

Proof. For any admissible strategy $\pi \in \Pi_p$, by applying Itô's formula to $Y_{1,t}\frac{(X_t^\pi + G_{1,t})^\gamma}{\gamma}$ and using some localization techniques, we obtain

$$\begin{aligned} &\mathrm{E}_0\left[Y_{1,\tau_n\wedge T}\frac{(X_{\tau_n\wedge T}^\pi + G_{1,\tau_n\wedge T})^\gamma}{\gamma}\right] \\ =&\frac{1}{2}(\gamma-1)\mathrm{E}_0\Bigg[\int_0^{\tau_n\wedge T} Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-2}\Big[\big(X_t^\pi\left(\pi_t^I - 1\right)\sigma_p + \Lambda_{0,t}\big) \\ &+\frac{1}{\gamma-1}(X_t^\pi + G_{1,t})\left(\frac{Z_{0,t}}{Y_{1,t}} + \frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\right)\Big]^2\, dt\Bigg] \\ &+\frac{1}{2}(\gamma-1)\mathrm{E}_0\Bigg[\int_0^{\tau_n\wedge T} Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-2}\Big[(X_t^\pi\pi_t^S\sigma_t + \Lambda_{1,t}) + \frac{1}{\gamma-1} \\ &\times(X_t^\pi + G_{1,t})\left(\frac{Z_{1,t}}{Y_{1,t}} + \lambda\sqrt{\alpha_t}\right)\Big]^2\, dt\Bigg] + Y_{1,0}\frac{(x_0 + G_{1,0})^\gamma}{\gamma}, \end{aligned} \tag{7.F.1}$$

where $\{\tau_n\}_{n\in\mathbb{N}}$ is the localizing sequence. Recalling from Definition 7.2.6 that for any admissible strategy $\pi \in \Pi_p$, the corresponding inflation-adjusted wealth satisfies $X_t^\pi + G_{1,t} > 0$, P_0 almost surely, then since the risk aversion parameter γ lies in $(0,1)$ and $Y_{1,t}$ is strictly positive for $t \in [0,T]$, we have the following inequality from (7.F.1):

$$\mathrm{E}_0\left[Y_{1,\tau_n\wedge T}\frac{(X^\pi_{\tau_n\wedge T}+G_{1,\tau_n\wedge T})^\gamma}{\gamma}\right] \le Y_{1,0}\frac{(x_0+G_{1,0})^\gamma}{\gamma}. \tag{7.F.2}$$

Sending n to infinity in (7.F.2) and using Fatou's lemma yield

$$\mathrm{E}_0\left[\frac{(X_T^\pi)^\gamma}{\gamma}\right] \le \liminf_{n\to\infty} \mathrm{E}_0\left[Y_{1,\tau_n\wedge T}\frac{(X^\pi_{\tau_n\wedge T}+G_{1,\tau_n\wedge T})^\gamma}{\gamma}\right] \le Y_{1,0}\frac{(x_0+G_{1,0})^\gamma}{\gamma},$$

for any admissible strategy $\pi \in \Pi_p$. Thus, recalling the explicit expressions of $Y_{1,t}$ and $G_{1,t}$ given by (7.3.4) and (7.3.13), we find that

$$\sup_{\pi\in\Pi_p} \mathrm{E}_0\left[\frac{(X_T^\pi)^\gamma}{\gamma}\right] \le \frac{(x_0+b_1(0)l_0)^\gamma}{\gamma}\exp\left(f_1(0)+g_1(0)\alpha_0\right). \tag{7.F.3}$$

Particularly, the right-hand side of (7.F.3) corresponds to the value function when the strategy (π_t^{S*}, π_t^{I*}) given in (7.3.15) is adopted. Denote by X_t^* the inflation-adjusted wealth process associated with the strategy π_t^{S*} and π_t^{I*}, we have

$$\begin{aligned} d\left(Y_{1,t}\frac{(X_t^*+G_{1,t})^\gamma}{\gamma}\right) =& Y_{1,t}\frac{(X_t^*+G_{1,t})^\gamma}{\gamma}\Bigg[\frac{\gamma}{1-\gamma}\frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\,dW_t^0 \\ &+\left(\frac{1}{1-\gamma}\rho_1 g_1(t)+\frac{\gamma}{1-\gamma}\lambda\right)\sqrt{\alpha_t}\,dW_t^1+\rho_2 g_1(t)\sqrt{\alpha_t}\,dW_t^2\Bigg]. \end{aligned}$$

As a result of Lemma 7.3.6 and Theorem 2.4 in Cherny (2006), it is clear that $Y_{1,t}\frac{(X_t^*+G_{1,t})^\gamma}{\gamma}$ is an $(\mathbb{F}, \mathrm{P}_0)$-martingale. Therefore, we have

$$\mathrm{E}_0\left[\frac{(X_T^*)^\gamma}{\gamma}\right] = \frac{(x_0+b_1(0)l_0)^\gamma}{\gamma}\exp\left(f_1(0)+g_1(0)\alpha_0\right). \tag{7.F.4}$$

We finally show that the strategy $\left(\pi_t^{S*}, \pi_t^{I*}\right)$ given by (7.3.15) is admissible. For this, we need to show $\pi^* = \left(\left\{\pi_t^{S*}\right\}_{t\in[0,T]}, \left\{\pi_t^{I*}\right\}_{t\in[0,T]}\right)$ is $\mathbb{F}$-progressively measurable and $X_t^* + G_{1,t} > 0$, P_0 almost surely when $x_0 + b_1(0)l_0 > 0$ holds. In fact, plugging the strategy $\left(\pi_t^{S*}, \pi_t^{I*}\right)$ given in (7.3.15) back into the inflation adjusted wealth process (7.2.9) and solving the corresponding SDE explicitly lead to

$$\begin{aligned} X_t^* =& \exp\left\{\int_0^t r-\frac{1}{\gamma-1}(\rho_1 g_1(s)\lambda+\lambda^2)\alpha_s-\frac{1}{\gamma-1}\frac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2}\,ds\right\} \\ &\times\exp\left\{-\frac{1}{2}\int_0^t\frac{1}{(\gamma-1)^2}(\rho_1 g_1(s)+\lambda)^2\,ds-\int_0^t\frac{1}{\gamma-1}(\rho_1 g_1(s)+\lambda)\sqrt{\alpha_s}\,dW_s^1\right\} \\ &\times\exp\left\{-\frac{1}{2}\int_0^t\frac{1}{(\gamma-1)^2}\frac{(r+\mu_p-R-\sigma_p^2)^2}{\sigma_p^2}\,ds-\int_0^t\frac{1}{\gamma-1}\frac{r+\mu_p-R-\sigma_p^2}{\sigma_p}\,dW_s^0\right\} \\ &\times(x_0+b_1(0)l_0)-b_1(t)L_t. \end{aligned} \tag{7.F.5}$$

Combining (7.3.15) and (7.F.5), we see that π^* is $\mathbb{F}$-progressively measurable, and $X_t^* + G_{1,t} > 0$ always holds P_0 almost surely whenever the initial data (α_0, x_0, l_0) satisfies $x_0 + b_1(0)l_0 > 0$. Therefore, we can conclude that $\pi^* \in \Pi_p$, and from (7.F.3), we can say that π^* is the optimal strategy of problem (7.2.11) and the optimal value function is given by (7.3.16). □

7.G Proof of Theorem 7.3.20

Proof. For any admissible strategy $\pi \in \Pi_l$, using Itô's formula to $\log\left(X_t^\pi + G_{2,t}\right) + Y_{2,t}$ and using some localization techniques, we obtain

$$
\begin{aligned}
&\mathrm{E}_0\left[\log\left(X^\pi_{T\wedge\tau_n} + G_{2,T\wedge\tau_n}\right) + Y_{2,T\wedge\tau_n}\right] \\
=&\mathrm{E}_0\left[\int_0^{T\wedge\tau_n} \frac{-1}{2(X_t^\pi + G_{2,t})^2}\left[\left(X_t^\pi\left(\pi_t^I - 1\right)\sigma_p + \Gamma_{0,t}\right) - \frac{(X_t^\pi + G_{2,t})(r + \mu_p - R - \sigma_p^2)}{\sigma_p}\right]^2 dt\right] \\
&+ \mathrm{E}_0\left[\int_0^{T\wedge\tau_n} \frac{-1}{2(X_t^\pi + G_{2,t})^2}\left[\left(X_t^\pi \pi_t^S \sigma_t + \Gamma_{1,t}\right) - (X_t^\pi + G_{2,t})\lambda\sqrt{\alpha_t}\right]^2 dt\right] \\
&+ \left(\log(x_0 + G_{2,0}) + Y_{2,0}\right).
\end{aligned}
\tag{7.G.1}
$$

Since for any admissible strategy $\pi \in \Pi_l$, the family $\left\{\log\left(X^\pi_{T\wedge\tau_n} + G_{2,T\wedge\tau_n}\right) + Y_{2,T\wedge\tau_n}\right\}_{n\in\mathbb{N}}$ is uniformly integrable for any $\mathbb{F}$-stopping time sequences $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, P_0 almost surely, and the terms in the expectations on the right-hand side of (7.G.1) are non-positive and decreasing with respect to $n \in \mathbb{N}$. By using the monotone convergence theorem to the right-hand side of (7.G.1) and the equivalence between uniform integrability and $\mathcal{L}^1$ convergence to the left-hand side of (7.G.1), sending n to ∞ gives

$$
\begin{aligned}
&\mathrm{E}_0\left[\log\left(X_T^\pi\right)\right] \\
=&\mathrm{E}_0\left[\int_0^{T} \frac{-1}{2(X_t^\pi + G_{2,t})^2}\left[\left(X_t^\pi\left(\pi_t^I - 1\right)\sigma_p + \Gamma_{0,t}\right) - \frac{(X_t^\pi + G_{2,t})(r + \mu_p - R - \sigma_p^2)}{\sigma_p}\right]^2 dt\right] \\
&+ \mathrm{E}_0\left[\int_0^{T} \frac{-1}{2(X_t^\pi + G_{2,t})^2}\left[\left(X_t^\pi \pi_t^S \sigma_t + \Gamma_{1,t}\right) - (X_t^\pi + G_{2,t})\lambda\sqrt{\alpha_t}\right]^2 dt\right] \\
&+ \log(x_0 + G_{2,0}) + Y_{2,0},
\end{aligned}
$$

which implies that

$$
\sup_{\pi\in\Pi_l} \mathrm{E}_0\left[\log\left(X_T^\pi\right)\right] \le \log\left(x_0 + G_{2,0}\right) + Y_{2,0} = \log\left(x_0 + b_2(0)l_0\right) + f_2(0) + g_2(0)\alpha_0.
\tag{7.G.2}
$$

In particular, when we opt for the strategy π_t^{S*} and π_t^{I*} given by (7.3.25) and denote by X_t^* the corresponding inflation-adjusted wealth process, we have the following SDE:

$$
\begin{aligned}
d\left(\log(X_t^* + G_{2,t}) + Y_{2,t}\right) =& \frac{r + \mu_p - R - \sigma_p^2}{\sigma_p}\, dW_t^0 + \left(\lambda + \rho_1 g_2(t)\right)\sqrt{\alpha_t}\, dW_t^1 \\
&+ \rho_2 g_2(t)\sqrt{\alpha_t}\, dW_t^2.
\end{aligned}
\tag{7.G.3}
$$

Note that for all $t \in [0,T]$, the following expectation value for square-root factor process α_t

$$\mathrm{E}_0[\alpha_t] = \alpha_0 e^{-\kappa t} + \kappa\theta \int_0^t e^{-\kappa(t-s)}\, ds$$

is non-negative and uniformly bounded. Therefore, by using Fubini's theorem, it can be shown that

$$\mathrm{E}_0\left[\int_0^T \left(\frac{(r+\mu_p - R - \sigma_p^2)^2}{\sigma_p^2} + \left(\lambda + \rho_1 g_2(t)\right)^2 + \rho_2^2 g_2^2(t)\right)\alpha_t\right) dt\right] < \infty,$$

which implies from (7.G.3) that $\log(X_t^* + G_{2,t}) + Y_{2,t}$ is an $(\mathbb{F}, \mathrm{P}_0)$-martingale. Hence, we have

$$\mathrm{E}_0\left[\log(X_T^*)\right] = \log(x_0 + G_{2,0}) + Y_{2,0} = \log(x_0 + b_2(0)l_0) + f_2(0) + g_2(0)\alpha_0 < \infty. \tag{7.G.4}$$

Moreover, for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, P_0 almost surely as $n \to \infty$, $\tau_n \wedge T$ and T are two bounded stopping times. Then by Doob's optional sampling theorem for bounded stopping times (refer to Corollary 3.23 in Le Gall (2016)), we obtain

$$\log(X_{\tau_n\wedge T}^* + G_{2,\tau_n\wedge T}) + Y_{2,\tau_n\wedge T} = \mathrm{E}\left[\log(X_T^*)\middle|\, \mathcal{F}_{\tau_n\wedge T}\right].$$

Since $\{\mathcal{F}_{\tau_n\wedge T}\}_{n\in\mathbb{N}}$ is a family of sub σ-algebra of $\mathcal{F}_T$ and $\mathrm{E}_0\left[\log(X_T^*)\right] < \infty$, by Theorem 4.6.1 in Durrett (2019), we find that $\left\{\log(X_{\tau_n\wedge T}^* + G_{2,\tau_n\wedge T}) + Y_{2,\tau_n\wedge T}\right\}_{n\in\mathbb{N}}$ is a uniformly integrable family. This confirms condition (1) in Definition 7.2.7.

We next show that $X_t^* + G_{2,t} > 0$ holds for all $t \in [0,T]$, P_0 almost surely, if the initial data satisfies $x_0 + b_2(0)l_0 > 0$. In fact, applying Itô's formula to $X_t^* + G_{2,t}$ and solving the corresponding SDE explicitly give us

$$\begin{aligned} X_t^* + G_{2,t} =& (x_0 + b_2(0)l_0)\exp\left\{\int_0^t \left(r + \frac{\lambda^2\alpha_s}{2} + \frac{(r+\mu_p - R - \sigma_p^2)^2}{2\sigma_p^2}\right) ds \right.\\ &\left. + \int_0^t \frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}\, dW_s^0 + \int_0^t \lambda\sqrt{\alpha_s}\, dW_s^1\right\} > 0. \end{aligned} \tag{7.G.5}$$

Noticing from BSDE (7.3.19) that $G_{2,T} = 0$, we, in particular, have $X_T^* > 0$ at time T from (7.G.5). Thus, condition (2) in Definition 7.2.7 is verified. Moreover, we see that π_t^{S*} and π_t^{I*} given by (7.3.25) are $\mathbb{F}$-progressively measurable since processes $X_t^*, G_{2,t}, \Lambda_{0,t}$ and α_t are all clearly $\mathbb{F}$-progressively measurable. Therefore, we know that conditions (3) in Definition 7.2.7 is satisfied and the strategy $\pi^* = \left(\left\{\pi_t^{S*}\right\}_{t\in[0,T]}, \left\{\pi_t^{I*}\right\}_{t\in[0,T]}\right)$ given by (7.3.25) is admissible, i.e. $\pi^* \in \Pi_l$. Finally, from (7.G.2) and (7.G.4), we can conclude that π^* is the optimal strategy and the optimal value function is given by (7.3.26). □

7.H Proof of Proposition 7.3.27

Proof. We conjecture that the first component of the solution to BSDE (7.3.31), $Y_{3,t}$, has an affine form $Y_{3,t} = f_3(t) + g_3(t)\alpha_t + b_3(t)\tilde{L}_t$, where $f_3(t), g_3(t)$ and $b_3(t)$ are undetermined differentiable functions of t with boundary conditions $f_3(T) = g_3(T) = b_3(T) = 0$. Applying Itô's formula to $Y_{3,t}$ yields

$$dY_{3,t} = \left[\frac{dg_3(t)}{dt}\alpha_t + \kappa(\theta - \alpha_t)g_3(t) + \tilde{L}_t\left(\frac{db_3(t)}{dt} + \left(\mu_l - \mu_p + \sigma_p^2 - \sigma_l\sigma_p - r\right)b_3(t)\right)\right. \\ \left. + \frac{df_3(t)}{dt}\right] dt + (\sigma_l - \sigma_p)b_3(t)\tilde{L}_t\, dW_t^0 + \rho_1\sqrt{\alpha_t}g_3(t)\, dW_t^1 + \rho_2\sqrt{\alpha_t}g_3(t)\, dW_t^2. \tag{7.H.1}$$

Let $M_{0,t} = (\sigma_l - \sigma_p)b_3(t)\tilde{L}_t$, $M_{1,t} = \rho_1\sqrt{\alpha_t}g_3(t)$ and $M_{2,t} = \rho_2\sqrt{\alpha_t}g_3(t)$, then the generator of BSDE (7.3.31) turns out to be:

$$\left(\frac{\lambda^2}{2\tilde{q}} + \lambda\rho_1 g_3(t) - \frac{\tilde{q}}{2}\rho_2^2 g_3^2(t)\right)\alpha_t + \left(\frac{r + \mu_p - R - \sigma_p^2}{\sigma_p}(\sigma_l - \sigma_p)b_3(t) + \eta\right)\tilde{L}_t \\ + \frac{(r + \mu_p - R - \sigma_p^2)^2}{2\tilde{q}\sigma_p^2}. \tag{7.H.2}$$

Comparing the drift of SDE (7.H.1) and (7.H.2) and separating the dependence on α_t and $\tilde{L}_t$, we find that functions $f_3(t), g_3(t)$ and $b_3(t)$ must satisfy the following ODEs:

$$\begin{cases} \dfrac{dg_3(t)}{dt} = -\dfrac{\tilde{q}}{2}\rho_2^2 g_3^2(t) + (\kappa + \lambda\rho_1)g_3(t) + \dfrac{\lambda^2}{2\tilde{q}}, \; g_3(T) = 0; \\ \dfrac{db_3(t)}{dt} = -\left(\mu_l - R - \sigma_l\sigma_p - \dfrac{r + \mu_p - R - \sigma_p^2}{\sigma_p}\sigma_l\right)b_3(t) + \eta, \; b_3(T) = 0; \\ \dfrac{df_3(t)}{dt} = -\kappa\theta g_3(t) + \dfrac{(r + \mu_p - R - \sigma_p^2)^2}{2\tilde{q}\sigma_p^2}, \; f_3(T) = 0. \end{cases}$$

By solving the above ODEs explicitly, we arrive at (7.3.34)-(7.3.36). □

7.I Proof of Proposition 7.3.28

Proof. Differentiating $g_3(t)$ with respect to t gives

$$\frac{dg_3(t)}{dt} = \begin{cases} \dfrac{2\frac{\lambda^2}{\tilde{q}}\Delta_{g_3}e^{\sqrt{\Delta_{g_3}}(T-t)}}{\left[-(\kappa + \lambda\rho_1) + \sqrt{\Delta_{g_3}} + \left(\kappa + \lambda\rho_1 + \sqrt{\Delta_{g_3}}\right)e^{\sqrt{\Delta_{g_3}}(T-t)}\right]^2}, & \text{if } \Delta_{g_3} > 0 \text{ and } \rho_2 \neq 0; \\ \dfrac{\lambda^2}{2\tilde{q}}e^{(\kappa+\lambda\rho_1)(t-T)}, & \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda\rho_1 \neq 0; \\ \dfrac{-2\tilde{q}\rho_2^2 n_{g_3}^2}{(\tilde{q}\rho_2^2(T-t)n_{g_3} + 2)^2}, & \text{if } \Delta_{g_3} = 0 \text{ and } \rho_2 \neq 0. \end{cases}$$

The first two cases reveal that function $g_3(t)$ is strictly increasing over $[0, T]$, while the last case shows that $g_3(t)$ is strictly decreasing over $[0, T]$. This, in turn, implies that $g_3(t)$ is bounded over $[0, T]$, and more precisely, we have $|g_3(t)| \leq |g_3(0)|$. □

7.J Proof of Theorem 7.3.30

Proof. For any admissible strategy $\pi \in \Pi_e$, using Itô's formula to $-e^{\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}$ and applying some localization techniques lead to

$$\begin{aligned}
&\mathrm{E}_0\left[-e^{-\tilde{q}(\tilde{X}^\pi_{T\wedge\tau_n} - Y_{3,T\wedge\tau_n})}\right] \\
&= -\frac{1}{2}\tilde{q}^2\mathrm{E}_0\left[\int_0^{T\wedge\tau_n} e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}\left(\tilde{X}_t^\pi\left(\pi_t^I - 1\right)\sigma_p - M_{0,t} - \frac{r+\mu_p - R - \sigma_p^2}{\tilde{q}\sigma_p}\right)^2 dt\right] \\
&\quad - \frac{1}{2}\tilde{q}^2\mathrm{E}_0\left[\int_0^{T\wedge\tau_n} e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}\left(\tilde{X}_t^\pi\pi_t^S\sigma_t - M_{1,t} - \frac{\lambda\sqrt{\alpha_t}}{\tilde{q}}\right)^2 dt\right] - e^{-\tilde{q}(x_0 - Y_{3,0})},
\end{aligned} \tag{7.J.1}$$

where $\{\tau_n\}_{n\in\mathbb{N}}$ is the localizing sequence. Since for any admissible strategy $\pi \in \Pi_e$, the term in the expectation on the left-hand side of (7.J.1) is uniformly integrable, and the terms in the expectations on the right-hand side of (7.J.1) are non-negative and increasing with respect to $n \in \mathbb{N}$. Hence, by applying the monotone convergence theorem to the right-hand side of (7.J.1) and the equivalence between uniform integrability and $\mathcal{L}^1$-convergence to the left-hand side of (7.J.1) upon recalling that $\tilde{q}\tilde{X}_T^\pi = qX_T^\pi$, we have

$$\begin{aligned}
&\mathrm{E}_0\left[-e^{-qX_T^\pi}\right] \\
&= -\frac{1}{2}\tilde{q}^2\mathrm{E}_0\left[\int_0^{T} e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}\left(\tilde{X}_t^\pi\left(\pi_t^I - 1\right)\sigma_p - M_{0,t} - \frac{r+\mu_p - R - \sigma_p^2}{\tilde{q}\sigma_p}\right)^2 dt\right] \\
&\quad - \frac{1}{2}\tilde{q}^2\mathrm{E}_0\left[\int_0^{T} e^{-\tilde{q}(\tilde{X}_t^\pi - Y_{3,t})}\left(\tilde{X}_t^\pi\pi_t^S\sigma_t - M_{1,t} - \frac{\lambda\sqrt{\alpha_t}}{\tilde{q}}\right)^2 dt\right] - e^{-qe^{rT}(x_0 - Y_{3,0})} \\
&\leq -e^{-qe^{rT}(x_0 - Y_{3,0})}.
\end{aligned}$$

This means that

$$\sup_{\pi\in\Pi_e}\mathrm{E}_0\left[-e^{-qX_T^\pi}\right] \leq -\exp\left\{-qe^{rT}(x_0 - f_3(0) - g_3(0)\alpha_0 - b_3(0)l_0)\right\}. \tag{7.J.2}$$

When adopting the strategy π_t^{S*} and π_t^{I*} given in (7.3.37) and denoting by X_t^* and $\tilde{X}_t^*$ the associated (discounted) inflation-adjusted wealth processes respectively, we have

$$\begin{aligned}
d\left(-e^{-\tilde{q}(\tilde{X}_t^* - Y_{3,t})}\right) = &-e^{-\tilde{q}(\tilde{X}_t^* - Y_{3,t})}\Bigg(-\frac{r+\mu_p - R - \sigma_p^2}{\sigma_p}dW_t^0 - \lambda\sqrt{\alpha_t}\,dW_t^1 \\
&+ \tilde{q}\rho_2 g_3(t)\sqrt{\alpha_t}\,dW_t^2\Bigg).
\end{aligned} \tag{7.J.3}$$

Solving SDE (7.J.3) explicitly, we have

$$
\begin{aligned}
&- e^{-\tilde{q}(\tilde{X}_t^* - Y_{3,t})} \\
&= - e^{-qe^{rT}(x_0 - Y_{3,0})} \exp\left\{ - \int_0^t \frac{r + \mu_p - R - \sigma_p^2}{\sigma_p} \, dW_s^0 - \frac{1}{2} \int_0^t \frac{(r + \mu_p - R - \sigma_p^2)^2}{\sigma_p^2} \, ds \right\} \\
&\quad \times \exp\left\{ - \int_0^t \lambda \sqrt{\alpha_s} \, dW_s^1 + \int_0^t \tilde{q} \rho_2 g_3(s) \sqrt{\alpha_s} \, dW_s^2 - \frac{1}{2} \int_0^t \left(\lambda^2 + \tilde{q}^2 \rho_2^2 g_3^2(s) \right) \alpha_s \, ds \right\}.
\end{aligned}
$$

This shows us the path-wise unique strong solution of $\tilde{X}_t^*$ as well as of X_t^*. Moreover, it follows from Lemma 7.3.6 and Theorem 2.4 in Cherny (2006) that $-e^{-\tilde{q}(\tilde{X}_t^* - Y_{3,t})}$ is an $(\mathbb{F}, \mathrm{P}_0)$-martingale. Thus, we have

$$
\mathrm{E}_0 \left[-e^{-qX_T^*} \right] = - \exp\left\{ -qe^{rT} (x_0 - f_3(0) - g_3(0)\alpha_0 - b_3(0) l_0) \right\}. \tag{7.J.4}
$$

Therefore, combining (7.J.2) and (7.J.4), we can conclude that the optimal investment strategy and optimal value function of problem (7.2.13) are given by (7.3.37) and (7.3.38), respectively. The proof of the admissibility of the optimal strategy $\pi^* = \left(\left\{ \pi_t^{S*} \right\}_{t \in [0,T]}, \left\{ \pi_t^{I*} \right\}_{t \in [0,T]} \right)$ is similar to that of Theorem 7.3.20 above, so we omit here. □

Chapter 8

Optimal investment strategies for asset-liability management with affine diffusion factor processes and HARA preferences

Abstract

This paper investigates an optimal asset-liability management problem within the expected utility maximization framework. The general hyperbolic absolute risk aversion (HARA) utility is adopted to describe the risk preference of the asset-liability manager. The financial market comprises a risk-free asset and a risky asset. The market price of risk depends on an affine diffusion factor process, which includes, but is not limited to, the constant elasticity of variance (CEV), Stein-Stein, Schöbel and Zhu, Heston, 3/2, 4/2 models, and some non-Markovian models, as exceptional examples. The accumulative liability process is featured by a generalized drifted Brownian motion with possibly unbounded and non-Markovian drift and diffusion coefficients. Due to the sophisticated structure of HARA utility and the non-Markovian framework of the incomplete financial market, a backward stochastic differential equation (BSDE) approach is adopted. By solving a recursively coupled BSDE system, closed-form expressions for both the optimal investment strategy and optimal value function are derived. Moreover, explicit solutions to some particular cases of our model are provided. Finally, numerical examples are presented to illustrate the effect of model parameters on the optimal investment strategies in several particular cases.

Keywords: Asset-liability management; HARA utility; Affine diffusion factor

process; Drifted Brownian motion; Backward stochastic differential equation.

8.1 Introduction

Asset-liability management (ALM) is of great importance not only for investment institutions such as banks, pension funds, and life insurance companies but also for individual investors who can ensure the match between assets and liabilities. The ALM problem under Markowitz (1952)'s mean-variance criterion was first studied by Sharpe and Tint (1990) in a single-period setting, and the multi-period case was then investigated by Leippold, Trojani, and Vanini (2004). By employing the stochastic linear-quadratic control theory and solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation, Chiu and Li (2006) extended the work of Leippold, Trojani, and Vanini (2004) to a continuous-time framework with uncontrollable liabilities driven by a geometric Brownian motion. Alternatively, Xie, Li, and Wang (2008) studied a continuous-time mean-variance ALM problem with liabilities driven by a drifted Brownian motion process. The works of Leippold, Trojani, and Vanini (2004) and Chiu and Li (2006) were further extended to the case with Markovian regime-switching market by Chen, Yang, and Yin (2008) and Chen and Yang (2011), respectively. By applying a backward stochastic differential equation (BSDE) approach, Chiu and Wong (2014a) investigated the case with cointegrated risky assets. For other research outcomes of ALM problems under the continuous-time mean-variance framework, one can refer to Zeng and Li (2011), Yu (2014), Chang (2015), Pan and Xiao (2017c), Peng and Chen (2021), and references therein. Besides the mean-variance criterion, ALM problems under the framework of expected utility maximization have attracted the attention of quite a few researchers over the last several years. For example, Chiu and Wong (2014b) studied an ALM problem with stochastic interest rates for an insurer under power and logarithmic utility, where the interest rate was modeled by an extended Cox–Ingersoll–Ross (CIR) process and the liability followed a risk model of compound Poisson process. Liang and Ma (2015) incorporated mortality and salary risks into an ALM problem under power and exponential utility and derived the optimal approximation investment strategy by using the martingale approach and the dynamic programming approach. Pan and Xiao (2017b) considered an ALM problem with inflation risk and stochastic interest rates, where the liability was governed by a geometric Brownian motion and the interest rate followed the Hull-White model. For other relevant works along this line, one may refer to Pan and Xiao (2017a) and Chen, Huang, and Li (2022), to name but only a few.

Although ALM problems under the utility maximization and mean-variance criteria have been extensively studied, two aspects are worthy of being further explored. Firstly, most of the above-mentioned literature generally assumes that the risky asset price follows a geometric Brownian motion. In other words, the volatility of risky asset price is a constant or a deterministic function, which cannot explain the empirical observation that the implied volatility in option price data displays the so-called volatility smile/skew. To articulate this issue, various stochastic (local) volatility models have been proposed over the last several decades, such as the constant elasticity of variance (CEV) model (Cox (1996)), Stein-Stein model (Stein and Stein (1991)), Heston model (Heston (1993)), Schöbel and Zhu model (Schöbel and Zhu (1999)), 3/2 model (Lewis (2000)), and 4/2 model (Grasselli (2017)). Recently, there has been emerging interest in ALM problems under stochastic volatility models. Using a BSDE approach, Zhang and Chen (2016) considered a mean-variance ALM problem under the CEV model with multiple risky assets. Li, Shen, and Zeng (2018) investigated the effect of derivatives trading on a mean-variance ALM problem under the Heston model, where the liability process was given by a generalized geometric Brownian motion. In Pan, Zhang, and Zhou (2018), the liability was modeled by a drifted Brownian motion, and the explicit solutions were derived under the mean-variance criterion for two special cases where the two fundamental risk factors in the Heston model were perfectly correlated or anti-correlated. Sun, Zhang, and Yuen (2020) stepped further by studying a mean-variance ALM problem in a complete market with multiple risky assets, where the volatility of risky assets was driven by an affine diffusion equation and the analytical solutions were obtained by using a BSDE approach. Zhang (2023) studied a mean-variance ALM problem under the CIR short rate of interest and state-of-the-art 4/2 stochastic volatility model with derivatives trading. Besides the mean-variance criterion, by using the dynamic programming approach, Pan, Hu, and Zhou (2019) considered an ALM problem with exponential utility under the Heston model.

Secondly, few papers on the ALM problems considers the hyperbolic absolute risk aversion (HARA) utility function. Indeed, due to the flexibility of capturing risk aversion preference, the HARA utility function includes power utility, exponential utility, and logarithmic utility as exceptional cases in the utility theory. It is also worth mentioning that the HARA utility is closely related to the mean-variance criterion since the quadratic loss minimization (benchmark) problem embedded in the mean-variance problem is a special case of HARA utility. Given the complicated structure of HARA utility, two main approaches, the martingale approach and the dynamic programming approach along with Legendre transform-dual theory (Jonsson and Sircar (2002)), are generally applied to solving utility maximization problems with HARA utility in recent years. For example, Tepla (2001) studied an optimal portfolio selection problem with minimum performance constraints for a HARA-utility investor and derived the explicit solution by using the martingale approach. Grasselli (2003) investigated a HARA utility maximization problem with stochastic interest rates in a complete market, where the interest rates were described by the Cox-Ingersoll-Ross (CIR) model. Alternative to Grasselli (2003), Chang, Chang, and Lu (2014) studied an ALM problem with stochastic interest rates, where the liability was driven by a drifted Brownian motion and the interest rates followed an affine diffusion process. Jung and Kim (2012) considered an optimal investment problem under the CEV model. Chang and Chang (2017) focused on an optimal consumption-investment problem with multiple risky assets and Vasicek interest rates. Escobar, Neykova, and Zagst (2017) studied a HARA utility maximization problem in a Markov-switching bond-stock market, where the stochastic volatility and stochastic interest rates were described by a Markov-modulated Heston model and Markov-modulated Vasicek model, respectively. More recently, Chang et al. (2020) solved a defined contribution pension problem with an affine interest rate and mean-reverting returns under HARA utility explicitly. Zhang and Zhao (2020) investigated an optimal reinsurance-investment problem under the CEV model and HARA utility by using Legendre transform-dual technique. For other previous works on HARA utility maximization problems, one can consult Çanakoğlu and Özekici (2012), Zhang, Zhao, and Kou (2021), Liu et al. (2023), and references therein.

To the best of our knowledge, there is no existing literature addressing the ALM problem under stochastic volatility models and HARA utility preferences in an incomplete market setting. The present paper aims to fill the gap. We assume that the asset-liability manager has access to a financial market consisting of one risk-free asset and one risky asset, and meanwhile, is subject to an uncontrollable random liability. The liability process is modeled by a generalized drifted Brownian motion with unspecified drift and diffusion coefficients and can be understood as a subtraction of the real liability and stochastic income. In this sense, a negative liability means that the real liability is smaller than the stochastic income. Unlike most of the aforementioned literature, it is not a prerequisite to suppose that the risky asset's return rate and volatility are specifically Markovian processes. Instead, we only assume that the market price of risk depends on an affine diffusion factor process (Duffie and Kan (1996)). The general modeling framework includes not only some well-known Markovian models, such as the Black-Scholes model, CEV model, Stein-Stein model, Schöbel and Zhu model, Heston model, 3/2 model, and 4/2 model but also some non-Markovian models, as exceptional examples (see Examples 8.2.2-8.2.5). The incomplete market setting and the potentially non-Markovian structures of risky asset price and random liability lead to the failure of an application of the martingale representation theorem and Bellman's optimality principle. In this sense, neither the martingale approach nor the dynamic programming approach along with Legendre transform-dual technique can be applied in the present paper directly. We, therefore, adopt a BSDE approach to solve the ALM problem under the general stochastic volatility model and HARA preference. Under an assumption on the model parameters (see Assumption 8.3.6), we discuss the solvability of a recursively coupled BSDE system consisting of a backward stochastic Riccati equation (BSRE) and two linear BSDEs and derive, in closed form, their respective solutions. Explicit expressions for the optimal investment strategy and optimal value function are then obtained via the solutions to the BSDEs. Moreover, the results for several special cases of our model are provided (see Corollary 8.4.3-8.4.7 and Remark 8.4.2-8.4.6). Finally, some numerical examples are given to illustrate the effect of model parameters on the optimal investment strategies under two extensively studied models, the CEV model and 4/2 model.

The main contributions of this paper are as follows: (i) we pioneer to study an ALM problem with a general class of stochastic volatility models and the HARA utility in an incomplete financial market setting, in which the market price of risk relies on an affine diffusion factor process but the return rate and volatility coefficient are unspecified, while in Sun, Zhang, and Yuen (2020) the financial market is complete and an investment-reinsurance problem under the mean-variance criterion is considered; (ii) the liability process is modeled by a generalized drifted Brownian motion rather than a geometric Brownian motion, which can be understood as a subtraction of the real liability and stochastic income and distinguishes this paper from some existing literature on ALM problems, such as Zhang and Chen (2016), Li, Shen, and Zeng (2018), Sun, Zhang, and Yuen (2020), and etc; (iii) a BSDE approach is applied to overcome the possibly incomplete and non-Markovian financial market, which differentiates this paper from some published works on HARA utility maximization problems by using either the martingale approach or the Legendre transform-dual theory, such as Grasselli (2003), Jung and Kim (2012), Chang et al. (2020), Zhang and Zhao (2020), Zhang, Zhao, and Kou (2021), Liu et al. (2023), and etc.

The remainder of this paper is organized as follows. Section 8.2 formulates the market model and ALM problem. Section 8.3 discusses the solvability of BSDEs associated with the ALM problem and derives closed-form solutions to the BSDEs. In Section 8.4, explicit expressions for the optimal investment strategy and optimal value function are presented, and the results for several particular cases of our model are recovered. Section 8.5 gives some numerical examples to illustrate theoretical results. Finally, Section 8.6 concludes the paper.

8.2 General formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $T > 0$ be a finite constant standing for the decision-making horizon, and $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ be the right-continuous, $\mathbb{P}$-complete filtration generated by a two one-dimensional, mutually independent standard Brownian motions $\{W_{1,t}\}_{t\in[0,T]}$ and $\{W_{2,t}\}_{t\in[0,T]}$. Denote by $\mathbb{E}[\cdot]$ the expectation under $\mathbb{P}$. In what follows, we introduce several spaces on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

- $\mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes with $\mathbb{P}$-a.s. continuous sample paths;
- $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{P}\left(\int_0^T |f_t|^2\,dt < \infty\right) = 1$;
- $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}\left[\int_0^T |f_t|^2\,dt\right] < \infty$;
- $\mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}\left[\sup_{t\in[0,T]}|f_t|^2\right] < \infty$;
- $\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted uniformly bounded processes with $\mathbb{P}$-a.s. continuous sample paths.

8.2.1 The financial market and liability process

We consider a financial market consisting of a risk-free asset (money account) and a risky asset (stock). The price process of the money account $\{B_t\}_{t\in[0,T]}$ evolves according to

$$dB_t = rB_t\,dt,\ B_0 = 1,$$

where $r \in \mathbb{R}$ is the risk-free interest rate. The risky asset price $\{S_t\}_{t\in[0,T]}$ follows the dynamics

$$dS_t = S_t\left(\mu_{S,t}\,dt + \sigma_{S,t}\,dW_{1,t}\right),\ S_0 = s_0 > 0, \tag{8.2.1}$$

where $\mu_{S,t}$ and $\sigma_{S,t} > 0$ are two potentially unbounded and non-Markovian $\mathbb{F}$-adapted stochastic processes, which stand for the risky asset's return rate and volatility at time t, respectively. We denote the market price of volatility risk by

$$\theta_t := \frac{\mu_{S,t} - r}{\sigma_{S,t}} \tag{8.2.2}$$

for $t \in [0,T]$, and assume that the market price of volatility risk process $\{\theta_t\}_{t\in[0,T]}$ depends on a stochastic factor process $\{V_t\}_{t\in[0,T]}$ in the following way:

$$\theta_t = \lambda\sqrt{\eta_1 + \eta_2 V_t},\ \lambda \in \mathbb{R}\setminus\{0\} = \mathbb{R}_0,$$

where the stochastic factor process $\{V_t\}_{t\in[0,T]}$ follows an affine diffusion equation (Duffie and Kan (1996)):

$$dV_t = (a - bV_t)\,dt + \sqrt{\eta_1 + \eta_2 V_t}(\rho_1\,dW_{1,t} + \rho_2\,dW_{2,t}),\ V_0 = v_0 > 0. \tag{8.2.3}$$

To make our framework more general, no further conditions on parameters $a, b, \rho_1, \rho_2 \in \mathbb{R}$ and $\eta_1 \geq 0, \eta_2 \geq 0$ are imposed at the present stage. Instead, we only assume that the solution to the affine diffusion equation (8.2.3) is well-defined, i.e., $\eta_1 + \eta_2 V_t$ is non-negative $\mathbb{P}$ almost surely, for $t \in [0,T]$.

Remark 8.2.1. For the case when $\eta_2 > 0$, it can be checked that $\{\eta_1 + \eta_2 V_t\}_{t\in[0,T]}$ follows a square-root diffusion process:

$$d(\eta_1 + \eta_2 V_t) = (a\eta_2 + b\eta_1 - b(\eta_1 + \eta_2 V_t))\, dt + \eta_2 \sqrt{\eta_1 + \eta_2 V_t}\,(\rho_1\, dW_{1,t} + \rho_2\, dW_{2,t})\,.$$

It follows from Chapter 6.3 of Jeanblanc, Chesney, and Yor (2009) that the process $\{\eta_1 + \eta_2 V_t\}_{t\in[0,T]}$ is non-negative $\mathbb{P}$ almost surely, if the parameters satisfy $a\eta_2 + b\eta_1 \geq 0$. Moreover, it is worth mentioning that although the diffusion coefficient of the process $\{\eta_1 + \eta_2 V_t\}_{t\in[0,T]}$ does not satisfy the uniform Lipschitz continuity, a unique strong solution such that $\mathbb{E}\left[\sup_{t\in[0,T]}(\eta_1 + \eta_2 V_t)^2\right] \leq C_T$ exists, where the upper bound C_T depends on $a, b, \eta_1, \eta_2, \rho_1, \rho_2$, and T. For the simple case when $\eta_2 = 0$, the affine diffusion equation (8.2.3) is reduced to an Ornstein-Uhlenbeck (OU) process:

$$dV_t = (a - bV_t)\, dt + \sqrt{\eta_1}\,(\rho_1\, dW_{1,t} + \rho_2\, dW_{2,t})\,,$$

of which the solution does exist (refer to Chapter 2.6 of Jeanblanc, Chesney, and Yor (2009)).

It is worth mentioning that the above modeling framework is general and embraces not only a wide class of stochastic (local) volatility models, such as the CEV model, Stein and Stein model, Schöbel and Zhu model, Heston model, 3/2 model, and 4/2 model (see Examples 8.2.2-8.2.4) but also some non-Markovian models (Example 8.2.5), as particular cases.

Example 8.2.2 (CEV model)**.** If we set $\mu_{S,t} = \mu$ and $\sigma_{S,t} = \sigma S_t^{\gamma}$, where $\mu \neq r$ and σ are positive constants and $\gamma \leq -\frac{1}{2}$ is the elasticity parameter, then the risky asset price follows the CEV model:

$$dS_t = S_t\,(\mu\, dt + \sigma S_t^{\gamma}\, dW_{1,t})\,,\ S_0 = s_0 > 0. \tag{8.2.4}$$

By setting $V_t = S_t^{-2\gamma}, a = \gamma(2\gamma+1)\sigma^2, b = 2\gamma\mu, \eta_1 = 0, \eta_2 = 1, \rho_1 = -2\gamma\sigma, \rho_2 = 0, \lambda = \frac{\mu - r}{\sigma}$ and applying Itô's formula to $S_t^{-2\gamma}$, we have

$$\begin{aligned} dV_t =& \left[\gamma(2\gamma+1)\sigma^2 - 2\gamma\mu S_t^{-2\gamma}\right]\, dt - 2\gamma\sigma S_t^{-\gamma}\, dW_{1,t} \\ =&(a - bV_t)\, dt + \sqrt{\eta_1 + \eta_2 V_t}\,(\rho_1\, dW_{1,t} + \rho_2\, dW_{2,t})\,. \end{aligned}$$

This shows that the CEV model (8.2.4) is a special case of the model given by (8.2.1)-(8.2.3). In particular, when $\gamma = 0$, the condition $a\eta_2 + b\eta_1 \geq 0$ still holds, and the CEV model (8.2.4) is reduced to the Black-Scholes model.

Example 8.2.3 (The 4/2 model)**.** If we set $\mu_{S,t} = r + \lambda\sqrt{\eta_2}(c_1 V_t + c_2)$, $\sigma_{S,t} = c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}$, $\eta_1 = 0$, $\rho_1 = \rho \in [-1, 1]$, and $\rho_2 = \sqrt{1-\rho^2}$, where $c_1 \geq 0$, $c_2 \geq 0$, and $\eta_2 \in \mathbb{R}^+$, then the risky asset price is governed by the family of 4/2 models (Grasselli Grasselli, 2017):

$$\begin{cases} dS_t = S_t \left[(r + \lambda\sqrt{\eta_2}(c_1 V_t + c_2))\, dt + \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}} \right) dW_{1,t} \right], \; S_0 = s_0 > 0, \\ dV_t = (a - bV_t)\, dt + \sqrt{\eta_2 V_t} \left(\rho\, dW_{1,t} + \sqrt{1-\rho^2}\, dW_{2,t} \right), \; V_0 = v_0 > 0. \end{cases} \tag{8.2.5}$$

In this case, $b \in \mathbb{R}^+$ is the mean-reversion speed, $a/b \in \mathbb{R}^+$ is the long-run mean, and $\sqrt{\eta_2}$ is the volatility of the variance driver process V_t. The Feller condition $2a \geq \eta_2$ is required, so that V_t is strictly positive $\mathbb{P}$ almost surely, for $t \in [0, T]$. Particularly, by further specifying $(c_1, c_2) = (1, 0)$, the 4/2 model (8.2.5) degenerates to the Heston model (Heston (1993)), while the case $(c_1, c_2) = (0, 1)$ corresponds to the 3/2 model (Lewis (2000)).

Example 8.2.4 (The Schöbel and Zhu model)**.** If we set $\mu_{S,t} = r + \lambda\sqrt{\eta_1}V_t$, $\sigma_{S,t} = V_t$, $\eta_2 = 0$, $\rho_1 = \rho \in [-1, 1]$, $\rho_2 = \sqrt{1-\rho^2}$, and $\eta_1 \in \mathbb{R}^+$, then the risky asset price follows the Schöbel and Zhu model (Schöbel and Zhu (1999)):

$$\begin{cases} dS_t = S_t \left[(r + \lambda\sqrt{\eta_1}V_t)\, dt + V_t\, dW_{1,t} \right], \; S_0 = s_0 > 0, \\ dV_t = (a - bV_t)\, dt + \sqrt{\eta_1} \left(\rho\, dW_{1,t} + \sqrt{1-\rho^2}\, dW_{2,t} \right), \; V_0 = v_0 > 0. \end{cases} \tag{8.2.6}$$

Here, $b \in \mathbb{R}^+$ is the mean-reversion speed, $a/b \in \mathbb{R}^+$ is the long-run mean, and $\sqrt{\eta_1}$ is the volatility of the instantaneous volatility process V_t. In particular, the case $\rho = 0$ in (8.2.6) is known as the Stein-Stein model (Stein and Stein (1991)).

Example 8.2.5 (A path-dependent stochastic volatility model)**.** If we set $\mu_{S,t} = r + \lambda\sqrt{\eta_2}\sqrt{V_t}\hat{\sigma}(V_{[0,t]})$, $\sigma_{S,t} = \hat{\sigma}(V_{[0,t]})$ for some functional $\hat{\sigma} : \mathcal{C}([0, t]; \mathbb{R}) \mapsto \mathbb{R}^+$, $\eta_1 = 0$, $\eta_2 \in \mathbb{R}^+$, $\rho_1 = \rho \in [-1, 1]$, and $\rho_2 = \sqrt{1-\rho^2}$, where $V_{[0,t]} := (V_s)_{s\in[0,t]}$ is the restriction of $V \in \mathcal{C}([0, T]; \mathbb{R})$ to $\mathcal{C}([0, t]; \mathbb{R})$, i.e. the space of real-valued, continuous functions defined on $[0, t]$, then the risky asset price is governed by the following path-dependent model:

$$\begin{cases} dS_t = S_t \left[\left(r + \lambda\sqrt{\eta_2 V_t}\hat{\sigma}(V_{[0,t]}) \right) dt + \hat{\sigma}(V_{[0,t]})\, dW_{1,t} \right], \; S_0 = s_0 > 0, \\ dV_t = (a - bV_t)\, dt + \sqrt{\eta_2 V_t} \left(\rho\, dW_{1,t} + \sqrt{1-\rho^2}\, dW_{2,t} \right), \; V_0 = v_0 > 0. \end{cases} \tag{8.2.7}$$

which is clearly a special case of non-Markovian models because the risky asset's return rate and volatility are path-dependent. For more details on model (8.2.7), one may consult Siu (2012).

Denote by $\pi := \{\pi_t\}_{t\in[0,T]}$ the money amount invested in the risky asset and X_t^π the asset value process under the strategy π. Suppose that there are no transaction costs as well as other restrictions in the financial market. Under a self-financing condition, the asset process X_t^π evolves according to the following stochastic differential equation (SDE):

$$dX_t^\pi = \pi_t \frac{dS_t}{S_t} + (X_t^\pi - \pi_t) \frac{dB_t}{B_t} = (rX_t^\pi + (\mu_{S,t} - r)\pi_t)\, dt + \pi_t \sigma_{S,t}\, dW_{1,t}, \quad (8.2.8)$$

with initial asset value $X_0 = x_0 \in \mathbb{R}^+$. Apart from investing in the above financial market, we consider that the asset-liability manager is subject to an uncontrollable liability commitment and assume that the accumulative liability process L_t follows a generalized drifted Brownian motion, for $t \in [0, T]$

$$dL_t = \mu_{L,t}\, dt + \sigma_{L,t}\, dW_{1,t}, \;\; L_0 = l_0 \in \mathbb{R}. \quad (8.2.9)$$

In the liability process (8.2.9), the drift and diffusion coefficients $\mu_{L,t} \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$ and $\sigma_{L,t} \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}_0)$, and they are related to each other via

$$\frac{\mu_{L,t} - \mu_l}{\sigma_{L,t}} = \theta_t,$$

where the non-negative constant $\mu_l \geq 0$ is the drift coefficient. Notice that the processes $\mu_{L,t}$ and $\sigma_{L,t}$ are potentially unbounded and non-Markovian, and the liability process adopted in this paper is in a general sense that it can be understood as a subtraction of the real liability and stochastic income of the manager. Therefore, negative liabilities are allowed, which means that the stochastic income is larger than the real liability. Similar specifications on the liability process can be found in some literature, such as Xie, Li, and Wang (2008), Pan, Zhang, and Zhou (2018), and Pan, Hu, and Zhou (2019).

Definition 8.2.6 (Admissible strategy)**.** *An investment strategy π is said to be admissible if it satisfies the following conditions:*
(i) π_t is a real-valued, $\mathbb{F}$-adapted process such that $\pi_t \sigma_{S,t} \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$;
(ii) given $(x_0, l_0, v_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ fixed and given, the SDE (8.2.8) associated with π_t has a unique strong solution $X_t^\pi \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$ such that $\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t} > 0$ for $t \in [0, T]$;
(iii) the family $\left\{ \frac{1-p}{qp} \left(\frac{q}{1-p}(X^\pi_{T\wedge\tau_n} - G_{1,T\wedge\tau_n} L_{T\wedge\tau_n}) + \beta G_{2,T\wedge\tau_n} \right)^p Y_{T\wedge\tau_n} \right\}_{n\in\mathbb{N}}$ is uniformly integrable, for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, $\mathbb{P}$ almost surely as $n \to \infty$, where $Y_t, G_{1,t}$, and $G_{2,t}$ are given by (8.3.5), (8.3.12), and (8.3.15), respectively.

The set of all admissible strategies is denoted by $\mathcal{A}$.

8.2.2 The optimization problem

The asset-liability manager aims to obtain an admissible strategy $\pi \in \mathcal{A}$ to maximize the expected utility of the terminal surplus $X_T^\pi - L_T$ at time T with the initial surplus $x_0 - l_0$. More specifically, the optimization problem can be formulated as follows:

$$\sup_{\pi \in \mathcal{A}} J(x_0, l_0, v_0; \pi) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^\pi - L_T)\right], \tag{8.2.10}$$

where the utility function $U(\cdot)$ is supposed to be a strictly increasing and concave function characterizing the manager's risk-averse preference, and $J(\cdot)$ is referred to as the value function in the literature. In this paper, we investigate the solution of ALM problem (8.2.10) in a general utility framework by considering the following HARA utility function with parameters p, q, and β:

$$U_{HARA}(x) = \frac{1-p}{qp}\left(\frac{q}{1-p}x + \beta\right)^p, \quad q > 0, \ p < 0, \ \beta \in \mathbb{R} \tag{8.2.11}$$

such that $\frac{q}{1-p}x + \beta > 0$. It can be checked that the above general HARA utility maximization framework (8.2.10)-(8.2.11) recovers the power, exponential, and logarithmic utility functions as special cases. In particular,

- if we take $\beta = 0$ and $q = 1-p$ in (8.2.11), we obtain the power utility function case, i.e.,
$$U_{power}(x) = \frac{1}{p}x^p;$$
- if we take $\beta = 1$ and compute the limit as $p \to -\infty$ in (8.2.11), we have the exponential utility function:
$$U_{exp}(x) = -\frac{1}{q}e^{-qx};$$
- if we compute the limit as $p \to 0$ of the following modified HARA function
$$U_{mHARA}(x) = \frac{1-p}{qp}\left[\left(\frac{q}{1-p}x + \beta\right)^p - 1\right]$$
and take $\beta = 0$ and $q = 1-p$, we have the logarithmic utility function:
$$U_{log}(x) = \log(x).$$

It is worth mentioning that since the limit procedures in the above exponential utility and logarithmic utility cases are only formal, the solutions for these two cases cannot be derived as an immediate result of the HARA utility case (see, for example, Grasselli (2003)).

Remark 8.2.7. Given the possibly non-Markovian structures of risky asset price process (8.2.1), asset process (8.2.8), and liability process (8.2.9) as well as the incomplete market setting, neither the dynamic programming (HJB) approach along with Legendre transform-dual technique nor the martingale approach can be applied in the present paper. Therefore, we solve problem (8.2.10) in Section 8.3 and 8.4 by means of BSDE, which distinguishes this paper from some existing literature on HARA utility maximization problems, for example, Grasselli (2003), Jung and Kim (2012), Escobar, Neykova, and Zagst (2017), Chang et al. (2020), Zhang and Zhao (2020), Zhang, Zhao, and Kou (2021), Liu et al. (2023), and etc.

8.3 Backward stochastic differential equations

In this section, we discuss the solvability of a recursively coupled BSDE system, including a BSRE and two linear BSDEs, based on which ALM problem (8.2.10) under HARA utility will be solved completely in the next section.

To find the BSDEs associated with problem (8.2.10), we first introduce the following three continuous $(\mathbb{F},\mathbb{P})$-semi-martingales, $Y_t, G_{1,t}$, and $G_{2,t}$, with canonical decomposition:

$$\begin{cases} dY_t = \Psi_t\,dt + Z_{1,t}\,dW_{1,t} + Z_{2,t}\,dW_{2,t}, \\ dG_{1,t} = H_{1,t}\,dt + \Lambda_{1,t}\,dW_{1,t} + \Lambda_{2,t}\,dW_{2,t}, \\ dG_{2,t} = H_{2,t}\,dt + \Gamma_{1,t}\,dW_{1,t} + \Gamma_{2,t}\,dW_{2,t}, \end{cases}$$

where $\Psi_t, H_{1,t}$, and $H_{2,t}$ are some $\mathbb{F}$-adapted processes to be determined, and $Z_{1,t}, Z_{2,t}, \Lambda_{1,t}, \Lambda_{2,t}, \Gamma_{1,t}$, and $\Gamma_{2,t}$ lie in $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. We expect that the process $\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^p Y_t$ is a (local) supermartingale for any admissible strategy $\pi \in \mathcal{A}$, and a (local) martingale for the optimal strategy by determining the processes $\Psi_t, H_{1,t}$, and $H_{2,t}$ in what follows. An application of Itô's formula shows that

$$\begin{aligned}
& d\left(\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^p Y_t\right) \\
=& \frac{1-p}{qp}\left[Y_t p\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-1}\left(\frac{q}{1-p}\left(\pi_t\sigma_{S,t} - L_t\Lambda_{1,t} - G_{1,t}\sigma_{L,t}\right)\right.\right. \\
& \left.\left. + \beta\Gamma_{1,t}\right) + \left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^p Z_{1,t}\right] dW_{1,t} \\
& - \frac{1-p}{qp}\left[Y_t p\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-1}\left(\frac{q}{1-p}L_t\Lambda_{2,t} - \beta\Gamma_{2,t}\right) - \left(\frac{q}{1-p}(X_t^\pi\right.\right. \\
& \left.\left. - G_{1,t}L_t) + \beta G_{2,t}\right)^p Z_{2,t}\right] dW_{2,t} - \frac{(p-1)^2}{2q} Y_t\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-2}\left[\beta\Gamma_{1,t}\right. \\
& \left. + \frac{q}{1-p}\left(\pi_t\sigma_{S,t} - L_t\Lambda_{1,t} - G_{1,t}\sigma_{L,t}\right) + \frac{\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}}{p-1}\left(\frac{Z_{1,t}}{Y_t} + \theta_t\right)\right]^2 dt \\
& - \frac{(p-1)^2}{2q} Y_t\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-2}\left(\frac{q}{1-p}L_t\Lambda_{2,t} - \beta\Gamma_{2,t}\right)^2 dt
\end{aligned}$$

$$
-\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^p \left[\left(\frac{1}{2(p-1)}\left(\frac{Z_{1,t}}{Y_t}+\theta_t\right)^2 - r\right)pY_t - \Psi_t\right] dt
$$
$$
+\frac{1-p}{q}\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-1} Y_t \left[\frac{q}{1-p}L_t\left(rG_{1,t}+\theta_t\Lambda_{1,t} - \frac{Z_{2,t}}{Y_t}\Lambda_{2,t}\right.\right.
$$
$$
\left.\left. -H_{1,t}\right) + \beta\left(H_{2,t} - rG_{2,t} - \theta_t\Gamma_{1,t} + \frac{Z_{2,t}}{Y_t}\Gamma_{2,t}\right) - \frac{q}{1-p}\left(\mu_l G_{1,t} + \sigma_{L,t}\Lambda_{1,t}\right)\right] dt.
\tag{8.3.1}
$$

We expect that the above drift coefficients are non-positive for arbitrary $\pi \in \mathcal{A}$ and zero for the optimal strategy. Therefore, the processes Ψ_t, $H_{1,t}$, and $H_{2,t}$ can be determined by formally letting the last two terms on the right-hand side of (8.3.1) be zero. As a result, we obtain the following BSRE of $(Y_t, Z_{1,t}, Z_{2,t})$ and two linear BSDEs of $(G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t})$ and $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$:

$$
\begin{cases}
dY_t = \left[\left(\frac{p}{2(p-1)}\theta_t^2 - rp\right)Y_t + \frac{p}{2(p-1)}\frac{Z_{1,t}^2}{Y_t} + \frac{p}{p-1}\theta_t Z_{1,t}\right] dt + Z_{1,t}\,dW_{1,t} \\
\qquad + Z_{2,t}\,dW_{2,t}, \\
Y_T = 1, \\
Y_t > 0, \text{ for all } t \in [0,T],
\end{cases}
\tag{8.3.2}
$$

$$
\begin{cases}
dG_{1,t} = \left(rG_{1,t} + \theta_t\Lambda_{1,t} - \frac{Z_{2,t}}{Y_t}\Lambda_{2,t}\right) dt + \Lambda_{1,t}\,dW_{1,t} + \Lambda_{2,t}\,dW_{2,t}, \\
G_{1,T} = 1,
\end{cases}
\tag{8.3.3}
$$

and

$$
\begin{cases}
dG_{2,t} = \left[rG_{2,t} + \theta_t\Gamma_{1,t} - \frac{Z_{2,t}}{Y_t}\Gamma_{2,t} + \frac{q}{\beta(1-p)}\left(\mu_l G_{1,t} + \sigma_{L,t}\Lambda_{1,t}\right)\right] dt \\
\qquad + \Gamma_{1,t}\,dW_{1,t} + \Gamma_{2,t}\,dW_{2,t} \\
G_{2,T} = 1.
\end{cases}
\tag{8.3.4}
$$

Throughout the paper, by a solution to BSRE (8.3.2), we mean a triplet of stochastic processes $(Y_t, Z_{1,t}, Z_{2,t}) \in \mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ and verifies (8.3.2). In the same vein, solutions to linear BSDEs (8.3.3) and (8.3.4) are two triplets of stochastic processes $(G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t}) \in \mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ and $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t}) \in \mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, respectively.

Remark 8.3.1. Notice that BSDEs (8.3.2)-(8.3.4) constitute a coupled BSDE system, and this coupling is recursive. More specifically, the generator of linear BSDE (8.3.3) involves the solution of $(Y_t, Z_{1,t}, Z_{2,t})$ to BSRE (8.3.2), while the generator of linear BSDE (8.3.4) includes both the solutions of $(Y_t, Z_{1,t}, Z_{2,t})$ to BSRE (8.3.2) and of $(G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t})$ to BSDE (8.3.3). This observation implies that BSDEs (8.3.2)-(8.3.4) shall be solved recursively forwards.

Next, we derive respective solutions to BSDEs (8.3.2)-(8.3.4), and prove that these solutions are unique.

Proposition 8.3.2. *One candidate solution triplet* $(Y_t, Z_{1,t}, Z_{2,t})$ *to BSRE (8.3.2) is given by*

$$Y_t = \exp\left\{f_1(t) + f_2(t)V_t\right\}, \tag{8.3.5}$$

and

$$\begin{cases} Z_{1,t} = f_2(t)\sqrt{\eta_1 + \eta_2 V_t}\rho_1 Y_t, \\ Z_{2,t} = f_2(t)\sqrt{\eta_1 + \eta_2 V_t}\rho_2 Y_t, \end{cases} \tag{8.3.6}$$

where $f_1(t)$ *and* $f_2(t)$ *are solutions to the following system of ordinary differential equations (ODEs):*

$$\begin{cases} \dfrac{df_1(t)}{dt} + \dfrac{1}{2}\left(\rho_2^2 - \dfrac{1}{p-1}\rho_1^2\right)\eta_1 f_2^2(t) - \left(\dfrac{\lambda\rho_1\eta_1 p}{p-1} - a\right)f_2(t) - \dfrac{\lambda^2\eta_1 p}{2(p-1)} + rp = 0; \\ \dfrac{df_2(t)}{dt} + \dfrac{1}{2}\left(\rho_2^2 - \dfrac{1}{p-1}\rho_1^2\right)\eta_2 f_2^2(t) - \left(\dfrac{\lambda\rho_1\eta_2 p}{p-1} + b\right)f_2(t) - \dfrac{\lambda^2\eta_2 p}{2(p-1)} = 0, \end{cases} \tag{8.3.7}$$

with boundary conditions $f_1(T) = f_2(T) = 0$.

Proof. See Appendix 8.A. □

In the following proposition, the solutions of functions $f_1(t)$ and $f_2(t)$ to ODE system (8.3.7) are derived in closed form.

Proposition 8.3.3. *Closed-form solutions to ODE system (8.3.7) are given by*

$$f_2(t) = \begin{cases} 0, \ \eta_2 = 0; \\ \dfrac{n_1 n_2\left(1 - e^{\sqrt{\Delta}(T-t)}\right)}{n_1 - n_2 e^{\sqrt{\Delta}(T-t)}}, \ \eta_2 \neq 0 \ \textit{and} \ \Delta > 0; \\ \dfrac{\eta_2\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)(T-t)n_0^2}{\eta_2\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)(T-t)n_0 - 2}, \ \eta_2 \neq 0 \ \textit{and} \ \Delta = 0; \\ n_0 + \dfrac{\sqrt{-\Delta}}{\eta_2\left(\frac{\rho_1^2}{p-1} - \rho_2^2\right)}\tan\left(\arctan\left(\dfrac{b + \lambda\eta_2\rho_1\frac{p}{p-1}}{\sqrt{-\Delta}}\right) \right. \\ \left. -\dfrac{\sqrt{-\Delta}}{2}(T-t)\right), \ \eta_2 \neq 0 \ \textit{and} \ \Delta < 0; \end{cases} \tag{8.3.8}$$

and

$$\begin{aligned} f_1(t) = &\int_t^T \left\{\frac{1}{2}\left(\rho_2^2 - \frac{1}{p-1}\rho_1^2\right)\eta_1 f_2^2(s) - \left(\frac{\lambda\rho_1\eta_1 p}{p-1} - a\right)f_2(s)\right\}ds \\ &+ (T-t)\left(rp - \frac{\lambda^2\eta_1 p}{2(p-1)}\right), \end{aligned} \tag{8.3.9}$$

where $\Delta, n_0, n_1,$ *and* n_2 *are given by*

$$\begin{cases} \Delta = \left(b + \dfrac{\lambda\rho_1\eta_2 p}{p-1}\right)^2 + \left(\rho_2^2 - \dfrac{\rho_1^2}{p-1}\right)\dfrac{\lambda^2\eta_2^2 p}{p-1}, \ n_0 = \dfrac{-\left(b + \frac{\lambda\rho_1\eta_2 p}{p-1}\right)}{\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)\eta_2}, \\ n_1 = \dfrac{-\left(b + \frac{\lambda\rho_1\eta_2 p}{p-1}\right) + \sqrt{\Delta}}{\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)\eta_2}, \ n_2 = \dfrac{-\left(b + \frac{\lambda\rho_1\eta_2 p}{p-1}\right) - \sqrt{\Delta}}{\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)\eta_2}. \end{cases} \tag{8.3.10}$$

Proof. See Appendix 8.B. □

Remark 8.3.4. From Proposition 8.3.3, it is straightforward to verify that $f_2(t)$ is a monotonically increasing function for the case when $\eta_2 \neq 0$ and $\Delta > 0$, while $f_2(t)$ is monotonically decreasing for the case when $\eta_2 \neq 0$ and $\Delta \leq 0$. These results imply that $|f_2(t)|$ is bounded by $|f_2(0)|$ for $t \in [0, T]$.

To give the space where the candidate solution triplet $(Y_t, Z_{1,t}, Z_{2,t})$ presented in Proposition 8.3.2 lies in, let us recall the following result on the Laplace transform of an integrated square-root diffusion process; see, for example, Pitman and Yor (1982) or Zeng and Taksar (2013).

Lemma 8.3.5. *Consider process* $\{\alpha_t\}_{t\in[0,T]}$ *with the following square-root diffusion dynamics:*

$$d\alpha_t = \kappa_\alpha(\theta_\alpha - \alpha_t)\, dt + \sigma_\alpha\sqrt{\alpha_t}\, dW_t,$$

where W_t *is a one-dimensional Brownian motion under* $\mathbb{P}$ *measure. When* $c_\alpha \leq \kappa_\alpha^2/2\sigma_\alpha^2$*, the Laplace transform is well-defined, i.e.,*

$$\mathbb{E}\left[\exp\left\{c_\alpha \int_0^T \alpha_t\, dt\right\}\right] < \infty.$$

Inspired by the uniform boundedness of function $f_2(t)$, Lemma 8.3.5 and the form of square-root diffusion process $\{\eta_1 + \eta_2 V_t\}_{t\in[0,T]}$, we impose the following technical condition on the model parameters throughout out the rest of the paper.

Assumption 8.3.6. *When* $\eta_2 \neq 0$*, the model parameters satisfy*

$$\left[\left(1 - \frac{1}{p-1}\right)^2 \rho_1^2 + 2\rho_2^2\right] f_2^2(0) + \lambda^2\left(\frac{4p^2}{(p-1)^2} + \frac{2p}{p-1} + \frac{1}{2}\right) \leq \frac{b^2}{2\eta_2^2}.$$

Remark 8.3.7. It is worth mentioning that the feasibility of Assumption 8.3.6 is guaranteed by the monotonicity of function $f_2(t)$. In particular, when the investment horizon T is small enough, $f_2^2(0)$ decreases and converges to zero. This assumption essentially assures the uniqueness results of BSRE (8.3.2) and linear BSDEs (8.3.3)-(8.3.4) as well as the admissibility of the optimal investment strategy π_t^* given in

(8.4.1). From an economic point of view, the above assumption gives an upper bound for the slope λ of the market price of risk process $\theta_t = \lambda\sqrt{\eta_1 + \eta_2 V_t}$. Due to the unboundedness of θ_t, taking risk might be rewarded too much by the market, if there is no any restriction on the slope λ. As stated in Korn and Kraft (2004) and Kraft (2005), in such a case, neither the finiteness of terminal utility nor the uniqueness of optimal solution can be ensured.

Remark 8.3.8. For the simple case when $\eta_2 = 0$, we observe that the market price of risk reduces to the constant $\lambda\sqrt{\eta_1}$. Consequently, the validity of Girsanov's measure change holds without any technical conditions on the model parameters in the proof of Proposition 8.3.9, Theorem 8.3.10, and Proposition 8.3.12 and 8.3.13 below.

Proposition 8.3.9. *Suppose Assumption 8.3.6 holds. The solution* $(Y_t, Z_{1,t}, Z_{2,t})$ *given in Proposition 8.3.2 lies in* $\mathcal{S}^{\infty}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+) \times \mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \times \mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. *More precisely,* $0 < Y_t < e^{rp(T-t)}$ *for all* $t \in [0,T]$, $\mathbb{P}$ *almost surely.*

Proof. See Appendix 8.C. □

Having verified that the candidate solution $(Y_t, Z_{1,t}, Z_{2,t})$ is indeed a solution triplet to BSRE (8.3.2), we next prove the uniqueness of the solution using the results of Kobylanski (2000).

Theorem 8.3.10. *Suppose Assumption 8.3.6 holds. BSRE (8.3.2) admits a unique solution* $(Y_t, Z_{1,t}, Z_{2,t})$*, which is given by (8.3.5) and (8.3.6).*

Proof. See Appendix 8.D. □

Remark 8.3.11. In the proof of Theorem 8.3.10, the generator of quadratic BSDE (8.D.2) is concave in the control components $Z_{1,t}/Y_t$ and $Z_{2,t}/Y_t$ of the solution, so the uniqueness theorem for quadratic BSDEs in Briand and Hu (2008) can be used if the random variable $\int_0^T \theta_t^2\,dt$ has exponential moments of all order (see Assumption (A.2) in Briand and Hu (2008)), i.e., for sufficiently large constant $c \in \mathbb{R}^+$, it holds that

$$\mathbb{E}\left[\exp\left\{c\int_0^T \theta_t^2\,dt\right\}\right] < \infty.$$

This assumption, however, violates the explosion criteria of integrated square-root diffusion processes, namely, Lemma 8.3.5 above. Therefore, a comparison method is applied in the above proof to eliminate the singular term $\frac{p}{2(p-1)}\theta_t^2 - rp$ within the generator of quadratic BSDE (8.D.2).

Having solved BSRE (8.3.2) completely, we can simplify linear BSDE (8.3.3) of $(G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t})$ as follows:

$$\begin{cases} dG_{1,t} = \left(rG_{1,t} + \theta_t \Lambda_{1,t} - \frac{\rho_2}{\lambda} f_2(t)\theta_t \Lambda_{2,t}\right) dt + \Lambda_{1,t}\, dW_{1,t} + \Lambda_{2,t}\, dW_{2,t}, \\ G_{1,T} = 1. \end{cases} \tag{8.3.11}$$

Proposition 8.3.12. *Suppose Assumption 8.3.6 holds. The unique solution to linear BSDE (8.3.11) is given by*

$$G_{1,t} = e^{-r(T-t)}, \tag{8.3.12}$$

and

$$(\Lambda_{1,t}, \Lambda_{2,t}) = (0, 0), \text{ for all } t \in [0, T]. \tag{8.3.13}$$

Proof. See Appendix 8.E. □

Relying on the preceding results on the solutions to BSRE (8.3.2) and linear BSDE (8.3.3), linear BSDE (8.3.4) of $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ can be reformulated as

$$\begin{cases} dG_{2,t} = \left(rG_{2,t} + \theta_t \Gamma_{1,t} - \frac{\rho_2}{\lambda} f_2(t)\theta_t \Gamma_{2,t} + \frac{q}{\beta(1-p)}\mu_l e^{-r(T-t)}\right) dt \\ \qquad\qquad + \Gamma_{1,t}\, dW_{1,t} + \Gamma_{2,t}\, dW_{2,t}, \\ G_{2,T} = 1. \end{cases} \tag{8.3.14}$$

Proceeding as in the proof of Proposition 8.3.12, we can obtain the unique solution to BSDE (8.3.14) in Proposition 8.3.13 below.

Proposition 8.3.13. *Suppose Assumption 8.3.6 holds. The unique solution to linear BSDE (8.3.14) is given by*

$$G_{2,t} = e^{-r(T-t)} \left(1 - \frac{q\mu_l}{\beta(1-p)}(T-t)\right), \tag{8.3.15}$$

and

$$(\Gamma_{1,t}, \Gamma_{2,t}) = (0, 0). \tag{8.3.16}$$

Remark 8.3.14. It is of importance to identify that the control components $\Lambda_{2,t}$ and $\Gamma_{2,t}$ of linear BSDEs (8.3.3)-(8.3.4) are zeros, because of which the second drift term on the right-hand side of (8.3.1) can be removed.

8.4 Solution to the optimization problem

In this section, we derive the main result of this paper, the optimal investment strategy and optimal value function for ALM problem (8.2.10) under HARA utility, which are represented in terms of the solutions to BSDEs (8.3.2)-(8.3.4) with explicit expressions given in Proposition 8.3.2, Proposition 8.3.12 and 8.3.13, respectively.

Theorem 8.4.1. *Suppose Assumption 8.3.6 holds, then for any initial value of data (x_0, l_0) satisfying $\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0} > 0$, the optimal investment strategy and optimal value function of ALM problem (8.2.10) under HARA utility are respectively given by*

$$\pi_t^* = \frac{\left(\frac{1}{1-p}(X_t^* - G_{1,t}L_t) + \frac{\beta}{q}G_{2,t}\right)\left(\frac{Z_{1,t}}{Y_t} + \theta_t\right) + \sigma_{L,t}G_{1,t}}{\sigma_{S,t}}, \tag{8.4.1}$$

and

$$J(x_0, l_0, v_0; \pi^*) = \frac{1-p}{qp} Y_0 \left(\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0}\right)^p, \tag{8.4.2}$$

where $Y_t, Z_{1,t}, G_{1,t}$, and $G_{2,t}$ are explicitly given by (8.3.5), (8.3.6), (8.3.12), and (8.3.15). Moreover, the optimal investment strategy is admissible, i.e., $\pi^* \in \mathcal{A}$.

Proof. See Appendix 8.F. □

Remark 8.4.2. If we set $\mu_l = l_0 = \mu_{L,t} = \sigma_{L,t} = 0$ for all $t \in [0, T]$ in Theorem 8.4.1, we obtain the closed-form solution to the case without random liability. Instead, the specifications $q = 1 - p$ and $\beta = 0$ lead to the results for the ALM problem under power utility.

The next three corollaries provide the results for the CEV model, 4/2 model, and Schöbel and Zhu model in Example 8.2.2-8.2.4, respectively.

Corollary 8.4.3 (CEV model)**.** *Suppose Assumption 8.3.6 holds, then for any initial value of data (x_0, l_0) satisfying $\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0} > 0$, if the risky asset price process follows the CEV model (8.2.4) in Example 8.2.2, the optimal investment strategy of ALM problem (8.2.10) under HARA utility is given by*

$$\pi_t^* = \frac{\left(\frac{1}{1-p}(X_t^* - G_{1,t}L_t) + \frac{\beta}{q}G_{2,t}\right)\left(\frac{\mu-r}{\sigma} - 2\gamma\sigma\tilde{f}_2(t)\right)S_t^{-\gamma} + \sigma_{L,t}G_{1,t}}{\sigma S_t^{\gamma}},$$

and the optimal value function is given by

$$J(x_0, l_0, s_0; \pi^*) = \frac{1-p}{qp}\left(\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0}\right)^p \exp\left\{\tilde{f}_1(0) + \tilde{f}_2(0)s_0^{-2\gamma}\right\},$$

where $G_{1,t}$ and $G_{2,t}$ are explicitly given by (8.3.12) and (8.3.15), and functions $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ are given by

$$\tilde{f}_2(t) = \frac{\tilde{n}_1\tilde{n}_2\left(1 - e^{\sqrt{\tilde{\Delta}}(T-t)}\right)}{\tilde{n}_1 - \tilde{n}_2 e^{\sqrt{\tilde{\Delta}}(T-t)}}$$

and

$$\tilde{f}_1(t) = (rp + \gamma(2\gamma+1)\sigma^2\tilde{n}_2)(T-t) - \frac{(2\gamma+1)(p-1)}{2\gamma}\log\left(\frac{\tilde{n}_1-\tilde{n}_2}{\tilde{n}_1-\tilde{n}_2 e^{\sqrt{\tilde{\Delta}}(T-t)}}\right),$$

with $\tilde{\Delta}, \tilde{n}_1, \tilde{n}_2$ *given by*

$$\begin{cases} \tilde{\Delta} = \dfrac{4\gamma^2}{(p-1)^2}\left[(rp-\mu)^2 - (\mu-r)^2 p\right], \\ \tilde{n}_1 = \dfrac{2\gamma(\mu-rp)+(p-1)\sqrt{\tilde{\Delta}}}{4\gamma^2\sigma^2}, \\ \tilde{n}_2 = \dfrac{2\gamma(\mu-rp)-(p-1)\sqrt{\tilde{\Delta}}}{4\gamma^2\sigma^2}. \end{cases}$$

Proof. Substituting the specified parameters in Example 8.2.2 into Theorem 8.4.1 yields the above results. □

Remark 8.4.4. When $\mu_l = l_0 = \mu_{L,t} = \sigma_{L,t} = 0$ in Corollary 8.4.3, the optimal investment strategy and optimal value function under the CEV model and HARA utility without liability are provided, which are the same as Theorem 4.1 in Zhang and Zhao (2020) (function $\tilde{I}(t)$ therein reduces to zero if no reinsurance is considered) and Theorem 3.2 in Jung and Kim (2012). If we further set $q = 1-p$ and $\beta = 0$, then Corollary 8.4.3 provides the results under power utility without liability, which are consistent with Proposition 4.1 in Gao (2009).

Corollary 8.4.5 (4/2 model)**.** *Suppose Assumption 8.3.6 holds, then for any initial value of data* (x_0, l_0) *satisfying* $\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0} > 0$*, if the risky asset price process follows the 4/2 stochastic volatility model (8.2.5) in Example 8.2.3, the optimal investment strategy of ALM problem (8.2.10) under HARA utility is given by*

$$\pi_t^* = \frac{\left(\frac{1}{1-p}(X_t^* - G_{1,t}L_t) + \frac{\beta}{q}G_{2,t}\right)\left(\bar{f}_2(t)\rho + \lambda\right)\sqrt{\eta_2 V_t} + \sigma_{L,t}G_{1,t}}{c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}}},$$

and the optimal value function is given by

$$J(x_0, l_0, v_0; \pi^*) = \frac{1-p}{p}\left(\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0}\right)^p \exp\left\{\bar{f}_1(0) + \bar{f}_2(0)v_0\right\},$$

where $G_{1,t}$ *and* $G_{2,t}$ *are explicitly given by (8.3.12) and (8.3.15), and functions*

$\bar{f}_1(t)$ and $\bar{f}_2(t)$ are given by

$$\bar{f}_2(t)=\begin{cases}\dfrac{\bar{n}_1\bar{n}_2\left(1-e^{\sqrt{\bar{\Delta}}(T-t)}\right)}{\bar{n}_1-\bar{n}_2e^{\sqrt{\bar{\Delta}}(T-t)}}, & \bar{\Delta}>0;\\[2ex] \dfrac{\eta_2\left(\frac{p}{p-1}\rho^2-1\right)(T-t)\bar{n}_0^2}{\eta_2\left(\frac{p}{p-1}\rho^2-1\right)(T-t)\bar{n}_0-2}, & \bar{\Delta}=0;\\[2ex] \dfrac{\sqrt{-\bar{\Delta}}}{\eta_2\left(\frac{p}{p-1}\rho^2-1\right)}\tan\left(\arctan\left(\dfrac{b+\lambda\eta_2\rho\frac{p}{p-1}}{\sqrt{-\bar{\Delta}}}\right)-\dfrac{\sqrt{-\bar{\Delta}}}{2}(T-t)\right)+\bar{n}_0, & \bar{\Delta}<0;\end{cases}$$

and

$$\bar{f}_1(t)=\begin{cases}(rp+a\bar{n}_2)(T-t)+\dfrac{a(\bar{n}_2-\bar{n}_1)}{\sqrt{\bar{\Delta}}}\log\left(\dfrac{\bar{n}_1-\bar{n}_2}{\bar{n}_1-\bar{n}_2e^{\sqrt{\bar{\Delta}}(T-t)}}\right), & \bar{\Delta}>0;\\[2ex] (rp+a\bar{n}_0)(T-t)-\dfrac{2a}{\left(\frac{p}{p-1}\rho^2-1\right)\eta_2}\log\left(\dfrac{2}{2-\eta_2\left(\frac{p}{p-1}\rho^2-1\right)(T-t)\bar{n}_0}\right), & \bar{\Delta}=0;\\[2ex] (rp+a\bar{n}_0)(T-t)-\dfrac{2a}{\left(\frac{p}{p-1}\rho^2-1\right)\eta_2}\left[\log\left(\cos\left(\arctan\left(\dfrac{b+\lambda\eta_2\rho\frac{p}{p-1}}{\sqrt{-\bar{\Delta}}}\right)\right)\right)\right.\\ \left.-\log\left(\cos\left(\arctan\left(\dfrac{b+\lambda\eta_2\rho\frac{p}{p-1}}{\sqrt{-\bar{\Delta}}}\right)-\dfrac{\sqrt{-\bar{\Delta}}(T-t)}{2}\right)\right)\right], & \bar{\Delta}<0,\end{cases}$$

with $\bar{\Delta},\bar{n}_0,\bar{n}_1,\bar{n}_2$ given by

$$\begin{cases}\bar{\Delta}=b^2+\lambda\eta_2(2b\rho+\lambda\eta_2)\dfrac{p}{p-1}, \quad \bar{n}_0=\dfrac{-\left(b+\frac{\lambda\rho\eta_2p}{p-1}\right)}{\left(\frac{p}{p-1}\rho^2-1\right)\eta_2},\\[2ex] \bar{n}_1=\dfrac{-\left(b+\frac{\lambda\rho\eta_2p}{p-1}\right)+\sqrt{\bar{\Delta}}}{\left(\frac{p}{p-1}\rho^2-1\right)\eta_2}, \quad \bar{n}_1=\dfrac{-\left(b+\frac{\lambda\rho\eta_2p}{p-1}\right)-\sqrt{\bar{\Delta}}}{\left(\frac{p}{p-1}\rho^2-1\right)\eta_2}.\end{cases}$$

Proof. Plugging the specified parameters in Example 8.2.3 into Theorem 8.4.1 leads to the above results immediately. □

Remark 8.4.6. Notice that the case $(c_1,c_2)=(1,0)$ in Corollary 8.4.5 leads to the optimal investment strategy and optimal value function under the Heston model. If we further get rid of random liability by specifying $\mu_l=l_0=\mu_{L,t}=\sigma_{L,t}=0$, the degenerated results are the same as Lemma 4.2 and Eq. (67) in Zhang, Zhao, and Kou (2021) (function $I(t)$ therein degenerates to zero when reinsurance is ignored). Instead, the specification $(c_1,c_2)=(0,1)$ corresponds to the 3/2 model. If we specify $q=1-p$ and $\beta=0$, explicit expressions for the optimal investment strategy and optimal value function of ALM problem under power utility are obtained. In this sense, our solutions generalize the results of Kraft (2005), Zeng and Taksar (2013), and Cheng and Escobar (2021a) from the optimal investment problem under power utility to the ALM problem under the more general HARA utility.

Corollary 8.4.7 (Schöbel and Zhu model). *Suppose Assumption 8.3.6 holds, then for any initial value of data* (x_0, l_0) *satisfying* $\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0} > 0$*, if the risky asset price process follows the Schöbel and Zhu model (8.2.6) in Example 8.2.4, the optimal investment strategy of ALM problem (8.2.10) under HARA utility is given by*

$$\pi_t^* = \frac{\left(\frac{1}{1-p}(X_t^* - G_{1,t}L_t) + \frac{\beta}{q}G_{2,t}\right)\lambda\sqrt{\eta_1} + \sigma_{L,t}G_{1,t}}{V_t},$$

and the optimal value function is given by

$$J(x_0, l_0; \pi^*) = \frac{1-p}{qp}\left(\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0}\right)^p \exp\left\{\left(rp - \frac{\lambda^2\eta_1 p}{p-1}\right)T\right\},$$

where $G_{1,t}$ *and* $G_{2,t}$ *are explicitly given by (8.3.12) and (8.3.15).*

Proof. Substituting the specifications in Example 8.2.4 into Theorem 8.4.1 gives the above results. □

Remark 8.4.8. It is interesting to see from Corollary 8.4.7 that although the optimal investment strategy π_t^* hinges upon the instantaneous volatility V_t within the Schöbel and Zhu model (8.2.6), the optimal value function $J(x_0, l_0; \pi^*)$ is not affected by V_t. This finding can be explained by our specification of the market price of risk process $\theta_t = \lambda\sqrt{\eta_1 + \eta_2 V_t}$. Indeed, when the Schöbel and Zhu model (8.2.6) is considered, η_2 turns out to be zero so that a constant market price of risk $\lambda\sqrt{\eta_1}$ is obtained over $t \in [0, T]$. In other words, taking risks is rewarded the same by the market. However, as discussed in Kraft (2005), we should be aware that a constant market price of risk is a rather exceptional case. In the paper of Schöbel and Zhu (1999), the market price of risk is assumed to be proportional to the volatility process, i.e., $\theta_t = \lambda\sqrt{\eta_1}V_t$, but this specification cannot lead to explicit expressions for both the optimal investment strategy and optimal value function under the HARA utility framework. Our specification, however, retains the mathematical tractability of ALM problem (8.2.10) as well as the form of the risk-neutralized process under the risk-adjusted martingale measure; one may consult Eqs. (1)-(2) in Schöbel and Zhu (1999).

8.5 Numerical examples

This section provides numerical examples to examine the effect of some model parameters on the behavior of optimal investment strategies. We mainly focus on two extensively studied models in the literature, the CEV model (8.2.4) and 4/2 stochastic volatility model (8.2.5). For simplicity but without loss of generality, the diffusion coefficient of liability process (8.2.9) is chosen to be a non-zero constant, i.e., $\sigma_{L,t} = \sigma_l \in \mathbb{R}_0$. Unless otherwise stated, the hypothetical values of model parameters are as follows: $p = -\frac{1}{2}, q = 2, \beta = 1, r = 0.05, \mu_l = 0.02, \sigma_l = 0.15, x_0 =$

$1, l_0 = 0.2, T = 5$; in the CEV model $\mu = 0.12, \sigma = 0.2, \gamma = -0.7$, and $s_0 = 0.5$; in the 4/2 model $c_1 = 0.9051, c_2 = 0.023, \eta_2 = 0.4356, \lambda = 2.9428, b = 7.3479, a = 0.24, \rho = 0.3$, and $v_0 = 0.04$. In the following numerical illustrations, we vary the value of one parameter with others fixed each time.

8.5.1 Effect of parameters in the CEV model on the optimal investment strategy

In this subsection, we focus on the results for the CEV model (8.2.4) given in Corollary 8.4.3. Because π_t^* depends on the stochastic processes S_t, L_t, and X_t^*, we use the Monte Carlo simulation technique to analyze the effect of some parameters in the CEV model on the optimal investment strategy π_t^*. More precisely, one sample path of π_t^* is given in the following Figure 8.1-8.2.

Figure 8.1 shows the relationship between the parameters μ, σ, γ and the optimal investment strategy π_t^*. We can observe that the optimal amount of money invested in the risk asset is positively correlated with the parameter μ. The reason is that in the CEV model (8.2.4), μ represents the return rate of the risky asset price. When other parameters remain unchanged, a greater value of μ implies a higher premium return of the risky asset. Consequently, the asset-liability manager is more willing to invest in the risky asset. From Figure 8.1, we also find that the optimal investment strategy decreases with respect to the parameter σ. This can be explained by the fact that σ characterizes the risky asset's local volatility. The greater the volatility parameter σ is, the higher the risk of the risky asset becomes. Therefore, the asset-liability manager will decrease the amount of money invested in the risky asset to avoid volatility risk. We can draw from the right panel of Figure 8.1 that the optimal amount of money invested in the risky asset has positive relationships with the elasticity parameter γ. From the economic implication of γ, the negativeness of γ means that the leverage effect is present, which, in turn, leads to a more significant volatility risk since the instantaneous volatility increases as the risky asset's price decreases. Hence, when γ becomes less negative from -0.7 to -0.5, the volatility risk becomes less significant and the asset-liability manager would increase the investment in the risky asset.

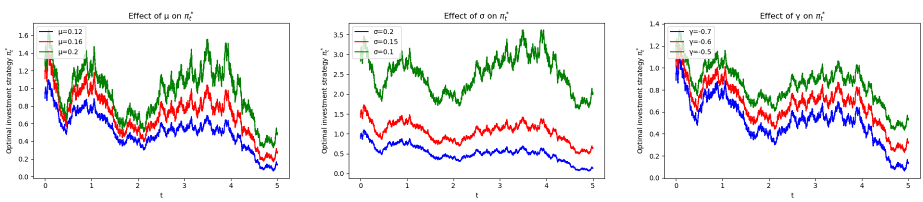


Figure 8.1: *Effect of μ, σ, and γ on the optimal strategy π_t^* under the CEV model*

Figure 8.2 depicts the effect of the parameters r, σ_l, μ_l on the optimal investment strategy π_t^*. From the left panel of Figure 8.2, we see that the optimal amount of money invested in the risky asset decreases as the risk-free interest rate r increases from 0.02 to 0.07. This is because the greater value of r means that the expected return of the money account becomes higher. Therefore, the asset-liability manager is willing to put more money into the risk-free money account to reduce the overall risks. We also vary σ_l from 0.1 to 0.2, and find that the optimal amount of money invested in the risky asset increases as the value of parameter σ_l becomes larger. This is consistent with our intuition. Indeed, by Eq. (8.2.9), a greater value of σ_l implies higher volatility of the uncontrollable liability. To hedge against the increased volatility risk of liability, the asset-liability manager tends to adopt a more aggressive investment strategy. In contrast, from the right panel of Figure 8.2, we notice that μ_l exerts a negative effect on the optimal investment strategy. One of the possible reasons is that when μ_l increases, the random liability's drift rate becomes larger while the volatility remains unchanged. Hence, the asset-liability manager opts for a more conservative investment strategy instead of putting more money into the risky asset, which might lead to mis-hedging against the volatility risk.

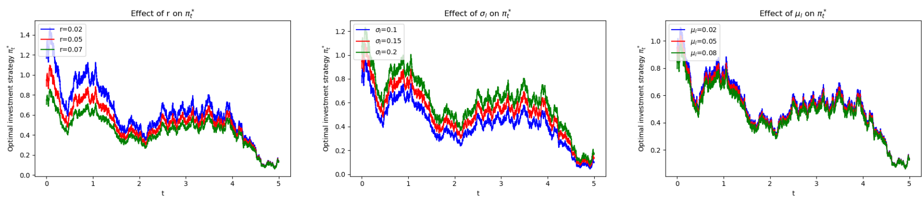


Figure 8.2: *Effect of r, σ_l, and μ_l on the optimal strategy π_t^* under the CEV model*

8.5.2 Effect of parameters in the 4/2 model on the optimal investment strategy

In this subsection, we are interested in the effect of some parameters in the 4/2 stochastic volatility model (8.2.5) on the optimal investment strategy π_t^* given in Corollary 8.4.5. By using some Monte Carlo simulation techniques, one sample path of π_t^* with respect to time is presented in Figure 8.3-8.4 below.

Figure 8.3 illustrates the effect of the parameters λ and b on the optimal investment strategy π_t^*. It is shown that as λ increases, the optimal amount of money invested in the risky asset increases. This can be explained by the fact that λ partially reflects the market price of volatility risk, and the asset-liability manager can derive a higher volatility risk premium from the risky asset when λ becomes larger. As such, the manager tends to invest more in the risky asset. We vary b from 5.3479 to 9.3479 in the right panel of Figure 8.3. As b increases, the instantaneous variance driver of the 4/2 model reverts faster towards the long-run mean a/b. In this case, the variance driver would stay in a smaller level for a longer period of time, which reduces the volatility risk and makes the asset-liability manager more willing to invest in the risky asset.

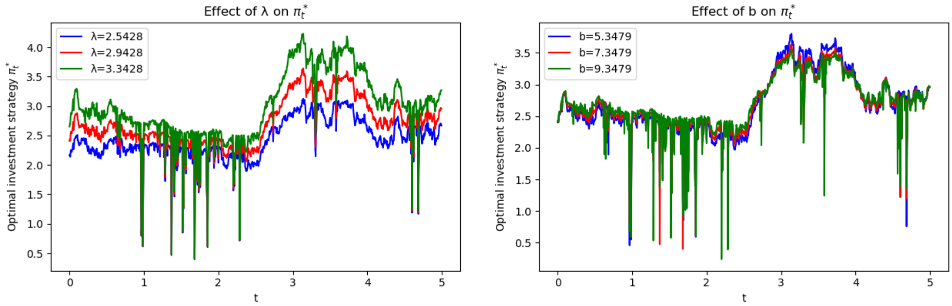


Figure 8.3: *Effect of λ and b on the optimal strategy π_t^* under the 4/2 model*

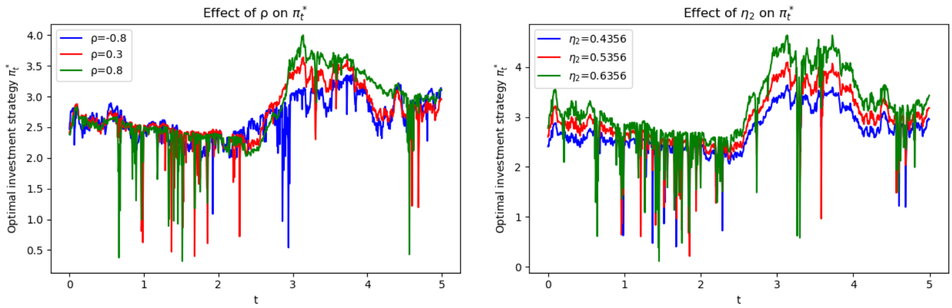


Figure 8.4: *Effect of ρ and η_2 on the optimal strategy π_t^* under the 4/2 model*

Figure 8.4 shows how the optimal investment strategy π_t^* changes with respect to the parameters ρ and η_2. We vary ρ from -0.8 to 0.8 in the left panel of Figure 8.4, and find that the optimal investment strategy increases as ρ becomes larger. One explanation is that as ρ increases, the risky asset price is less negatively correlated and more positively correlated with the instantaneous variance driver. In this case, the offset between the uncertainties of two fundamental risk factors, i.e., $W_{1,t}$ and $W_{2,t}$, is reduced, which amplifies the asset-liability manager's exposure to the overall risks. Therefore, the asset-liability manager needs to invest more in the risky asset to hedge against the overall risks. It can be seen from the right panel of Figure 8.4 that the optimal amount of money invested in the risky asset increases with respect to η_2 which varies from 0.4356 to 0.6356. This can be possibly explained by the 4/2 model dynamic (8.2.5), from which we see that η_2 influences not only the volatility of the variance driver process but also the market price of volatility risk. On one hand, as η_2 increases, the fluctuation of the variance driver process becomes more volatile. On the other hand, a greater value of η_2 allows the asset-liability manager to acquire a higher risk premium by bearing the same amount of volatility risk. In other words, the overall effect of η_2 on the optimal investment strategy takes on these two opposite sides. The increment of the optimal amount of money invested in the risky asset shows that compared with the variance driver process itself, the risk premium is more sensitive to the change of η_2. Therefore, the asset-liability manager tends to invest more money in the risky asset when η_2 increases.

8.6 Conclusion

In this paper, we investigate an optimal ALM problem under the HARA utility framework in the presence of stochastic volatility. The asset-liability manager has access to a financial market consisting of a risk-free asset and a risky asset, in which the market price of risk is described by an affine diffusion factor process and the uncontrollable liability is featured by a generalized drifted Brownian motion. The general modeling framework includes not only a wide class of Markovian models, such as the CEV model, Stein-Stein model, Schöbel and Zhu model, Heston model, 3/2 model, and 4/2 model but also some non-Markovian models, as exceptional cases. The asset-liability manager aims to determine the optimal investment strategy to maximize the utility of terminal surplus. Given the potentially non-Markovian and incomplete market setting, we apply a BSDE approach to solve the problem. By solving a system of three related BSDEs, explicit expressions for the optimal investment strategy and optimal value function are derived. Furthermore, closed-form solutions to some particular cases of our model are obtained. Finally, we provide some numerical experiments for two extensively studied models, the CEV model and 4/2 model, to illustrate the effects of model parameters on the optimal investment strategies. As far as we know, there is no literature discussing ALM problems under the HARA utility preferences and stochastic volatility models in such a non-Markovian and incomplete market setting. In this sense, this paper extends the existing results and models on ALM problems.

In future research, some extensions of the present paper are worthy of being further explored. For instance, since this paper only considers the case with a single risky asset, one may extend the current framework to that with multiple risky assets, where an appropriate adoption of a multi-dimensional market price of risk would be critical. Once the relevant market price of risk is delicately chosen, the associated BSDEs are expected to have resemblant structures to the ones considered in the present paper. It is noteworthy that the complicated case with multiple risky assets may lead to an ODE system with high dimensions, thus, resulting in no analytical solutions to the problem in general. One may also introduce model ambiguity into ALM problems in a non-Markovian market setting. In such a case, the generally applied Hamilton-Jacobi-Bellman-Issacs (HJBI) approach in the literature cannot be used, and therefore, a novel BSDE approach may be disentangled.

Acknowledgment(s)

We are grateful to Prof. Jesper Lund Pedersen, the handling editors, and two anonymous reviewers for their valuable comments and suggestions which have improved this work substantially.

8.A Proof of Proposition 8.3.2

Proof. We conjecture that the first component Y_t of solution to BSRE (8.3.2) has an exponential-affine form:

$$Y_t = \exp\{f_1(t) + f_2(t)V_t\},$$

where $f_1(t)$ and $f_2(t)$ are two undetermined differentiable functions with boundary condition $f_1(T) = f_2(T) = 0$. An application of Itô's formula to Y_t yields

$$\begin{aligned} dY_t = &\left(\frac{df_1(t)}{dt} + \frac{df_2(t)}{dt}V_t + (a - bV_t)f_2(t) + \frac{1}{2}(\eta_1 + \eta_2 V_t)\left(\rho_1^2 + \rho_2^2\right)f_2^2(t)\right) Y_t\,dt \\ &+ f_2(t)\sqrt{\eta_1 + \eta_2 V_t}\rho_1 Y_t\,dW_{1,t} + f_2(t)\sqrt{\eta_1 + \eta_2 V_t}\rho_2 Y_t\,dW_{2,t}. \end{aligned} \tag{8.A.1}$$

Comparing (8.A.1) and the first equation of BSRE (8.3.2) and separating the dependence on V_t, we find that $f_1(t)$ and $f_2(t)$ must solve ODE system (8.3.7). This verifies that (8.3.5) and (8.3.6) form one solution to BSDE (8.3.2). □

8.B Proof of Proposition 8.3.3

Proof. We first solve the ODE of $f_2(t)$ because $f_1(t)$ can be immediately obtained given that we know $f_2(t)$. When $\eta_2 = 0$, the Riccati equation of $f_2(t)$ reduces to the following first-order linear equation:

$$\frac{df_2(t)}{dt} - bf_2(t) = 0, \;\; f_2(T) = 0,$$

from which we obtain $f_2(t) = 0$. For the case when $\eta_2 \neq 0$, set $\Delta = \left(b + \frac{\lambda\rho_1\eta_2 p}{p-1}\right)^2 + \left(\rho_2^2 - \frac{1}{p-1}\rho_1^2\right)\frac{\lambda^2\eta_2^2 p}{p-1}$. If $\Delta > 0$, we can reformulate the Riccati ODE of $f_2(t)$ as follows:

$$\frac{df_2(t)}{dt} = \frac{\eta_2}{2}\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)(f_2(t) - n_1)(f_2(t) - n_2), \tag{8.B.1}$$

where n_1 and n_2 are given by (8.3.10). By taking integration on both sides from t to T, we obtain

$$f_2(t) = \frac{n_1 n_2\left(1 - e^{\sqrt{\Delta}(T-t)}\right)}{n_1 - n_2 e^{\sqrt{\Delta}(T-t)}}.$$

If $\Delta = 0$, we can simplify (8.B.1) as follows:

$$\frac{df_2(t)}{(f_2(t)-n_0)^2} = \frac{\eta_2}{2}\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right) dt$$

with n_0 given in (8.3.10). Then implementing an integral calculation leads to

$$f_2(t) = \frac{\eta_2\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)(T-t)n_0^2}{\eta_2\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)(T-t)n_0 - 2}.$$

If $\Delta < 0$, the terms within the Riccati equation of $f_2(t)$ can be rewritten as follows:

$$\frac{df_2(t)}{(f_2(t)-n_0)^2 + \frac{-\Delta}{\eta_2^2\left(\frac{1}{p-1}\rho_1^2-\rho_2^2\right)^2}} = \frac{\eta_2}{2}\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right) dt.$$

After doing some tedious calculations and noticing the boundary condition that $f_2(T) = 0$, we obtain

$$f_2(t) = \frac{\sqrt{-\Delta}}{\eta_2\left(\frac{1}{p-1}\rho_1^2 - \rho_2^2\right)} \tan\left(\arctan\left(\frac{b + \lambda\eta_2\rho_1\frac{p}{p-1}}{\sqrt{-\Delta}}\right) - \frac{\sqrt{-\Delta}}{2}(T-t)\right) + n_0.$$

Finally, it follows from direct differentiation that $f_1(t)$ is given by (8.3.9). This completes the proof. $\square$

8.C Proof of Proposition 8.3.9

Proof. Consider the reciprocal process of Y_t and denote by $P_t = \frac{1}{Y_t}$. Applying Itô's formula to P_t, we obtain the following linear BSDE of $(P_t, Q_{1,t}, Q_{2,t})$:

$$\begin{cases} dP_t = \left[\left(rp - \frac{p}{2(p-1)}\theta_t^2\right)P_t + \left(\frac{p}{p-1}\theta_t - \frac{p-2}{2(p-1)}\frac{Z_{1,t}}{Y_t}\right)Q_{1,t} - \frac{Z_{2,t}}{Y_t}Q_{2,t}\right] dt \\ \qquad + Q_{1,t}\, dW_{1,t} + Q_{2,t}\, dW_{2,t}, \\ P_T = 1, \end{cases} \tag{8.C.1}$$

where $Q_{1,t} = -\frac{Z_{1,t}}{Y_t}$ and $Q_{2,t} = -\frac{Z_{2,t}}{Y_t}$. Under Assumption 8.3.6, it is straightforward to check that Novikov's condition holds for the following Radon-Nikodym derivative

$$\begin{aligned} \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} &= \exp\Bigg\{-\frac{1}{2}\int_0^T\left(\frac{p}{p-1}\theta_t - \frac{p-2}{2(p-1)}\frac{Z_{1,t}}{Y_t}\right)^2 dt - \int_0^T\Bigg(\frac{p}{p-1}\theta_t \\ &\quad - \frac{p-2}{2(p-1)}\frac{Z_{1,t}}{Y_t}\Bigg) dW_{1,t} - \frac{1}{2}\int_0^T \frac{Z_{2,t}^2}{Y_t^2}\, dt + \int_0^T \frac{Z_{2,t}}{Y_t}\, dW_{2,t}\Bigg\} \\ &= \exp\Bigg\{-\frac{1}{2}\int_0^T\left[\left(\frac{p}{p-1} - \frac{p-2}{2(p-1)}\frac{\rho_1}{\lambda}f_2(t)\right)^2 + \frac{\rho_2^2}{\lambda^2}f_2^2(t)\right]\theta_t^2\, dt - \int_0^T\Bigg(\frac{p}{p-1} \\ &\quad - \frac{p-2}{2(p-1)}\frac{\rho_1}{\lambda}f_2(t)\Bigg)\theta_t\, dW_{1,t} + \int_0^T \frac{\rho_2}{\lambda}f_2(t)\theta_t\, dW_{2,t}\Bigg\} := M_{1,T}, \end{aligned}$$

so that $\hat{\mathbb{P}}$ measure is well-defined on $\mathcal{F}_T$, where the second equality follows from the expressions of $(Y_t, Z_{1,t}, Z_{2,t})$ given in Proposition 8.3.2. By Girsanov's theorem, the two processes given by

$$\hat{W}_{1,t} = W_{1,t} + \int_0^t \left(\frac{p}{p-1}\theta_s - \frac{p-2}{2(p-1)} \frac{Z_{1,s}}{Y_s} \right) ds$$

and

$$\hat{W}_{2,t} = W_{2,t} - \int_0^t \frac{Z_{2,s}}{Y_s} \, ds$$

are standard $(\mathbb{F}, \hat{\mathbb{P}})$-Brownian motions. Therefore, linear terms within the generator of BSDE (8.C.1) can be removed, and we have

$$\begin{cases} dP_t = \left(rp - \dfrac{p}{2(p-1)}\theta_t^2 \right) P_t \, dt + Q_{1,t} \, d\hat{W}_{1,t} + Q_{2,t} \, d\hat{W}_{2,t}, \\ P_T = 1. \end{cases} \tag{8.C.2}$$

Define

$$\hat{P}_t = P_t \exp\left\{ \int_0^t \left(-rp + \frac{p}{2(p-1)}\theta_s^2 \right) ds \right\}$$

and

$$\hat{Q}_{i,t} = Q_{i,t} \exp\left\{ \int_0^t \left(-rp + \frac{p}{2(p-1)}\theta_s^2 \right) ds \right\}, \text{ for } i = 1, 2.$$

It follows from (8.C.2) that

$$\begin{cases} d\hat{P}_t = \hat{Q}_{1,t} \, d\hat{W}_{1,t} + \hat{Q}_{2,t} \, d\hat{W}_{2,t}, \\ \hat{P}_T = \exp\left\{ \displaystyle\int_0^T \left(-rp + \frac{p}{2(p-1)}\theta_t^2 \right) dt \right\}. \end{cases} \tag{8.C.3}$$

BSDE (8.C.3) is clearly a standard linear BSDE with uniformly Lipschitz continuity (El Karoui, Peng, and Quenez (1997)) since the generator is zero. Moreover, by Cauchy-Schwarz inequality along with Assumption 8.3.6, we see that the boundary value $\hat{P}_T$ is square-integrable under measure $\hat{\mathbb{P}}$, i.e.,

$$\begin{aligned} &\hat{\mathbb{E}}\left[\exp\left\{ \int_0^T \left(-2rp + \frac{p}{p-1}\theta_t^2 \right) dt \right\} \right] \\ =&\mathbb{E}\left[M_{1,T} \exp\left\{ \int_0^T \left(-2rp + \frac{p}{p-1}\theta_t^2 \right) dt \right\} \right] \\ \leq& c \Bigg\{ \mathbb{E}\Bigg[\exp\Bigg\{ -2\int_0^T \left[\left(\frac{p}{p-1} - \frac{p-2}{2(p-1)} \frac{\rho_1}{\lambda} f_2(t) \right)^2 + \frac{\rho_2^2}{\lambda^2} f_2^2(t) \right] \theta_t^2 \, dt - 2\int_0^T \Bigg(\frac{p}{p-1} \\ &- \frac{p-2}{2(p-1)} \frac{\rho_1}{\lambda} f_2(t) \Bigg) \theta_t \, dW_{1,t} + 2\int_0^T \frac{\rho_2}{\lambda} f_2(t)\theta_t \, dW_{2,t} \Bigg\} \Bigg] \Bigg\}^{1/2} \\ &\times \left\{ \mathbb{E}\left[\exp\left\{ \int_0^T \left[\left(\frac{p}{p-1} - \frac{p-2}{2(p-1)} \frac{\rho_1}{\lambda} f_2(t) \right)^2 + \frac{\rho_2^2}{\lambda^2} f_2^2(t) + \frac{2p}{p-1} \right] \theta_t^2 \, dt \right\} \right] \right\}^{1/2} < \infty, \end{aligned}$$

where c is a positive constant. Therefore, it follows from Proposition 2.2 in El Karoui, Peng, and Quenez (1997) that

$$\hat{P}_t = \hat{\mathbb{E}}\left[\exp\left\{ \int_0^T \left(-rp + \frac{p}{2(p-1)}\theta_s^2 \right) ds \right\} \middle| \mathcal{F}_t \right],$$

and thus,

$$Y_t = \frac{1}{\hat{\mathbb{E}}\left[\exp\left\{\int_t^T \left(-rp + \frac{p}{2(p-1)}\theta_s^2\right) ds\right\} \middle| \mathcal{F}_t\right]} < e^{rp(T-t)},$$

for any $t \in [0,T]$, $\mathbb{P}$-almost surely. As a result, we find that

$$\mathbb{E}\left[\sup_{t\in[0,T]} Z_{i,t}^2\right] \le K_i \mathbb{E}\left[\sup_{t\in[0,T]} (\eta_1 + \eta_2 V_t)\right] \le K_i C_T^{\frac{1}{2}} < \infty,$$

where $K_i = f_2^2(0)e^{2|rp|T}\rho_i^2$, for $i = 1, 2$. This completes the proof. □

8.D Proof of Theorem 8.3.10

Proof. Applying Itô's formula to $\log(Y_t)$ shows that $(\log(Y_t), Z_{1,t}/Y_t, Z_{2,t}/Y_t)$ solves the following quadratic BSDE:

$$\begin{cases} d\log(Y_t) = \left[\left(\frac{p}{2(p-1)}\theta_t^2 - rp\right) + \frac{1}{2(p-1)}\left(\frac{Z_{1,t}}{Y_t}\right)^2 - \frac{1}{2}\left(\frac{Z_{2,t}}{Y_t}\right)^2 + \frac{p}{p-1}\theta_t \frac{Z_{1,t}}{Y_t}\right] dt \\ \qquad\qquad + \frac{Z_{1,t}}{Y_t}\, dW_{1,t} + \frac{Z_{2,t}}{Y_t}\, dW_{2,t}, \\ \log(Y_T) = 0. \end{cases} \tag{8.D.1}$$

By Assumption 8.3.6 and Lemma 8.3.5, it is clear that Novikov's condition holds for the following Radon-Nikodym derivative:

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\int_0^T \frac{p}{p-1}\theta_t\, dW_{1,t} - \frac{1}{2}\int_0^T \frac{p^2}{(p-1)^2}\theta_t^2\, dt\right\} := M_{2,T}.$$

Thus, $\bar{\mathbb{P}}$ measure is well-defined and equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. From Girsanov's theorem, BSDE (8.D.1) can be rewritten under $\bar{\mathbb{P}}$ measure:

$$\begin{cases} d\log(Y_t) = \left[\left(\frac{p}{2(p-1)}\theta_t^2 - rp\right) + \frac{1}{2(p-1)}\left(\frac{Z_{1,t}}{Y_t}\right)^2 - \frac{1}{2}\left(\frac{Z_{2,t}}{Y_t}\right)^2\right] dt \\ \qquad\qquad + \frac{Z_{1,t}}{Y_t}\, d\bar{W}_{1,t} + \frac{Z_{2,t}}{Y_t}\, d\bar{W}_{2,t}, \\ \log(Y_T) = 0. \end{cases} \tag{8.D.2}$$

where $\bar{W}_{1,t} = W_{1,t} + \int_0^t \frac{p}{p-1}\theta_s\, ds$ and $\bar{W}_{2,t} = W_{2,t}$ are two standard $(\mathbb{F}, \bar{\mathbb{P}})$-Brownian motions. Suppose that $(\bar{Y}_t, \bar{Z}_{1,t}, \bar{Z}_{2,t})$ is another solution triplet to BSRE (8.3.2), which might be different from $(Y_t, Z_{1,t}, Z_{2,t})$ given in Proposition 8.3.2. Define the difference between $(\log(Y_t), Z_{1,t}/Y_t, Z_{2,t}/Y_t)$ and $\left(\log(\bar{Y}_t), \bar{Z}_{1,t}/\bar{Y}_t, \bar{Z}_{2,t}/\bar{Y}_t\right)$ by

$$(\Delta\log(Y_t), \Delta Z_{1,t}, \Delta Z_{2,t}) := \left(\log(Y_t) - \log(\bar{Y}_t), \frac{Z_{1,t}}{Y_t} - \frac{\bar{Z}_{1,t}}{\bar{Y}_t}, \frac{Z_{2,t}}{Y_t} - \frac{\bar{Z}_{2,t}}{\bar{Y}_t}\right).$$

Then it follows from (8.D.2) that $(\Delta\log(Y_t), \Delta Z_{1,t}, \Delta Z_{2,t})$ solves the following BSDE under $\bar{\mathbb{P}}$ measure:

$$\begin{cases} d\Delta\log(Y_t) = \left[\frac{1}{2(p-1)}\left(\frac{Z_{1,t}^2}{Y_t^2} - \frac{\bar{Z}_{1,t}^2}{\bar{Y}_t^2}\right) - \frac{1}{2}\left(\frac{Z_{2,t}^2}{Y_t^2} - \frac{\bar{Z}_{2,t}^2}{\bar{Y}_t^2}\right)\right] dt + \Delta Z_{1,t}\, d\bar{W}_{1,t} \\ \qquad\qquad + \Delta Z_{2,t}\, d\bar{W}_{2,t}, \\ \Delta\log(Y_T) = 0. \end{cases} \tag{8.D.3}$$

Define another probability measure $\tilde{\mathbb{P}}$ by

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}}{d\bar{\mathbb{P}}}\bigg|_{\mathcal{F}_T} = \exp\bigg\{ & -\frac{1}{2}\int_0^T \frac{1}{(p-1)^2}\frac{Z_{1,t}^2}{Y_t^2}\, dt - \int_0^T \frac{1}{p-1}\frac{Z_{1,t}}{Y_t}\, d\bar{W}_{1,t} \\ & -\frac{1}{2}\int_0^T \frac{Z_{2,t}^2}{Y_t^2}\, dt + \int_0^T \frac{Z_{2,t}}{Y_t}\, d\bar{W}_{2,t}\bigg\}. \end{aligned}$$

Indeed, by Assumption 8.3.6, Lemma 8.3.5, and Cauchy-Schwarz inequality, we see that the following Novikov's condition holds

$$\begin{aligned} & \bar{\mathbb{E}}\left[\exp\left\{\frac{1}{2}\int_0^T \left[\frac{1}{(p-1)^2}\frac{Z_{1,t}^2}{Y_t^2} + \frac{Z_{2,t}^2}{Y_t^2}\right] dt\right\}\right] \\ = & \mathbb{E}\left[M_{2,T}\exp\left\{\frac{1}{2}\int_0^T \left[\frac{1}{(p-1)^2}\frac{Z_{1,t}^2}{Y_t^2} + \frac{Z_{2,t}^2}{Y_t^2}\right] dt\right\}\right] \\ \le & \left\{\mathbb{E}\left[\exp\left\{-2\int_0^T \frac{p}{p-1}\theta_t\, dW_{1,t} - 2\int_0^T \frac{p^2}{(p-1)^2}\theta_t^2\, dt\right\}\right]\right\}^{1/2} \\ & \times\left\{\mathbb{E}\left[\exp\left\{\int_0^T \left[\frac{p^2}{(p-1)^2} + \frac{f_2^2(t)}{\lambda^2}\left(\frac{\rho_1^2}{(p-1)^2} + \rho_2^2\right)\right]\theta_t^2\, dt\right\}\right]\right\}^{1/2} < \infty. \end{aligned}$$

Therefore, $\tilde{\mathbb{P}}$ measure is well-defined and equivalent to $\bar{\mathbb{P}}$ on $\mathcal{F}_T$, and we can reformulate BSDE (8.D.3) as follows:

$$\begin{cases} d\Delta\log(Y_t) = -\left[\frac{1}{2(p-1)}\Delta Z_{1,t}^2 - \frac{1}{2}\Delta Z_{2,t}^2\right] dt + \Delta Z_{1,t}\, d\tilde{W}_{1,t} + \Delta Z_{2,t}\, d\tilde{W}_{2,t}, \\ \Delta\log(Y_T) = 0, \end{cases} \tag{8.D.4}$$

where $\tilde{W}_{1,t} = \bar{W}_{1,t} + \int_0^t \frac{1}{p-1}\frac{Z_{1,s}}{Y_s}\, ds$ and $\tilde{W}_{2,t} = \bar{W}_{2,t} - \int_0^t \frac{Z_{2,s}}{Y_s}\, ds$ are two standard $(\mathbb{F}, \tilde{\mathbb{P}})$-Brownian motions. Notice that BSDE (8.D.4) of $(\Delta\log(Y_t), \Delta Z_{1,t}, \Delta Z_{2,t})$ is a standard quadratic BSDE satisfying all the regularity conditions in Kobylanski (2000). As a result of Theorem 2.3 and 2.6 in Kobylanski (2000), quadratic BSDE (8.D.4) admits a unique solution triplet $(\Delta\log(Y_t), \Delta Z_{1,t}, \Delta Z_{2,t}) = (0, 0, 0)$. This implies

$$Y_t = \bar{Y}_t,\ \ Z_{i,t} = \bar{Z}_{i,t},\ \text{ for } i = 1, 2.$$

In other words, the solution triplet $(Y_t, Z_{1,t}, Z_{2,t})$ given in Proposition 8.3.2 is the unique solution to BSRE (8.3.2). $\square$

8.E Proof of Proposition 8.3.12

Proof. By Assumption 8.3.6, the following Radon-Nikodym derivative

$$\frac{d\check{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = \exp\left\{-\frac{1}{2}\int_0^T \left(1+\frac{\rho_2^2 f_2^2(t)}{\lambda^2}\right)\theta_t^2\,dt - \int_0^T \theta_t\,dW_{1,t} + \int_0^T \frac{\rho_2}{\lambda} f_2(t)\theta_t\,dW_{2,t}\right\}$$

is well-defined such that the equivalent probability measure $\check{\mathbb{P}}$ is well-defined on $\mathcal{F}_T$. Consequently, the following two processes defined by

$$\check{W}_{1,t} = \int_0^t \theta_s\,ds + W_{1,t}$$

and

$$\check{W}_{2,t} = -\int_0^t \frac{\rho_2}{\lambda} f_2(s)\theta_s\,ds + W_{2,t}$$

are two standard $(\mathbb{F}, \check{\mathbb{P}})$-Brownian motions according to Girsanov's theorem. Hence, linear BSDE (8.3.11) can be reformulated under $\check{\mathbb{P}}$ as follows:

$$\begin{cases} dG_{1,t} = rG_{1,t}\,dt + \Lambda_{1,t}\,d\check{W}_{1,t} + \Lambda_{2,t}\,d\check{W}_{2,t}, \\ G_{1,T} = 1, \end{cases} \tag{8.E.1}$$

which is clearly a linear BSDE with standard data (refer to El Karoui, Peng, and Quenez (1997)). Hence, by Theorem 2.1 and Proposition 2.2 in El Karoui, Peng, and Quenez (1997), (8.3.12) and (8.3.13) form the unique solution to linear BSDE (8.3.11). This completes the proof. $\square$

8.F Proof of Theorem 8.4.1

Proof. It follows from (8.3.1)-(8.3.4) and results obtained in Proposition 8.3.2, 8.3.12 and 8.3.13 that

$$\begin{aligned}
& d\left(\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^p Y_t\right) \\
=& \Big[Y_t\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-1}\left(\pi_t\sigma_{S,t} - \sigma_{L,t}G_{1,t}\right) + \frac{1-p}{qp}\Big(\frac{q}{1-p}(X_t^\pi \\
& - G_{1,t}L_t) + \beta G_{2,t}\Big)^p Z_{1,t}\Big]\,dW_{1,t} + \frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^p Z_{2,t}\,dW_{2,t} \\
& - \frac{(p-1)^2}{2q}Y_t\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-2}\Bigg[\frac{q}{1-p}\Big(\pi_t\sigma_{S,t} - G_{1,t}\sigma_{L,t}\Big) \\
& + \frac{\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}}{p-1}\left(\frac{Z_{1,t}}{Y_t} + \theta_t\right)\Bigg]^2\,dt.
\end{aligned} \tag{8.F.1}$$

Due to the pathwise continuity of $Y_t, X_t^\pi, \pi_t\sigma_{S,t}, L_t, \sigma_{L,t}, G_{1,t}, G_{2,t}, \theta_t, \Lambda_{1,t}, \Lambda_{2,t}, \Gamma_{1,t}$, and $\Gamma_{2,t}$, two stochastic integrals on the right-hand side of (8.F.1) are $(\mathbb{F},\mathbb{P})$-local martingales, and thus, there exists a sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$, $\mathbb{P}$ almost surely as $n\to\infty$, and the aforementioned local martingales are indeed $(\mathbb{F},\mathbb{P})$-martingales when stopped by $\{\tau_n\}_{n\in\mathbb{N}}$. Integrating both sides of (8.F.1) from 0 to $T\wedge\tau_n$ and taking expectations, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\frac{1-p}{qp}\left(\frac{q}{1-p}(X^\pi_{T\wedge\tau_n} - G_{1,T\wedge\tau_n}L_{T\wedge\tau_n}) + \beta G_{2,T\wedge\tau_n}\right)^p Y_{T\wedge\tau_n}\right]\\
&= -\mathbb{E}\left[\int_0^{T\wedge\tau_n} \frac{(p-1)^2}{2q} Y_t\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-2}\right.\\
&\quad\left.\times\left[\frac{q}{1-p}\Big(\pi_t\sigma_{S,t} - G_{1,t}\sigma_{L,t}\Big) + \frac{\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t)+\beta G_{2,t}}{p-1}\left(\frac{Z_{1,t}}{Y_t}+\theta_t\right)\right]^2 dt\right]\\
&\quad + \frac{1-p}{qp}Y_0\left(\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0}\right)^p.
\end{aligned} \tag{8.F.2}$$

Observe from Definition 8.2.6 that for any $\pi\in\mathcal{A}$, the term in the expectation on the left-hand side of (8.F.2) is uniformly integrable and the term in the expectation on the right-hand side of (8.F.2) is non-negative and increasing with respect to n. Then applying the equivalence between uniform integrability and $\mathcal{L}^1$ convergence to the left-hand side of (8.F.2) and the monotone convergence theorem to the right-hand side of (8.F.2), we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{1-p}{qp}\left(\frac{q}{1-p}(X^\pi_T - L_T) + \beta\right)^p\right]\\
&= -\mathbb{E}\left[\int_0^{T} \frac{(p-1)^2}{2q} Y_t\left(\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t) + \beta G_{2,t}\right)^{p-2}\right.\\
&\quad\left.\times\left[\frac{q}{1-p}\Big(\pi_t\sigma_{S,t} - G_{1,t}\sigma_{L,t}\Big) + \frac{\frac{q}{1-p}(X_t^\pi - G_{1,t}L_t)+\beta G_{2,t}}{p-1}\left(\frac{Z_{1,t}}{Y_t}+\theta_t\right)\right]^2 dt\right]\\
&\quad + \frac{1-p}{qp}Y_0\left(\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0}\right)^p,
\end{aligned}$$

which reads the optimal investment strategy π_t^* given by

$$\pi_t^* = \frac{\left(\frac{1}{1-p}(X_t^* - G_{1,t}L_t) + \frac{\beta}{q}G_{2,t}\right)\left(\frac{Z_{1,t}}{Y_t}+\theta_t\right) + \sigma_{L,t}G_{1,t}}{\sigma_{S,t}},$$

where X_t^* is the asset process associated with π_t^*, and the optimal value function is given by (8.4.2).

We next verify the admissibility of optimal strategy π_t^* given in (8.4.1). To this end, we insert π_t^* into (8.F.1) and obtain

$$\frac{d\left(\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^*-G_{1,t}L_t)+\beta G_{2,t}\right)^p Y_t\right)}{\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^*-G_{1,t}L_t)+\beta G_{2,t}\right)^p Y_t} = \left(\frac{1}{1-p}\frac{Z_{1,t}}{Y_t}+\frac{p}{1-p}\theta_t\right)dW_{1,t}+\frac{Z_{2,t}}{Y_t}dW_{2,t}$$
$$=\frac{f_2(t)\rho_1+\lambda p}{\lambda(1-p)}\theta_t\,dW_{1,t}+\frac{\rho_2}{\lambda}f_2(t)\theta_t\,dW_{2,t}. \tag{8.F.3}$$

Under Assumption 8.3.6, the following Novikov's condition holds

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left(\frac{(f_2(t)\rho_1+\lambda p)^2}{\lambda^2(1-p)^2}+\frac{\rho_2^2f_2^2(t)}{\lambda^2}\right)\theta_t^2\,dt\right\}\right]<\infty$$

for the process $\frac{1-p}{qp}\left(\frac{q}{1-p}(X_t^*-G_{1,t}L_t)+\beta G_{2,t}\right)^p Y_t$, so that it is an $(\mathbb{F},\mathbb{P})$-uniformly integrable martingale. Therefore, for any sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n\uparrow\infty$ $\mathbb{P}$ almost surely as $n\to\infty$, by Doob's optional sampling theorem (see, for example, Theorem 3.22 in Le Gall (2016)), we have

$$\frac{1-p}{qp}\left(\frac{q}{1-p}(X^*_{T\wedge\tau_n}-G_{1,T\wedge\tau_n}L_{T\wedge\tau_n})+\beta G_{2,T\wedge\tau_n}\right)^p Y_{T\wedge\tau_n}$$
$$=\mathbb{E}\left[\frac{1-p}{qp}\left(\frac{q}{1-p}(X_T^\pi-L_T)+\beta\right)^p\middle|\mathcal{F}_{T\wedge\tau_n}\right]$$

Note that $\{\mathcal{F}_{T\wedge\tau_n}\}_{n\in\mathbb{N}}$ is a family of sub-σ-algebra of $\mathcal{F}_T$, i.e., $\mathcal{F}_{T\wedge\tau_n}\subseteq\mathcal{F}_T$ for $n\in\mathbb{N}$. Then it follows from Theorem 4.6.1 in Durrett (2019) that the family

$$\left\{\frac{1-p}{qp}\left(\frac{q}{1-p}(X^*_{T\wedge\tau_n}-G_{1,T\wedge\tau_n}L_{T\wedge\tau_n})+\beta G_{2,T\wedge\tau_n}\right)^p Y_{T\wedge\tau_n}\right\}_{n\in\mathbb{N}}$$

is uniformly integrable. This confirms (iii) in Definition 8.2.6. Moreover, by combining the asset equation (8.2.8) with π_t replaced by π_t^* and linear BSDEs (8.3.3)-(8.3.4), we have

$$\frac{d\left(\frac{q}{1-p}(X_t^*-G_{1,t}L_t)+\beta G_{2,t}\right)}{\frac{q}{1-p}(X_t^*-G_{1,t}L_t)+\beta G_{2,t}}=\left(r+\frac{1}{1-p}\left(1+\frac{f_2(t)\rho_1}{\lambda}\right)\theta_t^2\right)dt$$
$$+\frac{1}{1-p}\left(1+\frac{f_2(t)\rho_1}{\lambda}\right)\theta_t\,dW_{1,t}. \tag{8.F.4}$$

Solving linear SDE (8.F.4) explicitly, we find that

$$\frac{q}{1-p}(X_t^*-G_{1,t}L_t)+\beta G_{2,t}$$
$$=\exp\left\{\int_0^t\left[r+\left(\frac{1}{1-p}\left(1+\frac{f_2(s)\rho_1}{\lambda}\right)-\frac{1}{2(1-p)^2}\left(1+\frac{f_2(s)\rho_1}{\lambda}\right)^2\right)\theta_s^2\right]ds+\right.$$
$$\left.\int_0^t\frac{1}{1-p}\left(1+\frac{f_2(s)\rho_1}{\lambda}\right)\theta_s\,dW_{1,s}\right\}\left(\frac{q}{1-p}(x_0-G_{1,0}l_0)+\beta G_{2,0}\right)>0,\ \mathbb{P}-a.s. \tag{8.F.5}$$

whenever the initial value of (x_0, l_0) satisfies $\frac{q}{1-p}(x_0 - G_{1,0}l_0) + \beta G_{2,0} > 0$. This result confirms condition (ii) in Definition 8.2.6. Finally, from (8.4.1), we know that π_t^* is $\mathbb{F}$-adapted and $\pi_t^* \sigma_{S,t} \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$, namely, condition (i) in Definition 8.2.6 is verified. This ends the proof. □

Chapter 9

Robust optimal asset-liability management under square-root factor processes and model ambiguity: a BSDE approach

Abstract

This paper studies robust optimal asset-liability management problems for an ambiguity-averse manager in a possibly non-Markovian environment with stochastic investment opportunities. The manager has access to one risk-free asset and one risky asset in a financial market. The market price of risk relies on a stochastic factor process satisfying an affine-form, square-root, Markovian model, whereas the risky asset's return rate and volatility are potentially given by general non-Markovian, unbounded stochastic processes. This financial framework includes, but is not limited to, the constant elasticity of variance (CEV) model, the family of 4/2 stochastic volatility models, and some path-dependent non-Markovian models, as exceptional cases. As opposed to most of the papers using the Hamilton-Jacobi-Bellman-Issacs (HJBI) equation to deal with model ambiguity in the Markovian cases, we address the non-Markovian case by proposing a backward stochastic differential equation (BSDE) approach. By solving the associated BSDEs explicitly, we derive, in closed form, the robust optimal controls and robust optimal value functions for power and exponential utility, respectively. In addition, analytical solutions to some particular cases of our model are provided. Finally, the effects of model ambiguity and market parameters on the robust optimal investment strategies are illustrated under the CEV model and 4/2 model with numerical examples.

Keywords: Ambiguity aversion; Asset-liability management; Non-Markovian

model; Square-root factor process; Backward stochastic differential equation

9.1 Introduction

Asset-liability management (ALM) is one of the important concerns not only for financial institutions, such as pension funds, banks, and insurance companies but also for individual investors who coordinate the existing and future assets and liabilities to earn an adequate return. Based on Markowitz (1952)'s mean-variance criterion, Sharpe and Tint (1990) first investigated the ALM problem in a single-period setting, and Leippold, Trojani, and Vanini (2004) extended the results to a multi-period setting. By applying the linear-quadratic control theory, Chiu and Li (2006) and Xie, Li, and Wang (2008) considered the continuous-time mean-variance ALM problems with uncontrollable liabilities described by a geometric Brownian motion and a drifted Brownian motion, respectively. Chen, Yang, and Yin (2008) and Chen and Yang (2011) further extended the results of Chiu and Li (2006) and Leippold, Trojani, and Vanini (2004) to the case with Markovian regime-switching markets. Chiu and Wong (2014a) studied an ALM problem with asset correlation driven by a multivariate Wishart process. Under the framework of expected utility maximization, Liang and Ma (2015) considered the ALM problems under power and exponential utility with mortality and salary risks, and the optimal approximation investment strategies were derived. Pan and Xiao (2017a,b) studied an ALM problem with inflation risks and liquidity constraints, respectively. For other relevant works on ALM problems, readers may refer to Zeng and Li (2011), Chang (2015), Pan and Xiao (2017c), Peng and Chen (2021), and references therein.

The motivation for this paper is three-fold. First, most of the above-mentioned literature on the ALM problems assumes that the volatility of risky asset's price is a constant or deterministic function, which violates the well-documented evidence to support the existence of stochastic (local) volatility, mainly referred to French, Schwert, and Stambaugh (1987), Heston (1993), Cox (1996), Lewis (2000), and Grasselli Grasselli (2017). In the last decade, some papers have emerged that investigated the optimal ALM problems with various stochastic investment opportunities. For example, Zhang and Chen (2016) studied a mean-variance ALM problem under the constant elasticity of variance (CEV) model with multiple risky assets. Li, Shen, and Zeng (2018) considered the derivative-based optimal investment strategy for a mean-variance ALM problem under the Heston model. Zhang (2023) stepped further by incorporating the Cox-Ingersoll-Ross (CIR) interest rate and the family of 4/2 model (Grasselli (2017)) into an ALM problem with derivative trading. Sun, Zhang, and Yuen (2020) studied a mean-variance ALM problem with a reinsurance option in a complete market under an affine diffusion equation. Besides the mean-variance criterion, Pan, Hu, and Zhou (2019) considered an ALM problem for the exponential utility function under the Heston model. Zhang (2022d) investigated an ALM problem in an incomplete market setting with an affine diffusion factor process for the hyperbolic absolute risk aversion utility function.

Second, most of the literature mentioned above on ALM problems assumes that the asset-liability manager knows exactly the true probability measure. In many situations, however, economic agents are skeptical about the true model, because, for instance, as shown by Merton (1980) and Cochrane (1997), the drift parameters are difficult to estimate with precision. In addition, experimental evidence from Ellsberg (1961) and Bossaerts et al. (2010) demonstrate that economic agents display not only risk aversion but also ambiguity aversion. In this sense, it is plausible to incorporate model ambiguity into portfolio choice problems. In the pioneering work of Andersen, Hansen, and Sargent (2003), a robust control approach was proposed to address model ambiguity in continuous-time stochastic control problems, where the agent regards a particular probability measure as a reference measure and considers a set of alternative probability measures which are close to the reference measure in terms of relative entropy. Maenhout (2004) refined the robust control approach by proposing the homothetic robustness, and Uppal and Wang (2003) extended the analysis of Maenhout (2004) by allowing different levels of ambiguity aversion about the state variables. Maenhout (2006) considered a robust portfolio selection problem with a mean-reverting expected stock return. Flor and Larsen (2014) studied a robust investment problem in a setting with stochastic interest rates. Munk and Rubtsov (2014) extended the work of Flor and Larsen (2014) by incorporating an unobservable inflation rate. Escobar, Ferrando, and Rubtsov (2015) considered a robust investment problem with derivatives trading under the Heston model. Zeng et al. (2018) analyzed a robust derivative-based pension investment problem with stochastic income and volatility. Recently, Cheng and Escobar (2021b) investigated robust investment under the state-of-the-art 4/2 model. In the field of ALM, Yuan and Mi (2022b) considered a robust investment problem for maximizing the minimal expected utility of terminal wealth and minimizing the maximal cumulative deviation, respectively. Chen, Huang, and Li (2022) studied a robust ALM problem in a regime-switching market. As the literature on robust investment problems is abundant, the above review is not exhaustive. Other works considering robust investment problems under various scenarios include Yi et al. (2013), Zheng, Zhou, and Sun (2016), Wang and Li (2018), Wang, Li, and Sun (2021), Chang, Li, and Zhao (2022), Baltas et al. (2022), Wei, Yang, and Zhuang (2023), to name but only a few.

Third, although robust investment problems have been extensively studied over the last decade, one common feature shared by most of the existing works is that the exogenous parameter processes are assumed to be only constants or Markovian diffusion processes. In the Markovian case, such problems can be studied by using the dynamic programming principle and solving the corresponding Hamilton-Jacobi-

Bellman-Issacs (HJBI) equations (see, for example, Mataramvura and Øksendal (2008)). These methods, however, cannot be applied directly to the non-Markovian setting because the dynamic programming principle no longer works. To handle the non-Markovian case, Øksendal and Sulem (2011) studied an optimal investment problem under model ambiguity by proposing a backward stochastic differential equation (BSDE) approach, where the performance functional (value function) is written as the solution of an associated controlled BSDE and the comparison theorem for BSDEs plays a key role. But this approach is strongly linked to the exponential utility function. Øksendal and Sulem (2014) extended the analysis of Øksendal and Sulem (2011) for general utility functions by developing a forward-backward stochastic differential equation (FBSDE) approach. Following the methodology of Øksendal and Sulem (2014), Peng, Chen, and Hu (2014) considered an optimal investment-consumption and reinsurance problem under model ambiguity.

In this paper, we investigate a robust optimal ALM problem under model ambiguity in the presence of stochastic volatility. The risk- and ambiguity-averse manager has access to a financial market consisting of one risk-free asset (money account) and one risky asset (stock) and is subject to an uncontrollable random liability. Unlike most of the preceding literature on robust decision problems, it is not a prerequisite to assume that the risky asset's return rate and volatility are specifically Markovian processes as they may depend on past values. Inspired by Shen and Zeng (2015) and Zhang (2022c), we only suppose that the market price of risk relies on an affine-form, square-root, Markovian process, which includes, but is not limited to, the Black-Scholes model, CEV model, Heston model, 3/2 model, 4/2 model, and some non-Markovian models, as exceptional cases (see Example 9.2.1-9.2.4). In the spirit of Maenhout (2004) and Uppal and Wang (2003), the manager is assumed to have different levels of ambiguity aversion about the risky's asset price and volatility and aims to maximize the terminal surplus under the worst-case scenario for power and exponential utility, respectively. Given the potentially non-Markovian setting, the HJBI equation approach does not work, and a BSDE approach is disentangled. Different from Øksendal and Sulem (2011, 2014), where the value function is written as the value at time zero of the solution to a controlled FBSDE and the comparison theorem for solutions to BSDEs is applied, we propose to construct a stochastic process hinging upon any admissible control, and such that its value at time zero does not depend on any admissible control and its terminal value equals the utility of the terminal surplus penalized by model ambiguity. The proposed stochastic process is shown to be either sub-martingale or super-martingale for any admissible control, and even martingale for a particular control under the reference measure, which then leads to the associated uncontrolled BSDEs. By solving the BSDEs explicitly, we derive the analytical expressions for the robust optimal controls and robust optimal value functions for the above two utility maximization problems. Furthermore, several special cases of our model are discussed and the corresponding results are provided in closed form. Finally, the economic effects of model ambiguity and model parameters on the behavior of robust optimal investment strategies are analyzed by giving numerical examples. To sum up, we think that this paper has three main contributions:

1. In the literature on the ALM problems, model ambiguity and stochastic volatility are simultaneously taken into consideration in a potentially non-Markovian modeling framework for the very first time, whereas in Yuan and Mi (2022b) and Chen, Huang, and Li (2022), only Markovian cases were investigated and stochastic volatility was not taken into account; Zhang and Chen (2016), Li, Shen, and Zeng (2018), Pan, Hu, and Zhou (2019), Sun, Zhang, and Yuen (2020), and Zhang (2023, 2022d) considered the presence of stochastic volatility but not model uncertainty.

2. At the mathematical level, compared with the literature on the robust investment problems considering the Markovian models and using the HJBI equation approach, such as Yi et al. (2013), Flor and Larsen (2014), Escobar, Ferrando, and Rubtsov (2015), Zheng, Zhou, and Sun (2016), Wang and Li (2018), Wang, Li, and Sun (2021), Cheng and Escobar (2021b), Chang, Li, and Zhao (2022), Baltas et al. (2022), and Wei, Yang, and Zhuang (2023), a novel BSDE approach, which has distinct differences with the FBSDE approach proposed in Øksendal and Sulem (2011, 2014), is disentangled to deal with the non-Markovian setting.

3. A general class of stochastic volatility models is considered for modeling the risky asset's price and volatility, embracing the CEV model, Heston model, 3/2 model, 4/2 model, and some path-dependent models, as particular cases. Furthermore, closed-form expressions for the robust optimal controls and robust optimal value functions are derived for the power and exponential utility functions, and explicit solutions to some special cases of our model are recovered, such as Gao (2009), Zheng, Zhou, and Sun (2016), Sun, Yong, and Gao (2020), Cheng and Escobar (2021a,b), and Zhang (2022c).

The remainder of this paper is organized as follows. In Section 9.2, we formulate the model and establish the robust optimal ALM problems for the power and exponential utility functions. Section 9.3 and 9.4 derive the robust optimal solutions to the power and exponential utility cases, respectively. Section 9.5 discusses the effects of model ambiguity and model parameters on the robust optimal investment strategies with numerical analysis. Section 9.6 concludes our work. All proofs are given in the Appendix.

9.2 General formulation

In this paper, we consider the optimal ALM problems for an asset-liability manager with ambiguity aversion under the expected utility maximization framework. We assume that assets can be traded continuously, infinite short-selling and leverage are allowed, and no transaction costs or taxes are involved. Let $T > 0$ be a fixed constant describing the decision-making horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions on which are defined two one-dimensional, mutually independent Brownian motions $\{W_{1,t}\}_{t\in[0,T]}$ and $\{W_{2,t}\}_{t\in[0,T]}$. The filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be generated by the two Brownian motions, $\mathbb{P}$ stands for a real-world probability measure, and $\mathbb{E}^{\mathbb{P}}[\cdot]$ denotes the expectation associated with measure $\mathbb{P}$. In what follows, we introduce several spaces on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

- $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{P}\left(\int_0^T |f_t|^2\, dt < \infty\right) = 1$;
- $\mathcal{L}^{2}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |f_t|^2\, dt\right] < \infty$;
- $\mathcal{S}^{2p}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} |f_t|^{2p}\right] < \infty$, $p = 1, 2$;
- $\mathcal{S}^{\infty}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted uniformly bounded processes with $\mathbb{P}$-a.s. continuous sample paths.

9.2.1 Financial market and random liability

Assume that the financial market consists of one risk-free asset (money account) and one risky asset (stock). The price process $\{B_t\}_{t\in[0,T]}$ of the risk-free asset evolves according to

$$dB_t = rB_t\, dt,\ B_0 = 1,$$

where the constant $r \in \mathbb{R} \setminus \{0\} = \mathbb{R}_0$ is the risk-free interest rate. The price process $\{S_t\}_{t\in[0,T]}$ of the risky asset is described by the following stochastic differential equation (SDE):

$$dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dW_{1,t},\ S_0 = s_0 \in \mathbb{R}^+, \tag{9.2.1}$$

where μ_t and $\sigma_t \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+)$ are two potentially unbounded and non-Markovian $\mathbb{F}$-adapted stochastic processes describing the risky asset's return rate and volatility at time t, respectively. Assume that the market price of risk is related to an affine form, square-root factor process $\{\alpha_t\}_{t\in[0,T]}$ as follows:

$$\frac{\mu_t - r}{\sigma_t} = \lambda\sqrt{\alpha_t},\ \lambda \in \mathbb{R}_0, \tag{9.2.2}$$

where the dynamics of α_t are given by

$$d\alpha_t = \kappa(\theta - \alpha_t)\,dt + \sqrt{\alpha_t}\,(\rho_1\,dW_{1,t} + \rho_2\,dW_{2,t})\,,\ \alpha_0 \in \mathbb{R}^+ \tag{9.2.3}$$

with the speed of mean reversion κ, long-run level θ, and volatility $\sqrt{\rho_1^2+\rho_2^2}$. In line with Chapter 6.3 in Jeanblanc, Chesney, and Yor (2009), we assume that the constants $\kappa, \theta \in \mathbb{R}$ satisfy $\kappa\theta \in \mathbb{R}^+$ to ensure the process $\alpha_t \geq 0$ for all $t \in [0,T]$, $\mathbb{P}$ almost surely, while no specific conditions are imposed on the constants $\rho_1, \rho_2 \in \mathbb{R}$. Notice that we do not impose the Feller condition for strict positivity of α_t, i.e., $2\kappa\theta \geq \rho_1^2 + \rho_2^2$ in our case.

The above financial modeling framework (9.2.1)-(9.2.3) was studied in Shen (2015) and Zhang (2022c) in the context of solving a mean-variance investment-reinsurance problem and a defined contribution pension investment problem with stochastic income and stochastic inflation, respectively. It is also worth mentioning that this modeling framework is generally embracing not only a wide class of stochastic (local) volatility models, such as the CEV model, Heston model, 3/2 model, and 4/2 model (see Examples 9.2.1 and 9.2.2) but also some non-Markovian models (Example 9.2.4), as exceptional cases.

Example 9.2.1 (CEV model)**.** If $\mu_t = \mu$ and $\sigma_t = \sigma S_t^{\beta}$, where $\mu \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$, and $\beta \leq -\frac{1}{2}$ such that $\mu \neq r$, then the risky asset price S_t is given by the CEV model:

$$dS_t = S_t\left(\mu\,dt + \sigma S_t^{\beta}\,dW_{1,t}\right),\ S_0 = s_0 \in \mathbb{R}^+, \tag{9.2.4}$$

where β is called the elasticity parameter. By setting $\alpha_t = S_t^{-2\beta}$, $\kappa = 2\beta\mu$, $\theta = (\beta + \frac{1}{2})\frac{\sigma^2}{\mu}$, $\rho_1 = -2\beta\sigma$, $\rho_2 = 0$ and $\lambda = \frac{\mu - r}{\sigma}$, we have

$$\begin{aligned} d\alpha_t &= 2\beta\mu\left[\left(\beta + \frac{1}{2}\right)\frac{\sigma^2}{\mu} - S_t^{-2\beta}\right]dt - 2\beta\sigma S_t^{-\beta}\,dW_{1,t} \\ &= \kappa(\theta - \alpha_t)\,dt + \sqrt{\alpha_t}\,(\rho_1\,dW_{1,t} + \rho_2\,dW_{2,t})\,. \end{aligned}$$

For the particular case when $\beta = 0$, the condition $\kappa\theta \geq 0$ is still met and the CEV model degenerates to the Black-Scholes model.

Example 9.2.2 (The family of 4/2 models)**.** If $\mu_t = r + \lambda(c_1\alpha_t + c_2)$, $\sigma_t = c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$, $V_t = \alpha_t$, $\kappa \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$, $\rho_1 = \sigma_v\rho$ and $\rho_2 = \sigma_v\sqrt{1-\rho^2}$, where $c_1 \geq 0, c_2 \geq 0, \sigma_v \in \mathbb{R}^+$, and $\rho \in [-1,1]$, then the risky asset price process S_t is governed by the family of 4/2 stochastic volatility models (Grasselli (2017)):

$$\begin{cases} dS_t = S_t \left[(r + \lambda(c_1 V_t + c_2))\, dt + \left(c_1\sqrt{V_t} + \dfrac{c_2}{\sqrt{V_t}} \right) dW_{1,t} \right],\ S_0 = s_0 \in \mathbb{R}^+, \\ dV_t = \kappa(\theta - V_t)\, dt + \sigma_v \sqrt{V_t} \left(\rho\, dW_{1,t} + \sqrt{1-\rho^2}\, dW_{2,t} \right),\ V_0 = v_0 = \alpha_0 \in \mathbb{R}^+, \end{cases} \tag{9.2.5}$$

where V_t is the variance driver process with mean-reversion rate κ, long-run mean θ, volatility σ_v, and correlation coefficient between the risky asset price and its variance driver ρ. For the 4/2 model (9.2.5), we impose the Feller condition, i.e., $2\kappa\theta \geq \sigma_v^2$ to keep the process V_t strictly positive for $t \in [0, T]$, $\mathbb{P}$ almost surely.

Remark 9.2.3. The 4/2 model (9.2.5) is featured by two embedded parsimonious models, the Heston model (Heston (1993)) and 3/2 model (Lewis (2000)) via the constants c_1 and c_2. Particularly, the case $(c_1, c_2) = (1, 0)$ corresponds to the Heston model, while the specification $(c_1, c_2) = (0, 1)$ is known as the 3/2 model.

Example 9.2.4 (A path-dependent stochastic volatility model)**.** If $\mu_t = r + \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]})$ and $\sigma_t = \hat{\sigma}(\alpha_{[0,t]})$ for some functional $\hat{\sigma} : \mathcal{C}([0, t]; \mathbb{R}) \mapsto \mathbb{R}^+$, where $\alpha_{[0,t]} := (\alpha_s)_{s\in[0,t]}$ is the restriction of $\alpha \in \mathcal{C}([0, T]; \mathbb{R})$ to $\mathcal{C}([0, t]; \mathbb{R})$, i.e., the space of real-valued, continuous functions defined on $[0, t]$. In this case, the risky asset price process S_t is given by the following path-dependent stochastic volatility model:

$$\begin{cases} dS_t = S_t \left[\left(r + \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]}) \right) dt + \hat{\sigma}(\alpha_{[0,t]})\, dW_{1,t} \right],\ S_0 = s_0 \in \mathbb{R}^+, \\ d\alpha_t = \kappa(\theta - \alpha_t)\, dt + \sqrt{\alpha_t} \left(\rho_1\, dW_{1,t} + \rho_2\, dW_{2,t} \right),\ \alpha_0 \in \mathbb{R}^+. \end{cases} \tag{9.2.6}$$

Due to the path-dependence of the return rate and volatility of the risky asset price, the model (9.2.6) is a special case of the non-Markovian stochastic volatility models. For more details on (9.2.6), readers may consult Siu (2012).

Consider an asset-liability manager who is subject to an uncontrollable liability commitment with an initial value l_0. Similar to Zhang and Chen (2016) and Sun, Zhang, and Yuen (2020), we assume that the liability process L_t is driven by the following SDE:

$$dL_t = L_t \left[\mu_l\, dt + \sigma_l \left(\lambda\alpha_t\, dt + \sqrt{\alpha_t}\, dW_{1,t} \right) \right],\ L_0 = l_0 \in \mathbb{R}^+, \tag{9.2.7}$$

where $\mu_l \in \mathbb{R}$ is the drift coefficient and the constant $\sigma_l \in \mathbb{R}$ is a volatility scale factor measuring how the risk source of the risky asset affects the random liability. In the following subsection, we will formulate the robust optimal ALM problems from the point of view of the manager.

9.2.2 Ambiguity and optimization problem

In the traditional framework of ALM problems, the asset-liability manager is assumed to be ambiguity-neutral and completely convinced by the above dynamics of the available risky asset price, factor process, and random liability under the real-world probability measure $\mathbb{P}$. However, the fact is that the manager may not know exactly the true model in many cases, for example, due to parameter uncertainty, and thus, any particular probability measure used to describe the model may lead to potential model misspecification. For this reason, it is desirable to take model uncertainty into account for an ambiguity-averse manager when he/she makes investment decisions. To incorporate model ambiguity, we assume that the ambiguity-averse manager's knowledge of ambiguity is characterized by the measure $\mathbb{P}$, which is referred to as the reference measure. The ambiguity-averse manager is skeptical about the reference measure $\mathbb{P}$ and only regards it as an approximation to the truly real-world measure. Therefore, he/she considers some adverse alternative measures to seek robust optimal investment strategies. In line with Andersen, Hansen, and Sargent (2003), the alternative measures are assumed to be equivalent to, i.e., mutually absolutely continuous with the reference measure $\mathbb{P}$, and we denote by $\mathcal{Q}$ such a class of alternative measures $\mathbb{Q}$, i.e., $\mathcal{Q} := \{\mathbb{Q}|\ \mathbb{Q} \sim \mathbb{P}\}$. More specifically, for each $\mathbb{Q} \in \mathcal{Q}$, there is a two-dimensional $\mathbb{F}$-adapted process $\phi = (\phi_1, \phi_2) := \left(\{\phi_{1,t}\}_{t\in[0,T]}, \{\phi_{2,t}\}_{t\in[0,T]}\right)$, which can be referred as the probability distortion process, such that the following Radon-Nikodym derivative process φ_t^{ϕ}:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \varphi_t^{\phi} = \exp\left\{\int_0^t \phi_{1,s}\, dW_{1,s} + \int_0^t \phi_{2,s}\, dW_{2,s} - \frac{1}{2}\int_0^t \left(\phi_{1,s}^2 + \phi_{2,s}^2\right)\, ds\right\} \tag{9.2.8}$$

is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale. For this, we shall only consider the distortion process ϕ satisfying the following Novikov's condition:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\phi_{1,t}^2 + \phi_{2,t}^2\right)\, dt\right\}\right] < +\infty, \tag{9.2.9}$$

and denote by Φ the space of all distortion process ϕ such that (9.2.9) holds. According to Girsanov's theorem, the dynamics of the standard Brownian motions $W_{1,t}^{\mathbb{Q}}$ and $W_{2,t}^{\mathbb{Q}}$ under the alternative measure $\mathbb{Q} \in \mathcal{Q}$ are given by

$$dW_{1,t}^{\mathbb{Q}} = dW_{1,t} - \phi_{1,t}\, dt,\ dW_{2,t}^{\mathbb{Q}} = dW_{2,t} - \phi_{2,t}\, dt.$$

Suppose that the asset-liability manager has an initial wealth $x_0 \in \mathbb{R}^+$. Denote by π_t the proportion of wealth invested in the risky asset at time t, then the process $\pi := \{\pi_t\}_{t\in[0,T]}$ represents the investment strategy. Let $X^{\pi} := \{X_t^{\pi}\}_{t\in[0,T]}$ be the wealth process associated with strategy π. Under a self-financing condition, the dynamics of X_t^{π} are then given by

$$\begin{aligned} dX_t^{\pi} =& (1-\pi_t)X_t^{\pi}\frac{dB_t}{B_t} + \pi_t X_t^{\pi}\frac{dS_t}{S_t} \\ =& \left[r + (\mu_t - r)\pi_t\right] X_t^{\pi}\, dt + \sigma_t\pi_t X_t^{\pi}\, dW_{1,t} \\ =& [r + (\mu_t - r + \sigma_t\phi_{1,t})\pi_t]X_t^{\pi}\, dt + \sigma_t\pi_t X_t^{\pi}\, dW_{1,t}^{\mathbb{Q}},\ X_0^{\pi} = x_0. \end{aligned} \tag{9.2.10}$$

In this paper, we will subsequently consider two utility maximization problems when the risk preferences of the ambiguity-averse manager are characterized by a power utility function $U(x) = x^\gamma/\gamma$ with the relative risk aversion $\gamma \in \mathbb{R}^-$ and an exponential utility function $U_2(x) = -e^{-qx}/q$ with the absolute risk aversion $q \in \mathbb{R}^+$. To this end, we give below the formal definitions of the admissible strategies for these two utility maximization problems, respectively.

Definition 9.2.5 (Admissible strategy for power utility). *A control (π, ϕ) is said to be admissible if the following conditions are satisfied:*

1. *π is $\mathbb{F}$-adapted and $\phi \in \Phi$;*
2. *for any initial data $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ such that $x_0 + \bar{G}_{1,0} l_0 \in \mathbb{R}^+$, the associated asset process (9.2.10) admits a pathwise unique solution such that $X_t^\pi + \bar{G}_{1,t} L_t > 0$, $\mathbb{P}$ almost surely, for all $t \in [0, T]$, where $\bar{G}_{1,t}$ is given by (9.3.14) below;*
3. *either the family of random variables*

$$\left\{ \varphi^{\hat{\phi}}_{\tau_n \wedge T} \left(Y_{1,\tau_n \wedge T} \frac{(X^\pi_{\tau_n \wedge T} + \bar{G}_{1,\tau_n \wedge T} L_{\tau_n \wedge T})^\gamma}{\gamma} + \int_0^{\tau_n \wedge T} \frac{\hat{\phi}^2_{1,t}}{2\psi_{1,t}} + \frac{\hat{\phi}^2_{2,t}}{2\psi_{2,t}} \, dt \right) \right\}_{n \in \mathbb{N}}$$

is uniformly integrable under $\mathbb{P}$ measure for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow +\infty$ as $n \to +\infty$, where $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2) \in \Phi$ is given in (9.3.17) with X_t^* and π_t^* replaced by X_t^π and π_t, respectively, and $\psi_{1,t}, \psi_{2,t}$, and $Y_{1,t}$ are given by (9.2.12) and (9.3.6) below, or the family of random variables

$$\left\{ \varphi^{\phi}_{\tau_n \wedge T} \left(Y_{1,\tau_n \wedge T} \frac{(X^{\hat{\pi}}_{\tau_n \wedge T} + \bar{G}_{1,\tau_n \wedge T} L_{\tau_n \wedge T})^\gamma}{\gamma} + \int_0^{\tau_n \wedge T} \frac{\phi^2_{1,t}}{2\hat{\psi}_{1,t}} + \frac{\phi^2_{2,t}}{2\hat{\psi}_{2,t}} \, dt \right) \right\}_{n \in \mathbb{N}}$$

is uniformly integrable under $\mathbb{P}$ measure for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow +\infty$ as $n \to +\infty$, where $\hat{\pi}_t = \pi_t^*$ is given in (9.3.17) and $\hat{\psi}_{1,t}$ and $\hat{\psi}_{2,t}$ are given in (9.2.12) with X_t^π replaced by $X_t^{\hat{\pi}}$.

The set of all admissible controls is denoted by $\Pi_p \otimes \Phi$.

Remark 9.2.6. The technical condition 3 in Definition 9.2.5 implies that both the controls $(\pi, \hat{\phi})$ and $(\hat{\pi}, \phi)$ are immediately admissible whenever there exists at least one control $(\pi, \phi) \in \Pi_p \otimes \Phi$. For the sake of tractability, we suppose that the set of admissible controls is not empty throughout the rest of the paper.

For the power utility case, the ambiguity-averse manager aims to seek a robust investment strategy π to maximize the expected utility from the terminal surplus $X_T^\pi - L_T$ under the worst-case alternative measure. Inspired by Maenhout (2004), the robust optimal ALM problem for the ambiguity-averse manager is formulated as

$$\sup_{\pi\in\Pi_p}\inf_{\phi\in\Phi} J_p(\pi,\phi) := \sup_{\pi\in\Pi_p}\inf_{\phi\in\Phi} \mathbb{E}^{\mathbb{Q}}\left[\frac{(X_T^\pi - L_T)^\gamma}{\gamma} + \int_0^T \left(\frac{\phi_{1,t}^2}{2\psi_{1,t}} + \frac{\phi_{2,t}^2}{2\psi_{2,t}}\right) dt\right], \tag{9.2.11}$$

where $J_p(\pi,\phi)$ denotes the value function associated with admissible control (π,ϕ), the minimization over $\phi \in \Phi$ reflects the asset-liability manager's aversion to ambiguity, and $\psi_{1,t}$ and $\psi_{2,t}$ are two $\mathbb{R}^+$-valued, $\mathbb{F}$-adapted stochastic processes capturing the level of ambiguity aversion with respect to model misspecification. The larger the levels of ambiguity $\psi_{1,t}$ and $\psi_{2,t}$ are, the more skeptical the manager is about the reference measure $\mathbb{P}$, and the smaller the penalty for a given deviation from the reference measure is. For the extreme case where $\psi_{1,t} = \psi_{2,t} = +\infty$, the integral term within (9.2.11) vanishes and the manager considers all alternative measures equally. For the other extreme case where $\psi_{1,t} = \psi_{2,t} = 0$, i.e., the manager is completely confident that the reference measure $\mathbb{P}$ is the true measure, any alternative measure deviating from the reference measure $\mathbb{P}$ will be severely penalized. In this case, $\phi_{1,t} = \phi_{2,t} = 0$ must be required such that the integral term within (9.2.11) disappears, and thus, the robust ALM problem (9.2.11) reduces to the traditional ALM problem without model ambiguity, i.e., $\sup_{\pi\in\Pi_p} \mathbb{E}^{\mathbb{P}}\left[\frac{(X_T^\pi - L_T)^\gamma}{\gamma}\right]$.

For analytical tractability, we assume $\psi_{1,t}$ and $\psi_{2,t}$ are state-dependent. Similar to the existing works, such as Maenhout (2004), Escobar, Ferrando, and Rubtsov (2015), and Wang and Li (2018), we set

$$\psi_{i,t} = \frac{\beta_i}{Y_{1,t}(X_t^\pi + G_{1,t})^\gamma}, \ i = 1, 2, \tag{9.2.12}$$

where $G_{1,t}$ and $Y_{1,t}$ are given by (9.3.5) and (9.3.6), respectively, and positive constants $\beta_i \in \mathbb{R}^+$, $i = 1, 2$, are called the ambiguity-aversion parameters. In particular, β_1 can be interpreted as the level of ambiguity about the risky asset dynamics, while β_2 represents the ambiguity aversion about the stochastic factor process.

Definition 9.2.7 (Admissible strategy for exponential utility)**.** *A control* (π,ϕ) *is said to be admissible if the following conditions are met:*

1. π *is* $\mathbb{F}$*-adapted and* $\phi \in \Phi$*;*
2. *for any initial data* $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$*, the associated asset process (9.2.10) admits a pathwise unique solution;*
3. *either the family of random variables*

$$\left\{\varphi_{\tau_n\wedge T}^{\check{\phi}}\left(-\frac{e^{-q(X_{\tau_n\wedge T}^\pi Y_{2,\tau_n\wedge T} + G_{2,\tau_n\wedge T})}}{q} + \int_0^{\tau_n\wedge T} \frac{\check{\phi}_{1,t}^2}{2\eta_{1,t}} + \frac{\check{\phi}_{2,t}^2}{2\eta_{2,t}}\, dt\right)\right\}_{n\in\mathbb{N}}$$

is uniformly integrable under $\mathbb{P}$ measure for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow +\infty$ as $n \to +\infty$, where $\check{\phi} = (\check{\phi}_1, \check{\phi}_2) \in \Phi$ is given by (9.4.17) with X_t^* and π_t^* replaced by X_t^π and π_t, respectively, and $\eta_{1,t}, \eta_{2,t}, Y_{2,t}$, and $G_{2,t}$ are given by (9.2.14), (9.4.4), and (9.4.7) below, or the family of

$$\left\{\varphi^{\phi}_{\tau_n\wedge T}\left(-\frac{e^{-q(X^{\check{\pi}}_{\tau_n\wedge T}Y_{2,\tau_n\wedge T}+G_{2,\tau_n\wedge T})}}{q}+\int_0^{\tau_n\wedge T}\frac{\phi_{1,t}^2}{2\check{\eta}_{1,t}}+\frac{\phi_{2,t}^2}{2\check{\eta}_{2,t}}\right)\right\}_{n\in\mathbb{N}}$$

is uniformly integrable under $\mathbb{P}$ measure for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow +\infty$ as $n \to +\infty$, where $\check{\pi}_t = \pi_t^*$ is given in (9.4.17) and $\check{\eta}_{1,t}$ and $\check{\eta}_{2,t}$ are given in (9.2.14) with X_t^π replaced by $X_t^{\check{\pi}}$.

Denote by $\Pi_e \otimes \Phi$ the set of all admissible controls.

Remark 9.2.8. Similar to the above power utility case, in the rest of the paper, we suppose that the set of admissible controls is not empty, i.e., there exists at least a control $(\pi, \phi) \in \Pi_e \otimes \Phi$. As a result, both controls $(\check{\pi}, \phi)$ and $(\pi, \check{\phi})$ are admissible as well based on condition 3 in Definition 9.2.7.

The robust optimal ALM problem under the exponential utility case is formally written as follows:

$$\sup_{\pi\in\Pi_e}\inf_{\phi\in\Phi} J_e(\pi,\phi) := \sup_{\pi\in\Pi_e}\inf_{\phi\in\Phi}\mathbb{E}^{\mathbb{Q}}\left[-\frac{e^{-q(X_T^\pi - L_T)}}{q}+\int_0^T\left(\frac{\phi_{1,t}^2}{2\eta_{1,t}}+\frac{\phi_{2,t}^2}{2\eta_{2,t}}\right)dt\right], \tag{9.2.13}$$

where $J_e(\pi, \phi)$ denotes the value function associated with admissible control (π, ϕ), and the two $\mathbb{R}^+$-valued, $\mathbb{F}$-adapted stochastic processes $\eta_{1,t}$ and $\eta_{2,t}$ characterize the level of ambiguity aversion with respect to model ambiguity. Again, for the sake of tractability, we assume that $\eta_{i,t},\ i = 1, 2$ are state-dependent. More specifically, we make the following assumption on $\eta_{i,t},\ i = 1, 2$:

$$\eta_{i,t} = \frac{\beta_i}{e^{-q(X_t^\pi Y_{2,t}+G_{2,t})}},\ i = 1, 2, \tag{9.2.14}$$

where $Y_{2,t}$ and $G_{2,t}$ are given by (9.4.4) and (9.4.7), respectively, and positive constants $\beta_i \in \mathbb{R}^+,\ i = 1, 2$, denote the ambiguity-aversion parameters. For an ambiguity-neutral manager, the robust optimization problem (9.2.13) degenerates to finding an admissible investment strategy such that $\sup_{\pi\in\Pi_e}\mathbb{E}^{\mathbb{P}}\left[-\frac{e^{-q(X_T^\pi - L_T)}}{q}\right]$ is attained.

Remark 9.2.9. Given the possibly non-Markovian structures of the market model, the dynamic programming approach along with the HJBI equation (refer to Mataramvura and Øksendal (2008)) is not applicable in our case. We, therefore, solve the above two utility maximization problems (9.2.11) and (9.2.13) in Section 9.3 and 9.4 by proposing a novel BSDE approach. This distinguishes our paper from the published works considering robust investment problems in the Markovian settings and using the HJBI approach; see, for example, Yi et al. (2013), Flor and Larsen (2014), Munk and Rubtsov (2014), Escobar, Ferrando, and Rubtsov (2015), Zheng, Zhou, and Sun (2016), Zeng et al. (2018), Wang and Li (2018), Wang, Li, and Sun (2021), Cheng and Escobar (2021b), Chen, Huang, and Li (2022), Chang, Li, and Zhao (2022), Baltas et al. (2022), Wei, Yang, and Zhuang (2023), and etc.

9.3 Optimal investment strategies for the power utility case

This section is dedicated to deriving the robust optimal investment strategies for the power utility maximization problem (9.2.11) by using a BSDE approach. To this end, we introduce two continuous $(\mathbb{F},\mathbb{P})$-semi-martingales $Y_{1,t}$ and $G_{1,t}$ with the following canonical decomposition:

$$dY_{1,t} = P_{1,t}\,dt + Z_{1,t}\,dW_{1,t} + Z_{2,t}\,dW_{2,t}$$

and

$$dG_{1,t} = H_{1,t}\,dt + \Lambda_{1,t}\,dW_{1,t} + \Lambda_{2,t}\,dW_{2,t},$$

where $P_{1,t}$ and $H_{1,t}$ are two undetermined $\mathbb{F}$-adapted processes, and $Z_{1,t}, Z_{2,t}, \Lambda_{1,t}$, and $\Lambda_{2,t}$ lie in $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. Applying Itô's formula to $\varphi^\phi_t\Big(Y_{1,t}\frac{(X^\pi_t+G_{1,t})^\gamma}{\gamma} + \int_0^t \frac{\phi^2_{1,s}}{2\psi_{1,s}}+\frac{\phi^2_{2,s}}{2\psi_{2,s}}\,ds\Big)$ under $\mathbb{P}$ measure and using the method of completion of squares, we have

$$\begin{aligned}
&d\varphi^\phi_t\left(Y_{1,t}\frac{(X^\pi_t+G_{1,t})^\gamma}{\gamma} + \int_0^t \frac{\phi^2_{1,s}}{2\psi_{1,s}} + \frac{\phi^2_{2,s}}{2\psi_{2,s}}\,ds\right)\\
=&\varphi^\phi_t\bigg[(Y_{1,t}\phi_{1,t}+Z_{1,t})\frac{(X^\pi_t+G_{1,t})^\gamma}{\gamma} + Y_{1,t}(X^\pi_t+G_{1,t})^{\gamma-1}(X^\pi_t\pi_t\sigma_t+\Lambda_{1,t}) + \bigg(\int_0^t \frac{\phi^2_{1,s}}{2\psi_{1,s}}\\
&+\frac{\phi^2_{2,s}}{2\psi_{2,s}}\,ds\bigg)\phi_{1,t}\bigg]\,dW_{1,t} + \varphi^\phi_t\bigg[(Y_{1,t}\phi_{2,t}+Z_{2,t})\frac{(X^\pi_t+G_{1,t})^\gamma}{\gamma} + Y_{1,t}(X^\pi_t+G_{1,t})^{\gamma-1}\Lambda_{2,t} + \bigg(\int_0^t \frac{\phi^2_{1,s}}{2\psi_{1,s}}\\
&+\frac{\phi^2_{2,s}}{2\psi_{2,s}}\,ds\bigg)\phi_{2,t}\bigg]\,dW_{2,t} + \frac{\varphi^\phi_t}{2\psi_{1,t}}\bigg[\phi_{1,t} + \bigg(\frac{(X^\pi_t+G_{1,t})^\gamma}{\gamma}Z_{1,t} + Y_{1,t}\,(X^\pi_t+G_{1,t})^{\gamma-1}\,(X^\pi_t\pi_t\sigma_t\\
&+\Lambda_{1,t})\bigg)\psi_{1,t}\bigg]^2\,dt + \frac{\varphi^\phi_t}{2\psi_{2,t}}\bigg[\phi_{2,t} + \bigg(\frac{(X^\pi_t+G_{1,t})^\gamma}{\gamma}Z_{2,t} + Y_{1,t}(X^\pi_t+G_{1,t})^{\gamma-1}\Lambda_{2,t}\bigg)\psi_{2,t}\bigg]^2\,dt\\
&+\frac{(\gamma-1-\beta_1)\varphi^\phi_t}{2}Y_{1,t}(X^\pi_t+G_{1,t})^{\gamma-2}\bigg[X^\pi_t\pi_t\sigma_t + \Lambda_{1,t} + \frac{X^\pi_t+G_{1,t}}{\gamma-1-\beta_1}\bigg(\frac{\gamma-\beta_1}{\gamma}\frac{Z_{1,t}}{Y_{1,t}} + \lambda\sqrt{\alpha_t}\bigg)\bigg]^2\,dt\\
&+\frac{(\gamma-1-\beta_2)\varphi^\phi_t}{2}Y_{1,t}(X^\pi_t+G_{1,t})^{\gamma-2}\Lambda^2_{2,t}\,dt - \frac{\beta_2\varphi^\phi_t}{\gamma}(X^\pi_t+G_{1,t})^{\gamma-1}Z_{2,t}\Lambda_{2,t}\,dt\\
&+\varphi^\phi_t(X^\pi_t+G_{1,t})^\gamma\bigg[\frac{P_{1,t}}{\gamma} + rY_{1,t} - \frac{\beta_1}{2\gamma^2}\frac{Z^2_{1,t}}{Y_{1,t}} - \frac{\beta_2}{2\gamma^2}\frac{Z^2_{2,t}}{Y_{1,t}} - \frac{1}{2(\gamma-1-\beta_1)}Y_{1,t}\bigg(\frac{\gamma-\beta_1}{\gamma}\frac{Z_{1,t}}{Y_{1,t}}\\
&+\lambda\sqrt{\alpha_t}\bigg)^2\bigg]\,dt + \varphi^\phi_tY_{1,t}(X^\pi_t+G_{1,t})^{\gamma-1}\bigg(H_{1,t} - rG_{1,t} - \lambda\sqrt{\alpha_t}\Lambda_{1,t} + \frac{Z_{2,t}}{Y_{1,t}}\Lambda_{2,t}\bigg)\,dt.
\end{aligned}\tag{9.3.1}$$

We expect that by introducing the two continuous semi-martingales $Y_{1,t}$ and $G_{1,t}$, the stochastic process $\varphi_t^{\phi}\left(Y_{1,t}\frac{(X_t^\pi+G_{1,t})^\gamma}{\gamma}+\int_0^t \frac{\phi_{1,s}^2}{2\psi_{1,s}}+\frac{\phi_{2,s}^2}{2\psi_{2,s}}\,ds\right)$ is a local $(\mathbb{F},\mathbb{P})$-martingale under an admissible control $(\pi^*,\phi^*)\in\Pi_p\otimes\Phi$, a local $(\mathbb{F},\mathbb{P})$-super-martingale for $(\pi,\hat{\phi})\in\Pi_p\otimes\Phi$, and a local $(\mathbb{F},\mathbb{P})$-sub-martingale for $(\hat{\pi},\phi)\in\Pi_p\otimes\Phi$, respectively. For this, we can determine the process $P_{1,t}$ and $H_{1,t}$ by formally letting the last two terms on the right-hand side of (9.3.1) be zeros. Inspired by this result, we propose the following backward stochastic Riccati equation (BSRE) of $(Y_{1,t},Z_{1,t},Z_{2,t})$:

$$\begin{cases} dY_{1,t} = \Bigg[\left(-r\gamma+\frac{\gamma}{2(\gamma-1-\beta_1)}\lambda^2\alpha_t\right)Y_{1,t}+\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\lambda\sqrt{\alpha_t}Z_{1,t} \\ \qquad +\frac{1}{2\gamma}\left(\beta_1+\frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}\right)\frac{Z_{1,t}^2}{Y_{1,t}}+\frac{\beta_2}{2\gamma}\frac{Z_{2,t}^2}{Y_{1,t}}\Bigg]\,dt+Z_{1,t}\,dW_{1,t}+Z_{2,t}\,dW_{2,t}, \\ Y_{1,T}=1, \\ Y_{1,t}>0,\ \text{for all } t\in[0,T), \end{cases} \tag{9.3.2}$$

and the linear BSDE of $(G_{1,t},\Lambda_{1,t},\Lambda_{2,t})$:

$$\begin{cases} dG_{1,t}=\left(rG_{1,t}+\lambda\sqrt{\alpha_t}\Lambda_{1,t}-\frac{Z_{2,t}}{Y_{1,t}}\Lambda_{2,t}\right)dt+\Lambda_{1,t}\,dW_{1,t}+\Lambda_{2,t}\,dW_{2,t}, \\ G_{1,T}=-L_T. \end{cases} \tag{9.3.3}$$

Moreover, by separating the dependence of BSDE (9.3.3) on the liability value L_T and applying Itô's formula, we can decompose the BSDE of $(G_{1,t},\Lambda_{1,t},\Lambda_{2,t})$ into the following linear BSDE of $(\bar{G}_{1,t},\bar{\Lambda}_{1,t},\bar{\Lambda}_{2,t})$:

$$\begin{cases} d\bar{G}_{1,t}=\left((r-\mu_l)\bar{G}_{1,t}+(\lambda-\sigma_l)\sqrt{\alpha_t}\bar{\Lambda}_{1,t}-\frac{Z_{2,t}}{Y_{1,t}}\bar{\Lambda}_{2,t}\right)dt+\bar{\Lambda}_{1,t}\,dW_{1,t}+\bar{\Lambda}_{2,t}\,dW_{2,t}, \\ \bar{G}_{1,T}=-1, \end{cases} \tag{9.3.4}$$

and the solutions $(G_{1,t},\Lambda_{1,t},\Lambda_{2,t})$ and $(\bar{G}_{1,t},\bar{\Lambda}_{1,t},\bar{\Lambda}_{2,t})$ are related via the following linear formulation:

$$(G_{1,t},\Lambda_{1,t},\Lambda_{2,t})=\left(\bar{G}_{1,t}L_t,(\bar{\Lambda}_{1,t}+\bar{G}_{1,t}\sigma_l\sqrt{\alpha_t})L_t,\bar{\Lambda}_{2,t}L_t\right). \tag{9.3.5}$$

Throughout this section, by a solution to BSRE (9.3.2), we mean a triplet of stochastic processes $(Y_{1,t},Z_{1,t},Z_{2,t})\in\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+)\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ and satisfies (9.3.2). Similarly, the solution to linear BSDE (9.3.4) is a triplet of stochastic process $(\bar{G}_{1,t},\bar{\Lambda}_{1,t},\bar{\Lambda}_{2,t})\in\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.

Remark 9.3.1. Note that the generator of BSRE (9.3.2) depends on the market price of risk rather than the return rate and volatility of the risky asset price, which implies that the solvability of BSRE (9.3.2) is completely determined by the square-root factor process α_t (9.2.3), and it is, therefore, unnecessary to specify the return rate μ_t and volatility σ_t as Markovian processes. However, due to the unboundedness of α_t, the established theory of BSDEs (see, for example, El Karoui, Peng, and Quenez (1997), Bender and Kohlmann (2000), Kobylanski (2000), Briand and Hu (2008)) cannot be applied to (9.3.2) directly. Similar to Shen and Zeng (2015) and Zhang (2022c), we first propose one explicit solution to (9.3.2) by trial and verify its uniqueness by using Girsanov's measure change technique and the standard results of quadratic BSDE with bounded terminal condition (Kobylanski (2000)).

In this section, we impose the following assumption on the model parameters. This guarantees that the factor process α_t preserves affinity and square-root structure under an equivalent probability measure $\tilde{\mathbb{P}}$ which is well-defined in the proof of Theorem 9.3.7.

Assumption 9.3.2. $\kappa + \rho_1 \lambda \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \neq 0$.

Proposition 9.3.3. *One solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ to BSRE (9.3.2) is given by*

$$Y_{1,t} = \exp\left\{ f_1(t) + g_1(t)\alpha_t \right\}, \tag{9.3.6}$$

and

$$(Z_{1,t}, Z_{2,t}) = \left(\rho_1 g_1(t)\sqrt{\alpha_t} Y_{1,t}, \rho_2 g_1(t)\sqrt{\alpha_t} Y_{1,t}\right), \tag{9.3.7}$$

where functions $f_1(t)$ and $g_1(t)$ solve the following ordinary differential equations (ODEs):

$$\frac{dg_1(t)}{dt} = \left(\frac{\gamma - \beta_1}{2\gamma(\gamma - 1 - \beta_1)}\rho_1^2 - \frac{\gamma - \beta_2}{2\gamma}\rho_2^2 \right) g_1^2(t) + \left(\kappa + \frac{(\gamma - \beta_1)\lambda\rho_1}{\gamma - 1 - \beta_1} \right) g_1(t) + \frac{\gamma\lambda^2}{2(\gamma - 1 - \beta_1)}, \tag{9.3.8}$$

and

$$\frac{df_1(t)}{dt} = -\kappa\theta g_1(t) - r\gamma \tag{9.3.9}$$

with boundary conditions $f_1(T) = g_1(T) = 0$. Moreover, the closed-form solutions to ODEs (9.3.8) and (9.3.9) are given by

$$g_1(t) = \frac{n_{g_1^+} n_{g_1^-} \left(1 - e^{\sqrt{\Delta_{g_1}}(T-t)}\right)}{n_{g_1^+} - n_{g_1^-} e^{\sqrt{\Delta_{g_1}}(T-t)}}, \tag{9.3.10}$$

and

$$f_1(t) = \left(r\gamma + \kappa\theta n_{g_1^-}\right)(T-t) + \frac{\kappa\theta\left(n_{g_1^-} - n_{g_1^+}\right)}{\sqrt{\Delta_{g_1}}} \log\left(\frac{n_{g_1^+} - n_{g_1^-}}{n_{g_1^+} - n_{g_1^-} e^{\sqrt{\Delta_{g_1}}(T-t)}} \right), \tag{9.3.11}$$

where $\Delta_{g_1}, n_{g_1^+}$, and $n_{g_1}^-$ are given by

$$\begin{cases} \Delta_{g_1} = \left(\kappa + \dfrac{(\gamma - \beta_1)\lambda\rho_1}{\gamma - 1 - \beta_1}\right)^2 - \dfrac{\lambda^2}{\gamma - 1 - \beta_1}\left(\dfrac{\gamma - \beta_1}{\gamma - 1 - \beta_1}\rho_1^2 - (\gamma - \beta_2)\rho_2^2\right), \\ n_{g_1^+} = \dfrac{-\left(\kappa + \frac{(\gamma-\beta_1)\lambda\rho_1}{\gamma-1-\beta_1}\right) + \sqrt{\Delta_{g_1}}}{\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}\rho_1^2 - \frac{\gamma-\beta_2}{\gamma}\rho_2^2}, \quad n_{g_1^-} = \dfrac{-\left(\kappa + \frac{(\gamma-\beta_1)\lambda\rho_1}{\gamma-1-\beta_1}\right) - \sqrt{\Delta_{g_1}}}{\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}\rho_1^2 - \frac{\gamma-\beta_2}{\gamma}\rho_2^2}. \end{cases} \tag{9.3.12}$$

Proof. See Appendix 9.A. □

Remark 9.3.4. $\frac{dg_1(t)}{dt} = \frac{2\gamma\lambda^2\Delta_{g_1} e^{\sqrt{\Delta_{g_1}}(T-t)}}{(\gamma-1-\beta_1)\left(\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}\rho_1^2 - \frac{\gamma-\beta_2}{\gamma}\rho_2^2\right)^2\left(n_{g_1^+} - n_{g_1^-} e^{\sqrt{\Delta_{g_1}}(T-t)}\right)^2} > 0$ due to $\gamma \in \mathbb{R}^-$. In other words, function $g_1(t)$ is strictly increasing over $[0,T]$, and thus, we have $g_1(t) \in [g_1(0), 0]$ and $f_1(t) \le r\gamma(T-t)$.

Proposition 9.3.5. *The solution* $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ *proposed in Proposition 9.3.3 lies in* $\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. *More precisely,* $Y_t \le e^{r\gamma(T-t)}$, *for* $t \in [0,T]$, $\mathbb{P}$ *almost surely.*

Proof. See Appendix 9.B. □

Before verifying that the proposed solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in Proposition 9.3.3 is the unique solution to BSRE (9.3.2) in the space $\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, we present the following auxiliary result on the stochastic exponential process of the square-root factor process α_t (refer to Lemma A1 in Shen and Zeng (2015)).

Lemma 9.3.6 (Bona-fide martingale property)**.** *If* $m_1(t)$ *and* $m_2(t)$ *are two bounded functions over* $[0,T]$, *the following stochastic exponential process*

$$\exp\left\{\int_0^t m_1(s)\sqrt{\alpha_s}\,dW_{1,s} + \int_0^t m_2(s)\sqrt{\alpha_s}\,dW_{2,s} - \frac{1}{2}\int_0^t \left(m_1^2(s) + m_2^2(s)\right)\alpha_s\,ds\right\}$$

is an $(\mathbb{F},\mathbb{P})$*-martingale.*

Theorem 9.3.7. *Suppose Assumption 9.3.2 holds true. The solution* $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ *given in (9.3.6) and (9.3.7) is the unique solution to BSRE (9.3.2).*

Proof. See Appendix 9.C. □

After deriving the closed-form expression for the unique solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ to BSRE (9.3.3), the linear BSDE (9.3.4) of $(\bar{G}_{1,t}, \bar{\Lambda}_{1,t}, \bar{\Lambda}_{2,t})$ can be reformulated as follows:

$$\begin{cases} d\bar{G}_{1,t} = \left((r-\mu_l)\bar{G}_{1,t} + (\lambda-\sigma_l)\sqrt{\alpha_t}\bar{\Lambda}_{1,t} - \rho_2 g_1(t)\sqrt{\alpha_t}\bar{\Lambda}_{2,t}\right)\,dt + \bar{\Lambda}_{1,t}\,dW_{1,t} \\ \qquad\qquad + \bar{\Lambda}_{2,t}\,dW_{2,t}, \\ \bar{G}_{1,T} = -1. \end{cases} \tag{9.3.13}$$

Proposition 9.3.8. *The unique solution to linear BSDE (9.3.13) is given by*

$$\bar{G}_{1,t} = -e^{(r-\mu_l)(t-T)}, \tag{9.3.14}$$

and

$$(\bar{\Lambda}_{1,t}, \bar{\Lambda}_{2,t}) = (0,0)\,. \tag{9.3.15}$$

Proof. See Appendix 9.D. □

Remark 9.3.9. It is crucial to identify that the second control component $\Lambda_{2,t}$ of the solution to linear BSDE (9.3.3) is zero from Proposition 9.3.8 and the relationship between $\Lambda_{2,t}$ and $\bar{\Lambda}_{2,t}$ given in (9.3.5) above, which, in turn, allows us to remove the fourth and fifth drift terms on the right-hand side of (9.3.1).

After solving the associated BSDEs explicitly and deriving their unique results, now we are ready to state our first main result.

Theorem 9.3.10. *For any initial data $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ such that $x_0 + \bar{G}_{1,0} l_0 \in \mathbb{R}^+$, suppose that Assumption 9.3.2 and the following conditions hold*

$$\max\{k_0, k_1, k_2\} \leq \frac{\kappa^2}{\rho_1^2 + \rho_2^2} \tag{9.3.16}$$

with $k_0, k_1,$ and k_2 given by

$$\begin{cases} k_0 = \sup\limits_{t\in[0,T]} (120 + 32\sqrt{14}) \left[\dfrac{(\beta_1 \rho_1 g_1(t) + \lambda\gamma\beta_1)^2}{\gamma^2(\gamma - 1 - \beta_1)^2} + \dfrac{\beta_2^2 \rho_2^2 g_1^2(t)}{\gamma^2} \right], \\ k_1 = \sup\limits_{t\in[0,T]} (2 + \sqrt{2}) \dfrac{(8(\gamma - \beta_1)\rho_1 g_1(t) + 8\gamma\lambda)^2}{(\beta_1 + 1 - \gamma)^2}, \\ k_2 = \sup\limits_{t\in[0,T]} \dfrac{32(\gamma - \beta_1)\lambda\rho_1 g_1(t) + 32\lambda^2\gamma}{\beta_1 + 1 - \gamma} + \left(128\gamma^2 - 16\gamma\right) \dfrac{(\frac{\gamma - \beta_1}{\gamma}\rho_1 g_1(t) + \lambda)^2}{(\beta_1 + 1 - \gamma)^2}. \end{cases}$$

Then, for the following control (π^*, ϕ^*)

$$\begin{cases} \pi_t^* = \dfrac{1}{X_t^* \sigma_t} \left[\dfrac{X_t^* + \bar{G}_{1,t} L_t}{\beta_1 + 1 - \gamma} \left(\dfrac{\gamma - \beta_1}{\gamma} \dfrac{Z_{1,t}}{Y_{1,t}} + \lambda\sqrt{\alpha_t} \right) - \sigma_l \sqrt{\alpha_t} \bar{G}_{1,t} L_t \right], \\ \phi_{1,t}^* = - \left(\dfrac{\beta_1}{\gamma} \dfrac{Z_{1,t}}{Y_{1,t}} + \dfrac{\beta_1 (X_t^* \pi_t^* \sigma_t + \sigma_l \sqrt{\alpha_t} \bar{G}_{1,t} L_t)}{X_t^* + \bar{G}_{1,t} L_t} \right) = \dfrac{\beta_1}{\gamma(\gamma - 1 - \beta_1)} \dfrac{Z_{1,t}}{Y_{1,t}} \\ \qquad + \dfrac{\lambda\beta_1}{\gamma - 1 - \beta_1} \sqrt{\alpha_t}, \\ \phi_{2,t}^* = = -\dfrac{\beta_2}{\gamma} \dfrac{Z_{2,t}}{Y_{1,t}}, \end{cases} \tag{9.3.17}$$

where X_t^* is the asset process associated with π_t^*, and the closed-form expressions for $Y_{1,t}, Z_{1,t},$ and $\bar{G}_{1,t}$ are given by (9.3.6), (9.3.7), and (9.3.14), respectively, we have

(i) the distortion process $\phi^* \in \Phi$, i.e., the Radon-Nikodym derivative process $\varphi_t^{\phi^*}$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale, and $X_t^* + \bar{G}_{1,t} L_t > 0$, $\mathbb{P}$ almost surely, for all $t \in [0, T]$;

(ii) $\varphi_t^{\phi^*} Y_{1,t} \frac{(X_t^* + \bar{G}_{1,t} L_t)^\gamma}{\gamma} \in \mathcal{S}^4_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}^+)$;

(iii) $\varphi_t^{\phi^*} \left(\int_0^t \frac{(\phi_{1,s}^*)^2}{2\psi_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*} \, ds \right) \in \mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}^+)$, where $\psi_{i,t}^* = \frac{\beta_i}{Y_{1,t}(X_t^* + \bar{G}_{1,t} L_t)^\gamma}$, for $i = 1, 2$.

In the affirmative, the control $(\pi^*, \phi^*) \in \Pi_p \otimes \Phi$ *is the optimal control of the robust ALM problem (9.2.11), and the optimal value function is given by*

$$J_p(\pi^*, \phi^*) = Y_{1,0}\frac{(x_0 + \bar{G}_{1,0}l_0)^\gamma}{\gamma}. \tag{9.3.18}$$

Proof. See Appendix 9.E. □

Remark 9.3.11. The feasibility of the technical condition (9.3.16) is guaranteed by the monotonicity of function $g_1(t)$, and in essence, this sufficient condition is imposed to show that (π^*, ϕ^*) is a saddle point of the value function $J_p(\pi, \phi)$ for the robust control problem (9.2.11). More specifically, it helps prove that (i)-(iii) in Theorem 9.3.10 hold for the control (π^*, ϕ^*) and then verifies the admissibility of (π^*, ϕ^*) by confirming conditions 1-3 in Definition 9.2.5 above.

Remark 9.3.12. Note that although robust investment problems using BSDE approaches have been considered in the literature (see, for example, Øksendal and Sulem (2011, 2014)), our BSDE approach is different from that in Øksendal and Sulem (2011, 2014), where the value function is written as the solution to an associated controlled BSDE and the comparison theorem for the solutions to BSDEs is applied to show that the optimal control is a saddle point of the value function. Recalling (9.3.1) and the proof of Theorem 9.3.10 above, we propose to construct a stochastic process depending on any admissible control (π, ϕ) instead, and such that its terminal value equals the sum of the utility of the terminal surplus and the penalty term within the expectation of (9.2.11), which then leads to two uncontrolled BSDEs (9.3.2)-(9.3.3).

Remark 9.3.13. To our knowledge, the results provided in Theorem 9.3.10 are not reported in the existing literature. If we further set $l_0 = \mu_l = \sigma_l = 0$ in Theorem 9.3.10, then we derive the explicit solutions to the robust portfolio selection problem under power utility and square-root factor processes. If we specify $\beta_1 = \beta_2 = 0$, the analytical solutions to the optimal ALM problem without model uncertainty under power utility and square-root factor processes are provided. In other words, the benefits of Theorem 9.3.10 are two-fold.

The next three corollaries provide the explicit results for three particular cases of our model, the CEV model (9.2.4), the family of 4/2 models (9.2.5), and non-Markovian stochastic volatility model (9.2.6), respectively.

Corollary 9.3.14 (CEV model)**.** *If the risky asset price* S_t *follows the CEV model (9.2.4) with any initial data* $(x_0, s_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ *such that* $x_0 + \bar{G}_{1,0}l_0 \in \mathbb{R}^+$, *and suppose that* $\mu - r(\gamma - \beta_1) \neq 0$ *and the following conditions hold*

$$\max\left\{\tilde{k}_0, \tilde{k}_1, \tilde{k}_2\right\} \leq \frac{\mu^2}{\sigma^2}$$

with $\tilde{k}_0, \tilde{k}_1$, and $\tilde{k}_2$ given by

$$\begin{cases} \tilde{k}_0 = \sup_{t\in[0,T]} (120 + 32\sqrt{14}) \dfrac{(\gamma \frac{\mu-r}{\sigma} - 2\beta\sigma\tilde{g}_1(t))^2 \beta_1^2}{\gamma^2(\gamma - 1 - \beta_1)^2}, \\ \tilde{k}_1 = \sup_{t\in[0,T]} (2 + \sqrt{2}) \dfrac{(8\gamma \frac{\mu-r}{\sigma} - 16(\gamma - \beta_1)\beta\sigma\tilde{g}_1(t))^2}{(\beta_1 + 1 - \gamma)^2}, \\ \tilde{k}_2 = \sup_{t\in[0,T]} \dfrac{32\lambda^2\gamma - 64(\gamma - \beta_1)(\mu - r)\beta\tilde{g}_1(t)}{\beta_1 + 1 - \gamma} + (128\gamma^2 - 16\gamma) \dfrac{(\frac{\mu-r}{\sigma} - 2\beta\sigma\frac{\gamma-\beta_1}{\gamma}\tilde{g}_1(t))^2}{(\beta_1 + 1 - \gamma)^2}, \end{cases}$$

then, the optimal control and optimal value function of the robust ALM problem (9.2.11) are, respectively, given by

$$\begin{cases} \pi_t^* = \dfrac{(\mu - r)\left(X_t^* + \bar{G}_{1,t}L_t\right)}{X_t^*\sigma^2 S_t^{2\beta}(\beta_1 + 1 - \gamma)} - \dfrac{2(\gamma - \beta_1)\beta\sigma\tilde{g}_1(t)(X_t^* + \bar{G}_{1,t}L_t) + (\beta_1 + 1 - \gamma)\gamma\sigma_l\bar{G}_{1,t}L_t}{X_t^*\sigma S_t^{2\beta}(\beta_1 + 1 - \gamma)\gamma}, \\ \phi_{1,t}^* = \dfrac{\beta_1}{(\gamma - 1 - \beta_1)S_t^\beta} \left(\dfrac{\mu - r}{\sigma} - \dfrac{2\beta\sigma\tilde{g}_1(t)}{\gamma}\right), \\ \phi_{2,t}^* = 0, \end{cases}$$

and

$$J_p(\pi^*, \phi^*) = \frac{(x_0 - \bar{G}_{1,0}l_0)^\gamma}{\gamma} \exp\left\{\tilde{f}_1(0) + \tilde{g}_1(0)s_0^{-2\beta}\right\},$$

where $\bar{G}_{1,t}$ is given by (9.3.14), and functions $\tilde{f}_1(t)$ and $\tilde{g}_1(t)$ are given by

$$\tilde{f}_1(t) = \left(r\gamma + (2\beta^2 + \beta)\sigma^2 n_{\tilde{g}_1^-}\right)(T-t) + \frac{(2\beta^2 + \beta)\sigma^2\left(n_{\tilde{g}_1^-} - n_{\tilde{g}_1^+}\right)}{\sqrt{\Delta_{\tilde{g}_1}}} \log\left(\frac{n_{\tilde{g}_1^+} - n_{\tilde{g}_1^-}}{n_{\tilde{g}_1^+} - n_{\tilde{g}_1^-} e^{\sqrt{\Delta_{\tilde{g}_1}}(T-t)}}\right),$$

and

$$\tilde{g}_1(t) = \frac{n_{\tilde{g}_1^+} n_{\tilde{g}_1^-}\left(1 - e^{\sqrt{\Delta_{\tilde{g}_1}}(T-t)}\right)}{n_{\tilde{g}_1^+} - n_{\tilde{g}_1^-} e^{\sqrt{\Delta_{\tilde{g}_1}}(T-t)}}$$

with $\Delta_{\tilde{g}_1}, n_{\tilde{g}_1^+}$, and $n_{\tilde{g}_1^-}$ given by

$$\begin{cases} \Delta_{\tilde{g}_1} = 4\beta^2 \left[\left(\mu - \dfrac{(\gamma - \beta_1)(\mu - r)}{\gamma - 1 - \beta_1}\right)^2 - \dfrac{\gamma - \beta_1}{(\gamma - 1 - \beta_1)^2}(\mu - r)^2\right], \\ n_{\tilde{g}_1^+} = \dfrac{2\beta\left(\frac{\gamma-\beta_1}{\gamma-1-\beta_1}(\mu - r) - \mu\right) + \sqrt{\Delta_{\tilde{g}_1}}}{\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}4\beta^2\sigma^2}, \quad n_{\tilde{g}_1^-} = \dfrac{2\beta\left(\frac{\gamma-\beta_1}{\gamma-1-\beta_1}(\mu - r) - \mu\right) - \sqrt{\Delta_{\tilde{g}_1}}}{\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}4\beta^2\sigma^2}. \end{cases}$$

Proof. Substituting the parameters specified in Example 9.2.1 into (9.3.16)-(9.3.18) leads to the results immediately. ☐

Corollary 9.3.15 (The family of 4/2 models)**.** *If the risky asset price process S_t and the variance driver process V_t are governed by the family of 4/2 models (9.2.5) with any initial data $(x_0, v_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ such that $x_0 + \bar{G}_{1,0}l_0 \in \mathbb{R}^+$, and suppose that $\kappa + \sigma_v\rho\lambda\frac{\gamma-\beta_1}{\gamma-1-\beta_1} \neq 0$ and the following conditions hold*

$$\max\left\{\bar{k}_0, \bar{k}_1, \bar{k}_2\right\} \leq \frac{\kappa^2}{\sigma_v^2}$$

with $\bar{k}_0, \bar{k}_1$*, and* $\bar{k}_2$ *given by*

$$\begin{cases} \bar{k}_0 = \sup_{t\in[0,T]} (120+32\sqrt{14}) \left[\frac{(\beta_1\sigma_v\rho\bar{g}_1(t)+\lambda\gamma\beta_1)^2}{\gamma^2(\gamma-1-\beta_1)^2} + \frac{\beta_2^2\sigma_v^2(1-\rho^2)\bar{g}_1^2(t)}{\gamma^2} \right], \\ \bar{k}_1 = \sup_{t\in[0,T]} (2+\sqrt{2}) \frac{(8(\gamma-\beta_1)\sigma_v\rho\bar{g}_1(t)+8\gamma\lambda)^2}{(\beta_1+1-\gamma)^2}, \\ \bar{k}_2 = \sup_{t\in[0,T]} \frac{32(\gamma-\beta_1)\lambda\sigma_v\rho\bar{g}_1(t)+32\lambda^2\gamma}{\beta_1+1-\gamma} + \left(128\gamma^2-16\gamma\right) \frac{(\frac{\gamma-\beta_1}{\gamma}\sigma_v\rho\bar{g}_1(t)+\lambda)^2}{(\beta_1+1-\gamma)^2}, \end{cases}$$

then, the optimal control and optimal value function of the robust ALM problem (9.2.11) are, respectively, given by

$$\begin{cases} \pi_t^* = \frac{V_t}{X_t^*(c_1V_t+c_2)} \left[\frac{X_t^*+\bar{G}_{1,t}L_t}{\beta_1+1-\gamma} \left(\frac{\gamma-\beta_1}{\gamma}\sigma_v\rho\bar{g}_1(t)+\lambda \right) - \sigma_l\bar{G}_{1,t}L_t \right], \\ \phi_{1,t}^* = \left[\frac{\beta_1}{\gamma(\gamma-1-\beta_1)}\sigma_v\rho\bar{g}_1(t) + \frac{\lambda\beta_1}{\gamma-1-\beta_1} \right] \sqrt{V_t}, \\ \phi_{2,t}^* = -\frac{\beta_2}{\gamma}\sigma_v\sqrt{1-\rho^2}\bar{g}_1(t)\sqrt{V_t}, \end{cases}$$

and

$$J_p(\pi^*,\phi^*) = \frac{(x_0-\bar{G}_{1,0}l_0)^\gamma}{\gamma} \exp\left\{ \bar{f}_1(0)+\bar{g}_1(0)v_0 \right\},$$

where $\bar{G}_{1,t}$ *is given by (9.3.14), and functions* $\bar{f}_1(t)$ *and* $\bar{g}_1(t)$ *are given by*

$$\bar{f}_1(t) = \left(r\gamma+\kappa\theta n_{\bar{g}_1^-} \right)(T-t) + \frac{\kappa\theta\left(n_{\bar{g}_1^-}-n_{\bar{g}_1^+}\right)}{\sqrt{\Delta_{\bar{g}_1}}} \log\left(\frac{n_{\bar{g}_1^+}-n_{\bar{g}_1^-}}{n_{\bar{g}_1^+}-n_{\bar{g}_1^-}e^{\sqrt{\Delta_{\bar{g}_1}}(T-t)}} \right)$$

and

$$\bar{g}_1(t) = \frac{n_{\bar{g}_1^+}n_{\bar{g}_1^-}\left(1-e^{\sqrt{\Delta_{\bar{g}_1}}(T-t)}\right)}{n_{\bar{g}_1^+}-n_{\bar{g}_1^-}e^{\sqrt{\Delta_{\bar{g}_1}}(T-t)}}$$

with $\Delta_{\bar{g}_1}, n_{\bar{g}_1^+}$*, and* $n_{\bar{g}_1^-}$ *given by*

$$\begin{cases} \Delta_{\bar{g}_1} = \left(\kappa+\frac{(\gamma-\beta_1)\lambda\sigma_v\rho}{\gamma-1-\beta_1}\right)^2 - \frac{\lambda^2\sigma_v^2}{\gamma-1-\beta_1}\left(\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\rho^2-(\gamma-\beta_2)(1-\rho^2)\right), \\ n_{\bar{g}_1^+} = \frac{-\left(\kappa+\frac{(\gamma-\beta_1)\lambda\sigma_v\rho}{\gamma-1-\beta_1}\right)+\sqrt{\Delta_{\bar{g}_1}}}{\sigma_v^2\left(\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}\rho^2-\frac{\gamma-\beta_2}{\gamma}(1-\rho^2)\right)}, \; n_{\bar{g}_1^-} = \frac{-\left(\kappa+\frac{(\gamma-\beta_1)\lambda\sigma_v\rho}{\gamma-1-\beta_1}\right)-\sqrt{\Delta_{\bar{g}_1}}}{\sigma_v^2\left(\frac{\gamma-\beta_1}{\gamma(\gamma-1-\beta_1)}\rho^2-\frac{\gamma-\beta_2}{\gamma}(1-\rho^2)\right)}. \end{cases}$$

Proof. Plugging the specified parameters of the 4/2 model (9.2.5) into (9.3.16)-(9.3.18) yields the above results. □

Remark 9.3.16. Setting either $(c_1,c_2)=(1,0)$ or $(c_1,c_2)=(0,1)$ in the 4/2 model (9.2.5), Corollary 9.3.15 provides the explicit expressions for the optimal controls and optimal value functions of the robust ALM problem (9.2.11) under the Heston model and 3/2 model, respectively, and neither of them is considered in the published works.

Remark 9.3.17. It is worth mentioning that Cheng and Escobar (2021b) recently solves the robust portfolio selection problem under the 4/2 model in a complete market. In this sense, Corollary 9.3.15 extends the results of Cheng and Escobar (2021b) to the case with random liabilities in an incomplete market setting. Moreover, if we ignore model ambiguity by imposing $\beta_1 = \beta_2 = 0$ in Corollary 9.3.15, it can be verified that our result generalizes that of Cheng and Escobar (2021a) to the case with random liabilities.

Corollary 9.3.18 (Non-Markovian path-dependent model)**.** *If the risky asset price process S_t and its volatility driver process α_t are governed by the path-dependent stochastic volatility model (9.2.6) with initial data $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ such that $x_0 + \bar{G}_{1,0} l_0 \in \mathbb{R}^+$, and suppose that Assumption 9.3.2 and condition (9.3.16) hold true, then the optimal control and optimal value function of the robust ALM problem (9.2.11) are, respectively, given by*

$$\begin{cases} \pi_t^* = \dfrac{1}{X_t^* \hat{\sigma}(\alpha_{[0,t]})} \left[\dfrac{X_t^* + \bar{G}_{1,t} L_t}{\beta_1 + 1 - \gamma} \left(\dfrac{\gamma - \beta_1}{\gamma} \dfrac{Z_{1,t}}{Y_{1,t}} + \lambda \sqrt{\alpha_t} \right) - \sigma_l \sqrt{\alpha_t} \bar{G}_{1,t} L_t \right], \\ \phi_{1,t}^* = \dfrac{\beta_1}{\gamma(\gamma - 1 - \beta_1)} \dfrac{Z_{1,t}}{Y_{1,t}} + \dfrac{\lambda \beta_1}{\gamma - 1 - \beta_1} \sqrt{\alpha_t}, \\ \phi_{2,t}^* = -\dfrac{\beta_2}{\gamma} \dfrac{Z_{2,t}}{Y_{1,t}}, \end{cases}$$

and

$$J_p(\pi^*, \phi^*) = Y_{1,0} \frac{(x_0 + \bar{G}_{1,0} l_0)^\gamma}{\gamma},$$

where the closed-form expressions for $Y_{1,t}, Z_{1,t}, Z_{2,t}$ and $\bar{G}_{1,t}$ are given by (9.3.6), (9.3.7), and (9.3.14), respectively.

Proof. Replacing σ_t in Theorem 9.3.10 by $\hat{\sigma}(\alpha_{[0,t]})$ leads to the above results immediately. □

9.4 Optimal investment strategies for the exponential utility case

In this section, we investigate the robust optimization problem under exponential utility (9.2.13) by using a BSDE approach. Similar to the previous section, to find the BSDEs associated with problem (9.2.13), we introduce the following continuous $(\mathbb{F}, \mathbb{P})$-semi-martingales $Y_{2,t}$ and $G_{2,t}$ with canonical decomposition as follows:

$$dY_{2,t} = P_{2,t}\, dt + M_{1,t}\, dW_{1,t} + M_{2,t}\, dW_{2,t},$$

and

$$dG_{2,t} = H_{2,t}\, dt + \Gamma_{1,t}\, dW_{1,t} + \Gamma_{2,t}\, dW_{2,t},$$

where $P_{2,t}$ and $H_{2,t}$ are two undetermined $\mathbb{F}$-adapted processes, and $M_{1,t}, M_{2,t}, \Gamma_{1,t}$, and $\Gamma_{2,t}$ belong to $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$. An application of Itô's formula to $\varphi_t^\phi \Big(-$

$\frac{e^{-q(X_t^\pi Y_{2,t}+G_{2,t})}}{q} + \int_0^t \left(\frac{\phi_{1,s}^2}{2\eta_{1,s}} + \frac{\phi_{2,s}^2}{2\eta_{2,s}} \right) ds \Big)$ leads to

$$
\begin{aligned}
& d\varphi_t^\phi \left(-\frac{e^{-q(X_t^\pi Y_{2,t}+G_{2,t})}}{q} + \int_0^t \left(\frac{\phi_{1,s}^2}{2\eta_{1,s}} + \frac{\phi_{2,s}^2}{2\eta_{2,s}} \right) ds \right) \\
=& \varphi_t^\phi \left[e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} \left((\sigma_t \pi_t Y_{2,t} + M_{1,t}) X_t^\pi + \Gamma_{1,t} \right) + \phi_{1,t} \Big(\int_0^t \left(\frac{\phi_{1,s}^2}{2\eta_{1,s}} + \frac{\phi_{2,s}^2}{2\eta_{2,s}} \right) ds \right. \\
& \left. - \frac{e^{-q(X_t^\pi Y_{2,t}+G_{2,t})}}{q} \Big) \right] dW_{1,t} + \varphi_t^\phi \Big[e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} \left(M_{2,t} X_t^\pi + \Gamma_{2,t} \right) + \Big(\int_0^t \Big(\frac{\phi_{1,s}^2}{2\eta_{1,s}} \\
& + \frac{\phi_{2,s}^2}{2\eta_{2,s}} \Big) ds - \frac{e^{-q(X_t^\pi Y_{2,t}+G_{2,t})}}{q} \Big) \phi_{2,t} \Big] dW_{2,t} + \frac{\varphi_t^\phi}{2\eta_{1,t}} \Big[\phi_{1,t} + e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} \Big((\sigma_t \pi_t Y_{2,t} \\
& + M_{1,t}) X_t^\pi + \Gamma_{1,t} \Big) \eta_{1,t} \Big]^2 dt + \frac{\varphi_t^\phi}{2\eta_{2,t}} \left[\phi_{2,t} + e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} \left(M_{2,t} X_t^\pi + \Gamma_{2,t} \right) \eta_{2,t} \right]^2 dt \\
& - \frac{q+\beta_1}{2} \varphi_t^\phi e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} \left[(\sigma_t \pi_t Y_{2,t} + M_{1,t}) X_t^\pi + \Gamma_{1,t} - \frac{1}{q+\beta_1} \left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) \right]^2 dt \\
& - \frac{q+\beta_2}{2} \varphi_t^\phi e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} M_{2,t}^2 (X_t^\pi)^2 \, dt - (q+\beta_2) \varphi_t^\phi e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} M_{2,t} X_t^\pi \Gamma_{2,t} \, dt \\
& + \varphi_t^\phi e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} \left[H_{2,t} + \frac{\left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right)^2}{2(q+\beta_1)} - \frac{q+\beta_2}{2} \Gamma_{2,t}^2 - \left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) \Gamma_{1,t} \right] dt \\
& + \varphi_t^\phi e^{-q(X_t^\pi Y_{2,t}+G_{2,t})} X_t^\pi \left[r Y_{2,t} + P_{2,t} - \left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) M_{1,t} \right] dt.
\end{aligned} \tag{9.4.1}
$$

We expect $\varphi_t^\phi \left(-\frac{e^{-q(X_t^\pi Y_{2,t}+G_{2,t})}}{q} + \int_0^t \left(\frac{\phi_{1,s}^2}{2\eta_{1,s}} + \frac{\phi_{2,s}^2}{2\eta_{2,s}} \right) ds \right)$ is a local $(\mathbb{F}, \mathbb{P})$-martingale for an admissible control $(\pi^*, \phi^*) \in \Pi_e \otimes \Phi$, a local $(\mathbb{F}, \mathbb{P})$-super-martingale for $(\pi, \check{\phi}) \in \Pi_e \otimes \Phi$, and a local $(\mathbb{F}, \mathbb{P})$-sub-martingale for $(\breve{\pi}, \phi) \in \Pi_e \otimes \Phi$, respectively. Inspired by this, the stochastic processes $P_{2,t}$ and $H_{2,t}$ can be determined by simply letting the last two terms on the right-hand side of (9.4.1) be zeros. In other words, we find the following BSRE of $(Y_{2,t}, M_{1,t}, M_{2,t})$:

$$
\begin{cases}
dY_{2,t} = \left[-rY_{2,t} + \left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) M_{1,t} \right] dt + M_{1,t} \, dW_{1,t} + M_{2,t} \, dW_{2,t}, \\
Y_{2,T} = 1,
\end{cases} \tag{9.4.2}
$$

and quadratic BSDE of $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$:

$$
\begin{cases}
dG_{2,t} = \left[\left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) \Gamma_{1,t} + \frac{q+\beta_2}{2} \Gamma_{2,t}^2 - \frac{1}{2(q+\beta_1)} \left(\lambda\sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right)^2 \right] dt \\
\qquad\qquad + \Gamma_{1,t} \, dW_{1,t} + \Gamma_{2,t} \, dW_{2,t}, \\
G_{2,T} = - L_T.
\end{cases} \tag{9.4.3}
$$

Here, a solution to BSRE (9.4.2) is a triplet of $\mathbb{F}$-adapted processes $(Y_{2,t}, M_{1,t}, M_{2,t})$ such that $(Y_{2,t}, M_{1,t}, M_{2,t}) \in \mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^+) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$; a solution to quadratic BSDE (9.4.3) is a triplet of $\mathbb{F}$-adapted processes $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t}) \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.

Similar to Section 9.3, we make the following assumptions on the model parameters to ensure that α_t is an affine-form, square-root factor process under the well-defined probability measure $\widehat{\mathbb{P}}$ in Theorem 9.4.4.

Assumption 9.4.1. $\kappa + \lambda\rho_1 \neq 0$.

Proposition 9.4.2. *The unique solution to BSRE (9.4.2) is given by*

$$Y_{2,t} = e^{r(T-t)}, \tag{9.4.4}$$

and

$$(M_{1,t}, M_{2,t}) = (0,0). \tag{9.4.5}$$

Proof. See Appendix 9.F. □

After solving BSRE (9.4.2) explicitly, we can simplify quadratic BSDE (9.4.3) to the following form:

$$\begin{cases} dG_{2,t} = \left(\lambda\sqrt{\alpha_t}\Gamma_{1,t} + \dfrac{q+\beta_2}{2}\Gamma_{2,t}^2 - \dfrac{1}{2(q+\beta_1)}\lambda^2\alpha_t\right) dt + \Gamma_{1,t}\, dW_{1,t} + \Gamma_{2,t}\, dW_{2,t}, \\ G_{2,T} = -\, L_T. \end{cases} \tag{9.4.6}$$

Proposition 9.4.3. *One solution $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ to quadratic BSDE (9.4.6) is given by*

$$G_{2,t} = f_2(t) + g_2(t)\alpha_t + h_2(t)L_t \tag{9.4.7}$$

and

$$(\Gamma_{1,t}, \Gamma_{2,t}) = \left((\rho_1 g_2(t) + \sigma_l h_2(t) L_t)\sqrt{\alpha_t}, \rho_2 g_2(t)\sqrt{\alpha_t}\right), \tag{9.4.8}$$

where functions $f_2(t), g_2(t)$, and $h_2(t)$ solve the following ODEs:

$$\frac{dg_2(t)}{dt} = \frac{q+\beta_2}{2}\rho_2^2 g_2^2(t) + (\kappa + \lambda\rho_1)\, g_2(t) - \frac{1}{2(q+\beta_1)}\lambda^2, \;\; g_2(T) = 0, \tag{9.4.9}$$

$$\frac{df_2(t)}{dt} = -\kappa\theta g_2(t), \;\; f_2(T) = 0, \tag{9.4.10}$$

and

$$\frac{dh_2(t)}{dt} = -\mu_l h_2(t), \;\; h_2(T) = -1. \tag{9.4.11}$$

Furthermore, the closed-form expressions for $g_2(t), f_2(t)$, and $h_2(t)$ are given by

$$g_2(t) = \begin{cases} -\dfrac{\lambda^2}{2(q+\beta_1)}(t-T), \;\; \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda\rho_1 = 0; \\ -\dfrac{\lambda^2}{2(q+\beta_1)(\kappa+\lambda\rho_1)}\left(e^{(\kappa+\lambda\rho_1)(t-T)} - 1\right), \;\; \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda\rho_1 \neq 0; \\ \dfrac{n_{g_2^+} n_{g_2^-}\left(1 - e^{\sqrt{\Delta_{g_2}}(T-t)}\right)}{n_{g_2^+} - n_{g_2^-} e^{\sqrt{\Delta_{g_2}}(T-t)}}, \;\; \text{if } \rho_2 \neq 0, \end{cases} \tag{9.4.12}$$

$$f_2(t) = \begin{cases} \dfrac{\kappa\theta\lambda^2}{4(q+\beta_1)}(t-T)^2, \ \textit{if } \rho_2 = 0 \textit{ and } \kappa+\lambda\rho_1 = 0; \\ \dfrac{-\lambda^2\kappa\theta}{2(q+\beta_1)(\kappa+\lambda\rho_1)}\left(\dfrac{1-e^{(\kappa+\lambda\rho_1)(t-T)}}{\kappa+\lambda\rho_1}+t-T\right), \ \textit{if } \rho_2 = 0 \textit{ and } \kappa+\lambda\rho_1 \neq 0; \\ \kappa\theta n_{g_2^-}(T-t) + \dfrac{\kappa\theta\left(n_{g_2^-}-n_{g_2^+}\right)}{\sqrt{\Delta_{g_2}}}\log\left(\dfrac{n_{g_2^+}-n_{g_2^-}}{n_{g_2^+}-n_{g_2^-}e^{\sqrt{\Delta_{g_2}}(T-t)}}\right), \ \textit{if } \rho_2 \neq 0, \end{cases} \tag{9.4.13}$$

and

$$h_2(t) = -e^{\mu_l(T-t)}, \tag{9.4.14}$$

where $\Delta_{g_2}, n_{g_2^+}$, *and* $n_{g_2^-}$ *are given by*

$$\Delta_{g_2} = (\kappa+\lambda\rho_1)^2 + \frac{q+\beta_2}{q+\beta_1}\rho_2^2\lambda^2, \ n_{g_2^+} = \frac{-(\kappa+\lambda\rho_1)+\sqrt{\Delta_{g_2}}}{(q+\beta_2)\rho_2^2}, \ n_{g_2^-} = \frac{-(\kappa+\lambda\rho_1)-\sqrt{\Delta_{g_2}}}{(q+\beta_2)\rho_2^2}. \tag{9.4.15}$$

Proof. See Appendix 9.G. □

In the next theorem, we show that the candidate solution $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ presented in Proposition 9.4.3 lies in $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ and is the unique solution to quadratic BSDE (9.4.6).

Theorem 9.4.4. *Suppose Assumption 9.4.1 holds true. The solution* $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ *given by (9.4.7) and (9.4.8) is the unique solution to BSDE (9.4.6) and belongs to* $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.

Proof. See Appendix 9.H. □

Having derived the closed-form solutions to BSDEs (9.4.2) and (9.4.3), we are ready to state our second main result below.

Theorem 9.4.5. *For any initial data* $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$, *suppose that Assumption 9.4.1 and the following conditions hold*

$$\max\{b_0, b_1, b_2\} \leq \frac{\kappa^2}{\rho_1^2+\rho_2^2} \tag{9.4.16}$$

with b_0, b_1, *and* b_2 *given by*

$$\begin{cases} b_0 = \sup\limits_{t\in[0,T]}(120+32\sqrt{14})\left(\dfrac{\beta_1^2\lambda^2}{(q+\beta_1)^2}+\beta_2^2\rho_2^2g_2^2(t)\right), \\ b_1 = \sup\limits_{t\in[0,T]} 64(2+\sqrt{2})\left(\dfrac{\lambda^2}{(q+\beta_1)^2}+\rho_2^2g_2^2(t)\right)q^2, \\ b_2 = \sup\limits_{t\in[0,T]} \dfrac{128q^2\lambda^2}{(q+\beta_1)^2} - \dfrac{16q\lambda^2}{q+\beta_1} + (112q^2-16q\beta_2)\rho_2^2g_2^2(t). \end{cases}$$

Then, for the following control (π^*, ϕ^*)

$$\begin{cases} \pi_t^* = \dfrac{\frac{\lambda}{q+\beta_1}\sqrt{\alpha_t} - \Gamma_{1,t}}{X_t^* \sigma_t Y_{2,t}}, \\ \phi_{1,t}^* = -\beta_1 \left(\sigma_t \pi_t^* Y_{2,t} X_t^* + \Gamma_{1,t}\right) = -\dfrac{\beta_1}{q+\beta_1}\lambda\sqrt{\alpha_t}, \\ \phi_{2,t}^* = -\beta_2 \Gamma_{2,t}, \end{cases} \tag{9.4.17}$$

where X_t^* *is the asset process associated with* π_t^**, and the closed-form expressions for* $Y_{2,t}, \Gamma_{1,t}$*, and* $\Gamma_{2,t}$ *are given by (9.4.4) and (9.4.8), respectively, we have*
(i) the distortion process $\phi^* \in \Phi$*, i.e., the Radon-Nikodym derivative process* $\varphi_t^{\phi^*}$ *is a uniformly integrable* $(\mathbb{F}, \mathbb{P})$*-martingale;*
(ii) $-\varphi_t^{\phi^*} \frac{e^{-q(X_t^* Y_{2,t} + G_{2,t})}}{q} \in \mathcal{S}^4_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$*;*
(iii) $\varphi_t^{\phi^*} \left(\int_0^t \frac{(\phi_{1,s}^*)^2}{2\eta_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\eta_{2,s}^*} ds \right) \in \mathcal{S}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$*, where* $\eta_{i,t}^* = \frac{\beta_i}{e^{-q(X_t^* Y_{2,t} + G_{2,t})}}$*, for* $i = 1, 2$*.*
In the affirmative, the control $(\pi^*, \phi^*) \in \Pi_e \otimes \Phi$ *is the optimal control of the robust ALM problem (9.2.13), and the optimal value function is given by*

$$J_e(\pi^*, \phi^*) = -\frac{e^{-q(x_0 Y_{2,0} + G_{2,0})}}{q}, \tag{9.4.18}$$

where the explicit expression for $G_{2,t}$ *is given by (9.4.7).*

Proof. See Appendix 9.I. □

Remark 9.4.6. To our knowledge, the results shown in Theorem 9.4.5 are not reported in the existing literature. If we further consider the special case without model ambiguity by setting $\beta_1 = \beta_2 = 0$, we obtain the explicit expressions for the optimal investment strategy and optimal value function of the ALM problem under exponential utility and square-root factor process. If we plug $l_0 = \mu_l = \sigma_l = 0$ into Theorem 9.4.5, the analytical solutions to the robust optimal portfolio selection problems under exponential utility are derived.

In the next three corollaries, we present the explicit expressions for the robust optimal controls and robust optimal value functions under the CEV model, 4/2 model, and non-Markovian path-dependent model given in Example 9.2.1-9.2.4, respectively.

Corollary 9.4.7 (CEV model)**.** *If the risky asset price* S_t *follows the CEV model (9.2.4) with any initial data* $(x_0, s_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$*, and suppose that the following conditions hold*

$$\max\left\{\tilde{b}_0, \tilde{b}_1\right\} \leq \frac{\mu^2}{\sigma^2}$$

with $\tilde{b}_0$ *and* $\tilde{b}_1$ *given by*

$$\begin{cases} \tilde{b}_0 = (120 + 32\sqrt{14})\dfrac{(\mu - r)^2\beta_1^2}{\sigma^2(q+\beta_1)^2}, \\ \tilde{b}_1 = 64(2+\sqrt{2})\dfrac{(\mu - r)^2 q^2}{\sigma^2(q+\beta_1)^2}, \end{cases}$$

then, the optimal control and optimal value function of the robust ALM problem (9.2.13) are, respectively, given by

$$(\pi_t^*, \phi_{1,t}^*, \phi_{2,t}^*) = \left(\frac{\frac{\mu - r}{\sigma(q+\beta_1)} + 2\beta\sigma\tilde{g}_2(t) - \sigma_l h_2(t) L_t}{X_t^* Y_{2,t}\sigma S_t^{2\beta}}, -\frac{\beta_1}{q+\beta_1}\frac{\mu - r}{\sigma S_t^{\beta}}, 0 \right),$$

and

$$J_e(\pi^*, \phi^*) = -\frac{e^{-q\left(x_0 Y_{2,0} + \tilde{f}_2(0) + \tilde{g}_2(0) S_0^{-\beta} + h_2(0) l_0\right)}}{q},$$

where $Y_{2,t}$ *and* $h_2(t)$ *are given by (9.4.4) and (9.4.14), respectively, and functions* $\tilde{f}_2(t)$ *and* $\tilde{g}_2(t)$ *are as follows:*

$$\tilde{f}_2(t) = -\frac{(\mu - r)^2\left(\beta + \frac{1}{2}\right)}{2(q+\beta_1)r}\left(\frac{1 - e^{2\beta r(t-T)}}{2\beta r} + t - T\right)$$

and

$$\tilde{g}_2(t) = \frac{(\mu - r)^2}{4\beta r(q+\beta_1)\sigma^2}\left(1 - e^{2\beta r(t-T)}\right).$$

Proof. Substituting the specified parameters of the CEV model (9.2.4) in Example 9.2.1 into Theorem 9.4.5 yields the above results. In addition, it is easy to see that Assumption 9.4.1 always holds for the CEV model since $2\beta r \neq 0$. □

Remark 9.4.8. If we ignore random liabilities by imposing $l_0 = \mu_l = \sigma_l = 0$, then the optimal value function and optimal control for the robust portfolio selection problem under the CEV model and exponential utility are provided, which are the identical to that presented in Theorem 3.4 in Zheng, Zhou, and Sun (2016) when no reinsurance is involved. If we further ignore model ambiguity by setting $\beta_1 = 0$, the optimal investment strategy recovers the results provided in Proposition 2 in Gao (2009) and Sun, Yong, and Gao (2020).

Corollary 9.4.9 (The family of 4/2 model)**.** *If the risky asset price process* S_t *and the variance driver process* V_t *follow the family of 4/2 models (9.2.5) with any initial data* $(x_0, v_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$, *and suppose that* $\kappa + \sigma_v\rho\lambda \neq 0$ *and the following conditions hold*

$$\max\left\{\bar{b}_0, \bar{b}_1, \bar{b}_2\right\} \le \frac{\kappa^2}{\sigma_v^2}$$

with $\bar{b}_0, \bar{b}_1$, and $\bar{b}_2$ given by

$$\begin{cases} \bar{b}_0 = \sup_{t\in[0,T]} (120 + 32\sqrt{14}) \left(\dfrac{\beta_1^2\lambda^2}{(q+\beta_1)^2} + \beta_2^2\sigma_v^2(1-\rho^2)\bar{g}_2^2(t) \right), \\ \bar{b}_1 = \sup_{t\in[0,T]} 64(2+\sqrt{2}) \left(\dfrac{\lambda^2}{(q+\beta_1)^2} + \sigma_v^2(1-\rho^2)\bar{g}_2^2(t) \right) q^2, \\ \bar{b}_2 = \sup_{t\in[0,T]} \dfrac{128q^2\lambda^2}{(q+\beta_1)^2} - \dfrac{16q\lambda^2}{q+\beta_1} + (112q^2 - 16q\beta_2)\sigma_v^2(1-\rho^2)\bar{g}_2^2(t), \end{cases}$$

then, the optimal control and optimal value function of the robust ALM problem (9.2.13) are, respectively, given by

$$\left(\pi_t^*, \phi_{1,t}^*, \phi_{2,t}^*\right) = \left(\frac{V_t\left(\frac{\lambda}{q+\beta_1} - \sigma_v\rho\bar{g}_2(t) - \sigma_l h_2(t) L_t\right)}{(c_1V_t + c_2)X_t^* Y_{2,t}}, -\frac{\beta_1\lambda}{q+\beta_1}\sqrt{V_t}, -\beta_2\sigma_v\sqrt{1-\rho^2}\bar{g}_2(t)\sqrt{V_t} \right),$$

and

$$J_e(\pi^*, \phi^*) = -\frac{e^{-q(x_0Y_{2,0} + \bar{f}_2(0) + \bar{g}_2(0)v_0 + h_2(0)l_0)}}{q},$$

where $Y_{2,t}$ and $h_2(t)$ are given by (9.4.4) and (9.4.14), respectively, and functions $\bar{f}_2(t)$ and $\bar{g}_2(t)$ are given as follows:

$$\bar{f}_2(t) = \begin{cases} -\dfrac{\lambda^2\kappa\theta}{2(q+\beta_1)(\kappa+\lambda\sigma_v\rho)} \left(\dfrac{1 - e^{(\kappa+\lambda\sigma_v\rho)(t-T)}}{\kappa+\lambda\sigma_v\rho} + t - T \right), & \text{if } \rho = \pm 1; \\ \kappa\theta n_{\bar{g}_2^-}(T-t) + \dfrac{\kappa\theta(n_{\bar{g}_2^-} - n_{\bar{g}_2^+})}{\sqrt{\Delta_{\bar{g}_2}}} \log\left(\dfrac{n_{\bar{g}_2^+} - n_{\bar{g}_2^-}}{n_{\bar{g}_2^+} - n_{\bar{g}_2^-} e^{\sqrt{\Delta_{\bar{g}_2}}(T-t)}} \right), & \text{if } \rho \neq \pm 1, \end{cases}$$

and

$$\bar{g}_2(t) = \begin{cases} -\dfrac{\lambda^2}{2(q+\beta_1)(\kappa+\lambda\sigma_v\rho)} \left(e^{(\kappa+\lambda\sigma_v\rho)(t-T)} - 1 \right), & \text{if } \rho = \pm 1; \\ \dfrac{n_{\bar{g}_2^+} n_{\bar{g}_2^-}\left(1 - e^{\sqrt{\Delta_{\bar{g}_2}}(T-t)}\right)}{n_{\bar{g}_2^+} - n_{\bar{g}_2^-} e^{\sqrt{\Delta_{\bar{g}_2}}(T-t)}}, & \text{if } \rho \neq \pm 1 \end{cases}$$

with $\Delta_{\bar{g}_2}, n_{\bar{g}_2^+}$, and $n_{\bar{g}_2^-}$ given by

$$\Delta_{\bar{g}_2} = (\kappa+\lambda\sigma_v\rho)^2 + \frac{q+\beta_2}{q+\beta_1}\sigma_v^2(1-\rho^2)\lambda^2, \; n_{\bar{g}_2^+} = \frac{-(\kappa+\lambda\sigma_v\rho) + \sqrt{\Delta_{\bar{g}_2}}}{(q+\beta_2)\sigma_v^2(1-\rho^2)}, \; n_{\bar{g}_2^-} = \frac{-(\kappa+\lambda\sigma_v\rho) - \sqrt{\Delta_{\bar{g}_2}}}{(q+\beta_2)\sigma_v^2(1-\rho^2)}.$$

Proof. Substituting the specified parameters of the 4/2 model (9.2.5) in Example 9.2.2 into Theorem 9.4.5 leads to the results immediately. □

Remark 9.4.10. By specifying $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$ in Corollary 9.4.9, we obtain the corresponding results for the embedded Heston model and 3/2 model, respectively. Moreover, it is straightforward to verify that the optimal investment strategy is in line with that in Corollary 3.22 in Zhang (2022c) when we impose $\beta_1 = \beta_2 = l_0 = \mu_l = \sigma_l = 0$. In other words, Corollary 9.4.9 extends the recent results of Zhang (2022c) to the case with random liabilities and model ambiguity.

Corollary 9.4.11 (Non-Markovian path-dependent model)**.** *If the risky asset price process S_t and its volatility driver process α_t follow the path-dependent stochastic volatility model (9.2.6) with any initial data $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$, and suppose that condition (9.4.16) holds true, then optimal control and optimal value function of the robust ALM problem (9.2.13) are, respectively, given by*

$$(\pi_t^*, \phi_{1,t}^*, \phi_{2,t}^*) = \left(\frac{\frac{\lambda}{q+\beta_1}\sqrt{\alpha_t} - \Gamma_{1,t}}{X_t^* \hat{\sigma}(\alpha_{[0,t]}) Y_{2,t}}, -\frac{\beta_1}{q+\beta_1}\lambda\sqrt{\alpha_t}, -\beta_2 \Gamma_{2,t} \right),$$

and

$$J_e(\pi^*, \phi^*) = -\frac{e^{-q(x_0 Y_{2,0} + G_{2,0})}}{q},$$

where the closed-form expressions for $Y_{2,t}, G_{2,t}, \Gamma_{1,t}$, and $\Gamma_{2,t}$ are given by (9.4.4), (9.4.7), and (9.4.8), respectively.

Proof. Replacing σ_t in Theorem 9.4.5 by the specification $\hat{\sigma}(\alpha_{[0,t]})$ of the path-dependent model (9.2.6) in Example 9.2.4 yields the above results immediately. □

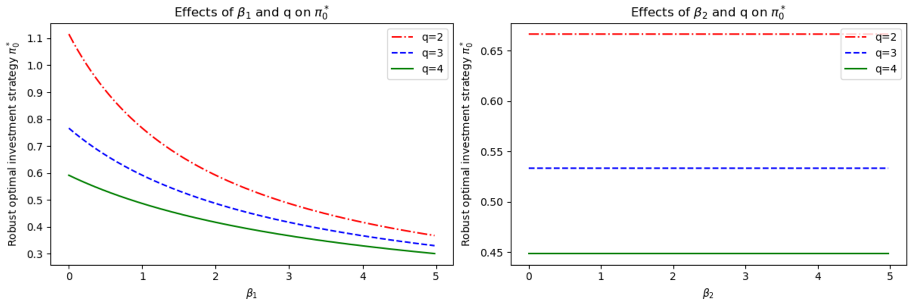
9.5 Numerical analysis

In this section, we devote ourselves to showing the effects of model parameters on the behavior of the robust optimal investment strategy by giving numerical examples. In the following numerical illustrations, we are mainly concerned about the exponential utility case under the CEV model (9.2.4) and 4/2 stochastic volatility model (9.2.5) since these two models are extensively studied in the literature in recent years and the power utility case can be conducted in a similar manner. Throughout this section, unless otherwise specified, the fundamental values of the model parameters are given as follows: $r = 0.02, \mu_l = 0.01, \sigma_l = 0.2, x_0 = 1, l_0 = 0.5, T = 0.1, \beta_1 = 1.5, \beta_2 = 1, q = 2$; in the 4/2 model, $\kappa = 7.3479, \theta = 0.0328, \sigma_v = 0.6612, \rho = -0.7689, \lambda = 2.9428, c_1 = 0.9051, c_2 = 0.023, v_0 = 0.04$, mainly referred to Cheng and Escobar (2021a); in the CEV model, $\mu = 0.05, \sigma = 0.25, \beta = -0.7, s_0 = 0.5$. For simplicity but without loss of generality, we focus on the analysis at time $t = 0$ and vary the value of one parameter with others fixed at each time. The range allowed for the parameters is the possibility that the conditions in Corollary 9.4.7 and 9.4.9 are respectively met.

9.5.1 Effects of parameters in the 4/2 model on the robust investment strategy

In this subsection, we are interested in the effects of some model parameters in the 4/2 stochastic volatility model (9.2.5) on the robust optimal investment strategy π^* given in Corollary 9.4.9.

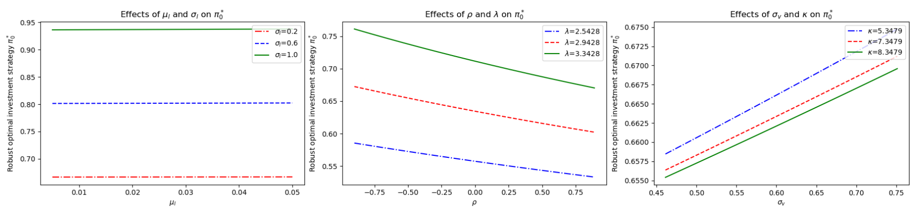
Figure 9.1 displays the effects of the ambiguity aversion parameters β_1 and β_2 and the risk aversion coefficient q on the robust optimal investment strategy π^*.



(a) **(b)**

Figure 9.1: *Effects of the ambiguity parameters β_1 and β_2 and the risk aversion coefficient q on the robust optimal investment strategy π^* under the 4/2 model (9.2.5)*

From Figure 9.1(a), we find that π^* decreases with respect to β_1 and q. Along with the growth of β_1, the asset-liability manager is more ambiguity averse about the risky asset dynamics. Hence, the manager is willing to reduce the investment proportion in the risky asset. With the increase of q, the manager becomes more risk averse and tends to accept a lower risk for the investment. So, less wealth will be invested in the risky asset. For a similar reason, as the ambiguity aversion parameter β_2 becomes larger, the manager is more ambiguity averse about the risky asset variance driver process. Therefore, the investment proportion in the risky asset is reduced, which is consistent with the results shown in Figure 9.1(b).



(a) **(b)** **(c)**

Figure 9.2: *Effects of parameters $\mu_l, \sigma_l, \lambda, \rho, \kappa,$ and σ_v on the robust optimal investment strategy π^* under the 4/2 model (9.2.5)*

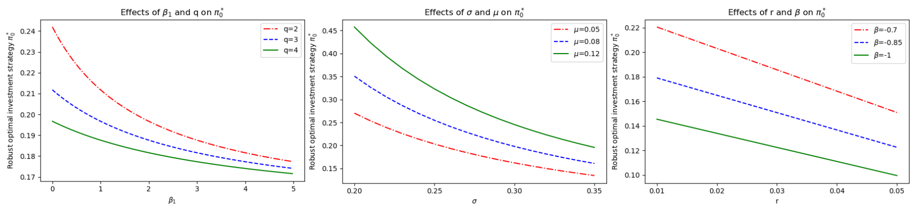
Figure 9.2 contributes to the evolution of the robust optimal investment strategy π^* with respect to the random liability parameters μ_l, σ_l, the slope of the market price of volatility risk λ, the correlation between the risky asset price and instantaneous variance driver ρ, and the mean-reversion rate and volatility of the variance driver process κ and σ_v. From Figure 9.2(a), we observe that π^* increases slightly with respect to μ_l. When μ_l is growing, the appreciation rate of the random liability becomes larger. In this case, the manager is willing to increase the investment proportion in the risky asset to obtain a higher terminal surplus. Figure 9.2(a) also indicates that the investment proportion in the risky asset increases along with the volatility scale factor σ_l. In fact, as σ_l increases, the volatility of the random liability caused by the risky asset becomes larger. As a result, the manager tends to invest more wealth into the risky asset as a hedging instrument to reduce the volatility risk of the random liability to an acceptable level. In addition, when ρ decreases from 0.9 to -0.9, the investment proportion in the risky asset increases as revealed by Figure 9.2(b). This is due to the fact that decreasing ρ leads the hedge demand $-\sigma_v \rho \bar{g}_2(t)$ ($\bar{g}_2(t) > 0$) to increase. Figure 9.2(b) also shows that the investment proportion in the risky asset increases with respect to λ. Since λ depicts the slope of the market price of volatility risk, a larger value of λ implies that the manager could obtain higher returns by investing in the risky asset. Finally, Figure 9.2(c) illustrates that π^* decreases along with κ but increases along with σ_v. Indeed, since κ stands for the mean-reversion rate of the variance driver process, the variance driver process moves faster towards the constant long-run level θ as κ increases. Along with the growth of κ, the volatility risk of the risky asset becomes smaller, and hence the investment proportion in the risky asset is reduced. Conversely, as the volatility coefficient σ_v increases, the instantaneous variance of the risky asset price fluctuates more dramatically and the manager faces a higher volatility risk. As a result, a larger investment proportion in the risky asset is necessary to hedge against the volatility risk.

9.5.2 Effects of parameters in the CEV model on the robust investment strategy

This subsection focuses on the effects of some model parameters in the CEV model (9.2.4) on the robust optimal investment strategy π^* given in Corollary 9.4.7. More specifically, Figure 9.3 describes how the robust optimal investment strategy π^* evolves with respect to $\beta_1, q, \sigma, \mu, r$, and β.

Similar to the results shown in Figure 9.1(a) under the 4/2 model, we can observe from Figure 9.3(a) that under the CEV model, the robust optimal investment strategy π^* has negative relationships with both the ambiguity aversion parameter β_1 and risk aversion coefficient q. In other words, the manager tends to put less wealth into the risky asset when he/she becomes either more ambiguity-averse or more risk-averse. Figure 9.3(b) shows that the robust optimal investment strategy π^* is positively correlated with the parameter μ under the CEV model. Along with the growth of μ, the manager can earn a higher risk premium from the risky asset as μ stands for the expected return rate of the risky asset. In this case, the manager is willing to increase the proportion of wealth invested in the risky asset to derive a higher terminal surplus. From Figure 9.3(b), we also find that π^* decreases with respect to σ, which can be interpreted by the fact that σ characterizes the risky asset's local volatility, and when σ becomes larger, the risky asset displays greater local volatility. Therefore, the manager has the motivation to decrease the amount of wealth invested in the risky asset to avoid amplified volatility risk. Finally, from

Figure 9.3(c), we observe that the robust optimal investment strategy π^* increases when the elasticity parameter β increases from -1 to -0.7. This can be explained by the economic implication of β; the negativeness of β indicates the existence of the leverage effect, and when β becomes less negative, the volatility risk turns out to be less significant, and thus, the manager would increase the investment in the risky asset. Figure 9.3(c) also reveals that the optimal proportion of wealth invested into the risky asset π^* has a negative relationship with the risk-free interest rate r. Varying r from 0.01 to 0.05, the expected rate return of the risk-free asset becomes higher. Hence, the manager would invest more in the risk-free asset and less in the risky asset to reduce the overall risk.



(a) **(b)** **(c)**

Figure 9.3: *Effects of parameters $\beta_1, q, \sigma, \mu, r$ and β on the robust optimal investment strategy π^* under the CEV model (9.2.4)*

9.6 Conclusion

In this paper, we investigate robust ALM problems for a manager with both risk and ambiguity aversion in the presence of stochastic volatility. The manager is subject to random liabilities and has access to a financial market consisting of one risk-free asset and one risky asset, where the market price of risk follows an affine-form, square-root, Markovian model, while the return rate and volatility are possibly non-Markovian, unbounded stochastic processes. The modeling framework embraces the CEV model, the family of state-of-the-art 4/2 stochastic volatility models, and some non-Markovian path-dependent models, as particular cases. The manager is allowed to have different levels of ambiguity about the risky asset price and volatility and aims to seek a robust optimal investment strategy against the worst-case measure among the class of alternative measures equivalent to the reference measure. In the non-Markovian case, the dynamic programming principle along with the HJBI equation approach no longer works, and thus, a novel BSDE approach is proposed. To find the associated BSDEs, we propose to construct a stochastic process depending on any admissible control, and such that its value at time zero does not rely on any admissible control and its terminal value equals the utility of the terminal surplus penalized by model ambiguity. By solving the BSDEs explicitly, we derive, in closed form, the robust optimal controls and robust optimal value functions for the power and exponential utility functions, respectively.

Moreover, analytical solutions to some special cases of our model are obtained. Finally, the economic impacts of model ambiguity and model parameters on the robust optimal investment strategies are analyzed with numerical examples, from which we find that (1) the levels of ambiguity aversion about the risky asset's price and volatility both reduce the robust optimal investment proportion in the risky asset; (2) the robust optimal investment strategy is more sensitive to the level of ambiguity about the risky asset dynamics than to that about its volatility. As far as we know, this paper is the first to address the ALM problems in the presence of model ambiguity as well as stochastic volatility, and more importantly, there is no existing literature using the above BSDE approach to study robust decision problems in the non-Markovian setting. So, this study is meaningful from both theoretical and practical perspectives.

Built on the current study, some promising directions for future research might be followed. For instance, (1) this paper investigates the robust ALM problems within the expected utility maximization framework. One may consider other non-utility criteria, such as the mean-variance criterion. (2) In addition to model ambiguity, the manager may also face partial information. (3) It may also be of interest to apply the proposed BSDE approach to address robust pension investment or investment-consumption problems in non-Markovian cases.

Acknowledgement(s)

The authors are very grateful to Prof. Jesper Lund Pedersen, the editor, and two anonymous reviewers for their constructive comments and suggestions, which greatly improve the quality of this paper.

9.A Proof of Proposition 9.3.3

Proof. Conjecture that the first component $Y_{1,t}$ of the solution to BSRE (9.3.2) has the following exponential-affine form:

$$Y_{1,t} = \exp\left\{f_1(t) + g_1(t)\alpha_t\right\},$$

where $f_1(t)$ and $g_1(t)$ are two differentiable functions which shall be determined later with terminal conditions $f_1(T) = g_1(T) = 0$. An application of Itô's formula to $Y_{1,t}$ then leads to

$$\begin{aligned} dY_{1,t} =& Y_{1,t}\left[\frac{df_1(t)}{dt} + \frac{dg_1(t)}{dt}\alpha_t + \kappa(\theta - \alpha_t)g_1(t) + \frac{1}{2}\left(\rho_1^2 + \rho_2^2\right)g_1^2(t)\alpha_t\right]dt \\ &+ \rho_1 g_1(t)\sqrt{\alpha_t}Y_{1,t}\,dW_{1,t} + \rho_2 g_1(t)\sqrt{\alpha_t}Y_{1,t}\,dW_{2,t}. \end{aligned} \tag{9.A.1}$$

Match the diffusive coefficients in (9.A.1) with BSRE (9.3.2) by letting $Z_{1,t} = \rho_1 g_1(t)\sqrt{\alpha_t}Y_{1,t}$ and $Z_{2,t} = \rho_2 g_1(t)\sqrt{\alpha_t}Y_{1,t}$. We can rewrite the generator of BSRE

(9.3.2) as follows:

$$Y_{1,t}\left[\left(\frac{\gamma\lambda^2}{2(\gamma-1-\beta_1)}+\frac{(\gamma-\beta_1)\lambda\rho_1 g_1(t)}{\gamma-1-\beta_1}+\frac{\rho_1^2 g_1^2(t)}{2\gamma}\left(\beta_1+\frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}\right)+\frac{\beta_2\rho_2^2 g_1^2(t)}{2\gamma}\right)\alpha_t-r\gamma\right]. \tag{9.A.2}$$

Comparing (9.A.2) with the drift coefficient of (9.A.1) leads to the ODEs (9.3.8) and (9.3.9) governing $g_1(t)$ and $f_1(t)$, respectively.

Denote by $\Delta_{g_1}=\left(\kappa+\frac{(\gamma-\beta_1)\lambda\rho_1}{\gamma-1-\beta_1}\right)^2-\frac{\lambda^2}{\gamma-1-\beta_1}\left(\frac{\gamma-\beta_1}{(\gamma-1-\beta_1)}\rho_1^2-(\gamma-\beta_2)\rho_2^2\right)>0$. We can reformulate the Riccati ODE (9.3.8) as follows:

$$\frac{dg_1(t)}{dt}=\left(\frac{\gamma-\beta_1}{2\gamma(\gamma-1-\beta_1)}\rho_1^2-\frac{\gamma-\beta_2}{2\gamma}\rho_2^2\right)\left(g_1(t)-n_{g_1^+}\right)\left(g_1(t)-n_{g_1^-}\right),$$

where $n_{g_1^+}$ and $n_{g_1^-}$ are given in (9.3.12). By using the separation method and some simple calculations, the closed-form expression of $g_1(t)$ is given in (9.3.10). Finally, noticing the boundary condition that $f_1(T)=0$ and substituting $g_1(t)$ back into ODE (9.3.9), we have the close-form expression of $f_1(t)$ given in (9.3.11). □

9.B Proof of Proposition 9.3.5

Proof. The uniform boundedness of the process $Y_{1,t}$ follows immediately from the negativeness of function $g_1(t)$ and the positiveness of the square-root factor process α_t, for $t\in[0,T]$. More precisely, we have $\mathbb{P}$ almost surely,

$$Y_{1,t}=\exp\{f_1(t)+g_1(t)\alpha_t\}\le\exp\left\{\int_t^T\kappa\theta g_1(s)\,ds+r\gamma(T-t)\right\}\le\exp\{r\gamma(T-t)\}<+\infty.$$

As a result, we find from (9.3.7) that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Z_{i,t}^2\,dt\right]\le C_i\int_0^T\mathbb{E}^{\mathbb{P}}[\alpha_t]\,dt=C_i\int_0^T\left(\alpha_0e^{-\kappa t}+\kappa\theta\int_0^t e^{-\kappa(t-s)}\,ds\right)dt<+\infty,$$

where $C_i=\rho_i^2g_1^2(0)e^{2|r\gamma|T}$, for $i=1,2$. This completes the proof. □

9.C Proof of Theorem 9.3.7

Proof. First of all, it follows from Lemma 9.3.6 that the probability measure $\tilde{\mathbb{P}}$ defined by

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T}=\exp\left\{-\int_0^T\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\lambda\sqrt{\alpha_t}\,dW_{1,t}-\frac{1}{2}\int_0^t\left(\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\right)^2\lambda^2\alpha_t\,dt\right\}$$

is equivalent to the reference measure $\mathbb{P}$. By Girsanov's theorem, the following processes $\tilde{W}_{1,t}$ and $\tilde{W}_{2,t}$

$$\tilde{W}_{1,t}=\int_0^t\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\lambda\sqrt{\alpha_s}\,ds+W_{1,t}\text{ and }\tilde{W}_{2,t}=W_{2,t}$$

are two standard Brownian motions under $\tilde{\mathbb{P}}$. In addition, the $\tilde{\mathbb{P}}$-dynamics of the factor process α_t is given by

$$d\alpha_t = \left(\kappa + \rho_1\lambda\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\right)\left(\frac{\kappa\theta}{\kappa+\rho_1\lambda\frac{\gamma-\beta_1}{\gamma-1-\beta_1}} - \alpha_t\right)dt + \sqrt{\alpha_t}\left(\rho_1\, d\tilde{W}_{1,t} + \rho_2\, d\tilde{W}_{2,t}\right),$$

which preserves the affine-form, square-root structure under Assumption 9.3.2. Moreover, the BSRE (9.3.2) can be rewritten under $\tilde{\mathbb{P}}$ measure as follows:

$$\begin{cases} dY_{1,t} = \left[\left(-r\gamma + \frac{\gamma}{2(\gamma-1-\beta_1)}\lambda^2\alpha_t\right)Y_{1,t} + \frac{1}{2\gamma}\left(\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}\right)\frac{Z_{1,t}^2}{Y_{1,t}} + \frac{\beta_2}{2\gamma}\frac{Z_{2,t}^2}{Y_{1,t}}\right]dt \\ \qquad\qquad + Z_{1,t}\, d\tilde{W}_{1,t} + Z_{2,t}\, d\tilde{W}_{2,t}, \\ Y_{1,T} = 1, \\ Y_{1,t} > 0, \ \text{for all } t\in[0,T). \end{cases} \tag{9.C.1}$$

Denote by $(\tilde{Y}_{1,t}, \tilde{Z}_{1,t}, \tilde{Z}_{2,t})$ another solution to BSRE (9.3.2), which is different from the proposed one given in Proposition 9.3.3. Define the following difference process:

$$(\Delta\log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t}) = \left(\log(Y_{1,t}) - \log(\tilde{Y}_{1,t}), \frac{Z_{1,t}}{Y_{1,t}} - \frac{\tilde{Z}_{1,t}}{\tilde{Y}_{1,t}}, \frac{Z_{2,t}}{Y_{1,t}} - \frac{\tilde{Z}_{2,t}}{\tilde{Y}_{1,t}}\right).$$

Apply Itô's formula to $\Delta\log(Y_{1,t})$, we have the BSDE of $(\Delta\log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t})$:

$$\begin{cases} d\Delta\log(Y_{1,t}) = \frac{1}{2}\left[\left(\frac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1\right)\left(\frac{Z_{1,t}^2}{Y_{1,t}^2} - \frac{\tilde{Z}_{1,t}^2}{\tilde{Y}_{1,t}^2}\right) + \left(\frac{\beta_2}{\gamma} - 1\right)\left(\frac{Z_{2,t}^2}{Y_{1,t}^2} - \frac{\tilde{Z}_{2,t}^2}{\tilde{Y}_{1,t}^2}\right)\right]dt \\ \qquad\qquad + \Delta Z_{1,t}\, d\tilde{W}_{1,t} + \Delta Z_{2,t}\, dW_{2,t}, \\ \Delta\log(Y_{1,T}) = 0. \end{cases} \tag{9.C.2}$$

Furthermore, since α_t is still an affine-form, square-root factor process under $\tilde{\mathbb{P}}$ measure, we can define the following equivalent probability measure $\hat{\mathbb{P}}$ on $\mathcal{F}_T$ as a result of Lemma 9.3.6 and the explicit expression of the proposed solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in Proposition 9.3.3:

$$\begin{aligned} \left.\frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}}\right|_{\mathbb{F}_T} &= \exp\Bigg\{-\int_0^T \left(\frac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1\right)\rho_1 g_1(t)\sqrt{\alpha_t}\, d\tilde{W}_{1,t} - \int_0^T \left(\frac{\beta_2}{\gamma} - 1\right)\rho_2 g_1(t)\sqrt{\alpha_t}\, d\tilde{W}_{2,t} \\ &\quad -\frac{1}{2}\int_0^T \left[\left(\frac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1\right)^2\rho_1^2 + \left(\frac{\beta_2}{\gamma} - 1\right)^2\rho_2^2\right] g_1^2(t)\alpha_t\, dt\Bigg\} \\ &= \exp\Bigg\{-\int_0^T \left(\frac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1\right)\frac{Z_{1,t}}{Y_{1,t}}\, d\tilde{W}_{1,t} - \int_0^T \left(\frac{\beta_2}{\gamma} - 1\right)\frac{Z_{2,t}}{Y_{1,t}}\, d\tilde{W}_{2,t} \\ &\quad -\frac{1}{2}\int_0^T \left[\left(\frac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1\right)^2\frac{Z_{1,t}^2}{Y_{1,t}^2} + \left(\frac{\beta_2}{\gamma} - 1\right)^2\frac{Z_{2,t}^2}{Y_{1,t}^2}\right] dt\Bigg\}. \end{aligned}$$

So, by Girsanov's theorem, the following processes $\hat{W}_{1,t}$ and $\hat{W}_{2,t}$:

$$\hat{W}_{1,t} = \int_0^t \left(\frac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1 \right) \frac{Z_{1,s}}{Y_{1,s}} \, ds + \tilde{W}_{1,t} \text{ and } \hat{W}_{2,t} = \int_0^t \left(\frac{\beta_2}{\gamma} - 1 \right) \frac{Z_{2,s}}{Y_{1,s}} \, ds + \tilde{W}_{2,t} \tag{9.C.3}$$

are two standard Brownian motions under $\hat{\mathbb{P}}$ measure. Therefore, it follows from (9.C.2) and (9.C.3) that $(\Delta \log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t})$ solves the following quadratic BSDE under $\hat{\mathbb{P}}$ measure:

$$\begin{cases} d\Delta \log(Y_{1,t}) = -\dfrac{1}{2} \left[\left(\dfrac{\beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma-1-\beta_1}}{\gamma} - 1 \right) \Delta Z_{1,t}^2 + \left(\dfrac{\beta_2}{\gamma} - 1 \right) \Delta Z_{2,t}^2 \right] dt \\ \qquad\qquad + \Delta Z_{1,t} \, d\hat{W}_{1,t} + \Delta Z_{2,t} \, d\hat{W}_{2,t}, \\ \Delta \log(Y_{1,T}) = 0. \end{cases}$$

This quadratic BSDE clearly satisfies all the regularity conditions in Kobylanski (2000). Thus, by Theorem 2.3 and 2.6 in Kobylanski (2000), we can conclude that the above quadratic BSDE admits a unique solution which is $(\Delta \log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t}) = (0, 0, 0)$. In other words, the proposed solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in (9.3.6) and (9.3.7) must be the unique solution to BSRE (9.3.2) in the space $\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}^+) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$. This completes the proof. □

9.D Proof of Proposition 9.3.8

Proof. By Lemma 9.3.6, the following Radon-Nikodym derivative

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{ -\int_0^T (\lambda - \sigma_l)\sqrt{\alpha_t} \, dW_{1,t} + \int_0^T \rho_2 g_1(t)\sqrt{\alpha_t} \, dW_{2,t} - \frac{1}{2} \int_0^T \left((\lambda - \sigma_l)^2 + \rho_2^2 g_1^2(t) \right) \alpha_t \, dt \right\}$$

is well-defined, and thus, the probability measure $\bar{\mathbb{P}}$ is also well-defined and equivalent to $\mathbb{P}$. By Girsanov's theorem, the following two processes

$$\bar{W}_{1,t} = \int_0^t (\lambda - \sigma_l)\sqrt{\alpha_s} \, ds + W_{1,t} \text{ and } \bar{W}_{2,t} = -\int_0^t \rho_2 g_1(s)\sqrt{\alpha_s} \, ds + W_{2,t}$$

are two standard Brownian motions under $\bar{\mathbb{P}}$. So, linear BSDE (9.3.13) can be reformulated under $\bar{\mathbb{P}}$ measure as follows:

$$\begin{cases} d\bar{G}_{1,t} = (r - \mu_l)\bar{G}_{1,t} \, dt + \bar{\Lambda}_{1,t} \, d\bar{W}_{1,t} + \bar{\Lambda}_{2,t} \, d\bar{W}_{2,t}, \\ \bar{G}_{1,T} = -1, \end{cases} \tag{9.D.1}$$

which is linear BSDE with standard data (refer to El Karoui, Peng, and Quenez (1997)) and has deterministic coefficients in the generator. Then by Theorem 2.1 and Proposition 2.2 in El Karoui, Peng, and Quenez (1997), we notice that (9.3.14) and (9.3.15) form the unique solution to (9.D.1) and to (9.3.13) as well. This completes the proof. □

9.E Proof of Theorem 9.3.10

Proof. In the first place, we show that $\phi^* = \left(\left\{\phi^*_{1,t}\right\}_{t\in[0,T]}, \left\{\phi^*_{2,t}\right\}_{t\in[0,T]}\right)$ given in (9.3.17) lies in Φ, i.e.,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T \left((\phi^*_{1,t})^2 + (\phi^*_{2,t})^2\right) dt\right\}\right] < +\infty.$$

Indeed, recalling from Remark 9.3.4 that $g_1(t)$ is bounded by $[g_1(0), 0]$ and using the Laplace transform of an integrated square-root diffusion process (see, for example, Theorem 5.1 in Zeng and Taksar (2013)) and condition (9.3.16), we have

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T \left((\phi^*_{1,t})^2 + (\phi^*_{2,t})^2\right) dt\right\}\right] \leq \mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{k_0}{2}\int_0^T \alpha_t\, dt\right\}\right] < +\infty,$$

where the constant k_0 is given in (9.3.16). Substituting π^*_t into X^*_t and applying Itô' formula to $X^*_t + \bar{G}_{1,t}L_t$ under the reference measure $\mathbb{P}$, we find that

$$\frac{d\left(X^*_t + \bar{G}_{1,t}L_t\right)}{X^*_t + \bar{G}_{1,t}L_t} = \left[r + \frac{\frac{\gamma-\beta_1}{\gamma}\lambda\rho_1 g_1(t) + \lambda^2}{\beta_1 + 1 - \gamma}\alpha_t\right] dt + \frac{\frac{\gamma-\beta_1}{\gamma}\rho_1 g_1(t) + \lambda}{\beta_1 + 1 - \gamma}\sqrt{\alpha_t}\, dW_{1,t}. \tag{9.E.1}$$

Solving the linear SDE (9.E.1) explicitly, we have

$$\begin{aligned} X^*_t + \bar{G}_{1,t}L_t = &\exp\left\{\int_0^t \left[r + \left(\frac{\frac{\gamma-\beta_1}{\gamma}\lambda\rho_1 g_1(s) + \lambda^2}{\beta_1 + 1 - \gamma} - \frac{(\frac{\gamma-\beta_1}{\gamma}\rho_1 g_1(s) + \lambda)^2}{2(\beta_1 + 1 - \gamma)^2}\right)\alpha_s\right] ds\right. \\ &\left. + \int_0^t \frac{\frac{\gamma-\beta_1}{\gamma}\rho_1 g_1(s) + \lambda}{\beta_1 + 1 - \gamma}\sqrt{\alpha_s}\, dW_{1,s}\right\} \times (x_0 + \bar{G}_{1,0}l_0) > 0, \end{aligned} \tag{9.E.2}$$

for the initial data (x_0, α_0, l_0) such that $x_0 + \bar{G}_{1,0}l_0 \in \mathbb{R}^+$. Then, it follows from (9.E.2) and Proposition 9.3.5 that

$$\begin{aligned} &\left|Y_{1,t}\frac{(X^*_t + \bar{G}_{1,t}L_t)^\gamma}{\gamma}\right|^8 \\ \leq &\underbrace{c\exp\left\{8\gamma\int_0^t \frac{\frac{\gamma-\beta_1}{\gamma}\rho_1 g_1(s) + \lambda}{\beta_1 + 1 - \gamma}\sqrt{\alpha_s}\, dW_{1,s} - 32\gamma^2\int_0^t \frac{(\frac{\gamma-\beta_1}{\gamma}\rho_1 g_1(s) + \lambda)^2}{(\beta_1 + 1 - \gamma)^2}\alpha_s\, ds\right\}}_{K_{1,t}} \\ &\times \underbrace{\exp\left\{\int_0^t \left(\frac{8(\gamma-\beta_1)\lambda\rho_1 g_1(s) + 8\lambda^2\gamma}{\beta_1 + 1 - \gamma} + (32\gamma^2 - 4\gamma)\frac{(\frac{\gamma-\beta_1}{\gamma}\rho_1 g_1(s) + \lambda)^2}{(\beta_1 + 1 - \gamma)^2}\right)\alpha_s\, ds\right\}}_{K_{2,t}}, \end{aligned} \tag{9.E.3}$$

where c is a positive constant. We observe that $K_{1,t}$ is the stochastic exponential process of continuous $(\mathbb{F}, \mathbb{P})$-local martingale $\int_0^t \frac{8(\gamma-\beta_1)\rho_1 g_1(s) + 8\gamma\lambda}{\beta_1 + 1 - \gamma}\sqrt{\alpha_s}\, dW_{1,s}$, and thus, it follows from Theorem 15.4.6 in Cohen and Elliott (2015), Theorem 5.1 in

Zeng and Taksar (2013) and condition (9.3.16) that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} K_{1,t}^2\right] &\leq 2\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{2+\sqrt{2}}{2}\int_0^T \frac{(8(\gamma-\beta_1)\rho_1 g_1(t)+8\gamma\lambda)^2}{(\beta_1+1-\gamma)^2}\alpha_t\,dt\right\}\right]\right\}^{\frac{\sqrt{2+\sqrt{2}}-1}{2+\sqrt{2}}} \\ &\leq 2\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{k_1}{2}\int_0^T \alpha_t\,dt\right\}\right]\right\}^{\frac{\sqrt{2+\sqrt{2}}-1}{2+\sqrt{2}}} < +\infty,\end{aligned} \tag{9.E.4}$$

where the constant k_1 is give in (9.3.16). Similarly, by using Theorem 5.1 in Zeng and Taksar (2013) and condition (9.3.16), we have

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} K_{2,t}^2\right] \leq \mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{k_2}{2}\int_0^T \alpha_t\,dt\right\}\right] < +\infty, \tag{9.E.5}$$

where the constant k_2 is given by (9.3.16). Then, it follows from Hölder's inequality and (9.E.3)-(9.E.5) that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|Y_{1,t}\frac{(X_t^*+\bar{G}_{1,t}L_t)^\gamma}{\gamma}\right|^8\right] < +\infty. \tag{9.E.6}$$

Hence, as a result of Theorem 15.4.6 in Cohen and Elliott (2015), Theorem 5.1 in Zeng and Taksar (2013), condition (9.3.16), the explicit expressions for $\phi_{1,t}^*$ and $\phi_{2,t}^*$ given in (9.3.17) and Hölder's inequality, we obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|\varphi_t^{\phi^*}Y_{1,t}\frac{(X_t^*+\bar{G}_{1,t}L_t)^\gamma}{\gamma}\right|^4\right] \\ &\leq \left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|\varphi_t^{\phi^*}|^8\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|Y_{1,t}\frac{(X_t^*+\bar{G}_{1,t}L_t)^\gamma}{\gamma}\right|^8\right]\right\}^{\frac{1}{2}} \\ &\leq \sqrt{\frac{8}{7}}\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{k_0}{2}\int_0^T\alpha_t\,dt\right\}\right]\right\}^{\frac{\sqrt{120+32\sqrt{14}}-1}{240+64\sqrt{14}}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|Y_{1,t}\frac{(X_t^*+\bar{G}_{1,t}L_t)^\gamma}{\gamma}\right|^8\right]\right\}^{\frac{1}{2}} < +\infty,\end{aligned} \tag{9.E.7}$$

More importantly, we find that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|\varphi_t^{\phi^*}\left(\int_0^t \frac{(\phi_{1,s}^*)^2}{2\psi_{1,s}^*}+\frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*}\,ds\right)\right|^2\right] \\ &\leq \left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|\varphi_t^{\phi^*}|^4\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}^{\mathbb{P}}\left[\left|\int_0^T \frac{(\phi_{1,t}^*)^2}{2\psi_{1,t}^*}+\frac{(\phi_{2,t}^*)^2}{2\psi_{2,t}^*}\,dt\right|^4\right]\right\}^{\frac{1}{2}} \\ &\leq c\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|\varphi_t^{\phi^*}|^4\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|Y_{1,t}(X_t^*+\bar{G}_{1,t}L_t)|^8\right]\right\}^{\frac{1}{4}}\left\{\mathbb{E}^{\mathbb{P}}\left[\left|\int_0^T\alpha_t\,dt\right|^8\right]\right\}^{\frac{1}{4}} \\ &\leq c\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|\varphi_t^{\phi^*}|^8\right]\right\}^{\frac{1}{4}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|Y_{1,t}(X_t^*+\bar{G}_{1,t}L_t)|^8\right]\right\}^{\frac{1}{4}}\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\int_0^T\alpha_t\,dt\right\}\right]\right\}^{\frac{1}{4}} < +\infty,\end{aligned} \tag{9.E.8}$$

where c is a positive constant which differs between lines, the first inequality follows from Hölder's inequality and the positiveness of $\psi_{1,t}^*$ and $\psi_{2,t}^*$ due to $X_t^*+\bar{G}_{1,t}L_t > 0$ as shown in (9.E.2), the second inequality makes use of the explicit expressions of $\phi_{1,t}^*$ and $\phi_{2,t}^*$ given in (9.3.17), the third inequality follows from the simple algebraic result that $x^8 \leq ae^x$, $x \in \mathbb{R}^+$ for some constant $a \in \mathbb{R}^+$, and the last strict inequality is due to (9.E.6) and (9.E.7), Theorem 5.1 in Zeng and Taksar (2013), and condition (9.3.16).

Based on (9.E.7) and (9.E.8), for any sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow +\infty$ as $n \to +\infty$, we know that

$$\sup_{\tau_n\wedge T} \mathbb{E}^{\mathbb{P}}\left[\left|\varphi^{\phi^*}_{\tau_n\wedge T} Y_{1,\tau_n\wedge T}\frac{(X^*_{\tau_n\wedge T}+\bar{G}_{1,\tau_n\wedge T}L_{\tau_n\wedge T})^\gamma}{\gamma}\right|^4\right] < +\infty,$$

and

$$\sup_{\tau_n\wedge T} \mathbb{E}^{\mathbb{P}}\left[\left|\varphi^{\phi^*}_{\tau_n\wedge T}\left(\int_0^{\tau_n\wedge T}\frac{(\phi^*_{1,s})^2}{2\psi^*_{1,s}}+\frac{(\phi^*_{2,s})^2}{2\psi^*_{2,s}}\,ds\right)\right|^2\right] < +\infty.$$

From the above results, we know that $\left\{\varphi^{\phi^*}_{\tau_n\wedge T} Y_{1,\tau_n\wedge T}\frac{(X^*_{\tau_n\wedge T}+\bar{G}_{1,\tau_n\wedge T}L_{\tau_n\wedge T})^\gamma}{\gamma}\right\}_{n\in\mathbb{N}}$ and $\left\{\varphi^{\phi^*}_{\tau_n\wedge T}\left(\int_0^{\tau_n\wedge T}\frac{(\phi^*_{1,s})^2}{2\psi^*_{1,s}}+\frac{(\phi^*_{2,s})^2}{2\psi^*_{2,s}}\,ds\right)\right\}_{n\in\mathbb{N}}$ are two uniformly integrable families under the reference measure $\mathbb{P}$ since both functions $t_1(x)=x^4$ and $t_2(x)=x^2$ are test functions of uniform integrability (see, for example, Proposition 11.7 in Zitkovic (2010)). Thus, we can conclude from the above results that the control $(\pi^*,\phi^*)\in\Pi_p\otimes\Phi$.

We next show that the control $(\pi^*,\phi^*)\in\Pi_p\otimes\Phi$ is the optimal control of the robust ALM problem (9.2.11). In fact, plugging $(\pi^*_t,\phi^*_{1,t},\phi^*_{2,t})$ into (9.3.1) leads to

$$\begin{aligned}
&d\varphi^{\phi^*}_t\left(Y_{1,t}\frac{(X^*_t+G_{1,t})^\gamma}{\gamma}+\int_0^t\frac{(\phi^*_{1,s})^2}{2\psi^*_{1,s}}+\frac{(\phi^*_{2,s})^2}{2\psi^*_{2,s}}\,ds\right)\\
&=\varphi^{\phi^*}_t\left[(Y_{1,t}\phi^*_{1,t}+Z_{1,t})\frac{(X^*_t+G_{1,t})^\gamma}{\gamma}+Y_{1,t}(X^*_t+G_{1,t})^{\gamma-1}(X^*_t\pi^*_t\sigma_t+\Lambda_{1,t})+\left(\int_0^t\frac{(\phi^*_{1,s})^2}{2\psi^*_{1,s}}\right.\right.\\
&\left.\left.+\frac{(\phi^*_{2,s})^2}{2\psi^*_{2,s}}\,ds\right)\phi^*_{1,t}\right]dW_{1,t}+\varphi^{\phi^*}_t\left[(Y_{1,t}\phi^*_{2,t}+Z_{2,t})\frac{(X^*_t+G_{1,t})^\gamma}{\gamma}+Y_{1,t}(X^*_t+G_{1,t})^{\gamma-1}\Lambda_{2,t}\right.\\
&\left.+\left(\int_0^t\frac{(\phi^*_{1,s})^2}{2\psi^*_{1,s}}+\frac{(\phi^*_{2,s})^2}{2\psi^*_{2,s}}\,ds\right)\phi^*_{2,t}\right]dW_{2,t}.
\end{aligned}\tag{9.E.9}$$

Due to the path-wise continuity of the stochastic integrals on the right-hand side of (9.E.9), we see that they are $(\mathbb{F},\mathbb{P})$-local martingales. Therefore, there exists a localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n\uparrow+\infty$ as $n\to+\infty$ and when stopped by such a sequence, the aforementioned local martingales are true $(\mathbb{F},\mathbb{P})$-martingales. Then, integrating both sides of (9.E.9) from 0 to $\tau_n\wedge T$ and taking expectations, we have

$$\mathbb{E}^{\mathbb{P}}\left[\varphi^{\phi^*}_{\tau_n\wedge T}\left(Y_{1,\tau_n\wedge T}\frac{(X^*_{\tau_n\wedge T}+G_{1,\tau_n\wedge T})^\gamma}{\gamma}+\int_0^{\tau_n\wedge T}\frac{(\phi^*_{1,s})^2}{2\psi^*_{1,s}}+\frac{(\phi^*_{2,s})^2}{2\psi^*_{2,s}}\,ds\right)\right]=Y_{1,0}\frac{(x_0+\bar{G}_{1,0}l_0)^\gamma}{\gamma}.\tag{9.E.10}$$

As we have shown that the term in the expectation on the left-hand side of (9.E.10) is uniformly integrable, by using the equivalence between $\mathcal{L}^1$ convergence and uniformly integrability and sending $n \to +\infty$, we have from (9.E.10)

$$\begin{aligned} J_p(\pi^*, \phi^*) =& \mathbb{E}^{\mathbb{Q}^*}\left[\frac{(X_T^* - L_T)^\gamma}{\gamma} + \int_0^T \frac{(\phi_{1,t}^*)^2}{2\psi_{1,t}^*} + \frac{(\phi_{2,t}^*)^2}{2\psi_{2,t}^*}\, dt\right] \\ =& \mathbb{E}^{\mathbb{P}}\left[\varphi_T^{\phi^*}\left(\frac{(X_T^* - L_T)^\gamma}{\gamma} + \int_0^T \frac{(\phi_{1,t}^*)^2}{2\psi_{1,t}^*} + \frac{(\phi_{2,t}^*)^2}{2\psi_{2,t}^*}\, dt\right)\right] = Y_{1,0}\frac{(x_0 + \bar{G}_{1,0} l_0)^\gamma}{\gamma}, \end{aligned} \tag{9.E.11}$$

where $\mathbb{Q}^*$ stands for the probability measure corresponding to the Radon-Nikodym derivative $\varphi_T^{\phi^*}$. In addition, on one hand, for the admissible strategy $(\pi, \hat{\phi}) \in \Pi_p \otimes \Phi$, by using some similar localization techniques, it follows from the first part of condition 3 in Definition 9.2.5 and (9.3.1) that

$$\begin{aligned} J_p(\pi, \hat{\phi}) =& \frac{(\gamma - 1 - \beta_1)}{2}\mathbb{E}^{\mathbb{P}}\Big[\int_0^T \varphi_t^{\hat{\phi}} Y_{1,t}(X_t^\pi + G_{1,t})^{\gamma-2}\Big[X_t^\pi \pi_t \sigma_t + \Lambda_{1,t} + \frac{X_t^\pi + G_{1,t}}{\gamma - 1 - \beta_1}\Big(\frac{\gamma - \beta_1}{\gamma}\frac{Z_{1,t}}{Y_{1,t}} \\ &+ \lambda\sqrt{\alpha_t}\Big)\Big]^2 dt\Big] + Y_{1,0}\frac{(x_0 + \bar{G}_{1,0} l_0)^\gamma}{\gamma} \le J_p(\pi^*, \phi^*), \end{aligned}$$

which implies that

$$\inf_{\phi \in \Phi} \sup_{\pi \in \Pi_p} J_p(\pi, \phi) \le \sup_{\pi \in \Pi_p} J_p(\pi, \hat{\phi}) \le J_p(\pi^*, \phi^*). \tag{9.E.12}$$

On the other hand, for the admissible strategy $(\hat{\pi}, \phi) \in \Pi_p \otimes \Phi$, from the second part of condition 3 in Definition 9.2.5 and (9.3.1), we also find that

$$\begin{aligned} & J_p(\hat{\pi}, \phi) \\ =& \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \frac{\varphi_t^\phi}{2\hat{\psi}_{1,t}}\Big[\phi_{1,t} + \Big(\frac{(X_t^{\hat{\pi}} + G_{1,t})^\gamma}{\gamma} Z_{1,t} + Y_{1,t}\left(X_t^{\hat{\pi}} + G_{1,t}\right)^{\gamma-1}(X_t^{\hat{\pi}}\hat{\pi}_t\sigma_t + \Lambda_{1,t})\Big)\hat{\psi}_{1,t}\Big]^2 dt\Big] \\ &+ \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \frac{\varphi_t^\phi}{2\hat{\psi}_{2,t}}\Big[\phi_{2,t} + \Big(\frac{(X_t^{\hat{\pi}} + G_{1,t})^\gamma}{\gamma} Z_{2,t} + Y_{1,t}(X_t^{\hat{\pi}} + G_{1,t})^{\gamma-1}\Lambda_{2,t}\Big)\hat{\psi}_{2,t}\Big]^2 dt\Big] \\ &+ Y_{1,0}\frac{(x_0 + \bar{G}_{1,0} l_0)^\gamma}{\gamma} \ge J_p(\pi^*, \phi^*). \end{aligned}$$

This result indicates that

$$J_p(\pi^*, \phi^*) \le \inf_{\phi \in \Phi} J_p(\hat{\pi}, \phi) \le \sup_{\pi \in \Pi_p} \inf_{\phi \in \Phi} J_p(\pi, \phi). \tag{9.E.13}$$

Since we always have $\inf(\sup) \ge \sup(\inf)$, we must have equality everywhere in (9.E.12)-(9.E.13). This proves that $\sup_{\pi \in \Pi_p} \inf_{\phi \in \Phi} J_p(\pi, \phi) = J_p(\pi^*, \phi^*) = Y_{1,0}\frac{(x_0 + \bar{G}_{1,0} l_0)^\gamma}{\gamma}$ and $(\pi^*, \phi^*) \in \Pi_p \otimes \Phi$ is the optimal control of the robust ALM problem (9.2.11). □

9.F Proof of Proposition 9.4.2

Proof. From Lemma 9.3.6 we know that the probability measure $\check{\mathbb{P}}$ defined by

$$\frac{d\check{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\left\{-\int_0^T \lambda\sqrt{\alpha_t}\, dW_{1,t} - \frac{1}{2}\int_0^T \lambda^2 \alpha_t\, dt\right\}$$

is equivalent to the reference measure $\mathbb{P}$. Then, the following processes $\check{W}_{1,t}$ and $\check{W}_{2,t}$

$$\check{W}_{1,t} = \int_0^t \lambda\sqrt{\alpha_s}\, ds + W_{1,t} \text{ and } \check{W}_{2,t} = W_{2,t}$$

are two standard Brownian motions under $\check{\mathbb{P}}$ due to Girsanov's theorem. Applying Itô's formula to $\log(Y_{2,t})$ under $\check{\mathbb{P}}$ measure, we have the following quadratic BSDE of $\left(\log(Y_{2,t}), \frac{M_{1,t}}{Y_{2,t}}, \frac{M_{2,t}}{Y_{2,t}}\right)$:

$$\begin{cases} d\log(Y_{2,t}) = \left[-r + \frac{1}{2}\left(\frac{M_{1,t}}{Y_{2,t}}\right)^2 - \frac{1}{2}\left(\frac{M_{2,t}}{Y_{2,t}}\right)^2\right] dt + \frac{M_{1,t}}{Y_{2,t}}\, d\check{W}_{1,t} + \frac{M_{2,t}}{Y_{2,t}}\, d\check{W}_{2,t}, \\ \log(Y_{2,T}) = 0. \end{cases} \tag{9.F.1}$$

Clearly, quadratic BSDE (9.F.1) satisfies all the regularity conditions in Kobylanski (2000). By Theorem 2.3 and 2.6 in Kobylanski (2000), we can conclude that quadratic BSDE (9.F.1) admits a unique solution. Hence, BSRE (9.4.2) admits a unique solution as well. Moreover, it is straightforward to verify that (9.4.4) and (9.4.5) form the unique solution to BSRE (9.4.2). This completes the proof. □

9.G Proof of Proposition 9.4.3

Proof. We conjecture that the first component $G_{2,t}$ of the solution to quadratic BSDE (9.4.6) has the following affine form:

$$G_{2,t} = f_2(t) + g_2(t)\alpha_t + h_2(t)L_t,$$

where $f_2(t), g_2(t)$, and $h_2(t)$ are three undetermined differentiable functions with terminal conditions $f_2(T) = g_2(T) = 0$ and $h_2(T) = -1$. Using Itô's formula to $G_{2,t}$, we derive

$$\begin{aligned} dG_{2,t} = &\left[\frac{df_2(t)}{dt} + \kappa\theta g_2(t) + \left(\frac{dg_2(t)}{dt} - \kappa g_2(t)\right)\alpha_t + \left((\mu_l + \lambda\sigma_l\alpha_t)h_2(t) + \frac{dh_2(t)}{dt}\right)L_t\right] dt \\ &+ \left(\rho_1 g_2(t) + \sigma_l h_2(t) L_t\right)\sqrt{\alpha_t}\, dW_{1,t} + \rho_2 g_2(t)\sqrt{\alpha_t}\, dW_{2,t}. \end{aligned} \tag{9.G.1}$$

Let $\Gamma_{1,t} = \left(\rho_1 g_2(t) + \sigma_l h_2(t) L_t\right)\sqrt{\alpha_t}$ and $\Gamma_{2,t} = \rho_2 g_2(t)\sqrt{\alpha_t}$ and substitute them into quadratic BSDE (9.4.6). Then, the generator of (9.4.6) can be rewritten as follows:

$$\left(\lambda\rho_1 g_2(t) + \frac{q+\beta_2}{2}\rho_2^2 g_2^2(t) - \frac{1}{2(q+\beta_1)}\lambda^2\right)\alpha_t + \lambda\sigma_l h_2(t)\alpha_t L_t. \tag{9.G.2}$$

Comparing (9.G.2) and the drift coefficient of (9.G.1) and separating the dependence on α_t, L_t, and $\alpha_t L_t$, we obtain the ODEs (9.4.9)-(9.4.11).

Moreover, when $\rho_2 \neq 0$, we denote by $\Delta_{g_2} = (\kappa + \lambda\rho_1)^2 + \frac{q+\beta_2}{q+\beta_1}\rho_2^2\lambda^2 > 0$ and rewrite the Riccati ODE (9.4.9) as follows:

$$\frac{dg_2(t)}{dt} = \frac{q+\beta_2}{2}\rho_2^2\left(g_2(t) - n_{g_2^+}\right)\left(g_2(t) - n_{g_2^-}\right),$$

where $n_{g_2^+}$ and $n_{g_2^-}$ are given by (9.4.15). After some tedious calculations, we derive the closed-form expression of $g_2(t)$ given in (9.4.12). When $\rho_2 = 0$ and $\kappa + \lambda\rho_1 \neq 0$, the Riccati ODE (9.4.9) degenerates to the following linear ODE:

$$\frac{dg_2(t)}{dt} = (\kappa + \lambda\rho_1)g_2(t) - \frac{1}{2(q+\beta_1)}\lambda^2,$$

and we immediately find that

$$g_2(t) = -\frac{\lambda^2}{2(q+\beta_1)(\kappa+\lambda\rho_1)}\left(e^{(\kappa+\lambda\rho_1)(t-T)} - 1\right).$$

For the case when $\rho_2 = 0$ and $\kappa + \lambda\rho_1 = 0$, we have from (9.4.9) that $g_2(t) = -\frac{\lambda^2}{2(q+\beta_1)}(t-T)$. Substituting (9.4.12) into the ODE (9.4.10) gives the closed-form expressions of $f_2(t)$ in (9.4.13). Finally, by a simple calculation, the explicit solution $h_2(t)$ to ODE (9.4.11) is given by (9.4.14). □

9.H Proof of Theorem 9.4.4

Proof. In the first part of the proof, we show that the proposed solution $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ lies in the space $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. For this, from the $\mathbb{P}$-dynamics of the random liabilities (9.2.7) we observe that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T L_t^4\,dt\right] =& l_0^4\int_0^T e^{4\mu_l t}\mathbb{E}^{\mathbb{P}}\left[\exp\left\{(4\lambda\sigma_l - 2\sigma_l^2)\int_0^t \alpha_s\,ds + 4\sigma_l\int_0^t\sqrt{\alpha_s}\,dW_{1,s}\right\}\right]dt\\ \leq& l_0^4\int_0^T e^{4\mu_l t}\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{(8\lambda\sigma_l + 28\sigma_l^2)\int_0^t\alpha_s\,ds\right\}\right]\right\}^{\frac{1}{2}}dt,\end{aligned} \tag{9.H.1}$$

where the inequality follows from the Hölder's inequality and the fact that the stochastic exponential process $\exp\left\{8\sigma_l\int_0^t\sqrt{\alpha_s}\,dW_{1,s} - 32\sigma_l^2\int_0^t\alpha_s\,ds\right\}$ is an $(\mathbb{F},\mathbb{P})$-martingale by Lemma 9.3.6. To calculate the term $\mathbb{E}^{\mathbb{P}}\left[\exp\left\{(8\lambda\sigma_l + 28\sigma_l^2)\int_0^t\alpha_s\,ds\right\}\right]$, let $\mathbb{E}^{\mathbb{P}}[\cdot|\mathcal{F}_u]$ be the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_u$, for $u \leq t$. By using the Markovian structure of the process α_t, we have

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{(8\lambda\sigma_l + 28\sigma_l^2)\int_u^t\alpha_s\,ds\right\}\Big|\mathcal{F}_u\right] = F(\alpha_u, u),\ \text{for } u \leq t, \tag{9.H.2}$$

where $F : \mathbb{R}^+ \otimes [0,t] \mapsto \mathbb{R}^+$ is an unknown function. By the Feynman-Kac theorem, we know that the function F is governed by the following partial differential equation (PDE):

$$\begin{cases}\frac{\partial F}{\partial u}(x,u) + \kappa(\theta - x)\frac{\partial F}{\partial x}(x,u) + \frac{1}{2}\left(\rho_1^2 + \rho_2^2\right)x\frac{\partial^2 F}{\partial x^2}(x,u) + (8\lambda\sigma_l + 28\sigma_l^2)xF(x,u) = 0,\\ F(x,t) = 1.\end{cases}$$

Furthermore, it can be shown that $F(x,u) = \exp\{M(u;t)x + N(u;t)\}$, for $u \in [0,t]$, where $M(u;t)$ and $N(u;t)$ satisfy the following ODEs:

$$\frac{dM(u;t)}{du} = -\frac{\rho_1^2+\rho_2^2}{2}M^2(u;t) + \kappa M(u;t) - \left(8\lambda\sigma_l + 28\sigma_l^2\right),\ M(t;t)=0,$$

and

$$\frac{dN(u;t)}{du} = -\kappa\theta M(u;t),\ N(t;t)=0.$$

As in Proposition 9.4.3, we can show the closed-form expressions for $M(u;t)$ and $N(u;t)$ as follows:

$$M(u;t) = \begin{cases} \dfrac{n_M^+ n_M^- \left(1-e^{\sqrt{\Delta_M}(t-u)}\right)}{n_M^+ - n_M^- e^{\sqrt{\Delta_M}(t-u)}}, \text{ if } \Delta_M > 0; \\ \dfrac{(\rho_1^2+\rho_2^2)(t-u)n_M^2}{(\rho_1^2+\rho_2^2)(t-u)n_M + 2}, \text{ if } \Delta_M = 0; \\ \dfrac{\sqrt{-\Delta_M}}{-(\rho_1^2+\rho_2^2)} \tan\left(\arctan\left(\dfrac{\kappa}{\sqrt{-\Delta_M}}\right) - \dfrac{\sqrt{-\Delta_M}}{2}(t-u)\right), \text{ if } \Delta_M < 0, \end{cases} \tag{9.H.3}$$

and

$$N(u;t) = \int_u^t \kappa\theta M(s;t)\,ds \tag{9.H.4}$$

where Δ_M, n_M^+, n_M^-, and n_M are given by

$$\Delta_M = \kappa^2 - 2(\rho_1^2+\rho_2^2)(8\lambda\sigma_l + 28\sigma_l^2),\ n_M = \frac{\kappa}{\rho_1^2+\rho_2^2},\ n_M^+ = \frac{-\kappa+\sqrt{\Delta_M}}{-(\rho_1^2+\rho_2^2)},\ n_M^- = \frac{-\kappa-\sqrt{\Delta_M}}{-(\rho_1^2+\rho_2^2)}.$$

Then, combining (9.H.1)-(9.H.4) and using the law of total expectation, we derive

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T L_t^4\,dt\right] \le l_0^4 \int_0^T \exp\left\{4\mu_l t + \frac{M(0;t)}{2}\alpha_0 + \frac{N(0;t)}{2}\right\} dt < +\infty.$$

Additionally, note that for all $t \in [0,T]$, the second moment of the square-root factor process α_t is explicitly given by

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\alpha_t^2\right] = \left(\alpha_0 e^{-\kappa t} + \theta\left(1-e^{-\kappa t}\right)\right)^2 + \alpha_0 \frac{(\rho_1^2+\rho_2^2)\left(e^{-\kappa t} - e^{-2\kappa t}\right)}{\kappa} \\ + \frac{\theta(\rho_1^2+\rho_2^2)\left(1-e^{-\kappa t}\right)^2}{2\kappa}.\end{aligned}$$

Therefore, from the explicit expressions for $G_{2,t}, \Gamma_{1,t}$ and $\Gamma_{2,t}$ given in (9.4.7) and (9.4.8) we derive

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T G_{2,t}^2 + \Gamma_{1,t}^2 + \Gamma_{2,t}^2\,dt\right] \le c\left[1 + \int_0^T \mathbb{E}^{\mathbb{P}}\left[\alpha_t^2\right]\,dt + \mathbb{E}^{\mathbb{P}}\left[\int_0^T L_t^4\,dt\right]\right] < +\infty,$$

where c is a positive constant. This finishes the first part of the proof.

Next, we show that the proposed solution given by (9.4.7)-(9.4.8) is the unique solution to quadratic BSDE (9.4.6). To this end, note from Proposition 9.4.2 that the probability measure $\check{\mathbb{P}}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and the $\check{\mathbb{P}}$-dynamics of the square-root factor process α_t is given by

$$d\alpha_t = (\kappa + \lambda\rho_1)\left(\frac{\kappa\theta}{\kappa + \lambda\rho_1} - \alpha_t\right) dt + \sqrt{\alpha_t}\left(\rho_1\, d\widehat{W}_{1,t} + \rho_2\, d\widehat{W}_{2,t}\right),$$

which preserves the affine-form, square-root structure under Assumption 9.4.1. Reformulate BSDE (9.4.6) of $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ under $\check{\mathbb{P}}$ as follows:

$$\begin{cases} dG_{2,t} = \left(\dfrac{q+\beta_2}{2}\Gamma_{2,t}^2 - \dfrac{1}{2(q+\beta_1)}\lambda^2\alpha_t\right) dt + \Gamma_{1,t}\, d\check{W}_{1,t} + \Gamma_{2,t}\, d\check{W}_{2,t}, \\ G_{2,T} = -\, L_T, \end{cases} \tag{9.H.5}$$

and suppose that there exists another solution $(\widehat{G}_{2,t}, \widehat{\Gamma}_{1,t}, \widehat{\Gamma}_{2,t})$ to (9.H.5), which is different from the proposed solution given in Proposition 9.4.3. Then, the difference process $(\Delta G_{2,t}, \Delta\Gamma_{1,t}, \Delta\Gamma_{2,t}) := (G_{2,t} - \widehat{G}_{2,t}, \Gamma_{1,t} - \widehat{\Gamma}_{1,t}, \Gamma_{2,t} - \widehat{\Gamma}_{2,t})$ must solve the following BSDE:

$$\begin{cases} d\Delta G_{2,t} = \dfrac{q+\beta_2}{2}\left(\Gamma_{2,t}^2 - \widehat{\Gamma}_{2,t}^2\right) dt + \Delta\Gamma_{1,t}\, d\check{W}_{1,t} + \Delta\Gamma_{2,t}\, d\check{W}_{2,t}, \\ \Delta G_{2,T} = 0. \end{cases} \tag{9.H.6}$$

Furthermore, we notice from the explicit expression for $\Gamma_{2,t}$ given in (9.4.8) and Lemma 9.3.6 that the following probability measure $\widetilde{\mathbb{P}}$ is well-defined and equivalent to $\check{\mathbb{P}}$ on $\mathcal{F}_T$:

$$\begin{aligned} \left.\frac{d\widetilde{\mathbb{P}}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_T} &= \exp\left\{-\int_0^T (q+\beta_2)\rho_2 g_2(t)\sqrt{\alpha_t}\, d\check{W}_{2,t} - \int_0^T \frac{(q+\beta_2)^2\rho_2^2 g_2^2(t)}{2}\alpha_t\, dt\right\} \\ &= \exp\left\{-\int_0^T (q+\beta_2)\Gamma_{2,t}\, d\check{W}_{2,t} - \frac{(q+\beta_2)^2}{2}\int_0^T \Gamma_{2,t}^2\, dt\right\}, \end{aligned}$$

and $\widetilde{W}_{2,t} = \int_0^t (q+\beta_2)\Gamma_{2,s}\, ds + \widehat{W}_{2,t}$ and $\widetilde{W}_{1,t} = \widehat{W}_{1,t}$ are two standard Brownian motions under $\widetilde{\mathbb{P}}$. Hence, BSDE (9.H.6) can be rewritten under $\widetilde{\mathbb{P}}$:

$$\begin{cases} d\Delta G_{2,t} = -\, \dfrac{q+\beta_2}{2}\Delta\Gamma_{2,t}^2\, dt + \Delta\Gamma_{1,t}\, d\widetilde{W}_{1,t} + \Delta\Gamma_{2,t}\, d\widetilde{W}_{2,t}, \\ \Delta G_{2,T} = 0, \end{cases}$$

which is a quadratic BSDE satisfying all the regularity conditions in Kobylanski (2000), and we can conclude that $(\Delta G_{2,t}, \Delta\Gamma_{1,t}, \Delta\Gamma_{2,t}) = (0, 0, 0)$ is the unique solution by Theorem 2.3 and 2.6 in Kobylanski (2000). This proves that the proposed solution given in Proposition 9.4.3 is the unique solution to BSDE (9.4.6). □

9.I Proof of Theorem 9.4.5

Proof. First of all, by Theorem 5.1 in Zeng and Taksar (2013), it follows from the explicit expressions for ϕ^* and $\Gamma_{2,t}$ given in (9.4.17) and (9.4.8) that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T (\phi^*_{1,t})^2 + (\phi^*_{2,t})^2\,dt\right\}\right] = \mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\frac{\beta_1^2\lambda^2}{(q+\beta_1)^2} + \beta_2^2\rho_2^2 g_2^2(t)\right)\alpha_t\,dt\right\}\right]$$
$$\leq \mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{b_0}{2}\int_0^T \alpha_t\,dt\right\}\right] < +\infty,$$

where the constant b_0 is given in (9.4.16). This shows that $\phi^* \in \Phi$. Plugging π^*_t into X^*_t and using Itô's formula to $X^*_t Y_{2,t} + G_{2,t}$ under the reference measure $\mathbb{P}$, we derive

$$X^*_t Y_{2,t} + G_{2,t} = \int_0^t \left(\frac{\lambda^2}{2(q+\beta_1)} + \frac{q+\beta_2}{2}\rho_2^2 g_2^2(s)\right)\alpha_s\,ds + \int_0^t \frac{\lambda}{q+\beta_1}\sqrt{\alpha_s}\,dW_{1,s}$$
$$+ \int_0^t \rho_2 g_2(s)\sqrt{\alpha_s}\,dW_{2,s} + x_0 Y_{2,0} + G_{2,0},$$

from which it follows that

$$e^{-8q(X^*_t Y_{2,t} + G_{2,t})}$$
$$= c\,\underbrace{\exp\left\{\int_0^t \frac{-8q\lambda}{q+\beta_1}\sqrt{\alpha_s}\,dW_{1,s} - \int_0^t 8q\rho_2 g_2(s)\sqrt{\alpha_s}\,dW_{2,s} - 32q^2\int_0^t\left(\frac{\lambda^2}{(q+\beta_1)^2} + \rho_2^2 g_2^2(s)\right)\alpha_s\,ds\right\}}_{K_{3,t}}$$
$$\times \underbrace{\exp\left\{\int_0^t\left(\frac{32q^2\lambda^2}{(q+\beta_1)^2} - \frac{4q\lambda^2}{q+\beta_1} + (28q^2 - 4q\beta_2)\rho_2^2 g_2^2(s)\right)\alpha_s\,ds\right\}}_{K_{4,t}},$$

where c is a positive constant. We notice that $K_{3,t}$ is the stochastic exponential process of continuous $(\mathbb{F},\mathbb{P})$-local martingale $\int_0^t \frac{-8q\lambda}{q+\beta_1}\sqrt{\alpha_s}\,dW_{1,s} - \int_0^t 8q\rho_2 g_2(s)\sqrt{\alpha_s}\,dW_{2,s}$. Then, applying the Hölder's inequality, Theorem 5.1 in Zeng and Taksar (2013) and Theorem 15.4.6 in Cohen and Elliott (2015), we derive

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} e^{-8q(X^*_t Y_{2,t} + G_{2,t})}\right]$$
$$\leq c\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} K^2_{3,t}\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} K^2_{4,t}\right]\right\}^{\frac{1}{2}}$$
$$\leq c\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{b_1}{2}\int_0^T \alpha_t\,dt\right\}\right]\right\}^{\frac{\sqrt{2+\sqrt{2}}-1}{4+2\sqrt{2}}}\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{b_2}{2}\int_0^T \alpha_t\,dt\right\}\right]\right\}^{\frac{1}{2}}$$
$$< +\infty,$$

where the constant c might differ between lines, and b_1 and b_2 are given in (9.4.16). Thus, using the explicit expressions for $\phi^*_{1,t}$ and $\phi^*_{2,t}$ given in (9.4.17) and applying Theorem 15.4.6 in Cohen and Elliott (2015) and Theorem 5.1 in Zeng and Taksar

(2013) again, we find that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|\varphi_t^{\phi^*}\frac{e^{-q(X_t^*Y_{2,t}+G_{2,t})}}{q}\right|^4\right]\\
\leq&\frac{1}{q^4}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}|\varphi_t^{\phi^*}|^8\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}e^{-8q(X_t^*Y_{2,t}+G_{2,t})}\right]\right\}^{\frac{1}{2}}\\
\leq&\frac{1}{q^4}\sqrt{\frac{8}{7}}\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{b_0}{2}\int_0^T\alpha_t\,dt\right\}\right]\right\}^{\frac{\sqrt{120+32\sqrt{14}}-1}{240+64\sqrt{14}}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}e^{-8q(X_t^*Y_{2,t}+G_{2,t})}\right]\right\}^{\frac{1}{2}}\\
<&+\infty.
\end{aligned}
$$

Then, it follows from condition (9.4.16), the Hölder's inequality, and the trivial algebraic result that $x^8\leq ae^x$, $x\in\mathbb{R}^+$ for some constant $a\in\mathbb{R}^+$ that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left|\varphi_t^{\phi^*}\left(\int_0^t\frac{(\phi_{1,s}^*)^2}{2\eta_{1,s}^*}+\frac{(\phi_{2,s}^*)^2}{2\eta_{2,s}^*}\,ds\right)\right|^2\right]\\
\leq&c\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}|\varphi_t^{\phi^*}|^8\right]\right\}^{\frac{1}{4}}\left\{\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]}e^{-8q(X_t^*Y_{2,t}+G_{2,t})}\right]\right\}^{\frac{1}{4}}\left\{\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\int_0^T\alpha_t\,dt\right\}\right]\right\}^{\frac{1}{4}}\\
<&+\infty,
\end{aligned}
$$

where c is a positive constant. The above results prove (i)-(iii) in the statement of Theorem 9.4.5, and thus, we know that the following two families of random variables

$$
\left\{\varphi_{\tau_n\wedge T}^{\phi^*}\frac{-e^{-q(X_{\tau_n\wedge T}^*Y_{2,\tau_n\wedge T}+G_{2,\tau_n\wedge T})}}{q}\right\}_{n\in\mathbb{N}}\text{ and }\left\{\varphi_{\tau_n\wedge T}^{\phi^*}\int_0^{\tau_n\wedge T}\frac{(\phi_{1,s}^*)^2}{2\eta_{1,s}^*}+\frac{(\phi_{2,s}^*)^2}{2\eta_{2,s}^*}\,ds\right\}_{n\in\mathbb{N}}
$$

are uniformly integrable under the reference measure $\mathbb{P}$, where $\{\tau_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence of $\mathbb{F}$-stopping times such that $\tau_n\uparrow+\infty$ as $n\to+\infty$. Hence, we can conclude that the control $(\pi^*,\phi^*)\in\Pi_e\otimes\Phi$. The proof for the optimality of the admissible control (π^*,ϕ^*) given in (9.4.17) is similar to Theorem 9.3.10, so we omit it here. □

Chapter 10

Non-zero-sum stochastic differential games for asset-liability management with stochastic inflation and stochastic volatility

Abstract

This paper investigates the optimal asset-liability management problems for two managers subject to relative performance concerns in the presence of stochastic inflation and stochastic volatility. The objective of the two managers is to maximize the expected utility of their relative terminal surplus with respect to that of their competitor. The problem of finding the optimal investment strategies for both managers is modeled as a non-zero-sum stochastic differential game. Both managers have access to a financial market consisting of a risk-free asset, a risky asset, and an inflation-linked index bond. The risky asset's price process and uncontrollable random liabilities are not only affected by the inflation risk but also driven by a general class of stochastic volatility models including the constant elasticity of variance model, the family of state-of-the-art 4/2 models, and some path-dependent models as particular cases. By adopting a backward stochastic differential equation (BSDE) approach to overcome the possibly non-Markovian setting, closed-form expressions for the equilibrium investment strategies and corresponding value functions are derived under power and exponential utility preferences. Moreover, explicit solutions to some special cases of our model are provided. Finally, we perform numerical studies to illustrate the impact of model parameters on the equilibrium strategies and draw some economic interpretations.

Keywords: Asset-liability management; Non-zero-sum game; Stochastic volatility; Stochastic inflation; Backward stochastic differential equation

10.1 Introduction

Asset-liability management (ALM) is an important concern not only to financial security systems such as banks, insurance complies, and pension funds but also to individual investors who aim to ensure the match between assets and liabilities and achieve management goals by continuously adjusting the investment amount in accordance with financial markets and regulatory requirements. In recent years, a great deal of research on the optimal ALM problems has been carried out under various scenarios and objectives including searching for the pre-commitment and time-consistent strategies under the mean-variance criterion, maximizing the expected utility of terminal surplus, and minimizing the cumulative deviation. Leippold, Trojani, and Vanini (2004) considered a multi-period ALM problem under the mean-variance criterion and derived the explicit expressions for the pre-commitment strategy and efficient frontier. Chiu and Li (2006) and Xie, Li, and Wang (2008) investigated the pre-commitment strategies for the continuous-time mean-variance ALM problems where the liability processes were driven by a geometric Brownian motion and a drifted Brownian motion, respectively. Zhang and Chen (2016) studied a mean-variance ALM problem under the constant elasticity of variance (CEV) model. Li, Shen, and Zeng (2018) incorporated derivatives trading into a mean-variance ALM problem where the price process of the risky asset along with its volatility was described by the Heston model (Heston (1993)). Zhang (2023) stepped further by investigating a derivative-based mean-variance ALM problem where the short rate of interest and stochastic volatility were driven by a Cox-Ingersoll-Ross (CIR) model and the state-of-the-art 4/2 model (Grasselli (2017)), respectively. Sun, Zhang, and Yuen (2020) considered a mean-variance ALM problem with an affine diffusion factor process and a reinsurance option. Apart from the pre-commitment strategies, Wei et al. (2013) and Wei and Wang (2017) derived the explicit solutions to the time-consistent strategies (Björk, Khapko, and Murgoci (2017)) of mean-variance ALM problems under a Markov regime-switching market and random coefficient setting, respectively. Zhang et al. (2017) considered the time-consistent strategy of a mean-variance ALM problem with state-dependent risk aversion and multiple risky assets. Within the framework of expected utility maximization, Pan and Xiao (2017a) investigated an ALM problem with liquidity constraints and stochastic interest rate. Pan, Hu, and Zhou (2019) studied an ALM problem under the Heston model and exponential utility function. Recently, some literature focuses on the impact of model ambiguity (Andersen, Hansen, and Sargent (2003) and Maenhout (2004)) on the optimal investment strategies of ALM problems. For example, Chen, Huang, and Li (2022) and Yuan and Mi (2022a) considered robust ALM problems in a regime-switching model and jump-diffusion market with delay, respectively, and derived the explicit solutions by solving the corresponding Hamilton-Jacobi-Bellman-Issac (HJBI) equations. By adopting a BSDE approach, Zhang (2022e) investigated robust ALM problems in a non-Markovian setting described by an affine-form, square-root factor process for both the power and exponential utility functions. As the literature on ALM problems is abundant, the above review is not exhaustive. For more related works, one may refer to Zeng, Li, and Wu (2013), Yuan and Mi (2022b), Peng and Chen (2021), and references therein.

Although ALM problems have been extensively studied over the last decade, one common feature shared by the above-mentioned literature is that none of them takes into account the strategic interaction among agents. However, as emphasized in economic studies, such as Garcia and Strobl (2010) and Basak and Makarov (2014), relative performance concerns play a key role in a competitive economy. To model such interaction, Espinosa and Touzi (2015) formulated a continuous-time optimal investment problem with relative performance concerns in the framework of non-zero-sum stochastic differential games and investigated the Nash equilibrium strategy for multiple agents. Their study proved the existence and uniqueness of the Nash equilibrium for the cases of unconstrained and constrained agents with exponential utilities in a Black-Scholes financial market. Recently, non-zero-sum stochastic differential games with applications to finance and insurance have received increasing attention. Bensoussan et al. (2014) considered a non-zero-sum stochastic differential investment and reinsurance game between two insurance companies whose surplus processes were driven by continuous-time Markov chains. Meng, Li, and Jin (2015) studied an optimal reinsurance problem for two insurers where the surplus processes were subject to quadratic risk processes. Guan and Liang (2016) investigated a non-zero-sum stochastic investment game with inflation risk for two DC pension funds. Kwok, Chiu, and Wong (2016) investigated the impact of relative performance concerns on the longevity risk transfer market in the presence of stochastic interest and mortality rates. Deng, Zeng, and Zhu (2018) studied a non-zero-sum stochastic differential investment and reinsurance game with default risk and Heston's stochastic volatility. Hu and Wang (2018) considered the optimal time-consistent investment and reinsurance strategies for two mean-variance insurance managers with relative performance concerns in a Black-Scholes market. Zhu, Cao, and Zhang (2019) and Zhu, Cao, and Zhu (2021) extended the results of Hu and Wang (2018) to the cases with the Heston model and CEV model, respectively. Other investigations regarding non-zero-sum stochastic differential games with applications to finance and insurance can be found in Pun and Wong (2016), Dong, Rong, and Zhao (2022), Savku and Weber (2022), and references therein.

Another aspect worthy of being further explored is to take inflation risk into account in that ALM plans may involve quite a long period and inflation risk has become one of the most anxious factors among the side effects of expansionary monetary policy. To hedge against inflation risk, there has been a high demand for inflation-linked securities in the financial market, such as Treasury Inflation-Protected Securities (TIPS) in the US and gilt-edged securities in the UK. In recent years, continuous-time portfolio optimization problems under inflation risk have attracted the attention of quite a few scholars. Campbell and Viceira (2000) and Brennan and Xia (2002) considered dynamic asset allocation problems with stochastic inflation and stochastic interest rates and probed the importance of inflation-linked index bonds for long-term and conservative investment. Battocchio and Menoncin (2004) and Zhang and Ewald (2010) investigated DC pension management problems under inflation risk and derived explicit solutions for exponential and power utility cases, respectively. Korn, Siu, and Zhang (2011) studied a DC pension management problem in a regime-switching environment under inflation risk. Kwak and Lim (2014) considered an optimal investment-consumption problem with life insurance under inflation risk and obtained the explicit solutions for the power utility case using the martingale method. Wang, Li, and Sun (2021) investigated a robust DC pension management problem with inflation risk and mean-reverting risk premium under model ambiguity. Zhang (2022c) studied a DC pension investment problem with stochastic income under inflation and volatility risks. Particularly, in the field of ALM, Pan and Xiao (2017b,c) considered ALM problems with stochastic interest rates and inflation risks under the mean-variance criterion and expected utility maximization framework, respectively, and derived the closed-form solutions by using the dynamic programming principle and solving the associated Hamilton-Jacobi-Bellman (HJB) equations.

Inspired by the above works, in this paper, we investigate the optimal ALM problems for two competitive managers with relative performance concerns under the risks of stochastic inflation and stochastic volatility and formulate the problems within a non-zero-sum stochastic differential game framework. In our problem setting, the managers are subject to two different stochastic liability processes and are allowed to invest in a risk-free asset, a risky asset, and an inflation-linked index bond. It is assumed that the risky asset's price process and uncontrollable liability processes are not only driven by the dynamics of stochastic inflation but also governed by a general class of stochastic volatility models, where the risk premium and volatility of the risky asset are general non-Markovian, unbounded stochastic processes, whereas the market price of risk satisfies an affine-form, square-root factor process including the CEV model, the family of state-of-the-art 4/2 models, and some path-dependent models as exceptional cases. Both managers aim at maximizing the expected utility of their relative terminal surplus after stripping out inflation concerning that of their competitor and searching the Nash equilibrium investment strategies. As opposed to most of the aforementioned literature investigating the portfolio selection problems with relative performance concerns via the dynamic programming approach developed by Mataramvura and Øksendal (2008), we extend and apply the BSDE techniques introduced by Hu, Imkeller, and Müller (2005) to overcome the potentially non-Markovian setting. To be more precise, we propose to construct a stochastic process that hinges upon any admissible strategy for manager i whenever his/her competitor j's strategy is fixed and given and is such that its value at time zero does not depend on both managers' investment strategies and its terminal value equals the utility of manager i's relative terminal surplus with respect to that of manager j, where $i \neq j \in \{1, 2\}$. The proposed stochastic process is shown to be a (local) super-martingale for any admissible response strategy taken by manager i, and even a (local) martingale for the optimal response strategy to manager j's strategy. The determination of such a stochastic process leads to the associated BSDEs. By solving the BSDEs explicitly, we derive the closed-form expressions for the Nash equilibrium strategies and value functions under the exponential and power utility preferences. Moreover, several particular cases of our modelling framework are discussed and the corresponding analytical results are provided. Finally, the sensitivities of the equilibrium strategies regarding model parameters and relative performance concerns are investigated with numerical experiments. To summarize, the contributions of this paper are as follows:

1. we pioneer to incorporate relative performance concerns, stochastic volatility, and stochastic inflation simultaneously into the optimal ALM problems in the framework of expected utility maximization;

2. comparing with the preceding literature on the optimal investment problems with relative performance concerns using the dynamic programming approach and solving a coupled HJB equation, such as Bensoussan et al. (2014), Guan and Liang (2016), Deng, Zeng, and Zhu (2018), Dong, Rong, and Zhao (2022), Savku and Weber (2022), etc, we apply a BSDE approach to solve non-zero-sum stochastic differential game problems explicitly in a possibly non-Markovian framework for both the exponential and power utility functions, which extends the techniques developed by Hu, Imkeller, and Müller (2005) for addressing single-agent optimization problems and differentiates from Espinosa and Touzi (2015) where the exogenous parameter processes were assumed to be uniformly bounded and only the existence and uniqueness of solutions to the BSDEs under the exponential utility preference were considered without presenting the closed-form solutions;

3. we derive the closed-form expressions for the Nash equilibrium strategies and the corresponding value functions represented in terms of the explicit solutions to the associated BSDEs, from which we find the herd effect on managers' decisions, that is, each manager mimics the competitor's strategy, and when

relative performance concerns are taken into account, each manager will adopt a riskier investment strategy than that without competition involved. Furthermore, analytical solutions to some special cases of our model are provided.

The remainder of this paper is organized as follows. Section 10.2 introduces the model setup of two competitive asset-liability managers. Section 10.3 establishes the optimal ALM problems with relative performance concerns within the framework of non-zero-sum stochastic differential games. Section 10.4 and 10.5 derive the closed-form expressions for the Nash equilibrium strategies and the corresponding value functions under the exponential and power utility functions, respectively. Section 10.6 provides detailed numerical experiments to discuss the impacts of model parameters and relative performance concerns on the equilibrium strategies. Section 10.7 concludes the paper.

10.2 Financial market and random liabilities

In this paper, let $[0, T]$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions, where three one-dimensional, mutually independent Brownian motions, $\{W_{i,t}\}_{t\in[0,T]}, i = 1, 2, 3$ are defined, the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be generated by the three Brownian motions, and $\mathbb{P}$ denotes the real-world probability measure. In what follows, we introduce several spaces on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

- $\mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes with $\mathbb{P}$-a.s. continuous sample paths;
- $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{P}\left(\int_0^T |f_t|^2\, dt < \infty\right) = 1$;
- $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}\left[\int_0^T |f_t|^2\, dt\right] < \infty$;
- $\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted uniformly bounded processes with $\mathbb{P}$-a.s. continuous sample paths.

10.2.1 The financial market

We consider a financial market that consists of three tradable assets: a risk-free asset (money account), a risky asset (stock), and an inflation-linked index bond. The price process of the risk-free asset B_t is given by

$$dB_t = RB_t\, dt,\;\; B_0 = 1,$$

where the constant $R \in \mathbb{R}$ denotes the nominal interest rate. As mentioned in the introduction, the risk of inflation cannot be ignored in a long-term investment cycle, which leads to the devaluation of the wealth of asset-liability managers. To describe the inflation risk, following some literature (see, for example, Zhang and Ewald (2010), Korn, Siu, and Zhang (2011), and Kwak and Lim (2014)), we assume that the price index P_t satisfies the following geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu_p \, dt + \sigma_p \, dW_{0,t}, \;\; P_0 = p_0 \in \mathbb{R}^+, \tag{10.2.1}$$

where the constants $\mu_p \in \mathbb{R}$ and $\sigma_p \in \mathbb{R}^+$ stand for the expected rate and volatility of the price level, respectively, and the price dynamics of the inflation-linked index bond I_t are given by the following diffusion process:

$$\frac{dI_t}{I_t} = r \, dt + \frac{dP_t}{P_t} = R \, dt + \sigma_p \left(\lambda_p \, dt + dW_{0,t}\right), \tag{10.2.2}$$

where the constant $r \in \mathbb{R}$ is the real interest rate and the constant $\lambda_p = \frac{r+\mu_p-R}{\sigma_p} \in \mathbb{R}$ is the market price of inflation risk. As supported by some empirical analysis (see, for example, Lee (2010)), the inflation rate usually has a direct or indirect impact on the price of risky assets, we, therefore, assume that the price dynamics of the risky assets S_t are given by

$$\frac{dS_t}{S_t} = R \, dt + \sigma_s(\lambda_p \, dt + dW_{0,t}) + (\mu_t \, dt + \sigma_t \, dW_{1,t}), \;\; S_0 = s_0 \in \mathbb{R}^+, \tag{10.2.3}$$

where the constant $\sigma_s \in \mathbb{R}$ characterizes how large the impact of the inflation risk on the risky asset's dynamics is, and μ_t and $\sigma_t > 0$ two potentially unbounded and non-Markovian $\mathbb{F}$-adapted stochastic processes describing the risky asset's risk premium and volatility generated by the fundamental risk source $W_{1,t}$ at time t. In addition, we assume that the market price of volatility risk is linear in the square root of an observable affine form, square-root factor process α_t, i.e.,

$$\frac{\mu_t}{\sigma_t} = \lambda\sqrt{\alpha_t}, \;\; \lambda \in \mathbb{R} \setminus \{0\}, \tag{10.2.4}$$

and

$$d\alpha_t = \kappa(\theta - \alpha_t) \, dt + \sqrt{\alpha_t}(\rho_1 \, dW_{1,t} + \rho_2 \, dW_{2,t}), \;\; \alpha_0 \in \mathbb{R}^+ \tag{10.2.5}$$

where the constants κ, θ, and $\sqrt{\rho_1^2 + \rho_2^2}$ represent the mean reversion rate, long-run level, and volatility of the factor process, respectively. In the spirit of Chapter 6.3 in Jeanblanc, Chesney, and Yor (2009), we suppose that the constants $\kappa, \theta \in \mathbb{R}$ satisfy $\kappa\theta \geq 0$, which guarantees that the process $\alpha_t \geq 0$ for all $t \in [0, T]$, $\mathbb{P}$ almost surely, whereas no specific conditions are introduced to the constants $\rho_1, \rho_2 \in \mathbb{R}$. Moreover, it is worth mentioning that the Feller condition, i.e., $2\kappa\theta \geq \rho_1^2 + \rho_2^2$ is not imposed in the current context to ensure that α_t is strictly positive.

It should be noted that the above price dynamics of the risky asset excluding the impact of the inflation rate, i.e., $\sigma_s = 0$ in (10.2.3) were considered in some existing literature for different research interests in recent years; for example, Shen and Zeng (2015), Li et al. (2022), and Zhang (2022c,e). The reason we opt for the above modeling framework for the risky asset's price S_t is that we can take a unified approach to address a class of stochastic (local) volatility models used in practice, such as the CEV model, Heston model, 3/2 model, and 4/2 model as well as some path-dependent models. For the reader's convenience, these specific models are listed below as examples.

Example 10.2.1 (CEV model)**.** If $\mu_t = \mu - R$, $\sigma_t = \sigma S_t^{\beta}$, and $\sigma_s = 0$ where $\mu \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$, and $\beta \leq -\frac{1}{2}$, then the risky asset price S_t is given by the CEV model:

$$dS_t = S_t \left(\mu\, dt + \sigma S_t^{\beta}\, dW_{1,t} \right), \ S_0 = s_0 \in \mathbb{R}^+, \tag{10.2.6}$$

where β is called the elasticity parameter. By setting $\alpha_t = S_t^{-2\beta}$, $\kappa = 2\beta\mu$, $\theta = (\beta + \frac{1}{2})\frac{\sigma^2}{\mu}$, $\rho_1 = -2\beta\sigma$, $\rho_2 = 0$ and $\lambda = \frac{\mu - r}{\sigma}$, we have

$$\begin{aligned} d\alpha_t &= 2\beta\mu \left[\left(\beta + \frac{1}{2} \right) \frac{\sigma^2}{\mu} - S_t^{-2\beta} \right] dt - 2\beta\sigma S_t^{-\beta}\, dW_{1,t} \\ &= \kappa(\theta - \alpha_t)\, dt + \sqrt{\alpha_t}\, (\rho_1\, dW_{1,t} + \rho_2\, dW_{2,t}) \,. \end{aligned}$$

For the particular case when $\beta = 0$, the condition $\kappa\theta \geq 0$ is still satisfied and the CEV model is reduced to the Black-Scholes model.

Example 10.2.2 (The family of 4/2 models)**.** If $\mu_t = \lambda(c_1\alpha_t + c_2)$, $\sigma_t = c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$, $\kappa \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$, $\rho_1 = \sigma_\alpha \rho$ and $\rho_2 = \sigma_\alpha \sqrt{1 - \rho^2}$, where $c_1 \geq 0, c_2 \geq 0, \sigma_\alpha \in \mathbb{R}^+$, and $\rho \in [-1, 1]$, then the price dynamics of the risky asset S_t are governed by the family of 4/2 stochastic volatility models:

$$\begin{cases} \dfrac{dS_t}{S_t} = (R + \lambda(c_1\alpha_t + c_2) + \lambda_p \sigma_s)\, dt + \sigma_s\, dW_{0,t} + \Bigg(c_1\sqrt{\alpha_t} \\ \qquad + \dfrac{c_2}{\sqrt{\alpha_t}} \Bigg) dW_{1,t}, \ S_0 = s_0 \in \mathbb{R}^+, \\ d\alpha_t = \kappa(\theta - \alpha_t)\, dt + \sigma_\alpha \sqrt{\alpha_t} \left(\rho\, dW_{1,t} + \sqrt{1 - \rho^2}\, dW_{2,t} \right), \ \alpha_0 \in \mathbb{R}^+, \end{cases} \tag{10.2.7}$$

where α_t is the variance driver process with mean-reversion rate κ, long-run mean θ, volatility σ_α, and correlation coefficient between the risky asset price and its variance driver ρ. For the 4/2 model (10.2.7), we impose the Feller condition, i.e., $2\kappa\theta \geq \sigma_\alpha^2$ to keep α_t strictly positive for $t \in [0, T]$, $\mathbb{P}$ almost surely.

Remark 10.2.3*.* The 4/2 model (10.2.7) embraces two embedded parsimonious models, the Heston model (Heston (1993)) and 3/2 model (Lewis (2000)) via the constants c_1 and c_2. Particularly, the case $(c_1, c_2) = (1, 0)$ corresponds to the Heston model, while the specification $(c_1, c_2) = (0, 1)$ is known as the 3/2 model.

Example 10.2.4 (A path-dependent model)**.** If $\mu_t = \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]})$ and $\sigma_t = \hat{\sigma}(\alpha_{[0,t]})$ for some functional $\hat{\sigma} : \mathcal{C}([0,t];\mathbb{R}) \mapsto \mathbb{R}^+$, where $\alpha_{[0,t]} := (\alpha_s)_{s\in[0,t]}$ is the restriction of $\alpha \in \mathcal{C}([0,T];\mathbb{R})$ to $\mathcal{C}([0,t];\mathbb{R})$, i.e., the space of real-valued, continuous functions defined on $[0,t]$. In this case, the price dynamics of the risky asset S_t are featured by the following path-dependent stochastic volatility model:

$$\begin{cases} \dfrac{dS_t}{S_t} = \left(R + \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]}) + \lambda_p\sigma_s\right)\, dt + \sigma_s\, dW_{0,t} + \hat{\sigma}(\alpha_{[0,t]})\, dW_{1,t},\;\; S_0 = s_0 \in \mathbb{R}^+, \\ d\alpha_t = \kappa(\theta - \alpha_t)\, dt + \sqrt{\alpha_t}\left(\rho_1\, dW_{1,t} + \rho_2\, dW_{2,t}\right),\;\; \alpha_0 \in \mathbb{R}^+. \end{cases} \tag{10.2.8}$$

In view of the path-dependence of the return rate and volatility of the risky asset price, the model (10.2.8) is a special case of the non-Markovian stochastic volatility models. For more details on (10.2.8), readers may refer to Siu (2012).

Suppose that there are two asset-liability managers with an initial nominal wealth $\tilde{x}_{i,0} \in \mathbb{R}^+, i = 1, 2$, at time zero, respectively. Denote by $\pi^S_{i,t}$ and $\pi^I_{i,t}$, $i = 1, 2$, the proportions of nominal wealth invested in the risky asset and inflation-linked index bond at time t. Then, $\pi_i := \left\{\pi^S_{i,t}, \pi^I_{i,t}\right\}_{t\in[0,T]}$ denotes the investment strategy for manager i. Let $\tilde{X}^{\pi_i}_t$ be the nominal wealth process associated with π_i, for $i = 1, 2$. Under a self-financing condition, the nominal wealth process of manager i is given by

$$\begin{aligned} \frac{d\tilde{X}^{\pi_i}_t}{\tilde{X}^{\pi_i}_t} &= \left[R + \pi^S_{i,t}\mu_t + (\pi^S_{i,t}\sigma_s\lambda_p + \pi^I_{i,t}\sigma_p\lambda_p)\right]\, dt + (\pi^S_{i,t}\sigma_s + \pi^I_{i,t}\sigma_p)\, dW_{0,t} \\ &\quad + \pi^S_{i,t}\sigma_t\, dW_{1,t},\;\; \tilde{X}^{\pi_i}_0 = \tilde{x}_{i,0}. \end{aligned}$$

Denote by $X^{\pi_i}_t := \tilde{X}^{\pi_i}_t / P_t$ the real wealth after stripping out inflation for manager i, $i = 1, 2$, whose dynamics can be expressed as follows by using Itô's formula:

$$\begin{aligned} \frac{dX^{\pi_i}_t}{X^{\pi_i}_t} &= \left[r + \pi^S_{i,t}\mu_t + (\lambda_p - \mu_p)\left(\pi^S_{i,t}\sigma_s + (\pi^I_{i,t} - 1)\sigma_p\right)\right]\, dt + \left(\pi^S_{i,t}\sigma_s\right. \\ &\quad \left. + (\pi^I_{i,t} - 1)\sigma_p\right)\, dW_{0,t} + \pi^S_{i,t}\sigma_t\, dW_{1,t},\;\; X^{\pi_i}_0 = x_{i,0} = \tilde{x}_{i,0}/p_0 \in \mathbb{R}^+. \end{aligned} \tag{10.2.9}$$

10.2.2 The asset-liability management

Apart from continuous investment in the above financial market, we consider that the asset-liability manager i is subject to a nominal liability commitment with an initial value $n_{i,0} \in \mathbb{R}^+$, for $i = 1, 2$. The nominal liability process $N_{i,t}$ follows

$$\frac{dN_{i,t}}{N_{i,t}} = \mu_i\, dt + \beta_i\, dW_{0,t} + \sigma_i\sqrt{\alpha_t}(\lambda\sqrt{\alpha_t}\, dt + dW_{1,t}),$$

where the constant $\mu_i \in \mathbb{R}$ is the drift coefficient, and $\beta_i, \sigma_i \in \mathbb{R}^+$ are the volatility coefficients measuring how large the impacts of inflation and volatility risks on the dynamics of random liability are, for $i = 1, 2$. In this paper, we consider the case that the random liability $N_{i,t}$ is uncontrollable, i.e., the manager i cannot decide the value of liability by changing his/her investment strategy. Denote by $L_{i,t} = N_{i,t}/P_t$ the inflation-adjusted liability. An application of Itô's formula leads to the following dynamics of $L_{i,t}$ for manager i:

$$\frac{dL_{i,t}}{L_{i,t}} = \left(\mu_i - \mu_p + \sigma_p^2 - \sigma_p\beta_i\right) dt + (\beta_i - \sigma_p)\, dW_{0,t} + \sigma_i\sqrt{\alpha_t}(\lambda\sqrt{\alpha_t}\, dt + dW_{1,t}), \tag{10.2.10}$$

with the initial level of real liability $l_{i,0} = n_{i,0}/p_0 \in \mathbb{R}^+$, for $i = 1, 2$.

10.3 Formulation of a non-zero-sum game

In this paper, the two asset-liability managers aim to choose an admissible strategy $(\pi_1, \pi_2) \in \Pi_1 \otimes \Pi_2$ to maximize their expected utility of terminal inflation-adjusted surplus at time T, where the set of admissible strategies $\Pi_1 \otimes \Pi_2$ will be defined later. In addition, each manager also cares about the difference between his/her terminal surplus and the other's and tries to fare better relative to his/her competitor. We formulate the optimization problem as a non-zero-sum stochastic differential game between the two competitive asset-liability managers. In line with some existing literature, such as Bensoussan et al. (2014), Deng, Zeng, and Zhu (2018), and Savku and Weber (2022), we focus on games with perfect observation, i.e., the managers' strategies are instantaneously revealed to their opponent.

Given the initial values of state variables as above, we define a non-zero-sum stochastic differential game with the following objective functions for $j \neq i \in \{1, 2\}$:

$$\begin{aligned}
&J_i^{(\pi_i,\pi_j)}(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0)\\
:=&\mathbb{E}\left[U_i\left((1 - w_i)(X_T^{\pi_i} - L_{i,T}) + w_i\left((X_T^{\pi_i} - L_{i,T}) - (X_T^{\pi_j} - L_{j,T})\right)\right)\right] \qquad (10.3.1)\\
=&\mathbb{E}\left[U_i\left(X_T^{\pi_i} - L_{i,T} - w_i(X_T^{\pi_j} - L_{j,T})\right)\right],
\end{aligned}$$

where $U_i(\cdot)$ is a strictly increasing and strictly concave smooth utility function for each manager i, and the parameter $w_i \in [0, 1]$, $i = 1, 2$, describes the degree to which manager i values the relevant performance with his/her competitor. A greater value of w_i implies that manager i cares more about his/her relative surplus. In particular, for the case when $w_i = 0$, manager i only considers his/her own terminal inflation-adjusted surplus. Conversely, the specification $w_i = 1$ corresponds to the case when manager i is only interested in the relative surplus.

Definition 10.3.1 (Nash equilibrium strategy)**.** *The classical non-zero-sum stochastic differential game problem is to find a Nash equilibrium strategy* $(\pi_1^*, \pi_2^*) \in \Pi_1 \otimes \Pi_2$ *such that*

$$J_1^{(\pi_1^*,\pi_2^*)}(x_{1,0}, x_{2,0}, l_{1,0}, l_{2,0}, s_0, \alpha_0) \geq J_1^{(\pi_1,\pi_2^*)}(x_{1,0}, x_{2,0}, l_{1,0}, l_{2,0}, s_0, \alpha_0), \tag{10.3.2}$$

and

$$J_2^{(\pi_2^*,\pi_1^*)}(x_{2,0},x_{1,0},l_{2,0},l_{1,0},s_0,\alpha_0) \geq J_2^{(\pi_2,\pi_1^*)}(x_{2,0},x_{1,0},l_{2,0},l_{1,0},s_0,\alpha_0). \quad (10.3.3)$$

If the above two inequalities (10.3.2) and (10.3.3) hold, then the value functions of managers 1 and 2 are given by

$$\begin{aligned} J_1(x_{1,0},x_{2,0},l_{1,0},l_{2,0},s_0,\alpha_0) :=& J_1^{(\pi_1^*,\pi_2^*)}(x_{1,0},x_{2,0},l_{1,0},l_{2,0},s_0,\alpha_0) \\ =& \sup_{\pi_1\in\Pi_1} J_1^{(\pi_1,\pi_2^*)}(x_{1,0},x_{2,0},l_{1,0},l_{2,0},s_0,\alpha_0), \end{aligned} \quad (10.3.4)$$

and

$$\begin{aligned} J_2(x_{2,0},x_{1,0},l_{2,0},l_{1,0},s_0,\alpha_0) :=& J_2^{(\pi_2^*,\pi_1^*)}(x_{2,0},x_{1,0},l_{2,0},l_{1,0},s_0,\alpha_0) \\ =& \sup_{\pi_2\in\Pi_2} J_2^{(\pi_2,\pi_1^*)}(x_{2,0},x_{1,0},l_{2,0},l_{1,0},s_0,\alpha_0), \end{aligned} \quad (10.3.5)$$

and the Nash equilibrium pair (π_1^*,π_2^*) is called the competitively optimal investment strategy.

To proceed, we consider two utility maximization problems when the risk preferences of the two asset-liability managers are characterized by the exponential and power utility functions, respectively, i.e,

$$U_i(x) = -\frac{1}{q_i}e^{-q_i x}, \; x\in\mathbb{R}, \; q_i\in\mathbb{R}^+, \; i=1,2, \quad (10.3.6)$$

and

$$U_i(x) = \frac{x^{\gamma_i}}{\gamma_i}, \; x\in\mathbb{R}^+, \; \gamma_i\in\mathbb{R}^-, \; i=1,2. \quad (10.3.7)$$

Definition 10.3.2. *For the exponential utility preference (10.3.6), the set of admissible strategies $\Pi_1\otimes\Pi_2$ is the set of $\mathbb{F}$-adapted processes (π_1,π_2) such that*

1. *SDE (10.2.9) has a unique strong solution $X_t^{\pi_i}\in\mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, for $i=1,2$;*
2. *the family $\left\{-\frac{1}{q_i}e^{-q_i\left(Y_{i,\tau_n\wedge T}(X^{\pi_i}_{\tau_n\wedge T}-w_iX^{\pi_j}_{\tau_n\wedge T})+G_{i,\tau_n\wedge T}\right)}\right\}_{n\in\mathbb{N}}$ is uniformly integrable, for any sequence of $\mathbb{F}$-stopping times such that $\tau_n\to\infty$, $\mathbb{P}$ almost surely as $n\to\infty$, where $i\neq j\in\{1,2\}$, and processes $Y_{i,t}$ and $G_{i,t}$ are given by (10.4.4) and (10.4.7), respectively.*

Similarly, the definition of the admissible strategies for the power utility case (10.3.7) is formally given as follows.

Definition 10.3.3. *For the power utility perference (10.3.7), the set of admissible strategies $\Pi_1\otimes\Pi_2$ is the set of $\mathbb{F}$-adapted processes (π_1,π_2) such that*

1. *the process* $X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t}$ *is positive,* $\mathbb{P}$ *almost surely, for* $t \in [0,T]$*, where* $\tilde{G}_{i,t}$ *is given by (10.5.9) and* $i \neq j \in \{1,2\}$*;*

2. *SDE (10.2.9) has a unique strong solution* $X_t^{\pi_i} \in \mathcal{L}^0_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$*, for* $i = 1,2$*;*

3. *the family* $\left\{\frac{\tilde{Y}_{i,\tau_n\wedge T}}{\gamma_i}(X^{\pi_i}_{\tau_n\wedge T} - w_i X^{\pi_j}_{\tau_n\wedge T} + \tilde{G}_{i,\tau_n\wedge T})^{\gamma_i}\right\}$ *is uniformly integrable, for any sequence of* $\mathbb{F}$*-stopping times such that* $\tau_n \to \infty$*,* $\mathbb{P}$ *almost surely as* $n \to \infty$*, where* $i \neq j \in \{1,2\}$ *and process* $\tilde{Y}_{i,t}$ *is given by (10.5.4).*

In a Markovian market setting, the Nash equilibrium strategy can be constructed as a solution to a system of coupled HJB partial differential equations; see, for example, Bensoussan et al. (2014), Guan and Liang (2016), Deng, Zeng, and Zhu (2018), and Savku and Weber (2022). However, due to the potentially non-Markovian structures of the market model induced by the two stochastic processes μ_t and σ_t in (10.2.3), the standard dynamic programming principle falls apart in the current context, and so does the associated HJB equation. We, therefore, manage to find the Nash equilibrium (π_1^*, π_2^*) for the problem (10.3.2)-(10.3.3) by opting for a BSDE approach in the Section 10.4 and 10.5.

10.4 Non-zero-sum game for the exponential utility case

In this section, we address the non-zero-sum stochastic differential game (10.3.2)-(10.3.3) for the exponential utility case (10.3.6) by means of BSDE and provide explicit expressions for the competitively optimal investment strategy (π_1^*, π_2^*) as well as the associated value functions (10.3.4) and (10.3.5).

We first introduce the following continuous semi-martingales $Y_{i,t}$ and $G_{i,t}$, for $i = 1,2$, with the canonical decomposition:

$$dY_{i,t} = \Psi_{i,t}\,dt + Z_{i,t}\,dW_{0,t} + M_{i,t}\,dW_{1,t} + P_{i,t}, dW_{2,t},$$

and

$$dG_{i,t} = Q_{i,t}\,dt + H_{i,t}\,dW_{0,t} + \Lambda_{i,t}\,dW_{1,t} + \Gamma_{i,t}\,dW_{2,t},$$

where $\Psi_{i,t}$ and $Q_{i,t}$ are some $\mathbb{F}$-adapted processes that shall be determined in what follows, and $Z_{i,t}, M_{i,t}, P_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t} \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, for $i = 1,2$. Then, applying Itô's formula to the process $-\frac{1}{q_i}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j}) + G_{i,t}\right)}$ shows us the following dynamics:

$$\begin{aligned}
&d\left(-\frac{1}{q_i}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\right)\\
=&e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\Big[(X_t^{\pi_i}-w_iX_t^{\pi_j})Z_{i,t}+Y_{i,t}\Big(X_t^{\pi_i}(\pi^S_{i,t}\sigma_s+(\pi^I_{i,t}-1)\sigma_p)\\
&-w_iX_t^{\pi_j}(\pi^S_{j,t}\sigma_s+(\pi^I_{j,t}-1)\sigma_p)\Big)+H_{i,t}\Big]\,dW_{0,t}+e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\Big[(X_t^{\pi_i}-w_iX_t^{\pi_j})M_{i,t}
\end{aligned}$$

$$
\begin{aligned}
&+ Y_{i,t}(X_t^{\pi_i}\pi_{i,t}^S - w_i X_t^{\pi_j}\pi_{j,t}^S)\sigma_t + \Lambda_{i,t}\Big]\, dW_{1,t} + e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\Big((X_t^{\pi_i} - w_i X_t^{\pi_j})P_{i,t}\\
&+\Gamma_{i,t})\, dW_{2,t} - \frac{q_i}{2}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\Big[(X_t^{\pi_i} - w_iX_t^{\pi_j})Z_{i,t} + Y_{i,t}\Big(X_t^{\pi_i}(\pi_{i,t}^S\sigma_s\\
&+(\pi_{i,t}^I - 1)\sigma_p) - w_iX_t^{\pi_j}(\pi_{j,t}^S\sigma_s + (\pi_{j,t}^I - 1)\sigma_p)\Big) + H_{i,t} - \frac{1}{q_i}\left(\lambda_p - \sigma_p + \frac{Z_{i,t}}{Y_{i,t}}\right)\Big]^2 dt\\
&- \frac{q_i}{2}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\Big[(X_t^{\pi_i} - w_iX_t^{\pi_j})M_{i,t} + Y_{i,t}(X_t^{\pi_i}\pi_{i,t}^S - w_iX_t^{\pi_j}\pi_{j,t}^S)\sigma_t + \Lambda_{i,t}\\
&-\frac{1}{q_i}\left(\lambda\sqrt{\alpha_t} + \frac{M_{i,t}}{Y_{i,t}}\right)\Big]^2 dt - \frac{q_i}{2}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}(X_t^{\pi_i} - w_iX_t^{\pi_j})^2P_{i,t}^2\, dt\\
&- q_ie^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}(X_t^{\pi_i} - w_iX_t^{\pi_j})P_{i,t}\Gamma_{i,t}\, dt + e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}(X_t^{\pi_i}\\
&- w_iX_t^{\pi_j})\left[\Psi_{i,t} + rY_{i,t} - Z_{i,t}\left(\lambda_p - \sigma_p + \frac{Z_{i,t}}{Y_{i,t}}\right) - M_{i,t}\left(\lambda\sqrt{\alpha_t} + \frac{M_{i,t}}{Y_{i,t}}\right)\right] dt\\
&+ e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}\left[Q_{i,t} + \frac{\left(\lambda_p - \sigma_p + \frac{Z_{i,t}}{Y_{i,t}}\right)^2}{2q_i} + \frac{\left(\lambda\sqrt{\alpha_t} + \frac{M_{i,t}}{Y_{i,t}}\right)^2}{2q_i}\right.\\
&\left.- H_{i,t}\left(\lambda_p - \sigma_p + \frac{Z_{i,t}}{Y_{i,t}}\right) - \Lambda_{i,t}\left(\lambda\sqrt{\alpha_t} + \frac{M_{i,t}}{Y_{i,t}}\right) - \frac{q_i\Gamma_{i,t}^2}{2}\right] dt, \text{ for } i \neq j \in \{1,2\}.
\end{aligned}
\tag{10.4.1}
$$

We expect given any $\pi_j \in \Pi_j$, the stochastic process $-\frac{1}{q_i}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i}-w_iX_t^{\pi_j})+G_{i,t}\right)}$ is a (local) super-martingale for any admissible strategy and is a (local) martingale for the optimal response strategy. In other words, we expect that the drift in (10.4.1) is non-positive for any $\pi_i \in \Pi_i$ and zero for the optimal response strategy denoted by $\tilde{\pi}_i \in \Pi_i$ whenever $\pi_j \in \Pi_j$ is fixed and given, where $i \neq j \in \{1,2\}$. Therefore, the two stochastic processes $\Psi_{i,t}$ and $Q_{i,t}$ can be determined by letting the last two drift terms on the right-hand side of (10.4.1) be zeros. As a result, we have the following BSRE of $(Y_{i,t}, Z_{i,t}, M_{i,t}, P_{i,t})$:

$$
\begin{cases}
dY_{i,t} = \left[-rY_{i,t} + \left(\lambda_p - \sigma_p + \dfrac{Z_{i,t}}{Y_{i,t}}\right)Z_{i,t} + \left(\lambda\sqrt{\alpha_t} + \dfrac{M_{i,t}}{Y_{i,t}}\right)M_{i,t}\right] dt\\
\qquad\qquad + Z_{i,t}\, dW_{0,t} + M_{i,t}\, dW_{1,t} + P_{i,t}\, dW_{2,t},\\
Y_{i,T} = 1,
\end{cases}
\tag{10.4.2}
$$

and quadratic BSDE of $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t})$:

$$
\begin{cases}
dG_{i,t} = \left[\left(\lambda_p - \sigma_p + \dfrac{Z_{i,t}}{Y_{i,t}}\right)H_{i,t} + \left(\lambda\sqrt{\alpha_t} + \dfrac{M_{i,t}}{Y_{i,t}}\right)\Lambda_{i,t} + \dfrac{q_i}{2}\Gamma_{i,t}^2 - \dfrac{\left(\lambda_p - \sigma_p + \frac{Z_{i,t}}{Y_{i,t}}\right)^2}{2q_i}\right.\\
\qquad\qquad \left.-\dfrac{\left(\lambda\sqrt{\alpha_t} + \frac{M_{i,t}}{Y_{i,t}}\right)^2}{2q_i}\right] dt + H_{i,t}\, dW_{0,t} + \Lambda_{i,t}\, dW_{1,t} + \Gamma_{i,t}\, dW_{2,t},\\
G_{i,T} = -L_{i,T} + w_iL_{j,T}.
\end{cases}
\tag{10.4.3}
$$

where $i \neq j \in \{1,2\}$. It is worth mentioning that the terminal conditions in (10.4.2) and (10.4.3) follow from the term in the expectation (10.3.1). Here, by a solution to

BSDE (10.4.2), we mean a quadruplet of stochastic processes $(Y_{i,t}, Z_{i,t}, M_{i,t}, P_{i,t}) \in \mathcal{S}^{\infty}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. Similarly, a solution to BSDE (10.4.3) is a quadruplet of stochastic processes $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t}) \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$.

Remark 10.4.1. Note that BSDEs (10.4.2)-(10.4.3) form a coupled BSDE system. The generator of quadratic BSDE (10.4.3) involves the solution to BSRE (10.4.2). This finding shows that the two BSDEs shall be addressed recursively forwards in the sequel. Lim (2004) shows the existence and uniqueness result for a BSRE similar to (10.4.2) for the case with uniformly bounded random parameters in the dynamics of asset prices. However, since the generator involves α_t which is not uniformly bounded, the results of Lim (2004) cannot be applied to our case. In the following proposition, we derive a closed-form solution to BSRE (10.4.2) and prove the uniqueness of the solution using the results of Kobylanski (2000) and Girsanov's measure change techniques.

For the reader's convenience, we present the following auxiliary result (refer to Lemma 4.3 in Zeng and Taksar (2013) or Lemma A1 in Shen and Zeng (2015)) assisting the proof in the main body of this paper.

Lemma 10.4.2 (Bona-fide martingale property)**.** *If $a_1(t)$ and $a_2(t)$ are two bounded functions over $[0,T]$, the following stochastic exponential process*

$$\exp\left\{\int_0^t a_1(s)\sqrt{\alpha_s}\, dW_{1,s} + \int_0^t a_2(s)\sqrt{\alpha_s}\, dW_{2,s} - \frac{1}{2}\int_0^t \left(a_1^2(s) + a_2^2(s)\right)\alpha_s\, ds\right\}$$

is an $(\mathbb{F},\mathbb{P})$-martingale.

Proposition 10.4.3. *The unique solution $(Y_{i,t}, Z_{i,t}, M_{i,t}, P_{i,t})$ to BSRE (10.4.2), for $i = 1, 2$, is given by*

$$Y_{i,t} = e^{r(T-t)}, \tag{10.4.4}$$

and

$$(Z_{i,t}, M_{i,t}, P_{i,t}) = (0, 0, 0). \tag{10.4.5}$$

Proof. See Appendix 10.A. □

Remark 10.4.4. It is interesting to identify that the control components of BSRE (10.4.2) are zeros, from which we find that the third and fourth drift terms on the right-hand side of (10.4.1) vanish. Moreover, quadratic BSDE (10.4.3) can be substantially simplified so that the generator of BSDE (10.4.3) does not involve the solution to BSRE (10.4.2).

Having solved BSRE (10.4.2) explicitly, we can rewrite BSDE (10.4.3) of $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t})$ as follows, for $i \neq j \in \{1, 2\}$,

$$\begin{cases} dG_{i,t} = \left((\lambda_p - \sigma_p)H_{i,t} + \lambda\sqrt{\alpha_t}\Lambda_{i,t} + \frac{q_i}{2}\Gamma_{i,t}^2 - \frac{(\lambda_p - \sigma_p)^2}{2q_i} - \frac{\lambda^2\alpha_t}{2q_i} \right) dt \\ \qquad\qquad + H_{i,t}\, dW_{0,t} + \Lambda_{i,t}\, dW_{1,t} + \Gamma_{i,t}\, dW_{2,t} \\ G_{i,T} = -L_{i,T} + w_i L_{j,T}. \end{cases} \tag{10.4.6}$$

In the following proposition, we derive one explicit solution to BSDE (10.4.6) by using the Markovian structures of factor process α_t and liability processes $L_{i,t}, i = 1, 2$.

Proposition 10.4.5. *One closed-form solution $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t})$ to quadratic BSDE (10.4.6) is given by*

$$G_{i,t} = f_i(t) + g_i(t)\alpha_t + h_i(t)L_{i,t} - w_i m_i(t)L_{j,t} \tag{10.4.7}$$

and

$$\begin{cases} H_{i,t} = (\beta_i - \sigma_p)h_i(t)L_{i,t} - w_i(\beta_j - \sigma_p)m_i(t)L_{j,t}, \\ \Lambda_{i,t} = (\rho_1 g_i(t) + \sigma_i h_i(t)L_{i,t} - w_i\sigma_j m_i(t)L_{j,t})\sqrt{\alpha_t}, \\ \Gamma_{i,t} = \rho_2 g_i(t)\sqrt{\alpha_t}, \end{cases} \tag{10.4.8}$$

where functions $f_i(t), g_i(t), h_i(t)$, and $m_i(t)$ solve the following ordinary differential equations (ODEs):

$$\begin{cases} \frac{dg_i(t)}{dt} = \frac{q_i}{2}\rho_2^2 g_i^2(t) + (\kappa + \lambda\rho_1)g_i(t) - \frac{\lambda^2}{2q_i}, \ g_i(T) = 0; \\ \frac{dh_i(t)}{dt} = (\lambda_p(\beta_i - \sigma_p) + \mu_p - \mu_i)\, h_i(t), \ h_i(T) = -1; \\ \frac{dm_i(t)}{dt} = (\lambda_p(\beta_j - \sigma_p) + \mu_p - \mu_j)\, m_i(t), \ m_i(T) = -1; \\ \frac{df_i(t)}{dt} = -\kappa\theta g_i(t) + \frac{(\lambda_p - \sigma_p)^2}{2q_i}, \ f_i(T) = 0, \end{cases} \tag{10.4.9}$$

for $i \neq j \in \{1, 2\}$.

Proof. See Appendix 10.B. □

Proposition 10.4.5 transforms the problem of finding a solution to quadratic BSDE (10.4.6) into the determination of solutions to ODEs (10.4.9). In the following proposition, we derive, in closed form, the solutions to ODEs (10.4.9) for $i \neq j \in \{1, 2\}$. Before that, we impose the following assumption on the model parameters through out the rest of this section:

Assumption 10.4.6. $\kappa + \lambda\rho_1 \neq 0$.

Remark 10.4.7. It is worth mentioning that Assumption 10.4.6 does not simplify the calculations for deriving the solutions to ODEs (10.4.9) but ensures that the factor process α_t preserves the affine-form, square-root structure under another equivalent probability measure as shown in the proof of Proposition 10.4.10 below.

Proposition 10.4.8. *Under Assumption 10.4.6, closed-form solutions to ODEs (10.4.9) are given by*

$$g_i(t) = \begin{cases} -\dfrac{\lambda^2}{2q_i(\kappa+\lambda\rho_1)}\left(e^{(\kappa+\lambda\rho_1)(t-T)}-1\right), & \text{if } \rho_2 = 0; \\ \dfrac{n_{g_i^+}n_{g_i^-}\left(1-e^{\sqrt{\Delta_{g_i}}(T-t)}\right)}{n_{g_i^+}-n_{g_i^-}e^{\sqrt{\Delta_{g_i}}(T-t)}}, & \text{if } \rho_2 \neq 0, \end{cases} \tag{10.4.10}$$

$$h_i(t) = -e^{(\lambda_p(\beta_i-\sigma_p)+\mu_p-\mu_i)(t-T)}, \tag{10.4.11}$$

$$m_i(t) = -e^{(\lambda_p(\beta_j-\sigma_p)+\mu_p-\mu_j)(t-T)}, \tag{10.4.12}$$

and

$$f_i(t) = \begin{cases} \dfrac{-\lambda^2\kappa\theta\left(1-e^{(\kappa+\lambda\rho_1)(t-T)}\right)}{2q_i(\kappa+\lambda\rho_1)^2} + \left(\dfrac{-\lambda^2\kappa\theta}{2q_i(\kappa+\lambda\rho_1)} + \dfrac{(\lambda_p-\sigma_p)^2}{2q_i}\right)(t-T), & \text{if } \rho_2 = 0; \\ \dfrac{\kappa\theta\left(n_{g_i^-}-n_{g_i^+}\right)}{\sqrt{\Delta_{g_i}}}\log\left(\dfrac{n_{g_i^+}-n_{g_i^-}}{n_{g_i^+}-n_{g_i^-}e^{\sqrt{\Delta_{g_i}}(T-t)}}\right) & \\ +\left(\dfrac{(\lambda_p-\sigma_p)^2}{2q_i}-\kappa\theta n_{g_i^-}\right)(t-T), & \text{if } \rho_2 \neq 0, \end{cases} \tag{10.4.13}$$

where $\Delta_{g_i}, n_{g_i^+}$, *and* $n_{g_i^-}$ *are given by*

$$\Delta_{g_i} = (\kappa+\lambda\rho_1)^2+\rho_2^2\lambda^2,\ n_{g_i^+} = \frac{-(\kappa+\lambda\rho_1)+\sqrt{\Delta_{g_i}}}{q_i\rho_2^2}.\ n_{g_i^-} = \frac{-(\kappa+\lambda\rho_1)-\sqrt{\Delta_{g_i}}}{q_i\rho_2^2}, \tag{10.4.14}$$

for $i \neq j \in \{1,2\}$.

Proof. See Appendix 10.C. □

Remark 10.4.9. From Proposition 10.4.8, it is easy to verify that $g_i(t)$ is a positive and bounded function on $[0,T]$, for $i = 1,2$, which, combined with Assumption 10.4.6, implies that the hypothesis in Lemma 10.4.2 is satisfied under another equivalent probability measure given in the proof of Proposition 10.4.10.

Combining Proposition 10.4.5 and 10.4.8, we have found one solution to BSDE (10.4.6) in a closed-form, which is given by (10.4.7)-(10.4.8). In the following proposition, we prove that it is the unique solution to BSDE (10.4.6).

Proposition 10.4.10. *Under Assumption 10.4.6, the solution* $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t})$ *given by (10.4.7)-(10.4.8) is the unique solution to quadratic BSDE (10.4.6) in* $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$*, for* $i \in \{1,2\}$.

Proof. See Appendix 10.D. □

We are now in the position to provide the explicit expressions of the competitively optimal investment strategy (π_1^*, π_2^*) and the value functions of manager i, $i = 1, 2$, for the non-zero-sum stochastic differential game problem (10.3.2)-(10.3.3), which are expressed in terms of the unique solution to BSRE (10.4.2) and BSDE (10.4.6).

Theorem 10.4.11 (Nash equilibrium to the ALM game)**.** *Under Assumption 10.4.6, the Nash equilibrium pair* (π_1^*, π_2^*) *for the ALM game with the exponential utility preference (10.3.6) is as follows:*

$$\begin{cases} \pi_{i,t}^{S*} = \dfrac{1}{\sigma_t X_t^{\pi_i^*}(1 - w_i w_j)} \left(\dfrac{\frac{\lambda}{q_i}\sqrt{\alpha_t} - \Lambda_{i,t}}{Y_{i,t}} + w_i \dfrac{\frac{\lambda}{q_j}\sqrt{\alpha_t} - \Lambda_{j,t}}{Y_{j,t}} \right), \\ \pi_{i,t}^{I*} = \dfrac{1}{\sigma_p X_t^{\pi_i^*}(1 - w_i w_j)} \left(\dfrac{\frac{\lambda_p - \sigma_p}{q_i} - H_{i,t}}{Y_{i,t}} + w_i \dfrac{\frac{\lambda_p - \sigma_p}{q_j} - H_{j,t}}{Y_{j,t}} \right) - \dfrac{\sigma_s}{\sigma_p} \pi_{i,t}^{S*} + 1, \end{cases} \tag{10.4.15}$$

and the value functions are given by

$$J_i(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0) = -\frac{1}{q_i} \exp\left\{ -q_i (Y_{i,0}(x_{i,0} - w_i x_{j,0}) + G_{i,0}) \right\} \tag{10.4.16}$$

for $i \neq j \in \{1,2\}$*, where the explicit expressions for* $Y_{i,t}, G_{i,t}, \Lambda_{i,t}$*, and* $H_{i,t}$ *are given by (10.4.4), (10.4.7), and (10.4.8), respectively. Moreover,* $(\pi_1^*, \pi_2^*) \in \Pi_1 \otimes \Pi_2$.

Proof. See Appendix 10.E. □

Remark 10.4.12. From the proof of Theorem 10.4.11, we find the herd effect on managers' decisions, that is, managers will mimic their competitor's strategy. Specifically, it can be found from (10.E.4)-(10.E.5) that the optimal response strategy given the competitor's strategy can be decomposed into two parts. The first part is the case where $w_i = 0$, i.e., the manager i is only concerned about the partial objective of maximizing the expectation of terminal surplus, which is similar to Merton-type solution consisting of a multiplier, a myopic (time-independent) component, and an inter-temporal hedging component. The second part shows the effect of competition on their investment strategies. More precisely, when the competitor j increases his/her risk exposure, i.e., $X_t^{\hat{\pi}_j}(\hat{\pi}_{j,t}^S \sigma_s + (\hat{\pi}_{j,t}^I - 1)\sigma_p)$ and $X_t^{\hat{\pi}_j} \hat{\pi}_{j,t}^S \sigma_t$ to the two fundamental risk factors $W_{0,t}$ and $W_{1,t}$, (10.E.4)-(10.E.5) reveal that the manager i will adopt a riskier investment strategy, for $i \neq j \in \{1,2\}$.

Remark 10.4.13. When $w_1 = w_2 = 0$, the equilibrium investment strategy $\pi_{i,t}^{S*}$ and $\pi_{i,t}^{I*}$ given in (10.4.15), for $i = 1, 2$, is reduced to the classical optimal investment strategy for ALM with inflation and volatility risks under exponential utility without relative performance concerns.

Corollary 10.4.14 (Nash equilibrium to the investment game)**.** *Under Assumption 10.4.6, the Nash equilibrium pair* (π_1^*, π_2^*) *for the pure investment game with the exponential utility preference (10.3.6) is given by*

$$\begin{cases} \pi_{i,t}^{S*} = \dfrac{1}{\sigma_t X_t^{\pi_i^*}(1 - w_i w_j)} \left(\dfrac{\frac{\lambda}{q_i} - \rho_1 g_i(t)}{Y_{i,t}} + w_i \dfrac{\frac{\lambda}{q_j} - \rho_1 g_j(t)}{Y_{j,t}} \right) \sqrt{\alpha_t}, \\ \pi_{i,t}^{I*} = \dfrac{1}{\sigma_p X_t^{\pi_i^*}(1 - w_i w_j)} \left(\dfrac{\lambda_p - \sigma_p}{q_i Y_{i,t}} + w_i \dfrac{\lambda_p - \sigma_p}{q_j Y_{j,t}} \right) - \dfrac{\sigma_s}{\sigma_p} \pi_{i,t}^{S*} + 1, \end{cases}$$

and the value functions are given by

$$J_i(x_{i,0}, x_{j,0}, s_0, \alpha_0) = -\frac{1}{q_i} \exp\left\{-q_i(Y_{i,0}(x_{i,0} - w_i x_{j,0}) + f_i(0) + g_i(0)\alpha_0)\right\},$$

for $i \neq j \in \{1, 2\}$*.*

Proof. Substituting the specification $\mu_i = \beta_i = \sigma_i = l_{i,0} = 0, i = 1, 2$, into (10.4.15) and (10.4.16) yields the above results. □

Corollary 10.4.15 (Nash equilibrium to the ALM game under the 4/2 model)**.** *If the risky asset price process* S_t *and the variance driver process* α_t *follow the family of 4/2 stochastic volatility model (10.2.7) and suppose that* $\kappa + \sigma_\alpha \rho \lambda \neq 0$*, the Nash equilibrium pair* (π_1^*, π_2^*) *for the ALM game with the exponential utility preference (10.3.6) is given by*

$$\begin{cases} \pi_{i,t}^{S*} = \dfrac{1}{(c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}) X_t^{\pi_i^*}(1 - w_i w_j)} \left(\dfrac{\frac{\lambda}{q_i}\sqrt{\alpha_t} - \bar{\Lambda}_{i,t}}{Y_{i,t}} + w_i \dfrac{\frac{\lambda}{q_j}\sqrt{\alpha_t} - \bar{\Lambda}_{j,t}}{Y_{j,t}} \right), \\ \pi_{i,t}^{I*} = \dfrac{1}{\sigma_p X_t^{\pi_i^*}(1 - w_i w_j)} \left(\dfrac{\frac{\lambda_p - \sigma_p}{q_i} - H_{i,t}}{Y_{i,t}} + w_i \dfrac{\frac{\lambda_p - \sigma_p}{q_j} - H_{j,t}}{Y_{j,t}} \right) - \dfrac{\sigma_s}{\sigma_p} \pi_{i,t}^{S*} + 1, \end{cases}$$

and the value functions are given by

$$J_i(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0) = -\frac{1}{q_i} \exp\left\{-q_i(Y_{i,0}(x_{i,0} - w_i x_{j,0}) + \bar{G}_{i,0})\right\},$$

where the explicit expressions for $\bar{G}_{i,t}$ *and* $\bar{\Lambda}_{i,t}$ *are as follows:*

$$\bar{G}_{i,t} = \bar{f}_i(t) + \bar{g}_i(t)\alpha_t + h_i(t)L_{i,t} - w_i m_i(t) L_{j,t},$$

and

$$\bar{\Lambda}_{i,t} = (\sigma_\alpha \rho \bar{g}_i(t) + \sigma_i h_i(t) L_{i,t} - w_i \sigma_j m_i(t) L_{j,t}) \sqrt{\alpha_t},$$

where functions $\bar{g}_i(t)$ *and* $\bar{f}_i(t)$ *are given by*

$$\bar{g}_i(t) = \begin{cases} -\dfrac{\lambda^2}{2q_i(\kappa+\lambda\sigma_\alpha\rho)}\left(e^{(\kappa+\lambda\sigma_\alpha\rho)(t-T)}-1\right), & \text{if } \rho = \pm 1; \\ \dfrac{n_{\bar{g}_i^+}n_{\bar{g}_i^-}\left(1-e^{\sqrt{\Delta_{\bar{g}_i}}(T-t)}\right)}{n_{\bar{g}_i^+}-n_{\bar{g}_i^-}e^{\sqrt{\Delta_{\bar{g}_i}}(T-t)}}, & \text{if } \rho \neq \pm 1, \end{cases}$$

and

$$\bar{f}_i(t) = \begin{cases} \dfrac{-\lambda^2\kappa\theta\left(1-e^{(\kappa+\lambda\sigma_\alpha\rho)(t-T)}\right)}{2q_i(\kappa+\lambda\sigma_\alpha\rho)^2}+\left(\dfrac{-\lambda^2\kappa\theta}{2q_i(\kappa+\lambda\sigma_\alpha\rho)}+\dfrac{(\lambda_p-\sigma_p)^2}{2q_i}\right)(t-T), & \text{if } \rho = \pm 1; \\ \dfrac{\kappa\theta\left(n_{\bar{g}_i^-}-n_{\bar{g}_i^+}\right)}{\sqrt{\Delta_{\bar{g}_i}}}\log\left(\dfrac{n_{\bar{g}_i^+}-n_{\bar{g}_i^-}}{n_{\bar{g}_i^+}-n_{\bar{g}_i^-}e^{\sqrt{\Delta_{\bar{g}_i}}(T-t)}}\right) & \\ +\left(\dfrac{(\lambda_p-\sigma_p)^2}{2q_i}-\kappa\theta n_{\bar{g}_i^-}\right)(t-T), & \text{if } \rho \neq \pm 1, \end{cases}$$

with $\Delta_{\bar{g}_i}, n_{\bar{g}_i^+}, n_{\bar{g}_i^-}$ *given by*

$$\Delta_{\bar{g}_i} = (\kappa+\lambda\sigma_\alpha\rho)^2+\sigma_\alpha^2\lambda^2(1-\rho^2), \; n_{\bar{g}_i^+} = \frac{-(\kappa+\lambda\sigma_\alpha\rho)+\sqrt{\Delta_{\bar{g}_i}}}{q_i\sigma_\alpha^2(1-\rho^2)}, \; n_{\bar{g}_i^-} = \frac{-(\kappa+\lambda\sigma_\alpha\rho)-\sqrt{\Delta_{\bar{g}_i}}}{q_i\sigma_\alpha^2(1-\rho^2)},$$

for $i \neq j \in \{1, 2\}$*.*

Proof. Plugging the specified parameters of the 4/2 model (10.2.7) in Example 10.2.2 into Theorem 10.4.11 yields the above results immediately. □

Remark 10.4.16. By specifying $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$ in Corollary 10.4.15, we derive the Nash equilibrium strategies and the value functions for the ALM games under the Heston model and 3/2 model with inflation risk, respectively. To the best of our knowledge, these results are not provided in the existing literature.

10.5 Non-zero-sum game for the power utility case

This section solves the non-zero-sum stochastic differential game (10.3.2)-(10.3.3) for the power utility case (10.3.7) by using a BSDE approach.

Similar to the previous section, we introduce the following two $(\mathbb{F}, \mathbb{P})$-semi-martingales $\tilde{Y}_{i,t}$ and $\tilde{G}_{i,t}$ to find the BSDEs associated with the problem:

$$d\tilde{Y}_{i,t} = \tilde{\Psi}_{i,t}\,dt + \tilde{Z}_{i,t}\,dW_{0,t} + \tilde{M}_{i,t}\,dW_{1,t} + \tilde{P}_{i,t}\,dW_{2,t},$$

and

$$d\tilde{G}_{i,t} = \tilde{Q}_{i,t}\,dt + \tilde{H}_{i,t}\,dW_{0,t} + \tilde{\Lambda}_{i,t}\,dW_{1,t} + \tilde{\Gamma}_{i,t}\,dW_{2,t},$$

where $\tilde{\Psi}_{i,t}$ and $\tilde{Q}_{i,t}$ are $\mathbb{F}$-adapted processes that will be determined later, and $\tilde{Z}_{i,t}, \tilde{M}_{i,t}, \tilde{P}_{i,t}, \tilde{H}_{i,t}, \tilde{\Lambda}_{i,t}, \tilde{\Gamma}_{i,t} \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$, for $i = 1, 2$. Then, an application of

Itô's formula to $\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}$ combined with the method of completion of squares yields

$$
\begin{aligned}
& d\left(\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}\right) \\
=& \Bigg[\frac{(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}}{\gamma_i}\tilde{Z}_{i,t} + \tilde{Y}_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 1}\Big(X_t^{\pi_i}(\pi_{i,t}^S \sigma_s + (\pi_{i,t}^I - 1)\sigma_p) \\
& - w_i X_t^{\pi_j}(\pi_{j,t}^S \sigma_s + (\pi_{j,t}^I - 1)\sigma_p) + \tilde{H}_{i,t}\Big)\Bigg] dW_{0,t} + \Bigg[\frac{(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}}{\gamma_i}\tilde{M}_{i,t} \\
& + \tilde{Y}_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 1}\Big((X_t^{\pi_i}\pi_{i,t}^S - w_i X_t^{\pi_j}\pi_{j,t}^S)\sigma_t + \tilde{\Lambda}_{i,t}\Big)\Bigg] dW_{1,t} \\
& + \Bigg[\frac{(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}}{\gamma_i}\tilde{P}_{i,t} + \tilde{Y}_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 1}\tilde{\Gamma}_{i,t}\Bigg] dW_{2,t} + \frac{\gamma_i - 1}{2}\tilde{Y}_{i,t} \\
& \times (X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 2}\Bigg[(X_t^{\pi_i}\pi_{i,t}^S - w_i X_t^{\pi_j}\pi_{j,t}^S)\sigma_t + \tilde{\Lambda}_{i,t} + \frac{X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t}}{\gamma_i - 1}\Big(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} \\
& + \lambda\sqrt{\alpha_t}\Big)\Bigg]^2 dt + \frac{\gamma_i - 1}{2}\tilde{Y}_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 2}\Bigg[X_t^{\pi_i}(\pi_{i,t}^S \sigma_s + (\pi_{i,t}^I - 1)\sigma_p) - w_i X_t^{\pi_j} \\
& \times (\pi_{j,t}^S \sigma_s + (\pi_{j,t}^I - 1)\sigma_p) + \tilde{H}_{i,t} + \frac{X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t}}{\gamma_i - 1}\Big(\frac{\tilde{Z}_{i,t}}{\tilde{Y}_{i,t}} + \lambda_p - \sigma_p\Big)\Bigg]^2 dt \\
& + \frac{\gamma_i - 1}{2}\tilde{Y}_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 2}\tilde{\Gamma}_{i,t}^2\, dt + (X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}\Bigg[\frac{\tilde{\Psi}_{i,t}}{\gamma_i} + r\tilde{Y}_{i,t} \\
& - \frac{1}{2(\gamma_i - 1)}\tilde{Y}_{i,t}\Big(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} + \lambda\sqrt{\alpha_t}\Big)^2 - \frac{1}{2(\gamma_i - 1)}\tilde{Y}_{i,t}\Big(\frac{\tilde{Z}_{i,t}}{\tilde{Y}_{i,t}} + \lambda_p - \sigma_p\Big)^2\Bigg] dt \\
& + \tilde{Y}_{i,t}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i - 1}\Big(\tilde{Q}_{i,t} - r\tilde{G}_{i,t} - \lambda\sqrt{\alpha_t}\tilde{\Lambda}_{i,t} - (\lambda_p - \sigma_p)\tilde{H}_{i,t} + \frac{\tilde{P}_{i,t}}{\tilde{Y}_{i,t}}\tilde{\Gamma}_{i,t}\Big) dt.
\end{aligned}
\tag{10.5.1}
$$

Given any admissible strategy π_j of manager j, we expect that the stochastic process $\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\pi_i} - w_i X_t^{\pi_j} + \tilde{G}_{i,t})^{\gamma_i}$ is a (local) super-martingale for any admissible strategy π_i, and even a (local) martingale for the optimal response strategy, which indicates that we should opt for $\tilde{\Psi}_{i,t}$ and $\tilde{Q}_{i,t}$ such that the last two drift terms on the right-hand side of (10.5.1) turn out to be zeros. This finding, combined with the boundary condition at time T, leads to the following BSRE of $(\tilde{Y}_{i,t}, \tilde{Z}_{i,t}, \tilde{M}_{i,t}, \tilde{P}_{i,t})$:

$$
\begin{cases}
d\tilde{Y}_{i,t} = \Bigg[\Big(-r\gamma_i + \frac{\gamma_i}{2(\gamma_i - 1)}(\lambda_p - \sigma_p)^2 + \frac{\gamma_i}{2(\gamma_i - 1)}\lambda^2\alpha_t\Big)\tilde{Y}_{i,t} + \frac{\gamma_i}{\gamma_i - 1}(\lambda_p - \sigma_p)\tilde{Z}_{i,t} \\
\qquad + \frac{\gamma_i}{\gamma_i - 1}\lambda\sqrt{\alpha_t}\tilde{M}_{i,t} + \frac{\gamma_i}{2(\gamma_i - 1)}\frac{\tilde{Z}_{i,t}^2 + \tilde{M}_{i,t}^2}{\tilde{Y}_{i,t}}\Bigg] dt + \tilde{Z}_{i,t}\, dW_{0,t} + \tilde{M}_{i,t}\, dW_{1,t} + \tilde{P}_{i,t}\, dW_{2,t}, \\
\tilde{Y}_{i,T} = 1, \\
\tilde{Y}_{i,t} > 0, \text{ for all } t \in [0, T),
\end{cases}
\tag{10.5.2}
$$

and linear BSDE of $(\tilde{G}_{i,t}, \tilde{H}_{i,t}, \tilde{\Lambda}_{i,t}, \tilde{\Gamma}_{i,t})$:

$$
\begin{cases}
d\tilde{G}_{i,t} = \Big(r\tilde{G}_{i,t} + \lambda\sqrt{\alpha_t}\tilde{\Lambda}_{i,t} + (\lambda_p - \sigma_p)\tilde{H}_{i,t} - \frac{\tilde{P}_{i,t}}{\tilde{Y}_{i,t}}\tilde{\Gamma}_{i,t}\Big) dt + \tilde{H}_{i,t}\, dW_{0,t} \\
\qquad + \tilde{\Lambda}_{i,t}\, dW_{1,t} + \tilde{\Gamma}_{i,t}\, dW_{2,t}, \\
\tilde{G}_{i,T} = -L_{i,T} + w_i L_{j,T}.
\end{cases}
\tag{10.5.3}
$$

For $i = 1, 2$, a solution to BSRE (10.5.2) is a quadruplet of stochastic processes $(\tilde{Y}_{i,t}, \tilde{Z}_{i,t}, \tilde{M}_{i,t}, \tilde{P}_{i,t}) \in \mathcal{S}_{\mathbb{F},\mathbb{P}}^{\infty}(0, T; \mathbb{R}) \otimes \mathcal{L}_{\mathbb{F},\mathbb{P}}^2(0, T; \mathbb{R}) \otimes \mathcal{L}_{\mathbb{F},\mathbb{P}}^2(0, T; \mathbb{R}) \otimes \mathcal{L}_{\mathbb{F},\mathbb{P}}^2(0, T; \mathbb{R})$;

similarly, a solution to BSDE (10.5.3) is a quadruplet of stochastic processes $(\tilde{G}_{i,t}, \tilde{H}_{i,t}, \tilde{\Lambda}_{i,t}, \tilde{\Gamma}_{i,t}) \in \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$. It is worth mentioning that the above BSRE (10.5.2) is essentially the same as Eq. (15) in Zhang (2022c). Therefore, following the proof of Proposition 3.2 in Zhang (2022c), we can immediately provide one explicit solution to BSRE (10.5.2) in the next proposition.

Proposition 10.5.1. *One closed-form solution* $(\tilde{Y}_{i,t}, \tilde{Z}_{i,t}, \tilde{M}_{i,t}, \tilde{P}_{i,t})$ *to BSRE (10.5.2), for* $i = 1, 2$, *is given by*

$$\tilde{Y}_{i,t} = \exp\left\{\tilde{f}_i(t) + \tilde{g}_i(t)\alpha_t\right\}, \tag{10.5.4}$$

and

$$\left(\tilde{Z}_{i,t}, \tilde{M}_{i,t}, \tilde{P}_{i,t}\right) = \left(0, \rho_1\tilde{g}_i(t)\sqrt{\alpha_t}\tilde{Y}_{i,t}, \rho_2\tilde{g}_i(t)\sqrt{\alpha_t}\tilde{Y}_{i,t}\right), \tag{10.5.5}$$

where $\tilde{g}_i(t)$ *and* $\tilde{f}_i(t)$ *are as follows:*

$$\tilde{g}_i(t) = \frac{n_{\tilde{g}_i^+}n_{\tilde{g}_i^-}\left(1 - e^{\sqrt{\Delta_{\tilde{g}_i}}(T-t)}\right)}{n_{\tilde{g}_i^+} - n_{\tilde{g}_i^-}e^{\sqrt{\Delta_{\tilde{g}_i}}(T-t)}}, \tag{10.5.6}$$

and

$$\begin{aligned}\tilde{f}_i(t) =& \int_t^T \left(r\gamma_i - \frac{\gamma_i}{2(\gamma_i - 1)}(\lambda_p - \sigma_p)^2 + \kappa\theta\tilde{g}_i(s)\right) ds \\ =& \frac{\kappa\theta\left(n_{\tilde{g}_i^-} - n_{\tilde{g}_i^+}\right)}{\sqrt{\Delta_{\tilde{g}_i}}} \log\left(\frac{n_{\tilde{g}_i^+} - n_{\tilde{g}_i^-}}{n_{\tilde{g}_i^+} - n_{\tilde{g}_i^-}e^{\sqrt{\Delta_{\tilde{g}_i}}(T-t)}}\right) \\ &+ \left(r\gamma_i - \frac{\gamma_i}{2(\gamma_i - 1)}(\lambda_p - \sigma_p)^2 + \kappa\theta n_{\tilde{g}_i^-}\right)(T - t),\end{aligned} \tag{10.5.7}$$

with $\Delta_{\tilde{g}_i}, n_{\tilde{g}_i^+}$, *and* $n_{\tilde{g}_i^-}$ *given by*

$$\begin{cases}\Delta_{\tilde{g}_i} = \left(\kappa + \lambda\rho_1\frac{\gamma_i}{\gamma_i - 1}\right)^2 - \left(\frac{1}{\gamma_i - 1}\rho_1^2 - \rho_2^2\right)\frac{\gamma_i}{\gamma_i - 1}\lambda^2, \\ n_{\tilde{g}_i^+} = \frac{-\left(\kappa + \lambda\rho_1\frac{\gamma_i}{\gamma_i - 1}\right) + \sqrt{\Delta_{\tilde{g}_i}}}{\frac{1}{\gamma_i - 1}\rho_1^2 - \rho_2^2}, \; n_{\tilde{g}_i^-} = \frac{-\left(\kappa + \lambda\rho_1\frac{\gamma_i}{\gamma_i - 1}\right) - \sqrt{\Delta_{\tilde{g}_i}}}{\frac{1}{\gamma_i - 1}\rho_1^2 - \rho_2^2}.\end{cases}$$

Remark 10.5.2. From (10.5.6), it is easy to show that $\tilde{g}_i(t)$ is strictly increasing in t, and thus, is a negative and bounded function over $[0, T]$. It follows from (10.5.7) that $\tilde{f}_i(t) \leq \left(r\gamma_i - \frac{\gamma_i}{2(\gamma_i - 1)}(\lambda_p - \sigma_p)^2\right)(T - t)$, for $i = 1, 2$.

Proposition 10.5.3. *The solution* $(\tilde{Y}_{i,t}, \tilde{Z}_{i,t}, \tilde{M}_{i,t}, \tilde{P}_{i,t})$ *given in (10.5.4)-(10.5.5) belongs to* $\mathcal{S}^\infty_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, *for* $i = 1, 2$.

Proof. It follows from the positivity of α_t, (10.5.4), and Remark 10.5.2 that for $i=1,2$,

$$Y_{i,t}=\exp\left\{\tilde{f}_i(t)+\tilde{g}_i(t)\alpha_t\right\}\leq \exp\left\{\left(r\gamma_i-\frac{\gamma_i}{2(\gamma_i-1)}(\lambda_p-\sigma_p)^2\right)(T-t)\right\}<\infty,$$

and thus, we find from (10.5.5) that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \tilde{Z}_{i,t}^2+\tilde{M}_{i,t}^2+\tilde{P}_{i,t}^2\,dt\right]\\
\leq&(\rho_1^2+\rho_2^2)\tilde{g}_i^2(0)e^{2|r\gamma_i-\frac{\gamma_i}{2(\gamma_i-1)}(\lambda_p-\sigma_p)^2|T}\int_0^T \mathbb{E}[\alpha_t]\,dt\\
=&(\rho_1^2+\rho_2^2)\tilde{g}_i^2(0)e^{2|r\gamma_i-\frac{\gamma_i}{2(\gamma_i-1)}(\lambda_p-\sigma_p)^2|T}\int_0^T\left(\alpha_0 e^{-\kappa t}+\kappa\theta\int_0^t e^{-\kappa(t-s)}\,ds\right)dt<\infty.
\end{aligned}$$

□

Through out the rest of this section, we impose the following assumptions on the model parameters, which are in line with Assumption 3.1 in Zhang (2022c).

Assumption 10.5.4. $\kappa+\rho_1\lambda\frac{\gamma_i}{\gamma_i-1}\neq 0$*, for* $i=1,2$.

Then, following almost the same arguments as in Theorem 3.6 in Zhang (2022c), it can be verified that the solution presented in Proposition 10.5.1 is the unique solution to BSRE (10.5.2). Therefore, we do not repeat the proof here.

Theorem 10.5.5. *Suppose that Assumption 10.5.4 holds. Then, the solution* $(\tilde{Y}_{i,t},\tilde{Z}_{i,t},\tilde{M}_{i,t},\tilde{P}_{i,t})$ *given in (10.5.4)-(10.5.5) is the unique solution to BSRE (10.5.2), for* $i=1,2$.

Having derived the unique solution to BSRE (10.5.2) in a closed form, we can rewrite linear BSDE (10.5.3) as follows:

$$\begin{cases} d\tilde{G}_{i,t}=\left(r\tilde{G}_{i,t}+\lambda\sqrt{\alpha_t}\tilde{\Lambda}_{i,t}+(\lambda_p-\sigma_p)\tilde{H}_{i,t}-\rho_2\tilde{g}_i(t)\sqrt{\alpha_t}\tilde{\Gamma}_{i,t}\right)dt+\tilde{H}_{i,t}\,dW_{0,t}\\ \qquad\qquad +\tilde{\Lambda}_{i,t}\,dW_{1,t}+\tilde{\Gamma}_{i,t}\,dW_{2,t},\\ \tilde{G}_{i,T}=-L_{i,T}+w_iL_{j,T}, \end{cases} \tag{10.5.8}$$

where $i\neq j\in\{1,2\}$. In the next proposition, we derive one explicit solution to BSDE (10.5.8) by trial and then show its uniqueness by using the Girsanov's measure change techniques combined with the standard results of linear BSDEs with uniformly Lipschitz continuity (refer to El Karoui, Peng, and Quenez (1997)).

Proposition 10.5.6. *The unique solution* $(\tilde{G}_{i,t},\tilde{H}_{i,t},\tilde{\Lambda}_{i,t},\tilde{\Gamma}_{i,t})$ *to linear BSDE (10.5.8) is given by*

$$\tilde{G}_{i,t}=\tilde{a}_i(t)L_{i,t}-w_i\tilde{a}_j(t)L_{j,t}, \tag{10.5.9}$$

and

$$\begin{cases} \tilde{H}_{i,t} = \tilde{a}_i(t)(\beta_i - \sigma_p)L_{i,t} - w_i \tilde{a}_j(t)(\beta_j - \sigma_p)L_{j,t}, \\ \tilde{\Lambda}_{i,t} = (\tilde{a}_i(t)\sigma_i L_{i,t} - w_i \tilde{a}_j(t)\sigma_j L_{j,t})\sqrt{\alpha_t}, \\ \tilde{\Gamma}_{i,t} = 0. \end{cases} \tag{10.5.10}$$

where $i \neq j \in \{1, 2\}$, *and* $\tilde{a}_i(t)$ *is as follows:*

$$\tilde{a}_i(t) = -e^{(r+\mu_p-\mu_i+\lambda_p(\beta_i-\sigma_p))(t-T)}, \ i = 1, 2. \tag{10.5.11}$$

Proof. See Appendix 10.F. □

Remark 10.5.7. It is crucial to find that the last control component of BSDE (10.5.3) $\tilde{\Gamma}_{i,t}, i = 1, 2$, is zero, which allows us to remove the third drift term from the right-hand side of (10.5.1).

Based on the unique solutions to BSDEs (10.5.2) and (10.5.3), we are now ready to state the second main result in this paper.

Theorem 10.5.8 (Nash equilibrium to the ALM game)**.** *Suppose that Assumption 10.5.4 holds and the initial data at time zero satisfies* $x_{i,0} + \tilde{a}_i(0)l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0)l_{j,0}) > 0$, *for* $i \neq j \in \{1, 2\}$. *Then, the Nash equilibrium pair* (π_1^*, π_2^*) *for the ALM game with the power utility preference (10.3.7) is as follows:*

$$\begin{cases} \pi_{i,t}^{S*} = \dfrac{\frac{X_t^{\pi_i^*} - w_i X_t^{\pi_j^*} + \tilde{G}_{i,t}}{1-\gamma_i}\left(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} + \lambda\sqrt{\alpha_t}\right) - \tilde{\Lambda}_{i,t} + w_i\left(\frac{X_t^{\pi_j^*} - w_j X_t^{\pi_i^*} + \tilde{G}_{j,t}}{1-\gamma_j}\left(\frac{\tilde{M}_{j,t}}{\tilde{Y}_{j,t}} + \lambda\sqrt{\alpha_t}\right) - \tilde{\Lambda}_{j,t}\right)}{\sigma_t X_t^{\pi_i^*}(1 - w_i w_j)}, \\ \pi_{i,t}^{I*} = \dfrac{\frac{X_t^{\pi_i^*} - w_i X_t^{\pi_j^*} + \tilde{G}_{i,t}}{1-\gamma_i}(\lambda_p - \sigma_p) - \tilde{H}_{i,t} + w_i\left(\frac{X_t^{\pi_j^*} - w_j X_t^{\pi_i^*} + \tilde{G}_{j,t}}{1-\gamma_j}(\lambda_p - \sigma_p) - \tilde{H}_{j,t}\right)}{\sigma_p X_t^{\pi_i^*}(1 - w_i w_j)} - \dfrac{\sigma_s}{\sigma_p}\pi_{i,t}^{S*} + 1, \end{cases} \tag{10.5.12}$$

and the value functions are given by

$$J_i(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0) = \frac{\tilde{Y}_{i,0}}{\gamma_i}\left(x_{i,0} + \tilde{a}_i(0)l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0)l_{j,0})\right)^{\gamma_i}, \tag{10.5.13}$$

for $i \neq j \in \{1, 2\}$, *where the explicit expressions for* $\tilde{Y}_{i,t}, \tilde{M}_{i,t}, \tilde{H}_{i,t}, \tilde{\Lambda}_{i,t}$, *and* $\tilde{a}_i(t)$ *are given by (10.5.4), (10.5.5), (10.5.10), and (10.5.11), respectively. Moreover,* $(\pi_1^*, \pi_2^*) \in \Pi_1 \otimes \Pi_2$.

Proof. See Appendix 10.G. □

Remark 10.5.9. The herd effect on managers' decisions can be found from the optimal response strategy given in (10.G.4)-(10.G.5), which shows that when the competitor j puts more wealth into the risky asset or the inflation-linked index bond, i.e., when $X_t^{\hat{\pi}_j}\hat{\pi}_{j,t}^S$ or $X_t^{\hat{\pi}_j}\hat{\pi}_{j,t}^I$ increases, the manager i tends to adopt a riskier investment strategy as well, for $i \neq j \in \{1, 2\}$. This finding is also reflected in the symmetric form of the Nash equilibrium strategy (10.5.12).

Remark 10.5.10. We should point out that different from the results for the exponential utility case (10.4.15), where the risk exposure under the Nash equilibrium strategy, i.e., $(\pi_{i,t}^{S*}\sigma_s + (\pi_{i,t}^{I*} - 1)\sigma_p)X_t^{\pi_i^*}$ and $\pi_t^{S*}\sigma_t X_t^{\pi_i^*}, i = 1, 2$ to the two fundamental risk factors $W_{0,t}$ and $W_{1,t}$ are independent of the wealth level of the two managers, the counterparts under power utility (10.5.12) depend on both the managers' wealth level $X_t^{\pi_i^*}$ and $X_t^{\pi_j^*}$, for $i \neq j \in \{1, 2\}$. This result can be featured by the constant absolute risk aversion coefficients q_i, $i = 1, 2$, under the exponential utility function, whereas the power utility function is characterized by relative risk aversion. Moreover, we require a technical condition on the initial data in Theorem 10.5.8 ensuring the admissibility of the Nash equilibrium (10.5.12). It is also worth mentioning that the admissibility condition (1) in Definition 10.3.3 serves as a constraint on the scope of the investment strategies, under which not only the regularity condition of power utility but also the martingale principle holds for any admissibility strategy.

Remark 10.5.11. For the special case when there do not exist any relative performance concerns, i.e, $w_1 = w_2 = 0$, the equilibrium investment strategies $\pi_{i,t}^{S*}$ and $\pi_{i,t}^{I*}$ given in (10.5.12), for $i = 1, 2$, degenerate to the standard optimal investment strategies for ALM with inflation and volatility risks, which, to our knowledge, are not reported in the existing literature.

Corollary 10.5.12 (Nash equilibrium to the investment game)**.** *Suppose that Assumption 10.5.4 holds and the initial data at time zero satisfies $x_{i,0} - w_i x_{j,0} > 0$, for $i \neq j \in \{1, 2\}$. Then, the Nash equilibrium pair (π_1^*, π_2^*) for the pure investment game with the power utility preference (10.3.7) is as follows:*

$$\begin{cases} \pi_{i,t}^{S*} = \dfrac{\frac{X_t^{\pi_i^*} - w_i X_t^{\pi_j^*}}{1-\gamma_i}\left(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} + \lambda\sqrt{\alpha_t}\right) + w_i \frac{X_t^{\pi_j^*} - w_j X_t^{\pi_i^*}}{1-\gamma_j}\left(\frac{\tilde{M}_{j,t}}{\tilde{Y}_{j,t}} + \lambda\sqrt{\alpha_t}\right)}{\sigma_t X_t^{\pi_i^*}(1 - w_i w_j)}, \\ \pi_{i,t}^{I*} = \dfrac{\frac{X_t^{\pi_i^*} - w_i X_t^{\pi_j^*}}{1-\gamma_i}(\lambda_p - \sigma_p) + w_i \frac{X_t^{\pi_j^*} - w_j X_t^{\pi_i^*}}{1-\gamma_j}(\lambda_p - \sigma_p)}{\sigma_p X_t^{\pi_i^*}(1 - w_i w_j)} - \dfrac{\sigma_s}{\sigma_p}\pi_{i,t}^{S*} + 1, \end{cases}$$

and the value functions are given by

$$J_i(x_{i,0}, x_{j,0}, s_0, \alpha_0) = \frac{\tilde{Y}_{i,0}}{\gamma_i}(x_{i,0} - w_i x_{j,0})^{\gamma_i},$$

for $i \neq j \in \{1, 2\}$.

Proof. Plugging the specification $\mu_i = \beta_i = \sigma_i = l_{i,0} = 0$, $i = 1, 2$ into Theorem 10.5.8 leads to the above results immediately. □

To end this section, we provide the equilibrium solution under the family of state-of-the-art 4/2 stochastic volatility models.

Corollary 10.5.13 (Nash equilibrium to the ALM game under the 4/2 model)**.** *If the risky asset price process S_t and the variance driver process α_t follow the family of 4/2 stochastic volatility model (10.2.7) and suppose that $\kappa + \sigma_\alpha \rho \lambda \frac{\gamma_i}{\gamma_i - 1} \neq 0$ and the initial data at time zero satisfies $x_{i,0} + \tilde{a}_i(0) l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0) l_{j,0}) > 0$, for $i \neq j \in \{1, 2\}$. Then, the Nash equilibrium pair (π_1^*, π_2^*) for the ALM game with the power utility preference (10.3.7) is given by*

$$\begin{cases} \pi_{i,t}^{S*} = \dfrac{\frac{X_t^{\pi_i^*} - w_i X_t^{\pi_j^*} + \tilde{G}_{i,t}}{1-\gamma_i} \left(\frac{\check{M}_{i,t}}{\check{Y}_{i,t}} + \lambda \sqrt{\alpha_t} \right) - \tilde{\Lambda}_{i,t} + w_i \left(\frac{X_t^{\pi_j^*} - w_j X_t^{\pi_i^*} + \tilde{G}_{j,t}}{1-\gamma_j} \left(\frac{\check{M}_{j,t}}{\check{Y}_{j,t}} + \lambda \sqrt{\alpha_t} \right) - \tilde{\Lambda}_{j,t} \right)}{\left(c_1 \sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}} \right) X_t^{\pi_i^*} (1 - w_i w_j)}, \\ \pi_{i,t}^{I*} = \dfrac{\frac{X_t^{\pi_i^*} - w_i X_t^{\pi_j^*} + \tilde{G}_{i,t}}{1-\gamma_i} (\lambda_p - \sigma_p) - \tilde{H}_{i,t} + w_i \left(\frac{X_t^{\pi_j^*} - w_j X_t^{\pi_i^*} + \tilde{G}_{j,t}}{1-\gamma_j} (\lambda_p - \sigma_p) - \tilde{H}_{j,t} \right)}{\sigma_p X_t^{\pi_i^*} (1 - w_i w_j)} - \dfrac{\sigma_s}{\sigma_p} \pi_{i,t}^{S*} + 1, \end{cases}$$

and the value functions are as follows:

$$J_i(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0) = \frac{\check{Y}_{i,0}}{\gamma_i} \left(x_{i,0} + \tilde{a}_i(0) l_{i,0} - w_i (x_{j,0} + \tilde{a}_j(0) l_{j,0}) \right)^{\gamma_i},$$

where the explicit expressions for $\check{Y}_{i,t}$ and $\check{M}_{i,t}$ are given by

$$\check{Y}_{i,t} = \exp \left\{ \check{f}_i(t) + \check{g}_i(t) \alpha_t \right\},$$

and

$$\check{M}_{i,t} = \sigma_\alpha \rho \check{g}_i(t) \sqrt{\alpha_t} \check{Y}_{i,t}, \tag{10.5.14}$$

where functions $\check{f}_i(t)$ and $\check{g}_i(t)$ are given by

$$\begin{aligned} \check{f}_i(t) = & \frac{\kappa\theta \left(n_{\check{g}_i^-} - n_{\check{g}_i^+} \right)}{\sqrt{\Delta_{\check{g}_i}}} \log \left(\frac{n_{\check{g}_i^+} - n_{\check{g}_i^-}}{n_{\check{g}_i^+} - n_{\check{g}_i^-} e^{\sqrt{\Delta_{\check{g}_i}}(T-t)}} \right) \\ & + \left(r\gamma_i - \frac{\gamma_i}{2(\gamma_i - 1)} (\lambda_p - \sigma_p)^2 + \kappa\theta n_{\check{g}_i^-} \right) (T - t), \end{aligned}$$

and

$$\check{g}_i(t) = \frac{n_{\check{g}_i^+} n_{\check{g}_i^-} \left(1 - e^{\sqrt{\Delta_{\check{g}_i}}(T-t)} \right)}{n_{\check{g}_i^+} - n_{\check{g}_i^-} e^{\sqrt{\Delta_{\check{g}_i}}(T-t)}},$$

with $\Delta_{\check{g}_i}, n_{\check{g}_i^+}$, and $n_{\check{g}_i^-}$ given by

$$\begin{cases} \Delta_{\check{g}_i} = \left(\kappa + \lambda \sigma_\alpha \rho \dfrac{\gamma_i}{\gamma_i - 1} \right)^2 - \left(\dfrac{\gamma_i}{\gamma_i - 1} \rho^2 - 1 \right) \dfrac{\gamma_i}{\gamma_i - 1} \sigma_\alpha^2 \lambda^2, \\ n_{\check{g}_i^+} = \dfrac{-\left(\kappa + \lambda \sigma_\alpha \rho \frac{\gamma_i}{\gamma_i - 1} \right) + \sqrt{\Delta_{\check{g}_i}}}{\frac{\gamma_i}{\gamma_i - 1} \rho^2 - 1}, \quad n_{\check{g}_i^-} = \dfrac{-\left(\kappa + \lambda \sigma_\alpha \rho \frac{\gamma_i}{\gamma_i - 1} \right) - \sqrt{\Delta_{\check{g}_i}}}{\frac{\gamma_i}{\gamma_i - 1} \rho^2 - 1}, \end{cases}$$

for $i \neq j \in \{1, 2\}$.

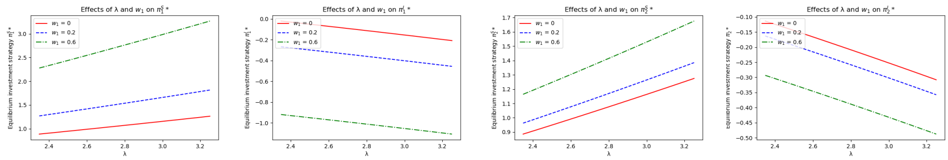
Proof. Substituting the specified parameters in the 4/2 model (10.2.7) in Example 10.2.2 into Theorem 10.5.8 leads to the above results. □

Remark 10.5.14. Specifications $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$ in Corollary 10.5.13 provide the Nash equilibrium strategies and value functions under the Heston model and 3/2 model, respectively. Moreover, it can be checked that Corollary 10.5.13 generalizes the results of Kraft (2005) and Cheng and Escobar (2021a) from pure investment problems for a single investor to ALM games for two competitive managers with relative performance concerns and inflation risk.

10.6 Sensitivity analysis

In this section, we implement numerical experiments to illustrate the sensitivities of the equilibrium strategies with respect to model parameters and relative performance concerns. We are concerned about the 4/2 stochastic volatility model (10.2.7) for the exponential utility case (10.3.6) in that the 4/2 model embracing the parsimonious Heston model and 3/2 model has revealed practical significance in recent years and numerical illustration for the power utility case can be conducted similarly. Through out this section, unless otherwise stated, the base parameters we adopt are as follows: $\kappa = 7.3479, \theta = 0.0328, \sigma_v = 0.6612, \rho = -0.7689, \lambda = 2.9428, c_1 = 0.9051, c_2 = 0.023, R = 0.05, r = 0.02, \mu_p = 0.01, \sigma_p = 0.4, \sigma_s = 0.2, T = 1, v_0 = 0.04$; for manager 1, $\mu_1 = 0.02, \beta_1 = 0.3, \sigma_1 = 0.5, q_1 = 2, w_1 = 0.2, x_{1,0} = 1, l_{1,0} = 0.1$; for manager 2, $\mu_2 = 0.03, \beta_2 = 0.2, \sigma_2 = 0.3, q_2 = 1.2, w_2 = 0.4, x_{2,0} = 2, l_{2,0} = 0.4$. For simplicity but without loss of generality, we focus on the analysis at time $t = 0$ and vary the value of one parameter with others fixed at each time.

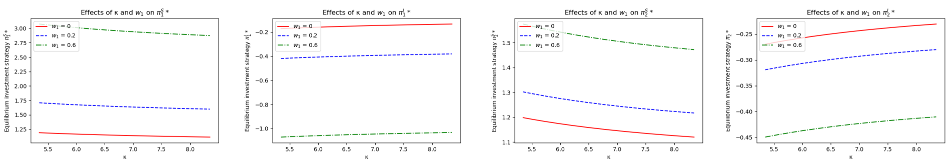
Figure 10.1 contributes to the evolution of the equilibrium strategy (π_1^*, π_2^*) with respect to λ and w_1. From Figure 10.1(a) and (b), we observe that as λ increases, π_1^{S*} increases, whereas π_1^{I*} decreases. Since λ represents the slope of the market price of risk induced by $W_{1,t}$, when λ is growing, manager 1 can derive higher returns and is willing to invest more in the risky asset. Figure 10.1(a) also demonstrates that a larger w_1 prompts manager 1 to invest more in the risky asset as a response to the competition. In this case, manager 1 can maximize the probability of generating a greater terminal surplus against his/her competitor at the terminal date T. In addition, the dynamics of risky asset price (10.2.7) show that the inflation risk can be partially hedged against by trading the risky asset, and thus, the inflation-linked index bond is less needed to hedge against the same amount of inflation risk when manager 1 invests more in the risky asset, which explains why π_1^{S*} and π_1^{I*} move in the opposite direction as shown in Figure 10.1(a) and (b). Figure 10.1(c) and (d) verify the existence of the herd effect (recall Remark 10.4.12), that is, manager 2 tends to take an investment strategy similar to his/her competitor. Figure 10.2 describes the sensitivity of the equilibrium investment



(a) **(b)** **(c)** **(d)**

Figure 10.1: *Effects of λ and w_1 on π_1^* and π_2^**

strategy (π_1^*, π_2^*) with respect to κ. From Figure 10.2(a) and (b), we find that the equilibrium investment strategy for the risky asset decreases as κ increases, while the strategy for the inflation-linked index bond increases as κ increases. Recall from (10.2.7) that κ denotes the mean-reversion rate of the variance driver process under the 4/2 model. Along with the growth of κ, the variance driver process V_t reverts faster to its long-term mean θ, and thus, both the risky asset and random liability have a more stable appreciation rate. In this case, manager 1 can invest less in the risky asset facing the reduced volatility risk. Conversely, since the amount of overall inflation risk is not affected by the changes in κ, manager 1 has to invest more in the inflation-linked index bond to hedge against the inflation risk. Again, the herd effect can be found by comparing Figure 10.2(c) and (d) with 10.2(a) and (b); manager 2 mimics manager 1's investment strategy in the competition.

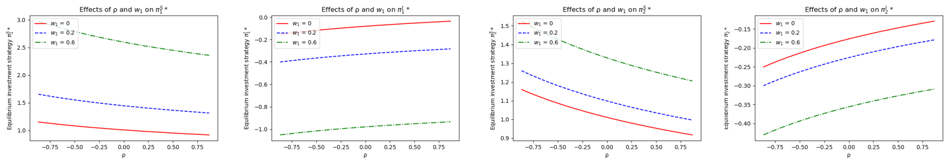


(a) **(b)** **(c)** **(d)**

Figure 10.2: *Effect of κ on π_1^* and π_2^**

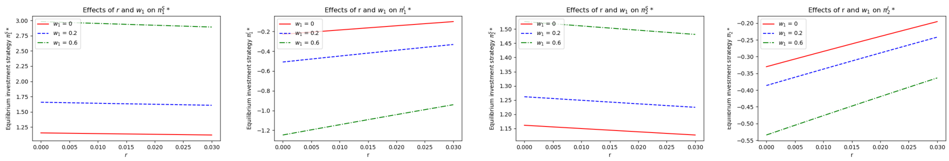
Figure 10.3 shows how the equilibrium strategy (π_1^*, π_2^*) changes with respect to ρ. It is shown that both managers 1 and 2 tend to put less wealth into the risky asset and more wealth into the inflation-linked index bond as ρ increases. One of the possible reasons is that since ρ is the correlation coefficient between the dynamics of the risky asset price and its variance driver, when ρ increases from -0.9 to 0.9, the risky asset price and the variance process become less negatively correlated and more positively correlated. As such, the offset between the risk caused by fluctuations in the risky asset price and its variance driver becomes less. Therefore, investing the same amount of wealth into the risky asset amplifies the two managers' exposure to volatility risk, and thus, they tend to decrease the investments in the risky asset and increase the investments in the inflation-linked index bond.

Figure 10.4 displays the effect of r on the equilibrium investment strategy (π_1^*, π_2^*).



(a) (b) (c) (d)

Figure 10.3: *Effect of ρ on π_1^* and π_2^**

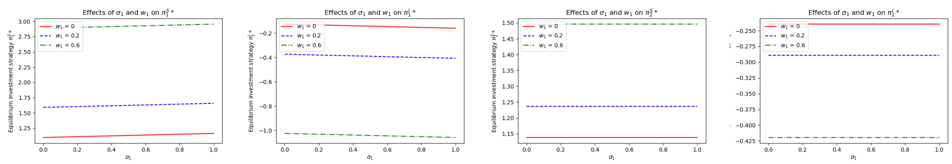


(a) (b) (c) (d)

Figure 10.4: *Effect of r on π_1^* and π_2^**

From Figure 10.4(a) and (c), we find that π_1^{S*} and π_2^{S*} move down as r becomes larger, while Figure 10.4(b) and (d) show that π_1^{I*} and π_2^{I*} move upwards as r becomes larger. These findings are in line with the economic implication of r. First of all, since r stands for the real short rate of interest, a greater value of r allows the two asset-liability managers to derive a higher expected return rate without investing in the risky asset and the inflation-linked index bond, which can be verified by setting $\pi_{i,t}^S$ and $\pi_{i,t}^I$ to zeros in (10.2.9). Second, r has a positive relationship with the market price of inflation risk $\lambda_p = (r + \mu_p - R)/\sigma_p$, i.e., λ_p increases as r increases, whereas the market price of volatility risk $\lambda\sqrt{\alpha_t}$ is not affected by the changes in r. Consequently, the two managers can acquire higher inflation risk premium bearing the same amount of inflation risk, and they are willing to put more wealth into the inflation-linked index bond.

Figure 10.5 provides graphical illustrations of the effect of σ_1 on the equilibrium investment strategy (π_1^*, π_2^*). We observe from Figure 10.5(a) and (b) that manager 1 increases (decreases) the proportion of wealth into the risky asset (inflation-linked index bond) as σ_1 increases. This can be explained by the economic implication of σ_1 that characterizes the impact of the risk caused by the fluctuations in the risky asset price on the instantaneous volatility of the uncontrollable liability $L_{1,t}$. A larger σ_1 amplifies manager 1's exposure to the risk. So, manager 1 has to put more wealth into the risky asset to hedge against the risk and to maximize the expectation of terminal surplus. Moreover, since investing in the risky asset can partially hedge against the inflation risk, manager 1 can decrease his/her investment in the inflation-linked index bond to hedge against the overall inflation risk that is not influenced by the changes in σ_1. It is shown from Figure 10.5(c) and (d) that the equilibrium investment strategy of manager 2 does not change when σ_1 varies from 0 to 1. On one hand, when facing a riskier approach taken by manager 1 to the risky asset, manager 2 tends to increase his/her investment in the risky asset as well, which can be verified by looking into the second term within the parenthesis in (10.E.4). On the other hand, when a larger σ_1 is perfectly revealed to manager 2, manager 2 realizes that his/her liability $L_{2,t}$ undertakes a relatively smaller risk caused by the fluctuation in the risky asset price than manager 1's liability $L_{1,t}$ does. In this case, manager 2 has the incentive to decrease the investment in the risky asset. The overall effect of σ_1 on the equilibrium strategy π_2^{S*} takes on these two opposite sides. Finally, since the overall inflation risk remains the same for manager 2, manager 2 will not change his/her trading position in the inflation-linked index bond, which is consistent with Figure 10.5(d).



(a) (b) (c) (d)

Figure 10.5: *Effect of σ_1 on π_1^* and π_2^**

10.7 Conclusion

In this paper, we investigate a class of non-zero-sum stochastic differential games for ALM between two competitive asset-liability managers in a financial market under the inflation and volatility risks. The two managers are subject to two different uncontrollable random liabilities and allowed to allocate their wealth to a financial market consisting of an inflation-linked index bond, a risk-free asset, and a risky asset whose price process is governed by a general class of stochastic volatility models, including the CEV model, the family of state-of-the-art 4/2 models, and some non-Markovian models as exceptional cases. The goal of each manager is to maximize the expected utility of his/her relative terminal surplus after stripping out inflation with respect to that of the competitor. By applying a BSDE approach to overcome the potentially non-Markovian market setting and solving the associated BSDEs explicitly, we derive the closed-form expressions for the Nash equilibrium strategies and the corresponding value functions for the games under the exponential and power utility preferences, respectively. Moreover, analytical solutions to some particular cases of our model are presented. Finally, numerical examples are provided to explore the economic impacts of model parameters and relative performance concerns on the equilibrium strategies. Results indicate that each manager mimics the competitor's strategy in the presence of relative performance concerns and tends to increase the amount invested in the risky asset. In other words, relative performance concerns lead managers to deviate from rational decisions without competition and become more risk-seeking.

Concerning future related works, there are several remaining interesting problems to discuss. For example, one may introduce jumps and more general stochastic volatility and stochastic inflation models to the optimal ALM problems with relative performance concerns. It may also be of interest to involve fixed or proportional transaction costs, although the derivation of closed-form solutions may not be easy.

Acknowledgement(s)

The authors are grateful to Prof. Jesper Lund Pedersen for his insightful comments and suggestions.

10.A Proof of Proposition 10.4.3

Proof. From Lemma 10.4.2, we know that the following stochastic exponential process

$$\exp\left\{-\int_0^t \lambda\sqrt{\alpha_s}\, dW_{1,s} - \frac{1}{2}\int_0^t \lambda^2\alpha_s\, ds\right\}$$

is an $(\mathbb{F}, \mathbb{P})$-martingale. In addition, since Novikov's condition obviously holds for the following stochastic exponential process

$$\exp\left\{-\int_0^t (\lambda_p - \sigma_p)\, dW_{0,s} - \frac{1}{2}\int_0^t (\lambda_p - \sigma_p)^2\, ds\right\}.$$

Due to the pathwise continuity of the above two processes and the independence between $W_{1,t}$ and $W_{2,t}$, it follows from Theorem 2.4 in Cherny (2006) that the product of these two stochastic exponential processes is also an $(\mathbb{F}, \mathbb{P})$-martingale. As a result, the probability measure denoted by $\tilde{\mathbb{P}}$ is well-defined and equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ via the following Radon-Nikodym derivative:

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\int_0^T (\lambda_p - \sigma_p)\, dW_{0,t} - \int_0^T \lambda\sqrt{\alpha_t}\, dW_{1,t} - \frac{1}{2}\int_0^T (\lambda_p - \sigma_p)^2 + \lambda^2\alpha_t\, dt\right\},$$

and the following three processes $\tilde{W}_{k,t}$, for $k = 0, 1, 2$,

$$\tilde{W}_{0,t} = \int_0^t (\lambda_p - \sigma_p)\, ds + W_{0,t},\ \ \tilde{W}_{1,t} = \int_0^t \lambda\sqrt{\alpha_s}\, ds + W_{1,t},\ \ \tilde{W}_{2,t} = W_{2,t}$$

are three standard Brownian motions under measure $\tilde{\mathbb{P}}$ by Girsanov's theorem. Therefore, reformulating BSRE (10.4.2) under measure $\tilde{\mathbb{P}}$ and applying Itô's formula to $\log(Y_{i,t})$ lead to the following quadratic BSDE of $\left(\log(Y_{i,t}), \frac{Z_{i,t}}{Y_{i,t}}, \frac{M_{i,t}}{Y_{i,t}}, \frac{P_{i,t}}{Y_{i,t}}\right)$:

$$\begin{cases} d\log(Y_{i,t}) = \left[-r + \frac{1}{2}\left(\frac{Z_{i,t}}{Y_{i,t}}\right)^2 + \frac{1}{2}\left(\frac{M_{i,t}}{Y_{i,t}}\right)^2 - \frac{1}{2}\left(\frac{P_{i,t}}{Y_{i,t}}\right)^2\right] dt + \frac{Z_{i,t}}{Y_{i,t}}\, d\tilde{W}_{0,t} \\ \qquad\qquad + \frac{M_{i,t}}{Y_{i,t}}\, d\tilde{W}_{1,t} + \frac{P_{i,t}}{Y_{i,t}}\, d\tilde{W}_{2,t}, \\ \log(Y_{i,T}) = 0. \end{cases} \tag{10.A.1}$$

Notice that quadratic BSDE (10.A.1) satisfies all the regularity conditions in Kobylanski (2000). Hence, we can conclude that it admits a unique solution by Theorem 2.3 and 2.6 in Kobylanski (2000), and so does BSRE (10.4.2). Furthermore, it can be easily checked that (10.4.4)-(10.4.5) form the unique solution to BSRE (10.4.2). This completes the proof. □

10.B Proof of Proposition 10.4.5

Proof. We conjecture that the first component $G_{i,t}$ of the solution to BSDE (10.4.6) admits an affine form as follows:

$$G_{i,t} = f_i(t) + g_i(t)\alpha_t + h_i(t)L_{i,t} - w_i m_i(t) L_{j,t},$$

where $i \neq j \in \{1,2\}$ and $f_i(t), g_i(t), h_i(t)$, and $m_i(t)$ are four differentiable functions to be determined in what follows with boundary conditions $f_i(T) = g_i(T) = 0$ and $h_i(T) = m_i(T) = -1$. Then, applying Itô's formula to $G_{i,t}$ reads

$$\begin{aligned} dG_{i,t} = &\left[\frac{df_i(t)}{dt} + \kappa\theta g_i(t) + \left(\frac{dg_i(t)}{dt} - \kappa g_i(t)\right)\alpha_t + L_{i,t}\left(\frac{dh_i(t)}{dt} + (\mu_i - \mu_p + \sigma_p^2 - \beta_i\sigma_p\right.\right. \\ &\left.\left. + \lambda\sigma_i\alpha_t)h_i(t)\right) - w_i L_{j,t}\left(\frac{dm_i(t)}{dt} + (\mu_j - \mu_p + \sigma_p^2 - \beta_j\sigma_p + \lambda\sigma_j\alpha_t)m_i(t)\right)\right] dt \\ &+ \left(L_{i,t}(\beta_i - \sigma_p)h_i(t) - w_i L_{j,t}(\beta_j - \sigma_p)m_i(t)\right) dW_{0,t} \\ &+ \left(\rho_1 g_i(t) + \sigma_i h_i(t) L_{i,t} - w_i\sigma_j m_i(t) L_{j,t}\right)\sqrt{\alpha_t}\, dW_{1,t} + \rho_2 g_i(t)\sqrt{\alpha_t}\, dW_{2,t}. \end{aligned} \tag{10.B.1}$$

Match the diffusion coefficients of (10.B.1) with the control components of BSDE (10.4.6), i.e.,

$$\begin{cases} H_{i,t} = (\beta_i - \sigma_p)h_i(t)L_{i,t} - w_i(\beta_j - \sigma_p)m_i(t)L_{j,t}, \\ \Lambda_{i,t} = \left(\rho_1 g_i(t) + \sigma_i h_i(t)L_{i,t} - w_i\sigma_j m_i(t)L_{j,t}\right)\sqrt{\alpha_t}, \\ \Gamma_{i,t} = \rho_2 g_i(t)\sqrt{\alpha_t}, \end{cases}$$

and substitute the above expressions into the generator of BSDE (10.4.6). Then, we have the reformulated generator as follows:

$$\begin{aligned} &\left(\lambda\rho_1 g_i(t) + \frac{q_i}{2}\rho_2^2 g_i^2(t) - \frac{\lambda^2}{2q_i}\right)\alpha_t + L_{i,t}\left((\lambda_p - \sigma_p)(\beta_i - \sigma_p) + \lambda\sigma_i\alpha_t\right)h_i(t) \\ &- w_i L_{j,t}\left((\lambda_p - \sigma_p)(\beta_j - \sigma_p) + \lambda\sigma_j\alpha_t\right)m_i(t) - \frac{(\lambda_p - \sigma_p)^2}{2q_i}. \end{aligned} \tag{10.B.2}$$

Comparing the drift coefficient of (10.B.1) with (10.B.2) and separating the dependence on $\alpha_t, L_{i,t}$, and $L_{j,t}$, we obtain the ODE system governing functions $f_i(t), g_i(t), h_i(t)$, and $m_i(t)$ given in (10.4.9), for $i \in \{1, 2\}$. $\square$

10.C Proof of Proposition 10.4.8

Proof. When $\rho_2 = 0$, Riccati equation of $g_i(t)$ is reduced to the following first-order linear equation:

$$\frac{dg_i(t)}{dt} = (\kappa + \lambda\rho_1)g_i(t) - \frac{\lambda^2}{2q_i}.$$

Using some standard methods of solving first-order linear ODE upon noticing the boundary condition that $g_i(T) = 0$, we have the first solution given in (10.4.10). For the case when $\rho_2 \neq 0$, denote by $\Delta_{g_i} = (\kappa + \lambda\rho_1)^2 + \rho_2^2\lambda^2 > 0$ and $n_{g_i^+}$ and $n_{g_i^-}$ given in (10.4.14). Then, we can rewrite Riccati ODE (10.4.9) of $g_i(t)$ as follows:

$$\frac{dg_i(t)}{dt} = \frac{q_i}{2}\rho_2^2 \left(g_i(t) - n_{g_i^+}\right)\left(g_i(t) - n_{g_i^-}\right).$$

By applying the separation variable method to the above ODE, we have

$$\frac{dg_i(t)}{g_i(t) - n_{g_i^+}} - \frac{dg_i(t)}{g_i(t) - n_{g_i^-}} = \sqrt{\Delta_{g_i}}\, dt. \tag{10.C.1}$$

Integrating (10.C.1) on both sides with respect to t, combined with the boundary condition $g_i(T) = 0$, we have the second solution given in (10.4.10). Substituting (10.4.10) into the ODE (10.4.9) of $f_i(t)$ leads to the explicit expressions of $f_i(t)$ given in (10.4.13). As for the first-order homogeneous linear equations of $h_i(t)$ and $m_i(t)$, we have

$$\frac{dh_i(t)}{h_i(t)} = (\lambda_p(\beta_i - \sigma_p) + \mu_p - \mu_i)\, dt,$$

and

$$\frac{dm_i(t)}{m_i(t)} = (\lambda_p(\beta_j - \sigma_p) + \mu_p - \mu_j)\, dt.$$

By integrating the above two equations with respect to t from 0 to T and taking into account the boundary conditions, we have the closed-form solutions given in (10.4.11)-(10.4.12), respectively. $\square$

10.D Proof of Proposition 10.4.10

Proof. The proof is similar to that of Theorem 4.4 in Zhang (2022e). For the reader's convenience, we provide the modifications of the proof here.

To show the proposed solution $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t})$ given in Proposition 10.4.5 belongs to the space $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$, for

$i = 1, 2$, we first recall from the dynamics of inflation-adjusted liability processes (10.2.10) that

$$L_{i,t} = l_{i,0} \exp\left\{ \int_0^t \left(\mu_i - \mu_p + \sigma_p^2 - \sigma_p \beta_i + \left(\lambda\sigma_i - \frac{\sigma_i^2}{2} \right) \alpha_s - \frac{(\beta_i - \sigma_p)^2}{2} \right) ds \right.$$
$$\left. + \int_0^t (\beta_i - \sigma_p)\, dW_{0,s} + \int_0^t \sigma_i \sqrt{\alpha_s}\, dW_{1,s} \right\}.$$

Combined with Hölder' inequality, it can be shown that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T L_{i,t}^4 \, dt \right] \\
\leq & c \int_0^T \left\{ \mathbb{E}\left[\exp\left\{ \int_0^t 8(\beta_i - \sigma_p)\, dW_{0,s} + \int_0^t 8\sigma_i \sqrt{\alpha_s}\, dW_{1,s} - 32 \int_0^t \left((\beta_i - \sigma_p)^2 \right. \right. \right. \right. \\
& \left. \left. \left. \left. + \sigma_i^2 \alpha_s \right) ds \right\} \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}\left[\exp\left\{ (8\lambda\sigma_i + 28\sigma_i^2) \int_0^t \alpha_s \, ds \right\} \right] \right\}^{\frac{1}{2}} dt \\
= & c \int_0^T \left\{ \mathbb{E}\left[\exp\left\{ (8\lambda\sigma_i + 28\sigma_i^2) \int_0^t \alpha_s \, ds \right\} \right] \right\}^{\frac{1}{2}} dt,
\end{aligned} \tag{10.D.1}$$

where $c \in \mathbb{R}^+$ and the equality follows the fact that the stochastic exponential process $\exp\left\{ \int_0^t 8(\beta_i - \sigma_p)\, dW_{0,s} + \int_0^t 8\sigma_i \sqrt{\alpha_s}\, dW_{1,s} - 32 \int_0^t \left((\beta_i - \sigma_p)^2 + \sigma_i^2 \alpha_s \right) ds \right\}$ is an $(\mathbb{F}, \mathbb{P})$-martingale by Lemma 10.4.2 and Theorem 2.4 in Cherny (2006). Hence, to ensure the right-hand side of (10.D.1) is finite, we need to calculate the term $\mathbb{E}\left[\exp\left\{ (8\lambda\sigma_i + 28\sigma_i^2) \int_0^t \alpha_s \, ds \right\} \right]$. To this end, denote by $\mathbb{E}[\cdot | \, \mathcal{F}_u]$ the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_u$, $\forall u \leq t$. Then, the Markovian structure of α_t leads to the following result:

$$\mathbb{E}\left[\exp\left\{ (8\lambda\sigma_i + 28\sigma_i^2) \int_u^t \alpha_s \, ds \right\} \middle| \, \mathcal{F}_u \right] = F_i(\alpha_u, u), \text{ for } u \leq t, \tag{10.D.2}$$

where $F_i : \mathbb{R}^+ \otimes [0, t] \mapsto \mathbb{R}^+$ is an undetermined differentiable function, for $i \in \{1, 2\}$. Then, we have the following PDE governing F_i from the Feynman-Kac theorem:

$$\begin{cases} \dfrac{\partial F_i}{\partial u}(x, u) + \kappa(\theta - x)\dfrac{\partial F_i}{\partial x}(x, u) + \dfrac{1}{2}\left(\rho_1^2 + \rho_2^2 \right) x \dfrac{\partial^2 F_i}{\partial x^2}(x, u) + (8\lambda\sigma_i + 28\sigma_i^2) x F_i(x, u) = 0, \\ F_i(x, t) = 1. \end{cases}$$

Conjecture that $F_i(x, u) = \exp\left\{ \tilde{F}_i(u; t)x + \bar{F}_i(u; t) \right\}$. The above PDE of $F_i(x, u)$ can be decomposed into the following two ODEs of functions $\tilde{F}_i(u; t)$ and $\bar{F}_i(u; t)$:

$$\frac{d\tilde{F}_i(u; t)}{du} = -\frac{\rho_1^2 + \rho_2^2}{2} \tilde{F}_i^2(u; t) + \kappa \tilde{F}_i(u; t) - (8\lambda\sigma_i + 28\sigma_i^2), \; \tilde{F}_i(t; t) = 0, \tag{10.D.3}$$

and

$$\frac{d\bar{F}_i(u; t)}{du} = -\kappa\theta \tilde{F}_i(u; t), \; \bar{F}_i(t; t) = 0,$$

for $i=1,2$. Notice that (10.D.3) is a Riccati equation. Denote by

$$\Delta_{\tilde{F}_i}=\kappa^2-2(\rho_1^2+\rho_2^2)(8\lambda\sigma_i+28\sigma_i^2),\ n_{\tilde{F}_i}=\frac{\kappa}{\rho_1^2+\rho_2^2},\ n_{\tilde{F}_i}^{+}=\frac{-\kappa+\sqrt{\Delta_{\tilde{F}_i}}}{-(\rho_1^2+\rho_2^2)},\ n_{\tilde{F}_i}^{-}=\frac{-\kappa-\sqrt{\Delta_{\tilde{F}_i}}}{-(\rho_1^2+\rho_2^2)}.$$

It can be shown that the closed-form expressions for $\tilde{F}_i(u;t)$ and $\bar{F}_i(u;t)$ are given by

$$\tilde{F}_i(u;t)=\begin{cases}\dfrac{n_{\tilde{F}_i}^{+}n_{\tilde{F}_i}^{-}\left(1-e^{\sqrt{\Delta_{\tilde{F}_i}}(t-u)}\right)}{n_{\tilde{F}_i}^{+}-n_{\tilde{F}_i}^{-}e^{\sqrt{\Delta_{\tilde{F}_i}}(t-u)}}, & \text{if } \Delta_{\tilde{F}_i}>0;\\ \dfrac{(\rho_1^2+\rho_2^2)(t-u)n_{\tilde{F}_i}^2}{(\rho_1^2+\rho_2^2)(t-u)n_{\tilde{F}_i}+2}, & \text{if } \Delta_{\tilde{F}_i}=0;\\ \dfrac{\sqrt{-\Delta_{\tilde{F}_i}}}{-(\rho_1^2+\rho_2^2)}\tan\left(\arctan\left(\dfrac{\kappa}{\sqrt{-\Delta_{\tilde{F}_i}}}\right)-\dfrac{\sqrt{-\Delta_{\tilde{F}_i}}}{2}(t-u)\right), & \text{if } \Delta_{\tilde{F}_i}<0,\end{cases}$$

and

$$\bar{F}_i(u;t)=\int_u^t\kappa\theta\tilde{F}_i(s;t)\,ds,$$

for $i=1,2$. Hence, from the above results, we find that

$$\mathbb{E}\left[\int_0^T L_{i,t}^4\,dt\right]\le c\int_0^T\exp\left\{\frac{\tilde{F}_i(0;t)}{2}\alpha_0+\frac{\bar{F}_i(0,t)}{2}\right\}dt<\infty,\ \text{for } i=1,2.$$

Moreover, note that the following second-order moment of α_t

$$\mathbb{E}\left[\alpha_t^2\right]=\left(\alpha_0e^{-\kappa t}+\theta\left(1-e^{-\kappa t}\right)\right)^2+\alpha_0\frac{(\rho_1^2+\rho_2^2)\left(e^{-\kappa t}-e^{-2\kappa t}\right)}{\kappa}+\frac{\theta(\rho_1^2+\rho_2^2)\left(1-e^{-\kappa t}\right)^2}{2\kappa}.$$

is continuous in t over $[0,T]$. Therefore, combining the above results with the closed-form expressions for $G_{i,t},H_{i,t},\Lambda_{i,t}$ and $\Gamma_{i,t}$ given in (10.4.7) and (10.4.8), we have

$$\mathbb{E}\left[\int_0^T G_{i,t}^2+H_{i,t}^2+\Lambda_{i,t}^2+\Gamma_{i,t}^2\,dt\right]\le c\left[1+\int_0^T\mathbb{E}\left[\alpha_t^2\right]dt+\mathbb{E}\left[\int_0^T L_{i,t}^4+L_{j,t}^4\,dt\right]\right]<\infty,$$

for $i\neq j\in\{1,2\}$, which means that $(G_{i,t},H_{i,t},\Lambda_{i,t},\Gamma_{i,t})\in\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})\otimes\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$ as desired.

In the second part of this proof, we show that the proposed solution given in Proposition 10.4.5 is the unique solution to BSDE (10.4.6). In fact, the linear terms within the generator of BSDE (10.4.6) can be eliminated by using Girsanov's measure change techniques. More precisely, define a new probability measure

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T}=\exp\left\{-\int_0^T(\lambda_p-\sigma_p)\,dW_{0,t}-\int_0^T\lambda\sqrt{\alpha_t}\,dW_{1,t}-\frac{1}{2}\int_0^T(\lambda_p-\sigma_p)^2+\lambda^2\alpha_t\,dt\right\}.$$

Using Lemma 10.4.2 and Theorem 2.4 in Cherny (2006) again, the stochastic exponential process associated with the above Radon-Nikodym derivative is a true $(\mathbb{F},\mathbb{P})$-martingale and thus, the newly-defined measure $\hat{\mathbb{P}}$ is well-defined and equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. It follows from Girsanov's theorem that

$$\hat{W}_{0,t} = \int_0^t (\lambda_p - \sigma_p)\, ds + W_{0,t},\ \hat{W}_{1,t} = \int_0^t \lambda\sqrt{\alpha_s}\, ds + W_{1,t},\ \hat{W}_{2,t} = W_{2,t}$$

are standard Brownian motions under $\hat{\mathbb{P}}$. Then, quadratic BSDE (10.4.6) can be rewritten as follows:

$$\begin{cases} dG_{i,t} = \left(\frac{q_i}{2}\Gamma_{i,t}^2 - \frac{(\lambda_p - \sigma_p)^2}{2q_i} - \frac{\lambda^2\alpha_t}{2q_i}\right) dt + H_{i,t}\, d\hat{W}_{0,t} + \Lambda_{i,t}\, d\hat{W}_{1,t} + \Gamma_{i,t}\, d\hat{W}_{2,t}, \\ G_{i,T} = -\, L_{i,T} + w_i L_{j,T}, \end{cases} \tag{10.D.4}$$

for $i \neq j \in \{1,2\}$, and the factor process α_t turns to be under $\hat{\mathbb{P}}$:

$$d\alpha_t = (\kappa + \lambda\rho_1)\left(\frac{\kappa\theta}{\kappa + \lambda\rho_1} - \alpha_t\right) dt + \sqrt{\alpha_t}\left(\rho_1\, d\hat{W}_{1,t} + \rho_2\, d\hat{W}_{2,t}\right),$$

which is still a square-root process due to Assumption 10.4.6. Let $(\hat{G}_{i,t}, \hat{H}_{i,t}, \hat{\Lambda}_{i,t}, \hat{\Gamma}_{i,t})$ be any solution to BSDE (10.4.6), for $i = 1,2$. Define

$$\Delta G_{i,t} = G_{i,t} - \hat{G}_{i,t},\ \Delta H_{i,t} = H_{i,t} - \hat{H}_{i,t},\ \Delta\Lambda_{i,t} = \Lambda_{i,t} - \hat{\Lambda}_{i,t},\ \Delta\Gamma_{i,t} = \Gamma_{i,t} - \hat{\Gamma}_{i,t}.$$

Then, the difference process $(\Delta G_{i,t}, \Delta H_{i,t}, \Delta\Lambda_{i,t}, \Delta\Gamma_{i,t})$ is a solution to the following BSDE under $\hat{\mathbb{P}}$:

$$\begin{cases} d\Delta G_{i,t} = \frac{q_i}{2}\left(\Gamma_{i,t}^2 - \hat{\Gamma}_{i,t}^2\right) dt + \Delta H_{i,t}\, d\hat{W}_{0,t} + \Delta\Lambda_{i,t}\, d\hat{W}_{1,t} + \Delta\Gamma_{i,t}\, d\hat{W}_{2,t}, \\ \Delta G_{i,T} = 0, \end{cases} \tag{10.D.5}$$

for $i = 1,2$. By using the closed-form expression for $\Gamma_{i,t}$ given in (10.4.8), uniform boundedness of function $g_i(t)$ on $[0,T]$ and Lemma 10.4.2, the stochastic exponential process associated with the following Radon-Nikodym derivative:

$$\begin{aligned} \left.\frac{d\check{\mathbb{P}}}{d\hat{\mathbb{P}}}\right|_{\mathcal{F}_T} &= \exp\left\{-\int_0^T q_i\rho_2 g_i(t)\sqrt{\alpha_t}\, d\hat{W}_{2,t} - \int_0^T \frac{q_i^2\rho_2^2 g_i^2(t)}{2}\alpha_t\, dt\right\} \\ &= \exp\left\{-\int_0^T q_i\Gamma_{i,t}\, d\hat{W}_{2,t} - \int_0^T \frac{q_i^2\Gamma_{i,t}^2}{2}\, dt\right\} \end{aligned}$$

is an $(\mathbb{F},\hat{\mathbb{P}})$-martingale, i.e., measure $\check{\mathbb{P}}$ is well-defined and equivalent to $\hat{\mathbb{P}}$ on $\mathcal{F}_T$. It follows from Girsanov's theorem that

$$\check{W}_{0,t} = \hat{W}_{0,t},\ \check{W}_{1,t} = \hat{W}_{1,t},\ \check{W}_{2,t} = \int_0^t q_i\Gamma_{i,s}\, ds + \hat{W}_{2,t}$$

are standard Brownian motions under measure $\check{\mathbb{P}}$, for $i = 1,2$. We then have the following quadratic BSDE of $(\Delta G_{i,t}, \Delta H_{i,t}, \Delta\Lambda_{i,t}, \Delta\Gamma_{i,t})$ from (10.D.5) satisfying all the regularity conditions in Kobylanski (2000):

$$\begin{cases} d\Delta G_{i,t} = -\dfrac{q_i}{2}\Delta\Gamma_{i,t}^2\,dt + \Delta H_{i,t}\,d\check{W}_{0,t} + \Delta\Lambda_{i,t}\,d\check{W}_{1,t} + \Delta\Gamma_{i,t}\,d\check{W}_{2,t} \\ \Delta G_{i,T} = 0, \end{cases} \tag{10.D.6}$$

for $i = 1, 2$. By Theorem 2.3 and 2.6 in Kobylanski (2000), we know that there is a unique solution to BSDE (10.D.6), and it is easy to check that $(0, 0, 0, 0)$ is the unique solution, which indicates that $(G_{i,t}, H_{i,t}, \Lambda_{i,t}, \Gamma_{i,t}) = (\hat{G}_{i,t}, \hat{H}_{i,t}, \hat{\Lambda}_{i,t}, \hat{\Gamma}_{i,t})$ for $i = 1, 2$. In other words, the proposed solution given in (10.4.7)-(10.4.8) must be the unique solution to quadratic BSDE (10.4.6).

□

10.E Proof of Theorem 10.4.11

Proof. Given the competitor's strategy $\hat{\pi}_j := \left\{\hat{\pi}_{j,t}^S, \hat{\pi}_{j,t}^I\right\} \in \Pi_j$ and the associated inflation-adjusted wealth process $X_t^{\hat{\pi}_j}$, it follows from (10.4.1), Remark 10.4.4 and Proposition 10.4.3 that for manager $i \neq j \in \{1, 2\}$,

$$\begin{aligned}
&\frac{d\left(-\frac{1}{q_i}e^{-q_i\left(Y_{i,t}(X_t^{\pi_i} - w_i X_t^{\hat{\pi}_j}) + G_{i,t}\right)}\right)}{e^{-q_i\left(Y_{i,t}(X_t^{\pi_i} - w_i X_t^{\hat{\pi}_j}) + G_{i,t}\right)}} \\
&= \underbrace{\left[Y_{i,t}\left(X_t^{\pi_i}(\pi_{i,t}^S\sigma_s + (\pi_{i,t}^I - 1)\sigma_p) - w_i X_t^{\hat{\pi}_j}(\hat{\pi}_{j,t}^S\sigma_s + (\hat{\pi}_{j,t}^I - 1)\sigma_p)\right) + H_{i,t}\right]}_{K_{0,t}} dW_{0,t} \\
&\quad + \underbrace{\left[Y_{i,t}(X_t^{\pi_i}\pi_{i,t}^S - w_i X_t^{\hat{\pi}_j}\hat{\pi}_{j,t}^S)\sigma_t + \Lambda_{i,t}\right]}_{K_{1,t}} dW_{1,t} + \Gamma_{i,t}\,dW_{2,t} - \frac{q_i}{2}\Bigg[Y_{i,t}\Bigg(X_t^{\pi_i}(\pi_{i,t}^S\sigma_s \\
&\quad + (\pi_{i,t}^I - 1)\sigma_p) - w_i X_t^{\hat{\pi}_j}(\hat{\pi}_{j,t}^S\sigma_s + (\hat{\pi}_{j,t}^I - 1)\sigma_p)\Bigg) + H_{i,t} - \frac{1}{q_i}(\lambda_p - \sigma_p)\Bigg]^2 dt \\
&\quad - \frac{q_i}{2}\left[Y_{i,t}(X_t^{\pi_i}\pi_{i,t}^S - w_i X_t^{\hat{\pi}_j}\hat{\pi}_{j,t}^S)\sigma_t + \Lambda_{i,t} - \frac{1}{q_i}\lambda\sqrt{\alpha_t}\right]^2 dt.
\end{aligned} \tag{10.E.1}$$

Define the following stopping time γ_n:

$$\gamma_n = \inf\left\{t \geq 0 : \int_0^t e^{-q_i\left(Y_{i,s}(X_s^{\pi_i} - w_i X_s^{\hat{\pi}_j}) + G_{i,s}\right)}(K_{0,s} + K_{1,s} + \Lambda_{i,s})^2\,ds \geq n\right\}.$$

We see that $\gamma_n \to \infty$, $\mathbb{P}$ almost surely as $n \to \infty$, and the stochastic integrals in (10.E.1) are true $(\mathbb{F}, \mathbb{P})$-martingales when stopped by $\{\gamma_n\}_{n\in\mathbb{N}}$. In other words, integrating (10.E.1) both sides from 0 to $\gamma_n \wedge T$ and taking expectations, we have

$$\begin{aligned}
&\mathbb{E}\left[-\frac{1}{q_i}e^{-q_i\left(Y_{i,\gamma_n\wedge T}(X^{\pi_i}_{\gamma_n\wedge T}-w_iX^{\hat{\pi}_j}_{\gamma_n\wedge T})+G_{i,\gamma_n\wedge T}\right)}\right]\\
=&-\mathbb{E}\left[\int_0^{\gamma_n\wedge T}\frac{q_ie^{-q_i\left(Y_{i,t}(X^{\pi_i}_t-w_iX^{\hat{\pi}_j}_t)+G_{i,t}\right)}}{2}\left[Y_{i,t}\left(X^{\pi_i}_t(\pi^S_{i,t}\sigma_s+(\pi^I_{i,t}-1)\sigma_p)-w_iX^{\hat{\pi}_j}_t(\hat{\pi}^S_{j,t}\sigma_s\right.\right.\right.\\
&\left.\left.\left.+(\hat{\pi}^I_{j,t}-1)\sigma_p\right)\right)+H_{i,t}-\frac{1}{q_i}(\lambda_p-\sigma_p)\right]^2dt\right]-\mathbb{E}\left[\int_0^{\gamma_n\wedge T}\frac{q_ie^{-q_i\left(Y_{i,t}(X^{\pi_i}_t-w_iX^{\hat{\pi}_j}_t)+G_{i,t}\right)}}{2}\right.\\
&\left.\times\left[Y_{i,t}(X^{\pi_i}_t\pi^S_{i,t}-w_iX^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t})\sigma_t+\Lambda_{i,t}-\frac{1}{q_i}\lambda\sqrt{\alpha_t}\right]^2dt\right]-\frac{1}{q_i}e^{-q_i(Y_{i,0}(x_{i,0}-w_ix_{j,0})+G_{i,0})}.
\end{aligned}\tag{10.E.2}$$

Note that the terms in the expectations on the right-hand side of (10.E.2) are non-negative and increasing with respect to n and the term in the expectation on the left-hand side of (10.E.2) is uniformly integrable for any admissible strategy by Definition 10.3.2. Applying the monotone convergence theorem and the equivalence between uniform integrability and $\mathcal{L}^1$ convergence to the right-hand side and left-hand side of (10.E.2), respectively, we obtain

$$\begin{aligned}
&\mathbb{E}\left[-\frac{1}{q_i}e^{-q_i\left(X^{\pi_i}_T-L_{i,T}-w_i(X^{\hat{\pi}_j}_T-L_{j,T})\right)}\right]\\
=&-\mathbb{E}\left[\int_0^{T}\frac{q_ie^{-q_i\left(Y_{i,t}(X^{\pi_i}_t-w_iX^{\hat{\pi}_j}_t)+G_{i,t}\right)}}{2}\left[Y_{i,t}\left(X^{\pi_i}_t(\pi^S_{i,t}\sigma_s+(\pi^I_{i,t}-1)\sigma_p)-w_iX^{\hat{\pi}_j}_t(\hat{\pi}^S_{j,t}\sigma_s\right.\right.\right.\\
&\left.\left.\left.+(\hat{\pi}^I_{j,t}-1)\sigma_p\right)\right)+H_{i,t}-\frac{1}{q_i}(\lambda_p-\sigma_p)\right]^2dt\right]-\mathbb{E}\left[\int_0^{T}\frac{q_ie^{-q_i\left(Y_{i,t}(X^{\pi_i}_t-w_iX^{\hat{\pi}_j}_t)+G_{i,t}\right)}}{2}\right.\\
&\left.\times\left[Y_{i,t}(X^{\pi_i}_t\pi^S_{i,t}-w_iX^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t})\sigma_t+\Lambda_{i,t}-\frac{1}{q_i}\lambda\sqrt{\alpha_t}\right]^2dt\right]-\frac{1}{q_i}e^{-q_i(Y_{i,0}(x_{i,0}-w_ix_{j,0})+G_{i,0})}\\
\le&-\frac{1}{q_i}e^{-q_i(Y_{i,0}(x_{i,0}-w_ix_{j,0})+G_{i,0})},
\end{aligned}\tag{10.E.3}$$

and the upper bound is attained when the terms in the expectations on the right-hand side of (10.E.3) are zeros, from which we find that given the competitor's strategy $\hat{\pi}_j$, the optimal response strategy $\hat{\pi}_i$ is given by

$$\hat{\pi}^S_{i,t}=\frac{1}{X^{\hat{\pi}_i}_t}\left(\frac{\frac{\lambda}{q_i}\sqrt{\alpha_t}-\Lambda_{i,t}}{\sigma_tY_{i,t}}+w_iX^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t}\right),\tag{10.E.4}$$

and

$$\hat{\pi}^I_{i,t}=\left[\frac{1}{X^{\hat{\pi}_i}_t}\left(\frac{\frac{\lambda_p-\sigma_p}{q_i}-H_{i,t}}{\sigma_pY_{i,t}}+w_iX^{\hat{\pi}_j}_t\frac{\hat{\pi}^S_{j,t}\sigma_s+(\hat{\pi}^I_{j,t}-1)\sigma_p}{\sigma_p}\right)-\frac{\sigma_s}{\sigma_p}\hat{\pi}^S_{i,t}\right]+1,\tag{10.E.5}$$

for $i \neq j \in \{1,2\}$. Then, solving (10.E.4) and (10.E.5) explicitly leads to the strategy $\pi_i^*, i=1,2$, given in (10.4.15). Moreover, substituting the optimal response strategy $\hat{\pi}_i$ into (10.E.1) whenever $\hat{\pi}_j \in \Pi_j$ is given, we have

$$\frac{d\left(-\frac{1}{q_i}e^{-q_i(Y_{i,t}(X_t^{\hat{\pi}_i}-w_iX_t^{\hat{\pi}_j})+G_{i,t})}\right)}{-\frac{1}{q_i}e^{-q_i(Y_{i,t}(X_t^{\hat{\pi}_i}-w_iX_t^{\hat{\pi}_j})+G_{i,t})}} = -(\lambda_p-\sigma_p)\,dW_{0,t} - \lambda\sqrt{\alpha_t}\,dW_{1,t} - q_i\rho_2 g_i(t)\sqrt{\alpha_t}\,dW_{2,t}.$$

Solving the above linear SDE explicitly yields

$$\begin{aligned}
&-\frac{1}{q_i}e^{-q_i(Y_{i,t}(X_t^{\hat{\pi}_i}-w_iX_t^{\hat{\pi}_j})+G_{i,t})}\\
&= -\frac{1}{q_i}e^{-q_i(Y_{i,0}(x_{i,0}-w_ix_{j,0})+G_{i,0})}\exp\bigg\{-\int_0^t(\lambda_p-\sigma_p)\,dW_{0,s} - \int_0^t\lambda\sqrt{\alpha_s}\,dW_{1,s}\\
&\quad -\int_0^t q_i\rho_2 g_i(s)\sqrt{\alpha_s}\,dW_{2,s} - \frac{1}{2}\int_0^t(\lambda_p-\sigma_p)^2 + (\lambda^2+q_i^2\rho_2^2g_i^2(s))\alpha_s\,ds\bigg\}.
\end{aligned}$$

In view of this result, combined with Lemma 10.4.2 and Theorem 2.4 in Cherny (2006), we know that $-\frac{1}{q_i}e^{-q_i(Y_{i,t}(X_t^{\hat{\pi}_i}-w_iX_t^{\hat{\pi}_j})+G_{i,t})}$ is an $(\mathbb{F},\mathbb{P})$-martingale. Therefore, for $i \neq j \in \{1,2\}$ we have

$$\begin{aligned}
J_i^{(\hat{\pi}_i,\hat{\pi}_j)}(x_{i,0},x_{j,0},l_{i,0},l_{j,0},s_0,\alpha_0) =& \mathbb{E}\left[-\frac{1}{q_i}e^{-q_i\left(X_T^{\hat{\pi}_i}-L_{i,T}-w_i(X_T^{\hat{\pi}_j}-L_{j,T})\right)}\right]\\
=& -\frac{1}{q_i}e^{-q_i(Y_{i,0}(x_{i,0}-w_ix_{j,0})+G_{i,0})}.
\end{aligned}\tag{10.E.6}$$

Combining (10.E.3) and (10.E.6), we find that given any competitor's strategy $\hat{\pi}_j$, for $i \neq j \in \{1,2\}$,

$$\sup_{\pi_i\in\Pi_i} J_i^{(\pi_i,\hat{\pi}_j)}(x_{i,0},x_{j,0},l_{i,0},l_{j,0},s_0,\alpha_0) \le J_i^{(\hat{\pi}_i,\hat{\pi}_j)}(x_{i,0},x_{j,0},l_{i,0},l_{j,0},s_0,\alpha_0).\tag{10.E.7}$$

In particular, when $\hat{\pi}_j = \pi_j^*$, we have $\hat{\pi}_i = \pi_i^*$. Then, it follows from (10.E.7) that the pair (π_1^*, π_2^*) given in (10.4.15) is the Nash equilibrium of the non-zero-sum stochastic differential game (10.3.2)-(10.3.3), and the values functions are given by (10.4.16).

To end this proof, it remains to show that the Nash equilibrium strategy is admissible. Clearly, the Nash equilibrium strategy given in (10.4.15) is $\mathbb{F}$-adapted. In addition, since we have known that the stochastic process $-\frac{1}{q_i}e^{-q_i(Y_{i,t}(X_t^{\hat{\pi}_i}-w_iX_t^{\hat{\pi}_j})+G_{i,t})}$ is an $(\mathbb{F},\mathbb{P})$-martingale, and for any sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \to \infty$, $\mathbb{P}$ almost surely as $n\to\infty$, $\tau_n \wedge T$ and T are two bounded stopping times, it follows from Doob's optimal sampling theorem for bounded stopping times (refer to Corollary 3.23 in Le Gall (2016)) that

$$-\frac{1}{q_i}e^{-q_i\left(Y_{i,\tau_n\wedge T}(X_{\tau_n\wedge T}^{\hat{\pi}_i}-w_iX_{\tau_n\wedge T}^{\hat{\pi}_j})+G_{i,\tau_n\wedge T}\right)} = \mathbb{E}\left[-\frac{1}{q_i}e^{-q_i\left(X_T^{\hat{\pi}_i}-L_{i,T}-w_i(X_T^{\hat{\pi}_j}-L_{j,T})\right)}\middle|\,\mathcal{F}_{\tau_n\wedge T}\right].$$

Note that $\{\mathcal{F}_{\tau_n\wedge T}\}_{n\in\mathbb{N}}$ is family of sub-algebra of $\mathcal{F}_T$, by Theorem 4.6.1 in Durrett (2019), we know that $\left\{-\frac{1}{q_i}e^{-q_i\left(Y_{i,\tau_n\wedge T}(X_{\tau_n\wedge T}^{\hat{\pi}_i}-w_iX_{\tau_n\wedge T}^{\hat{\pi}_j})+G_{i,\tau_n\wedge T}\right)}\right\}_{n\in\mathbb{N}}$ is uniformly integrable. Particularly, $\left\{-\frac{1}{q_i}e^{-q_i\left(Y_{i,\tau_n\wedge T}(X^{\pi_i^*}_{\tau_n\wedge T}-w_iX^{\pi_j^*}_{\tau_n\wedge T})+G_{i,\tau_n\wedge T}\right)}\right\}_{n\in\mathbb{N}}$ is also a uniformly integrable family, for $i \neq j \in \{1,2\}$. This verifies admissibility condition (2) in Definition 10.3.2. Finally, plugging (10.4.15) into the dynamics of the inflation-adjusted wealth process (10.2.9) for manager i and applying Itô's formula to $e^{r(T-t)}X_t^{\pi_i^*}$, for $i = 1,2$, we have

$$
\begin{aligned}
d\left(e^{r(T-t)}X_t^{\pi_i^*}\right) = &\left[\frac{\lambda\sqrt{\alpha_t}}{1-w_iw_j}\left(\frac{\lambda\sqrt{\alpha_t}}{q_i}-\Lambda_{i,t}+w_i\left(\frac{\lambda\sqrt{\alpha_t}}{q_j}-\Lambda_{j,t}\right)\right)\right.\\
&\left.+\frac{\lambda_p-\mu_p}{1-w_iw_j}\left(\frac{\lambda_p-\mu_p}{q_i}-H_{i,t}+w_i\left(\frac{\lambda_p-\sigma_p}{q_j}-H_{j,t}\right)\right)\right]dt\\
&+\frac{1}{1-w_iw_j}\left(\frac{\lambda_p-\mu_p}{q_i}-H_{i,t}+w_i\left(\frac{\lambda_p-\sigma_p}{q_j}-H_{j,t}\right)\right)dW_{0,t}\\
&+\frac{1}{1-w_iw_j}\left(\frac{\lambda\sqrt{\alpha_t}}{q_i}-\Lambda_{i,t}+w_i\left(\frac{\lambda\sqrt{\alpha_t}}{q_j}-\Lambda_{j,t}\right)\right)dW_{1,t}.
\end{aligned}
\tag{10.E.8}
$$

Integrating both sides of (10.E.8) with respect t leads to the following explicit expression for the inflation-adjusted wealth process governed by the Nash equilibrium strategy for manager $i = 1,2$:

$$
\begin{aligned}
X_t^{\pi_i^*} =&x_{i,0}e^{rt}+e^{r(t-T)}\int_0^t\left[\frac{\lambda\sqrt{\alpha_s}}{1-w_iw_j}\left(\frac{\lambda\sqrt{\alpha_s}}{q_i}-\Lambda_{i,s}+w_i\left(\frac{\lambda\sqrt{\alpha_s}}{q_j}-\Lambda_{j,s}\right)\right)\right.\\
&\left.+\frac{\lambda_p-\mu_p}{1-w_iw_j}\left(\frac{\lambda_p-\mu_p}{q_i}-H_{i,s}+w_i\left(\frac{\lambda_p-\sigma_p}{q_j}-H_{j,s}\right)\right)\right]ds\\
&+e^{r(t-T)}\int_0^t\frac{1}{1-w_iw_j}\left(\frac{\lambda_p-\mu_p}{q_i}-H_{i,s}+w_i\left(\frac{\lambda_p-\sigma_p}{q_j}-H_{j,s}\right)\right)dW_{0,s}\\
&+e^{r(t-T)}\int_0^t\frac{1}{1-w_iw_j}\left(\frac{\lambda\sqrt{\alpha_s}}{q_i}-\Lambda_{i,s}+w_i\left(\frac{\lambda\sqrt{\alpha_s}}{q_j}-\Lambda_{j,s}\right)\right)dW_{1,s},
\end{aligned}
$$

which verifies admissibility condition (1) in Definition 10.3.2. Hence, we can conclude that the Nash equilibrium strategy (π_1^*,π_2^*) given by (10.4.15) is admissible. □

10.F Proof of Proposition 10.5.6

Proof. Inspired by the affinity of the terminal condition of BSDE (10.5.8), we conjecture that the first component of the solution to BSDE (10.5.8) admits an affine form as well, i.e, for $i \neq j \in \{1,2\}$,

$$\tilde{G}_{i,t} = \tilde{a}_i(t)L_{i,t} - w_i\tilde{b}_i(t)L_{j,t}, \tag{10.F.1}$$

where $\tilde{a}_i(t)$ and $\tilde{b}_i(t)$ are two differentiable functions that will be determined later with boundary condition that $\tilde{a}_i(T) = \tilde{b}_i(T) = -1$. Applying Itô's formula to $\tilde{G}_{i,t}$

shows that

$$
\begin{aligned}
d\tilde{G}_{i,t} = \Bigg[& \left(\frac{d\tilde{a}_i(t)}{dt} + \tilde{a}_i(t)(\mu_i - \mu_p + \sigma_p^2 - \beta_i\sigma_p + \lambda\sigma_i\alpha_t) \right) L_{i,t} - \left(\frac{d\tilde{b}_i(t)}{dt} + \tilde{b}_i(t)(\mu_j - \mu_p + \sigma_p^2 \right. \\
& \left. -\beta_j\sigma_p + \lambda\sigma_j\alpha_t) \right) w_i L_{j,t} \Bigg] \, dt + \left(\tilde{a}_i(t)(\beta_i - \sigma_p)L_{i,t} - w_i\tilde{b}_i(t)(\beta_j - \sigma_p)L_{j,t} \right) dW_{0,t} \\
& + \left(\tilde{a}_i(t)\sigma_i L_{i,t} - w_i\tilde{b}_i(t)\sigma_j L_{j,t} \right) \sqrt{\alpha_t} \, dW_{1,t}.
\end{aligned}
\tag{10.F.2}
$$

Let

$$
\begin{aligned}
\tilde{H}_{i,t} = & \tilde{a}_i(t)(\beta_i - \sigma_p)L_{i,t} - w_i\tilde{b}_i(t)(\beta_j - \sigma_p)L_{j,t}, \\
\tilde{\Lambda}_{i,t} = & \left(\tilde{a}_i(t)\sigma_i L_{i,t} - w_i\tilde{b}_i(t)\sigma_j L_{j,t} \right) \sqrt{\alpha_t}, \\
\tilde{\Gamma}_{i,t} = & 0.
\end{aligned}
$$

Then, the generator of linear BSDE (10.5.8) turns out to be

$$
L_{i,t}\tilde{a}_i(t)\left(r + \lambda\sigma_i\alpha_t + (\lambda_p - \sigma_p)(\beta_i - \sigma_p)\right) - w_i L_{j,t}\tilde{b}_i(t)\left(r + \lambda\sigma_j\alpha_t + (\lambda_p - \sigma_p)(\beta_j - \sigma_p)\right), \tag{10.F.3}
$$

where $i \neq j \in \{1, 2\}$. A direct comparison between (10.F.3) and the drift coefficient of (10.F.2) leads to the following linear homogeneous ODEs:

$$
\begin{aligned}
\frac{d\tilde{a}_i(t)}{dt} &= (r + \mu_p - \mu_i + \lambda_p(\beta_i - \sigma_p))\,\tilde{a}_i(t), \;\; \tilde{a}_i(T) = -1, \\
\frac{d\tilde{b}_i(t)}{dt} &= (r + \mu_p - \mu_j + \lambda_p(\beta_j - \sigma_p))\,\tilde{b}_i(t), \;\; \tilde{b}_i(T) = -1,
\end{aligned}
$$

from which we know that $\tilde{a}_j(t) = \tilde{b}_i(t)$, $i \neq j \in \{1, 2\}$ and the analytical expression for $\tilde{a}_i(t)$ is then given by (10.5.11), for $i = 1, 2$.

So far, we have found one solution to linear BSDE (10.5.8) which is given in (10.5.9) and (10.5.10). Following almost the same arguments in the proof of Proposition 10.4.10, it is easy to show that the solution $(\tilde{G}_{i,t}, \tilde{H}_{i,t}, \tilde{\Lambda}_{i,t}, \tilde{\Gamma}_{i,t})$ given in (10.5.9) and (10.5.10) lies in the space $\mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R})$. So, we do not repeat them here.

To complete the proof, we now in the position to show that the solution (10.5.9)-(10.5.10) forms the unique solution to linear BSDE (10.5.8). By Lemma 10.4.2 and Theorem 2.4 in Cherny (2006), the following Radon-Nikodym derivative

$$
\begin{aligned}
\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\Bigg\{ & -\int_0^T (\lambda_p - \sigma_p)\, dW_{0,t} - \int_0^T \lambda\sqrt{\alpha_t}\, dW_{1,t} + \int_0^T \rho_2\tilde{g}_i(t)\sqrt{\alpha_t}\, dW_{2,t} \\
& -\frac{1}{2}\int_0^T (\lambda_p - \sigma_p)^2 + (\lambda^2 + \rho_2^2\tilde{g}_i^2(t))\alpha_t\, dt \Bigg\}
\end{aligned}
$$

is well-defined and hence the probability measure $\bar{\mathbb{P}}$ is well-defined on $\mathcal{F}_T$, where $i = 1, 2$. Due to the equivalence between $\bar{\mathbb{P}}$ and $\mathbb{P}$ and Girsanov's theorem, the processes $\bar{W}_{0,t}, \bar{W}_{1,t}, \bar{W}_{2,t}$ defined by

$$
\bar{W}_{0,t} = \int_0^t (\lambda_p - \sigma_p)\, ds + W_{0,t}, \;\; \bar{W}_{1,t} = \int_0^t \lambda\sqrt{\alpha_s}\, ds + W_{1,t}, \;\; \bar{W}_{2,t} = -\int_0^t \rho_2\tilde{g}_i(s)\sqrt{\alpha_s}\, ds + W_{2,t}
$$

are standard Brownian motions under $\bar{\mathbb{P}}$. Then, linear BSDE (10.5.8) can be simplified as follows:

$$\begin{cases} d\tilde{G}_{i,t} = & r\tilde{G}_{i,t}\,dt + \tilde{H}_{i,t}\,d\bar{W}_{0,t} + \tilde{\Lambda}_{i,t}\,d\bar{W}_{1,t} + \tilde{\Gamma}_{i,t}\,d\bar{W}_{2,t}, \\ \tilde{G}_{i,T} = & -L_{i,T} + w_i L_{j,T}. \end{cases} \tag{10.F.4}$$

Denoted by $\left(\check{G}_{i,t}, \check{H}_{i,t}, \check{\Lambda}_{i,t}, \check{\Gamma}_{i,t}\right)$ any solution to BSDE (10.5.8). Then, it follows from (10.F.4) that the difference process between $\left(\check{G}_{i,t}, \check{H}_{i,t}, \check{\Lambda}_{i,t}, \check{\Gamma}_{i,t}\right)$ and the solution given in (10.5.9)-(10.5.10) defined by

$$\Delta\tilde{G}_{i,t} = \tilde{G}_{i,t} - \check{G}_{i,t},\ \ \Delta\tilde{H}_{i,t} = \tilde{H}_{i,t} - \check{H}_{i,t},\ \ \Delta\tilde{\Lambda}_{i,t} = \tilde{\Lambda}_{i,t} - \check{\Lambda}_{i,t},\ \ \Delta\tilde{\Gamma}_{i,t} = \tilde{\Gamma}_{i,t} - \check{\Gamma}_{i,t}$$

must solve the following linear BSDE of $(\Delta\tilde{G}_{i,t}, \Delta\tilde{H}_{i,t}, \Delta\tilde{\Lambda}_{i,t}, \Delta\tilde{\Gamma}_{i,t})$ under measure $\bar{\mathbb{P}}$:

$$\begin{cases} d\Delta\tilde{G}_{i,t} = & r\Delta\tilde{G}_{i,t}\,dt + \Delta\tilde{H}_{i,t}\,d\bar{W}_{0,t} + \Delta\tilde{\Lambda}_{i,t}\,d\bar{W}_{1,t} + \Delta\tilde{\Gamma}_{i,t}\,d\bar{W}_{2,t}, \\ \Delta\tilde{G}_{i,T} = & 0, \end{cases}$$

for $i = 1, 2$. This is a linear BSDE with standard data (refer to El Karoui, Peng, and Quenez (1997)). Then, it follows from Theorem 2.1 in El Karoui, Peng, and Quenez (1997) that the above BSDE admits a unique solution, and particularly, we notice that $(0, 0, 0, 0)$ forms the unique solution, which indicates the solution presented in (10.5.9)-(10.5.10) is the unique solution to BSDE (10.5.8). □

10.G Proof of Theorem 10.5.8

Proof. The proof is similar to that of Theorem 10.4.11 for the exponential utility preferences. Given any competitor's investment strategy denoted by $\hat{\pi}_j := \left\{\hat{\pi}^S_{j,t}, \hat{\pi}^I_{j,t}\right\}$ and the associated inflation-adjusted wealth process $X^{\hat{\pi}_j}_t$, we know from (10.5.1), Remark 10.5.7, and the two unique solutions to BSDEs (10.5.2) and (10.5.3) that for manager i, $i \neq j \in \{1, 2\}$,

$$\begin{aligned}
& d\left(\frac{\tilde{Y}_{i,t}}{\gamma_i}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i}\right) \\
=& \Bigg[\frac{(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i}}{\gamma_i}\tilde{Z}_{i,t} + \tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i - 1}\Big(X^{\pi_i}_t(\pi^S_{i,t}\sigma_s + (\pi^I_{i,t} - 1)\sigma_p) \\
& - w_i X^{\hat{\pi}_j}_t(\hat{\pi}^S_{j,t}\sigma_s + (\hat{\pi}^I_{j,t} - 1)\sigma_p) + \tilde{H}_{i,t}\Big)\Bigg]\,dW_{0,t} + \Bigg[\frac{(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i}}{\gamma_i}\tilde{M}_{i,t} \\
& + \tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i - 1}\Big((X^{\pi_i}_t\pi^S_{i,t} - w_i X^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t})\sigma_t + \tilde{\Lambda}_{i,t}\Big)\Bigg]\,dW_{1,t} \\
& + \Bigg[\frac{(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i}}{\gamma_i}\tilde{P}_{i,t} + \tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i - 1}\tilde{\Gamma}_{i,t}\Bigg]\,dW_{2,t} + \frac{\gamma_i - 1}{2}\tilde{Y}_{i,t} \\
& \times (X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i - 2}\Bigg[(X^{\pi_i}_t\pi^S_{i,t} - w_i X^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t})\sigma_t + \tilde{\Lambda}_{i,t} + \frac{X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i - 1}\Big(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} \\
& + \lambda\sqrt{\alpha_t}\Big)\Bigg]^2 dt + \frac{\gamma_i - 1}{2}\tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i - 2}\Bigg[X^{\pi_i}_t(\pi^S_{i,t}\sigma_s + (\pi^I_{i,t} - 1)\sigma_p) - w_i X^{\hat{\pi}_j}_t \\
& \times (\hat{\pi}^S_{j,t}\sigma_s + (\hat{\pi}^I_{j,t} - 1)\sigma_p) + \tilde{H}_{i,t} + \frac{X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i - 1}\Big(\frac{\tilde{Z}_{i,t}}{\tilde{Y}_{i,t}} + \lambda_p - \sigma_p\Big)\Bigg]^2 dt.
\end{aligned} \tag{10.G.1}$$

Using some localization techniques, integrating both sides of (10.G.1) from 0 to $\tilde{\tau}_n \wedge T$, and taking expectations, where $\{\tilde{\tau}_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathbb{F}$-stopping times such that $\tilde{\tau}_n \to \infty$, $\mathbb{P}$ almost surely as $n \to \infty$ and the above three stochastic integrals in (10.G.1) are true $(\mathbb{F},\mathbb{P})$-martingales when stopped by $\tilde{\tau}_n$, we then obtain

$$\begin{aligned}
&\mathbb{E}\left[\frac{\tilde{Y}_{i,\tilde{\tau}_n\wedge T}}{\gamma_i}\left(X^{\pi_i}_{\tilde{\tau}_n\wedge T} - w_i X^{\hat{\pi}_j}_{\tilde{\tau}_n\wedge T} + \tilde{G}_{i,\tilde{\tau}_n\wedge T}\right)^{\gamma_i}\right]\\
=&\frac{\gamma_i-1}{2}\Bigg\{\mathbb{E}\Bigg[\int_0^{\tilde{\tau}_n\wedge T}\tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i-2}\Bigg[(X^{\pi_i}_t\pi^S_{i,t} - w_i X^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t})\sigma_t + \tilde{\Lambda}_{i,t}\\
&+\frac{X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i-1}\left(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} + \lambda\sqrt{\alpha_t}\right)\Bigg]^2 dt\Bigg] + \mathbb{E}\Bigg[\int_0^{\tilde{\tau}_n\wedge T}\tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i-2}\\
&\times\Bigg[X^{\pi_i}_t(\pi^S_{i,t}\sigma_s + (\pi^I_{i,t}-1)\sigma_p) - w_i X^{\hat{\pi}_j}_t(\hat{\pi}^S_{j,t}\sigma_s + (\hat{\pi}^I_{j,t}-1)\sigma_p) + \tilde{H}_{i,t} + \frac{X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i-1}\\
&\times\left(\frac{\tilde{Z}_{i,t}}{\tilde{Y}_{i,t}} + \lambda_p - \sigma_p\right)\Bigg]^2 dt\Bigg]\Bigg\} + \frac{\tilde{Y}_{i,0}}{\gamma_i}\left(x_{i,0} + \tilde{a}_i(0)l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0)l_{j,0})\right)^{\gamma_i}.
\end{aligned} \tag{10.G.2}$$

Since the terms in the expectations on the right-hand side of (10.G.2) are non-negative and increasing with respect to n and the term in the expectation on the left-hand side of (10.G.2) is uniformly integrable due to Definition 10.3.3, sending n to the limit and applying the monotone-convergence theorem and the equivalence between the uniform integrability and $\mathcal{L}^1$-convergence to the expectations on the right-hand side and left-hand side of (10.G.2) respectively yield:

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\gamma_i}\left(X^{\pi_i}_T - L_{i,T} - w_i(X^{\hat{\pi}_j}_T - L_{j,T})\right)^{\gamma_i}\right]\\
=&\frac{\gamma_i-1}{2}\Bigg\{\mathbb{E}\Bigg[\int_0^{T}\tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i-2}\Bigg[(X^{\pi_i}_t\pi^S_{i,t} - w_i X^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t})\sigma_t + \tilde{\Lambda}_{i,t}\\
&+\frac{X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i-1}\left(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} + \lambda\sqrt{\alpha_t}\right)\Bigg]^2 dt\Bigg] + \mathbb{E}\Bigg[\int_0^{T}\tilde{Y}_{i,t}(X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t})^{\gamma_i-2}\\
&\times\Bigg[X^{\pi_i}_t(\pi^S_{i,t}\sigma_s + (\pi^I_{i,t}-1)\sigma_p) - w_i X^{\hat{\pi}_j}_t(\hat{\pi}^S_{j,t}\sigma_s + (\hat{\pi}^I_{j,t}-1)\sigma_p) + \tilde{H}_{i,t} + \frac{X^{\pi_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i-1}\\
&\times\left(\frac{\tilde{Z}_{i,t}}{\tilde{Y}_{i,t}} + \lambda_p - \sigma_p\right)\Bigg]^2 dt\Bigg]\Bigg\} + \frac{\tilde{Y}_{i,0}}{\gamma_i}\left(x_{i,0} + \tilde{a}_i(0)l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0)l_{j,0})\right)^{\gamma_i}\\
\leq&\frac{\tilde{Y}_{i,0}}{\gamma_i}\left(x_{i,0} + \tilde{a}_i(0)l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0)l_{j,0})\right)^{\gamma_i}.
\end{aligned} \tag{10.G.3}$$

Therefore, the optimal response investment strategy $\hat{\pi}_i$ for manager i is given by

$$\hat{\pi}^S_{i,t} = \frac{1}{X^{\hat{\pi}_i}_t}\left[-\frac{\frac{X^{\hat{\pi}_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i-1}\left(\frac{\tilde{M}_{i,t}}{\tilde{Y}_{i,t}} + \lambda\sqrt{\alpha_t}\right) + \tilde{\Lambda}_{i,t}}{\sigma_t} + w_i X^{\hat{\pi}_j}_t\hat{\pi}^S_{j,t}\right], \tag{10.G.4}$$

and

$$\begin{aligned}
\hat{\pi}^I_{i,t} = &\Bigg[\frac{1}{X^{\hat{\pi}_i}_t}\left(-\frac{\frac{X^{\hat{\pi}_i}_t - w_i X^{\hat{\pi}_j}_t + \tilde{G}_{i,t}}{\gamma_i-1}(\lambda_p - \sigma_p) + \tilde{H}_{i,t}}{\sigma_p} + w_i X^{\hat{\pi}_j}_t\frac{\hat{\pi}^S_{j,t}\sigma_s + (\hat{\pi}^I_{j,t}-1)\sigma_p}{\sigma_p}\right)\\
&-\frac{\sigma_s}{\sigma_p}\hat{\pi}^S_{i,t}\Bigg] + 1,
\end{aligned} \tag{10.G.5}$$

for $i \neq j \in \{1,2\}$. Solving (10.G.4) and (10.G.5) then yields the strategy $\pi^*_{i,t}$, $i = 1,2$ given in (10.5.12). In addition, plugging the optimal response strategy $\hat{\pi}_i$ given in (10.G.4)-(10.G.5) into (10.G.1) whenever the competitor's strategy $\hat{\pi}_j$ is given, we find that

$$\begin{aligned}
&\frac{d\left(\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\hat{\pi}_i} - w_i X_t^{\hat{\pi}_j} + \tilde{G}_{i,t})^{\gamma_i}\right)}{\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\hat{\pi}_i} - w_i X_t^{\hat{\pi}_j} + \tilde{G}_{i,t})^{\gamma_i}}\\
=&\frac{\gamma_i}{1-\gamma_i}(\lambda_p - \sigma_p)\,dW_{0,t} + \frac{\rho_1 \tilde{g}_i(t) + \gamma_i \lambda}{1-\gamma_i}\sqrt{\alpha_t}\,dW_{1,t} + \rho_2 \tilde{g}_i(t)\sqrt{\alpha_t}\,dW_{2,t}.
\end{aligned}$$

Solving this linear SDE explicitly and using Theorem 2.4 in Cherny (2006) and Lemma 10.4.2 above, we know that

$$\begin{aligned}
&\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\hat{\pi}_i} - w_i X_t^{\hat{\pi}_j} + \tilde{G}_{i,t})^{\gamma_i}\\
=&\exp\left\{\int_0^t \frac{\gamma_i}{1-\gamma_i}(\lambda_p - \sigma_p)\,dW_{0,s} + \int_0^t \frac{\rho_1 \tilde{g}_i(s) + \gamma_i \lambda}{1-\gamma_i}\sqrt{\alpha_s}\,dW_{1,s} + \int_0^t \rho_2 \tilde{g}_i(s)\sqrt{\alpha_s}\,dW_{2,s}\right.\\
&\left.-\frac{1}{2}\int_0^t \frac{\gamma_i^2}{(1-\gamma_i)^2}(\lambda_p - \sigma_p)^2 + \left(\frac{(\rho_1 \tilde{g}_i(s) + \gamma_i\lambda)^2}{(1-\gamma_i)^2} + \rho_2^2 \tilde{g}_i^2(s)\right)\alpha_s\,ds\right\}\frac{\tilde{Y}_{i,0}}{\gamma_i}(x_{i,0} + \tilde{a}_i(0) l_{i,0}\\
&-w_i(x_{j,0} + \tilde{a}_j(0) l_{j,0}))^{\gamma_i}
\end{aligned}$$

is an $(\mathbb{F},\mathbb{P})$-martingale, for $i \neq j \in \{1,2\}$. Then, the martingale property of $\frac{\tilde{Y}_{i,t}}{\gamma_i}(X_t^{\hat{\pi}_i} - w_i X_t^{\hat{\pi}_j} + \tilde{G}_{i,t})^{\gamma_i}$ leads to

$$\begin{aligned}
J_i^{(\hat{\pi}_i,\hat{\pi}_j)}(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0) &= \mathbb{E}\left[\frac{1}{\gamma_i}\left(X_T^{\hat{\pi}_i} - L_{i,T} - w_i(X_T^{\hat{\pi}_j} - L_{j,T})\right)^{\gamma_i}\right]\\
&= \frac{\tilde{Y}_{i,0}}{\gamma_i}\left(x_{i,0} + \tilde{a}_i(0) l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0) l_{j,0})\right)^{\gamma_i}.
\end{aligned} \tag{10.G.6}$$

From (10.G.3) and (10.G.6), we see that given any competitor's strategy $\hat{\pi}_j$, for $i \neq j \in \{1,2\}$,

$$\sup_{\pi_i \in \Pi_i} J_i^{(\pi_i,\hat{\pi}_j)}(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0) \leq J_i^{(\hat{\pi}_i,\hat{\pi}_j)}(x_{i,0}, x_{j,0}, l_{i,0}, l_{j,0}, s_0, \alpha_0),$$

where the equalities are attained when the two managers opt for the optimal response strategies at the same time. In other words, the pair (π_1^*, π_2^*) given in (10.5.12) is the Nash equilibrium by the definition of the non-zero-sum stochastic differential game (10.3.2)-(10.3.3). Note from (10.5.12) that the Nash equilibrium strategy (π_1^*, π_2^*) is $\mathbb{F}$-adapted. Moreover, when the initial data satisfies $x_{i,0} + \tilde{a}_i(0) l_{i,0} - w_i(x_{j,0} + \tilde{a}_j(0) l_{j,0}) > 0$, for $i \neq j \in \{1,2\}$, it follows from the above results that the process $X_t^{\pi_i^*} - w_i X_t^{\pi_j^*} + \tilde{G}_{i,t} > 0$, $\mathbb{P}$ almost surely, for all $t \in [0,T]$, which means that the admissibility condition (1) in Definition 10.3.3 is verified. The admissibility conditions (2) and (3) for the Nash equilibrium strategy (π_1^*, π_2^*) can be checked by following almost the same arguments in the proof of Theorem 10.4.11 above, so we omit it here. □

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