Tensor Decompositions: Theory and Applications in Quantum Information

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Preface

Abstract

In this thesis, we shed light from various angles on ways of decomposing tensors. Our investigation consists of four parts.

In the first part, we study *entanglement structures*, which are a natural generalization of tensor networks: While tensor network states can be seen as locally transformed versions of a tensor "built" from two-party tensors laid out according to the geometry of a graph, entanglement structures are constructed from tensors put on the hyperedges of a hypergraph. We find constructions and obstructions for the conversion between entanglement structures in the sense of restriction and degeneration, and calculate the tensor rank of specific entanglement structures.

We then go on to study tensor networks in more depth. More precisely, we study the *quantum max-flow*, which quantifies the amount of entanglement between two regions of a tensor network. We relate the quantum max-flow in the so-called *bridge graph* to the theory of prehomogeneous tensor spaces and the representation theory of quivers, a connection that enables us to calculate the quantum max-flow in this graph in a large number of cases.

After that, we define and study *partial degeneration*, an intermediate version of restriction and degeneration, which are well-known preorders for tensors. By constructing various examples and showing obstructions, we demonstrate that partial degeneration is inequivalent to both restriction and degeneration. We also relate this concept to the notion of *aided rank*, a generalization of tensor rank. Here, we again highlight differences between the concepts of degeneration and partial degeneration.

Finally, we analyze *stabilizer rank decompositions*, which are relevant in the theory of simulating quantum circuits. In particular, we present a technique to lower bound stabilizer rank and approximate stabilizer rank. This technique yields, together with other interesting consequences, a strong lower bound on the stabilizer rank of tensor powers of the so-called T-state – a quantity gauging the efficiency of the simulation of quantum circuits built from Clifford+T gates using the Gottesman-Knill theorem.

Resumé

I denne afhandling kaster vi lys over måder at dekomponere tensorer. Vores undersøgelse består af fire dele.

I den første del studerer vi *sammenfiltringsstrukturer*, som er en naturlig generalisering af tensornetværk: Mens tensornetværkstilstande kan ses som lokalt transformerede versioner af en tensor "opbygget" af bipartite tensorer lagt på kanterne af en graf, er sammenfiltringsstrukturer den naturlige generalisering, hvor en tensor er konstrueret af mindre tensorer sat på hyperkanterne af en hypergraf. Vi finder konstruktioner og obstruktioner for omdannelse, i betydningen restriktion og degeneration, af sammenfiltringsstrukturer. Vi beregner også tensorrangen af visse sammenfiltringsstrukturer.

Vi fortsætter derefter med at dykke dybere ned i teorien om tensornetværk. Mere præcist studerer vi *det kvantemekaniske max-flow*, som kvantificerer graden af sammenfiltring mellem to områder i et tensornetværk. Vi relaterer det kvantemekaniske max-flow i en specifik graf til teorien om præhomogene tensorrum og repræsentationsteorien for koggere, som gør os i stand til at beregne størrelsen i mange tilfælde.

Derefter studerer vi *partiel degeneration*, som er en mellemversion af restriktion og degeneration. Ved at konstruere forskellige eksempler og vise obstruktioner viser vi, at partiel degeneration hverken er ækvivalent med restriktion eller degeneration. Vi relaterer også dette begreb til begrebet understøttet rang, en generalisering af tensor rank. Her fremhæver vi igen forskelle mellem sædvanlige begreber degeneration og partiel degeneration.

Til sidst analyserer vi *stabilisatorrangdekompositioner*, som er relevante for teorien om simulering af kvantekredsløb. Specielt præsenterer vi en teknik til at begrænse både stabilisatorrang og approximativ stabilisatorrang nedenfra. Teknikken giver, som en blandt flere interessante konsekvenser, en god nedre grænse for stabilisatorrangen af den såkaldte T-tilstand – en størrelse, der kvantificerer hvor effektiv man kan simulere en kvantekredsløb bygget af Clifford+T-porte ved hjælp af Gottesman-Knill-sætningen.

Contributions and structure

This thesis consists of **four** main parts following the general introduction in Chapter 1.

- Chapter 2 is based on ongoing work with Matthias Christandl.
- Chapter 3 is a partly modified version of the preprint [GLS22] which is joint work with Fulvio Gesmundo and Vladimir Lysikov:

  Gesmundo, Fulvio and Lysikov, Vladimir and Steffan, Vincent. Quantum max-flow in the bridge graph, 2022. doi:10.48550/arXiv.2212.09794
- Chapter 4 is a partly modified version of the preprint [CGLS22] which is joint work with Matthias Christandl, Fulvio Gesmundo and Vladimir Lysikov.

  Christandl, Matthias and Gesmundo, Fulvio and Lysikov, Vladimir and Steffan, Vincent. Partial degeneration of tensors, 2022. doi:10.48550/arXiv.2212.14095
- Chapter 5 is a partly modified version of [LS22] which is joint work with Benjamin Lovitz.

  Lovitz, Benjamin and Steffan, Vincent. New techniques for bounding stabilizer rank, Quantum, 6:692, 2022. doi:10.22331/q-2022-04-20-692

  Parts of this chapter also appear in Benjamin Lovitz's Ph.D. thesis [Lov22]. Previous to publication, the manuscript [LS22] has been uploaded to a preprint server:

  Lovitz, Benjamin and Steffan, Vincent. New techniques for bounding stabilizer rank, 2021. doi:10.48550/arXiv.2110.07781.

During my Ph.D., I also worked on the article [BCG+20] which is not included in this thesis.

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Contents

Chapter 1

Introduction

The main objects of study in this thesis are tensors. By that, we always mean elements $T \in U_1 \otimes \cdots \otimes U_k$ of the tensor product of finite-dimensional complex vector spaces $U_1, \ldots, U_k$ of dimension $\mathrm{u}_1, \ldots, \mathrm{u}_k$. Throughout this thesis, we will refer to k as the number of *parties* and call T a k-party tensor. By fixing a basis $\mathrm{e}_1, \ldots, \mathrm{e}_{\mathrm{u}_i}$ for each of the spaces U_i, a tensor T is essentially a multidimensional array $(T_{i_1 \ldots i_k})_{i_1 \ldots i_k}$ where the index i_j runs from 1 to $\mathrm{u}_j = \dim(U_j)$:

$$T = \sum_{i_1 \ldots i_k = 1}^{\mathrm{u}_1 \ldots \mathrm{u}_k} T_{i_1 \ldots i_k} \mathrm{e}_{i_1} \otimes \cdots \otimes \mathrm{e}_{i_k}$$

Tensors are used in various fields of applied and theoretical mathematics and are the main protagonists in many areas. One reason for their versatility is the sheer complexity of tensors: The number of parameters necessary to describe a general tensor scales exponentially in the number of parties. A key task for many applications is therefore, to find succinct descriptions of the relevant tensors. Often, such descriptions are achieved by ways of *decomposing* tensors. Typical examples that are frequently used and that we will study in this thesis are tensor rank decompositions, tensor network representations, and stabilizer rank decompositions.

One way to think about tensor rank decompositions and tensor network representations is by introducing certain preorders on the set of k-party tensors, for example, *restriction* and *degeneration*. The notions of restriction and degeneration can be thought of as ways to "compare" tensors: The fact that T restricts or degenerates to S reflects that T is more complex than S. For a tensor, having a tensor rank decomposition of a certain length resp. a particular tensor network representation can be equivalently stated as the tensor being a restriction of a special tensor (which depends on the setup). In that way, we can think of tensors as *resources* using restriction and degeneration. We will review all of the mentioned concepts in Section 1.1, Section 1.3 and Section 1.4.

Another way of decomposing tensors comes from the simulation of quantum circuits. It turns out that quantum circuits with a computational basis state as input and that are composed of gates from the so-called Clifford gate set followed by measurements in the computational basis can be simulated efficiently on a classical device [Got98]. States that can be prepared from a computational basis state by applying Clifford gates only are called stabilizer states. In this context, a natural way of decomposing tensors arises: The stabilizer rank of a tensor is the minimal number of stabilizer states whose linear span contains it. By so-called magic state injection, one can see that the stabilizer rank of a certain tensor quantifies the complexity of a certain simulation protocol [BSS16, BG16, BBC+19, QPG21]. We will review the theory of stabilizer rank decompositions in Section 1.5.

Each of the mentioned ways of decomposing tensors has its use cases in both application and theory. Let us outline a few areas in which the study of tensors is central.

- In *quantum many body physics* and *quantum information theory*, tensors $T \in U_1 \otimes \cdots \otimes U_k$ specify the *state* of a system of k *particles* where the i'th particle has u_i *degrees of freedom*. Many interesting problems in these areas can be translated to problems about the corresponding tensors [NC00, DVC00]. In this context, tensor network decompositions are an important tool. Tensors that admit efficient tensor network representations are believed to capture the *physical corner of the Hilbert space*, that is, the set of "physically reasonable" quantum states. More specifically, they are believed to parametrize the set of quantum states obeying an *area law* [Has07, ECP10]. Because of that, they have been useful both for theoretical insights and in applications, for example, to perform efficient calculations for complex quantum systems [FNW92, Whi92, PGVWC07, Vid07, Vid08, SCPG10, Sch11, Orú14, LVV15, ALVV17, HP18]. We also mention that tensor network techniques lie at the heart of many state-of-the-art quantum circuit simulators [FSC+18, Orú19, ZSW20, PCZ22, PZ22]. Tensor network decompositions are moreover frequently used in areas like holography [Swi12, MNS+15, PYHP15, HNQ+16, Eve17], quantum chemistry [CS11, KDTR15, SPM+15, CKN+16, ZC21] and numerical computations for complex quantum systems [HP18, CGFW21]. More specifically, the theory of restrictions and degenerations can lead to insights into *quantum entanglement*. A breakthrough result in quantum information theory was the fact that three-party quantum systems can be entangled in two genuinely different ways. This observation was merely a translation of the fact that restriction and degeneration are inequivalent notions as soon as the number of tensor factors is at least three [DVC00]. To mention another example, the theory of *prehomogeneous tensor spaces*, more precisely, of so-called *matrix pencils* has been applied to improve the understanding of entanglement in special three-party systems [CdTP06, CMS10]. Finally, in the theory of quantum circuit simulation the minimal lengths, of stabilizer rank decompositions of tensor powers of the so-called T-state are an important indicator of the efficiency of certain simulation protocols. Simulation schemes using the Gottesman-Knill theorem to simulate quantum circuits built from the Clifford+T gate set scale polynomially in this quantity [BSS16, BG16, BBC+19, QPG21].

- In *algebraic complexity theory*, tensors are used to study the complexity of multilinear maps, most prominently, the complexity of matrix multiplication [BCS97, Blä13]. Here, the minimal size of a tensor rank decomposition of T is a natural measure of complexity for the multilinear map associated with T: Roughly speaking, the minimal length of a tensor rank decomposition of T tells us how many multiplications need to be performed to calculate the corresponding multilinear map. Finding explicit (border) rank decompositions is equivalent to finding explicit (approximate) algorithms to calculate this multilinear map. For example, the study of tensor rank and tensor border rank decompositions of certain special tensors led to a sequence of breakthrough results on the complexity of matrix multiplication [Str69, Pan78, BCLR79, Bin80, Sch81, CW81, Rom82, Str87, CW87, Wil12, LG14, AW21]. For a comprehensive introduction to this area, we refer to [BCS97, Blä13], for a discussion from the viewpoint of algebraic geometry and representation theory, see for example [Lan12].

- Various other areas of applied mathematics, like machine learning, and theoretical areas, like combinatorics, have employed the rich theory of tensors that has been developed in the last decades. In the realm of machine learning, tensor network decompositions like, for example, so-called *tensor train decompositions*, are used, see [Ose11, SS16, CWZ21, LLZ+21] for examples. Tensor rank decomposition is often referred to as *canonical polyadic decomposition* ("CANDECOMP"), *parallel factor analysis* ("PARAFAC") or *CP-decomposition* [KB09]. The highly developed theory of tensor (border) rank decompositions and closely related notions like, for example, *slice rank* have been successfully applied to make progress on combinatorial problems like the sunflower problem and cap sets [Tao16, EG17, CFTZ22].

In this thesis, we will study and generalize the mentioned ways of decomposing tensors from various different angles. Let us briefly summarize the main contributions of this thesis.

- In Chapter 2, we will study *entanglement structures* which are a natural generalization of tensor networks. We mention that similar concepts have been used recently in the context of tensor network representations, for example, to construct tensor network representations of the resonating valence bond (RVB) state [SPCPG12, CLVW20]. We will study to what extent entanglement structures can be transformed into one another in the sense of restriction and degeneration, respectively. We also study tensor rank decompositions of entanglement structures. Concretely, we derive novel constructions and obstructions for the conversion between entanglement structures and highlight by that how subtle the question about the conversion between entanglement structures is. To do so, we analyze the geometry of the underlying hypergraphs and relate our problem to asymptotic tensor restrictions and the asymptotic spectrum of tensors introduced in [Str86, Str88]. We also answer an open question in [CF18] by calculating the tensor rank of every possible entanglement structure built from two copies of the so-called W-tensor.

- In Chapter 3, we will study the so-called *quantum max-flow*, which is a quantity associated with tensor networks. Roughly speaking, the quantum max-flow specifies how entangled two regions of a tensor network can be. Work prior to [GLS22] showed certain upper bounds on the quantum max-flow, but calculating the quantum max-flow seemed intangible [CFS+16, GLW18]. We will see how one can calculate the quantum max-flow exactly for the smallest graph for which the quantum max-flow is nontrivial. We are able to calculate it in a wide range of cases by relating it to the theory of prehomogeneous tensor spaces and the representation theory of the so-called Kronecker quiver. Chapter 3 is an adjusted version of [GLS22] which is joint work with Fulvio Gesmundo and Vladimir Lysikov.

- In Chapter 4, we will introduce and study a novel, intermediate version of restriction and degeneration and shed some light on this new way of seeing tensors as a resource. We will see a plethora of examples as well as no-go and classification results highlighting the properties of this new notion and demonstrating that it is, in fact, inequivalent to both restriction and degeneration. We will also relate it to a notion called *aided rank*. Chapter 4 is an adjusted version of [CGLS22] which is joint work with Matthias Christandl, Fulvio Gesmundo and Vladimir Lysikov.

- In Chapter 5, we will present techniques from number theory that enable us to prove lower bounds on the stabilizer rank of a large number of tensors, in particular, tensor powers of the so-called T-state providing strong lower bounds on the simulation complexity of quantum circuits built from the Clifford+T gate set. Moreover, we develop a deeper understanding of the notion of stabilizer rank by constructing tensors with maximal possible stabilizer rank and tensors with multiplicative stabilizer rank. Chapter 5 is an adjusted version of [LS22] which is joint work with Benjamin Lovitz.

We will now introduce the most important concepts for this thesis to set the stage for the mentioned results.

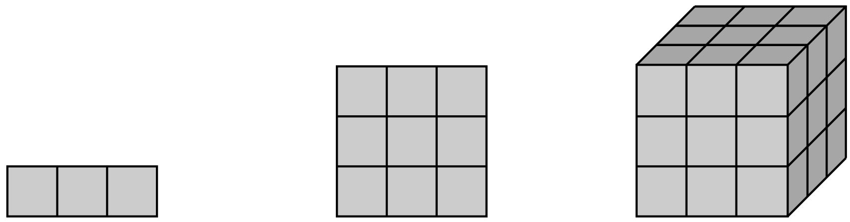
1.1 Tensors: Resources of entanglement and complexity

Let $U_1, \dots, U_k$ be finite-dimensional, complex vector spaces of dimensions $\mathrm{u}_1, \dots, \mathrm{u}_k$. For each of the spaces U_i, we fix a basis $\mathrm{e}_1 \dots \mathrm{e}_{\mathrm{u}_i}$ with a corresponding dual basis $\mathrm{e}_1^*, \dots, \mathrm{e}_{\mathrm{u}_i}^*$. Consider a tensor

$$T = \sum_{i_1 \dots i_k = 1}^{\mathrm{u}_1 \dots \mathrm{u}_k} T_{i_1 \dots i_k} \mathrm{e}_{i_1} \otimes \cdots \otimes \mathrm{e}_{i_k} \in U_1 \otimes \cdots \otimes U_k.$$

For $k = 1$, this is just a u_1-dimensional vector, and for $k = 2$, it can be identified with a $\mathrm{u}_1 \times \mathrm{u}_2$ matrix. We visualize k-party tensors for $k = 1, 2, 3$ in Figure 1.1. Note that we are being slightly sloppy by using the same symbols for basis vectors in all of the spaces U_i in order to reduce the number of indices needed. We will stick to this practice throughout this thesis.

Often, it is useful to think of a tensor in a resource-theoretic way.



(a) A one-party tensor. (b) A two-party tensor. (c) A three-party tensor.

Figure 1.1: Visualization of tensors in $U_1 \otimes \cdots \otimes U_k$ for $k = 1, 2, 3$. For $k = 1$, this is just a vector (Figure 1.1(a)), and for $k = 2$ a matrix (Figure 1.1(b)). For $k = 3$, we can imagine a cube in which the coefficients of the tensor are arranged (Figure 1.1(c)).

Definition 1.1.1. *For tensors $T \in U_1 \otimes \cdots \otimes U_k$ and $S \in V_1 \otimes \cdots \otimes V_k$, we say that T* restricts *to S and write $T \geq S$ if there are linear maps $A_i : U_i \to V_i$ such that $S = (A_1 \otimes \cdots \otimes A_k)T$.*

It is clear that restriction of tensors defines a preorder on the set of all k-party tensors. For $k = 1$, all tensors (except the zero tensor) are equivalent under this preorder. For $k = 2$, tensors T and S can be interpreted as matrices. It is a standard fact that $T \geq S$ holds if and only if the matrix rank of T is greater than or equal to the matrix rank of S. Since matrix rank is lower semicontinuous, the set

$$\{S \in V_1 \otimes V_2 : T \geq S\} \subset V_1 \otimes V_2$$

is a closed set for any choice of $T \in U_1 \otimes U_2$. It has been known at least since [Syl52] that this does not generalize to $k \geq 3$. The way to deal with this is to define an approximate version of restriction called *degeneration*.

Definition 1.1.2. *We say that T* degenerates *to S and write $T \trianglerighteq S$ if $S = \lim_{\epsilon \to 0} T_\epsilon$ is a limit of restrictions T_ϵ of T. Here, the limit is taken in the Zariski topology.*

It is a classical result that S is a degeneration of T if and only if there are linear maps $A_i(\epsilon) : U_i \to V_i$ depending polynomially on ϵ such that

$$(A_1(\epsilon) \otimes \cdots \otimes A_k(\epsilon))T = \epsilon^d S + \epsilon^{d+1} S_1 + \cdots + \epsilon^{d+e} S_e$$

for some natural numbers d, e called *approximation degree* and *error degree*, respectively, and some tensors $S_1, \ldots, S_e$ [Hil93]. For a degeneration $T \trianglerighteq S$, both the approximation degree and the error degree are not unique and depend on the specific maps $A_i(\epsilon)$. Sometimes, we write $T \trianglerighteq_d^e S$ if there are specific degeneration maps $A_i(\epsilon)$ realizing the degeneration in these degrees.

Example 1.1.3. *Consider a three-party tensor*

$$T = \sum_{i_1, i_2, i_3 = 1}^{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3} T_{i_1, i_2, i_3} \mathrm{e}_{i_1} \otimes \mathrm{e}_{i_2} \otimes \mathrm{e}_{i_3} \in U_1 \otimes U_2 \otimes U_3.$$

We can naturally associate with T a bilinear map

$$T : U_1^* \times U_2^* \to U_3, (\mathrm{e}_i^*, \mathrm{e}_j^*) \mapsto \sum_{k=1}^{\mathrm{u}_3} T_{ijk} \mathrm{e}_k,$$

or, equivalently, u_3 *bilinear forms. One bilinear map of particular interest is the multiplication of matrices. The tensor corresponding to the bilinear map multiplying an $m \times n$ matrix with an $n \times p$ matrix is given by*

$$\langle m, n, p \rangle = \sum_{i,j,k=1}^{m,n,p} (\mathrm{e}_i \otimes \mathrm{e}_j) \otimes (\mathrm{e}_j \otimes \mathrm{e}_k) \otimes (\mathrm{e}_k \otimes \mathrm{e}_i) \in (\mathbb{C}^m \otimes \mathbb{C}^n) \otimes (\mathbb{C}^n \otimes \mathbb{C}^p) \otimes (\mathbb{C}^p \otimes \mathbb{C}^m).$$

Another tensor that is important in this context is the r-th unit tensor

$$\langle r \rangle = \sum_{i=1}^{r} \mathrm{e}_i \otimes \mathrm{e}_i \otimes \mathrm{e}_i \in (\mathbb{C}^r)^{\otimes 3}.$$

In this basis, the bilinear map corresponding to the unit tensor is the entry-wise multiplication

$$\langle r \rangle : \mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}^r, \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \mapsto \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_r y_r \end{pmatrix}.$$

In this context, $\langle r \rangle \geq T$ means that one can calculate the u_3 *bilinear forms associated with T using r independent products of complex numbers. If $\langle r \rangle \trianglerighteq T$, then the bilinear forms corresponding to T can be approximated arbitrarily well using r products of complex numbers.*

Clearly, we can generalize the unit tensor from Example 1.1.3 to any number of parties k by writing

$$\langle r \rangle = \langle r \rangle_k = \sum_{i=1}^{r} \mathrm{e}_i \otimes \cdots \otimes \mathrm{e}_i \in (\mathbb{C}^r)^{\otimes k}.$$

Most of the time, it will be clear what k is, and we will just write $\langle r \rangle$. The following definition is motivated by Example 1.1.3.

Definition 1.1.4. *For a tensor $T \in U_1 \otimes \cdots \otimes U_k$, we define its* rank *as*

$$R(T) = \min\{r : \langle r \rangle \geq T\}.$$

Moreover, we define the border rank *of T as*

$$\underline{R}(T) = \min\{r : \langle r \rangle \trianglerighteq T\}.$$

From Definition 1.1.4, the following characterization of rank and border rank is immediate. We mention that often, instead of Definition 1.1.4, this equivalent characterization is used to define rank and border rank.

Proposition 1.1.5. *Let $T \in U_1 \otimes \cdots \otimes U_k$. It holds that $R(T) \leq r$ if and only if there are vectors $u_{i1}, \ldots, u_{ir} \in U_i$ for $i = 1, \ldots, k$ such that*

$$T = \sum_{i=1}^{r} u_{i1} \otimes \cdots \otimes u_{ik}.$$

Moreover, T has border rank $\underline{R}(T) \leq r$ if and only if there are vectors $u_{i1}(\epsilon) \ldots u_{ir}(\epsilon) \in U_i[\epsilon]$ depending polynomially on ϵ for $i = 1, \ldots, k$ such that

$$\epsilon^d T = \sum_{i=1}^{r} u_{i1}(\epsilon) \otimes \cdots \otimes u_{ik}(\epsilon) + \mathcal{O}(\epsilon^{d+1})$$

for some natural number d.

We will sometimes refer to tensors of rank 1 as *simple tensors*. For $k = 2$, T can be interpreted as a matrix, and Proposition 1.1.5 tells us that its rank and border rank coincide with its matrix rank.

Remark 1.1.6. *Let $T \in U_1 \otimes \cdots \otimes U_k$ be a tensor of rank at most r, that is, we can write*

$$T = \sum_{i=1}^{r} u_{i1} \otimes \cdots \otimes u_{ik}.$$

Consequently, we see that all tensors of rank at most r can be specified by $r \cdot (\mathrm{u}_1 + \cdots + \mathrm{u}_k)$ complex parameters in the fixed bases. Since the tensor product is multilinear with respect to scalar multiplication, we can actually reduce this number further, and every tensor of rank at most r can be specified by at most $r \cdot (\mathrm{u}_1 + \cdots + \mathrm{u}_k) - r(k-1)$ complex parameters. To specify any *tensor in $U_1 \otimes \cdots \otimes U_k$ we, of course, need $\mathrm{u}_1 \ldots \mathrm{u}_k$ parameters. Hence, there must exist tensors of rank at least $\lceil \frac{\mathrm{u}_1 \ldots \mathrm{u}_k}{r \cdot (\mathrm{u}_1 + \cdots + \mathrm{u}_k) - r(k-1)} \rceil$. In the language of algebraic geometry, the quantity $r \cdot (\mathrm{u}_1 + \cdots + \mathrm{u}_k) - r(k-1)$ is an upper bound on the dimension of the r-th* secant variety *of the* Segre variety *which is the set of all tensors with border rank at most r [Lan12, Section 4.3.6]. In that way, one can see that there must be tensors of border rank at least $\lceil \frac{\mathrm{u}_1 \ldots \mathrm{u}_k}{r \cdot (\mathrm{u}_1 + \cdots + \mathrm{u}_k) - r(k-1)} \rceil$ in $U_1 \otimes \cdots \otimes U_k$ [Lan12, Section 5.1.2]. Letting, for example, $k = 3$ and $\mathrm{u}_1 = \mathrm{u}_2 = \mathrm{u}_3 = \mathrm{u}$, we see that there are tensors in $U_1 \otimes U_2 \otimes U_3$ with border rank scaling quadratically in* u.

As mentioned before, rank and border rank do not coincide in general for $k \geq 3$. The following example demonstrates that and has been known at least since [Syl52].

Example 1.1.7. *Let $U_1 \cong U_2 \cong U_3 \cong \mathbb{C}^2$ with fixed bases $\mathrm{e}_1, \mathrm{e}_2$. Define the W-state as*

$$W = \mathrm{e}_1 \otimes \mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_1 \otimes \mathrm{e}_2 \otimes \mathrm{e}_1 + \mathrm{e}_2 \otimes \mathrm{e}_1 \otimes \mathrm{e}_1 \in U_1 \otimes U_2 \otimes U_3.$$

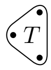
It turns out that $R(W) = 3$ *(see Example 1.3.3). On the other hand, applying the linear maps*

$$A_i(\epsilon) : \begin{cases} \mathrm{e}_1 \mapsto \mathrm{e}_1 + \epsilon \mathrm{e}_2 \\ \mathrm{e}_2 \mapsto -\mathrm{e}_1 \end{cases}$$

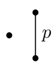
for $i = 1, 2, 3$*, one can see that* $\langle 2 \rangle \trianglerighteq W$ *and consequently* $\underline{R}(W) = 2$.

Our way of thinking will often be inspired by an interpretation of tensors in quantum many-body physics and quantum information theory.

Example 1.1.8. *Tensors specify the joint state of a* $k-$*particle quantum system. For that, fix for all* $i = 1 \dots k$ *a basis* $\mathrm{e}_1 \dots \mathrm{e}_{\mathrm{u}_i}$ *of the space* U_i *which induces a Euclidean product on* $U_1 \otimes \cdots \otimes U_k$*. Then, a* k*-particle quantum state is a unit vector* $\psi \in U_1 \otimes \cdots \otimes U_k$*. In this context, the dimension* u_i *of the space* U_i *is often referred to as the number of degrees of freedom of the* i*'th particle. Motivated by this, we will often depict tensors graphically. We might for example depict a tensor* $T \in U_1 \otimes U_2 \otimes U_3$ *by*

$T =$  $\in U_1 \otimes U_2 \otimes U_3.$

In this context, a tensor $u_1 \otimes \cdots \otimes u_k$ *of tensor rank* 1 *is often called a* product state*. A quantum state that is not a product state is called* entangled*. We will use the term entanglement in this sense frequently. Consider, for example, the matrix multiplication tensor* $\langle 1, 1, p \rangle \in \mathbb{C} \otimes \mathbb{C}^p \otimes \mathbb{C}^p$*. It is clear that this tensor is not a product state, so it is entangled. Interpreted as a bipartite tensor* $\langle 1, 1, p \rangle \in \mathbb{C} \otimes (\mathbb{C}^p \otimes \mathbb{C}^p)$ *though, it is a product state. In that sense, we can think of the second and the third party of this tensor being entangled, whereas there is no entanglement between the first party and the rest. We depict this situation as*

$\langle 1, 1, p \rangle =$  $\in \mathbb{C} \otimes \mathbb{C}^p \otimes \mathbb{C}^p.$

More generally, the tensor rank is known in the physics literature as Schmidt rank *and is commonly used as a barometer for entanglement in many-party quantum systems [EB01, HEB04, CDS08].*

There are various ways of *combining tensors*. Clearly, given $T \in U_1 \otimes \cdots \otimes U_k$ and $S \in V_1 \otimes \cdots \otimes V_l$, we can combine it to a $(k+l)$-party tensor $T \otimes S$ using the usual tensor product. When $k = l$, one often considers this tensor again as a k-party tensor in the following way.

Definition 1.1.9. *Let* $T \in U_1 \otimes \cdots \otimes U_k$ *and* $S \in V_1 \otimes \cdots \otimes V_k$ *be specified via*

$$T = \sum_{i_1 \dots i_k = 1}^{\mathrm{u}_1 \dots \mathrm{u}_k} T_{i_1 \dots i_k} \mathrm{e}_{i_1} \otimes \cdots \otimes \mathrm{e}_{i_k}, S = \sum_{j_1 \dots j_k = 1}^{\mathrm{v}_1 \dots \mathrm{v}_k} S_{j_1 \dots j_k} \mathrm{e}_{j_1} \otimes \cdots \otimes \mathrm{e}_{j_k}.$$

(a) A visualization of the Kronecker product of two three-party tensors.

(b) A visualization of the tensor product of two three-party tensors.

Figure 1.2: Illustrations of the Kronecker product in Figure 1.2(a) and the tensor product in Figure 1.2(b): Taking the Kronecker product of two three-party tensors yields a three-party tensor $T \boxtimes S$ while taking the tensor product yields a six-party tensor.

Their Kronecker product $T \boxtimes S \in (U_1 \otimes V_1) \otimes \cdots \otimes (U_k \otimes V_k)$ *is given by*

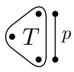
$$T \boxtimes S = \sum_{i_1 \dots i_k, j_1 \dots j_k = 1}^{\mathrm{u}_1 \dots \mathrm{u}_k, \mathrm{v}_1 \dots \mathrm{v}_k} T_{i_1 \dots i_k} \cdot S_{j_1 \dots j_k} (\mathrm{e}_{i_1} \otimes \mathrm{e}_{j_1}) \otimes \cdots \otimes (\mathrm{e}_{i_k} \otimes \mathrm{e}_{j_k}).$$

We will write $T^{\boxtimes n}$ *for the* n*-fold Kronecker product* $T \boxtimes \cdots \boxtimes T$.

The Kronecker product combines two tensors of the same order k to a tensor which again is of order k. Visually, one can, for example, think of the Kronecker product of two three-party tensors as depicted in Figure 1.2(a).

Clearly, one can imagine situations that are intermediate to tensor product and Kronecker product, that is, where only some of the local spaces are grouped together. These situations will be studied in Chapter 2.

Example 1.1.10. *The following trick was first observed in the context of quantum information theory [BBC+93] and is known as* (quantum) teleportation. *Recall from Example 1.1.8 that the matrix multiplication tensor* $\langle 1, 1, p \rangle \in \mathbb{C} \otimes \mathbb{C}^p \otimes \mathbb{C}^p$ *is essentially a two-party tensor, that is, a matrix. In the context of quantum information theory, this is known as an* entangled pair *or* EPR-pair *on* p levels *between the second and third party [EPR35]. For a tensor* $T \in U_1 \otimes U_2 \otimes U_3$, *define*

$$T^{\blacksquare p} := T \boxtimes \langle 1, 1, p \rangle =$$  $$\in U_1 \otimes (U_2 \otimes \mathbb{C}^p) \otimes (U_3 \otimes \mathbb{C}^p).$$

Here, the graphical notation reflects that the three parties share a quantum state (specified by the tensor T*) and, in addition, the second and the third party share an EPR-pair on* p *levels. We will frequently call this shared EPR-pair an* aiding matrix. *The overall tensor is given by*

$$T^{\blacksquare p} = \sum_{i_1, i_2, i_3, j = 1}^{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, p} T_{i_1, i_2, i_3} \mathrm{e}_{i_1} \otimes (\mathrm{e}_{i_2} \otimes \mathrm{e}_j) \otimes (\mathrm{e}_{i_3} \otimes \mathrm{e}_j) \in U_1 \otimes (U_2 \otimes \mathbb{C}^p) \otimes (U_3 \otimes \mathbb{C}^p).$$

Assume now $p = \mathrm{u}_3$ *and consider*

$$\Pi_{\mathrm{u}_3} : U_3 \otimes \mathbb{C}^{\mathrm{u}_3} \to \mathbb{C}, \mathrm{e}_i \otimes \mathrm{e}_j \mapsto \delta_{ij}.$$

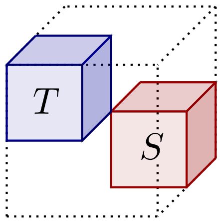


Figure 1.3: Visually, the direct sum of two tensors T and S is a block diagonal tensor with blocks T and S on the diagonal. This figure is borrowed from [CGLS22, Figure 1].

We observe that applying Π_{u_3} to the third tensor factor of $T^{\blacksquare \mathrm{u}_3}$ and the identity to the first two tensor factors yields a restriction

$$T^{\blacksquare \mathrm{u}_3} \geq \tilde{T} \in U_1 \otimes (U_2 \otimes U_3) \otimes \mathbb{C}$$

where $\tilde{T}$ is the tensor T considered as a two-party tensor $T \in U_1 \otimes (U_2 \otimes U_3)$. Hence, we have teleported the third party to the second party using the tensor $\langle 1, 1, \mathrm{u}_3 \rangle$.

Another way of combining tensors is the *direct sum*. Visually, taking the direct sum of two tensors corresponds to putting the two tensors on the diagonal of a bigger tensor, see Figure 1.3.

Definition 1.1.11. *Let tensors $T \in U_1 \otimes \cdots \otimes U_k$ and $S \in V_1 \otimes \cdots \otimes V_k$ be given via*

$$T = \sum_{i_1 \dots i_k = 1}^{\mathrm{u}_1 \dots \mathrm{u}_k} T_{i_1 \dots i_k} \mathrm{e}_{i_1} \otimes \cdots \otimes \mathrm{e}_{i_k}, \quad S = \sum_{j_1 \dots j_k = 1}^{\mathrm{v}_1 \dots \mathrm{v}_k} S_{j_1 \dots j_k} \mathrm{e}_{j_1} \otimes \cdots \otimes \mathrm{e}_{j_k}.$$

To define their direct sum, pick a basis $\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{u}_i + \mathrm{v}_i}$ of $U_i \oplus V_i$ for $i = 1, \dots, k$ and define the tensor $T \oplus S \in (U_1 \oplus V_1) \otimes \cdots \otimes (U_k \oplus V_k)$ via

$$(T \oplus S)_{i_1, \dots, i_k} = \begin{cases} T_{i_1, \dots, i_k} & \text{if } i_1 \leq \mathrm{u}_1, \dots, i_k \leq \mathrm{u}_k, \\ S_{i_1 - \mathrm{u}_1, \dots, i_k - \mathrm{u}_k} & \text{if } i_1 > \mathrm{u}_1, \dots, i_k > \mathrm{u}_k, \\ 0 & \text{else.} \end{cases}$$

Sometimes, it is insightful to group together parties of a tensor.

Definition 1.1.12. *Let $T \in U_1 \otimes \cdots \otimes U_k$ be a tensor and write $[k] = \{1, \dots, k\}$. Let $\mathcal{S} \subset [k]$ and denote $\mathcal{S}^c = [k] \setminus \mathcal{S}$. Writing $U_{\mathcal{S}} = \bigotimes_{i \in \mathcal{S}} U_i$ and $U_{\mathcal{S}^c} = \bigotimes_{i \in \mathcal{S}^c} U_i$ we can view $T \in U_{\mathcal{S}} \otimes U_{\mathcal{S}^c}$ as a bipartite tensor. We call the linear map associated with T via the identification*

$$U_{\mathcal{S}} \otimes U_{\mathcal{S}^c} \cong \mathrm{Hom}(U_{\mathcal{S}}^*, U_{\mathcal{S}^c})$$

a flattening map *with respect to $\mathcal{S}$. If $\mathcal{S}$ has exactly one element, we denote the associated linear map by $T_i : U_i^* \to U_1 \otimes \cdots \otimes \hat{U}_i \otimes \cdots \otimes U_k$.*

Sometimes it is important for technical reasons to make sure that a tensor uses all degrees of freedom. For that, the following definition is handy.

Definition 1.1.13. *A tensor $T \in U_1 \otimes \cdots \otimes U_k$ is called* concise *if all the maps $T_1, \ldots, T_k$ are injective.*

Intuitively, a tensor T is concise if it uses all degrees of freedom in each of the spaces $U_1, \ldots, U_k$.

Remark 1.1.14. *We will often restrict ourselves to three-party tensors $T \in U_1 \otimes U_2 \otimes U_3$. Note that we can, up to a change of basis on the space U_1, identify T with the subspace $T_1(U_1^*) \subset U_2 \otimes U_3$. By fixing bases of the spaces U_2 and U_3, we can interpret $U_2 \otimes U_3$, and with that also $T_1(U_1^*)$, as a space of matrices:*

$$\sum_{i_2, i_3 = 1}^{\mathrm{u}_2, \mathrm{u}_3} x_{i_2, i_3} \mathrm{e}_{i_2} \otimes \mathrm{e}_{i_3} \leftrightarrow (x_{i,j})_{i,j=1}^{\mathrm{u}_2, \mathrm{u}_3}.$$

For example, for the W-tensor from Example 1.1.7, we have

$$W_1(\mathrm{e}_1^*) = \mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_2 \otimes \mathrm{e}_1 \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W_1(\mathrm{e}_2^*) = \mathrm{e}_1 \otimes \mathrm{e}_1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In that way, we can interpret

$$W(U_1^*) = \left\{ \begin{pmatrix} x_2 & x_1 \\ x_1 & 0 \end{pmatrix} : x_1, x_2 \in \mathbb{C} \right\}.$$

1.2 Representation theory for tensors

The resource-theoretic interpretation of tensors that we have seen in Section 1.1 is often studied using tools from *representation theory*. We emphasize that the following discussion simplifies drastically since we work over complex numbers. Many concepts and results we will see are more subtle for fields with positive characteristics. We assume familiarity with some basic notation and terminology from commutative algebra and algebraic geometry. For an introduction, we refer to [Sha13]. For an in-depth introduction to the representation theory of linear algebraic groups, we refer to [Bor91]. Let us recall the notion of a group action.

Definition 1.2.1. *Let G be a (linear algebraic) group and V be a vector space. We say that G acts on V if there is a homomorphism $\rho : G \to \mathrm{GL}(V)$. The map ρ is called a* representation *of the group G. In this context, V is sometimes called a G-module.*

We will almost exclusively deal with the setup where G is a product of general linear groups. Since it will be clear in most cases what the map ρ is, we will most of the time drop ρ and use the shorthand notation $g.v = \rho(g)v$.

Note that we can combine two actions of a single group: If a group G acts on spaces V_1 and V_2 via $\rho_i : G \to \mathrm{GL}(V_i)$ then this induces an action of G on $V_1 \otimes V_2$ via $g.T = \rho_1(g) \otimes \rho_2(g)T$ for all $T \in V_1 \otimes V_2$. Similarly, this induces an action of G on $V_1 \oplus V_2$, and in the same way, one can, for two groups G and H acting on spaces V and W, construct a representation of $G \times H$ on $V \otimes W$. With this, we can see how representation theory connects to the notions presented in Section 1.1.

Example 1.2.2. *For $V_1, \ldots, V_k$ finite-dimensional, complex vector spaces, $\times_{i=1}^{k} \mathrm{GL}(V_i)$ acts naturally on $V_1 \otimes \cdots \otimes V_k$ by extending*

$$(g_1, \ldots, g_k).v_1 \otimes \cdots \otimes v_k = (g_1 v_1) \otimes \cdots \otimes (g_k v_k)$$

linearly.

The following concepts will be used throughout.

Definition 1.2.3. *Let G act on V as above and let $v \in V$. We call $G.v = \{g.v : g \in G\}$ the* orbit *of v (under G) and $\mathrm{Stab}_G(v) = \{g \in G : g.v = v\}$ the* stabilizer *of v (under G). The group $\mathrm{Stab}_G(v)$ is sometimes called the* isotropy group *of v in G.*

The following result is well-known, for a detailed discussion, we refer to, for example, [Bor91, Chapter I, Section 1.2].

Theorem 1.2.4. *Let G be a group acting on a space V. For every $v \in V$, the orbit $G.v$ is Zariski-open in $\overline{G.v}$. In particular, we have $\dim(G.v) = \dim(\overline{G.v})$ for every $v \in V$. Here, we refer to the dimension of an (irreducible, quasi-projective) variety in the sense of [Sha13, Chapter 1, Section 6]*

Example 1.2.5. *Consider the action introduced in Example 1.2.2. Let $T, S \in V_1 \otimes \cdots \otimes V_k$ be tensors such that $T \geq S$, that is, $S = (A_1 \otimes \cdots \otimes A_k)T$ for some linear maps A_i. Since the general linear group is Zariski-dense in the space of endomorphisms of V_i, it follows that $G.S \subset \overline{G.T}$. Hence, by Theorem 1.2.4 we see that $\dim(G.S) \leq \dim(G.T)$. In other words, we can find obstructions for tensor restriction by calculating orbit dimensions.*

If the tensor S is concise, we can make Example 1.2.5 more precise.

Lemma 1.2.6. *Let $T, S \in V_1 \otimes \cdots \otimes V_k$ be tensors and assume that S is concise. Then $T \geq S$ holds if and only if T and S lie in the same orbit under the action of $G = \times_{i=1}^{k} \mathrm{GL}(V_i)$.*

Proof. It is clear that $S \in G.T$ implies $T \geq S$. Assume now that $T \geq S$ and let $A_1, \ldots, A_k$ be linear maps such that $(A_1 \otimes \cdots \otimes A_k)T = S$. Clearly, such a tensor S cannot be concise if any of the maps A_i is not invertible. Hence, $S \in G.T$, which finishes the proof. □

The following result will be useful for calculating the dimension of an orbit. For a proof, see, for example, [Kra85, Chapter II, Section 2.2].

Proposition 1.2.7. *For a linear algebraic group G acting on a vector space V, it holds that*

$$\dim(\mathrm{Stab}_G(v)) = \dim(G) - \dim(G.v)$$

for all $v \in V$.

The following two definitions will be used frequently.

Definition 1.2.8. *Let G act on V as before. We call a linear subspace $W \subset V$ a G-*submodule *if $g.v \in W$ for all $v \in W$ and all $g \in G$. The module V is called* irreducible *under the action of G if it has no submodule except $\{0\}$ and V itself.*

Definition 1.2.9. *Let G be a group, and U and V be G-modules. Then, a linear map $f : U \to V$ is called a G-*module homomorphism *if for all $u \in U$ and $g \in G$ it holds that $f(g.u) = g.f(u)$, that is if f commutes with the action of G. The space of G-homomorphisms is denoted $\mathrm{Hom}_G(U, V)$. If f is an isomorphism, then the G-modules U and V are called isomorphic.*

The following result about irreducible representations is known as *Schur's lemma* [FH91, Lemma 1.7].

Theorem 1.2.10. *Let G be a group, and U and V be irreducible G-modules. Then, if U and V are not isomorphic (as G-modules), $\mathrm{Hom}_G(U, V) = \{0\}$. If, on the other hand, U and V are isomorphic as G-modules then $\mathrm{Hom}_G(U, V)$ is 1-dimensional. In particular, if $U = V$ then $\mathrm{Hom}_G(U, U) = \{\lambda \cdot \mathrm{id}_U : \lambda \in \mathbb{C}\}$.*

For a finite-dimensional vector space V, consider its k-th tensor power $V^{\otimes k}$. The symmetric group $\mathfrak{S}_k$ acts on $V^{\otimes k}$ by permuting tensor factors: For each $\sigma \in \mathfrak{S}_k$, we obtain a linear map by linearly extending

$$\sigma : v_1 \otimes \cdots \otimes v_k \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \tag{1.1}$$

to the whole space. As we have seen before, we also have an action of the group $\mathrm{GL}(V)$ on $V^{\otimes k}$ by extending

$$g.(v_1 \otimes \cdots \otimes v_k) = (g.v_1 \otimes \cdots \otimes g.v_k) \tag{1.2}$$

linearly. Note that clearly for any $\sigma \in \mathfrak{S}_k$ and any $g \in \mathrm{GL}(V)$, the actions in Equation (1.1) and Equation (1.2) commute. Using this observation, we can construct two submodules of $V^{\otimes k}$ for the action of $\mathrm{GL}(V)$. A more in-depth discussion of the following definitions and facts can be found in [Lan12, Section 2.6].

Definition 1.2.11. *Let V be a finite-dimensional, complex vector space and $k \in \mathbb{N}$. We define the symmetric subspace as*

$$S^k(V) = \{T \in V^{\otimes k} : \sigma T = T \text{ for all } \sigma \in \mathfrak{S}_k\} \subset U^{\otimes k}$$

and the antisymmetric subspace as

$$\bigwedge^k(V) = \{T \in V^{\otimes k} : \sigma T = \operatorname{sgn}(\sigma)T \text{ for all } \sigma \in \mathfrak{S}_k\} \subset V^{\otimes k}.$$

Clearly, the two spaces in Definition 1.2.11 are submodules of $V^{\otimes k}$ under the action of $\mathrm{GL}(V)$. It turns out that they are irreducible.

The projection in $V^{\otimes k}$ onto the subspace $S^k(V)$ in $V^{\otimes k}$ is given by the map

$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \dots v_k := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}^k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

It is well-known that, for $\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{v}}$ a basis of V, the set $\{\mathrm{e}_{i_1} \dots \mathrm{e}_{i_k} : i_1 \leq \cdots \leq i_k\}$ is a basis of $S_k(V)$. Consequently, the symmetric subspace has dimension $\binom{\mathrm{v}+k-1}{k}$.

The situation is similar for the antisymmetric subspace: Here, the projection onto $\bigwedge^k(V)$ is given by

$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \wedge \cdots \wedge v_k := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Moreover, the set $\{\mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k} : i_1 < \cdots < i_k\}$ is a basis of $\bigwedge^k(V)$. Consequently, this space has dimension $\binom{\mathrm{v}}{k}$. In particular, this space is trivial for $k \geq \mathrm{v} + 1$.

For $k = 2$, the submodules defined in Definition 1.2.11 are in fact all submodules that one can find in $V^{\otimes 2}$, in other words, $V^{\otimes 2} = S^2(V) \oplus \bigwedge^2(V)$. While for $k \geq 3$ this is not true any longer, the following still holds. The following result is classical and well-known, for a proof see, for example, [Kra85, Appendix II.5, Theorem 4].

Theorem 1.2.12. *Every finite-dimensional representation of* $\mathrm{GL}(V)$ *decomposes as a direct sum of irreducible representations of* $\mathrm{GL}(V)$*. The same holds for* $\mathrm{SL}(V)$ *and products of the form* $\times_{i=1}^k \mathrm{GL}(V_i)$*.*

Remark 1.2.13. *In the language of linear algebraic groups, Theorem 1.2.12 says that any general linear group is "*reductive*".*

Remark 1.2.14. *Let, for example,* $G = \mathrm{GL}(V)$ *act on a space* W *and let* $W = \bigoplus_i W_i$ *be a decomposition of* W *into irreducible submodules. Then, using Theorem 1.2.10, we observe that any irreducible* G*-module* U *appears exactly* $\dim(\mathrm{Hom}_G(U, W))$*-many times in the decomposition.*

1.3 Lower bounding rank and border rank

We have seen in Example 1.1.3 that upper bounds on the minimal length of tensor (border) rank decompositions correspond to more efficient (approximate) algorithms to compute bilinear maps. More generally, they promise more succinct descriptions of tensors. Lower bounds, on the other hand, are important as they provide barriers to efficiency. A large amount of research has been done to find methods to lower bound the minimal size of tensor rank and tensor border rank decompositions. In this section, we explain examples of such methods that will be important throughout. The most basic technique of lower bounding rank and border rank is simple and comes from the *flattening maps* that we have already seen in Definition 1.1.12.

Lemma 1.3.1. *For $T \in U_1 \otimes \cdots \otimes U_k$ and any $\mathcal{S} \subset [k]$, the rank of the associated flattening map lower bounds $\underline{R}(T)$.*

It is clear that for three-party tensors $T \in U_1 \otimes U_2 \otimes U_3$, the best lower bound on the border rank of T coming from Lemma 1.3.1 is $\max\{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\}$. Comparing this with Remark 1.1.6, where we saw that there are tensors with border rank which grows quadratically in the dimensions of the tensor factors, this is not very satisfying. A more involved method to lower bound tensor rank is the so-called *substitution method.* The technique of substituting variables to find lower bounds on computational complexity was originally introduced in [Pan66] and further developed in [Win70]. This method has been reformulated and applied in various places. We will use a formulation similar to the one presented in [AFT11].

Theorem 1.3.2. *Let $T \in U_1 \otimes U_2 \otimes U_3$. Fixing a basis $\mathrm{e}_1 \dots \mathrm{e}_{\mathrm{u}_1}$ of U_1 we can write*

$$T = \sum_{i=1}^{\mathrm{u}_1} \mathrm{e}_i \otimes M_i$$

for matrices $M_i \in U_2 \otimes U_3$. Assume that $M_1 \neq 0$. For any complex numbers $\lambda_2, \dots, \lambda_{\mathrm{u}_1}$, define

$$\hat{T}(\lambda_2, \dots, \lambda_{\mathrm{u}_1}) = \sum_{j=2}^{\mathrm{u}_1} \mathrm{e}_j \otimes (M_j - \lambda_j M_1).$$

Then, there exist $\lambda_2, \dots, \lambda_{\mathrm{u}_1} \in \mathbb{C}$ such that

$$R(\hat{T}(\lambda_2, \dots, \lambda_{\mathrm{u}_1})) \leq R(T) - 1.$$

If the matrix M_1 has rank 1, then, for all $\lambda_2, \dots, \lambda_{\mathrm{u}_1}$ it holds that

$$R(\hat{T}(\lambda_2, \dots, \lambda_{\mathrm{u}_1})) \geq R(T) - 1.$$

Consequently, if M_1 has rank 1, we always find $\lambda_2, \dots, \lambda_{\mathrm{u}_1}$ such that

$$R(\hat{T}(\lambda_2, \dots, \lambda_{\mathrm{u}_1})) = R(T) - 1.$$

Proof. The proof is identical to the proof of Theorem 4.3.8 ([CGLS22, Theorem 4.11]), which we will present later. □

Again, it is clear that the substitution method can only provide lower bounds that are linear in the dimensions of the tensor factors.

Example 1.3.3. *An example for which the substitution method works well is the W-tensor from Example 1.1.7. Recall that there, the space $W(U_1^*)$ is spanned by*

$$M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since M_2 has rank 1, we can find λ such that

$$R(\mathrm{e}_1 \otimes \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}) = R(W) - 1. \tag{1.3}$$

But clearly, for any choice of λ, the quantity in Equation (1.3) is greater or equal to 2. Consequently, $R(W) \geq 3$, and since W is given as a sum of three rank-1 tensors, equality holds.

To lower bound border rank, one often uses generalizations of Lemma 1.3.1. This has been done, for example, in [CJZ18, Section 4].

Proposition 1.3.4. *Let $F : U_1 \otimes \cdots \otimes U_k \to X \otimes Y$ be a linear map and let*

$$r_0 = \max\{\operatorname{rank}(F(u_1 \otimes \cdots \otimes u_k)) : u_1 \in U_1, \ldots, u_k \in U_k\}$$

be the maximal rank of the image of a rank-1 tensor. Then,

$$\underline{R}(T) \geq \frac{\operatorname{rank}(F(T))}{r_0}.$$

Such a map F is called a generalized flattening.

It has been shown in [EGOW17, Section 4] that lower bounds on tensor border rank from generalized flattenings can be at most linear in the local dimensions. We will now review the so-called *Koszul flattenings* introduced in [LO11] based on ideas from [Str83] which are special cases of generalized flattenings. Roughly speaking, they are modified versions of the standard flattenings from Lemma 1.3.1 with a little twist involving the antisymmetric space we have seen in Definition 1.2.11. We refer to [Lan12, Chapters 3 & 7] for a more in-depth discussion of these concepts in relation to algebraic geometry and representation theory and only discuss how to obtain specific lower bounds on border rank using Koszul flattenings.

Definition 1.3.5. *Let $T \in U_1 \otimes U_2 \otimes U_3$ and let k be a natural number. The map*

$$T_{U_1}^{\wedge k} : \left(\bigwedge^k U_1\right) \otimes U_2^* \to \left(\bigwedge^{k+1} U_1\right) \otimes U_3$$

which is the composition of the induced linear map $T : U_2^* \to U_1 \otimes U_3$ *with the projection* $\bigwedge^k U_1 \otimes U_1 \to \bigwedge^{k+1} U_1$*, is called a (k-th) Koszul flattening of* T.

For example, let $\dim(U_1) = 3$ and $k = 2$. Here, we interpret $T(U_1^*)$ as a space of $\mathrm{u}_2 \times \mathrm{u}_3$ matrices spanned by

$$M_1 = T(\mathrm{e}_1^*),\ M_2 = T(\mathrm{e}_2^*),\ M_3 = T(\mathrm{e}_3^*).$$

Then, choosing bases appropriately, the linear map $T_{U_1}^{\wedge 2}$ is represented by the matrix

$$T_{U_1}^{\wedge 2} = \begin{pmatrix} M_2 & -M_1 & 0 \\ M_1 & 0 & M_3 \\ 0 & -M_3 & -M_2 \end{pmatrix}. \tag{1.4}$$

In general, if $T = \mathrm{e}_1 \otimes M_1 + \cdots + \mathrm{e}_{\mathrm{u}_1} \otimes M_{\mathrm{u}_1}$, we always can think of $T_{U_1}^{\wedge k}$ as a $\binom{\mathrm{u}_1}{k} \times \binom{\mathrm{u}_1}{k+1}$ block matrix where the blocks are of the form $\pm M_i$ for some $i = 1, \dots, \mathrm{u}_1$.

The following proposition, which is the specialization of Proposition 1.3.4 to Koszul flattenings, is the key to using Koszul flattenings for lower bounding border rank.

Proposition 1.3.6. *For a simple tensor* $T = u_1 \otimes u_2 \otimes u_3$*, the map* $T_{U_1}^{\wedge k}$ *has rank* $\binom{2k}{k}$*. Consequently,* $\underline{R}(T) \geq r$ *implies that the rank of* $T_{U_1}^{\wedge k}$ *is at most* $r\binom{2k}{k}$*, in other words,*

$$\underline{R}(T) \geq \left\lceil \frac{\operatorname{rank}(T_{U_1}^{\wedge 2})}{\binom{2k}{k}} \right\rceil.$$

For a proof of this result, we refer to [Lan17, Section 2.4.2]. Here, the reader can also find an in-depth discussion on the best choice of k and why a similar construction using the symmetric instead of the antisymmetric subspace does not yield better lower bounds on tensor rank.

Let us see an example of how to show lower bounds on border rank using Koszul flattenings. Consider the Bini tensor

$$T_{Bini} = \mathrm{e}_{11} \otimes (\mathrm{e}_{11} \otimes \mathrm{e}_{11} + \mathrm{e}_{12} \otimes \mathrm{e}_{21}) + \mathrm{e}_{12} \otimes (\mathrm{e}_{11} \otimes \mathrm{e}_{12} + \mathrm{e}_{12} \otimes \mathrm{e}_{22}) + \mathrm{e}_{21} \otimes (\mathrm{e}_{21} \otimes \mathrm{e}_{11} + \mathrm{e}_{22} \otimes \mathrm{e}_{21})$$

The Bini tensor was introduced in [BCLR79] and corresponds to the bilinear map calculating three entries of the product of two 2×2 matrices. Note that the rank of each of the three flattening maps of the Bini tensor is at most 4. Using Koszul flattenings, one can show the border rank of the Bini tensor is at least 5. Indeed, plugging the three matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

spanning $T_{Bini}(U_1^*)$ into the block matrix in Equation (1.4) yields a matrix of rank 9. Consequently, this implies

$$\underline{R}(T_{Bini}) \geq \left\lceil \frac{9}{2} \right\rceil = 5$$

by Proposition 1.3.6.

Remark 1.3.7. *All methods we have seen can only provide lower bounds linear in the dimensions of the tensor factors for three-party tensors. In fact, there are no methods known which go beyond linear lower bounds. The largest known border rank tensor can be found in [LM19].*

1.4 Tensor network representations

We have seen that in the context of algebraic complexity theory, tensor rank decompositions are a natural way to break up a tensor into smaller building blocks. Another way of constructing tensors in a systematic way is the *tensor network ansatz.* Loosely speaking, we "build" a tensor $T \in U_1 \otimes \cdots \otimes U_k$ from k "smaller" tensors according to the geometry of a graph.

Definition 1.4.1. *Let $\Gamma = (V, E)$ be a directed graph and consider weight functions*

$$\mathbf{m} : E \to \mathbb{N},\ \mathbf{n} : V \to \mathbb{N}.$$

A tensor network state *associated with the graph Γ and the weight functions $\mathbf{m}$ and $\mathbf{n}$ is a tensor*

$$T \in \bigotimes_{v \in V} \mathbb{C}^{\mathbf{n}(v)}$$

that arises by choosing for each $v \in V$ a tensor

$$T_v \in \left(\bigotimes_{e=(v,x)} \mathbb{C}^{\mathbf{m}(e)} \right) \otimes \left(\bigotimes_{e=(x,v)} (\mathbb{C}^{\mathbf{m}(e)})^* \right) \otimes \mathbb{C}^{\mathbf{n}(v)}.$$

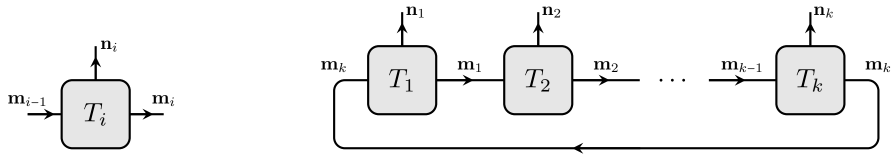
and contracting the tensors T_v over the edges of Γ.

Here, *contracting* tensors $T \in U \otimes V$ and $S \in V^* \otimes W$ is essentially the operation $\lrcorner$ resulting from linearly extending

$$\lrcorner : (U \otimes V) \times (V^* \otimes W) \to U \otimes W, (\mathrm{e}_i \otimes \mathrm{e}_j) \lrcorner (\mathrm{e}_k^* \otimes \mathrm{e}_l) = \delta_{jk} \mathrm{e}_i \otimes \mathrm{e}_l.$$

We will sometimes write $\mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})$ for the set of tensor network states associated with Γ, $\mathbf{m}$ and $\mathbf{n}$.

Remark 1.4.2. *The edge weights $\mathbf{m}(e)$ for $e \in E$ are often called the* virtual dimensions *or* bond dimensions. *In the context of many-body physics, the bond dimension $\mathbf{m}(e)$ is a parameter with which we can control the entanglement between the particles at the two*



(a) A tensor T_i.

(b) A matrix product state with periodic boundary conditions.

Figure 1.4: A building block for a tensor network state and a matrix product state with periodic boundary conditions: In Figure 1.4(a), we show how one visually depicts a tensor $T_i \in (\mathbb{C}^{\mathbf{m}_{i-1}})^* \otimes \mathbb{C}^{\mathbf{n}_i} \otimes \mathbb{C}^{\mathbf{m}_i}$. Contracting k of these fundamental building blocks along the edges of a cyclic graph yields a matrix product state on k parties as depicted in Figure 1.4(b).

vertices that the edge connects. The weights $\mathbf{n}(v)$ are frequently called physical dimensions *and are the number of degrees of freedom that the particle at vertex v has. Also, note that in physics literature, the contraction is often done by defining an inner product. In this way, one can avoid dealing with directed graphs and dual spaces at the incoming edges.*

Prominent examples of tensor network states are the so-called *matrix product states with periodic boundary conditions.* For that, consider a graph Γ_k with vertex set $\{1, \dots, k\}$ and edges (i, j) whenever $j = i \oplus 1$ where $\oplus$ denotes addition modulo k. Fix some physical dimensions $\mathbf{n}_i = \mathbf{n}(i)$ and bond dimensions $\mathbf{m}_i = \mathbf{m}(i, i \oplus 1)$. Then, a matrix product state arises by picking tensors

$$T_i \in (\mathbb{C}^{\mathbf{m}_{i-1}})^* \otimes \mathbb{C}^{\mathbf{m}_i} \otimes \mathbb{C}^{\mathbf{n}_i} \text{ for } i = 1, \dots, k$$

and building a k-party tensor from it according to the recipe in Definition 1.4.1. For a visualization, see Figure 1.4. The name *matrix product state* comes from the following well-known and simple technical result.

Lemma 1.4.3. *A tensor $T \in V_1 \otimes \dots \otimes V_k$ is a matrix product state for the graph Γ_k if and only if for each $j = 1, \dots, k$ there are matrices*

$$M_{j,i} \in \mathbb{C}^{\mathbf{m}_{j-1} \times \mathbf{m}_j} \textit{ for } i = 1, \dots, \mathrm{v}_j$$

such that

$$T = \sum_{i_1 \dots i_k = 1}^{\mathrm{v}_1 \dots \mathrm{v}_k} \mathrm{tr}(M_{1,i_1} \dots M_{k,i_k}) \mathrm{e}_{i_1} \otimes \dots \otimes \mathrm{e}_{i_k}.$$

In Section 1.1, we have seen that tensor rank decompositions are just special cases of the restriction preorder on tensors. In fact, tensor network states also fit into the framework of restriction of tensors. While we could characterize tensor rank-r tensors as restrictions of the unit tensor $\langle r \rangle$, we need to construct an alternative for tensor network states. We can construct this tensor from the respective graph.

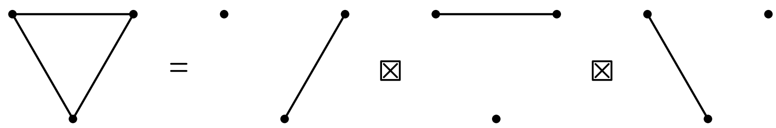


Figure 1.5: Construction of the graph tensor for the graph Γ_3: For each edge, we take a unit tensor between the two vertices that the edge connects. Grouping together the spaces at the same vertex yields a three-party tensor. A simple calculation shows that the graph tensor $\mathcal{T}_{\Gamma,\mathbf{m}} = \langle \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \rangle = \langle 1,1,\mathbf{m}_3 \rangle \boxtimes \langle 1,\mathbf{m}_2,1 \rangle \boxtimes \langle \mathbf{m}_1,1,1 \rangle$ is the matrix multiplication tensor.

Definition 1.4.4. *Let $\Gamma = (V, E)$ be a graph together with a weight function $\mathbf{m} : E \to \mathbb{N}$ and assume $|V| = k$. We construct the* graph tensor *associated with Γ and $\mathbf{m}$ as follows. For each edge $e = \{v_1, v_2\} \in E$, define the k-party tensor*

$$T_e = \langle \mathbf{m}(e) \rangle_2 \otimes \bigotimes_{v \in V \setminus \{v_1, v_2\}} \mathrm{e}_0 \ \in \ \mathbb{C}^{\mathbf{m}(e)} \otimes \mathbb{C}^{\mathbf{m}(e)} \otimes \bigotimes_{v \in V \setminus \{v_1, v_2\}} \mathbb{C}$$

where each party of the tensor is associated with a vertex of Γ. Grouping together the parties at the same vertex for all $v \in V$, we can interpret

$$\mathcal{T}_{\Gamma,\mathbf{m}} = \boxtimes_{e \in E} T_e \in H_1 \otimes \cdots \otimes H_k$$

as a k-party tensor where for each $i = 1, \ldots, k$, we let $H_i = \bigotimes_{e : v \in e} \mathbb{C}^{\mathbf{m}(e)}$.

For the graph Γ_3, this construction yields a familiar tensor. In fact, it is not hard to see that the resulting graph tensor is just the matrix multiplication tensor $\langle \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 \rangle$. We visualize the construction of the graph tensor for this particular example in Figure 1.5.

Using the graph tensor, we can now give an equivalent characterization of tensor network states in terms of restriction. We omit the proof which is straightforward.

Proposition 1.4.5. *Let $\Gamma = (V, E)$ be a directed graph together with associated weight functions $\mathbf{m} : E \to \mathbb{N}$ and $\mathbf{n} : V \to \mathbb{N}$. Assume, $V = \{1, \ldots, k\}$ and fix spaces V_i of dimension* $\dim(V_i) = \mathbf{n}(i)$. *A tensor $T \in V_1 \otimes \cdots \otimes V_k$ is a tensor network state for Γ, $\mathbf{m}$ and $\mathbf{n}$ if and only if the associated graph tensor restricts to T, that is, $\mathcal{T}_{\Gamma,\mathbf{m}} \geq T$.*

Remark 1.4.6. *It has been first observed in [LQY11] that the set $\mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})$ is in general not closed if the graph contains cycles. Taking the Zariski or, equivalently, the Euclidean closure, one obtains*

$$\overline{\mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})} = \{T \in U_1 \otimes \cdots \otimes U_k : \mathcal{T}_{\Gamma,\mathbf{m}} \trianglerighteq T\}.$$

It turns out that the study of $\overline{\mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})}$ can lead to more efficient algorithms by extending the ansatz class one optimizes over from the set of tensor network states to its closure [CGFW21].

1.5 Stabilizer states and the simulation of quantum circuits

In this section (as well as in Chapter 5), we will switch notation and use Greek letters like ψ and ϕ instead of T and S for tensors. In that way, we match the standard notation in this area. We also mention that in the context of this section, it will be important to consider tensors that are normalized with respect to the Euclidean inner product induced by a choice of bases. Hence, Greek letters may be used as an indicator that we are dealing with a normalized tensor. Moreover, we will refer in this section (as well as in Chapter 5) to normalized tensors as *quantum states*. We saw that such normalized tensors $\psi \in U_1 \otimes \cdots \otimes U_k$ can describe the joint state of a quantum system made up of k particles. In that way, tensor decompositions can give insight into the physical processes going on in *quantum circuits*. One important way of decomposing tensors that is especially relevant for the simulation of quantum circuits is via so-called *stabilizer rank decompositions*. In this section, all physical systems will be built up from a number of *qubits*, that is, we will consider only quantum states $\psi \in (\mathbb{C}^2)^{\otimes k}$ where all of the tensor factors are two-dimensional. For each of the copies of $\mathbb{C}^2$, we fix a basis $\mathrm{e}_0, \mathrm{e}_1$. We refer to this basis as the *computational basis*. The following way of expressing computational basis vectors in $(\mathbb{C}^2)^{\otimes k}$ will be handy.

Definition 1.5.1. *For a vector $x \in \mathbb{F}_2^k$, write $\mathrm{e}_x := \mathrm{e}_{x_1} \otimes \cdots \otimes \mathrm{e}_{x_k} \in (\mathbb{C}^2)^{\otimes k}$.*

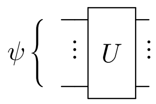
Recall the *Pauli matrices*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For any k, we define the k-qubit Pauli group $\mathcal{P}_k$ as the group generated by all unitaries of the form $A_1 \otimes \cdots \otimes A_k$ for $A_i \in \{X, Y, Z\}$. Denoting by $\mathcal{U}_k$ the set of unitary operators on $(\mathbb{C}^2)^{\otimes k}$, we define $\mathcal{C}_k$ to be the *normalizer* of $\mathcal{P}_k$ in $\mathcal{U}_k$, that is,

$$\mathcal{C}_k = \{U \in \mathcal{U}_k : UPU^\dagger \in \mathcal{P}_k \text{ for all } P \in \mathcal{P}_k\}.$$

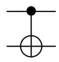
Recall that a *quantum circuit* is a diagram specifying a unitary by applying unitaries on subsystems (so-called *gates*) in a fixed order to an input state $\psi \in (\mathbb{C}^2)^{\otimes k}$. A circuit applying a unitary U to a quantum state ψ is depicted in the following way:



Frequently used gates are the one-qubit gates

$$-\boxed{H}- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } -\boxed{S}- = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

as well as the two-qubit $CNOT$-gate



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It turns out that these gates are in the Clifford group and that all Clifford unitaries – up to a global phase – can be constructed from quantum circuits built from H, S, and $CNOT$ gates. Hence, we will call these three gates the *Clifford gates*. States that can be produced by applying these gates to a computational basis state are called *stabilizer states*.

Definition 1.5.2. *A quantum state $\psi \in (\mathbb{C}^2)^{\otimes k}$ is a* stabilizer state *if it is of the form $C(\mathrm{e}_0 \otimes \cdots \otimes \mathrm{e}_0)$ for some $C \in \mathcal{C}_k$.*

We have the following characterization of stabilizer states [DDM03, VdN10].

Proposition 1.5.3. *A quantum state ψ is a* stabilizer state *if and only there is an affine subspace $A \subset \mathbb{F}_2^k$, a linear form $l : \mathbb{F}_2^k \to \mathbb{F}_2$, a quadratic form $q : \mathbb{F}_2^k \to \mathbb{F}_2$ and some (normalizing) constant $c \in \mathbb{C}$ such that*

$$\psi = c \cdot \sum_{x \in A} i^{l(x)} \cdot (-1)^{q(x)} \cdot \mathrm{e}_x.$$

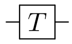
We observe that – up to a normalizing prefactor – all coefficients of a stabilizer state in the computational basis are in $\{0, \pm 1, \pm i\}$. Note also that all computational basis vectors e_x for $x \in \mathbb{F}_2^k$ are stabilizer states, in other words, the set of stabilizer states spans $(\mathbb{C}^2)^{\otimes k}$. In particular, every quantum state ψ can be written as a linear combination of stabilizer states. This motivates the following definition.

Definition 1.5.4. *For $\psi \in (\mathbb{C}^2)^{\otimes k}$, we define the stabilizer rank of ψ as the minimal r such that*

$$\psi = \sum_{i=1}^{r} c_i \sigma_i$$

for some $c_i \in \mathbb{C}$ and stabilizer states σ_i. We denote the stabilizer rank of ψ by $\chi(\psi)$.

It turns out that adding one particular additional gate to the set $\{H, S, CNOT\}$ makes this gate set *universal*, that is, any unitary can be approximated arbitrarily well with a circuit built from this extended gate set. The gate is the so-called T-gate



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}.$$

We only mention briefly that using a trick called *magic state injection*, one can relate the simulation cost of a circuit built from Clifford gates and k T-gates to the stabilizer rank of

$T^{\otimes k}$ where $T = \frac{1}{\sqrt{2}}(\mathrm{e}_0 + e^{i\frac{\pi}{4}}\mathrm{e}_1)$, and refer for details to Appendix 5.A. We mention briefly that for the simulation of quantum circuits, it is often enough to consider an approximate version of stabilizer rank.

Definition 1.5.5. *Fix $\delta > 0$ and a quantum state $\psi \in (\mathbb{C}^2)^{\otimes k}$. The* δ-approximate stabilizer rank *of ψ is given by*

$$\chi_\delta(\psi) = \min\{\chi(\phi) : \phi \in (\mathbb{C}^2)^{\otimes k} \textit{ quantum state with } \|\phi - \psi\| \leq \delta\}.$$

Chapter 2

Transforming entanglement structures

In this chapter, we will study the conversion between entanglement structures. Intuitively, an entanglement structure results from "putting" tensors on a hypergraph and grouping together parties at the same vertex. Entanglement structures have been investigated before: In [CLVW20], the authors find more efficient tensor network representations from degenerations of the single tensors from which the entanglement structures are built. In [MGSC18], the authors investigate under which conditions acting in a translationally invariant manner on entanglement structures yields an equivalent entanglement structure. In this chapter, we investigate entanglement structures from a slightly different angle. We are particularly interested in the extent to which one entanglement structure can be transformed into another via restriction or degeneration, even if the single plaquette tensors from which the entanglement structures are built cannot be transformed in this way.

We will develop a formal language enabling us to discuss and highlight the subtleties of this question. On the one hand, we will do so by constructing specific, non-trivial examples. On the other hand, we will develop powerful tools to find obstructions to the conversion between entanglement structures yielding a deeper understanding of the multipartite entanglement properties of entanglement structures.

Closely related to this setup is the question of how to calculate the tensor rank of an entanglement structure. We make a first step in this direction and answer an open question in [CF18] by calculating the rank of two copies of the W-tensor put on every possible two-plaquette hypergraph. We also show a more general result about the stabilizers of so-called *tree hypergraphs*, leading us to the conjecture that the rank of entanglement structures on such graphs does not differ from the rank of disconnected entanglement structures.

2.1 Overview

The starting point of this chapter is the characterization of tensor network states in Proposition 1.4.5. There, we have seen that tensor network states associated with a graph Γ and a weight function $\mathbf{m}$ can be seen as restrictions of the graph tensor $\mathcal{T}_{\Gamma,\mathbf{m}}$. The graph tensor was essentially constructed by putting two-party tensors on the edges of the graph. In this chapter, we study a natural generalization of this ansatz where – instead of a graph Γ – we pick a *hypergraph* H. Now, instead of putting two-party tensors on the edges, we can associate with every k-vertex hyperedge a k-party tensor. We will call a tensor constructed from a hypergraph H by "putting" tensors from a suitable family of tensors $T = (t_1, \dots, t_n)$ on the hyperedges an *entanglement structure* which we will denote by T_H. For this overview section, we will stick to this rough intuition and refer for a more precise mathematical definition to Section 2.2.

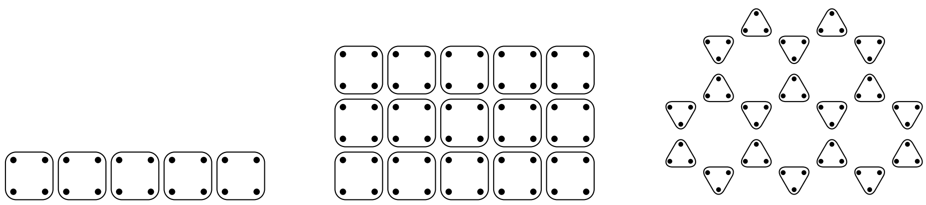
In this chapter, we will study three questions about entanglement structures.

(1) Do there exist a hypergraph H and suitable families of tensors $T = (t_1, \dots, t_n)$ and $S = (s_1, \dots, s_n)$ such that for all $i = 1, \dots, n$, it holds that $t_i \not\geq s_i$, but for the resulting entanglement structures, it holds that $T_H \geq S_H$?

(2) Under which conditions on the hypergraph resp. the families of tensors are such examples impossible?

(3) What is the tensor rank of an entanglement structure $R(T_H)$ for a hypergraph H and a suitable family of tensors T?

In points (1) and (2), we ask under which conditions $T_H \geq S_H$ resp. $T_H \trianglerighteq S_H$ can hold. Examples for similar phenomena have been constructed before: Already since [Str69] we know examples where both $t \not\geq s$ and $t^{\boxtimes 2} \geq s^{\boxtimes 2}$ hold, see also [CJZ18, Example 3] for a discussion. Motivated by the most frequently used tensor network approaches like matrix product states that we saw in Section 1.4 and generalizations like *projected entangled pair states* (PEPS), we are more interested in the setup where the hypergraph describes a "1D-" or "2D-structure", see Figure 2.1 for an illustration.



(a) A strip of plaquettes. (b) A regular lattice. (c) A kagome lattice with triangle plaquettes.

Figure 2.1: Three different types of hypergraphs. The strip graph (Figure 2.1(a)) reflects a 1D-structure, whereas the regular rectangular lattice (Figure 2.1(b)) and the kagome lattice (Figure 2.1(c)) reflect 2D-structures.

2.1.1 Constructions and Obstructions

In Section 2.3, we will demonstrate several examples of the conversion between entanglement structures. More precisely, we will find examples for a hypergraph H and tensors t and s with $t \ngeq s$ such that the entanglement structure T_H obtained from putting a copy of t on each of the hyperedges of H restricts to the entanglement structure S_H constructed in the same way from copies of s. To do so, we will use the teleportation technique introduced in Example 1.1.10 and results about the border subrank of matrix multiplication tensors from [CLVW20]. The hypergraphs in these examples will be the so-called *ring graph* and the so-called *strip graph*, respectively. Visualizations of these graphs can be found in Figure 2.3 and Figure 2.4.

After we have found concrete examples, we will derive obstructions to the conversion between entanglement structures in Section 2.4. The first kind of obstruction comes from the geometry of the hypergraph: We will see that certain modifications of the graph preserve restriction. Using this, we will, for a wide range of hypergraphs H, find that a restriction $T_H \geq S_H$ for any families of tensors $T = (t_1, \dots, t_n)$ and $S = (s_1, \dots, s_n)$ already implies $t_i \geq s_i$ for all $i = 1, \dots, n$.

In Section 2.4.2, we then go on and relate the conversion between entanglement structures to the notion of *asymptotic restriction*. Recall, that for tensors t and s, we say that t restricts asymptotically to s and write $t \gtrsim s$ if $t^{\boxtimes n + o(n)} \geq s^{\boxtimes n}$ for all $n \in \mathbb{N}$. When all hyperedges of the hypergraph H have the same number of vertices and the geometry of H allows us to "fold" the hyperedges on top of each other, we will see that for families of tensors $T = (t, \dots, t)$ and $S = (s, \dots, s)$ which just contain copies of symmetric tensors t and s, respectively, $T_H \geq S_H$ implies $t \gtrsim s$.

We generalize this idea in Section 2.4.3 and demonstrate how one can use the so-called *asymptotic spectral points* which were introduced in [Str86, Str88], and the so-called *quantum functionals* introduced in [CVZ18] to find further obstructions for the conversion between entanglement structures.

2.1.2 Tensor rank of entanglement structures

In Section 2.5, we study tensor ranks of entanglement structures. Note that it is easy to see that if an entanglement structure T_H is built from a family of tensors $T = (t_1, \dots, t_n)$, then the rank of T_H is at most $R(t_1 \otimes \cdots \otimes t_n)$. It has been noticed that for the Kronecker product, which, in principle, can be considered an entanglement structure, there are examples where the tensor rank of T_H is strictly less than $R(t_1 \otimes \cdots \otimes t_n)$. In particular, it has been shown in [YCGD10] that for the W-tensor from Example 1.1.7, the tensor rank of $W \boxtimes W$ equals 7, whereas in [CF18], it has been shown that $R(W \otimes W) = 8$. In [CF18], it has been posed as an open question what the tensor rank of other entanglement structures built from two copies of the W-tensor is. We will refine the method from [CF18] to calculate these tensor ranks for all possible entanglement structures.

2.2 Structured hypergraphs

In order to formally argue about entanglement structures, we will develop a precise, mathematical terminology tailored to describe entanglement structures in this section. The central notion is that of a structured hypergraph.

Definition 2.2.1. *A structured hypergraph $H = (W, E, V, m)$ consists of the following data:*

(♣) *A set of vertices W.*

(♡) *A set of (hyper)edges E. Each hyperedge $e \in E$ is an ordered tuple of elements in W. We require that each $w \in W$ can be a coordinate of at most one edge $e \in E$.*

(♠) *A set of structure vertices V.*

(♢) *A structure map $m\colon W \to V$.*

We will frequently abuse notation and write $w \in e$ for some $w \in W$ and $e \in E$ such that w is a coordinate of e. More generally, we will often write e for the set of coordinates of e. If not otherwise specified, we will assume that the structure map cannot map two vertices that lie on the same edge to the same structure vertex to avoid pathological examples.

Example 2.2.2. *To see an example of a structured hypergraph, consider the diamond graph $H = (W, E, V, m)$ given by the following data:*

(♣) $W = \{w_1, w_2, w_3, w_2', w_3', w_4\}$

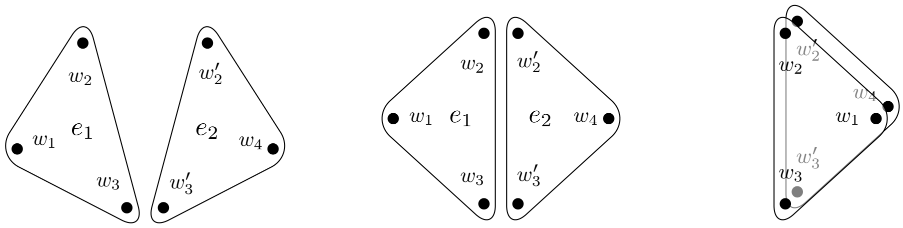
(♡) $E = \{e_1 = (w_1, w_2, w_3), e_2 = (w_2', w_3', w_4)\}$

(♠) $V = \{v_1, v_2, v_3, v_4\}$

(♢) $m\colon w_i^{(\prime)} \mapsto v_i$ *for* $i = 1, \ldots, 4$

We visualize the diamond graph in Figure 2.2(b). A similarly structured hypergraph is the butterfly graph. It only differs from the diamond graph in the way that w_2 and w_2' correspond to different structure vertices. A visualization of the butterfly graph can be found in Figure 2.2(a).

Let H be a structured hypergraph, and for $e \in E$ define $l(e)$ to be the length of the tuple e. Let $T = (t(e) : e \in E)$ be a tuple of tensors where $t(e) \in \bigotimes_{w \in e} U_w$ is a $l(e)$-party tensor for each edge $e \in E$. In this case, we say the family of tensors T *fits the shape of* H. Define the tensor $T_H = \bigotimes_{e \in E} t(e)$ associated to H and T. We say that the i'th party *lies at* vertex w_j where w_j is the i'th coordinate of e. We define the associated *entanglement structure* T_H by grouping together all parties lying at vertices that get mapped to the same structure vertex under m. Let us illustrate this for the diamond graph.



(a) The butterfly graph (b) The diamond graph. (c) The folded diamond graph.

Figure 2.2: Some examples of structured hypergraphs. In Figure 2.2(a), we illustrate the butterfly graph. In Figure 2.2(b), we show the diamond graph. In Figure 2.2(c), we illustrate the folded diamond graph resulting from the structure-modifying map in Example 2.2.4.

Example 2.2.3. *(Continuing Example 2.2.2) Every tuple* $T = (t_1, t_2)$ *of three-party tensors* $t_1 \in U_{w_1} \otimes U_{w_2} \otimes U_{w_3}$ *and* $t_2 \in U_{w_2'} \otimes U_{w_3'} \otimes U_{w_4}$ *fits the shape of the diamond graph* H*. The corresponding graph tensor is then*

$$t_1 \otimes t_2 \in U_{w_1} \otimes U_{w_2} \otimes U_{w_3} \otimes U_{w_2'} \otimes U_{w_3'} \otimes U_{w_4}.$$

The associated entanglement structure will be a four-party tensor

$$T_H \in U_{w_1} \otimes (U_{w_2} \otimes U_{w_2'}) \otimes (U_{w_3} \otimes U_{w_3'}) \otimes U_{w_4}$$

because w_2 *and* w_2' *resp.* w_3 *and* w_3' *get mapped to the same structure vertex* v_2 *resp.* v_3 *under the structure map* m*.*

Take a structured hypergraph $H = (W, E, V, m)$. We call a surjective map $\mu: V \to U$ a *structure-modifying map for* H. The map μ induces a new structured hypergraph which we will denote by $\mu.H = (W, E, U, \mu \circ m)$ whose set of structure vertices is U and whose structure map is $\mu \circ m$.

Example 2.2.4. *(Continuing Example 2.2.3) We can fold the two hyperedges of the diamond graph on top of each other. This can be done via*

$$\mu: V \to \{u_1, u_2, u_3\},$$
$$v_1 \mapsto u_1,\ v_2 \mapsto u_2,\ v_3 \mapsto u_3,\ v_4 \mapsto u_1.$$

This does precisely what visually corresponds to folding the two hyperedges on top of each other. A visualization can be found in Figure 2.2(c).

Motivated by Example 2.2.4, we say that a structured hypergraph $H = (W, E, V, m)$ is *foldable* if $l(e) = k$ for all $e \in E$ and if there is a structure-modifying map $\mu : V \to \{1, \dots, k\}$ such that $(\mu \circ m)|_e$ is bijective for all $e \in E$. It is clear that such a map folds the edges of H on top of each other, and we will refer to such a structure-modifying map as a *folding*. We emphasize that for a structured hypergraph, there can be inequivalent structure-modifying maps that are foldings.

For a structured hypergraph $H = (W, E, V, m)$, we call a sequence

$$(w_{1,1}, w_{1,2}, e_1, \dots, e_{n-1}, w_{n,1}, w_{n,2})$$

a *path* if for all $i = 1, \dots, n$, the vertices $w_{i,1}$ and $w_{i,2}$ have the same structure vertex, that is, $m(w_{i,1}) = m(w_{i,2})$, and for all $i = 1, \dots, n-1$ we have $w_{i,2}, w_{i+1,1} \in e_i$.

For an edge $e \in E$, we say that e *is contained in a cycle* if there are $w, \tilde{w} \in e$ such that there is a path from w to $\tilde{w}$ in $H \setminus \{e\} = (W, E \setminus \{e\}, V, m)$. We call a hypergraph connected if for any pair of vertices, there is a path containing both.

For $w \in e$ for some edge e, we define the *affix* $a(w)$ of w as the set of vertices that can be contained in a path containing w but not containing e. It is clear from the definition that e is contained in a cycle in H if and only if there are w and $\tilde{w}$ both in e such that $\tilde{w}$ is contained in the affix $a(w)$ of w. That is, if e is not contained in a cycle and H is connected, W is a disjoint union of affixes $W = \bigsqcup_{w \in e} a(w)$ of the vertices in e.

We finish this section by mentioning that the Kronecker product and the tensor product we have seen in Section 1.1 are special cases of entanglement structures: The Kronecker product is the folded version of the diamond graph that we saw in Example 2.2.4. The tensor product of two tensors can be constructed as an entanglement structure by giving each vertex its own structure vertex.

2.3 Constructions

In this section, we will give several explicit constructions to highlight how subtle the question of conversion between entanglement structures is. Our examples are natural generalizations of matrix product states, namely, the ring graph (see Figure 2.3) and the strip graph (see Figure 2.4).

2.3.1 Entanglement structures on a ring

Our first examples are obtained by placing copies of the same tensor on the plaquettes of the *ring graph.*

Definition 2.3.1. *The ring graph on n plaquettes H_n is the structured hypergraph constructed from the following data.*

(♣) *The set of vertices is given by* $W_n = \{w_{i,j} : i = 1, \dots, n, j = 1 \dots, 4\}$.

(♡) *The set of hyperedges is* $E_n = \{e_1, \dots, e_n\}$ *where* $e_i = (w_{i,1}, \dots, w_{i,4})$.

(♠) *The set V_n of structure vertices is some set of cardinality $2n$.*

(♢) *The structure map $m_n : W_n \to V_n$ is a surjective map such that $m_n(i,1) = m_n(i \oplus 1, 4)$ and $m_n(i,2) = m_n(i \oplus 1, 3)$ for all $i = 1, \dots, n$ where $\oplus$ is addition modulo n. This defines m uniquely up to permutations of elements of V.*

We visualize the ring graph H_4 in Figure 2.3.

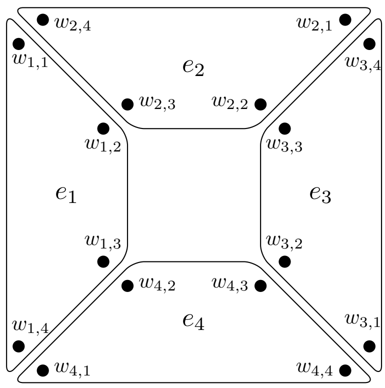
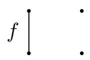
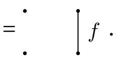


Figure 2.3: An example of a ring graph on 4 plaquettes.

We record a first and simple construction that has already been introduced in [MGSC18].

Example 2.3.2. *In this example, we construct the single tensors as graph tensors as in Definition 1.4.4. Let $\Gamma_1 = (\{1,2,3,4\}, \{(3,4)\})$ and $\Gamma_2 = (\{1,2,3,4\}, \{(1,2)\})$. Consider edge weight functions $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively, such that $\mathbf{m}_1(\{3,4\}) = \mathbf{m}_2(\{1,2\}) = f$ for some $f \in \mathbb{N}$. Define*

$t = \mathcal{T}_{\Gamma_1, \mathbf{m}_1} =$  , $s = \mathcal{T}_{\Gamma_2, \mathbf{m}_2} =$  .

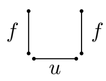
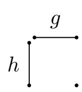
It is clear that $t \not\geq s$, but for the ring graph H_n, it is obvious that the entanglement structures T_{H_n} and S_{H_n} are equivalent under restriction where $T = (t, \dots, t)$ and $S = (s, \dots, s)$.

The following example is more interesting. In particular, the resulting entanglement structures will not be equivalent.

Example 2.3.3. *Again, we construct tensors as graph tensors. For that, define two graphs $\Gamma_1 = (\{1,2,3,4\}, \{(1,2),(2,3),(3,4)\})$ and $\Gamma_2 = (\{1,2,3,4\}, \{(3,4),(4,1)\})$. For the edge weights, let $g \geq 2$, $h \geq 2$, $u \geq g$ and $f = gh - 1$ be natural numbers and define*

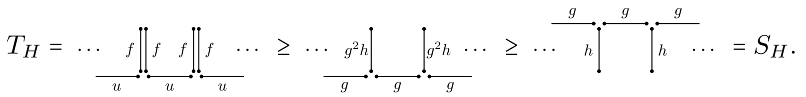
$$\mathbf{m}_1((1,2)) = \mathbf{m}_1((3,4)) = f, \;\; \mathbf{m}_1((2,3)) = u, \;\; \mathbf{m}_2((3,4)) = h, \;\; \mathbf{m}_2((4,1)) = g.$$

Define tensors t and s via

$t = \mathcal{T}_{\Gamma_1, \mathbf{m}_1} =$  , $s = \mathcal{T}_{\Gamma_2, \mathbf{m}_2} =$ 

and consider flattening maps of these tensors. Interpreting the tensors t and s as bipartite tensors by grouping the second, third, and fourth party together, we see that $t \not\geq s$ since $gh > f$.

For $T = (t : e \in E)$ and $S = (s : e \in E)$, it is clear that t and s fit the shape of H_n. We can use a teleportation argument as in Example 1.1.10 to see that, in fact, $T_H \geq S_H$: Noting that $f^2 \geq g^2 h$, we see that

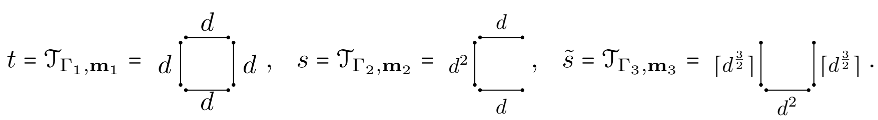


Using similar ideas, we can also construct examples for degeneration by using a fact about the *border subrank* of the matrix multiplication tensor from [CLVW20].

Example 2.3.4. *Let $d \in \mathbb{N}$ and define three graphs Γ_i with associated edge weight functions $\mathbf{m}_i$ for $i = 1,2,3$, all with vertex set $V = \{1,2,3,4\}$.*

- $\Gamma_1 = (V, \{(1,2),(2,3),(3,4),(4,1)\})$, $\mathbf{m}_1 \equiv d$.
- $\Gamma_2 = (V, \{(2,3),(3,4),(4,1)\})$, $\mathbf{m}_2((3,4)) = d^2$ *and* $\mathbf{m}_2((2,3)) = \mathbf{m}_2((4,1)) = d$.
- $\Gamma_3 = (V, \{(1,2),(2,3),(3,4)\})$, $\mathbf{m}_3((1,2)) = \mathbf{m}_3((3,4)) = \lceil d^{\frac{3}{2}} \rceil$ *and* $\mathbf{m}_3((2,3)) = d^2$.

With that, consider the three graph tensors



It has been shown that $t \trianglerighteq \langle \lceil \frac{d^2}{2} \rceil \rangle$ for all $d \in \mathbb{N}$ [CLVW20, Section 5.2]. Hence, we know that

$$(t, \dots, t)_{H_n} \trianglerighteq (\langle \lceil \frac{d^2}{2} \rceil \rangle, \dots, \langle \lceil \frac{d^2}{2} \rceil \rangle)_{H_n}$$

where H_n is again the ring graph on n plaquettes. On the other hand, a flattening argument shows that $s \ntrianglerighteq \langle \lceil \frac{d^2}{2} \rceil \rangle$ and for d sufficiently large, $\tilde{s} \ntrianglerighteq \langle \lceil \frac{d^2}{2} \rceil \rangle$. Similar to Example 2.3.2, we get

$$(s, \dots, s)_{H_n} \geq (t, \dots, t)_{H_n} \trianglerighteq (\langle \lceil \frac{d^2}{2} \rceil \rangle, \dots, \langle \lceil \frac{d^2}{2} \rceil \rangle)_{H_n}.$$

By applying the same teleportation trick as in Example 2.3.3, we also obtain

$$(\tilde{s}, \dots, \tilde{s})_{H_n} \geq (t, \dots, t)_{H_n} \trianglerighteq (\langle \lceil \frac{d^2}{2} \rceil \rangle, \dots, \langle \lceil \frac{d^2}{2} \rceil \rangle)_{H_n}.$$

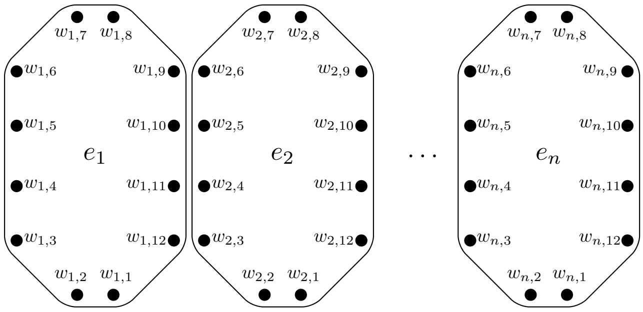


Figure 2.4: A strip graph. Each edge contains 12 vertices.

2.3.2 Entanglement structure on a strip

It is clear that all constructions in Section 2.3.1 crucially rely on the "periodic boundary conditions" of the graph. In this section, we will see an example of the conversion between entanglement structures on a *strip graph* which does not rely on periodic boundary conditions.

Definition 2.3.5. *The strip graph $H_n = (W_n, E_n, V_n, m_n)$ is defined by the following data.*

(♣) *The set of vertices is $W_n = \{w_{i,j} : i = 1, \ldots, n, j = 1, \ldots, 12\}$.*

(♡) *The set of hyperedges is $E_n = \{e_1, \ldots, e_n\}$, where $e_i = (w_{i,1}, \ldots, w_{i,12})$ for $i = 1, \ldots, n$.*

(♠) *The set of structure vertices is a set of cardinality $|V_n| = 12 + 8(n-1)$.*

(♢) *The structure map is a surjective map $m_n : W \to V$ such that for all $i = 1, \ldots, n-1$, it holds that*

$$m_n(w_{i,11}) = m_n(w_{i+1,2}), \ldots, m_n(w_{i,8}) = m(w_{i+1,5}).$$

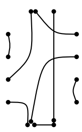
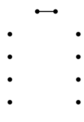
We visualize the strip graph in Figure 2.4.

As in Section 2.3.1, we will construct graph tensors to put on the single hyperedges of the graph.

Example 2.3.6. *Let p be a natural number and define $V = \{1, \ldots, 12\}$. Define a graph $\Gamma_1 = (V, E_1)$ where*

$$E_1 = \{(1,2), (1,8), (2,3), (2,10), (4,7), (5,6), (7,9), (11,12)\}.$$

Define a second graph $\Gamma_2 = (V, E_2)$ with $E_2 = \{(7,8)\}$. Define constant weight functions $\mathbf{m}_1 \equiv p$ and $\mathbf{m}_2 \equiv p$. With this, define the graph tensors

$t = \mathcal{T}_{\Gamma_1, \mathbf{m}_1} =$  $, s = \mathcal{T}_{\Gamma_2, \mathbf{m}_2} =$ 

and let $T = (t, \dots, t)$ and $S = (s, \dots, s)$. Since t is a simple tensor when considered as a bipartite tensor

$$t \in (H_1 \otimes H_2 \otimes H_3 \otimes H_8 \otimes H_{10}) \otimes (H_4 \otimes H_5 \otimes H_6 \otimes H_7 \otimes H_9 \otimes H_{11} \otimes H_{12}),$$

and s is not, it is clear that $t \not\geq s$. On the other hand, we can apply a similar teleportation trick as in Example 2.3.3 to see that for any n, it holds that $T_{H_n} \geq S_{H_n}$.

2.4 Obstructions

Our results in Section 2.3 demonstrate that conversion between entanglement structures is a subtle topic. We want to find criteria under which the conversion between entanglement structures is impossible. Since we have seen that only looking at the single tensors that are to be put on the hyperedges of the graph does, in general, not suffice, we will start in Section 2.4.1 by showing for a large number of special hypergraphs that the conversion between entanglement structures is equivalent to the conversion between each of the single tensors. We then go on in Section 2.4.2 and relate the conversion between entanglement structures to the well-studied notion of asymptotic restriction as well as, in Section 2.4.3, to the so-called quantum functionals.

2.4.1 Obstructions from the geometry of the graph

To find obstructions from the geometry of the hypergraph, we first observe the following, easy fact without proof.

Lemma 2.4.1. *Let $H = (W, E, V, m)$ be a hypergraph and $\mu: V \to U$ be a structure-modifying map. Moreover, let $T = (t(e) : e \in E)$ and $S = (s(e) : e \in E)$ be two families of tensors fitting the shape of H. Then, $T_H \geq S_H$ implies $T_{\mu.H} \geq S_{\mu.H}$. The same holds when we replace all restrictions with degenerations.*

Intuitively this means that operations like folding the hypergraph and grouping together parties preserve both restriction and degeneration.

Example 2.4.2 (Bipartitioning)**.** *Let U have two elements and take a structure-modifying map $\mu : V \to U$. Then, Lemma 2.4.1 implies that if T_H restricts resp. degenerates to another entanglement structure S_H (or, more generally, to any tensor S), then T_H must also restrict resp. degenerate to S_H seen as bipartite tensors by grouping the parties into two groups (note that every bipartition can be realized by such a map μ). In particular, the flattening rank over any bipartition of T_H must be greater than that of S_H. This rank can be deduced easily from the structure of the graph, and the single tensors $t(e)$ resp. $s(e)$ and gives a first obstruction which is easy to compute. Note that this is nothing but Lemma 1.3.1 applied to T_H and S_H.*

Next, we observe that we can drop hyperedges under certain conditions. Here, it will be useful to deviate for a moment from our convention that vertices contained in the same edge cannot map to the same structure vertex.

Lemma 2.4.3. *Let $H = (W, E, V, m)$ be a hypergraph and consider two families of tensors $T = (t(e) : e \in E)$ and $S = (s(e) : e \in E)$ fitting the shape of H. Assume further that for some edge $e_0 \in E$, all coordinates of e_0 map to the same structure vertex v. Then, $T_H \geq S_H$ implies*

$$(T \setminus \{t(e_0)\})_{H \setminus \{e_0\}} \geq (S \setminus \{s(e_0)\})_{H \setminus \{e_0\}}.$$

Proof. We can use the local map at v to achieve

$$(T \setminus \{t(e_0)\})_{H \setminus \{e_0\}} \geq T_H \geq S_H \geq (S \setminus \{s(e_0)\})_{H \setminus \{e_0\}}.$$

□

We are now ready to state the following implication of Lemma 2.4.1 and Lemma 2.4.3.

Corollary 2.4.4. *Let $H = (W, E, V, m)$ be a hypergraph and let $e_0 \in E$ be an edge that is not contained in a cycle. Assume furthermore that all vertices in e_0 map to different structure vertices under m. Let $T = (t(e) : e \in E)$ and $S = (s(e) : e \in E)$ be two families of tensors fitting the shape of H. Then, $T_H \geq S_H$ implies $t(e_0) \geq s(e_0)$.*

Proof. Define a structure-modifying map $\mu: W \mapsto \{w \in e_0\}$ by mapping a vertex v to the unique vertex w such that $v \in a(w)$ is in the affix of w. By Lemma 2.4.1, $T_{\mu.H} \geq S_{\mu.H}$. By assumption, it holds that for all edges $e \neq e_0$, all its coordinates map to the same structure vertex in $\mu.H$. With that, we can remove all edges except e_0 from the graph preserving the restriction using Lemma 2.4.3, which implies $t(e_0) \geq s(e_0)$. □

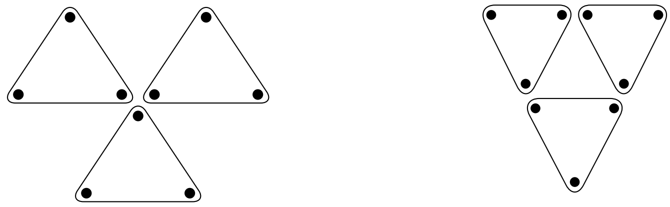
We finish this discussion by giving an example for graphs where Lemma 2.4.3 is applicable.

Definition 2.4.5. *We say that a structured hypergraph $H = (W, E, V, m)$ with $E = \{e_1, \dots, e_n\}$ is a tree if no edge is contained in a cycle.*

We visualize the concept of a tree in Figure 2.5.

Corollary 2.4.6. *Let $H = (W, V, E, m)$ be a tree and consider $T = (t(e) : e \in E)$ and $S = (s(e) : e \in E)$ fitting the shape of H. Then, $T_H \geq S_H$ holds if and only if $t(e) \geq s(e)$ for all $e \in E$. The same statement holds when we replace all restrictions with degenerations.*

Proof. By definition, it is clear that a tree contains no cycles. In particular, no edge is contained in a cycle. Hence, applying Corollary 2.4.4 finishes the proof. □



(a) A tree. (b) A hypergraph that is not a tree.

Figure 2.5: A tree in Figure 2.5(a) and a graph that is not a tree in Figure 2.5(b).

2.4.2 Folding and the asymptotic restriction of tensors

We now go on to derive further obstructions to the conversion between entanglement structures by relating it to the concept of asymptotic restriction of tensors.

Definition 2.4.7. *For k-party tensors t and s, we say that t restricts asymptotically to s and write* $t \gtrsim s$ *if we have* $t^{\boxtimes n+o(n)} \geq s^{\boxtimes n}$ *for all* $n \in \mathbb{N}$.

Let H be foldable and define families of tensors $T = (t(e) : e \in E)$ and $S = (s(e) : e \in E)$ where all $t(e)$ are the same, and similar for the $s(e)$. Denote by μ a structure-modifying map folding H onto one hyperedge. We notice that if t and s are symmetric under permuting the indices of the tensors, that is, in the notation of Section 1.3, $t \in S^k(U)$ and $s \in S^k(V)$ for some spaces U and V, then $T_H \geq S_H$ implies $t^{\boxtimes|E|} = T_{\mu.H} \geq S_{\mu.H} = s^{\boxtimes|E|}$. Again, we refer to Figure 2.2(c) for a visualization. In particular, we get $t \gtrsim s$. The same argument holds even when we only assume $T_H \gtrsim S_H$. Hence, we have proved the following.

Proposition 2.4.8. *Let* $H = (W, E, V, m)$ *be a foldable hypergraph where every edge consists of* k *vertices, and let* t, s *be symmetric tensors such that* $t \not\gtrsim s$. *Then,* $T_H \not\gtrsim S_H$ *where* T *and* S *are families consisting of* $|E|$ *copies of* t *resp.* s.

For example, it has been shown in [CVZ18] that the W-tensor does not asymptotically restrict to the unit tensor $\langle 2 \rangle$, that is, $W \not\gtrsim \langle 2 \rangle$. Hence, for any foldable hypergraph where every edge consists of three vertices, we know that the entanglement structure arising from putting W-tensors on each of the edges cannot restrict to the same entanglement structure with unit tensors instead of W-tensors. We note that the other direction holds as well. In Proposition 4.3.9, we will see a proof for the well-known fact that for no $n \in \mathbb{N}$, it holds that $\langle 2 \rangle^{\boxtimes n} \geq W^{\boxtimes n}$. Hence, by the discussion before Proposition 2.4.8, there is no foldable hypergraph such that the entanglement structure arising from putting copies of $\langle 2 \rangle$ on it restricts to the same structure built from W-tensors. In summary, we see that entanglement structures built from unit tensors and W-tensors, respectively, on foldable graphs are incomparable under restriction.

2.4.3 Quantum functionals

In this section, we will demonstrate how one can use *quantum functionals*, which are important tools in the study of asymptotic entanglement conversion introduced in [CVZ18] to find obstructions for the conversion between entanglement structures. For fixed k, call $\mathcal{X}(k)$ the set of all functions

$$f : \{k\text{-party tensors}\} \to \mathbb{R}_{\geq 0}$$

that satisfy all of the following four conditions:

1. f is monotone under restriction, that is, $t \geq s$ implies $f(t) \geq f(s)$.
2. f is normalized on the unit tensor, that is, $f(\langle r \rangle) = r$.
3. f is multiplicative under the Kronecker product, that is, $f(t \boxtimes s) = f(t)f(s)$ for all t and s.
4. f is additive under the direct sum, that is, $f(t \oplus s) = f(t) + f(s)$ for all t and s.

Then, $\mathcal{X}(k)$ is usually referred to as the *asymptotic spectrum of tensors*, and a function $f \in \mathcal{X}(k)$ is called a *point* in the asymptotic spectrum. The points in the asymptotic spectrum of tensors play an important role in the study of asymptotic restriction.

Theorem 2.4.9 ([Str86, Str88])**.** *Let $t \in U_1 \otimes \cdots \otimes U_k$ and $s \in V_1 \otimes \cdots \otimes V_k$ be k-party tensors. Then, $t \gtrsim s$ if and only if $f(t) \geq f(s)$ for all $f \in \mathcal{X}(k)$.*

All bipartite flattening ranks are examples of asymptotic spectral points. In [CVZ18], many more spectral points called *quantum functionals* were constructed.

Definition 2.4.10. *Let $t \in U_1 \otimes \cdots \otimes U_k$ and consider a probability distribution $\theta = (\theta_1, \ldots, \theta_k)$. We define the* quantum functional *associated with θ to be $F_\theta = 2^{E_\theta}$ where*

$$E_{\theta_1,\ldots,\theta_k}(t) = \sup_{t \trianglerighteq t'} \sum_{i=1}^{k} \theta_i H(t^{(i)}).$$

Here, $t^{(i)}$ is the matrix obtained by tracing out all but the i'th system from tt^* (where we, for a moment, interpret t as a column vector) and $H(t^{(i)}) = \operatorname{tr}(t^{(i)} \log(t^{(i)}))$ denotes the von Neumann entropy.

In fact, the quantum functionals give non-trivial examples of asymptotic spectral points.

Theorem 2.4.11 ([CVZ18])**.** *For any probability distribution $\theta = (\theta_1, \ldots, \theta_k)$, the quantum functional F_θ is a point in the asymptotic spectrum of k-party tensors.*

Note that the maps F_θ can also be used to rule out asymptotic restrictions in the case where the θ_i are any non-negative, real numbers that do not necessarily sum to 1. We will later also use quantum functionals for non-normalized θ.

Example 2.4.12. *We already used that the W-tensor does not asymptotically restrict to $\langle 2\rangle$. This was shown in [CVZ18] by showing that $F_{(1/3,1/3,1/3)}(\langle 2\rangle) > F_{(1/3,1/3,1/3)}(W)$.*

In fact, we can apply the quantum functionals more directly to obtain obstructions for the conversion between entanglement structures.

Theorem 2.4.13. *Let $H = (W, E, V, m)$ be a structured hypergraph and $T = (t_e : e \in E)$ and $S = (s_e : e \in E)$ be families of tensors fitting the shape of H. Moreover, let $\theta = (\theta_1, \dots, \theta_{|W|})$ be a probability distribution. Then, $T_H \gtrsim S_H$ implies*

$$\prod_{e\in E} F_{(\theta_{m(v)}:v\in e)}(t_e) \geq \prod_{e\in E} F_{(\theta_{m(v)}:v\in e)}(s_e).$$

Proof. This is an immediate consequence of the fact that the quantum functionals are multiplicative under taking the Kronecker product. □

In particular, Theorem 2.4.13 can be used to reprove that entanglement structures built from W-tensors and unit tensors $\langle 2\rangle$, respectively, are incomparable, see Example 2.4.12. Note that, unlike the results in Section 2.4.2, Theorem 2.4.11 does not require all hyperedges to comprise the same number of vertices. In particular, it does not require the hypergraph to be foldable.

2.5 Tensor rank of entanglement structures

In this section, we ask for the minimal size of rank decompositions of entanglement structures. In Section 2.5.1, we calculate the rank of a specific entanglement structure arising from putting two copies of the W-tensor on any possible two-edge hypergraph. In Section 2.5.2, we calculate the stabilizer group of entanglement structures where the underlying hypergraph is a tree.

2.5.1 Tensor rank of two copies of the W-tensor

Let $H = (W, V, E, m)$ be a hypergraph and let $T = (t_1, \dots, t_k)$ a family of tensors fitting the shape of H. It is interesting to ask about the rank of the entanglement structure $R(T_H)$. The following observation is immediate.

Lemma 2.5.1. *For $H = (W, V, E, m)$ any hypergraph and $T = (t_1, \dots, t_k)$ a family of tensors fitting the shape of H, it holds that*

$$R(T_H) \leq R(t_1 \otimes \dots \otimes t_k) \leq R(t_1) \dots R(t_k). \tag{2.1}$$

We ask under which conditions the inequalities in Equation (2.1) can be strict. Again, it is already known that both inequalities can be strict.

(a) The tensor W_5.

(b) The tensor W_4.

Figure 2.6: Illustrations of the tensors W_5 in Figure 2.6(a) and W_4 in Figure 2.6(b).

Example 2.5.2. *Consider the W-tensor. Recall from Example 1.3.3 that $R(W) = 3$ and consequently, $R(W)^2 = 9$. It has been shown in [CJZ18] that $R(W \otimes W) \leq 8$ and in [CF18] that equality holds. Moreover, it has been shown in [YCGD10] that $R(W \boxtimes W) = 7$. Since we have seen that the Kronecker product of tensors can be realized as an entanglement structure, this provides an example where all inequalities in Equation (2.1) are strict.*

In this section, we consider two setups that are intermediate to $W \otimes W$ and $W \boxtimes W$. Consider hypergraphs Γ_1 and Γ_2 where Γ_1 is the butterfly graph and Γ_2 the diamond graph as we defined them in Example 2.2.2. We will write $W_5 = (W, W)_{\Gamma_1}$ and $W_4 = (W, W)_{\Gamma_2}$. In other words, let $U_1, \ldots, U_6$ be two-dimensional and consider two copies of $W \in U_1 \otimes U_2 \otimes U_3$ and $W \in U_4 \otimes U_5 \otimes U_6$. The tensor W_5 is now $W \otimes W$ considered as an element of

$$U_1 \otimes U_2 \otimes (U_3 \otimes U_4) \otimes U_5 \otimes U_6$$

and W_4 is the same tensor considered as an element of $U_1 \otimes (U_2 \otimes U_5) \otimes (U_3 \otimes U_4) \otimes U_6$. We illustrate these tensors in Figure 2.6. We will now calculate the rank of both W_4 and W_5.

Proposition 2.5.3. *It holds that $R(W_4) = R(W_5) = 8$.*

Proof. Since $8 = R(W \otimes W) \geq R(W_5) \geq R(W_4)$, it suffices to show that $R(W_4) = 8$. Assume that there is a decomposition

$$W_4 = \sum_{j=1}^{7} u_{1,j} \otimes u_{25,j} \otimes u_{34,j} \otimes u_{6,j} \tag{2.2}$$

where $u_{25,j} \in U_2 \otimes U_5$ and $u_{34,j} \in U_3 \otimes U_4$. Note that $(\mathrm{e}_2^* \otimes \mathrm{e}_2^*)_{2,5} W_4 \neq 0$ where $(\mathrm{e}_2^* \otimes \mathrm{e}_2^*)_{2,5}$ acts as the linear functional e_2^* on U_2 and U_5 and as an identity on the remaining spaces. Hence, we can without loss of generality assume that $(\mathrm{e}_2^* \otimes \mathrm{e}_2^*) u_{25,7} = 1$. With that,

$$u_{25,7} = a\mathrm{e}_1 \otimes \mathrm{e}_1 + b\mathrm{e}_1 \otimes \mathrm{e}_2 + c\mathrm{e}_2 \otimes \mathrm{e}_1 + \mathrm{e}_2 \otimes \mathrm{e}_2$$

for some coefficients $a, b, c \in \mathbb{C}$.

Note that the W-tensor has symmetries: Defining the linear transformation

$$A_p : \mathbb{C}^2 \to \mathbb{C}^2, \mathrm{e}_1 \mapsto \mathrm{e}_1, \mathrm{e}_2 \mapsto \mathrm{e}_2 + p\mathrm{e}_1$$

we have that $(A_\alpha \otimes A_\beta \otimes A_\gamma)W = W$ for all α, β and γ such that $\alpha + \beta + \gamma = 0$. Applying the map $\mathrm{id}_1 \otimes (A_{-b})_2 \otimes (A_b)_3 \otimes \mathrm{id}_4 \otimes (A_{-c})_5 \otimes (A_c)_6$ to the decomposition in Equation (2.2), we see that there is a decomposition

$$W_4 = \sum_{j=1}^{7} v_{1,j} \otimes v_{25,j} \otimes v_{34,j} \otimes v_{6,j} \tag{2.3}$$

such that $v_{25,7} = a'\mathrm{e}_1 \otimes \mathrm{e}_1 + \mathrm{e}_2 \otimes \mathrm{e}_2$ for some $a' \in \mathbb{C}$.

Note that W_4 has additional symmetries. Define for any $q \in \mathbb{C}$ the linear transformation

$$\begin{aligned} B_q : \left(\mathbb{C}^2 \otimes \mathbb{C}^2\right) \to \left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \mathrm{e}_1 \otimes \mathrm{e}_1 &\mapsto \mathrm{e}_1 \otimes \mathrm{e}_1, \\ \mathrm{e}_1 \otimes \mathrm{e}_2 &\mapsto \mathrm{e}_1 \otimes \mathrm{e}_2, \\ \mathrm{e}_2 \otimes \mathrm{e}_1 &\mapsto \mathrm{e}_2 \otimes \mathrm{e}_1, \\ \mathrm{e}_2 \otimes \mathrm{e}_2 &\mapsto \mathrm{e}_2 \otimes \mathrm{e}_2 + q\mathrm{e}_1 \otimes \mathrm{e}_1. \end{aligned}$$

One easily checks that

$$(\mathrm{id} \otimes B_{25,q} \otimes B_{34,-q} \otimes \mathrm{id})W_4 = W_4. \tag{2.4}$$

Applying Equation (2.4) with $q = -a'$ to the rank-7 decomposition in Equation (2.3) transforms it into a decomposition

$$W_4 = \sum_{j=1}^{7} w_{1,j} \otimes w_{25,j} \otimes w_{34,j} \otimes w_{6,j} \tag{2.5}$$

with $w_{25,7} = \mathrm{e}_2 \otimes \mathrm{e}_2$. With that, the remainder of the proof is identical to the proof that the rank of $W \otimes W$ is 8 in [CF18]:

Applying the projectors $(\mathrm{e}_1^*)_2$ and $(\mathrm{e}_1^*)_5$ to the rank-7 decomposition of W_4 yields rank-6 decompositions of

$$\begin{aligned} (\mathrm{e}_1^*)_2 W_4 = M \otimes W = \sum_{j=1}^{6} w_{1,j} \otimes (\mathrm{e}_1^* \otimes \mathrm{id}) w_{25,j} \otimes w_{34,j} \otimes w_{6j} \in U_1 \otimes (U_3 \otimes U_4) \otimes U_5 \otimes U_6, \\ (\mathrm{e}_1^*)_5 W_4 = W \otimes N = \sum_{j=1}^{6} w_{1j} \otimes (\mathrm{id} \otimes \mathrm{e}_1^*) w_{25,j} \otimes w_{34,j} \otimes w_{6j} \in U_1 \otimes U_2 \otimes (U_3 \otimes U_4) \otimes U_6. \end{aligned}$$

where $M = N = \mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_2 \otimes \mathrm{e}_1$ is a rank-2 matrix. It is easy to verify using the substitution method from Theorem 1.3.2 that $W \otimes N$, considered as an element of $U_1 \otimes (U_2 \otimes U_6) \otimes (U_3 \otimes U_4)$, has rank 6. Defining

$$\mathcal{N} = \{w_{1,j} \otimes w_{34,j} \text{ for } j = 1, \ldots, 6\},$$

it follows that the set $\mathcal{N}$ is linearly independent. Consider moreover the set

$$\mathcal{M} = \{(\mathrm{e}_1^* \otimes \mathrm{id})w_{25,j} \otimes w_{6,j} \text{ for } j = 1, \dots, 6\}.$$

It is clear that all elements of $\mathcal{M}$ are simple tensors and span $W(U_4^*)$. Hence, $\mathcal{M}$ must contain at least three linearly independent vectors.

View the tensor $M \otimes W \in (U_1 \otimes U_3 \otimes U_4) \otimes (U_5 \otimes U_6)$ as a bipartite tensor. On the one hand, we can write

$$M \otimes W = (M \otimes \mathrm{e}_1) \otimes (\mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_2 \otimes \mathrm{e}_1) + (M \otimes \mathrm{e}_2) \otimes \mathrm{e}_1 \otimes \mathrm{e}_1.$$

Consequently, viewed as a bipartite tensor, $M \otimes W$ has rank at most 2. On the other hand, the preceding discussion implies that

$$M \otimes W = \sum_{j=1}^{6} n_j \otimes m_j$$

where $n_j = w_{1,j} \otimes w_{34,j}$ are the elements of $\mathcal{N}$ and $m_j = (\mathrm{e}_1^* \otimes \mathrm{id})w_{25,j} \otimes w_{6,j}$ are the elements of $\mathcal{M}$. Since the m_j are linearly independent, and at least three of the n_j are linearly independent, this tensor must have rank at least 3 – a contradiction. □

2.5.2 Stabilizers of entanglement structures

We will now turn our attention to the stabilizer group of entanglement structures. Writing $G = \mathrm{GL}(U_1) \times \cdots \times \mathrm{GL}(U_k)$, recall the definition

$$\mathrm{Stab}_G(t) = \{(A_1, \dots, A_k) \in G : (A_1 \otimes \cdots \otimes A_k)t = t\}$$

for the group of stabilizers in G of a tensor $t \in U_1 \otimes \cdots \otimes U_k$ from Definition 1.2.3.

Let H be a hypergraph and $T = (t_1, \dots, t_k)$ be a family of tensors fitting the shape of H. It is clear that if A_i is a stabilizer of t_i for all $i = 1, \dots, k$, then $\bigotimes_{i=1}^k A_i$ grouped according to the structure vertices is a stabilizer of T_H. In the proof of Proposition 2.5.3, we have seen that entanglement structures can have stabilizers that do not arise in this way. In this section, we will see that for tree hypergraphs, all stabilizers arise in this way.

Let $t \in U_1 \otimes U_2 \otimes U_3$ and $s \in U_4 \otimes U_5 \otimes U_6$. To simplify the notation, we will write $t \bowtie s$ for the entanglement structure that arises by putting t and s on a butterfly graph, that is, $t \otimes s$ considered as an element of $U_1 \otimes U_2 \otimes (U_3 \otimes U_4) \otimes U_5 \otimes U_6$. We record the following result.

Proposition 2.5.4. *Let $t, \tilde{t} \in U_1 \otimes U_2 \otimes U_3$ and $s, \tilde{s} \in U_4 \otimes U_5 \otimes U_6$ be concise and consider a linear map $M : U_3 \otimes U_4 \to U_3 \otimes U_4$ such that $(\mathrm{id} \otimes \mathrm{id} \otimes M \otimes \mathrm{id} \otimes \mathrm{id}) t \bowtie s = \tilde{t} \bowtie \tilde{s}$. Then, there are $M_1 \in \mathrm{GL}(U_3)$ and $M_2 \in \mathrm{GL}(U_4)$ such that $M = M_1 \otimes M_2$.*

Proof. Pick bases $\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{u}_3}$ and $\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{u}_4}$ of U_3 and U_4. Note that we can think of t and s as bipartite tensors

$$t = \sum_{j=1}^{\mathrm{u}_3} \rho_j \otimes \mathrm{e}_j \in (U_1 \otimes U_2) \otimes U_3, \; s = \sum_{k=1}^{\mathrm{u}_4} \mathrm{e}_k \otimes \sigma_k \in U_4 \otimes (U_5 \otimes U_6).$$

By conciseness we know that the sets $\{\rho_j, j = 1, \dots, \mathrm{u}_3\}$ and $\{\sigma_k, k = 1, \dots, \mathrm{u}_4\}$ are linearly independent. Define the linear functionals ρ_j^* for $j = 1, \dots, w$ such that $\rho_j^*(\rho_i) = \delta_{ij}$ and similarly, σ_k^* with $\sigma_k^*(\sigma_i) = \delta_{ik}$.

Let now $M : U_3 \otimes U_4 \to U_3 \otimes U_4$ be a map with $M_{34} t \bowtie s = \tilde{t} \bowtie \tilde{s}$ where we write M_{34} for $\mathrm{id} \otimes \mathrm{id} \otimes M \otimes \mathrm{id} \otimes \mathrm{id}$. Clearly, M_{34} commutes with $\rho_j^* \otimes \mathrm{id} \otimes \sigma_k^*$ for any j, k in the sense that $(\rho_j^* \otimes \mathrm{id} \otimes \sigma_k^*) M_{34} = M(\rho_j^* \otimes \mathrm{id} \otimes \sigma_k^*)$. Hence,

$$\begin{aligned} M(\mathrm{e}_j \otimes f_k) = M(\rho_j^* \otimes \mathrm{id} \otimes \sigma_k^*) t \bowtie s = \\ (\rho_j^* \otimes \mathrm{id} \otimes \sigma_k^*) M_{34} t \bowtie s = \\ (\rho_j^* \otimes \mathrm{id} \otimes \sigma_k^*) \tilde{t} \bowtie \tilde{s} = \rho_j^* \tilde{t} \otimes \sigma_k^* \tilde{s}. \end{aligned}$$

In other words, defining M_1 and M_2 via $M_1(\mathrm{e}_j) = \rho_j^* \tilde{t}$ and $M_2(\mathrm{e}_k) = \sigma_k^* \tilde{s}$, respectively, we see that $M = M_1 \otimes M_2$. □

Applying Proposition 2.5.4 yields insights into the stabilizers of tensors of the form $t \bowtie s$.

Corollary 2.5.5. *Say,* $(A_1 \otimes A_2 \otimes M \otimes A_5 \otimes A_6) t \bowtie s = t \bowtie s$ *for some concise tensors* t *and* s*. Then,* M *is of the form* $M_1 \otimes M_2$*.*

Proof. Note that the maps A_1, A_2, A_5, and A_6 must be invertible by conciseness of the tensors. The claim now follows immediately from Proposition 2.5.4 by choosing $\tilde{t} = (A_1^{-1} \otimes A_2^{-1} \otimes \mathrm{id}) t$ and $\tilde{s} = (\mathrm{id} \otimes A_5^{-1} \otimes A_6^{-1}) s$. □

Our proof did not use that the tensors t and s were three-party tensors. The same holds for any number of parties, that is, $s \in U_1 \otimes \cdots \otimes U_m$ and $t \in V_1 \otimes \cdots \otimes V_n$ for any m and n which "touch" in only one party. From this, one easily gets the following stronger result.

Corollary 2.5.6. *Tensors on a tree hypergraph only have the product of stabilizers of the single tensors as stabilizers.*

This leads us to the following conjecture, compare also the discussion in Example 1.2.5.

Conjecture 2.5.7. *Let* $H = (V, W, E, m)$ *be a tree and* $T = (t_1, \dots, t_k)$ *a family of tensors fitting the shape of* H*. Then,*

$$R(T_H) = R(t_1 \otimes \cdots \otimes t_k).$$

Chapter 3

Quantum max-flow in the bridge graph

The quantum max-flow introduced in [CFW10] quantifies the maximal possible entanglement between two regions of a tensor network for a fixed graph and fixed bond dimensions. Understanding the quantum max-flow provides valuable insights into the entanglement properties of the physical states one can obtain from such tensor networks.

Some progress in understanding this quantity was achieved in the past: In [CFS⁺16], the authors relate the quantum max-flow to the classical max-flow of a graph and introduce the notion of *quantum min-cut* which is the information-theoretic analog of the classical min-cut in graph theory [EFS56, FF56]. They show that the quantum max-flow is always bounded from above by the quantum min-cut and demonstrate for a few examples with fixed, small bond dimensions that the inequality can be sharp. In [GLW18], families of examples presenting big separations between quantum max-flow and quantum min-cut were given, but computing the quantum max-flow exactly seemed out of reach. Moreover, it was shown in [Has17] that asymptotically, quantum min-cut and quantum max-flow are the same.

In this chapter, we study this problem for a specific graph, which we call the bridge graph. We relate the problem of computing the quantum max-flow in the bridge graph to the theory of prehomogeneous tensor spaces and the representation theory of quivers which allows us to exactly compute the quantum max-flow in the bridge graph for essentially all choices of bond dimensions.

This chapter is a partly modified version of [GLS22].

3.1 Overview

In this chapter, we focus on tensor network representations. Understanding properties of tensor network states from the point of view of geometry and information theory provides valuable insights into the physical states that they define. However, usually, quantities associated with tensor network states are difficult to compute. One such quantity is the quantum max-flow which was introduced in [CFW10] in the context of the positivity of certain tensor operators. The quantum max-flow is an information-theoretic version of the classical max-flow of a graph, and the exact definition will be given in Section 3.2.1. Informally, given the data of a tensor network $\Gamma = (V, E)$ with bond dimensions $\mathbf{m}$, and two disjoint subsets $\mathcal{S}, \mathcal{T} \subseteq V$ of the set of vertices of Γ, the associated quantum max-flow, denoted $\mathrm{QMaxFlow}(\Gamma, \mathbf{m}, \mathcal{S}, \mathcal{T})$, is the maximum possible rank across the bipartition $(\mathcal{S}, \mathcal{T})$ of a tensor arising as a tensor network state associated with Γ and $\mathbf{m}$.

In [CFS+16], the authors relate the quantum max-flow to the notion of *quantum min-cut*, denoted $\mathrm{QMinCut}(\Gamma, \mathbf{m}, \mathcal{S}, \mathcal{T})$, that is, the information-theoretic analog of the classical min-cut in graph theory [EFS56, FF56], see Section 3.2.1. They prove the inequality $\mathrm{QMaxFlow}(\Gamma, \mathbf{m}, \mathcal{S}, \mathcal{T}) \leq \mathrm{QMinCut}(\Gamma, \mathbf{m}, \mathcal{S}, \mathcal{T})$ for all instances of Γ, $\mathbf{m}$, $\mathcal{S}$ and $\mathcal{T}$ and construct examples where this inequality is strict. They highlight connections between the quantum max-flow and entropy of entanglement as well as the quantum satisfiability problem and suggest further connections to spin systems in condensed matter and quantum gravity. Further progress was achieved in [GLW18] where the authors construct families of examples where the gap between the quantum min-cut and the quantum max-flow can be arbitrarily large. In [Has17], Hastings showed that asymptotically, quantum min-cut and quantum max-flow are the same.

As of today, the exact value of the quantum max-flow was computed only in a limited number of cases with small fixed bond dimensions, and computing the quantum max-flow exactly for a given graph seemed out of reach. In this chapter, we study this problem for a specific graph, the bridge graph in Figure 3.1.

The study of the quantum max-flow in the bridge graph has two interesting connections to other areas that we briefly outline.

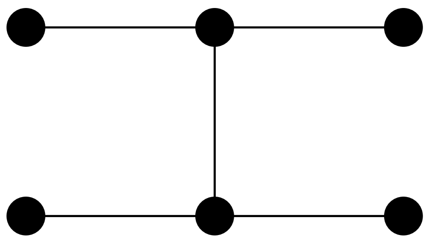


Figure 3.1: The bridge graph.

In the representation theory of quivers, the value of the quantum max-flow reveals the existence (or the nonexistence) of certain covariants for the so-called Kronecker quiver. Quiver representation theory is the main tool used in the study of the invariant theory of families of matrices [DM21], which has, among others, applications in the study of matrix product states [MS19, DMS22]. In fact, the invariant theory of the Kronecker quiver is the main tool used in the proof of Theorem 3.3.13, where we rely on the existence of certain invariants introduced in [Sch91] and studied in [DZ01, SV01, DM17].

The second connection we briefly mention is the one to algebraic statistics and the study of maximum likelihood estimation, a widely used method to determine the free parameters of a probability distribution that best explains given data. In this context, the maximum likelihood threshold is the smallest sample size allowing one to completely reconstruct the model from the given sample, see, for example, [DKH21]. The existence of a maximum likelihood threshold has strong connections with classical invariant theory, as explained in [AKRS21, DM21]. In particular, [DM21, DMW22] study invariant theoretic properties of matrix and tensor spaces in the context of maximum likelihood estimation. Moreover, [AKRS21] studies the maximum likelihood threshold in relation to the *cut-and-paste rank* introduced in [BD06]. In the case of matrices, the cut-and-paste rank is exactly the quantum max-flow on the bridge graph. In particular, the results of this chapter can be read in terms of cut-and-paste rank.

3.1.1 The quantum max-flow and the bridge graph: summary of the results

Associate to the bridge graph of Figure 3.1 bond dimensions a, b, w, a', b' on the edges and consider the two disjoint subsets $\mathcal{S}$ and $\mathcal{T}$ of the vertices as depicted in Figure 3.2.

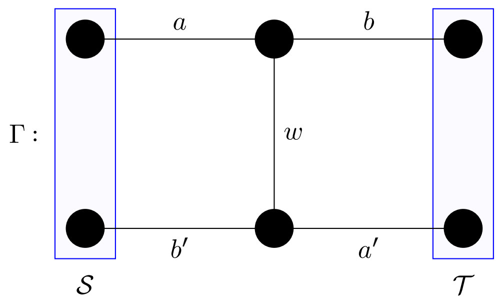


Figure 3.2: The bridge graph with bond dimensions a, b, w, b', a', set of sources $\mathcal{S}$ and set of targets $\mathcal{T}$.

Fix vector spaces A, B, A', B', W of dimension a, b, a', b', w respectively. A flow map on the bridge graph is defined as follows. Let $T \in A \otimes B \otimes W$ and $T' \in A' \otimes B' \otimes W^*$ be tensors. Pictorially one can think of them as placed on the top and bottom central vertex of the bridge graph, with the "legs" corresponding to the three edges incident to each vertex. Let

$F_{T,T'} \in A \otimes B \otimes A' \otimes B'$ be the tensor obtained by contracting the factor W of T with the factor W^* of T'. Note that this is up to change of bases on the spaces A, B, A', B' exactly a tensor network state in the sense of Definition 1.4.1. We explain this in more detail in Lemma 3.2.3. The tensor $F_{T,T'}$ can be regarded as a linear map $F_{T,T'} : (A \otimes B')^* \to (B \otimes A')$, namely a bipartite tensor between the vertices in $\mathcal{S}$ and the vertices in $\mathcal{T}$. The quantum flow associated with T and T' is the rank of this linear map. The quantum max-flow is the maximum possible value of $\operatorname{rank}(F_{T,T'})$ as T and T' vary in the respective space. We write

$$\operatorname{QMaxFlow}\left(\begin{smallmatrix} a & & b \\ & |w & \\ b' & & a' \end{smallmatrix}\right) = \max\left\{\operatorname{rank}(F_{T,T'}) : \begin{array}{l} T \in A \otimes B \otimes W \\ T' \in A' \otimes B' \otimes W^* \end{array}\right\}.$$

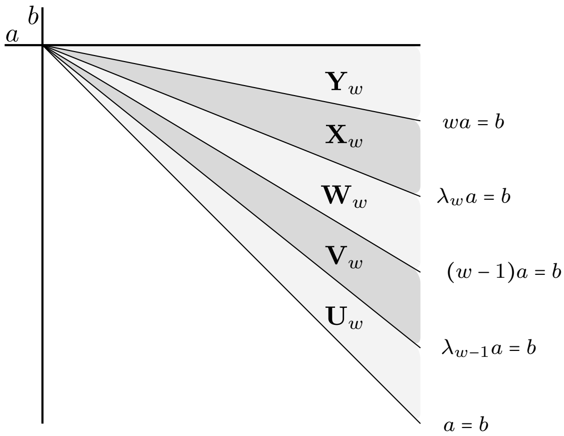
Note that quantum max-flow in the bridge graph is symmetric, in the sense that

$$\operatorname{QMaxFlow}\left(\begin{smallmatrix} a & & b \\ & |w & \\ b' & & a' \end{smallmatrix}\right) = \operatorname{QMaxFlow}\left(\begin{smallmatrix} b & & a \\ & |w & \\ a' & & b' \end{smallmatrix}\right).$$

Moreover, if $a \le b$ and $b' \le a'$, then the quantum max-flow in the bridge graph is ab' which coincides with the quantum min-cut that will be introduced in Section 3.2.1. Hence, we will restrict our analysis to the case $a \le b$ and $a' \le b'$.

The constant $\lambda_w = \frac{w+\sqrt{w^2-4}}{2}$ will play a crucial role in the behavior of the quantum max-flow. In order to summarize our results, we consider a partition of the set $\{(a,b) \in \mathbb{N}^2 : b \ge a\}$ into five regions:

$$\begin{aligned} \mathbf{Y}_w &= \{(a,b) : wa \le b\} \\ \mathbf{X}_w &= \{(a,b) : \lambda_w a < b < wa\} \\ \mathbf{W}_w &= \{(a,b) : (w-1)a \le b \le \lambda_w a\} \\ \mathbf{V}_w &= \{(a,b) : \lambda_{w-1} a < b < (w-1)a\} \\ \mathbf{U}_w &= \{(a,b) : a \le b \le \lambda_{w-1} a\} \end{aligned}$$



Note that $\mathbf{X}_w = \mathbf{V}_{w+1}$ and $\mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w = \mathbf{U}_{w+1}$. Moreover, $\mathbf{W}_2 = \mathbf{U}_3 = \{(a,a) : a \in \mathbb{N}\}$ and $\mathbf{U}_2, \mathbf{V}_2 = \varnothing$. Using this notation, we now record the main results of this chapter.

Theorem 3.1.1. *The quantum max-flow in the bridge graph*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right)$$

is given by the value specified in Table 3.1. In particular,

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right)$$

holds except possibly in the cases marked with $\bigstar, \clubsuit, \diamondsuit$. *More precisely, the following holds.*

($\bigstar$) *Let* $(a,b),(a',b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$. *If* $(a,b) = q(a',b')$ *for some* $q \in \mathbb{Q}$, *then*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right) = ab' = a'b.$$

($\clubsuit$) *Let* $(a,b),(a',b') \in \mathbf{W}_w$. *If* $\mathrm{depth}(a,b) \geq \mathrm{depth}(a',b')$, *then*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right) = ab',$$

where depth *denotes the* castling depth *defined in Definition 3.3.5.*

($\diamondsuit$) *Let* $(a,b),(a',b') \in \mathbf{X}_w$. *Write*

$$\begin{aligned} a &= z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta, & b &= z_{p+1}^{(w)}\alpha + z_{p+2}^{(w)}\beta \\ a' &= z_{p'}^{(w)}\alpha' + z_{p'+1}^{(w)}\beta', & b' &= z_{p'+1}^{(w)}\alpha' + z_{p'+2}^{(w)}\beta', \end{aligned}$$

as described in Lemma 3.2.13. Then,

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b'}{\underline{\quad}} & \underset{a'}{\underline{\quad}} \end{array}\right) = \begin{cases} ab' & \text{if } p' > p, \\ ab' - \beta\alpha' & \text{if } p = p', \\ a'b & \text{if } p' < p. \end{cases}$$

In particular, we have a full characterization of the quantum max-flow in the bridge graph for *symmetric* bond dimension, that is, in the case $(a,b) = (a',b')$.

Corollary 3.1.2. *Let* $w, a, b \geq 1$.

- *If* $(a,b) \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$, *then*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} \overset{a}{\overline{\quad}} & \overset{b}{\overline{\quad}} \\ & w \\ \underset{b}{\underline{\quad}} & \underset{a}{\underline{\quad}} \end{array}\right) = ab.$$

		(a,b)				
		$\mathbf{U}_w$	$\mathbf{V}_w$	$\mathbf{W}_w$	$\mathbf{X}_w$	$\mathbf{Y}_w$
(a',b')	$\mathbf{U}_w$	★	ab'	ab'	ab'	ab'
	$\mathbf{V}_w$	$a'b$	★	ab'	ab'	ab'
	$\mathbf{W}_w$	$a'b$	$a'b$	♣ + ★	ab'	ab'
	$\mathbf{X}_w$	$a'b$	$a'b$	$a'b$	$\diamondsuit$	ab'
	$\mathbf{Y}_w$	$a'b$	$a'b$	$a'b$	$a'b$	$aa'w$

Table 3.1: The quantum max-flow in the bridge graph with bond dimension a, b, a', b', w. The orange cases are solved in Theorem 3.3.6. The cyan-colored cases are solved in Theorem 3.3.7. The purple cases are solved in Theorem 3.3.8, the precise formulation of $\diamondsuit$ can be found in Theorem 3.1.1. The green cases are solved in Corollary 3.3.9.
The case marked with ♣ is partly solved in Theorem 3.3.10. The cases marked with ★ are partly solved in Theorem 3.3.13. A precise formulation of the results is in Theorem 3.1.1.

- *If* $(a,b) \in \mathbf{X}_w$*, let* α, β, p *be such that*

$$a = z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta \qquad b = z_{p+1}^{(w)}\alpha + z_{p+2}^{(w)}\beta$$

as in Lemma 3.2.13. Then,

$$\mathrm{QMaxFlow}\left(\begin{smallmatrix} a & b \\ & w \\ b & a \end{smallmatrix}\right) = ab - \alpha\beta.$$

- *If* $(a,b) \in \mathbf{Y}_w$*, then*

$$\mathrm{QMaxFlow}\left(\begin{smallmatrix} a & b \\ & w \\ b & a \end{smallmatrix}\right) = wa^2.$$

Theorem 3.1.1 is almost complete. The cases that remain open lie in the regions marked with ★ in Table 3.1. We conjecture that the result can be extended in full generality.

Conjecture 3.1.3. *Let* $(a,b),(a',b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$*. Then,*

$$\mathrm{QMaxFlow}\left(\begin{smallmatrix} a & b \\ & w \\ b' & a' \end{smallmatrix}\right) = \mathrm{QMinCut}\left(\begin{smallmatrix} a & b \\ & w \\ b' & a' \end{smallmatrix}\right) = \min\{ab', a'b\}.$$

In Section 3.3.4, we prove a reduction argument allowing one to deduce Conjecture 3.1.3 for any $w \in \mathbb{N}$ from the case $w = 3$.

3.1.2 Quantum max-flow and castling transform

A key ingredient in the proof of the results described in Section 3.1.1 is that the quantum max-flow is easy to control under the so-called *castling transform*. The castling transform was introduced in [SK77] in the study of prehomogeneous tensor spaces. In general, it defines a correspondence between orbits in tensor spaces under particular group actions. We refer to Section 3.2.2 for the precise definition. In the framework of the castling transform, we will prove the following result:

Theorem 3.1.4. *Let a, b, a', b', w be natural numbers such that $a \le bw$ and $a' \le b'w$. Then,*

$$ab' - \mathrm{QMaxFlow}\left(\begin{smallmatrix} a & & b \\ & w & \\ b' & & a' \end{smallmatrix}\right) = b(b'w - a') - \mathrm{QMaxFlow}\left(\begin{smallmatrix} b & & bw-a \\ & w & \\ b'w-a' & & b' \end{smallmatrix}\right).$$

Moreover, if we have $b \le aw$ and $b' \le a'w$, then we also have

$$a \cdot b' - \mathrm{QMaxFlow}\left(\begin{smallmatrix} a & & b \\ & w & \\ b' & & a' \end{smallmatrix}\right) = (wa - b)a' - \mathrm{QMaxFlow}\left(\begin{smallmatrix} wa-b & & a \\ & w & \\ a' & & wa'-b' \end{smallmatrix}\right).$$

Let us briefly outline how Theorem 3.1.4 helps to calculate the quantum max-flow in the bridge graph. For a detailed discussion, we refer to Section 3.3. On a high level, we use Theorem 3.1.4 to reduce calculating the quantum max-flow in the bridge graph to one of the following two easy cases:

(i) For $b \ge aw$, that is, $(a, b) \in \mathbf{Y}_w$, it holds that

$$\mathrm{QMaxFlow}\left(\begin{smallmatrix} a & & b \\ & w & \\ b' & & a' \end{smallmatrix}\right) = \min\{ab', aa'w\}.$$

(ii) For $a \le b$ and $b' \le a'$, it holds that

$$\mathrm{QMaxFlow}\left(\begin{smallmatrix} a & & b \\ & w & \\ b' & & a' \end{smallmatrix}\right) = ab'.$$

For pairs of natural numbers, we say that (a, b) and $(\tilde{a}, \tilde{b})$ are *castling equivalent* if (a, b) can be transformed into $(\tilde{a}, \tilde{b})$ by a number of operations $(a, b) \mapsto (wa - b, a)$ or $(a, b) \mapsto (b, wb - a)$. We show in Lemma 3.3.4 that every pair $(a, b) \in \mathbf{X}_w$ is castling equivalent to a pair in $\mathbf{Y}_w$. Consequently, we can apply Theorem 3.1.4 multiple times and use (i) to calculate the quantum max-flow. This procedure is applied in the proofs of Theorem 3.3.7 and Theorem 3.3.8. We also show in Lemma 3.3.4 that a pair $(a, b) \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$ is always castling equivalent to a pair $(\tilde{a}, \tilde{b})$ where $\tilde{a} > \tilde{b}$. This – in many cases – reduces calculating the quantum max-flow in the bridge graph to the easier case (ii). This procedure is applied in the proofs of Theorem 3.3.7, Corollary 3.3.9, and Theorem 3.3.10.

3.2 Preliminaries

In this section, we provide the exact definition of the quantum max-flow for a general graph, and we prove an alternative description in the case of the bridge graph. Moreover, we introduce the castling framework and describe the connections to the theory of prehomogeneous tensor spaces that will be useful in the proofs of the main results.

3.2.1 The quantum min-cut/max-flow problem

Let $\Gamma = (V, E)$ be a graph and consider an assignment of bond dimensions $\mathbf{m} : E \to \mathbb{N}$ defining the data for a graph tensor as in Definition 1.4.4. Let $\mathcal{S}, \mathcal{T} \subseteq V$ be two disjoint subsets. For $v \in \mathcal{S} \cup \mathcal{T}$, set $\mathbf{n}(v) = \prod_{e \ni v} \mathbf{m}(e)$ and for $v \notin \mathcal{S} \cup \mathcal{T}$, set $\mathbf{n}(v) = 1$ yielding an assignment $\mathbf{n}$ of physical dimensions. Now, every element $T \in \mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})$ can be regarded as a bipartite tensor $T \in \left(\bigotimes_{v\in\mathcal{S}} \mathbb{C}^{\mathbf{n}(v)}\right) \otimes \left(\bigotimes_{v\in\mathcal{T}} \mathbb{C}^{\mathbf{n}(v)}\right)$, or equivalently a linear map $F_T : \bigotimes_{v\in\mathcal{S}} \mathbb{C}^{\mathbf{n}(v)^*} \to \bigotimes_{v\in\mathcal{T}} \mathbb{C}^{\mathbf{n}(v)}$. The *quantum max-flow* of $(\Gamma, \mathbf{m})$ relative to the subsets $\mathcal{S}, \mathcal{T}$ is the maximum possible rank of this bipartite tensor, namely

$$\mathrm{QMaxFlow}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T}) = \max\left\{\mathrm{rank}(F_T) : T \in \mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})\right\}.$$

A cut of Γ relative to the sets $\mathcal{S}, \mathcal{T}$ is a partition $V = \mathcal{A} \sqcup \mathcal{B}$ of V with $\mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{T} \subseteq \mathcal{B}$. The quantum capacity of a cut is

$$\mathrm{qcap}(\mathcal{A}, \mathcal{B}) = \prod_{\substack{\{v_1, v_2\} \in E \\ v_1 \in \mathcal{A}, v_2 \in \mathcal{B}}} \mathbf{m}(\{v_1 v_2\}).$$

The *quantum min-cut* of $(\Gamma, \mathbf{m})$ relative to the sets $\mathcal{S}, \mathcal{T}$ is

$$\mathrm{QMinCut}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T}) = \min\left\{\mathrm{qcap}(\mathcal{A}, \mathcal{B}) : \mathcal{S} \subseteq \mathcal{A}, \mathcal{T} \subseteq \mathcal{B}, \mathcal{A} \cap \mathcal{B} = \varnothing\right\}. \tag{3.1}$$

Remark 3.2.1. *The quantum min-cut can alternatively be defined as the minimal possible rank of a flattening map as defined in Definition 1.1.12: Every choice of $\mathcal{A}$ and $\mathcal{B}$ induces a flattening for the graph tensor $\mathfrak{T}_{\Gamma,\mathbf{m}}$. The quantum capacity of this cut is just the rank of the induced flattening map.*

We state two immediate but crucial properties of the quantum min-cut and the quantum max-flow. They were first stated in [CFS⁺16] and are immediate consequences of the definition.

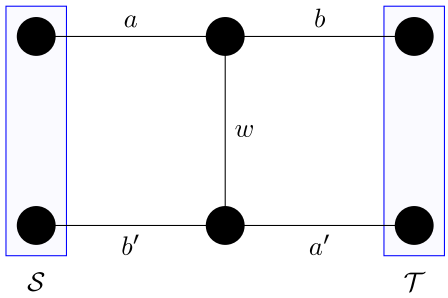
Proposition 3.2.2. *For every network Γ with bond dimensions $\mathbf{m}$, the following holds:*

- $\mathrm{QMaxFlow}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T}) \leq \mathrm{QMinCut}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T})$.

- *The set of* $T \in \mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})$ *such that* $\operatorname{rank}(F_T) = \operatorname{QMaxFlow}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T})$ *is a dense and Zariski-open subset of* $\mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n})$.

In general, computing the quantum max-flow is not trivial. In fact, in most cases, even proving a separation between quantum max-flow and quantum min-cut is challenging. We want to compute $\operatorname{QMinCut}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T})$ and $\operatorname{QMaxFlow}(\Gamma, \mathbf{m}; \mathcal{S}, \mathcal{T})$ in the case of the *bridge graph*

$\Gamma :$



for integers a, b, w, b', a'. For the four vertices of degree 1, the local tensors T_v in the definition of the tensor network state have no effect. This yields the following immediate result

Lemma 3.2.3. *Let* $\mathbf{m} = (a, b, w, b', a')$ *be bond dimensions on the bridge graph. Let* $\mathbf{n}$ *be the physical dimensions induced by the choice of* $\mathcal{S}$ *and* $\mathcal{T}$ *as before. Write* $A = \mathbb{C}^a$ *and similarly for* B, W, B', A'. *Then, regarded as a subset of* $A \otimes B \otimes A' \otimes B'$,

$$\mathcal{TNS}(\Gamma, \mathbf{m}, \mathbf{n}) = \{T \lrcorner T' \in A \otimes B \otimes A' \otimes B' : T \in A \otimes B \otimes W,\ T' \in A' \otimes B' \otimes W^*\}$$

where $\lrcorner : W \otimes W^* \to \mathbb{C}$ *is the tensor contraction as described in Section 1.4. Writing* $F_{T,T'} = T \lrcorner T' : (A \otimes B')^* \to A' \otimes B$ *for the tensor network state interpreted as a linear map, we have*

$$\operatorname{QMaxFlow}\left(\begin{smallmatrix} \overset{a}{\text{—}} & \overset{b}{\text{—}} \\ & |\, w & \\ \underset{b'}{\text{—}} & \underset{a'}{\text{—}} \end{smallmatrix}\right) = \max\left\{\operatorname{rank}(F_{T,T'}) : \begin{array}{l} T \in A \otimes B \otimes W, \\ T' \in A' \otimes B' \otimes W^* \end{array}\right\}.$$

The quantum max-flow in the bridge graph can be characterized only using linear algebra: Fix a basis $\mathrm{e}_1, \dots, \mathrm{e}_w$ of W with dual basis $\mathrm{e}_1^*, \dots, \mathrm{e}_w^*$. For $T \in A \otimes B \otimes W$, the *slices* $T(\mathrm{e}_j^*) \in A \otimes B$ can be regarded as matrices of size $a \times b$. Similarly for $T' \in A' \otimes B' \otimes W^*$, the elements $T'(\mathrm{e}_j) \in A' \otimes B'$ can be regarded as matrices of size $b' \times a'$. Then, the linear map

$$F_{T,T'} : (A \otimes B')^* \to A' \otimes B$$

is represented by the matrix $T(\mathrm{e}_1^*) \boxtimes T'(\mathrm{e}_1) + \dots + T(\mathrm{e}_w^*) \boxtimes T'(\mathrm{e}_w)$. As T and T' are arbitrary, we obtain the following characterization.

(a) The graph Γ_1 with bond dimensions $\mathbf{m}_1$. (b) The graph Γ_2 with bond dimensions $\mathbf{m}_2 \equiv r$.

Figure 3.3: The two graphs Γ_1 and Γ_2 with associated bond dimensions. The quantum max-flow in the bridge graph specifies the maximal possible r such that $\mathcal{T}_{\Gamma_1,\mathbf{m}_1} \geq \mathcal{T}_{\Gamma_2,\mathbf{m}_2}$ for $\mathbf{m}_2 \equiv r$.

Lemma 3.2.4. *The quantum max-flow in the bridge graph satisfies*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \max\left\{\mathrm{rank}\Big(\sum_{i=1}^{w} M_i \boxtimes N_i\Big) : \begin{array}{l} M_1 \ldots M_w \;\; \textit{matrices of size } a \times b, \\ N_1 \ldots N_w \;\; \textit{matrices of size } b' \times a' \end{array}\right\}.$$

We can also characterize the quantum max-flow in the bridge graph using the language presented in Section 1.4. For that, consider the two graphs Γ_1 and Γ_2 with bond dimensions $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively, as depicted in Figure 3.3. Then the following is immediate.

Lemma 3.2.5. *The quantum max-flow in the bridge graph satisfies*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \max\left\{r : \mathcal{T}_{\Gamma_1,\mathbf{m}_1} \geq \mathcal{T}_{\Gamma_2,\mathbf{m}_2} \textit{ for } \mathbf{m}_2 \equiv r\right\}$$

where the bond dimensions $\mathbf{m}_1$ *are specified as in Figure 3.3(a).*

3.2.2 Pehomogeneous tensor spaces and the castling transform

Motivated by Proposition 3.2.2, we will now review some facts from the theory of *prehomogeneous tensor spaces*. We will only present a small part of this theory and restrict ourselves to the results that we will use in this thesis. For a general introduction to the representation theory of linear algebraic groups, we refer to [Bor91]. Prehomogeneous tensor spaces have been studied extensively in, for example, [SK77, Kim02, Man13, Ven19, DMW22]. We start by defining the main concept of this section.

Definition 3.2.6. *Let G be a linear algebraic group acting on a vector space V. The space V is* prehomogeneous *for the action of G if the action has a dense orbit, that is, if there exists $v \in V$ such that $\overline{G.v} = V$.*

Recall that Theorem 1.2.4 ensures that if an orbit is dense in the Zariski topology, then it is also dense in the Euclidean topology. In particular, if V is prehomogeneous for the action of G and the orbit of $v \in V$ dense, then $G.v$ is Zariski-open in V and consequently dense in the Euclidean topology. Moreover, it is unique, and it coincides with the orbit of any generic element.

Note that if a group G acts on a vector space V, then this induces an action on the dual space V^* via $g.f(-) = f(g^{-1}.-)$ for all $g \in G$ and $f \in V^*$. This is often called the *contragredient action* or *dual action*. For general linear groups over the complex numbers (or more generally, for so-called *reductive groups*), prehomogeneity carries over to the contragredient action [Kim02, Propositions 2.21 & 7.40].

Theorem 3.2.7. *Let $G = \times_{i=1}^{k} \mathrm{GL}(V_i)$ be a product of general linear groups and W any finite-dimensional vector space. Consider the natural action of G on a space $V = \bigotimes_{i=1}^{k} V_i \otimes W$. Then V is prehomogeneous for the action of G if and only if V^* is prehomogeneous for the contragredient action of G.*

In this section, we are interested in tensor spaces $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ which are prehomogeneous for the action of $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$ on the first two factors. A fundamental ingredient in the theory of prehomogeneous spaces is the notion of *castling transform*, introduced in [SK77] and extensively used in the geometric study of prehomogeneous tensor spaces, see, e.g., [Man13, Ven19, DMW22]. To present the proof of the castling transform, it is handy to introduce the *Grassmanian of m-planes*.

Let $Gr(m, V) \subset \bigwedge^m V$ be the Grassmannian of m-planes in V, that is, the set of $v_1 \wedge \cdots \wedge v_m$ such that $v_1 \ldots v_m$ are linearly independent. Up to scalar prefactors, we can identify the element $v_1 \wedge \cdots \wedge v_m$ with the m-dimensional space E it spans. Note that if a group G acts on V, then it also acts on $Gr(m, V)$ by mapping $E = \mathrm{span}(v_1 \ldots v_m)$ to $g.E = \mathrm{span}(g.v_1 \ldots g.v_m)$. Similarly, the action of G on V induces an action on $Gr(m, V^*)$ by using the contragredient action. We note that Grassmanians, in general, are not vector spaces. However, one can, in a natural way, give them the structure of an algebraic variety, see, for example [Lan17, Section 2.3.3]. The following simple observation lies at the core of the castling transform.

Lemma 3.2.8. *Let V be a vector space of dimension v and $m \leq \mathrm{v}$. There is a natural bijective map $Gr(m, V) \to G(\mathrm{v} - m, V^*)$ mapping $E \mapsto E^\perp$. Moreover, if G acts on V, then this map commutes with the group action, that is, $g.E^\perp = (g.E)^\perp$.*

With that, we are ready to state the result called castling transform which will be an important tool throughout this thesis.

Theorem 3.2.9. *Let G be a linear algebraic group acting on a vector space V of dimension v and let $m \in \mathbb{N}$. Then the space $V \otimes \mathbb{C}^m$ is prehomogeneous for the action of $G \times \mathrm{GL}(\mathbb{C}^m)$ if and only if $V^* \otimes \mathbb{C}^{\mathrm{v}-m}$ is prehomogeneous for the action of $G \times \mathrm{GL}(\mathbb{C}^{\mathrm{v}-m})$.*

Proof. Fix a basis $\mathrm{e}_1 \ldots \mathrm{e}_m$ of $\mathbb{C}^m$ and let $\Omega \subset V \otimes \mathbb{C}^m$ be the subset of elements of the form $T = \mathrm{e}_1 \otimes M_1 + \cdots + \mathrm{e}_m \otimes M_m$ such that the M_i are linearly independent. Consider the surjective, Zariski-continuous map $\phi : \Omega \to Gr(m, V), T \mapsto \mathrm{span}(M_1 \ldots M_m)$ commuting with the action of G. If the orbit of an element $v \in V$ is dense, then the orbit of $\phi(v)$ must be dense as well. On the other hand, it is clear that $\mathrm{GL}(\mathbb{C}^m)$ acts transitively on $\phi^{-1}(E)$ for all E in $Gr(k, E)$.

Hence, prehomogeneity of the action of G on $Gr(m,V)$ implies prehomogeneity of V under $G \times \mathrm{GL}(\mathbb{C}^m)$.

Using Lemma 3.2.8 and applying the same line of arguments to $G(\mathrm{v}-m, V^*)$ yields the claim. □

Corollary 3.2.10. *Let $b \le aw$. The space $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ is prehomogeneous for the action of $\mathrm{GL}(\mathbb{C}^a) \otimes \mathrm{GL}(\mathbb{C}^b)$ if and only if the space $\mathbb{C}^{wa-b} \otimes \mathbb{C}^a \otimes \mathbb{C}^w$ is prehomogeneous for the action of $\mathrm{GL}(\mathbb{C}^{wa-b}) \otimes \mathrm{GL}(\mathbb{C}^a)$.*

Proof. This is a direct consequence of Theorem 3.2.7 and Theorem 3.2.9. □

In fact, one can use Corollary 3.2.10 to derive a condition that only depends on the involved dimensions. For that, define $\lambda_w = \frac{w+\sqrt{w^2-4}}{2}$ for every $w \ge 2$. The central result of this section is the following.

Proposition 3.2.11. *Let $w \ge 2$ and let a, b be integers such that $\lambda_w a < b$. Then, $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ has a dense $(\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b))$-orbit.*

In preparation to prove Proposition 3.2.11, we need to introduce some technical results. First, note that we have $\lambda_2 = 1$, $\lambda_w \in (w-1, w)$ for $w \ge 3$ and $\lambda_w^{-1} = \frac{w-\sqrt{w^2-4}}{2}$. In particular, λ_w and λ_w^{-1} are the two roots of the equation $\lambda^2 - w\lambda + 1 = 0$. For every $w \ge 2$, define recursively the generalized Fibonacci sequence

$$z_0^{(w)} = 0, \;\; z_1^{(w)} = 1, \;\; z_{p+1}^{(w)} = w \cdot z_p^{(w)} - z_{p-1}^{(w)}.$$

By resolving the recursion, one obtains

$$z_p^{(2)} = p, \quad z_p^{(w)} = \frac{\lambda_w^p - \lambda_w^{-p}}{\sqrt{w^2-4}} \quad \text{for } w \ne 2.$$

We record an immediate fact, which will be useful multiple times throughout:

Lemma 3.2.12. *For every $w \ge 2$ and $p \ge 0$, we have*

$$\det \begin{pmatrix} z_{p+1}^{(w)} & z_{p+2}^{(w)} \\ z_p^{(w)} & z_{p+1}^{(w)} \end{pmatrix} = {z_{p+1}^{(w)}}^2 - z_{p+2}^{(w)} \cdot z_p^{(w)} = 1.$$

Proof. The proof is by induction on p. If $p = 0$, then

$$\det \begin{pmatrix} z_{p+1}^{(w)} & z_{p+2}^{(w)} \\ z_p^{(w)} & z_{p+1}^{(w)} \end{pmatrix} = \det \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} = 1.$$

If $p \geq 1$, we have

$$\det \begin{pmatrix} z_{p+1}^{(w)} & z_{p+2}^{(w)} \\ z_p^{(w)} & z_{p+1}^{(w)} \end{pmatrix} = {z_{p+1}^{(w)}}^2 - z_{p+2}^{(w)} \cdot z_p^{(w)} =$$

$$(w \cdot z_p^{(w)} - z_{p-1}^{(w)}) z_{p+1}^{(w)} - (w \cdot z_{p+1}^{(w)} - z_p^{(w)}) z_p^{(w)} =$$

$$w \cdot z_p^{(w)} \cdot z_{p+1}^{(w)} - z_{p-1}^{(w)} \cdot z_{p+1}^{(w)} - w \cdot z_{p+1}^{(w)} \cdot z_p^{(w)} + {z_p^{(w)}}^2 =$$

$${z_p^{(w)}}^2 - z_{p-1}^{(w)} \cdot z_{p+1}^{(w)} = 1.$$

□

We also prove the following technical result, which appears in [Kac80] without proof.

Lemma 3.2.13. *Let $w \geq 2$ and let $a, b \in \mathbb{N}$ with $\lambda_w a < b \leq wa$. Then there exist unique $\alpha, \beta \geq 0$, $\beta \neq 0$, and $p \geq 1$ such that*

$$a = z_p^{(w)} \alpha + z_{p+1}^{(w)} \beta$$
$$b = z_{p+1}^{(w)} \alpha + z_{p+2}^{(w)} \beta.$$

Proof. In the case $w = 2$, we have $z_p^{(w)} = p$, and the result is straightforward.

Consider the case $w > 2$. Define recursively the sequence

$$c_0 = b, \quad c_1 = a, \quad c_s = w c_{s-1} - c_{s-2}.$$

We show by induction that $c_s > \lambda_w c_{s+1}$, which in particular implies that c_s is strictly decreasing because $\lambda_w > 1$. The hypothesis guarantees this is true for $s = 0$. If $s \geq 1$, the induction hypothesis guarantees $c_{s-1} > \lambda_w c_s$ so that

$$\lambda_w c_{s-1} > \lambda_w^2 c_s = (w \lambda_w - 1) c_s = w \lambda_w c_s - c_s$$

and therefore $c_s > \lambda_w (w c_s - c_{s-1}) = \lambda_w c_{s+1}$, as desired.

Since c_s is strictly decreasing, there exists $s^* \geq 0$ such that $c_{s^*+1} \leq 0$. Define $\beta = c_{s^*}$ and $\alpha = c_{s^*-1} - w c_{s^*} = -c_{s^*+1}$. In particular, $\beta > 0$ and $\alpha \geq 0$. An immediate induction argument shows

$$c_{s^*-\ell} = \alpha z_\ell^{(w)} + \beta z_{\ell+1}^{(w)}.$$

Clearly, this is true if $\ell = 0, 1$. If $\ell \geq 2$, we have

$$\begin{aligned} c_{s^*-\ell} &= w c_{s^*-\ell-1} - c_{s^*-\ell-2} = \\ &= w \left(\alpha z_{\ell-1}^{(w)} + \beta z_\ell^{(w)} \right) - \left(\alpha z_{\ell-2}^{(w)} + \beta z_{\ell-1}^{(w)} \right) = \\ &= \alpha \left((w z_{\ell-1}^{(w)} - z_{\ell-2}^{(w)} \right) + \beta \left(w z_\ell^{(w)} - z_{\ell-1}^{(w)} \right) = \alpha z_\ell^{(w)} + \beta z_{\ell+1}^{(w)}. \end{aligned}$$

Setting $p = s^* - 1$, we obtain

$$\begin{aligned} b = c_0 = c_{s^*-(p+1)} &= \alpha z_{p+1}^{(w)} + \beta z_{p+2}^{(w)} \\ a = c_1 = c_{s^*-p} &= \alpha z_p^{(w)} + \beta z_{p+1}^{(w)}. \end{aligned}$$

This shows the existence of α, β and p as desired.

To show uniqueness, suppose

$$\begin{aligned} a &= \alpha' z_{p'}^{(w)} + \beta' z_{p'+1}^{(w)} \\ b &= \alpha' z_{p'+1}^{(w)} + \beta' z_{p'+2}^{(w)} \end{aligned}$$

for some p', $\alpha' \geq 0$ and $\beta' > 0$. We first show that $p = p'$ must hold. Assume by contradiction $p \neq p'$ and without loss of generality consider the case $p' > p$, equivalent to the condition $p' \geq p + 1$. With a similar induction argument as before, we observe that for $s \leq p' + 1$,

$$c_s = \alpha' z_{p'-(s-1)}^{(w)} + \beta' z_{p'+1-(s-1)}^{(w)}.$$

By assumption, this is true for $s = 0, 1$. For $s \geq 2$, we observe that

$$\begin{aligned} c_s &= wc_{s-1} - c_{s-2} \\ &= w\left(\alpha' z_{p'-(s-2)}^{(w)} + \beta' z_{p'+1-(s-2)}^{(w)}\right) - \left(\alpha' z_{p'-(s-3)}^{(w)} + \beta' z_{p'+1-(s-3)}^{(w)}\right) \\ &= \alpha' z_{p'-(s-1)}^{(w)} + \beta' z_{p'+1-(s-1)}^{(w)} \end{aligned}$$

In particular, setting $s = s^* + 1 = p$, we obtain

$$\begin{aligned} 0 \geq c_{s^*+1} &= \alpha' z_{p'-s^*}^{(w)} + \beta' z_{p'+1-s^*}^{(w)} \\ &= \alpha' z_{p'-p-1}^{(w)} + \beta' z_{p'-p}^{(w)}. \end{aligned}$$

By assumption, $\beta' > 0$ and $\alpha, z_{p'-p-1}^{(w)} \geq 0$. Therefore, $c_{s^*+1} \leq 0$ implies that $z_{p'-p}^{(w)} = 0$, that is $p = p'$. This contradicts the condition $p' \geq p + 1$. Therefore, we deduce $p = p'$.

Since $p = p'$, we have that (α, β) and (α', β') satisfy

$$\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} z_{p+1}^{(w)} & z_{p+2}^{(w)} \\ z_p^{(w)} & z_{p+1}^{(w)} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} z_{p+1}^{(w)} & z_{p+2}^{(w)} \\ z_p^{(w)} & z_{p+1}^{(w)} \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}.$$

By Lemma 3.2.12, we deduce $(\alpha, \beta) = (\alpha', \beta')$. ☐

Lemma 3.2.12 and Lemma 3.2.13 allow us to show that the tensor space $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ is prehomogeneous for the action of $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$ whenever $\lambda_w a < b$.

Proof of Proposition 3.2.11. First, assume that $b \geq wa$. Considering the space $\mathbb{C}^b \otimes (\mathbb{C}^a \otimes \mathbb{C}^w)$ as a space of $b \times aw$ matrices, it is clear that any full rank matrix has a dense orbit under the action of $\mathrm{GL}(\mathbb{C}^b)$. In particular, the space is $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$-prehomogeneous.

Now, take any $b \geq \lambda_w a$ and write

$$a = z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta, \quad b = z_{p+1}^{(w)}\alpha + z_{p+2}^{(w)}\beta$$

as in Lemma 3.2.13.

By appying Corollary 3.2.10 recursively p times, we see that $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ is prehomogeneous for $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$ if and only if $\mathbb{C}^{a'} \otimes \mathbb{C}^{b'} \otimes \mathbb{C}^w$ is prehomogeneous for $\mathrm{GL}(\mathbb{C}^{a'}) \times \mathrm{GL}(\mathbb{C}^{b'})$ where

$$\begin{aligned} a' &= z_0^{(w)}\alpha + z_1^{(w)}\beta = \beta \\ b' &= z_1^{(w)}\alpha + z_2^{(w)}\beta = \alpha + w\beta. \end{aligned}$$

Observing that $b' \geq wa'$ finishes the proof. □

Remark 3.2.14. *The proof of Proposition 3.2.11 gives a recursive way of constructing elements with dense orbit in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ by choosing a full rank matrix in $\mathbb{C}^{a'} \otimes \mathbb{C}^{b'} \otimes \mathbb{C}^w$ and applying Lemma 3.2.8. We note that we do not know about a closed formula for an element with dense orbit in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^w$ for general $\lambda_w a \leq b$ when $w \geq 3$.*

3.2.3 Representation theory of quivers and invariants

The quantum max-flow in the bridge graph is related to the representation theory of the *Kronecker quiver*. We refer to [DW17] for a comprehensive exposition of the representation theory of quivers. We will outline a series of results from [Sch91, DZ01, DW00] which will be crucial for certain cases of the quantum max-flow in the bridge graph.

Definition 3.2.15. *A quiver is a finite directed graph $Q = (Q_0, Q_1, t, s)$, where Q_0 is the set of vertices and Q_1 is the set of arrows. Here, the t and s are maps specifying the source resp. target of an arrow e, that is, $s(e) = i$ and $t(e) = j$ for an arrow $e = (i, j)$.*

Example 3.2.16. *The Kronecker quiver $\mathcal{K}_w$ is the quiver that has two vertices α and β and w arrows from α to β. In other words, $s(e) = \alpha$ and $t(e) = \beta$ for all arrows e. See Figure 3.4 for a visualization.*

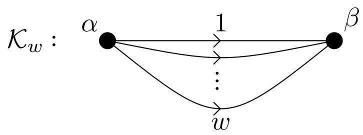


Figure 3.4: The Kronecker quiver $\mathcal{K}_w$.

Quivers are often used to study several linear maps simultaneously. This is done by introducing *representations* of quivers.

Definition 3.2.17. *A representation of a quiver Q is a pair $V = ((V_i)_{i\in Q_0}, (V_e)_{e\in Q_1})$. For each $i \in Q_0$, V_i is a finite-dimensional vector space and if $e = (i,j)$ is an arrow in Q_1 from vertex i to vertex j, then $V_e : V_i \to V_j$ is a linear map. We associate to a quiver representation V its dimension vector* $\mathrm{v} = \dim V = (\dim V_i)_{i\in Q_0}$.

For a quiver Q with fixed dimension vector v, its space of representations is

$$\mathcal{R}(Q,\mathrm{v}) = \bigoplus_{e\in Q_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)})$$

where $\dim(V_i) = \mathrm{v}_i$.

Example 3.2.18. *For $a, b \geq 1$, the space of representations of $\mathcal{K}_w$ with dimension vector (a,b) can be identified with the space of w-tuples of linear maps $\mathbb{C}^a \to \mathbb{C}^b$, in other words, $\mathcal{R}(\mathcal{K}_w, (a,b)) \cong A \otimes B \otimes W$ with $\dim A = a$, $\dim B = b$ and $\dim W = w$. For a fixed basis of $\mathrm{e}_1, \dots, \mathrm{e}_w$ of W^*, the linear map on the arrow j is $T(\mathrm{e}_j) \in A \otimes B$.*

Let Q be a quiver and v a dimension vector of Q. The group $\mathrm{GL}(\mathrm{v}) = \times_{i\in Q_0} \mathrm{GL}(\mathbb{C}^{\mathrm{v}_i})$ acts on the space of representations of Q with dimension vector v: Let $V = ((V_i)_{i\in Q_0}, (V_e)_{e\in Q_1})$ be such a representation. Then $g = (g_i)_{i\in Q_0}$ acts on it by simultaneous basis change, that is,

$$(V, g) \mapsto ((V_i)_{i\in Q_0}, (g_{s(e)} V_e g_{t(e)}^{-1})_{e\in Q_1}).$$

In particular, $\times_{i\in Q_0} \mathrm{GL}(\mathbb{C}^{\mathrm{v}_i})$ acts on the coordinate ring of $\mathcal{R}(Q,\mathrm{v})$, that is, the set of polynomials on $\mathcal{R}(Q,\mathrm{v})$: If $f \in \mathbb{C}[\mathcal{R}(Q,\mathrm{v})]$ is a polynomial, then $g \in \times_{i\in Q_0} \mathrm{GL}(\mathbb{C}^{\mathrm{v}_i})$ acts via $g.f(-) = f(g^{-1}.-)$.

Definition 3.2.19. *Let Q be a quiver with dimension vector* v*. The ring of invariants* $\mathrm{I}(Q,\mathrm{v})$ *for the quiver Q with respect to* v *is the set of $f \in \mathbb{C}[\mathcal{R}(Q,\mathrm{v})]$ invariant under the action of any $g \in \times_{i\in Q_0} \mathrm{GL}(\mathbb{C}^{\mathrm{v}_i})$. Its ring of semi-invariants* $\mathrm{SI}(Q,\mathrm{v})$ *is the ring of polynomial functions invariant under the natural action of any $g \in \times_{i\in Q_0} \mathrm{SL}(\mathbb{C}^{\mathrm{v}_i})$.*

Most of the time we will be interested in semi-invariants. It is clear that the action of $\times_{i\in Q_0} \mathrm{SL}(\mathbb{C}^{\mathrm{v}_i})$ does not change the degree of a monomial. Consequently, $\mathrm{SI}(Q,\mathrm{v})$ is a graded ring

$$\mathrm{SI}(Q,\mathrm{v}) = \bigoplus_{\delta\in\mathbb{N}} \mathrm{SI}(Q,\mathrm{v})^{[\delta]}$$

where $\mathrm{SI}(Q,\mathrm{v})^{[\delta]}$ is the component of degree δ. The ring of semi-invariants for the Kronecker quiver is directly related to the quantum max-flow problem.

Example 3.2.20. *For the Kronecker quiver $\mathcal{K}_w$, one can construct a semi-invariant in the following way. Recall that $\mathcal{R}(\mathcal{K}_w,(a,b)) \cong A \otimes B \otimes W$. Pick spaces B' and A' and matrices $Z_1 \dots Z_w \in B' \otimes A'$ such that for the dimension of the space, it holds that $ab' = a'b$. Then,*

$$T \mapsto \det(T(\mathrm{e}_1^*) \otimes Z_1 + \dots + T(\mathrm{e}_w^*) \otimes Z_w)$$

is a semi-invariant for the action of $\mathrm{GL}(A) \times \mathrm{GL}(B)$. This invariant is either the zero polynomial or a semi-invariant of homogeneous degree $\delta = ab' = a'b$, that is, an element of $SI(Q,\mathrm{v})^{[\delta]}$.

The following result is a special case of [SV01, Theorem 2.3] to Kronecker quivers and shows that the semi-invariants in Example 3.2.20 are essentially all.

Proposition 3.2.21. *Let $w \geq 3$ and let (a,b) be a dimension vector for the Kronecker quiver $\mathcal{K}_w$. Then, $\mathrm{SI}(\mathcal{K}_w,(a,b))^{[\delta]}$ is spanned by the polynomials on $A \otimes B \otimes W$ of the form*

$$T \mapsto \det(T(\mathrm{e}_1^*) \boxtimes Z_1 + \dots + T(\mathrm{e}_w^*) \boxtimes Z_w)$$

where $T \in A \otimes B \otimes W$, and $Z_1, \dots, Z_w$ are matrices of size $b_1 \times a_1$ such that $\delta = ab_1 = ba_1$. In particular, if $\mathrm{SI}(\mathcal{K}_w,(a,b))^{[\delta]} \neq 0$, then δ is a multiple of $\mathrm{lcm}(a,b)$.

We will next recall a result from [DW00, Theorem 3], see also [DW17, Theorem 10.7.8].

Proposition 3.2.22. *Let $w \geq 3$ and let (a,b) be a dimension vector for the Kronecker quiver $\mathcal{K}_w$. Let $\mathrm{SI}(\mathcal{K}_w,(a,b))^{[\delta]}$ be the component of the ring of semi-invariants $\mathrm{SI}(\mathcal{K}_w,(a,b))$ of degree δ. Then, $\mathrm{SI}(\mathcal{K}_w,(a,b))^{[\delta]} \neq 0$ for some δ implies $\mathrm{SI}(\mathcal{K}_w,(a,b))^{[\mathrm{lcm}(a,b)]} \neq 0$.*

Proof sketch: The proof follows from [DW00, Theorem 3], see also [DW17, Theorem 10.7.8]. Write $\delta = ab_1 = a_1 b$ for uniquely determined a_1, b_1. In particular, there exists δ_1 such that $a_1 = \delta_1 a_2$ and $b_1 = \delta_1 b_2$ where $a_2 = \frac{a}{\gcd(a,b)}$ and $b_2 = \frac{b}{\gcd(a,b)}$.

To match the notation from [DW00] and [DW17], recall that as a representation of $\mathrm{GL}(\mathrm{v})$, the ring of semi-invariants decomposes into weight spaces

$$\mathrm{SI}(Q,\mathrm{v}) = \bigoplus_{\sigma \in \mathbb{Z}^{Q_0}} \mathrm{SI}(Q,\mathrm{v})_\sigma$$

where

$$\mathrm{SI}(Q,\mathrm{v})_\sigma = \{f \in \mathcal{R}(Q,\mathrm{v}) : g.f = \prod_{i \in Q_0} \det(g_i)^{\sigma_i} f \text{ for all } g \in \mathrm{GL}(\mathrm{v})\},$$

see [DW17, Section 10] for an in-depth discussion. Using Proposition 3.2.21, we see that $\mathrm{SI}(\mathcal{K}_w,(a,b))^{[\delta]} = \mathrm{SI}(\mathcal{K}_w,(a,b))_\sigma$ for $\sigma = (b_1, a_1)$. In [DW00, Theorem 3], the authors show that the set

$$\Sigma(Q,\mathrm{v}) = \{\sigma : \mathrm{SI}(Q,\mathrm{v})_\sigma \neq 0\}$$

is saturated, that is, if $n\sigma \in \Sigma(Q, \mathrm{v})$ for any $n \in \mathbb{N}$, then also $\sigma \in \Sigma(Q, \mathrm{v})$. Consequently, if $\mathrm{SI}(\mathcal{K}_w, (a, b))_\sigma \neq 0$, then also $\mathrm{SI}(\mathcal{K}_w, (a, b))_{\sigma_2} \neq 0$ must hold. Passing to the degrees, this guarantees $\mathrm{SI}(\mathcal{K}_w, (a, b))^{[\delta_2]} \neq 0$, where $\delta_2 = ab_2 = a_2 b = \mathrm{lcm}(a, b)$. □

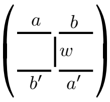
3.3 The quantum max-flow in the bridge graph

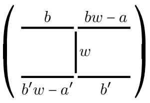
In this section, we will compute the quantum max-flow for the bridge graph for a wide range of parameters. In Section 3.3.1, we characterize the behavior of the quantum max-flow under the castling transform. This yields the main result of this section, Theorem 3.3.1, which will allow us to deduce the results which we summarized in the Table 3.1 and Theorem 3.1.1.

3.3.1 The castling transform and the quantum max-flow

The quantum max-flow behaves well under the castling transform in the following sense.

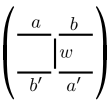
Theorem 3.3.1. *Let a, b, a', b', w be natural numbers such that $a \leq bw$ and $a' \leq b'w$. Then,*

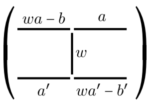
$ab' - \mathrm{QMaxFlow}$ 

$= b(b'w - a') - \mathrm{QMaxFlow}$ 

.

Moreover, if we have $b \leq aw$ and $b' \leq a'w$, then we also have

$ab' - \mathrm{QMaxFlow}$ 

$= (wa - b)a' - \mathrm{QMaxFlow}$ 

.

In the following, assume $a \leq bw$ and $a' \leq b'w$. Given $(E, E') \in \mathrm{Gr}(a, B \otimes W) \times \mathrm{Gr}(a', B' \otimes W^*)$, consider the linear map $F^1_{E,E'}$ defined by

$$\begin{aligned} F^1_{E,E'} : E \otimes B'^* &\to [E' \otimes B^*]^* \\ e \otimes \beta' &\mapsto (e' \otimes \beta \mapsto \beta(e) \lrcorner \beta'(e')), \end{aligned}$$

and extended by linearity.

Proposition 3.3.2. *Let $a \leq bw$ and $a' \leq b'w$ and let $T \in A \otimes B \otimes W$ and $T' \in A' \otimes B' \otimes W^*$ be such that the induced maps*

$$T : A^* \to B \otimes W \text{ and } T' : A'^* \to B' \otimes W^*$$

are injective. Write $E_T = \mathrm{im}(T : A^* \to B \otimes W)$, $E_{T'} = \mathrm{im}(T : A'^* \to B' \otimes W^*)$. Then,

$$\mathrm{rank}(F_{T,T'}) = \mathrm{rank}(F^1_{E_T, E_{T'}})$$

and

$$\dim\ker(F_{T,T'}) = \dim\ker(F^1_{E_T,E_{T'}}).$$

Proof. Fixing bases, it is straightforward to verify that both linear maps can be represented by the same $ab' \times ba'$–matrix

$$\left(\sum_{k=1}^{w} T_{i_1,i_2,k}T'_{j_1,j_2,k}\right)^{(a,b'),(b,a')}_{(i_1,j_2),(i_2,j_1)=(1,1),(1,1)}$$

labeled by double indices. This clearly implies the claim. □

The following result shows that the construction of the map $F^1_{E,E'}$ is, in some sense, equivariant under castling.

Theorem 3.3.3. *For $a \le bw$ and $a' \le b'w$ let $(E,E') \in \mathrm{Gr}(a, B\otimes W)\times \mathrm{Gr}(a', B'\otimes W^*)$, so that $(E'^{\perp}, E^{\perp}) \in \mathrm{Gr}(b'w-a', B'^*\otimes W)\times \mathrm{Gr}(bw-a, B^*\otimes W^*)$. Then,*

$$\ker(F^1_{E,E'}) = \ker({F^1_{E^{\perp},E'^{\perp}}}^t),$$

regarded as a subspace of $(B\otimes W)\otimes B'^* \cong B\otimes(W^*\otimes B')^*$.

Proof. Notice $F^1_{E^{\perp},E'^{\perp}} : E^{\perp}\otimes B' \to [E'^{\perp}\otimes B]^*$, so ${F^1_{E^{\perp},E'^{\perp}}}^t : E'^{\perp}\otimes B \to [E^{\perp}\otimes B']^*$. In particular, the domain $E\otimes B'^*$ of $F^1_{E,E'}$ and the domain $E'^{\perp}\otimes B$ of ${F^1_{E^{\perp},E'^{\perp}}}^t$ can both be regarded as subspaces of $(B\otimes W)\otimes B'^* \cong (B'\otimes W^*)^*\otimes B$.

Let $\theta \in \ker(F^1_{E,E'}) \subseteq (B\otimes W)\otimes B'^*$. First, we show that $\theta \in E'^{\perp}\otimes B$ regarded as a subspace of $(B'\otimes W^*)^*\otimes B$. Indeed, since $\theta \in \ker(F^1_{E,E'})$, the element $F^1_{E,E'}(\theta) \in [E'\otimes B^*]^*$ is identically 0 as a map $F^1_{E,E'}(\theta) : E'\otimes B^* \to \mathbb{C}$. Since $F^1_{E,E'}(\theta)$ is defined by the contraction of θ against the elements of $E'\otimes B^*$, we deduce $\theta \in (E'\otimes B^*)^{\perp} = E'^{\perp}\otimes B$. In particular ${F^1_{E^{\perp},E'^{\perp}}}^t$ is well defined on θ.

Now observe ${F^1_{E^{\perp},E'^{\perp}}}^t(\theta) = 0$. Indeed, ${F^1_{E^{\perp},E'^{\perp}}}^t(\theta) \in [E^{\perp}\otimes B']^*$ is the map $E^{\perp}\otimes B' \to \mathbb{C}$ defined by contraction of θ against the elements of $E^{\perp}\otimes B'$. By assumption $\theta \in E\otimes B'^* = (E^{\perp}\otimes B')^{\perp}$, hence this contraction is identically 0.

This shows the inclusion $\ker(F^1_{E,E'}) \subseteq \ker({F^1_{E^{\perp},E'^{\perp}}}^t)$. Reversing the roles of (E,E') with $(E^{\perp},E'^{\perp})$, one has the other inclusion, hence equality. □

Theorem 3.3.3 allows us to prove the main result of this section, namely Theorem 3.3.1.

Proof of Theorem 3.3.1. Let $a \le bw$. It is clear that we can choose tensors $T \in A\otimes B\otimes W$ and $T' \in A'\otimes B'\otimes W^*$ with the property that they maximize the quantum flow and that the flattening maps

$$T\colon A^* \to B\otimes W \text{ and } T'\colon A'^* \to B'\otimes W^*$$

are injective. This can be seen from the fact that these are open conditions in the Zariski topology. By Proposition 3.3.2 and Theorem 3.3.3, we obtain

$$
\begin{aligned}
ab' - \mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) &= \dim(\ker(F_{T,T'})) \\
&= \dim(\ker(F^{1}_{E_T,E_{T'}})) \\
&= \dim(\ker((F^{1}_{E_T^{\perp},E_{T'}^{\perp}})^{\mathbf{t}})) \\
&= \dim(\ker(F_{S,S'})) \geq b(b'w - a') - \mathrm{QMaxFlow}\left(\begin{array}{c|c} b & bw-a \\ & w \\ b'w-a' & b' \end{array}\right)
\end{aligned}
$$

where the tensors $S \in B^* \otimes \mathbb{C}^{bw-a} \otimes W^*$ and $S' \in \mathbb{C}^{b'w-a'} \otimes B'^* \otimes W$ are chosen so that the image of their flattening maps are $E_T^{\perp}$ and $E_{T'}^{\perp}$, respectively. The same argument, in the opposite direction, yields equality.

Now, assume $b \leq wa$ and $b' \leq wa'$. Set $x = wa - b, y = a, x' = wa' - b'$ and $y' = a'$ so that $x \leq wy$ and $x' \leq wy'$. Then, we obtain

$$
\begin{aligned}
&(wa-b)a' - \mathrm{QMaxFlow}\left(\begin{array}{c|c} wa-b & a \\ & w \\ a' & wa'-b' \end{array}\right) = xy' - \mathrm{QMaxFlow}\left(\begin{array}{c|c} x & y \\ & w \\ y' & x' \end{array}\right) = \\
&y(y'w-x') - \mathrm{QMaxFlow}\left(\begin{array}{c|c} y & yw-x \\ & w \\ y'w-x' & y' \end{array}\right) = ab' - \mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right). \qquad \square
\end{aligned}
$$

3.3.2 Calculating the quantum max-flow

Theorem 3.3.1 enables us to calculate the quantum max-flow in a wide range of cases. In this section, we compute the max-flow in a series of results: Theorem 3.3.6, Theorem 3.3.7, Theorem 3.3.8, and Corollary 3.3.9. These are obtained via arithmetic arguments: starting from pairs $(a,b),(a',b')$, one reduces via castling to "easier pairs" and obtains the result using Theorem 3.3.1. More precisely, we will use the following observations about the behavior of tuples (a,b) under castling operations which we visualize in Figure 3.5.

Lemma 3.3.4. *Recall the regions $\mathbf{U}_w, \mathbf{V}_w, \mathbf{W}_w, \mathbf{X}_w$ and $\mathbf{Y}_w$ from Section 3.1 and consider a tuple $(a,b) \in \mathbb{N}^2$. As long as $a_n w \geq b_n$, define recursively*

$$
(a_0, b_0) = (a,b), (a_{n+1}, b_{n+1}) = (wa_n - b_n, a_n). \tag{3.2}
$$

(a) Let $(a,b) \in \mathbf{Y}_w$. Then, $(b, bw-a) \in X_w$, that is, with one castling step, we move to the region $\mathbf{X}_w$.

(b) Let $(a,b) \in \mathbf{X}_w$ and write

$$
a = z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta, \; b = z_{p+1}^{(w)}\alpha + z_{p+2}^{(w)}\beta.
$$

as in Lemma 3.2.13. Then, for all $n = 1, \dots, p-1$, we have $(a_n, b_n) \in \mathbf{X}_w$ as well as $(a_p, b_p) \in \mathbf{Y}_w$. That is, with p castling steps, we reach $\mathbf{Y}_w$ and stay until then in $\mathbf{X}_w$.

(c) *Let $(a,b) \in \mathbf{W}_w$ and consider the recursive sequence in Equation (3.2). Then, for some $n \in \mathbb{N}$, we have $(a_n, b_n) \in \mathbf{U}_w \cup \mathbf{V}_w$. Moreover, we have $(b, bw - a) \in \mathbf{W}_w$, in other words, castling "in the other direction" lets us stay in $\mathbf{W}_w$.*

(d) *Let $(a,b) \in \mathbf{U}_w \cup \mathbf{V}_w$. Then, $(b, wb - a) \in \mathbf{W}_w$ and $(a, wa - b) \in \mathbf{U}_w \cup \mathbf{V}_w$. In particular, for $w \geq 4$ and $(a,b) \in V_w$, it holds $(a, wa - b) \in \mathbf{U}_w$ and similarly, for $(a,b) \in U_w$, it holds $(a, wa - b) \in \mathbf{V}_w$.*

For any fixed w, we say that two pairs (a,b) and (a', b') are *castling equivalent* if there is a sequence as in Equation (3.2) with $(a_0, b_0) = (a,b)$ and $(a_n, b_n) = (a', b')$.

Proof. Note that in all cases, $a \neq 0$. Start with (a) and let $(a,b) \in \mathbf{Y}_w$. Then, $bw - a < bw$ as well as $(w - \lambda_w)b \geq (w^2 - w\lambda_w)a \geq a$ which shows $(bw - a, b) \in \mathbf{X}_w$. For (b), we observe that if $(a,b) \in \mathbf{X}_w$ can be written as

$$a = z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta,\ b = z_{p+1}^{(w)}\alpha + z_{p+2}^{(w)}\beta$$

as in Lemma 3.2.13, then $wa - b = z_{p-1}^{(w)}\alpha + z_p^{(w)}\beta$ and $a = z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta$. Since $z_i^{(w)} \neq 0$ as long as $i \neq 0$, the claim follows by applying this observation p times. For (c), assume $(a,b) \in \mathbf{W}_w$. We have $a = (\lambda_w w - \lambda_w^2)a = \lambda_w(wa - \lambda_w a) < \lambda_w(wa - b)$ and $wa - b \leq wa - (w-1)a = a$, in other words, $(wa - b, a) \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$. Moreover, we see that, by assumption, the second coordinate strictly decreases in the castling step, that is, $a < b$. This guarantees that the sequence cannot stay in $\mathbf{W}_w$ forever and, in particular, that there is an n such that $(a_n, b_n) \in \mathbf{U}_w \cup \mathbf{V}_w$. Finally, for $(a,b) \in \mathbf{U}_w \cup \mathbf{V}_w$, we clearly have $(w-1)b = wb - b \leq wb - a$. From (a) and (b), we know that $(b, wb - a) \notin \mathbf{X}_w \cup \mathbf{Y}_w$ and therefore, $(b, wb - a) \in \mathbf{W}_w$. Moreover, if $w \geq 4$ and $(a,b) \in \mathbf{V}_w$, we have $a = wa - (w-1)a \leq wa - b$ as well as $wa - b \leq wa - \lambda_{w-1}a = (w - \lambda_{w-1})a \leq \lambda_{w-1}a$ where the last inequality follows from the fact $\lambda_{w-1} \geq w - 2$. The case $(a,b) \in \mathbf{U}_w$ is similar. This finishes (d). □

Definition 3.3.5. *Let $(a,b) \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$ and consider the sequence (a_n, b_n) in Equation (3.2). Call the minimal n_0 such that $(a_{n_0}, b_{n_0}) \in \mathbf{U}_w \cup \mathbf{V}_w$ the* castling depth *of (a,b), denoted* $\text{depth}(a,b)$.

The existence of such a minimal n_0 is guaranteed by Lemma 3.3.4. In particular, for all $(a,b) \in \mathbf{U}_w \cup \mathbf{V}_w$, we have $\text{depth}(a,b) = 0$.

We will now calculate the quantum max-flow in the bridge graph. We start with the easiest case when $(a,b) \in \mathbf{Y}_w$, that is, $b \geq aw$. This corresponds to the orange cells in Table 3.1.

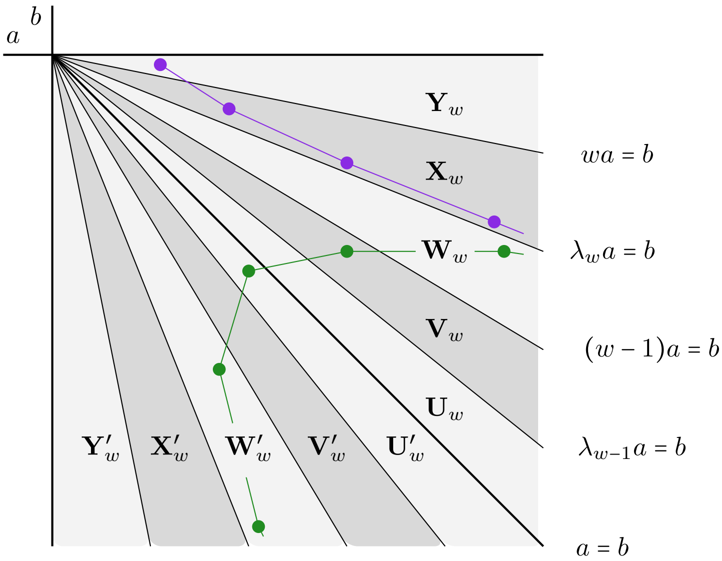


Figure 3.5: The regions $\mathbf{U}_w, \mathbf{V}_w, \mathbf{W}_w, \mathbf{X}_w$ and $\mathbf{Y}_w$ with their counterparts "on the other side" of the line $a = b$. In purple, we visualize the cases (a) and (b) in Lemma 3.3.4: Any pair in $\mathbf{Y}_w$ castles directly to the $\mathbf{X}_w$ region. For any pair in the $\mathbf{X}_w$ region, we eventually land in the $\mathbf{Y}_w$ region by castling. Cases (c) and (d) are visualized in green. Here, we see that any pair in $\mathbf{W}_w$ castles eventually to a pair in $\mathbf{U}_w \cup \mathbf{V}_w$ and then "flips side" to $\mathbf{U}'_w \cup \mathbf{V}'_w$.

Theorem 3.3.6. *Let a, b, a', b', w be natural numbers such that $aw \le b$. We have*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \begin{cases} ab' & \text{if } (a', b') \notin \mathbf{Y}_w, \\ aa'w & \text{if } (a', b') \in \mathbf{Y}_w. \end{cases}$$

In other words, if T and T' are tensors realizing the quantum max-flow, the resulting linear map $F_{T,T'}$ has a no kernel if $b' \le a'w$ and a kernel of dimension $a \cdot (b' - a'w)$ if $b' \ge a'w$.

Proof. Let T be a tensor such that the map $T : (A \otimes W)^* \to B$ is injective. If $(a', b') \notin \mathbf{Y}_w$, let T' be a tensor such that the map $T' : B'^* \to W^* \otimes A'$ is injective. If $(a', b') \in \mathbf{Y}_w$, let T' be a tensor such that the map $T' : B'^* \to W^* \otimes A'$ is surjective. In both cases, we obtain the desired result. □

Next we consider the case $(a, b) \in \mathbf{W}_w \cup \mathbf{X}_w$ and $(a', b') \in \mathbf{U}_w \cup \mathbf{V}_w$, corresponding to the cyan cells in Table 3.1.

Theorem 3.3.7. *Let $(a, b) \in \mathbf{W}_w \cup \mathbf{X}_w$ and $(a', b') \in \mathbf{U}_w \cup \mathbf{V}_w$, that is, $(w-1)a \le b \le wa$ and $a' \le b' \le (w-1)a'$. Then, we have*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = ab'.$$

Proof. Since $ab' \le aa'(w-1) \le ba'$, it is clear the quantum min-cut, in this case, is ab'. Moreover since $b \le aw$ and $b' \le a'w$, we can apply Theorem 3.3.1 yielding

$$ab' - \text{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = (wa-b)a' - \text{QMaxFlow}\left(\begin{array}{c|c} wa-b & a \\ & w \\ a' & wa'-b' \end{array}\right). \tag{3.3}$$

We notice that $wa - b \le wa - (w-1)a = a$ and $wa' - b' \ge wa' - (w-1)a' = a'$ (see also Figure 3.5). This guarantees that the quantum max-flow with bond dimensions $(wa-b, a)$ and $(wa'-b', a')$ equals $(wa-b)a'$. In particular, the right-hand side of Equation (3.3) is 0, showing that the left-hand side is 0, as well. This concludes the proof. $\square$

Let us turn our attention to the case $(a, b) \in \mathbf{X}_w$. This is the most delicate case and the only one where we can prove the existence of sets of bond dimensions for which the quantum max-flow is strictly smaller than the quantum min-cut. From Theorem 3.3.7, we already know the quantum max-flow for $(a', b') \in \mathbf{U}_w \cup \mathbf{V}_w$. From Theorem 3.3.6, we know the quantum max-flow when $(a', b') \in \mathbf{Y}_w$. The following result provides an answer when $(a', b') \in \mathbf{W}_w \cup \mathbf{X}_w$. These are the purple cases of Table 3.1.

Theorem 3.3.8. *Let $(a, b) \in \mathbf{X}_w$, that is, $\lambda_w a < b \le wa$, and write*

$$a = z_p^{(w)}\alpha + z_{p+1}^{(w)}\beta, \quad b = z_{p+1}^{(w)}\alpha + z_{p+2}^{(w)}\beta$$

as in Lemma 3.2.13.

1. *If $(a', b') \in \mathbf{W}_w$, then*

$$\text{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \text{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = ab'.$$

2. *If $(a', b') \in \mathbf{X}_w$, write*

$$a' = z_{p'}^{(w)}\alpha' + z_{p'+1}^{(w)}\beta', \quad b' = z_{p'+1}^{(w)}\alpha' + z_{p'+2}^{(w)}\beta'.$$

   Then,

$$\text{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \begin{cases} ab' & \text{if } p' > p, \\ ab' - \beta\alpha' & \text{if } p = p', \\ a'b & \text{if } p' < p. \end{cases}$$

Proof. Consider the sequence (a_n, b_n) in Equation (3.2) consisting of castling equivalent pairs and define a similar sequence (a'_n, b'_n). We know that $a_p = \beta$ and $b_p = \alpha + w\beta$. If $(a', b') \in \mathbf{W}_w$, we certainly have $b'_p \leq wa'_p$ and consequently

$$ab' - \text{QMaxFlow}\left(\begin{array}{cc} a & b \\ & w \\ b' & a' \end{array}\right) = a_p b'_p - \text{QMaxFlow}\left(\begin{array}{cc} a_p & b_p \\ & w \\ b'_p & a'_p \end{array}\right) = 0$$

by Theorem 3.3.6. This shows the first claim.

Now, let $(a', b') \in \mathbf{X}_w$ and assume first that $p' > p$. By Lemma 3.3.4, we see that in this case, $wa'_p \geq b'_p$ holds. The same argument as for $(a', b') \in \mathbf{W}_w$ finishes this case. We note that the case $p' < p$ follows by symmetry.

Finally, assume $(a', b') \in \mathbf{X}_w$ and $p = p'$. In this case, applying Theorem 3.3.1 p times yields

$$ab' - \text{QMaxFlow}\left(\begin{array}{cc} a & b \\ & w \\ b' & a' \end{array}\right) = \beta(\alpha' + w\beta') - \text{QMaxFlow}\left(\begin{array}{cc} \beta & \alpha + w\beta \\ & w \\ \alpha' + w\beta' & \beta' \end{array}\right) = \beta\alpha'$$

where the last equality is Theorem 3.3.6. This finishes the proof. □

We conclude this section with an immediate consequence of the previous results, which allows us to compute the quantum max-flow when $(a, b) \in \mathbf{V}_w$ and $(a', b') \in \mathbf{U}_w$. These are the green cases of Table 3.1.

Corollary 3.3.9. *Let* $(a, b) \in \mathbf{V}_w$ *and* $(a', b') \in \mathbf{U}_w$. *Then,*

$$\text{QMaxFlow}\left(\begin{array}{cc} a & b \\ & w \\ b' & a' \end{array}\right) = \text{QMinCut}\left(\begin{array}{cc} a & b \\ & w \\ b' & a' \end{array}\right) = ab'.$$

Proof. Since $\mathbf{V}_w = \mathbf{X}_{w-1}$ and $\mathbf{U}_w = \mathbf{U}_{w-1} \cup \mathbf{V}_{w-1} \cup \mathbf{W}_{w-1}$, the result follows from Theorem 3.3.7 and Theorem 3.3.8 applied to the case $w - 1$. □

3.3.3 Partial solution of the remaining cases

In this section, we see a series of partial results that cover a great number of cases in which (a, b) and (a', b') are both in $\mathbf{W}_w$, in $\mathbf{V}_w$ or in $\mathbf{U}_w$.

Our first observation is that for pairs with different castling depths as defined in Definition 3.3.5, the quantum max-flow and the quantum min-cut in the bridge graph coincide.

Theorem 3.3.10. *Let* $(a, b), (a', b') \in \mathbf{W}_w$ *be such that* $\text{depth}(a, b) > \text{depth}(a', b')$. *Then,*

$$\text{QMaxFlow}\left(\begin{array}{cc} a & b \\ & w \\ b' & a' \end{array}\right) = \text{QMinCut}\left(\begin{array}{cc} a & b \\ & w \\ b' & a' \end{array}\right) = ab'.$$

Proof. Let $p = \mathrm{depth}(a', b')$. We apply iteratively Theorem 3.3.1 to obtain

$$ab' - \mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = a_p b'_p - \mathrm{QMaxFlow}\left(\begin{array}{c|c} a_p & b_p \\ & w \\ b'_p & a'_p \end{array}\right).$$

By assumption, $(a_p, b_p) \in \mathbf{W}_w$ and $(a'_p, b'_p) \in \mathbf{U}_w \cup \mathbf{V}_w$. By Theorem 3.3.7, we deduce that the right-hand side of the equation above is 0. So is the left-hand side, which provides the assertion. □

We can also calculate the quantum max-flow for some instances of (a, b) and (a', b') where both have the same castling depth by generalizing Corollary 3.3.9.

Lemma 3.3.11. *Fix a, b, w, a', b' such that $(a, b), (a', b') \in \mathbf{U}_w$ and let $\overline{w} \in \{2, \dots, w-1\}$. Assume that $a\overline{w} \leq b$ and $a'\overline{w} \geq b'$. Then,*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = ab'. \tag{3.4}$$

Proof. It follows from Theorem 3.3.6 that Equation (3.4) holds even when we replace the bridge width w by $\overline{w}$. This implies the claim. □

For example, recall the recursively defined sequence (a_n, b_n) in Equation (3.2) and define $(a', b') = (a, wa - b)$. It is clear that $\mathrm{depth}(a_n, b_n) = \mathrm{depth}(a'_n, b'_n)$ for all $n \in \mathbb{N}$. It turns out that we can calculate the quantum max-flow for these pairs.

Corollary 3.3.12. *Let w be even, $(a, b) \in \mathbf{U}_w \cup \mathbf{V}_w$ and consider (a_n, b_n) and (a'_n, b'_n) as before. Then,*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a_n & b_n \\ & w \\ b'_n & a'_n \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} a_n & b_n \\ & w \\ b'_n & a'_n \end{array}\right) = \min(a_n b'_n, a'_n b_n).$$

Proof. Using Theorem 3.3.1 it is clear that it suffices to show the claim for $n = 0$. There, both pairs are in $\mathbf{U}_w \cup \mathbf{V}_w$. Assuming $\frac{w}{2}a \leq b$, that is, $aw - b \leq \frac{w}{2}a$, the claim is exactly Lemma 3.3.11. The case $\frac{w}{2}a \geq b$ follows by symmetry. □

We conclude this section with a result that determines the quantum max-flow for proportional pairs of bond dimensions in the region $\mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$.

Theorem 3.3.13. *Let $w \geq 3$ and let $(a, b), (a', b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$ and suppose there exists $q \in \mathbb{Q}$ with $(a, b) = q(a', b')$. Then,*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = ab'.$$

Proof. The hypothesis guarantees $\dim(A \otimes B') = \dim(B \otimes A')$. In particular, the statement is equivalent to the fact that for some $T \in A \otimes B \otimes W$ and $T' \in A' \otimes B' \otimes W^*$, the flow map $F_{T,T'}$ has nonzero determinant.

If $(a,b) \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$, a generic tensor $T \in A \otimes B \otimes W$ is *semistable* for the action of $\mathrm{GL}(A) \otimes \mathrm{GL}(B)$ in the sense of geometric invariant theory, see, for example, [AKRS21, Theorem 4.1 & Corollary 4.10], [DKH21, DMW22, Proposition 1.3], or [DM21, Theorem 1.2]. In particular, semistability of generic elements guarantees that the ring of invariants

$$\mathbb{C}[A \otimes B \otimes W]^{\mathrm{SL}(A)\times \mathrm{SL}(B)}$$

is nontrivial. By the discussion of Section 3.2.3, this ring of invariants coincides with $\mathrm{SI}(\mathcal{K}_w, (a,b))$. By Proposition 3.2.21 together with Proposition 3.2.22, this ring has nontrivial elements in degree ab' and they arise as determinants

$$\det(T(\mathrm{e}_1) \boxtimes Z_1 + \cdots + T(\mathrm{e}_w) \boxtimes Z_w)$$

for certain, generic enough, matrices Z_j of size $b' \times a'$. Regarding the w-tuple $(Z_1, \ldots, Z_w)$ as a tensor in $A' \otimes B' \otimes W^*$, we see that $T(\mathrm{e}_1) \boxtimes Z_1 + \cdots + T(\mathrm{e}_w) \boxtimes Z_w$ can be identified with the flow map $F_{T,T'}$. Since this determinant is nonzero for generic enough T and $Z_1, \ldots, Z_w$, we conclude that $F_{T,T'}$ has full rank. This yields the desired result. □

3.3.4 Conjectural behavior of the quantum max-flow in the bridge graph

We expect the quantum max-flow to be always equal to the quantum min-cut with the only exception discussed in case ♢ of Theorem 3.1.1, which was analyzed in Theorem 3.3.8. This has been proved for a large number of cases, as shown in Table 3.1. When both (a,b) and (a',b') belong to $\mathbf{U}_w$, $\mathbf{V}_w$ or $\mathbf{W}_w$ and the results of Section 3.3.3 do not apply, calculating the quantum max-flow remains an open problem. We propose the following conjecture.

Conjecture 3.3.14. *For all $(a,b), (a',b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$ such that $ab' \leq a'b$ we have that*

$$\mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \mathrm{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = ab'.$$

In this section, we prove a reduction result showing that Conjecture 3.3.14 is equivalent to its specialization to the case $w = 3$.

Conjecture 3.3.15. *Let $w = 3$ and $(a,b), (a',b') \in \mathbf{U}_3 \cup \mathbf{V}_3 \cup \mathbf{W}_3$ such that $ab' \leq a'b$. Then,*

$$\mathrm{QMinCut}\left(\begin{array}{c|c} a & b \\ & 3 \\ b' & a' \end{array}\right) = \mathrm{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & 3 \\ b' & a' \end{array}\right) = ab'.$$

It is clear that Conjecture 3.3.14 implies Conjecture 3.3.15.

Using Lemma 3.3.4, one can reduce the computation of the max-flow for pairs in $\mathbf{V}_w$ and $\mathbf{W}_w$ to pairs in $\mathbf{U}_w$. This is recorded in the following result.

Lemma 3.3.16. *Let $w \geq 4$. If, for all $(a, b), (a', b') \in \mathbf{U}_w$,*

$$\operatorname{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \operatorname{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \min\{a'b, ab'\}, \tag{3.5}$$

then Equation (3.5) also holds for all $(a, b), (a', b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$.

Proof. The case $(a, b) \in \mathbf{U}_w \cup \mathbf{V}_w$ and $(a', b') \in \mathbf{W}_w$ follows fom Theorem 3.3.7, the case $(a, b) \in \mathbf{U}_w$ and $(a', b') \in \mathbf{V}_w$ from Corollary 3.3.9. Let $(a, b), (a', b') \in \mathbf{V}_w$. Then, using (d) from Lemma 3.3.4, we see that by castling once we reduce to the case $(a, b), (a', b') \in \mathbf{U}_w$. So assume $(a, b), (a', b') \in \mathbf{W}_w$. If they have different castling depth, the case is solved by Theorem 3.3.10. Else, we can castle both pairs simultaneously to $\mathbf{U}_w \cup \mathbf{V}_w$ and the claim follows by the preceding discussion. □

We can now easily complete the proof of the equivalence between Conjecture 3.3.15 and Conjecture 3.3.14.

Proposition 3.3.17. *If*

$$\operatorname{QMinCut}\left(\begin{array}{c|c} a & b \\ & 3 \\ b' & a' \end{array}\right) = \operatorname{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & 3 \\ b' & a' \end{array}\right)$$

holds for all $(a, b), (a', b') \in \mathbf{U}_3 \cup \mathbf{V}_3 \cup \mathbf{W}_3$, then

$$\operatorname{QMinCut}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right) = \operatorname{QMaxFlow}\left(\begin{array}{c|c} a & b \\ & w \\ b' & a' \end{array}\right)$$

holds for all $(a, b), (a', b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$.

Proof. We use induction on w. The case $w = 3$ coincides with the hypothesis of the claim. Suppose $w \geq 4$. Let $(a, b), (a', b') \in \mathbf{U}_w \cup \mathbf{V}_w \cup \mathbf{W}_w$. By Lemma 3.3.16, it suffices to consider the case $(a, b), (a', b') \in \mathbf{U}_w$. Now, $\mathbf{U}_w = \mathbf{U}_{w-1} \cup \mathbf{V}_{w-1} \cup \mathbf{W}_{w-1}$. Therefore the statement holds by the induction hypothesis. This concludes the proof. □

3.A Pedestrian-style proofs for the bridge 2 case

When $w = 2$, the regions $\mathbf{U}_2, \mathbf{V}_2$ are empty and the region $\mathbf{W}_2$ reduces to cases where $a = b$. Therefore, the only interesting behavior, in this case, is for $(a, b), (a', b') \in \mathbf{X}_2$. This is covered by Theorem 3.3.8 whose proof relies on the castling transform and on Theorem 3.3.1. In this appendix, we propose two alternative proofs for this specific case, which do not rely on castling transform. A deep understanding of this initial case might provide insights on how to obtain a proof for the $w = 3$ case of Conjecture 3.3.15 and in turn of Conjecture 3.3.14.

The first proof that we present in Appendix 3.A.1 relies on the fact that when $w = 2$, we know explicit examples of tensors $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^2$ having a dense $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$-orbit. This relies on the Kronecker classification of matrix pencils [Gan59, Chapter XIII], and more precisely on the results of [Pok86].

The second proof presented in Appendix 3.A.2 is a variation of the first one, where the explicit calculation of the rank of certain maps relies on the representation theory of $\mathrm{GL}(\mathbb{C}^2)$.

3.A.1 A proof using explicit tensors with dense orbit

The main tool for this section is the following result by Pokrzywa [Pok86]. Recall that $z_p^{(2)} = p$ for all $p \in \mathbb{N}$.

Lemma 3.A.1. *Let $(a, b) \in \mathbf{X}_2$, in other words, $a < b \leq 2a$. Write*

$$a = p\alpha + (p+1)\beta$$
$$b = (p+1)\alpha + (p+2)\beta$$

as in Lemma 3.2.13. Let $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^2$ be regarded as a 2-dimensional subspace of $\mathbb{C}^a \otimes \mathbb{C}^b \cong \mathrm{Mat}_{a \times b}$ via

$$T((\mathbb{C}^2)^*) = \{(\mathrm{id}_\alpha \boxtimes R_p(\xi_1, \xi_2) \oplus (\mathrm{id}_\beta \boxtimes R_{p+1}(\xi_1, \xi_2)) : \xi_1, \xi_2 \in \mathbb{C}\}$$

where

$$R_p(\xi_1, \xi_2) = \begin{bmatrix} \xi_1 & \xi_2 & & & \\ & \xi_1 & \xi_2 & & \\ & & \ddots & \ddots & \\ & & & \xi_1 & \xi_2 \end{bmatrix} \in \mathbb{C}^p \otimes \mathbb{C}^{p+1}$$

and id_α is an $\alpha \times \alpha$ identity matrix. Then the $(\mathrm{GL}_a \times \mathrm{GL}_b)$-orbit of T is dense in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^2$.

The proof of Lemma 3.A.1 can be obtained by computing directly the stabilizer of the tensor T in $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$ and using Proposition 1.2.7. The direct sum structure allows one to reduce to the case $\alpha = 0$, $\beta = 1$, for which the calculation is straightforward, see e.g. [CGL+21, Theorem 4.1].

Lemma 3.A.1, together with semicontinuity of matrix rank, guarantees that the quantum max-flow in the bridge graph equals the rank of the flow map for tensors T, T' having dense orbit. The rest of this section performs the calculation of this rank.

Fix $(a,b),(a',b') \in \mathbf{X}_2$ and write

$$\begin{aligned} a &= p\alpha + (p+1)\beta, & b &= (p+1)\alpha + (p+2)\beta \\ a' &= p'\alpha' + (p'+1)\beta', & b' &= (p'+1)\alpha' + (p'+2)\beta', \end{aligned}$$

as in Lemma 3.2.13. Let $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^2$ and $T' \in \mathbb{C}^{b'} \otimes \mathbb{C}^{a'} \otimes \mathbb{C}^2$ be the tensors described in Lemma 3.A.1.

Fix a basis $\mathrm{e}_1, \dots, \mathrm{e}_k$ of $\mathbb{C}^k$ for any k. For $j = 1,2$, let $M_j = T(\mathrm{e}_j^*) \in \mathbb{C}^a \otimes \mathbb{C}^b$, and similarly, $M'_j = T'(\mathrm{e}_j^*) \in \mathbb{C}^{b'} \otimes \mathbb{C}^{a'}$. By Lemma 3.2.4, we have

$$F_{T,T'} = M_1 \boxtimes M'_1 + M_2 \boxtimes M'_2.$$

A direct calculation shows

$$\begin{aligned} M_1 &= \mathrm{id}_\alpha \boxtimes R_p(1,0) \oplus \mathrm{id}_\beta \boxtimes R_{p+1}(1,0), \\ M_2 &= \mathrm{id}_\alpha \boxtimes R_p(0,1) \oplus \mathrm{id}_\beta \boxtimes R_{p+1}(0,1), \\ M'_1 &= \mathrm{id}_\alpha \boxtimes R_q(1,0)^t \oplus \mathrm{id}_\beta \boxtimes R_{q+1}(1,0)^t, \\ M'_2 &= \mathrm{id}_\alpha \boxtimes R_q(0,1)^t \oplus \mathrm{id}_\beta \boxtimes R_{q+1}(0,1)^t. \end{aligned}$$

After reordering the Kronecker factors, we obtain that $F_{T,T'}$ can be represented in diagonal block form with

$$\begin{aligned} &\alpha\alpha' \text{ blocks} & &R_p(1,0) \boxtimes R_q(1,0)^t + R_p(0,1) \boxtimes R_q(0,1)^t, \\ &\alpha\beta' \text{ blocks} & &R_p(1,0) \boxtimes R_{q+1}(1,0)^t + R_p(0,1) \boxtimes R_{q+1}(0,1)^t, \\ &\beta\alpha' \text{ blocks} & &R_{p+1}(1,0) \boxtimes R_q(1,0)^t + R_{p+1}(0,1) \boxtimes R_q(0,1)^t, \\ &\beta\beta' \text{ blocks} & &R_{p+1}(1,0) \boxtimes R_{q+1}(1,0)^t + R_{p+1}(0,1) \boxtimes R_{q+1}(0,1)^t. \end{aligned}$$

Consequently, the rank of $F_{T,T'}$ coincides with the sum of the ranks of these blocks.

Define

$$K_{x,y} = R_x(1,0) \boxtimes R_y(1,0)^t + R_x(0,1) \boxtimes R_y(0,1)^t \in (\mathbb{C}^x \otimes \mathbb{C}^{y+1}) \otimes (\mathbb{C}^{x+1} \otimes \mathbb{C}^y).$$

The four block types appearing in $F_{T,T'}$ are $K_{p,p'}$, $K_{p,p'+1}$, $K_{p+1,p'}$ and $K_{p+1,p'+1}$.

Consider $K_{x,y}$ as a linear map $K_{x,y}\colon \mathbb{C}^x \otimes \mathbb{C}^{y+1} \to \mathbb{C}^{x+1} \otimes \mathbb{C}^y$. On the basis vector, it acts as follows:

$$K_{x,y}(\mathrm{e}_i \otimes \mathrm{e}_j) = \begin{cases} \mathrm{e}_i \otimes \mathrm{e}_j & j = 1 \\ \mathrm{e}_i \otimes \mathrm{e}_j + \mathrm{e}_{i+1} \otimes \mathrm{e}_{j-1}, & 2 \le j \le y \\ \mathrm{e}_{i+1} \otimes \mathrm{e}_{j-1} & j = y+1. \end{cases}$$

Denote by U_c the subspace of $\mathbb{C}^x \otimes \mathbb{C}^{y+1}$ spanned by the basis vectors $\mathrm{e}_i \otimes \mathrm{e}_j$ with $i + j = c + 1$, and by V_c the analogous subspace of $\mathbb{C}^{x+1} \otimes \mathbb{C}^y$. The map $K_{x,y}$ maps U_c to V_c. The matrix of $K_{x,y}$ restricted to U_c has a simple structure with 1's on two diagonals. Depending on the relation between x, y and c we have the following 4 cases.

$$\begin{cases} \operatorname{rank} K_{x,y}|_{U_c} = \dim(U_c) = \dim(V_c) = c, & \text{if } c \le x,\ c \le y, \\ \operatorname{rank} K_{x,y}|_{U_c} = \dim(U_c) = x,\ \dim(V_c) = y+1, & \text{if } x < c \le y, \\ \operatorname{rank} K_{x,y}|_{U_c} = \dim(V_c) = y,\ \dim(U_c) = y+1, & \text{if } y < c \le x, \\ \operatorname{rank} K_{x,y}|_{U_c} = \dim(U_c) = \dim(V_c) = x + y + 1 - c, & \text{if } c > x,\ c > y. \end{cases}$$

If $x \le y$, then $\operatorname{rank}(K_{x,y}|_{U_c}) = \dim(U_c)$ for all c and, therefore,

$$\operatorname{rank} K_{x,y} = \sum_c \dim(U_c) = x(y+1).$$

Similarly, if $x \ge y$, then $\operatorname{rank} K_{x,y} = y(x+1)$.

With these considerations, we can now compute $\operatorname{rank}(F_{T,T'})$. Clearly, if $p < q$, then $p + 1 \le q$. Consequently, all appearing maps $K_{x,y}$ have rank $x(y+1)$ and we obtain

$$\begin{aligned} \operatorname{rank} F_{T,T'} &= \alpha\alpha' \operatorname{rank} K_{p,p'} + \alpha\beta' \operatorname{rank} K_{p,p'+1} + \beta\alpha' \operatorname{rank} K_{p+1,p'} + \beta\beta' \operatorname{rank} K_{p+1,p'+1} \\ &= \alpha\alpha' p(p'+1) + \alpha\beta' p(p'+2) + \beta\alpha'(p+1)(p'+1) + \beta\beta'(p+1)(p'+2) = ab'. \end{aligned}$$

This shows that if $p < p'$, then the quantum max-flow equals the quantum min-cut, and both are ab'. Similarly, if $p > p'$, the quantum max-flow and quantum min-cut coincide and are equal to $a'b$.

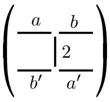
If $p = q$, we calculate the rank of $F_{T,T'}$ as

$$\begin{aligned} \operatorname{rank} F_{T,T'} &= \alpha\alpha' \operatorname{rank} K_{p,p'} + \alpha\beta' \operatorname{rank} K_{p,p'+1} + \beta\alpha' \operatorname{rank} K_{p+1,p'} + \beta\beta' \operatorname{rank} K_{p+1,p'+1} \\ &= \alpha\alpha' p(p'+1) + \alpha\beta' p(p'+2) + \beta\alpha'(p+2)p' + \beta\beta'(p+1)(p'+2). \end{aligned}$$

This quantity differs from

$$ab' = \alpha\alpha' p(p'+1) + \alpha\beta' p(p'+2) + \beta\alpha'(p+1)(p'+1) + \beta\beta'(p+1)(p'+2)$$

by $(p - p' + 1)\beta\alpha' = \beta\alpha'$. Hence, we have reproduced the result in Theorem 3.3.8



$$\mathrm{QMaxFlow}\left(\;\; \right) = \begin{cases} ab' & \text{if } p < p', \\ ab' - \beta\alpha' & \text{if } p = p', \\ a'b & \text{if } p > p'. \end{cases}$$

3.A.2 Bridge 2 via representation theory

One can also use representation theory of $\mathrm{GL}(\mathbb{C}^2)$ to perform the rank calculation in Appendix 3.A.1. Most of the presented theory follows [Lan17, Section 8.1]. We mention that this idea in principle lets us calculate quantum max-flows for higher bridge dimensions in particular cases.

For a vector space V, recall the symmetric and antisymmetric subspaces from Section 1.2. Let V be a finite-dimensional vector space with $\mathrm{v} = \dim V$ and fix a basis $\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{v}$ of V and a corresponding dual basis $x_1, \ldots, x_\mathrm{v}$ of V^*. The space $S^d(V^*)$ can be identified with the space of homogeneous polynomials of degree d on V. The basis elements $x_1, \ldots, x_\mathrm{v}$ can be thought as coordinates on V. In a similar way, $S^k(V)$ can be thought of as the space of order k differential operators (with constant coefficients) via the natural contraction map

$$\begin{aligned} S^d(V^*) \otimes S^k(V) &\to S^{d-k}(V^*) \\ f \otimes \mathrm{e}_{i_1} \ldots \mathrm{e}_{i_k} &\mapsto \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \end{aligned}$$

for any $d \geq k$.

The group $\mathrm{GL}(V)$ acts on $V^{\otimes d}$ and its elements can be represented as matrices in the fixed basis. Let $\mathbf{T}$ be the abelian subgroup of invertible diagonal matrices (called a *torus*) and $\mathbf{B}$ be the subgroup of invertible upper triangular matrices (called the *Borel subgroup*). A tensor $v \in V^{\otimes d}$ is a *weight vector* with *weight* $(p_1, \ldots, p_\mathrm{v})$ if for all elements in $\mathbf{T}$, one has

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_\mathrm{v} \end{pmatrix} v = t_1^{p_1} \ldots t_\mathrm{v}^{p_\mathrm{v}} v.$$

Moreover, v is a *highest weight vector* if $\mathbf{B}$ preserves the line through v. In this case, the weight $(p_1 \ldots p_\mathrm{v})$ is called a *highest weight*.

Two irreducible representations of $\mathrm{GL}(V)$ are isomorphic if and only if they have the same highest weight, that is, contain a highest weight vector for the same highest weight. The irreducible representations appearing in the tensor algebra $V^{\otimes} = \bigoplus_{d=0}^{\infty} V^{\otimes d}$ are labeled by Young diagrams or, equivalently, by tuples of natural numbers $\pi = (p_1, \ldots, p_k)$ with $p_i \leq p_{i+1}$ where $k \leq \mathrm{v}$. Write $\mathbb{S}_\pi(V)$ for the irreducible representation corresponding to π. Moreover, $\mathbb{S}_\pi(V)$ is an irreducible representation appearing in $V^{\otimes d}$ if and only if $p_1 + \cdots + p_k = d$. Let

$\pi' = (q_1 \dots q_{p_1})$ be the conjugate Young diagram to π, that is the Young diagram whose i-th column has p_i boxes. Then,

$$(\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_{q_1}) \otimes \cdots \otimes (\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_{q_{p_1}}) \in V^{\otimes d}$$

is a highest weight vector of highest weight π in $V^{\otimes d}$.

We have seen in Theorem 1.2.12 that every finite-dimensional representation of $\mathrm{GL}(V)$ splits into a direct sum of irreducible representations. The Pieri formula controls which irreducible representations appear in $\mathbb{S}_\pi(V) \otimes S^d(V)$. For a proof of the Pieri formula, see, for example, [RG10, Corollary 9.2.4].

Theorem 3.A.2. *The representation $\mathbb{S}_\pi(V) \otimes S^d(V)$ decomposes into irreducible representations of $\mathrm{GL}(V)$ as*

$$\mathbb{S}_\pi(V) \otimes S^d(V) = \bigoplus_\mu \mathbb{S}_\mu(V)$$

where the direct sum ranges over Young diagrams μ obtained by adding d boxes to π with no two boxes added to the same column. In particular, this decomposition is multiplicity free.

For example, let W be a 2-dimensional space and consider the action of $\mathrm{GL}(W^*)$ on the space $S^k(W^*) \otimes S^m(W^*)$ with $k \geq m$. By Pieri's formula, this decomposes as

$$S^k(W^*) \otimes S^m(W^*) = \bigoplus_{r=0}^{m} \mathbb{S}_{(k+r,m-r)}(W^*),$$

because the Young diagram $(k+r, m-r)$ is obtained by adding m boxes to the Young diagram (k), with r boxes in the first row and $m-r$ boxes in the second row.

Define $h^r_{k,m} = (x_0y_1 - x_1y_0)^{m-r} x_0^{k-m+r} y_0^r$. Identify elements of $S^k(W^*) \otimes S^m(W^*)$ as polynomials in the variables x_0, x_1, y_0, y_1 which are bi-homogeneous of bi-degree (k,m) in the two sets of variables $\{x_0, x_1\}, \{y_0, y_1\}$. In particular, $h^r_{k,m}$ is an element of $S^k(W^*) \otimes S^m(W^*)$. One can verify that this is, in fact, a highest weight vector for highest weight $(k+r, m-r)$.

Similarly, for $k \leq m$, the Pieri formula gives

$$S^k(W^*) \otimes S^m(W^*) = \bigoplus_{r=0}^{m} \mathbb{S}_{(m+r,k-r)}(W^*).$$

Define $h'^r_{k,m} = (x_0y_1 - x_1y_0)^{k-r} x_0^{m-k+r} y_0^r$, which turns out to be a highest weight vector in $\mathbb{S}_{(m+r,k-r)}(W^*)$ in $S^k(W^*) \otimes S^m(W^*)$.

In the study of the bridge graph, with $w = 2$, we may interpret the spaces A, B, A', B' as representations for the group $\mathrm{GL}(W^*)$. Suppose preliminarily $\dim(A) = p+1$, $\dim(B) = p+2$, $\dim(A') = p'+1$ and $\dim(B') = p'+2$, and consider

$$A \cong S^p(W),\ B \cong S^{p+1}(W^*),\ A' \cong S^{p'}(W^*),\ B' \cong S^{p'+1}(W).$$

By Schur's lemma, there is an (up to scaling) unique $\mathrm{GL}(W^*)$-equivariant projection

$$S^p(W^*) \otimes W^* \to S^{p+1}(W^*),$$

see Theorem 1.2.10 and Remark 1.2.14. This corresponds to a tensor $T \in A \otimes B \otimes W$ that can be interpreted as the multiplication map $T(\ell \otimes g) = \ell g$.

By Pieri's formula, $W^* \otimes S^{p'}(W^*)$ contains a copy of $S^{p'+1}(W^*)$. By Schur's Lemma, there is a unique, up to scaling, $\mathrm{GL}(W^*)$-equivariant embedding $S^{p'+1}(W^*) \to W^* \otimes S^{p'}(W^*)$. Let $T' \in A' \otimes B' \otimes W^*$ be the tensor corresponding to this embedding, regarded as an element of $S^{p'}(W^*) \otimes S^{p'+1}(W) \otimes W^*$. This map is called *polarization* in [Lan12, Section 2.6.4] and is explicitly given by $T'(f) = x_0 \otimes \frac{\partial}{\partial x_0} f + x_1 \otimes \frac{\partial}{\partial x_1} f$.

The flow map $F_{T,T'}$ in the bridge graph with this choice of T and T' is

$$\begin{aligned} F_{T,T'} : S^p(W^*) \otimes S^{p'+1}(W^*) &\to S^{p+1}(W^*) \otimes S^{p'}(W^*) \\ f \otimes g &\mapsto \textstyle\sum_{i=0,1} f \cdot x_i \cdot \dfrac{\partial}{\partial y_i} g \end{aligned} \tag{3.6}$$

where, again, the tensor product of two symmetric powers is interpreted as a space of bi-homogeneous polynomials.

Using Pieri's formula and Schur's Lemma, we compute the rank of this map for every choice of p and p'.

Lemma 3.A.3. *The flow map $F_{T,T'}$ from Equation (3.6) is $\mathrm{GL}(W^*)$-equivariant. Moreover, the map is injective if $p \le p'$ and surjective if $p \ge p'$. In particular, $F_{T,T'}$ is an isomorphism if and only if $p = p'$.*

Proof. Since the maps defined by T and T' are $\mathrm{GL}(W^*)$-equivariant, the flow map $F_{T,T'}$ is as well. Suppose $p \le p'$. By Pieri's formula, the domain is

$$S^p(W^*) \otimes S^{p'+1}(W^*) = \bigoplus_{r=0}^{p} \mathbb{S}_{(p'+1+i,p-i)}(W^*).$$

Since the decomposition is multiplicity free, by Schur's Lemma, it suffices to check that the highest weight vectors of the domain are not mapped to 0. The highest weight vectors of the domain are $h'^r_{p,p'+1} \in \mathbb{S}_{(p'+1+r,p-r)}(W^*) \subset S^p(W^*) \otimes S^{p'+1}(W^*)$.

It is easy to verify that $F_{T,T'}(h'^r_{p,p'+1})$ is a nonzero multiple of $h^r_{p+1,p'}$. This guarantees that $F_{T,T'}$ is injective.

If $p \ge p'$, the same calculation can be performed to show that the transpose map $F^t_{T,T'}$ is injective. Hence $F_{T,T'}$ is surjective.

We conclude that if $p = p'$, $F_{T,T'}$ is an isomorphism. On the other hand, if $p \neq p'$, domain and codomain do not have the same dimension, hence the map is not an isomorphism. □

To calculate the quantum max-flow for general bond dimensions a, b, a' and b', we use the same method as in Appendix 3.A.1. Write

$$\begin{aligned} a &= p\alpha + (p+1)\beta, & b &= (p+1)\alpha + (p+2)\beta \\ a' &= p'\alpha' + (p'+1)\beta', & b' &= (p'+1)\alpha' + (p'+2)\beta', \end{aligned}$$

and assume without loss $p \leq p'$. We can define tensors in $A \otimes B \otimes W$ resp. $A' \otimes B' \otimes W^*$ as block tensors where the block elements are the distinguished tensors T and T' defined before. Now, using Lemma 3.A.3, one can repeat the same calculation as in Appendix 3.A.1 to calculate the quantum max-flow.

Chapter 4

Partial degeneration of tensors

In this chapter, we shed light on a variation of the notions of restriction and degeneration. Recall that tensor restriction was defined as a transformation of the tensors by *local* linear maps on its tensor factors. Degeneration, on the other hand, was defined as a transformation where the local linear maps may vary along a curve, and the resulting tensor is expressed as a limit along this curve. We also saw that for two tensor factors, these notions are equivalent, whereas for three tensor factors, they are not (see Example 1.1.7 and Example 1.3.3). It is therefore natural to define *partial degeneration*, a special version of degeneration where one of the local linear maps is constant, whereas the others vary along a curve.

We will use this chapter to study this novel notion in detail. Motivated by algebraic complexity, quantum entanglement, and tensor networks, we present constructions based on matrix multiplication tensors and find examples by making a connection to the theory of prehomogeneous tensor spaces. We also highlight the subtleties of this new notion by showing obstruction and classification results for the unit tensor.

Moreover, we study the notion of aided rank, a natural generalization of tensor rank. It turns out that the existence of partial degenerations gives strong upper bounds on the aided rank of the tensor, which in turn allows one to turn degenerations into restrictions. In particular, we present several examples based on the W-tensor and the Coppersmith-Winograd-tensors, where lower bounds on aided rank provide obstructions to the existence of certain partial degenerations.

This chapter is a partly modified version of [CGLS22].

4.1 Overview

This chapter consists of two parts. In the first part (Section 4.2), we introduce the notion of partial degeneration and show that restriction, partial degeneration, and degeneration are mutually inequivalent notions. To show a separation between the notions of restriction and partial degeneration, we present a number of constructions based on tensors motivated by algebraic complexity theory and tensor networks. We also draw a connection to the theory of prehomogeneous tensor spaces which allows us to derive further examples manifesting this separation. To show a separation between partial degenerations and degenerations, we prove a no-go result for the unit tensor which moreover allows us to classify certain families of partial degenerations.

In the second part (Section 4.3), we introduce the notion of *aided restriction*, which is performed on a version of the tensor augmented via an *aiding matrix*. This raises the question of how large the rank of such an aiding matrix should be in order to allow certain restrictions. We study upper and lower bounds, highlighting the role of degeneration and partial degeneration.

4.1.1 Partial degeneration

In Section 4.2, we introduce and study the notion of *partial degeneration*: For three-party tensors $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$, we say that T partially degenerates to S, and write $T \trianglerighteq S$, if T degenerates to S where the degeneration map $A_1(\epsilon) = A_1$ can be chosen constant in ϵ. Analogous notions can be defined by assuming that $A_2(\epsilon)$ or $A_3(\epsilon)$ are constant. For simplicity, we will always assume that this map is the first one. Hence, we have $T \trianglerighteq S$ if and only if there are linear maps $A_1, A_2(\epsilon)$ and $A_3(\epsilon)$, with $A_2(\epsilon)$ and $A_3(\epsilon)$ depending polynomially in ϵ such that $\lim_{\epsilon\to 0} \frac{1}{\epsilon^d}(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))T = S$ for some d. As in the case of degeneration, we sometimes write $T \trianglerighteq_d^e S$ to keep track of the approximation and error degrees. We point out that allowing only one of the three maps to depend on ϵ provides a notion of degeneration which is equivalent to restriction, see Remark 4.2.2. A priori, it is unclear whether the notion of partial degeneration is indeed non-trivial, or whether one might always reduce a degeneration to a partial degeneration or a partial degeneration to a restriction. We will show this is not the case. We point out that an example of partial degeneration has been known since [Str87], and it was used to achieve a breakthrough result in the study of the complexity of matrix multiplication: the tensor

$$\mathrm{Str}_q = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes \mathrm{e}_q \otimes \mathrm{e}_i + \mathrm{e}_i \otimes \mathrm{e}_i \otimes \mathrm{e}_q \in \mathbb{C}^{q-1} \otimes \mathbb{C}^q \otimes \mathbb{C}^q \tag{4.1}$$

has tensor rank equal to $2q-2$, but it is a partial degeneration of $\langle q \rangle$.

In this chapter, we study partial degenerations in depth. In Section 4.2, we construct various families of examples of honest partial degenerations. We also study the question under which circumstances partial degenerations cannot exist and provide a no-go result for the unit tensor.

In Section 4.2.3, we construct a family of partial degenerations of the matrix multiplication tensor. Let $\langle m, n, p \rangle$ be the matrix multiplication tensor associated with the bilinear map multiplying an $m \times n$ matrix with an $n \times p$ matrix. To construct a family of partial degenerations of the matrix multiplication tensor $\langle 2, 2, 2 \rangle$, the challenge is to show that these are not actually restrictions of $\langle 2, 2, 2 \rangle$. To see this, we resort to the notion of tensor compressibility in the sense of [LM18].

In Section 4.2.4, we study the notion of partial degeneration in the setting of prehomogeneous tensor spaces where we can find many more examples of honest partial degenerations. Recall the notion of prehomogeneity from Section 3.2.2 and consider the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ on $U_1 \otimes U_2 \otimes U_3$. If this action is prehomogeneous , with the tensor T having a dense orbit, then every tensor in $U_1 \otimes U_2 \otimes U_3$ is a partial degeneration of T. As we have seen in Section 3.2.2, the prehomogeneity of the action is determined by simple arithmetic relations among the dimensions of the tensor factors. In Section 4.2.4, we provide examples of tensors that cannot be a restriction of a tensor with dense orbit for every instance where the space $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous under the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$. We emphasize that while it is well understood under which conditions $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous, there are, in general, no closed formulas for elements having a dense orbit. If $\dim(U_1) = 2$, that is, in the case of *matrix pencils*, explicit elements with closed orbit are known, see, e.g., [Gan59, Chapter XIII] and [Pok86]. In Section 4.2.4, we use these examples to provide explicit partial degenerations for matrix pencils.

In Section 4.2.5, we study situations in which partial degenerations cannot occur. More precisely, we show that every partial degeneration of the unit tensor $\langle r \rangle$ to a concise tensor $T \in U_1 \otimes U_2 \otimes U_3$ with $\dim(U_1) = r$ can be reduced to a restriction. We use this result to show that for $\dim(U_1) = r - 1$, tensors as in Equation (4.1) are essentially all honest partial degenerations that can occur.

Note that the constructed examples, together with the no-go result for the unit tensor, shows that restriction, partial degeneration, and degeneration are, in fact, three different notions.

4.1.2 Aided restriction and aided rank

The starting point of the second part of this chapter is the fact that any degeneration can be turned into a restriction using *interpolation*. It is known that if $T \trianglerighteq_d S$, then $T \boxtimes \langle d+1 \rangle \geq S$. In Section 4.3, we study the case where the supporting tensor is a matrix instead of a unit tensor. More precisely, for a tensor $T \in U_1 \otimes U_2 \otimes U_3$, define

$$T^{\blacksquare p} = T \boxtimes \langle 1, 1, p \rangle \in U_1 \otimes (U_2 \otimes \mathbb{C}^p) \otimes (U_3 \otimes \mathbb{C}^p).$$

In Lemma 4.3.2, we show that if $T \trianglerighteq S$ then there exists a p such that $T^{\blacksquare p} \geq S$. We call $\langle 1, 1, p \rangle$ the *aiding matrix* and p the rank of the aiding matrix. It is interesting to ask how large the rank of the aiding matrix has to be.

In Section 4.3.1, we provide upper bounds on this rank in the case of partial degenerations. In particular, we show that for partial degenerations of approximation degree d and error degree e, the rank of the aiding matrix can be chosen to be $\min\{d+1, e+1\}$, in other words, if $T \blacktriangleright_d^e S$ then $T^{\blacksquare d+1} \geq S$ and $T^{\blacksquare e+1} \geq S$. Even more strikingly, we show that if $T \in U_1 \otimes U_2 \otimes U_3$, where the space $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous under the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$, and if the orbit of T is dense, then $T^{\blacksquare 2} \geq S$ for any other tensor $S \in U_1 \otimes U_2 \otimes U_3$.

It turns out that these findings are in strong contrast to the case of degenerations that are not partial degenerations. To see this, we develop in Section 4.3.2 a method to lower bound the minimal possible rank of an aiding matrix. This relies on a variant of the *substitution method* from [AFT11], see Theorem 1.3.2. For that, we define in Section 4.3.2 the notion of *aided rank* as

$$R^{\blacksquare p}(T) = \min\{r : \langle r \rangle^{\blacksquare p} \geq T\}.$$

When $p = 1$, this reduces to the notion of tensor rank [Lan17, Proposition 5.1.2.1]. We show that one can generalize the substitution method to give lower bounds on aided rank and on the minimal possible rank of an aiding matrix for several examples of degeneration. For example, for the degeneration

$$\langle 2^k \rangle \trianglerighteq W^{\boxtimes k} \tag{4.2}$$

we show that $R^{\blacksquare 2^k - 1}(W^{\boxtimes k}) \geq 2^k + 1$. In other words, the minimal rank p of an aiding matrix turning the degeneration in Equation (4.2) into a restriction is 2^k. Note that for this example, the no-go result for partial degenerations (Proposition 4.2.10) gives that the degeneration cannot be realized as a partial degeneration.

4.2 Partial degeneration

In this section, we introduce and study the notion of partial degeneration, a natural intermediate notion of restriction and degeneration. In Section 4.2.1, we will define partial degeneration. After that, we review in Section 4.2.2 a known example of partial degeneration. In Section 4.2.3, we will recall a property of tensors called *compressibility* and demonstrate with an example how this can be used to rule out restriction. We will see more examples in Section 4.2.4 using the theory of matrix pencils. Finally, in Section 4.2.5, we will study situations where no honest partial degeneration exists.

4.2.1 Definition and motivation

The main concept of this section is the following special version of degeneration, intermediate between restriction and fully general degeneration.

Definition 4.2.1. *Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ be tensors. We say that T degenerates partially to S and write $T \trianglerighteq S$ if there is a linear map $A_1 : U_1 \to V_1$ and linear maps $A_2(\epsilon), A_3(\epsilon)$ with entries in the polynomial ring $\mathbb{C}[\epsilon]$ such that*

$$(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))T = \epsilon^d S + \epsilon^{d+1} S_1 + \cdots + \epsilon^{d+e} S_e.$$

We sometimes write $T \trianglerighteq_d^e S$ to keep track of d and e. We call a partial degeneration $T \trianglerighteq S$ an honest partial degeneration if T does not restrict to S.

It is clear that every restriction is a partial degeneration and every partial degeneration is a degeneration. This raises the following two questions:

(1) Can every partial degeneration $T \trianglerighteq S$ be realized as a restriction $T \geq S$?

(2) Can every degeneration $T \unrhd S$ be realized as a partial degeneration $T \trianglerighteq S$?

We point out that only allowing one of the three linear maps to depend on ϵ provides the same notion as restriction.

Remark 4.2.2. *Let $T \in U_1 \otimes U_2 \otimes U_3$, $S \in V_1 \otimes V_2 \otimes V_3$ be tensors and suppose there are linear maps $A_1, A_2, A_3(\epsilon)$ with $A_i : U_i \to V_i$, and $A_3(\epsilon)$ depending polynomially in ϵ such that $S = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d}((A_1 \otimes A_2 \otimes A_3(\epsilon))T)$. Then S is a restriction of T. To see this, write $A_3(\epsilon) = \epsilon^d A_{3,d} + \cdots + \epsilon^{d+e} A_{3,e}$ with $A_{3,j} : U_3 \to V_3$. It is immediate that $S = (A_1 \otimes A_2 \otimes A_{3,d})T$. This expresses S as a restriction of T.*

In Section 4.2.2, Section 4.2.3 and Section 4.2.4, we will see families of examples demonstrating that the answer to the question in (1) is no. In Section 4.2.5, we will see that the answer to question (2) is also no.

4.2.2 Strassen's tensor

In [Str87], a first example of a partial degeneration was found: Let $U_1 \cong \mathbb{C}^{q-1}$ and $U_2 \cong U_3 \cong \mathbb{C}^q$ and consider the tensor

$$\mathrm{Str}_q = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes \mathrm{e}_i \otimes \mathrm{e}_q + \mathrm{e}_i \otimes \mathrm{e}_q \otimes \mathrm{e}_i \in U_1 \otimes U_2 \otimes U_3.$$

Using the substitution method, it is not hard to see that this tensor has rank $2q - 2$. On the other hand, it is a partial degeneration of the unit tensor $\langle q \rangle$ via

$$\epsilon \mathrm{Str}_q = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes (\mathrm{e}_q + \epsilon \mathrm{e}_i) \otimes (\mathrm{e}_q + \epsilon \mathrm{e}_i) - \left(\sum_{i=1}^{q-1} \mathrm{e}_i \right) \otimes \mathrm{e}_q \otimes \mathrm{e}_q + \mathcal{O}(\epsilon^2).$$

In Section 4.2.5, we will show that these are essentially all partial degenerations of $\langle r \rangle$ that can be found in $U_1 \otimes U_2 \otimes U_3$ with $\dim U_1 = r - 1$.

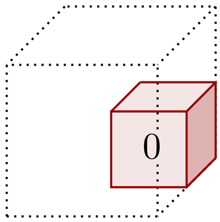


Figure 4.1: Visualization of an (a_1, a_2, a_3)-compressible tensor: This large $u_1 \times u_2 \times u_3$-cube is the tensor $T \in U_1 \otimes U_2 \otimes U_3$. The entries of T – specified in some fixed basis – can be written in the cells of this cube. The smaller, red $a_1 \times a_2 \times a_3$-cube depicts a block of size $a_1 \times a_2 \times a_3$ where each entry of T equals zero. By choosing the linear maps as projectors onto the last $u_1 - a_1$ resp. $u_2 - a_2$ resp. $u_3 - a_3$ coordinates, we see that each such tensor is (a_1, a_2, a_3)-compressible.

4.2.3 Compressibility of tensors and matrix multiplication

In this section, we will find a family of examples of partial degeneration of the 2×2-matrix multiplication tensor. Note that one challenge of finding an honest partial degeneration is to show that this partial degeneration is actually not a restriction. To achieve that, we recall the notion of *compressibility* introduced in [LM18].

Definition 4.2.3. *A tensor $T \in U_1 \otimes U_2 \otimes U_3$ is (a_1, a_2, a_3)-compressible if there are linear maps $A_i : U_i \to U_i$ of rank a_i such that $(A_1 \otimes A_2 \otimes A_3)T = 0$.*

Equivalently, T is (a_1, a_2, a_3)-compressible if there are bases of the spaces U_1, U_2 and U_3 such that in these bases, $T_{i_1,i_2,i_3} = 0$ for all $i_j \geq \dim(U_j) - a_j$. We visualize the concept of an (a_1, a_2, a_3)-compressible tensor in Figure 4.1. The following technical result will become handy later.

Lemma 4.2.4. *Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$. Let $T \geq S$ and let S be concise. Moreover, assume that S is (a_1, a_2, a_3)-compressible. Then, T is also (a_1, a_2, a_3)-compressible.*

Proof. By assumption, there are maps A_i with rank a_i such that $(A_1 \otimes A_2 \otimes A_3)T = 0$. As S is concise, the restriction maps M_i must be surjective where $S = (M_1 \otimes M_2 \otimes M_3)T$. Therefore, the maps A_1M_1, A_2M_2 and A_3M_3 also have rank a_1, a_2 and a_3, respectively. Since $(A_1M_1 \otimes A_2M_2 \otimes A_3M_3)T = (A_1 \otimes A_2 \otimes A_3)S = 0$ the claim follows. □

Lemma 4.2.4 can be used to exclude restrictions $T \geq S$ if T is less compressible than S. An example of a tensor that is not “very compressible” is the matrix multiplication tensor.

Lemma 4.2.5. *The 2×2 matrix multiplication tensor $\langle 2, 2, 2 \rangle$ is not $(2, 3, 3)$-compressible.*

Proof. Note that any 4×4 matrix $M = (M_{(u,v),(x,y)})_{u,v,x,y=1}^2$ (labeled by double indices) induces a linear endomorphism of the space of 2×2 matrices via

$$M : x \mapsto M.x = (M.x)_{i,j=1}^2, \; (M.x)_{i,j} = \sum_{k,l=1,2} M_{(i,j),(k,l)} x_{k,l}.$$

Recall that $\langle 2,2,2 \rangle$ corresponds to calculating the four bilinear forms $z_{j,i} = x_{i1}y_{1j} + x_{i2}y_{2j}$ for $i,j = 1,2$, that is, the entries of $(x \cdot y)^T$ where the entries of x and y are indeterminates.

Let $S = (A_1 \otimes A_2 \otimes A_3)\langle 2,2,2 \rangle$ be a restriction of the two-by-two matrix multiplication tensor. Interpreting A_1, A_2 and A_3 as 4×4 matrices, an easy calculation shows that the four bilinear forms corresponding to the tensor S are the four entries of the transpose of

$$A_3.\left((A_1.x) \cdot (A_2.y)\right). \tag{4.3}$$

Now, let the rank of A_1 and A_2 be at least 3 and the rank of A_3 be at least 2. It is clear that the space of all $A_1.x$ for $x \in M_{2\times 2}$ is at least three-dimensional (the same holds for A_2). It is well-known that every subspace of $M_{2\times 2}$ of dimension at least 3 must contain an invertible matrix. Choosing $x_0 \in M_{2\times 2}$ such that $A_1.x_0$ is invertible, we see that the space of matrices of the form $(A_1.x_0)\cdot(A_2.y)$ for $y \in M_{2\times 2}$ contains three linearly independent matrices. Hence, since we assumed that A_3 has rank at least 2, we see that Equation (4.3) cannot be identical zero. This finishes the proof. □

The following technical result is a simple generalization of Proposition 1.4.5 and will help us to construct partial degenerations of the matrix multiplication tensor.

Lemma 4.2.6. *Let V_1, V_2, V_3 be vector spaces with dimensions $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ and consider a tensor $S \in V_1 \otimes V_2 \otimes V_3$. Then, we have $\langle m,n,p \rangle \trianglerighteq S$ if and only if there are a natural number d and matrices*

$$\alpha_1(\epsilon), \dots, \alpha_{\mathrm{v}_1}(\epsilon) \in \mathbb{C}[\epsilon]^{m\times n},\; \beta_1(\epsilon), \dots, \beta_{\mathrm{v}_2}(\epsilon) \in \mathbb{C}[\epsilon]^{n\times p},\; \gamma_1(\epsilon), \dots, \gamma_{\mathrm{v}_3}(\epsilon) \in \mathbb{C}[\epsilon]^{p\times m} \tag{4.4}$$

such that $\epsilon^d S_{i,j,k} = \mathrm{tr}(\alpha_i(\epsilon)\beta_j(\epsilon)\gamma_k(\epsilon)) + \mathcal{O}(\epsilon^{d+1})$.

Moreover, if the matrices $\alpha_1(\epsilon), \dots, \alpha_{\mathrm{v}_1}(\epsilon)$ can be chosen constant in ϵ we have $T \blacktriangleright S$. If all matrices in Equation (4.4) can be chosen constant in ϵ, we have $\langle m,n,p \rangle \geq S$.

With this, we are now ready to find honest partial degenerations of $\langle 2,2,2 \rangle$.

Proposition 4.2.7. *Every concise tensor $S \in \mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ that is $(3,3,3)$-compressible is an honest partial degeneration of $\langle 2,2,2 \rangle$.*

Proof. Fixing bases, we can write our tensor S as

$$S = \sum_{i,j,k=1}^{3,4,4} S_{i,j,k} \mathrm{e}_i \otimes \mathrm{e}_j \otimes \mathrm{e}_k$$

such that $S_{i,j,k} = 0$ whenever both j and k are greater or equal than 2. From Lemma 4.2.6, it suffices to find 2×2 matrices

$$\alpha_1, \dots, \alpha_3 \in \mathbb{C}^{m \times n},\ \beta_1(\epsilon), \dots, \beta_4(\epsilon) \in \mathbb{C}[\epsilon]^{n \times p},\ \gamma_1(\epsilon), \dots, \gamma_4(\epsilon) \in \mathbb{C}[\epsilon]^{p \times m}$$

such that

$$\epsilon S_{i,j,k} = \mathrm{tr}(\alpha_i \beta_j \gamma_k) + \mathcal{O}(\epsilon^2). \tag{4.5}$$

Choosing matrices

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \beta_1 = \begin{pmatrix} \epsilon(S_{1,1,1} - 1) + 1 & S_{2,1,1} \\ S_{3,1,1} & 1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} \epsilon + 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \beta_2 = \begin{pmatrix} \epsilon S_{1,2,1} & \epsilon S_{2,2,1} \\ \epsilon S_{3,2,1} & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} \epsilon\, S_{1,1,2} & \epsilon\, S_{2,1,2} \\ \epsilon\, S_{3,1,2} & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \beta_3 = \begin{pmatrix} \epsilon S_{1,3,1} & \epsilon S_{2,3,1} \\ \epsilon S_{3,3,1} & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} \epsilon\, S_{1,1,3} & \epsilon\, S_{2,1,3} \\ \epsilon\, S_{3,1,3} & 0 \end{pmatrix}$$

$$\beta_4 = \begin{pmatrix} 0 & \epsilon S_{2,4,1} \\ \epsilon S_{3,4,1} & -\epsilon S_{1,4,1} \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & \epsilon\, S_{2,1,4} \\ \epsilon\, S_{3,1,4} & -\epsilon\, S_{1,1,4} \end{pmatrix},$$

one easily verifies that Equation (4.5) is fullfilled.

Since S is concise and is $(3,3,3)$-compressible we conclude with Lemma 4.2.4 and Lemma 4.2.5 that S is an honest partial degeneration of $\langle 2,2,2 \rangle$. □

Remark 4.2.8. *Lemma 4.2.5 implies that no concise tensor which is* $(2,3,3)$*-compressible is a restriction of* $\langle 2,2,2 \rangle$*. One might ask if Proposition 4.2.7 still holds if we relax the condition on S to being* $(2,3,3)$*-compressible. This turns out to be not true: In fact, it has been shown that the set of all degenerations of* $\langle 2,2,2 \rangle$ *in* $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ *has dimension 31 [BLG21]. A simple calculation – the code for which can be found in Appendix 4.A – shows that the orbit closure of a generic element of* $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ *which is* $(2,3,3)$*-compressible has dimension at least 37.*

4.2.4 Prehomogeneous spaces

In this section, we will see more examples of partial degenerations by making a connection to the theory of prehomogeneous tensor spaces which we reviewed in Section 3.2.2. Recall that we saw in Proposition 3.2.11 that one can read off the dimensions $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$ of the involved spaces if $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous under the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$: Defining $\lambda_{\mathrm{u}_1} = (\mathrm{u}_1 + \sqrt{\mathrm{u}_1^2 - 4})/2$ we saw that the space is prehomogeneous whenever $\mathrm{u}_3 > \lambda_{\mathrm{u}_1}\mathrm{u}_2$. Hence, for any choices of $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$ satisfying this condition there is an element $T \in U_1 \otimes U_2 \otimes U_3$ such that for all $S \in U_1 \otimes U_2 \otimes U_3$ it holds $T \trianglerighteq S$. To show that there exists S which is not a restriction of T, we will use Lemma 1.2.6.

Theorem 4.2.9. *Let U_1, U_2, U_3 have dimensions $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$ such that $\lambda_{\mathrm{u}_1}\mathrm{u}_2 < \mathrm{u}_3 < \mathrm{u}_1\mathrm{u}_2$. Then, there exist tensors $T, S \in U_1 \otimes U_2 \otimes U_3$ such that $T \trianglerighteq S$, but $T \not\geq S$.*

Proof. We know from Proposition 3.2.11 that the space $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous under $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$. Let T be a tensor in the dense $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$-orbit, so that $T \trianglerighteq S$ for every $S \in U_1 \otimes U_2 \otimes U_3$.

Let $p = \mathrm{u}_1\mathrm{u}_2 - \mathrm{u}_3$. Note that $\mathrm{u}_1 - 1 \leq \lambda_{\mathrm{u}_1} < \mathrm{u}_1$, so $\lambda_{\mathrm{u}_1}\mathrm{u}_2 < \mathrm{u}_3 < \mathrm{u}_1\mathrm{u}_2$ implies that $0 < p < \mathrm{u}_2$. Define the tensor $S \in U_1 \otimes U_2 \otimes U_3$ as

$$S = \sum_{i=1}^{\mathrm{u}_1 - 1} \mathrm{e}_i \otimes \left(\sum_{j=1}^{\mathrm{u}_2} \mathrm{e}_j \otimes \mathrm{e}_{(i-1)\mathrm{u}_2 + j} \right) + \mathrm{e}_{\mathrm{u}_1} \otimes \left(\sum_{j=1}^{\mathrm{u}_2 - p} \mathrm{e}_j \otimes \mathrm{e}_{(\mathrm{u}_1 - 1)\mathrm{u}_2 + j} \right)$$

It is not hard to see that the tensor S is concise.

We will now show that T and S lie in different $\mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$-orbits and use Lemma 1.2.6. For that, we will compute the dimensions of these orbits. We denote $G = \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$.

For T we have $U_1 \otimes U_2 \otimes U_3 \supset \overline{G \cdot T} \supset \overline{[\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)] \cdot T} = U_1 \otimes U_2 \otimes U_3$, which implies $\overline{G \cdot T} = U_1 \otimes U_2 \otimes U_3$ and $\dim(G \cdot T) = \mathrm{u}_1\mathrm{u}_2\mathrm{u}_3$.

For S, the dimension of the orbit $G \cdot S$ can be found as $\dim(G \cdot S) = \dim(G) - \dim(\mathrm{Stab}_G(S))$ (see Proposition 1.2.7). The stabilizer $\mathrm{Stab}_G(S)$ is isomorphic to $P(1, \mathrm{u}_1) \times P(\mathrm{u}_2 - p, \mathrm{u}_2)$ where $P(a, b) \subset \mathrm{GL}_b$ is the parabolic group preserving a subspace of dimension a. Indeed, let $S_i \in U_2 \otimes U_3$ be the slices of S corresponding to the standard basis, that is, $S = \sum_{i=1}^{\mathrm{u}_1} \mathrm{e}_i \otimes S_i$. Note that $\mathrm{rank}(S_i) = \mathrm{u}_2$ for $i < \mathrm{u}_1$ and $\mathrm{rank}(S_{\mathrm{u}_1}) = \mathrm{u}_2 - p$. Moreover, a nonzero linear combination $\sum_{i=1}^{\mathrm{u}_1} \alpha_i S_i$ has rank $\mathrm{u}_2 - p$ if and only if $\alpha_i = 0$ for $i \leq \mathrm{u}_1 - 1$. It follows that $(A \otimes B \otimes C)S = S$, then A preserves the one-dimensional subspace $\mathrm{span}(\mathrm{e}_{\mathrm{u}_1})$. Therefore, we have $a_{\mathrm{u}_1,\mathrm{u}_1}(B \otimes C)S_{\mathrm{u}_1} = S_{\mathrm{u}_1}$ and it follows that B preserves the $(\mathrm{u}_1 - p)$-dimensional subspace $\mathrm{span}(\mathrm{e}_1, \ldots, \mathrm{e}_{\mathrm{u}_2 - p + 1})$, which is the image of S_{u_1} considered as a linear map $U_3^* \to U_2$. Now, given A and B which preserve the required subspaces, the map C such that $(A \otimes B \otimes C)S = S$

always exists and is unique. To prove this, note that S considered as a linear map $U_3^* \to U_1 \otimes U_2$ is an isomorphism between U_3^* and the subspace

$$(\operatorname{span}(\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{u}_1 - 1}) \otimes U_2 \oplus \operatorname{span}(\mathrm{e}_{\mathrm{u}_1}) \otimes \operatorname{span}(\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{u}_2 - p + 1})) \subset U_1 \otimes U_2.$$

With this, we see that there is a unique choice for the map C which is the contragredient map to $A \otimes B$ restricted to this subspace.

From the description of $\operatorname{Stab}_G(S)$ it follows that

$$\dim(\operatorname{Stab}_G(S)) = (\mathrm{u}_1^2 - \mathrm{u}_1 + 1) + (\mathrm{u}_2^2 - p(\mathrm{u}_2 - p))$$

and

$$\begin{aligned}\dim(G \cdot S) = \mathrm{u}_3^2 + (\mathrm{u}_1 - 1) + p(\mathrm{u}_2 - p) = \mathrm{u}_3(\mathrm{u}_1\mathrm{u}_2 - p) + (\mathrm{u}_1 - 1) + p(\mathrm{u}_2 - p) = \\ \mathrm{u}_1\mathrm{u}_2\mathrm{u}_3 - p(\mathrm{u}_3 - \mathrm{u}_2) + \mathrm{u}_1 - 1 - p^2 < \mathrm{u}_1\mathrm{u}_2\mathrm{u}_3 - \mathrm{u}_3 + \mathrm{u}_2 + \mathrm{u}_1 - 2 < \mathrm{u}_1\mathrm{u}_2\mathrm{u}_3.\end{aligned}$$

The last inequality holds because u_2 cannot be equal to 1 under the assumptions of the theorem, and thus, $\mathrm{u}_3 \geq (\mathrm{u}_1 - 1)\mathrm{u}_2 > \mathrm{u}_1 - 2 + \mathrm{u}_2$.

It follows that the orbits of T and S are distinct and thus, $T \not\trianglerighteq S$ by Lemma 1.2.6. □

We have seen that one, in principle, can recursively construct elements with dense orbit, see Remark 3.2.14. This, in principle, enables us to construct explicit examples of partial degenerations in Theorem 4.2.9. A closed formula for elements of $U_1 \otimes U_2 \otimes U_3$ that have dense orbit, on the other hand, is not known. To see more concrete examples of partial degenerations, we now focus on tensors $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$. Clearly, this space is prehomogeneous for $\mathrm{GL}(\mathbb{C}^m) \times \mathrm{GL}(\mathbb{C}^n)$ whenever $m \neq n$. Fixing as basis $\mathrm{e}_1, \mathrm{e}_2$ of $\mathbb{C}^2$, we can write our tensor as $T = \mathrm{e}_1 \otimes T_1 + \mathrm{e}_2 \otimes T_2$ where $T_1, T_2 \in \mathbb{C}^m \otimes \mathbb{C}^n$ can be thought of as $m \times n$ matrices. In that way, our tensor is uniquely specified by a tuple of matrices $[T_1, T_2]$, which one often calls the *matrix pencil* associated with T. We have seen an explicit formula of the dense orbit in this setup in Lemma 3.A.1. By exchanging columns and rows, we see that the tensor associated with the matrix pencil $[I_1, I_2]$, where

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{4.6}$$

has a dense orbit. Letting now, for example, $n = m + 1$, we know from the proof of Theorem 4.2.9 that the pencil

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

is an honest partial degeneration of the dense orbit element in Equation (4.6).

4.2.5 A no-go result for the unit tensor

In this section, we will see that under certain circumstances, partial degenerations do not exist even when degenerations do. We first show that there are no proper partial degenerations of the unit tensor if the constant map has full rank. Note that the rank condition on the constant map cannot be dropped: Already in Section 4.2.2, we saw an example of an honest partial degeneration where the constant map has rank $r - 1$. We will use our no-go result to prove a classification result for this setup.

Proposition 4.2.10. *Let $S \in V_1 \otimes V_2 \otimes V_3$ be any tensor. If $\langle r \rangle \trianglerighteq S$ via degeneration maps $A_1, A_2(\epsilon)$ and $A_3(\epsilon)$ where the constant map A_1 is of full rank r then $\langle r \rangle \geq S$.*

Proof. It is clear that we can assume $\dim(V_1) = r$ and that A_1 is invertible.

Assume

$$S = \lim_{\epsilon \to 0} (\mathrm{id} \otimes A_2(\epsilon) \otimes A_3(\epsilon)) \langle r \rangle$$

is a degeneration where the first map is the identity. That is, we have

$$S = \mathrm{e}_1 \otimes M_1 + \dots \mathrm{e}_r \otimes M_r \tag{4.7}$$

where $M_i = \lim_{\epsilon \to 0} A_2(\epsilon)\mathrm{e}_i \otimes A_3(\epsilon)\mathrm{e}_i$. Hence, it is clear that for all i, M_i must be a rank 1 matrix as the limit of rank 1 matrices. However, a tensor of the form in Equation (4.7) where the M_i are rank 1 is a restriction of $\langle r \rangle$.

Now, let $S = \lim_{\epsilon \to 0} (A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon)) \langle r \rangle$ be any partial degeneration of $\langle r \rangle$. From before, we know that $\tilde{S} = (A_1^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) S = \lim_{\epsilon \to 0} (\mathrm{id} \otimes A_2(\epsilon) \otimes A_3(\epsilon)) \langle r \rangle$ is a restriction of $\langle d \rangle$. Hence, the same holds for $S = (A_1 \otimes \mathrm{id} \otimes \mathrm{id}) \tilde{S}$. This finishes the proof. ☐

Remark 4.2.11. *We note that the result in Proposition 4.2.10 does not apply to degenerations. Proposition 4.2.10 in particular says that if V_1 has dimension r and $S \in V_1 \otimes V_2 \otimes V_3$ is concise we cannot have an honest partial degeneration $\langle r \rangle \trianglerighteq S$ (else the constant map would be invertible by conciseness of S). However, for example, the unit tensor $\langle 2 \rangle$ does not restrict but degenerates to $W = \mathrm{e}_1 \otimes \mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_1 \otimes \mathrm{e}_2 \otimes \mathrm{e}_1 + \mathrm{e}_2 \otimes \mathrm{e}_1 \otimes \mathrm{e}_1$ which is concise in the same space* *as* $\langle 2\rangle$ *(see Example 1.1.7 and Example 1.3.3). Hence,* W *is an honest degeneration of* $\langle 2\rangle$ *but not a partial degeneration. We note that the same holds for the degenerations* $\langle 2^k\rangle \trianglerighteq W^{\boxtimes k}$ *for all* k*.*

It is clear that one cannot drop the condition that A_1 has full rank: in Section 4.2.2, we saw that Strassen's tensor Str_r is an example of a partial degeneration of $\langle r\rangle$ where A_1 has rank $r-1$. In fact, we can use Proposition 4.2.10 to prove the following characterization of all partial degenerations of $\langle r\rangle$ where the constant map has rank $r-1$.

Proposition 4.2.12. *Let* $T \in U_1 \otimes U_2 \otimes U_3$ *with* $\dim U_1 = r-1$ *be a concise tensor such that* $\langle r\rangle \blacktrianglerighteq T$ *and* $\langle r\rangle \not\geq T$*. Then, for some* q *such that* $3 \leq q \leq r$*, the tensor* T *decomposes as*

$$T = S_q + X_{r-q}$$

where $\mathrm{Str}_q \geq S_q$ *and* $\langle r-q\rangle \geq X_{r-q}$*.*

Proof. Suppose $\langle r\rangle \blacktrianglerighteq T$ via a partial degeneration

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^d}(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\langle r\rangle = T \in U_1 \otimes U_2 \otimes U_3.$$

Since T is concise, the map A_1 has rank equal to $\dim U_1 = r-1$. Note that A_1 can be factored as $A_1 = AM_qDP$ where $A\colon \mathbb{C}^{r-1} \to U_1$ is invertible, $M_q\colon \mathbb{C}^r \to \mathbb{C}^{r-1}$ is defined as

$$M_q\colon \begin{cases} \mathrm{e}_i \mapsto \mathrm{e}_i & \text{for } 1 \leq i \leq r-1,\\ \mathrm{e}_r \mapsto \mathrm{e}_1 + \cdots + \mathrm{e}_{q-1} & \end{cases}$$

with $1 \leq q \leq r$, $D\colon \mathbb{C}^r \to \mathbb{C}^r$ is diagonal, and $P\colon \mathbb{C}^r \to \mathbb{C}^r$ is a permutation matrix. Indeed, suppose $\pi \in \mathfrak{S}_r$ is a permutation such that $A_1\mathrm{e}_{\pi(1)}, \ldots A_1\mathrm{e}_{\pi(r-1)}$ are linearly independent and $A_1\mathrm{e}_{\pi(r)} = \lambda_1 A_1\mathrm{e}_1 + \cdots + \lambda_{q-1}A_1\mathrm{e}_{q-1}$ with nonzero $\lambda_1, \ldots, \lambda_{q-1}$. Defining $A\colon \mathrm{e}_i \mapsto \lambda_i A_1 \mathrm{e}_{\pi(i)}$, $D = \mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_{q-1}^{-1}, 1, \ldots, 1)$, and P the permutation matrix corresponding to π^{-1}, we get the required factorization.

Note that $(DP \otimes \mathrm{id} \otimes \mathrm{id})\langle r\rangle = (\mathrm{id} \otimes DP^{-1} \otimes P^{-1})\langle r\rangle$. Now, we can rearrange the partial degeneration $(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\langle r\rangle$ as

$$(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\langle r\rangle = (A \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes A_2(\varepsilon)DP^{-1} \otimes A_3(\varepsilon)P^{-1})(M_q \otimes \mathrm{id} \otimes \mathrm{id})\langle r\rangle.$$

This means that if $\langle r\rangle \blacktrianglerighteq T$, then up to a change of basis T is a partial degeneration of

$$(M_q \otimes \mathrm{id} \otimes \mathrm{id})\langle r\rangle = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes (\mathrm{e}_i \otimes \mathrm{e}_i + \mathrm{e}_r \otimes \mathrm{e}_r) + \sum_{i=q}^{r-1} \mathrm{e}_i \otimes \mathrm{e}_i \otimes \mathrm{e}_i$$

with an identity map on the first factor.

Define $H_q = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes (\mathrm{e}_i \otimes \mathrm{e}_i + \mathrm{e}_q \otimes \mathrm{e}_q) \in \mathbb{C}^{q-1} \otimes \mathbb{C}^q \otimes \mathbb{C}^q$. We have $(M_q \otimes \mathrm{id} \otimes \mathrm{id})\langle r\rangle = H_q + \langle r-q\rangle$. Using Proposition 4.2.10, we see that $T = S_q + X_{r-q}$ where S_q is a partial degeneration of H_q and X_{r-q} is a restriction of $\langle r-q\rangle$. It remains to analyze partial degenerations of H_q.

So, consider a partial degeneration

$$S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d} (\mathrm{id} \otimes B(\epsilon) \otimes C(\epsilon)) H_q.$$

Define $b_i(\epsilon) = B(\epsilon)\mathrm{e}_i$ and $c_i(\epsilon) = C(\epsilon)\mathrm{e}_i$. Suppose $b_q(\epsilon) = b_{q,\mu}\epsilon^\mu + b_{q,\mu+1}\epsilon^{\mu+1} + \dots$ After a basis change, we may assume $b_{q,\mu} = \mathrm{e}_q$. Define

$$E(\epsilon)\colon \begin{cases} \mathrm{e}_i \mapsto \mathrm{e}_i, & i < q, \\ \mathrm{e}_q \mapsto \epsilon^{-\mu} b_q(\epsilon). \end{cases}$$

We have $\lim_{\epsilon\to 0} E(\epsilon) = \mathrm{id}$, so by changing $B(\epsilon)$ to $E(\epsilon)^{-1}B(\epsilon)$ we obtain a partial degeneration for the same tensor S_q with $b_q(\epsilon) = \epsilon^\mu \mathrm{e}_q$. Using the same argument, we can assume without loss of generality that $c_q(\epsilon) = -\epsilon^\nu \mathrm{e}_q$. In this situation, we have

$$S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d} (\mathrm{id} \otimes B(\epsilon) \otimes C(\epsilon)) H_p = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes \left(\frac{1}{\epsilon^d} b_i(\epsilon) \otimes c_i(\epsilon) - \epsilon^{\mu+\nu-d} \mathrm{e}_q \otimes \mathrm{e}_q \right).$$

In case $\mu + \nu > d$, we clearly have

$$S_q = \lim_{\epsilon \to 0} \sum_{i=1}^{q} \mathrm{e}_i \otimes \left(\frac{1}{\epsilon^d} b_i(\epsilon) \otimes c_i(\epsilon) \right).$$

In this case, S_q is a partial degeneration of $\langle q-1\rangle$ and by Proposition 4.2.10, we can choose the $b_i(\epsilon)$ and $c_i(\epsilon)$ constant in ϵ and obtain $\langle q-1\rangle \geq S_q$ which yields $T \leq \langle r-1\rangle \leq \langle r\rangle$.

If $\mu + \nu < d$, we must have

$$\begin{aligned} b_i(\epsilon) &= \epsilon^\sigma \mathrm{e}_q + \tilde{b}_i(\epsilon) \\ c_i(\epsilon) &= \epsilon^\tau \mathrm{e}_q + \tilde{c}_i(\epsilon) \end{aligned}$$

with $\sigma + \tau = \mu + \nu$ so that

$$S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d} \sum_{i=1}^{q-1} \mathrm{e}_i \otimes \left(\epsilon^\sigma \tilde{b}_i(\epsilon) \otimes \mathrm{e}_q + \epsilon^\tau \mathrm{e}_q \otimes \tilde{c}_i(\epsilon) \right).$$

For each $i = 1, \dots, q-1$, the limit

$$\mathrm{e}^i S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d} \left(\epsilon^\sigma \tilde{b}_i(\epsilon) \otimes \mathrm{e}_q + \epsilon^\tau \mathrm{e}_q \otimes \tilde{c}_i(\epsilon) \right)$$

must exist and is of the form $b_i \otimes \mathrm{e}_q + \mathrm{e}_q \otimes c_i$ for some $b_i \in U_2$ and $c_i \in U_3$. Consequently, $S_q = \sum_{i=1}^{q-1} \mathrm{e}_i \otimes (b_i \otimes \mathrm{e}_q + \mathrm{e}_q \otimes c_i)$ is a restriction of Str_q.

Finally, consider the case $\lambda + \mu = d$. Here it holds that

$$S_q = \lim_{\epsilon\to 0} \sum_{i=1}^{q-1} \mathrm{e}_i \otimes \left(\frac{1}{\epsilon^d} b_i(\epsilon) \otimes c_i(\epsilon) - \epsilon^{\lambda+\mu-d} \mathrm{e}_q \otimes \mathrm{e}_q \right) =$$

$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon^d} \sum_{i=1}^{q-1} \mathrm{e}_i \otimes (b_i(\epsilon) \otimes c_i(\epsilon)) - \left(\sum_{i=1}^{q-1} \mathrm{e}_i \right) \otimes \mathrm{e}_q \otimes \mathrm{e}_q$$

In this case, S_q is a partial degeneration of $\langle q \rangle$, and applying Proposition 4.2.10, we see that $\langle q \rangle \geq S_q$ and $\langle r \rangle \geq T$.

We obtain that the only case where $\langle r \rangle \ngeq T$ is when $T = S_q + X_{r-q}$ with $S_q \leq \mathrm{Str}_q$ and $X_{r-q} \leq \langle r - q \rangle$ for some q such that $1 \leq q \leq r$. We can exclude cases $q = 1$ and $q = 2$ because in these cases $\mathrm{Str}_q \leq \langle q \rangle$. □

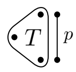
4.3 Aided restriction and aided rank

A related notion to partial degeneration is the notion of aided rank which we will introduce in Section 4.3.1. In Section 4.3.2, we will present a generalization of the method to lower bound rank in [AFT11] and use it in Section 4.3.3 to calculate the aided rank for tensor powers of the W-tensor.

4.3.1 Aided restriction and interpolation

In this section, we will introduce the notion of aided rank and show its relation to partial degeneration. For any tensor $T \in U_1 \otimes U_2 \otimes U_3$ recall the notation

$$T^{\blacksquare p} = T \boxtimes \langle 1, 1, p \rangle =$$



from Example 1.1.10.

Recall the following interpolation result, which is based on ideas introduced in [BCLR79].

Theorem 4.3.1. *Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ such that $T \trianglerighteq_d^e S$. Then, $T \boxtimes \langle e + 1 \rangle \geq S$ and $T \boxtimes \langle 2d + 1 \rangle \geq S$.*

We start by observing that one can use a unit matrix instead of a unit tensor to interpolate degenerations. We use notation from matrix multiplication in order to write this matrix as $\langle 1, 1, p \rangle$ where p is the rank of the unit matrix.

Lemma 4.3.2. *Consider tensors $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ and assume $T \trianglerighteq S$. Then,*

$$T^{\blacksquare \mathrm{u}_3 \mathrm{v}_3} \geq S.$$

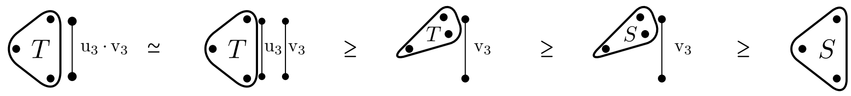


Figure 4.2: A visualization of the proof of Lemma 4.3.2: For the matrix multiplication tensor it holds that $\langle 1,1,\mathrm{u}_3\mathrm{v}_3\rangle = \langle 1,1,\mathrm{u}_3\rangle \boxtimes \langle 1,1,\mathrm{v}_3\rangle$. Using the supporting matrix $\langle 1,1,\mathrm{u}_3\rangle$, we can teleport the third party to the second party and view T as a two-party tensor. After that, $T \geq S$ and $T \trianglerighteq S$ are equivalent. Again, using $\langle 1,1,\mathrm{v}_3\rangle$, we can recover the three-party version of S.

Proof. The proof uses the teleportation trick Example 1.1.10, and we visualize the steps of this proof in Figure 4.2. Using the teleportation trick we observe that $T^{\blacksquare \mathrm{u}_3} \geq \tilde{T}$ where the tensor $\tilde{T} \in U_1 \otimes (U_2 \otimes U_3) \otimes \mathbb{C}$ is T considered as a two-party tensor. We can also interpret S as bipartite tensor $\tilde{S} \in V_1 \otimes (V_2 \otimes V_3) \otimes \mathbb{C}$ and, by assumption, know $\tilde{T} \trianglerighteq \tilde{S}$. In fact, since degeneration and restriction are equivalent for tensors on two factors, we have $\tilde{T} \geq \tilde{S}$. Now, applying the teleportation trick again, it is not hard to see that $\tilde{S}^{\blacksquare \mathrm{v}_3} \geq S$. Here, we teleport the third party of S which, when we consider $\tilde{S}$, at the second party back to the third party. After all,

$$T^{\blacksquare(\mathrm{u}_3 \cdot \mathrm{v}_3)} = (T^{\blacksquare \mathrm{u}_3})^{\blacksquare \mathrm{v}_3} \geq \tilde{T}^{\blacksquare \mathrm{v}_3} \geq \tilde{S}^{\blacksquare \mathrm{v}_3} \geq S$$

which finishes the proof. $\square$

The main question we ask is for a degeneration $T \trianglerighteq S$, how big must p be such that $T^{\blacksquare p} \geq S$. We will find that the minimal rank of an aiding matrix necessary to turn a degeneration into a restriction can be chosen drastically smaller if the degeneration is a partial degeneration. On the other hand, we will calculate p precisely for the degeneration $\langle 2^k\rangle \trianglerighteq W^{\boxtimes k}$ where we know from Proposition 4.2.10 that no partial degeneration exists. As it will turn out, here, the minimal p differs from the naive bound in Lemma 4.3.2 only by a factor of $\frac{1}{2}$. To simplify further discussions, let us introduce the following notation.

Definition 4.3.3. *Let $S \in V_1 \otimes V_2 \otimes V_3$ and fix $p \geq 1$. We define the* aided rank *of S as*

$$R^{\blacksquare p}(S) = \min\{r \colon \langle r\rangle^{\blacksquare p} \geq S\}.$$

Clearly, we have $R^{\blacksquare 1}(S) = R(S)$. Lemma 4.3.2 shows that $\underline{R}(S) = r$ implies that there is some q such that $R^{\blacksquare p}(S) = r$. To find better bounds on the minimal p, we now show a variation of Theorem 4.3.1.

Proposition 4.3.4. *Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ and assume $T \blacktriangleright_d^e S$. Then,*

$$T^{\blacksquare d+1} \geq S \text{ and } T^{\blacksquare e+1} \geq S.$$

Proof. Say, the partial degeneration is given by

$$(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\,T = \epsilon^d S + \sum_{i=1}^{e} \epsilon^{d+i} S_i. \tag{4.8}$$

Note that we can discard powers of ϵ higher than d and write

$$A_2(\epsilon) = \sum_{i=0}^{d} \epsilon^i A_{2,i}, \quad A_3(\epsilon) = \sum_{i=0}^{d} \epsilon^i A_{3,i}.$$

We then observe

$$S = A_1 \otimes \left(\sum_{i=0}^{d} A_{2,i} \otimes A_{3,d-i} \right) T$$

and therefore

$$A_1 \otimes \left(\sum_{i=0}^{d} A_{2,i} \otimes \mathrm{e}_i^* \right) \otimes \left(\sum_{i=0}^{d} A_{3,d-i} \otimes \mathrm{e}_i^* \right) T^{\blacksquare d+1} = S$$

which shows $T^{\blacksquare d+1} \geq S$.

In order to see $T^{\blacksquare e+1} \geq S$, note that for $\epsilon > 0$, we can rewrite Equation (4.8) as

$$\big(A_1 \otimes (A_2(\epsilon)/\epsilon^d) \otimes A_3(\epsilon)\big)\,T = S + \epsilon S_1 + \cdots + \epsilon^e S_e =: q(\epsilon).$$

Using Langrangian interpolation, we can pick $\alpha_0, \ldots, \alpha_e \neq 0$ such that

$$q(\epsilon) = \sum_{j=0}^{e} q(\alpha_j) \prod_{m \neq j} \frac{\epsilon - \alpha_m}{\alpha_j - \alpha_m}.$$

By writing $\mu_j := \prod_{m \neq j} \frac{\alpha_m}{\alpha_m - \alpha_j}$, we therefore get $S = q(0) = q(\alpha_0)\mu_0 + \cdots + q(\alpha_e)\mu_e$. Note that the $q(\alpha_j)$ are all restrictions of T where the first restriction map can be chosen to be A_1. With that,

$$S = q(0) = \left(A_1 \otimes (\sum_{j=0}^{e} \frac{\mu_j}{\alpha_j^d} A_2(\alpha_j) \otimes \mathrm{e}_j^*) \otimes (\sum_{j=0}^{e} A_3(\alpha_j) \otimes \mathrm{e}_j^*) \right) T^{\blacksquare e+1}$$

which finishes the proof. □

In particular, we can exclude partial degeneration with a certain degeneration degree if we can lower bound the aided rank of a tensor. Note that in the case of prehomogeneous spaces, we can find an even better bound.

Proposition 4.3.5. *Assume that $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous under the action of* $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ *and let T be an element with dense orbit. Then, for all $S \in U_1 \otimes U_2 \otimes U_3$ it holds that*

$$T^{\blacksquare 2} \geq S.$$

Proof. Consider the affine degree-1 curve L parametrized by $L(\epsilon) = T + \epsilon(S - T)$. It is clear that both T and S lie on L. Clearly, the linear span of any two distinct points on L contains all points on L. The orbit of T is open in $U_1 \otimes U_2 \otimes U_3$. Therefore, the intersection of L and the complement of the orbit of T is a closed subset of L, that is, a finite collection of points. In particular, there exists a second point $\tilde{T}$ in the orbit of T on L. Writing $\tilde{T} = (\mathrm{id} \otimes M_2 \otimes M_3)T$, and $S = \lambda T + \mu \tilde{T}$, we observe

$$[\mathrm{id} \otimes (\lambda \mathrm{id} \otimes \mathrm{e}_1^* + \mu M_2 \otimes \mathrm{e}_2^*) \otimes (\mathrm{id} \otimes \mathrm{e}_1^* + M_3 \otimes \mathrm{e}_2^*)] T^{\blacksquare 2} = S$$

which proves the claim. □

Note that Proposition 4.3.5 supports the intuition that in the case of partial degenerations, the minimal aiding rank q turning it into a restriction is small. In Section 4.2.4 we saw that whenever $T \in U_1 \otimes U_2 \otimes U_3$ has a dense orbit under the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ it holds for all $S \in U_1 \otimes U_2 \otimes U_3$ that $T \trianglerighteq S$.

4.3.2 A substitution method for aided rank

In this section, we will give a method to calculate aided ranks precisely. Our method builds on a known method from [AFT11] which we recalled in Theorem 1.3.2. We will use it to calculate aided ranks of powers of the W-tensor. We start by mentioning the following easy technical fact without proof.

Lemma 4.3.6. *Let V be a vector space and U be a finite-dimensional subspace of dimension u of V. If U is contained in the span of vectors $u_1, \ldots, u_{\mathrm{u}}$, then all u_i must be elements of U.*

The second lemma gives a useful characterization of restrictions of $\langle n \rangle^{\blacksquare p}$ in terms of flattenings and is a simple generalization of [Lan12, Theorem 3.1.1.1].

Lemma 4.3.7. *Let $S \in V_1 \otimes V_2 \otimes V_3$ be any tensor and fix some natural number p. Then we have*

$$R^{\blacksquare p}(S) = \min\{r : S(V_1^*) \subseteq V_2 \otimes V_3 \text{ spanned by } r \text{ matrices of rank } \leq p\}.$$

Proof. If $\langle r \rangle^{\blacksquare p} \geq S$ we can write $S = a_1 \otimes N_1 + \cdots + a_r \otimes N_r$ for matrices N_i of rank at most p, in other words, $S(V_1^*)$ is spanned by r matrices $N_1, \ldots, N_r$ of rank at most p.

On the other hand, assume $S(V_1^*) = \mathrm{span}(N_1, \ldots, N_r)$ for matrices N_i of rank at most p. Fixing a basis of V_1, the tensor S is given by $S = \sum_{i=1}^{\mathrm{v}_1} \mathrm{e}_i \otimes M_i$ where $M_i = S(\mathrm{e}_i^*)$. Since $S(V_1^*)$ is spanned by the N_j for $j = 1, \ldots, r$, we can find coefficients λ_{ij} such that $M_i = \sum_{j=1}^{r} \lambda_{ij} N_j$ for all $i = 1, \ldots, \mathrm{v}_1$. Hence,

$$S = \sum_{j=1}^{r} \left(\sum_{i=1}^{\mathrm{v}_1} \lambda_{i,j} \mathrm{e}_i \right) \otimes N_j.$$

Noticing that this is a restriction of $\langle r \rangle^{\blacksquare p}$ finishes the proof. □

We can use Lemma 4.3.7 to generalize the substitution method we saw in Theorem 1.3.2.

Theorem 4.3.8. *Let* $S \in V_1 \otimes V_2 \otimes V_3$ *and say,* $\dim(V_1) = \mathrm{v}_1$. *Fixing a basis* $\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{v}_1}$ *of* V_1*, we write*

$$S = \sum_{i=1}^{\mathrm{v}_1} \mathrm{e}_i \otimes M_i$$

for matrices $M_i \in V_2 \otimes V_3$ *and assume* $M_1 \neq 0$. *Moreover, for complex numbers* $\lambda_2, \dots, \lambda_{\mathrm{v}_1}$ *define*

$$\hat{S}(\lambda_2, \dots, \lambda_{\mathrm{v}_1}) = \sum_{j=2}^{\mathrm{v}_1} \mathrm{e}_j \otimes (M_j - \lambda_j M_1).$$

Then, the following hold.

(i) There exist $\lambda_2, \dots, \lambda_{\mathrm{v}_1} \in \mathbb{C}$ *such that*

$$R^{\blacksquare p}(\hat{S}(\lambda_2, \dots, \lambda_{\mathrm{v}_1})) \leq R^{\blacksquare p}(S) - 1.$$

(ii) Assume that M_1 *has rank at most* p. *Then, for all* $\lambda_2, \dots, \lambda_{\mathrm{v}_1}$

$$R^{\blacksquare p}(\hat{S}(\lambda_2, \dots, \lambda_{\mathrm{v}_1})) \geq R^{\blacksquare p}(S) - 1.$$

Hence, if M_1 *has rank at most* p*, we always find* $\lambda_2, \dots, \lambda_{\mathrm{v}_1}$ *such that*

$$R^{\blacksquare p}(\hat{S}(\lambda_2, \dots, \lambda_{\mathrm{v}_1})) = R^{\blacksquare p}(S) - 1.$$

Proof. Let $r = R^{\blacksquare p}(T)$, that is, $S(V_1^*)$ is contained in the span of r matrices of rank at most p. Denote these matrices by $X_1, \dots, X_r$ and write

$$M_i = \sum_{j=1}^{r} \mu_{i,j} X_j \text{ for } i = 1, \dots, \mathrm{v}_1.$$

Without loss of generality, assume that $\mu_{1,1} \neq 0$ and set $\lambda_i = \frac{\mu_{1,i}}{\mu_{1,1}}$. We easily see that $\hat{S}(\lambda_2, \dots, \lambda_a)(V_1^*) \subset \mathrm{span}(X_2, \dots, X_r)$, and therefore

$$R^{\blacksquare p}(\hat{S}(\lambda_2, \dots, \lambda_{\mathrm{v}_1})) \leq R^{\blacksquare p}(S) - 1.$$

That shows the first claim.
On the other hand, if M_1 has rank at most p and $Y_1, \dots, Y_s$ span $\hat{S}(\lambda_2, \dots, \lambda_{\mathrm{v}_1})$, then clearly $M_1, Y_1, \dots, Y_s$ will span $S(V_1^*)$, which shows the second claim. □

In the next section, we will see how one can use Theorem 4.3.8 to calculate aided ranks.

4.3.3 Aided rank of Kronecker powers of the W-tensor

Let V_1, V_2 and V_3 be two-dimensional with fixed bases $\mathrm{e}_1, \mathrm{e}_2$. In this section, we will use the method developed in Section 4.3.2 to calculate the aided rank of powers of the W-tensor

$$W = \mathrm{e}_1 \otimes \mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_1 \otimes \mathrm{e}_2 \otimes \mathrm{e}_1 + \mathrm{e}_2 \otimes \mathrm{e}_1 \otimes \mathrm{e}_1 \in V_1 \otimes V_2 \otimes V_3.$$

The main result of this section is the following.

Proposition 4.3.9. *For the k'th Kronecker power of the W tensor* $W^{\boxtimes k} \in V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes V_3^{\otimes k}$ *it holds that*

$$R^{\blacksquare p}(W^{\boxtimes k}) \begin{cases} = 2^k & \text{if } p \geq 2^k \\ > 2^k & \text{if } p < 2^k. \end{cases}$$

Proof. It is also clear that for any $r < 2^k$, $\langle r \rangle^{\blacksquare p} \ngeq W^{\boxtimes k}$. On the other hand, $W(V_1^*)$ is the span of $\mathrm{e}_1 \otimes \mathrm{e}_2 + \mathrm{e}_2 \otimes \mathrm{e}_1$ and $\mathrm{e}_1 \otimes \mathrm{e}_1$, in other words, $R^{\blacksquare 2}(W) = 2$. Consequently, we know that $\langle 2^k \rangle^{\blacksquare 2^k} \geq W^{\boxtimes k}$ for all k, in other words, $R^{\blacksquare 2^k}(W^{\boxtimes k}) \leq 2^k$.

We will now use Theorem 4.3.8 to show that $\langle 2^k \rangle^{\blacksquare 2^k - 1} \ngeq W^{\boxtimes k}$ which will finish the proof. One can verify that – thinking of the elements of $V_2^{\otimes k} \otimes V_3^{\otimes k}$ as $2^k \times 2^k$ matrices – all matrices in $W^{\boxtimes k}((V_1^{\otimes k})^*)$ are of the form

$$\begin{pmatrix} * & & x_0 \\ & \iddots & \\ x_0 & & 0 \end{pmatrix}. \tag{4.9}$$

That is, all matrices in $W^{\boxtimes k}((V_1^{\otimes k})^*)$ have the same entry x_0 in all antidiagonal entries and zeros in all entries below the antidiagonal. Now, assume for some p that $\langle 2^k \rangle^{\blacksquare p} \geq W^{\boxtimes k}$. By Lemma 4.3.7, there are matrices N_i of rank at most p such that

$$W^{\boxtimes k}\left((V_1^{\otimes k})^*\right) \subseteq \langle N_1, \dots, N_{2^k} \rangle. \tag{4.10}$$

As $W^{\boxtimes k}$ is concise, $W^{\boxtimes k}\left(V_1^{\otimes k})^*\right)$ has dimension 2^k. Therefore, by Lemma 4.3.6, the N_i are elements of $W^{\boxtimes k}\left((V_1^{\otimes k})^*\right)$. We observe that a matrix of the form Equation (4.9) with $x_0 \neq 0$ has full rank 2^k. That is, if the matrices N_i have rank $p < 2^k$ and are elements of $W^{\boxtimes k}\left((V_1^{\otimes k})^*\right)$, their span only contains matrices with zeros on the antidiagonal. That is, Equation (4.10) cannot be satisfied if all N_i have rank at most $p < 2^k$, that is,

$$\langle 2^k \rangle^{\blacksquare p} \ngeq W^{\boxtimes k} \text{ if } p < 2^k.$$

In other words, $R^{\blacksquare p}(W^{\boxtimes k}) > 2^k$ for $p < 2^k$. □

In particular, we see that the minimal rank of an aiding matrix turning the degeneration $\langle 2^k \rangle \trianglerighteq W^{\boxtimes k}$ into a restriction differs from the bound in Lemma 4.3.2 only by a factor of $\frac{1}{2}$.

4.A Code for Remark 4.2.8

The following Macaulay2 [GS] code gives a lower bound of 37 on the dimension of the orbit of a generic tensor in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ which is $(2, 3, 3)$-compressible. This code is an adjustment of code that can be found at `https://fulges.github.io/code/BDG-DimensionTNS.html`.

```
V_1 = QQ[v_(1,1)..v_(1,3)]
V_2 = QQ[v_(2,1)..v_(2,4)]
V_3 = QQ[v_(3,1)..v_(3,4)]
W_1 = QQ[w_(1,1)..w_(1,3)]
W_2 = QQ[w_(2,1)..w_(2,4)]
W_3 = QQ[w_(3,1)..w_(3,4)]

ALL = V_1**V_2**V_3**W_1**W_2**W_3

M_1 = sub(random(QQ^4,QQ^4),ALL)
M_2 = mutableMatrix(ALL,4,4)
M_3 = mutableMatrix(ALL,4,4)

for i from 0 to 3 do(
    M_2_(0,i)=random(QQ);
    M_2_(i,0)=random(QQ);
    M_3_(0,i)=random(QQ);
    M_3_(i,0)=random(QQ);
    )
M_2 = matrix M_2
M_3 = matrix M_3

T = 0
for i from 1 to a do(
    for j from 1 to b do(
        for k from 1 to c do(
            T = T + M_i_(j-1,k-1)*w_(1,i)*w_(2,j)*w_(3,k);
            );
        );
     )--T is now (2,3,3)-compressible with random entries

-- a random point in Hom(W1,V1) + Hom(W2,V2) + Hom(W3,V3)
-- the rank of the differential of the parametrization map at randHom
-- will provide a lower bound on dim of the orbit closure of our tensor

randHom =flatten flatten apply(3,j->
toList apply(1..di_(j+1),i ->w_(j+1,i)=>sub(random(1,V_(j+1)),ALL)))

-- compute the image of the differential
-- LL will be a list of elements of multidegree (1,1,1),
-- which are to be interpreted as elements of V1 \otimes V2 \otimes V3
-- generating the image of the differential of the parametrization map
LL = flatten for i from 1 to 3 list (
     ww = sub(vars(W_i),ALL);
     vv = sub(vars(V_i),ALL);
     flatten entries (sub( (vv ** diff(ww,Tused)),randHom)));
minGen = mingens (ideal LL);
orbitdim = numcols(minGen) --37
```

4.B Partial degenerations of the unit tensor

We have seen in Proposition 4.2.10 that the unit tensor $\langle r\rangle$ does not admit partial degenerations where the constant map A_1 is full rank. However, we also saw that in the case that A_1 has rank $r-1$ there are honest partial degenerations which we classify in Proposition 4.2.12. In this appendix, we see that also in the realm of matrix pencils, examples exist. For that, we consider for simplicity only matrix pencils in $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^{m+1}$. Recall that the matrix pencil T given in Equation (4.6) has a dense orbit under the action of $\mathrm{GL}_m \times \mathrm{GL}_{m+1}$. It is well known (see, for example, [Lan12, Theorem 3.11.1.1]) that this pencil has rank $m+1$. On the other hand, it is known that the maximal rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^{m+1}$ is $\lceil \frac{3m}{2} \rceil$. Hence, we can find tensors S in $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^{m+1}$ with $\langle m+1\rangle \geq T \trianglerighteq S$, $R(S) > m+1$ and consequently $\langle m+1\rangle \trianglerighteq S$ but $\langle m+1\rangle \not\geq S$.

To see an explicit example, let us construct for every m a matrix pencil of rank greater or equal to $m+1$ to which $\langle m\rangle$ degenerates partially. For this, we recall the following well-known result about the rank of matrix pencils [Gri78, Já79].

Proposition 4.B.1. *Consider $p_1 \times q_1$ matrices T_1', T_2' and $p_2 \times q_2$ matrices T_1'', T_2''. Let T' be the tensor corresponding to the matrix pencil $[T_1', T_2']$ and similar for T'' and write $T \in \mathbb{C}^2 \otimes \mathbb{C}^{p_1+p_2} \otimes \mathbb{C}^{q_1+q_2}$ for the tensor corresponding to the matrix pencil*

$$\left[\begin{pmatrix} T_1' & \\ & T_1'' \end{pmatrix}, \begin{pmatrix} T_2' & \\ & T_2'' \end{pmatrix}\right].$$

Then, it holds that

$$R(T) = R(T') + R(T'').$$

We will now construct a partial degeneration of $\langle m\rangle$ and will show using Proposition 4.B.1 that it has rank at least $m+1$. Applying the linear map

$$A_1 : U \to \mathbb{C}^2, \mathrm{e}_k \mapsto \mathrm{e}_1 + k\mathrm{e}_2$$

we see that $\langle m\rangle$ restricts to the tensor corresponding to the matrix pencil $[\mathrm{id}_m, \mathrm{diag}(1,\dots,m)]$. Since the matrix

$$M = \begin{pmatrix} 1 & 1 & & & \\ & 2 & 1 & & \\ & & \ddots & \ddots & \\ & & & m-1 & 1 \\ & & & & m \end{pmatrix}$$

has m different eigenvalues $1,\dots,m$, we deduce that also the tensor associated with the matrix pencil $[\mathrm{id}_m, M]$ is a restriction of $\langle m\rangle$.

For any $1 < k < m$ define $S_{k,m}$ to be the tensor corresponding to the matrix pencil

$$\left[\begin{pmatrix} \begin{pmatrix}\mathrm{id}_{k-1}, & 0\end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 \\ \mathrm{id}_{m-k}\end{pmatrix}\end{pmatrix}, \begin{pmatrix} J_1 & 0 \\ 0 & J_2\end{pmatrix}\right]. \tag{4.11}$$

where

$$J_1 = \begin{pmatrix} 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & k-1 & 1 \end{pmatrix}, J_2 = \begin{pmatrix} 1 & & & & \\ k+1 & 1 & & & \\ & \ddots & \ddots & & \\ & & m-1 & 1 & \\ & & & m & \end{pmatrix}$$

One verifies that applying the degeneration maps

$$A_2(\epsilon) = \mathrm{diag}(\underbrace{1,\dots,1}_{k},\underbrace{\epsilon,\dots,\epsilon}_{m-k}),\quad A_3(\epsilon) = \mathrm{diag}(\underbrace{\epsilon,\dots\epsilon}_{k+1},\underbrace{1,\dots,1}_{m-k-1})$$

the tensor corresponding to the matrix pencil $[\mathrm{id}_k, \mathrm{diag}(1,\dots,m)]$ results in $\epsilon S_{k,m} + \mathcal{O}(\epsilon^2)$. In particular, $\langle m\rangle \trianglerighteq S_{k,m}$.

From Proposition 4.B.1, we know that

$$R(S_{k,m}) = R(S^1_{k,m}) + R(S^2_{k,m}) \tag{4.12}$$

where $S^1_{k,m}$ corresponds to $[(\mathrm{id}_{k-1}, 0), J_1])$ and $S^2_{k,m}$ to $[\begin{pmatrix}0 & \mathrm{id}_{m-k}\end{pmatrix}^t, J_2]$, respectively. Using flattenings, one can now verify that the two pencils in Equation (4.12) have ranks k and $m-k+1$, respectively, which shows $R(S_{k,m}) \geq m+1$. Hence, it is not a restriction of $\langle m\rangle$.

4.C The aided rank of the Coppersmith-Winograd-tensor

In this appendix, we demonstrate with further examples how to calculate aided rank using Theorem 4.3.8. In particular, we are going to calculate the aided ranks of the Coppersmith-Winograd-tensors (CW-tensors). The study of these tensors was a crucial tool in the breakthrough result [CW87] bounding the exponent of matrix multiplication ω from above by 2.376.

Definition 4.C.1. *Let $V_1 \cong V_2 \cong V_3 \cong \mathbb{C}^{q+2}$ and fix a basis $\mathrm{e}_0,\dots,\mathrm{e}_{q+1}$. The q'th CW-tensor is the tensor*

$$T_{CW,q} = \sum_{i=1}^{q} \mathrm{e}_0 \otimes \mathrm{e}_i \otimes \mathrm{e}_i + \mathrm{e}_{q+1} \otimes \mathrm{e}_0 \otimes \mathrm{e}_0 +$$

$$\sum_{j=1}^{q} \mathrm{e}_j \otimes \mathrm{e}_0 \otimes \mathrm{e}_j + \mathrm{e}_0 \otimes \mathrm{e}_{q+1} \otimes \mathrm{e}_0 +$$
$$\sum_{k=1}^{q} \mathrm{e}_k \otimes \mathrm{e}_k \otimes \mathrm{e}_0 + \mathrm{e}_0 \otimes \mathrm{e}_0 \otimes \mathrm{e}_{q+1} \in V_1 \otimes V_2 \otimes V_3.$$

We want to calculate $R^{\blacksquare p}(T_{CW,q})$ for any p and q.

Proposition 4.C.2. *For $p \geq 2$, the p-aided rank of the q'th Coppersmith-Winograd-tensor is given by*

$$R^{\blacksquare p}(T_{CW,q}) = q + 1 + \left\lceil \frac{q+2}{p} \right\rceil.$$

Proof. Writing

$$M(x_0, \dots, x_{q+1}) = \begin{pmatrix} x_{q+1} & x_1 & \dots & x_q & x_0 \\ x_1 & x_0 & & & \\ \vdots & & \ddots & & \\ x_q & & & x_0 & \\ x_0 & & & & 0 \end{pmatrix}$$

we have $T_{CW,q}(V_1^*) = \{M(x_0, \dots x_{q+1}) : x_0, \dots x_{q+1} \in \mathbb{C}\}$. Note that $T_{CW,q}$ is concise. Hence, we have $R^{\blacksquare p}(T_{CW,q}) \geq q+2$ for any $p \in \mathbb{N}$. Moreover, it is clear that $R^{\blacksquare p}(T_{CW,q}) \leq q+2$ whenever $p \geq q+2$ which gives $R^{\blacksquare p}(T_{CW,q}) = q+2$ for all $p \geq q+2$.

For $p \leq q+1$, we will use Theorem 4.3.8.

Say, $p \geq 2$. Interpreting $V_2 \otimes V_3$ as space of $(q+2) \times (q+2)$ matrices, we have

$$T_{CW,q} = \sum_{i=0}^{q+1} \mathrm{e}_i \otimes M(x_i = 1, x_j = 0 \text{ for } i \neq j),$$

The matrix $M(0, \dots, 0, 1)$ has rank 1, hence we can find $\lambda_0^{(1)}, \dots, \lambda_q^{(1)}$ using Theorem 4.3.8 such that

$$T_{CW,q}^{(1)} = \sum_{i=0}^{q} \mathrm{e}_i \otimes \underbrace{\Big(M(x_i = 1, x_j = 0 \text{ for } i \neq j) - \lambda_i^{(1)} M(0, \dots, 0, 1)\Big)}_{=:M^{(1)}(x_i=1, x_j=0 \text{ for } i \neq j)}$$

satisfies

$$R^{\blacksquare p}(T_{CW,q}^{(1)}) = R^{\blacksquare p}(T_{CW,q}) - 1.$$

Note that the matrices $M^{(1)}(x_0, \dots x_q)$ have the form

$$M^{(1)}(x_0, \dots, x_q) = \begin{pmatrix} * & x_1 & \dots & x_q & x_0 \\ x_1 & x_0 & & & \\ \vdots & & \ddots & & \\ x_q & & & x_0 & \\ x_0 & & & & 0 \end{pmatrix}.$$

Still, $M^{(1)}(0,\dots,0,1)$ has only non-zero entries in the first column or in the first row. That is, it has rank less than or equal to p, hence we can apply Theorem 4.3.8 again and obtain $\lambda_0^{(2)},\dots,\lambda_q^{(2)}$ such that

$$T^{(2)}_{CW,q} = \sum_{i=0}^{q-1} \mathrm{e}_{x_i} \otimes \underbrace{\left(M^{(1)}(x_i = 1, x_j = 0 \text{ for } i \neq j) - \lambda_i^{(2)} M^{(1)}(0,\dots,0,1)\right)}_{=:M^{(2)}(x_i=1,x_j=0 \text{ for } i\neq j)}$$

satisfies

$$R^{\blacksquare p}(T^{(2)}_{CW,q}) = R^{\blacksquare p}(T^{(1)}_{CW,q}) - 1.$$

Again, we see that the elements of $T^{(2)}_{CW,q}(V_1^*)$ have the form

$$M^{(2)}(x_0,\dots,x_{q-1}) = \begin{pmatrix} * & x_1 & \dots & * & x_0 \\ x_1 & x_0 & & & \\ \vdots & & \ddots & & \\ * & & & x_0 & \\ x_0 & & & & 0 \end{pmatrix}$$

Repeating this procedure $q+2$ times leads to

$$T^{(q+1)}_{CW,q} = \mathrm{e}_0 \otimes \underbrace{\left(M^{(q)}(1,0) - \lambda^{(q+1)} M^{(q)}(0,1)\right)}_{=:M^{(q+1)}}.$$

with

$$M^{(q+1)} = \begin{pmatrix} * & * & \dots & * & 1 \\ * & 1 & & & \\ \vdots & & \ddots & & \\ * & & & 1 & \\ 1 & & & & 0 \end{pmatrix}.$$

By Theorem 4.3.8 we reduced the aided rank by exactly 1 in each step yielding

$$R^{\blacksquare p}(T^{(q+1)}_{CW,q}) = R^{\blacksquare p}(T^{(q)}_{CW,q}) - 1 = \dots = R^{\blacksquare p}(T_{CW,q}) - (q+1).$$

As $M^{(q+1)}$ has rank $q+2$ it follows $R^{\blacksquare p}(T^{(q+1)}_{CW,q}) = \lceil \frac{q+2}{p} \rceil$. and with that

$$R^{\blacksquare p}(T_{CW,q}) = q + 1 + \left\lceil \frac{q+2}{p} \right\rceil.$$

□

We can also find the following upper bound on the aided rank of $T^{\boxtimes 2}_{CW,q}$.

Proposition 4.C.3. *It holds that*

$$R^{\blacksquare p^2}(T_{CW,q}^{\boxtimes 2}) \leq q^2 + 4q + 3 + \left\lceil \frac{(q+2)^2}{p^2} \right\rceil.$$

In particular, there there are choices of m, p and q such that both $\langle m\rangle^{\blacksquare p} \ngeq T_{CW,q}$ and $(\langle m\rangle^{\blacksquare p})^{\boxtimes 2} \geq (T_{CW,q})^{\boxtimes 2}$ hold.

Proof. Let us write

$$N(x_0,\dots,x_{q+1},y_0,\dots,y_{q+1}) = \begin{pmatrix} x_{q+1}\cdot M(y) & x_1\cdot M(y) & \dots & x_q\cdot M(y) & x_0\cdot M(y) \\ x_1\cdot M(y) & x_0\cdot M(y) & & & \\ \vdots & & \ddots & & \\ x_q\cdot M(y) & & & x_0\cdot M(y) & \\ x_0\cdot M(y) & & & & 0 \end{pmatrix}$$

where the matrices $M(y)$ are as in the proof of Proposition 4.C.2 given by

$$M(y) = M(y_0,\dots,y_{q+1}) = \begin{pmatrix} y_{q+1} & y_1 & \dots & y_q & y_0 \\ y_1 & y_0 & & & \\ \vdots & & \ddots & & \\ y_q & & & y_0 & \\ y_0 & & & & 0 \end{pmatrix}.$$

With this, we have

$$(T_{CW,q})^{\boxtimes 2} = \sum_{i,j=0}^{q+1} (\mathrm{e}_i \otimes \mathrm{e}_j) \otimes N(x_i = 1, y_j = 1).$$

and consequently,

$$(T_{CW,q})^{\boxtimes 2}\left(\left(V_1^{\otimes 2}\right)^*\right) = \{N(x,y) : x,y \in \mathbb{C}^{q+2}\}.$$

The rank of the matrix $N(x,y)$ depends on these vectors x and y.

(i) If $x = y = \mathrm{e}_0$, the matrix $N(x,y)$ has rank $(q+2)^2$.

(ii) If $x = \mathrm{e}_0$ and $y = \mathrm{e}_{q+1}$ or if $x = \mathrm{e}_{q+1}$ and $y = \mathrm{e}_0$, the matrix $N(x,y)$ has rank $q+2$.

(iii) If $x = \mathrm{e}_0$ and $y \in \{\mathrm{e}_1,\dots,\mathrm{e}_q\}$ or if $x \in \{\mathrm{e}_1,\dots,\mathrm{e}_q\}$ and $y = \mathrm{e}_0$ the matrix $N(x,y)$ has rank $2(q+2)$.

(iv) If $x = \mathrm{e}_{q+1}$ and $y \in \{\mathrm{e}_1,\dots,\mathrm{e}_q\}$ or if $x \in \{\mathrm{e}_1,\dots,\mathrm{e}_q\}$ and $y = \mathrm{e}_{q+1}$ the matrix $N(x,y)$ has rank 2.

(v) If $x = y = \mathrm{e}_{q+1}$, the matrix $N(x,y)$ has rank 1.

(vi) If $x \in \{\mathrm{e}_1,\dots,\mathrm{e}_q\}$ and $y \in \{\mathrm{e}_1,\dots,\mathrm{e}_q\}$ the matrix $N(x,y)$ has rank 4.

Hence, to generate $(T_{CW,q})^{\boxtimes 2}(A^*)$, we need to generate one matrix of rank 1, $2q$ matrices of rank 2, q^2 matrices of rank 4, two matrices of rank $q+2$, $2q$ matrices of rank $2(q+2)$ and one matrix of rank $(q+2)^2$. Assuming $p^2 \geq 2(q+2)$, we will need at most

$$q^2 + 4q + 3 + \left\lceil \frac{(q+2)^2}{p^2} \right\rceil$$

matrices of rank p^2 to generate $(T_{CW,q})^{\boxtimes 2}((V_1^{\otimes 2})^*)$. In other words,

$$R^{\blacksquare p^2}(T_{CW,q}^{\boxtimes 2}) \leq q^2 + 4q + 3 + \left\lceil \frac{(q+2)^2}{p^2} \right\rceil \tag{4.13}$$

To find m,p and q such that $\langle m\rangle^{\blacksquare p} \ngeq T_{CW,q}$ but $\langle m^2\rangle^{\blacksquare p^2} \geq (T_{CW,q})^{\boxtimes 2}$, we choose p and q such that $\left\lceil \frac{q+2}{p+1} \right\rceil < \left\lceil \frac{q+2}{p} \right\rceil$ and $m = q + 1 + \left\lceil \frac{q+2}{p} \right\rceil$. By construction, we have $\langle m\rangle^{\blacksquare p} \ngeq T_{CW,q}$. We have found an example whenever

$$q^2 + 4q + 3 + \left\lceil \frac{(q+2)^2}{p^2} \right\rceil \leq \left(q + 1 + \left\lceil \frac{q+2}{p+1} \right\rceil \right)^2.$$

To see an explicit example, pick $q = 11$ and $p = 6$. We have

$$R^{\blacksquare 6}(T_{CW,11}) = 11 + 1 + \left\lceil \frac{11+2}{6} \right\rceil = 15$$
$$R^{\blacksquare 7}(T_{CW,11}) = 11 + 1 + \left\lceil \frac{11+2}{7} \right\rceil = 14,$$

that is, $\langle 14\rangle^{\blacksquare 7} \geq T_{CW,11}$ but $\langle 14\rangle^{\blacksquare 6} \ngeq T_{CW,11}$. From Equation (4.13), we get

$$R^{\blacksquare 6^2}(T_{CW,11}^{\boxtimes 2}) \leq 11^2 + 4 \cdot 11 + 3 + \left\lceil \frac{13^2}{6^2} \right\rceil = 173 \leq 14^2 = 196.$$

That gives

$$\left(\langle 14\rangle^{\blacksquare 6}\right)^{\boxtimes 2} = \langle 196\rangle^{\blacksquare 36} \geq \langle 173\rangle^{\blacksquare 36} \geq T_{CW,11}^{\boxtimes 2}.$$

□

4.D Catalysis with an EPR pair

The observations in this chapter are closely related to the question of so-called *catalytic restrictions* and *catalytic degenerations*. In [CCD+10], the authors find examples of tensors $T \in U_1 \otimes \cdots \otimes U_k$ and $S \in V_1 \otimes \cdots \otimes V_k$ with $T \ngeq S$ such that there exists some other tensor $H \in W_1 \otimes \cdots \otimes W_k$ with $T \boxtimes H \geq S \boxtimes H$. This phenomenon is called *catalytic restriction*, the tensor H is called *catalyst*. Note that the set of k-party tensors together with restriction is a preordered semiring. It has been shown later that the example in [CCD+10] can be generalized to preordered semirings fulfilling certain properties [Fri20, Theorem 7.15].

We raise the question if an honest catalytic entanglement conversion is possible where the *catalyst* H is only an aiding matrix. In other words, we ask if we can find $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ such that $T \ngeq S$, but for some p, $T^{\boxtimes p} \geq S^{\boxtimes p}$. We are also interested in a similar question replacing restriction with degeneration.

In this appendix, we present some partial progress on that question. In particular, we will construct non-trivial examples that are reminiscent of catalysis with an aiding matrix by refining the rank resp. border rank decompositions of the matrix multiplication tensor and the Bini tensor. Recall the following celebrated result by Strassen [Str69].

Theorem 4.D.1. *The rank of the matrix multiplication tensor $\langle 2,2,2\rangle$ is 7.*

Proof. Recall that

$$\langle 2,2,2\rangle((\mathbb{C}^2 \otimes \mathbb{C}^2)^*) = \operatorname{span}(N_1, N_2, N_3, N_4)$$

where the N_i can be interpreted as matrices

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Defining the 7 rank-1 matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, M_5 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

one verifies easily that

$$\begin{array}{ccccccccc} M_1 & +M_2 & & -M_4 & & +M_6 & & = N_1 \\ & & & M_4 & & & +M_7 & = N_2 \\ & & M_3 & & & +M_6 & & = N_3 \\ M_1 & & -M_3 & & +M_5 & & +M_7 & = N_4. \end{array} \tag{4.14}$$

Hence, $\langle 2,2,2\rangle((\mathbb{C}^2 \otimes \mathbb{C}^2)^*)$ is spanned by seven rank-1 matrices which shows that the rank of matrix multiplication is at most 7. The lower bound is involved and was first shown in [Lan04]. For a more modern proof using a technique called *border apolarity*, we refer to [CHL19]. □

We will also need the following result about the Bini tensor [BLR80].

Theorem 4.D.2. *The border rank of the Bini tensor is* 5*, in other words,* $\underline{R}(T_{Bini}) = 5$*.*

Proof. For the Bini tensor, $T_{Bini}((\mathbb{C}^3)^*)$ is spanned by the three matrices

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the following border rank decomposition: For $\epsilon > 0$, define the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix},$$

$$L_4 = \begin{pmatrix} -1 & 1 & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L_5 = \begin{pmatrix} 0 & 1 & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \epsilon & \epsilon^2 & 0 \end{pmatrix}$$

They are rank 1 and we have

$$\begin{array}{cccccc} \epsilon L_1 & -\epsilon L_2 & & & & = \epsilon N_1 + \mathcal{O}(\epsilon^2) \\ & L_2 & & -L_4 & +L_5 & = \epsilon N_2 + \mathcal{O}(\epsilon^2) \\ L_1 & & -L_3 & & +L_5 & = \epsilon N_3 + \mathcal{O}(\epsilon^2) \end{array}$$

which shows that $\underline{R}(T_{Bini}) \leq 5$. From the discussion after Proposition 1.3.6, we know that the border rank of the Bini must be at least 5 and we conclude. □

We will now refine these two constructions and derive restrictions and degenerations that are reminiscent of catalysis.

Proposition 4.D.3. *For the matrix multiplication tensor it holds that* $\langle 6\rangle^{\boxtimes 3} \geq \langle 2,2,2\rangle^{\boxtimes 2}$ *and* $\langle 6\rangle^{\boxtimes 4} \geq \langle 2,2,2\rangle^{\boxtimes 3}$*. For the Bini tensor, we have* $\langle 4\rangle^{\boxtimes 5} \trianglerighteq T_{Bini}^{\boxtimes 3}$.

Proof. We start by showing $\langle 6\rangle^{\boxtimes 3} \geq \langle 2,2,2\rangle^{\boxtimes 2}$. The space $\langle 2,2,2\rangle^{\boxtimes 2}((\mathbb{C}^2 \otimes \mathbb{C}^2)^*)$ is spanned by matrices

$$\nu_1 = \begin{pmatrix} N_1 & \\ & N_1 \end{pmatrix}, \nu_2 = \begin{pmatrix} N_2 & \\ & N_2 \end{pmatrix}, \nu_3 = \begin{pmatrix} N_3 & \\ & N_3 \end{pmatrix}, \nu_4 = \begin{pmatrix} N_4 & \\ & N_4 \end{pmatrix}.$$

Consider the following matrices which are constructed blockwise from the matrices of Strassen's algorithm:

$$\mu_2 = \begin{pmatrix} M_2 + M_1 & \\ & M_2 \end{pmatrix}, \mu_3 = \begin{pmatrix} M_3 & \\ & M_3 \end{pmatrix}, \mu_4 = \begin{pmatrix} M_4 & \\ & M_4 - M_1 \end{pmatrix},$$
$$\mu_5 = \begin{pmatrix} M_5 + M_1 & \\ & M_5 \end{pmatrix}, \mu_6 = \begin{pmatrix} M_6 & \\ & M_6 \end{pmatrix}, \mu_7 = \begin{pmatrix} M_7 & \\ & M_7 + M_1 \end{pmatrix}.$$

Clearly, these matrices all have rank at most 3. Moreover, it is easy to see from Equation (4.14) that

$$\begin{aligned}
\mu_2 \quad -\mu_4 \quad +\mu_6 \quad &= \nu_1\\
\mu_4 \quad +\mu_7 \quad &= \nu_2\\
\mu_3 \quad +\mu_6 \quad &= \nu_3\\
-\mu_3 \quad +\mu_5 \quad +\mu_7 \quad &= \nu_4.
\end{aligned}$$

Applying a similar trick, we can also show $\langle 6\rangle^{\boxtimes 4} \geq \langle 2,2,2\rangle^{\boxtimes 3} = \langle 2,2,6\rangle$. Here, we need to consider the space $\langle 2,2,2\rangle^{\boxtimes 3}((\mathbb{C}^2 \otimes \mathbb{C}^2)^*)$ which is spanned by

$$\nu_1 = \begin{pmatrix} N_1 & & \\ & N_1 & \\ & & N_1 \end{pmatrix}, \nu_2 = \begin{pmatrix} N_2 & & \\ & N_2 & \\ & & N_2 \end{pmatrix}, \nu_3 = \begin{pmatrix} N_3 & & \\ & N_3 & \\ & & N_3 \end{pmatrix}, \nu_4 = \begin{pmatrix} N_4 & & \\ & N_4 & \\ & & N_4 \end{pmatrix}.$$

Again, we define the block matrices which this time are of rank ≤ 4

$$\mu_2 = \begin{pmatrix} M_2 + M_1 & & \\ & M_2 & \\ & & M_2 \end{pmatrix}, \mu_3 = \begin{pmatrix} M_3 & & \\ & M_3 & \\ & & M_3 - M_1 \end{pmatrix}, \mu_4 = \begin{pmatrix} M_4 & & \\ & M_4 - M_1 & \\ & & M_4 \end{pmatrix},$$
$$\mu_5 = \begin{pmatrix} M_5 + M_1 & & \\ & M_5 & \\ & & M_5 \end{pmatrix}, \mu_6 = \begin{pmatrix} M_6 & & \\ & M_6 & \\ & & M_6 + M_1 \end{pmatrix}, \mu_7 = \begin{pmatrix} M_7 & & \\ & M_7 + M_1 & \\ & & M_7 \end{pmatrix}.$$

From Equation (4.14), we see

$$\begin{array}{ccccccc} \mu_2 & & -\mu_4 & & +\mu_6 & & = \nu_1 \\ & & \mu_4 & & & +\mu_7 & = \nu_2 \\ & \mu_3 & & & +\mu_6 & & = \nu_3 \\ & -\mu_3 & & +\mu_5 & & +\mu_7 & = \nu_4 \end{array}$$

which shows that $\langle 2 \rangle^{\blacksquare 4} \geq \langle 2,2,2 \rangle^{\blacksquare 3}$.

Finally, we want to show that $\langle 4 \rangle^{\blacksquare 5} \geq T_{Bini}^{\blacksquare 3}$. With that, we see that for

$$\lambda_1 = \begin{pmatrix} L_1 + L_5 & & \\ & L_1 + L_5 & \\ & & L_1 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} L_2 + L_5 & & \\ & L_2 + L_5 & \\ & & L_2 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} L_3 & & \\ & L_3 & \\ & & L_3 - L_5 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} L_4 & & \\ & L_4 & \\ & & L_4 - L_5 \end{pmatrix}$$

we have

$$\begin{array}{ccccc} \epsilon\lambda_1 & -\epsilon\lambda_2 & & & = \epsilon\nu_1 + \mathcal{O}(\epsilon^2) \\ & \lambda_2 & & -\lambda_4 & = \epsilon\nu_2 + \mathcal{O}(\epsilon^2) \\ \lambda_1 & & -\lambda_3 & & = \epsilon\nu_3 + \mathcal{O}(\epsilon^2) \end{array}$$

which in particular implies $\langle 4 \rangle^{\blacksquare 5} \geq T_{Bini}^{\blacksquare 3}$. □

We will now show two results indicating that it is unlikely to actually find explicit, provable examples of catalysis with an aiding matrix as catalyst. Indeed, finding an example requires us to find tensors T and S such that $T \not\geq S$ resp. $T \not\trianglerighteq S$. For T being a unit tensor, this is equivalent to showing rank resp. border rank lower bounds. Only few methods to show lower bounds are known and the next two results show that well-known methods to show rank and border rank lower bounds actually "lift" to the catalytic setting.

Proposition 4.D.4. *Let $T \in U_1 \otimes U_2 \otimes U_3$ be a tensor such that $R(T) \geq r$ holds by the substitution method. Then, $R^{\blacksquare p}(T^{\blacksquare p}) \geq r$ holds. In particular, for all $p \geq 1$ it holds that $\langle r-1 \rangle^{\blacksquare p} \not\geq t^{\blacksquare p}$.*

Proof. Comparing Theorem 1.3.2 with Theorem 4.3.8 we see that we can step by step obtain the same lower bound on p-aided rank using the aided rank substitution method in Theorem 4.3.8 that was obtained for the rank of T. □

Also, the Koszul flattenings that we introduced in Section 1.3 lift to the aided setup.

Proposition 4.D.5. *Let* $T \in U_1 \otimes U_2 \otimes U_3$ *and consider a Koszul flattening* $T_{U_1}^{\wedge k}$. *It holds that*

$$\operatorname{rank}((T^{\blacksquare p})_{U_1}^{\wedge k}) = p \cdot \operatorname{rank}(T_{U_1}^{\wedge k}).$$

In particular, the the border rank lower bound $\underline{R}(T) \geq \lceil \frac{r}{\binom{\mathrm{u}_1}{k}} \rceil$ *lifts to* $\underline{R}(T^{\blacksquare p}) \geq \lceil \frac{pr}{\binom{\mathrm{u}_1}{k}} \rceil \geq p \lceil \frac{r}{\binom{\mathrm{u}_1}{k}} \rceil$.

Proof. Write $T = \mathrm{e}_1 \otimes M_1 + \cdots + \mathrm{e}_{\mathrm{u}_1} \otimes M_{\mathrm{u}_1}$ are think of $M_1, \ldots, M_{\mathrm{u}_1}$ as matrices. As we have seen in Section 1.3 the linear map $T_{U_1}^{\wedge k}$ can be represented by a matrix which has block structure with the blocks being the M_i. Since $T^{\blacksquare p}(U_1^*)$ is spanned by the matrices

$$\begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_1 \end{pmatrix}, \ldots, \begin{pmatrix} M_{\mathrm{u}_1} & & \\ & \ddots & \\ & & M_{\mathrm{u}_1} \end{pmatrix}$$

we see that by permuting rows and columns we can transform $(T^{\blacksquare p})_{U_1}^{\wedge k}$ into a block-diagonal matrix, the p diagonal blocks being the matrix representing $T_{U_1}^{\wedge k}$. This finishes the proof. □

In particular, this implies that for the Bini tensor, we will not find an example $\langle 4 \rangle^{\blacksquare p} \geq T_{Bini}^{\blacksquare p}$. The techniques to prove that matrix multiplication has border rank 7 go far beyond Koszul flattenings, see [Lan04, CHL19]. We do not know if they lift to the catalytic setup.

Chapter 5

New techniques for bounding stabilizer rank

In this final chapter, we introduce and discuss yet another way of decomposing tensors motivated by the task of simulating quantum circuits built from Clifford+T gates using the Gottesman-Knill theorem. In this specific setup, the relevant way of decomposing tensors is via so-called *stabilizer rank decompositions*. We call the minimal length of a stabilizer rank decomposition of a tensor its *stabilizer rank*. It turns out that the efficiency of this simulation technique scales polynomially in the stabilizer rank of tensor powers of the so-called T-state.

Upper and lower bounds on the stabilizer rank can provide more efficient algorithms and barriers for such, respectively. Yet, before [LS22], only few methods to lower bound stabilizer rank have been known [BSS16, PSV22]. In this chapter, we will see how one can apply techniques from number theory to investigate stabilizer rank. In particular, we will construct tensors with maximal possible stabilizer rank and see alternate proofs of the best-known asymptotic lower bounds on stabilizer rank and approximate stabilizer rank of tensor powers of the T-state. Using basic facts from algebra will also construct non-trivial examples of tensors with multiplicative stabilizer rank under the tensor product.

Warning: In this chapter, we will denote tensors using greek letters like ψ and ϕ. In the context of this chapter, we will denote by T the so-called T-state, which is given by $T = \frac{1}{\sqrt{2}}(\mathrm{e}_0 + e^{\frac{\pi}{4}}\mathrm{e}_1)$. Note that we used the letter T for any choice of tensors in Chapter 3 and Chapter 4. Different than in Chapter 2, Chapter 3 and Chapter 4, it will be important to consider *normalized tensors* where the norm comes naturally from the Euclidean inner product induced by a choice of *computational bases*. We will use the term *quantum state* to refer to a normalized tensor.

This chapter is a partly modified version of [LS22]. Parts of this chapter also appear in Benjamin Lovitz's Ph.D. thesis [Lov22].

5.1 Overview

It is of great practical importance to determine the classical simulation cost of quantum computations. Indeed, lower bounds on the simulation cost indicate quantum speedups, while upper bounds can help us to understand the limitations of quantum computation. Among many proposals for constructing a universal quantum computer like, for example, measurement-based and adiabatic quantum computation [RB01, RBB03, AL18], one of the earliest and still one of the most promising is the quantum circuit approach. Here, the quantum circuits are constructed from a universal gate set like, for example, the so-called *Clifford+T* gate set consisting of the gates H, S and $CNOT$ as well as the T-gate which we have introduced in Section 1.5 [DP89, BBC+95, BMP+99]. While it has been shown that circuits built from Clifford gates only can be simulated efficiently using the Gottesman-Knill theorem, this does not hold once we allow for T-gates in addition. The *stabilizer rank* is a useful barometer for the computational cost of classically simulating such quantum circuits under the stabilizer formalism [BSS16]. A *stabilizer state* is a quantum state in the orbit of a computational basis state under the action of the Clifford group. For a quantum state ψ, we define its *stabilizer rank*, denoted $\chi(\psi)$, to be the smallest integer r for which ψ can be written as a linear combination of r stabilizer states. The stabilizer rank is motivated by the fact that the classical simulation cost of a quantum circuit that applies Clifford gates to a computational basis state and measures in computational basis under current state-of-the-art simulation protocols scales polynomially in the number of Clifford gates and $\chi(\psi)$ (see Appendix 5.A, and [BSS16, BG16, BBC+19, QPG21]). For a real number $\delta > 0$, the *δ-approximate stabilizer rank*, $\chi_\delta(\psi)$, is defined as the minimum stabilizer rank over all quantum states that are δ-close to ψ, and similarly quantifies the classical simulation cost of approximating the application of Clifford gates and computational basis measurements to ψ under the stabilizer formalism.

Despite the practical importance of the stabilizer rank, few techniques are known for bounding this quantity [BSS16, PSV22]. In this chapter, we analyze stabilizer rank from a number-theoretic perspective which, in particular, will provide us with lower bounds on the stabilizer rank of copies of the T-state.

5.1.1 Lower bounds on stabilizer rank and approximate stabilizer rank

We start in Section 5.2 by refining a number-theoretic theorem of Moulton to prove lower bounds on stabilizer rank and approximate stabilizer rank [Mou01].

Definition 5.1.1. *Let $[n] = \{1, \dots, n\}$ when n is a positive integer. For integers $q \geq 2$ and $r \geq 1$, and tuples of non-zero complex numbers*

$$\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{C}^q$$
$$\beta = (\beta_1, \dots, \beta_r) \in \mathbb{C}^r,$$

we say that β *is a* subset-sum representation *of* α *if, for all* $i \in [q]$*, there exists a subset* $R_i \subseteq [r]$ *for which* $\sum_{j \in R_i} \beta_j = \alpha_i$*. We refer to the integer* r *as the* length *of the subset-sum representation* $\beta \in \mathbb{C}^r$*.*

It is interesting to ask for a fixed tuple α for the minimal possible length of a subset-sum representation. To find examples of tuples that require subset sum representations with large length, the following property will be useful.

Definition 5.1.2. *For an integer* $2 \le p \le q$*, we say that* $\alpha \in \mathbb{C}^q$ *has an* exponentially increasing subsequence *of length* p *if there exists* $i_1, \ldots, i_p \in [q]$ *for which*

$$\left|\alpha_{i_{j+1}}\right| \ge 2\left|\alpha_{i_j}\right| \quad \textit{for all} \quad j \in [p].$$

Moulton showed that any subset-sum representation of a q-tuple containing the subsequence $(1, 2, 4, \ldots, 2^{p-1})$ has length at least $p/\log_2 p$ [Mou01]. We will refine this result in Theorem 5.2.1 and prove that the same bound holds for any q-tuple that contains an exponentially increasing subsequence of length p.

Since stabilizer states have coordinates in $\{0, \pm 1, \pm i\}$ in the computational basis (see Proposition 1.5.3), any decomposition of a state ψ into a superposition of r stabilizer states can be converted into a subset-sum representation of length $4r$ of the coordinates of ψ. Using that, we will show in Theorem 5.2.4 that if the coordinates of ψ contain an exponentially increasing subsequence of length p, then $\chi(\psi) \ge p/(4\log_2 p)$. In particular, since T is Clifford-equivalent to the H-*state* $H \propto \mathrm{e}_0 + \frac{1}{\sqrt{2}-1}\mathrm{e}_1$, and the coordinates of $H^{\otimes n}$ contain an exponentially increasing subsequence of length $n+1$, we obtain $\chi(T^{\otimes n}) \ge \frac{n+1}{4\log_2(n+1)}$. More generally, we prove in Theorem 5.2.4 that $\chi(\psi^{\otimes n}) = \Omega(n/\log_2 n)$ for any non-stabilizer qubit state ψ.

We further use Theorem 5.2.1, along with standard concentration inequalities for the binomial distribution, to prove in Theorem 5.2.8 that for any non-stabilizer qubit state ψ, there exists a constant $\delta > 0$ for which it holds that $\chi_\delta(\psi^{\otimes n}) \ge \sqrt{n}/(2\log_2 n)$ for all $n \ge 2$.

We note that similar results have been obtained in [PSV22]: Here, the authors prove that $\chi(T^{\otimes n}) \ge n/100$, and that there exists $\delta > 0$ for which $\chi_\delta(T^{\otimes n}) = \Omega(\sqrt{n}/\log_2 n)$ [PSV22]. Asymptotically, our bounds match theirs up to a log factor, and we suggest that our proof technique is much simpler. While both of our bounds follow quite quickly from our refinement of Moulton's theorem mentioned above, the two bounds in [PSV22] use two different approaches from the analysis of boolean functions and complexity theory: For their lower bound on $\chi(T^{\otimes n})$, they analyze directional derivatives of quadratic polynomials, and for their lower bound on $\chi_\delta(T^{\otimes n})$, they use Razborov-Smolensky low-degree polynomial approximations and correlation bounds against the majority function [Raz87, Smo87, Smo93]. It is interesting that the vastly different approaches of ours and [PSV22] yield such similar results.

As a further application of our refinement of Moulton's theorem, we explicitly construct a sequence of n-qubit product states $\psi^{\otimes n}$ for which it holds that $\chi(\psi^{\otimes n}) \ge \frac{2^n}{4n}$ and $\chi_\delta(\psi^{\otimes n}) = \mathcal{O}(1)$

for any $\delta > 0$, simply by writing down a product state with exponentially increasing coordinate amplitudes. Using different techniques, we construct in Proposition 5.2.13 a sequence of n-qubit product states $\psi^{\otimes n}$ for which $\chi(\psi^{\otimes n}) = 2^n$ (the largest possible) and $\chi_\delta(\psi^{\otimes n}) = 1$ (the smallest possible). It is interesting to compare this situation to tensor rank and tensor border rank: For example, we have seen that there must exist tensors in $(\mathbb{C}^d)^{\otimes 3}$ with border rank scaling quadratically in d, see Remark 1.1.6. On the other hand, all methods we have seen (and, in fact, all known methods, see Remark 1.3.7) to lower bound tensor rank and tensor border rank can only prove linear lower bounds.

5.1.2 States with multiplicative stabilizer rank under the tensor product

It is a standard fact that the stabilizer rank is *sub-multiplicative* under the tensor product, i.e., $\chi(\psi \otimes \psi) \leq \chi(\psi)^2$ for any quantum state ψ [Qas21, Section 2.1.3]. In [Qas21, Section 4.4], it was remarked that there are no known examples of quantum states ψ of stabilizer rank greater than one for which equality holds. In Section 5.3, we explicitly construct two-qubit states ψ for which $\chi(\psi) = 2$ and $\chi(\psi \otimes \psi) = 4$. This is the smallest possible example of such a state, since for any single-qubit state ϕ it holds that $\chi(\phi \otimes \phi) \leq 3$.

5.2 Lower bounds on stabilizer rank and approximate stabilizer rank

In this section, we will present a method to lower bound stabilizer rank using a result from number theory. In Section 5.2.1, we will refine a result of Moulton about the minimal length of a subset-sum representation of a tuple containing a long exponentially increasing sequence. We then use our refinement in Section 5.2.2 to prove lower bounds on the stabilizer rank of quantum states whose coordinates contain a long exponentially increasing subsequence. As an application, we will see that $\chi(\psi^{\otimes n}) = \Omega(n/\log_2 n)$ for any non-stabilizer qubit state ψ which, in particular, gives a strong lower bound on the stabilizer rank of n copies of the T state. We finally combine this technique in Section 5.2.3 with a standard concentration inequality to see that for any non-stabilizer qubit state ψ, there exists $\delta > 0$ for which $\chi_\delta(\psi^{\otimes n}) \geq \sqrt{n}/(2\log_2 n)$ for all $n \in \mathbb{N}$. Moreover, our techniques will allow us to construct in Section 5.2.4 states in $(\mathbb{C}^2)^{\otimes n}$ with stabilizer rank exponential in n.

5.2.1 A refinement of Moulton's theorem

The following refinement of [Mou01, Theorem 1] will be used throughout this chapter.

Theorem 5.2.1. *Let $2 \leq p \leq q$ be integers, and let $\alpha \in \mathbb{C}^q$ be a q-tuple of non-zero complex numbers. If α contains an exponentially increasing subsequence of length p, then any subset-sum representation of α has length at least $p/\log_2(p)$.*

Proof. It suffices to consider the case $p = q$ and $2|\alpha_i| \leq |\alpha_{i+1}|$ for all $i \in [q-1]$. Let $\beta \in \mathbb{C}^r$ be a subset-sum representation of α. Then for each $i \in [q]$, there exists $c_i \in \{0,1\}^r$ such that $\alpha_i = \beta^t c_i$ Suppose that, for some $u_1, \ldots, u_q, v_1, \ldots, v_q \in \{0,1\}$, we have

$$\sum_{i=1}^{q} u_i c_i = \sum_{i=1}^{q} v_i c_i.$$

Applying β^t to both sides gives

$$\sum_{i=1}^{q} u_i \alpha_i = \sum_{i=1}^{q} v_i \alpha_i.$$

It follows that $u_i = v_i$ for all $i \in [q]$. Indeed, it suffices to prove that $|\alpha_{i+1}| > |\alpha_1 + \cdots + \alpha_i|$ for all $i \in [q-1]$, which in turn can be easily verified by an inductive argument. By assumption, $|\alpha_2| > |\alpha_1|$, and by induction,

$$|\alpha_1 + \cdots + \alpha_i| \leq |\alpha_1 + \cdots + \alpha_{i-1}| + |\alpha_i| < 2|\alpha_i| \leq |\alpha_{i+1}|.$$

The remainder of the proof is identical to that of [Mou01]. There are at most $2^q - 1$ choices of $u_1, \ldots, u_q \in \{0,1\}$, excluding the case $u_1 = \cdots = u_q = 1$. For each of these choices, the sum $\sum_{i=1}^{q} u_i c_i$ can take one of $q^r - 1$ possible values in $\{0, 1, \ldots, q-1\}^{\times r}$ (note that the vector $(q-1, q-1, \ldots, q-1)^t$ is excluded since the u_i are not all equal to 1). Since each choice of $u_1, \ldots, u_q$ yields a different vector, we must have $q^r - 1 \geq 2^q - 1$, i.e. $r \geq q/\log_2(q)$. ☐

5.2.2 Lower bounds on stabilizer rank

In this section, we use Theorem 5.2.1 to prove lower bounds on stabilizer rank. For that, we record the following technical results.

Lemma 5.2.2. *Let $\psi \in (\mathbb{C}^2)^{\otimes n}$ be any quantum state and C a Clifford unitary. Then, we have $\chi(\psi) = \chi(C\psi)$ and $\chi_\delta(\psi) = \chi_\delta(C\psi)$ for all $\delta > 0$.*

Proof. If $\psi \in \operatorname{span}(\sigma_1, \ldots, \sigma_r)$ is contained in the span of stabilizer states $\sigma_1, \ldots, \sigma_r$, then $C\psi$ is contained in the span of the stabilizer states $C\sigma_1, \ldots, C\sigma_r$, that is, $\chi(\psi) \geq \chi(C\psi)$. Applying the same argument with C^{-1} proves the other direction. Observing that the Clifford unitary C preserves the norm finishes the proof. ☐

To apply Theorem 5.2.1, the following technical result will be helpful.

Lemma 5.2.3. *Let $\psi \in \mathbb{C}^2$ not be a stabilizer state. Then, there is a Clifford unitary C such that*

$$C\psi \propto \frac{1}{\sqrt{1 + |\alpha|^2}} (e_0 + \alpha e_1)$$

with $|\alpha| > 1$.

Proof. Clearly,

$$\psi \propto \frac{1}{\sqrt{1+|\beta|^2}}(\mathrm{e}_0 + \beta \mathrm{e}_1)$$

for some β. We note that for $|\beta| > 1$, the result follows with C being the identity. For $|\beta| < 1$, the result holds with $C = X$. It remains to consider the case $|\beta| = 1$. Note that for $\beta \in \{0, \pm 1, \pm i\}$, the tensor ψ would be a stabilizer state. Else, one easily verifies that the claim of the lemma holds for either $C = H$ or $C = HX$. □

Theorem 5.2.4. *Let $p \geq 2$ be an integer, and let $\psi \in (\mathbb{C}^2)^{\otimes n}$ be a quantum state. If the coordinates of ψ contain an exponentially increasing subsequence of length p, then we have $\chi(\psi) \geq p/(4 \log_2 p)$.*

Proof. Let $r = \chi(\psi)$, let $x_1, \ldots, x_p \in \mathbb{F}_2^n$ be such that $|\psi_{x_i}| \leq 2|\psi_{x_{i+1}}|$ for all $i \in [p-1]$, and let $\alpha = (\psi_{x_1}, \ldots, \psi_{x_p}) \in \mathbb{C}^p$. Without loss of generality, there exist complex numbers c_i and (unnormalized) stabilizer states σ_i for $i \in [r]$ such that for all $i \in [r]$, every coordinate of σ_i is an element of $\{0, \pm 1, \pm i\}$, and $\psi = \sum_{i=1}^r c_i \sigma_i$. Let

$$S = (\sigma_1, \ldots, \sigma_r) \in \{0, \pm 1, \pm i\}^{\{0,1\}^n \times r}$$

be a matrix whose columns are the σ_i and

$$c = (c_1, \ldots, c_r) \in \mathbb{C}^r,$$

so that $Sc = \psi$. In particular, there exists a $p \times r$ submatrix T of S for which $Tc = \alpha$. Let $T_1, T_2, T_3, T_4 \in \{0,1\}^{p \times r}$ be such that

$$T = T_1 - T_2 + i(T_3 - T_4).$$

Then

$$(T_1, T_2, T_3, T_4)(c, -c, ic, -ic)^t = Tc = \alpha,$$

so $(c, -c, ic, -ic)$ is a subset-sum representation of α of length $4r$. It follows from Theorem 5.2.1 that $4r \geq p/(\log_2 p)$. This completes the proof. □

Theorem 5.2.4 also implies the following lower bound on $\chi(T^{\otimes n})$, and more generally, on $\chi(\psi^{\otimes n})$ for any non-stabilizer qubit state ψ.

Corollary 5.2.5. *For any state $\psi \in (\mathbb{C}^s)^{\otimes n}$ that is not a stabilizer state, $\chi(\psi^{\otimes n}) = \Omega(n/\log_2 n)$. In particular,*

$$\chi(T^{\otimes n}) \geq \frac{n+1}{4\log_2(n+1)}.$$

Proof. Since ψ is not a stabilizer state, we can assume by Lemma 5.2.2 and Lemma 5.2.3 that $\psi = \frac{1}{\sqrt{1+|\alpha|^2}}(\mathrm{e}_0 + \alpha \mathrm{e}_1)$ with $|\alpha| > 1$. Consequently, there exists $k \in \mathbb{N}$ for which $|\alpha|^k \geq 2$. (When $\psi = T$, we can take $k = 1$.) Now observe that the complex numbers $1, \alpha^k, \alpha^{2k}, \dots, \alpha^{\lfloor n/k \rfloor k}$ all appear as coordinates of $\psi^{\otimes n}$. By Theorem 5.2.4, it follows that

$$\chi(\psi^{\otimes n}) \geq \frac{\lfloor n/k \rfloor + 1}{4 \log_2(\lfloor n/k \rfloor + 1)}.$$

This completes the proof. □

5.2.3 Lower bounds on approximate stabilizer rank

In this section, we want to apply Theorem 5.2.1 to obtain lower bounds on approximate stabilizer rank of states of the form $\psi^{\otimes n}$ for non-stabilizer states $\psi \in \mathbb{C}^2$. To do so, we will apply the De Moivre-Laplace Theorem. For a proof and a more in-depth discussion of the De Moivre-Laplace Theorem, we refer to [Fel91, Section VII, Theorem 1] and [PSV22, Claim 4.6].

Theorem 5.2.6. *Let $p \in [0,1]$ and $C \geq 0$. Then, there exists $c > 0$ such that*

$$\binom{n}{k} p^{n-k} (1-p)^k \geq \frac{c}{\sqrt{n}} \tag{5.1}$$

for all $k \in [pn - C\lceil\sqrt{n}\rceil, pn + C\lceil\sqrt{n}\rceil]$ and for all $n \in \mathbb{N}$. The quantity c might depend on p and C, but it is independent of n.

Remark 5.2.7. *Say, we conduct n independent experiments where in each experiment, we get outcome "1" with probability p, and outcome "0" with probability $1-p$. The binomial distribution $B(n,p)$ describes the probability of seeing k instances of "1" for $k = 0, \dots, n$. This probability is exactly the quantity in Equation (5.1). In this context, Theorem 5.2.6 essentially says that the binomial distribution $B(n,p)$ is heavily concentrated on an interval $[pn - \mathcal{O}(\sqrt{n}), pn + \mathcal{O}(\sqrt{n})]$.*

We can apply Theorem 5.2.6 and recover an exponentially increasing subsequence of length $\sqrt{n}$ in the coordinates of any quantum state sufficiently close to $\psi^{\otimes n}$ where ψ is not a stabilizer state.

Theorem 5.2.8. *For any non-stabilizer qubit state $\psi \in (\mathbb{C}^2)^{\otimes n}$, there exists a constant $\delta > 0$ such that, for every integer $n \geq 2$,*

$$\chi_\delta(\psi^{\otimes n}) \geq \frac{\sqrt{n}}{2 \log_2 n}.$$

Proof. We know from Lemma 5.2.3 that we can assume $\psi = \frac{1}{\sqrt{1+|\alpha|^2}}(\mathrm{e}_0 + \alpha \mathrm{e}_1)$ for some $|\alpha| > 1$. Let $\beta = \frac{1}{\sqrt{1+|\alpha|^2}}$ and $\gamma = \frac{\alpha}{\sqrt{1+|\alpha|^2}}$, so that $\psi = \beta \mathrm{e}_0 + \gamma \mathrm{e}_1$.

Let $l \in \mathbb{N}$ be the smallest integer for which $|\alpha|^l > 2$, and let $\lambda = \frac{2}{|\alpha|^l}$. If $\psi = T$, then we can take $\alpha = \frac{1}{\sqrt{2}-1}$ and $l = 1$. Let

$$\mathcal{I} = \{q \in [n] : |\gamma|^2 n - l\lceil\sqrt{n}\rceil \leq q \leq |\gamma|^2 n + l\lceil\sqrt{n}\rceil\}$$

be the set of integers in the interval $[|\gamma|^2 n - l\lceil\sqrt{n}\rceil, |\gamma|^2 n + l\lceil\sqrt{n}\rceil]$. Note that $|\mathcal{I}| \geq 2l\sqrt{n}$. By Theorem 5.2.6, there exists a constant $\tilde{c} > 0$ (which may depend on $|\alpha|$, but does not depend on n) for which

$$\binom{n}{q}|\beta^{n-q}\gamma^q|^2 \geq \tilde{c}/\sqrt{n}$$

for all $q \in \mathcal{I}$. Let

$$c = \left(\frac{1-\lambda}{1+\lambda}\right)^2 \tilde{c},$$

so that

$$\left(\frac{1-\lambda}{1+\lambda}\right)^2 \binom{n}{q}|\beta^{n-q}\gamma^q|^2 \geq c/\sqrt{n} \tag{5.2}$$

for all $q \in \mathcal{I}$. Note that c only depends on $|\alpha|$.

For $\phi \in (\mathbb{C}^2)^{\otimes n}$ a quantum state, define $S \subset \mathcal{I}$ to be the set of $q \in \mathcal{I}$ such that for all $x \in \mathbb{F}_2^n$ of Hamming weight $|x| = q$, it holds that

$$\left|\psi_x^{\otimes n} - \phi_x\right| \geq \left(\frac{1-\lambda}{1+\lambda}\right)|\beta|^{n-q}|\gamma|^q. \tag{5.3}$$

With that,

$$\left\|\psi^{\otimes n} - \phi\right\|^2 \geq \sum_{q \in S}\binom{n}{q}\left(\frac{1-\lambda}{1+\lambda}\right)^2|\beta^{n-q}\gamma^q|^2 \geq |S|\frac{c}{\sqrt{n}}. \tag{5.4}$$

Define $\delta = \sqrt{cl}$ and assume $\left\|\psi^{\otimes n} - \phi\right\| < \delta$. With that, Equation (5.4) implies $|S| \leq l\sqrt{n}$. Since $|\mathcal{I}| \geq 2l\sqrt{n}$, we obtain $|\mathcal{I} \setminus S| \geq l\sqrt{n}$. By definition we can find for each $q \in \mathcal{I} \setminus S$ some $x_q \in \mathbb{F}_2^n$ with $|x_q| = q$ such that

$$\left|\psi_{x_q}^{\otimes n} - \phi_{x_q}\right| \leq \left(\frac{1-\lambda}{1+\lambda}\right)|\beta|^{n-q}|\gamma|^q. \tag{5.5}$$

Let now $q, q' \in \mathcal{I} \setminus S$ with $q < q'$ and pick $x_q, x_{q'}$ as in Equation (5.5). Then,

$$\frac{\left|\phi_{x_{q'}}\right|}{\left|\phi_{x_q}\right|} = \frac{\left|\phi_{x_{q'}} - \psi_{x_{q'}}^{\otimes n} + \psi_{x_{q'}}\right|}{\left|\phi_{x_q} - \psi_{x_q}^{\otimes n} + \psi_{x_q}\right|}$$

$$
\begin{aligned}
&\geq \frac{\left|\psi_{x_{q'}}\right| - \left|\phi_{x_{q'}} - \psi^{\otimes n}_{x_{q'}}\right|}{\left|\psi_{x_q}\right| + \left|\phi_{x_q} - \psi^{\otimes n}_{x_q}\right|} \\
&\geq \frac{(1 - \frac{1-\lambda}{1+\lambda})\left|\beta^{n-q'}\gamma^{q'}\right|}{(1 + \frac{1-\lambda}{1+\lambda})|\beta^{n-q}\gamma^{q}|} \\
&= \lambda|\alpha|^{q'-q},
\end{aligned}
$$

where the second line is the triangle inequality, the third line follows from Equation (5.5) and the fact that $\left|\psi_x\right| = \left|\beta^{n-|x|}\gamma^{|x|}\right|$ for all $x \in \mathbb{F}_2^n$, and the fourth from the definition of β, γ and λ, respectively. In particular, if $q' - q \geq l$, then $\frac{\left|\phi_{x_{q'}}\right|}{\left|\phi_{x_q}\right|} \geq 2$. Since $|\mathcal{I} \setminus S| \geq l\lceil\sqrt{n}\rceil$, there exists a subset $Q \subseteq \mathcal{I} \setminus S$ of size $|Q| \geq \sqrt{n}$ for which $q' - q \geq l$ for all $q, q' \in Q$ with $q < q'$. In particular, we see that ϕ must have an exponentially increasing subsequence of length $\sqrt{n}$ in its coordinates. By Theorem 5.2.4,

$$
\chi(\phi) \geq \frac{\sqrt{n}}{2\log_2(n)}.
$$

This completes the proof. □

5.2.4 Product states with exponential stabilizer rank and constant approximate stabilizer rank

Recall that, up to global phase, there are only finitely many stabilizer states in $(\mathbb{C}^2)^{\otimes n}$. In other words, there is a finite set S of stabilizer states such that for all stabilizer states $\sigma \in (\mathbb{C}^2)^{\otimes n}$ there is some $\tilde{\sigma} \in S$ with $\sigma \propto \tilde{\sigma}$. This, in particular, implies the following simple observation.

Lemma 5.2.9. *For all $r < 2^n$ the set of quantum states $\psi \in (\mathbb{C}^2)^{\otimes n}$ with $\chi(\psi) \leq r$ is a finite union of proper linear subspaces of $(\mathbb{C}^2)^{\otimes n}$. In particular, a generic quantum state in $(\mathbb{C}^2)^{\otimes n}$ has stabilizer rank 2^n.*

Let $\psi \in (\mathbb{C}^2)^{\otimes n}$ be a state with stabilizer rank 1, e.g., $\psi = \mathrm{e}_0^{\otimes n}$. By Lemma 5.2.9, any δ-ball around ψ will contain states with stabilizer rank 2^n, that is, for any $\delta > 0$, there exist examples of states with maximal possible stabilizer rank and minimal possible δ-approximate stabilizer rank. In this section, we will explicitly construct examples of quantum states with maximal possible stabilizer rank. In Lemma 5.2.10, we use Theorem 5.2.4 to provide a simple proof that a particular sequence of product states has stabilizer rank at least $\frac{2^n}{4n}$ and δ-approximate stabilizer rank $\mathcal{O}(1)$. In Proposition 5.2.13, we use independent, field-theoretic techniques to construct a different sequence of product states of stabilizer rank 2^n and δ-approximate stabilizer rank 1. The following lemma will allow us to upper-bound the δ-approximate stabilizer rank of product states with exponentially increasing coordinates.

Lemma 5.2.10. *Let $\delta > 0$ be a positive real number. For a complex number $\theta \in \mathbb{C}$ and natural number $n \in \mathbb{N}$, let*

$$\psi_n^\theta = \sqrt{\frac{|\theta|^2 - 1}{|\theta|^{2^{n+1}} - 1}} \bigotimes_{i=1}^{n} (\mathrm{e}_0 + \theta^{2^{i-1}} \mathrm{e}_1) \in (\mathbb{C}^2)^{\otimes n}.$$

If $|\theta| > 1$, then $\chi_\delta(\psi_n^\theta) = \mathcal{O}(1)$. Furthermore, there exists a positive real number $\lambda_\delta > 0$ such that for any $\theta \in \mathbb{C}$ with $|\theta| > \lambda_\delta$, it holds that $\chi_\delta(\psi_n^\theta) = 1$ for all $n \in \mathbb{N}$.

Proof. For each $i \in \{0, 1, \ldots, 2^n - 1\}$, let

$$c_i = \sum_{j=0}^{i} |\theta|^{2j} = \frac{|\theta|^{2i+2} - 1}{|\theta|^2 - 1}, \tag{5.6}$$

and observe that for any positive integer k, the tensor $\phi_{n,k}^\theta \in (\mathbb{C}^2)^{\otimes n}$ obtained by setting all but the k largest coordinates of ψ_n^θ to zero satisfies

$$\begin{aligned}
\left\| \psi_n^\theta - \frac{\phi_{n,k}^\theta}{\left\| \phi_{n,k}^\theta \right\|} \right\|^2 &= \left\| \frac{1}{\sqrt{c_{2^n-1}}} \sum_{i=0}^{2^n-1} \theta^i \mathrm{e}_i - \frac{1}{\sqrt{c_{2^n-1} - c_{2^n-k-1}}} \sum_{i=2^n-k}^{2^n-1} \theta^i \mathrm{e}_i \right\|^2 \\
&= \frac{c_{2^n-k-1}}{c_{2^n-1}} + \left(\frac{1}{\sqrt{c_{2^n-1}}} - \frac{1}{\sqrt{c_{2^n-1} - c_{2^n-k-1}}} \right)^2 (c_{2^n-1} - c_{2^n-k-1}) \\
&= \frac{c_{2^n-k-1}}{c_{2^n-1}} + \left[\sqrt{1 - \frac{c_{2^n-k-1}}{c_{2^n-1}}} - 1 \right]^2, \\
&= \frac{|\theta|^{-2k} - |\theta|^{-2^{n+1}}}{1 - |\theta|^{-2^{n+1}}} + \left[\sqrt{\frac{1 - |\theta|^{-2k}}{1 - |\theta|^{-2^{n+1}}}} - 1 \right]^2,
\end{aligned} \tag{5.7}$$

where we have re-indexed the computational basis of $(\mathbb{C}^2)^{\otimes n}$ as $\mathrm{e}_0, \ldots, \mathrm{e}_{2^n-1}$ for clarity in this proof. Since $|\theta| > 1$, the quantity Equation (5.7) can be set to less than δ^2 by appropriate choice of $k = \mathcal{O}(1)$. Since $\chi(\phi_{n,k}^\theta) \le k$, this shows that $\chi_\delta(\psi_n^\theta) = \mathcal{O}(1)$. It is clear that we can set $k = 1$ if $|\theta|$ is large enough. This completes the proof. □

Note that, using Theorem 5.2.4 and Lemma 5.2.10, we can easily find a sequence of product states of stabilizer rank at least $\frac{2^n}{4n}$ and δ-approximate stabilizer rank $\mathcal{O}(1)$.

Corollary 5.2.11. *For any $n \in \mathbb{N}$, let*

$$\psi_n = \sqrt{\frac{3}{4^{2^n} - 1}} \bigotimes_{i=1}^{n} (\mathrm{e}_0 + 2^{2^{i-1}} \mathrm{e}_1) \in (\mathbb{C}^2)^{\otimes n} \tag{5.8}$$

be a quantum state. Then $\chi(\psi_n) \ge \frac{2^n}{4n}$ and for any constant $\delta > 0$, $\chi_\delta(\psi_n) = \mathcal{O}(1)$.

Proof. For each n, the coordinates of ψ_n form an exponentially increasing sequence of length 2^n. It follows from Theorem 5.2.4 that $\chi(\psi_n) \geq \frac{2^n}{4n}$. The bound $\chi_\delta(\psi_n) = \mathcal{O}(1)$ follows from Lemma 5.2.10. □

We can also apply field-theoretic techniques to construct a sequence of product states of stabilizer rank 2^n and δ-approximate stabilizer rank 1. For that, recall the following basic notions from field theory.

Definition 5.2.12. *Let $\theta \in \mathbb{C}$ be a complex number and $\mathbb{K} \subset \mathbb{C}$ be a subfield. We call θ* algebraic *(over $\mathbb{K}$) if it is the root of a polynomial with coefficients in $\mathbb{K}$. The minimal possible degree of such a polynomial is called the degree of θ. If there is no such polynomial, we call θ* transcendental *over $\mathbb{K}$.*

It is clear that if $\theta \in \mathbb{C}$ has degree n over $\mathbb{K}$, then the set $\{1, \dots, \theta^{n-1}\}$ is linearly independent over $\mathbb{K}$, that is, $\mathrm{span}_{\mathbb{K}}(1, \dots, \theta^{n-1})$ has dimension n as a $\mathbb{K}$-vector space. We will be mostly interested in the case $\mathbb{K} = \mathbb{Q}$ and $\mathbb{K} = \mathbb{Q}(i)$.

Proposition 5.2.13. *Let $\delta > 0$ be a positive real number. If $\theta \in \mathbb{C}$ is a complex number of degree at least 2^n over $\mathbb{Q}(i)$, then the state*

$$\psi_n^\theta = \sqrt{\frac{|\theta|^2 - 1}{|\theta|^{2^{n+1}} - 1}} \bigotimes_{i=1}^{n} (\mathrm{e}_0 + \theta^{2^{i-1}} \mathrm{e}_1) \in (\mathbb{C}^2)^{\otimes n}$$

has stabilizer rank $\chi(\psi_n^\theta) = 2^n$. If it furthermore holds that $|\theta| > 1$, then $\chi_\delta(\psi_n^\theta) = \mathcal{O}(1)$. Finally, there exists a positive real number λ_δ such that for every $\theta \in \mathbb{C}$ of degree at least 2^n over $\mathbb{Q}(i)$ for which $|\theta| > \lambda_\delta$, it holds that $\chi(\psi_n^\theta) = 2^n$ and $\chi_\delta(\psi_n^\theta) = 1$.

Proof. Note that the coordinates of ψ_n^θ are (up to the normalization factor) $1, \theta, \dots, \theta^{2^n - 1}$. By the degree assumption on θ, the coordinates of ψ_n^θ are linearly independent over $\mathbb{Q}(i)$. Let $\psi_n^\theta = \sum_{i=1}^r c_i \sigma_i$ be a stabilizer decomposition of ψ_n^θ, where σ_i is a stabilizer state and $c_i \in \mathbb{C}$ for each $i \in [r]$. Since the coordinates of each σ_i are contained in $\mathbb{Q}(i)$, it follows that

$$\mathrm{span}_{\mathbb{Q}(i)}\{c_1, \dots, c_r\} \supseteq \{1, \theta, \dots, \theta^{2^n - 1}\}, \tag{5.9}$$

so $r \geq 2^n$. This proves that $\chi(\psi_n^\theta) = 2^n$. The remaining statements follow from Lemma 5.2.10. □

For example, to obtain a product state with stabilizer rank 2^n and approximate stabilizer rank $\mathcal{O}(1)$, one can choose any transcendental number $\theta \in \mathbb{C}$, e.g., the circumference-to-diameter ratio π. Note also that there are $\theta \in \mathbb{C}$ with $|\theta| = 1$ such that Proposition 5.2.13 applies. It is worth mentioning that in this case, the coordinates of ψ_n^θ contain no exponentially increasing sequence, that is, Theorem 5.2.4 does not yield an immediate lower bound on $\chi(\psi_n^\theta)$.

5.3 States with multiplicative stabilizer rank under the tensor product

It is a standard fact that the stabilizer rank is *sub-multiplicative* under the tensor product, in other words, $\chi(\psi \otimes \psi) \le \chi(\psi)^2$ for any quantum state ψ [Qas21, Section 2.1.3]. In [Qas21, Section 4.4], it was remarked that there are no known examples of quantum states ψ – that are not stabilizer states – for which equality holds. In this section, we explicitly construct two-qubit states ψ for which $\chi(\psi) = 2$ and $\chi(\psi \otimes \psi) = 4$. The following observation tells us that our example is, in that sense, minimal.

Lemma 5.3.1. *For every quantum state $\psi \in \mathbb{C}^2$, we have $\chi(\psi^{\otimes 2}) \le 3$. In particular, the stabilizer rank of ψ is multiplicative if and only if $\psi \in \mathbb{C}^2$ is a stabilizer state, that is, $\chi(\psi) = 1$.*

Proof. For a state $\psi \propto \mathrm{e}_0 + \alpha \mathrm{e}_1$, it holds that $\psi^{\otimes 2} \propto \mathrm{e}_{00} + \alpha(\mathrm{e}_{01} + \mathrm{e}_{10}) + \alpha^2 \mathrm{e}_{11}$. This is already a stabilizer rank decomposition of length 3. ☐

For two-qubit quantum states, on the other hand, we can find examples where the stabilizer rank is multiplicative. For that, the following simple lemma will be helpful.

Lemma 5.3.2. *Consider the subsets*

$$S_0 = \{0000\}, \ \ S_1 = \{x \in \mathbb{F}_2^4 : |x| = 1\}, \ \ S_2 = \{0101, 0110, 1001, 1010\}$$

of $\mathbb{F}_2^4$ and let $S = S_0 \cup S_1 \cup S_2$. Then, S does not contain an affine subspace of cardinality larger than 4, and $S_0 \cup S_1$ does not contain an affine subspace of cardinality larger than 2.

Proof. Note that the cardinality of an affine subspace in $\mathbb{F}_2^4$ is a power of 2. Assume $A \subset S$ is an affine subspace of cardinality 8. Then, it must contain at least three elements of S_1 and, consequently, an element of Hamming weight 3 – a contradiction. Similarly, if $B \subset S_0 \cup S_1$ is an affine subspace with four elements, it must contain an element with Hamming weight at least 2 – also a contradiction. ☐

Equipped with Lemma 5.3.2, we are ready to construct quantum states with multiplicative stabilizer rank.

Theorem 5.3.3. *Consider the quantum state*

$$\psi_\alpha = \frac{1}{\sqrt{1 + 2|\alpha|^2}} (\mathrm{e}_{00} + \alpha(\mathrm{e}_{01} + \mathrm{e}_{10})) \in (\mathbb{C}^2)^{\otimes 2} \tag{5.10}$$

where $\alpha \in \mathbb{C}^\times$ is a non-zero complex number. Then $\chi(\psi_\alpha) = 2$, and for all but finitely many α, it holds that $\chi(\psi_\alpha^{\otimes 2}) = 4$. In particular, $\chi(\psi_\alpha^{\otimes 2}) = 4$ if α is transcendental over $\mathbb{Q}$.

Proof. By definition, it is clear that ψ_α has stabilizer rank $\chi(\psi_\alpha) = 2$.

Now, assume that

$$\psi_\alpha^{\otimes 2} \propto (\mathrm{e}_{00} + \alpha(\mathrm{e}_{01} + \mathrm{e}_{10}))^{\otimes 2} = \mathrm{e}_{00} + \alpha\left(\sum_{x:|x|=1} \mathrm{e}_x\right) + \alpha^2(\mathrm{e}_{01} + \mathrm{e}_{10})^{\otimes 2}$$

has stabilizer rank at most 3. From the standard form of stabilizer states (Proposition 1.5.3), we know that there are affine subspaces $A_1^\alpha, A_2^\alpha, A_3^\alpha$ as well as $\mathbb{Z}_4$-valued functions $f_\alpha, g_\alpha, h_\alpha$ and coefficients $\beta_\alpha, \gamma_\alpha, \lambda_\alpha \in \mathbb{C}$ such that

$$(\mathrm{e}_{00} + \alpha(\mathrm{e}_{01} + \mathrm{e}_{10}))^{\otimes 2} = \beta_\alpha(\sum_{x \in A_1^\alpha} i^{f_\alpha(x)} \mathrm{e}_x) + \gamma_\alpha(\sum_{x \in A_2^\alpha} i^{g_\alpha(x)} \mathrm{e}_x) + \lambda_\alpha(\sum_{x \in A_3^\alpha} i^{h_\alpha(x)} \mathrm{e}_x). \quad (5.11)$$

In particular, for any such $\alpha \in \mathbb{C}^\times$, we can find $a_{ij} \in \{0, \pm 1, \pm i\}$ such that

$$A \begin{pmatrix} \beta_\alpha \\ \gamma_\alpha \\ \lambda_\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}. \quad (5.12)$$

where A is the matrix with entries a_{ij}. We will now show that for any choice of the a_{ij}, there are only finitely many α such that there exist $\beta_\alpha, \gamma_\alpha, \lambda_\alpha$ satisfying Equation (5.12). Since the α_{ij} are taken from a finite set, this will imply that for only finitely many α, the quantum state $\psi_\alpha^{\otimes 2}$ has stabilizer rank 3.

First, assume that we chose the a_{ij} in a way such that A is singular. Consequently, we can find a non-zero vector $\mu \in \mathbb{Q}(i)^3$ such that $\mu^t A = 0$. Applying that to both sides of Equation (5.12) yields

$$\mu_0 + \mu_1 \alpha + \mu_2 \alpha^2 = 0.$$

Therefore there are at most two possible choices for α for any such choice of the a_{ij}. Since the μ_j are in $\mathbb{Q}(i)$, any such choice of α must be algebraic over $\mathbb{Q}(i)$.

Now, assume that A is non-singular. In that case, Equation (5.12) can be read as

$$\begin{pmatrix} \beta_\alpha \\ \gamma_\alpha \\ \lambda_\alpha \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}.$$

In particular, β_α, γ_α and λ_α are linearly independent polynomials in $\mathbb{Q}(i)[\alpha]$. Recall the sets S_0, S_1, S_2 from Lemma 5.3.2. The set $S = S_0 \cup S_1 \cup S_2$ is exactly the set of $x \in \mathbb{F}_2^4$ such that $\psi_\alpha^{\otimes 2}$ has a non-zero coefficient for e_x. If there exists $x \in A_1 \setminus S$, then

$$i^{f_\alpha(x)} \beta_\alpha + \star \gamma_\alpha + \star \lambda_\alpha = 0 \quad (5.13)$$

is a non-trivial polynomial equation that α has to satisfy. Here, $\star$ is a placeholder for some elements in $\{0, \pm 1, \pm i\}$. Hence, in that case, there are only finitely many α such that $\chi(\psi_\alpha^{\otimes 2}) = 3$ can hold. Also note that the polynomial in Equation (5.13) is in $\mathbb{Q}(i)[\alpha]$. Hence, all α satisfying Equation (5.13) must be algebraic over $\mathbb{Q}(i)$.

By symmetry, it remains to consider the case where $A_j \subset S$ for all $j = 1, 2, 3$. Assume that there exists $x \in A_1 \cap S_2$ and $y \in S_2 \setminus A_1$. In that case, Equation (5.11) implies that

$$i^{f(x)}\beta_\alpha + \star\,\gamma_\alpha + \star\lambda_\alpha = \alpha^2$$
$$\star\,\gamma_\alpha + \star\lambda_\alpha = \alpha^2,$$

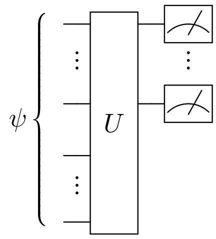
where we again write $\star$ as a placeholder for elements of $\{0, \pm 1, \pm i\}$. In particular, subtracting the two equations yields a non-zero polynomial in $\mathbb{Q}(i)[\alpha]$, of which α is a zero. Again, this can only hold for finitely many α which are algebraic over $\mathbb{Q}(i)$. By symmetry, the same holds for A_2 and A_3.

Finally, consider the case where for all $j = 1, 2, 3$ it holds that either $A_j = S_2$ or $A_j \cap S_2 = \varnothing$. If two of the A_j's are equal to S_2, the third space would need to contain at least five elements contradicting Lemma 5.3.2. Hence, we can assume without loss of generality that $A_1 = S_2$ and $A_2 \cap S_2 = A_3 \cap S_2 = \varnothing$. But then, A_2 and A_3 must have cardinality at most 2 by Lemma 5.3.2 since they are affine spaces contained in $S_0 \cup S_1$. This is a contradiction to $A_2 \cup A_3 \supset S_0 \cup S_1$ since $|S_0 \cup S_1| = 5$.

Summarizing, we have seen that $\chi(\psi_\alpha^{\otimes 2}) = 3$ can only hold for finitely many α. Moreover, we have seen that in all such cases, α must be algebraic over $\mathbb{Q}(i)$. Noticing that since the imaginary unit i is algebraic over $\mathbb{Q}$ any $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ if and only if it is algebraic over $\mathbb{Q}(i)$ finishes the proof. □

5.A Motivation behind stabilizer rank

In this appendix, we will motivate the definition of stabilizer rank in more depth and explain how it relates to the simulation cost of quantum circuits. In particular, we will review in detail how the stabilizer rank of a quantum state quantifies the classical simulation cost of applying Clifford gates and computational basis measurements to that state and review how the stabilizer rank of n copies of the so-called T-state quantifies the simulation cost of Clifford+T circuits utilizing n T-gates. We use the standard graphical notation for quantum circuits which we already saw in Section 1.5: Fixing some natural numbers $m \leq k$, we depict with



(5.14)

a circuit applying a k-qubit unitary U to an input state $\psi \in (\mathbb{C}^2)^{\otimes k}$ and measuring an output register consisting of m qubits in computational basis. Without loss of generality, we will always take the first m qubits as output register. In the following, the number of qubits in a circuit as in Equation (5.14) will always be k, and the number of measured qubits m unless specified differently.

Let us start by recalling some basic notions of the simulation of quantum circuits. Simulating a quantum circuit as in Equation (5.14) with a k-qubit input state ψ *weakly* means to draw m classical bits according to the output distribution of the circuit. Simulating the circuit *strongly*, on the other hand, means to be able to compute the probability of a bitstring $x_1 \dots x_m$ being the output of the circuit. By Born's rule, this probability is given by

$$P(x_1 \dots x_m) = \psi^* U^\dagger \left(\Pi_{x_1 \dots x_m} \otimes \mathbb{1}_2^{\otimes k-m}\right) U\psi$$

where $\Pi_{x_1 \dots x_m}$ is the orthogonal projection onto $\mathrm{span}\{\mathrm{e}_{x_1} \otimes \cdots \otimes \mathrm{e}_{x_m}\}$.

Another standard notion of simulation is the so-called *ϵ-strong simulation*. For a bitstring $x_1 \dots x_m$, the task is to approximate the output probability $P(x_1, \dots, x_m)$ up to relative error ϵ. More precisely, fix a relative error ϵ. Given a bitstring $x_1 \dots x_m$, we then want to get an output ξ such that

$$(1 - \epsilon) P(x_1 \dots x_m) \leq \xi \leq (1 + \epsilon) P(x_1 \dots x_m)$$

where P is the output distribution of the circuit.

Recall the *(affine) Clifford gates* H, S, and $CNOT$ from Section 1.5. We will later also need the controlled Z gate defined via

$$CZ = (\mathbb{1}_2 \otimes H) CNOT (\mathbb{1}_2 \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Clifford unitaries $U \in \mathcal{C}_k$ are exactly those unitaries composed from $H, S, CNOT$, and global phase gates only. As done in Section 1.5, one can equivalently define the group $\mathcal{C}_k$ as the normalizer of the k-qubit Pauli group $\mathcal{P}_k$ (also defined in Section 1.5): Conjugating a Pauli unitary with a Clifford unitary yields another Pauli unitary [Got97].

As a consequence of this fact, we can work very efficiently on certain quantum states using the *stabilizer formalism*, which was introduced in [Got97]. We will now briefly recall some basics about the stabilizer formalism and refer the reader to [NC00] for an in-depth discussion and detailed proofs. To start, note that $\mathrm{e}_0^{\otimes k}$ is, up to global phase, the unique quantum state invariant under applying (or *stabilized by*) Z_i for all $i \in [k]$, where Z_i is the tensor product of a Pauli Z on the i'th qubit with identities on all other qubits. In fact, one can show that if $P_1, \dots, P_k$ are independent Pauli unitaries (that is, none of them is a product of the others), and the group generated by them does not contain $-\mathbb{1}_2^{\otimes k}$, then there exists up to global phase a unique state σ stabilized by $P_1, \dots, P_k$ (see [NC00, Chapter 10.5.1] for details). It turns out that if a state σ can be uniquely specified as above, the same holds for the state resulting from applying a Clifford unitary U to σ: In fact, one can show that the Pauli unitaries $UP_1U^\dagger, \dots, UP_kU^\dagger$ are independent again, do not generate $-\mathbb{1}_2^{\otimes k}$ and it is easy to see that they stabilize $U\sigma$ (see [NC00, Chapter 10.5.2] for details). Conversely, for every state of the form $U\mathrm{e}_0^{\otimes k}$, there are k independent Paulis stabilizing it, see [Gro06] for a proof. Summarizing, we see that the states of the form $\sigma = U\mathrm{e}_0^{\otimes k}$ for a Clifford unitary U are precisely the ones uniquely specified up to global phase by k independent Pauli operators stabilizing it. This explains why we defined a *stabilizer state* as a state of the form $U\mathrm{e}_0^{\otimes k}$ where U is a Clifford unitary in Section 1.5.

The celebrated Gottesman-Knill theorem states that a circuit as in Equation (5.14) where U is a Clifford unitary can be simulated efficiently if the input state is a stabilizer state $\sigma \in (\mathbb{C}^2)^{\otimes k}$.

Theorem 5.A.1 (Gottesman, Knill [Got98])**.** *Let σ be a k-qubit stabilizer state specified by k independent Paulis that stabilize it. A quantum circuit U composed only from Clifford gates acting on σ followed by measuring an output register consisting of $m \leq k$ qubits in the computational basis can be efficiently simulated both strongly and weakly. More precisely, the complexity of simulating the quantum computation scales quadratically in the number of qubits and linearly in the number of Clifford gates and measurements applied.*

A proof of Theorem 5.A.1 can be found in [NC00, Section 10.5.4]. We mention that one essentially has to update the Pauli stabilizers gate-by-gate, which yields k independent Paulis stabilizing $U\sigma$. One can read in detail in [NC00, Section 10.5.3] how one can then efficiently obtain the outcome probabilities from this. It is also worth mentioning that the theorem holds more generally if we measure any observable from the Pauli group, not only for measuring in the computational basis.

Note that if a quantum state $\sigma \in (\mathbb{C}^2)^{\otimes k}$ is stabilized by a Pauli unitary P, all states of the form $e^{i\theta}\sigma$ for $\theta \in \mathbb{R}$ are also stabilized by P. While a global phase does not influence the outcome probabilities of computational basis measurements, it changes the amplitudes of the state. This means that Theorem 5.A.1, as stated above, does not let us calculate amplitudes but only outcome probabilities. In [BBC+19], the authors describe a way of keeping track of the global phase: They define a classical data format for stabilizer states which they call the

CH-form. We now briefly describe their approach and refer to [BBC+19] for more details. Another in-depth discussion can be found in [Qas21, Section 2.1.4] from where we also borrow our notation. Essentially, one can show that every stabilizer state σ is of the form

$$\sigma = \sigma(w, U, h, s) = wUH(h)(\mathrm{e}_{s_1} \otimes \cdots \otimes \mathrm{e}_{s_k})$$

where $w \in \mathbb{C}$, $s, h \in \mathbb{F}_2^k$ and U is a Clifford unitary composed from gates $CNOT, CZ$ and S only. The unitary $H(h)$ is a tensor product of H gates acting on the qubits i such that $h_i = 1$ and $\mathbb{1}_2$ elsewhere. As in the proof of Theorem 5.A.1, one can now simulate a circuit built from Clifford gates only by updating w, U, h and s gate-by-gate. The authors show in [BBC+19] that this update can be done with computational cost at most quadratic in k: Updating the CH form after applying a global phase has constant computational cost, a $CNOT$, CZ, or S has linear cost in k, and a Hadamard gate has quadratic cost in k. Moreover, they show that given a stabilizer state $\sigma \in (\mathbb{C}^2)^{\otimes k}$ in CH-form, one can calculate the amplitude $(\mathrm{e}_{x_1}^* \otimes \cdots \otimes \mathrm{e}_{x_k}^*)\sigma$ with computational cost quadratic in the number of qubits. Here, we again used the notation e_x^* for the dual of e_x. Consequently, one can calculate the overlap

$$\sigma(w, U, h, s)^* \sigma(w', U', h', s') = \overline{w} \cdot w' \cdot (\mathrm{e}_{s_1}^* \otimes \cdots \otimes \mathrm{e}_{s_k}^*) H(h) U^\dagger \sigma(w', U', h', s')$$

of two stabilizer states given in CH-form with computational cost at most cubic in the number of qubits by first updating the CH-form of $H(h)U^\dagger\sigma(w', U', h', s')$ and then calculating the overlap with $(\mathrm{e}_{s_1} \otimes \cdots \otimes \mathrm{e}_{s_k})$. Finally, for a stabilizer state $\sigma(w, U, h, s)$ given in CH-form, an integer $m \leq k$, and an orthogonal projector $\Pi_{x_1,\ldots,x_m}$ onto $\operatorname{span}\{\mathrm{e}_{x_1} \otimes \cdots \otimes \mathrm{e}_{x_m}\}$ for some bitstring $x_1 \ldots x_m$, one can calculate the CH-form of the stabilizer state

$$\left(\Pi_{x_1,\ldots,x_m} \otimes \mathbb{1}_2^{\otimes k-m}\right) \sigma(w, U, h, s)$$

with computational cost quadratic in k.

With that, we can now understand the importance of the stabilizer rank as a measure of the computational cost of strong simulation of quantum circuits with general input states. Say, we want to strongly simulate a quantum circuit as in Equation (5.14) where ψ is a k-qubit quantum state and U is a Clifford unitary. Say furthermore that we can decompose

$$\psi = \sum_{i=1}^{r} c_i \sigma_i \tag{5.15}$$

where the σ_i are stabilizer states each specified in CH-form and the c_i are complex numbers. Given a bitstring $x_1, \ldots, x_m$, we want to calculate

$$P(x_1, \ldots, x_m) = \psi^* U^\dagger \left(\Pi_{x_1,\ldots,x_m} \otimes \mathbb{1}\right) U \psi = \sum_{i,j=1}^{r} \overline{c_i} \cdot c_j \cdot \sigma_i^* U^\dagger \left(\Pi_{x_1,\ldots,x_m} \otimes \mathbb{1}_2^{\otimes k-m}\right) U \sigma_j.$$

We can do so by first updating the CH-form of $U\sigma_i$ and $(\Pi_{x_1,\ldots,x_m} \otimes \mathbb{1}_2^{\otimes k-m})U\sigma_j$ followed by calculating r^2 overlaps between stabilizer states. Since for each of the summands, the computational cost is linear in the number of gates of which U is composed and cubic in the number of qubits, the whole cost is cubic in the number of qubits, linear in the number of Clifford gates and quadratic in the number of terms r appearing in the stabilizer decomposition.

As already done in Section 1.5, we call the minimal r such that there exists a decomposition as in equation Equation (5.15) the *stabilizer rank* of ψ and denote it $\chi(\psi)$. It follows that lower bounds on $\chi(\psi)$ imply lower bounds on the complexity of simulating a circuit of the form Equation (5.14) using the approach we have just described. On the other hand, finding upper bounds on $\chi(\psi)$ by providing a decomposition as in Equation (5.15) can give us better simulation algorithms for circuits of the form Equation (5.14).

For a discussion of a weak simulation protocol and an ϵ-strong simulation protocol making use of stabilizer rank decompositions, see [BBC+19, Section 4]. We mention that there, the authors deduce that the computational cost of weak simulation scales linearly in the approximate stabilizer rank of ψ and polynomially in the number of qubits and the number of gates applied and that the cost of ϵ-strong simulation scales linearly in the exact stabilizer rank of ψ and polynomially in the number of qubits and the number of gates applied.

As an application of the connection between the stabilizer rank of a state ψ and the classical simulation cost of applying Clifford circuits and computational basis measurements to ψ, we will now see how low-rank stabilizer decompositions of n copies of the so-called T-state can also be used to simulate circuits built from a universal gate set. This strategy has been used, for instance, in [BG16]. For simplicity, we will only consider the case where all qubits in the circuit in Equation (5.14) are measured in computational basis, that is, the size m of the output register is equal to the number of qubits k.

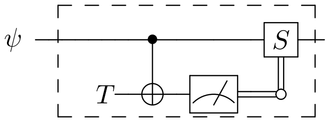
Recall that a gate set $\mathcal{G}$ is called *universal* if every unitary U on k qubits can be approximated arbitrarily well by unitaries $U_{\mathcal{G}}$ composed solely of gates in $\mathcal{G}$. It is well-known that the set of Clifford gates is not universal, but, together with the so-called T-gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix},$$

they form a universal gate set [BMP+99].

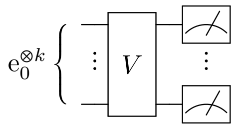
Under the assumption that our quantum device can only prepare a computational basis state $\mathrm{e}_0^{\otimes k}$, apply Clifford+T operations and measure the qubits in computational basis, the T-gates appear, by the preceding discussion, to be responsible for the potential superiority of the quantum device. It is therefore an interesting question how efficiently we can simulate a quantum computation on k qubits which, in addition to Clifford gates, uses n single-qubit T-gates.

A standard way of approaching this question is via the study of *magic state injection* [BK05]. For this, we note that for the magic state $T = \frac{1}{\sqrt{2}}\left(\mathrm{e}_0 + e^{i\frac{\pi}{4}}\mathrm{e}_1\right)$, applying a T-gate to any qubit state ψ is the same as applying the circuit



(5.16)

where the double wire denotes a *classical control*: The S gate on the first system is applied if and only if the outcome of the computational basis measurement on the second system is 1. Therefore, any k-qubit quantum circuit



(5.17)

where V is composed of Clifford gates and n T-gates, can be implemented with a circuit composed from Clifford gates and classical controls acting on the state $\mathrm{e}_0^{\otimes k} \otimes T^{\otimes n}$ by replacing each T-gate with the gadget in Equation (5.16). By *postselecting* outcomes 0 for each measurement on a T-state in this circuit, we see that

$$V\mathrm{e}_0^{\otimes k} = 2^{n/2}\left(\mathbb{1}^{\otimes k} \otimes (\mathrm{e}_0^*)^{\otimes n}\right) U \left(\mathrm{e}_0^{\otimes k} \otimes T^{\otimes n}\right)$$

where U is composed of Clifford gates only (more precisely, the Clifford gates from V plus an additional $CNOT$ gate for each injected T-state).

With that, it follows that the probability of outcome $x_1, \ldots, x_k$ in the circuit in Equation (5.17) is given by $|p_{x_1 \ldots x_k}|^2$, where

$$\begin{aligned} p_{x_1,\ldots,x_k} &= \left(\mathrm{e}_{x_1}^* \otimes \cdots \otimes \mathrm{e}_{x_k}^*\right) V \mathrm{e}_0^{\otimes k} \\ &= 2^{n/2}\left(\mathrm{e}_{x_1}^* \otimes \cdots \otimes \mathrm{e}_{x_k}^* \otimes (\mathrm{e}_0^*)^{\otimes n}\right) U \left((\mathrm{e}_0)^{\otimes k} \otimes T^{\otimes n}\right). \end{aligned} \tag{5.18}$$

Note that if there are stabilizer states $\sigma_1, \ldots, \sigma_r$ and complex numbers $c_1, \ldots, c_r \in \mathbb{C}$ such that $\psi = \sum_{i=1}^r c_i \sigma_i$, then the quantity in Equation (5.18) decomposes by linearity as

$$p_{x_1 \ldots x_k} = 2^{n/2} \sum_{i=1}^{r} c_i \left(\mathrm{e}_{x_1}^* \otimes \cdots \otimes \mathrm{e}_{x_k}^* \otimes (\mathrm{e}_0^*)^{\otimes n}\right) U \left((\mathrm{e}_0)^{\otimes k} \otimes \sigma_i\right). \tag{5.19}$$

All summands in Equation (5.19) can be calculated efficiently using the CH-form as discussed before: They are amplitudes of the outcome of a quantum circuit composed of Clifford gates acting on a stabilizer state.

Summarizing, we saw that every stabilizer decomposition of $T^{\otimes n}$ yields a way to strongly simulate Clifford+T circuits consisting of n T-gates. The complexity of this simulation scales linearly in the number of terms appearing in the stabilizer decomposition and polynomially in all other parameters. With this, it follows that finding decompositions of $T^{\otimes n}$ into few stabilizer states can reduce the complexity of simulating such circuits. Lower bounds on $\chi(T^{\otimes n})$, on the other hand, translate directly into lower bounds on the cost of simulating quantum circuits using these methods, namely, stabilizer decompositions and CH-forms.

Bibliography

[AFT11] B. Alexeev, M. A. Forbes, and J. Tsimerman. Tensor rank: Some lower and upper bounds. In *IEEE 26th Annual Conference on Computational Complexity*, page 283–291, 2011. doi:10.1109/CCC.2011.28.

[AKRS21] C. Améndola, K. Kohn, P. Reichenbach, and A. Seigal. Invariant Theory and Scaling Algorithms for Maximum Likelihood Estimation. *SIAM J. Appl. Alg. Geom.*, 5(2):304–337, 2021. doi:10.1137/20M1328932.

[AL18] T. Albash and D. A. Lidar. Adiabatic quantum computation. *Rev. Mod. Phys.*, 90:015002, 2018. doi:10.1103/RevModPhys.90.015002.

[ALVV17] I. Arad, Z. Landau, U. Vazirani, and T. Vidick. Rigorous RG Algorithms and Area Laws for Low Energy Eigenstates in 1D. *Communications in Mathematical Physics*, 356(1):65–105, 2017. doi:10.1007/s00220-017-2973-z.

[AW21] J. Alman and V. V. Williams. A refined laser method and faster matrix multiplication. In *Proc. 2021 ACM-SIAM Symp. Disc. Alg. (SODA)*, page 522–539. SIAM, 2021. doi:10.1137/1.9781611976465.32.

[BBC+93] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, 1993. doi:10.1103/PhysRevLett.70.1895.

[BBC+95] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J.A. Smolin, and H. Weinfurter. Elementary gates for quantum computation. *Phys. Rev. A*, 52:3457–3467, 1995. doi:10.1103/PhysRevA.52.3457.

[BBC+19] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard. Simulation of quantum circuits by low-rank stabilizer decompositions. *Quantum*, 3:181, 2019. doi:10.22331/q-2019-09-02-181.

[BCG+20] A. Bluhm, M. Christandl, F. Gesmundo, F. Ravn Klausen, Laura Mančinska, V. Steffan, D. Stilck França, and A. H. Werner. Sars-cov-2 transmission routes from genetic data: A danish case study. *PLOS ONE*, 15(10):1–15, 10 2020. doi:10.1371/journal.pone.0241405.

[BCLR79] D. Bini, M. Capovani, G. Lotti, and F. Romani. $O(n^{2.7799})$ complexity for $n \times n$ approximate matrix multiplication. *Inform. Process. Lett.*, 8(5):234–235, 1979. doi:10.1016/0020-0190(79)90113-3.

[BCS97] P. Bürgisser, M. Clausen, and M. A. Shokrollahi. *Algebraic complexity theory*. Springer-Verlag, Berlin, 1997. doi:10.1007/978-3-662-03338-8.

[BD06] M. Bürgin and J. Draisma. The Hilbert Null-cone on Tuples of Matrices and Bilinear Forms. *Math. Z.*, 254(4):785–809, 2006. doi:10.1007/s00209-006-0008-0.

[BG16] S. Bravyi and D. Gosset. Improved classical simulation of quantum circuits dominated by Clifford gates. *Phys. Rev. Lett.*, 116:250501, 2016. doi:10.1103/PhysRevLett.116.250501.

[Bin80] D. Bini. Relations between exact and approximate bilinear algorithms. Applications. *Calcolo*, 17(1):87–97, 1980. doi:10.1007/BF02575865.

[BK05] S. Bravyi and A. Kitaev. Universal quantum computation with ideal clifford gates and noisy ancillas. *Phys. Rev. A*, 71:022316, 2005. doi:10.1103/PhysRevA.71.022316.

[Blä13] M. Bläser. *Fast Matrix Multiplication*. Number 5 in Graduate Surveys. Theory of Computing Library, 2013. doi:10.4086/toc.gs.2013.005.

[BLG21] A. Bernardi, C. De Lazzari, and F. Gesmundo. Dimension of Tensor Network Varieties. *Comm. Cont. Math.*, page 2250059, 2021. doi:10.1142/S0219199722500596.

[BLR80] D. Bini, G. Lotti, and F. Romani. Approximate Solutions for the Bilinear Form Computational Problem. *SIAM Journal on Computing*, 9(4):692–697, 1980. doi:10.1137/0209053.

[BMP+99] P. O. Boykin, T. Mor, M. Pulver, V. Roychowdhury, and F. Vatan. On universal and fault-tolerant quantum computing: A novel basis and a new constructive proof of universality for Shor's basis. In *Proceedings of the 40th Annual Symposium on Foundations of Computer Science*, 1999. doi:10.1109/SFFCS.1999.814621.

[Bor91] A. Borel. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer, New York, 1991. doi:10.1007/978-1-4612-0941-6.

[BSS16] S. Bravyi, G. Smith, and J.A. Smolin. Trading classical and quantum computational resources. *Phys. Rev. X*, 6:021043, 2016. doi:10.1103/PhysRevX.6.021043.

[CCD+10] L. Chen, E. Chitambar, R. Duan, Z. Ji, and A. Winter. Tensor rank and stochastic entanglement catalysis for multipartite pure states. *Phys. Rev. Lett.*, 105:200501, 2010. doi:10.1103/PhysRevLett.105.200501.

[CDS08] E. Chitambar, R. Duan, and Y. Shi. Tripartite entanglement transformations and tensor rank. *Phys. Rev. Lett.*, 101:140502, 2008. doi:10.1103/PhysRevLett.101.140502.

[CdTP06] M. F. Cornelio and A. F. R. de Toledo Piza. Classification of tripartite entanglement with one qubit. *Phys. Rev. A*, 73:032314, 2006. doi:10.1103/PhysRevA.73.032314.

[CF18] L. Chen and S. Friedland. The tensor rank of tensor product of two three-qubit w states is eight. *Linear Algebra and its Applications*, 543:1–16, 2018. doi:10.1016/j.laa.2017.12.015.

[CFS+16] S. X. Cui, M. H. Freedman, O. Sattath, R. Stong, and G. Minton. Quantum max-flow/min-cut. *J. Math. Phys.*, 57(6):062206, 2016. doi:10.1063/1.4954231.

[CFTZ22] M. Christandl, O. Fawzi, H. Ta, and J. Zuiddam. Larger Corner-Free Sets from Combinatorial Degenerations. In *13th Innovations in Theoretical Computer Science Conference*, volume 215, page 48:1–48:20, 2022. doi:10.4230/LIPIcs.ITCS.2022.48.

[CFW10] D. Calegari, M. Freedman, and K. Walker. Positivity of the universal pairing in 3 dimensions. *Journal of the American Mathematical Society*, 23:107–188, 2010. doi:10.1090/S0894-0347-09-00642-0.

[CGFW21] M. Christandl, F. Gesmundo, D. Stilck França, and A. H. Werner. Optimization at the boundary of the tensor network variety. *Phys. Rev. B*, 103(19):195139, 2021. doi:10.1103/PhysRevB.103.195139.

[CGL+21] A. Conner, F. Gesmundo, J. M. Landsberg, E. Ventura, and Y. Wang. Towards a geometric approach to Strassen's asymptotic rank conjecture. *Collect. Math.*, 72(1):63–86, 2021. doi:10.1007/s13348-020-00280-8.

[CGLS22] M. Christandl, F. Gesmundo, V. Lysikov, and V. Steffan. Partial degeneration of tensors. *arXiv preprint*, math.AG/arXiv:2212.14095, 2022. doi:10.48550/arXiv.2212.14095.

[CHL19] A. Conner, A. Harper, and J. M. Landsberg. New lower bounds for matrix multiplication and the 3x3 determinant. *arXiv preprint*, math.AG/1911.07981, 2019. doi:10.48550/arxiv.1911.07981.

[CJZ18] M. Christandl, A. K. Jensen, and J. Zuiddam. Tensor rank is not multiplicative under the tensor product. *Linear Algebra and its Applications*, 543:125–139, 2018. doi:10.1016/j.laa.2017.12.020.

[CKN+16] G. Kin-Lic Chan, A. Keselman, N. Nakatani, Z. Li, and S. R. White. Matrix product operators, matrix product states, and ab initio density matrix renormalization group algorithms. *The Journal of Chemical Physics*, 145(1):014102, 2016. doi:10.1063/1.4955108.

[CLVW20] M. Christandl, A. Lucia, P. Vrana, and A. H. Werner. Tensor network representations from the geometry of entangled states. *SciPost Phys.*, 9:42, 2020. doi:10.21468/SciPostPhys.9.3.042.

[CMS10] E. Chitambar, C. A. Miller, and Y. Shi. Matrix pencils and entanglement classification. *Journal of Mathematical Physics*, 51(7):072205, 2010. doi:10.1063/1.3459069.

[CS11] G. K.-L. Chan and S. Sharma. The Density Matrix Renormalization Group in Quantum Chemistry. *Annual Review of Physical Chemistry*, 62(1):465–481, 2011. doi:10.1146/annurev-physchem-032210-103338.

[CVZ18] M. Christandl, P. Vrana, and J. Zuiddam. Universal points in the asymptotic spectrum of tensors. New York, NY, USA, 2018. Association for Computing Machinery. doi:10.1145/3188745.3188766.

[CW81] D. Coppersmith and S. Winograd. On the asymptotic complexity of matrix multiplication. In *22nd Annual Symposium on Foundations of Computer Science (sfcs 1981)*, pages 82–90, 1981. doi:10.1109/SFCS.1981.27.

[CW87] D. Coppersmith and S. Winograd. Matrix multiplication via arithmetic progressions. In *Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing*, page 1–6, 1987. doi:10.1145/28395.28396.

[CWZ21] S. Cheng, L. Wang, and P. Zhang. Supervised learning with projected entangled pair states. *Phys. Rev. B*, 103(12):125117, 2021. doi:10.1103/PhysRevB.103.125117.

[DDM03] J. Dehaene and B. De Moor. Clifford group, stabilizer states, and linear and quadratic operations over GF(2). *Phys. Rev. A*, 68:042318, 2003. doi:10.1103/PhysRevA.68.042318.

[DKH21] M. Drton, S. Kuriki, and P. Hoff. Existence and uniqueness of the Kronecker covariance MLE. *Ann. Stat.*, 49(5):2721–2754, 2021. doi:10.1214/21-AOS2052.

[DM17] H. Derksen and V. Makam. Polynomial degree bounds for matrix semi-invariants. *Adv. Math.*, 310:44–63, 2017. doi:10.1016/j.aim.2017.01.018.

[DM21] H. Derksen and V. Makam. Maximum Likelihood Estimation for Matrix Normal Models via Quiver Representations. *SIAM J. Appl. Alg. Geom.*, 5(2):338–365, 2021. doi:10.1137/20M1369348.

[DMS22] C. De Lazzari, H. J. Motwani, and T. Seynnaeve. The linear span of uniform matrix product states. *SIGMA*, 18:099 – 117, 2022. doi:10.3842/SIGMA.2022.099.

[DMW22] H. Derksen, V. Makam, and M. Walter. Maximum likelihood estimation for tensor normal models via castling transforms. *Forum of Mathematics, Sigma*, 10:e50, 2022. doi:10.1017/fms.2022.37.

[DP89] D. E. Deutsch and R. Penrose. Quantum computational networks. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 425(1868):73–90, 1989. doi:10.1098/rspa.1989.0099.

[DVC00] W. Dür, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A*, 62:062314, 2000. doi:10.1103/PhysRevA.62.062314.

[DW00] H. Derksen and J. Weyman. Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients. *J. AMS*, 13(3):467–479, 2000. doi:10.1090/S0894-0347-00-00331-3.

[DW17] H. Derksen and J. Weyman. *An introduction to quiver representations*, volume 184 of *Graduate Studies in Mathematics*. AMS, Providence, 2017. doi:10.1090/gsm/184.

[DZ01] M. Domokos and A. N. Zubkov. Semi-invariants of quivers as determinants. *Transform. Groups*, 6(1):9–24, 2001. doi:10.1007/BF01236060.

[EB01] J. Eisert and H. J. Briegel. Schmidt measure as a tool for quantifying multiparticle entanglement. *Phys. Rev. A*, 64:022306, 2001. doi:10.1103/PhysRevA.64.022306.

[ECP10] J. Eisert, M. Cramer, and M. B. Plenio. Colloquium: Area laws for the entanglement entropy. *Rev. Mod. Phys.*, 82:277–306, 2010. doi:10.1103/RevModPhys.82.277.

[EFS56] P. Elias, A. Feinstein, and C. Shannon. A note on the maximum flow through a network. *IRE Transactions on Information Theory*, 2(4):117–119, 1956. doi:10.1109/TIT.1956.1056816.

[EG17] J. Ellenberg and D. Gijswijt. On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression. *Annals of Mathematics*, 185(1):339 – 343, 2017. doi:10.4007/annals.2017.185.1.8.

[EGOW17] K. Efremenko, A. Garg, R. Oliveira, and A. Wigderson. Barriers for rank methods in arithmetic complexity. *arXiv preprint*, cs.CC/1710.09502, 2017. doi:10.48550/arxiv.1710.09502.

[EPR35] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47:777–780, 1935. doi:10.1103/PhysRev.47.777.

[Eve17] O. Evenbly. Hyperinvariant Tensor Networks and Holography. *Phys. Rev. Lett.*, 119:141602, 2017. doi:10.1103/PhysRevLett.119.141602.

[Fel91] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. John Wiley and Sons, third edition, 1991.

[FF56] L. R. Ford and D. R. Fulkerson. Maximal Flow Through a Network. *Canadian Journal of Mathematics*, 8:399–404, 1956. doi:10.4153/CJM-1956-045-5.

[FH91] W. Fulton and J. Harris. *Representation Theory: a First Course*, volume 129 of *Graduate Texts in Mathematics*. Springer, New York, 1991. doi:10.1007/978-1-4612-0979-9.

[FNW92] M. Fannes, B. Nachtergaele, and R. F. Werner. Finitely correlated states on quantum spin chains. *Communications in Mathematical Physics*, 144(3):443–490, 1992. URL: https://doi.org/10.1007%2Fbf02099178.

[Fri20] T. Fritz. Abstract vergleichsstellensätze for preordered semifields and semirings I. *arXiv preprint*, math.RA/2003.13835, 2020. doi:10.48550/arXiv.2003.13835.

[FSC+18] E. S. Fried, N. P. D. Sawaya, Y. Cao, I. D. Kivlichan, J. Romero, and A. Aspuru-Guzik. qTorch: The quantum tensor contraction handler. *PloS one*, 13(12):e0208510, 2018. doi:10.1371/journal.pone.0208510.

[Gan59] F. R. Gantmacher. *The theory of matrices. Vols. 1, 2*. Chelsea Publishing Co., New York, 1959.

[GLS22] F. Gesmundo, V. Lysikov, and V. Steffan. Quantum max-flow in the bridge graph. *arXiv preprint*, quant-ph/2212.09794, 2022. doi:10.48550/arXiv.2212.09794.

[GLW18] F. Gesmundo, J. M. Landsberg, and M. Walter. Matrix product states and the quantum max-flow/min-cut conjectures. *J. Math. Phys.*, 55(10):102205, 2018. doi:10.1063/1.5026985.

[Got97] D. Gottesman. *Stabilizer codes and quantum error correction*. PhD thesis, California Institute of Technology, 1997.

[Got98] D. Gottesman. The Heisenberg representation of quantum computers. In *22nd International Colloquium on Group Theoretical Methods in Physics*, pages 32–43, 1998. arXiv:quant-ph/9807006.

[Gri78] D. Grigoriev. Multiplicative complexity of a pair of bilinear forms and of the polynomial multiplication. In *MFCS*, volume 64 of *Lecture Notes in Computer Science*, page 250–256. Springer, 1978. doi:10.1007/3-540-08921-7_72.

[Gro06] D. Gross. Hudson's theorem for finite-dimensional quantum systems. *Journal of Mathematical Physics*, 47(12):122107, 2006. doi:10.1063/1.2393152.

[GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry - v.1.17.1. Available at http://www.math.uiuc.edu/Macaulay2/.

[Has07] M. B. Hastings. An area law for one-dimensional quantum systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08024, 2007. doi:10.1088/1742-5468/2007/08/p08024.

[Has17] M. B. Hastings. The Asymptotics of Quantum Max-Flow Min-Cut. *Comm. in Math. Ph.*, 351(1):387–418, 2017. doi:10.1007/s00220-016-2791-8.

[HEB04] M. Hein, J. Eisert, and H. J. Briegel. Multiparty entanglement in graph states. *Phys. Rev. A*, 69:062311, 2004. doi:10.1103/PhysRevA.69.062311.

[Hil93] D. Hilbert. Ueber die vollen invariantensysteme. *Mathematische Annalen*, 42:313–373, 1893. URL: http://eudml.org/doc/157652.

[HNQ+16] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang. Holographic duality from random tensor networks. *Journal of High Energy Physics*, 2016(11), 2016. doi:10.1007/JHEP11(2016)009.

[HP18] J. Hauschild and F. Pollmann. Efficient numerical simulations with Tensor Networks: Tensor Network Python (TeNPy). *SciPost Physics Lecture Notes*, 2018. doi:10.21468/SciPostPhysLectNotes.5.

[Já79] J. F. JáJá. Optimal evaluation of pairs of bilinear forms. *SIAM J. Comput.*, 8(3):443–462, 1979. doi:10.1137/0208037.

[Kac80] V. G. Kac. Infinite root systems, representations of graphs and invariant theory. *Inventiones mathematicae*, 56(1):57–92, 1980. doi:10.1007/BF01403155.

[KB09] T. G. Kolda and B. W. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, 2009. doi:10.1137/07070111X.

[KDTR15] S. Keller, M. Dolfi, M. Troyer, and M. Reiher. An efficient matrix product operator representation of the quantum chemical Hamiltonian. *The Journal of Chemical Physics*, 143(24):244118, 2015. doi:10.1063/1.4939000.

[Kim02] T. Kimura. Introduction to prehomogeneous vector spaces. Translations of Mathematical Monographs. American Mathematical Society, 2002. `doi:10.1090/mmono/215`.

[Kra85] H. Kraft. *Geometrische Methoden in der Invariantentheorie*. Vieweg+Teubner Verlag Wiesbaden, 1 edition, 1985. `doi:10.1007/978-3-322-83813-1`.

[Lan04] J. M. Landsberg. The border rank of the multiplication of two by two matrices is seven. *J. Amer. Math. Soc.*, 19:447–459, 2004. `doi:10.1090/S0894-0347-05-00506-0`.

[Lan12] J. M. Landsberg. *Tensors: Geometry and Applications*, volume 128 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012. `doi:10.1090/gsm/128`.

[Lan17] J. M. Landsberg. *Geometry and complexity theory*, volume 169 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. `doi:10.1017/9781108183192`.

[LG14] F. Le Gall. Algebraic complexity theory and matrix multiplication. In *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, page 23, New York, 2014. Association for Computing Machinery. `doi:10.1145/2608628.2627493`.

[LLZ+21] Y. Liu, W.-J. Li, X. Zhang, M. Lewenstein, G. Su, and S.-J. Ran. Entanglement-based feature extraction by tensor network machine learning. *Frontiers in Applied Mathematics and Statistics*, 7:716044, 2021. `doi:10.3389/fams.2021.716044`.

[LM18] J. M. Landsberg and M. Michałek. A $2n^2-\log(n)-1$ lower bound for the border rank of matrix multiplication. *Int. Math. Res. Not.*, 2018(15):4722–4733, 2018. `doi:10.1093/imrn/rnx025`.

[LM19] J. Landsberg and M. Michałek. Towards finding hay in a haystack: explicit tensors of border rank greater than 2.02m in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$. *arXiv preprint*, cs.CC/1912.11927, 2019. `doi:10.48550/arXiv.1912.11927`.

[LO11] J. M. Landsberg and G. Ottaviani. New lower bounds for the border rank of matrix multiplication. *arXiv preprint*, cs.CC/1112.6007, 2011. `doi:10.48550/arxiv.1112.6007`.

[Lov22] B. Lovitz. *Tensors: Entanglement, Geometry, and Combinatorics*. PhD thesis, University of Waterloo, 2022. URL: `http://hdl.handle.net/10012/18172`.

[LQY11] J. M. Landsberg, Y. Qi, and K. Ye. On the geometry of tensor network states. *Quantum Inf. Comput.*, 12:346–354, 2011. `doi:10.26421/QIC12.3-4-12`.

[LS22] B. Lovitz and V. Steffan. New techniques for bounding stabilizer rank. *Quantum*, 6:692, 2022. `doi:10.22331/q-2022-04-20-692`.

[LVV15] Z. Landau, U. Vazirani, and T. Vidick. A polynomial time algorithm for the ground state of one-dimensional gapped local Hamiltonians. *Nature Physics*, 11(7):566–569, 2015. `doi:10.1038/nphys3345`.

[Man13] L. Manivel. Prehomogeneous spaces and projective geometry. *Rend.*

Sem. Mat. Univ. Politec. Torino, 71(1):35–118, 2013. URL: http://www.seminariomatematico.polito.it/rendiconti/71-1/35.pdf.

[MGSC18] A. Molnar, Y. Ge, N. Schuch, and J. I. Cirac. A generalization of the injectivity condition for projected entangled pair states. *Journal of Mathematical Physics*, 59(2):021902, 2018. doi:10.1063/1.5007017.

[MNS+15] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi, and K. Watanabe. Continuous Multiscale Entanglement Renormalization Ansatz as Holographic Surface-State Correspondence. *Phys. Rev. Lett.*, 115:171602, 2015. doi:10.1103/PhysRevLett.115.171602.

[Mou01] D. P. Moulton. Representing powers of numbers as subset sums of small sets. *Journal of Number Theory*, 89(2):193–211, 2001. URL: https://www.sciencedirect.com/science/article/pii/S0022314X00926465, doi:https://doi.org/10.1006/jnth.2000.2646.

[MS19] M. Michałek and Y. Shitov. Quantum Version of Wielandt's Inequality Revisited. *IEEE Trans. Inf. Theory*, 65(8):5239–5242, 2019. doi:10.1109/TIT.2019.2897772.

[NC00] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000. doi:10.1017/CBO9780511976667.

[Orú14] R. Orús. A practical introduction to tensor networks: Matrix product states and projected entangled pair states. *Annals of Physics*, 349:117–158, 2014. doi:10.1016/j.aop.2014.06.013.

[Orú19] R. Orús. Tensor networks for complex quantum systems. *Nature Reviews Physics*, 1(9):538–550, 2019. doi:10.1038/s42254-019-0086-7.

[Ose11] I. Oseledets. Tensor-train decomposition. *SIAM J. Scientific Computing*, 33:2295–2317, 01 2011. doi:10.1137/090752286.

[Pan66] V. Y. Pan. Methods of computing values of polynomials. *Russian Mathematical Surveys*, 21(1):105, 1966. doi:10.1070/RM1966v021n01ABEH004147.

[Pan78] V. Y. Pan. Strassen's algorithm is not optimal trilinear technique of aggregating, uniting and canceling for constructing fast algorithms for matrix operations. In *19th Annual Symposium on Foundations of Computer Science (sfcs 1978)*, pages 166–176, 1978. doi:10.1109/SFCS.1978.34.

[PCZ22] F. Pan, K. Chen, and . Zhang. Solving the Sampling Problem of the Sycamore Quantum Circuits. *Phys. Rev. Lett.*, 129:090502, 2022. doi:10.1103/PhysRevLett.129.090502.

[PGVWC07] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix Product State Representations. *Quantum Info. Comput.*, 7(5):401–430, 2007. doi:10.26421/QIC7.5-6-1.

[Pok86] A. Pokrzywa. On perturbations and the equivalence orbit of a matrix pencil. *Linear Algebra and its Applications*, 82:99–121, 1986. doi:10.1016/0024-3795(86)90144-8.

[PSV22] S. Peleg, A. Shpilka, and B. L. Volk. Lower Bounds on Stabilizer Rank. *Quan-*

tum, 6:652, 2022. doi:10.22331/q-2022-02-15-652.

[PYHP15] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill. Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. *Journal of High Energy Physics*, 2015(6), 2015. doi:10.1007/JHEP06(2015)149.

[PZ22] F. Pan and P. Zhang. Simulation of quantum circuits using the big-batch tensor network method. *Phys. Rev. Lett.*, 128(3):030501, 2022. doi:10.1103/PhysRevLett.128.030501.

[Qas21] H. Qassim. *Classical simulations of quantum systems using stabilizer decompositions*. PhD thesis, University of Waterloo, 2021. URL: http://hdl.handle.net/10012/16814.

[QPG21] H. Qassim, H. Pashayan, and D. Gosset. Improved upper bounds on the stabilizer rank of magic states. *Quantum*, 5:606, 2021. doi:10.22331/q-2021-12-20-606.

[Raz87] A. A. Razborov. Lower bounds on the size of bounded depth circuits over a complete basis with logical addition. *Mathematical notes of the Academy of Sciences of the USSR*, 41:333–338, 1987. doi:10.1007/BF01137685.

[RB01] R. Raussendorf and H. J. Briegel. A one-way quantum computer. *Phys. Rev. Lett.*, 86:5188–5191, 2001. doi:10.1103/PhysRevLett.86.5188.

[RBB03] R. Raussendorf, D. E. Browne, and H. J. Briegel. Measurement-based quantum computation on cluster states. *Phys. Rev. A*, 68:022312, 2003. doi:10.1103/PhysRevA.68.022312.

[RG10] N. R. Wallach R. Goodman. *Symmetry, Representations, and Invariants*. Springer New York, 1 edition, 2010. doi:10.1007/978-0-387-79852-3.

[Rom82] F. Romani. Some properties of disjoint sums of tensors related to matrix multiplication. *SIAM Journal on Computing*, 11(2):263–267, 1982. doi:10.1137/0211020.

[Sch81] A. Schönhage. Partial and total matrix multiplication. *SIAM Journal on Computing*, 10(3):434–455, 1981. doi:10.1137/0210032.

[Sch91] A. Schofield. Semi-invariants of quivers. *J. London Math. Soc. (2)*, 43(3):385–395, 1991. doi:10.1112/jlms/s2-43.3.385.

[Sch11] U. Schollwöck. The density-matrix renormalization group in the age of matrix product states. *Annals of Physics*, 326(1):96–192, 2011. doi:10.1016/j.aop.2010.09.012.

[SCPG10] N. Schuch, I. Cirac, and D. Pérez-García. PEPS as ground states: Degeneracy and topology. *Annals of Physics*, 325(10):2153–2192, 2010. doi:10.1016/j.aop.2010.05.008.

[Sha13] I. R. Shafarevich. *Basic Algebraic Geometry 1*. Springer Berlin, 3 edition, 2013. doi:10.1007/978-3-642-37956-7.

[SK77] M. Sato and T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. J.*, 65:1–155, 1977. doi:10.1017/S0027763000017633.

[Smo87] R. Smolensky. Algebraic methods in the theory of lower bounds for boolean circuit complexity. In *Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing*, pages 77–82, 1987. doi:10.1145/28395.28404.

[Smo93] R. Smolensky. On representations by low-degree polynomials. In *Proceedings of the 1993 IEEE 34th Annual Foundations of Computer Science*, pages 130–138, 1993. doi:10.1109/SFCS.1993.366874.

[SPCPG12] N. Schuch, D. Poilblanc, J. I. Cirac, and D. Perez-Garcia. Resonating valence bond states in the PEPS formalism. *Phys. Rev. B*, 86(11), 2012. doi:10.1103/physrevb.86.115108.

[SPM+15] S. Szalay, M. Pfeffer, V. Murg, G. Barcza, F. Verstraete, R. Schneider, and O. Legeza. Tensor product methods and entanglement optimization for ab initio quantum chemistry. *International Journal of Quantum Chemistry*, 115(19):1342–1391, 2015. doi:10.1002/qua.24898.

[SS16] E. Stoudenmire and D. J. Schwab. Supervised learning with tensor networks. *Advances in Neural Information Processing Systems*, 29, 2016. doi:10.48550/arXiv.1605.05775.

[Str69] V. Strassen. Gaussian elimination is not optimal. *Numerische mathematik*, 13(4):354–356, 1969. doi:10.1007/BF02165411.

[Str83] V. Strassen. Rank and optimal computation of generic tensors. *Linear Algebra and its Applications*, 52-53:645–685, 1983. doi:10.1016/0024-3795(83)80041-X.

[Str86] V. Strassen. The asymptotic spectrum of tensors and the exponent of matrix multiplication. *27th Annual Symposium on Foundations of Computer Science (sfcs 1986)*, pages 49–54, 1986. doi:10.1109/SFCS.1986.52.

[Str87] V. Strassen. Relative bilinear complexity and matrix multiplication. *Journal für die reine und angewandte Mathematik*, 0375_0376:406–443, 1987. doi:10.1515/crll.1987.375-376.406.

[Str88] V. Strassen. The asymptotic spectrum of tensors. *Journal für die reine und angewandte Mathematik*, 384:102–152, 1988. doi:10.1515/crll.1988.384.102.

[SV01] A. Schofield and M. Van den Bergh. Semi-invariants of quivers for arbitrary dimension vectors. *Indagationes Mathematicae*, 12(1):125–138, 2001. doi:10.1016/S0019-3577(01)80010-0.

[Swi12] B. Swingle. Entanglement renormalization and holography. *Phys. Rev. D*, 86:065007, 2012. doi:10.1103/PhysRevD.86.065007.

[Syl52] J. J. Sylvester. On the principles of the calculus of forms. *Cambridge and Dublin Math. J.*, 7:52–97, 1852.

[Tao16] T. Tao. A symmetric formulation of the Croot–Lev–Pach–Ellenberg–Gijswijt capset bound. https://terrytao.wordpress.com, 2016.

[VdN10] M. Van den Nest. Classical simulation of quantum computation, the Gottesman-Knill theorem, and slightly beyond. *Quantum Information and Computation*, 10(3):258–271, 2010. doi:10.26421/QIC10.3-4-6.

[Ven19] F. Venturelli. Prehomogeneous tensor spaces. *Lin. and Mult. Algebra*, 67(3):510–526, 2019. doi:10.1080/03081087.2018.1424112.

[Vid07] G. Vidal. Entanglement Renormalization. *Phys. Rev. Lett.*, 99(22), 2007. doi:10.1103/PhysRevLett.99.220405.

[Vid08] G. Vidal. Class of Quantum Many-Body States That Can Be Efficiently Simulated. *Phys. Rev. Lett.*, 101(11), 2008. URL: https://doi.org/10.1103%2Fphysrevlett.101.110501.

[Whi92] S. R. White. Density matrix formulation for quantum renormalization groups. *Phys. Rev. Lett.*, 69(19):2863–2866, 1992. doi:10.1103/physrevlett.69.2863.

[Wil12] V. V. Williams. Multiplying matrices faster than coppersmith-winograd. In *Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing*, STOC '12, page 887–898, New York, 2012. Association for Computing Machinery. doi:10.1145/2213977.2214056.

[Win70] S. Winograd. On the number of multiplications necessary to compute certain functions. *Communications on Pure and Applied Mathematics*, 23(2):165–179, 1970. doi:10.1002/cpa.3160230204.

[YCGD10] N. Yu, E. Chitambar, C. Guo, and R. Duan. Tensor rank of the tripartite state $|w\rangle^{\otimes n}$. *Phys. Rev. A*, 81:014301, 2010. doi:10.1103/PhysRevA.81.014301.

[ZC21] H. Zhai and G. Kin-Lic Chan. Low communication high performance ab initio density matrix renormalization group algorithms. *The Journal of Chemical Physics*, 154(22):224116, 2021. doi:10.1063/5.0050902.

[ZSW20] Y. Zhou, E. M. Stoudenmire, and X. Waintal. What Limits the Simulation of Quantum Computers? *Phys. Rev. X*, 10:041038, 2020. doi:10.1103/PhysRevX.10.041038.