PhD Thesis

Definability of maximal discrete sets

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Abstract

In this thesis we study set-theoretic definability of maximal objects originating from various branches of mathematics, encompassing set theory, combinatorics, group theory, measure theory and operator algebras.

In Part II, which is based on joint work with Asger Törnquist, we study definability of maximal almost disjoint families. With a simple tree derivative process, we first give a new proof of the classical theorem, due to Mathias, stating that there are no infinite analytic maximal almost disjoint families. With small adjustments, the process can be carried out and terminates in $L_{\omega_1^{\mathrm{CK}}}$, which proves that for every infinite Σ^1_1 almost disjoint family $\mathcal{A}$ there is a Δ^1_1 infinite subset x of ω such that $x \cap z$ is finite for every $z \in \mathcal{A}$. Our argument can be adapted to prove that if $\aleph_1^{L[a]} < \aleph_1$, then there are no infinite $\Sigma^1_2[a]$ maximal almost disjoint families. A small modification of the derivative process can also be used to prove that under $\mathsf{MA}(\kappa)$ there are no infinite κ-Suslin maximal almost disjoint families.

Part III is a reproduction of a preprint on definability of maximal cofinitary groups, authored jointly with David Schrittesser. We give a construction of a closed (even Π^0_1) set which freely generates an F_σ (even Σ^0_2) maximal cofinitary group. In this isomorphism class, this is the lowest possible complexity of a maximal cofinitary group. Additionally, we discuss obstructions to potential constructions of G_δ maximal cofinitary groups and introduce (maximal) finitely periodic groups.

In Part IV, which is also a reproduction of a preprint, we study maximal orthogonal families. We begin by giving a new, short and elementary proof of a theorem by Preiss and Rataj, stating that there are no analytic maximal orthogonal families of Borel probability measures on a Polish space. In case when the underlying space is compact and perfect, we establish that the set of witnesses to non-maximality is comeagre. The idea of our argument is based on the original proof by Preiss and Rataj, but with significant simplifications. Our proof generalises to show that under $\mathsf{MA} + \neg\mathsf{CH}$ there are no $\boldsymbol{\Sigma}^1_2$ maximal orthogonal families, that under PD there are no projective maximal orthogonal families and that under AD there are no maximal orthogonal families at all. Finally, we introduce a notion of strong orthogonality for states on separable C*-algebras and generalise a theorem due to Kechris and Sofronidis, stating that for every analytic orthogonal family of Borel probability measures there is a product measure orthogonal to all measures in the family, to states on a certain class of C*-algebras.

Resumé

I denne afhandling undersøger vi mængdeteoretisk definérbarhed af maksimale objekter som stammer fra forskellige grene af matematikken, herunder mængdelære, kombinatorik, gruppeteori, målteori og operator-algebra.

I Del II, som er baseret på arbejde sammen med Asger Törnquist, undersøger vi definérbarhed af maksimale næsten disjunkte familier. Med en simpel træafledningsproces giver vi et nyt bevis for den klassiske sætning af Mathias, som siger, at der ikke findes uendelige analytiske maksimale næsten disjunkte familier. Med små justeringer kan proceduren realiseres og afsluttes i $L_{\omega_1^{\mathrm{CK}}}$, hvilket beviser at for hver uendelige Σ^1_1 næsten disjunkt familie $\mathcal{A}$ findes der en Δ^1_1-uendelig delmængde x af ω, således at $x \cap z$ er endelig for hver $z \in \mathcal{A}$. Vores bevis kan tilpasses til at vise, at hvis $\aleph_1^{L[a]} < \aleph_1$, så findes der ingen uendelige $\Sigma^1_2[a]$ maksimale næsten disjunkte familier. En modifikation af afledningsprocessen kan bruges til at bevise, at under antagelse af $\mathsf{MA}(\kappa)$ findes der ingen uendelige κ-Suslin maksimale næsten disjunkte familier.

Del III er en reproduktion af et fortryk om definérbarhed af maksimale kofinitære grupper, skrevet i fællesskab med David Schrittesser. Vi giver en konstruktion af en lukket (endnu Π^0_1) mængde, som frit genererer en F_σ (endnu Σ^0_2) maksimal kofinitær gruppe. Det er den optimale kompleksitet af en maksimal kofinitær gruppe i denne isomorfiklasse. Ydermere diskuterer vi hindringer for potentielle konstruktioner af G_δ maksimale kofinitære grupper og introducerer (maksimale) endeligt periodiske grupper.

I Del IV, som er også en gengivelse af et fortryk, undersøger vi maksimale ortogonale familier. Vi begynder med at give et nyt, kort og elementært bevis af en sætning af Preiss og Rataj, som siger at der ikke findes analytiske maksimale ortogonale familier af Borel-sandsynlighedsmål på et polsk rum. Under antagelse af at det underliggende rum er kompakt og perfekt, viser vi at samlingen af vidner til ikke-maksimalitet er komagre. Idéen i vores bevis er baseret på det originale bevis af Preiss og Rataj, men med betydelige forenklinger. Vores bevis kan generaliseres til at vise, at under antagelse af $\mathsf{MA}+\neg\mathsf{CH}$ findes der ingen $\boldsymbol{\Sigma^1_2}$ maksimale ortogonale familier, under antagelse af PD findes der ingen projektive maksimale ortogonale familier, og at under antagelse af AD findes der slet ikke nogen maksimale ortogonale familier. Afslutningsvis introducerer vi stærk ortogonalitet for tilstande på separable C*-algebraer og generaliserer en sætning af Kechris og Sofronidis, som siger, at for enhver analytisk ortogonal familie af Borel-sandsynlighedsmål findes der et produktmål, som er ortogonalt til alle mål i familien, til tilstande på en bestemt klasse af C*-algebraer.

Povzetek

V tej doktorski dizertaciji obravnavamo definabilnost maksimalnih objektov iz različnih vej matematike, obsegajoč teorijo množic, kombinatoriko, teorijo grup, teorijo mere in operatorske algebre.

V Delu II, ki je osnovan na skupnem delu z Asgerjem Törnquistom, obravnavamo definabilnost maksimalnih skoraj-disjunktnih družin. S preprostim odvodnim postopkom na drevesu podamo nov dokaz za klasični Mathiasov izrek, ki trdi, da ni nobenih neskončnih analitičnih maksimalnih skoraj-disjunktnih družin. Z majhnimi spremembami lahko postopek izvedemo v $L_{\omega_1^{\mathrm{CK}}}$, kar dokaže, da za vsako neskončno Σ^1_1 skoraj-disjunktno družino $\mathcal{A}$ obstaja Δ^1_1 neskončna podmnožica naravnih števil x, tako da je $x \cap z$ končen za vsak $z \in \mathcal{A}$. Adaptacija našega argumenta pokaže, da ob predpostavki $\aleph_1^{L[a]} < \aleph_1$ ni nobenih neskončnih $\Sigma^1_2[a]$ maksimalnih skoraj-disjunktnih družin. Z majhno modifikacijo odvodnega postopka lahko dokažemo tudi, da ob predpostavki $\mathsf{MA}(\kappa)$ ni nobenih neskončnih κ-Suslinovih maksimalnih skoraj-disjunktnih družin.

Del III je reprodukcija prednatisa o definabilnosti maksimalnih kofinitarnih grup, napisanega skupaj z Davidom Schrittesserjem. V njem podamo konstrukcijo Π^0_1 množice, ki prosto generira Σ^0_2 maksimalno kofinitarno grupo. To je najboljša možna kompleksnost maksimalne kofinitarne grupe v izomorfnostnem razredu prosto generiranih grup. Prav tako obravnavamo rezultate, zaradi katerih je potencialna konstrukcija G_δ maksimalnih kofinitarnih grup otežena in uvedemo (maksimalne) končno-periodične grupe.

V Delu IV, ki je prav tako reprodukcija prednatisa, obravnavamo maksimalne ortogonalne družine. Začnemo z novim kratkim elementarnim dokazom izreka Preissa in Rataja, ki trdi, da ni nobene analitične maksimalne družine Borelovih verjetnostnih mer na polskem prostoru. Ko je prostor kompakten in perfekten, dokažemo, da je množica prič k nemaksimalnosti enaka komplementu množice prve kategorije. Naša ideja je osnovana na originalnem dokazu Preissa in Rataja, vendar s precejšnjimi poenostavitvami. Posplošitev dokaza pokaže, da ob predpostavki $\mathsf{MA} + \neg\mathsf{CH}$ ni nobenih $\boldsymbol{\Sigma}^1_2$ maksimalnih ortogonalnih družin, ob predpostavki PD nobenih projektivnih maksimalnih ortogonalnih družin in ob predpostavki AD prav nobenih maksimalnih ortogonalnih družin. Za zaključek uvedemo pojem krepke ortogonalnosti za stanja na separabilnih C*-algebrah in posplošimo izrek Kechrisa in Sofronidisa, ki trdi, da za vsako analitično ortogonalno družino Borelovih verjetnostnih mer obstaja produktna mera, ki je ortogonalna na vse mere iz družine, na stanja na določenem razredu C*-algeber.

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Contents

Part I.

Introduction

Structure of the thesis

The thesis consists of Introduction, three independent parts, each written in the form of a preprint and including its own respective introduction, acknowledgments and bibliography, and Conclusion.

Part II, Maximal almost disjoint families: In order to match with the formatting of the rest of this thesis, the content of this part, composed of work by Severin Mejak and Asger Törnquist, has the format of a preprint with the title *An effective strengthening of Mathias' theorem*, even though it has not yet been uploaded to any preprint server, as there is hope that the ongoing work by Severin Mejak, David Schrittesser and Asger Törnquist will materialise in more general results.

Part III, Maximal cofinitary groups: The content of this part is an exact reproduction of a preprint [MS22] by Severin Mejak and David Schrittesser, titled *Definability of maximal cofinitary groups*, uploaded to the preprint repository `arXiv.org` in 2022 with

`doi:10.48550/arXiv.2212.05318`

and is accessible at

`arXiv:2212.05318v1 [math.GR]`.

The preprint is intended to be submitted to a research journal in the near future.

Part IV, Maximal orthogonal families: The content of this part is an exact reproduction of a preprint [Mej22] by Severin Mejak, titled *Orthogonality of measures and states*, uploaded to the preprint repository `arXiv.org` in 2022 with

`doi:10.48550/arXiv.2204.02767`

and is accessible at

`arXiv:2204.02767v5 [math.OA]`.

The preprint has been submitted to Fundamenta Mathematicae.

In the following Contextual background we re-organise, amplify and incorporate portions of respective introductions to Parts II, III and IV, and supply them with additional content in order to provide both a historical background and state-of-the-art of the studied mathematical area, as well as an overview of the thesis.

Contextual background

(Descriptive) set theory

Set theory was born of Georg Cantor's quest into the infinite and developed alongside logical and mathematical formalisation, undertaken by Gottlob Frege, Bertrand Russell and others, into a mathematical discipline with a dual role: on one hand the one of forming the foundations of all of mathematics (even though this has since been contested by other formal systems, set theory undoubtedly remains the most widespread and accepted one), and on the other hand the one of being a branch of mathematics on its own. Modern set theory is divided into the following complemental subfields: descriptive set theory, infinitary combinatorics, forcing, large cardinals, inner model theory and set-theoretic topology. This thesis concerns itself with the first.

Descriptive set theory has its origins in the early years of general topology and measure theory. It has since developed into an area of set theory, where the questions about *definability* and *regularity properties* of subsets of Polish spaces (e.g., $\mathbb{R}$, 2^ω or ω^ω, where ω denotes the set of all natural numbers) are studied. Roughly speaking, a subset of a Polish space is more definable (less complex) if there is a simple formula describing when an element is a member of this subset. One then assigns subsets of any fixed Polish space to classes according to their definability, so that the classes form a hierarchy under inclusion (the more definable a class is, the less subsets it contains). Some of these classes are well known to all mathematicians, such us the class of all open subsets (very definable) and the class of all Borel subsets (still quite definable), others, such us the class of *analytic* subsets (a class strictly larger than the class of Borel sets), appear rarely outside the realm of descriptive set theory and the branches of mathematics, where descriptive set theory has applications, such as measure theory, dynamical systems or operator algebras. Different definability classes satisfy different regularity properties, hence the more definable a subset is (and consequently belongs to more classes), the nicer properties hold for it.

Many natural objects considered across mathematics are subsets of some Polish space and are moreover definable in the above sense. This means that many objects considered in everyday mathematical practice satisfy certain regularity properties. As an example, Georg Cantor's famous Continuum hypothesis asks whether every uncountable set of reals has cardinality $\mathfrak{c}$ (called continuum; the cardinality of the set of all real numbers). Paul Cohen, introducing the groundbreaking technique of forcing, proved that this does not hold in all models of set theory. But if one restricts the question to analytic subsets, the Continuum hypothesis does hold (in every model of set theory). In particular, it holds for many objects of interest across mathematics, since many of these are analytic.

Maximal discrete sets

Considering objects of a certain kind, for instance linearly independent sets of vectors in a fixed vector space, non-principal algebraic ideals on a fixed commutative ring, filters on a fixed set, etc., we say that a given object is *maximal*, if it is not strictly contained (as a subset) in another such object. Since a maximal linearly independent set of vectors in a fixed vector space is a Hamel basis, a maximal non-principal algebraic ideal on a fixed commutative ring is a prime ideal and a maximal filter is an ultrafilter, every mathematician would agree that maximal objects play distinctly important roles, as they are employed in numerous constructions and arguments.

In this thesis we will focus on a specific kind of maximal objects, called *maximal discrete sets*. Fix a Polish space X. A *hypergraph* on X is a subset $H \subseteq [X]^{<\omega} \setminus \{\emptyset\}$, where $[X]^{<\omega}$ denotes the collection of finite subsets of X. To each hypergraph H on X we associate the following notion: a subset $Y \subseteq X$ is *H-discrete*, if $[Y]^{<\omega} \cap H = \emptyset$. In the literature, H-discrete sets are usually called *H-independent sets*, but as we will also consider a different notion of independent families, we rather use the name discrete sets (introduced by Benjamin Miller in [Mil] and used in e.g. [ST18] and [Sch20]). As already defined in the previous paragraph, a *maximal H-discrete set* is an H-discrete set, which is maximal with respect to inclusion among all H-discrete sets.

Invoking the axiom of choice in the form of Zorn's lemma, we see that for every Polish space X and any hypergraph H on X there are maximal H-discrete sets. When X is uncountable, there are two types of natural questions one may ask about maximal discrete sets. The first asks what the possible cardinalities of infinite maximal discrete sets are (of course, assuming that the continuum hypothesis fails), and the second what is the best possible complexity of infinite maximal discrete sets. In this thesis we concern ourselves with the second kind of questions. A survey on research concerning both sorts of inquiries is given in David Schrittesser's [Sch20].

It is to be expected that for a fixed hypergraph H, one can obtain better result than for general hypergraphs. Nevertheless, there are some general results, which we present before discussing certain important types of maximal discrete sets in detail.

The following is a special case (our presentation of it follows the one given in [Sch20]) of a result by Zoltán Vidnyánszky from [Vid14], generalising the technique introduced and applied to certain examples of maximal discrete sets by Arnold Miller in [Mil89]. The coding idea can be traced back even further to [EKM81] by Paul Erdős, Kenneth Kunen and Daniel Mauldin.

Theorem (Vidnyánszky)**.** *Assume that $V = L$ and that H is a $\Sigma^1_1[a]$ hypergraph on a Polish space X. Furthermore, assume that for any $z \in 2^\omega$, any countable $C \subseteq X$ and any $x \in X$, for which $C \cup \{x\}$ is H-discrete, it holds that there are $y_0, \dots, y_n \in X$ so that:*

- *$C \cup \{y_0, \dots, y_n\}$ is H-discrete;*
- *$C \cup \{x, y_0, \dots, y_n\}$ is not H-discrete;*
- *for every $i \in n + 1$ it holds that $z \in \Delta^1_1(y_i)$.*

Then there is a $\Pi^1_1[a]$ maximal H-discrete set.

As we will soon see, for many of the hypergraphs H meeting the assumptions of this theorem, Π^1_1 is the best possible complexity of a maximal H-discrete set.

General results without the assumption $V = L$ require some other additional assumptions. David Schrittesser and Asger Törnquist proved in [ST18] that for any $\Sigma^1_1[a]$ hypergraph H there is a $\Delta^1_2[a]$ predicate which defines a maximal H-discrete set in both $L[a]$ and in Sacks and Miller extensions of $L[a]$. In [Sch16], David Schrittesser proved that in the extension of L by an ω_2-length countable support iteration or a finite product of Sacks forcing, every analytic hypergraph H admits a $\boldsymbol{\Delta}^1_2$ maximal H-discrete set. In particular, this shows that the existence of $\boldsymbol{\Delta}^1_2$ maximal discrete sets is consistent with the negation of the continuum hypothesis. Jonathan Schilhan ([Sch22c]) generalised the statement to splitting forcing.

Regarding the consistency strength of the non-existence of maximal discrete sets (of course without the axiom of choice), Haim Horowitz and Saharon Shelah constructed in [HS19] a Borel graph H, for which the theory $\mathsf{ZF} + \mathsf{DC} +$ "there is no maximal H-discrete set" is equiconsistent with $\mathsf{ZFC} +$ "there exists an inaccessible cardinal". For other results on maximal discrete sets in the absence of choice see [Sch22b].

We next first present a few types of maximal discrete sets and briefly recall some results on their definability, after which we focus on the maximal discrete sets examined in detail in this thesis: maximal almost disjoint families, maximal cofinitary groups and maximal orthogonal families.

Hamel basis

One of the first considerations of definability of maximal objects was Wacław Sierpiński's [Sie20] from 1920, where he proved that there can be no Borel Hamel basis for $\mathbb{R}$, when seen as a vector space over $\mathbb{Q}$. Burton Jones later proved (see [Jon42]) that no Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ can be analytic. On the other hand, Arnold Miller's seminal [Mil89] establishes that under the assumption $V = L$ there is a Π^1_1 Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$.

To see that Hamel basis is a type of maximal discrete set, define

$$H := \left\{ A \in [\mathbb{R}]^{<\omega} \;\middle|\; (\exists v_0 \in A)\,(\exists v_1, \dots, v_n \in A \setminus \{v_0\}) \right.$$
$$\left. (\exists \lambda_1, \dots, \lambda_n \in \mathbb{Q})\, v_0 = \sum_{k=1}^{n} \lambda_k v_k \right\} \subseteq [\mathbb{R}]^{<\omega} \setminus \{\emptyset\}$$

and observe that H-discrete sets are exactly $\mathbb{Q}$-linearly independent subsets of $\mathbb{R}$.

Ultrafilters

Wacław Sierpiński proved in [Sie38] that no non-principal ultrafilter on ω can be Borel. By Kolmogorov's zero-one law it is easy to see that there can be no Haar measurable (identifying $\mathcal{P}(\omega)$, the powerset of ω, with the locally compact Polish group $\mathbb{Z}_2^\omega$) non-principal ultrafilter on ω (and taking category analogues to measurability notions shows that there can be no non-principal ultrafilter with the Baire property), so in particular there are no analytic nor coanalytic non-principal ultrafilters on ω. On the other hand, if $V = L$ the standard argument using the L-ordering of the reals shows that there are Δ^1_2 non-principal ultrafilters.

The H-discrete sets, when we define

$$H := \{A \in [\mathcal{P}(\omega)]^{<\omega} \setminus \{\emptyset\} \mid \cap A \text{ is finite}\},$$

do not exactly correspond to non-principal filters, but it is not difficult to see that maximal H-discrete sets are non-principal ultrafilters.

Related research considers definability (of basis) of P-points, Q-points and Ramsey ultrafilters, see e.g. [Sch22a].

Maximal independent sets

A family $\mathcal{I} \subseteq [\omega]^\omega$ (we use $[\omega]^\omega$ to denote the set of all infinite subsets of ω and equip it with the Polish topology inherited from 2^ω) is *independent*, if for any disjoint finite subsets I, J of $\mathcal{I}$ it holds that $\cap I \setminus (\cup J)$ is infinite. Defining

$$H := \{A \in [[\omega]^\omega]^{<\omega} \setminus \{\emptyset\} \mid (\exists I \subseteq A)\, (\exists J \subseteq A \setminus I)\, |\cap I \setminus (\cup J)| < \infty\},$$

we can see that independent families are exactly H-discrete sets.

It was proved by Arnold Miller in [Mil89] that there are no analytic maximal independent families and that if $V = L$ there is a Π^1_1 maximal independent family. More recently, Jörg Brendle, Vera Fischer and Yurii Khomskii (see [BFK18]) proved that the existence of a Σ^1_2 maximal independent family is equivalent to the existence of a Π^1_1 maximal independent family.

Maximal eventually different families

Two functions $f, g \in \omega^\omega$ are *eventually different*, if there is some $n \in \omega$, so that for all $m > n$ it holds that $f(m) \neq g(m)$. A subset of ω^ω is *eventually different* if its elements are pairwise eventually different, and it is *maximal eventually different*, if it is maximal with respect to inclusion among eventually different families. It is easy to define a hypergraph $H \subseteq [\omega^\omega]^{<\omega}$ for which eventually different families are exactly H-discrete sets.

Quite opposite to the non-definability results of the so far mentioned maximal discrete sets, there are very definable maximal eventually different families. In 2010s, Haim Horowitz and Saharon Shelah (see [HS16b]) constructed a Δ^1_1-maximal eventually different family. Soon afterwards, this was improved by David Schrittesser ([Sch17]) to a construction of a Π^0_1 maximal eventually different family.

Maximal almost disjoint families

Arguably the most studied maximal objects in set theory are maximal almost disjoint families, to which we devote Part II of this thesis.

Two subsets x, y of ω are said to be *almost disjoint* if their intersection is finite. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if for every $x \neq y$ from $\mathcal{A}$ it holds that they are almost disjoint. A *maximal almost disjoint family* is any almost disjoint family which is maximal with respect to inclusion among all almost disjoint families. As with eventually different families, since almost disjoint families are defined in terms of a property being true for each pair of distinct elements from the family, it is immediate to define a hypergraph H for which almost disjoint families correspond to H-discrete sets. We use the acronym *mad* in place of "maximal almost disjoint".

Any partition of ω into finitely many infinite sets clearly constitutes a mad family. For this reason we focus only on infinite mad families. It is not difficult to see that every infinite mad family is uncountable.

During the late 1960s Adrian Mathias (published in [Mat77]) proved that there are no infinite analytic mad families, introducing what is now called Mathias forcing. Around 1988, Arnold Miller (see [Mil89]) showed that if $V = L$ then there is an infinite Π^1_1 mad family. In 2010s, Asger Törnquist first proved (in [Tör09]) that existence of infinite Π^1_1 mad families is equivalent to existence of infinite Σ^1_2 mad families. Not long after (see [Tör18]), he came up with a tree derivative proof of the fact that there are no infinite analytic mad families, and extended the idea to prove that there are no infinite mad families in Solovay's model. This was followed by Itay Neeman's and Zach Norwood's [NN18], in which they showed that there are no infinite mad families under AD^+. Around the same time, Asger Törnquist and David Schrittesser proved in [ST19] that assuming that all sets of reals are Ramsey and that Ramsey uniformisation holds, there are no infinite mad families. The line of research on consistency of non-existence of infinite mad families culminated with [HS19], where Haim Horowitz and Saharon Shelah established that the theory $\mathsf{ZF} + \mathsf{DC} +$ "there are no infinite mad families" is equiconsistent with ZFC. Note also that David Schrittesser and Asger Törnquist proved in [ST20] that if x is Laver-generic over L, then there is an infinite Π^1_1 mad family in $L[x]$. Finally, Vera Fischer, David Schrittesser and Thilo Weinert proved in [FSW21] that if bounded proper forcing axiom holds and ω_1 is not remarkable in L, then there is an infinite $\boldsymbol{\Pi^1_2}$ mad family.

In Part II, we simplify the derivative process introduced in [Tör18] to get a new proof of the classical Mathias theorem.

Theorem (Mathias)**.** *There are no infinite analytic mad families.*

We observe that by making small adjustments to the argument, we are able to run it in $L_{\omega_1^{\mathrm{CK}}}$. Moreover, we can prove that the process terminates in $L_{\omega_1^{\mathrm{CK}}}$. In this way, we obtain the following.

Theorem. *For every infinite Σ^1_1 almost disjoint family $\mathcal{A}$ there is a Δ^1_1 witness to non-maximality, i.e., there is some $x \in \Delta^1_1([\omega]^\omega)$ so that $x \cap z$ is finite for all $z \in \mathcal{A}$.*

Furthermore, with minor modifications, our method can be used to provide new and simpler proofs of the following facts established in [Tör18].

Theorem (Törnquist)**.** *Let $a \in \omega^\omega$ and suppose that $\aleph_1^{L[a]} < \aleph_1$. Then there are no infinite $\Sigma^1_2[a]$ mad families.*

Theorem (Törnquist)**.** *If $\mathsf{MA}(\kappa)$ holds for some $\kappa < 2^{\aleph_0}$ then there are no infinite κ-Suslin mad families.*

Iterating the Fubini product of the ideal Fin of finite subsets of ω, one obtains ideals Fin^α for $\alpha \in \omega_1 \setminus \{0\}$, defined on countable sets, which we respectively denote by M^α. For $x, y \subseteq M^\alpha$ we say that they are Fin^α-*almost disjoint*, if $x \cap y \in \mathrm{Fin}^\alpha$.

In [BST22], Karen Bakke Haga, David Schrittesser and Asger Törnquist proved that for every $\alpha \in \omega_1 \setminus \{0\}$, there are no infinite analytic maximal Fin^α-almost disjoint families.

There is every indication that our result on effective witnesses for the ideal Fin should generalise to the ideals Fin^α for $\alpha \in \omega_1^{\mathrm{CK}} \setminus \{0\}$, however, despite considerable progress, the proof has so far eluded us.

Maximal cofinitary groups

In [Cam96], Peter Cameron introduced the notion of a cofinitary subgroup of S_∞ (the group of all permutations of ω) as follows. An element $g \in S_\infty \setminus \{\mathrm{id}_\omega\}$ is called *cofinitary* if it has only finitely many fixed points, i.e., there is some $n \in \omega$ so that for every $m > n$ it holds that $g(m) \neq m$. Then a subgroup $G \leq S_\infty$ is *cofinitary* if every $g \in G \setminus \{\mathrm{id}_\omega\}$ is cofinitary. We write "cofinitary group" in place of the longer "cofinitary subgroup of S_∞". Cofinitary groups were studied before [Cam96] under different names; e.g., in [Ade88] they are called *sharp groups*. For early results on embeddings of cofinitary groups see [Tru87] and [Ade88]. In [Cam96], Peter Cameron discusses combinatorial properties of cofinitary groups and conjectures that every closed cofinitary group is locally compact, which was refuted by Greg Hjorth in [Hjo98] soon afterwards.

To see that cofinitary groups are discrete sets, define the hypergraph

$$H := \{A \in [S_\infty]^{<\omega} \mid (\exists g \in \langle A \rangle \setminus \{1_\omega\})\, g \text{ has infinitely many fixed points}\},$$

where $\langle A \rangle$ is the subgroup of S_∞ generated by A.

A cofinitary group is *maximal* if it is maximal among all cofinitary groups, i.e., it is not strictly contained in any cofinitary group. We use the acronym *mcg* to refer to maximal cofinitary groups. Mcgs were first considered in [Tru87] and [Ade88]. The first result which made mcgs interesting to set-theorists was established by Adeleke in [Ade88], asserting that mcgs are always uncountable.

The first breakthrough in the study of definability of mcgs was when Su Gao and Yi Zhang (see [GZ08]) established that if $V = L$ then there is a Π^1_1 set which generates an mcg. This was improved by Bart Kastermans in [Kas08] to the existence of a Π^1_1 mcg under the assumption $V = L$. Kastermans also proved the beautiful results that no mcg can be K_σ (see [Kas08]) and that no mcg can have infinitely many orbits (see [Kas09]).

Kastermans' result on definability inspired [FST17], in which Vera Fischer, David Schrittesser and Asger Törnquist constructed a Cohen-indestructible Π^1_1 mcg, assuming that $V = L$. The next milestone was achieved (to some surprise) soon afterwards, when Haim Horowitz and Saharon Shelah established in [HS16a] that it is provable in ZF (without the axiom of choice) that there is a Borel mcg. Using ideas of [HS16a], David Schrittesser proved (without use of the axiom of choice) in [Sch21] that there is actually an arithmetical mcg.

In Part III, we first use the idea of the construction from [Sch21] and enhance it with ideas from [Sch17] to get a construction (again without using the axiom of choice) proving the following.

Theorem. *There is a Π^0_1 subset of ω^ω which freely generates a Σ^0_2 mcg.*

Since the topological interior of any cofinitary group is clearly empty, Π^0_1 is the best possible complexity of a set generating an mcg.

By a result of Richard Dudley ([Dud61]), there is no Polish topology on a group freely generated by continuum many generators. This clearly implies that Σ^0_2 is the best possible complexity of a freely generated mcg. Additional difficulties on potential constructions of G_δ mcgs are imposed by [Slu12], in which Konstantin Slutsky improved on Dudley's result and proved that if G is a free product of groups and carries a Polish topology, then G is countable. Since all presently known constructions of definable mcgs produce groups which decompose into free products, this means that current ideas are insufficient to produce a G_δ mcg.

We conclude Part III by introducing maximal finitely periodic groups. Say that $g \in S_\infty \setminus \{\mathrm{id}_\omega\}$ is *finitely periodic* if $\langle g \rangle \subseteq S_\infty$ has finitely many finite orbits. A subgroup $G \leq S_\infty$ is then called *finitely periodic* if every $g \in G \setminus \{\mathrm{id}_\omega\}$ is finitely periodic (it is again easy to see that finitely periodic groups are H-discrete sets for an appropriate choice of H), and *maximal finitely periodic* if it is maximal among finitely periodic groups. Clearly, any finite periodic group is also cofinitary. By adapting a proof by Kastermans from [Kas09], we prove the following.

Theorem. *There is no maximal finitely periodic group with infinitely many orbits.*

Maximal orthogonal families

For a Polish space X, we denote the Polish space of all Borel probability measures on X by $P(X)$. For $\mu, \nu \in P(X)$, we say that μ and ν are *orthogonal* (often also called *singular*), which we denote by $\mu \perp \nu$, if there is a Borel subset $B \subseteq X$ such that $\mu(B) = 0$ and $\nu(B) = 1$. A family $\mathcal{A} \subseteq P(X)$ is *orthogonal*, if every $\mu \neq \nu \in \mathcal{A}$ are orthogonal. It is again effortless to define a hypergraph H for which orthogonal families are precisely H-discrete sets.

In 1985 David Preiss and Jan Rataj proved the following theorem with $X = [0, 1]$, see [PR85].

Theorem. *Suppose that X is an uncountable Polish space. Then there is no analytic maximal orthogonal family of Borel probability measures on X.*

This answered an open question from [MPV82]. The idea of the proof from [PR85] is to use a Baire category argument. However, once the authors prepared the scene for the application of the Baire category theorem, they resorted to a couple of technical lemmas, which relied on restricting Borel probability measures on $[0, 1]$ to finite unions of closed subintervals. For the proof of one of the lemmas they also employed Banach–Mazur games. Consequently, the question whether there is a shorter and simpler proof remained open.

In 1999, Alexander Kechris and Nikolaos Sofronidis (see Thoerem 3.1 in [KS01]) found an alternative short proof which uses the theory of *turbulence* (see Greg Hjorth's [Hjo00] for a great guide to turbulence). As part of their proof, they defined an embedding of the Cantor space 2^ω into the space of Borel probability measures (using the work of Shizuo Kakutani from [Kak48]), assigning to every $x \in 2^\omega$ a product measure $\mu_{\alpha(x)}$. They proved that for every analytic orthogonal family, there is some $x \in 2^\omega$ so that $\mu_{\alpha(x)}$ is a witness to non-maximality. Their proof has as a consequence that the relation $\sim$ of measure equivalence between Borel probability measures is not classifiable by countable structures.

Almost two decades later David Schrittesser and Asger Törnquist used the same embedding of 2^ω into the space of measures to prove (see Theorem 5.5 of [ST18]) that an argument using a weaker form of turbulence suffices. Since the theory of turbulence requires some background knowledge, one might argue that even thought the proofs from [KS01] and [ST18] are *shorter*, they are not necessarily *simpler*.

In Part IV, we first go back to the original idea of Preiss and Rataj to use a Baire category argument. We were able to use the Kuratowski–Ulam theorem and some elementary convexity theory, to give a short and straightforward proof of the above theorem. The argument works to show the following strengthening, where for $\mathcal{A} \subseteq P(X)$, we let $\mathcal{A}^\perp := \{\nu \in P(X) \mid (\forall \mu \in \mathcal{A})\, \nu \perp \mu\}$.

Theorem. *Suppose that X is a compact perfect Polish space. Then for every analytic orthogonal family $\mathcal{A} \subseteq P(X)$, the set $\mathcal{A}^\perp$ is comeagre. In particular, when $\mathcal{A} \subseteq P(2^\omega)$ is a Σ^1_1 orthogonal family, there is a Δ^1_1-witness to non-maximality.*

Actually, under additional assumptions, our method yields the following.

Theorem. *Suppose that X is an uncountable Polish space.*

1. *Assume* MA *and* $\neg\mathsf{CH}$. *Then no $\mathbf{\Sigma^1_2}$ orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*
2. *Assume* PD. *Then no projective orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*
3. *Assume* AD. *Then no orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*

If moreover X is compact perfect, then in each of the above cases $\mathcal{A}^\perp$ is comeagre.

Concerning positive results on definability of maximal orthogonal families, Vera Fischer and Asger Törnquist proved in [FT10] that under the assumption $V = L$ there is a Π^1_1 maximal orthogonal family in $P(2^\omega)$. On the other hand, David Schrittesser and Asger Törnquist proved in [ST18] that if all $\Sigma^1_2[a]$ sets of reals are completely Ramsey, then there are no $\Sigma^1_2[a]$ maximal orthogonal families (so as a consequence there are no $\Sigma^1_2[a]$ maximal orthogonal families in $L[a][x]$, where x is Mathias-generic over $L[a]$).

It is well-known that via the Riesz–Markov–Kakutani representation theorem, Borel probability measures on a compact Polish space X are precisely states on the commutative C*-algebra $C(X)$ of complex-valued continuous functions on X. For any separable C*-algebra A it holds that $S(A)$, the collection of all states on A, carries a natural Polish topology. In [Dye52], Henry Dye introduced the notion of absolute continuity for states on C*-algebras. Being a pre-order, it naturally gives rise to a notion of orthogonality. However, it turns out that this notion is ill-behaved even for states on the matrix algebra $M_2(\mathbb{C})$.

There is another natural notion of orthogonality for states, which we call *strong orthogonality* and denote by $\perp\!\!\!\perp$. This notion of orthogonality shares many nice properties with orthogonality of measures (among other things, $\perp\!\!\!\perp$-orthogonal families are H-discrete sets for the appropriate choice of a hypergraph H), with which it coincides when the C*-algebra is commutative. Hence it is natural to ask ourselves whether analogues to above statements hold for non-commutative separable unital C*-algebras and strong orthogonality as well.

Since the original proof by Preiss and Rataj relied on restrictions of measures to compact subspaces, it is not clear how to generalise that proof. The idea from our proof seems more promising, but there are still some steps for which we do not know if they hold for strong orthogonality for states.

On the other hand, it turns out that the idea of Kechris and Sofronidis from [KS01] can easily be extended to a class of separable unital C*-algebras, as the following theorem from Part IV shows.

Theorem. *Suppose A is a separable unital C*-algebra, which contains a copy of $C(2^\omega)$ as a subalgebra and for which there is a conditional expectation*

$$E : A \to C(2^\omega).$$

Then for every strongly orthogonal $\mathcal{A} \subseteq S(A)$ there is $\alpha \in (0,1)^\omega$ so that $\tilde{\mu}_\alpha \perp\!\!\!\perp \psi$ for every $\psi \in \mathcal{A}$, where $\tilde{\mu}_\alpha$ is the extension of the state, corresponding to the product measure

$$\prod_{n\in\omega}(\alpha(n)\delta_0 + (1-\alpha(n))\delta_1),$$

from $C(2^\omega)$ to A.

As in [KS01], along the way of proving this theorem we also get that for C*-algebras A, satisfying the assumptions of the theorem, the equivalence relation $\sim$ on $S(A)$, arising from Dye's notion of absolute continuity for states, is not classifiable by countable structures.

Natural examples of C*-algebras, for which the assumptions of the above theorem are satisfied, include the CAR algebra M_{2^∞} and the Cuntz algebra $\mathcal{O}_2$. Moreover, for any A satisfying these assumptions, also the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ (for any countable discrete group Γ and any homomorphism $\alpha : \Gamma \to \mathrm{Aut}(A)$) and the tensor product $A \otimes B$ (for any separable unital C*-algebra B) satisfy the assumptions.

In 1969 Donald Bures (see [Bur69]) proved an extension of Kakutani's result from [Kak48] to semi-finite von Neumann algebras. Instead of absolute continuity and orthogonality between states, Bures considered when two product states give rise to isomorphic tensor products of von Neumann algebras. This was extended to all von Neumann algebras by David Promislow in [Pro71].

As a consequence of the main ingredient of the proof of the last theorem, we get the following version of Kakutani's theorem for states, involving absolute continuity and strong orthogonality.

Proposition. *Suppose that* $(\alpha_n)_{n\in\omega}, (\beta_n)_{n\in\omega} \in [\frac{1}{4}, \frac{3}{4}]^\omega$ *and let*

$$\phi_n := \alpha_n \operatorname{ev}_{1,1} + (1-\alpha_n) \operatorname{ev}_{2,2} \quad \text{and} \quad \psi_n := \beta_n \operatorname{ev}_{1,1} + (1-\beta_n) \operatorname{ev}_{2,2}$$

be states on $M_2(\mathbb{C})$*. Let also* $\phi := \otimes_{n=0}^\infty \phi_n$ *and* $\psi := \otimes_{n=0}^\infty \psi_n$ *be the product states on* M_{2^∞}*. Then in* $S(M_{2^\infty})$*, either* $\phi \sim \psi$ *or* $\phi \perp \psi$ *according to whether*

$$\sum_{n\in\omega} (\alpha_n - \beta_n)^2$$

converges or diverges respectively.

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Part II.

Maximal almost disjoint families

AN EFFECTIVE STRENGTHENING OF MATHIAS' THEOREM

SEVERIN MEJAK AND ASGER TÖRNQUIST

Abstract. We present a tree derivative process which can be carried out (and terminates) in $L_{\omega_1^{\mathrm{CK}}}$ and use it to prove that for every infinite Σ^1_1 almost disjoint family there is a Δ^1_1 witness to non-maximality. The argument also gives a new elementary proof of the classical theorem due to Mathias that there are no infinite analytic maximal almost disjoint families. The derivative idea is generalised to provide new straightforward derivative proofs of the theorems due to the second author asserting that under $\aleph_1^{L[a]} < \aleph_1$ there are no infinite $\Sigma^1_2[a]$ maximal almost disjoint families, and that under $\mathsf{MA}(\kappa)$ there are no infinite κ-Suslin maximal almost disjoint families.

Introduction

Two subsets x, y of $\omega = \{0, 1, 2, \dots\}$ are said to be *almost disjoint* if their intersection is finite. A family $\mathcal{A} \subseteq [\omega]^\omega$ (the collection of infinite subsets of ω) is *almost disjoint* if for every $x \neq y$ from $\mathcal{A}$ it holds that they are almost disjoint. An almost disjoint family $\mathcal{A}$ is *maximal* if there is no almost disjoint family $\mathcal{B}$ with $\mathcal{A} \subsetneq \mathcal{B}$. We use the acronym *mad* in place of "maximal almost disjoint".

Any partition of ω into finitely many infinite sets clearly constitutes a mad family. For this reason we focus only on infinite (maximal) almost disjoint families. Invoking the axiom of choice in the form of Zorn's lemma, one can for any almost disjoint family $\mathcal{A}$ find a maximal almost disjoint family $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$.

It is not difficult to see (in fact, this follows from Lemma 1.1) that every infinite mad family is uncountable. The *almost disjointness number* $\mathfrak{a}$ denotes the least size of an infinite mad family. It is known that $\mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$, where $\mathfrak{b}$ denotes the *bounding number* and $\mathfrak{c}$ the *continuum*. Consistency of relations of $\mathfrak{a}$ to other cardinal invariants is not completely charted out, e.g., an open question, first posed by Roitman, asks whether $\mathfrak{d} = \omega_1$ implies that $\mathfrak{a} = \omega_1$, where $\mathfrak{d}$ denotes the *dominating number*.

In this note we focus on another kind of questions about mad families, asking how definable such families can be. During the late 1960s Mathias (published in [Mat77]) proved that there are no analytic infinite mad families, introducing what is now called Mathias forcing. Around 1988, Miller (in [Mil89]) showed that if $V = L$ then there is an infinite Π^1_1 mad family. In 2010s, the second author first proved (in [Tö09]) that existence of infinite Π^1_1 mad families is equivalent to existence of infinite Σ^1_2 mad families. Not long after (see [Tö18]), he came up with a tree derivative proof of the fact that there are no analytic infinite mad families, and extended the idea to prove that there are no infinite mad families in Solovay's model. This was followed by Neeman's and Norwood's [NN18], in which they established that there are no infinite mad families under AD^+. Around the same time, Schrittesser and the second author proved in [ST19] that assuming that all sets of reals are Ramsey and that Ramsey uniformisation holds, there are no infinite mad families. The line of research on consistency of non-existence of infinite mad families culminated with [HS19], where Horowitz and Shelah established that

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the theory $\mathsf{ZF}+\mathsf{DC}+$"there are no infinite mad families" is equiconsistent with ZFC. Note also that Schrittesser and the second author proved in [ST20] that if x is Laver-generic over L, then there is an infinite Π^1_1 mad family in $L[x]$. Finally, Fischer, Schrittesser and Weinert established in [FSW21] that if bounded proper forcing axiom holds and ω_1 is not remarkable in L, then there is an infinite $\boldsymbol{\Pi}^1_2$ mad family.

In this note we simplify the derivative process introduced in [Tö18] (in a similar fashion as is done in [CM]) to get a new elementary proof of Mathias' classical theorem.

Theorem 0.1 (Mathias)**.** *There are no infinite analytic mad families.*

We observe that by making small adjustments to the argument, we are able to run it in $L_{\omega_1^{\mathrm{CK}}}$. Moreover, we can prove that the process terminates in $L_{\omega_1^{\mathrm{CK}}}$. In this way, we obtain the following.

Theorem 0.2. *For every infinite Σ^1_1 almost disjoint family $\mathcal{A}$ there is a Δ^1_1 witness to non-maximality, i.e., there is some $x \in \Delta^1_1([\omega]^\omega)$ so that $x \cap z$ is finite for all $z \in \mathcal{A}$.*

Furthermore, with minor modifications, our method can be used to provide new and simpler proofs of the following facts established in [Tö18].

Theorem 0.3 (Törnquist)**.** *Let $a \in \omega^\omega$ and suppose that $\aleph_1^{L[a]} < \aleph_1$. Then there are no infinite $\Sigma^1_2[a]$ mad families.*

Theorem 0.4 (Törnquist)**.** *If $\mathsf{MA}(\kappa)$ holds for some $\kappa < 2^{\aleph_0}$ then there are no infinite κ-Suslin mad families.*

Structure of the note. This note has been written specifically for the purpose of constituting a part of the first author's PhD dissertation. For this reason it contains more background and details.

In Section 1, we introduce some notation and provide a simple proof of Theorem 0.1. We continue with Section 2, where we recall the fundamentals of effective descriptive set theory and prove Theorem 0.2. In Sections 3 and 4 we prove Theorems 0.3 and 0.4 respectively. We conclude the note with a discussion of open problems in Section 5.

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1. There are no analytic mad families

First we introduce some notation. We identify $\mathcal{P}(\omega)$, the powerset of ω, and the Cantor space 2^ω in the usual way (via characteristic functions). We use $[\omega]^\omega$ to denote the set of all infinite subsets of ω, equipped with the Polish topology inherited from 2^ω. Let Fin denote the ideal of all finite subsets of ω, i.e.,

$$\mathrm{Fin} = \{a \in \mathcal{P}(\omega) \mid a \text{ is finite}\}.$$

For $\mathcal{C} \subseteq \mathcal{P}(\omega)$ we denote the ideal generated by $\mathcal{C}$ and Fin by $\mathcal{I}_{\mathcal{C}}$. If $A, B \in \mathcal{P}(\omega)$, we let $A \subseteq^* B$ mean that $A \setminus B \in \mathrm{Fin}$.

For any set X (e.g., ω, ω_1 or κ), we use p to denote the projection from $2^\omega \times X^\omega$ onto the first component 2^ω. We use $\sqsubseteq$ to denote end-extension of finite sequences and lh to denote their length. Following the notation from [Kec95], for a tree T on $2 \times \omega$ (so $T \subseteq (2 \times \omega)^{<\omega}$) we use

$$T_{[t]} := \{s \in T \mid t \sqsubseteq s \vee s \sqsubseteq t\}$$

and

$$T_t := \{s \in T \mid t \sqsubseteq s\}.$$

Suppose that for a tree T on $2\times\omega$ the set of projections of branches $p[T]$ is almost disjoint and let $\mathcal{B}$ be a countable family of infinite subsets of ω. Then define

$$T^{\mathcal{B}} := \{t \in T \mid (\exists w \in [T_{[t]}])\, p(w) \notin \mathcal{I}_{\mathcal{B}}\}.$$

Note that $T^{\mathcal{B}}$ is a subtree of T. For $t \in T^{\mathcal{B}}$ we write $T^{\mathcal{B}}_{[t]}$ and $T^{\mathcal{B}}_t$ in place of $(T^{\mathcal{B}})_{[t]}$ and $(T^{\mathcal{B}})_t$ respectively. For $t \in (2\times\omega)^{<\omega}$ let t_* be the *first component of* t, i.e., for every $n < \mathrm{lh}(t)$, if $t(n) = (t^n_0, t^n_1)$ then $t_*(n) = t^n_0$, so that $t_* \in 2^{\mathrm{lh}(t)}$.

The following simple but crucial lemma is a slight modification of Lemma 2.2 from [Tö18] (and Proposition 1 from [CM]), and will form the core of our argument.

Lemma 1.1. *Suppose that $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $\mathcal{B} \subseteq \mathcal{P}(\omega)$ with $|\mathcal{B}| \leq \aleph_0$, so that:*

(1) $(\forall z \in \mathcal{A})\,(\exists m \in \omega)\,(\exists B_0,\dots,B_{m-1} \in \mathcal{B})\, z \subseteq^* \cup_{n<m} B_n$*;*
(2) $(\forall m \in \omega)\,(\forall B_0,\dots,B_{m-1} \in \mathcal{B})\, |\omega \setminus \cup_{n<m} B_n| = \infty$*.*

Then there is some $x \in [\omega]^\omega$ so that $x \cap z \in \mathrm{Fin}$ for all $z \in \mathcal{A}$.

For reader's convenience, we briefly repeat the proof.

Proof. In case $\mathcal{B}$ is finite, let $x := \omega \setminus \cup\mathcal{B}$. Then $x \in [\omega]^\omega$ by (2), and (1) clearly implies that for every $z \in \mathcal{A}$ the intersection $x \cap z$ is finite.

In case $\mathcal{B}$ is infinite, enumerate it as $(B_n)_{n\in\omega}$. Then inductively pick a sequence $(n_k)_{k\in\omega} \in \omega^\omega$ so that for every $k \in \omega$ it holds that

- $n_k < n_{k+1}$ and
- $n_k \in \omega \setminus (\cup_{i\leq k} B_i)$,

which is possible by assumption (2). Then set

$$x := \{n_k \mid k \in \omega\}$$

and use assumption (1) to conclude that $(\forall z \in \mathcal{A})\, x \cap z \in \mathrm{Fin}$. □

We next state and prove the claim from Lemma 2.4 in [Tö18]. Recall that for $s, t \in (2\times\omega)^{<\omega}$ we say that they are *incomparable in the first component* if $s_* \not\sqsubseteq t_*$ and $t_* \not\sqsubseteq s_*$. Note that every $a \in 2^{<\omega}$ corresponds to a unique element of Fin, namely to

$$\{n \in \omega \mid n \in \mathrm{dom}(a) \wedge a(n) = 1\}.$$

In this way we can use $\cup$, $\cap$, etc. on elements of $2^{<\omega}$.

Lemma 1.2. *Suppose S is a tree on $2\times\omega$ such that $p[S]$ is almost disjoint. Suppose also that $s, t \in S$ are incomparable in the first component. Then there are $s' \in S_s$ and $t' \in S_t$ so that for all $s'' \in S_{s'}$ and all $t'' \in S_{t'}$ it holds that $s''_* \cap t''_* = s'_* \cap t'_*$.*

Proof. Suppose for contradiction that $s, t \in S$ are incomparable in the first component, but for every $s' \in S_s$ and $t' \in S_t$ there are $s'' \in S_{s'}$ and $t'' \in S_{t'}$ for which $s''_* \cap t''_* \supsetneq s'_* \cap t'_*$. Then we can inductively define sequences $(s^n)_{n\in\omega}$ and $(t^n)_{n\in\omega}$ so that:

- $s^0 = s$ and $t^0 = t$,
- for every $n \in \omega$ it holds that $s^n \sqsubseteq s^{n+1}$ and $t^n \sqsubseteq t^{n+1}$, and
- for every $n \in \omega$ we have that $s^n_* \cap t^n_* \subsetneq s^{n+1}_* \cap t^{n+1}_*$.

Letting x be the infinite branch through S which extends all s^n and y be the infinite branch through S which extends all t^n, we have that $p(x) \neq p(y)$ and $p(x) \cap p(y)$ is infinite. This is in contradiction with the assumption that $p[S]$ is almost disjoint. □

We introduce the following notation which will be crucial in the next section. For a tree S on $2 \times \omega$ and $s \in S$, set

$$x_s^S := \cup\{t_* \mid t \in S_{[s]}\} \in 2^\omega.$$

Then the property of s', t' in Lemma 1.2 can be written as

$$x_{s'}^S \cap x_{t'}^S = s'_* \cap t'_*.$$

Theorem 0.1 (Mathias)**.** *There are no infinite analytic mad families.*

Proof. Let T be a tree on $2 \times \omega$ so that $\mathcal{A} := p[T]$ is an infinite almost disjoint family. We will inductively define a countable family $\mathcal{B}$ of infinite subsets of ω, which will satisfy conditions of Lemma 1.1. Set $\mathcal{B}_0 := \emptyset$. Suppose we have defined $\mathcal{B}_\alpha$ for $\alpha \leq \gamma$ so that

(i) if $\alpha \leq \beta \leq \gamma$ then $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ and both are countable;
(ii) for all $\alpha \leq \gamma$ it holds that $\omega \notin \mathcal{I}_{\mathcal{A} \cup \mathcal{B}_\alpha}$.

We will now define $\mathcal{B}_{\gamma+1}$. If there are $s, t \in T^{\mathcal{B}_\gamma}$ which are incomparable in the first component and so that $x_s^{T^{\mathcal{B}_\gamma}} \cap x_t^{T^{\mathcal{B}_\gamma}} = s_* \cap t_*$, then consider the following cases:

- if $x_s^{T^{\mathcal{B}_\gamma}} \notin \mathcal{I}_{\mathcal{A} \cup \mathcal{B}_\gamma}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_t^{T^{\mathcal{B}_\gamma}}\}$;
- else if $x_t^{T^{\mathcal{B}_\gamma}} \notin \mathcal{I}_{\mathcal{A} \cup \mathcal{B}_\gamma}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_s^{T^{\mathcal{B}_\gamma}}\}$;
- else put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_s^{T^{\mathcal{B}_\gamma}}, x_t^{T^{\mathcal{B}_\gamma}}\}$.

It is clear that in all three cases $\mathcal{B}_{\gamma+1}$ still satisfies that $\omega \notin \mathcal{I}_{\mathcal{A} \cup \mathcal{B}_{\gamma+1}}$, since the intersection of the two potential new sets ($x_s^{T^{\mathcal{B}_\gamma}}$ and $x_t^{T^{\mathcal{B}_\gamma}}$) is finite and since the condition held for $\mathcal{B}_\gamma$. In case there are no such $s, t \in T^{\mathcal{B}_\gamma}$ we stop the process and set $\alpha^* := \gamma$ and $\mathcal{B}^* := \mathcal{B}^\gamma$.

Suppose we have defined $\mathcal{B}_\alpha$ for all $\alpha < \lambda$, where λ is countable limit, so that the above conditions (i) and (ii) hold. Then let

$$\mathcal{B}_\lambda := \cup_{\alpha < \lambda} \mathcal{B}_\alpha$$

and observe that conditions (i) and (ii) are preserved.

The process terminates with a countable α^*, since at each step we use some pair s, t which has not yet been used up to that point and since there are only countably many pairs (the tree T is countable), so the process cannot last for uncountably many steps.

Claim 1.3. *Any two $s, t \in T^{\mathcal{B}^*}$ are comparable in the first component.*

Proof. Suppose for contradiction that the process has stopped, but there are some $s, t \in T^{\mathcal{B}^*}$ which are incomparable in the first component. Applying Lemma 1.2 with $T^{\mathcal{B}^*}$ in place of S, we get some $s' \in T_s^{\mathcal{B}^*}$ and $t' \in T_t^{\mathcal{B}^*}$ which satisfy that

$$x_{s'}^{T^{\mathcal{B}^*}} \cap x_{t'}^{T^{\mathcal{B}^*}} = s'_* \cap t'_*.$$

This means that we can continue the process, which is a contradiction. ⊣

Consider two further cases:

- If $x_\emptyset^{T^{\mathcal{B}^*}} \notin \mathcal{I}_\mathcal{A}$, then it must hold that $p[T^{\mathcal{B}^*}] = \emptyset$. In this case let $\mathcal{B} := \mathcal{B}^*$.
- If $x_\emptyset^{T^{\mathcal{B}^*}} \in \mathcal{I}_\mathcal{A}$, then let $\mathcal{B} := \mathcal{B}^* \cup \{x_\emptyset^{T^{\mathcal{B}^*}}\}$.

Note that $\mathcal{B}$ satisfies that

$$(\forall m \in \omega)\,(\forall B_0, \dots, B_{m-1} \in \mathcal{B})\,\omega \setminus \cup_{n<m} B_n \notin \mathcal{I}_\mathcal{A}.$$

In particular, $\mathcal{B}$ satisfies condition (2) of Lemma 1.1.

Claim 1.4. *For all $z \in \mathcal{A}$ there are $n \in \omega$ and $B_0, \dots, B_{n-1} \in \mathcal{B}$ so that $z \subseteq^* \cup_{k<n} B_k$.*

Proof. In case $z \in p[T^{\mathcal{B}^*}]$, it holds that $z \in \mathcal{B}$. So suppose that $z \notin p[T^{\mathcal{B}^*}]$. Then there is some $\alpha < \alpha^*$ so that $z \in p[T^{\mathcal{B}_\alpha}] \setminus p[T^{\mathcal{B}_{\alpha+1}}]$. But this means that there are $n \in \omega$ and $B_0, \dots, B_{n-1} \in \mathcal{B}_{\alpha+1}$ so that $z \subseteq^* \cup_{k<n} B_k$. ⊣

Since $\mathcal{B}$ is countable and conditions (1) and (2) of Lemma 1.1 are fulfilled, the proof is completed by an application of Lemma 1.1. □

An observant reader might have noticed similarities between the above proof and the proof of Theorem 3 given in [CM]. The main difference is the stopping criterion; in [CM] the process stops when the analogue of $p[T^{\mathcal{B}_\gamma}]$, called $\mathscr{A}^\gamma$, is countable. Moreover, the property which is maintained for the analogue of $\mathcal{B}_\gamma$, called $\mathscr{C}^\gamma$ in [CM], is

$$(\forall n \in \omega)\,(\forall B_0, \ldots, B_{n-1} \in \mathcal{B}_\gamma)\,(\exists A \in \mathcal{A})\, A \not\subseteq^* \cup_{k<n} B_k.$$

This condition, although not pointed out explicitly, holds in our construction as well. Hence the requirement we impose on $\mathcal{B}_\gamma$ is stronger than in [CM]. Furthermore, our process runs until the branches through the remaining tree project to at most one element of 2^ω. Because of this one may say that our process is more informative. In the next section, we will use this additional information and combine it with some tricks from effective descriptive set theory in order to be able to carry out the procedure in a small segment of set theory, resulting in an effective witness to non-maximality.

2. An effective strengthening of Mathias' theorem

For X being one of ω^ω, 2^ω or $[\omega]^\omega$, we let

$$\Delta^1_1(X) := \{x \in X \mid x \text{ is a } \Delta^1_1 \text{ real}\}.$$

In this section we prove the following effective strengthening of Mathias' theorem.

Theorem 0.2. *For every infinite Σ^1_1 almost disjoint family $\mathcal{A}$ there is a Δ^1_1 witness to non-maximality, i.e., there is some $x \in \Delta^1_1([\omega]^\omega)$ so that $x \cap z$ is finite for all $z \in \mathcal{A}$.*

To prove this theorem we will work in $L_{\omega_1^{\mathrm{CK}}}$, which is a model of Kripke–Platek set theory. The reader acquainted with the fundamentals of effective descriptive set theory should not feel guilty for skipping the following subsection, in which we briefly recall some basic facts.

2.1. **Effective descriptive set theory.** Recall that a formula is Δ_0 if all of its quantifiers are bounded, viz., all universal quantifiers are of the form $(\forall v \in w)$ and all existential quantifiers are of the form $(\exists v \in w)$. The class of all formulas, built from atomic formulas and their negations using $\wedge$, $\vee$, $(\exists v \in w)$, $(\forall v \in w)$ and $(\exists v)$ is denoted by Σ.

For the purposes of this note, following the phrasing of [MWS85], *Kripke–Platek set theory* (abbreviated to KP) is a fragment of ZF, consisting of the following axioms and axioms schemes:

(I) Extensionality: $(\forall x, y)\,[x = y \longleftrightarrow (\forall z)\,(z \in x \longleftrightarrow z \in y)]$.
(II) Regularity/Foundation: if $\phi(x)$ is a (not necessarily Δ_0) formula, in which all occurrences of y are bounded, then the following is an axiom
$((\exists x)\,\phi(x)) \to (\exists x)\,(\phi(x) \wedge (\forall y \in x)\,\neg\phi(y))$.
(III) Pairing: $(\forall x, y)\,(\exists z)\,(x \in z \wedge y \in z)$.
(IV) Union: $(\forall x)\,(\exists u)\,(\forall y \in x)\,(\forall z \in y)\, z \in u$.
(V) Δ_0-Separation: for every Δ_0 formula $\psi(z)$ the following is an axiom
$(\forall x)\,(\exists y)\,(\forall z)\,(z \in y \longleftrightarrow z \in x \wedge \psi(z))$.
(VI) Δ_0-Collection: for every Δ_0 formula $\chi(y, z)$ the following is an axiom
$(\forall x)\,[((\forall y \in x)\,(\exists z)\,\chi(y, z)) \to (\exists w)\,(\forall y \in x)\,(\exists z \in w)\,\chi(y, z)]$.
(VII) Infinity: $(\exists x)\,[((\exists e)\,(e \in x \wedge (\forall y)\, y \notin e)) \wedge (\forall y)\,(y \in x \to y \cup \{y\} \in x)]$.

An *ω-model* of KP is a model $\mathfrak{A}$ of KP, for which $\omega^{\mathfrak{A}} = \omega^V$, i.e., the first infinite ordinal in $\mathfrak{A}$ is equal to the standard ω.

Remarkably, one can deduce from the axioms of KP the following results, which make KP stronger than it appears at first sight. Note that the results cited from [Bar17] are established in KPU, the *Kripke–Platek set theory with urelements*, but the proofs can be easily adapted to provide the analogues for KP.

Theorem 2.1 (Σ-Collection)**.** *For any Σ formula $\chi(y,z)$, the following is a theorem of* KP*:*

$$(\forall x)\,[((\forall y \in x)\,(\exists z)\,\chi(y,z)) \to (\exists w)\,(\forall y \in x)\,(\exists z \in w)\,\chi(y,z)].$$

See [Bar17, Theorem 4.4] for a proof.

Theorem 2.2 (Δ-Separation)**.** *For any Σ formulas $\phi(z)$ and $\psi(z)$, the following is a theorem of* KP*:*

$$(\forall x)\,[((\forall z \in x)\,(\phi(z) \longleftrightarrow \neg\psi(z))) \to (\exists y)\,(\forall z)\,(z \in y \longleftrightarrow z \in x \wedge \phi(z))].$$

The proof can be found in [Bar17] below Theorem 4.5. In the following theorem (see Theorem 4.6 in [Bar17]), $\mathrm{func}(f)$ stands for "f is a function".

Theorem 2.3 (Σ-Replacement)**.** *For every Σ formula $\chi(y,z)$, the following is a theorem of* KP*:*

$$(\forall x)\,\Big[((\forall y \in x)\,(\exists! z)\,\chi(y,z)) \to (\exists f)\,\Big(\mathrm{func}(f) \wedge \mathrm{dom}(f) = x \wedge (\forall y \in x)\,\chi(y,f(y))\Big)\Big].$$

Suppose that $\phi(v_0,\dots,v_{n-1},u)$ is a Σ formula and that

$$\mathsf{KP} \vdash (\forall x_0,\dots,x_{n-1})\,(\exists! y)\,\phi(x_0,\dots,x_{n-1},y).$$

Then, using Theorem 2.3, we can define an n-ary function symbol F by specifying that

$$(\forall x_0,\dots,x_{n-1})\,(\forall y)\,F(x_0,\dots,x_{n-1}) = y \longleftrightarrow \phi(x_0,\dots,x_{n-1},y).$$

Since ϕ is a Σ formula, we say that F is a Σ function symbol.

Theorem 2.4 (Σ Recursion)**.** *Suppose we are given an $(n+2)$-ary Σ function symbol G. Then we can introduce a new Σ function symbol F for which the following is a theorem of* KP*:*

$$(\forall x_0,\dots,x_{n-1})\,(\forall y)$$
$$F(x_0,\dots,x_{n-1},y) = G\Big(x_0,\dots,x_{n-1},y,\{\langle z, F(x_0,\dots,x_{n-1},z)\rangle \mid z \in \mathrm{TC}(y)\}\Big).$$

See [Bar17, Theorem 6.4] for the proof of this theorem and [Bar17, Theorem 6.1] for more on TC (transitive closure) in the context of KP. With Theorem 2.4, we are able to carry out definable recursive processes inside models of KP.

Theorem 2.5. *Suppose that $\mathfrak{A}$ is an ω-model of* KP *and that $T \in \mathfrak{A}$ is a well-founded (in V) tree. Then the height of T is an ordinal in $\mathfrak{A}$.*

This is Theorem 5.5 in [MWS85]. It asserts that any ω-model $\mathfrak{A}$ of KP is strong enough to see that a tree it contains is well-founded, given that tree is well-founded from the perspective of V. Moreover, $\mathfrak{A}$ correctly calculates the height of that tree. Of course, the statement also holds for ranks of well-founded strict partial orders on countable sets.

Recall that ω_1^{CK} denotes the *Church–Kleene ordinal*, the smallest non-recursive ordinal (an ordinal α is *recursive*, if there is some recursive well-founded tree with height α).

Theorem 2.6. *$L_{\omega_1^{\mathrm{CK}}}$ is an ω-model of* KP*.*

The proof can be found in Chapter 5 of [MWS85]. The theorem asserts that we can use the axioms and theorems of KP when we are reasoning inside the model $L_{\omega_1^{\mathrm{CK}}}$. The following is a restatement of Corollary 5.19 from [MWS85] and justifies our wish to work in $L_{\omega_1^{\mathrm{CK}}}$, as the reals one obtains in $L_{\omega_1^{\mathrm{CK}}}$ are particularly nice.

Proposition 2.7. $\omega^\omega \cap L_{\omega_1^{\mathrm{CK}}} = \Delta^1_1(\omega^\omega)$.

Of course it also holds that $2^\omega \cap L_{\omega_1^{\mathrm{CK}}} = \Delta^1_1(2^\omega)$ and $[\omega]^\omega \cap L_{\omega_1^{\mathrm{CK}}} = \Delta^1_1([\omega]^\omega)$. Below we state a well-known result (see the proof of Theorem 6.3 in [MWS85] for a proof), which is very useful in effective descriptive set theory.

Theorem 2.8 (Effective perfect set theorem)**.** *Every Σ^1_1 subset of ω^ω not contained in $\Delta^1_1(\omega^\omega)$ has a perfect subset.*

We conclude the subsection on fundamentals of effective descriptive set theory with another convenient statement, the proof of which can be found below the statement of Corollary 4.19 in [MWS85].

Theorem 2.9 (Spector–Gandy theorem)**.** *When calculating complexity of a given formula, the existential quantifier $(\exists x \in \Delta^1_1([\omega]^\omega))$ can be considered to be $(\forall x \in \omega^\omega)$.*

Note that the statement is significant, as $\Delta^1_1([\omega]^\omega)$ is a Π^1_1 subset of $[\omega]^\omega$.

2.2. **Proof of Theorem 0.2.** We next adapt some definitions so that we avoid use of infinite branches, which will enable us to describe the procedure in $L_{\omega_1^{\mathrm{CK}}}$. Let T be a tree on $2 \times \omega$ so that $p[T]$ is almost disjoint, let $\mathcal{B}$ be a countable family of infinite subsets of ω and let $s \in T$. Recall, that we defined

$$x^T_s := \cup\{t_* \mid t \in T_{[s]}\}$$

and define the subtrees

$$T^{\mathcal{B},e} := \{s \in T \mid x^T_s \notin \mathcal{I}_{\mathcal{B}}\}.$$

Note that for $T \in L_{\omega_1^{\mathrm{CK}}}$ and $s \in T$ it holds that $x^T_s \in L_{\omega_1^{\mathrm{CK}}} \cap 2^\omega$. Hence we added e in the superscript of $T^{\mathcal{B}}$, previously defined in Section 1, to point out that we are not using infinite branches through T and have thus made things more effective. As before, if $s \in T^{\mathcal{B},e}$ we write $T^{\mathcal{B},e}_{[s]}$ in place of $(T^{\mathcal{B},e})_{[s]}$ and $T^{\mathcal{B},e}_s$ in place of $(T^{\mathcal{B},e})_s$. Before finally commencing with the proof, note that Lemma 1.2 is a theorem of KP, and that the diagonalisation argument of the proof of Lemma 1.1 can be performed in $L_{\omega_1^{\mathrm{CK}}}$ (given that $\mathcal{B} \in L_{\omega_1^{\mathrm{CK}}}$) by reasoning in KP.

Proof of Theorem 0.2. Suppose for contradiction that there is an infinite Σ^1_1 almost disjoint family $\mathcal{A}$ so that for every $x \in \Delta^1_1([\omega]^\omega)$ there is some $z \in \mathcal{A}$ with $x \cap z$ infinite. Then it holds that for any $x \in \Delta^1_1([\omega]^\omega)$, the assertion "$x \in \mathcal{A}$" is Δ^1_1, as

$$x \notin \mathcal{A} \iff (\exists z \in \mathcal{A})\, |x \cap z| = \infty \wedge x \neq z.$$

Furthermore, we can show that for $x \in \Delta^1_1([\omega]^\omega)$, the assertion "$x \in \mathcal{I}_{\mathcal{A}}$" is Δ^1_1, as the following claim establishes.

Claim 2.10. *There is a Π^1_1 predicate φ such that if $x \in \Delta^1_1([\omega]^\omega)$ then*

$$x \in \mathcal{I}_{\mathcal{A}} \iff \varphi(x).$$

In particular, for $x \in \Delta^1_1([\omega]^\omega)$, it is Δ^1_1 to say "$x \in \mathcal{I}_{\mathcal{A}}$".

Proof. Let $x \in \Delta^1_1([\omega]^\omega)$ and assume that $x \in \mathcal{I}_{\mathcal{A}}$. Then

$$\{z \in \mathcal{A} \mid x \cap z \text{ is infinite}\}$$

is a finite Σ^1_1 set, so by the Effective perfect set theorem 2.8 it consists of finitely many Δ^1_1 reals. It follows that for $x \in \Delta^1_1([\omega]^\omega)$,

$$x \in \mathcal{I}_{\mathcal{A}} \iff \Big(\exists \vec{z} \in (\Delta^1_1([\omega]^\omega))^{<\omega}\Big)\Big(\forall i < \mathrm{lh}(\vec{z})\Big)\, \vec{z}_i \in \mathcal{A} \wedge x \subseteq^* \bigcup_{i<\mathrm{lh}(\vec{z})} \vec{z}_i,$$

and the right hand side is a Π^1_1 predicate by the Spector–Gandy theorem 2.9 and the observation that "$\vec{z}_i \in \mathcal{A}$" is Δ^1_1 since $\vec{z}_i \in \Delta^1_1([\omega]^\omega)$. $\dashv$

For a countable set $\mathcal{B} \in L_{\omega_1^{\mathrm{CK}}}$ with $\mathcal{B} \subseteq \Delta^1_1(2^\omega)$ and any $x \in \Delta^1_1([\omega]^\omega)$ it holds that the assertion "$x \in \mathcal{I}_{\mathcal{A}\cup\mathcal{B}}$" is Δ^1_1, as

$$x \in \mathcal{I}_{\mathcal{A}\cup\mathcal{B}} \iff (\exists n \in \omega)\,(\exists B_0, \dots, B_{n-1} \in \mathcal{B})\, x \setminus \cup_{k<n} B_k \in \mathcal{I}_{\mathcal{A}}$$

and since $x \setminus \cup_{k<n} B_k \in \Delta^1_1(2^\omega)$. This means that for $\mathcal{B}$ as above we can make inquiries whether $x \in \Delta^1_1([\omega]^\omega)$ is in $\mathcal{I}_{\mathcal{A}\cup\mathcal{B}}$ as part of a recursive process in $L_{\omega_1^{\mathrm{CK}}}$ (see Theorem 2.4 above). With this we are fairly close to using the argument from the proof of Theorem 0.1.

Since in $L_{\omega_1^{\mathrm{CK}}}$ we have to work without the luxury of inquiring whether there are infinite branches which avoid $\mathcal{B}_\alpha$, we will have to pass to subtrees of T. Hence we will recursively (in $L_{\omega_1^{\mathrm{CK}}}$) define not only countable sets $\mathcal{B}_\alpha$, but also subtrees T_α. In order to make it clear that the process stops from the perspective of $L_{\omega_1^{\mathrm{CK}}}$, we will at certain steps α in the process (the first α at which this happens is $\alpha = 0$) compile a list of pairs $s, t \in T_\alpha$ which are:

- incomparable in the first component, and
- satisfy that $x_s^{T_\alpha} \cap x_t^{T_\alpha} = s_* \cap t_*$.

We will then at each successive step γ use the pair which has not been already used, still lies in the subtree T_γ and appears the first among such in the list. After some steps (possibly finitely many, and maximally ω-many), we will run out of such pairs. Then we compile a new list (note that there might be some new pair, since the tree got smaller). When we run out of pairs and are unable to compile a new list because there are no more pairs satisfying the condition, we stop the process. We keep track of the ordinals at which we compile a new list of pairs and call the ordinal at which we compiled a new list for the α-th time β_α. These ordinals will be important later, when we will prove that the process terminates (from the perspective of $L_{\omega_1^{\mathrm{CK}}}$) and that moreover we can estimate in advance when the process will stop.

We now describe the recursive process. Let $T_0 := T$, $\mathcal{B}_0 := \emptyset$, $\beta_0 = 0$ and compile the first list of pairs as described above. Suppose we have defined trees T_α and countable collections (of subsets of ω) $\mathcal{B}_\alpha$ so that the process has not yet terminated and:

(i) if $\alpha \le \alpha' \le \gamma$ then $T_{\alpha'} \subseteq T_\alpha \subseteq T$ and $\mathcal{B}_\alpha \subseteq \mathcal{B}_{\alpha'}$ are countable;
(ii) for all $\alpha \le \gamma$ it holds that $\omega \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_\alpha}$;
(iii) for all $\alpha < \gamma$ we have $T_{\alpha+1} = (T_\alpha)^{\mathcal{B}_\alpha, e}$;
(iv) for limit $\lambda \le \gamma$ we have $T_\lambda = \cap_{\alpha<\lambda} T_\alpha$ and $\mathcal{B}_\lambda = \cup_{\alpha<\lambda}\mathcal{B}_\alpha$.

Set $T_{\gamma+1} := (T_\gamma)^{\mathcal{B}_\gamma}$ and let $s, t \in T_{\gamma+1}$ be the pair as described above (or we compile a new list if we have run out of pairs in the previously compiled list; or else if there are no more pairs the process terminates at this stage). Now consider the following cases:

- if $x_s^{T_{\gamma+1}} \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_\gamma}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_t^{T_{\gamma+1}}\}$;
- else if $x_t^{T_{\gamma+1}} \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_\gamma}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_s^{T_{\gamma+1}}\}$;
- else put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_s^{T_{\gamma+1}}, x_t^{T_{\gamma+1}}\}$.

Same as in the non-effective case, we see that in all three cases it holds that $\omega \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_{\gamma+1}}$.

Suppose we have defined $\mathcal{B}_\alpha$ for all $\alpha < \lambda$, where λ is countable limit, so that the above conditions (i) to (iv) hold. Then let $T_\lambda := \cap_{\alpha<\lambda} T_\alpha$ and $\mathcal{B}_\lambda := \cup_{\alpha<\lambda}\mathcal{B}_\alpha$. Clearly the conditions are preserved.

We now prove that the process stops. To this end, define the relation $\prec$ on $T \times T$ by

$$(s,t) \prec (s',t') \;:\!\iff\; s' \sqsubseteq s \wedge t' \sqsubseteq t \wedge s'_* \cap t'_* \subsetneq s_* \cap t_*.$$

Since $p[T]$ is almost disjoint in V, it follows that $\prec$ is well founded in V. By Theorem 2.5 (and the comments below it), $L_{\omega_1^{\mathrm{CK}}}$ also believes that $\prec$ is well-founded and correctly calculates its rank. Let

$$\Gamma := \mathrm{rk}(\prec) < \omega_1^{\mathrm{CK}}.$$

To prove that the process stops we first prove the following claim.

Claim 2.11. *Suppose that (s,t) was listed at step β_γ. Then*

$$\mathrm{rk}\left(\prec\restriction\{(s',t') \mid (s',t') \prec (s,t)\}\right) \geq \gamma.$$

Proof. By induction on γ. The statement is clear for $\gamma = 0$. Suppose that the claim holds for all $\alpha < \gamma$ and that $\gamma > 0$. Since (s,t) was listed at step β_γ, it could not have been listed at steps β_α for any $\alpha < \gamma$. Fix $\alpha < \gamma$. Then there must be some $s' \in (T_{\beta_\alpha})_s$ and $t' \in (T_{\beta_\alpha})_t$ so that $s_* \cap t_* \subsetneq s'_* \cap t'_*$. Applying Lemma 1.2 on T_{β_α}, s' and t', we get some $s'' \in (T_{\beta_\alpha})_{s'}$ and some $t'' \in (T_{\beta_\alpha})_{t'}$ so that (s'',t'') is listed at step β_α and so that $(s'',t'') \prec (s,t)$. Let $(s(\alpha),t(\alpha))$ be the least such (s'',t'') according to some in advanced fixed well-order of $T \times T$.

By the inductive hypothesis it holds for every $\alpha < \gamma$ that

$$\mathrm{rk}\left(\prec\restriction\{(s',t') \mid (s',t') \prec (s(\alpha),t(\alpha))\}\right) \geq \alpha.$$

Thus clearly

$$\mathrm{rk}\left(\prec\restriction\{(s',t') \mid (s',t') \prec (s,t)\}\right) \geq \mathrm{rk}\left(\prec\restriction\{(s',t') \mid (s',t') \prec (s(\alpha),t(\alpha))\}\right) + 1 \geq \alpha + 1.$$

Since this holds for every $\alpha < \gamma$, the proof of the claim is complete. ⊣

Now we are ready to show that the process terminates in $L_{\omega_1^{\mathrm{CK}}}$. Suppose for contradiction, that it does not. Then $\beta_{\Gamma+1}$ is defined and is less than ω_1^{CK}. Let (s,t) be some pair listed at step $\beta_{\Gamma+1}$. Then by Claim 2.11

$$\mathrm{rk}\left(\prec\restriction\{(s',t') \mid (s',t') \prec (s,t)\}\right) \geq \Gamma + 1,$$

which is in contradiction with $\mathrm{rk}(\prec) = \Gamma$. Moreover, this tells us that the process stops before $\omega \cdot (\Gamma + 1) < \omega_1^{\mathrm{CK}}$. So we can estimate in advance how long it will take for the process to terminate! If the process terminates at step γ, let $\alpha^* := \gamma$, $T^* := T_\gamma$ and $\mathcal{B}^* := \mathcal{B}_\gamma$. Note that γ is by design a successor ordinal.

Claim 2.12. *Every two $s,t \in T^*$ are comparable in the first component.*

Proof. Suppose for contradiction that there are $s,t \in T^*$ with s_*, t_* not comparable. Then apply Lemma 1.2 on T^*, s and t to conclude that the process should have continued, which is a contradiction. ⊣

Finally, as in the non-effective case, consider the two possibilities:

- If $x_\emptyset^{T^*} \notin \mathcal{I}_\mathcal{A}$, then it must be the case that $p[T^*] = \emptyset$ holds in V, so let $\mathcal{B} := \mathcal{B}^*$.
- If $x_\emptyset^{T^*} \in \mathcal{I}_\mathcal{A}$, then let $\mathcal{B} := \mathcal{B}^* \cup \{x_\emptyset^{T^*}\}$.

Note that $\mathcal{B}$ still satisfies that $\omega \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}}$. As in the classical case, this implies that $\mathcal{B}$ satisfies condition (2) of Lemma 1.1. The following claim is true in V.

Claim 2.13. *For all $z \in \mathcal{A}$ there are $n \in \omega$ and $B_0, \ldots, B_{n-1} \in \mathcal{B}$ so that $z \subseteq^* \cup_{k<n} B_k$.*

Proof. In case $z \in p[T^*]$, we have that $z \in \mathcal{B}$, so suppose that $z \notin p[T^*]$. Then there is some $\alpha < \alpha^*$ so that $z \in p[T_\alpha] \setminus p[T_{\alpha+1}]$. But this means that there are $n \in \omega$ and $B_0, \ldots, B_{n-1} \in \mathcal{B}_\alpha$ for which $z \subseteq^* \cup_{k<n} B_k$. ⊣

Since $\mathcal{B}$ is countable in $L_{\omega_1^{\mathrm{CK}}}$, we can apply the proof of Lemma 1.1 in $L_{\omega_1^{\mathrm{CK}}}$ to obtain $x \in [\omega]^\omega \cap L_{\omega_1^{\mathrm{CK}}}$ so that (since Claim 2.13 is true in V) it holds in V that for every $z \in \mathcal{A}$ the intersection $x \cap z$ is finite. By Theorem 2.7 we have that $x \in \Delta^1_1([\omega]^\omega)$. This contradicts the assumption that there is no such witness and with this the proof is complete. □

3. $\aleph_1^{L[a]} < \aleph_1$ implies that there are no infinite $\Sigma^1_2[a]$ mad families

In this section we give a new proof of the following theorem.

Theorem 0.3. *Let $a \in \omega^\omega$ and suppose that $\aleph_1^{L[a]} < \aleph_1$. Then there are no infinite $\Sigma^1_2[a]$ mad families.*

This was first proved by the second author in [Tö18] (see Theorem 3.6) with use of forcing. Our goal in this section is to prove it using the derivative method from the proof of Theorem 0.1. Note that we can go back to using $T^{\mathcal{B}}$ (which makes reference to infinite branches), thanks to Σ^1_2 absoluteness. Additionally, note that for $\mathcal{A} \subseteq [\omega]^\omega$ which is $\Sigma^1_2[a]$, memberships of $\mathcal{A}$, $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{A}\cup\mathcal{B}}$ (for $\mathcal{B} \in L[a]$) are absolute for $L[a]$ by Σ^1_2 absoluteness.

Proof. We suppress the parameter a and prove that if $\aleph_1^L < \aleph_1$ then there are no infinite Σ^1_2 mad families. The proof can then be easily adapted to include the parameter.

Suppose that $\mathcal{A}$ is an infinite Σ^1_2 almost disjoint family. Then there is a tree T on $2 \times \omega_1$ which lies in L so that $p[T] = \mathcal{A}$ holds in V (see e.g. Theorem 13.14 in [Kan09]).

We will work in L and inductively define a family $\mathcal{B} \subseteq [\omega]^\omega \cap L$, which will satisfy conditions of Lemma 1.1. Set $\mathcal{B}_0 := \emptyset$. Suppose we have defined $\mathcal{B}_\alpha$ for $\alpha \leq \gamma$ so that

(i) if $\alpha \leq \beta \leq \gamma$ then $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ and $|\mathcal{B}_\alpha|^L \leq |\mathcal{B}_\beta|^L \leq \aleph_1^L$;
(ii) for all $\alpha \leq \gamma$ it holds that $\omega \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_\alpha}$.

We now define $\mathcal{B}_{\gamma+1}$. If there are $s, t \in T^{\mathcal{B}_\gamma}$ so that s, t are incomparable in the first component and so that $x_s^{T^{\mathcal{B}_\gamma}} \cap x_t^{T^{\mathcal{B}_\gamma}} = s_* \cap t_*$, then consider the following cases:

- if $x_s^{T^{\mathcal{B}_\gamma}} \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_\gamma}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_t^{T^{\mathcal{B}_\gamma}}\}$;
- else if $x_t^{T^{\mathcal{B}_\gamma}} \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_\gamma}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_s^{T^{\mathcal{B}_\gamma}}\}$;
- else put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{x_s^{T^{\mathcal{B}_\gamma}}, x_t^{T^{\mathcal{B}_\gamma}}\}$.

It is clear that in all three cases $\mathcal{B}_{\gamma+1}$ still satisfies that $\omega \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}_{\gamma+1}}$, since the intersection of the two potential new sets ($x_s^{T^{\mathcal{B}_\gamma}}$ and $x_t^{T^{\mathcal{B}_\gamma}}$) is finite and since the condition held at step γ. If there are no such s, t stop the process and set $\alpha^* := \gamma$ and $\mathcal{B}^* := \mathcal{B}^\gamma$.

Suppose we have defined $\mathcal{B}_\alpha$ for all $\alpha < \lambda$, where $|\lambda|^L \leq \aleph_1^L$ and λ is limit, so that the above conditions (i) and (ii) hold. Then by letting

$$\mathcal{B}_\lambda := \cup_{\alpha<\lambda}\mathcal{B}_\alpha$$

the conditions are clearly preserved.

To see that the process stops from the perspective of L, observe that at each step γ we add at least one new member of $[\omega]^\omega \cap L$ to $\mathcal{B}_\gamma$. Thus the process terminates from the perspective of L. Furthermore, it holds that $|\alpha^*|^L \leq \aleph_1^L$ and $|\mathcal{B}^*|^L \leq \aleph_1^L$. So by assumption, in V the ordinal α^* and the set $\mathcal{B}^*$ are countable.

Claim 3.1. *Any two $s, t \in T^{\mathcal{B}^*}$ are comparable in the first component.*

Proof. Suppose for contradiction that the process stopped, but that there are some $s, t \in T^{\mathcal{B}^*}$ which are incomparable in the first component. Applying Lemma 1.2, we get some $s' \in T_s^{\mathcal{B}^*}$ and $t' \in T_t^{\mathcal{B}^*}$ for which it holds that $x_{s'}^{T^{\mathcal{B}^*}} \cap x_{t'}^{T^{\mathcal{B}^*}} = s'_* \cap t'_*$. This is clearly a contradiction. ⊣

Finally, consider two further cases:

- If $x_\emptyset^{T^{\mathcal{B}^*}} \notin \mathcal{I}_\mathcal{A}$, then $p[T^{\mathcal{B}^*}] = \emptyset$ must hold in V. In this case let $\mathcal{B} := \mathcal{B}^*$.
- If $x_\emptyset^{T^{\mathcal{B}^*}} \in \mathcal{I}_\mathcal{A}$, then let $\mathcal{B} := \mathcal{B}^* \cup \{x_\emptyset^{T^{\mathcal{B}^*}}\}$.

Note that it holds for $\mathcal{B}$ that $\omega \notin \mathcal{I}_{\mathcal{A}\cup\mathcal{B}}$, so that $\mathcal{B}$ satisfies condition (2) of Lemma 1.1. The following claim is true both in L (with $\mathcal{A} \cap L$ in place of $\mathcal{A}$) and V.

Claim 3.2. *For every $z \in \mathcal{A}$ there are $n \in \omega$ and $B_0, \dots, B_{n-1} \in \mathcal{B}$ so that $z \subseteq^* \cup_{k<n} B_k$.*

Proof. In case $z \in p[T^{\mathcal{B}^*}]$, we have that $z \in \mathcal{B}$. So suppose that $z \notin p[T^{\mathcal{B}^*}]$. Then there is some $\alpha < \alpha^*$ so that $z \in p[T^{\mathcal{B}_\alpha}] \setminus p[T^{\mathcal{B}_{\alpha+1}}]$. But this means that there are $n \in \omega$ and $B_0, \dots, B_{n-1} \in \mathcal{B}_{\alpha+1}$ for which $z \subseteq^* \cup_{k<n} B_k$. ⊣

We conclude the proof by applying Lemma 1.1 in V, where $\mathcal{B}$ is countable by assumption. □

Remark 3.3. Note that since in L there are infinite Σ^1_2 mad families (even Π^1_1 ones, see [Mil89] or [Tö09]), we cannot in general expect that $|\alpha^*|^L < \aleph_1^L$. Nevertheless, the argument shows that L falls just a little bit short of proving non-maximality (namely seeing that $\mathcal{B}$ is countable).

4. $\mathsf{MA}(\kappa)$ implies that there are no infinite κ-Suslin mad families

Recall that a set $\mathcal{A} \subseteq 2^\omega$ is *κ-Suslin* if there is a tree T on $2 \times \kappa$ for which $\mathcal{A} = p[T]$. The aim of this section is to provide a new proof of the following statement.

Theorem 0.4. *If $\mathsf{MA}(\kappa)$ holds for some $\kappa < 2^{\aleph_0}$ then there are no infinite κ-Suslin mad families.*

The above theorem was proved by the second author in [Tö18] (see Theorem 2.5) by generalising the method from the proof of Theorem 2.1 (which is our Theorem 0.1) and using Lemma 2.6, restated below in a slightly modified version.

Lemma 4.1. *Suppose that $\mathsf{MA}(\kappa)$ holds for some $\kappa < 2^{\aleph_0}$ and suppose that $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $\mathcal{B} \subseteq [\omega]^\omega$ with $|\mathcal{B}| \le \kappa$, so that:*

(1) for all $z \in \mathcal{A}$ there are $n \in \omega$ and $B_0, \dots, B_{n-1} \in \mathcal{B}$ so that $z \subseteq^* \cup_{k<n} B_k$;
(2) for all $n \in \omega$ and $B_0, \dots, B_{n-1} \in \mathcal{B}$ it holds that $\omega \setminus \cup_{k<n} B_k$ is infinite.

Then there is some $x \in [\omega]^\omega$ such that for all $z \in \mathcal{A}$ the intersection $x \cap z$ is finite.

For completeness and reader's convenience we repeat the short proof from [Tö18].

Proof. If $|\mathcal{B}| \le \aleph_0$, then the proof is completed by an application of Lemma 1.1. So suppose that $|\mathcal{B}| > \aleph_0$. We let $\mathbb{P}$ be the forcing poset, whose set of conditions is

$$2^{<\omega} \times [\mathcal{B}]^{<\omega}$$

(where $[\mathcal{B}]^{<\omega}$ denotes the collection of all finite subsets of $\mathcal{B}$) and where for two conditions $(s, \mathcal{F})$ and $(s', \mathcal{F}')$ we set that

$$(s, \mathcal{F}) \le_\mathbb{P} (s', \mathcal{F}') \;\; :\Longleftrightarrow \;\; s' \sqsubseteq s \wedge \mathcal{F}' \subseteq \mathcal{F} \wedge s \setminus s' \subseteq \omega \setminus \cup \mathcal{F}'.$$

Clearly, $\mathbb{P}$ has the ccc property, so invoking $\mathsf{MA}(\kappa)$ we get a filter G meeting all dense sets

$$D_{\mathcal{F}'} := \Big\{(s, \mathcal{F}) \,\Big|\, \mathcal{F}' \subseteq \mathcal{F}\Big\},$$

where $\mathcal{F}'$ ranges over $[\mathcal{B}]^{<\omega}$, and all dense sets

$$D_n := \Big\{(s, \mathcal{F}) \,\Big|\, n \le |\{m \in \mathrm{dom}(s) \mid s(m) = 1\}|\Big\},$$

where n ranges over ω. Then set

$$x := \cup\{s \in 2^{<\omega} \mid (\exists \mathcal{F})\, (s, \mathcal{F}) \in G\}$$

and notice that $x \in [\omega]^\omega$ and that for every $z \in \mathcal{A}$ the intersection $x \cap z$ is finite. □

We now generalise the proof of Theorem 0.1 with the aim of using Lemma 4.1 instead of Lemma 1.1.

Sketch of proof of Theorem 0.4. Suppose that $\mathcal{A}$ is an infinite κ-Suslin almost disjoint family. Let T be a tree on $2 \times \kappa$ for which $\mathcal{A} = p[T]$. Now we use the argument from the proof of Theorem 0.1 (and perform it in V), with the following changes:

- The sets $\mathcal{B}_\alpha$ are no longer required to be countable but rather satisfying $|\mathcal{B}_\alpha| \leq \kappa$.
- The process terminates with $|\alpha^*| \leq \kappa$, since $|T| \leq \kappa$, and since at each step we remove at least one node from T.
- The set $\mathcal{B}$ will thus satisfy $|\mathcal{B}| \leq \kappa$. To conclude the proof use Lemma 4.1 in place of Lemma 1.1.

We leave the details to the reader. □

Note that since every $\mathbf{\Sigma}^1_2$ set is $\aleph_1$-Suslin (see e.g. [Mos09] or [Kan09]), the following is an immediate corollary of Theorem 0.4 (this is Corollary 2.8 in [Tö18]).

Corollary 4.2. *Under the assumption* $\mathsf{MA}(\aleph_1)$ *there are no infinite* $\mathbf{\Sigma}^1_2$ *mad families.*

5. Open questions and further work

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are two ideals on sets X and Y respectively. Then the *Fubini product of* $\mathcal{I}$ *with* $\mathcal{J}$ is an ideal on $X \times Y$, denoted by $\mathcal{I} \otimes \mathcal{J}$ and defined as

$$\mathcal{I} \otimes \mathcal{J} := \Big\{ A \subseteq X \times Y \,\Big|\, \{x \in X \mid A(x) \notin \mathcal{J}\} \in \mathcal{I} \Big\},$$

where for $A \subseteq X \times Y$ we let

$$A(x) := \{y \in Y \mid (x, y) \in A\}.$$

We inductively define M^α for $0 < \alpha < \omega_1$ as follows. Set $M^1 := \omega$, and given M^α define

$$M^{\alpha+1} := \omega \times M^\alpha.$$

For $\lambda < \omega_1$ limit, fix a cofinal sequence $(\alpha^\lambda_n)_{n\in\omega}$ in λ (to be used again later) and set

$$M^\lambda := \bigcup_{n\in\omega} \{n\} \times M^{\alpha^\lambda_n}.$$

Next, we for every $0 < \alpha < \omega_1$ inductively define the ideal Fin^α on M^α. Set $\mathrm{Fin}^1 := \mathrm{Fin}$, and given Fin^α define

$$\mathrm{Fin}^{\alpha+1} := \mathrm{Fin} \otimes \mathrm{Fin}^\alpha,$$

which is an ideal on $M^{\alpha+1}$. For limit $\lambda < \omega_1$ set

$$\mathrm{Fin}^\lambda := \Big\{ A \in M^\lambda \,\Big|\, \{n \in \omega \mid A(n) \notin \mathrm{Fin}^{\alpha^\lambda_n}\} \in \mathrm{Fin} \Big\}.$$

The ideals Fin^α are called *iterated Fubini products* of the ideal Fin.

Two subsets $A, B \subseteq M^\alpha$ are said to be Fin^α*-almost disjoint* if $A \cap B \in \mathrm{Fin}^\alpha$. In [BHST22], Bakke Haga, Schrittesser and the second author proved that for every $\alpha \in \omega_1 \setminus \{0\}$ there are no infinite analytic maximal Fin^α-almost disjoint families. David Schrittesser and the authors of this note believe that the following has a positive answer for every $\alpha \in \omega_1^{\mathrm{CK}} \setminus \{0\}$ and made some progress towards proving it.

Question 5.1. *Let* $\alpha \in \omega_1^{\mathrm{CK}} \setminus \{0\}$*. Is it the case that for every infinite* Σ^1_1 Fin^α*-almost disjoint family there is a* Δ^1_1 *witness to non-maximality?*

It is unknown whether AD (the *axiom of determinacy*; see [Kan09] or [Kec95] for great overviews) and AD^+ (see [Woo10]) are equivalent (they are known to be equivalent under $V = L(\mathbb{R})$). Furthermore, it is also unknown whether AD implies that all sets of reals are Ramsey. Hence, in spite of the results from [NN18], [BHST22] and [ST19], the following question, originally posed by the second author, remains open.

Question 5.2. *Does* AD *(if needed, together with* DC*) imply that there are no infinite mad families?*

The authors consider Theorem 0.2 a significant indication that the above question has an affirmative answer.

The $\mathbb{G}_0$ dichotomy from the seminal [KST99] by Kechris, Solecki and Todorčević has since been used in providing new proofs of many result in descriptive set theory (see e.g. [Mil12]).

Question 5.3. *Can the $\mathbb{G}_0$ dichotomy be used to provide a simple proof of Theorem 0.1, i.e., that there are no infinite analytic mad families?*

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Part III.

Maximal cofinitary groups

DEFINABILITY OF MAXIMAL COFINITARY GROUPS

SEVERIN MEJAK AND DAVID SCHRITTESSER

Abstract. We present a proof of a result, previously announced by the second author, that there is a closed (even Π^0_1) set generating an F_σ (even Σ^0_2) maximal cofinitary group (short, mcg) which is isomorphic to a free group. In this isomorphism class, this is the lowest possible definitional complexity of an mcg.

Introduction

In [Cam96], Cameron introduced the notion of a cofinitary subgroup of S_∞ as follows. An element $g \in S_\infty \setminus \{\mathrm{id}_\omega\}$ is called *cofinitary* if it has only finitely many fixed points, i.e., there is some $n \in \omega$ so that for every $m > n$ it holds that $g(m) \neq m$. Then a subgroup $G \leq S_\infty$ is *cofinitary* if every $g \in G \setminus \{\mathrm{id}_\omega\}$ is cofinitary. We write "cofinitary group" in place of the longer "cofinitary subgroup of S_∞". Cofinitary groups were studied before [Cam96] under different names; e.g., in [Ade88] they are called *sharp*. See [Cam96] for combinatorial properties of cofinitary groups and [Tru87] and [Ade88] for results on their embeddings.

A cofinitary group is *maximal* (we write *mcg* for short) if it is not strictly contained in any cofinitary group. Mcgs were first considered in [Tru87] and [Ade88]. The first result which made mcgs interesting to set-theorists was established by Adeleke in [Ade88] and asserted that mcgs are always uncountable. Current research on mcgs is divided between answering two questions about mcgs. The first question asks what are the possible cardinalities of mcgs, and in particular, what is the least possible size of an mcg. See e.g. [BSZ00] and [HSZ01] for early results which answer parts of this question. The second question asks what is the least achievable complexity of an mcg. This paper concerns itself with the latter.

The first breakthrough in the study of definability of mcgs was when Gao and Zhang (see [GZ08]) established that if $V = L$ there is a Π^1_1 set which generates an mcg. This was improved by Kastermans in [Kas08] to the existence of a Π^1_1 mcg under the assumption $V = L$. Kastermans also proved the beautiful results that no mcg can be K_σ (see [Kas08]) and that no mcg can have infinitely many orbits (see [Kas09]).

Kastermans' result on definability inspired [FST17], in which Fischer, Törnquist and the second author constructed a Cohen-indestructible Π^1_1 mcg. The next milestone was achieved (to some surprise) soon afterwards, when Horowitz and Shelah established in [HS16] that it is provable in ZF (without the axiom of choice) that there is a Borel mcg. Using ideas of [HS16], the second author proved (again without use of the axiom of choice) in [Sch21] that there is actually an arithmetical mcg, and announced that the construction from [Sch21] can be improved to produce an F_σ (actually even Σ^0_2) mcg.

In this paper we use the idea of the construction from [Sch21] and enhance it with ideas from [Sch17] to get a construction (not employing the axiom of choice) proving the following.

Theorem 0.1. *There is a Π^0_1 subset of ω^ω which freely generates a Σ^0_2 mcg.*

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Since the topological interior of any cofinitary group is clearly empty, Π^0_1 is the best possible complexity of a set generating an mcg.

By a result of Dudley ([Dud61]), there is no Polish topology on a group freely generated by continuum many generators. This clearly implies that Σ^0_2 is the best possible complexity of a freely generated mcg. Additional difficulties on potential constructions of G_δ mcgs are imposed by [Slu12], in which Slutsky improved on Dudley's result and proved that if G is a free product of groups and carries a Polish topology, then G is countable. Since all presently known constructions of definable mcgs produce groups which decompose into free products, this means that current ideas are insufficient to produce a G_δ mcg.

Finally, we introduce *finitely periodic groups* (a relative to cofinitary groups) and prove an analogue to a well known theorem for mcgs.

Structure of the paper. We begin by briefly recalling the notation we will be using. In Section 2 we prove Theorem 0.1. We proceed with Section 3, where we discuss known obstructions to constructing G_δ mcgs. We conclude the paper with Section 4, where we introduce finitely periodic groups and discuss open problems related to mcgs and maximal finitely periodic groups.

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1. Notation

We use ω to denote the set of all natural numbers. We think of natural numbers as ordinals, so for $n \in \omega$, $n = \{0, 1, \ldots, n-1\}$. No other properties of ordinals will be used and no knowledge of ordinals is required. For a set A, we let $|A|$ denote its cardinality.

For a finite sequence s, we denote its length by $\mathrm{lh}(s)$. We also use $s(-1)$ to denote the last entry of s, i.e., $s(-1) := s(\mathrm{lh}(s)-1)$. For a set A, we denote the set of all finite sequences in A by $A^{<\omega}$, the set of all finite or infinite sequences in A by $A^{\leq\omega}$ and the set of all infinite sequences in A by A^ω. We denote the subsets of these sets consisting only of *injective sequences*, viz., sequences where no two entries are the same, by $(A)^{<\omega}$, $(A)^{\leq\omega}$ and $(A)^\omega$ respectively. For α in $\omega+1 = \omega \cup \{\omega\}$ we let $[A]^\alpha$ denote the collection of all subsets of A of size α, $[A]^{<\alpha}$ denote the collection of all subsets of A of size less than α and $[A]^{\leq\alpha}$ denote the collection of all subsets of A of size at most α. For sequences $s, t \in A^{<\omega}$ (or in $(A)^{<\omega}$) we denote t strictly end-extending s by $s \sqsubset t$, and non-strictly by $s \sqsubseteq t$. Generally, we use $\bar{(\cdot)}$ to denote finitary versions of infinite objects, e.g., if $f \in (\omega)^\omega$, then $\bar{f} \in (\omega)^{<\omega}$ represents some initial segment of f.

If $s \in 2^{\leq\omega}$, we use $\hat{s}$ to denote the strictly increasing function enumerating $s^{-1}[\{1\}]$, the set which s maps to 1. Sometimes, we write finite sequences $s \in 2^{<\omega}$ in "binary format", e.g., $(0, 1, 0, 0, 1)$ is written as 01001_2. This is convenient when there are many consecutive zeros in a sequence, e.g., for $n, m \in \omega$,

$$(\underbrace{0, \ldots, 0}_{n}, 1, \underbrace{0, \ldots, 0}_{m}, 1)$$

is then written simply as $0^n 1 0^m 1_2$.

As usually, the expression $(\forall^\infty n \in \omega)\, \phi(n)$ means that $\phi(n)$ holds for all but finitely many n, and the expression $(\exists^\infty n \in \omega)\, \phi(n)$ means that $\phi(n)$ holds for infinitely many n.

For a group G and $A \subseteq G$, we denote the subgroup of G, generated by A, with $\langle A \rangle$. To avoid confusion, we use the round brackets for tuples (and sequences), e.g., instead of the standard $\langle a, b \rangle$, we write (a, b).

We use S_∞ to denote the Polish group of all permutations of ω and let id_ω denote the unit of this group. For $g_0, \ldots, g_l \in S_\infty$ and $m \in \omega$, we define the *path of m under $g_l \circ \cdots \circ g_0$* to be the sequence $(m(i))_{i \in l+1}$, inductively defined by setting $m(0) := m$ and for $i \leq l$ by $m(i+1) := g_i(m(i))$. We will sometimes omit $\circ$ and write just $g_l \cdots g_0$.

We denote the free group generated by a set A with $\mathbb{F}(A)$. For $x_0, \ldots, x_k \in A$ and $i_0, \ldots, i_k \in \{-1, 1\}$, we occasionally use the vector notation $\vec{x}$ for the reduced word $x_k^{i_k} \cdots x_0^{i_0} \in \mathbb{F}(A)$ in order to emphasise that $\vec{x}$ is not necessarily a generator but a proper word (in case $k > 0$).

Otherwise, we use the standard notation of [Kec12] and introduce any non-standard notation on the way.

We use $\dashv$, instead of the otherwise used $\square$, to denote that the proof of a nested claim or a subclaim has been completed.

2. Constructing a Σ^0_2 mcg

In this section we present a construction of a Σ^0_2 mcg. Our construction and presentation is based on [Sch21], from which we also borrow some notation. Note that many of these notions have a slightly different definition in order to accommodate new coding components and in this way make the construction more definable. The construction of the definable mcg and the proofs of maximality and cofinitariness closely follow the respective proofs in [Sch21]. The main improvement over the construction from [Sch21] is that we augment it by ideas from [Sch17]. This will be apparent in the definition of our group and when proving that the constructed group is definable.

It is not necessary for the reader to be familiar with either [Sch21] or [Sch17]. However, [Sch21] does occasionally contain more details, which the reader might find helpful in understanding this proof and [Sch17] contains the original idea of how maximal objects can be very definable in a simpler setting of maximal eventually different families, possibly making it easier for the reader to understand the present construction.

2.1. **Cofinitary action, freely generated by continuum many generators.** We start by defining an injective map

$$e : 2^\omega \times 2^\omega \times 2^\omega \to S_\infty,$$

such that $\mathrm{range}(e) \subseteq S_\infty$ freely generates a cofinitary group. We follow the idea of the construction from [Sch21], additionally making sure that the intervals are wider, allowing us to code the second and third component of the domain as well.

Recall that we denote the free group, generated by $2^\alpha \times 2^\alpha \times 2^\alpha$, by $\mathbb{F}(2^\alpha \times 2^\alpha \times 2^\alpha)$ for any $\alpha \in \omega + 1\ (= \omega \cup \{\omega\})$. For a word

$$w = (x_k, d^0_k, d^1_k)^{i_k} \cdots (x_0, d^0_0, d^1_0)^{i_0} \in \mathbb{F}(2^\alpha \times 2^\alpha \times 2^\alpha)$$

we sometimes write $w = (w_0, w_1, w_2)$, where we implicitly define

$$w_0 := x_k^{i_k} \cdots x_0^{i_0} \in \mathbb{F}(2^\alpha) \quad \text{and} \quad w_{j+1} := (d_k^j)^{i_k} \cdots (d_0^j)^{i_0} \in \mathbb{F}(2^\alpha)$$

for $j \in 2$. For a word w, we denote its length by $\mathrm{lh}(w)$. If $m \leq \alpha$ we let r_m^α be the group homomorphism

$$r_m^\alpha : \mathbb{F}(2^\alpha \times 2^\alpha \times 2^\alpha) \to \mathbb{F}(2^m \times 2^m \times 2^m),$$

which is defined on a generator $(x, d^0, d^1) \in \mathbb{F}(2^\alpha \times 2^\alpha \times 2^\alpha)$ as

$$r_m^\alpha(x, d^0, d^1) = (x \restriction m, d^0 \restriction m, d^1 \restriction m).$$

We will define a sequence of finite groups

$$(G_n)_{n\in\omega}$$

and group homomorphisms

$$e_n : \mathbb{F}(2^n \times 2^n \times 2^n) \to G_n,$$

along with transitive faithful actions

$$\sigma_n : G_n \curvearrowright I_n,$$

where each $I_n = [m_n, m_{n+1})$ is a non-empty interval in ω, with $m_0 = 0$ and so that the intervals constitute a partition of ω. We also define for each $n \in \omega$ the set

$$W_n := \{w \in \mathbb{F}(2^n \times 2^n \times 2^n) \mid \mathrm{lh}(w) \leq n\}.$$

The requirements that we wish to satisfy are the following. We will inductively construct $(G_n)_{n\in\omega}$, $(e_n)_{n\in\omega}$ and $(\sigma_n : G_n \curvearrowright I_n)_{n\in\omega}$ so that in addition to the above, for every $n \in \omega$ it holds that

(1) $\sum_{m<n} |I_m| < |I_n|$;
(2) $e_n \restriction W_n$ is injective;
(3) $|I_0| \geq 7$.

Since $2^0 = \{\emptyset\}$ and $W_0 = \{\emptyset\}$, it is easy to define G_0, e_0, I_0 and $\sigma_0 : G_0 \curvearrowright I_0$ so that (3) is satisfied.

Suppose that we have already defined G_n, e_n, I_n and $\sigma_n : G_n \curvearrowright I_n$. Let $(w^i)_{i\in l}$ be some enumeration of W_{n+1} for which $w^0 = \emptyset$, the empty word. We let e_{n+1}^0 be a function from $2^{n+1} \times 2^{n+1} \times 2^{n+1}$ to the set of partial injections from $l = \{0, \ldots, l-1\}$ to itself, as follows. For $(x, d^0, d^1) \in 2^{n+1} \times 2^{n+1} \times 2^{n+1}$ and $i, j \in l$ we set

$$e_{n+1}^0(x, d^0, d^1)(i) = j \quad \text{if and only if} \quad w^j = (x, d^0, d^1)w^i.$$

Let $e_{n+1}^1(x, d^0, d^1)$ be some extension of $e_{n+1}^0(x, d^0, d^1)$ to a permutation of l and let

$$G_{n+1}^0 := \langle e_{n+1}^1(x, d^0, d^1) \mid (x, d^0, d^1) \in 2^{n+1} \times 2^{n+1} \times 2^{n+1} \rangle \leq S_l,$$

where S_l denotes the permutation group of l. By the universal property of the free group, e_{n+1}^1 extends uniquely to a group homomorphism from $\mathbb{F}(2^{n+1} \times 2^{n+1} \times 2^{n+1})$ onto G_{n+1}^0, which we denote in the same way. Since $e_{n+1}^1(w^i)(0) = i$ for every $i \in l$, e_{n+1}^1 is injective on W_{n+1}.

For some sufficiently large $k \in \omega$ let

$$G_{n+1} := G_{n+1}^0 \times S_k.$$

By "sufficiently large" we mean large enough so that when I_{n+1} is defined to be of the same size as G_{n+1}, condition (1) holds. Moreover, let

$$e_{n+1} : \mathbb{F}(2^{n+1} \times 2^{n+1} \times 2^{n+1}) \to G_{n+1}$$

be defined on a generator $(x, d^0, d^1) \in 2^{n+1} \times 2^{n+1} \times 2^{n+1}$ by

$$e_{n+1}(x, d^0, d^1) := (e_{n+1}^1(x, d^0, d^1), 1_{S_k}).$$

Let I_{n+1} be the interval immediately to the right of the previously defined I_n, which has the same size as G_{n+1}. Fix some bijection $\Phi_{n+1} : G_{n+1} \to I_{n+1}$ and define the action

$$\sigma_{n+1} : G_{n+1} \curvearrowright I_{n+1}$$

by

$$\sigma_{n+1}(g)(k) = \Phi_{n+1}(g \cdot (\Phi_{n+1})^{-1}(k)).$$

Clearly, all desired conditions have been met and with this the inductive construction is completed.

With the sequences $(G_n)_{n\in\omega}$, $(e_n)_{n\in\omega}$ and $(\sigma_n : G_n \curvearrowright I_n)_{n\in\omega}$ at our disposal define

$$e : \mathbb{F}(2^\omega \times 2^\omega \times 2^\omega) \to S_\infty$$

by defining it on a generator $(x, d^0, d^1) \in 2^\omega \times 2^\omega \times 2^\omega$ as

$$e(x, d^0, d^1) \restriction I_n := \sigma_n(e_n(r^\omega_n(x, d^0, d^1))).$$

The following proposition is an elaboration of Propositions 1.3 and 1.4 from [Sch21].

Proposition 2.1. *The map e is a continuous injective homomorphism, whose range is a cofinitary group. Moreover, e is Δ^0_1 (in terms of that for $w \in \mathbb{F}(2^\omega \times 2^\omega \times 2^\omega)$ and $n \in \omega$, we can calculate $e(w)(n)$ in finite time by analysing $r^\omega_m(w)$ for some $m \in \omega$, which can in turn be calculated in finite time from n; all this using w as an oracle), its range is Π^0_1 and the sequences $(G_n)_{n\in\omega}$ and $(\sigma_n)_{n\in\omega}$ are Δ^0_1.*

Proof. Clear by construction. $\square$

2.2. **The definition of B_0.** In this subsection we define the assignment B_0, which will form the core of our construction.

We first introduce the map $\chi : (\omega)^{\leq\omega} \to 2^{\leq\omega}$ defined on $h \in (\omega)^{\leq\omega}$ by

$$\chi(h) := (\underbrace{0, \dots, 0}_{h(0)}, 1, \underbrace{0, \dots, 0}_{h(1)}, 1, \underbrace{0, \dots, 0}_{h(2)}, 1, \dots) \in 2^{\leq\omega}.$$

Of course, $\mathrm{lh}(\chi(h))$ is infinite if and only if $\mathrm{lh}(h)$ is infinite.

We define the map $\chi^\dagger : 2^{\leq\omega} \to (\omega)^{\leq\omega}$ on $x \in 2^{\leq\omega}$ by

$$\chi^\dagger(x) := \begin{cases} h & \text{if } x \in \mathrm{range}(\chi) \text{ and } \chi(h) = x; \\ s & \text{if } x \notin \mathrm{range}(\chi) \text{ and } n \text{ is maximal such that} \\ & x \restriction n \in \mathrm{range}(\chi) \text{ and } \chi(s) = x \restriction n. \end{cases}$$

Of course, $\chi^\dagger$ is a left inverse of χ. There are two ways for $x \in 2^\omega$ not to be in $\mathrm{range}(\chi)$. The first one is that x is of the form

$$(\dots, 1, \underbrace{0, 0, \dots, 0}_{n}, 1, \dots, 1, \underbrace{0, 0, \dots, 0}_{n}, 1, \dots),$$

in which case it clearly cannot be the image of any $g \in (\omega)^{\leq\omega}$, since it violates injectivity. The second option is that x only has finitely many 1s (but does not fall under the first option). Then if $n > 0$, $n - 1$ is the last index of a 1 in x; and $n = 0$ if and only if x is constant with value 0, in which case $\chi^\dagger(x)$ is the empty function.

Next, define for every $g \in (\omega)^{\leq\omega}$ a function $\vartheta_g \in (\omega)^{\leq\omega}$ with the following properties:

(i) if $g \in (\omega)^\omega$, then as k increases, the maps $\vartheta_{g\restriction k} \in (\omega)^{<\omega}$ approach $\vartheta_g \in (\omega)^\omega$;
(ii) if $g \neq g'$, then $\mathrm{range}(\vartheta_g)$ and $\mathrm{range}(\vartheta_{g'})$ are almost disjoint (this will be used in Claim 2.3);
(iii) for any $n \in \mathrm{range}(\vartheta_g)$, if $n \in I_m$ and $g(n) \in I_{m'}$ then $m \leq m'$ (this will be important in Subclaim 2.12 and in Proposition 2.17).

(iv) each set $\operatorname{range}(\vartheta_g)$ is *g-spaced*, i.e., for any distinct $n, n' \in \operatorname{range}(\vartheta_g)$, it holds that n' is not in the same interval as any of n, $g^{-1}(n)$ or $g(n)$. This will be used in Fact 2.7.

These properties are not difficult to arrange, but since our aim is to do this in such a way that in the end the definition of the set of generators is simple, we shall go into the details.

Let

$$\# : (\omega)^{<\omega} \to \omega$$

be some fixed Δ^0_1 bijection, which we use to introduce an auxiliary function F, defined for $g \in (\omega)^{<\omega}$ and $k \in \omega$ by

$$F(g,k) := 2^{\#(g)} \cdot 3^k.$$

The only property we require of F is that it is Δ^0_1 and injective. We also define an auxiliary family of functions which satisfy all of the properties (i) to (iv) above, except possibly (iii): given $g \in (\omega)^{\leq\omega}$, define $\xi_g \in (\omega)^{\leq\omega}$ inductively as follows. Suppose that we have defined ξ_g on all $k < n$ and that for every $k < n$ we have

$$I_{F(g\restriction(k+1),\xi_g(k))} \subseteq \operatorname{dom}(g) \cup \operatorname{range}(g).$$

(If the above requirement is not fulfilled for some $k < n$, then $\operatorname{dom}(\xi_g) = n$, and our inductive definition terminates.) Then we set $\xi_g(n)$ to be the smallest such that

$$\min I_{F(g\restriction(n+1),\xi_g(n))}$$

is strictly larger than any member of

$$\bigcup \Big\{ \{l, g(l), g^{-1}(l)\} \,\Big|\, (\exists k < n)\, l \in I_{F(g\restriction(k+1),\xi_g(k))} \Big\}.$$

With ξ_g at our disposal, we finally define the function $\vartheta_g \in (\omega)^{\leq\omega}$, defined on those $n \in \omega$, for which

$$\operatorname{dom}(g) \geq \max \Big\{ n+1, \min I_{F(g\restriction(n+1),\xi_g(n))+1} \Big\}$$

by

$$\vartheta_g(n) := \min\Big(I_m \setminus \cup\{g^{-1}[I_k] \mid k < m\} \Big),$$

where $m = F(g \restriction (n+1), \xi_g(n))$. It is clear that properties (i) to (iv) are fulfilled.

Now we define for every $g \in (\omega)^{\leq\omega}$ and $c^0, c^1 \in 2^{\leq\omega}$ with $\operatorname{lh}(g) \leq \operatorname{lh}(c^0) = \operatorname{lh}(c^1)$ the sets

$$D(g) := \operatorname{dom}(g) \cap \operatorname{range}(\vartheta_g)$$

and

$$B_0(g, c^0, c^1) := \operatorname{dom}(g) \cap \Big\{ \vartheta_g\Big(\widehat{c^0}(n)\Big) \,\Big|\, n \in \omega \wedge c^1(n) = 1 \wedge \widehat{c^0}(n) \in \operatorname{dom}(\vartheta_g) \Big\} \Big\backslash$$
$$\Big\{ m \in \omega \,\Big|\, g(m) = e(g, c^0, c^1)(m) \Big\}.$$

Recall that $\widehat{c^0}$ is the function enumerating $\operatorname{range}(c^0)$. Basically, what we are doing is that we are using c^1 to pick certain elements from $\operatorname{range}(c^0)$, and then using these picked elements as inputs of ϑ_g. This way we have passed to a subset $B_0(g, c^0, c^1)$ of $D(g)$ in such a way that case (2) of Claim 2.3 (establishing a strong form of almost disjointness) holds.

Above we have defined the notion of when a set is g-spaced. The notion is based on the notion of (f, g)-spacedness, taken from Subsection 1.2 of [Sch21]. Note that since $e(\cdot,\cdot,\cdot)$ respects the interval partition, our notion of g-spaced agrees with the notion of $(e(\chi(g), c^0, c^1), g)$-spaced from [Sch21] for any c^0 and c^1. Instead of saying that for each $g \in (\omega)^{\leq\omega}$ and any $c^0, c^1 \in 2^{\leq\omega}$ (with $\operatorname{lh}(g) \leq \operatorname{lh}(c^0) = \operatorname{lh}(c^1)$) the set $B_0(g, c^0, c^1)$ is g-spaced, we rather simply say that B_0 is *spaced*.

Finally, observe that $D(g)$ and $B_0(g, c^0, c^1)$ are $\Delta^0_1(g)$ and $\Delta^0_1(g, c^0, c^1)$ respectively (in terms of that their membership is decidable using their respective oracles).

2.3. **Relations coding extensions.** We continue with a discussion of binary relations which will, by abstracting certain technical aspects, simplify some of the technical steps in the proof of the main theorem. The definitions of the relations $<^f_0$ and $<^f_1$ are inspired by [Sch21].

Following [Sch17], for $c \in 2^{\leq\omega}$ we say that it is *good*, if for every two successive $n_0 < n_1$ from $c^{-1}[\{1\}]$ we have that

$$n_1 \equiv \sum_{i \leq n_0} c(i) \cdot 2^i \pmod{2^{n_0+1}},$$

and in case

$$|\{n \in \omega \mid c(n) \text{ is defined and equals } 1\}| < \omega$$

the sequence c is finite.

Let $\mathcal{C}$ denote the set of all finite good sequences c, with either $c(-1) = 1$ or $c = \emptyset$. For $c, c' \in \mathcal{C}$, define the relation $c \lhd c'$ by

$$c \lhd c' \quad :\Longleftrightarrow \quad \text{for some } n \in \omega,\ c' = c^\frown 0^n 1_2.$$

Good sequences will be crucial for our construction (see the beginning of Subsection 2.5) and for the nice properties used e.g. in Claim 2.3.

Define now a uniformly computable family

$$\{<^f_0 \ \mid f \in (\omega)^{\leq\omega}\}$$

of strict partial orders as follows: given $m, m' \in I_n$ define

$$\delta_n(m, m') = \begin{cases} \text{the unique element } (w_0, w_1, w_2) \in W_n \text{ such that} \\ e_n(w_0, w_1, w_2)(m) = m', \text{ if such exists; or} \\ \uparrow \text{ (remains undefined) otherwise.} \end{cases}$$

For $f \in (\omega)^{\leq\omega}$ and $m \in \omega$, we let

$$\delta(f, m) := \delta_n(m, f(m))$$

for the unique n such that $m \in I_n$, when $m \in \operatorname{dom}(f)$ and $\delta_n(m, f(m))$ is defined; otherwise we let $\delta(f, m)$ remain undefined. Finally, set

$$m <^f_0 m'$$

precisely when $m < m'$, $m' \in \operatorname{dom}(f)$, $\delta(f, m)$ and $\delta(f, m')$ are defined and

$$\delta(f, m) = r^{m'}_m(\delta(f, m')).$$

Of course, by "uniformly computable" we mean that there is some $a \in \omega$ such that the family is precisely the set

$$\Big\{\{a\}^f \mid f \in (\omega)^{\leq\omega}\Big\},$$

which is clearly the case.

Next, we introduce another uniformly computable family of strict partial orders

$$\{<^f_1 \ \mid f \in (\omega)^{\leq\omega}\}$$

as follows. For $m, m' \in \omega$ set $m <^f_1 m'$ if and only if it holds that $m < m'$, $f(m) < f(m')$ are both defined and there is some $g \in (\omega)^{<\omega}$ so that $\{f(m), f(m')\} \subseteq D(g)$. It is not hard to see that $<^f_1$ is a strict partial order (to prove transitivity use the nice properties of ϑ_g). Note also that the family is uniformly computable, as the existence of the appropriate $g \in (\omega)^{<\omega}$ can be determined in finitely many steps.

We now present an amplification of Claim 2.7 from [Sch17], which will be essential for the proof of maximality of our constructed group (see Proposition 2.9).

Lemma 2.2. *For any $g \in (\omega)^\omega$ one of the following holds:*

(1) There is some $I \in [D(g)]^\infty$, which is linearly ordered by $<^g_0$.

(2) There are good sequences $d^0, d^1 \in 2^\omega$, such that no two elements of $B_0(g, d^0, d^1)$ are $<^g_0$-comparable and either:

(a) $B_0(g, d^0, d^1)$ is linearly ordered by $<^g_1$; or

(b) no two elements of $B_0(g, d^0, d^1)$ are $<^g_1$-comparable.

Proof. Fix any $g \in (\omega)^\omega$. Assume first that

$$(*) \qquad (\exists c_0 \in \mathcal{C})\,(\forall c_1 \in \mathcal{C})\, c_0 \lhd c_1 \to (\exists c_2 \in \mathcal{C})\, c_1 \lhd c_2 \wedge \vartheta_g(\mathrm{lh}(c_1) - 1) <^g_0 \vartheta_g(\mathrm{lh}(c_2) - 1)$$

holds. Then fix c_0 witnessing the first existential quantifier, let $c_1 \in \mathcal{C}$ be such that $c_0 \lhd c_1$ and set $n_0 := \vartheta_g(\mathrm{lh}(c_1) - 1)$. Using the second existential quantifier, we get some c_2 with $c_1 \lhd c_2$ and $n_0 <^g_0 \vartheta_g(\mathrm{lh}(c_2) - 1)$. Put $n_1 := \vartheta_g(\mathrm{lh}(c_2) - 1)$ and let $d_2 \in \mathcal{C}$ be unique such that $\mathrm{lh}(d_2) = \mathrm{lh}(c_2)$ and $c_0 \lhd d_2$. Applying $(*)$ with d_2 instantiating the universal quantifier, we get $c_3 \in \mathcal{C}$ with $d_2 \lhd c_3$ and $n_1 <^g_0 \vartheta_g(\mathrm{lh}(c_3) - 1)$, so we set $n_2 := \vartheta_g(\mathrm{lh}(c_3) - 1)$. Clearly, we can proceed inductively to obtain an infinite sequence $(n_i)_{i \in \omega}$ contained in $D(g)$ and such that for every $i \in \omega$ it holds that $n_i <^g_0 n_{i+1}$. Then set

$$I := \{n_i \,|\, i \in \omega\},$$

to obtain (1).

Suppose now that $(*)$ does not hold. Then its negation

$$(\neg *) \quad (\forall c_0 \in \mathcal{C})\,(\exists c_1 \in \mathcal{C})\, c_0 \lhd c_1 \wedge (\forall c_2 \in \mathcal{C})\, c_1 \lhd c_2 \to \neg\Big(\vartheta_g(\mathrm{lh}(c_1) - 1) <^g_0 \vartheta_g(\mathrm{lh}(c_2) - 1)\Big)$$

must be true. Now, first instantiate the leftmost universal quantifier of $(\neg *)$ with $c_0 := \emptyset$, to get $d_0 \in \mathcal{C}$ with $\emptyset \lhd d_0$. Next, instantiate the same quantifier with $c_0 := d_0$, to get $d_1 \in \mathcal{C}$ with $d_0 \lhd d_1$. Inductively, we get a sequence $(d_i)_{i \in \omega}$, such that for every $i \in \omega$ it holds that $d_i \lhd d_{i+1}$. With this, we define an infinite good sequence $d^0 \in 2^\omega$ by

$$d^0 := \cup\{d_i \,|\, i \in \omega\}.$$

Now we basically repeat the argument of case (1) to get the second infinite good sequence d^1. Assume first that

$$(**) \quad (\exists c_0 \in \mathcal{C})\,(\forall c_1 \in \mathcal{C})\, c_0 \lhd c_1 \to (\exists c_2 \in \mathcal{C})\, c_1 \lhd c_2 \wedge \vartheta_g(\widehat{d^0}(\mathrm{lh}(c_1) - 1)) <^g_1 \vartheta_g(\widehat{d^0}(\mathrm{lh}(c_2) - 1))$$

holds. Then as in the case when $(*)$ is true, we can get a good sequence $d^1 \in 2^\omega$ such that the set

$$I := \{\vartheta_g(\widehat{d^0}(n)) \,|\, n \in \omega \wedge d^1(n) = 1\}$$

is linearly ordered by $<^g_1$. By definition it holds that $B_0(g, d^0, d^1) \subseteq I$, so $B_0(g, d^0, d^1)$ is linearly ordered by $<^g_1$ as well. Due to the property in the right part of $(\neg *)$, we have made sure that no two elements of $B_0(g, d^0, d^1)$ are $<^g_0$-comparable. We have thus obtained case (a) of (2).

If on the other hand the negation of $(**)$,

$$(\neg **) \qquad (\forall c_0 \in \mathcal{C})\,(\exists c_1 \in \mathcal{C})\, c_0 \lhd c_1 \wedge (\forall c_2 \in \mathcal{C})\, c_1 \lhd c_2 \to \\ \neg\Big(\vartheta_g(\widehat{d^0}(\mathrm{lh}(c_1) - 1)) <^g_1 \vartheta_g(\widehat{d^0}(\mathrm{lh}(c_2) - 1))\Big)$$

holds, we repeat the procedure by which we have constructed d^0 to get an infinite good d^1. By definition of $B_0(g, d^0, d^1)$ and the right parts of $(\neg *)$ and $(\neg **)$, we have made sure that no two elements of $B_0(g, d^0, d^1)$ are comparable with respect to either one of $<^g_0$ or $<^g_1$, obtaining case (b) of (2). ∎

2.4. **Further restriction of B_0.** The definitions of D and B_0 make sure that the sets $D(\cdot)$ and $B_0(\cdot,\cdot,\cdot)$ are mutually sparse, as the following claim shows (this is a stronger analogue to Claim 2.4 from [Sch17] and Lemma 1.9 from [Sch21]).

Claim 2.3. *Suppose that $g, h \in (\omega)^\omega$ and $c^0, c^1, d^0, d^1 \in 2^\omega$ are all good.*

(1) If $g \neq h$, then the set
$$\{n \in \omega \mid D(g) \cap I_n \neq \emptyset \wedge D(h) \cap I_n \neq \emptyset\}$$
is finite.

(2) If $(g, c^0, c^1) \neq (h, d^0, d^1)$, then the set
$$\{n \in \omega \mid B_0(g, c^0, c^1) \cap I_n \neq \emptyset \wedge B_0(h, d^0, d^1) \cap I_n \neq \emptyset\}$$
is finite.

Proof. For (1), let $m \in \omega$ be such that $g \restriction m \neq h \restriction m$. Then for all $k_0, k_1 > m$ it holds that $\vartheta_g(k_0)$ and $\vartheta_h(k_1)$ are not in the same interval I_n, and thus the proof of (1) is complete.

Suppose for (2) that $g = h$ and that $c^j \neq d^j$ for some $j \in 2$. Let $m \in \omega$ be such that $c^j \restriction m \neq d^j \restriction m$. By definition of goodness, there is at most one $k \geq m$, for which it holds that both $c^j(k) = 1$ and $d^j(k) = 1$. Then a quick look at the definition of B_0 (left to the reader) completes the proof of (2). □

Unfortunately, the sets $B_0(\cdot,\cdot,\cdot)$ are still not sparse enough for the construction to succeed, so we now restrict them further.

We will define a function
$$B : \{(g, c^0, c^1) \in (\omega)^{\leq\omega} \times 2^{\leq\omega} \times 2^{\leq\omega} \mid \mathrm{lh}(g) \leq \mathrm{lh}(c^0) = \mathrm{lh}(c^1)\} \to [\omega]^{\leq\omega},$$
such that for every $(g, c^0, c^1) \in \mathrm{dom}(B)$ it holds that $B(g, c^0, c^1) \subseteq B_0(g, c^0, c^1)$. Moreover, we will require that the following property, called *superspacedness* (since it is reminiscent of the notion of being *spaced*, borrowed from [Sch21] and reintroduced above), holds for B (note that since B is pointwise contained in B_0, it is automatically spaced).

Definition 2.4 (Superspacedness)**.** A function
$$B : \{(g, c^0, c^1) \in (\omega)^{\leq\omega} \times 2^{\leq\omega} \times 2^{\leq\omega} \mid \mathrm{lh}(g) \leq \mathrm{lh}(c^0) = \mathrm{lh}(c^1)\} \to [\omega]^{\leq\omega}$$
is *superspaced* if for every $g \in (\omega)^\omega$, for which the set
$$I := \{n \in D(g) \mid g(n) = e(\vec{x}, \vec{d^0}, \vec{d^1})(n)\}$$
is infinite for some $(\vec{x}, \vec{d^0}, \vec{d^1}) \in \mathbb{F}(2^\omega \times 2^\omega \times 2^\omega)$, with $\vec{x} = x_k^{i_k} \cdots x_0^{i_0}$ and $\vec{d^j} = (d_k^j)^{i_k} \cdots (d_0^j)^{i_0}$ for $j \in 2$, defining
$$J := \{j \in k+1 \mid x_j \in \mathrm{range}(\chi) \wedge \chi^\dagger(x_j) \neq g \wedge d_j^0, d_j^1 \text{ are good}\},$$
it holds that there are infinitely many $m \in I$, so that
$$(\forall j \in J)\, \chi^\dagger(x_j)[B(\chi^\dagger(x_j), d_j^0, d_j^1)] \cap I_{n(m)} = \emptyset,$$
where $n(m)$ is the unique such that $m \in I_{n(m)}$.

The notion of superspacedness is similar to the notion of being *cooperative* from [Sch21]. The purpose of the definition is the following: we will be catching elements $f \in (\omega)^\omega$ on $D(f)$ by words in range(e), so we have to ensure that with the definition of $\dot{e}$ (using the definition of e in certain cases; see Subsection 2.5) we still catch the elements (now with words in range$(\dot{e})$). This is formalised in Proposition 2.9.

We continue by defining B. We first define a set
$$T \subseteq (\omega)^{<\omega} \times \{-1, 1\}^{<\omega} \times (2^{<\omega})^{<\omega} \times (2^{<\omega})^{<\omega} \times (2^{<\omega})^{<\omega},$$

equipped with a tree like order $<$, and a function

$$\psi : T \to 2^{<\omega}.$$

They will store information which we will then use to define B from B_0. The process of defining ψ will be algorithmic in nature and is similar to the algorithm called "semaphore" in [Sch21], where it is also explained in detail (which the reader might find helpful). Let

$$\begin{aligned} T := \Big\{ (s, \vec{i}, \vec{x}, \vec{d^0}, \vec{d^1}) \in (\omega)^{<\omega} \times \{-1, 1\}^{<\omega} \times (2^{<\omega})^{<\omega} \times (2^{<\omega})^{<\omega} \times (2^{<\omega})^{<\omega} \;\Big| \\ \mathrm{lh}(\vec{i}) = \mathrm{lh}(\vec{x}) = \mathrm{lh}(\vec{d^0}) = \mathrm{lh}(\vec{d^1}) \wedge \\ (\exists k \in \omega)\ \mathrm{lh}(s) = \sum_{m \le k} |I_m| \wedge \\ (\forall j \in \mathrm{lh}(\vec{x}))\ \mathrm{lh}(\vec{x}(j)) = \mathrm{lh}(\vec{d^0}(j)) = \mathrm{lh}(\vec{d^1}(j)) = k \Big\}. \end{aligned}$$

Here, the first component s is an approximation to an element of $(\omega)^\omega$, and the last four components, $\vec{i}, \vec{x}, \vec{d^0}, \vec{d^1}$ determine the word

$$w(\vec{i}, \vec{x}, \vec{d^0}, \vec{d^1}) \in \mathbb{F}(2^{\mathrm{lh}(\vec{x}(0))} \times 2^{\mathrm{lh}(\vec{d^0}(0))} \times 2^{\mathrm{lh}(\vec{d^1}(0))}),$$

defined by

$$w(\vec{i}, \vec{x}, \vec{d^0}, \vec{d^1}) := \Big(\vec{x}(-1), \vec{d^0}(-1), \vec{d^1}(-1)\Big)^{\vec{i}(-1)} \cdots \Big(\vec{x}(0), \vec{d^0}(0), \vec{d^1}(0)\Big)^{\vec{i}(0)}.$$

For $(s, \vec{i}, \vec{x}, \vec{d^0}, \vec{d^1}) \in T$ let $k(s)$ be the unique k for which $\mathrm{lh}(s) = \sum_{m \le k} |I_m|$.

Clearly, T is recursive. We next define a recursive tree-like ordering $<$ on T as follows:

$$\begin{aligned} (s_0, \vec{i}_0, \vec{x}_0, \vec{d^0_0}, \vec{d^1_0}) < (s_1, \vec{i}_1, \vec{x}_1, \vec{d^0_1}, \vec{d^1_1}) :\Longleftrightarrow \\ s_0 \sqsubset s_1 \wedge \vec{i}_0 = \vec{i}_1 \wedge r^{k(s_1)}_{k(s_0)}\Big(w(\vec{i}_1, \vec{x}_1, \vec{d^0_1}, \vec{d^1_1})\Big) = w(\vec{i}_0, \vec{x}_0, \vec{d^0_0}, \vec{d^1_0}). \end{aligned}$$

Finally, we inductively define ψ. The idea is that for every $(f, p^0, p^1) \in \mathrm{dom}(B_0)$, we will look at the value of ψ on the relevant elements of T and based on the value of ψ on those elements we will decide whether to remove elements from $B_0(f, p^0, p^1)$ when defining $B(f, p^0, p^1)$ (see Equation 1 below), so that the desired superspacedness condition holds (we will verify this in Claim 2.6).

Let $t = (s, \vec{i}, \vec{x}, \vec{d^0}, \vec{d^1}) \in T$ and suppose we have defined ψ on all $<$-predecessors of t in T. Set $n := \mathrm{lh}(\vec{i})$ and for $j \in n$ define $g_j := \chi^\dagger(\vec{x}(j))$. We will define $\psi(t)$, which will be an element of 2^n. Let $t_* = (s_*, \vec{i}_*, \vec{x}_*, \vec{d^0_*}, \vec{d^1_*}) \in T$ be the strict predecessor of t (note that when there is a strict predecessor it is unique; in case when t does not have a strict predecessor, set

$$t_* := (\emptyset, \vec{i}, (\underbrace{\emptyset, \ldots, \emptyset}_{\mathrm{lh}(\vec{i})}), (\underbrace{\emptyset, \ldots, \emptyset}_{\mathrm{lh}(\vec{i})}), (\underbrace{\emptyset, \ldots, \emptyset}_{\mathrm{lh}(\vec{i})}))$$

and use 0 for any occurrence of $\psi(t_*)(j)$ below). For $j \in n$ set $g^*_j := \chi^\dagger(\vec{x}_*(j))$. We now consider two cases.

- If there is some $j \in n$ for which

$$\psi(t_*)(j) = 0$$

and

$$
\begin{aligned}
(\exists m \leq k)\,(\exists l \in I_m)\, l \in D(s) \wedge &\\
&\delta(s,l) = r^k_m\Big(w(\vec{i},\vec{x},\vec{d^0},\vec{d^1})\Big) \wedge\\
&g_j[B_0(g_j,\vec{d^0}(j),\vec{d^1}(j))] \cap I_m \neq \emptyset \wedge\\
&g^*_j[B_0(g^*_j,\vec{d^0_*}(j),\vec{d^1_*}(j))] \cap I_m = \emptyset,
\end{aligned}
$$

then for all such j put

$$\psi(t)(j) = 1$$

and for all other $j' \in n$ keep

$$\psi(t)(j') = \psi(t_*)(j').$$

- If there is no such j, put

$$\psi(t)(j) = 0$$

for all $j \in n$.

It is not hard to see that ψ is recursive.

Finally, we use T and ψ to define B. Take any $(f,p^0,p^1) \in \mathrm{dom}(B_0)$ and suppose that $m \in B_0(f,p^0,p^1)$. Then let

$$
\begin{aligned}
(1) \quad m \notin B(f,p^0,p^1) \quad :\Longleftrightarrow \quad & (\exists (s,\vec{i},\vec{x},\vec{d^0},\vec{d^1}) \in T)\,(\exists j \in \mathrm{lh}(\vec{i}\,))\\
& \chi^\dagger(\vec{x}(j)) \sqsubseteq f \wedge \vec{d^0}(j) = p^0 \upharpoonright k(s) \wedge \vec{d^1}(j) = p^1 \upharpoonright k(s) \wedge\\
& m \in \mathrm{dom}(\chi^\dagger(\vec{x}(j))) \wedge (\exists l \in I_{n(f(m))})\, l \in D(s) \wedge\\
& \delta(s,l) = r^{k(s)}_{n(f(m))}\Big(w(\vec{i},\vec{x},\vec{d^0},\vec{d^1})\Big) \wedge\\
& \psi(s,\vec{i},\vec{x},\vec{d^0},\vec{d^1})(j) = 0 \wedge\\
& (\forall (s_*,\vec{i}_*,\vec{x}_*,\vec{d^0_*},\vec{d^1_*}) < (s,\vec{i},\vec{x},\vec{d^0},\vec{d^1}))\\
& m \notin \mathrm{dom}(\chi^\dagger(\vec{x}_*(j))) \vee k(s_*) < n(f(m)),
\end{aligned}
$$

where $n(f(m))$ is the unique n such that $f(m) \in I_n$. Note that even though (1) makes reference to an existential quantifier over T, the existence of the $(s,\vec{i},\vec{x},\vec{d^0},\vec{d^1}) \in T$ satisfying the required properties can be determined in finitely many steps. This is because $D(\cdot)$ are mutually almost disjoint, and so the requirements on the right side of (1) leave us only with finitely many potential candidates $(s,\vec{i},\vec{x},\vec{d^0},\vec{d^1}) \in T$ which we need to consider. We leave the details to the reader.

Hence, since B_0, T and ψ are all Δ^0_1, B is Δ^0_1 as well. Moreover, it holds by design that $B(f,p^0,p^1) \subseteq B_0(f,p^0,p^1)$ for every $(f,p^0,p^1) \in \mathrm{dom}(B_0)$.

The following claim identifies sufficient conditions for $B(f,p^0,p^1)$ to be infinite. This will be used later in Proposition 2.9, where we prove that the constructed group is maximal.

Claim 2.5. *Suppose that $f \in (\omega)^\omega$ and $p^0, p^1 \in 2^\omega$ are good so that $B_0(f,p^0,p^1)$ is infinite, no two elements of $B_0(f,p^0,p^1)$ are comparable with respect to $<^f_0$ and so that one of the following holds:*

(a) $B_0(f,p^0,p^1)$ is linearly ordered by $<^f_1$; or
(b) no two elements of $B_0(f,p^0,p^1)$ are $<^f_1$-comparable.

Then $B(f,p^0,p^1)$ is infinite.

Proof. In case (b) holds, it is easy to verify that $B(f,p^0,p^1) = B_0(f,p^0,p^1)$. So assume that (a) holds. The definition of ψ makes sure that for every $m \in B_0(f,p^0,p^1) \setminus B(f,p^0,p^1)$, there is some $m' > m$ with $m' \in B(f,p^0,p^1)$. In particular, $B(f,p^0,p^1)$ is infinite. The details are left to the reader. □

Finally, we prove that B is superspaced.

Claim 2.6. *Suppose that for $g \in (\omega)^\omega$, $\vec{x} = x_k^{i_k} \cdots x_0^{i_0} \in \mathbb{F}(2^\omega)$ and $\vec{d^j} = (d_k^j)^{i_k} \cdots (d_0^j)^{i_0} \in \mathbb{F}(2^\omega)$ for $j \in 2$ it holds that the set*

$$I := \{n \in D(g) \mid g(n) = e(\vec{x}, \vec{d^0}, \vec{d^1})(n)\}$$

is infinite. Then, setting

$$J := \{j \in k+1 \mid x_j \in \operatorname{range}(\chi) \wedge \chi^\dagger(x_j) \neq g \wedge d_j^0, d_j^1 \text{ are good}\},$$

there are infinitely many $m \in I$ for which

$$(\forall j \in J)\, \chi^\dagger(x_j)[B(\chi^\dagger(x_j), d_j^0, d_j^1)] \cap I_{n(m)} = \emptyset.$$

Proof. Set $g_j := \chi^\dagger(x_j)$ for every $j \in k+1$. Then let

$$J_0 := \{j \in J \mid (\exists^\infty m \in I)\, g_j[B(g_j, d_j^0, d_j^1)] \cap I_{n(m)} \neq \emptyset\}.$$

Let $m_0 \in \omega$ be such that for every $m \in I \setminus m_0$ it holds that

$$(\forall j \in J \setminus J_0)\, g_j[B(g_j, d_j^0, d_j^1)] \cap I_{n(m)} = \emptyset.$$

Let $\vec{i} := (i_0, \ldots, i_k)$ and for $n \in \omega$ use $\vec{x} \restriction n$ to denote $(x_0 \restriction n, \ldots, x_k \restriction n)$, $\vec{d^0} \restriction n$ to denote $(d_0^0 \restriction n, \ldots, d_k^0 \restriction n)$ and $\vec{d^1} \restriction n$ to denote $(d_0^1 \restriction n, \ldots, d_k^1 \restriction n)$. For $l \in \omega$, we also let $N(l) := \min I_{n(l)+1}$ (recall that $n(l)$ is the unique n such that $l \in I_n$). By case analysis of the definition of ψ on

$$\left(g \restriction N(l), \vec{i}, \vec{x} \restriction n(l), \vec{d^0} \restriction n(l), \vec{d^1} \restriction n(l)\right),$$

one can see from the definition of B that there are infinitely many $m \in I \setminus m_0$, so that for every $j \in J_0$ it holds that

$$g_j[B(g_j, d_j^0, d_j^1)] \cap I_{n(m)} = \emptyset.$$

With this the proof is complete. □

2.5. **The construction.** We finally begin the construction. Define $\dot{e} : 2^\omega \times 2^\omega \times 2^\omega \to S_\infty$ on $(x, c^0, c^1) \in 2^\omega \times 2^\omega \times 2^\omega$ as follows. With $g := \chi^\dagger(x) \in (\omega)^{\leq\omega}$ let

$$\dot{e}(x,c^0,c^1)(n) := \begin{cases} g(n) & \text{if } n \in B(g,c^0,c^1),\ c^0 \restriction (n+1),\ c^1 \restriction (n+1) \text{ are} \\ & \text{good and } \neg(\exists n_0, n_1 \in B_0(g,c^0,c^1) \cap n)\, n_0 <_0^g n_1; \\ e(x,c^0,c^1)(m) & \text{if } m := g^{-1}(n) \text{ is defined and every condition from} \\ & \text{case 1 is satisfied when we replace } n \text{ with } m; \\ e(x,c^0,c^1)^2(n) & \text{if } m := (g^{-1} \circ e(x,c^0,c^1))(n) \text{ is defined and every} \\ & \text{condition from case 1 is satisfied when we replace} \\ & n \text{ with } m; \\ e(x,c^0,c^1)(n) & \text{otherwise.} \end{cases}$$

Fact 2.7. *The map $\dot{e}$ is well-defined and $\dot{e}(x,c^0,c^1)$ is a permutation for every $(x,c^0,c^1) \in 2^\omega \times 2^\omega \times 2^\omega$.*

Proof. Note that the cases in the above definition are mutually exclusive, thanks to the definition of $D(g)$ (we are still using $g := \chi^\dagger(x)$), which makes sure that $D(g)$ and hence also $B(g, c^0, c^1)$ are spaced. When one of the first three cases takes place, the other two of the first three cases make sure that $\dot{e}(x, c^0, c^1)$ is a permutation. When this happens, we say that we are performing a surgery operation (since we are surgically joining two orbits; see Figure 1). The details are left to the reader. $\square$

Note that in case $x \notin \mathrm{range}(\chi)$ it holds that $g = \chi^\dagger(x) \in (\omega)^{<\omega}$. In this case, we only perform finitely many surgeries and afterwards only use e. The definition of $\dot{e}$ is elucidated in Figure 1 below (adapted after Figure 1 from [Sch21, p. 8]), where the orbit structure is shown before (in black) and after (in wine red) the surgery operation.

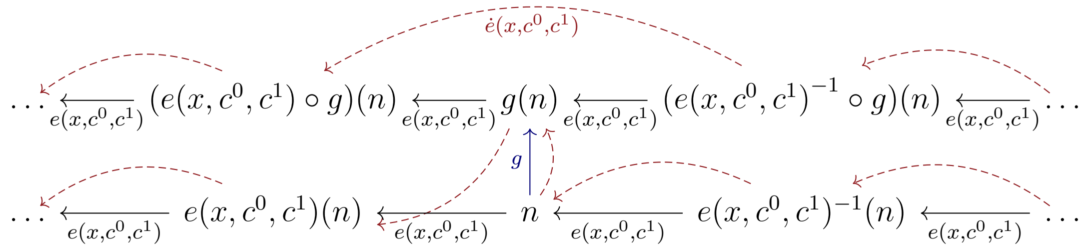

FIGURE 1. The definition of $\dot{e}(x, c^0, c^1)$, where $g = \chi^\dagger(x)$, $n \in B(g, c^0, c^1)$, $\neg(\exists n_0, n_1 \in B_0(g, c^0, c^1) \cap n)\, n_0 <_0^g n_1$ and $c^0 \restriction n$, $c^1 \restriction n$ are both good.

The following is a reformulation of Theorem 0.1.

Theorem 2.8. *The set* $\mathrm{range}(\dot{e})$ *is a* Π^0_1 *subset of* ω^ω *and freely generates a* Σ^0_2 *mcg.*

Set $\mathcal{G} := \langle \mathrm{range}(\dot{e}) \rangle \leq S_\infty$. As in [Sch21], we split the long proof into propositions with shorter proofs. We first prove that $\mathcal{G}$ satisfies a strong form of maximality. The idea is based on the proof of Proposition 1.13 from [Sch21].

Proposition 2.9. *For every* $g \in (\omega)^\omega$ *there is some* $h \in \mathcal{G}$ *such that*

$$\{n \in \omega \mid g(n) = h(n)\}$$

is infinite.

Proof. If g has infinitely many fixed points, then clearly $\mathrm{id}_\omega \in \mathcal{G}$ agrees with g on infinitely many places. So let $m \in \omega$ be such that for every $n \geq m$ it holds that $g(n) \neq n$. Applying Lemma 2.2 to g, we consider the following two cases:
(1) We have some $I \in [D(g)]^\infty$ which is linearly ordered by $<_0^g$. By definition of $<_0^g$, there is some $(\vec{x}, \vec{d^0}, \vec{d^1}) \in \mathbb{F}(2^\omega \times 2^\omega \times 2^\omega)$, with

$$\vec{x} = x_k^{i_k} \dots x_0^{i_0} \quad \text{and} \quad \vec{d^j} = (d_k^j)^{i_k} \dots (d_0^j)^{i_0}$$

for $j \in 2$, so that

$$e(\vec{x}, \vec{d^0}, \vec{d^1}) \restriction I = g \restriction I.$$

For every $j \in k+1$ set also $g_j := \chi^\dagger(x_j) \in (\omega)^{\leq\omega}$. If there is some $j \in k+1$, for which $g_j = g$, d_j^0, d_j^1 are good, no two elements of $B_0(g, d_j^0, d_j^1)$ are $<_0^g$-comparable and $B(g, d_j^0, d_j^1)$ is infinite, it follows that

$$\dot{e}(g, d_j^0, d_j^1) \restriction B(g, d_j^0, d_j^1) = g \restriction B(g, d_j^0, d_j^1),$$

so we are done. Otherwise define the set $I' \supseteq I$ by

$$I' := \{n \in D(g) \mid g(n) = e(\vec{x}, \vec{d^0}, \vec{d^1})(n)\}$$

and the sets

$$J_0 := \{j \in k+1 \,|\, g_j = g \wedge d_j^0, d_j^1 \text{ are good } \wedge (\exists n_0, n_1 \in B_0(g, d_j^0, d_j^1))\, n_0 <_0^g n_1\}$$
$$J_1 := \{j \in k+1 \,|\, x_j \in \operatorname{range}(\chi) \wedge (d_j^0 \text{ is not good} \vee d_j^1 \text{ is not good})\}$$
$$J_2 := \{j \in k+1 \,|\, x_j \notin \operatorname{range}(\chi) \vee B(g_j, d_j^0, d_j^1) \text{ is finite}\}$$
$$J_3 := \{j \in k+1 \,|\, x_j \in \operatorname{range}(\chi) \wedge g_j \neq g \wedge d_j^0, d_j^1 \text{ are good}\}.$$

Let $m_0 \in \omega$ be such that for every $j \in J_0$ it holds that there are some $n_0, n_1 \in B_0(g, d_j^0, d_j^1) \cap m_0$ for which $n_0 <_0^g n_1$. Next, let $m_1 \in \omega$ be such that for every $j \in J_1$, it holds that either $d_j^0 \restriction m_1$ is not good or $d_j^1 \restriction m_1$ is not good. Finally, let $m_2 \in \omega$ be such that for every $j \in J_2$ it holds that either $\operatorname{lh}(g_j) < m_2$ or $B(g, d_j^0, d_j^1) \subseteq m_2$. Then define $m_3 \in \omega$ to be the smallest element of $\{\min I_n \,|\, n \in \omega\}$ such that for every element q of

$$\bigcup \Big\{ \{l, g_j(l), g_j^{-1}(l)\} \,\Big|\, l \leq \max\{m_0, m_1, m_2\} \wedge j \in J_0 \cup J_1 \cup J_2 \Big\}$$

it holds that $q < m_3$. Note that some of the $g_j(l)$ and $g_j^{-1}(l)$ above might be undefined, in which case we ignore them.

By the superspacedness property of B it holds that there is some infinite $I'' \subseteq I' \setminus m_3$ so that for every $m \in I''$ it holds that

$$(\forall j \in J_3)\, g_j[B(g_j, d_j^0, d_j^1)] \cap I_{n(m)} = \emptyset,$$

where $n(m)$ is the unique such that $m \in I_{n(m)}$.

If $j \in J_0 \cup J_1 \cup J_2$, then

$$\dot{e}(x_j, d_j^0, d_j^1) \restriction (\omega \setminus m_3) = e(x_j, d_j^0, d_j^1) \restriction (\omega \setminus m_3).$$

Finally, if $j \in J_3$, the definition of $\dot{e}$ ensures that for every $m \in I''$ and every $m' \in I_{n(m)}$ it holds that

$$\dot{e}(x_j, d_j^0, d_j^1)(m') = e(x_j, d_j^0, d_j^1)(m').$$

Thus we have shown that

$$\dot{e}(\vec{x}, \vec{d}^0, \vec{d}^1) \restriction I'' = e(\vec{x}, \vec{d}^0, \vec{d}^1) \restriction I'' = g \restriction I''.$$

(2) There are good $d^0, d^1 \in 2^\omega$ such that no two elements of $B_0(g, d^0, d^1)$ are $<_0^g$-comparable and

(a) $B_0(g, d^0, d^1)$ is linearly ordered by $<_1^g$; or
(b) no two elements of $B_0(g, d^0, d^1)$ are comparable with respect to $<_1^g$.

If $B_0(g, d^0, d^1)$ is infinite, then so is $B(g, d^0, d^1)$ by Claim 2.5, so we get by definition that

$$\dot{e}(\chi(g), d^0, d^1) \restriction B(g, d^0, d^1) = g \restriction B(g, d^0, d^1).$$

If on the other hand $B_0(g, d^0, d^1)$ is finite, the definition of $B_0(g, d^0, d^1)$ implies that

$$(\exists^\infty m \in \omega)\, e(\chi(g), d^0, d^1)(m) = g(m).$$

For every such m it holds by the definition of $\dot{e}$ that

$$\dot{e}(\chi(g), d^0, d^1)(m) = e(\chi(g), d^0, d^1)(m).$$

Since in both cases (1) and (2) we got an element of $\mathcal{G}$, agreeing with g on an infinite set, the proof is complete. $\square$

Next we move to the proof of cofinitariness, which is a bit more involved. The proof described below closely follows the proof of Proposition 1.14 from [Sch21].

Proposition 2.10. *$\mathcal{G}$ is cofinitary.*

Proof. Let
$$c = (c_l)^{i_l} \circ \cdots \circ (c_0)^{i_0},$$
for $\{c_j\}_{j \in l+1} \subseteq \mathrm{range}(\dot{e})$ and $\{i_j\}_{j \in l+1} \subseteq \{-1, 1\}$, be some reduced word in $\mathrm{range}(\dot{e})$, which has infinitely many fixed points. For $j \in l+1$ let $x_j \in 2^\omega$ and $d_j^0, d_j^1 \in 2^\omega$ be such that $\dot{e}(x_j, d_j^0, d_j^1) = c_j$, and put $g_j := \chi^\dagger(x_j) \in (\omega)^{\leq \omega}$. Note that
$$w := (x_l, d_l^0, d_l^1)^{i_l} \cdots (x_0, d_0^0, d_0^1)^{i_0}$$
is a reduced word in $\mathbb{F}(2^\omega \times 2^\omega \times 2^\omega)$.

Let $F \subseteq \mathrm{fix}(c)$ be a tail segment, for which it holds that for every $m \in F$ and every m' in the path of m under c, m' lies in the same interval I_k with at most one $B(g_j, d_j^0, d_j^1)$, for which $d_j^0 \restriction (\min I_{k+1}), d_j^1 \restriction (\min I_{k+1})$ are good. This is possible by Claim 2.3.

For every $m \in F$ there is some $l(m) \in \omega$, so that
$$c(m) = (a_{l(m)}^m \circ \cdots \circ a_0^m)(m),$$
where for $k \in l(m)+1$, each a_k^m is either $(g_k^m)^{i_k^m}$ or $e(x_k^m, (d_k^m)^0, (d_k^m)^1)^{i_k^m}$, where $i_k^m \in \{-1, 1\}$ and $g_k^m = g_j$ or $(x_k^m, (d_k^m)^0, (d_k^m)^1) = (x_j, d_j^0, d_j^1)$ for some $j \in l+1$. This holds by unfolding the definition of $\dot{e}$ (and is left to the reader). Write also
$$w^m := a_{l(m)}^m \cdots a_0^m.$$
Note that $l(m) \leq 2l+2$ by definition of $\dot{e}$. We can write F as a finite union of sets, on each of which $w^{(\cdot)}$ is constant. Let F^* be one of these sets, which is infinite. We replace every superscript m with $*$, so that we have $l^* = l(m)$, $w^* = w^m$ and $a_k^* = a_k^m$, where a_k^* is now either $(g_k^*)^{i_k^*}$ or $e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{i_k^*}$, for all $k \in l^*+1$.

Claim 2.11. *The word w^* reduces to the empty word in $\mathbb{F}(\{(a_k^*)^{i_k^*} \mid k \in l^*+1\})$.*

We consider elements $(a_k^*)^{i_k^*}$, which are either g_k^* or $e(x_k^*, (d_k^*)^0, (d_k^*)^1)$, as abstract generators in the above statement (we used $(a_k^*)^{i_k^*}$ and not a_k^*, so that the abstract generators g_k^* or $e(x_k^*, (d_k^*)^0, (d_k^*)^1)$ are without powers).

Proof of Claim. Suppose for contradiction that when seen as an abstract element of $\mathbb{F}(\{(a_k^*)^{i_k^*} \mid k \in l^*+1\})$, the word w^* reduces to $v \neq \emptyset$. Then there is some $r \in \omega$ and a sequence $k(0), k(1), \ldots, k(r)$, for which
$$v = a_{k(r)}^* \cdots a_{k(0)}^*.$$
Define for $m \in F^*$ the sequence $(m(u))_{u=0}^r$ inductively by $m(0) := m$, and for $u < r$ by
$$m(u+1) := a_{k(u)}^*(m(u)).$$
We can assume without loss of generality that for any proper subword
$$a_{k(r_1)}^* \cdots a_{k(r_0)}^*$$
of v, where $0 \leq r_0 < r_1 \leq r$, it holds that $m(r_0) \neq m(r_1)$. Indeed, if this is not already the case, we use the Pigeonhole principle to find an infinite subset F' of F^* and a subword
$$v' := a_{k(r_1)}^* \cdots a_{k(r_0)}^*$$
of w, so that for every $m \in F'$ it holds that $m(r_0) = m(r_1)$ and so that for any proper subword
$$v'' := a_{k(r_3)}^* \cdots a_{k(r_2)}^*$$
of v', with $r_0 \leq r_2 < r_3 \leq r_1$ it holds that $m(r_2) \neq m(r_3)$.

Define
$$n(m) := \min\{n \in \omega \mid (\exists u \in r+1)\, m(u) \in I_n\},$$
i.e., $n(m)$ is the least index of an interval which we pass through with $(m(u))_{u=0}^r$. Note that this is slightly different from how we previously defined $n(m)$.

Subclaim 2.12. *For all $u \in r+1$ it holds that $m(u) \in I_{n(m)}$.*

Proof of Subclaim. Suppose not. Then at least one of the following happens:

(1) there is $0 < u_0 < r$ so that $m(u_0 - 1) \notin I_{n(m)}$ and $m(u_0) \in I_{n(m)}$; and $u_0 < u_1 < r$ so that $m(u_1) \in I_{n(m)}$ and $m(u_1 + 1) \notin I_{n(m)}$; or

(2) there is $0 < u_0 < r$ so that $m(u_0 - 1) \in I_{n(m)}$ and $m(u_0) \notin I_{n(m)}$; and $u_0 < u_1 < r$ so that $m(u_1) \notin I_{n(m)}$ and $m(u_1 + 1) \in I_{n(m)}$.

We can assume without loss of generality that case (1) happens. Note that it must hold that $a^*_{k(u_0-1)}$ is equal to $(g^*_j)^{-1}$ for some $j \in l+1$ and $a^*_{k(u_1)}$ is equal to g^*_j. This is by property (iii) of ϑ_g, the definition of $\dot{e}$, the fact that e stays inside the intervals and the definition of F. But since $B(g^*_j, (d^*_j)^0, (d^*_j)^1)$ intersects $I_{n(m)}$ in exactly one point, it must be that $m(u_0 - 1) = m(u_1 + 1)$. This contradicts our assumption on v. ⊣

Subclaim 2.13. *For at most one $u \in r+1$ is it the case that $a^*_{k(u)}$ equals g^*_j or $(g^*_j)^{-1}$ for some $j \in l+1$.*

Proof of Subclaim. This holds by the definition of F (which used Claim 2.3), by the fact that $B(g^*_j, (d^*_j)^0, (d^*_j)^1)$ intersects each I_n in at most one point, the definition of $\dot{e}$ and by the assumption that v has no subwords with infinitely many fixed points on the paths starting with elements of F^*. We leave it to the reader to provide the details. ⊣

Subclaim 2.14. *It cannot be the case that any $a^*_{k(u)}$ equals g^*_j or $(g^*_j)^{-1}$ for any $j \in l+1$.*

Proof of Subclaim. By the previous subclaim we know that there is at most one $a^*_{k(u)}$ which equals g^*_j or $(g^*_j)^{-1}$. For contradiction suppose that v is of the form (the argument with $(g^*_j)^{-1}$ is analogous):

$$e\Big(x^*_{k(r)}, (d^*_{k(r)})^0, (d^*_{k(r)})^1\Big)^{i^*_{k(r)}} \cdots \cdots e\Big(x^*_{k(\bar r+1)}, (d^*_{k(\bar r+1)})^0, (d^*_{k(\bar r+1)})^1\Big)^{i^*_{k(\bar r+1)}} g^*_j \, e\Big(x^*_{k(\bar r-1)}, (d^*_{k(\bar r-1)})^0, (d^*_{k(\bar r-1)})^1\Big)^{i^*_{k(\bar r-1)}} \cdots \cdots e\Big(x^*_{k(0)}, (d^*_{k(0)})^0, (d^*_{k(0)})^1\Big)^{i^*_{k(0)}}$$

for some $\bar r \in r+1$. But then it holds for infinitely many $n \in \omega$ that

$$(2) \quad g^*_j(n) = \Big(e\Big(x^*_{k(\bar r+1)}, (d^*_{k(\bar r+1)})^0, (d^*_{k(\bar r+1)})^1\Big)^{-i^*_{k(\bar r+1)}} \cdots \cdots e\Big(x^*_{k(r)}, (d^*_{k(r)})^0, (d^*_{k(r)})^1\Big)^{-i^*_{k(r)}} e\Big(x^*_{k(0)}, (d^*_{k(0)})^0, (d^*_{k(0)})^1\Big)^{-i^*_{k(0)}} \cdots \cdots e\Big(x^*_{k(\bar r-1)}, (d^*_{k(\bar r-1)})^0, (d^*_{k(\bar r-1)})^1\Big)^{-i^*_{k(\bar r-1)}}\Big)(n).$$

Let I be the infinite set of all such $n \in \omega$. Since also

$$g^*_j \restriction I = \dot{e}(\chi(g^*_j), (d^*_j)^0, (d^*_j)^1) \restriction I,$$

(this is the only way in which $a^*_{k(u)}$ was able to be g^*_j), the definition of $\dot{e}$ (see the first case) implies that elements of I are pairwise $<_0^{g^*_j}$-incomparable. On the other hand, equation (2) implies that I forms a $<_0^{g^*_j}$-increasing chain. This is a contradiction. ⊣

Subclaim 2.15. *It is the case that $v = \emptyset$.*

Proof of Subclaim. By the previous subclaim it holds that

$$v = e\Big(x^*_{k(r)}, (d^*_{k(r)})^0, (d^*_{k(r)})^1\Big)^{i^*_{k(r)}} \cdots e\Big(x^*_{k(0)}, (d^*_{k(0)})^0, (d^*_{k(0)})^1\Big)^{i^*_{k(0)}}.$$

But since v has infinitely many fixed points and the range of e generates a cofinitary subgroup it must hold that $v = \emptyset$. ⊣

This is a contradiction with our assumption that $v \neq \emptyset$, and with this the proof of Claim 2.11 is complete. ⊣

Claim 2.16. *It must be the case that w^* is the empty word.*

Proof of Claim. Suppose for contradiction that w^* is not the empty word. By considering all possible cases we will conclude that this is a contradiction. Suppose first that w^* contains a subword of the form

$$(g_k^*)^{-1}\ g_k^*$$

for some $k \in l^* + 1$. By definition of $\dot{e}$, this subword can only arise from a subword of w (recall that w is reduced), in two ways. The first option is that $(g_k^*)^{-1}\ g_k^*$ arose from

$$(x_k^*, (d_k^*)^0, (d_k^*)^1)\ (x_k^*, (d_k^*)^0, (d_k^*)^1),$$

and where the corresponding right $\dot{e}(x_k^*, (d_k^*)^0, (d_k^*)^1)$ was substituted by g_k^* and the left $\dot{e}(x_k^*, (d_k^*)^0, (d_k^*)^1)$ by $e(x_k^*, (d_k^*)^0, (d_k^*)^1)\ (g_k^*)^{-1}$, so that we actually have the following subword of w^*

$$e(x_k^*, (d_k^*)^0, (d_k^*)^1)\ (g_k^*)^{-1}\ g_k^*.$$

But as we have observed that w^* reduces to $\emptyset$, the occurrence of $e(x_k^*, (d_k^*)^0, (d_k^*)^1)$ must cancel out, so actually, there must be a subword of w^* of the form

$$e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}\ e(x_k^*, (d_k^*)^0, (d_k^*)^1)\ (g_k^*)^{-1}\ g_k^*$$

or of the form

$$e(x_k^*, (d_k^*)^0, (d_k^*)^1)\ (g_k^*)^{-1}\ g_k^*\ e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}.$$

In both cases the occurrence of $e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}$ arose by substitution of

$$(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}.$$

This is a contradiction, as this means that w contains a subword of the form

$$(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}\ (x_k^*, (d_k^*)^0, (d_k^*)^1)\ (x_k^*, (d_k^*)^0, (d_k^*)^1)$$

or of the form

$$(x_k^*, (d_k^*)^0, (d_k^*)^1)\ (x_k^*, (d_k^*)^0, (d_k^*)^1)\ (x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1},$$

implying that w is not reduced. The second option is that $(g_k^*)^{-1}\ g_k^*$ arose from

$$(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}\ (x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1},$$

and where the corresponding left $\dot{e}(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}$ was substituted by $(g_k^*)^{-1}$ and the right $\dot{e}(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}$ by $g_k^*\ e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}$, so that we actually have the following subword of w^*

$$(g_k^*)^{-1}\ g_k^*\ e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}.$$

A contradiction is then established in a similar manner as in the first case.

Next, suppose that w^* contains a subword of the form

$$e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}\ e(x_k^*, (d_k^*)^0, (d_k^*)^1).$$

The only way this subword can appear in w^* is (by applying the definition of $\dot{e}$) that $(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}\ (x_k^*, (d_k^*)^0, (d_k^*)^1)$ was a subword of w. This is clearly a contradiction.

By a similar argument, one can easily prove that w^* cannot have subwords of the form $g_k^*\ (g_k^*)^{-1}$ or $e(x_k^*, (d_k^*)^0, (d_k^*)^1)\ e(x_k^*, (d_k^*)^0, (d_k^*)^1)^{-1}$, hence the proof is complete. ⊣

Claim 2.16 implies that $w = \emptyset$ and so $c = \mathrm{id}_\omega$. Since c was an arbitrary element of $\mathcal{G}$ with infinitely many fixed points, the proof of Proposition 2.10 is complete. □

Before we prove the next proposition, we introduce the following useful notion. For $\bar{f} \in (\omega)^{<\omega}$, we say that $\bar{f}$ is of *interval length k*, if $\mathrm{lh}(\bar{f}) = \sum_{m \leq k} |I_m|$. For $\bar{f}$ of interval length k, we say that $(\bar{x}, \bar{d}^0, \bar{d}^1) \in 2^k \times 2^k \times 2^k$ is *recovered from $\bar{f}$*, if for every $m \leq k$ it holds that

$$|\{n \in I_m \,|\, e(\bar{x}, \bar{d}^0, \bar{d}^1)(n) \neq \bar{f}(n)\}| \leq 3.$$

Next, for $\bar{x} \in 2^{<\omega}$ and $\bar{g} \in (\omega)^{\leq \mathrm{lh}(\bar{x})}$, we say that $\bar{g}$ is *$\bar{x}$-compatible*, if $\chi^\dagger(\bar{x}) \sqsubseteq \bar{g}$ and

(I) if $\bar{x}$ is of the form

$$(\dots, 1, \underbrace{0, 0, \dots, 0}_{n}, 1, \dots, 1, \underbrace{0, 0, \dots, 0}_{n}, 1, \dots)$$

for some $n \in \omega$ (and is thus not in range(χ) as it violates injectivity), then

$$\bar{g} = \chi^\dagger(\bar{x});$$

(II) if there is some $n \in \omega$ such that

$$\bar{x} = \chi(\chi^\dagger(\bar{x}))^\frown(\underbrace{0, \dots, 0}_{n}),$$

i.e., $\bar{x}$ is not in range(χ) because of the second reason described below the definition of $\chi^\dagger$, namely that it does not end with a 1, then in case $\mathrm{lh}(\bar{g}) > \mathrm{lh}(\chi^\dagger(\bar{x}))$, it must hold that

$$\bar{g}(\mathrm{lh}(\chi^\dagger(\bar{x}))) \geq n;$$

(III) otherwise (when $\bar{x} \in \mathrm{range}(\chi)$) we impose no further requirements.

Finally, for $\bar{f}$ of interval length k and $(\bar{x}, \bar{d}^0, \bar{d}^1)$ recovered from $\bar{f}$ we define when $\bar{g} \in (\omega)^{\leq \mathrm{lh}(\bar{x})}$ is *$(\bar{f}, \bar{x}, \bar{d}^0, \bar{d}^1)$-matching*. For $n < k$ define temporarily

$$\begin{aligned} \varphi(n) \;:\Longleftrightarrow\;& \text{no two } n_0, n_1 \in B_0(\bar{g} \restriction (n+1), \bar{d}^0 \restriction (n+1), \bar{d}^1 \restriction (n+1)) \cap n \\ & \text{are comparable w.r.t. } <_0^{\bar{g}}, \bar{d}^0 \restriction (n+1), \bar{d}^1 \restriction (n+1) \text{ are good and} \\ & n \in B(\bar{g} \restriction (n+1), \bar{d}^0 \restriction (n+1), \bar{d}^1 \restriction (n+1)). \end{aligned}$$

Then $\bar{g} \in (\omega)^{\leq \mathrm{lh}(\bar{x})}$ is *$(\bar{f}, \bar{x}, \bar{d}^0, \bar{d}^1)$-matching*, if $\bar{g}$ is $\bar{x}$-compatible and for every $n < k$ it holds that

(i) if $\varphi(n)$, then $n \in \mathrm{dom}(\bar{g})$ and $\bar{f}(n) = \bar{g}(n)$;
(ii) if $k := \bar{g}^{-1}(n)$ is defined, $k \in \mathrm{dom}(\bar{f})$ and $\varphi(k)$, then $\bar{f}(n) = e(\bar{x}, \bar{d}^0, \bar{d}^1)(k)$;
(iii) if $k := (\bar{g}^{-1} \circ e(\bar{x}, \bar{d}^0, \bar{d}^1))(n)$ is defined, $k \in \mathrm{dom}(\bar{f})$ and $\varphi(k)$, then $\bar{f}(n) = e(\bar{x}, \bar{d}^0, \bar{d}^1)^2(n)$;
(iv) if none of the "if" parts of the "if ... then" statements from (i), (ii) and (iii) are true, then $\bar{f}(n) = e(\bar{x}, \bar{d}^0, \bar{d}^1)(n)$.

Of course, the idea behind the definition of $(\bar{f}, \bar{x}, \bar{d}^0, \bar{d}^1)$-matching is that it captures the requirements imposed by the definition of $\dot{e}$. This will be made more precise in Proposition 2.17.

With this we are ready to establish that the set of generators of the constructed group is definable.

Proposition 2.17. *range$(\dot{e})$ is a Π^0_1 subset of ω^ω.*

Proof. We will define a Δ^0_1 set $U \subseteq (\omega)^{<\omega}$, so that

$$\mathrm{range}(\dot{e}) = \Big\{ f \in \omega^\omega \,\Big|\, (\forall k \in \omega)\, f \restriction \Big(\sum_{m \leq k} |I_m|\Big) \in U \Big\}. \tag{3}$$

Then clearly range($\dot{e}$) will be Π^0_1. For $\bar{f} \in (\omega)^{<\omega}$ of interval length k, we let

$$\begin{aligned}\bar{f} \in U \;:\Longleftrightarrow\; & (\exists(\bar{x}, \bar{d^0}, \bar{d^1}) \in 2^k \times 2^k \times 2^k) \\ & (\bar{x}, \bar{d^0}, \bar{d^1}) \text{ is recovered from } \bar{f} \wedge \\ & (\exists \bar{g} \in (\omega)^{\leq \mathrm{lh}(\bar{x})})\, \bar{g} \text{ is } (\bar{f}, \bar{x}, \bar{d^0}, \bar{d^1})\text{-matching.}\end{aligned}$$

Note that even though we are using two existential quantifiers in the above definition, U is Δ^0_1, as we can verify whether there are witnesses to the existential quantifiers in finitely many steps. We leave the details to the reader.

We now verify that (3) holds. For $k \in \omega$, let $l(k) := \sum_{m \leq k} |I_m|$. Suppose first that $f \in \mathrm{range}(\dot{e})$. Then there are $(x, d^0, d^1) \in 2^\omega \times 2^\omega \times 2^\omega$ for which $f = \dot{e}(x, d^0, d^1)$. Clearly, for each k it holds that $(x \upharpoonright k, d^0 \upharpoonright k, d^1 \upharpoonright k)$ is recovered from $f \upharpoonright l(k)$. Let also $g := \chi^\dagger(x)$. We consider the following two cases:

(1) If $g \in (\omega)^\omega$, then clearly each $g \upharpoonright k$ is $(f \upharpoonright l(k), x \upharpoonright k, d^0 \upharpoonright k, d^1 \upharpoonright k)$-matching. In particular, for every $k \in \omega$ it holds that $f \upharpoonright l(k) \in U$.

(2) If $g \in (\omega)^{<\omega}$, then for every k with $k \leq \mathrm{lh}(g)$ let $g_k := g \upharpoonright k$ and for every k with $k > \mathrm{lh}(g)$ define $g_k := g$. Then each g_k is clearly $(x \upharpoonright k)$-compatible and $(f \upharpoonright l(k), x \upharpoonright k, d^0 \upharpoonright k, d^1 \upharpoonright k)$-matching. We have thus established that for every $k \in \omega$, $f \upharpoonright l(k) \in U$.

Conversely, assume that for $f \in \omega^\omega$ it holds that $(\forall k \in \omega)\, f \upharpoonright l(k) \in U$. For each $k \in \omega$, let $(\bar{x}_k, \bar{d^0_k}, \bar{d^1_k})$ be recovered from $f \upharpoonright l(k)$ and let $\bar{g}_k$ be $(f \upharpoonright l(k), \bar{x}_k, \bar{d^0_k}, \bar{d^1_k})$-matching. Clearly, for every $k < k'$ it must hold that $\bar{x}_k \sqsubset \bar{x}_{k'}$ and $\bar{d^j_k} \sqsubset \bar{d^j_{k'}}$ for $j \in 2$. Let $x \in 2^\omega$ be unique such that $x \upharpoonright k = \bar{x}_k$ and let d^j be unique such that $d^j \upharpoonright k = \bar{d^j_k}$. On the other hand, it is not necessarily the case that the $(\bar{g}_k)_k$ line up. Nevertheless, setting $g := \chi^\dagger(x)$, it holds that whenever $\bar{g}_k(m)$ is used in cases (i) to (iv) in the definition of matching, we have that $m \in \mathrm{dom}(g)$ and $\bar{g}_k(m) = g(m)$. To see this, note that if $\bar{g}_k(m)$ is used in cases (i) to (iv) in the definition of matching, then for every $k' > k$ it must hold that $\bar{g}_{k'}(m) = \bar{g}_k(m)$, so if $\bar{g}_k(m) \neq g(m)$, there would be some $k' > k$ for which $\bar{g}_{k'}$ is not $(f \upharpoonright l(k'), \bar{x}_{k'}, \bar{d^0_{k'}}, \bar{d^1_{k'}})$-matching. Hence we may assume that $\bar{g}_k = g \upharpoonright k$ for every $k \in \omega$. The reader can routinely verify that indeed, the conditions imposed on x, d^0, d^1 and g guarantee that $\dot{e}(x, d^0, d^1) = f$. Note that when $g \in (\omega)^{<\omega}$, the conditions (i) to (iv) from the definition of matching make sure that $f(n) = e(x, d^0, d^1)(n)$ for all $n \geq m$ for an appropriate m (in the same way as in the definition of $\dot{e}$). □

Proposition 2.18. *The group $\mathcal{G}$ is Σ^0_2 and is freely generated by* range($\dot{e}$).

Proof. The proof of Proposition 2.10 shows that $\mathcal{G} = \langle \mathrm{range}(\dot{e}) \rangle$ is freely generated.

For $h \in S_\infty$, h is in $\mathcal{G}$ precisely when there is a word $w(x_0, \dots, x_k)$, and some $g_0, \dots, g_k \in \mathrm{range}(\dot{e})$ so that

$$h = w(g_0, \dots, g_k).$$

The idea is to use one existential quantifier to guess the word $w(x_0, \dots, x_k)$ (note that there are countably many words) and then use δ and the argument from the previous proposition to recover each g_i for $i \in k+1$ in a Π^0_1 way so that $h = w(g_0, \dots, g_k)$. Together, the complexity of membership of $\mathcal{G}$ is Σ^0_2. We leave the details to the reader. □

This was the last ingredient needed to complete the proof of Theorem 2.8.

3. Limitations to the construction of a G_δ mcg

Since S_∞ is G_δ in ω^ω, the notions of being G_δ in S_∞ and in ω^ω are the same. Moreover, by [Kec12, Exercise 9.6], for $G \leq S_\infty$ being closed in S_∞ is equivalent to being G_δ. From now on, we will use the expression "G_δ mcg" for $G \leq S_\infty$, which are G_δ (in either of the spaces) and "closed mcg" for $G \leq S_\infty$, which is closed as a subset of ω^ω.

Since the construction of the previous section produces a freely generated mcg, the following result of Dudley (see [Dud61] for the original paper and Rosendal's [Ros09] for an updated overview) implies that among freely generated mcgs our construction achieves the best possible complexity.

Theorem 3.1 (Dudley)**.** *There is no Polish topology on the free group with continuum many generators.*

Actually, the work of Slutsky (see Theorem 1.6 of [Slu12]) gives an even stronger statement.

Theorem 3.2 (Slutsky)**.** *The only Polish group topology on any free product $G * H$ is the discrete topology (in which case, of course, $G * H$ must be countable).*

This means that if there is a G_δ mcg G, all its elements are "non-free" from the rest of the group, i.e., for any $g \in G$, there is no $H \leq G$ for which $G = \langle g \rangle * H$. Hence any construction of a G_δ mcg would have to ensure that there are non-trivial relations between its elements. Achieving this while also making sure that the group is cofinitary seems fairly difficult, but perhaps not impossible. Note that on the other hand it is possible to adapt the techniques in this paper to produce mcgs which are isomorphic to

$$\mathop{\ast}_{x \in \omega^\omega} \mathbb{Z}/2\mathbb{Z},$$

that is, the free product of continuum many copies of the two-element group, $\mathbb{Z}/2\mathbb{Z}$. Nevertheless, all currently known constructions of definable mcgs produce groups which decompose into free products.

The authors thank Asger Törnquist for directing our attention to Dudley's result, through which we discovered Slutsky's work. Previously, we developed a method of *definable gluing of orbits*. The idea was the following. It is not hard to prove that there can be no G_δ cofinitary group G which has exactly one k-orbit for all $k \geq 1$ (here we count as k-orbits of G the orbits under the diagonal action of G on $(\omega)^k$). We developed a method of gluing k-orbits (for all $k \geq 1$) while preserving a certain notion of definability and while also preserving cofinitariness. Then we proved that for an mcg, which has a nice catching function (this is a natural condition based on the known constructions of mcgs, and which implies that the group is G_δ) and which has a small degree of freeness, we can define from it a G_δ mcg with just one k-orbit for all $k \geq 1$, resulting in a contradiction. We do not describe this idea in further detail, as Theorem 3.2 implies a stronger statement.

4. Open problems

The most important open question is the longstanding:

Question 4.1. *Is there a G_δ mcg?*

If one is convinced that the answer to Question 4.1 is negative, then a great step forward is to first show that the answer to the next open question is negative.

Question 4.2. *Is there a closed (in the Baire space) mcg?*

Note that in [Kas09], Kastermans proved that every mcg has only finitely many orbits. For $k \geq 1$ and $G \leq S_\infty$, the *k-orbits* of G are the orbits of the diagonal action of G on $(\omega)^k$. The usual orbits are hence 1-orbits. It is unknown whether the following is possible (this was already asked in [Kas09]).

Question 4.3. *Can there be an mcg with infinitely many k-orbits for some $k > 1$?*

Call $x \in (\omega \cup \{\omega\})^\omega$ *increasing* if for every $n \in \omega$ it holds that $x(n+1) \geq x(n)$. For an mcg G, define $x_G \in \omega^\omega$ by setting $x_G(0) := 0$ and for $n \geq 1$ that

$$x_G(n) := \text{number of } n\text{-orbits of } G.$$

Question 4.4. *Which increasing $x \in (\omega \cup \{\omega\})^\omega$ arise as x_G for some mcg G?*

4.1. **Maximal finitely periodic groups.** Say that $g \in S_\infty \setminus \{\mathrm{id}_\omega\}$ is *finitely periodic* if $\langle g \rangle \subseteq S_\infty$ has finitely many finite orbits. A subgroup $G \leq S_\infty$ is then called *finitely periodic*, if every $g \in G \setminus \{\mathrm{id}_\omega\}$ is finitely periodic. Clearly, every finite periodic group is also cofinitary. We use the abbreviation *mpg* to refer to maximal finitely periodic groups. Of course, the first question asked by any descriptive-set-theorist is the following.

Question 4.5. *Is there a Borel mpg?*

Note that our construction of a Σ^0_2 mcg cannot be immediately adapted to this setting, as the construction relies on the sequence of finite intervals $(I_n)_{n\in\omega}$. In particular, for any $x \in 2^\omega \setminus \mathrm{range}(\chi)$ and any $d^0, d^1 \in 2^\omega$, we have that $\dot{e}(x, d^0, d^1)$ is not finitely periodic. Nevertheless, the authors believe that the answer to the question is positive.

We next present a slight variant of a notion appearing in Kastermans' [Kas08] and [Kas09]. For $G \leq S_\infty$ and $A \subseteq G$ with $\mathrm{id}_\omega \in A$ (for convenience), we write $W_A(x)$ for the set of all finite words in the variable x and elements of A. Formally,

$$W_A(x) = \Big\{ g_k x^{i_k} g_{k-1} x^{i_{k-1}} \cdots x^{i_0} g_0 \,\Big|\, g_0, \dots, g_k \in A, i_0, \dots, i_k \in \mathbb{Z} \setminus \{0\} \Big\}.$$

For $h \in S_\infty$ and $w(x) \in W_A(x)$, we let $w(h)$ be the permutation obtained by substituting all occurrences of x by h.

Theorem 4.6. *There is no mpg with infinitely many orbits.*

Proof. The proof is based on the proof of Kastermans' Theorem 13 from [Kas09]. Due to the similarity, we omit some details.

Let $(O_i)_{i\in\omega}$ be some fixed enumeration of all orbits of a finitely periodic group G. We define $h \in S_\infty$ inductively through increasingly larger finite approximations. Let $h_0 := \emptyset$ and suppose we have defined a finite partial injective function h_k for some $k \in \omega$. Then let

$$n := \min \Big((\omega \setminus \mathrm{dom}(h_k)) \cup (\omega \setminus \mathrm{range}(h_k)) \Big),$$

j be the least such that $O_j \cap (\mathrm{dom}(h_k) \cup \mathrm{range}(h_k) \cup \{n\}) = \emptyset$ and $m := \min O_j$. If $n \notin \mathrm{dom}(h_k)$, set $h_{k+1} := h_k \cup \{(n, m)\}$, otherwise set $h_{k+1} := h_k \cup \{(m, n)\}$. By definition,

$$h := \cup \{ h_k \,|\, k \in \omega \} \in S_\infty$$

and since the orbit structure of $\langle G \cup \{h\} \rangle$ is different from the orbit structure of G, it follows that $h \notin G$.

Take any reduced $w(x) \in W_G \setminus \{\emptyset\}$; we will show that $w(h)$ is finitely periodic. Note that when we reduce $w(x)^2$, strictly less than half of each instance of $w(x)$ gets annihilated (as otherwise $w(x) = \emptyset$). We consider the following cases:

1. $w(x)$ is of odd length $2l + 1$, where $g \in G$ is at the $(l+1)$-th place and when reducing $w(x)^2$, we annihilate l characters from each instance of the word, so that now there is g^2 at the $(l+1)$-th place in the reduced $w(x)^2$;
2. same as case (1), but with x^i (where $i \in \mathbb{Z} \setminus \{0\}$) in place of g;
3. when we reduce $w(x)^2$, the resulting word is strictly longer than $w(x)$.

In order so that we do not have to separately consider all three different cases, we observe that the following captures all of them: there is a finite set $A \subseteq G$ and $g \in G$ so that for every $k \in \omega$, the reduced form of $w(x)^k$ is equal to some word $w_k(x) \in W_{A \cup \{g^k\}}$.

Suppose that $n \in \omega$ is in a finite orbit of $\langle w(h)\rangle$. Then there is some k_n, so that n is a fixed point of $w_{k_n}(h)$. Using the argument of Theorem 13 from [Kas09], there is some $g_n \in A \cup \{g\}$, so that if $g_n \in A$ then it has a fixed point m_n, and if $g_n = g$ then m_n is a fixed point of g^{k_n}. Similarly as in [Kas09], we observe that for n_0, n_1 which belong to different finite orbits of $\langle w(h)\rangle$, either $g_{n_0} \neq g_{n_1}$, or $g_{n_0} = g_{n_1}$ and $m_{n_0} \neq m_{n_1}$ (to be precise, like in [Kas09], we need to consider each g_{n_i} together with its position in $w_{k_{n_i}}(x)$). By the Pigeonhole principle, if there are infinitely many finite orbits of $\langle w(h)\rangle$, there must either be some $f \in A$ which has infinitely many fixed points or g has infinitely many finite orbits. This contradicts the assumption on G. Since $w(x)$ was arbitrary, we have proved that $G \cup \{h\}$ generates a finitely periodic group. In particular, G is not maximal. $\square$

Recall, that a set $A \subseteq \omega^\omega$ is *eventually bounded* if there is some $x \in \omega^\omega$, so that for every $y \in A$ there is some $n \in \omega$ such that for every $m > n$ it holds that $y(m) < x(m)$. Kastermans proved in [Kas08] (see Theorem 10) that no mcg can be eventually bounded.

Question 4.7. *Can an mpg be eventually bounded?*

At first sight it seems likely that an adaptation of the proof of Theorem 10 from [Kas08], similar to the way we adapted Kastermans' argument in the proof of Theorem 4.6, would be successful in proving that no mpg can be eventually bounded. However, it turns out that the straightforward modification does not work this time. The authors nevertheless believe the answer should be negative.

If G is an mcg, which is finitely periodic, it is clearly an mpg. The authors believe however that it is possible to find an example, affirmatively answering the following question.

Question 4.8. *Is there an mpg, which is not an mcg?*

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Part IV.

Maximal orthogonal families

ORTHOGONALITY OF MEASURES AND STATES

SEVERIN MEJAK

Abstract. We give a short proof of the theorem due to Preiss and Rataj stating that there are no analytic maximal orthogonal families (mofs) of Borel probability measures on a Polish space. When the underlying space is compact and perfect, we show that the set of witnesses to non-maximality is comeagre. Our argument is based on the original proof by Preiss and Rataj, but with significant simplifications. The proof generalises to show that under $\mathsf{MA} + \neg\mathsf{CH}$ there are no $\boldsymbol{\Sigma}^1_2$ mofs, that under PD there are no projective mofs and that under AD there are no mofs at all. We also generalise a result due to Kechris and Sofronidis, stating that for every analytic orthogonal family of Borel probability measures there is a product measure orthogonal to all measures in the family, to states on a certain class of C*-algebras.

Introduction

In this paper we consider orthogonality, first for Borel probability measures on Polish spaces and then for states on separable C*-algebras. In 1985 Preiss and Rataj proved the following theorem with $X = [0, 1]$, see [PR85].

Theorem 0.1. *Suppose that X is an uncountable Polish space. Then there is no analytic maximal orthogonal family of Borel probability measures on X.*

This answered an open question from [MPVW82]. The idea of the proof from [PR85] is to use a Baire category argument. However, once the authors prepared the scene for the application of the Baire category theorem, they resorted to a couple of technical lemmas, which relied on restricting Borel probability measures on $[0, 1]$ to finite unions of closed subintervals. For the proof of one of the lemmas they also employed Banach–Mazur games. Consequently, the question whether there is a shorter and simpler proof remained open.

In 1999, Kechris and Sofronidis (see Thoerem 3.1 in [KS01]) found an alternative short proof which uses the theory of *turbulence* (see [Hjo00] for a great introduction to turbulence). As part of their proof, they defined an embedding of the Cantor space $2^{\mathbb{N}}$ into the space of Borel probability measures (using the work of Kakutani from [Kak48]), assigning to every $x \in 2^{\mathbb{N}}$ a product measure $\mu_{\alpha(x)}$. They proved that for every analytic orthogonal family, there is some $x \in 2^{\mathbb{N}}$ so that $\mu_{\alpha(x)}$ is a witness to non-maximality. Their proof has as a consequence that the relation $\sim$ of measure equivalence between Borel probability measures is not classifiable by countable structures.

Almost two decades later Schrittesser and Törnquist used the same embedding of $2^{\mathbb{N}}$ into the space of measures to prove (see Theorem 5.5 of [ST18]) that an argument using a weaker form of turbulence suffices to prove Theorem 0.1. Since the theory of turbulence requires some background knowledge, one might argue that even thought the proofs from [KS01] and [ST18] are *shorter*, they are not necessarily *simpler*.

In this article, we first go back to the original idea of Preiss and Rataj to use a Baire category argument to prove Theorem 0.1. We were able to use the Kuratowski–Ulam

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theorem and some elementary convexity theory, to give a short and straightforward proof of Theorem 0.1. The argument works to show the following strengthening, where for $\mathcal{A} \subseteq P(X)$ (here $P(X)$ denotes the space of Borel probability measures on X), we let $\mathcal{A}^{\perp} := \{\nu \in P(X) : (\forall \mu \in \mathcal{A})\, \nu \perp \mu\}$.

Theorem 0.2. *Suppose that X is a compact perfect Polish space. Then for every analytic orthogonal family $\mathcal{A} \subseteq P(X)$, the set $\mathcal{A}^{\perp}$ is comeagre. In particular, when $\mathcal{A} \subseteq P(2^{\mathbb{N}})$ is a Σ^1_1 orthogonal family, there is a Δ^1_1-witness to non-maximality.*

Actually, under additional assumptions, our method yields the following.

Theorem 0.3. *Suppose that X is an uncountable Polish space.*

(1) Assume MA *and* $\neg$CH*. Then no $\mathbf{\Sigma^1_2}$ orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*
(2) Assume PD*. Then no projective orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*
(3) Assume AD*. Then no orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*

If moreover X is compact perfect, then in each of the above cases $\mathcal{A}^{\perp}$ is comeagre.

It is well-known that via the Riesz–Markov–Kakutani representation theorem, Borel probability measures on a compact Polish space X are precisely states on the commutative C*-algebra of complex-valued continuous functions on X. In [Dye52], Dye introduced the notion of absolute continuity for states on C*-algebras. Being a pre-order, it naturally gives rise to a notion of orthogonality. However, it turns out that this notion is ill-behaved even for states on the matrix algebra $M_2(\mathbb{C})$.

There is another natural notion of orthogonality for states, which we call strong orthogonality and denote by $\perp\!\!\!\perp$. This notion of orthogonality shares many nice properties with orthogonality of measures, with which it coincides when the C*-algebra is commutative. Hence it is natural to ask ourselves whether Theorem 0.1 holds for non-commutative separable unital C*-algebras and strong orthogonality as well.

Since the original proof by Preiss and Rataj relied on restrictions of measures to compact subspaces, it is not clear how to generalise that proof. The idea from the proof of Theorem 0.1 seems more promising, but there are still some steps for which we do not know if they hold for strong orthogonality for states.

On the other hand, it turns out that the idea of Kechris and Sofronidis from [KS01] can easily be extended to a class of separable unital C*-algebras.

Theorem 0.4. *Suppose A is a separable unital C*-algebra, which contains a copy of $C(2^{\mathbb{N}})$ as a subalgebra and for which there is a conditional expectation $E : A \to C(2^{\mathbb{N}})$. Then for every strongly orthogonal $\mathcal{A} \subseteq S(A)$ there is $\alpha \in (0,1)^{\mathbb{N}}$ so that $\tilde{\mu}_\alpha \perp\!\!\!\perp \psi$ for every $\psi \in \mathcal{A}$, where $\tilde{\mu}_\alpha$ is the extension of the state, corresponding to the product measure*

$$\prod_{n\in\mathbb{N}} (\alpha(n)\delta_0 + (1-\alpha(n))\delta_1),$$

from $C(2^{\mathbb{N}})$ to A.

As in [KS01], along the way of proving this theorem we also get that for C*-algebras A, satisfying the assumptions of the theorem, the relation $\sim$ on $S(A)$ is not classifiable by countable structures.

Natural examples of C*-algebras, for which the assumptions of Theorem 0.4 are satisfied, include the CAR algebra M_{2^∞} and the Cuntz algebra $\mathcal{O}_2$. Moreover, for any A satisfying assumptions of Theorem 0.4 also the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ (for any countable discrete group Γ and any homomorphism $\alpha : \Gamma \to \mathrm{Aut}(A)$) and the tensor product $A \otimes B$ (for any separable unital C*-algebra B) satisfy the assumptions of Theorem 0.4.

In 1969 Bures (see [Bur69]) proved an extension of Kakutani's result from [Kak48] to semi-finite von Neumann algebras. Instead of absolute continuity and orthogonality between states, Bures considered when two product states give rise to isomorphic tensor products of von Neumann algebras. This was extended to all von Neumann algebras by Promislow in [Pro71].

As a consequence of the main ingredient of the proof of Theorem 0.4, we get the following version of Kakutani's theorem for states, involving absolute continuity and strong orthogonality.

Proposition 0.5. *Suppose that* $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \in [\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}}$ *and let*

$$\phi_n := \alpha_n \operatorname{ev}_{1,1} + (1-\alpha_n) \operatorname{ev}_{2,2} \quad \textit{and} \quad \psi_n := \beta_n \operatorname{ev}_{1,1} + (1-\beta_n) \operatorname{ev}_{2,2}$$

be states on $M_2(\mathbb{C})$*. Let also* $\phi := \otimes_{n=0}^{\infty}\phi_n$ *and* $\psi := \otimes_{n=0}^{\infty}\psi_n$ *be the product states on* M_{2^∞}*. Then in* $S(M_{2^\infty})$*, either* $\phi \sim \psi$ *or* $\phi \perp\!\!\!\perp \psi$ *according to whether*

$$\sum_{n\in\mathbb{N}} (\alpha_n - \beta_n)^2$$

converges or diverges respectively.

Structure of the paper. The paper aims to target interested readers from descriptive set theory, measure theory and C*-algebras. Due to different backgrounds, we try to give as much details as possible and add references to literature containing more information about the discussed topics. Readers not familiar with set-theoretic notions such us MA or $\mathbf{\Sigma^1_2}$, can skip the parts where we consider them, with no effect to understanding the rest of the paper.

In section 1, we give proofs of Theorems 0.1, 0.2 and 0.3. This is followed by section 2, where we first present absolute continuity and two notions of orthogonality for states. Subsection 2.2 recalls the idea and some notions from [KS01] and proves Theorem 0.4 and Proposition 0.5. We conclude the paper by discussing related topics and listing some open problems in section 3.

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1. Borel probability measures

In this section we give a very short proof of the classical result due to Preiss and Rataj by simplifying some steps of their proof from [PR85]. The setup of using a Baire category argument is the same, what is new is that we replace the technical part of the proof from [PR85], which uses extensions of measures defined on subspaces with a more straightforward argument. We start by recalling some basic properties of Borel probability measures.

Let X be a Polish space. We denote by $C(X)$ the set of continuous complex-valued functions on X. With $P(X)$, we denote the collection of Borel probability measures on X, endowed with the topology generated by maps $\mu \mapsto \int f d\mu$ for f ranging over $C_b(X, \mathbb{R}) := \{f : X \to \mathbb{R} \,|\, f \text{ continuous and bounded}\}$. Recall that when X is Polish, then so is $P(X)$, and that if moreover X is compact, so is $P(X)$. See section 17.E of [Kec95] for more about $P(X)$.

For two Borel measures μ, ν on X, we denote μ being *absolutely continuous* with respect to ν (i.e., for every Borel subset $B \subseteq X$, if $\nu(B) = 0$, then $\mu(B) = 0$) by $\mu \ll \nu$. We say that $\mu, \nu \in P(X)$ are *measure equivalent*, denoted by $\mu \sim \nu$, if $\mu \ll \nu \wedge \nu \ll \mu$ and that $\mu, \nu \in P(X)$ are *orthogonal* (another term often used is *singular*), denoted by $\mu \perp \nu$, if there is no $\rho \in P(X)$ with $\rho \ll \mu$ and $\rho \ll \nu$. Observe that $\mu \perp \nu$ is equivalent to existence of a Borel $B \subseteq X$ with $\mu(B) = 1$ and $\nu(B) = 0$. Recall (see e.g. [KS01]) that $\ll$ satisfies the *ccc-below property*, i.e., for every $\mu \in P(X)$ there is no uncountable family $\{\nu_i : i \in I\} \subseteq P(X)$ with the property that for $i \neq j \in I$ it holds that $\nu_i \perp \nu_j$ and $\nu_i \ll \mu$.

For a signed Borel measure σ on X,

$$||\sigma|| := \sup\{|\sigma(B)| : B \subseteq X \text{ Borel}\}$$

defines a norm on the space of signed Borel measures. Then the map $P(X) \times P(X) \to \mathbb{R}_{\geq 0}$, defined by $(\mu, \mu) \mapsto ||\mu - \nu||$ is lower semicontinuous, when $P(X)$ is equipped with the Polish topology defined above. Consequently, for $\varepsilon \in (0, 1)$ the set

$$\{(\mu, \nu) \in P(X) \times P(X) : ||\mu - \nu|| < \varepsilon\}$$

is F_σ (i.e., it is a countable union of closed sets). It is also immediate to see that for $\mu, \nu \in P(X)$ we have

$$\mu \perp \nu \text{ if and only if } ||\mu - \nu|| = 1,$$

so that the relation $\mu \perp \nu$ is G_δ (i.e., a countable intersection of open sets). We continue with a lemma from convexity theory.

Lemma 1.1. *Suppose that V is an open convex subset of a locally convex topological vector space E. Then the map $V \times V \times [0, 1] \to V$, defined by $(x, y, t) \mapsto tx + (1 - t)y$ is continuous and open.*

Proof. Continuity holds because E is a topological vector space. To check that the map is open, take $U_0, U_1 \subseteq V$ convex open, $O \subseteq [0, 1]$ convex open and x, y, t in U_0, U_1, O respectively. Then let

$$U_2 := ((1 - t)(y - x) + U_0) \cap (t(x - y) + U_1),$$

which is convex open and contains $tx + (1 - t)y$. Clearly U_2 is contained in the image of $U_0 \times U_1 \times O$, hence this completes the proof that the map is open. □

We are now ready to prove Theorem 0.1, which we restate for reader's convenience.

Theorem 0.1. *Suppose that X is an uncountable Polish space. Then there is no analytic maximal orthogonal family of Borel probability measures on X.*

Proof. Suppose for contradiction that $\mathcal{A}$ is an analytic maximal pairwise orthogonal family of Borel probability measures on X. Observe that we can without loss of generality assume that X is perfect and compact. Indeed, for a general uncountable Polish space X, there is a compact perfect subspace Y of X (see Theorems 6.4 and 6.2 of [Kec95]). Then

$$\mathcal{A}' := \left\{ \frac{1}{\mu(Y)} \mu \restriction Y : \mu \in \mathcal{A} \wedge \mu(Y) > 0 \right\}$$

is clearly a maximal analytic family of pairwise orthogonal Borel probability measures on the uncountable perfect compact Polish space Y.

The first steps follow the proof of Theorem 0.1 from [PR85]. We include these steps with a little more detail for reader's convenience. For every $k \in \mathbb{N}$ denote by E_k the space of k-element subsets of $\mathcal{A}$, equipped with the usual topology in which it is clearly analytic. Fix some $\varepsilon \in (0,1)$ and define

$$H_{k,\varepsilon} := \{\nu \in P(X) : (\exists F \in E_k)\, (\forall \mu \in F)\, ||\nu - \mu|| < \varepsilon\},$$

which is evidently analytic, and for a fixed $\tau \in (0, \varepsilon)$ define also

$$U^{\tau}_{k,\varepsilon} := H_{k,\varepsilon-\tau} \setminus H_{k+1,\varepsilon},$$

which thus has the property of Baire. Since $\mathcal{A}$ is maximal orthogonal, we have that for every $\nu \in P(X)$ there is some $\mu \in \mathcal{A}$ with $||\nu - \mu|| < 1$. Moreover, since $\mathcal{A}$ consists of pairwise orthogonal measures, it holds that for any $0 \leq \sigma < 1$ and any $\nu \in P(X)$ the set

$$\{\mu \in \mathcal{A} : ||\nu - \mu|| < \sigma\}$$

is finite. Indeed, if it were infinite find $n \geq 1$, such that $1 - \sigma > 1/n$. Then there are some $\mu_0, \ldots, \mu_n \in \mathcal{A}$ with $||\nu - \mu_j|| < \sigma < 1 - 1/n$ for all $0 \leq j \leq n$. Therefore there are pairwise disjoint Borel subsets $D_0, \ldots, D_n$ of X such that for all $0 \leq i, j \leq n$, $i \neq j$ it holds that $\mu_i(D_i) = 1$ and $\mu_i(D_j) = 0$. Thus we have for all $0 \leq i \leq n$ that $\nu(D_i) > 1/n$. But then

$$\nu(X) \geq \nu\left(\bigcup_{i=0}^{n} D_i\right) > (n+1)\frac{1}{n} > 1,$$

which is of course a contradiction.

It is clear then that every $\nu \in P(X)$ is in some $U^{1/m}_{k,1/n}$ for some $k \geq 1, n > 1$ and $m > n$. Hence we have that

$$P(X) = \bigcup_{k\geq 1} \bigcup_{n>1} \bigcup_{m>n} U^{1/m}_{k,1/n},$$

and since $P(X)$ is a Baire space, it must hold that for some $k, \varepsilon := 1/n$ and $\tau := 1/m$ it holds that $U^{\tau}_{k,\varepsilon}$ is comeagre in a non-empty convex open set $V \subseteq P(X)$ (we can assume convexity of V, since $P(X)$ is locally convex). From now on, our proof diverges from the path taken in [PR85].

Claim 1.2. *There is $\nu \in U^{\tau}_{k,\varepsilon} \cap V$ and $C \subseteq U^{\tau}_{k,\varepsilon} \cap V$, which is comeagre in V, so that for every $\mu \in C$ the set*

$$M_\mu := \{t \in [0,1] : t\nu + (1-t)\mu \in U^{\tau}_{k,\varepsilon}\}$$

is comeagre in $[0,1]$.

Proof of Claim. The map $V \times V \times [0,1] \to V$, defined by $(\nu, \mu, t) \mapsto t\nu + (1-t)\mu$ is continuous and open by Lemma 1.1, so the preimage of $U^{\tau}_{k,\varepsilon} \cap V$ under this map is also a comeagre subset of $V \times V \times [0,1]$. Now the Kuratowski-Ulam theorem (see iii of Theorem 8.41 from [Kec95]) implies the desired result. $\dashv$

Let ν and C be as in the claim above. Next, we introduce the following notation: for $\rho \in U^{\tau}_{k,\varepsilon}$ write

$$N_\rho := \{\mu \in \mathcal{A} : ||\rho - \mu|| < \varepsilon - \tau\},$$

and by definition of $U^{\tau}_{k,\varepsilon}$ observe that $N_\rho = \{\mu \in \mathcal{A} : ||\rho - \mu|| < \varepsilon\}$ and has precisely k elements.

Claim 1.3. *For every $\rho \in C$ we have that $N_\nu = N_\rho$.*

Proof of Claim. Fix any $\rho \in C$ and put

$$T := \{t \in M_\rho : N_{t\nu+(1-t)\rho} = N_\nu\}.$$

and $s := \sup T$. We will show that $s = 1$ and that $s \in T$.

First, suppose for contradiction that $s < 1$. Find $t \in T$, such that $s - t < \tau/5$. Since $M_\rho \subseteq [0, 1]$ is comeagre, find $u \in M$ with $u \geq s$ and $u - s < \tau/5$. Then

$$\begin{aligned}||(u\nu + (1-u)\rho) - (t\nu + (1-t)\rho)|| &= ||(u-t)\nu + (t-u)\rho|| \\ &\leq |u-t|\,||\nu|| + |u-t|\,||\rho|| = 2(u-t) < \frac{4\tau}{5}.\end{aligned}$$

This implies that $N_{u\nu+(1-u)\rho} = N_{t\nu+(1-t)\rho} = N_\nu$, and thus $u \in T$, which is a contradiction.

So we have that $s = 1$. Find $t \in T$ with $|1 - t| < \tau/3$, and observe by the same reasoning as before that since $1 \in M_\rho$, we have that $N_\rho = N_{t\nu+(1-t)\rho} = N_\nu$, completing the proof. ⊣

Let now $\mu_0, \dots, \mu_{k-1} \in \mathcal{A}$ be such that $N_\nu = \{\mu_0, \dots, \mu_{k-1}\}$. But since it holds that for any $\mu \in P(X)$ the set $\mu^\perp := \{\rho \in P(X) : \mu \perp \rho\}$ is comeagre (see Proposition 4.1 from [KS01] and note that this is where we need that X is perfect compact), also the set

$$B := \bigcap_{j=0}^{k-1} \mu_j^\perp$$

is comeagre and in particular comeagre in V. So both B and C are comeagre in the open set V, which is of course a contradiction. □

For a (pairwise orthogonal) family $\mathcal{A} \subseteq P(X)$, observe that the set of witnesses to non-maximality

$$\mathcal{A}^\perp := \{\nu \in P(X) : (\forall \mu \in \mathcal{A})\, \nu \perp \mu\}$$

is co-analytic (in particular, it has the Baire property) and by Theorem 0.1 it is non-empty. When X is a perfect compact Polish space, we have the following strengthening.

Theorem 0.2. *Suppose that X is a compact perfect Polish space. Then for every analytic orthogonal family $\mathcal{A} \subseteq P(X)$, the set $\mathcal{A}^\perp$ is comeagre. In particular, when $\mathcal{A} \subseteq P(2^{\mathbb{N}})$ is a Σ^1_1 orthogonal family, there is a Δ^1_1-witness to non-maximality.*

Proof. Suppose for contradiction that $\mathcal{A}^\perp$ is not comeagre. Then there is a non-empty convex open set $O \subseteq P(X)$, in which $\mathcal{A}^\perp$ is meagre. Let Z be a dense (in O) G_δ subset of $O \setminus \mathcal{A}^\perp$; in particular, Z is a Polish subspace of $P(X)$. Note that $\mathcal{A}$ is maximal orthogonal in Z, i.e., for every $\nu \in Z$ there is some $\mu \in \mathcal{A}$ with $||\nu - \mu|| < 1$. Now we follow the proof of Theorem 0.1, but this time we use the Baire category theorem in Z.

We use the above defined E_k and redefine $H_{k,\varepsilon}$ and $U^{\tau}_{k,\varepsilon}$ as follows. Fix some $\varepsilon \in (0, 1)$ and set

$$H_{k,\varepsilon} := \{\nu \in Z : (\exists F \in E_k)\, (\forall \mu \in F)\, ||\nu - \mu|| < \varepsilon\},$$

which is clearly analytic, and for a fixed $\tau \in (0, \varepsilon)$ define also

$$U^{\tau}_{k,\varepsilon} := H_{k,\varepsilon-\tau} \setminus H_{k+1,\varepsilon},$$

which thus has the property of Baire. It is clear then that every $\nu \in Z$ is in some $U^{1/m}_{k,1/n}$ for some $k \geq 1, n > 1$ and $m > 1$. Hence we have that

$$Z = \bigcup_{k \geq 1} \bigcup_{n > 1} \bigcup_{m > n} U^{1/m}_{k,1/n},$$

and so it must hold that for some $k, \varepsilon := 1/n$ and $\tau := 1/m$ the set $U^{\tau}_{k,\varepsilon}$ is comeagre in a non-empty convex open set $V \subseteq O$. The proofs of Claims 1.2 and 1.3 work without changes to show the following two claims respectively.

Claim 1.4. *There is $\nu \in U^{\tau}_{k,\varepsilon} \cap V$ and $C \subseteq U^{\tau}_{k,\varepsilon} \cap V$, which is comeagre in V, so that for every $\mu \in C$ the set*

$$M_{\mu} := \{t \in [0, 1] : t\nu + (1 - t)\mu \in U^{\tau}_{k,\varepsilon}\}$$

is comeagre in $[0, 1]$.

Claim 1.5. *For every $\rho \in C$ we have that $N_{\nu} = N_{\rho}$.*

Finally, let $\mu_0, \ldots, \mu_{k-1} \in \mathcal{A}$ be such that $N_{\nu} = \{\mu_0, \ldots, \mu_{k-1}\}$. Again, the set

$$B := \bigcap_{j=0}^{k-1} \mu_j^{\perp}$$

is comeagre and in particular comeagre in V. So both B and C are comeagre in the open set V, which is again a contradiction.

For the "in particular" part of the theorem, note first that $P(2^{\mathbb{N}})$ is a recursively presentable Polish space (see [Mos09] for the definition of the notion and [FT10] for why $P(2^{\mathbb{N}})$ is recursively presentable), so it makes sense to talk about lightface pointclasses in $P(2^{\mathbb{N}})$. To get a Δ^1_1-witness to non-maximality of a Σ^1_1 orthogonal family $\mathcal{A} \subseteq P(2^{\mathbb{N}})$, use Corollary 4.1.2 of [Kec73] on $\mathcal{A}^{\perp}$, which, as we have just proved, is a comeagre Π^1_1 set. □

In [MS70], Martin and Solovay show that if Martin's axiom (MA) holds and Continuum hypothesis (CH) fails, then all $\mathbf{\Sigma^1_2}$ sets of reals have the Baire property. Recall also that the Axiom of projective determinacy (PD) implies that all projective sets of reals have the Baire property and that the Axiom of determinacy (AD) implies that all sets of reals have the Baire property (see e.g. Theorem 33.3 in [Jec03]). It is clear that we can substitute sets of reals with subsets of $P(X)$ for a Polish space X.

Theorem 0.3. *Suppose that X is an uncountable Polish space.*

(1) Assume MA *and* ¬CH*. Then no $\mathbf{\Sigma^1_2}$ orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*
(2) Assume PD*. Then no projective orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*
(3) Assume AD*. Then no orthogonal family $\mathcal{A} \subseteq P(X)$ is maximal.*

If moreover X is compact perfect, then in each of the above cases $\mathcal{A}^{\perp}$ is comeagre.

Proof. Repeat the proof of Theorem 0.1 (or 0.2 in case X is compact perfect), using the respective assumed axiom to get that the sets $U^{\tau}_{k,\varepsilon}$ have the Baire property. The rest of the proof is the same. □

Remark 1.6. Let $a \in \mathbb{N}^{\mathbb{N}}$ and note that $\omega_1^{L[a]} < \omega_1$ implies that for every $\Sigma^1_2[a]$ orthogonal family $\mathcal{A} \subseteq P(2^{\mathbb{N}})$, the set $\mathcal{A}^{\perp}$ is comeagre. The reason is again that $\omega_1^{L[a]} < \omega_1$ implies that every $\Sigma^1_2[a]$ subset of $P(2^{\mathbb{N}})$ has the Baire property (see Corollary 14.3 from [Kan09]).

2. States on separable C*-algebras

2.1. **Absolute continuity and orthogonality for states.** All C*-algebras considered will be unital and separable. For a C*-algebra A, we denote by $S(A)$ the collection of states on A (i.e., positive linear functionals, which map the unit of A to 1) and by $PS(A)$ the collection of pure states on A (i.e., the states that are the extreme points of the compact convex set $S(A)$). With $\mathrm{proj}(A)$ we denote the set of projections in A. See e.g. [Bla06] and [BO08] for other standard notions from C*-algebras theory.

It is well-known that every commutative C*-algebra A is *-isomorphic (via the Gelfand transform) to $C(M_A)$, where M_A is the maximal ideal space of A (which can in turn be described as the space of characters on A (i.e., non-zero algebra homomorphisms from A to $\mathbb{C}$)). Furthermore, M_A is compact Polish, being contained in $B_1(A^*)$. So when considering commutative separable C*-algebras we can restrict our attention to $C(X)$ for X compact Polish.

By Riesz–Markov–Kakutani representation theorem we know that $S(C(X))$ is actually the same as $P(X)$, and indeed, this is how $P(X)$ got its topology. Note here that a state $\phi \in S(C(X))$ is determined by its values on real-valued functions (real-valued functions are the self-adjoint elements in $C(X)$).

So it is natural to try to generalise notions from measure theory to states on C*-algebras. In [Dye52], Dye defined the notion of absolute continuity for states on σ-finite von Neumann algebras and proved a version of Radon-Nikodym theorem. There is an abundance of alternative formulations of absolute continuity for states on C*-algebras, some of them equivalent to the one given here, some weaker and some stronger. Standard results from measure theory like Lebesgue decomposition theorem generalise to states (this is the content of [Dye52]). See also e.g. [Hen72], [Hia84] or [Ino83] for some different formulations of absolute continuity for states and various results on generalisations.

For any *pre-order* (i.e., a reflexive and transitive relation) $\preccurlyeq$ on a set X one says that $x, y \in X$ are *orthogonal*, in symbols $x \perp y$, if there is no $z \in X$ with $z \preccurlyeq x, y$. Note that orthogonality of measures from the previous section is just orthogonality associated with the pre-order $\ll$. Accordingly, we define orthogonality for states, associated to Dye's notion of absolute continuity.

It is natural to ask whether Theorem 0.1 can be generalised to non-commutative unital C*-algebras A. As we will see, the orthogonality relation associated with Dye's notion of absolute continuity in general does not satisfy the nice properties of orthogonality for measures; in Example 2.6 we will show that $\ll$ on $S(M_2(\mathbb{C}))$ does not satisfy the ccc-below property and moreover we will construct an analytic mof $\mathcal{A} \subseteq S(M_2(\mathbb{C}))$. On the other hand, we are still able to use the idea of [KS01] to prove that $\sim$ for states is not classifiable by countable structures (this is Proposition 2.16).

There is however an alternative notion of orthogonality (which does not come from a pre-order) and for which it turns out that the argument of [KS01], using product measures, can be generalised to a class of C*-algebras; this is our Theorem 0.4. Furthermore, we are also able to generalise the result of Kakutani from product measures to product states; this is Proposition 0.5.

For reader's convenience we start by presenting in detail the definition of absolute continuity for states and the stronger notion of orthogonality. Since we are using descriptive set theoretic methods, we always require A to be separable.

So let A be a separable unital C*-algebra. By the Banach-Alaoglu theorem $(A^*)_1$, the closed unit ball in the dual space of A, is compact Polish in the weak*-topology. Moreover, $S(A) \subseteq (A^*)_1$ is compact convex, so convex compact Polish on its own.

Now let H be the Hilbert space from the universal representation of A in $B(H)$. Recall that we identify $A'' \subseteq B(H)$, the double commutant of A, which is a von Neumann algebra, with the double dual A^{**}, and call it the *enveloping von Neumann algebra* (see [BO08]). Denote $\mathcal{M} := A^{**}$, and recall that we can identify $S(A)$ with $(\mathcal{M}_*)^+_1$, where we identify $\phi \in S(A)$ with its normal extension $\phi^{**} \in S(\mathcal{M})$, and where $\mathcal{M}_*$ is the predual of $\mathcal{M}$. By definition, ϕ being normal means that ϕ is ultraweakly continuous on $\mathcal{M}$. We will employ Theorem 7.1.12 of [KR86], saying that a state ϕ is normal precisely when it is *completely additive*, i.e., for every orthogonal family of projections $\{p_i\}_{i\in I}$ of $\mathcal{M}$ it holds that $\phi(\sum_{i\in I} p_i) = \sum_{i\in I} \phi(p_i)$. See e.g. [BO08] or [KR86] for more details.

Recall that projections in a von Neumann algebra form a complete complemented lattice with 0 and 1. We use $\vee$ and $\wedge$ to denote *suprema* and *infima* respectfully. So for $\phi \in S(A)$ it holds that

$$\bigvee\{p \in \operatorname{proj}(\mathcal{M}) : \phi(p) = 0\}$$

is a projection in $\mathcal{M}$. Moreover, observe that it also holds that

$$\phi\left(\bigvee\{p \in \operatorname{proj}(\mathcal{M}) : \phi(p) = 0\}\right) = 0.$$

This is because $\vee\{p \in \operatorname{proj}(\mathcal{M}) : \phi(p) = 0\} = \sum_{i\in I} p_i$ for any maximal orthogonal family of projections $\{p_i\}_{i\in I} \subseteq \mathcal{M}$, satisfying that $\phi(p_i) = 0$ for every $i \in I$. For $\phi \in S(A)$, we define its *support* (sometimes called *carrier*) by

$$\operatorname{supp}\phi := 1 - \bigvee\{p \in \operatorname{proj}(\mathcal{M}) : \phi(p) = 0\},$$

which is again a projection in $\mathcal{M}$. Notice that by definition $\phi(\operatorname{supp}\phi) = 1$ (we think of $\operatorname{supp}\phi$ as the largest projection where ϕ is everywhere non-zero). Then for $\phi, \psi \in S(A)$ put

$$\psi \ll \phi \text{ if and only if } \operatorname{supp}\psi \leq \operatorname{supp}\phi,$$

and say that ψ is *absolutely continuous* with respect to ϕ. Observe that

$$\psi \ll \phi \text{ if and only if } (\forall p \in \operatorname{proj}(\mathcal{M}))\, \phi(p) = 0 \to \psi(p) = 0.$$

Set also $\phi \sim \psi$ if and only if $\phi \ll \psi$ and $\psi \ll \phi$. We continue with the following useful description of absolute continuity for states.

Claim 2.1. *For states ϕ, ψ it holds that $\psi \ll \phi$ if and only if for every positive $a \in \mathcal{M}$ we have that $\phi(a) = 0$ implies that $\psi(a) = 0$.*

Proof. The direction from right to left is immediate. For the other direction note that for every positive element $a \in \mathcal{M}$ there is a sequence $(\sum_{i=0}^{k_n} \lambda_i^n p_i^n)_{n\in\mathbb{N}}$, where $\lambda_i^n \in (0,\infty)$ and $p_i^n \in \operatorname{proj}(\mathcal{M})$ for every $n \in \mathbb{N}$ and $0 \leq i \leq k_n$, so that $\sum_{i=0}^{k_n} \lambda_i^n p_i^n \leq a$ for every $n \in \mathbb{N}$ and so that $||a - \sum_{i=0}^{k_n} \lambda_i^n p_i^n|| \to 0$, as $n \to \infty$. Indeed, for a positive (in particular normal), let $\mathcal{N}$ be the Abelian von Neumann subalgebra generated by a. Since the statement clearly holds in $\mathcal{N}$ (and the norm on $\mathcal{N}$ is the restriction of the one on $\mathcal{M}$), the same sequence also converges to a from below in $\mathcal{M}$.

Now, if $\phi(a) = 0$, then $\phi(p_i^n) = 0$ for every $n \in \mathbb{N}$ and every $0 \leq i \leq k_n$. So by assumption also $\psi(p_i^n) = 0$ for all $n \in \mathbb{N}$ and all $0 \leq i \leq k_n$. Hence $\psi(a) = 0$, which completes the proof. □

We proceed with the following result, which tells us that in order to check whether one state is absolutely continuous with respect to the other we do not need to go to the large enveloping von Neumann algebra.

Claim 2.2. *Let $\phi, \psi \in S(A)$ for a separable unital C^*-algebra A. Suppose that $\pi : A \to B(K)$ is a faithful representation of A on a Hilbert space K, so that ϕ and ψ have unique normal extensions to $\mathcal{N} := A'' \subseteq B(K)$, which we also denote by ϕ and ψ respectively.*

Then $\psi \ll \phi$ if and only if for every $p \in \operatorname{proj}(\mathcal{N})$ we have that $\phi(p) = 0$ implies that $\psi(p) = 0$.

Proof. By the universal property of the enveloping von Neumann algebra $\mathcal{M}$, there is a normal *-epimorphism (i.e., a *-homomorphism which is onto) $\alpha : \mathcal{M} \to \mathcal{N}$, which is equal to identity on A. Then since every element of $\mathcal{M}$ is a limit of an ultraweakly converging net $\{x_\xi\}$ from A, and since ϕ and ψ are normal on both $\mathcal{M}$ and $\mathcal{N}$, it holds for every $x \in \mathcal{M}$ that

$$\phi(x) = \phi(\alpha(x)) \quad \text{and} \quad \psi(x) = \psi(\alpha(x)).$$

Suppose first that $\psi \ll \phi$ and that $\phi(p) = 0$ for $p \in \operatorname{proj}(\mathcal{N})$. Then since α is onto there is some positive $b \in \mathcal{M}$ so that $\alpha(b) = p$. Hence $\phi(b) = 0$ and by Claim 2.1 also $\psi(b) = 0$, which in turn implies that $\psi(p) = \psi(\alpha(b)) = \psi(b) = 0$.

Conversely, suppose that for every $p \in \operatorname{proj}(\mathcal{N})$ it holds that $\phi(p)$ implies that $\psi(p) = 0$. Let $q \in \operatorname{proj}(\mathcal{M})$ be such that $\phi(q) = 0$. Then $\alpha(q)$ is a projection in $\mathcal{N}$, so $\psi(\alpha(q)) = 0$ and hence also $\psi(q) = \psi(\alpha(q)) = 0$. $\square$

Remark 2.3. Observe that for any projection $p \in \mathcal{N}$, there is some projection $q \in \mathcal{M}$ for which $\alpha(q) = p$. To see this, take any positive $a \in \mathcal{M}$ for which $\alpha(a) = p$. Then the sequence $(b_n)_n$, defined by

$$b_n := ||a||^{\frac{1}{n}} \left(\frac{a}{||a||} \right)^{\frac{1}{n}},$$

converges (strongly and ultraweakly) to some projection $q \in \mathcal{M}$ (see the proof of Theorem 17.5 from [Zhu93]), and also satisfies that $\alpha(b_n) = p$ for every $n \in \mathbb{N}$. Hence also $\alpha(q) = p$. So we could have proven the claim without using Claim 2.1.

As a consequence of this fact, we actually have that $\alpha(\operatorname{supp}\psi) = \operatorname{supp}\psi$, since clearly

$$\alpha\left(\bigvee\{p \in \operatorname{proj}(\mathcal{M}) : \psi(p) = 0\}\right) \leq \bigvee\{p \in \operatorname{proj}(\mathcal{N}) : \psi(p) = 0\},$$

but also if $q \in \operatorname{proj}(\mathcal{M})$ is such that $\alpha(q) = \vee\{p \in \operatorname{proj}(\mathcal{M}) : \psi(p) = 0\}$, then $\psi(q) = 0$.

Remark 2.4. Note that Claim 2.2 holds more generally (with essentially the same proof) for non-degenerate (i.e., unital) representations $\pi : A \to B(K)$, which are not necessarily faithful, and for $\phi, \psi \in S(A)$, for which there are $x, y \in K$, so that for every $a \in A$ it holds that $\phi(a) = \langle \pi(a)x, x\rangle_K$ and $\psi(a) = \langle \pi(a)y, y\rangle_K$.

Using (the proof of) Claim 2.1, we can replace the requirement in the statement of Claim 2.2 that $p \in \operatorname{proj}(\mathcal{N})$ with $p \in \mathcal{N}$ being positive.

The reason why we are allowed to call $\ll$ absolute continuity for states is that for commutative C*-algebras it coincides with the classical notion defined for measures.

Proposition 2.5. *If A is commutative, $\ll$ defined for states coincides with $\ll$ defined for measures.*

Proof. Suppose that $A = C(X)$ for a compact Polish space X. Let $\mu, \nu \in P(X)$ and let ϕ_μ, ϕ_ν be the corresponding states on A (via the Riesz–Markov–Kakutani representation) and also on $\mathcal{M} = A^{**}$ (via the unique normal extension). Let $\mathcal{N} := L^\infty(X, \frac{1}{2}(\mu + \nu))$ and observe that both ϕ_μ and ψ_ν admit unique normal extensions to $\mathcal{N}$, again denoted with ϕ_μ and ψ_ν respectively. Actually, it holds that the extensions to $\mathcal{N}$ are

$$\phi_\mu = \int \mathrm{d}\mu \quad \text{and} \quad \phi_\nu = \int \mathrm{d}\nu.$$

Recalling that projections in $\mathcal{N}$ are of the form χ_B for $B \subseteq X$ Borel, an application of Claim 2.2 completes the proof. $\square$

We now move to different notions of orthogonality. As already alluded to above, we say that $\phi, \psi \in S(A)$ are *orthogonal*, which we denote with $\phi \perp \psi$, if there is no $\rho \in S(A)$ for which $\rho \ll \phi$ and $\rho \ll \psi$. In particular, if $\operatorname{supp}\phi \wedge \operatorname{supp}\psi = 0$, then $\phi \perp \psi$. Hence, it is very easy for two states to be orthogonal. Consequently, $\perp$ for states fails to satisfy the nice properties of $\perp$ for measures.

Example 2.6. Let $A := M_2(\mathbb{C})$. Note that since A is finite dimensional the norm topology coincides with the ultraweak operator topology (and also with the weak/strong/ultrastrong operator topologies). Hence all states on A are normal, so using Claim 2.2, we can calculate $\ll$ (and $\perp$) by considering supports in the von Neumann algebra A. A rank 1 projection $p \in \operatorname{proj}(A)$ (i.e., when p is seen as a projection onto a subspace E_p of $\mathbb{C}^2$, the dimension of E_p is 1), gives rise to a state $\phi_p \in S(A)$, defined for $a \in M_2(\mathbb{C})$ by

$$\phi_p(a) := \operatorname{tr}(p\,a\,p),$$

where tr is the usual non-normalised trace on $M_2(\mathbb{C})$. Note that $\operatorname{supp}\phi_p = p$ and since for two distinct rank 1 projections p, q it holds that $p \wedge q = 0$, we get that $\phi_p \perp \phi_q$. Hence

$$\mathcal{A} := \{\phi_p : p \in \operatorname{proj}(A) \text{ has rank } 1\} \subseteq S(A)$$

is an orthogonal family. Take now any $\psi \in S(A)$. If $\operatorname{supp}\psi = \mathbb{C}^2$, then we get that $\phi \ll \psi$ for all $\phi \in \mathcal{A}$. Since there clearly are such ψ (e.g., take $\psi = \frac{1}{2}(\mathrm{ev}_{1,1} + \mathrm{ev}_{2,2})$), the ccc-below property fails for $\ll$. Suppose now that $\operatorname{supp}\psi$ is of rank 1. Then $\psi \sim \phi_{\operatorname{supp}\psi}$. Thus $\mathcal{A}$ is maximal. Moreover, $\mathcal{A}$ is clearly analytic, so we have an analytic mof in $S(A)$.

Next we introduce another notion of orthogonality. Let A be a separable unital C*-algebra. For $\phi, \psi \in S(A)$, we say that they are *strongly orthogonal*, denoted by $\phi \underline{\perp} \psi$ if $\operatorname{supp}\phi \operatorname{supp}\psi = 0$. Clearly, $\phi \underline{\perp} \psi$ implies $\phi \perp \psi$. Moreover, for $\phi \sim \psi$ and any $\chi \in S(A)$ it holds that $\phi \underline{\perp} \chi$ if and only if $\psi \underline{\perp} \chi$. We proceed with a useful characterisation of strong orthogonality.

Fact 2.7. *For $\phi, \psi \in S(A)$ the following are equivalent*

(1) $\phi \underline{\perp} \psi$*;*
(2) $\phi(\operatorname{supp}\psi) = 0$*;*
(3) $(\exists p \in \operatorname{proj}(A^{**}))\, \phi(p) = 0 \wedge \psi(p) = 1$.

Proof. (1) $\Leftrightarrow$ (2) follows from the definition of support and the fact that for projections p, q it holds that $p\,q = 0$ if and only if $p \leq 1 - q$.
(2) $\Leftrightarrow$ (3) is clear. $\square$

With this characterisation, Claim 2.2 and Remark 2.3, we obtain the following, which enables us to decide $\underline{\perp}$ in smaller representations.

Claim 2.8. *Let $\phi, \psi \in S(A)$ for a separable unital C*-algebra A. Suppose that $\pi : A \to B(K)$ is a faithful representation of A on a Hilbert space K, so that ϕ and ψ have unique normal extensions to $\mathcal{N} := A'' \subseteq B(K)$, which we also denote by ϕ and ψ respectively. Then $\phi \underline{\perp} \psi$ if and only if $\phi(\operatorname{supp}\psi) = 0$ holds in $\mathcal{N}$.*

Proof. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be as in the proof of Claim 2.2. By Fact 2.7, $\phi \underline{\perp} \psi$ if and only if $\phi(\operatorname{supp}\psi) = 0$ holds in $\mathcal{M}$. But by the proof of Claim 2.2 and by Remark 2.3, $\phi(\operatorname{supp}\psi) = 0$ is true in $\mathcal{M}$ precisely when it is true in $\mathcal{N}$. $\square$

Remark 2.9. As with Claim 2.2, we actually have that Claim 2.8 holds more generally for non-degenerate representations $\pi : A \to B(K)$, which are not necessarily faithful, and for $\phi, \psi \in S(A)$, for which there are $x, y \in K$, so that for every $a \in A$ it holds that $\phi(a) = \langle \pi(a)x, x\rangle_K$ and $\psi(a) = \langle \pi(a)y, y\rangle_K$.

Notice also that (3) of Fact 2.7 implies that $\perp\!\!\!\perp$ is analytic. Next we note that the notion of strong orthogonality extends orthogonality for measures.

Claim 2.10. *Suppose that A is a commutative separable unital C*-algebra and take any $\phi, \psi \in S(A)$. Then $\phi \perp\!\!\!\perp \psi$ if and only if $\phi \perp \psi$.*

Proof. The forward direction is obvious. So assume that $\phi \perp \psi$ and denote $p := \operatorname{supp} \phi$ and $q := \operatorname{supp} \psi$. Suppose for contradiction that $\phi(q) > 0$. Then define χ by setting

$$\chi(a) := \frac{1}{\phi(q)} \phi(p\, q\, a)$$

for $a \in A$. Clearly, $\chi \in S(A)$ and $\chi \ll \phi, \psi$ which is a contradiction. □

As opposed to $\perp$ for states, $\perp\!\!\!\perp$ shares some nice properties with orthogonality for measures. There is a version of the ccc-below property which is true for $\perp\!\!\!\perp$. A set $\mathcal{A} \subseteq S(A)$ is a $\perp\!\!\!\perp$-*antichain* if for every $\psi \neq \chi \in \mathcal{A}$ it holds that $\psi \perp\!\!\!\perp \chi$.

Claim 2.11. *For any $\phi \in S(A)$ and any $\perp\!\!\!\perp$-antichain $\mathcal{A} \subseteq S(A)$ there are only countably many $\psi \in \mathcal{A}$, for which $\neg \phi \perp\!\!\!\perp \psi$.*

Proof. Suppose for contradiction that there is an uncountable set $\{\psi_i : i \in I\}$ of pairwise strongly orthogonal states, such that for all $i \in I$ it holds that $\neg \phi \perp\!\!\!\perp \psi_i$. Then $\phi(\operatorname{supp} \psi_i) > 0$ for all $i \in I$. But since $\operatorname{supp} \psi_i$ are pairwise orthogonal and since ϕ is completely additive on A^{**}, we have that

$$\phi\left(\sum_{i \in I} \operatorname{supp} \psi_i\right) = \sum_{i \in I} \phi(\operatorname{supp} \psi_i).$$

But since I is uncountable, this is a contradiction, as the sum on the right diverges. □

In [Bur69], Bures defined ρ and d for normal states on von Neumann algebras, which generalise the identically denoted notions defined for measures in [Kak48]; see Proposition 2.7 of [Bur69]. We reintroduce these notions for states on a separable C*-algebra A. Let $\phi, \psi \in S(A)$ and put

$$\begin{aligned} Q(\phi, \psi) := \{(\pi, x, y) : \ & \pi \text{ is a faithful representation of } A \text{ on } H, \\ & x, y \in H \text{ respectively induce } \phi, \psi \text{ relative to } \pi\}. \end{aligned}$$

Here, x *induces* ϕ *relative to* π, means that for every $a \in A$ it holds that $\phi(a) = \langle \pi(a) x, x \rangle_H$. Then define

$$\rho(\phi, \psi) := \sup\{|\langle x, y \rangle| : (\pi, x, y) \in Q(\phi, \psi)\}$$

and

$$d(\phi, \psi) = \inf\{||x - y|| : (\pi, x, y) \in Q(\phi, \psi)\}.$$

Observe that it holds that $d(\phi, \psi)^2 = 2(1 - \rho(\phi, \psi))$. The same argument as the one in the proof of Proposition 1.7 from [Bur69], shows that d, defined for A (not necessarily a von Neumann algebra), is a metric. Let H_S be some separable Hilbert space, for which there is a faithful representation $\pi_S : A \to B(H_S)$. The proof of Proposition 1.6 from [Bur69] shows that for $\phi, \psi \in S(A)$, we can calculate $\rho(\phi, \psi)$ and $d(\phi, \psi)$ by ranging over the tuples $(\pi, x, y) \in Q(\phi, \psi)$, for which π is fixed to be

$$\pi : A \to B(H_\phi \oplus H_\psi \oplus H_S \oplus H_\phi \oplus H_\psi \oplus H_S),$$

defined for $a \in A$ by $\pi(a) := \pi_\phi(a) \oplus \pi_\psi(a) \oplus \pi_S(a) \oplus \pi_\phi(a) \oplus \pi_\psi(a) \oplus \pi_S(a)$, where $\pi_\phi : A \to B(H_\phi)$ and $\pi_\psi : A \to B(H_\psi)$ are the GNS representations. Fixing some countable dense set $D \subseteq A$, observe that unfolding the definitions of the GNS representations and direct sums of Hilbert spaces, yields that

$$\begin{aligned}\rho(\phi,\psi) = \sup\{&\langle x_0,u_0\rangle_\phi + \langle y_0,v_0\rangle_\psi + \langle z_0,w_0\rangle_{H_S} + \langle x_1,u_1\rangle_\phi + \langle y_1,v_1\rangle_\psi + \langle z_1,w_1\rangle_{H_S} :\\ &x_0,x_1,u_0,u_1 \in H_\phi; y_0,y_1,v_0,v_1 \in H_\psi; z_0,z_1,w_0,w_1 \in H_S \text{ and}\\ &(\forall d \in D)\, \phi(d) = \langle \pi_\phi(d)x_0,x_0\rangle_\phi + \langle \pi_\psi(d)y_0,y_0\rangle_\psi + \langle \pi_S(d)z_0,z_0\rangle_{H_S}\\ &\qquad + \langle \pi_\phi(d)x_1,x_1\rangle_\phi + \langle \pi_\psi(d)y_1,y_1\rangle_\psi + \langle \pi_S(d)z_1,z_1\rangle_{H_S} \wedge\\ &\psi(d) = \langle \pi_\phi(d)u_0,u_0\rangle_\phi + \langle \pi_\psi(d)v_0,v_0\rangle_\psi + \langle \pi_S(d)w_0,w_0\rangle_{H_S}\\ &\qquad + \langle \pi_\phi(d)u_1,u_1\rangle_\phi + \langle \pi_\psi(d)v_1,v_1\rangle_\psi + \langle \pi_S(d)w_1,w_1\rangle_{H_S}\}.\end{aligned}$$

In particular, for a fixed $\varepsilon \geq 0$ the sets

$$\{(\phi,\psi) \in S(A)^2 : \rho(\phi,\psi) > \varepsilon\}$$

and

$$\{(\phi,\psi) \in S(A)^2 : d(\phi,\psi) < \varepsilon\}$$

are analytic.

The proof of Lemma 1.2 from Promislow's [Pro71] proves that $\phi \underline{\perp} \psi$ if and only if $d(\phi,\psi) = \sqrt{2}$. In particular, $\neg\phi\underline{\perp}\psi$ is analytic, which since we have already observed that $\phi\underline{\perp}\psi$ is analytic, implies that $\underline{\perp}$ is Borel.

2.2. **No analytic maximal strongly orthogonal families.** In this subsection we prove Theorem 0.4.

In [KS01], Kechris and Sofronidis used the theory of turbulence to prove the following (which is Theorem 3.1 in [KS01]).

Theorem 2.12. *For any analytic orthogonal $\mathcal{A} \subseteq P(2^{\mathbb{N}})$, there exists $\alpha \in (0,1)^{\mathbb{N}}$ such that $\mu_\alpha \perp \mu$ for every $\mu \in \mathcal{A}$, where*

$$\mu_\alpha := \prod_{n\in\mathbb{N}} (\alpha(n)\delta_0 + (1-\alpha(n))\delta_1).$$

The idea of their proof is to build on Kakutani's [Kak48] and define a continuous map $f : 2^{\mathbb{N}} \to P(2^{\mathbb{N}})$, satisfying that for every $x,y \in 2^{\mathbb{N}}$ it holds that xE_Iy implies that $f(x) \sim f(y)$ and $\neg xE_Iy$ implies that $f(x) \perp f(y)$, where

$$xE_Iy \quad \text{if and only if} \quad \sum_{n\in x\Delta y} \frac{1}{n} < \infty.$$

(We use the notation E_I because it is used in [KS01]; other more common notations are either I_2 or E_2.) Recall that a Borel equivalence relation E on a Polish space Y is *generically S_∞-ergodic* if every E-class is meagre and for any standard Borel space Z, equipped with a Borel action of S_∞, and any Baire measurable $f : Y \to Z$, with the property that xEy implies that $(\exists g \in S_\infty)\, g \cdot f(x) = f(y)$, there is an E-invariant comeagre set $C \subseteq Y$, such that f maps C to a single class in Z.

If E is generically S_∞-ergodic, then E is not classifiable by countable structures. Here a relation E on a standard Borel space X is said to be *classifiable by countable structures* if there is a countable language L and a Borel map $f : X \to X_L$ (where X_L is the space of countable structures for L), so that for all $x,y \in X$ it holds that xEy if and only if $f(x) \cong f(y)$. See [Hjo00] for more about classification by countable structures, generic ergodicity and turbulence.

The relation E_I defined above is generically S_∞-ergodic (see [Hjo00] and [KS01]) and so with the above reduction of E_I to $\sim$, Kechris and Sofronidis establish that $\sim$ is not classifiable by countable structures. Then they prove the following lemma (see Lemma 3.3 in [KS01]), which gives Theorem 2.12 (since $\ll$ for measures has the ccc-below property).

Lemma 2.13. *Let $\preccurlyeq$ be an analytic partial pre-ordering on a Polish space X which satisfies the ccc-below property and assume that there exists a generically S_∞-ergodic equivalence relation E on a Polish space Y and a Borel measurable function $f : Y \to X$ with the properties that $zEy \implies f(z) \sim f(y)$ and $\neg zEy \implies f(z) \perp f(y)$, whenever z, y are in Y. Then, given any orthogonal analytic subset $\mathcal{A}$ of X, there exists $y \in Y$ such that $f(y) \perp x$ for every $x \in \mathcal{A}$.*

We will prove that the same idea can be generalised to a class of C*-algebras. Recall that for a C*-algebra A and its subalgebra B, a linear map $E : A \to B$ is a *conditional expectation*, when E is a contractive completely positive projection, such that for every $a \in A$ and $b, b' \in B$ it holds that $E(bab') = bE(a)b'$. By Tomiyama's theorem (see Theorem 1.5.10 in [BO08]), a projection $E : A \to B$ is a conditional expectation precisely when it is contractive. Notice that conditional expectations are closed under composition.

Suppose now that $B \subseteq A$ are unital C*-algebras (with possibly different units) and that $E : A \to B$ is a conditional expectation (note that $E(1_A) = 1_B E(1_A) 1_B = E(1_B) = 1_B$). For a state $\phi \in S(B)$, there is an extension $\tilde{\phi} \in S(A)$, defined by $\tilde{\phi}(a) = \phi(E(a))$ for $a \in A$. Clearly the map $\tilde{(\cdot)} : S(B) \to S(A)$, defined by $\phi \mapsto \tilde{\phi}$ is continuous. Note also that $E^{**} : A^{**} \to B^{**}$ is again a conditional expectation, extending E. We next list some nice properties of $\tilde{(\cdot)}$.

Claim 2.14. *For $\phi, \psi \in S(B)$ it holds that $\psi \ll \phi$ if and only if $\tilde{\psi} \ll \tilde{\phi}$.*

Proof. Suppose that $\psi \ll \phi$ and that $\tilde{\phi}(a) = 0$ for a positive $a \in A^{**}$. This means that $\phi(E^{**}(a)) = 0$, and consequently $\psi(E^{**}(a)) = 0$. Hence by definition, $\tilde{\psi}(a) = 0$.

The other direction is obvious. ☐

Claim 2.15. *For $\phi, \psi \in S(B)$ it holds that $\phi \underline{\perp} \psi$ implies $\tilde{\phi} \underline{\perp} \tilde{\psi}$.*

Proof. Suppose that $\phi \underline{\perp} \psi$. We will show that $\tilde{\phi}(\operatorname{supp} \tilde{\psi}) = 0$, which implies that $\tilde{\phi} \underline{\perp} \tilde{\psi}$ by Fact 2.7. Note that

$$\operatorname{supp} \tilde{\psi} = 1_A - \bigvee \{p \in \operatorname{proj}(A^{**}) : \psi(E^{**}(p)) = 0\}.$$

Since

$$\bigvee \{q \in \operatorname{proj}(B^{**}) : \psi(q) = 0\} \leq \bigvee \{p \in \operatorname{proj}(A^{**}) : \psi(E^{**}(p)) = 0\},$$

and since E^{**} is monotone, it holds that

$$E^{**}(\operatorname{supp} \tilde{\psi}) \leq 1_B - \bigvee \{q \in \operatorname{proj}(B^{**}) : \psi(q) = 0\} = \operatorname{supp} \psi.$$

But then

$$\tilde{\phi}(\operatorname{supp} \tilde{\psi}) = \phi(E^{**}(\operatorname{supp} \tilde{\psi})) \leq \phi(\operatorname{supp} \psi) = 0$$

and hence $\tilde{\phi}(\operatorname{supp} \tilde{\psi}) = 0$, which completes the proof. ☐

With this we are ready to prove Theorem 0.4.

Theorem 0.4. *Suppose A is a separable unital C*-algebra, which contains a copy of $C(2^{\mathbb{N}})$ as a subalgebra and for which there is a conditional expectation $E : A \to C(2^{\mathbb{N}})$. Then for every strongly orthogonal $\mathcal{A} \subseteq S(A)$ there is $\alpha \in (0,1)^{\mathbb{N}}$ so that $\tilde{\mu}_\alpha \underline{\perp} \psi$ for every $\psi \in \mathcal{A}$, where $\tilde{\mu}_\alpha$ is the extension of the state, corresponding to the product measure*

$$\prod_{n \in \mathbb{N}} (\alpha(n)\delta_0 + (1 - \alpha(n))\delta_1),$$

from $C(2^{\mathbb{N}})$ to A.

Proof. As in [KS01], define $\alpha : 2^{\mathbb{N}} \to [\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}}$ by

$$\alpha(x)(n) := \begin{cases} \frac{1}{4}\left(1 + \frac{1}{\sqrt{n+1}}\right) & \text{if } n \in x \\ \frac{1}{4} & \text{if } n \in \mathbb{N} \setminus x \end{cases}$$

for all $x \in 2^{\mathbb{N}}$, where we identify $2^{\mathbb{N}}$ with $\mathcal{P}(\mathbb{N})$, the powerset of $\mathbb{N}$. Now let $f : 2^{\mathbb{N}} \to S(A)$ be defined by $f(x) = \tilde{\mu}_{\alpha(x)}$ for $x \in 2^{\mathbb{N}}$. Since the maps α, $\tilde{(\cdot)}$ and the map $[\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}} \to S(2^{\mathbb{N}})$, defined by $h \mapsto \mu_h$ are all continuous, so is f.

In [KS01], it is established that for $x, y \in 2^{\mathbb{N}}$ it holds that

$$\sum_{n \in \mathbb{N}} (\alpha(x)(n) - \alpha(y)(n))^2 = \sum_{n \in x \Delta y} \frac{1}{16(n+1)},$$

so by Corollary 1 from [Kak48] and by Claims 2.14 and 2.15 we have for every $x, y \in 2^{\mathbb{N}}$ that

$$xE_I y \implies f(x) \sim f(y) \quad \text{and} \quad \neg xE_I y \implies f(x) \perp f(y).$$

Lemma 2.13 still holds (with the same proof) with ccc-below replaced with the property from Claim 2.11 and with $\preccurlyeq$ being analytic replaced with $\perp$ co-analytic (in our case it is even Borel). Thus the proof is complete. $\square$

The function f from the above proof is a continuous reduction of E_I to $\sim$ for states, and hence we have the following consequence.

Corollary 2.16. *Suppose A is a separable unital C*-algebra, which contains a copy of $C(2^{\mathbb{N}})$ as a subalgebra and for which there is a conditional expectation $E : A \to C(2^{\mathbb{N}})$. Then $\sim$ on $S(A)$ is not classifiable by countable structures.*

The following examples of nice C*-algebras satisfying assumptions of Theorem 0.4 (and hence also of Corollary 2.16) were suggested to the author by Magdalena Musat and Mikael Rørdam.

Fix any A satisfying assumptions of Theorem 0.4. Let B be any separable unital C*-algebra and pick some $\phi \in S(A \otimes B)$. Then there is a conditional expectation $E : A \otimes B \to A$, induced by

$$E(a \otimes b) = \phi(b)\, a.$$

Hence $A \otimes B$ also satisfies assumptions of Theorem 0.4.

Recalling Proposition 4.1.9 from [BO08], we get that for any countable discrete group Γ and any homomorphism $\alpha : \Gamma \to \mathrm{Aut}(A)$, the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ also satisfies the assumptions of Theorem 0.4. Note that if A is simple and the action is *outer* (i.e., Γ acts by outer automorphisms), then by [Kis81] $A \rtimes_{\alpha,r} \Gamma$ is simple.

Consider the following diagram

$$\begin{array}{ccccccccccc}
\mathbb{C} & \xrightarrow{\phi_0} & M_2(\mathbb{C}) & \xrightarrow{\phi_1} & M_4(\mathbb{C}) & \xrightarrow{\phi_2} & M_8(\mathbb{C}) & \xrightarrow{\phi_3} & \cdots & \longrightarrow & M_{2^\infty} \\
\downarrow{\scriptstyle E_0} & & \downarrow{\scriptstyle E_1} & & \downarrow{\scriptstyle E_2} & & \downarrow{\scriptstyle E_3} & & & & \downarrow{\scriptstyle E} \\
\mathbb{C} & \xrightarrow{\psi_0} & \mathbb{C}^2 & \xrightarrow{\psi_1} & \mathbb{C}^4 & \xrightarrow{\psi_2} & \mathbb{C}^8 & \xrightarrow{\psi_3} & \cdots & \longrightarrow & C(2^{\mathbb{N}}),
\end{array}$$

where $\phi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ is defined as $\phi_n(X) := \mathrm{id}_2 \otimes X$, $\psi_n : \mathbb{C}^{2^n} \to \mathbb{C}^{2^{n+1}}$ as $\psi_n(a_1, a_2, \ldots, a_{2^n}) := (a_1, a_1, a_2, a_2, \ldots, a_{2^n}, a_{2^n})$ and $E_n : M_{2^n}(\mathbb{C}) \to \mathbb{C}^{2^n}$ as $E([a_{i,j}]) = (a_{1,1}, a_{2,2}, \ldots, a_{2^n,2^n})$. On the right we have the inductive limits of the respective sequences: the CAR algebra M_{2^∞} and $C(2^{\mathbb{N}})$. Note that all E_n are conditional expectations and that the diagram commutes, which gives a conditional expectation $E : M_{2^\infty} \to C(2^{\mathbb{N}})$. Since M_{2^∞} is also separable, it satisfies the assumptions of Theorem 0.4.

Since one can view the Cuntz algebra $\mathcal{O}_2$ as a crossed product of M_{2^∞} with integers (see [Cun77], [AK02] and [ANS14]), $\mathcal{O}_2$ also satisfies the assumptions of Theorem 0.4.

2.3. **Kakutani's theorem for states.** In this subsection we use results from the previous subsection to prove Proposition 0.5.

Recall that we may view the CAR algebra M_{2^∞} as $\otimes_{n=0}^{\infty} M_2(\mathbb{C})$. If $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of states on $M_2(\mathbb{C})$, then $\otimes_{n=0}^{\infty}\phi_n$ denotes the unique state (called *product state*) with the property that for every sequence $(a_n)_{n\in\mathbb{N}}$, where for all but finitely many n it holds that $a_n = 1_{M_2(\mathbb{C})}$, we have that

$$\left(\bigotimes_{n=0}^{\infty}\phi_n\right)\left(\bigotimes_{n=0}^{\infty}a_n\right) = \prod_{n\in\mathbb{N}}\phi_n(a_n).$$

Observe that $\mathrm{ev}_{1,1}$ and $\mathrm{ev}_{2,2}$ are states on $M_2(\mathbb{C})$. These will be our non-commutative analogues of the Dirac measures δ_0 and δ_1 on $2 = \{0,1\}$, used in [Kak48] and [KS01].

In [Bur69], Bures extended Kakutani's theorem to semi-finite von Neumann algebras. This was improved by Promislow (see [Pro71]) to general von Neumann algebras. However, their statements do not mention absolute continuity nor (strong) orthogonality for states, which on the other hand are central to Kakutani's statement. Using Claims 2.14 and 2.15 and Corollary 1 of [Kak48], we provide an extension of Kakutani's result about absolute continuity and orthogonality to the special case for product states on the CAR algebra.

Proposition 0.5. *Suppose that* $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \in [\frac{1}{4},\frac{3}{4}]^{\mathbb{N}}$ *and let*

$$\phi_n := \alpha_n\,\mathrm{ev}_{1,1} + (1-\alpha_n)\,\mathrm{ev}_{2,2} \quad \text{and} \quad \psi_n := \beta_n\,\mathrm{ev}_{1,1} + (1-\beta_n)\,\mathrm{ev}_{2,2}$$

be states on $M_2(\mathbb{C})$. *Let also* $\phi := \otimes_{n=0}^{\infty}\phi_n$ *and* $\psi := \otimes_{n=0}^{\infty}\psi_n$ *be the product states on* M_{2^∞}. *Then in* $S(M_{2^\infty})$, *either* $\phi \sim \psi$ *or* $\phi \perp\!\!\!\perp \psi$ *according to whether*

$$\sum_{n\in\mathbb{N}}(\alpha_n - \beta_n)^2$$

converges or diverges respectively.

Proof. Let

$$\mu := \prod_{n\in\mathbb{N}}\alpha_n\delta_0 + (1-\alpha_n)\delta_1 \quad \text{and} \quad \nu := \prod_{n\in\mathbb{N}}\beta_n\delta_0 + (1-\beta_n)\delta_1$$

be product measures and note that $\phi = \tilde{\mu}$ and $\psi = \tilde{\nu}$ (using the conditional expectation E defined in the end of the previous subsection).

Now by Corollary 1 of [Kak48], we get that either $\mu \sim \nu$ or $\mu \perp \nu$ (which is the same as $\mu \perp\!\!\!\perp \nu$) according to whether

$$\sum_{n\in\mathbb{N}}(\alpha_n - \beta_n)^2$$

converges or diverges respectively. But then by Claims 2.14 and 2.15, we get that $\phi \sim \psi$ if and only if $\mu \sim \nu$ and $\phi \perp\!\!\!\perp \psi$ if and only if $\mu \perp \nu$, which completes the proof. $\square$

Remark 2.17. The main ingredient of [Kak48] are properties of ρ and d, defined for measures. Since for every $a \in M_2(\mathbb{C})$ it holds that

$$\phi_n(a) = \mathrm{tr}\left(\begin{pmatrix}\sqrt{\alpha_n} & 0\\ 0 & \sqrt{1-\alpha_n}\end{pmatrix} a \begin{pmatrix}\sqrt{\alpha_n} & 0\\ 0 & \sqrt{1-\alpha_n}\end{pmatrix}\right)$$

and

$$\psi_n(a) = \mathrm{tr}\left(\begin{pmatrix}\sqrt{\beta_n} & 0\\ 0 & \sqrt{1-\beta_n}\end{pmatrix} a \begin{pmatrix}\sqrt{\beta_n} & 0\\ 0 & \sqrt{1-\beta_n}\end{pmatrix}\right),$$

Proposition 2.3 from [Bur69] implies that

$$\rho(\phi_n,\psi_n) = \mathrm{tr}\left(\begin{pmatrix}\sqrt{\alpha_n} & 0\\ 0 & \sqrt{1-\alpha_n}\end{pmatrix}\begin{pmatrix}\sqrt{\beta_n} & 0\\ 0 & \sqrt{1-\beta_n}\end{pmatrix}\right) = \sqrt{\alpha_n\beta_n} + \sqrt{(1-\alpha_n)(1-\beta_n)},$$

which is equal to $\rho(\mu_n, \nu_n)$ (see paragraph above Corollary 1 in [Kak48]). So we actually have that either $\phi \sim \psi$ or $\phi \perp \psi$ according to whether $\prod_{n\in\mathbb{N}} \rho(\phi_n, \psi_n)$ is positive or equal to 0. Thus in absence of Claims 2.14 and 2.15, one could prove Proposition 0.5 by combining the proofs of [Kak48] and [Bur69] (and [Pro71]).

3. Conclusion and open problems

We conclude the paper with discussions about related topics and open questions.

3.1. **Abstract theorem.** The proof of Theorem 0.2 can be used to prove a more general fact.

Theorem 3.1. *Suppose that there is a semi-normed vector space $(E, ||\cdot||)$, which has a convex subset X, contained in the closed unit ball of E, so that for any $x, y \in X$ it holds that $||x-y|| \leq 1$. Moreover, X carries a Polish topology τ, which has a basis consisting of convex sets, so that for any $\varepsilon \in (0,\infty)$ the set $\{(x,y) \in X \times X : ||x-y|| < \varepsilon\}$ is analytic with respect to τ. A subset $A \subseteq X$ is called an* antichain *if for any two $y \neq z \in A$ we have that $||y-z|| = 1$. Suppose finally that the following properties are satisfied:*

(1) for every $x \in X$, the set $\{y \in X : ||x-y|| < 1\}$ does not contain an uncountable antichain;
(2) for every $\varepsilon \in (0,1)$, $x \in X$ and any antichain $A \subseteq X$ the set $\{y \in A : ||x-y|| < \varepsilon\}$ is finite;
(3) for every $x \in X$ the set $x^\perp := \{y \in X : ||x-y|| = 1\}$ is comeagre in (X,τ).

Then for any analytic antichain $\mathcal{A}$, it holds that

$$\mathcal{A}^\perp := \{x \in X : (\forall y \in \mathcal{A})\, ||x-y|| = 1\}$$

is comeagre.

Sketch of proof. Suppose for contradiction that $\mathcal{A}$ is an analytic antichain, for which $\mathcal{A}^\perp$ is not comeagre. Then there is a non-empty convex open $O \subseteq X$ in which $\mathcal{A}^\perp$ is meagre. Let $Z \subseteq O \setminus \mathcal{A}^\perp$ be a dense (in O) G_δ set.

For $k \in \mathbb{N}$ set E_k to be the space of k-element subsets of $\mathcal{A}$ and for fixed $\varepsilon \in (0,1)$ and $\tau \in (0,\varepsilon)$ define

$$H_{k,\varepsilon} := \{x \in Z : (\exists F \in E_k)\, (\forall y \in F) ||x-y|| < \varepsilon\}$$

and

$$U^\tau_{k,\varepsilon} := H_{k,\varepsilon-\tau} \setminus H_{k+1,\varepsilon},$$

which have the property of Baire. By the same reasoning as for measures, we get that

$$Z = \bigcup_{k\geq 1} \bigcup_{n>1} \bigcup_{m>n} U^{1/m}_{k,1/n}$$

and so for some $k, \varepsilon := 1/n$ and $\tau := 1/m$ it holds that $U^\tau_{k,\varepsilon}$ is comeagre in a nonempty open convex set $V \subseteq O$. Then by an application of Kuratowski-Ulam theorem (see the proof of Claim 1.2), there is $x \in U^\tau_{k,\varepsilon} \cap V$ and a comeagre $C \subseteq U^\tau_{k,\varepsilon} \cap V$ so that for every $y \in C$ the set

$$M_y := \{t \in [0,1] : tx + (1-t)y \in U^\tau_{k,\varepsilon}\}$$

is comeagre in $[0,1]$. Defining for $y \in U^\tau_{k,\varepsilon}$ the set

$$N_y := \{z \in \mathcal{A} : ||y-z|| < \varepsilon - \tau\},$$

we observe as in the proof of Claim 1.3 that for every $z \in C$ it holds that $N_x = N_z$. Then let $y_0, \ldots, y_{k-1} \in \mathcal{A}$ be such that $N_x = \{y_0, \ldots, y_{k-1}\}$. Since for every $y \in X$ it holds that $y^\perp$ is comeagre, we get a contradiction, since

$$B := \bigcap_{i=0}^{k-1} y_i^\perp$$

and C are both comeagre in V. ☐

Remark 3.2. Note that if $X \times X \to \mathbb{R}_{\geq 0}$, defined by $(x, y) \mapsto ||x - y||$ is lower-semicontinuous with respect to τ (so that in particular $\perp$ is G_δ) and if the space of extreme points of X (which is Polish by Proposition 2.1 from [CT18]) is perfect and an antichain, then item (3) follows by an argument similar to the proof of Proposition 4.1 from [KS01].

Remark 3.3. Assuming $\mathsf{MA} + \neg\mathsf{CH}$, PD or AD we can replace "analytic" from the statement of Theorem 3.1 with "$\mathbf{\Sigma}^1_2$", "projective" or "any" respectively.

For Theorem 3.1 to have any value, other examples than measures are needed.

Question 3.4. *Are there other natural examples beside Borel probability measures which satisfy assumptions of Theorem 3.1?*

In the next subsection we discuss a possible candidate.

3.2. **Comeagreness of witnesses to non-maximality for states.** Let A be a separable unital C*-algebra. As we have seen above, there is a metric d on $S(A)$, defined by Bures in [Bur69], which satisfies that for $\phi, \psi \in S(A)$ it holds that $\phi \underline{\perp} \psi$ if and only if $d(\phi, \psi) = \sqrt{2}$.

This gives us hope that Theorem 3.1 might apply to $\frac{1}{\sqrt{2}} d$. However, we do not know whether items (2) and (3) of Theorem 3.1 are satisfied. Even if this approach fails, it might still be the case that the following has a positive answer.

Question 3.5. *Is it the case for a separable unital C*-algebra A (or for a class of C*-algebras satisfying some additional properties), that for any analytic strongly orthogonal family $\mathcal{A} \subseteq S(A)$, the set*

$$\mathcal{A}^{\underline{\perp}} := \{\psi \in S(A) : (\forall \phi \in \mathcal{A})\, \psi \underline{\perp} \phi\}$$

is comeagre?

3.3. **Measure on the space of measures.** For this subsection we work on the Cantor space $2^{\mathbb{N}}$. Observe that $P(2^{\mathbb{N}})$ is homeomorphic to

$$p(2^{\mathbb{N}}) := \{f \in [0,1]^{2^{<\mathbb{N}}} : f(\emptyset) = 1 \wedge (\forall s \in 2^{<\mathbb{N}})\, f(s) = f(s^\frown 0) + f(s^\frown 1)\}.$$

Actually, there is even an isometric bijection between the two spaces when one defines natural metrics on both spaces, which generate the respective Polish topologies, see [FT10].

Furthermore, there is a surjective continuous map $\Phi : [0,1]^{2^{<\mathbb{N}}} \to p(2^{\mathbb{N}})$, defined recursively by

$$\Phi(f)(\emptyset) := 1$$
$$\Phi(f)(s^\frown 0) := \Phi(f)(s) \cdot f(s)$$
$$\Phi(f)(s^\frown 1) := \Phi(f)(s) \cdot (1 - f(s))$$

for $f \in [0,1]^{2^{<\mathbb{N}}}$ and $s \in 2^{<\mathbb{N}}$. Let λ denote the Lebesgue measure on $[0,1]$. Then $\prod_{s \in 2^{<\mathbb{N}}} \lambda$ is a Borel probability measure on $[0,1]^{2^{<\mathbb{N}}}$ and Φ is injective on a set of measure 1. We denote the pushforward of this measure to $P(2^{\mathbb{N}})$ (via the identifications above) with Λ. Given Theorem 0.2, it is natural to ask the following question.

Question 3.6. *Suppose that $\mathcal{A} \subseteq P(2^{\mathbb{N}})$ is an analytic orthogonal family. Does it hold that $\Lambda(\mathcal{A}^{\perp}) > 0$?*

One should not get one's hopes too high and wonder whether it could be that $\Lambda(\mathcal{A}^{\perp}) = 1$, as this turns out to be false.

Claim 3.7. *For any $\mu \in P(2^{\mathbb{N}})$, $\Lambda(\mu^{\perp}) < 1$.*

Sketch of proof. Fix any $\mu \in P(2^{\mathbb{N}})$ and suppose for contradiction that $\Lambda(\mu^{\perp}) = 1$. Fix also some arbitrary small $\varepsilon \in (0, 1/2)$. For $s \in 2^{<\mathbb{N}}$, let $U_s := \{x \in 2^{\mathbb{N}} : s \subseteq x\}$. By repeated use of Fubini's theorem and the fact that for $B \subseteq [0,1]$, $\lambda(B) = 1$ implies that B is dense in $[0,1]$, we get that there is some $\nu \in \mu^{\perp}$ so that for all $s \in 2^{<\mathbb{N}}$ it holds that

$$\nu(U_s) \in ((1-\varepsilon)\,\mu(U_s), (1+\varepsilon)\,\mu(U_s)).$$

Since open sets are disjoint unions of basic open sets, the same holds for all open $U \subseteq 2^{\mathbb{N}}$. But then ν and μ are not orthogonal, which is a contradiction. $\square$

3.4. **Definable maximal orthogonal families in forcing extensions.** The original motivation for trying to find a short and simple proof of Theorem 0.1 was that maybe a new proof would help us answer the following open question (which is a reformulation of Open problem 1) from [ST18]).

Question 3.8. *Are there any Π^1_1 maximal orthogonal families $\mathcal{A} \subseteq P(2^{\mathbb{N}})$ in Laver extensions?*

The hope was also that a new proof of Theorem 0.1, would shed some light onto why some arboreal forcing notions (see [Löw98] for the definition and results on arboreal forcing) admit Π^1_1 mofs in their forcing extensions (Sacks and Miller forcing) and some do not (Mathias forcing); see [ST18].

3.5. **Nice and bad subsets.** Let X be a Polish space and A a separable unital C*-algebra. Call an analytic subset $Y \subseteq P(X)$ (respectively $Y \subseteq S(A)$) *nice*, if for every analytic pairwise orthogonal (respectively strongly orthogonal) family $\mathcal{A} \subseteq Y$, there is $\mu \in Y \cap \mathcal{A}^{\perp}$. Otherwise, call Y *bad*.

For example Theorems 0.1 and 0.4 imply that $P(X)$ and $S(A)$ (where A satisfies assumptions of Theorem 0.4) are nice. Moreover, when X is compact perfect Theorem 0.2 implies that all non-meagre analytic $Y \subseteq P(X)$ are nice and Theorems 2.12 and 0.4 imply that the sets of product measures and product states are nice.

On the other hand $\partial_{\mathrm{e}} P(X)$ is clearly bad, since $\partial_{\mathrm{e}} P(X)$ is an orthogonal family (by Proposition 4.1 of [KS01]).

Question 3.9. *Are there other natural examples of nice/bad sets?*

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Part V.

Conclusion

In this final part we collect the main open questions that appeared across the thesis and discuss prospects for future research on definability of maximal discrete sets.

Maximal almost disjoint families

For $\alpha \in \omega_1 \setminus \{0\}$, the ideals Fin^α (see Part II for a brief presentation) form a sequence of increasingly more complex Borel ideals on countable sets. As already remarked, it is our conjecture that an appropriate adaptation of the derivative technique of Part II can be used to affirmatively answer the following.

Question. *Let $\alpha \in \omega_1^{\mathrm{CK}} \setminus \{0\}$. Is it the case that for every infinite Σ^1_1 Fin^α-almost disjoint family there is a Δ^1_1 witness to non-maximality?*

It is unknown whether AD and AD^+ are equivalent (they are known to be equivalent under $V = L(\mathbb{R})$). Furthermore, it is also unknown whether AD implies that all sets of reals are Ramsey. Hence, in spite of the exceptional results recalled in Part I, the following question, originally posed by Asger Törnquist, remains open.

Question. *Does AD (if needed, together with DC) imply that there are no infinite mad families?*

We believe that the results of Part II indicate that the answer to this question should be positive.

The $\mathbb{G}_0$ dichotomy has since its introduction been used in providing alternative proofs of many result in descriptive set theory (see Part II for references).

Question. *Can the $\mathbb{G}_0$ dichotomy be used to provide a simple proof of the fact that there are no infinite analytic mad families?*

Maximal cofinitary groups

With the construction of a Σ^0_2 mcg, the following longstanding open question has gained even more weight, as the negative answer would imply that Σ^0_2 is the best possible complexity of an mcg.

Question. *Is there a G_δ mcg?*

In case the answer to the above question is negative, a significant step forward is to first refute the following.

Question. *Is there an mcg, which is closed as a subset of ω^ω?*

Although not explicitly concerning itself with definability, the following open question is crucial for a better understanding of mcgs.

Question. *Can there be an mcg with infinitely many k-orbits for some $k > 1$?*

In Part III we introduced the notion of a maximal finitely periodic group (abbreviated as mpg), which is a close relative to maximal cofinitary groups. The most important question, which we conjecture has an affirmative answer, is the following.

Question. *Is there a Borel mpg?*

Since being eventually bounded is equivalent to being contained in a K_σ set, the following is important for the study of definability of mpgs.

Question. *Can an mpg be eventually bounded?*

Our prediction is that there can be no eventually bounded mpg.

Maximal orthogonal families

In Part IV we present a very abstract Theorem 3.1, which generalises our argument from the proof of Theorem 0.2.

Problem. *Find other natural applications of Theorem 3.1.*

We believe (but have not come up with a counterexample) that assumptions of Theorem 3.1 do not hold for general separable unital C*-algebras. Nevertheless, the following might still have an affirmative answer using a different method.

Question. *Is it the case for a separable unital C*-algebra A (or for a class of C*-algebras satisfying some additional properties), that for any analytic strongly orthogonal family $\mathcal{A} \subseteq S(A)$, the set*

$$\mathcal{A}^{\perp} := \{\psi \in S(A) \mid (\forall \phi \in \mathcal{A})\, \psi \perp \phi\}$$

is comeagre?

The question that originally inspired us to search for new methods for analysing maximal orthogonal families still remains open.

Question. *Suppose that x is Laver-generic over $L[a]$. Are there any $\Pi^1_1[a]$ maximal orthogonal families $\mathcal{A} \subseteq P(2^\omega)$ in $L[a][x]$?*