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*Til Ellen*

## Abstract

In this thesis I demonstrate how certain quantum group structures can be realised from Wilson line operators in perturbative Chern-Simons theory. The relevant theory for this purpose is a so called split Chern-Simons theory on a manifold with boundaries, for which the associated Lie algebra $\mathfrak{g}$ admits a decomposition into the direct sum of a dual pair of maximal isotropic subalgebras (a Manin triple). The thesis consists of an introduction and two papers.

The first paper is joint work with Dani Kaufman. We study the limit when two parallel Wilson lines carrying different representations of $\mathfrak{g}$ come together. By explicitly computing the corresponding Feynman integrals we show, up to leading order in perturbation theory, that this operation produces a single Wilson line associated to the tensor product representation in the quantized universal enveloping algebra $\mathcal{U}_\hbar(\mathfrak{g})$. In combination with a known result of recovering the classical $r$-matrix from crossing Wilson lines in the same theory, this suggests that the category of Wilson lines is equivalent to the category of representations of $\mathcal{U}_\hbar(\mathfrak{g})$ as a braided tensor category. We point out a connection of this theory with the Fock-Goncharov moduli spaces of local systems.

In the second paper, I study Chern-Simons theory for a Lie algebra that decomposes into the direct sum of a pair of dual subalgebras of which one of them is abelian. The resulting theory can be identified with a topologically twisted 3d $\mathcal{N} = 4$ gauge theory. For this Lie algebra Feynman diagrams become particularly simple and I show, at all orders in perturbation theory, that the expectation value of a pair of crossing Wilson line operators is a solution to the quantum Yang-Baxter equation. My proof is based on a known technique for constructing knot invariants from Wilson loops in Chern-Simons perturbation theory, using the Axelrod-Singer compactification of the configuration space of Feynman diagram vertices.

# Resumé

I denne afhandling demonstrerer jeg hvordan bestemte kvantegruppestrukturer kan realiseres fra Wilson linje operatorer i perturbativ Chern-Simons teori. Den relevante teori til dette formål er en såkaldt split Chern-Simons teori på en mangfoldighed med rand, for hvilken den tilhørende Lie algebra tillader en dekomposition som en direkte sum af et dualt par af maksimale isotropiske delalgebraer. Afhandlingen består af en introduktion og to artikler.

Den første artikel er et samarbejde med Dani Kaufman. Vi studerer grænsen når to Wilson linjer, der bærer på forskellige representationer af $\mathfrak{g}$, kommer tæt på hinanden. Ved eksplicit at udregne de tilhørende Feynman integraler, viser vi, op til første orden i perturbationsteori, at denne operation producerer en enkelt Wilson linje associeret til tensorprodukt-representationen i den kvantiserede universelle omsluttende algebra $U_\hbar(\mathfrak{g})$. I kombination med et kendt resultat om udledningen af en klassisk $r$-matrix fra krydsende Wilson linjer i den samme teori, peger dette på at kategorien af Wilson linjer er ækvivalent med kategorien af representationer af $U_\hbar(\mathfrak{g})$ som en flettet tensorkategori. Vi påpeger en forbindelse af denne teori med Fock-Goncharov moduli rum a lokale systemer.

I den anden artikel studerer jeg Chern-Simons teori i en opsætning tilsvarende den første artikel, men for en Lie algebra der kan dekomponeres til en direkte sum af et par af duale delalgebraer, af hvilken den ene er abelsk. Den resulterende teori kan identificeres med en topologisk tvisted 3d $\mathcal{N} = 4$ gauge teori. For denne Lie algebra bliver Feynman diagrammer særligt simple, og jeg viser, til alle ordner i perturbationsteori, at forventningsværdien af et par krydsende Wilson linje operatorer er en løsning på Yang-Baxter ligningen. Mit bevis er baseret på en velkendt teknik til at konstruere knudeinvarianter fra Wilson loop i Chern-Simons perturbationsteori, som bruger Axelrod-Singer kompaktificeringen af konfigurationsrummet af knuder i Feynman diagrammer.

## Acknowledgements

My gratitude goes to a number of people whose support in various ways has been invaluable during my time as a PhD student: First, I would like to thank my advisors Nathalie Wahl and Kevin Costello; Nathalie for providing careful and patient guidance in navigating at the interface between physics and mathematics, and Kevin for suggesting me interesting problems to work on and for sharing with me his insights during many inspiring discussions. Another central figure in the creation of this thesis is my friend and collaborator Dani Kaufman, who pulled me out of post-pandemic apathy with an infectious enthusiasm for exploring the connections between our fields. Dani, I am deeply grateful. I also owe great thanks to Ryszard Nest for generously enlightening me with his expertise and for hosting me in Boulder. I have highly valued the company and moral support of my friends and colleagues at the math department. In particular, I wish to thank my office mates: Alexander Frei, Alexis Aumonier, Calista Bernhard, Francesco Campagna, Jeroen van der Meer, Jingxuan Zhang, Kaif Muhammad Borhan Tan, Luigi Pagano and Severin Mejak, for creating a warm and friendly atmosphere and for entertaining office conversations. A special thanks to Nena Batenburg for caring friendship and for sharing a cozy office space during weekends. Finally, thank you to my family and friends outside the university for being an immense support and a constant source of love and care – also to Nokka for fluffy therapy.

## Thesis Statement

This thesis consists of an introduction and two papers.

The first paper is joint work with Dani Kaufman and is publicly available on arXiv: 2307.10830.

The second paper is partially based on material included in my master's thesis as part of the 4+4 PhD program at the University of Copenhagen. I submitted my master's thesis in August 2021 and defended in September 2021. The paper is publicly available on arXiv: 2309.15833.

# Contents

# Part I

# Introduction

## Chern-Simons perturbation theory

The underlying gauge theory which forms the foundation for the work done in this thesis a three-dimensional topological quantum field theory known as Chern-Simons. In this introduction I give a basic non-formal introduction to the concepts of the theory that will be relevant for the topics studied in the two papers.

Chern-Simons theory is defined for a general three-manifold $M$, a Lie group $G$ and a principal $G$ bundle $E \to M$ by the action functional

$$S_{\mathrm{CS}}(A) = \frac{1}{2\pi} \int_M \mathrm{Tr}(A \wedge dA) + \frac{1}{3}\,\mathrm{Tr}(A \wedge [A, A]), \tag{1}$$

which encodes all dynamics of the theory. Here, the gauge field $A$ is a connection on $E$ given locally as a one-form on $M$ taking value in the Lie algebra $\mathfrak{g}$ of $G$, and $\mathrm{Tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ denotes an invariant non-degenerate bilinear pairing on $\mathfrak{g}$. The above gauge theory was developed by Edward Witten in a seminal paper [15], where he showed through exact (non-perturbative) methods that it provides a framework for realising the Jones knot polynomials as the expectation value of at set of observables called Wilson loops. Chern-Simons theory has since then proven widely successful for its capacity of producing non-trivial invariants of three-manifolds.

In this thesis I study Chern-Simons theory in the formalism of perturbation theory. This means that the expectation value of observables are treated as perturbative series in an expansion parameter $\hbar$ around a "free field theory" with no interactions. Interactions are encoded in Feynman diagrams which in the present theory are weighted three-valent graphs with each edge representing a closed two-form on $M \times M \backslash \mathrm{diag}$ known as the propagator. Computing the weight (amplitude) of Feynman diagrams entails integrating the associated differential form over the space of embeddings of its vertices into $M$. However, since propagators are singular along the diagonals, such integrals are ill-defined in the ultraviolet limit where vertices come close together in $M$. This subject was treated by Axelrod and Singer in [2], [3] using the following idea: Let $\mathrm{Conf}_V(M)$ denote the (configuration) space of embeddings $V \hookrightarrow M$ and let $M^V$ be the space of all maps from $V$ to $M$. Then there is an embedding

$$\mathrm{Conf}_V(M) \subset M^V.$$

For a given subset $S \subset V$, Axelrod and Singer defined a compactification of $\mathrm{Conf}_V(M)$ in the direction where the vertices in $S$ come together by replacing the diagonal $\Delta_S \subset M^V$ with the spherical blowup of $M^V$ along $\Delta_S$. Since propagators extend smoothly to the boundary of the compactified space this accounts for all ultraviolet singularities of the theory.

## A new perspective on knot invariants

The construction of Axelrod and Singer led to a way of understanding Witten's knot invariants from a perspective of perturbation theory, put forward by Kontsevich [13] and formally elaborated by Bott and Taubes [4]. In particular, Bott and Taubes extended the definitions of Axelrod and Singer to include Wilson loops in the manifold. Given a closed loop $K \subset M$ and a representation $\rho$ of the gauge group $G$, a Wilson loop $W_\rho(K)$ is a gauge invariant observable defined as the trace of holonomy of the gauge field around $K$. This is often written somewhat informally as

$$W_\rho(K) = \mathrm{Tr}_\rho\left(\mathcal{P}\exp\int_K A\right), \tag{2}$$

where $\mathcal{P}$ denotes the path ordering of the exponential and $\mathrm{Tr}_\rho$ is the trace taken in the representation $\rho$. In perturbation theory one can think of a Wilson loop simply as a smooth embedding $K : S^1 \hookrightarrow M$ along with a Feynman rule for the coupling of the gauge field to $K$. In this setting Feynman diagrams are graphs in $M$ with one-valent "external" vertices along the Wilson loop $K$ and three-valent "internal" vertices in the ambient space (figure 1).

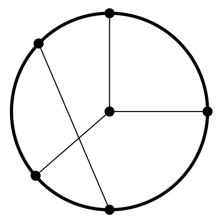

Figure 1: A Feynman diagrams in the theory with Wilson loops.

The study of Bott and Taubes was Wilson loops in $M = \mathbb{R}^3$. I here sketch the main ideas of their construction. Let $\mathscr{K}$ be a smooth one-parameter family (isotopy) of embeddings $K_t : S^1 \hookrightarrow \mathbb{R}^3$ for $t \in [0,1]$. An element $K_t \in \mathscr{K}$ induces an embedding of configuration spaces $\mathrm{Conf}_W(S^1) \hookrightarrow \mathrm{Conf}_W(\mathbb{R}^3)$ and hence there is a map

$$\mathrm{Conf}_W(S^1) \times \mathscr{K} \to \mathrm{Conf}_W(\mathbb{R}^3).$$

Bott and Taubes defined the configuration space associated to a graph with internal vertices $V$ and external vertices $W$ as the pullback

$$\begin{array}{ccc} \mathrm{Conf}_{V,W} & \longrightarrow & \mathrm{Conf}_{V\cup W}(\mathbb{R}^3) \\ \downarrow & & \downarrow \\ \mathrm{Conf}_W(S^1)\times\mathscr{K} & \longrightarrow & \mathrm{Conf}_W(\mathbb{R}^3)\ . \end{array} \tag{3}$$

Notice that $\mathrm{Conf}_{V,W}$ includes the parameter $t$ which continuously deforms the embedding. In fact, the left column in the above diagram gives a projection

$$\mathrm{Conf}_{V,W} \to \mathscr{K},$$

and the configuration space corresponding to a fixed embedding $K_t$ is the fiber $\mathrm{Conf}^t_{V,W}$ over $K_t \in \mathscr{K}$ via this projection. Now, for a given embedding $K_t \in \mathscr{K}$, the expectation value of the Wilson loop $W_\rho(K_t)$ is given by the sum over all Feynman diagrams as the one in figure 1:

$$\langle W_\rho(K_t)\rangle = \sum_\Gamma \hbar^{\mathrm{ord}(\Gamma)} I_t(\Gamma),$$

where, for each Feynman diagram $\Gamma$ the associated amplitude $I_t(\Gamma)$ is the integral of a closed differential form $\theta(\Gamma)$ over the fiber $\mathrm{Conf}^t_{V,W}$:

$$I_t(\Gamma) = \int_{\mathrm{Conf}^t_{V,W}} \theta(\Gamma).$$

To see how knot invariants can be deduced from the above construction, write $\Delta I_t(\Gamma) = I_1(\Gamma) - I_0(\Gamma)$. By Stokes' theorem it holds that:

$$\int_{\mathrm{Conf}_{V,W}} d\theta(\Gamma) = \Delta I_t(\Gamma) + \int_{\partial\,\mathrm{Conf}_{V,W}} \theta(\Gamma),$$

where $\partial\,\mathrm{Conf}_V$ is the co-dimension one boundary in the Axelrod-Singer compactification extended to include Wilson loops via the pullback diagram (3). Now, since $\theta(\Gamma)$ is a closed form, the term on the left-hand side of the above equality is zero and it follows that

$$\Delta \langle W_\rho(K_t)\rangle = \sum_\Gamma \hbar^{\mathrm{ord}(\Gamma)} \int_{\partial\,\mathrm{Conf}_{V,W}} \theta(\Gamma). \tag{4}$$

Thus, for the expectation value of Wilson loops to give knot invariants, the contributions from all boundary integrals on the right-hand side of equation (4) must either vanish identically or mutually cancel out. Bott and Taubes showed in [4], via a series of vanishing theorem, that $\langle W_\rho(K)\rangle$ defines a knot invariant up to subtracting an appropriate multiple of the self-linking number of $K$. The appearance of the self-linking number is due to an inherent feature of Chern-Simons theory known as the framing anomaly.

## Split Chern-Simons theory and quantum groups

Starting from a quasi-triangular Hopf-algebra, Reshetikhin and Turaev constructed *quantum* invariants of framed knots [14], which they suggested to be a purely mathematical realisation of Witten's knot invariants. Making this claim explicit would entail showing that similar quantum group structures can be derived in the setting of Chern-Simons theory.

In [8] and [9] Costello, Witten and Yamazaki derived quantum group structures in a four dimensional extension of Chern-Simons theory defined on a product manifold $\mathbb{R}^2\times\mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ is equipped with a holomorphic 1-form that has poles at 0 and $\infty$. A key to their constructions is the choice of appropriate boundary condition on the gauge field at the 0 and $\infty$ poles in $\mathbb{C}^\times$ which breaks the global gauge symmetry of the theory. To obtain such boundary conditions, the semi-simple Lie algebra $\mathfrak{g}$ of the theory is extended with an extra copy $\tilde{\mathfrak{h}}$ of the Cartan subalgebra to arrive at a Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}\oplus\tilde{\mathfrak{h}}$ that admits a decomposition into maximal isotropic subalgebras $\mathfrak{g} = \mathfrak{l}_- \oplus \mathfrak{l}_+$ (called a Manin triple). The gauge field is then restricted to take value in $\mathfrak{l}_-$ at $\mathbb{R}^2\times\{0\}$ and in $\mathfrak{l}_+$ at $\mathbb{R}^2\times\{\infty\}$. This theory permits a set of gauge invariant observables called Wilson lines, obtained from omitting the trace in equation (2) and replacing the closed loop $K$ by a line supported at a point in $\mathbb{C}^\times$ and extending to infinity along $\mathbb{R}^2$. To a Wilson line supported at $z \in \mathbb{C}^\times$ is associated a representation $\rho$ of $\mathfrak{g}[[z]]$ acting on a vector space $V$, and the expectation value of a set of Wilson lines $L_1, \ldots, L_k$ associated to representations $(\rho_1, V_1), \ldots, (\rho_k, V_k)$ is an element

$$\langle L_1, \ldots, L_k \rangle \in \mathrm{End}(V_1 \otimes \cdots \otimes V_k).$$

In this setting, Costello et al. realised the leading order Yangian deformation of $\mathfrak{g}[[z]]$ from operations on Wilson lines via explicit Feynman diagram computations. In particular, they derived the leading order deformation of the co-product from the operation of merging parallel Wilson lines and solutions to the Yang-Baxter equation (with spectral parameter) from the expectation value of crossing Wilson lines. It was conjectured in their second paper that similar structures can be recovered from the setting of usual three-dimensional Chern-Simons theory on a manifold with boundaries $\mathbb{R}^2 \times [-1, 1]$, where boundary conditions on the gauge field are given by restricting to $\mathfrak{l}_-$ (resp. $\mathfrak{l}_+$) on the lower (resp. upper) boundary. In a paper [1] the present author showed explicitly that this is the case by computing leading order Feynman integrals to recover solutions to the Yang-Baxter equation as the expectation value of crossing Wilson lines in the three-dimensional setting. This corresponding three dimensional gauge theory agrees with the "split" Chern-Simons theory that was studied by Cattaneo et al. [6, 5].

## Results of this thesis

The aim of this thesis is to expand on the connection between line operators in Chern-Simons perturbation theory on a three-manifold with boundaries and quantum groups.

**Paper 1** In the first paper, with Dani Kaufman, we study Wilson lines in the three dimensional setting described above. By computing the leading order Feynman diagrams contributions, we show that the operation of merging parallel Wilson lines in this theory produced the leading order deformation of the co-product in the quantized universal enveloping algebra $\mathcal{U}_\hbar(\mathfrak{g})$. Together with the result in [1], this justifies, up to leading order in perturbation theory, that the category of Wilson lines is equivalent to the category of representation $\mathrm{Rep}\,\mathcal{U}_\hbar(\mathfrak{g})$.

We point out a connection between split Chern-Simons theory with boundaries, including Wilson lines, and the moduli spaces of local systems introduced by Fock and Goncharov. In fact, Fock and Goncharov [10] introduced a set of coordinates on the moduli space of framed $G$-local systems on surfaces with punctures, marked points on the boundaries, and flags (parameterised by Borel subgroups of $G$) assigned to each marked point. These spaces are strikingly similar the setup of split Chern-Simons theory with boundaries, where boundary conditions on the gauge field are (up to an extra copy of the Cartan) given by Borel subalgebras of $\mathfrak{g}$. In work by Goncharov and Shen [12] the quantum group is constructed as part of the quantized ring of functions on the moduli space of local systems on a punctured disk with two marked points on the boundary. In light of the quantum group realisation from Wilson lines, in my work with Kaufman, we can interpret the disk as the cross section of $\mathbb{R}^2 \times I$ with a Wilson line operator extending to infinity in the "time" direction along $\mathbb{R}^2$. This is illustrated in figure 2. Understanding more extensively how the two frameworks are connected seems an interesting topic to explore.

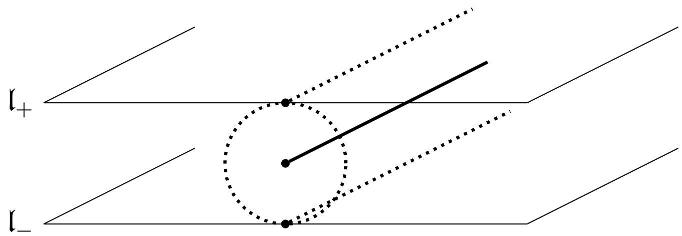


Figure 2: Cross section of a Wilson line in $\mathbb{R}^2 \times I$.

**Paper 2** In the second paper I study Chern-Simons theory for a Lie algebra that admits a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$, where $\mathfrak{a}^*$ is an abelian subalgebra of $\mathfrak{g}$, dual to $\mathfrak{a}$ with respect to the invariant pairing. For this algebra, the Chern-Simons action can be identified with the action for a three-dimensional topological BF theory. Moreover, work of Costello and Gaiotto [7] and Garner [11] shows this theory to be equivalent to a topologically twisted 3d $\mathcal{N} = 4$ gauge theory. I show that the expectation value of crossing Wilson lines in this theory solves the Yang-Baxter equation at all orders in perturbation theory. The essential idea for showing this to all orders is the use of Stokes' theorem and vanishing arguments similar to those of Bott and Taubes in [4]. Initially, my hope was to implement this technique for recovering Yang-Baxter solutions at all orders in same setup as paper 1. However, this appears to be too ambitious, as the arguments of Bott and Taubes rely on a full rotation symmetry of the propagator which in this case is broken by the boundary conditions – in fact, the broken symmetry is what allows for the recovering quantum group structures. In the appendix I give a detailed account of where the vanishing arguments seem to fail, specialising to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. In the case of 3d topological BF theory, the only contributing Feynman diagrams are trees, which turns out to account for the vanishing of all "problematic" boundary integrals.

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# Part II

# Papers

# A Wilson Line Realisation of Quantum Groups

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**Abstract**

We study Wilson line operators in 3-dimensional Chern-Simons theory on a manifold with boundaries and prove to leading order, through a direct calculation of Feynman integrals, that the merging of parallel Wilson lines reproduces the coproduct on the quantum group $U_\hbar(\mathfrak{g})$. We outline a connection of this theory with the moduli spaces of local systems defined by Goncharov and Shen.

Keywords: **Perturbation Theory, Chern-Simons Theory, Quantum Group, Wilson Line**

## 1 Introduction

Topological quantum field theories like Chern-Simons theory have long been known to have connections to quantum groups and their representations. The goal of this paper is to give an explicit realization of this connection in the setting of perturbative 3-dimensional Chern-Simons theory with boundary conditions, by showing that the tensor product in the category of representations of the quantum group can be realized from the operation of merging two parallel Wilson lines.

In recent papers [4], [5] Costello, Witten and Yamazaki constructed a 4-dimensional conformal Chern-Simons theory in which they realized the representation theory of the Yangian. They suggest in section 7.8 of their second paper that a 3-dimensional theory with boundaries can be constructed from their 4-dimensional theory by restricting to $U(1)$-invariant fields. The corresponding 3-dimensional theory was constructed explicitly by the first author in [1]. Concretely, the set-up is Chern-Simons theory on a manifold $M = \mathbb{R}^2 \times I$ with a semi-simple Lie algebra $\mathfrak{g}$. The relevant set of boundary conditions on the gauge field $A \in \Omega^1(M, \mathfrak{g})$ comes from defining a Manin triple $(\mathfrak{l}_-, \mathfrak{l}_+, \mathfrak{g})$ and restricting $A$ to take value in subalgebras $\mathfrak{l}_+$ (resp. $\mathfrak{l}_-$) on the upper (resp. lower) boundary. Since the gauge symmetry of the action is broken by the boundary conditions, this theory permits a set of gauge invariant operators given by open Wilson lines associated to representations of $\mathfrak{g}$ and extending to infinity along $\mathbb{R}^2$.

In the present paper, we consider a product on the set of Wilson lines in the above setting coming from merging two parallel lines. By computing the leading order Feynman amplitude for a gauge boson coupling to the pair of merging lines, we show that this product agrees with the leading order deformation of the tensor product in $\mathrm{Rep}(U_\hbar(\mathfrak{g}))$. It was argued in [1] that, in the same theory, the leading order contribution to the expectation value of a pair of crossing Wilson lines is given by the classical $r$-matrix. Together these results suggest that the category of Wilson line operators is equivalent to the category $\mathrm{Rep}(U_\hbar(\mathfrak{g}))$ as a braided monoidal category.

One motivation for considering this 3d Chern-Simons theory with boundary conditions instead of the 4d conformal version is its close connection to the moduli spaces of local systems on punctured surfaces considered by Goncharov and Shen, [9]. In fact, as we will discuss in the final section of this paper, our construction can be seen as a realization through perturbation theory of the "geometric avatar of a TQFT" described in section 5 of the paper of Goncharov and Shen.

# 2 The Gauge Theory

The gauge theory that we will be concerned with is 3-dimensional Chern-Simons theory defined by the action:

$$S_{\mathrm{CS}}(A) = \frac{1}{2\pi}\int_M \left\langle A \wedge dA + \frac{1}{3} A \wedge [A, A] \right\rangle \tag{2.1}$$

where the gauge field (connection) $A \in \Omega^1(M, \mathfrak{g})$ is a 1-form on $M$ taking values in the Lie algebra $\mathfrak{g}$ of the gauge group and $\langle\,,\rangle$ is an invariant symmetric bilinear form on $\mathfrak{g}$. In the present paper we will take $M = \mathbb{R}^2 \times I$ where $I = [-1, 1]$ and we take $\mathfrak{g}$ to be a complex semi-simple Lie algebra.

In order to have a well-defined theory in the presence of boundaries, we must impose boundary conditions on the gauge field. Specifically, when varying the action with respect to the gauge field, $A \to A + \delta A$ where $\delta A$ is an exact 1-form, we pick up a boundary term:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2\times\{-1,1\}} \langle A \wedge \delta A\rangle\,, \tag{2.2}$$

and we must impose boundary conditions on $A$ and $\delta A$ ensuring that this term vanishes on each boundary. At the same we want that the restriction of the gauge theory to each boundary component is in itself a well-defined, gauge invariant theory (see [4] section 9.1 for more elaboration on this). By the second requirement, choosing a set of boundary conditions amounts to specifying subalgebras $\mathfrak{l}_+, \mathfrak{l}_- \subset \mathfrak{g}$ and imposing that $A$ and $\delta A$ take value in $\mathfrak{l}_+$ (resp. $\mathfrak{l}_-$) at the upper (resp. lower) boundary. It was argued in [4] that a we get a valid set of boundary conditions giving rise to quantum group structures by choosing $\mathfrak{l}_+$ and $\mathfrak{l}_-$ so that the triple $(\mathfrak{g}, \mathfrak{l}_+, \mathfrak{l}_-)$ is a Manin triple. In other words $\mathfrak{l}_+$ and $\mathfrak{l}_-$ must be non-intersecting, half-dimensional, isotropic subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$.

# 3 Manin Triples and Quantum Groups

## 3.1 Constructing a Manin Triple

Not all semi-simple Lie algebras admit the structure of a Manin triple (for example if the Lie algebra has odd dimension). Following the construction of [4] (section 9.2) we can modify $\mathfrak{g}$ to accommodate for this by adding another copy of the Cartan subalgebra. We give here the construction in full detail.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and consider the root system $\Phi$ of $\mathfrak{g}$ relative to $\mathfrak{h}$ equipped with a polarization $\Phi = \Phi_+ \sqcup \Phi_-$. We write $\mathfrak{n}_+$ and $\mathfrak{n}_-$ for the sum over root spaces $\mathfrak{g}_\alpha$ corresponding to the set of positive roots $\alpha \in \Phi_+$ and negative roots $-\alpha \in \Phi_-$, respectively. Then $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are isotropic subalgebras and we get a decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Now add to $\mathfrak{g}$ another copy $\tilde{\mathfrak{h}}$ of the Cartan subalgebra:

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \tilde{\mathfrak{h}}$$

with the bracket on $\mathfrak{g}$ trivially extended to $\tilde{\mathfrak{g}}$, i.e. $[a, \tilde{b}] = 0$ for $a \in \mathfrak{g}$ and $\tilde{b} \in \tilde{\mathfrak{h}}$. We can extend the Killing form on $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$ as follows: $\langle a, \tilde{b} \rangle = 0$ for all $a \in \mathfrak{g}$, $\tilde{b} \in \tilde{\mathfrak{h}}$ and $\langle \tilde{a}, \tilde{b} \rangle = \langle a, b \rangle$ for all $a, b \in \mathfrak{h}$. This gives an invariant symmetric bilinear form on $\tilde{\mathfrak{g}}$. Define

$$\begin{aligned} \mathfrak{h}_+ &= \left\{ h + \mathbf{i}\tilde{h} \,\middle|\, h \in \mathfrak{h} \right\} \subset \mathfrak{h} \oplus \tilde{\mathfrak{h}} \\ \mathfrak{h}_- &= \left\{ h - \mathbf{i}\tilde{h} \,\middle|\, h \in \mathfrak{h} \right\} \subset \mathfrak{h} \oplus \tilde{\mathfrak{h}}\,. \end{aligned}$$

With this definition $\langle \, , \rangle$ vanishes on $\mathfrak{h}_+$ and $\mathfrak{h}_-$ and thereby choosing

$$\mathfrak{l}_- = \mathfrak{n}_- \oplus \mathfrak{h}_- \quad \text{and} \quad \mathfrak{l}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}_+. \tag{3.1}$$

the triple $(\tilde{\mathfrak{g}}, \mathfrak{l}_+, \mathfrak{l}_-)$ is a Manin triple.

**Conventions** Let $r$ be the rank of $\mathfrak{g}$. We fix a choice of simple roots $\Delta_+ = \{\alpha_i\}_{i=1}^r \subset \Phi_+$ along with a basis $\{H_i\}_{i=1}^r$ of $\mathfrak{h}$. Furthermore, we let $X_\alpha$ be a generator of the root space $\mathfrak{g}_\alpha$ normalized so that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ and $[X_{\alpha_i}, X_{-\alpha_i}] = H_i$. Using standard notation, we write $E_i, F_i := X_{\alpha_i}, X_{-\alpha_i}$ for each simple root $\alpha_i \in \Delta_+$ and we write $H_i^+ = H_i + \mathbf{i}\tilde{H}_i$ and $H_i^- = H_i - \mathbf{i}\tilde{H}_i$. A basis $\mathcal{B}_+$ for $\mathfrak{l}_+$ can now be given as:

$$\mathcal{B}_+ = \{X_\alpha, H_i^+\}_{\alpha \in \Phi_+, i=1,\ldots,r}.$$

Let $\mathcal{B}_-$ be the basis for $\mathfrak{l}_-$ dual to $\mathcal{B}_+$ with respect to the Killing form. Then $\mathcal{B} = \mathcal{B}_+ \cup \mathcal{B}_-$ is a basis for $\tilde{\mathfrak{g}}$. We write

$$\mathcal{B} = \{t_a\}_{a=1,\ldots,\dim \tilde{\mathfrak{g}}}\;, \quad \mathcal{B}_+ = \{t_a\}_{a=1,\ldots,\dim \tilde{\mathfrak{g}}/2}\,.$$

Finally, we denote by $t^a$ the dual element of $t_a \in \mathcal{B}$.

## 3.2 Quantization

In this subsection we briefly recall the construction of a quantum double via the Drinfel'd double construction. For a detailed exposition we refer the reader to e.g. [8] section 4.

Let $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h} \subset \mathfrak{g}$ be the Borel subalgebra relative the to setup of section 3.1 and let $(a_{ij})$ be the Cartan matrix of $\mathfrak{g}$. The Drinfel'd double $\mathfrak{D}_\hbar(\mathfrak{b})$ is the algebra over $\mathbb{C}[[\hbar]]$ with generators:

$$\{E_i, F_i, H_i^+, H_i^- \mid i = 1, \dots, r\}$$

and relations

$$\begin{aligned} [H_i^\pm, E_j] &= a_{ij} E_j & \qquad [H_i^\pm, H_j^\pm] &= [H_i^\pm, H_j^\mp] = 0 \\ [H_i^\pm, F_j] &= -a_{ij} F_j & \qquad [E_i, F_j] &= \delta_{ij} \frac{e^{\hbar H_i^+/2} - e^{-\hbar H_i^-/2}}{e^{\hbar/2} - e^{-\hbar/2}} \end{aligned} \tag{3.2}$$

along with the quantum Serre relations for $i \neq j$. In the case of $\mathfrak{sl}_n(\mathbb{C})$ these relations take the form

$$\begin{aligned} E_i^2 E_j - (e^{\hbar/2} + e^{-\hbar/2}) E_i E_j E_i + E_i E_j^2 &= 0 \\ F_i^2 F_j - (e^{\hbar/2} + e^{-\hbar/2}) F_i F_j F_i + F_i F_j^2 &= 0. \end{aligned}$$

For the general case see e.g. [8] section 4.2. The quantized universal enveloping algebra of $\mathfrak{g}$ is constructed from the double as

$$U_\hbar(\mathfrak{g}) \coloneqq \mathfrak{D}_\hbar(\mathfrak{b}) / \langle H_i^+ - H_i^- \rangle .$$

It holds that $\mathfrak{D}_\hbar(\mathfrak{b})$ has the structure of a quasi-triangular Hopf algebra with co-product:

$$\begin{aligned} \Delta E_i &= 1 \otimes E_i + E_i \otimes 1 + \frac{\hbar}{4}(E_i \otimes H_i^+ - H_i^+ \otimes E_i) + \mathcal{O}(\hbar^2) \\ \Delta F_i &= 1 \otimes F_i + F_i \otimes 1 + \frac{\hbar}{4}(F_i \otimes H_i^- - H_i^- \otimes F_i) + \mathcal{O}(\hbar^2) \\ \Delta H_i^\pm &= 1 \otimes H_i^\pm + H_i^\pm \otimes 1. \end{aligned} \tag{3.3}$$

Notice that this realizes the usual co-product on the universal enveloping algebra as the limit $\hbar \to 0$ and we have that $\mathfrak{D}_\hbar(\mathfrak{b}) \cong \mathfrak{D}(\mathfrak{b})[[\hbar]]$ as $\mathbb{C}[[\hbar]]$ modules, where $\mathfrak{D}(\mathfrak{b}) = U(\tilde{\mathfrak{g}})$.

*Remark* 1. Often in the theory of quantum groups one defines

$$K_i^\pm = q^{H_i^\pm}, \quad q = \exp(\hbar/2)$$

for which the (non-perturbative) co-product takes the form

$$\Delta E_i = (K_i^+)^{-1/2} \otimes E_i + E_i \otimes (K_i^+)^{1/2}.$$

Since we are realizing the co-product in the setting of perturbation theory, it will be more convenient to use equation (3.2) and (3.3) as our convention.

**Lemma 1.** The leading order correction to the co-product on a general basis element $t_a \in \mathcal{B}_+$ is given by

$$\Delta_{(1)} t_a = \frac{1}{2} \sum_{b,c=1}^{n/2} \big( f_a{}^{bc} \, t_{b,V} \otimes t_{c,V'} \big). \tag{3.4}$$

Recall the definition of the structure constant:

$$[t_a, t_b] = \sum_{c=1}^{n} f_{ab}{}^{c} t_c \ , \quad f_{abc} = \langle [t_a, t_b], t_c \rangle \, . \tag{3.5}$$

*Proof.* One checks that this formula agrees with the co-product in equation (3.3) on the algebra generators $E_i, H_i^+$ and that it commutes with the bracket. □

**The $R$-matrix** Another part of the quasi-triangular Hopf algebra structure is an $R$-matrix element $R \in \mathfrak{D}_\hbar(\mathfrak{b}) \otimes \mathfrak{D}_\hbar(\mathfrak{b})$ given by

$$R = 1 + \sum \hbar^k r^{(k)}, \tag{3.6}$$

where each $r^{(k)}$ is an element of $\mathfrak{D}(\mathfrak{b}) \otimes \mathfrak{D}(\mathfrak{b})$. The element $r := r^{(1)}$ is known as the classical $R$-matrix and is given by

$$r = \sum_{a=1}^{n/2} t_a \otimes t^a \, . \tag{3.7}$$

The category of representations of $\mathfrak{D}_\hbar(\mathfrak{b})$ is a braided monoidal category with monoidal product coming from the co-product in equation (3.3) and braiding coming the $R$-matrix in equation (3.7).

# 4 Perturbation Theory

## 4.1 The Propagator

The remainder of this paper studies the expectation value of operators in the theory in the setting of perturbation theory. An essential ingredient for this is constructing a propagator, which can be thought of as the probability distribution for a gauge boson traveling between two points on the manifold. The propagator in the present setting is a Lie algebra valued two-form $P \in \Omega^2\big((M \times M) \setminus \mathrm{diag},\, \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}\big)$ such that $P$ is a Green's function for the differential operator $d$, that is

$$dP(x, y) = \delta^{(3)}(x, y)\, \mathscr{C}(\tilde{\mathfrak{g}}), \tag{4.1}$$

where $\delta^{(3)}(x, y)$ is the 3-dimensional Dirac delta distribution localized at $x = y$ and $\mathscr{C}(\tilde{\mathfrak{g}}) \in \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ is the Casimir element of $\tilde{\mathfrak{g}}$. Moreover we impose the a set of boundary conditions on the propagator coming from the boundary conditions on the gauge field in equation (3.1): Write $\partial_+ M$ for the upper boundary $\partial_+ M = \mathbb{R}^2 \times \{1\}$ and $\partial_- M$ for the lower boundary $\partial_- M = \mathbb{R}^2 \times \{-1\}$. We require that

(i) the restrictions $P\big|_{\partial_+ M \times M}$ and $P\big|_{M \times \partial_- M}$ takes value in $\mathfrak{l}_+ \otimes \mathfrak{l}_-$,

(ii) the restrictions $P\big|_{\partial_- M \times M}$ and $P\big|_{M \times \partial_+ M}$ takes value in $\mathfrak{l}_+ \otimes \mathfrak{l}_-$.

A two-form satisfying the equation (4.1) along with the above boundary constraints can be constructed as follows: Let $\omega = f\, \mathrm{vol}_{S^2} \in \Omega^2(S^2)$ where $\mathrm{vol}_{S^2}$ is the unit volume form on $S^2$ given in terms of the coordinates on $\mathbb{R}^3$ by

$$\mathrm{vol}_{S^2} = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy,$$

and $f \in C^\infty(S^2)$ satisfying the following properties:

(i) $f$ is only supported in a small neighbourhood of "the north pole" $x_{np} = (0,0,1)$

(ii) $f$ is symmetric under rotations around the axis through $x_{np} = (0,0,1)$ and $x_{sp} = (0,0,-1)$

(iii) $\int_{S^2} f \,\mathrm{vol}_{S^2} = 1$.

Furthermore, let $\phi : M \times M \setminus \mathrm{diag} \to S^2$ be the map

$$\phi(x,y) = \frac{y-x}{|y-x|} \tag{4.2}$$

and let $R$ be the orientation reversing map, $R : S^2 \to S^2$, $R(x) = -x$. We now define the propagator as the pull back

$$P = \phi^*\big(\omega\ r^+ - R^*\omega\ r^-\big), \tag{4.3}$$

where $r^+ \in \mathfrak{l}_+ \otimes \mathfrak{l}_-$ and $r^- \in \mathfrak{l}_- \otimes \mathfrak{l}_+$ are uniquely determined by the constraint in equation (4.1). To see this, notice first that since $P$ is the pull back of a top-dimensional form on $S^2$ it holds that $dP(x,x')$ vanishes for all $x, x'$ with $x \neq x'$. Now fix $x' = 0$ and consider the integral of $dP(x,0)$ when $x$ is in the unit ball around 0. by Stokes' theorem we have

$$\int_{x\in B} dP(x,0) = \int_{S^2} P(x,0) = \int_{S^2} \big(\omega(x)\, r^+ - R^*\omega(x)\, r^-\big) = r^+ + r^- \,.$$

This fixes $r^+$ and $r^-$, namely

$$r^+ = r \ , \quad r^- = T \circ r\,. \tag{4.4}$$

where $r$ is the classical $R$-matrix given in equation (3.7) and $T$ is the map that swaps the tensor factors.

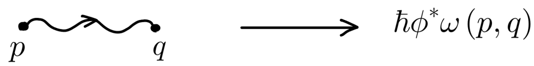
## 4.2 Feynman Diagrams

In perturbation theory, the expectation value of an observable is computed as an expansion in the parameter $\hbar$ in terms of a set of weighted graphs (Feynman diagrams). The weight of a given graph is determined from a set of Feynman rules derived from the Chern-Simons action in equation (2.1). By a Feynman diagram in the present setting we mean the following:

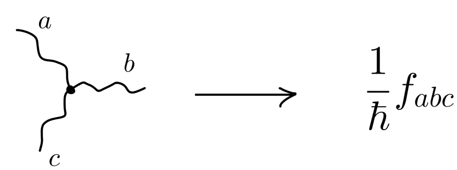
**Definition 1.** A Feynman diagram is a directed trivalent graph with leaves (external half-edges) and with the half-edges decorated by elements of $\mathcal{B}$ such that: A half-edge labeled by $t_a \in \mathcal{B}_+$ is connected by an edge to a half-edge labeled by $t^a \in \mathcal{B}_-$ with the edge orientation going from $\mathcal{B}_-$ to $\mathcal{B}_+$.

**Feynman rules** The Feynman rules outlined below associates to any Feynman diagram $\Gamma$ a differential form on the space of embeddings of the vertices of $\Gamma$ into $\mathbb{R}^2 \times I$. The weight of a Feynman diagram is computed as the integral of the associated differential form over the space of embeddings.

(1) An edge going from a vertex at $p \in \mathbb{R}^2 \times I$ to a vertex at $q \in \mathbb{R}^2 \times I$ contributes a two-form $\hbar\, \phi^* \omega(p, q)$ coming from the propagator (see remark 2 below).




(2) An internal vertex with incident edges labeled by basis elements $t_a, t_b, t_c \in \mathcal{B}$ contributes a factor structure constant $\frac{1}{\hbar} f_{abc}$.




Recall that the structure constant is given by $f_{abc} = \langle [t_a, t_b], t_c \rangle$.

(3) An external half-edge labeled by $t_a \in \mathcal{B}$ and connected a vertex at $p \in \mathbb{R}^2 \times I$ contributes a gauge field $A^a(p)$.

*Remark* 2. Write $P = \sum_{ab} P^{ab} t_a \otimes t_b$. We note that, in a free Chern-Simons theory (with no boundary conditions), one would consider Feynman diagrams with unoriented edges and with half-edges labeled by general elements of $\mathcal{B}$. To an edge with half-edges labeled by $t_a$ and $t_b$, the Feynman rules would associate the component $P^{ab}(x, y)$. However, as seen from equation (4.3), the boundary conditions in the present theory split the propagator into two parts corresponding to the two edge orientations, and we can therefore choose as a convention to define Feynman diagrams with oriented and sum over all edge orientations.

### 4.3 Wilson Lines in perturbation theory

A common set of gauge invariant observables to study in Chern-Simons theory is the so called Wilson loops. Given a closed loop $\gamma \subset M$ and a representation $V$ of $\mathfrak{g}$ the associated Wilson loop is defined as the trace of the holonomy of the gauge field around $\gamma$:

$$\begin{aligned} W_V(\gamma) &= \mathrm{Tr}_V \left( \mathscr{P} \exp \int_\gamma A \right) \\ &\coloneqq \mathrm{Tr}(1_V) + \int_\gamma dx^i A_i^a(x)\ \mathrm{Tr}(t_{a,V}) + \int_\gamma dx^i \int^x dx'^j A_i^a(x) A_j^b(x')\ \mathrm{Tr}(t_{a,V}\, t_{b,V}) + \dots \end{aligned}$$

where $\mathscr{P}$ means the path ordering of the exponential and we use the notation $t_{a,V}$ to denote the basis element $t_a$ acting in the representation $V$. In this paper, we consider instead a set of operators called Wilson lines coming from omitting the trace and replacing the closed loop $\gamma$ with an open line $L$ extending to infinity along $\mathbb{R}^2$. In the setting of perturbation theory, we think of a Wilson line $L(V)$ simply as a pair $(L, V)$, and we allow a gauge field $A^a$ to couple to $L(V)$ by inserting a basis element $t_{a,V}$ at the corresponding point on $L$. In other words, we expand the definition of Feynman diagrams to include graphs with univalent vertices along $L$, with the additional Feynman rule that a vertex on $L$ with incident half-edge labeled by $t_a$ contributes an element $t_{a,V}$.

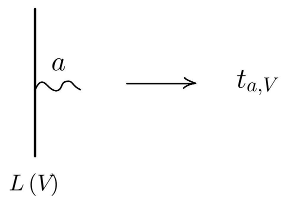



Figure 1: Feynman rule for the coupling of a gauge field $A^a$ to the Wilson line $L(V)$.

# 5 Quantum Groups and Wilson lines

## 5.1 Merging parallel Wilson lines

The study of the remainder of the paper will be the product of two parallel Wilson line operators in the limit when the lines come close together. We fix a set of coordinates $(x, y, z)$ on $\mathbb{R}^2 \times I$ with $(x, y)$ coordinates in $\mathbb{R}^2$ and $z$ the coordinate along $I$. Let $L(V)$ be a Wilson line supported at $x = z = 0$ and $L_\varepsilon(V')$ a Wilson line supported at $x = \varepsilon$, $z = 0$. We write $L(V)L_\varepsilon(V')$ to mean the disjoint union of the lines $L$ and $L_\varepsilon$ such that a gauge field $A^a$ couples to the line $L$ by inserting an element $t_{a,V} \otimes 1_{V'}$ and to the line $L'$ by inserting an element $1_V \otimes t_{a,V'}$. In general, the coupling of an external gauge field to the two Wilson lines is given by a perturbative expansion in $\hbar$ using the Feynman rules in section 4.2:

$$\mathcal{A}_a\big(L(V)L_\varepsilon(V')\big) = \sum_{k=0}^{\infty} \hbar^k \mathcal{A}_a^{(k)}\big(L(V)L_\varepsilon(V')\big) \in \mathrm{End}(V \otimes V') \tag{5.1}$$

where each element $\mathcal{A}_a^{(k)}\big(L(V)L_\varepsilon(V')\big) \in \mathrm{End}(V \otimes V')$ is computed as the weighted sum over Feynman diagrams with a single external half-edge (leaf) labeled by $t_a$ and with the number of internal edges minus the number of internal vertices equal to $k$. In the limit $\varepsilon \to 0$ one would expect equation (5.1) to reproduce the expression for an external gauge field coupling to a single Wilson line at $L$. It is however not immediately clear what representation should be associated to the merged Wilson line.

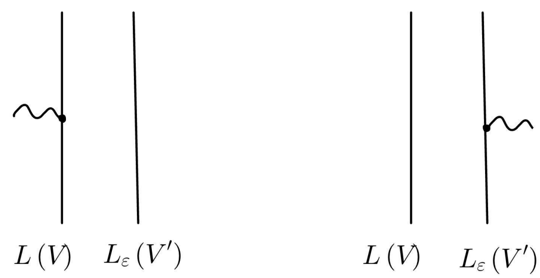



Figure 2: The classical level Feynman diagrams for an external gauge field coupling to the two Wilson lines.

At the classical level the gauge field simply couples to each line individually as shown in figure 2, and the corresponding Feynman amplitude is given by

$$\mathcal{A}_a^{(0)}\big(L(V)L_\varepsilon(V')\big) = \int_{q\in L} A^a(q)\, t_{a,V}\otimes 1_{V'} + \int_{q'\in L_\varepsilon} A^a(q')\, 1_V \otimes t_{a,V'}.$$

Taking the limit $\varepsilon \to 0$ on the right-hand side in the above we get

$$\int_{q\in L} A^a(x)\left(t_{a,V}\otimes 1_{V'} + 1_V\otimes t_{a,V'}\right),$$

which is the expression for a gauge field coupling to a single Wilson line at $L$ in the tensor product representation $V\otimes V'$. Hence, at the classical level we have

$$\lim_{\varepsilon\to 0}\mathcal{A}_a^{(0)}\big(L(V)L_\varepsilon(V')\big) = \mathcal{A}_a^{(0)}\big(L(V\otimes V')\big). \tag{5.2}$$

The object of the remainder of this paper is to carry out the computation of the leading order contribution $\mathcal{A}_a^{(1)}\big(L(V)L_\varepsilon(V')\big)$ in the limit $\varepsilon\to 0$. As we shall see, this gives a correction to the tensor product $V\otimes V'$ in equation (5.2) which agrees with the leading order quantum deformation of the tensor product in $\mathfrak{D}_\hbar(\mathfrak{b})$ given in equation (3.3). This is expressed in the following theorem:

**Theorem 1.** It holds that

$$\lim_{\varepsilon\to 0}\mathcal{A}_a^{(1)}\big(L(V)L_\varepsilon(V')\big) = \mathcal{A}_a^{(1)}\big(L(V\otimes_\hbar V')\big),$$

where $V\otimes_\hbar V'$ is the tensor product in $\operatorname{Rep}\mathfrak{D}_\hbar(\mathfrak{b})$ defined via the co-product in equation (3.3).

Notice that lemma 1 in section 3.2 defines the relevant co-product on a general basis element $t_a \in \mathcal{B}_+$ and it follows that:

$$\mathcal{A}_a^{(1)}\big(L(V\otimes_\hbar V')\big) = \frac{1}{2}\sum_{b,c=1}^{\dim\tilde{\mathfrak{g}}/2}\left(f_a{}^{bc}\; t_{b,V}\otimes t_{c,V'}\right)\int_{q\in L} A^a(q). \tag{5.3}$$

We conjecture theorem 1 to hold at all orders in perturbation theory. However, explicitly computing the contributing Feynman integrals at higher orders appears to be too difficult a task, and a proof would therefore require different techniques.

## 5.2 The Configuration Space of Vertices

The differential form associated to a Feynman diagram $\Gamma$ is defined on the configuration space of vertices of $\Gamma$. We here give a definition of the relevant configuration space in the presence of Wilson lines $L, L_\varepsilon$, and we refer the reader to [3] for a more general definition. As we are interested in studying the limit when $\varepsilon$ goes to 0 it will be convenient to think of $\varepsilon$ as a parameter in the configuration space.

**Definition 2.** For $n_1, n_2, m \in \mathbb{Z}_{\geq 0}$ define $\mathrm{Conf}_{n_1,n_2,m}$ to be the space of points

$$\{\varepsilon, q_1, \ldots q_{n_1}, q'_1, \ldots, q'_{n_2}, p_1, \ldots, p_m\},$$

where $\varepsilon \in [0, \infty)$ and

$$\begin{aligned} &q_1, \ldots q_{n_1} \in L \quad \text{with} \quad q_i \neq q_j\,, \\ &q'_1, \ldots, q'_{n_2} \in L_\varepsilon \quad \text{with} \quad q'_i \neq q'_j\,, \\ &p_1, \ldots, p_m \in (\mathbb{R}^2 \times I) \setminus \{q_1, \ldots, q_{n_1}, q'_1, \ldots, q'_{n_2}\} \quad \text{with} \quad p_i \neq p_j\,. \end{aligned}$$

Furthermore, consider the projection onto the first factor

$$\mathrm{Conf}_{n_1,n_2,m} \to (0, \infty).$$

We denote by $\mathrm{Conf}^{\varepsilon}_{n_1,n_2,m}$ the fiber of this projection over a fixed $\varepsilon \in (0, \infty)$.

## 5.3 Contributing Diagrams at Leading Order

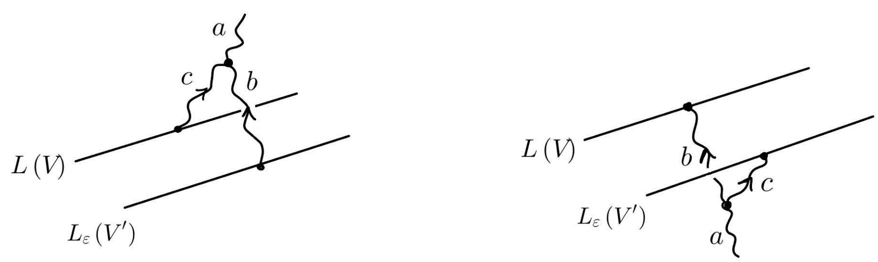



Figure 3: Contributing diagrams.

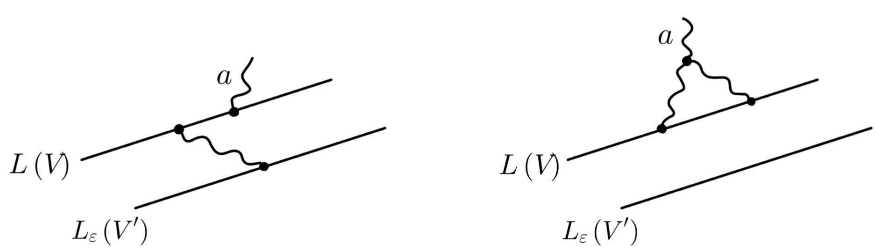



Figure 4: Vanishing diagrams.

**Lemma 2.** The only Feynman diagrams contributing to $\mathcal{A}_a^{(1)}\big(L(V)L_\varepsilon(V')\big)$ are the ones shown in figures 3.

*Proof.* Recall that the diagrams contributing to $\mathcal{A}_a^{(1)}\big(L(V)L_\varepsilon(V)\big)$ has the number of internal edges minus the number of internal vertices equal to one. This gives precisely the diagrams shown in figure 3 and 4. Consider first the diagram on the left-hand side of figure 4. Since the lines are in the same plane parallel to the boundary, the form vanishes due to the propagator only being non-zero in a small neighbourhood of the north pole. Consider now diagram on the right-hand side of figure 4. The associated configuration space is $\mathrm{Conf}_{2,0,1}$. Let $G < \mathrm{Homeo}(\mathbb{R}^3)$ be the subgroup of scalings and translations along $L$ and consider the quotient map:

$$\mathrm{Conf}_{2,0,1} \to \mathrm{Conf}_{2,0,1}/G \tag{5.4}$$

The subgroup $G$ is two-dimensional and hence the space $\mathrm{Conf}_{2,0,1}/G$ has dimension $5-2=3$. On the other hand, let $P_1 \wedge P_2 \in \Omega^4(\mathrm{Conf}_{2,0,1})$ be the product of propagators associated to the internal edges, that is,

$$P_1 \wedge P_2\,(q_1, q_2, p) \coloneqq \phi^*\omega(q_1, p) \wedge \phi^*\omega(q_2, p).$$

By definition the propagator is invariant under scalings and translations along the $L$ and hence the form $P_1 \wedge P_2$ factors through the quotient map in equation (5.4). By dimensional counting, this implies that $P_1 \wedge P_2$ vanishes. □

Consider therefore the diagrams in figure 3. We can assume that $t_a \in \mathcal{B}_+$ since the computation for $t_a \in \mathcal{B}_-$ is entirely analogous. In this case, the only contribution to the expectation value comes from the diagram on the right-hand side of figure 3. In fact, the internal vertex of diagram on the left-hand side of figure 3 has all three incident half edges labeled by elements $t_a, t_b, t_c \in \mathcal{B}_+$. By the Feynman rules in section 4.2 this vertex is assigned a structure constant $f_{abc} = \langle [t_a, t_b], t_c\rangle$ which is zero since the Killing form vanishes on $\mathfrak{l}_+$. The Feynman amplitude coming from the diagram on the right-hand side of figure 3 takes the form

$$\mathcal{A}_a^{(1)}\big(L(V)L_\varepsilon(V')\big) = \sum_{b,c=1}^{\dim\tilde{\mathfrak{g}}/2} \big(f_a{}^{bc}\; t_{b,V} \otimes t_{c,V'}\big)\, \mathcal{I}_\varepsilon\,, \tag{5.5}$$

where

$$\mathcal{I}_\varepsilon \coloneqq \int_{\mathrm{Conf}^\varepsilon_{1,1,1}} A^a(p) \wedge \phi^*\omega(p,q) \wedge \phi^*\omega(p,q'). \tag{5.6}$$

## 5.4 A Configuration Space Compactification

Since the propagator is only defined away from the diagonal, it is not clear what will happen to the integral $\mathcal{I}_\varepsilon$ in equation (5.6) when $p \to q, q'$. In order compute $\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon$, we must therefore define a partial compactification $\overline{\mathrm{Conf}}_{1,1,1}$ of $\mathrm{Conf}_{1,1,1}$ in the direction $\varepsilon \to 0$ such that the integrand extends smoothly to the corresponding boundary $\partial_\varepsilon \overline{\mathrm{Conf}}_{1,1,1}$. Then we have

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon = \int_{\partial_\varepsilon\overline{\mathrm{Conf}}_{1,1,1}} A^a(p) \wedge \phi^*\omega(p,q) \wedge \phi^*\omega(p,q')\,. \tag{5.7}$$

To this aim, we use the so called Fulton-MacPherson configuration space compactification. This compactification was originally due to Fulton and MacPherson [7] and applied to Chern-Simons perturbation theory by Axelrod and Singer [2] and (in the presence of Wilson loops) Bott and Taubes [3]. In this compactification $\partial_\varepsilon \overline{\mathrm{Conf}}_{1,1,1}$ can be divided into the disjoint union of strata

$$\partial_\varepsilon \overline{\mathrm{Conf}}_{1,1,1} = \bigcup_{i=1}^{3} \partial_i \overline{\mathrm{Conf}}_{1,1,1}$$

corresponding to the following cases:

(a) The internal vertex $p$ remains far from the lines compared to $\varepsilon$ as $\varepsilon \to 0$.

(b) The internal vertex $p$ moves close to the lines and at least one vertex $q$ or $q'$ remains far from $\varepsilon \to 0$.

(c) All three vertices move close to each other as $\varepsilon \to 0$.

**Lemma 3.** We get no contribution to equation (5.7) coming from the boundary stratum $\partial_1 \overline{\mathrm{Conf}}_{1,1,1}$ corresponding to case (a) in the above.

*Proof.* When $p$ is far from the lines the integrand in equation (5.6) extends smoothly to the boundary coming from allowing $\varepsilon \to 0$, and the corresponding boundary stratum takes the form

$$\partial_1 \overline{\mathrm{Conf}}_{1,1,1} = \left\{ (q, q', p) \in L \times L \times (\mathbb{R}^2 \times I \setminus L) \;\middle|\; p \neq q, q' \right\}.$$

The contribution to equation (5.7) is given by

$$\int_{p \in (\mathbb{R}^2 \times I) \setminus L} A(p) \wedge \left( \int_{q \in L} \phi^* \omega(q, p) \right) \wedge \left( \int_{q' \in L} \phi^* \omega(q', p) \right), \tag{5.8}$$

which is zero since the last two factors are identical one forms. □

**Lemma 4.** We get no contribution to equation (5.7) coming from the boundary stratum $\partial_2 \overline{\mathrm{Conf}}_{1,1,1}$ corresponding to case (b) in the above.

*Proof.* This follows from the property that $\omega$ is only non-zero in a small neighbourhood of the north pole. In fact, assume that $p$ is approaching some point $q \in L$ and that $q'$ remains far from $p$ as $\varepsilon \to 0$. Because the Wilson lines are in the same plane parallel to the boundary it holds that, given any $\eta > 0$ there is a $\delta > 0$ such that, if we define $U \subset \mathrm{Conf}_{1,1,1}$ to be the neighbourhood where $|p - q'| > \eta$ and $|p - q| < \delta$ then $\phi^* \omega(p, q') = 0$ for all $(p, q, q') \in U$. The situation is illustrated in figure 5 □

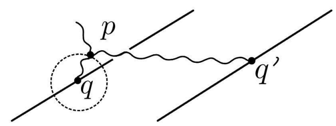



Figure 5: Neighbourhood of $\mathrm{Conf}_{1,1,1}$ where $p$ is close to $q$ and far from $q'$.

By lemma 3 and 4, the only contribution to equation (5.7) comes from the boundary stratum $\partial_3 \mathrm{Conf}_{1,1,1}$ corresponding to all three vertices coming together as $\varepsilon \to 0$. To define the corresponding boundary stratum we need the following definition:

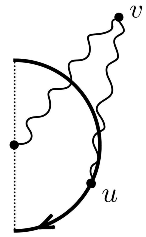



Figure 6: The space $\mathcal{H}$.

**Definition 3.** Let $S_r$ be the "right" half of the unit circle with center $(0,0)$, i.e.

$$S_r = \left\{(x, y, 0) \in \mathbb{R}^2 \times I \;\middle|\; x^2 + y^2 = 1\,,\; x > 0\right\}$$

We define

$$\mathcal{H} = \left\{(u, v) \in S_r \times \mathbb{R}^3 \setminus \{0\} \;\middle|\; v \neq u\right\}.$$

The space $\mathcal{H}$ is illustrated in figure 6.

**Lemma 5.** We can define a partial compactification of $\mathrm{Conf}_{1,1,1}$ in the direction where all three vertices come together, such that the compactified space is a manifold with boundary and corresponding boundary stratum $\partial_3 \overline{\mathrm{Conf}}_{1,1,1}$ is given by

$$\partial_3 \overline{\mathrm{Conf}}_{1,1,1} = L \times \mathcal{H}\,.$$

*Proof.* For some small $\eta > 0$, define $U \subset \mathrm{Conf}_{1,1,1}$ by

$$U = \left\{(\varepsilon, q, q', p) \in \mathrm{Conf}_{1,1,1} \;\middle|\; |p - q| < \eta \text{ and } |q' - q| < \eta\right\}.$$

Furthermore, define $V \subset (0, \eta) \times L \times \mathcal{H}$ by

$$V = \left\{(t, q_0, (u, v)) \in (0, \eta) \times L \times \mathcal{H} \;\middle|\; |v|t < \eta\right\}, \tag{5.9}$$

There exists a diffeomorphism $\varphi : V \to U$ defined by $(t, q_0, (v, u)) \mapsto (\varepsilon, q, q', p)$, where

$$\varepsilon = u_y \ , \quad q = q_0 \ , \quad q' = q_0 + tu \ , \quad p = q_0 + tv \, , \tag{5.10}$$

with $u_y$ denoting the $y$-coordinate of $u$. This implies that

$$\overline{\mathrm{Conf}}_{1,1,1} \coloneqq \mathrm{Conf}_{1,1,1} \cup_V \overline{V},$$

where

$$\overline{V} = \big\{(t, q_0, (u, v)) \in [0, \eta] \times L \times \mathcal{H} \ \big| \ |v|t < \eta\big\},$$

is a manifold with boundary. Letting all three vertices come together in $\mathrm{Conf}_{1,1,1}$ corresponds to letting $t \to 0$ in $\overline{V}$ and the lemma follows. □

## 5.5 Proof of Theorem 1

We are now equipped to prove theorem 1. Notice first that with the change of coordinates given in equation (5.10), we have

$$A(p) = A(q + tv) \ , \quad \phi^*\omega(p, q) = \phi^*\omega(0, v) \ , \quad \phi^*\omega(p, q') = \phi^*\omega(v, u).$$

All of the above forms extends continuously to the boundary $\partial_3 \overline{\mathrm{Conf}}_{1,1,1}$ corresponding to the limit $t \to 0$. Hence, by equation (5.7) and lemma 3, 4 and 5 in the previous subsection, it holds that

$$\lim_{\varepsilon \to 0} \mathcal{I}(\varepsilon) = \int_{q \in L} A^a(q) \int_{(u,v) \in \mathcal{H}} \phi^*\omega(v, 0) \wedge \phi^*\omega(v, u). \tag{5.11}$$

By equation (5.3) and (5.5), proving theorem 1 now amounts to showing that the second integral in equation (5.11) contributes a factor of 1/2. This is the goal of the present subsection.

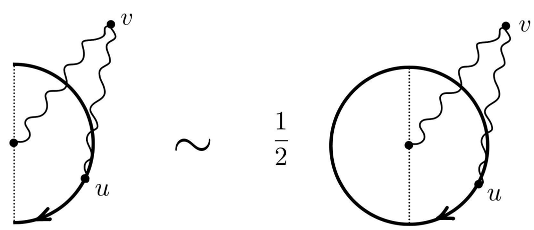



Figure 7: The space $\mathcal{C}$.

*Proof of theorem 1.* Let $\mathcal{C} \supset \mathcal{H}$ be the spaces obtained from allowing $u$ to be in the full circle $S \subset \mathbb{R}^2 \times \{0\}$. That is, we define

$$\mathcal{C} = \big\{(u, v) \in S \times \mathbb{R}^3\big\} \, .$$

Due to the rotation symmetry of $\omega$ (see section 4.1), it holds that

$$\int_{(u,v) \in \mathcal{H}} \phi^*\omega(v, 0) \wedge \phi^*\omega(v, u) = \frac{1}{2} \int_{(u,v) \in \mathcal{C}} \phi^*\omega(v, 0) \wedge \phi^*\omega(v, u). \tag{5.12}$$

In fact, we are going to modify the space of integration even further: Recall from section 4.1 that $\omega$ is only supported in a small neighbourhood of the north pole. Hence, the only contribution to the integral in the right-hand side of equation (5.12) comes from when $v \in \mathbb{R}^3_- := \mathbb{R}^2 \times (-\infty, 0)$. Defining $\mathcal{C}_- \subset \mathcal{C}$ by

$$\mathcal{C}_- = \{(u,v) \in S \times \mathbb{R}^3_-\},$$

we can therefore replace $\mathcal{C}$ with $\mathcal{C}_-$ in equation (5.12):

$$\int_{(u,v)\in\mathcal{H}} \phi^*\omega(v,0) \wedge \phi^*\omega(v,u) = \frac{1}{2}\int_{(u,v)\in\mathcal{C}_-} \phi^*\omega(v,0) \wedge \phi^*\omega(v,u). \tag{5.13}$$

The integral on the right-hand side of equation (5.13) can be computed using purely geometric arguments. Let $S^2_+$ be the upper half of the unit sphere and consider the map $\Phi : \mathcal{C}_- \to S^2_+ \times S^2_+ \setminus \text{diag}$, given by

$$\Phi(u,v) = (\phi(v,0), \phi(u,v)) = \left(-\frac{v}{|v|}, \frac{u-v}{|u-v|}\right).$$

**Lemma 6.** The map $\Phi$ is a diffeomorphism.

*Proof.* An inverse map $\Phi^{-1}$ is constructed as follows: Let $(a,b) \in S^2_+ \times S^2_+ \setminus \text{diag}$. For any $u \in S$ write $r^u_a$ for the ray going out from $u$ and pointing along the vector $-a$ and write $r_b$ for the ray going out from 0 and pointing along the vector $-b$. Because $a \neq b$, as we move $u$ around the circle we encounter exactly one point $u_{ab}$ for which the rays $r^u_a$ and $r_b$ intersect. Denoting the corresponding point of intersection by $v_{ab}$ we obtain an inverse map by defining $\Phi^{-1}(a,b) = (u_{ab}, v_{ab})$. □

From lemma 6 and the property that $\omega$ integrates to one on $S^2$ it now follows that

$$\int_{\mathcal{C}_-} \phi^*\omega(v,0) \wedge \phi^*\omega(v,u) = \int_{S^2_+ \times S^2_+} \omega(a) \wedge \omega(b) = 1.$$

Notice that we can include the diagonal $\text{diag} \subset S^2_+ \times S^2_+$ in the integral because $\omega$ extends continuously to the diagonal which is a subspace of co-dimension one. Inserting this back into equation (5.11) we get

$$\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon = \frac{1}{2}\int_{q\in L} A(q).$$

By equation (5.3) and (5.5) this completes the proof of theorem 1. □

# 6 Outlook to moduli spaces of local systems

To a Lie group $G$ and a surface $S$ with punctures, boundaries, and marked points on the boundaries, Goncharov and Shen [9] construct a moduli space $Loc_{S,G}$ which parameterizes $G$ local systems on $S$ along with some extra data at the punctures, boundaries, and marked points of $S$. These spaces are closely related to the $\mathcal{X}$ moduli spaces of "framed" local systems on $S$ originally constructed by Fock and Goncharov [6], with a slight modification to allow for cutting and gluing of surfaces. In particular, one associates to each marked point on the boundary the conjugacy class of the Borel subgroup $B \subset G$.

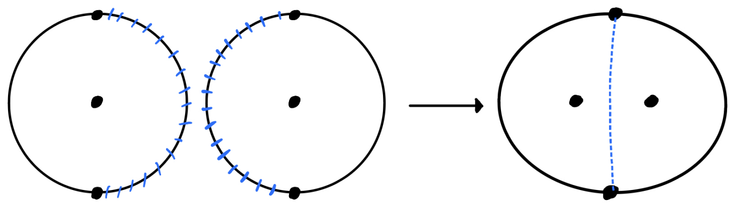


Figure 8: The gluing map on surfaces.

There is a quantization of the ring of regular functions on $Loc_{S,G}$ which is denoted by $\mathcal{O}_\hbar(Loc_{S,G})$. Goncharov and Shen construct a natural gluing operation on these quantized spaces coming from gluing surfaces along the boundary segments between two marked points; when $S$ is obtained by gluing $S'$ and $S''$ along boundary segments between marked points one obtains a map

$$\mathcal{O}_\hbar(Loc_{S,G}) \xrightarrow{Glue} \mathcal{O}_\hbar(Loc_{S',G}) \otimes \mathcal{O}_\hbar(Loc_{S'',G}).$$

To see how this construction relates that of the present paper, consider a disk with one puncture and two marked points on its boundary. There is a map

$$\kappa : \mathfrak{D}_\hbar(\mathfrak{b}) \to \mathcal{O}_\hbar(Loc_{\odot,G})$$

which is given explicitly on generators, see [10] for a very nice exposition on this in the $\mathfrak{sl}_n$ case and see [11] for the general ADE case. The coproduct is given in $\mathcal{O}_\hbar(Loc_{\odot,G})$ as follows: Take two copies of the punctured disk and glue them along boundary segments between marked points to obtain a twice punctured disk with two marked points on its boundary (see figure 8). We denote the twice punctured disk by $T$. By bringing the two punctures close together and cutting out a small circle around the two punctures one obtains a new once punctured disk (see figure 9). This construction gives a map

$$\mathcal{O}_\hbar(Loc_{\odot,G}) \xrightarrow{Cut} \mathcal{O}_\hbar(Loc_{T,G}) \xrightarrow{Glue} \mathcal{O}_\hbar(Loc_{\odot,G}) \otimes \mathcal{O}_\hbar(Loc_{\odot,G}). \tag{6.1}$$

which agrees with the coproduct in $\mathfrak{D}_\hbar(\mathfrak{b})$. Similarly, the braiding on $\mathfrak{D}_\hbar(\mathfrak{b})$ is given on $\mathcal{O}_\hbar(Loc_{\odot,G})$ as the map twisting the two punctures around each other:

$$\mathcal{O}_\hbar(Loc_{T,G}) \xrightarrow{Braid} \mathcal{O}_\hbar(Loc_{T,G}) \tag{6.2}$$

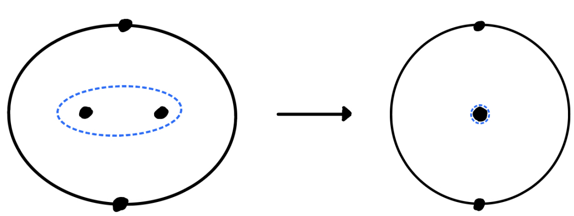


Figure 9: The cutting map.

In other words there are commutative diagrams:

$$
\begin{array}{ccc}
\mathfrak{D}_\hbar(\mathfrak{b}) & \xrightarrow{\quad\Delta\quad} & \mathfrak{D}_\hbar(\mathfrak{b}) \otimes \mathfrak{D}_\hbar(\mathfrak{b}) \\
\downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa\otimes\kappa} \\
\mathcal{O}_\hbar(Loc_{\odot,G}) & \xrightarrow{Glue} & \mathcal{O}_\hbar(Loc_{\odot,G}) \otimes \mathcal{O}_\hbar(Loc_{\odot,G})
\end{array}
$$

$$
\begin{array}{ccccccc}
\mathfrak{D}_\hbar(\mathfrak{b}) & \xrightarrow{\Delta} & \mathfrak{D}_\hbar(\mathfrak{b}) \otimes \mathfrak{D}_\hbar(\mathfrak{b}) & & \xrightarrow{\qquad\qquad R \qquad\qquad} & & \mathfrak{D}_\hbar(\mathfrak{b}) \otimes \mathfrak{D}_\hbar(\mathfrak{b}) \\
\downarrow{\scriptstyle\kappa} & & & & & & \downarrow{\scriptstyle\kappa\otimes\kappa} \\
\mathcal{O}_\hbar(Loc_{\odot,G}) & \xrightarrow{Cut} & \mathcal{O}_\hbar(Loc_{T,G}) & \xrightarrow{Braid} & \mathcal{O}_\hbar(Loc_{T,G}) & \xrightarrow{Glue} & \mathcal{O}_\hbar(Loc_{\odot,G}) \otimes \mathcal{O}_\hbar(Loc_{\odot,G})
\end{array}
$$

Thus the analogy of this in 3-dimension Chern-Simons theory should now be apparent from the construction in this paper: The punctures in the moduli spaces of Goncharov and Shen translate into Wilson line operators in our theory extending to infinity in the time direction. Cutting a disk around each Wilson line tangent to the boundaries gives a punctured disk two marked points on its boundary. The opposite Borel subgroups assigned to the marked points at the top and bottom can then be thought of as coming from the boundary conditions in the theory and the operation of merging two Wilson lines corresponds to the operation of gluing punctured disks together. We expect that much of the formalism described by Goncharov and Shen (the modular functor conjectures of sections 2.5 and 5 of [9]) can be realized within Chern-Simons theory by exploring this connection further.

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# The $R$-Matrix in 3d Topological BF Theory

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**Abstract**

In this paper I study Wilson line operators in a certain type of "split" Chern-Simons theory for a Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ on a manifold with boundaries. The resulting gauge theory is a 3d topological BF theory equivalent to a topologically twisted 3d $\mathcal{N} = 4$ theory. I show that this theory realises solutions to the quantum Yang-Baxter equation all orders in perturbation theory as the expectation value of crossing Wilson lines.

# Contents

# 1 Introduction

The perturbative framework for Chern-Simons theory on a general three-manifold $M$ was formalised by Axelrod and Singer in [4]. To account for ultraviolet singularities in Feynman integrals they used a Fulton-MacPherson like compactification of the configuration space of Feynman diagram vertices in $M$. The compactified space has the form of a stratified space with boundary strata defined from spherical blow-ups along the diagonals where subsets of vertices come together. This has led to a technique for recovering manifold invariants from Chern-Simons theory implemented in a series of notable works, see e.g. [3, 5, 6, 15]. In particular, Bott and Taubes [6] constructed knot invariants from Wilson loops in $S^3$. The essential ingredient in this work is the use of Stokes' theorem: Since propagators in the theory are closed forms, proving invariance of the expectation value of Wilson loops under continuously displacing loop strands amounts to proving a series of vanishing theorems for Feynman integrals on the boundary of the configuration space. The objective of this paper is to implement the same type of arguments for the purpose of recovering a solution to the Yang-Baxter equation (an $R$-matrix) from the expectation value of crossing Wilson lines at all orders in perturbation theory.

In [2] the present author carried out leading order Feynman diagram computations to realise the classical Yang-Baxter equation from Wilson lines in Chern-Simons theory for a semi-simple Lie algebra $\mathfrak{g}$, on a manifold with boundaries $\mathbb{R}^2 \times [-1, 1]$. In order to obtain Yang-Baxter solutions, one must place boundary condition on the gauge field to break the full gauge symmetry of the theory. This is achieved by extending the Lie algebra by an extra copy of the Cartan subalgebra to admit a decomposition into maximal isotropic subalgebras $\mathfrak{g} = \mathfrak{l}_- \oplus \mathfrak{l}_+$, restricting the gauge field to $\mathfrak{l}_-$ (resp. $\mathfrak{l}_+$) on the upper (resp. lower) boundary. This work was inspired by a construction of Costello, Witten and Yamazaki [10], [11] in a 4-dimensional analogue of Chern-Simons theory. In this framework, the Yang-Baxter equation states the equivalence between the diagrams on the left- and right-hand side of figure 1, where the lines represent Wilson lines extending to infinity along $\mathbb{R}^2$ and supported at different points in $[-1, 1]$. The corresponding expectation value is an element in $\mathcal{U}(\mathfrak{g})^{\otimes 3}[[\hbar]]$.

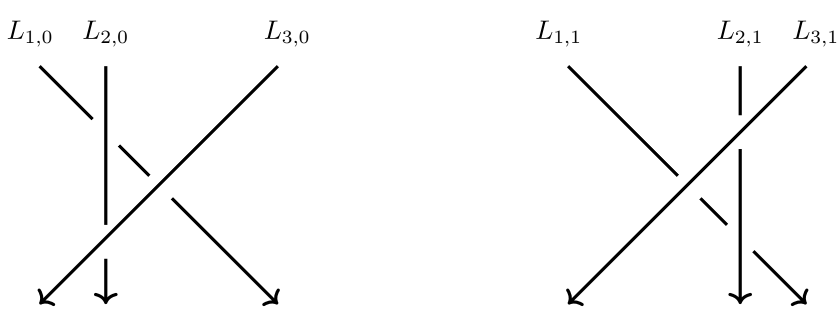



Figure 1: The Yang-Baxter equation for crossing Wilson lines.

Directly implementing vanishing arguments similar to those of Bott and Taubes to the above theory appears too ambitious as the vanishing theorems rely on a full rotational symmetry of the propagator which in this case is broken by the boundary conditions. However, things become easier if we instead consider a Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ with relations $[a, b^*] = [a, b]^*$ and $[a^*, b^*] = 0$ for $a, b \in \mathfrak{a}$. Chern-Simons theory for this Lie algebra is equivalent to a $B$-twisted 3d $\mathcal{N} = 4$ theory; see e.g. [9, 13, 14]. For this theory Feynman diagrams become particularly simple. In fact, the gauge field decomposes into two parts $\mathbf{A} \in \Omega^1(M) \otimes \mathfrak{a}$ and $\mathbf{B} \in \Omega^1(M) \otimes \mathfrak{a}^*$, and the only type of interaction vertices permitted by the theory has one incoming $\mathbf{B}$-field and two outgoing $\mathbf{A}$-fields. It turns out that this accounts for the problematic boundary faces and we can therefore prove the following theorem:

**Theorem 1.** Let $\langle L_t \rangle$ be the expectation value of the product of Wilson lines in figure 1, where the parameter $t$ corresponds to moving the middle line continuously to the right. In the theory described above it holds that $\langle L_1 \rangle - \langle L_0 \rangle = 0$.

This entails proving a series of vanishing theorems in line with those of Bott and Taubes. The perturbative formalism for this "split" Chern-Simons theory on a manifold with boundaries was first studied in work of Cattaneo et al. [7, 8], from where the term originates.

## 2 The Quantum Yang-Baxter Equation

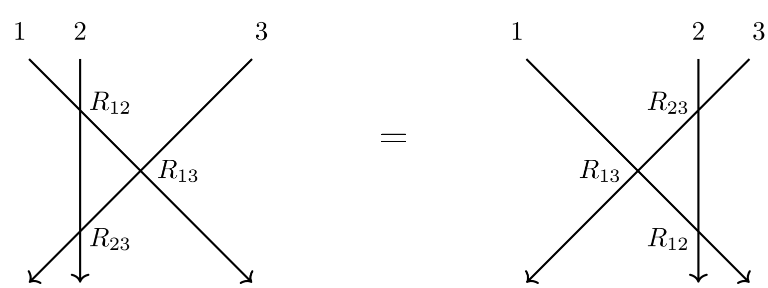
We begin by briefly recalling some basic notions relating to the quantum Yang-Baxter equation. Let $\mathfrak{g}$ be a Lie algebra that can be quantized via the Drinfel'd double construction and let $\mathcal{U}_\hbar(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra of $\mathfrak{g}$. For each $i, j \in \{1, 2, 3\}$ with $i \neq j$ define $\rho_{ij} : \mathcal{U}_\hbar(\mathfrak{g})^{\otimes 2} \to \mathcal{U}_\hbar(\mathfrak{g})^{\otimes 3}$ by

$$\rho_{12}(a \otimes b) = a \otimes b \otimes 1\,, \;\; \rho_{13}(a \otimes b) = a \otimes 1 \otimes b\,, \;\; \rho_{23}(a \otimes b) = 1 \otimes a \otimes b$$

Given an element $R_\hbar \in \mathcal{U}_\hbar(\mathfrak{g}) \otimes \mathcal{U}_\hbar(\mathfrak{g})$, write $R_{ij} = \rho_{ij}(R_\hbar)$. We say that $R_\hbar$ is a quantum $R$-matrix if it is invertible and it satisfies the following relation known as the Yang-Baxter equation:

$$R_{23} R_{13} R_{12} = R_{12} R_{13} R_{23}\,, \tag{2.1}$$

This equation is commonly represented graphically by the diagram shown below.



To interpret this diagram, we imagine that each line carries a vector space $V_i$, $i \in \{1,2,3\}$ corresponding to some representation of $\mathfrak{g}$. At the crossing between line $i$ and line $j$ the incoming vector spaces are transformed by the element $R_{ij} \in \mathrm{End}(V_i \otimes V_j)$ acting in the given representation.Reading the figure from up to down in the direction of the arrow reproduces the Yang-Baxter equation. The existence of an $R$-matrix gives a braiding structure on $\mathcal{U}_\hbar(\mathfrak{g})$, and hence in particular it allows for the construction of invariants of knots and braids.

# 3 Split Chern-Simons Theory with Boundaries

## 3.1 The basic setup

Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant pairing $\mathrm{Tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ and assume that $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ where $\mathfrak{a}^*$ is dual to $\mathfrak{a}$ with respect Tr. Moreover, let $\mathcal{B}(\mathfrak{a}) = \{\xi^a\}_{a=1,\ldots,\dim \mathfrak{a}}$ be a basis for $\mathfrak{a}$ and $\mathcal{B}(\mathfrak{a}^*) = \{\zeta_a\}_{a=1,\ldots,\dim \mathfrak{a}}$ be the dual basis for $\mathfrak{a}^*$. The gauge theory that we study in this paper is Chern-Simons for the Lie algebra $\mathfrak{g}$ described above, with relations

$$[\xi_a, \xi_b] = f^c{}_{ab}\xi_c \ , \quad [\zeta^a, \xi_b] = f^a{}_{bc}\zeta^c \ , \quad [\zeta^a, \zeta^b] = 0, \tag{3.1}$$

where $f^c{}_{ab}$ are the structure constants of $\mathfrak{a}$. Notice that, with this definition, $\mathfrak{a}$ and $\mathfrak{a}^*$ are maximal isotropic subalgebras of $\mathfrak{g}$ and hence the triple $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$ is a Manin triple. This is, in essence, what allows us to derive quantum groups structured in the theory. The above gauge theory is defined by the Chern-Simons action:

$$S_{\mathrm{CS}}(\mathbf{C}) = \frac{1}{2\pi}\int_M \mathrm{Tr}(\mathbf{C} \wedge d\mathbf{C}) + \frac{1}{3}\mathrm{Tr}([\mathbf{C}, \mathbf{C}] \wedge \mathbf{C}), \tag{3.2}$$

where the gauge field $\mathbf{C}$ is a one-form on a manifold $M$ taking values in $\mathfrak{g}$, i.e. $\mathbf{C} \in \Omega^1(M) \otimes \mathfrak{g}$. We will decompose $\mathbf{C}$ into a part $\mathbf{A}$ taking value in $\mathfrak{a}$ and a part $\mathbf{B}$ taking value in $\mathfrak{a}^*$. That is, we write

$$\mathbf{C} = \mathbf{A} + \mathbf{B},$$

where $\mathbf{A} \in \Omega^1(M) \otimes \mathfrak{a}$ and $\mathbf{B} \in \Omega^1(M) \otimes \mathfrak{a}^*$. Observe that, when inserting this into the Chern-Simons action, the terms containing only $\mathbf{A}$'s or $\mathbf{B}$'s vanish since the subalgebras $\mathfrak{a}$ and $\mathfrak{a}^*$ are isotropic. Similarly the term $\mathrm{Tr}([\mathbf{A}, \mathbf{B}], \mathbf{B})$ vanish by the relations in equation (3.1). Thus the resulting action takes the form:

$$S_{\mathrm{CS}}(\mathbf{A} + \mathbf{B}) = \frac{1}{2\pi}\int_M \mathrm{Tr}(\mathbf{A} \wedge d\mathbf{B} + \mathbf{B} \wedge d\mathbf{A}) + \frac{1}{3}\mathrm{Tr}([\mathbf{A}, \mathbf{A}] \wedge \mathbf{B}), \tag{3.3}$$

which we identify with the action of a 3d topological BF theory. The first term in the above action is a kinetic term and represents the free propagation of a gauge field between states $\mathbf{A}$ and $\mathbf{B}$. We use a convention where the corresponding propagator is represented by an oriented edge going from $\mathbf{A}$ to $\mathbf{B}$. The form of the cubic interaction term then implies that the only allowed interaction vertices in the theory are the of the form shown in figure 2, with one incoming **B**-edge and two outgoing **A**-edges. We will say more on this in section 4.4.

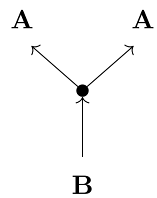



Figure 2: Interaction vertices in the relevant split Chern-Simons theory

In what follows we take $M$ to be a manifold with boundaries, $M = \mathbb{R}^2 \times I$, where $I = [-1, 1]$. In this setting, when varying the action with respect to the gauge field, i.e. $\mathbf{A} \to \mathbf{A} + d\chi_{\mathbf{A}}$ and $\mathbf{B} \to \mathbf{B} + d\chi_{\mathbf{B}}$, we pick up a boundary term:

$$\delta S_{\text{CS}} = \cdots + \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \{-1,1\}} \text{Tr}(d\chi_{\mathbf{A}} \wedge d\mathbf{B} + d\chi_{\mathbf{B}} \wedge d\mathbf{A}).$$

Therefore, in order to have a consistent theory in the presence of boundaries, we must impose boundary conditions on the gauge field such that this term vanishes (see e.g. [10]). We accommodate for this by requiring that $\mathbf{A} = 0$ on the upper boundary $\mathbb{R}^2 \times \{1\}$ and $\mathbf{B} = 0$ on the lower boundary $\mathbb{R}^2 \times \{-1\}$.

## 3.2 The propagator

As explained above the gauge field can propagate between states $\mathbf{A}^a(x)$ and $\mathbf{B}_b(y)$ for some $x, y \in M$ and $a, b \in \{1, \dots, \dim \mathfrak{a}\}$. The corresponding probability distribution is a two form $P^a{}_b(x, y)$ known as the propagator. It satisfies the following defining relations:

$$P^a{}_b(x, y) = -P_b{}^a(y, x) \tag{3.4}$$

$$dP^a{}_b(x, y) = \delta^a{}_b \delta^{(3)}(x, y). \tag{3.5}$$

where $d$ is the differential operator and $\delta^{(3)}(x, y)$ is the Dirac delta function. Furthermore, the boundary conditions on the gauge field translate to the following constraint on the propagator:

$$P^a{}_b(x, y) = 0 \text{ when } x \in \mathbb{R}^2 \times \{1\} \text{ or } y \in \mathbb{R}^2 \times \{-1\}. \tag{3.6}$$

Let $\phi : (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \Delta \to S^2$ be the map

$$\phi(x, y) = \frac{y - x}{|y - x|}\,.$$

and define $\omega \in \Omega^2(S^2)$ by

$$\omega \coloneqq f \, \text{vol}_{S^2} \in \Omega^2(S^2),$$

where $\mathrm{vol}_{S^2}$ is the unit volume form on $S^2$ given in terms of the coordinates on $\mathbb{R}^3$ by: $\mathrm{vol}_{S^2} = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ and $f : S^2 \to \mathbb{R}$ is a smooth function supported in a small neighbourhood around $x_{np} = (0,0,1)$ and normalized so that $\omega$ integrates to one on $S^2$.

**Proposition 3.1.** If we define $P \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta)$ by

$$P = \phi^* \omega, \tag{3.7}$$

then $P^a{}_b(x,y) := P(x,y)\delta^a{}_b$ satisfies the constraints in equation (3.4)-(3.6).

*Proof.* Since $\omega$ is a top-dimension form on $S^2$ it holds that $dP(x,y) = 0$ away from the diagonal $x = y$. To see that $dP$ it is in fact the Dirac delta function we use Stokes' theorem: Fix some $x \in \mathbb{R}^3$ and let $B_x$ be the unit ball centered at $x$

$$\int_{y \in B_x} dP(x,y) = \int_{y \in B} dP(0,y) = \int_{y \in S^2} P(0,y) = \int_{y \in S^2} \omega(y) = 1. \qquad \square$$

## 3.3 Wilson lines

With our choice of boundary conditions the global gauge symmetry of the action is completely broken. As a consequence, the theory admits a set of gauge invariant operators known as Wilson lines (see [10] for more details). For the present purpose we will think of a Wilson line simply as a proper embedding in $L : \mathbb{R} \hookrightarrow \mathbb{R}^2 \times I$ parallel to the boundary, along with a rule that a gauge field $\mathbf{A}^a$ (resp. $\mathbf{B}_a$) couples to $L$ by inserting a basis element $\xi_a$ (resp. $\zeta^a$) at the corresponding point in $L$.

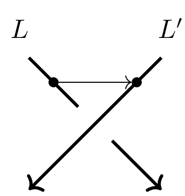



Figure 3: The projection onto $\mathbb{R}^2$ of a pair of crossing Wilson lines.

Consider for example a pair of Wilson lines $L$ and $L'$ supported at different points in $I$ and crossing in $\mathbb{R}^2$ as shown in figure 3. The two Wilson lines interact by exchanging gauge bosons. The simplest (leading order) interaction corresponds to a single gauge boson propagating between the lines. This interaction is illustrated in figure 3, where the oriented edge represents a propagator. The corresponding amplitude is given by

$$\hbar \int_{x \in L, y \in L'} P(x,y)\delta^a{}_b \; \xi_a \otimes \zeta^b, ,$$

where $\hbar$ is a small expansion parameter. At higher orders in $\hbar$ we get interactions coming from the cubic interaction term in the Chern-Simons action in equation (3.2). Each interaction is represented by a directed graph (Feynman diagram) with three-valent interaction vertices in the bulk and one-valent vertices along the Wilson lines. The expectation value for the interaction is an element $\langle LL' \rangle \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$ given as an perturbative expansion in $\hbar$ in terms of the set of Feynman diagrams:

$$\langle LL' \rangle = \sum_{\Gamma} \hbar^{\mathrm{ord}(\Gamma)} \mathcal{M}(\Gamma),$$

where $\mathrm{ord}(\Gamma)$ is the number of edges of $\Gamma$ minus the number internal vertices and the weight (amplitude) $\mathcal{M}(\Gamma)$ is determined by the Feynman rules.

*Remark* 1. On the surface it appears that the expectation value $\langle LL' \rangle$ depends on the angle of crossing between the lines $L$ and $L'$. We will argue in section 8 that $\langle LL' \rangle$ is in fact independent of the angle. For now we take this for given and define $\mathcal{R} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$ by

$$\mathcal{R} := \langle LL' \rangle . \tag{3.8}$$

## 3.4 The $R$-matrix from crossing Wilson lines

The goal of the remainder of this paper is to show that the element $\mathcal{R}$ is a quantum $R$-matrix, that is, it satisfies the Yang-Baxter equation (2.1). In this framework, the lines in the Yang-Baxter picure should be thought of as representing Wilson line operators supported at different points in $I$. With this as our motivation we define the following smooth family of proper embeddings:

**Definition 3.1.** Let $L_t$ be a family of embeddings

$$L_t : \coprod_{\alpha=1,2,3} \mathbb{R}_\alpha \hookrightarrow \mathbb{R}^2 \times I,$$

parametrized by $t \in [0,1]$, where $L_t|_{\mathbb{R}_\alpha} = L_{\alpha,t} : \mathbb{R} \hookrightarrow \mathbb{R}^2 \times I$ is given by

$$L_{1,t} : s \mapsto (-s/\sqrt{2}, s/\sqrt{2}, -1/2) \ , \quad L_{2,t} : s \mapsto (t, s, 0) \ , \quad L_{3,t} : s \mapsto (s/\sqrt{2}, s/\sqrt{2}, 1/2).$$

The family of embeddings defined above is illustrated in figure 4 which shows the projection onto $\mathbb{R}^2$. As $t$ increases, the lines $L_{1,t}$ and $L_{3,t}$ are held fixed while $L_{2,t}$ is dragged continuously over the crossing between the other two lines.

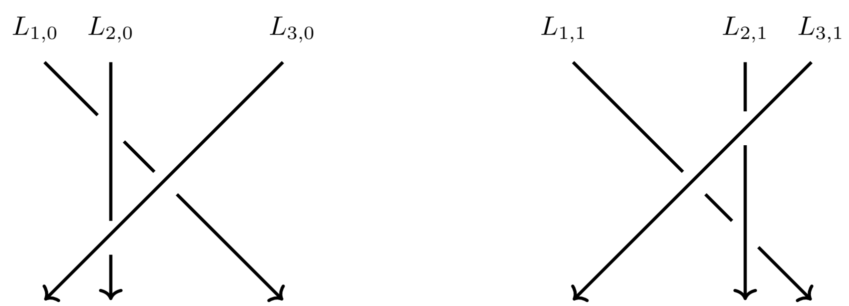



Figure 4: The embedding space $L_t$ where $t \in [0,1]$.

For each $t \in [0,1]$, the corresponding expectation value is an element

$$\langle L_t \rangle \coloneqq \langle L_{1,t} L_{2,t} L_{3,t} \rangle \in \mathcal{U}(\mathfrak{g})^{\otimes 3}.$$

The following section is dedicated to giving a precise definition of $\langle L_t \rangle$, which on the surface appears to depend on the parameter $t \in [0,1]$. The main objective of this paper is to show that $\langle L_t \rangle$ is in fact independent on $t$. Since the form of the propagator ensures that interactions only take place in a small neighbourhood around each crossing, this will imply the the expectation value of a pair of crossing Wilson lines is an $R$-matrix. A formal argument for this is given in section 4.4 below.

# 4 Chern-Simons Perturbation Theory

In this section we give a definition of the expectation value $\langle L_t \rangle$ in the formalism of perturbation theory. As mentioned, $\langle L_t \rangle$ is given by an expansion in $\hbar$ in terms of a set of weighted graphs called Feynman graphs which we define in subsection 4.1 below.

## 4.1 Feynman graphs

We here define the relevant set of graphs contributing to the expectation value $\langle L_t \rangle$. Given $m \in \mathbb{Z}_{\geq 0}$ and $\underline{n} = (n_1, n_2, n_3)$ a tuple of integers $n_\alpha \in \mathbb{Z}_{\geq 0}$, we first fix the data corresponding to the sets of $m$ internal (bulk) vertices and of $n_\alpha$ external vertices on the Wilson line $L_{\alpha,t}$, along with a set of half-edges incident on each vertex:

Let $n = \sum_\alpha n_\alpha$. We define a set $\mathcal{V}$ of vertices consisting of:

1. A set of internal vertices $V = \{v_1, \ldots, v_m\}$.
2. An set of external vertices $W_\alpha$ in each Wilson line $L_{\alpha,t}$, given by:

$$W_1 = \{w_1, \ldots, w_{n_1}\}\ ,\quad W_2 = \{w_{n_1+1}, \ldots, w_{n_1+n_2}\}\ ,\quad W_3 = \{w_{n_1+n_2+1}, \ldots, w_n\}.$$

   We write $W = \bigcup_{\alpha=1}^{3} W_\alpha$ and $\underline{W} = (W_1, W_2, W_3)$.

Moreover, we define a set $\mathcal{H}$ of half-edges consisting of:

1. A set of half-edges $\{h_i^1, h_i^2, h_i^3\}$ for each internal vertex $v_i \in V$.
2. A single half-edge $h_j$ for each external vertex $w_j \in W$.

Finally, we denote by $s : \mathcal{H} \to \mathcal{V}$ the source map $s(h_i^k) = v_i$ and $s(h_j) = w_j$.

With the above data fixed, the only data needed to define a graph is an involution of the set of half-edges to form edges. In addition, we want the definition of a Feynman graph to include an orientation of the edges and a Lie algebra labeling of the half-edges. This leads to the following definition:

**Definition 4.1** (Feynman graphs)**.** A Feynman graph $\Gamma \in \mathcal{G}_{m,\underline{n}}$ is defined by the following data:

(i) A free involution $\iota : \mathcal{H} \to \mathcal{H}$ such that, if $\iota(h_i^k) = h_j^l$ then $i \neq j$. A pair $\{h, \iota(h)\}$ is called an edge and we denote the set of edges by $E(\Gamma)$.

(ii) An orientation of the edges corresponding to an ordering $(h, h')$ of each pair $\{h, h'\} \in E(\Gamma)$.

(iii) An assignment $\tau : \mathcal{H} \to \mathcal{B}(\mathfrak{a}) \cup \mathcal{B}(\mathfrak{a}^*)$ such that if $(h, h') \in E(\Gamma)$ then $\tau(h) \in \mathcal{B}(\mathfrak{a})$ and $\tau(h') = \tau(h)^* \in \mathcal{B}(\mathfrak{a}^*)$.

We write $\mathcal{G} = \bigcup_{m,\underline{n}} \mathcal{G}_{m,\underline{n}}$ for the collection of all Feynman graphs. When writing the expectation $\langle L_t \rangle$ we only wish to sum over isomorphism classes of Feynman graphs. Let us therefore make precise what it means for two Feynman graphs to be isomorphic.

**Definition 4.2.** Two graphs $\Gamma, \Gamma' \in \mathcal{G}_{m,\underline{n}}$ are said to be isomorphic, and we write $\Gamma \sim \Gamma'$, if there are bijections

$$F_{\mathcal{V}} : \mathcal{V} \to \mathcal{V} \ , \quad F_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$$

such that:

(i) $F_{\mathcal{V}}$ acts as the identity map on the set of external vertices.

(ii) $(F_{\mathcal{V}}, F_{\mathcal{H}})$ is a graph isomorphism: $F_{\mathcal{V}} \circ s = s \circ F_{\mathcal{H}}$ and $F_{\mathcal{H}} \circ \iota = \iota' \circ F_{\mathcal{H}}$.

(iii) $(F_{\mathcal{V}}, F_{\mathcal{H}})$ preserves the edge orientation: If $(h, h') \in E(\Gamma)$ then $(F_{\mathcal{H}}(h), F_{\mathcal{H}}(h')) \in E(\Gamma')$.

(iv) $(F_{\mathcal{V}}, F_{\mathcal{H}})$ preserves the Lie algebra decoration of edges: $\tau(h, h') = \tau'(F_{\mathcal{H}}(h), F_{\mathcal{H}}(h'))$.

## 4.2 The configuration space of vertices

We wish to consider the space of embeddings of the vertices $V \cup W$ of Feynman graphs into $\mathbb{R}^2 \times I$, such that for each $\alpha \in \{1, 2, 3\}$ the set of external vertices $W_\alpha$ maps to the Wilson line $L_{\alpha,t}$. We here give a formal definition of the space in question, following the definition given by Bott and Taubes in [6].

Let $S$ be some ordered set. We denote by $\mathrm{Conf}_S(\mathbb{R}^2 \times I)$ the configuration space of $|S|$ ordered points in $\mathbb{R}^2 \times I$, i.e. the space of injections $S \hookrightarrow \mathbb{R}^2 \times I$. Moreover, we denote by $\mathrm{Conf}_S(\mathbb{R})$ the space of injections $S \hookrightarrow \mathbb{R}$ such that the points in $S$ are placed in increasing order along $\mathbb{R}$. Recall definition 3.1 and observe that an embedding $L_{t,\alpha} : \mathbb{R} \hookrightarrow \mathbb{R}^2 \times I$ induces an embedding of configuration spaces $\mathrm{Conf}_{W_\alpha}(\mathbb{R}) \hookrightarrow \mathrm{Conf}_{W_\alpha}(\mathbb{R}^2 \times I)$. Hence we have a map:

$$\mathscr{L} : \prod_{\alpha=1}^{k} \mathrm{Conf}_{W_\alpha}(\mathbb{R}) \times [0,1] \longrightarrow \mathrm{Conf}_W(\mathbb{R}^2 \times I)\,. \tag{4.1}$$

The relevant configuration space $\mathrm{Conf}_{V,\underline{W}}$ is now defined as the pullback:

$$\begin{array}{ccc} \mathrm{Conf}_{V,\underline{W}} & \longrightarrow & \mathrm{Conf}_{V\cup W}(\mathbb{R}^2\times I) \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \prod_{\alpha=1}^{3}\mathrm{Conf}_{W_\alpha}(\mathbb{R})\times[0,1] & \xrightarrow{\mathscr{L}} & \mathrm{Conf}_W(\mathbb{R}^2\times I)\ . \end{array} \tag{4.2}$$

In particular, we can describe $\mathrm{Conf}_{V,\underline{W}}$ as the set of points $(t,q,p)$, where $t\in[0,1]$, $q\in\prod_{\alpha=1}^{3}\mathrm{Conf}_{W_\alpha}(\mathbb{R})$ and $p\in\mathrm{Conf}_V\left(\mathbb{R}^2\times I\setminus\left\{\mathscr{L}(q,t)(w_i)\right\}_{w_i\in W}\right)$.

Notice that we have a projection

$$\mathrm{Conf}_{V,\underline{W}}\to[0,1]$$

via the map on the left-hand side of the diagram (4.2). We write $\mathrm{Conf}^t_{V,\underline{W}}$ for the fiber of this map over $t\in[0,1]$.

## 4.3 The expectation value

We are now equipped to present the Feynman rules that determines the amplitude $\mathcal{M}_t(\Gamma)$ associated to any $\Gamma\in\mathcal{G}$ and $t\in[0,1]$. Our first step is to define a differential form of $\lambda(\Gamma)$ on $\mathrm{Conf}_{V,\underline{W}}$ as follows: For each edge $e=(h,h')\in E(\Gamma)$, let $\phi_e:\mathrm{Conf}_{V\cup W}(\mathbb{R}^2\times I)\to S^2$ be the map

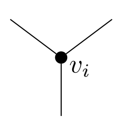
$$\phi_e(x)=\frac{x(s(h'))-x(s(h))}{|x(s(h'))-x(s(h))|},$$

where $s:\mathcal{H}\to\mathcal{V}$ is the source map (see section 4.1). Furthermore, let $\Phi_e:\mathrm{Conf}_{V,\underline{W}}\to S^2$ be the pull back of $\phi_e$ to $\mathrm{Conf}_{V,\underline{W}}$ along the map in the top row of diagram (4.2) and write $P_e=\Phi_e^*\omega\in\Omega^2(\mathrm{Conf}_{V,\underline{W}})$. We define

$$\lambda(\Gamma)\coloneqq\bigwedge_{e\in E(\Gamma)}P_e. \tag{4.3}$$

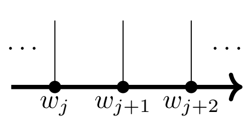
Notice that the degree of $\lambda(\Gamma)$ is $2|E|=3|V|+|W|$ and hence $\lambda(\Gamma)$ is a form of co-dimension one on $\mathrm{Conf}_{V,\underline{W}}$. Moreover, we associate to $\Gamma$ a Lie-algebra factor $c(\Gamma)\in\mathcal{U}(\mathfrak{g})^{\otimes3}$ as follows:

(i) For each internal vertex $v_i$ we multiply by a factor:




$$\sim\quad\left\langle[\tau(h_i^1),\tau(h_i^2)],\tau(h_i^3)\right\rangle$$

(ii) For each Wilson line $L_\alpha$ we get an element of $\mathcal{U}(\mathfrak{g})$ given by:




$$\sim\qquad\cdots\tau(h_j)\tau(h_{j+1})\tau(h_{j+2})\cdots$$

In other words, $c(\Gamma)$ is given by

$$c(\Gamma) = \prod_{i=1}^{m} \big\langle [\tau(h_i^1), \tau(h_i^2)], \tau(h_i^3) \big\rangle \prod_{j=1}^{n_1} \tau(h_j) \otimes \prod_{k=n_1+1}^{n_2} \tau(h_k) \otimes \prod_{l=n_1+n_2}^{n} \tau(h_l). \tag{4.4}$$

Given $t \in [0,1]$ and $\Gamma \in \mathcal{G}$ we now wish to define the amplitude $\mathcal{M}_t(\Gamma)$ as the integral of the element $\lambda(\Gamma)c(\Gamma)$ over the configuration space of vertices $\mathrm{Conf}^t_{V,\underline{W}}$. However, to properly define such an integral we must equip the configuration space with a suitable orientation form. Specifically, the orientation form in question must ensure that integrals are invariant under isomorphisms of $\Gamma \in \mathcal{G}$. Furthermore, the anti-symmetry relation in equation (3.4) implies that changing the orientation of an edge must reverse the sign of orientation of the configurations space.

Given a point $(q,p) \in \mathrm{Conf}^t_{V,\underline{W}}$ we write $p_i = p(v_i) \in \mathbb{R}^2 \times I$ and $q_j = q(w_j) \in \mathbb{R}$. Then, a small neighbourhood of $(p,q) \in \mathrm{Conf}^t_{V,\underline{W}}$ has local coordinates $t \in \mathbb{R}$, $(p_i^1, p_i^2, p_i^3) \in \mathbb{R}^3$ for each internal vertex $v_i \in V$ and $q_j \in \mathbb{R}$ for each external vertex $w_j \in W$.

**Definition 4.3.** Let $g : \mathcal{H} \to \mathbb{R}$ be the map $g(h_i^k) = p_i^k$ and $g(h_j) = q_j$. For each $\Gamma \in \mathcal{G}$ we define an orientation form on $\mathrm{Conf}^t_{V,\underline{W}}$ by

$$\mathrm{Or}(\Gamma) = \bigwedge_{(h,h') \in E(\Gamma)} \big( dg(h) \wedge dg(h') \big).$$

In the following we use the notation $\mathrm{Conf}^t(\Gamma)$ to denote the configuration space $\mathrm{Conf}^t_{V,\underline{W}}$ equipped with the orientation form $\mathrm{Or}(\Gamma)$. Similarly we denote by $\mathrm{Conf}(\Gamma)$ the configuration space $\mathrm{Conf}_{V,\underline{W}}$ equipped with the orientation form $\mathrm{Or}(\Gamma) \wedge dt$. We now define

$$\mathcal{M}_t(\Gamma) = \int_{\mathrm{Conf}^t(\Gamma)} \lambda(\Gamma)\, c(\Gamma). \tag{4.5}$$

**Proposition 4.1.** The Feynman amplitude $\mathcal{M}_t(\Gamma)$ in equation (4.5) is invariant under isomorphisms of $\Gamma$.

*Proof.* By definition 4.2, any isomorphism of $\Gamma$ is given by relabeling the internal vertices and permuting the set of half-edges at each internal vertex. Since the definition of $\mathcal{M}_t(\Gamma)$ does not depend on the labeling of vertices, we consider an isomorphism that permutes the half-edges $\{h_i^1, h_i^2, h_i^3\}$ incident to some $v_i \in V$. If the permutation is odd then the sign of $\mathrm{Or}(\Gamma)$ is reversed. On the other hand, since the structure constants are totally anti-symmetric, also $c(\Gamma)$ reverses its sign, thus leaving the overall sign of $\mathcal{M}_t(\Gamma)$ unchanged. □

We are now finally ready to give a precise definition of the expectation value $\langle L_t \rangle$

**Definition 4.4.** We define

$$\langle L_t \rangle = \sum_{\Gamma \in \mathcal{G}/\sim} \hbar^{\mathrm{ord}(\Gamma)} \mathcal{M}_t(\Gamma) \in \mathcal{U}(\mathfrak{g})^{\otimes 3}, \tag{4.6}$$

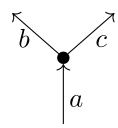
where the sum runs over isomorphism classes of Feynman graphs, and $\mathrm{ord}(\Gamma)$ is the number of edges minus the number of internal vertices of $\Gamma$.

## 4.4 Admissible Feynman graphs

Only a limited set of Feynman graphs has a non-vanishing contribution to the sum in equation (4.6). In fact, recall that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ is defined by the following non-trivial brackets:

$$[\xi_a, \xi_b] = f^c{}_{ab}\xi_c \ , \quad [\zeta^a, \xi_b] = f^a{}_{cb}\zeta^c .$$

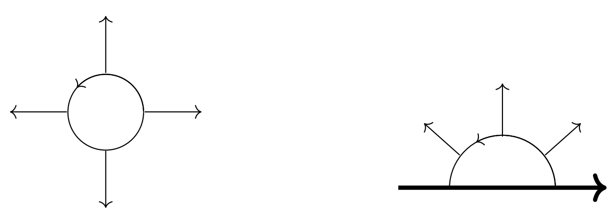
With this definition, the coefficient $\langle[\tau(h_i^1), \tau(h_i^2)], \tau(h_i^3)\rangle$ associated to an internal vertex $v_i$ is only non-zero when $v_i$ has exactly one incoming and two outgoing edges, as shown in figure 5.




$$\sim \quad \langle[\zeta^a, \xi_b], \xi_c\rangle = f^a{}_{bc}$$

Figure 5: The only allowed internal vertex

Moreover, we get no contributions from graphs that have an oriented cycles as shown in figure 6 (a) or from graphs that have an oriented path that ends and begins on the same Wilson line as shown in figure 6 (b). This follows from the definition of the propagator $P_e = \phi_e^*\omega$. In fact, because $\omega$ is only non-zero in a small neighbourhood of the north pole, $\lambda(\Gamma)$ is only supported in a neighbourhood of $\mathrm{Conf}_{V,\underline{W}}$ where all edges in $\mathbb{R}^2 \times I$ point strictly upwards along $I$. Hence $\lambda(\Gamma)$ vanishes everywhere for the graphs in figure 6.



(a) (b)

Figure 6: Non-contributing Feynman graphs

The above discussion can be summarized to give the following proposition:

**Proposition 4.2.** The only Feynman diagrams contributing to the sum in equation (4.6) are forests with edges in $\mathbb{R}^2 \times I$ pointing strictly upwards along $I$ and roots and leafs connected to the Wilson lines (see figure 7).

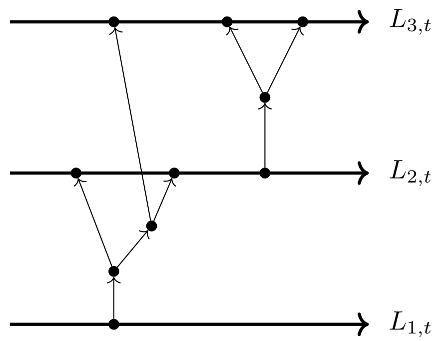



Figure 7: Example of an admissible Feynman graph

It follows from proposition 4.2 that a given *connected* Feynman graph $\Gamma$ connects at least two Wilson lines. Again using the fact that $\omega$ is only non-zero in a small neighbourhood of the north pole, it follows that the associated differential form $\lambda(\Gamma)$ only has support in a small neighbourhood of $\mathbb{R}^2$ around the crossing between the corresponding Wilson lines. Recall from remark 1 of section 3.3 that we denoted the (angle independent) expectation value of a pair of crossing Wilson lines by $\mathcal{R} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$. For each $i, j \in \{1, 2, 3\}$ with $i \neq j$ let $\rho_{ij} : \mathcal{U}(\mathfrak{g})^{\otimes 2} \to \mathcal{U}(\mathfrak{g})^{\otimes 3}$ be the map defined in section 2, i.e.

$$\rho_{12}(a \otimes b) = a \otimes b \otimes 1\,,\ \rho_{13}(a \otimes b) = a \otimes 1 \otimes b\,,\ \rho_{23}(a \otimes b) = 1 \otimes a \otimes b.$$

and write $\mathcal{R}_{ij} = \rho_{ij}(\mathcal{R}) \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$. By the above discussion we now have the following lemma:

**Lemma 4.1.** $\langle L_0 \rangle$ and $\langle L_1 \rangle$ takes the form

$$\langle L_0 \rangle = \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} \quad \text{and} \quad \langle L_1 \rangle = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,.$$

The situation is illustrated in figure 8. The dotted circle indicates the area where the interaction matrix $\mathcal{R}_{ij}$ acts.

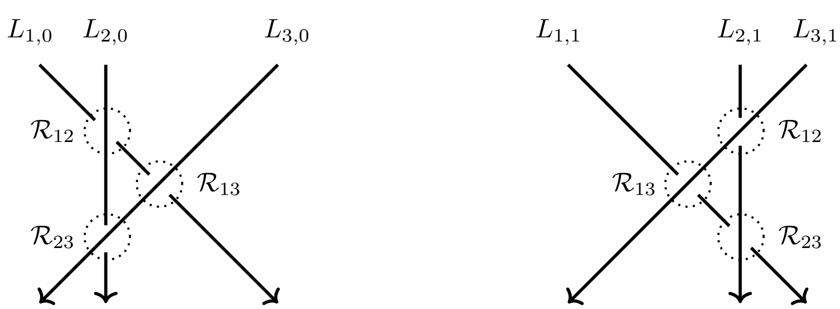



Figure 8: A graphical illustration of lemma 4.1.

# 5 Finiteness of the Integrals

Because the propagator $P_e = \phi_e^* \omega$ is only defined away from the diagonal it is not immediately clear that the Feynman integrals in equation (4.5) converges in the limit when vertices come together. In fact, the finiteness of Feynman integrals in Chern-Simons theory on a general three-manifold was proven in [4] by Axelrod and Singer, using a configuration space compactification closely related to the Fulton-MacPherson compactification [12], and in [6] this was extended by Bott and Taubes to Chern-Simons theory in the presence of Wilson lines. For the present purpose these results can be assembled to give the following theorem:

**Theorem 2.** There is a partial compactification $\overline{\mathrm{Conf}}_{V,\underline{W}}$ of the configuration spaces $\mathrm{Conf}_{V,\underline{W}}$ for subsets of vertices coming together, such that the compactified space is a manifold with corners and the differential forms $\lambda(\Gamma)$ are smooth forms with compact support on $\mathrm{Conf}_{V,\underline{W}}$.

In this compactification boundary strata are defined using spherical blow-ups along diagonals where subsets of vertices come together. In subsections 5.1 and 5.2 below we give a full description of the corresponding boundary strata of co-dimension one, each coming from a single subset of vertices all coming together at the same speed. Denoting by $\partial \mathrm{Conf}_{V,\underline{W}}$ the corresponding co-dimension one boundary it holds that $\partial \mathrm{Conf}_{V,\underline{W}}$ is given by the disjoint union of the following strata:

- For each $S \subset V$ we get boundary stratum $\partial_S \mathrm{Conf}_{V,\underline{W}}$ corresponding to the vertices $S$ coming together.
- For each $\alpha \in \{1,2,3\}$, $S \subset V$ and $T \subset W_\alpha$ with $T \neq \emptyset$ we get a boundary stratum $\partial_{S,T} \mathrm{Conf}_{V,\underline{W}}$ corresponding to vertices $S \cup T$ coming together on the line $L_{\alpha,t}$.

The reader is referred to [4], [16] and the appendix of [6] for details on the strata of higher co-dimension, which correspond to collapsing nested subsets of vertices.

## 5.1 Boundary strata for internal collisions

We begin by describing the boundary strata corresponding to a subset $S \subset V$ of internal vertices coming together. Recall that given a point $(t,q,p) \in \mathrm{Conf}_{V,\underline{W}}$ we use the notation $p_i = p(v_i)$ and $q_j = q(w_j)$. Let $i_0 \coloneqq \min\{i\}_{v_i \in S}$ and write $v_0 \coloneqq v_{i_0}$ and $p_0 \coloneqq p_{i_0}$. Furthermore, let $d_{\min}$ be the minimal distance between $p_0$ and a vertex in $\{p_i\}_{v_i \in V \setminus S} \cup \{q_j\}_{w_j \in W}$. We can define a neighbourhood $U \subset \mathrm{Conf}_{V,\underline{W}}$ where the vertices in $S$ are close together and far from all other vertices as follows:

$$U = \left\{ (t,q,p) \in \mathrm{Conf}_{V,\underline{W}} \;\middle|\; \left( \sum_{v_i \in S} |p_0 - p_i|^2 \right)^{1/2} < \eta d_{\min} \right\},$$

where $\eta > 0$ is small. Given any point $(t,q,p) \in U$ we can now write

$$p_i = p_0 + r d_{\min} u_\alpha \ , \quad v_i \in S \setminus \{v_0\}, \tag{5.1}$$

where $u_\alpha \in \mathbb{R}^3$ and $r \in (0, \eta)$ are uniquely determined by the conditions:

- $\sum_i |u_i|^2 = 1$,
- $u_i \neq u_j$ for $i \neq j$.

**Definition 5.1.** Let $G < \mathrm{Homeo}(\mathbb{R}^3)$ to be group of scalings and translations in $\mathbb{R}^3$. We define

$$C_S \coloneqq \mathrm{Conf}_S(\mathbb{R}^3)/G$$

where $G$ acts on $\mathrm{Conf}_S(\mathbb{R}^3)$ by translating and/or scaling all points simultaneously.

The points $(u_i)_{i \in S \setminus \{v_0\}}$ then determines a set of coordinates on the space $C_S$ and hence the change of coordinates in equation (5.1) determines a diffeomorphism

$$U \cong C_S \times \mathrm{Conf}_{(V \setminus S) \cup \{v_0\}, \underline{W}} \times (0, \eta). \tag{5.2}$$

The boundary stratum corresponding to the vertices $\{p_i\}_{i \in S}$ coming together is obtained by including the $r = 0$ in the interval on the right-hand side of equation (5.2). Hence

$$\partial_S \mathrm{Conf}_{V, \underline{W}} = C_S \times \mathrm{Conf}_{(V \setminus S) \cup \{v_0\}, \underline{W}}\ . \tag{5.3}$$

## 5.2 Boundary strata for external collisions

We now describe the boundary strata corresponding to a subset of both internal and external vertices coming together on one of the Wilson lines. Let $S \subset V$ and $T \subset W_\alpha$ for some $\alpha \in \{1, 2, 3\}$ and let $\mathbf{e}_\alpha$ be the unit vector pointing along $L_{\alpha,t}$ (notice that $\mathbf{e}_\alpha$ does not depend on $t$). Given a point $(t, q, p) \in \mathrm{Conf}_{V, \underline{W}}$ we use the following notation:

- $\langle p_i, \mathbf{e}_\alpha \rangle$ is the projection of $p_i$ onto $L_{\alpha,t}$,
- $j_0 = \min\{j\}_{w_j \in T}$ and we write $w_0 \coloneqq w_{j_0}$ and $q_0 \coloneqq q_{j_0}$
- $d_{\min}$ is the minimal distance between $L_{\alpha,t}(q_0)$ and a vertex in $(V \setminus S) \cup (W \setminus T)$.

We can define a neighbourhood $V \subset \mathrm{Conf}_{V, \underline{W}}$ where the vertices in $S \cup T$ are close together and far from all other vertices as follows:

$$V = \left\{ (t, q, p) \in \mathrm{Conf}_{V, \underline{W}} \;\middle|\; \left( \sum_{v_i \in T} |q_0 - \langle p_i, \mathbf{e}_i \rangle\,|^2 + \sum_{w_j \in T} |q_0 - q_j|^2 \right)^{1/2} < \eta d_{\min} \right\}.$$

Given any $(t, q, p) \in V$, $v_i \in S$ and $w_j \in T$ we can write:

$$\begin{aligned} p_i &= L_{\alpha,t}(q_0) + r d_{\min} u_i\ , \quad v_i \in S \\ q_j &= q_0 + r d_{\min} a_j\ , \quad w_\beta \in T \setminus \{w_0\} \end{aligned} \tag{5.4}$$

for unique $r \in (0, \eta)$, $u_i \in \mathbb{R}^3$ and $a_j \in \mathbb{R}$ subject to the conditions:

- $\sum_i |\,\langle u_i, \mathbf{e}_\alpha \rangle\,|^2 + \sum_j |a_j|^2 = 1$,

- $u_i \neq u_j$, $a_i \neq a_j$ and $u_i \neq a_j \mathbf{e}_\alpha$ when $i \neq j$.

**Definition 5.2.** Let $\mathrm{Conf}_{S,T}(L, \mathbb{R}^3)$ be the configuration space with points in the bulk and along the line $L \subset \mathbb{R}^3$. Concretely, $\mathrm{Conf}_{S,T}(L, \mathbb{R}^3)$ is defined as the pullback:

$$\begin{array}{ccc} \mathrm{Conf}_{S,T}(L,\mathbb{R}^3) & \longrightarrow & \mathrm{Conf}_{S\cup T}(\mathbb{R}^3) \\ \downarrow & & \downarrow \\ \mathrm{Conf}_T(\mathbb{R}) & \xhookrightarrow{L} & \mathrm{Conf}_T(\mathbb{R}^3)\ . \end{array} \tag{5.5}$$

Moreover, let $G' < \mathrm{Homeo}(\mathbb{R}^3)$ be the subgroup of scalings and translations along $L$. We define

$$C_{S,T} \coloneqq \mathrm{Conf}_{S,T}(L, \mathbb{R}^3)/G'$$

where $G'$ acts on $\mathrm{Conf}_{T,S}$ by translating and/or scaling all points simultaneously.

The points $\left\{(u_i)_{v_i \in S}, (a_j)_{w_j \in T \setminus \{w_0\}}\right\}$ determine a set of coordinates on the space $C_{S,T}$ defined above and hence the change of coordinates in equation (5.4) determines a diffeomorphism

$$V \cong \mathrm{Conf}_{V \setminus S, \underline{W'}} \times C_{S,T} \times (0, \eta), \tag{5.6}$$

where $\underline{W'}$ is obtained from $\underline{W}$ by substituting $W_\alpha$ with $(W_\alpha \setminus T) \cup \{w_0\}$. The boundary stratum corresponding to the vertices $S \cup T$ coming together is obtained by including the $r = 0$ in the interval on the right-hand side of equation (5.6). Hence

$$\partial_{S,T} \mathrm{Conf}_{V,\underline{W}} \cong \mathrm{Conf}_{V \setminus S, \underline{W'}} \times C_{S,T}. \tag{5.7}$$

# 6 Stokes' Theorem

The remainder of this paper is dedicated to proving theorem 1, namely that $\Delta_t \langle L_t \rangle = \langle L_1 \rangle - \langle L_0 \rangle = 0$. To this aim we will use the below proposition.

**Proposition 6.1.** Let $\partial\,\mathrm{Conf}(\Gamma)$ be the co-dimension one boundary in the Axelrod-Singer compactification. Then

$$\Delta_t \langle L_t \rangle = \sum_\Gamma \hbar^{\mathrm{ord}(\Gamma)} \int_{\partial\,\mathrm{Conf}(\Gamma)} \lambda(\Gamma)\, c(\Gamma)\,. \tag{6.1}$$

*Proof.* Observe that the total co-dimension one boundary of $\overline{\mathrm{Conf}}(\Gamma)$ is given by the union of boundary components coming from

1. The boundary $\partial\,\mathrm{Conf}(\Gamma)$ corresponding to subsets of vertices coming together.
2. The boundaries $\mathrm{Conf}^1(\Gamma)$ and $\mathrm{Conf}^0(\Gamma)$ corresponding to $t = 0$ and $t = 1$.
3. The boundaries coming from an internal vertex reaching $\mathbb{R}^2 \times \{-1\}$ or $\mathbb{R}^2 \times \{1\}$.

By proposition 4.2 and lemma 4.1 in section 4.4 it holds for any $\Gamma \in \mathcal{G}$ that $\lambda(\Gamma)$ has compact support in $\overline{\mathrm{Conf}}(\Gamma)$ and vanishes on the boundary corresponding to case 3 in the above. Moreover, since the propagator is a closed form on the interior of $\overline{\mathrm{Conf}}(\Gamma)$ it holds that $d\lambda(\Gamma) = 0$. The following version of Stokes' theorem now applies:

$$0 = \int_{\overline{\mathrm{Conf}}(\Gamma)} d\lambda(\Gamma) = \int_{\partial\,\mathrm{Conf}(\Gamma)} \lambda(\Gamma) + \int_{\mathrm{Conf}^0(\Gamma)} \lambda(\Gamma) - \int_{\mathrm{Conf}^1(\Gamma)} \lambda(\Gamma)\,. \tag{6.2}$$

Inserting equation (6.2) into the expression for $\langle L_t \rangle$ in equation (4.6) the proposition follows. □

Proving theorem 1 therefore amounts to showing that the sum of all boundary integrals in equation (6.1) vanishes. By the construction in the previous section we have

$$\int_{\partial\,\mathrm{Conf}(\Gamma)} \lambda(\Gamma)\, c(\Gamma) = \sum_S \int_{\partial_S\,\mathrm{Conf}(\Gamma)} \lambda(\Gamma)\, c(\Gamma) + \sum_{S,T} \int_{\partial_{S,T}\,\mathrm{Conf}(\Gamma)} \lambda(\Gamma)\, c(\Gamma)\,.$$

# 7 Vanishing Theorems

This section contains the proof of theorem 1 via a series of vanishing results for the boundary integrals in equation (6.1). These results are variations of the vanishing theorems of Bott and Taubes [6]. Concretely, in section 7.1 we prove the vanishing of boundary integrals coming from internal collisions and in section 7.2 we prove the vanishing of boundary integrals coming from external collisions (collisions along a Wilson line) .

## 7.1 Vanishing theorems for internal collisions

**Theorem 3.** The boundary integrals contributing to equation (6.1) coming from internal collisions vanishes, that is

$$\sum_\Gamma \hbar^{\mathrm{ord}(\Gamma)} \sum_S \int_{\partial_S\,\mathrm{Conf}(\Gamma)} \lambda(\Gamma)\, c(\Gamma) = 0\,. \tag{7.1}$$

**Notation:** Given $\Gamma \in \mathcal{G}$ and $S \subset V$, we denote by $\Gamma_S$ the sub-graph of $\Gamma$ spanned by the vertices in $S$ and by $\delta_S\Gamma$ the graph obtained from $\Gamma$ by collapsing $\Gamma_S$ to a single internal vertex $v_0$. Then

$$\partial_S\,\mathrm{Conf}(\Gamma) = C_S \times \mathrm{Conf}(\delta_S\Gamma).$$

Observe that $\lambda(\Gamma)$ splits into a product $\lambda(\Gamma) = \lambda_1 \wedge \lambda_2$ where $\lambda_1$ is constructed from edges in $\Gamma_S$ and $\lambda_2$ is constructed from the remaining edges. In order to prove theorem 3 we will need the following lemma.

**Lemma 7.1.** Upon restricting to $\partial_S\,\mathrm{Conf}(\Gamma)$, the form $\lambda_1$ factors through the projection

$$\pi_1 : \partial_S\,\mathrm{Conf}(\Gamma) \to C_S$$

and the form $\lambda_2$ factors through the projection

$$\pi_2 : \partial_S\,\mathrm{Conf}(\Gamma) \to \mathrm{Conf}(\delta_S\Gamma).$$

*Proof.* The proposition follows from the change of coordinates in equation (5.1). In fact, if $e$ connects two internal vertices $v_i, v_j \in S$ we have

$$\Phi_e(x) = \frac{p_j - p_i}{|p_j - p_i|} = \frac{u_j - u_i}{|u_j - u_i|}\,,$$

which implies that $\Phi_e$ and thereby $P_e$ factors through the projection $\pi_1$. On the other hand, if $e$ connects a vertex $v_i \in V \setminus S$ and a vertex $v_j \in S$ we have

$$\Phi_e(x) = \frac{p_j - p_i}{|p_j - p_i|} = \frac{p_0 + ru_j - p_i}{|p_0 + ru_j - p_i|} \to \frac{p_0 - p_i}{|p_0 - p_i|} \text{ when } r \to 0,$$

and hence $P_e$ factors through the projection $\pi_2$. □

We write

$$\lambda_1|_{\partial_S \operatorname{Conf}(\Gamma)} = \pi_1^* \,\widetilde{\lambda}(\Gamma_S) \quad\text{and}\quad \lambda_2|_{\partial_S \operatorname{Conf}(\Gamma)} = \pi_2^* \,\lambda(\delta_S \Gamma).$$

**Corollary 7.1.** Given $\Gamma \in \mathcal{G}$ and $S \subset V$, let $\eta_S(\Gamma)$ be the number of edges connecting a vertex in $S$ with a vertex in $(V \cup W) \setminus S$. The contribution to equation (7.1) from the boundary stratum $\partial_S \operatorname{Conf}(\Gamma)$ vanishes unless $\eta_S(\Gamma) = 4$.

*Proof.* By counting the number of edges connecting vertices in $S$ one finds $\deg \widetilde{\lambda}(\Gamma_S) = 3|S| - \eta_S(\Gamma)$. On the other hand, $\dim C_S = 3|S| - 4$, and hence $\widetilde{\lambda}(\Gamma_S)$ vanishes unless $\eta_S(\Gamma) \geq 4$. By a similar argument $\lambda(\Gamma_{S,T})$ vanishes on the boundary stratum unless $\eta_S(\Gamma) \leq 4$. □

**Lemma 7.2.** The contribution to equation (6.1) coming from boundary strata where more than two internal vertices come together vanishes.

*Proof.* This follows directly from corollary 7.1 and proposition 4.2, since collapsing more than two internal vertices in a forest creates a vertex of valence greater than four. □

The following lemma is known as the IHX relations.

**Lemma 7.3.** The contribution to equation (7.1) coming from boundary strata where two internal vertices come together vanishes.

*Proof.* Let $\Gamma_0$ be a graph which has a single four-valent internal vertex $v_0$ with one incoming and three outgoing edges, and with all other vertices three- and one-valent. There are exactly three graphs $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{G}$ that identify with $\Gamma_0$ when collapsing two internal vertices. These graphs are shown in figure 9, where we imagine that all vertices and edges outside the encircled area are held fixed:

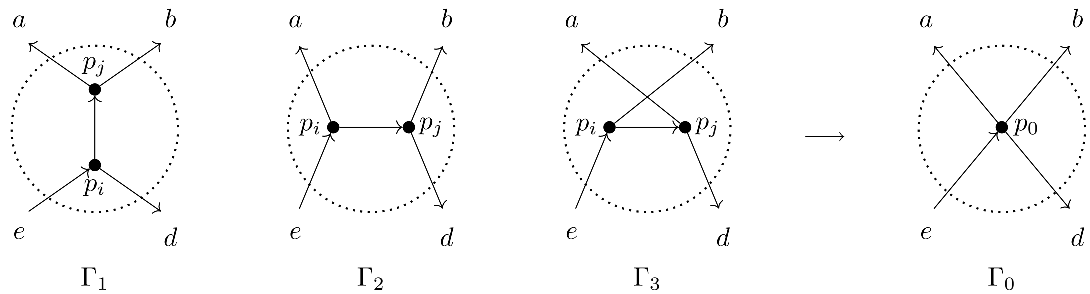



Figure 9: The IHX-relation

The boundary stratum corresponding to collapsing $p_i$ and $p_j$ is given by:

$$\partial_{\{v_i,v_j\}} \operatorname{Conf}(\Gamma_k) = C_{\{v_i,v_j\}} \times \operatorname{Conf}(\Gamma_0) \cong S^2 \times \operatorname{Conf}(\Gamma_0),$$

for $k = 1, 2, 3$. If we choose the ordering of half edges in each graph to be clockwise, it follows from definition 4.3 that

$$\operatorname{Or}(\Gamma_1) = -\operatorname{Or}(\Gamma_2) = \operatorname{Or}(\Gamma_3).$$

Hence, the contribution to the sum in equation (7.1) coming from this boundary stratum takes the form:

$$\int_{S^2} \omega \int_{\operatorname{Conf}(\Gamma_0)} \lambda(\Gamma_0)\, c(\Gamma_0), \tag{7.2}$$

where $c(\Gamma_0)$ obtained from applying the usual Feynman rules to all three- and one-valent vertices of $\Gamma_0$, and assigning to the four-valent vertex $p_0$ the factor:

$$({f^c}_{ab}{f^e}_{cd} - {f^c}_{bd}{f^e}_{ac} + {f^c}_{ad}{f^e}_{bc}),$$

which vanishes by Jacobi identity for the structure constants. This proves the theorem. □

By combining lemma 7.2 and 7.3 we have now proved theorem 3. In section 7.2 below we show the similar vanishing theorems for external collisions. Many of the arguments are repetitions of those given above.

## 7.2 Vanishing theorems for external collisions

**Theorem 4.** The boundary integrals contributing to equation (6.1) coming from external collisions vanishes, that is

$$\sum_{\Gamma} \hbar^{\operatorname{ord}(\Gamma)} \sum_{S,T} \int_{\partial_{S,T} \operatorname{Conf}(\Gamma)} \lambda(\Gamma)\, c(\Gamma) = 0. \tag{7.3}$$

**Notation:** Given $\Gamma \in \mathcal{G}$, $S \subset V$ and $T \subset W_\alpha$ for some $\alpha \in \{1,2,3\}$, denote by $\Gamma_{S,T}$ the subgraph of $\Gamma$ spanned by the vertices in $S \cup T$ and by $\delta_{S,T}\Gamma$ the graph obtained from $\Gamma$ by collapsing $\Gamma_{S,T}$ to a single external vertex $w_0$. Then

$$\partial_{S,T}\operatorname{Conf}(\Gamma) = C_{S,T} \times \operatorname{Conf}(\Gamma_{S,T}).$$

We begin by proving the equivalents of lemma 7.1 and corollary 7.1 in the case of external collisions. As in section 7.1 we can write $\lambda(\Gamma) = \lambda_1 \wedge \lambda_2$ where $\lambda_1$ is constructed from the edges in $\Gamma_{S,T}$ and $\lambda_2$ is constructed from the remaining edges.

**Lemma 7.4.** Upon restricting to $\partial_{S,T}\operatorname{Conf}(\Gamma)$, the form $\lambda_1$ factors through the projection

$$\widetilde{\pi}_1 : \partial_{S,T}\operatorname{Conf}(\Gamma) \to C_{S,T}$$

and the form $\lambda_2$ factors through the projection

$$\widetilde{\pi}_2 : \partial_{S,T}\operatorname{Conf}(\Gamma) \to \operatorname{Conf}(\delta_{S,T}\Gamma).$$

*Proof.* Let $e$ be an edge connecting a vertices $v_i \in S$ and $w_j \in T$. Then with the coordinate change in equation (5.4) we have

$$\Phi_e(x) = \frac{L_{\alpha,t}(q_j) - p_i}{\big|L_{\alpha,t}(q_j) - p_i\big|} = \frac{a_j\mathbf{e}_\alpha - u_i}{|a_j\mathbf{e}_\alpha - u_i|}\,,$$

which implies that $\Phi_e$ and thereby $P_e$ factors through the projection $\widetilde{\pi}_1$. On the other hand, if $e$ connects a vertex $v_i \in V \setminus S$ and a vertex $v_j \in S$ we have

$$\Phi_e(x) = \frac{p_j - p_i}{|p_j - p_i|} = \frac{L_{\alpha,t}(q_0) + ru_j - p_i}{\big|L_{\alpha,t}(q_0) + ru_j - p_i\big|} \to \frac{L_{\alpha,t}(q_0) - p_i}{|L_{\alpha,t}(q_0) - p_i|} \text{ when } r \to 0$$

and hence $\Phi_e$ factors through the projection $\widetilde{\pi}_2$. The remaining cases are similar. □

We write

$$\lambda_1|_{\partial_{S,T}\operatorname{Conf}(\Gamma)} = \widetilde{\pi}_1^*\,\widetilde{\lambda}(\Gamma_{S,T}) \quad\text{and}\quad \lambda_2|_{\partial_{S,T}\operatorname{Conf}(\Gamma)} = \widetilde{\pi}_2^*\,\lambda(\delta_{S,T}\Gamma).$$

**Corollary 7.2.** Given $\Gamma \in \mathcal{G}$, $S \subset V$ and $T \subset W_\alpha$, let $\eta_{S,T}(\Gamma)$ be the number of edges connecting a vertex in $S \cup T$ with a vertex in $(V \cup W) \setminus (S \cup T)$. The contribution to equation (7.3) from the boundary stratum $\partial_{S,T}\operatorname{Conf}(\Gamma)$ vanishes unless $\eta_{S,T}(\Gamma) = 2$.

*Proof.* By counting the number of edges connecting vertices in $S \cup T$ one finds $\deg\widetilde{\lambda}(\Gamma_{S,T}) = 3|S| + |T| - \eta_{S,T}(\Gamma)$. On the other hand, $\dim C_{S,T} = 3|S| + |T| - 2$, and hence $\widetilde{\lambda}(\Gamma_{S,T})$ vanishes unless $\eta_{S,T}(\Gamma) \geq 2$. By a similar argument $\lambda(\delta_{S,T}\Gamma)$ vanishes on the boundary stratum unless $\eta_{S,T}(\Gamma) \leq 2$. □

The following lemma is known as the STU relations.

**Lemma 7.5.** The contribution to equation (7.3) corresponding to two vertices coming together where at least one is external vanishes.

*Proof.* Let $\Gamma_0$ be a graph with a single two-valent external vertex $v_0$ that has an incoming and an outgoing edge, and with all other vertices three- and one-valent. There are exactly three graphs $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{G}$ that maps to $\Gamma_0$ upon collapsing two vertices. These graphs are shown in figure 10.

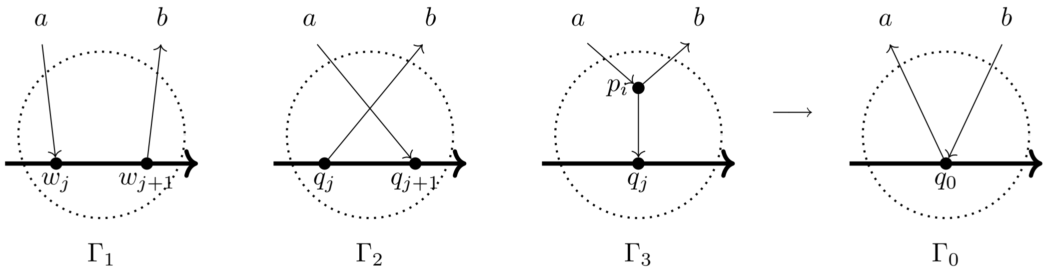



Figure 10: The STU-relation

Collapsing the vertices $q_j$ and $q_{j+1}$ in $\Gamma_1$ and $\Gamma_2$ and the vertices $q_j$ and $p_i$ in $\Gamma_3$ into a single vertex $q_0$ we obtain a graph $\Gamma_0$ with a single two-valent external vertex as shown on the right-hands side of figure 10. The corresponding boundary strata are given by

$$\partial_{\emptyset,\{w_j,w_{j+1}\}} \operatorname{Conf}(\Gamma_k) = \{*\} \times \operatorname{Conf}(\Gamma_0)$$

for $k = 1, 2$ and

$$\partial_{\{v_i\},\{w_j\}} \operatorname{Conf}(\Gamma_3) = C_{\{v_i\},\{w_j\}} \times \operatorname{Conf}(\Gamma_0) \cong S^2 \times \operatorname{Conf}(\Gamma_0).$$

We now determine the induced orientation on $\operatorname{Conf}(\Gamma_0)$ coming from each $\Gamma_k$, $k \in \{1, 2, 3\}$. By definition 4.3 we can write

$$\operatorname{Or}(\Gamma_1) = -\operatorname{Or}(\Gamma_2) = dq_j \wedge dq_{j+1} \wedge X \tag{7.4}$$

and

$$\operatorname{Or}(\Gamma_3) = (dq_j \wedge dp_i^1) \wedge dp_i^2 \wedge dp_i^3 \wedge X\,, \tag{7.5}$$

where $X$ is the same for all $\Gamma_k$, $k \in \{1, 2, 3\}$. Inserting $q_{j+1} = q_j + r$ for some $r > 0$ into equation (7.4) we get

$$\operatorname{Or}(\Gamma_1) = -\operatorname{Or}(\Gamma_2) = -dq_j \wedge dr \wedge X\,.$$

Similarly, we can write $p_i = L_{t,\alpha}(q_j) + ru$ for some $r > 0$ and unit vector $u \in \mathbb{R}^3$, and inserting this into (7.5) we get

$$\operatorname{Or}(\Gamma_3) = dq_j \wedge d^3(ru) \wedge X = dq_j \wedge \mathrm{vol}_{S^2} \wedge r^2 dr \wedge X\,.$$

In each case, the vector $r$ is orthogonal to the boundary and pointing into the configuration space. Thus, fixing an orientation $\mathrm{Or}(\Gamma_0) = dq_j \wedge dX$ on $\mathrm{Conf}(\Gamma_0)$, the contribution to equation (7.3) from the three boundary integrals takes the form:

$$\int_{\mathrm{Conf}(\Gamma_0)} \lambda(\Gamma_0)\, c(\Gamma_0),$$

where $c(\Gamma_0)$ is the Lie algebra factor obtained from applying the usual Feynman rules to $\Gamma_0$ at each three- and one-valent vertex, and assigning to the two-valent vertex $q_0$ the factor:

$$\zeta^a \xi_b - \xi_b \zeta^a - f^a{}_{bc} \zeta^c \int_{S^2} \omega.$$

Recall that from the definition in section 3.2 that $\omega$ integrates to one on $S^2$, and hence the above factor vanish by the Lie algebra relations:

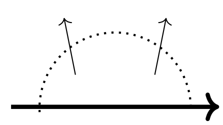
$$[\zeta^a, \xi_b] = f^a{}_{bc} \zeta^c\,.$$

Similar arguments would apply had we started from a graph $\Gamma_0$ with two incoming or two outgoing edges. Hence the theorem follows. □

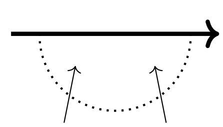
**Lemma 7.6.** Let $S \subset V$ and $T \subset W_\alpha$ such that $T \neq \emptyset$ and $|S \cup T| > 2$. Then contribution to equation (7.3) from the boundary stratum where the vertices in $S \cup T$ come together vanishes.

*Proof.* Recall from corollary 7.2 that we only get a contribution to equation (7.3) when $\Gamma$ has exactly two edges "leaving the stratum", that is, connecting a vertex in $S \cup T$ with a vertex not in $S \cup T$. We consider the following three cases separately:

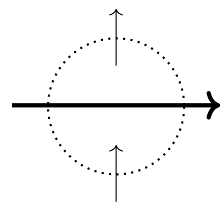
(a) Both of the edges leaving the stratum have orientations pointing out of $S \cup T$:



(b) Both of the edges leaving the stratum have orientations pointing into $S \cup T$:



(c) One of the edges leaving the stratum has orientation point into $S \cup T$ and the other edge has orientation pointing out of $S \cup T$:



**Case (a):** Since, by proposition 4.2, all contributing graphs are trees, this situation can only occur when $|S \cup T| = 2$.

**Case (b):** We can assume that at least one of the edges leaving the stratum is connected to an internal vertex $v \in S$ since otherwise $|S| = \emptyset$ and $|T| = 2$. Let $\Gamma_v$ be the disconnected sub-graph of $\Gamma_{S,T}$ spanned by the vertices $S' \cup T'$ connected by a path to $v$ as illustrated in figure 11. We write

$$\widetilde{\lambda}(\Gamma_{S,T}) = \widetilde{\lambda}_1 \wedge \widetilde{\lambda}_2$$

where $\widetilde{\lambda}_1$ is constructed from edges in $\Gamma_v$ and $\widetilde{\lambda}_2$ is the contribution from the remaining edges in $\Gamma_{S,T}$. It then holds that $\widetilde{\lambda}_1$ factors through the projection

$$p : C_{S,T} \to C_{S',T'}$$

which forgets about the vertices not in $\Gamma_v$. By counting the number of edges and vertices in $\Gamma_v$ one finds that $\widetilde{\lambda}_1$ vanishes by the same dimensional arguments as used in the proof of corollary 7.2.

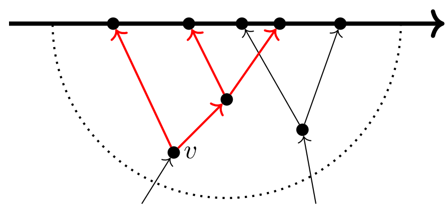



Figure 11: Boundary stratum with two incoming edges. The red edges form the sub-graph $\Gamma_v$.

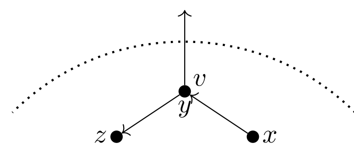
**Case (c):** We will further divide case (c) into two subcases:

(c1) Either one of the edges leaving the stratum is connected an external vertex $w \in T$ or both edges leaving the stratum are connected to the same internal vertex $v \in S$.

(c2) Both edges leaving the stratum are connected to internal vertices $v, v' \in S$ and $v \neq v'$.

**Case (c1):** In this case $\widetilde{\lambda}(\Gamma_{S,T})$ vanishes on dimensional grounds by arguments completely analogous to those for case (b).

**Case (c2):** Assume that the outgoing edge is connected $v \in S$. By assumption $\Gamma$ has two edges connecting $v$ to two different vertices in $S \cup T$. The situation is illustrated below where we have assigned coordinates $x, y$ and $z$ to the three vertices. Notice that $x$ and $z$ may be coordinates along the Wilson line.




We can now use a well known coordinate change originally due to Kontsevich [15] to show the vanishing of the integral

$$\int_{C_{S,T}} \widetilde{\lambda}(\Gamma_{S,T}). \tag{7.6}$$

In fact, integrating over $y$ in equation (7.6) while keeping all other vertices fixed produces the integral

$$\int_{y \in \mathbb{R}^3} \phi^*\omega(x, y) \wedge \phi^*\omega(y, z). \tag{7.7}$$

We now make the following change of coordinates: $y = x + z - y'$.

$$\begin{aligned} \int_y \phi^*\omega(x, y) \wedge \phi^*\omega(y, z) &= -\int_{y'} \phi^*\omega(x, x + z - y') \wedge \phi^*\omega(x + z - y', z) \\ &= -\int_{y'} \phi^*\omega(y', z) \wedge \phi^*\omega(x, y'). \end{aligned} \tag{7.8}$$

The minus since comes from this coordinate change being orientation reversing and the last equality uses translation invariance of $\phi$. This implies that the integral in equation (7.7) equals minus itself and hence must be zero. □

Lemma 7.5 and 7.6 proves theorem 4, and together with theorem 3 this completes the proof of theorem 1. By lemma 4.1, this implies that the expectation value $\mathcal{R}$ of a pair of crossing Wilson lines is a solution to the Yang-Baxter equation. In the following section we argue that $\mathcal{R}$ is in fact an $R$-matrix in the sence of section 2. In particular, we show that $\mathcal{R}$ is independent of the angle of crossing between the Wilson lines and that it satisfies a so called unitarity relation, implying that it is invertible.

# 8 Angle Independence and Unitarity

**Proposition 8.1.** Let $L$ and $L'$ be two (non-parallel) lines in $\mathbb{R}^2 \times I$ supported at different points in $I$. Then expectation value $\mathcal{R} = \langle LL' \rangle$ is independent of the angle of crossing between the lines.

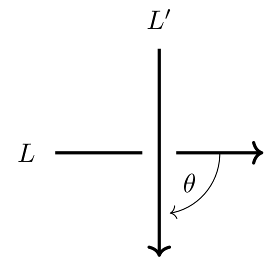
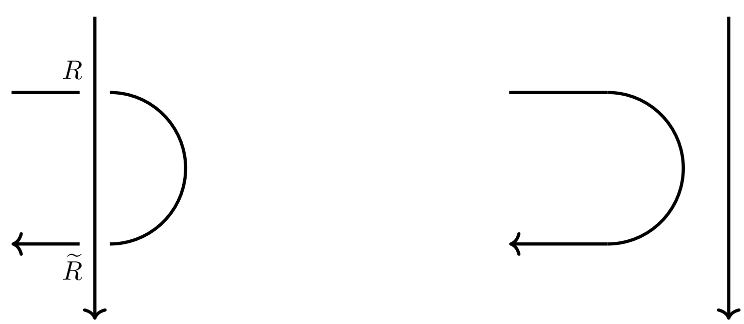



Figure 12

*Proof.* Consider changing the angle $\theta$ at the crossing in figure 12 by keeping $L'$ fixed while rotating $L$. We can apply the same vanishing arguments as in section 7 to check that the expectation value is unchanged under this operation. Notice that in this case the tangent vector to $L$ dependents on $\theta$. We therefore get the following weaker version of corollary 7.2: Let $\Gamma \in \mathcal{G}$ and let $S$ be a subset of internal vertices and $T$ a subset of external vertices on $L$. It then holds that $\lambda(\Gamma)$ vanishes on $\partial_{S,T} \operatorname{Conf}(\Gamma)$ unless $\eta_{S,T}(\Gamma) \leq 2$. On the other hand, since by proposition 4.2 the only contributing Feynman graphs in are forests with roots on $L$ and leafs on $L'$, it holds that $\eta_{S,T}(\Gamma) \geq 2$ for any choice of $\Gamma$, and hence the vanishing arguments carry through regardless. □

**Proposition 8.2.** The element $\mathcal{R}$ is invertible, that is, it satisfies the relation shown in figure 13.




(a) (b)

Figure 13

*Proof.* We here use the exact same arguments as for the angle independence of $\mathcal{R}$ in proposition 8.1. In fact, if we start from the diagram in figure 13 (a) and keep the top line fixed while continuously moving the bottom line to the left we obtain diagram in figure 13 (b). By the same argument as above, the expectation value is invariant under this operation. □

# 9 Conclusion

We have proved that the expectation value $\mathcal{R} = \langle LL' \rangle$ of a pair of crossing Wilson lines is an $R$-matrix. In [1] Kaufman and the present author showed that the leading order deformation of the co-product in $\mathcal{U}_\hbar(\mathfrak{g})$ can be realised from the operation of merging two parallel Wilson lines. As in [2], computations are here carried out in the setting of Chern-Simons theory for a semi-simple Lie algebra extended by an extra copy of the Cartan subalgebra. The arguments however translate directly into the present context. Together these results give a Wilson line realisation of the co-product and $R$-matrix in the quasi-triangular Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g})$, thus supporting the claim that the category of Wilson line operators is equivalent to the category of representations of $\mathcal{U}_\hbar(\mathfrak{g})$ as a braided tensor category.

A final remark worth noting: As mentioned in the introduction, the theory we have studied is equivalent to a topologically twisted 3d $\mathcal{N} = 4$ gauge theory. Moreover, if we take $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ to be a Lie super-algebra this would also cover Chern-Simons theory as a 3d $\mathcal{N} = 4$ gauge theory with matter. We have here only considered the case when $\mathfrak{g}$ is a classical Lie algebra but nothing in the arguments should change significantly if one instead considers the super-algebra case.

## Acknowledgements

I am grateful to Nathalie Wahl, Kevin Costello and Dani Kaufman for helpful discussions.
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# Part III

# Appendix

## Problematic boundary integrals for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

The broken symmetry of the propagator prevents us from directly implementing the vanishing theorems of Bott and Taubes [4] to recover solutions to the Yang-Baxter equation for Chern-Simons theory in the setup of paper 1 (section 2 and 3.1). In paper 2, this is accounted for by considering a Lie algebra for which the only contributing Feynman diagrams are trees. Recall from proposition (6.1) of paper 2 that, to recover solutions to the Yang-Baxter equation, we need the following sum to vanish:

$$\Delta_t \langle L_t \rangle = \sum_{\Gamma} \hbar^{\mathrm{ord}(\Gamma)} \int_{\partial \, \mathrm{Conf}(\Gamma)} \lambda(\Gamma) \, c(\Gamma) \, . \tag{1}$$

In this appendix I describe the boundary strata contributing to this sum starting from the setup of paper 1.

For simplicity, consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ extended by a copy $\widetilde{H}$ of the Cartan. Then $\tilde{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C}) \oplus \widetilde{H}$ has generators $\{E, F, H_+, H_-\}$, where $H_+ = H + \mathbf{i}\widetilde{H}$ and $H_- = H - \mathbf{i}\widetilde{H}$, and we impose boundary conditions by restricting the gauge field to $\mathfrak{l}_+ = \mathrm{span}\{E, H^+\}$ on the upper boundary $\mathbb{R}^2 \times \{1\}$ and to $\mathfrak{l}_- = \mathrm{span}\{F, H^-\}$ on the lower boundary $\mathbb{R}^2 \times \{-1\}$. We will use the definition of Feynman diagrams given in section 5 of paper 2, with the convention that edges have orientation going from $\mathfrak{l}_-$ to $\mathfrak{l}_+$. By definition of the structure constant, the only allowed internal vertices in the theory are the ones shown below:

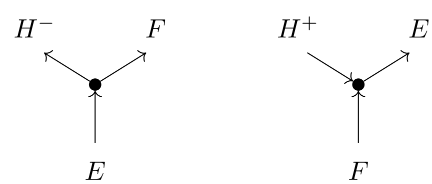



Figure 3: The allowed Feynman diagram vertices

Observe first that the contribution to equation (1) from principal faces (two vertex collisions) vanishes due to the IHX and STU relations just as in paper 2. We therefore focus in the contribution from hidden faces (more-than-two vertex collisions). For internal collisions the vanishing arguments carry through and we have the following proposition:

**Proposition A.** The contribution to equation (1) from the boundary strata corresponding to internal collisions vanishes.

*Proof.* We only give a sketch of the proof. Let $\Gamma \in \mathcal{G}$ and $S \subset V$ be a subset of internal vertices. By corollary 7.1 of paper 2 the associated differential form $\lambda(\Gamma)$ vanishes on $\partial_S \, \mathrm{Conf}_{V,\underline{W}}$ unless $\Gamma$ has exactly four half-edges "leaving $S$", as illustrated in the two examples below:

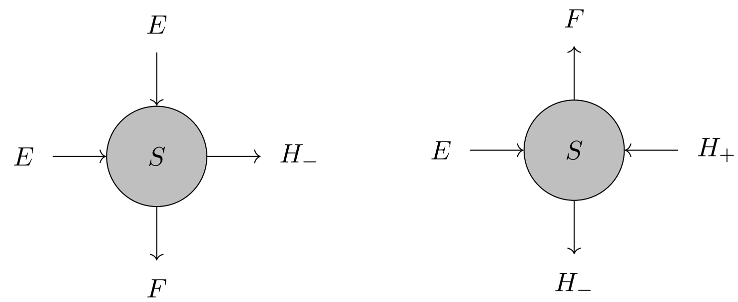



Figure 4: Examples of boundary strata for internal collisions

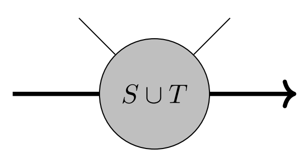
Assume that $\Gamma$ has at least two of the half-edges leaving $S$ labeled by $E$ (or equivalently by $F$), as in the diagram on the left-hand side of figure 4. Now, let $\Gamma'$ be the graph obtained from $\Gamma$ permuting these two half-edges. Then $c(\Gamma) = c(\Gamma')$ and the similarly $\lambda(\Gamma) = \lambda(\Gamma')$ upon restricting to the boundary $\partial_S \operatorname{Conf}_{V,\underline{W}}$. However, by definition 4.3 of paper 2, the orientation form changes sign: $\operatorname{Or}(\Gamma) = -\operatorname{Or}(\Gamma')$. Since both graphs contribute to the sum in equation (1), the total contribution coming from this boundary stratum cancels out.

Assume instead that at least two of the half-edges leaving $S$ are labeled by either $H_-$ or $H_+$ as on the right-hand side of figure 4. Notice that

$$\operatorname{Tr}([E,F],H_+) = \operatorname{Tr}([E,F],H_-) = 1.$$

Hence, the Lie algebra factor $c(\Gamma)$ is unchanged under permuting the two half-edges labeled by $H_\pm$ and the vanishing follows in the same way as above. This accounts for all possible labelings of the half-edges leaving $S$ and so the proposition follows. □

We now turn to the boundary integrals for external collisions. Let $T \subset W_\alpha$ for $\alpha \in \{1,2,3\}$, with $T \neq \emptyset$ and $|S \cup T| > 2$. By corollary 7.2 of paper 2 the differential form $\lambda(\Gamma)$ vanishes on $\partial_{S,T} \operatorname{Conf}_{V,\underline{W}}$ unless $\Gamma$ has exactly two half-edges leaving $S \cup T$.




**Proposition B.** The contribution to equation (1) from the stratum $\partial_{S,T} \operatorname{Conf}_{V,\underline{W}}$ vanishes for graphs with both of the half-edges leaving $S \cup T$ labeled by the same generator of $\mathfrak{g}$.

*Proof.* The corresponding boundary integrals cancel out by the same arguments as in proposition A. □

**Proposition C.** The contribution to equation (1) from the stratum $\partial_{S,T}\,\mathrm{Conf}_{V,\underline{W}}$ vanishes for graphs with at least one of the half-edges leaving the stratum labeled by $H_-$ or $H_+$.

*Proof.* Assume first that the half-edge labeled by $H_\pm$ is incident to a vertex on the Wilson line. Then $\lambda(\Gamma)$ vanishes on $\partial_{S,T}\,\mathrm{Conf}(\Gamma)$ on dimensional grounds (see case (b) of lemma 7.6 in paper 2). We can therefore assume that the half-edge labeled by $H_\pm$ is incident to an internal vertex in $v \in S$. Then the situation is as in figure 5.

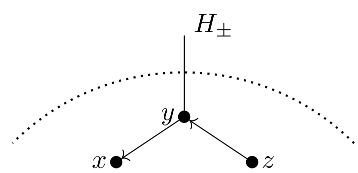



Figure 5: Boundary stratum with a red edge leaving the stratum.

The Kontsevich change of coordinates in case (c2) of lemma 7.6 now ensures that the integral over $C_{S,T}$ vanishes. □

The only remaining case is therefore the contribution to equation (1) from $\partial_{S,T}\,\mathrm{Conf}_{V,\underline{W}}$ for graphs $\Gamma$ with two half-edges leaving $S \cup T$, one labeled by $E$ and the other by $F$. Observe that this situation only occurs when $T$ is a set of vertices on the middle Wilson line. An example of such a boundary integral, for which it appears there is no easy vanishing argument, is illustrated in figure 6.

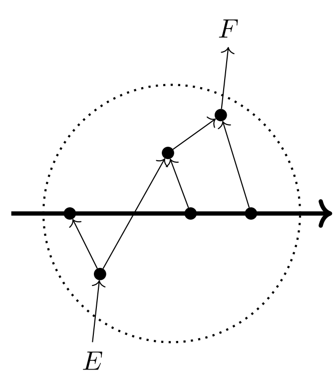



Figure 6: Problematic boundary integrals