Entropy bounds for self-shrinkers with symmetries and applications

Ali Muhammad

This thesis has been submitted to the PhD School of the
Faculty of Science, University of Copenhagen

Ali Muhammad
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen Ø, Denmark
ajhm@math.ku.dk

Date of submission:	31st of August, 2023
Date of defense:	9th of October, 2023
Advisor:	**Niels Martin Møller** University of Copenhagen
Assessment committee:	**Søren Fournais** (chair) University of Copenhagen **Anna Siffert** University of Münster **Shengwen Wang** Queen Mary University of London

ISBN: 978-87-7125-219-4

Abstract

In this thesis, we derive various entropy upper bounds for self-shrinkers of the mean curvature flow which admit a symmetry, including several applications.

In our first paper, which is a joint work with Niels Martin Møller and John Ma, we study the space of complete embedded rotationally symmetric self-shrinkers. We first derive explicit entropy upper bounds for this class of self-shrinkers. The proof is purely geometric and relies on an application of the general Toponogov's theorem from metric geometry to derive length upper bounds on simple closed geodesics in an incomplete surface with curvature bounded from below by a positive constant. We then apply the entropy bounds to first prove a smooth compactness theorem for this space of self-shrinkers. Second, we show that there are finitely many such self-shrinkers which additionally are symmetric with respect to the hyperplane perpendicular to the axis of rotation.

In our second paper, which is a joint work with John Ma, we generalize the entropy bounds obtained in our first work in two directions. We modify the proof of the embedded class to include entropy upper bounds for compact non-spherical immersed rotationally symmetric self-shrinkers. We also obtain entropy bounds for a larger class of embedded self-shrinkers which are constructed through the theory of isoparametric foliations of the sphere and which contain the space of complete embedded rotationally symmetric self-shrinkers as a special case.

Resumé

I denne afhandling udleder vi adskillige entropi øvre grænser for middelkrumningsflow selv-skrumpere med en vis symmetri, inklusive flere anvendelser.

I vores første artikel, som er udført i samarbejde med Niels Martin Møller og John Ma, studerer vi rummet af fuldstændige indlejrede rotationssymmetriske self-skrumpere. Først udleder vi eksplicitte entropi øvre grænser for denne klasse af selv-skrumpere. Beviset er rent geometrisk og er baseret på en anvendelse af den generelle Toponogovs sætning fra metrisk geometri for at udlede længde øvre grænser på simple, lukkede geodæter i en ufuldstændig flade hvor krumningen er nedre-begrænset af en positiv konstant. Først anvender vi entropi grænserne for at bevise en glat kompakthedssætning for dette rum af selv-skrumpere. Dernæst viser vi at der findes endelige mange sådanne selv-skrumpere som derudover er symmetriske med hensyn til den plan som er vinkelret på rotationsaksen.

I vores anden artikel, som er udført i samarbejde med John Ma, generaliserer vi de entropi grænser fra vores første arbjede i to retninger. Vi modificerer beviset af den indlejrede klasse til at inkludere entropi øvre grænser for kompakte, ikke-sfæriske immerserede rotationssymmetriske selv-skrumpere. Vi finder også entropi grænser for en større klasse af selv-skrumpere som er konstrueret igennem teorien for isoparametriske foliationer af kuglen og som indeholder rummet af fuldstændige indlejrede rotationssymmetriske self-skrumpere som et specialtilfælde.

Acknowledgements

I would like to thank my advisor Niels Martin Møller for his guidance, constant support and many fruitful discussions for the last three years.

Despite being a relatively small group at each instance of time, I have been very lucky to make the acquaintance of many geometric analysts in Copenhagen throughout my time as a student. Thanks to Francesco Chini for encouraging me to learn about mean curvature flow. Many thanks to John Ma for stimulating discussions, a valuable collaboration and from whom I have learnt a great deal of mathematics. Thanks also to Alexander Friedrich, Felix Lubbe, Jingxuan Zhang, Alex Mramor and Eric Ling for numerous conversations on mathematics and beyond.

I would also like to thank my warrior office mates Adriano Cordova and Vignesh Subramanian for making life at the office lively and rather enjoyable.

Thanks to my family who as neighbours to the department have provided me with supper and shelter on countless occasions - it has saved me many hours of cumbersome commuting. Finally, I'd like to express my deepest gratitude to Emilia who has provided me with much needed love and support.

Papers included in this thesis

This thesis is based on the following two papers:

- Paper I: [MMM22] John Man Shun Ma, Ali Muhammad, and Niels Martin Møller. Entropy bounds, compactness and finiteness theorems for embedded self-shrinkers with rotational symmetry. Journal für die reine und angewandte Mathematik (Crelles Journal), 2022(793):239–259, 2022.
- Paper II: [MM23] John Man Shun Ma and Ali Muhammad. Entropy bounds for self-shrinkers with symmetries. arXiv preprint arXiv:2306.12171, 2023.

Paper I is a joint work with Niels Martin Møller and John Ma, while Paper II is a joint work with John Ma.

Besides a few corrections to the bibliography sections and minor changes to the layout, the content of the two papers is unaltered before being included in this thesis.

Contents

Chapter 1

Introduction

The mean curvature flow is the most studied extrinsic geometric curvature flow. In a classical setting, it is the study of how immersed hypersurfaces Σ^n in $\mathbb{R}^{n+1}$ evolve in the normal direction as prescribed by the mean curvature. A most motivating reason to study the mean curvature flow is exhibited in the fact that it is the (negative) gradient flow of the area functional, and on a first look its equation bears a resemblance to the classical heat equation - but deep inside there are important differences that arise from the inherent nonlinearity. On a short time-scale the flow acts by smoothing and evening of the solution, and this opens up for numerous physical and mathematical applications: how can we understand a complicated geometric object by deforming it in a natural way - by minimizing it's area as efficiently as possible - so that we end up with a simpler object which is easier to understand. However, with the passage of time the flow inevitably develops singularities, and hence to better understand and appreciate the flow one must understand the singularities it goes through. Indeed, much of the work done in the field is in one way or another related to the singularities of the flow.

The equation of the mean curvature flow made its first appearance in the work of Mullins [Mul56] in the context of material science. Mullins used the flow to study the motion of idealized grain boundaries in the annealing process of metals. He also wrote down some of the first known solutions to the flow such as the famous translating grim reaper. In a fundamental work, Ken Brakke [Bra78] studied a weak notion of the flow using the language of geometric measure theory. A few years later, inspired by the work of Richard Hamilton [Ham82] on the intrinsic counterpart, the Ricci flow, Gerhard Huisken [Hui84] studied the flow from the PDEs point of view and proved the first important theorem in the field which states that convex hypersurfaces evolve under the flow into a spherical singularity in finite time.

Almost four decades since the work of Husiken [Hui84], the mean curvature flow remains an active and rich field in geometric analysis. The four decades have led into many important mathematical results including applications to topology and geometry, such as the proof of the Riemannian Penrose inequality using inverse mean curvature flow by Husiken-Ilmanen [HI01], the classification of two-convex hypersurfaces by Huisken-Sinestrari [HS09], the proof of path-connectedness of the moduli spaces of two-convex embedded tori and spheres by Buzano-Haslhofer-Hershkovits [BHH19], [BHH21], the classification of low-entropy hypersurfaces by Bernstein-Wang [BW18] and the optimal isoperimetric inequalities for surfaces in Cartan-Hadamard manifolds by Schulze [Sch20], just to mention a few.

As one of the most important models of singularities for the mean curvature flow, self-shrinkers have been extensively studied ever since the early work of Huisken [Hui84], with so much yet to be understood. This thesis is devoted to the study of self-shrinkers which admit a symmetry. In particular, the space of rotationally symmetric self-shrinkers constitute a main topic in this thesis.

Structure of the thesis

This thesis is structured as follows. In chapter 2 we briefly present some background to mean curvature flow, including a discussion on singularities with special focus on self-shrinkers and the entropy functional.

In chapter 3 we give a short overview of our results. First we briefly touch on the space of rotationally symmetric self-shrinkers. We state Toponogov's theorem from comparison geometry which is a main tool in the results obtained in this thesis, and then we provide a summary of paper I [MMM22] and paper II [MM23].

Paper I and II are contained as separate and independent chapters in this thesis as chapter 4 and chapter 5, respectively.

Chapter 2

Preliminaries

The goal of this chapter is to provide a brief overview on singularity formation in hypersurfaces flowing by mean curvature flow in $\mathbb{R}^{n+1}$, with special emphasis on self-shrinkers as an important class of singulariy models for the flow. It is not our aim to be comprehensive or to provide details to well-known results, as many excellent books and surveys provide all the details of what we are about to present here, e.g. the books [Man11], [Eck04] and [ACGL20], and the nice survey paper [CMP15]. Instead, the aim is to provide a selective storyline which serves as preliminaries to the work of this thesis. We will restrict to the setting of hypersurfaces in $\mathbb{R}^{n+1}$, although numerous results have been established in higher codimension and on mean curvature flow in more general ambient Riemannian manifolds.

Notation

In this thesis, Σ^n will by default denote a smooth orientable immersed hypersurface in $\mathbb{R}^{n+1}$. A unit normal vector field will be denoted by $\vec{n}$ and sometimes ν, and the second fundamental form will be denoted by A. The scalar mean curvature will be denoted by H and is given by the trace of A. The *mean curvature vector* is given by $\vec{H} = -\vec{n}H$. Sometimes we add a Σ-subscript to geometric quantities when needed. A hypersurface Σ is called *mean convex* if there exists a choice of unit normal $\vec{n}$ such that $H \geq 0$, and *convex* if there exists a choice of unit normal $\vec{n}$ such that A is positive semidefinite. Following [CM12a], a hypersurface $\Sigma \subseteq \mathbb{R}^{n+1}$ is said to have *polynomial volume growth* if for any fixed point $x \in \mathbb{R}^{n+1}$ there exist constants C and d such that for all $r \geq 1$

$$\mathrm{Vol}\left(B_r(x) \cap \Sigma\right) \leq Cr^d,$$

where $B_r(x)$ denotes the Euclidean ball centered at x and with radius r.

2.1 Mean curvature flow

Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be smooth 1-parameter family of immersions. We say that X is a *mean curvature flow* if it satisfies

$$\left(\frac{\partial X}{\partial t}(x,t)\right)^{\perp} = \vec{H}(x,t) \quad \text{for all } (x,t) \in M^n \times I \tag{1}$$

Where $\vec{H}(\cdot,t)$ is the mean curvature vector of the immersion $X(\cdot,t)$. We will sometimes write $M_t := X(M,t)$. By the identity $\Delta X = \vec{H}$, the mean curvature flow can be seen as a natural nonlinear analogue to the classical heat equation.

By a choice of gauge to break the diffeomorphism invariance of (1), together with standard parabolic theory for quasilinear equations, one obtains short time existence. For example, in the compact case we have [ACGL20, Proposition 6.8]:

Proposition 1. *Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed manifold M. There exists $\varepsilon > 0$ and a unique smooth solution $X : M \times [0,\varepsilon) \to \mathbb{R}^{n+1}$ to the mean curvature flow equation* (1) *with $X(\cdot,0) = X_0$.*

Example 1. Let M_0 be a minimal hypersurface in $\mathbb{R}^{n+1}$. Since the mean curvature vanishes, this gives rise to a stationary solution for (1) defined on $I = \mathbb{R}$.

Example 2. Let $M_0 = \mathbb{S}^n_R$, i.e. the n-sphere of radius R. One can show that the mean curvature flow is given by homothetically shrinking spheres with radius $R(t) = \sqrt{R_0^2 - 2nt}$. The flow is defined on $[0, T_{\max})$, where $T_{\max} = R_0^2/(2n)$.

A powerful tool for studying mean curvature flow is the *avoidance principle*. See e.g. [Man11, Theorem 2.2.1] for a proof.

Theorem 2 (Avoidance Principle)**.** *Let $X_i : M_i \times I \to \mathbb{R}^{n+1}$ for $i = 1,2$ be mean curvature flows on compact hypersurfaces. Then the distance between $X_1(M_1,t)$ and $X_2(M_2,t)$ is non-increasing in t.*

The theorem also holds if one of $X_i(M_i,t)$ is properly immersed for all $t \in I$. The avoidance principle demonstrates the abundance of singularities in mean curvature flow, as one expects of a nonlinear evolution equation. Indeed, any closed hypersurface M becomes singular in finite time under the mean curvature flow $X : M \times [0,T) \to \mathbb{R}^{n+1}$. This can be seen by enclosing M by a sufficiently large sphere, so $T \leq T_{\max}$, where $T_{\max}$ is from example 2.

Theorem 2 above utilizes a maximum principle argument. This is a very important tool in the study of the mean curvature flow. Further important consequences of the maximum principle include the preservation of embeddedness, convexity and mean convexity under the flow.

We say that a solution $X : M \times [0,T) \to \mathbb{R}^{n+1}$ is *maximal* if it has no smooth extension to a larger time interval, and we often denote the maximal time T by $T_{\max}$ as in example 2. The following theorem gives a useful characterization of the maximal time $T_{\max}$, see e.g. [ACGL20, Theorem 6.20] for a proof.

Theorem 3 (Long time existence)**.** *Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a solution to* (1)*, where M is compact. If X is a maximal solution and $T < \infty$, then*

$$\sup_{M\times[0,T)} |A| = \infty.$$

Weak formulations

Ultimately one would like to understand the structure of the singular set of the flow. As we will see in the next section, this set is quite complicated. A weak formulation of the flow allows to continue the flow across the singularities it encounters. One way to formulate a weak notion of mean curvature flow is through the pioneering work of Ken Brakke [Bra78]. He introduced *the Brakke flow* in the language of varifolds within geometric measure theory. This formulation makes it possible to take various limits through compactness theorems, as done in for example the existence of weak tangent flows (see Theorem 8 below). Furthermore, Brakke's general regularity theorem [Bra78], [Ton19] is a powerful tool in studying the mean curvature flow. A version of this due to White [Whi05] can also be proven in the smooth setting, see also [Eck04].

Another important weak formulation is the *level-set flow* developed independently by Evans and Spruck [ES91], and Chen, Giga and Goto [CGG91].

2.2 Singularities

Before taking a closer look at the study of singularities for $n \geq 2$, let us pause for a moment at the $n = 1$ case. The flow is then commonly called *the curve shortening flow*, and example 2 is simply a shrinking circle. the following theorem by Grayson [Gra87] shows that any embedded closed curve becomes extinct at a "round point", i.e. rescalings of the curve converge smoothly to an embedding $\tilde{X}$ whose image is a unit circle:

Theorem 4 (Grayson's theorem)**.** *Let $X_0 : M \to \mathbb{R}^2$ be a smooth embedding of a closed curve. Then the solution of* (1) *with initial data X_0 exists on* *the maximal interval* $[0, T_{\max})$ *and converges to a round point* $x_0 \in \mathbb{R}^2$ *as* $t \to T_{\max}$.

The theorem was first proven by Gage and Hamilton [GH86] when the initial embedding X_0 is convex. Grayson then improved it by showing that any closed curve eventually becomes convex under the flow.

For $n \geq 2$, Huisken proved in [Hui84] a similar result for compact, convex hypersurfaces in $\mathbb{R}^{n+1}$: the hypersurface remains convex and contracts to a round point in finite time. The general case for $n \geq 2$ is however quite more complicated, and Grayson's theorem is false in higher dimensions. This was demonstrated first by Grayson [Gra89] by constructing a particular dumbbell where a *pinching singularity* arises before the extinction of the entire hypersurface. Later, Angenent [Ang92] used his celebrated doughnut self-shrinking solution to prove the same statement using theorem 2.

Blow-up analysis: rescaling and the monotonicity formula

We have seen how two hypersurfaces diffeomorphic to $\mathbb{S}^n$ can develop different types of singularities under the mean curvature flow. Convex hypersurfaces develop so-called *spherical singularities*, while the first singularity arising in Grayson's dumbbell is an example of a *cylindrical singularity*. *Blow-up analysis* allows for a more systematic study of singularities and is based on two elements: rescaling and a monotonicity formula.

Let Φ be the function defined on $\mathbb{R}^{n+1} \times (-\infty, 0)$ and which is given by

$$\Phi(x,t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}} \tag{2}$$

The function Φ looks like the backward heat kernel on $\mathbb{R}^n$, but it is extended to $\mathbb{R}^{n+1}$. We also define its space-time translations $\Phi_{(x_0,t_0)}$ on $\mathbb{R}^{n+1} \times (-\infty, t_0)$ by the expression

$$\Phi_{(x_0,t_0)}(x,t) = \Phi(x - x_0, t - t_0) = \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4(t_0-t)}}, \tag{3}$$

where $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$. The following *monotonicity formula* due to Huisken [Hui90] is a fundamental tool in blow-up analysis:

Theorem 5 (Monotonicity formula)**.** *Let* $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$*, and assume that* $(M_t)_{t \in I}$ *is a mean curvature flow such that* $\int_{M_t} \Phi_{(x_0,t_0)} < \infty$ *for all*

$t < t_0$. *Then*

$$\frac{d}{dt}\int_{M_t}\Phi_{(x_0,t_0)} = -\int_{M_t}\left|\vec{H} - \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \Phi_{(x_0,t_0)} \tag{4}$$

To study singularities in a more general setting, one often introduces the following terminology on the rate at which singularities form:

Definition 6. *Let $X : M \times [0, T_{\max}) \to \mathbb{R}^{n+1}$ be a mean curvature flow, where $T_{\max} < \infty$. X is said to develop a type-*I *singularity if*

$$\sup_{M\times[0,T_{\max})} (T_{\max} - t)|A|^2 < \infty,$$

*and a type-*II *singularity if*

$$\sup_{M\times[0,T_{\max})} (T_{\max} - t)|A|^2 = \infty.$$

We shall now introduce the idea behind blow-up analysis. Let $(x_j, t_j) \in \mathbb{R}^{n+1} \times \mathbb{R}$ be a sequence of space-time points, and let (λ_j) be a sequence of scales $\lambda_j \to \infty$. Note that equation (1) is invariant under space-time translations and parabolic rescaling. Hence if $(M_t)_{t\in[0,T)}$ solves the mean curvature flow, then the rescaled family

$$M_{t,j} := \lambda_j(M_{\lambda_j^{-2}t+t_j} - x_j) \tag{5}$$

is also a solution to the mean curvature flow. Such a sequence (x_j, t_j) is called a *blow-up sequence*, and any flow arising as a subsequential limit of such a blow-up sequence is called a *limit flow*. A particular important case is when the blow-up sequence is constant, i.e. $(x_j, t_j) = (x_0, t_0)$. In this case, a limit flow is called a *tangent flow*.

The idea behind blow-up analysis is that it provides us with a zooming mechanism as the flow approaches a singularity. Under some conditions, Huisken's monotonicity formula (Theorem 5) can then be used to show that the limit flow is a simpler solution to mean curvature flow: *a self-similar flow* which can be regarded as a singularity model.

Singularity models

First, under the type-I hypothesis, one can prove the following theorem [ACGL20, Theorem 11.26].

Theorem 7. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a compact solution to* (1) *which develops a type-*I *singularity. Let $(x_j, t_j) \in M \times [0, T)$ be a sequence such* *that* $\limsup_{j\to\infty} |A|(x_j, t_j) \to \infty$*, then the sequence of rescaled solutions* $X_j : M \times I_j \to \mathbb{R}^{n+1}$ *given by*

$$X_j(x,t) = \lambda_j \left(X(x, \lambda_j^{-2}t + t_j) - X(x_j, t_j) \right), \quad I_j = [-\lambda_j^2 t_j, 0),$$

and $\lambda_j = (T - t_j)^{-\frac{1}{2}}$*, has a subsequence which converges locally uniformly in the smooth topology to a self-similar shrinking solution* X_∞*.*

We will review self-shrinkers in the next section. One can furthermore prove that all tangent flows on type-I singularities are self-shrinkers. Huisken [Hui90] proved a version of this theorem using a smart choice of rescaling which gives what is usually called the rescaled (or normalized) mean curvature flow $\tilde{X}$. The rough idea is that by the type-I hypothesis, the rescaled flow $\tilde{X}$ will have uniformly bounded second fundamental form. This allows an application of a compactness theorem to subsequentially obtain a limit. Finally the monotonicity formula then dictates that the limit has to be self-similar shrinking solution.

Without the type-I hypothesis, it is still true that tangent flows are self-similarly shrinking solutions, although one needs to take the convergence and the resulting limit in the weak sense of Brakke flow. The following theorem is proven in the work of Ilmanen [Ilm95b], see also [Whi97].

Theorem 8 (Weak tangent flows)**.** *Let* $(M_t)_{t\in[0,T)}$ *be a mean curvature flow on a compact hypersurface. Let* $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ *and let* $(\lambda_j)_{j\in\mathbb{N}}$ *such that* $\lambda_j \to \infty$*. Define the sequence* $(M_{t,j})_{t\in I_j}$

$$M_{t,j} = \lambda_j (M_{\lambda_j^{-2}t + t_0} - x_0), \quad I_j = [-\lambda_j^2 t_0, 0)$$

Then there is a subsequence of $(M_{t,j})$ *which converges to a limit integral self-similarly shrinking Brakke flow* $\{\mu_t\}_{t\in(-\infty,0)}$*.*

Note that the two theorems above say nothing about the uniqueness of the obtained limits, i.e. given different choices of the scale sequence $(\lambda_j)_{j\in\mathbb{N}}$, is the resulting tangent flow the same? This is a fundamental problem in the field and it has been settled in only a few special cases. Of course, by the combined work of Gage-Hamilton-Grayson [GH86], [Gra87] and Huisken [Hui84], the uniqueness of tangent flows is established for the curve shortening flow and convex mean curvature flow in $\mathbb{R}^{n+1}$ for $n \geq 2$, respectively. Furthermore, by the combined work of Schulze [Sch14] and Colding-Minicozzi [CM15], every tangent flow arising from mean curvature flow of an embedded, compact mean convex hypersurface is unique. Furthermore, Schulze and Chodosh [CS21] proved uniqueness of multiplicity-one asymptotically conical tangent flows.

Tangent flows constitute a very important example of limit flows, but there are situations when they do not provide the most accurate description of a singularity formation. A classical example of this is seen in the degenerate rotationally symmetric neckpinches of Angenent and Velázquez [AV97]. Huisken and Sinestrari [HS99] studied type-II singularities in the context of mean convex mean curvature flows. In this setting, they proved the existence of a blow-up sequence such that the singularity is modelled by a convex self-similar translating soliton. A hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is called a *translator* if it is a time-slice of a mean curvature flow that evolves by translation, i.e.

$$H_\Sigma(x) = -\langle e, \vec{n}(x)\rangle, \quad x \in \Sigma, \tag{6}$$

for some $e \in \mathbb{R}^{n+1}$. More precisely, we have the following theorem on type-II singularities, see [HS99] or [ACGL20, Theorem 11.32]:

Theorem 9. *Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a compact mean convex solution to* (1) *which develops a type-*II *singularity at time T. Then there is a blow-up sequence $(x_j, t_j) \in M \times [0,T)$ and a sequence of scales $(\lambda_j)_{j\in\mathbb{N}}$ such that the sequence of rescaled solutions $X_j : M \times I_j \to \mathbb{R}^{n+1}$, where*

$$X_j(x,t) = \lambda_j \left(X(x, \lambda_j^{-2} t + t_j) - X(x_j, t_j) \right), \quad I_j = [-\lambda_j^2 t_j, \lambda_j^2 (T - t_j - j^{-1})),$$

has a subsequence which - up to rigid motion - converges locally uniformly in the smooth topology to $\{\mathbb{R}^m \times \Sigma_t^{n-m}\}_{t\in(-\infty,\infty)}$ for some $m \in \{0, 1, \cdots, n-1\}$, where $\{\Sigma_t^{n-m}\}$ is a locally uniformly convex self-similar translator.

Theorem 7, 8 and 9 tell us that self-shrinkers and translators are models for singularity formation for the mean curvature flow. These are examples of self-similar flows, also called *solitons*. As exhibited by these theorems, to obtain these self-similar flows as limit flows one has to impose some assumptions, e.g. type-I assumption in theorem 7 and mean convexity for the type-II assumption in theorem 9. While it is true that limit flows at a singularity are always *ancient solutions*, i.e. flows that exist for the infinite past, it is unknown whether these will in general be self-similar. Nevertheless, self-shrinkers and translators still cover a large class of singularities and much of the field is devoted to their study.

The entropy

We finish this section with the definition of a very important and useful quantitiy. Let $\Sigma^n \subseteq \mathbb{R}^{n+k}$ be a submanifold, and let $x_0 \in \mathbb{R}^{n+k}$, $t_0 > 0$. The F-functional is defined as the weighted area functional

$$F_{x_0,t_0}(\Sigma) := \frac{1}{(4\pi t_0)^{n/2}} \int_\Sigma e^{\frac{-|x-x_0|^2}{4t_0}} d\mu_\Sigma. \tag{7}$$

The *entropy* functional $\lambda(\Sigma)$ is defined by taking the supremum of F_{x_0,t_0} over $x_0 \in \mathbb{R}^{n+1}$, $t_0 > 0$:

$$\lambda(\Sigma) = \sup_{(x_0,t_0)\in\mathbb{R}^{n+k}\times\mathbb{R}_{>0}} F_{x_0,t_0}(\Sigma). \tag{8}$$

This functional was introduced by Colding and Minicozzi [CM12a] (see also [MM09]) and has since become a powerful tool in the study of singularities. The entropy is invariant under dilations, translations and rotations. Furthermore, by Theorem 5, it follows that the entropy is nonincreasing along a mean curvature flow $(M_t)_{t\in I}$, i.e.

$$\lambda(M_{t_1}) \geq \lambda(M_{t_2})$$

for all $t_1 \leq t_2$ in I. We also note that the definition of the entropy extends to Radon measures. In the next section we will briefly review how the entropy was used in [CM12a] to study generic singularities.

The following theorem relates the entropy to a more geometric quantity which is typically denoted the *area growth* or the *maximal density ratio* (the supremum quantity in equation (9) below), see e.g. [Whi21, Theorem 9.1] for a proof.

Theorem 10. *Let M^m be an m-dimensional submanifold in $\mathbb{R}^n$. There is a constant $c_m > 0$ such that*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\mathscr{H}^m(M\cap B_r(x))}{V_m r^m} \geq \lambda(M) \geq c_m \sup_{x\in\mathbb{R}^n, r>0} \frac{\mathscr{H}^m(M\cap B_r(x))}{V_m r^m}. \tag{9}$$

Where $\mathscr{H}^m$ denotes the m-dimensional Hausdorff measure, and V_m is the volume of the unit m-ball.

Hence, bounded entropy is equivalent to bounded area growth.

2.3 Self-shrinkers

The previous section demonstrates the importance of self-shrinkers in the study of singularities of the mean curvature flow. A hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is called a *self-shrinker* if

$$H_\Sigma(x) = \frac{1}{2}\langle x, \vec{n}(x)\rangle, \quad x \in \Sigma. \tag{10}$$

A self-shrinker Σ is a time-slice of a one-parameter family of a homothetically shrinking solution to (1) given by

$$\Sigma_t = \sqrt{-t}\Sigma, \quad t \in (-\infty, 0).$$

One can study self-shrinkers from a variational point of view: they are critical points of the F-functional $\Sigma \mapsto F_{0,1}(\Sigma)$ as defined in (7). Equivalently, they are minimal hypersurfaces in $\mathbb{R}^{n+1}$ with respect to the conformal metric

$$g_{\text{ß}} = e^{-\frac{\|x\|^2}{2n}}\delta, \tag{11}$$

where δ denotes the usual Euclidean metric. We note that this metric is incomplete, and furthermore the scalar curvature does not have a sign and diverges for $|x| \to \infty$.

Generic self-shrinkers

In the influential work [CM12a], Colding and Minicozzi studied the stability properties of self-shrinkers. Let Σ_s be any normal variation of a self-shrinker Σ given by $f\vec{n}$. Then the second variation of $F_{0,1}$ at a self-shrinker Σ is given by

$$\frac{\partial^2}{\partial s^2}F_{0,1}(\Sigma_s)\bigg|_{s=0} = -(4\pi)^{-\frac{n}{2}}\int_\Sigma fLfe^{-\frac{|x|^2}{4}}d\mu_\Sigma,$$

where

$$L = \Delta + |A|^2 - \frac{1}{2}\langle x, \nabla(\cdot)\rangle + \frac{1}{2},$$

is the corresponding stability (or Jacobi) operator. Recall that as a minimal hypersurface in $(\mathbb{R}^{n+1}, g_{\text{ß}})$, we say that Σ is *L-stable* if for all compactly supported functions u we have

$$-\int_\Sigma (uLu)\, e^{-\frac{|x|^2}{4}} d\mu_\Sigma \geq 0.$$

Due to the the constant $\frac{1}{2}$ term in L, one can prove the following lemma. See e.g. [CM12b] for a proof.

Lemma 11. *There are no L-stable, smooth and complete self-shrinkers without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$.*

As argued in [CM12a], this apparent instability comes from the fact that in the second variation formula above we only take variations in Σ, while x_0, t_0 are fixed. A more natural variation will include all the three variations: let Σ_s be a normal variation, and let x_s, t_s be variations with $x_0 = 0, t_0 = 1$, and letting

$$\partial_s\big|_{s=0}\, \Sigma_s = f\vec{n}, \quad \partial_s\big|_{s=0}\, x_s = y, \quad \partial_s\big|_{s=0}\, t_s = h,$$

then we obtain the following second variation formula at a critical point Σ

$$F'' = -(4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}} \left(fLf + 2fhH - h^2H^2 + f\langle y, \vec{n}\rangle - \frac{\langle y, \vec{n}\rangle^2}{2} \right) d\mu_\Sigma$$

Where F'' denotes $\partial_{ss}|_{s=0} F_{x_s,t_s}(\Sigma_s)$. This gives rise to a new and more suitable notion of stability which is called *F-stability*. The reason for this is that the additional variations x_s and t_s take account of space-time translations, and we will now have F-stable self-shrinkers. Another related notion of stability is the so-called entropy stability. A self-shrinker Σ is called *entropy-stable* if it is a local minimum for the entropy functional (8). It is shown in [CM12a] that the entropy is attained for any self-shrinker Σ with polynomial volume growth:

$$\lambda(\Sigma) = F_{0,1}(\Sigma),$$

and that furthermore F-stability and entropy-stability are equivalent for self-shrinkers that do not split off a line isometrically.

Using the considerations above, Colding and Minicozzi were able to characterize the self-shrinking singularities of *generic* mean curvature flows, i.e. the self-shrinkers that arise as tangent flows and which cannot be perturbed away. Note that by the propreties of the entropy functional, if Σ is a self-shrinker which arises as a tangent flow from a mean curvature flow $(M_t)_{t\in I}$ starting from M_0, then $\lambda(M) \leq \lambda(M_0)$. Let $\mathcal{C}$ denote the set of generalized cylinders $\mathcal{S}^k \times \mathbb{R}^{n-k}$, where $\mathcal{S}^k$ is a sphere of radius $\sqrt{2k}$. One of the main results of [CM12a] is the following theorem:

Theorem 12. *Let Σ^n be a smooth complete embedded self-shrinker with $\partial\Sigma = \emptyset$ and with polynomial volume growth.*

1. *If $\Sigma \notin \mathcal{C}$, then for any m there is a graph $\tilde{\Sigma}$ over Σ of a function with arbitrarily small C^m norm so that $\lambda(\tilde{\Sigma}) < \lambda(\Sigma)$.*
2. *If Σ is not $\mathcal{S}^n$ and does not split off a line, the function in (1) can be taken to have compact support.*

In either case, Σ cannot arise as a tangent flow to the mean curvature flow starting from $\tilde{\Sigma}$.

The theorem above tells us that the only entropy stable self-shrinkers are the set of generalized cylinders $\mathcal{C}$, and hence the set $\mathcal{C}$ represents the set of generic self-shrinkers, i.e. ones that cannot be perturbed away. A key element in the proof is the classification of mean convex self-shrinkers, initiated by Abresch and Langer [AL86] for $n = 1$ and Huisken [Hui90], [Hui93] for $n \geq 2$, and finalized in [CM12a]. Under the same assumptions stated in Theorem 12, mean convex self-shrinkers are given by the generalized cylinders $\mathcal{C}$, see [CM12a, Theorem 0.17].

Huisken famously conjectured (see [Ilm03, #8]) that generic embedded mean curvature flow should only encounter spherical and cylindrical singularities. In a series of papers [CCMS20], [CCMS21], [CCS23] further advancemenets have been made towards this conjecture. In particular it was shown in [CCS23] that for generic surfaces $M \subset \mathbb{R}^3$, the mean curvature flow starting at M encounters only a spherical or cylindrical singularity until the first time it encounters a singularity with multiplicity $N \geq 2$. The higher multiplicity $N \geq 2$ is conjectured not to occur, see [Ilm03].

Examples beyond the generalized cylinders

The generalized cylinders are in a sense the simplest examples of self-shrinkers. By now there are a wealth of examples of self-shrinkers constructed using various techniques, both embedded and immersed. Angenent [Ang92] was the first to construct an example beyond the generalized cylinders. The *Angenent's doughnuts* are embedded, rotationally symmetric, diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and are constructed using shooting method in ODEs. For a long time these were the only known examples until the construction of noncompact, high-genus embedded examples independently by Kapouleas-Kleene-Møller [KKM18] and Nguyen [Ngu14] by gluing techniques. Further examples by gluing techniques are given in [Møl11] and [KM23], and examples using min-max techniques are constructed in [Ket16] and [BNS21]. Drugan and Nguyen [DN18] used a modified curvature flow to construct embedded $\mathbb{S}^{n-1} \times \mathbb{S}^1$ self-shrinkers similar to Angenent's construction, leaving open the question whether the two constructions coincide. More examples constructed using shooting method can be found in [Rie23], [DK17]; see also [DLN18] for a survey on those.

Many of these examples, together with numerical examples such as those by Ilmanen [Ilm95a] and Chopp [Cho94] (see also [AIC95]), indicate that a complete classification of self-shrinkers in dimension $n \geq 2$ is not to be expected, even in the embedded case.

The entropy and some rigidity results

Knowing the numerical value of the entropy for self-shrinkers is often very useful. Recall that for self-shrinkers, the supremum in (8) is attained. The hyperplanes have the least entropy $\lambda(\mathbb{R}^n) = 1$, and by applying Brakke's regularity theorem (see [Whi05]) one can show that there is $\varepsilon = \varepsilon(n) > 0$ such that if Σ is any non-flat self-shrinker in $\mathbb{R}^{n+1}$, then $\lambda(\Sigma) \geq 1 + \varepsilon$, i.e. there is a gap to the next lowest attained value for the entropy by a self-shrinker. By a computation due to Stone [Sto94], the entropy of the n-spheres is a strictly decreasing sequence in n and their numerical values are given by

$$\lambda(\mathbb{S}^n) = \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} \omega_n(\mathbb{S}^n)$$

Where ω_n is the surface area of $\mathbb{S}^n$. Furthermore, by the product property for the entropy

$$\lambda(N_1 \times N_2) = \lambda(N_1)\lambda(N_2),$$

for any submanifods N_1, N_2 in $\mathbb{R}^{n+1}$, the entropy of the family of generalized cylinders $\mathcal{C}$ are also determined.

Using theorem 12, it was shown by Colding, Minicozzi, Ilamanen and White [CIMW13] that the round sphere has the least entropy among all closed self-shrinkers, and that there is a gap to the next lowest attained value by a closed self-shrinker. Furthermore, by the work of [BW16] and [Zhu20], any closed hypersurface in $\mathbb{R}^{n+1}$ has entropy bounded from below by $\lambda(\mathbb{S}^n)$, settling a conjecture in [CIMW13].

Similar to the study of minimal and constant-H hypersurfaces, being able to prove *rigidity* theorems is a gateway to obtaining further insight into the space of self-shrinkers. Even in the case of complete embedded self-shrinkers this proves very challenging, except perhaps for the $n = 1$ case where Abresch and Langer [AL86] have proven that the circle is the only simple closed self-shrinker. One typically needs to impose some assumptions to obtain such results, and bounds on the entropy often prove useful in this regard.

As we have already seen, the mean convex self-shrinkers with polynomial volume growth are generalized cylinders. Colding-Minicozzi proved in [CM12b] a smooth compactness theorem on the space of embedded self-shrinkers $\Sigma^2 \subset \mathbb{R}^3$ with bounded genus and bounded entropy. The assumption on genus was later relaxed to a fixed genus By Sun and Wang [SW20]. Wang [Wan14] proved the uniqueness of embedded self-shrinkers with conical ends, while Brendle [Bre16] proved that the round sphere is the only embedded genus zero self-shrinker in $\mathbb{R}^3$.

Mramor and Wang [MW20] proved that any genus g closed embedded self-shrinker is isotopic to the standard genus g surface in $\mathbb{R}^3$. The result has since been generalized by Mramor [Mra20] to include noncompact self-shrinkers which contain a single asymptotically conical end. Furthermore Mramor [Mra21b], [Mra20] has proven various rigidity and topological results on asymptotically conical self-shrinkers.

In general, if a certain class of solitons (such as translators or self-shrinkers) are known to possess explicit entropy bounds, it may be used to obtain a rigidity or a classification result on that class as demonstrated in some of the results mentioned above. More examples where entropy bounds are utilized include the classification of ancient low entropy flows in $\mathbb{R}^3$ by Choi, Haslhofer and Hershkovits [CHH22] and the classification of low-entropy closed hypersurfaces in $\mathbb{R}^4$ by Bernstein and Wang [BW18]. In the pursue of classification theorems on translating solitons, entropy bounds are often used as assumptions, see e.g. [Chi20], [GMM22], [MOP22], [IMR23], just to mention a few.

Chapter 3

Entropy bounds: an overview

3.1 Rotationally symmetric self-shrinkers

The study of the space of complete, embedded rotationally symmetric self-shrinkers $\Sigma_\sigma^n \subset \mathbb{R}^{n+1}$ with profile curve $\sigma : I \to \mathbb{H}$ was initiated with the construction of the $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ diffeomorphic Angenent solutions [Ang92]. Kleene and Møller [KM14] obtained a partial classification of this space: it is made of the hyperplane, the round sphere, the round cylinder and doughnuts $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. The latter category is still not well understood. In particular it is still unknown whether Angenent's solution is the only member in this category. This is related to the following more general open problem in $n = 2$, see [DLN18, section 7]:

Open problem: Is Angenet's torus the only closed, embedded, genus 1 self-shrinker in $\mathbb{R}^3$?

By the work of Drugan and Kleene [DK17], we note that infinitely many immersed examples exist of each one of the topological types included in the classification theorem of [KM14] mentioned above.

The rotational symmetry reduces the minimality of Σ_σ^n in $(\mathbb{R}^{n+1}, g_{ß})$, where $g_{ß}$ is given in (11), to the minimality of the profile curve σ in $(\mathbb{H}, g_A)$, where $\mathbb{H}$ is the upper half-plane and g_A is derived from $g_{ß}$:

$$g_A = r^{2(n-1)} e^{-\frac{x^2+r^2}{2}} (dx^2 + dr^2), \quad \mathbb{H} = \{(x, r) \in \mathbb{R}^2 : r > 0\}.$$

Even though the 2-manifold $(\mathbb{H}, g_A)$ is incomplete, it does have one good geometric property: the Gaussian curvature K_A is positive and is bounded from below by a constant $\kappa > 0$. Spaces of sectional curvature bounded from below (or above) allow the use of theorems from comparison geometry, although without some notion of completeness not much can be done. Nevertheless, as we will see in the summary below, the following general theorem ([BBI01, Theorem 10.3.1]) plays a vital role in the results obtained in [MMM22] and [MM23].

Theorem 13 (Toponogov's Theorem)**.** *Let X be a complete length space of curvature $\geq \kappa$. Then X has curvature $\geq \kappa$ in the large.*

3.2 Summary of paper I and II

Paper I [MMM22]:

In this paper, we first derive explicit entropy bounds for the space of complete embedded self-shrinkers with rotational symmetry [MMM22, Theorem 1.1]:

Theorem 14. *For each $n \geq 2$, there is a positive number E_n such that*

$$1 \leq \lambda(\Sigma) \leq E_n.$$

for any complete embedded rotationally symmetric self-shrinker $\Sigma^n \subseteq \mathbb{R}^{n+1}$.

It is worthwhile to note that the proof of this theorem does not directly use the self-shrinker equation. It is rather a reflection of the geometric structure of $(g_A, \mathbb{H})$. Theorem 14 is a direct consequence of the following theorem [MMM22, Theorem 3.1].

Theorem 15. *Let (M, g) be a 2-dimensional Riemannian manifold so that M is homeomorphic to $\mathbb{R}^2$ and $K_g \geq \kappa_g > 0$, where K_g is the Gauss curvature of (M, g). Then every simple closed geodesic in (M, g) has length at most $\leq 2\pi/\sqrt{\kappa_g}$.*

The proof of this theorem utilizes Toponogov's theorem (theorem 13 above). Most of the work goes into showing that any simple closed geodesic in M bounds a compact region Ω which is a complete length space of curvature $\geq \kappa$. This relies on two facts: first is the fact that the boundary of Ω is a simple closed geodesic, and hence any two points in Ω can be connected by a geodesic in Ω which is a shortest path among C^1-curves inside Ω. Second is the fact that the curvature bound $K \geq \kappa > 0$ of M implies a similar one on Ω. We then use a clever argument inspired by [Kli95] to finalize the proof.

Equipped with the entropy bounds we are then able to show the following smooth compactness theorem [MMM22, Theorem 1.2]:

Theorem 16. *For each $n \geq 2$, the space of complete embedded rotationally symmetric self-shrinkers $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is compact in the C^∞_{loc}-topology.*

The proof is by contradiction and we sketch the idea here: Let (Σ_{σ_k}) be a sequence in this space. By the classification result of Kleene-Møller [KM14], one may assume that (Σ_{σ_k}) are doughnuts, i.e. of the type $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. We then find a limit hypersurface Σ_∞ for which a subsequence converges to on compact sets. If Σ_∞ is a doughnut then the convergence is smooth and we are done. If not, then once again by [KM14], Σ_∞ must be the plane, the cylinder or the sphere. We show that in this case the convergence above is in multiplicity $N \geq 2$ and by using arguments from [CM12b] this will imply that the limit Σ_∞ is L-stable, which is not possible by lemma 11.

Finally, we show that there are finitely many self-shrinkers in this class if one furthermore assumes a symmetry with respect to the hyperplane perpendicular to the axis of rotation. We note that embeddedness is necessary here, as infinitely many immersed examples with this symmetry have been constructed in [DK17]. The proof is based on the ideas of Mramor [Mra21a]. We note that our compactness and finiteness theorems are an improvement to previously obtained results in [Mra21a].

Paper II [MM23]

In this paper we expand on the entropy bounds obtained in paper I [MMM22]. We generalize the entropy bounds in the rotational symmetric case to include entropy bounds on the set of immersed rotationally symmetric self-shrinkers of the doughnut type $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Examples of these have been constructed in [DK17] using shooting method. We prove the following theorem [MM23, Theorem 1.2]:

Theorem 17. *Let $\Sigma = \Sigma_\sigma$ be a compact immersed non-spherical rotationally symmetric self-shrinker in $\mathbb{R}^{n+1}$, where the profile curve σ has k self-intersection points counted with multiplicity. Then*

$$\lambda(\Sigma) \leq (k+1)E_n.$$

An immersed closed geodesic in $(g_A, \mathbb{H})$ will enclose a number of domains Ω_i. To prove Theorem 17 we first need to generalize Theorem 15 to include piecewise smooth geodesic curves which arise as boundaries of such domains Ω_i. The rest of the proof is to take care of technical details.

Riedler [Rie23] constructed new examples of embedded self-shrinkers using the theory of isoparametric foliations of $\mathbb{S}^n$ which include Angenent's solution [Ang92] in particular. The problem of finding theses self-shrinkers is by a reduction to finding geodesics in a Riemannian surface with similar properties to $(\mathbb{H}, g_A)$. We show the following theorem [MM23, Theorem 1.1] which expands on the family of entropy bounds obtained in Paper I.

Theorem 18. *Let $\mathscr{M} = \{M_\varphi\}_{\varphi \in (0,\pi/g)}$ be an isoparametric foliation of $\mathbb{S}^n$ of type (g, m, m), and let N be a f-invariant embedded closed self-shrinker diffeomorphic to $\mathbb{S}^1 \times M$, where M is diffeomorphic to a regular fiber of the foliation $\mathscr{M}$. Then there is a positive number $E_{g,m}$ such that $\lambda(N) \leq E_{g,m}$.*

Bibliography

[ACGL20] Ben Andrews, Bennett Chow, Christine Guenther, and Mat Langford. *Extrinsic geometric flows*, volume 206 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2020.

[AIC95] Sigurd Angenent, Tom Ilmanen, and David L Chopp. A computed example of nonuniqueness of mean curvature flow in $\mathbb{R}^3$. *Communications in Partial Differential Equations*, 20(11-12):1937–1958, 1995.

[AL86] Uwe Abresch and J. Langer. The normalized curve shortening flow and homothetic solutions. *J. Differ. Geom.*, 23:175–196, 1986.

[Ang92] Sigurd B. Angenent. Shrinking doughnuts. In *Nonlinear diffusion equations and their equilibrium states, 3. Proceedings from the third conference, held August 20-29, 1989 in Gregynog, Wales, United Kingdom*, pages 21–38. Boston, MA etc.: Birkhäuser, 1992.

[AV97] S. B. Angenent and J. J. L. Velázquez. Degenerate neckpinches in mean curvature flow. *J. Reine Angew. Math.*, 482:15–66, 1997.

[BBI01] D. Burago, Yu. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2001.

[BHH19] Reto Buzano, Robert Haslhofer, and Or Hershkovits. The moduli space of two-convex embedded tori. *Int. Math. Res. Not.*, 2019(2):392–406, 2019.

[BHH21] Reto Buzano, Robert Haslhofer, and Or Hershkovits. The moduli space of two-convex embedded spheres. *J. Differ. Geom.*, 118(2):189–221, 2021.

[BNS21] Reto Buzano, Huy The Nguyen, and Mario B Schulz. Noncompact self-shrinkers for mean curvature flow with arbitrary genus. *arXiv preprint arXiv:2110.06027*, 2021.

[Bra78] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Math. Notes (Princeton)*. Princeton University Press, Princeton, NJ, 1978.

[Bre16] Simon Brendle. Embedded self-similar shrinkers of genus 0. *Ann. Math. (2)*, 183(2):715–728, 2016.

[BW16] Jacob Bernstein and Lu Wang. A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. *Invent. Math.*, 206(3):601–627, 2016.

[BW18] Jacob Bernstein and Lu Wang. Topology of closed hypersurfaces of small entropy. *Geom. Topol.*, 22(2):1109–1141, 2018.

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. *arXiv preprint arXiv:2003.14344*, 2020.

[CCMS21] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic low-entropy initial data. *arXiv preprint arXiv:2102.11978*, 2021.

[CCS23] Otis Chodosh, Kyeongsu Choi, and Felix Schulze. Mean curvature flow with generic initial data ii. *arXiv preprint arXiv:2302.08409*, 2023.

[CGG91] Yun-Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differ. Geom.*, 33(3):749–786, 1991.

[CHH22] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits. Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness. *Acta Math.*, 228(2):217–301, 2022.

[Chi20] Francesco Chini. Simply connected translating solitons contained in slabs. *Geometric Flows*, 5(1):102–120, 2020.

[Cho94] David L. Chopp. Computation of self-similar solutions for mean curvature flow. *Exp. Math.*, 3(1):1–15, 1994.

[CIMW13] Tobias Holck Colding, Tom Ilmanen, William P. II. Minicozzi, and Brian White. The round sphere minimizes entropy among closed self-shrinkers. *J. Differ. Geom.*, 95(1):53–69, 2013.

[CM12a] Tobias H. Colding and William P. II Minicozzi. Generic mean curvature flow. I: Generic singularities. *Ann. Math. (2)*, 175(2):755–833, 2012.

[CM12b] Tobias H. Colding and William P. II Minicozzi. Smooth compactness of self-shrinkers. *Comment. Math. Helv.*, 87(2):463–475, 2012.

[CM15] Tobias Holck Colding and William P. II Minicozzi. Uniqueness of blowups and Łojasiewicz inequalities. *Ann. Math. (2)*, 182(1):221–285, 2015.

[CMP15] Tobias Holck Colding, William P. II Minicozzi, and Erik Kjær Pedersen. Mean curvature flow. *Bull. Am. Math. Soc., New Ser.*, 52(2):297–333, 2015.

[CS21] Otis Chodosh and Felix Schulze. Uniqueness of asymptotically conical tangent flows. *Duke Math. J.*, 170(16):3601–3657, 2021.

[DK17] Gregory Drugan and Stephen J. Kleene. Immersed self-shrinkers. *Trans. Am. Math. Soc.*, 369(10):7213–7250, 2017.

[DLN18] Gregory Drugan, Hojoo Lee, and Xuan Hien Nguyen. A survey of closed self-shrinkers with symmetry. *Result. Math.*, 73(1):32, 2018. Id/No 32.

[DN18] Gregory Drugan and Xuan Hien Nguyen. Shrinking doughnuts via variational methods. *J. Geom. Anal.*, 28(4):3725–3746, 2018.

[Eck04] Klaus Ecker. *Regularity theory for mean curvature flow*, volume 57 of *Prog. Nonlinear Differ. Equ. Appl.* Boston, MA: Birkhäuser, 2004.

[ES91] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differ. Geom.*, 33(3):635–681, 1991.

[GH86] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Differ. Geom.*, 23:69–96, 1986.

[GMM22] Eddygledson Souza Gama, Francisco Martín, and Niels Martin Møller. Finite entropy translating solitons in slabs. *arXiv preprint arXiv:2209.01640*, 2022.

[Gra87] Matthew A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differ. Geom.*, 26:285–314, 1987.

[Gra89] Matthew A. Grayson. A short note on the evolution of a surface by its mean curvature. *Duke Math. J.*, 58(3):555–558, 1989.

[Ham82] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differ. Geom.*, 17:255–306, 1982.

[HI01] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differ. Geom.*, 59(3):353–437, 2001.

[HS99] Gerhard Huisken and Carlo Sinestrari. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Math.*, 183(1):45–70, 1999.

[HS09] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow with surgeries of two-convex hypersurfaces. *Invent. Math.*, 175(1):137–221, 2009.

[Hui84] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differ. Geom.*, 20:237–266, 1984.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differ. Geom.*, 31(1):285–299, 1990.

[Hui93] Gerhard Huisken. Local and global behaviour of hypersurfaces moving by mean curvature. In *Differential geometry. Part 1: Partial differential equations on manifolds. Proceedings of a summer research institute, held at the University of California, Los Angeles, CA, USA, July 8-28, 1990*, pages 175–191. Providence, RI: American Mathematical Society, 1993.

[Ilm95a] Tom Ilmanen. Lectures on mean curvature flow and related equations. `https://people.math.ethz.ch/~ilmanen/papers/notes.pdf`, 1995.

[Ilm95b] Tom Ilmanen. Singularities of mean curvature flow of surfaces. `https://people.math.ethz.ch/~ilmanen/papers/sing.ps`, 1995.

[Ilm03] Tom Ilmanen. Problems in mean curvature flow. `https://people.math.ethz.ch/~ilmanen/classes/eil03/problems03.pdf`, 2003.

[IMR23] Debora Impera, Niels Martin Møller, and Michele Rimoldi. Rigidity and non-existence results for collapsed translators. *arXiv preprint arXiv:2304.11953*, 2023.

[Ket16] Daniel Ketover. Self-shrinking platonic solids. *arXiv preprint arXiv:1602.07271*, 2016.

[KKM18] Nikolaos Kapouleas, Stephen James Kleene, and Niels Martin Møller. Mean curvature self-shrinkers of high genus: non-compact examples. *J. Reine Angew. Math.*, 739:1–39, 2018.

[Kli95] Wilhelm P. A. Klingenberg. *Riemannian geometry.*, volume 1 of *De Gruyter Stud. Math.* Berlin: Walter de Gruyter, 2nd ed. edition, 1995.

[KM14] Stephen Kleene and Niels Martin Møller. Self-shrinkers with a rotational symmetry. *Trans. Am. Math. Soc.*, 366(8):3943–3963, 2014.

[KM23] Nikolaos Kapouleas and Peter McGrath. Generalizing the linearized doubling approach. I: General theory and new minimal surfaces and self-shrinkers. *Camb. J. Math.*, 11(2):299–439, 2023.

[Man11] Carlo Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Prog. Math.* Basel: Birkhäuser, 2011.

[MM09] Annibale Magni and Carlo Mantegazza. Some remarks on

Huisken's monotonicity formula for mean curvature flow. In *Singularities in nonlinear evolution phenomena and applications. Proceedings of the workshop, Pisa, Italy, May 2008*, pages 157–169. Pisa: Edizioni della Normale, 2009.

[MM23] John Man Shun Ma and Ali Muhammad. Entropy bounds for self-shrinkers with symmetries. *arXiv preprint arXiv:2306.12171*, 2023.

[MMM22] John Man Shun Ma, Ali Muhammad, and Niels Martin Møller. Entropy bounds, compactness and finiteness theorems for embedded self-shrinkers with rotational symmetry. *J. Reine Angew. Math.*, 793:239–259, 2022.

[Møl11] Niels Martin Møller. Closed self-shrinking surfaces in $\mathbb{R}^3$ via the torus. *arXiv preprint arXiv:1111.7318*, 2011.

[MOP22] John Man Shun Ma, Yuan Shyong Ooi, and Juncheol Pyo. Rigidity results for mean curvature flow graphical translators moving in non-graphical direction. *arXiv preprint arXiv:2210.03707*, 2022.

[Mra20] Alexander Mramor. An unknottedness result for noncompact self shrinkers. *arXiv preprint arXiv:2005.01688*, 2020.

[Mra21a] Alexander Mramor. Compactness and finiteness theorems for rotationally symmetric self shrinkers. *J. Geom. Anal.*, 31(5):5094–5107, 2021.

[Mra21b] Alexander Mramor. On self shrinkers of medium entropy in $\mathbb{R}^4$. *arXiv preprint arXiv:2106.10243, (to appear in Geometry and Topology)*, 2021.

[Mul56] William W Mullins. Two-dimensional motion of idealized grain boundaries. *Journal of Applied Physics*, 27(8):900–904, 1956.

[MW20] Alexander Mramor and Shengwen Wang. On the topological rigidity of compact self-shrinkers in $\mathbb{R}^3$. *Int. Math. Res. Not.*, 2020(7):1933–1941, 2020.

[Ngu14] Xuan Hien Nguyen. Construction of complete embedded self-similar surfaces under mean curvature flow. III. *Duke Math. J.*, 163(11):2023–2056, 2014.

[Rie23] Oskar Riedler. Closed embedded self-shrinkers of mean curvature flow. *J. Geom. Anal.*, 33(6):27, 2023. Id/No 172.

[Sch14] Felix Schulze. Uniqueness of compact tangent flows in mean curvature flow. *J. Reine Angew. Math.*, 690:163–172, 2014.

[Sch20] Felix Schulze. Optimal isoperimetric inequalities for surfaces in any codimension in Cartan-Hadamard manifolds. *Geom. Funct. Anal.*, 30(1):255–288, 2020.

[Sto94] Andrew Stone. A density function and the structure of singularities of the mean curvature flow. *Calc. Var. Partial Differ. Equ.*, 2(4):443–480, 1994.

[SW20] Ao Sun and Zhichao Wang. Compactness of self-shrinkers in $\mathbb{R}^3$ with fixed genus. *Adv. Math.*, 367:39, 2020. Id/No 107110.

[Ton19] Yoshihiro Tonegawa. *Brakke's mean curvature flow. An introduction*. SpringerBriefs Math. Singapore: Springer, 2019.

[Wan14] Lu Wang. Uniqueness of self-similar shrinkers with asymptotically conical ends. *J. Am. Math. Soc.*, 27(3):613–638, 2014.

[Whi97] Brian White. Stratification of minimal surfaces, mean curvature flows, and harmonic maps. *J. Reine Angew. Math.*, 488:1–35, 1997.

[Whi05] Brian White. A local regularity theorem for mean curvature flow. *Ann. Math. (2)*, 161(3):1487–1519, 2005.

[Whi21] Brian White. Mean curvature flow with boundary. *Ars Inven. Anal.*, 2021:34, 2021. Id/No 4.

[Zhu20] Jonathan J. Zhu. On the entropy of closed hypersurfaces and singular self-shrinkers. *J. Differ. Geom.*, 114(3):551–593, 2020.

Chapter 4

Paper I: Entropy Bounds, Compactness and Finiteness Theorems for Embedded Self-shrinkers with Rotational Symmetry

This chapter includes the paper:

[MMM22] John Man Shun Ma, Ali Muhammad, and Niels Martin Møller. Entropy bounds, compactness and finiteness theorems for embedded self-shrinkers with rotational symmetry.

Entropy Bounds, Compactness and Finiteness Theorems for Embedded Self-shrinkers with Rotational Symmetry

John Man Shun Ma, Ali Muhammad, Niels Martin Møller

Abstract

In this work, we study the space of complete embedded rotationally symmetric self-shrinking hypersurfaces in $\mathbb{R}^{n+1}$. First, using comparison geometry in the context of metric geometry, we derive explicit upper bounds for the entropy of all such self-shrinkers. Second, as an application we prove a smooth compactness theorem on the space of all such shrinkers. We also prove that there are only finitely many such self-shrinkers with an extra reflection symmetry.

1 Introduction

An n-dimensional smooth hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is a *self-shrinker* if

$$H_\Sigma(x) = \frac{1}{2}\langle x, \vec{n}\rangle, \quad \text{for all } x \in \Sigma.$$

Here H_Σ is the mean curvature of Σ with respect to the outward unit normal $\vec{n}$. Given a self-shrinker, one obtains by scaling a one parameter family of hypersurfaces

$$\Sigma_t = \sqrt{-t}\Sigma, \quad t \in (-\infty, 0)$$

which solves the mean curvature flow (MCF) equation,

$$\left(\frac{\partial \Sigma_t}{\partial t}\right)^\perp = \vec{H}_{\Sigma_t}, \tag{1.1}$$

The authors were partially supported by DFF Sapere Aude 7027-00110B, by CPH-GEOTOP-DNRF151 and by CF21-0680 from respectively the Independent Research Fund Denmark, the Danish National Research Foundation and the Carlsberg Foundation.

where $\vec{H}_\Sigma = -H_\Sigma \vec{n}$ denotes the mean curvature vector of Σ.

Most importantly, self-shrinkers serve as singularity models for the MCF: under the Type I condition on the singularity, Huisken [Hui90] showed that a rescaling of a MCF around a singularity converges locally smoothly subsequentially to a self-shrinker, and proved that closed shrinkers with positive mean curvature are round spheres. Later Ilmanen [Ilm94] proved the subsequential weak convergence of the tangent flow of any MCF to a self-shrinking solution.

For $n = 1$, all compact immersed self-shrinkers in $\mathbb{R}^2$ were found in [AL86], the circle being the only embedded example. For $n = 2$, Brendle [Bre16] proved the long-standing conjecture that the round sphere of radius 2 is the only closed embedded genus zero self-shrinker in $\mathbb{R}^3$. For higher genus, embedded examples are constructed in [Ang92], [DN18], [KKM18], [Ngu14], [Ket16], [Møl11], [BNS21], [KM23]. In general, the space of embedded self-shrinkers is not well-understood, even in the case of e.g. topological 2-tori in $\mathbb{R}^3$.

In this paper we direct our attention to the class of complete embedded self-shrinking hypersurfaces in $\mathbb{R}^{n+1}$ with a rotational symmetry, for $n \geq 2$. Using a shooting method in ODEs, Angenent constructed in [Ang92] the first nontrivial self-shrinkers in $\mathbb{R}^{n+1}$ besides the round sphere, the generalized cylinders and the plane. These self-shrinkers constructed in [Ang92] are rotationally symmetric, embedded, diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and are commonly called *Angenent doughnuts*. They were used in a parabolic maximum principle argument in [Ang92] to prove that, when $n \geq 2$, mean curvature flows may develop thin neck-pinch singularities.

In [KM14], Kleene and the third named author proved a partial classification of all complete embedded rotationally symmetric self-shrinkers in any dimension (see also [Son14]). Mramor proved in [Mra21] several compactness and finiteness results on the space of all such shrinkers. It is conjectured that, at least in dimension 2, the Angenent doughnut (which when $n = 2$ is topologically a torus) is unique and gives the only embedded self-shrinking torus in $\mathbb{R}^3$. This conjecture is still open, even in the rotationally symmetric case [DLN18].

The goal of this paper is to study various properties of the space of all complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$.

For any hypersurface Σ in $\mathbb{R}^{n+1}$, let $\lambda(\Sigma)$ be the entropy of Σ defined in [CM12a], [MM09] (see also Section 2 for the definition).

Theorem 1.1. *For each $n \geq 2$, there is a positive number E_n such that*

$$1 \leq \lambda(\Sigma) \leq E_n.$$

for any complete embedded rotationally symmetric self-shrinker $\Sigma^n \subseteq \mathbb{R}^{n+1}$.

The constants E_n we obtain in Theorem 1.1 are explicit (see (3.1)). For example, when $n = 2$ we have $E_2 \sim 2.24759$, while the 2-dimensional Angenent torus constructed in [Ang92] has entropy around 1.85122, as computed numerically in [BK21b],[GN21]. We remark that for $n = 1$, there is no upper entropy bound for the family of Abresch-Langer immersed self-shrinking curves [AL86]. We also remark that if we exclude the stationary plane, the lower bounds can be improved to $\lambda(\mathbb{S}^n)$ ([CIMW13], [BW16]); if we consider only self shrinking doughnuts, the lower bound can be improved to $\lambda(\mathbb{S}^1) = \sqrt{2\pi/e} \sim 1.52035$, since they all have non-trivial fundamental groups [HW19].

The proof of Theorem 1.1 will make essential use of the fact that Angenent's Riemannian metrics have Gaussian curvatures bounded below by strictly positive constants (see Section 2 for the definition of the Angenent metrics).

As an application, our next theorem gives a smooth compactness result for the space of embedded rotationally symmetric self-shrinkers.

Theorem 1.2. *For each $n \geq 2$, the space of complete embedded rotationally symmetric self-shrinkers $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is compact in the C^∞_{loc}-topology.*

Under some extra assumptions on the bounds on entropy and genus, there are already several smooth compactness results for self-shrinkers. Colding and Minicozzi proved in [CM12b] the smooth compactness of the set of all complete embedded self-shrinkers $\Sigma^2 \subseteq \mathbb{R}^3$ with bounded genus and Euclidean volume growth (see also [DX13, Theorem 1.4]). Later Sun and Wang proved in [SW20] a similar compactness theorem for embedded self-shrinkers in $\mathbb{R}^3$ with fixed genus and uniformly bounded entropy. In particular, the $n = 2$ case of Theorem 1.2 follows from the main theorem in [SW20] and Theorem 1.1. We remark that there are more compactness results for two dimensional self-shrinkers in general, even in higher codimension [CM18]. In the rotationally symmetric situation, Mramor [Mra21] proved several compactness results on the space of compact embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$ with various assumptions on n, bounds on entropy and convexity of the profile curves. Theorem 1.2 is a natural generalization of the results therein.

Theorem 1.2 has several consequences, which include an index upper bound (Corollary 4.7), finiteness of the set of possible entropy values (Corollary 4.8) and the following finiteness theorem.

Theorem 1.3. *For each $n \geq 2$, up to rigid motions there are only finitely many complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$ which are symmetric with respect to the hyperplane perpendicular to the axis of rotation.*

We remark that embeddedness is necessary: there are infinitely many immersed rotationally symmetric self-shrinkers constructed in [DK17] with this extra reflection symmetry.

In Section 2, we recall the basic definitions and results needed. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 and 1.3.

2 Background

2.1 Entropy and Self-shrinkers

We follow the notations in [CM12a]. Let $\Sigma \subset \mathbb{R}^{n+1}$ be an n-dimensional properly embedded hypersurface. For each $x_0 \in \mathbb{R}^{n+1}$, $t_0 > 0$, define the F-functional

$$F_{x_0,t_0}(\Sigma) := \frac{1}{(4\pi t_0)^{n/2}} \int_{\Sigma} e^{\frac{-|x-x_0|^2}{4t_0}} d\mu,$$

where $d\mu$ is the volume form of Σ. The entropy of Σ (see [MM09]) is defined as

$$\lambda(\Sigma) = \sup_{x_0,t_0} F_{x_0,t_0}(\Sigma).$$

Using Huisken's monotonicity formula [Hui90], it was proved in [CM12a] that if $\{\Sigma_t\}_{t\in I}$ satisfies the MCF equation (1.1), then the entropy $t \mapsto \lambda(\Sigma_t)$ is non-increasing, and is constant if and only if Σ_t is self-shrinking. We recall the following lemma proved in [CM12a, Section 7.2].

Proposition 2.1. *Let Σ be a properly embedded self-shrinker. Then $\lambda(\Sigma) = F_{0,1}(\Sigma)$.*

A hypersurface Σ is a self-shrinker if and only if Σ is critical with respect to the functional $\Sigma \mapsto F_{0,1}(\Sigma)$ [CM12a, Proposition 3.6]. The second variation of $F_{0,1}$ at a self-shrinker is calculated in [CM12a, Section 4]: for any normal variation Σ_s of Σ given by $f\vec{n}$, we have

$$\frac{\partial^2}{\partial s^2} F_{0,1}(\Sigma_s)\bigg|_{s=0} = -\int_\Sigma f L f e^{-\frac{|x|^2}{4}} d\mu_\Sigma,$$

where

$$L = \Delta + |A|^2 - \frac{1}{2}\langle x, \nabla(\cdot)\rangle + \frac{1}{2} \tag{2.1}$$

is the stability operator on Σ. It is also shown that all self-shrinkers in $\mathbb{R}^{n+1}$ are L-unstable ([CM12a], see also [CM12b, Theorem 0.5]).

2.2 Rotationally symmetric self-shrinkers; Angenent doughnuts

Let l be any line in $\mathbb{R}^{n+1}$ passing through the origin. A hypersurface Σ of $\mathbb{R}^{n+1}$ is *rotationally symmetric* with respect to l if $R\Sigma = \Sigma$ for all rotations $R \in SO(n+1)$ fixing l. Assume $n \geq 2$ and let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ be the $(n-1)$-dimensional unit sphere. We denote the upper half-plane by

$$\mathbb{H} = \{(x, r) \in \mathbb{R}^2 : r > 0\}.$$

Definition 2.2. *Let I be any interval in $\mathbb{R}$, or $I = \mathbb{S}^1$. Let $\sigma : I \to \mathbb{H}$ be a smooth embedding. Then the embedded hypersurface Σ_σ in $\mathbb{R}^{n+1}$ with profile curve $\sigma(s) = (x(s), r(s))$ is given by*

$$\Sigma_\sigma = \{(x(s), \omega r(s)) : \omega \in \mathbb{S}^{n-1}, s \in I\}.$$

The hypersurface Σ_σ is rotationally symmetric with respect to $\ell = \mathbb{R}e_1$.

Proposition 2.3. *Given a profile curve σ. Then Σ_σ is a self-shrinker if and only if σ is a geodesic in $(\mathbb{H}, g_A)$, where g_A is the incomplete* **Angenent metric** *given by*

$$g_A = r^{2(n-1)} e^{-\frac{x^2+r^2}{2}} (dx^2 + dr^2). \tag{2.2}$$

Direct calculations give

$$F_{0,1}(\Sigma_\sigma) = (4\pi)^{-n/2} \omega_{n-1} L_A(\sigma),$$

where ω_{n-1} is the surface area of $\mathbb{S}^{n-1}$ and $L_A(\sigma)$ is the length of σ in

$(\mathbb{H}, g_A)$. Hence if Σ_σ is a self-shrinker,

$$\lambda(\Sigma_\sigma) = (4\pi)^{-n/2}\omega_{n-1}L_A(\sigma) \tag{2.3}$$

by Proposition 2.1.

Using a shooting method in ODEs, Angenent constructed in [Ang92] a compact embedded rotationally symmetric self-shrinker in $\mathbb{R}^{n+1}$ for each $n \geq 2$. The profile curve of the examples in [Ang92] are convex and symmetric with respect to the r-axis. The self-shrinkers so constructed are called *Angenent doughnuts*. More recently, Drugan and Nguyen in [DN18] used a geometric flow to construct compact embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$ with the same property of the profile curve for all $n \geq 2$. It is not known if both constructions in [Ang92] and [DN18] resulted in the same self-shrinkers. On the other hand, Kleene and the third named author proved the following partial classification, which we will be making use of in the present paper:

Theorem 2.4. *[KM14, Theorem 2] Let Σ be a complete embedded rotationally symmetric self-shrinker in $\mathbb{R}^{n+1}$. Then up to rigid motion, Σ is either*

(i) the hyperplane $\mathcal{P} = \{0\} \times \mathbb{R}^n$,

(ii) the round sphere $\mathcal{S}$ of radius $\sqrt{2n}$,

(iii) the round cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{n-1}$ with radius $\sqrt{2(n-1)}$, or

(iv) diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

Note that in the last case of Theorem 2.4, $\Sigma = \Sigma_\sigma$ for some simple closed geodesic σ in $(\mathbb{H}, g_A)$. This theorem is essential to the proof of Theorem 1.2. As a first simple but useful consequence, note that if a complete rotationally symmetric self-shrinker is embedded, it is automatically properly embedded.

3 Entropy bound for embedded rotationally symmetric self-shrinkers

In this section we prove Theorem 1.1. By a direct calculation using (2.2), the Gauss curvature of the Angenent metric g_A is

$$K = \frac{r^2 + (n-1)}{r^{2n}} e^{\frac{x^2+r^2}{2}}.$$

Note that K is strictly positive. For each n, let κ_n be the (positive) minimum of K. By simple calculus, one can find

$$\kappa_n = \frac{y_n + (n-1)}{y_n^n} e^{\frac{y_n}{2}}, \quad \text{where } y_n = \frac{n-1+\sqrt{9(n-1)^2+8(n-1)}}{2}.$$

We will prove the following more general result.

Theorem 3.1. *Let (M,g) be a 2-dimensional Riemannian manifold so that M is homeomorphic to $\mathbb{R}^2$ and $K_g \geq \kappa_g > 0$, where K_g is the Gauss curvature of (M,g). Then every simple closed geodesic in (M,g) has length at most $\leq 2\pi/\sqrt{\kappa_g}$.*

Theorem 1.1 follows directly from Theorem 3.1 and (2.3). Indeed, the constants E_n in Theorem 1.1 are given by

$$E_n = \frac{2\pi\omega_{n-1}}{(4\pi)^{n/2}\sqrt{\kappa_n}}. \tag{3.1}$$

In the Appendix we show that $2 < E_n \leq E_2$ for all $n \geq 2$ and

$$\lim_{n\to\infty} E_n = \sqrt{\frac{4\pi}{3}} \sim 2.04665.$$

When M is homeomorphic to $\mathbb{S}^2$, Theorem 3.1 is a classical theorem in comparison geometry, where a proof can be found in [Kli95, Theorem 3.4.10]; in our situation, (M,g) is non-compact and incomplete, and we are not able to find an exact reference in this generality. As a result, we provide a proof of Theorem 3.1 using the globalization theorem in metric geometry [BBI01].

Let σ be a simple closed geodesic in (M,g). Since M is homeomorphic to $\mathbb{R}^2$, by the Jordan curve theorem σ divides M into two connected components, where exactly one of them has compact closure.

Definition 3.2. *Given a simple closed geodesic σ in (M,g), let Ω be the compact domain in (M,g) with $\partial\Omega = \operatorname{Im}(\sigma)$.*

Lemma 3.3. *Let $p,q \in \Omega$. Then there is a simple geodesic γ in Ω joining p,q, which is shortest among all piecewise C^1 curves in Ω joining p and q. Moreover,*

(i) if one of p,q is in the interior of Ω, γ also lies in the interior of Ω (except possibly at the other end point),

(ii) if both p, q are in $\mathrm{Im}(\sigma) = \partial\Omega$, then either γ lies completely in $\partial\Omega$, or the interior of γ lies inside the interior of Ω.

Proof. Let $p, q \in \Omega$. Since the case $p = q$ is trivial, we assume $p \neq q$. Let

$$d^{\Omega}(p,q) = \inf_{\gamma} L(\gamma),$$

where $L(\gamma) = \int \sqrt{g(\dot\gamma, \dot\gamma)}$ is the length of γ and the infimum is taken among all piecewise C^1 curves $\gamma : [0,1] \to \Omega$ so that $\gamma(0) = p$ and $\gamma(1) = q$. It is easy to check that d^{Ω} is a metric on Ω. In particular, $d^{\Omega}(p,q) > 0$.

Let $\gamma_j : [0,1] \to \Omega$ be a sequence of piecewise C^1 curves, parametrized proportional to arc length, joining p, q so that $L(\gamma_j) \to d^{\Omega}(p,q)$ as $j \to \infty$. Since Ω is compact, by passing to a subsequence if necessary, we may assume that (γ_j) converges uniformly in d^{Ω} to a Lipschitz continuous curve $\gamma : [0,1] \to \Omega$ as $j \to \infty$. If $\gamma(t_0) = q$ for some $t_0 < 1$, we replace γ by $\gamma|_{[0,t_0]}$. Hence we can assume $\gamma^{-1}(\{q\}) = \{1\}$.

First we prove (i). Assume that $p \in \Omega \setminus \partial\Omega$. We claim that

$$\gamma(t) \notin \partial\Omega, \quad \text{for all } t < 1. \tag{3.2}$$

We argue by contradiction: if not, let t_0 be the infimum of the set $\gamma^{-1}(\partial\Omega \setminus \{q\})$. Since $p \notin \partial\Omega$, we have $t_0 > 0$ and $\gamma([0,t_0))$ lies in the interior of Ω. This together with the definition of γ implies that $\gamma|_{[0,t_0]}$ is a geodesic. Let U be a small geodesically convex neighborhood in (M,g) centered at $\gamma(t_0)$ not containing q. Then there is $\epsilon > 0$ such that $\gamma(t)$ lies in U for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$ and $\gamma(t_0 \pm \epsilon) \neq \gamma(t_0)$. Let σ_0 be the shortest geodesic in U connecting $\gamma(t_0 \pm \epsilon)$. Since $\partial\Omega \cap U$ is a portion of the geodesic σ and U is geodesically convex, σ_0 does not intersect with $\partial\Omega \cap U$ at more than one points. Hence the image of σ_0 except possibly at $\gamma(t_0 + \epsilon)$ must lie in the connected component of $U \setminus \partial\Omega$ containing $\gamma(t_0 - \epsilon)$. As a result, the image of σ_0 also lies in $\Omega \cap U$. Since γ is length minimizing, up to reparametrization we have $\gamma|_{[t_0-\epsilon,t_0+\epsilon]} = \sigma_0$ and thus $\gamma|_{[t_0-\epsilon,t_0+\epsilon]}$ is a smooth geodesic. Since γ, σ are tangential at $\gamma(t_0)$ (recall that $\partial\Omega = \mathrm{Im}\sigma$) and both are geodesics, we have $\gamma = \sigma$ locally around $\gamma(t_0)$. This contradicts the choice of t_0 and thus (3.2) is shown. This immediately implies (i).

Next we prove (ii). Assume that $p, q \in \partial\Omega$. Let (p_k) be a sequence of points in the interior of Ω converging to p. For each $k \in \mathbb{N}$, let $\gamma_k : [0,1] \to \Omega$ be a shortest geodesic in Ω joining p_k to q constructed in (i). By the smooth dependence of solutions to the geodesic equation and picking a subsequence if necessary, (γ_k) converges smoothly to a geodesic $\gamma : [0,1] \to \Omega$ joining

p, q. Using the triangle inequality

$$L(\gamma_k) = d^\Omega(p_k, q) \le d^\Omega(p_k, p) + d^\Omega(p, q)$$

and taking $k \to \infty$, we have $L(\gamma) \le d^\Omega(p, q)$. Thus γ is a length minimizing geodesic in Ω. Then either γ lies completely inside the interior of Ω away from the endpoints, or γ touches $\partial\Omega$ at some point in $\partial\Omega \setminus \{p, q\}$, which implies that γ lies completely in $\partial\Omega$ since both of them are geodesics. This finishes the proof of (ii). □

We will need some definitions and notations from metric geometry. We use the reference [BBI01]. For the convenience of the reader, we summarize the basic facts that we shall need to prove Theorem 3.1.

A *length space* (X, d) is a metric space such that the metric d can be obtained as a distance function associated to a length structure (see [BBI01] for a definition). The metric d is called an *intrinsic metric* in this case. If every pair of points p, q in X can be joined by a (possibly non-unique) shortest path, then the metric d is called *strictly intrinsic*. We recall that a shortest path is a curve γ where the length $L(\gamma)$ is given by the distance between the endpoints of γ. A length space whose metric is strictly intrinsic is called a *complete length space*.

From Lemma 3.3 we conclude

Proposition 3.4. *(Ω, d^Ω) is a complete length space.*

A *triangle* Δpqr in (X, d) is a set of points $\{p, q, r\}$ together with three shortest paths $[pq], [qr], [rp]$. The length of a triangle is the sum of the lengths of its sides. For each $\kappa > 0$ and for each triangle Δpqr in X with length $< 2\pi/\sqrt{\kappa}$, we can associate a unique (up to an isometry) comparison geodesic triangle $\Delta\bar{p}\bar{q}\bar{r}$ in $\mathbb{S}^2_{1/\sqrt{\kappa}}$ with vertices $\{\bar{p}, \bar{q}, \bar{r}\}$ such that the corresponding sides of the geodesic triangle $\Delta\bar{p}\bar{q}\bar{r}$ have the same lengths as the sides of the triangle Δpqr. Here $\mathbb{S}^2_{1/\sqrt{\kappa}}$ is the 2-sphere with radius $1/\sqrt{\kappa}$. For a triangle Δpqr in X, we denote the angles by $\angle p, \angle q, \angle r$, and if confusion arises we will e.g write $\angle pqr$ for the angle $\angle q$. We shall not need the definition of an angle between two shortest paths in a length space, but one can show that on a Riemannian manifold M, if c_1 and c_2 are two geodesics starting at $p = c_1(0) = c_2(0)$, then the angle $\angle p \in [0, \pi]$ between the shortest paths c_1 and c_2 is equal to the usual Riemannian angle between c_1 and c_2. See Corollary 1A.7 in [BH99] for a proof. For the comparison triangle $\Delta\bar{p}\bar{q}\bar{r}$ in $\mathbb{S}^2_{1/\sqrt{\kappa}}$ we denote the angles by $\angle\bar{p}, \angle\bar{q}, \angle\bar{r}$.

There are several definitions of a *space of curvature* $\ge \kappa$. We shall use the following *angle comparison* definition.

Definition 3.5. *Let X be a complete length space, and let $\kappa > 0$. We say that X is a space of curvature $\geq \kappa$ if for any point $x \in X$ there is a neighborhood U_x of x such that for all triangles $\Delta pqr \subset U_x$ the corresponding angles satisfy the inequalities*

$$\angle p \geq \angle \bar{p}, \quad \angle q \geq \angle \bar{q}, \quad \angle r \geq \angle \bar{r}, \tag{3.3}$$

for a comparison triangle $\Delta \bar{p}\bar{q}\bar{r}$ in $\mathbb{S}^2_{1/\sqrt{\kappa}}$. Furthermore, for any two shortest paths $[ab]$ and $[cd]$ in X where c is an interior point on $[ab]$, the following holds.

$$\angle acd + \angle bcd = \pi$$

Remark 3.1. The last statement in Definition 3.5 above can be summarized as "the sum of adjacent angles equals π". It is needed to prove the equivalence to other definitions of spaces of curvature $\geq \kappa$. See [BBI01, Section 4.3].

Proposition 3.6. *The complete length space (Ω, d^Ω) has curvature $\geq \kappa_g$, where κ_g is the lower bound of the Gaussian curvature of (M, g) in Theorem 3.1.*

Proof. Let $x \in \Omega$ and let V_x be a geodesically convex neighborhood in (M, g) centered at x. When x is in the interior of Ω, we choose $V_x \subset \Omega$. Let $U_x = \Omega \cap V_x$. For any $p, q \in U_x$, let γ be the unique shortest geodesic in V_x joining p and q. Note that γ must lie inside U_x: this holds when x is in the interior of Ω, since $U_x = V_x$. When $x \in \partial\Omega$, if γ is not in U_x, there are $t_1 < t_2$ such that $\gamma|_{(t_1,t_2)}$ lies in $V_x \setminus U_x$ and $\gamma(t_1), \gamma(t_2)$ are in $\partial\Omega \cap V_x$. Since $\partial\Omega \cap V_x$ is also a geodesic passing through $\gamma(t_1), \gamma(t_2)$, this contradicts the fact that V_x is a geodesically convex neighborhood.

In particular, triangles of (Ω, d^Ω) in U_x are also triangles of (M, g) in V_x. Together with the assumption that (M, g) has Gaussian curvature $K \geq \kappa_g > 0$, we can argue as in the case for Riemannian manifolds without boundary to show that (Ω, d^Ω) has curvature $\geq \kappa_g$; see Theorem 6.5.6. in [BBI01]. □

We will denote the inequalities in (3.3) by *the angle comparison condition.* If X is a complete length space, then Toponogov's globalization theorem [BBI01, Theorem 10.3.1] globalizes this local curvature condition to the entirety of X, not only in a neighborhood U_x of each point x:

Theorem 3.7 (Globalization Theorem)**.** *Let X be a complete length space of curvature $\geq \kappa$ for some $\kappa > 0$. Then the angle comparison condition*

(3.3) *is satisfied for any triangle* Δpqr *in* X *for which there is a unique (up to isometry) comparison triangle* $\Delta\bar{p}\bar{q}\bar{r}$ *in* $\mathbb{S}^2_{1/\sqrt{\kappa}}$.

Let us remind that when $\kappa > 0$, a comparison triangle only exists if the length of Δpqr does not exceed $2\pi/\sqrt{\kappa}$. As mentioned earlier, we can associate a unique comparison triangle when the length of Δpqr is strictly less than $2\pi/\sqrt{\kappa}$. If the length is equal to $2\pi/\sqrt{\kappa}$, then we have two situations:

- All the sides have length strictly less than $\pi/\sqrt{\kappa}$. Then a comparison triangle is a unique great circle, i.e. all of its angles are equal to π.
- One of the sides has length equal to $\pi/\sqrt{\kappa}$, say $[pq]$. The sum of lengths of the two other sides $[qr]$ and $[rp]$ is then equal to $\pi/\sqrt{\kappa}$. In this case there does not exist a unique comparison triangle, but we can fix the comparison triangle $\Delta\bar{p}\bar{q}\bar{r}$ where the side $[\bar{p}\bar{q}]$ passes through the point $\bar{r}$.

Thus, with this convention, the conclusion of Theorem 3.7 holds for every triangle Δpqr in X with length $\leq 2\pi/\sqrt{\kappa}$. In fact, as a corollary of theorem 3.7 one can show that any triangle in X has length no greater than $2\pi/\sqrt{\kappa}$, where X is a space of curvature $\geq \kappa$ for some $\kappa > 0$. See [BBI01, Corollary 10.4.2.]. We shall use this fact in the proof of Theorem 3.1 below.

Before proving Theorem 3.1, we recall the following elementary lemma in spherical geometry. We recall that a *convex polygon* in the sphere is a polygon so that the interior angle at each vertex is less than or equal to π.

Lemma 3.8. *Let* P *be a convex* n*-gon in* $\mathbb{S}^2_{1/\sqrt{\kappa}}$*, and denote the length of* P *by* $|P|$*. Then* $|P| \leq 2\pi/\sqrt{\kappa}$.

Proof of Theorem 3.1. The idea of this proof is similar to the proof presented in [Kli95, Theorem 3.4.10]. Let Ω be the compact domain bounded by the geodesic $\sigma : [0,1] \to M$ as defined in Definition 3.2, so that $\sigma(0) = \sigma(1)$. By Proposition 3.4 and Proposition 3.6, (Ω, d^{Ω}) is a complete length space with curvature $\geq \kappa_g$.

Let $L_0 := L(\sigma)$ be the length of σ. Take numbers $t_1, t_2, t_3, t_4 \in I$, where $t_1 = 0, t_4 = 1$, such that each of the subarcs $\sigma|_{[t_1,t_2]}, \sigma|_{[t_2,t_3]}$ and $\sigma|_{[t_3,t_4]}$ has length $L_0/3$. Let $\Delta\sigma(t_1)\sigma(t_2)\sigma(t_3)$ be a triangle in (Ω, d^{Ω}). By Proposition 3.4 each of the sides of this triangle either lies completely in $\partial\Omega$ or has interior contained in the interior of Ω. Using [BBI01, Corollary 10.4.2.] we deduce that the length of this triangle is not greater than $2\pi/\sqrt{\kappa_g}$. If the sides of the triangle $\Delta\sigma(t_1)\sigma(t_2)\sigma(t_3)$ coincide with the images of the subarcs $\sigma|_{[t_1,t_2]}, \sigma|_{[t_2,t_3]}$ and $\sigma|_{[t_3,t_4]}$, then we are done.

If not, then build new triangles

- $\Delta\sigma(t_1)\sigma(t_{1,2})\sigma(t_2)$,
- $\Delta\sigma(t_2)\sigma(t_{2,3})\sigma(t_3)$ and
- $\Delta\sigma(t_3)\sigma(t_{3,1})\sigma(t_1)$,

where $t_{i,k} \in (t_i, t_k)$ are chosen such that $\sigma(t_{i,k})$ define midpoints of the corresponding subarcs. To each one of these triangles, we associate a unique comparison triangle in $\mathbb{S}^2_{1/\sqrt{\kappa_g}}$. We put the triangles together along their common sides and obtain a comparison 6-gon $\mathcal{O}_6$ in $\mathbb{S}^2_{1/\sqrt{\kappa_g}}$. From Theorem 3.7 we know that the angles in the comparison triangles are not greater than the corresponding angles in Ω. This implies that the angles of the vertices in $\mathcal{O}_6$ are not bigger than π. Hence $\mathcal{O}_6$ is convex. By Lemma 3.8, the length of $\mathcal{O}_6$ satisfies

$$|\mathcal{O}_6| \leq \frac{2\pi}{\sqrt{\kappa_g}}$$

If the sides of the constructed 6-gon in Ω coincide with the arcs of σ, then we are done.

If not, then we continue to construct more triangles as above, and build the corresponding comparison n-gons $\mathcal{O}_n$ for increasingly large $n \in \mathbb{N}$. By Lemma 3.8 again,

$$|\mathcal{O}_n| \leq \frac{2\pi}{\sqrt{\kappa_g}}$$

For n large enough, the arcs of σ will be the unique shortest paths between the vertices of the constructed n-gon on Ω. This follows from the fact that shortest paths in (Ω, d^{Ω}) are geodesics in (M, g), and that each side of the n-gon will be contained in a geodesically convex neighborhood in M. In such a neighborhood, every two points are connected by a unique shortest path. This implies the desired bound since for n large enough we have

$$L(\sigma) = |\mathcal{O}_n| \leq \frac{2\pi}{\sqrt{\kappa_g}}.$$

□

4 Compactness and Finiteness of Embedded Self-Shrinkers With Rotational Symmetry

In this section we prove Theorem 1.2, 1.3 and some other related results. We start with the following well known lemma.

Lemma 4.1. *Let Σ_1, Σ_2 be two properly embedded self-shrinkers in $\mathbb{R}^{n+1}$ such that one of them is compact. Then Σ_1 and Σ_2 must intersect.*

When both self-shrinkers are compact, this is proved in [WW09, Theorem 7.4]. Recently, it was proved in [IPR21] that any two properly embedded self-shrinkers that are sufficiently separated at infinity must intersect. See also [CCMS20, Corollary C.4] for the statement for F-stationary varifolds.

Next we restrict attention to complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$. By Theorem 2.4, there are only two types of non-compact examples - cylinders and hyperplanes through the origin, and all such pairs also intersect. Thus we have

Lemma 4.2. *Any two complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$ intersect.*

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let (Σ_k) be a sequence of complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$ each with axis of rotation ℓ_k. After taking a subsequence, a limit axis exists, to which each Σ_k can be rotated, thus it is enough to consider the case where all axes of rotation are identical. By a further rotation, we assume the limit axis is $\ell = \mathbb{R}e_1 \subseteq \mathbb{R}^{n+1}$.

By Theorem 2.4, it suffices to assume that Σ_k is a self-shrinking doughnut for each k. Thus $\Sigma_k = \Sigma_{\sigma_k}$, where

$$\sigma_k : [-d_k, d_k] \to \mathbb{H}$$

is a unit speed geodesic in $(\mathbb{H}, g_A)$ so that $L(\sigma_k) = 2d_k$ and $\sigma_k(-d_k) = \sigma_k(d_k)$. Since all self-shrinkers have entropy larger than or equal to $\lambda(\mathbb{R}^n) = 1$, we have $d_k \geq d > 0$ for all k by (2.3) for some dimensional constant d.

Let σ_a be the profile curve of the Angenent doughnut constructed in [Ang92]. By Lemma 4.2, each σ_k intersects σ_a. Reparametrizing each σ_k if necessary, we assume that $\sigma_k(0) \in \operatorname{Im}\sigma_a$ for each k. Taking a subsequence if necessary, since $\operatorname{Im}\sigma_a$ is compact, we have

$$\sigma_k(0) \to p, \quad \sigma_k'(0) \to v \tag{4.1}$$

as $k \to \infty$. Note that $\|v\| = 1$ since each σ_k is of unit speed. Let $\sigma_\infty : I \to \mathbb{H}$ be the maximally defined geodesic in $(\mathbb{H}, g_A)$ with $\sigma_\infty(0) = p$, $\sigma_\infty'(0) = v$, where I is an open interval.

For any $R > 1$, let

$$K_R = [-R, R] \times [R^{-1}, R] \subset \mathbb{H}.$$

Then there is $R_0 > 1$ so that $\mathrm{Im}(\sigma_a) \subset K_R$ for all $R \geq R_0$. For each $k \in \mathbb{N}$ and $R \geq R_0$, let $I_{k,R}$ be the connected component of $\sigma_k^{-1}(K_R)$ in $[-d_k, d_k]$ containing 0. Since each σ_k is a geodesic in $(\mathbb{H}, g_A)$, by (4.1) and the smooth dependence on initial data of the ODE within each K_R, $(\sigma_k|_{I_{k,R}})$ converges smoothly to $\sigma_\infty|_{I_R}$ in K_R, where I_R is the connected component of $\sigma_\infty^{-1}(K_R)$ containing 0.

Since each $\sigma_k|_{I_{k,R}}$ is embedded and $(\sigma_k|_{I_{k,R}})$ converges smoothly to $\sigma_\infty|_{I_R}$, it is clear that σ_∞ does not admit transverse self-intersection. Thus if $\sigma_\infty(s) = \sigma_\infty(t)$ for some $s \neq t$, then $\sigma_\infty'(s) = \pm\sigma_\infty'(t)$ and this implies that σ_∞ is periodic. In this case, σ_∞ is a simple closed geodesic in $(\mathbb{H}, g_A)$ and (σ_k) converges smoothly to σ_∞. Thus we are done.

From now on we may therefore assume that σ_∞ is not a closed geodesic. Since $(\sigma_k|_{I_{k,R}})$ converges smoothly to σ_∞ in each K_R and $\cup_{R \geq R_0} K_R = \mathbb{H}$, σ_∞ is properly immersed. Since σ_∞ is injective, Σ_{σ_∞} is a complete embedded rotationally symmetric self-shrinker in $\mathbb{R}^{n+1}$. By Theorem 2.4, Σ_{σ_∞} is either the plane $\mathcal{P}$, the sphere $\mathcal{S}$ or the cylinder $\mathcal{C}$.

First we assume that Σ_{σ_∞} is the sphere $\mathcal{S}$ and we will derive a contradiction. After that, we consider the hyperplane $\mathcal{P}$ and the cylinder $\mathcal{C}$ and point out the necessary changes for the contradiction argument.

We split the argument into several lemmas.

Lemma 4.3. *There is $R_1 \geq R_0$ so that the following holds: for all $R \geq R_1$, there is $k_1 = k_1(R)$ so that for all $k \geq k_1$, $\mathrm{Im}\,\sigma_k \cap K_R$ contains a connected component different from $\sigma_k(I_{k,R})$ which intersects K_{R_1}.*

Proof of Lemma 4.3. First let $s \in (0, 1)$ be small so that the scaling of the Angenent doughnut $s\Sigma_{\sigma_a}$ lies completely inside the sphere $\Sigma_{\sigma_\infty} = \mathcal{S}$: that is, $x^2 + r^2 < 2n$ for all $(x, r) \in s\,\mathrm{Im}\,\sigma_a$. Let $\delta = d_0(s\Sigma_{\sigma_a}, \Sigma_{\sigma_\infty} \cup \ell)$, where $\ell \subset \mathbb{R}^{n+1}$ is the axis of rotation and d_0 is the Euclidean distance. Note that $\delta > 0$. Let $R_1 = \max\{R_0, 2\delta^{-1}, 3\sqrt{2n}\}$. For any $R > R_1$, since $\sigma_k|_{I_{k,R}}$ converges to $\sigma_\infty|_{I_R}$ in K_R uniformly, there is $k_1(R) > 0$ so that

$$d_0(\sigma_k(I_{k,R}), \sigma_\infty(I_R)) < \frac{\delta}{2} \tag{4.2}$$

for all $k \geq k_1$. By the choice of δ, we have

$$\sigma_k(I_{k,R}) \cap s(\operatorname{Im} \sigma_a) = \emptyset, \quad \text{for all } k \geq k_1.$$

Now we fix $k \geq k_1$ and show that $\operatorname{Im} \sigma_k \cap K_R$ has more than one component which intersects K_{R_1}. Assume the contrary, then $\operatorname{Im} \sigma_k \cap K_{R_1} = \sigma_k(I_{k,R}) \cap K_{R_1}$. Let

$$K^\delta_{R_1} = K_{R_1} \setminus \{(x,r) \in \mathbb{H} \mid \sqrt{x^2+r^2} < \sqrt{2n} + \delta\}.$$

Note that $K^\delta_{R_1}$ is connected. By (4.2), $K^\delta_{R_1}$ is disjoint from $\operatorname{Im} \sigma_k$. Since σ_k is a simple closed curve, by the Jordan curve theorem, its image divides $\mathbb{H}$ into two connected components, where exactly one of them is compact. There are two cases:

(i) $K^\delta_{R_1}$ lies in the compact component (see Figure 1). Let

$$\tilde{\sigma}_a := s\sigma_a + (2\sqrt{2n}, 0)$$

be the horizontal translation of $s\sigma_a$ by $(2\sqrt{2n}, 0)$. Since $R_1 > 3\sqrt{2n}$ and $R_1^{-1} < \delta$, $\tilde{\sigma}_a$ lies completely inside $K^\delta_{R_1}$ and thus in the compact component (see Figure 1). The translation is horizontal, hence the MCF $\{\Sigma^t_{\tilde{\sigma}_a}\}$ starting at $\Sigma_{\tilde{\sigma}_a}$ is also self-shrinking, centered at $(2\sqrt{2n}, 0)$. Since $s < 1$, it becomes extinct before the MCF $\{\Sigma^t_{\sigma_k}\}$ starting at Σ_{σ_k} does. This implies that $\Sigma^t_{\tilde{\sigma}_a}$ intersects $\Sigma^t_{\sigma_k}$ for some $t > 0$, which contradicts the maximum principle since $\Sigma_{\tilde{\sigma}_a}$ and Σ_{σ_k} are disjoint.

(ii) $K^\delta_{R_1}$ lies in the non-compact component (see Figure 2): let

$$H^\delta_{R_1} = \{(x,r) : K_{R_1} \mid \sqrt{x^2+r^2} \leq \sqrt{2n} - \delta\}.$$

Since $H^\delta_{R_1}$ is a subset of K_{R_1} and is disjoint from $\sigma_k(I_{k,R})$, $H^\delta_{R_1}$ also lies in a component of $\mathbb{H} \setminus \operatorname{Im}(\sigma_k)$. Since $K^\delta_{R_1}$, $H^\delta_{R_1}$ are separated by $\sigma_k(I_{k,R})$, then $H^\delta_{R_1}$ and $K^\delta_{R_1}$ lie in different components of $\mathbb{H} \setminus \operatorname{Im} \sigma_k$. Thus $H^\delta_{R_1}$ is in the compact component bounded by $\operatorname{Im} \sigma_k$. By the choice of R_1 and since $R > R_1$, $s \operatorname{Im} \sigma_a$ lies in the compact component (see Figure 2). By considering the MCF starting at $s\Sigma_{\sigma_a}$ as in (i), we again arrive at a contradiction.

Hence both cases are impossible, and we conclude that $\operatorname{Im} \sigma_k \cap K_R$ must contain more than one component which intersects K_{R_1} for all $k \geq k_1$. This finishes the proof of Lemma 4.3. □

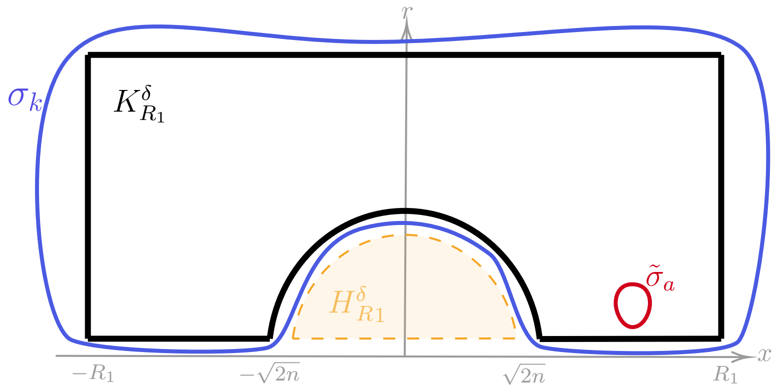

Figure 1: $\tilde{\sigma}_a$ is enclosed by σ_k.

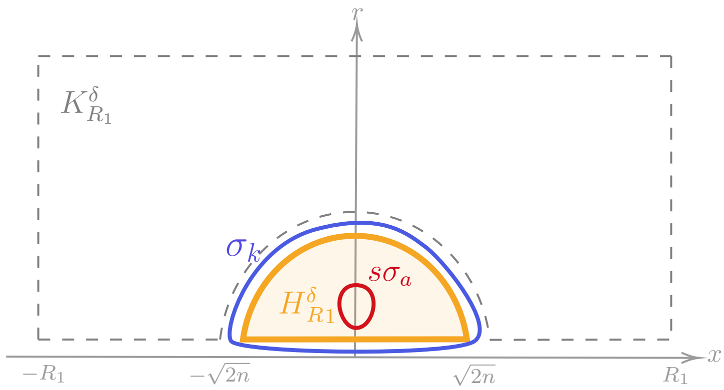

Figure 2: $s\sigma_a$ is enclosed by σ_k.

Let $R_1 < R_2 < \ldots < R_k < \ldots$ be any sequence such that $R_k \nearrow +\infty$ as $k \to \infty$. Using Lemma 4.3 and picking a subsequence of $(\sigma_k)_{k=1}^{\infty}$ if necessary, we may assume that $\operatorname{Im}\sigma_k \cap K_{R_k}$ has more than one component which intersects K_{R_1}. For each k, let $N_k \geq 2$ be the number of such connected components. Since there is a uniform positive lower bound on $d_A(K_{R_1}, \mathbb{H} \setminus K_{R_k})$ for all $k \geq 2$, N_k is uniformly bounded by Theorem 1.1 (see also (2.3)). Taking a further subsequence if necessary, we may assume that $N := N_k$ is constant.

Thus for each k, write $\operatorname{Im}\sigma_k \cap K_{R_k}$ as the disjoint union

$$\operatorname{Im}\sigma_k \cap K_{R_k} = \operatorname{Im}\sigma_k^1 \cup \cdots \cup \operatorname{Im}\sigma_k^N \cup \sigma_k^C, \tag{4.3}$$

where each σ_k^i, $i = 1, \cdots, N$ is a simple geodesic arc parametrizing a connected component of $\operatorname{Im}\sigma_k \cap K_{R_k}$ which intersects K_{R_1} and σ_k^C is the union of any connected components of $\operatorname{Im}\sigma_k \cap K_{R_k}$ which do not intersect K_{R_1}.

For each $k \in \mathbb{N}$, let $i_k \in \{1, \cdots, N\}$ be arbitrary. Up to reparametrization, there are $c_k > 0$ such that

$$\sigma_k^{i_k} : [-c_k, c_k] \to \mathbb{H}$$

is a unit speed geodesic. Since the image of $\sigma_k^{i_k}$ intersects K_{R_1}, one has

$$2c_k \geq d_A(K_{R_1}, \mathbb{H} \setminus K_{R_2}) > 0,$$

and there are $\bar{c}_k \in [-c_k, c_k]$ such that $\sigma_k^{i_k}(\bar{c}_k) \in K_{R_1}$. Taking a further subsequence if necessary, by compactness of K_{R_1}, we may assume that

$$\sigma_k^{i_k}(\bar{c}_k) \to q, \quad (\sigma_k^{i_k})'(\bar{c}_k) \to w$$

as $k \to \infty$. Let $\tilde{\sigma}_\infty : J \to \mathbb{H}$ be the unique complete maximal geodesic with $\tilde{\sigma}_\infty(0) = q$ and $\tilde{\sigma}'_\infty(0) = w$.

As in the construction of σ_∞, $(\sigma_k^{i_k})$ converges smoothly to $\tilde{\sigma}_\infty$ in K_R for all $R > R_1$ and $\tilde{\sigma}_\infty$ is an embedded geodesic in $(\mathbb{H}, g_A)$.

Lemma 4.4. *Up to reparametrization,* $\tilde{\sigma}_\infty = \sigma_\infty$.

Proof of Lemma 4.4. We may assume that $\sigma_k^{i_k} \neq \sigma_k|_{I_{k,R_k}}$ for all large k, since otherwise we have $\tilde{\sigma}_\infty = \sigma_\infty$ up to reparametrization. This implies that $\mathrm{Im}(\sigma_k|_{I_{k,R_k}})$ and $\mathrm{Im}\,\sigma_k^{i_k}$ have empty intersection since both of them are connected components of $\mathrm{Im}\,\sigma_k \cap K_{R_k}$. Assume the contrary, that $\tilde{\sigma}_\infty \neq \sigma_\infty$. By Lemma 4.2 and that $\tilde{\sigma}_\infty$, σ_∞ are both complete geodesics, they must intersect transversally. Assume that the intersection is in K_R for some $R > R_1$. Since $\sigma_k|_{I_{k,R}}$ converges locally smoothly to σ_∞ in K_R, $\sigma_k(I_{k,R})$ also intersects $\tilde{\sigma}_\infty$ for k large enough. On the other hand, $\sigma_k^{i_k}$ converges smoothly to $\tilde{\sigma}_\infty$, thus $\sigma_k^{i_k}$ also intersects $\sigma_k|_{I_{k,R}}$ in K_R for large k. This contradicts the assumption on $\sigma_k^{i_k}$ and hence the lemma is proved. □

Thus the union of subarcs $\mathrm{Im}\,\sigma_k^1 \cup \cdots \cup \mathrm{Im}\,\sigma_k^N$ of $\mathrm{Im}\,\sigma_k$ converge as $k \to \infty$ to σ_∞ locally smoothly with multiplicity N. While σ_k^C defined in (4.3) might not be empty, we can show that (by passing to a subsequence if necessary) it also stays close to σ_∞.

Lemma 4.5. *By passing to a subsequence of* (σ_k) *if necessary,*

$$|\sqrt{x^2 + r^2} - \sqrt{2n}| < R_k^{-1} \tag{4.4}$$

for all $(x, r) \in \mathrm{Im}(\sigma_k) \cap K_{R_k}$ *and for all* $k \in \mathbb{N}$.

Proof of Lemma 4.5. Let $k \in \mathbb{N}$ be fixed. First we show that there is n_k so that (4.4) holds for all $(x, r) \in \mathrm{Im}(\sigma_i) \cap K_{R_k}$ and for all $i \geq n_k$. To see this we argue by contradiction: if not, then there is a subsequence of (σ_{k_j}) of (σ_k) and $(x_j, r_j) \in \mathrm{Im}(\sigma_{k_j}) \cap K_{R_k}$ so that

$$|\sqrt{x_j^2 + r_j^2} - \sqrt{2n}| \geq R_k^{-1}$$

for all j. Since K_{R_k} is compact, we may assume that $(x_j, r_j) \to (x_\infty, r_\infty) \in K_{R_k}$, and thus there is a sequence $(\tilde{\sigma}_j)$ of subarcs of (σ_{k_j}) which converges locally smoothly to an embedded complete geodesic in $(\mathbb{H}, g_A)$, which is different from σ_∞ since $(x_\infty, r_\infty) \notin \mathrm{Im}(\sigma_\infty)$. Arguing similarly as in the proof of Lemma 4.4, this is impossible. Lastly, the lemma is proved by passing to a subsequence of (σ_k). □

A priori, as $k \to \infty$ some portions of $\mathrm{Im}\,\sigma_k$ might escape to infinity (as $x^2 + r^2 \to \infty$) or collapse to the rotational axis $r = 0$. The next lemma shows that this is not the case.

Lemma 4.6. *The sequence of self-shrinkers $(\Sigma_{\sigma_k})_{k=1}^\infty$ converges in Hausdorff distance to the round sphere Σ_{σ_∞} as $k \to \infty$.*

Proof of Lemma 4.6. We use a maximum principle argument similar to that in the proof of Lemma 4.3. Let $\epsilon > 0$. Let $t_0 \in (-1, 0)$ be given by

$$\sqrt{-t_0} = \frac{\sqrt{2n} + 0.5\epsilon}{\sqrt{2n} + \epsilon}. \tag{4.5}$$

Let $k_\epsilon \in \mathbb{N}$ be large such that $R_{k_\epsilon}^{-1} < \epsilon/2$ and the following holds: there are two horizontal translations and scalings of the Angenent torus $\Sigma_{\bar{a}_\pm}$ so that

- the image of $\bar{a}_+$ and $\bar{a}_-$ lie in $K_{R_{k_\epsilon}}$,
- $\sqrt{2n} + R_{k_\epsilon}^{-1} < \sqrt{x^2 + r^2} < \sqrt{2n} + \epsilon$ for all $(x, r) \in \mathrm{Im}\,\bar{a}_+$,
- $\sqrt{2n} - \epsilon < \sqrt{x^2 + r^2} < \sqrt{2n} - R_{k_\epsilon}^{-1}$ for all $(x, r) \in \mathrm{Im}\,\bar{a}_-$, and
- the MCF starting at $\Sigma_{\bar{a}_\pm}$ at $t = -1$ shrinks to $(\sqrt{2n} \pm 0.5\epsilon, 0)$ at time $t_e < t_0$.

By Lemma 4.5, $\bar{a}_\pm$ do not intersect with σ_k when $k \geq k_\epsilon$.

Now we claim that

$$\sqrt{2n} - \epsilon < \sqrt{x^2 + r^2} < \sqrt{2n} + \epsilon, \quad \text{for all } (x, r) \in \mathrm{Im}\,\sigma_k,\ k \geq k_\epsilon. \tag{4.6}$$

By Lemma 4.5 and that $R_{k_\epsilon}^{-1} < \epsilon$, it suffices to consider points outside $K_{R_{k_\epsilon}}$. Fix any $k \geq k_\epsilon$. If there is $(x, r) \in \mathrm{Im}(\sigma_k)$ so that $\sqrt{x^2 + r^2} \geq \sqrt{2n} + \epsilon$,

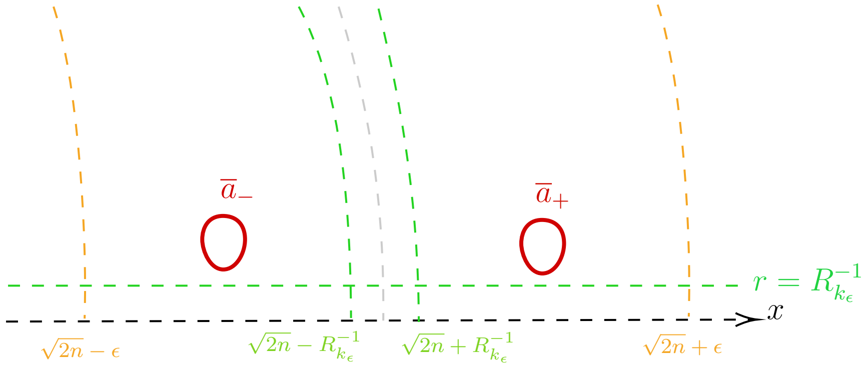

Figure 3: Choices of $\bar{a}_\pm$.

then since each σ_k is connected, there is a subarc β of σ_k connecting (x, r) and $\mathrm{Im}(\sigma_k) \cap K_{R_{k_\epsilon}}$ passing through the region

$$[\sqrt{2n}, \sqrt{2n} + \epsilon] \times (0, R_{k_\epsilon}].$$

But this is impossible: by (4.5), for all $-1 < t < t_0$, $\sqrt{-t}\beta$ contains a point $\{\sqrt{2n}+0.5\epsilon\} \times (0, R_{k_\epsilon})$. Hence the MCF starting at $\Sigma_{\bar{a}_+}$ (at time -1) would intersect with $\sqrt{-t}\beta$ for some $t \in (-1, t_e)$ and this contradicts the parabolic maximum principle, since Σ_{σ_k} and $\Sigma_{\bar{a}_+}$ are disjoint. Using $\Sigma_{\bar{a}_-}$ and arguing similarly, we conclude that $\sqrt{x^2 + r^2} \leq \sqrt{2n} - \epsilon$ is also impossible.

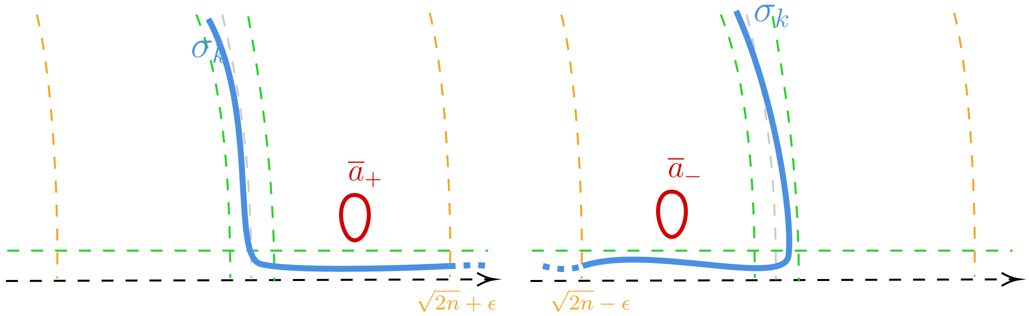

Figure 4: A subarc β of σ_k passing through $\{x = \sqrt{2n} + \epsilon\}$ (left) or $\{x = \sqrt{2n} - \epsilon\}$ (right).

Thus (4.6) is shown and this implies $d_0(p, \Sigma_{\sigma_\infty}) < \epsilon$ for all $k \geq k_\epsilon$ and $p \in \Sigma_{\sigma_k}$. On the other hand, since $(\sigma_k|_{I_{k,R_k}})$ converges smoothly to σ_∞ in $K_{2/\epsilon}$, by choosing a larger k_ϵ if necessary, we may assume that $d_0(p, \Sigma_{\sigma_k}) < \epsilon/2$ for all $p = (x, r\omega) \in \Sigma_{\sigma_\infty}$ with $(x, r) \in K_{2/\epsilon}$, $k \geq k_\epsilon$ and $\omega \in \mathbb{S}^{n-1}$. Thus $d_0(p, \Sigma_{\sigma_k}) < \epsilon$ for all $p \in \Sigma_{\sigma_\infty}$ and $k \geq k_\epsilon$. By the definition of the

Hausdorff distance $d^{\mathcal{H}}$,

$$d^{\mathcal{H}}(\Sigma_{\sigma_\infty}, \Sigma_{\sigma_k}) = \max\{ \sup_{p \in \Sigma_{\sigma_k}} d_0(p, \Sigma_{\sigma_\infty}), \sup_{p \in \Sigma_{\sigma_\infty}} d_0(p, \Sigma_{\sigma_k})\} < \epsilon$$

for all $k \geq k_\epsilon$. Since $\epsilon > 0$ is arbitrary, $(\Sigma_{\sigma_k})_{k=1}^\infty$ converges in Hausdorff distance to the round sphere Σ_{σ_∞} and this finishes the proof of Lemma 4.6. □

To summarize, we have shown that the sequence of self-shrinkers (Σ_{σ_k}) converges in Hausdorff distance to the sphere Σ_{σ_∞}. Moreover, for all $\epsilon > 0$, the set

$$\Sigma_{\sigma_k} \setminus (B_{(\sqrt{2n},0)}(\epsilon) \cup (B_{(-\sqrt{2n},0)}(\epsilon))$$

decomposes into N disjoint graphs $\Sigma_k^1, \cdots \Sigma_k^N$ over Σ_{σ_∞} for large enough k, where $N \geq 2$, and for each $i = 1, \cdots, N$, the convergence

$$\Sigma_k^i \to \Sigma_{\sigma_\infty} \setminus (B_{(\sqrt{2n},0)}(\epsilon) \cup (B_{(-\sqrt{2n},0)}(\epsilon)), \quad \text{as } k \to \infty,$$

is smooth graphical convergence.

Thus we can apply [CM12b, Proposition 3.2] to conclude that Σ_{σ_∞} is L-stable. Although [CM12b, Proposition 3.2] is stated only for $n = 2$, the same proof, which we now briefly describe, works for all $n \geq 2$ with only notational changes.

For each fixed k, since $\Sigma_k^1, \cdots, \Sigma_k^N$ are disjoint, we can order these N sheets by height (with respect to the outward unit normal of $\Sigma_{\sigma_\infty} = \mathcal{S}$). Let the top and the bottom layers be represented respectively by two functions w_k^+, w_k^- defined on $\Omega_k \subset \mathcal{S} \setminus \{(\pm\sqrt{2n}, 0)\}$ so that $w_k^+ > w_k^-$, $\Omega_k \subset \Omega_{k+1}$ and $\cup_k \Omega_k = \mathcal{S} \setminus \{(\pm\sqrt{2n}, 0)\}$. Fixing $x_0 \in \mathcal{S}$, then the sequence of functions

$$u_k = \frac{w_k^+ - w_k^-}{w_k^+(x_0) - w_k^-(x_0)}$$

converges locally smoothly on $\mathcal{S} \setminus \{(\pm\sqrt{2n}, 0)\}$ to a smooth function u which satisfies $Lu = 0$, where L is the stability operator (2.1). Using the Harnack inequality for linear second order elliptic equations and a maximum principle for minimal hypersurfaces (this is where Lemma 4.6 is used), one can bound u uniformly. Hence u extends across $\{(\pm\sqrt{2n}, 0)\}$ by the removable singularities lemma for L-harmonic functions [Ser64]. Thus we have constructed a positive function on $\mathcal{S}$ which satisfies $Lu = 0$ and this is sufficient to conclude that $\mathcal{S}$ is L-stable (see [CM12b] for more details and [FCS80] for a general statement). Since all properly embedded self-shrinkers in $\mathbb{R}^{n+1}$

are L-unstable ([CM12a], see also [CM12b, Theorem 0.5]), we have arrived at a contradiction and hence Σ_{σ_∞} is not the sphere $\mathcal{S}$.

Next we argue by contradiction that Σ_{σ_∞} is also not the plane $\mathcal{P}$ nor the cylinder $\mathcal{C}$. The arguments are similar to those for $\mathcal{S}$, thus we just point out the differences.

If Σ_{σ_∞} is the plane $\mathcal{P}$, then $\sigma_k|_{I_{k,R}}$ converges smoothly to the r-axis in K_R. Hence there is $k_1 = k_1(R) \in \mathbb{N}$ so that $\sqrt{x^2+r^2} < R^{-1}$ for all $(r,x) \in \operatorname{Im}\sigma_k \cap K_R$ and $k \geq k_1$. For any $R \geq R_1$, let

$$K_R^{\pm} := \{(x,r) \in K_R : \pm x \geq R^{-1}\}$$

One can argue that either one of $K_{R_1}^{\pm}$ must intersect the image of σ_k when $k \geq k_1$: if not, then either one of $K_{R_1}^{\pm}$ would lie in the compact region bounded by σ_k. This would lead to a contradiction by putting in suitably scaled and horizontally translated Angenent doughnuts in $K_{R_1}^{\pm}$.

Similar to the previous argument for the sphere $\mathcal{S}$, there is $N \geq 2$ so that for all $k \in \mathbb{N}$, $\operatorname{Im}\sigma_k \cap K_{R_k}$ contains N connected components which intersect K_{R_1}, and $\operatorname{Im}\sigma_k \cap K_{R_k}$ converges smoothly graphically to the r-axis with multiplicity N.

As in the proof of Lemma 4.6, for all $R > R_1$, one can show that $\operatorname{Im}\sigma_k \cap K_R$ converges (locally) in Hausdorff distance to $\operatorname{Im}\sigma_\infty \cap K_R$ as $k \to \infty$ (unlike the case of the sphere, there might be mass loss as $R \to \infty$). This is still sufficient for us to apply [CM12b, Proposition 3.2] to conclude that the plane is L-stable, which is impossible [CM12a].

For the remaining case for the cylinder $\mathcal{C}$, the curves $\sigma_k|_{I_{k,R}}$ converge smoothly to $\{r = \sqrt{2(n-1)}\}$ in K_R. Hence there is $k_1 = k_1(R) \in \mathbb{N}$ such that $|r - \sqrt{2(n-1)}| < R^{-1}$ for all $(x,r) \in \operatorname{Im}\sigma_k \cap K_R$ and $k \geq k_1$. By fitting Angenent doughnuts inside

$$K_{R_1}^{>} = \{(x,r) \in K_{R_1} : r > \sqrt{2(n-1)} + R^{-1}\},$$
$$K_{R_1}^{<} = \{(x,r) \in K_{R_1} : r < \sqrt{2(n-1)} - R^{-1}\}$$

respectively (in $K_{R_1}^{>}$ we insert a large Angenent doughnut), there is $N \geq 2$ so that $\operatorname{Im}\sigma_k \cap K_{R_k}$ converges smoothly graphically to $\{r = \sqrt{2(n-1)}\}$ with multiplicity N. Then again we apply [CM12b, Proposition 3.2].

To sum up, Σ_{σ_∞} is neither the plane $\mathcal{P}$ nor the cylinder $\mathcal{C}$ nor the sphere $\mathcal{S}$. By the classification Theorem 2.4, σ_∞ is an embedded closed geodesic in $(\mathbb{H}, g_A)$ and the convergence $\Sigma_{\sigma_k} \to \Sigma_{\sigma_\infty}$ is smooth. This finishes the proof of Theorem 1.2. $\square$

Remark 4.1. In the proof of Theorem 1.2, we argued by contradiction using L-stability that Σ_{σ_∞} is neither the sphere $\mathcal{S}$, nor the cylinder $\mathcal{C}$ nor the plane $\mathcal{P}$. In this remark we give an alternative argument ruling out $\mathcal{S}$ and $\mathcal{C}$ by instead using the entropy bound $E_n \leq E_2$ (see Lemma 5.1). By Lemma 4.3 and Lemma 4.4, for each k large, one can find N disjoint subarcs $\sigma_k^1 \cdots, \sigma_k^N$ of σ_k so that for each $i = 1, \cdots, N$, (σ_k^i) converges locally smoothly to σ_∞ as $k \to \infty$. In particular,

$$\lim_{k\to\infty} L_A(\sigma_k) \geq N L_A(\sigma_\infty).$$

By (2.3), we obtain

$$E_2 \geq E_n \geq \lim_{k\to\infty} \lambda(\Sigma_{\sigma_k}) \geq N\lambda(\Sigma_{\sigma_\infty}).$$

If $\Sigma_{\sigma_\infty} = \mathcal{S}$ or $\mathcal{C}$, then

$$\lambda(\Sigma_\infty) > \sqrt{2}$$

by [Sto94, A.4 Lemma]. Thus

$$N\sqrt{2} \leq E_2,$$

which is impossible as $N \geq 2$ and $E_2 \sim 2.2476 < 2\sqrt{2}$. Since $E_n > 2$ and $\lambda(\mathcal{P}) = 1$, this argument is however not sufficient to show $\Sigma_{\sigma_\infty} \neq \mathcal{P}$.

Next we prove several consequences of Theorem 1.2.

First, a simple contradiction argument using Theorem 1.2 shows that the profile curve of every embedded rotationally symmetric self-shrinking doughnut must lie in a fixed compact subset in $\mathbb{H}$. Together with Theorem 1.1 and [BK21a, Theorem 1.1], we obtain an upper bound on the index. (for the definition of index of a self-shrinker, see [BK21a, Definition 2.18].)

Corollary 4.7. *There is $I \in \mathbb{N}$ so that for any complete embedded rotationally symmetric self-shrinker Σ in $\mathbb{R}^3$, one has*

$$i(\Sigma) \leq I,$$

where $i(\Sigma)$ is the index of Σ.

Unlike our Theorem 1.1, the bound I in the above corollary is not explicit.

The next corollary and Theorem 1.3 generalize [Mra21, Theorem 1.3] to any dimension and by allowing rotationally symmetric self-shrinkers with non-convex profile curve and is a consequence of the fact that the Angenent metrics g_A are real analytic. See also [CM21] for a similar theorem for Lagrangian self-shrinking tori in $\mathbb{R}^4$.

Corollary 4.8. *For each $n \geq 2$, there is a finite set $S_n \subset [1, E_n]$ so that $\lambda(\Sigma) \in S_n$ for all complete embedded rotationally symmetric self-shrinkers Σ in $\mathbb{R}^{n+1}$.*

Proof. By Theorem 2.4, it suffices to consider only rotationally symmetric self-shrinkers that are diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Let σ be the profile curve of a rotationally symmetric self-shrinking doughnut in $\mathbb{R}^{n+1}$. Thus σ is a geodesic in $(\mathbb{H}, g_A)$. Since g_A is real analytic, the mapping

$$\gamma \mapsto E_A(\gamma) := \frac{1}{2} \int_{\mathbb{S}^1} g_A(\dot{\gamma}, \dot{\gamma})$$

is a real analytic functional defined on all closed curves γ close to σ. Note that $DE_A(\gamma) = 0$ if and only if γ is a geodesic, here $DE_A(\gamma)$ is the L^2-gradient of E_A at γ. By the celebrated Łojasiewicz-Simon gradient inequality [Sim83, equation (2.2)], there is $C_2 > 0$ and $\theta \in (0, 1/2)$ so that

$$|E_A(\gamma) - E_A(\sigma)|^{1-\theta} \leq C_2 \|DE_A(\gamma)\|_{L^2},$$

for all γ which are $C^{2,\alpha}$-close to σ. Hence if γ is another geodesic close to σ, we have $E_A(\gamma) = E_A(\sigma)$. Since geodesics are parametrized by constant length, one has

$$L_A(\gamma) = L_A(\sigma).$$

Thus the length functional $\gamma \mapsto L_A(\gamma)$ is locally constant on the space of all closed geodesics. Since the space of simple closed geodesics is compact by Theorem 1.2, the length functional has a finite image. Together with (2.3), this finishes the proof of the corollary. □

Lastly, we prove Theorem 1.3.

Proof of Theorem 1.3. In [Mra21, section 5], Mramor studies the Poincaré map of the (up to renormalization of length) geodesic equation in $(\mathbb{H}, g_A)$

$$x' = \cos\theta, \quad r' = \sin\theta, \quad \theta' = \frac{x}{2}\sin\theta + \left(\frac{n-1}{r} - \frac{r}{2}\right)\cos\theta. \tag{4.7}$$

For any $R > 0$ and θ, let $(x_{R,\theta}(t), y_{R,\theta}(t), \theta_{R,\theta}(t))$ be the maximal solution to (4.7) with initial value $(0, R, \theta)$. Let $T^* = T^*(R, \theta)$ be the second time at which $x_{R,\theta} = 0$ occurs.

By [Mra21, Lemma 5.1], the fixed points of the Poincaré map

$$P : \mathbb{R} \times (0, \infty) \to \mathbb{R} \times (0, \infty), \quad P(R, \theta) = (r_{R,\theta}(T^*), \theta_{R,\theta}(T^*))$$

are either isolated points or analytic curves in $\mathbb{R} \times (0, \infty)$. Arguing as in the proof of [Mra21, Theorem 1.3], the map

$$P_f : (0, \infty) \to (0, \infty), \quad P_f(R) = r_{R,0}(T^*(R, 0), 0)$$

has isolated fixed points. Since the profile curve of any embedded self-shrinker with reflectional symmetry must intersect the fixed points of P_f, together with Theorem 1.2, Theorem 1.3 has been proven. □

5 Appendix: the sequence (E_n)

In this appendix, we show the following lemma.

Lemma 5.1. *The sequence* (E_n) *defined in (3.1) satisfies* $2 < E_n \leq E_2$ *and*

$$\lim_{n\to\infty} E_n = \sqrt{\frac{4\pi}{3}}. \tag{5.1}$$

Proof. It is proved in [Sto94, A.4 Lemma] that the entropy of the n-sphere $\lambda(\mathbb{S}^n)$ satisfies

$$\lambda(\mathbb{S}^n) = \left(\frac{n}{2\pi e}\right)^{n/2} \omega_n \tag{5.2}$$

and the sequence $(\lambda(\mathbb{S}^n))$ is strictly decreasing. Also,

$$\lim_{n\to\infty} \lambda(\mathbb{S}^n) = \sqrt{2}. \tag{5.3}$$

From (3.1) and (5.2) we obtain

$$E_n = \sqrt{\frac{2\pi}{3}\frac{1 + x_n}{1 + 2x_n/3}} \left(\frac{1}{e}(1 + x_n)^{1/x_n}\right)^{a_n/4} \lambda(\mathbb{S}^{n-1}), \tag{5.4}$$

where

$$a_n = y_n - 2(n - 1), \quad x_n = \frac{a_n}{2(n - 1)}.$$

Direct calculations give

$$\frac{1}{2} < a_n < \frac{2}{3}, \quad 0 < x_n < 1 \tag{5.5}$$

and

$$\lim_{n\to\infty} a_n = \frac{2}{3}, \quad \lim_{n\to\infty} x_n = 0. \tag{5.6}$$

Using (5.4), (5.3) and (5.6), one obtains (5.1).

Next we show $2 < E_n \leq E_2$. By the Taylor expansion of $\ln(1+x)$, we have

$$x + x^3/3 > \ln(1+x) > x - x^2/2 \quad \text{for all } x \in (0,1).$$

Thus

$$e^{\frac{x^2}{3}} > \frac{1}{e}(1+x)^{1/x} > e^{-\frac{x}{2}}, \quad \text{for all } x \in (0,1).$$

Together with $\lambda(\mathbb{S}^{n-1}) > \sqrt{2}$, (5.5) and (5.4),

$$\frac{\sqrt{2\pi(3+(n-1)^{-1})}}{3} e^{\frac{1}{162(n-1)^2}} \lambda(\mathbb{S}^{n-1}) > E_n > \sqrt{\frac{4\pi}{3}} e^{-\frac{1}{36(n-1)}}. \tag{5.7}$$

Since $\lambda(\mathbb{S}^{n-1})$ is decreasing, the upper bound in (5.7) is strictly decreasing in n. Also, the lower bound in (5.7) is strictly increasing in n. Plugging in $n = 4$ in the upper and lower bound of (5.7) gives

$$2.21823 \sim \frac{\sqrt{10}\pi}{e^{1093/729}} > E_n > \sqrt{\frac{4\pi}{3}} e^{-\frac{1}{108}} \sim 2.02780, \quad \text{for all } n \geq 4.$$

The inequality implies $2 < E_n < E_2$ for all $n \geq 4$. The case $n = 2, 3$ can be checked directly.

□

References

[AL86] Uwe Abresch and J. Langer. The normalized curve shortening flow and homothetic solutions. *J. Differ. Geom.*, 23:175–196, 1986.

[Ang92] Sigurd B. Angenent. Shrinking doughnuts. In *Nonlinear diffusion equations and their equilibrium states, 3. Proceedings from the third conference, held August 20-29, 1989 in Gregynog, Wales, United Kingdom*, pages 21–38. Boston, MA etc.: Birkhäuser, 1992.

[BBI01] D. Burago, Yu. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2001.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren Math. Wiss.* Berlin: Springer, 1999.

[BK21a] Yakov Berchenko-Kogan. Bounds on the index of rotationally symmetric self-shrinking tori. *Geom. Dedicata*, 213:83–106, 2021.

[BK21b] Yakov Berchenko-Kogan. The entropy of the angenent torus is approximately 1.85122. *Exp. Math.*, 30(4):587–594, 2021.

[BNS21] Reto Buzano, Huy The Nguyen, and Mario B Schulz. Noncompact self-shrinkers for mean curvature flow with arbitrary genus. *arXiv preprint arXiv:2110.06027*, 2021.

[Bre16] Simon Brendle. Embedded self-similar shrinkers of genus 0. *Ann. Math. (2)*, 183(2):715–728, 2016.

[BW16] Jacob Bernstein and Lu Wang. A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. *Invent. Math.*, 206(3):601–627, 2016.

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. *arXiv preprint arXiv:2003.14344*, 2020.

[CIMW13] Tobias Holck Colding, Tom Ilmanen, William P. II. Minicozzi, and Brian White. The round sphere minimizes entropy among closed self-shrinkers. *J. Differ. Geom.*, 95(1):53–69, 2013.

[CM12a] Tobias H. Colding and William P. II Minicozzi. Generic mean curvature flow. I: Generic singularities. *Ann. Math. (2)*, 175(2):755–833, 2012.

[CM12b] Tobias H. Colding and William P. II Minicozzi. Smooth compactness of self-shrinkers. *Comment. Math. Helv.*, 87(2):463–475, 2012.

[CM18] Jingyi Chen and John Man Shun Ma. The space of compact self-shrinking solutions to the Lagrangian mean curvature flow in $\mathbb{C}^2$. *J. Reine Angew. Math.*, 743:229–244, 2018.

[CM21] Jingyi Chen and John Man Shun Ma. Geometry of Lagrangian self-shrinking tori and applications to the piecewise Lagrangian mean curvature flow. *Am. J. Math.*, 143(1):227–264, 2021.

[DK17] Gregory Drugan and Stephen J. Kleene. Immersed self-shrinkers. *Trans. Am. Math. Soc.*, 369(10):7213–7250, 2017.

[DLN18] Gregory Drugan, Hojoo Lee, and Xuan Hien Nguyen. A survey of closed self-shrinkers with symmetry. *Result. Math.*, 73(1):32, 2018. Id/No 32.

[DN18] Gregory Drugan and Xuan Hien Nguyen. Shrinking doughnuts via variational methods. *J. Geom. Anal.*, 28(4):3725–3746, 2018.

[DX13] Qi Ding and Y. L. Xin. Volume growth eigenvalue and compactness for self-shrinkers. *Asian J. Math.*, 17(3):443–456, 2013.

[FCS80] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non- negative scalar curvature. *Commun. Pure Appl. Math.*, 33:199–211, 1980.

[GN21] Harald Garcke and Robert Nürnberg. Numerical approximation of boundary value problems for curvature flow and elastic flow in Riemannian manifolds. *Numer. Math.*, 149(2):375–415, 2021.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differ. Geom.*, 31(1):285–299, 1990.

[HW19] Or Hershkovits and Brian White. Sharp entropy bounds for self-shrinkers in mean curvature flow. *Geom. Topol.*, 23(3):1611–1619, 2019.

[Ilm94] Tom Ilmanen. *Elliptic regularization and partial regularity for motion by mean curvature*, volume 520 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 1994.

[IPR21] Debora Impera, Stefano Pigola, and Michele Rimoldi. The Frankel property for self-shrinkers from the viewpoint of elliptic PDEs. *J. Reine Angew. Math.*, 773:1–20, 2021.

[Ket16] Daniel Ketover. Self-shrinking platonic solids. *arXiv preprint arXiv:1602.07271*, 2016.

[KKM18] Nikolaos Kapouleas, Stephen James Kleene, and Niels Martin Møller. Mean curvature self-shrinkers of high genus: non-compact examples. *J. Reine Angew. Math.*, 739:1–39, 2018.

[Kli95] Wilhelm P. A. Klingenberg. *Riemannian geometry.*, volume 1 of *De Gruyter Stud. Math.* Berlin: Walter de Gruyter, 2nd ed. edition, 1995.

[KM14] Stephen Kleene and Niels Martin Møller. Self-shrinkers with a rotational symmetry. *Trans. Am. Math. Soc.*, 366(8):3943–3963, 2014.

[KM23] Nikolaos Kapouleas and Peter McGrath. Generalizing the linearized doubling approach. I: General theory and new minimal surfaces and self-shrinkers. *Camb. J. Math.*, 11(2):299–439, 2023.

[MM09] Annibale Magni and Carlo Mantegazza. Some remarks on Huisken's monotonicity formula for mean curvature flow. In *Singularities in nonlinear evolution phenomena and applications. Proceedings of the workshop, Pisa, Italy, May 2008*, pages 157–169. Pisa: Edizioni della Normale, 2009.

[Møl11] Niels Martin Møller. Closed self-shrinking surfaces in $\mathbb{R}^3$ via the torus. *arXiv preprint arXiv:1111.7318*, 2011.

[Mra21] Alexander Mramor. Compactness and finiteness theorems for rotationally symmetric self shrinkers. *J. Geom. Anal.*, 31(5):5094–5107, 2021.

[Ngu14] Xuan Hien Nguyen. Construction of complete embedded self-similar surfaces under mean curvature flow. III. *Duke Math. J.*, 163(11):2023–2056, 2014.

[Ser64] J. Serrin. Removable singularities of solutions of elliptic equations. *Arch. Ration. Mech. Anal.*, 17:67–78, 1964.

[Sim83] Leon Simon. Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. *Ann. Math. (2)*, 118:525–571, 1983.

[Son14] Antoine Song. A maximum principle for self-shrinkers and some consequences. *arXiv preprint arXiv:1412.4755*, 2014.

[Sto94] Andrew Stone. A density function and the structure of singularities of the mean curvature flow. *Calc. Var. Partial Differ. Equ.*, 2(4):443–480, 1994.

[SW20] Ao Sun and Zhichao Wang. Compactness of self-shrinkers in $\mathbb{R}^3$ with fixed genus. *Adv. Math.*, 367:39, 2020. Id/No 107110.

[WW09] Guofang Wei and Will Wylie. Comparison geometry for the Bakry-Emery Ricci tensor. *J. Differ. Geom.*, 83(2):377–405, 2009.

John Man Shun Ma
Department of Mathematical Science, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
email: jm@math.ku.dk

Ali Muhammad
Department of Mathematical Science, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
email: ajhm@math.ku.dk

Niels Martin Møller
Department of Mathematical Science, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
email: nmoller@math.ku.dk

Chapter 5

Paper II: Entropy Bounds for Self-shrinkers with Symmetries

This chapter includes the paper:

[MM23] John Man Shun Ma and Ali Muhammad. Entropy bounds for self-shrinkers with symmetries

Entropy Bounds for Self-shrinkers with Symmetries

John Man Shun Ma, Ali Muhammad

Abstract

In this work we derive explicit entropy bounds for two classes of closed self-shrinkers: the class of embedded closed self-shrinkers recently constructed in [Rie23] using isoparametric foliations of spheres, and the class of compact non-spherical immersed rotationally symmetric self-shrinkers. These bounds generalize the entropy bounds found in [MMM22] on the space of complete embedded rotationally symmetric self-shrinkers.

1 Introduction

A smooth hypersurface Σ in $\mathbb{R}^{n+1}$ is called a *self-shrinker* if

$$H_\Sigma(x) = \frac{1}{2}\langle x, \nu\rangle, \quad \text{for all } x \in \Sigma. \tag{1.1}$$

Here ν is the unit normal vector field on Σ, $H_\Sigma = \operatorname{div} \nu$ is the mean curvature of Σ and x is the position vector. Self-shrinkers are central objects of study in the mean curvature flow (MCF) since they serve as singularity models [Hui90], [Ilm94]. Besides the generalized cylinders, different techniques have been used to construct explicit examples of self-shrinkers [AL86], [Ang92], [KKM18], [BNS21], [Ngu14], [DN18], [KM23], [Rie23].

In the seminal paper [CM12a] (see also [MM09]), Colding and Minicozzi defined the *entropy* $\lambda(\Sigma)$ of any smooth hypersurface Σ. The entropy is a scaling and translation invariant quantity. More importantly, if $\{\Sigma_t\}$ is a solution to the MCF, then $t \mapsto \lambda(\Sigma_t)$ is non-increasing and is constant

The authors were supported by the Danish National Research Foundation CPH-GEOTOP-DNRF151.

if and only if (up to rescaling and space-time translation) $\Sigma_t = \sqrt{-t}\Sigma$ for $t \in (-\infty, 0)$, where Σ is a self-shrinker.

The entropy of the generalized cylinders were computed in [Sto94], and the entropy of the Angenent doughnuts were numerically computed in [BK21], [GN21]. In general, upper bounds for the entropy are of great interest, since they normally imply surprising topological and geometrical consequences [Mra21], [BW16], [CHH22], [MMM22].

In [MMM22], the authors and Møller proved an entropy upper bound for complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$. Using that, several compactness and finiteness results for the space of such self-shrinkers were obtained. The goal of this paper is to generalize the entropy upper bounds obtained in [MMM22, Theorem 1.1] in two directions.

Using the theory of isoparametric foliations of spheres, Riedler constructed in [Rie23] new examples of closed embedded self-shrinkers in $\mathbb{R}^{n+1}$ with topology $\mathbb{S}^1 \times M$, where $M \subset \mathbb{S}^n$ is diffeomorphic to an isoparametric hypersurface of $\mathbb{S}^n$ for which the multiplicities of the principal curvatures agree. This is done by applying a reduction theorem from [PT86] to reduce the self-shrinker equation to a geodesic equation in an open subset of $\mathbb{R}^2$. Indeed, the construction generalizes the classical work by Angenent [Ang92] for rotationally symmetric self-shrinkers. Our first result is the following theorem which provides entropy bounds for this class of self-shrinkers.

Theorem 1.1. *Let $\mathscr{M} = \{M_\varphi\}_{\varphi \in (0,\pi/g)}$ be an isoparametric foliation of $\mathbb{S}^n$ of type (g, m, m), and let N be a f-invariant embedded closed self-shrinker diffeomorphic to $\mathbb{S}^1 \times M$, where M is diffeomorphic to a regular fiber of the foliation $\mathscr{M}$. Then there is a positive number $E_{g,m}$ such that $\lambda(N) \leq E_{g,m}$.*

We remark that the constants $E_{g,m}$ are explicit and depend only on the pair (g, m). We refer the reader to section 3 for the terminology on isoparametric foliations appearing in Theorem 1.1.

Our second result is Theorem 1.2 below which gives entropy bounds for immersed rotationally symmetric self-shrinkers diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Drugan and Kleene constructed in [DK17] infinitely many complete immersed rotationally symmetric self-shrinkers of various topological types. In the following theorem, we obtain an entropy upper bound for compact immersed non-spherical rotationally symmetric self-shrinkers in terms of the number of self-intersections of the corresponding profile curve. For each $n \geq 2$, let E_n denote the entropy upper bound for complete embedded rotationally symmetric self-shrinkers in $\mathbb{R}^{n+1}$ obtained in [MMM22, equation (3.1)].

Theorem 1.2. *Let $\Sigma = \Sigma_\sigma$ be a compact immersed non-spherical rotationally symmetric self-shrinker in $\mathbb{R}^{n+1}$, where the profile curve σ has k self-intersection points counted with multiplicity. Then*

$$\lambda(\Sigma) \leq (k+1)E_n. \tag{1.2}$$

We refer the reader to section 4 and 5 for the terminology on rotationally symmetric self-shrinkers appearing in Theorem 1.2. Both Theorem 1.1 and Theorem 1.2 are proven using comparison theorems ([MMM22, Theorem 3.1] and Theorem 4.8 respectively), which provide estimates for the length of closed geodesics σ in a simply connected Riemannian surface with strictly positive Gaussian curvature. Theorem 4.8 also implies an entropy bound for a class of closed immersed f-invariant self-shrinkers (see remark 5.1).

The organization of this paper is as follows. In section 2, we recall the definitions of self-shrinkers and the entropy of a hypersurface. In section 3, we recall the construction in [Rie23] and prove Theorem 1.1. In section 4, we prove a general comparison theorem (Theorem 4.8) and in the last section, we prove Theorem 1.2.

The first author would like to thank Hojoo Lee for their discussion on immersed rotationally symmetric self-shrinkers. The authors would also like to thank Alexander Mramor for fruitful discussions.

2 Self-shrinkers and the entropy

Let Σ be a properly immersed smooth hypersurface in $\mathbb{R}^{n+1}$. For any $x_0 \in \mathbb{R}^{n+1}$ and $t_0 > 0$, the functional F_{x_0,t_0} is defined in [CM12a] as

$$F_{x_0,t_0}(\Sigma) = \frac{1}{(4\pi t_0)^{n/2}} \int_\Sigma e^{-\frac{\|x-x_0\|^2}{4t_0}} d\mu, \tag{2.1}$$

Where $d\mu$ is the volume form of Σ. It is well known (e.g. [CM12b, section 1]) that the following statements are equivalent:

- Σ is a self-shrinker;
- Σ is a minimal hypersurface in $\mathbb{R}^{n+1}$ with respect to the conformal (Gaußian) metric $g_{ß}$, where

$$(g_{ß})_{ij} = e^{-\frac{\|x\|^2}{2n}} \delta_{ij} \tag{2.2}$$

and δ denotes the Euclidean metric on $\mathbb{R}^{n+1}$;

- Σ is a critical point of the F-functional $F = F_{0,1}$.

The *entropy* of a hypersurface Σ is defined by ([CM12a], [MM09])

$$\lambda(\Sigma) = \sup_{(x_0,t_0)\in\mathbb{R}^{n+1}\times\mathbb{R}_{>0}} F_{x_0,t_0}(\Sigma). \tag{2.3}$$

We also need the following fact about the entropy of self-shrinkers [CM12a, section 7.2].

Lemma 2.1. *Let Σ be a properly immersed self-shrinker in $\mathbb{R}^{n+1}$. Then $\lambda(\Sigma) = F_{0,1}(\Sigma)$.*

3 Closed embedded self-shrinkers constructed by isoparametric foliations

In this section we prove Theorem 1.1. First, we recall the construction of the closed embedded self-shrinkers in [Rie23] using the theory of isoparametric foliations of spheres, which generalizes the classical construction of rotationally symmetric self-shrinkers by Angenent [Ang92]. The definition of isoparametric foliations and the properties of such foliations of spheres, in particular, the structural theorems proved by Münzner in [Mün80], [Mün81], are neatly summarized in [Rie23, section 2] and we closely follow their notation. For a general introduction, see for example [CR15, Chapter 3]. We start by briefly recalling the necessary background on the theory. We then proceed to describe how the problem of finding self-shrinkers in the setting of [Rie23] is reduced to finding geodesics in a certain open subset of $\mathbb{R}^2$ equipped with a family of (incomplete) metrics $h_{g,m} = h$ with Gaussian curvature bounded from below by strictly positive numbers.

3.1 Background on Isoparametric Foliations of $\mathbb{S}^n$

A smooth function $F : \mathbb{S}^n \to \mathbb{R}$ is called *isoparametric* if there exist smooth functions $\phi_1, \phi_2 : F(\mathbb{S}^n) \to \mathbb{R}$ such that

$$\|\mathrm{grad}F\|^2 = \phi_1 \circ F, \quad \Delta F = \phi_2 \circ F,$$

where $\mathrm{grad}F$ and ΔF denote the gradient and the Laplacian of F, respectively. Foliations of $\mathbb{S}^n$ that arise from a family of level sets of an isoparametric function are called *isoparametric foliations* of $\mathbb{S}^n$. Such level sets are then the fibers of a given foliation. By the work of Münzner [Mün80], the isoparametric function of a given foliation is a restriction to $\mathbb{S}^n$ of a homogeneous polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree g which satisfies certain differential equations, where $g \in \{1, 2, 3, 4, 6\}$. Such a polynomial F is called the *Cartan-Münzner polynomial*. The foliation has exactly two singular fibers $V_\pm = (F|_{\mathbb{S}^n})^{-1}(\pm 1)$, and each regular fiber is given by

$$M_\varphi = \{x \in \mathbb{S}^n |\, \mathrm{dist}_0(x, V_-) = \varphi\}, \tag{3.1}$$

where $\varphi \in (0, \pi/g)$ and dist_0 is the distance function on the round $\mathbb{S}^n$. On each M_φ, there are g (the degree of the Cartan-Münzner polynomial F) distinct principal curvatures, and they each assume a constant value on a given regular fiber. Furthermore, there are at most two distinct multiplicities for the principal curvatures which we denote by m_1 and m_2, and we have the following relation $n - 1 = \frac{g}{2}(m_1 + m_2)$. Also, for each $\varphi \in (0, \pi/g)$ we have:

$$\mathrm{Vol}_0(M_\varphi) = c \sin\left(\frac{g}{2}\varphi\right)^{m_1} \cos\left(\frac{g}{2}\varphi\right)^{m_2}, \tag{3.2}$$

where c is some constant independent of φ and $\mathrm{Vol}_0(M_\varphi)$ is the volume of M_φ with respect to the round metric on $\mathbb{S}^n$.

We shall call an isoparametric foliation $\mathscr{M} = \{M_\varphi\}$ of type (g, m_1, m_2) if the fibers have g distinct principal curvatures with corresponding multiplicities m_1 and m_2. In the following we shall restrict to the case where the multiplicities agree, i.e. $m_1 = m_2$ and hence consider foliations of type (g, m, m). In this case we have $n = mg + 1$ and we shall use n and m interchangeably through this relation.

In the next lemma we determine the number c in (3.2) in terms of g and m. We start by recalling the following useful result: let $\pi : (E, g_E) \to (B, g_B)$ be a surjective Riemannian submersion between two Riemannian manifolds with compact fibers. Let $v : B \to \mathbb{R}$ be $v(x) = \mathrm{Vol}_{g_E}(\pi^{-1}(x))$. By the co-area formula [Nic11, Theorem 2.1], for all submanifolds M of B we have

$$\mathrm{Vol}_{g_E}(\widetilde{M}) = \int_M v(x) d\mu_M(x), \tag{3.3}$$

where $\widetilde{M} = \pi^{-1}(M)$ and $d\mu_M$ is the volume form of M in (B, g_B).

Lemma 3.1. *Given an isoparametric foliation $\{M_\varphi\}$ of $\mathbb{S}^n$ of type (g, m, m), we have*

$$\mathrm{Vol}_0(M_\varphi) = \frac{g\omega_n}{s(m)} \sin^m(g\varphi) \tag{3.4}$$

for every regular fiber M_φ, $\varphi \in (0, \pi/g)$. Here $n = mg + 1$, ω_n is the volume *of* $\mathbb{S}^n$ *and*

$$s(m) = \int_0^\pi \sin^m t \, dt. \tag{3.5}$$

Proof. Since $m_1 = m_2 = m$ we have from (3.2)

$$\mathrm{Vol}_0(M_\varphi) = \frac{c}{2^m} \sin^m(g\varphi), \tag{3.6}$$

hence it suffices to find c. By the relation (3.1) we know that the mapping

$$\mathbb{S}^n \setminus (V_- \cup V_+) \to \left(0, \frac{\pi}{g}\right), \quad x \in M_\varphi \mapsto \varphi,$$

is a Riemannian submersion, with respect to the standard metric on $\mathbb{S}^n$ and the Euclidean metric on the interval $(0, \pi/g)$. Using (3.3) and (3.6),

$$\omega_n := \mathrm{Vol}_0(\mathbb{S}^n) = \int_0^{\pi/g} \mathrm{Vol}_0(M_\varphi) d\varphi = \frac{c}{g 2^m} \int_0^\pi \sin^m t \, dt.$$

Hence $c = g 2^m \omega_n s(m)^{-1}$ and Lemma 3.1 is proved. □

For simplicity, from now on we write

$$c_{g,m} = \frac{g\omega_n}{s(m)}. \tag{3.7}$$

3.2 The Reduction Theory; Proof of Theorem 1.1

Equip $\mathbb{R}^{n+1} \backslash \mathbb{R}_{\geq 0} \cdot (V_+ \cup V_-)$ with the metric $g_ß$ defined in (2.2), and let

$$g_{\mathrm{S}} = e^{-\frac{r^2}{2n}}(dr^2 + r^2 d\varphi^2) \tag{3.8}$$

be a metric on $(0, \infty) \times (0, \pi/g)$. Define the map

$$f : \mathbb{R}^{n+1} \backslash \mathbb{R}_{\geq 0} \cdot (V_+ \cup V_-) \to (0, \infty) \times \left(0, \frac{\pi}{g}\right), \quad x \mapsto \left(\|x\|, \frac{\arccos(F(x/\|x\|))}{g}\right). \tag{3.9}$$

Here F is the corresponding Cartan-Münzner polynomial and $V_\pm = (F|_{\mathbb{S}^n})^{-1}(\pm 1)$ are the singular fibers of the foliation. It can be shown that f is a Riemannian submersion. A set $N \subset \mathbb{R}^{n+1}$ is called *f-invariant* if there exists a set $C \subset (0, \infty) \times (0, \pi/g)$ such that $N = f^{-1}(C)$.

Using [PT86, Theorem 4], it is proved in [Rie23, Proposition 2.4] that an f-invariant hypersurface $N \subset \mathbb{R}^{n+1}$ is a closed self-shrinker if and only if $C := f(N)$ is a closed geodesic in $(0,\infty) \times (0,\pi/g)$ with respect to the metric h defined by

$$h(r,\phi) := \mathrm{Vol}_{g_{\mathbb{B}}}(f^{-1}(r,\varphi))^2 g_{\mathbb{S}}(r,\phi). \tag{3.10}$$

Note that for any $(r,\varphi) \in (0,\infty) \times (0,\pi/g)$, we have by Lemma 3.1

$$\begin{aligned}\mathrm{Vol}_{g_{\mathbb{B}}}(f^{-1}(r,\varphi)) &= \mathrm{Vol}_{g_{\mathbb{B}}}(rM_\varphi)\\ &= e^{-\frac{(n-1)r^2}{4n}}\mathrm{Vol}_0(rM_\varphi)\\ &= r^{n-1}e^{-\frac{(n-1)r^2}{4n}}\mathrm{Vol}_0(M_\varphi)\\ &= c_{g,m}r^{n-1}e^{-\frac{(n-1)r^2}{4n}}\sin^m(g\varphi).\end{aligned}$$

Using (3.8) we hence obtain the following expression for the metric h[1]

$$h(r,\phi) = c_{g,m}^2 r^{2mg} e^{-r^2/2}\sin{(g\varphi)}^{2m}(dr^2 + r^2 d\varphi^2). \tag{3.11}$$

By a direct calculation, the Gaussian curvature $K = K_{g,m}$ of h is given by

$$K = \frac{e^{r^2/2}}{c_{g,m}^2 r^{2n}\sin{(g\varphi)}^{2m}}\left(r^2 + \frac{(n-1)g}{\sin^2(g\varphi)}\right).$$

It is clear that K is strictly positive and that $K \to +\infty$ as (r,φ) tend to the boundary of $(0,\infty) \times (0,\pi/g)$. By simple calculus, the Gaussian curvature K is bounded from below by $\kappa = \kappa_{g,m} > 0$, where

$$\kappa_{g,m} = \frac{e^{\frac{y_{g,m}}{2}}}{c_{g,m}^2 y_{g,m}^n}(y_{g,m} + (n-1)g) \tag{3.12}$$

and

$$y_{g,m} = \frac{(n-1)(2-g) + \sqrt{(g-2)^2(n-1)^2 + 8ng(n-1)}}{2}. \tag{3.13}$$

Moreover, by (3.10), (3.3) and (2.2), the length $L_h(C)$ of the curve C with

[1]In equation (4) of [Rie23], the metric has the term e^{-r^2} instead of $e^{-r^2/2}$ since they use a different scaling convention for self-shrinkers.

respect to the metric h is given by

$$\begin{aligned} L_h(C) &= \int_C \mathrm{Vol}_{g_{\mathbb{B}}}(f^{-1}(r,\varphi))d\mu_C(r,\phi) \\ &= \mathrm{Vol}_{g_{\mathbb{B}}}(f^{-1}(C)) \\ &= \mathrm{Vol}_{g_{\mathbb{B}}}(N) \\ &= \int_N e^{-\frac{\|x\|^2}{4}} d\mu_N. \end{aligned} \tag{3.14}$$

Where $d\mu_C$ is the volume form of C in $((0,\infty)\times(0,\pi/g), g_{\mathrm{S}})$.

Proof of Theorem 1.1. For each pair (g,m) let

$$E_{g,m} = \frac{2\pi}{(4\pi)^{n/2}\sqrt{\kappa_{g,m}}}. \tag{3.15}$$

Let N be a f-invariant embedded self-shrinker in $\mathbb{R}^{n+1}$. Then $C := f(N)$ is the image of an embedded geodesic in $(0,\infty)\times(0,\pi/g)$ with metric h given in (3.10). Note that N is diffeomorphic to $\mathbb{S}^1\times M$ if and only if $f(N)$ is an embedded closed geodesic. Hence we can apply [MMM22, Theorem 3.1] to conclude that

$$L_h(C) \leq \frac{2\pi}{\sqrt{\kappa_{g,m}}}.$$

Together with (3.14) and Lemma 2.1, one obtains the entropy bound $\lambda(N) \leq E_{g,m}$. □

Remark 3.1. When $g = 1$, the isoparametric foliation of $\mathbb{S}^n$ is given (up to congruence) by a family of $(n-1)$-spheres

$$M_\varphi = \{\cos\varphi\}\times \sin\varphi\ \mathbb{S}^{n-1}, \quad \varphi\in(0,\pi) \tag{3.16}$$

and f-invariant hypersurfaces are percisely the rotationally symmetric hypersurfaces. In [MMM22], the authors and Møller derived entropy bounds for complete embedded rotationally symmetric self-shrinkers. We remark that the entropy bounds found in [MMM22] are a special case of the ones obtained here: when $g = 1$, then $m = n-1$ and we obtain from (3.15) that

$$E_{1,n-1} = \frac{2\pi}{(4\pi)^{n/2}\sqrt{\kappa_{1,n-1}}}.$$

From (3.12) and κ_n found in [MMM22, section 3], we have

$$\sqrt{\kappa_{1,n-1}} = \frac{1}{c_{1,n-1}}\sqrt{\kappa_n}.$$

Together with the identity $\omega_n = s(n-1)\omega_{n-1}$, (3.7) and the expression of E_n in [MMM22, equation (3.1)], we have $E_{1,n-1} = E_n$ for all $n \geq 2$.

4 Length upper bound on immersed closed geodesics in Riemannian surfaces with positive curvature

In this section we prove Theorem 4.8, which gives an upper bound on the length of closed immersed geodesics in a simply connected Riemannian surface with Gaussian curvature bounded below by a strictly positive constant κ. We start with a short outline of the section.

An immersed closed geodesic σ encloses a number of domains. In Proposition 4.4 we determine the number of these domains in terms of the number of self-intersection points of σ. The proof of Theorem 4.8 then reduces to proving an estimate of the length of the boundary of each domain. This is done by showing that the domains are complete length spaces of curvature $\geq \kappa$, for which one can prove an analogous result as in [MMM22, Theorem 3.1]. The necessary technical details are provided in Lemma 4.5 and Proposition 4.6.

Let $\mathbb{V}$ be a Riemannian 2-manifold homeomorphic to $\mathbb{R}^2$.

Definition 4.1. *Let $\gamma : \mathbb{S}^1 \to \mathbb{V}$ be an immersed closed curve. Let* Image(γ) *denote the image of γ. A point p in* Image(γ) *is called a self-intersection point if $\gamma^{-1}(p)$ has more than one element. A self-intersection point p is transverse if for all $t, s \in \gamma^{-1}(p)$, the pair $\gamma'(s), \gamma'(t)$ are not parallel to each other. Let S be the set of all self-intersection points of γ and let $k \in \mathbb{N}$. We say that γ has k self-intersection points counted with multiplicity if*

$$\sum_{p\in S}(|\gamma^{-1}(p)| - 1) = k. \tag{4.1}$$

Definition 4.2. *Let γ be an immersed closed curve in $\mathbb{V}$ with only transverse self-intersections. A pre-compact connected component of* $\mathbb{V}\backslash$Image(γ) *is called a domain enclosed by γ.*

Proposition 4.3. *Let $\gamma : \mathbb{S}^1 \to \mathbb{V}$ be an immersed closed curve in $\mathbb{V}$ with only transverse self-intersections. Then the boundary ∂U of each domain U enclosed by γ is piece-wise smooth and at each corner of ∂U, the interior angle is less than π.*

Proof. It is clear that the boundary of U is piece-wise smooth. At each corner, the interior angle θ is not π since γ has transverse self-intersections. If $\theta > \pi$, then there is a sub-arc β of γ which lies in the interior of U and β is not part of ∂U. Let β be the boundary of a domain $\tilde{U}$ enclosed by γ. But since β lies in the interior of U, we must have $U = \tilde{U}$, which is a contradiction that β is not part of ∂U. Hence we have $\theta < \pi$. □

Proposition 4.4. *Let γ be an immersed closed curve in $\mathbb{V}$ with k transverse self-intersections counted with multiplicity. Then γ encloses exactly $k+1$ domains.*

Proof. Let $U_1, \cdots, U_f$ be the domains enclosed by γ. For every $j = 1, \cdots, f$ the closure of the domain U_j has a piece-wise smooth boundary consisting of subarcs of γ. Let $\alpha_1, \cdots, \alpha_e$ be the collection of those sub-arcs. Let $S = \{p_1, \cdots, p_\ell\}$ be the set of self-intersection points of γ and let

$$W = \overline{U_1 \cup \cdots \cup U_f}$$

. Since W is simply connected, the Euler formula implies

$$\chi(W) = 1 = \ell - e + f, \tag{4.2}$$

where $\chi(W)$ is the Euler characteristic of W. Let $s_i = |\gamma^{-1}(p_i)|$. By definition there are $2s_i$ sub-arcs with vertices p_i (in the case where there is a loop, i.e., a sub-arc which starts and ends at p_i, that sub-arc would be counted twice). Since each sub-arc has exactly two vertices (also counted with multiplicity), we have

$$e = \sum_{i=1}^{\ell} s_i$$

and hence

$$f = 1 + e - \ell = 1 + \sum_{i=1}^{\ell}(s_i - 1) = 1 + k \tag{4.3}$$

by (4.1). This finishes the proof of the proposition. □

Let h be a Riemannian metric on $\mathbb{V}$ such that the Gaussian curvature K satisfies $K \geq \kappa > 0$ for some positive constant κ, and let σ be an immersed closed geodesic in $(\mathbb{V}, h)$. By the uniqueness theorem of solutions to ODE, any self-intersection point of σ must be transverse. We shall now describe the type of domains $U \subset \mathbb{V}$ that can be realized as the domains enclosed by σ.

Let U be a precompact, connected open subset in $(\mathbb{V}, h)$ and let $\Omega := \overline{U} \subset \mathbb{V}$. Assume further that the boundary $\partial\Omega$ is parameterized by a curve $\beta : \mathbb{S}^1 \to (\mathbb{V}, h)$ consisting of piece-wise smooth geodesic arcs, and which furthermore satisfies the following properties: identifying $\mathbb{S}^1$ with $[0, d]$, with 0 and d identified, then there is a partition

$$\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_a = d(= 0)\}$$

so that

$$\begin{cases} \beta|_{\mathbb{S}^1 \setminus \mathscr{P}} \text{ is injective,} \\ \text{for each } j = 0, \cdots, a-1, \ \beta|_{[t_j, t_{j+1}]} \text{ is a geodesic in } (\mathbb{V}, h), \text{ and} \\ \text{for each } j = 0, \cdots, a-1 \text{ the interior angle at } q_j := \beta(t_j) \text{ is less than } \pi. \end{cases} \tag{4.4}$$

We remark that if β is injective, then each vertex q_j is joined by only two geodesic arcs $\beta|_{[t_{j-1}, t_j]}$, $\beta|_{[t_j, t_{j+1}]}$.

Assume first that $\partial\Omega$ is parametrized by an injective curve β which satisfies (4.4). Using Lemma 4.5 below, we will define as in [MMM22, Section 3] a metric structure d^Ω on Ω using the Riemannian metric h and show that (Ω, d^Ω) is a complete length space with curvature $\geq \kappa$, where κ is the strictly positive lower bound of the Gaussian curvature of $(\mathbb{V}, h)$. See [BBI01] or [MMM22, Section 3] for the terminology and notation needed from metric geometry.

For each $p, q \in \Omega$, let

$$d^\Omega(p, q) = \inf_\gamma L_h(\gamma),$$

where the infimum is taken among piece-wise C^1 curves $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = p$, $\gamma(1) = q$ and

$$L_h(\gamma) = \int_0^1 \sqrt{h(\gamma'(t), \gamma'(t))}dt \tag{4.5}$$

is the length of γ in $(\mathbb{V}, h)$. The following lemma is proved in [MMM22, Lemma 3.3] when $\partial\Omega$ has no corners.

Lemma 4.5. *Let Ω be as above, i.e. a compact connected set in $(\mathbb{V}, h)$ so* *that the boundary $\partial\Omega$ is parametrized by an injective curve β which satisfies (4.4). Let $p, q \in \Omega$. Then there is a simple geodesic γ in Ω joining p and q which is shortest among all piece-wise C^1 curves in Ω joining p and q. Moreover,*

(i) *$\gamma(t) \notin \{q_1, \cdots, q_a\}$ for all $t \in (0,1)$.*

(ii) *if one of p, q is in the interior of Ω, then γ also lies in the interior of Ω (except possibly at the other end point),*

(iii) *if both p, q are in $\mathrm{Image}(\beta) = \partial\Omega$, then either γ lies completely in the image of $\beta|_{[t_j,t_{j+1}]}$ for some $j = 0, \cdots, a-1$, or the interior of γ lies inside the interior of Ω.*

Proof. The Lemma follows from the same arguments used in the proof of [MMM22, Lemma 3.3]. We only sketch the minor modifications here.

Let $p, q \in \Omega$, $p \neq q$. As in the proof of [MMM22, Lemma 3.3], using a minimizing sequence we construct a Lipschitz length minimizing curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = p$, $\gamma(1) = q$, and we may assume that $\gamma^{-1}(q) = \{1\}$, $\gamma^{-1}(p) = \{0\}$.

First we show (i) by contradiction: if not, let $s_0 \in (0,1)$ be the first time such that $\gamma(s_0) = q_j$ for some j. Let B be the closed geodesic ball of radius ϵ in $(\mathbb{V}, h)$ centered at q_j. For ϵ small enough, p, q are not in B. Hence there are $s_1, s_2 \in (0,1)$ so that $s_1 < s_0 < s_2$ and $\gamma(s_i) \in \partial B$ for $i = 1, 2$. Note that since $\beta|_{[t_{j-1},t_j]}$ and $\beta_{[t_j,t_{j+1}]}$ are both geodesics, under geodesic polar coordinates centered at q_j, the set $B \cap \Omega$ is of the form

$$\{(r,\theta) : 0 \le r \le \epsilon, \theta_1 \le \theta \le \theta_2\} \tag{4.6}$$

with $\theta_2 - \theta_1 < \pi$ by the third assumption in (4.4). Assuming ϵ is small enough so that g is nearly Euclidean in U. Together with $\theta_2 - \theta_1 < \pi$, the (Euclidean) straight line ℓ joining $\gamma(s_1)$, $\gamma(s_2)$ has length

$$L_h(\ell) < d_h(\gamma(s_1), q_j) + d_h(q_j, \gamma(s_2)) \le L_h(\gamma|_{[s_1,s_2]}).$$

Thus γ cannot be length minimizing in Ω and hence (i) is proved. Part (ii) and (iii) of the Lemma can be proven similarly as in the proof of [MMM22, Lemma 3.3] and are skipped. □

From Lemma 4.5 it follows that (Ω, d^Ω) is a complete length space.

Proposition 4.6. *Let Ω be as in Lemma 4.5. Then the complete length space (Ω, d^Ω) has curvature $\ge \kappa$, where κ is the lower bound of the Gaussian curvature of $(\mathcal{S}, h)$.*

Proof. Let $x \in \Omega$ and let V_x be a geodesically convex neighborhood in $(\mathbb{V}, h)$ centered at x. When x is in the interior of Ω, we choose $V_x \subset \Omega$. Let $U_x = \Omega \cap V_x$. For any $p, q \in U_x$, let γ be the unique shortest geodesic in V_x joining p and q. Arguing as in the proof of [MMM22, Proposition 3.6], it suffices to show that γ lies in U_x. The argument there works in our case when $x \notin \{q_1, \cdots, q_a\}$. For the case $x \in \{q_1, \cdots, q_a\}$, one may assume that V_x is of the form (4.6) for some $r_0 > 0$ and $\theta_2 - \theta_1 < \pi$. Arguing as in the proof of [MMM22, Proposition 3.6], one shows that γ lies in H_{θ_0} for all $\theta_0 \in [\theta_2, \theta_1 + \pi]$, where in geodesic polar coordinates

$$H_{\theta_0} = \{(r, \theta) : 0 \le r \le r_0, \theta_0 - \pi \le \theta \le \theta_0\}.$$

Since

$$U_x = \bigcap_{\theta_0 \in (\theta_2, \theta_1 + \pi)} H_{\theta_0},$$

one concludes that γ lies in U_x. □

Using exactly the same proof, by approximating the boundary of Ω by geodesic polygons, one has the following generalization of [MMM22, Theorem 3.1].

Theorem 4.7. *Let Ω be as in Lemma 4.5. Then the boundary $\partial\Omega$ satisfies*

$$|\partial\Omega| := L_h(\beta) \le \frac{2\pi}{\sqrt{\kappa}}. \tag{4.7}$$

Now we are ready to prove the main theorem in this section.

Theorem 4.8. *Let σ be an immersed geodesic loop in $(\mathbb{V}, h)$ with k self-intersection points counted with multiplicity. Then the length $L_h(\sigma)$ of σ satisfies*

$$L_h(\sigma) \le (k+1)\frac{2\pi}{\sqrt{\kappa}}. \tag{4.8}$$

Proof. By Proposition 4.4, the immersed geodesic loop σ encloses $k+1$ domains, denote the closure of these domains by $\Omega_1, \cdots, \Omega_{k+1}$. Fix each $i = 1, \cdots, k+1$ and write $\Omega = \Omega_i$ for simplicity. The boundary $\partial\Omega$ of Ω is parametrized by a closed curve $\beta : \mathbb{S}^1 \to (\mathbb{V}, h)$ consisting of piece-wise smooth geodesic arcs which satisfies (4.4), but which may not be injective. Let q be any vertex of $\partial\Omega$ (i.e. $q = \beta(t_j)$ for some $j = 0, \cdots, a$). Let $s = |\beta^{-1}(q)|$. For any small $\epsilon > 0$, let $B = B_\epsilon(q)$ be the closed geodesic ball in $(\mathbb{V}, h)$ centered at q with radius ϵ. Then β intersects ∂B at $2s$ points, given by $\exp_q(\epsilon e^{i\theta_0}), \cdots, \exp_q(\epsilon e^{i\theta_{2s-1}})$ in geodesic polar coordinates (r, θ) centered at q. Moreover, we have

$$\Omega \cap B = \{\exp_q(re^{i\theta}) : 0 \leq r \leq \epsilon, \quad \theta_{2m} \leq \theta \leq \theta_{2m+1}, m = 0, \cdots, s-1\}.$$

For each m, let β_m be the shortest geodesic in B joining $\exp_q(\epsilon e^{i\theta_{2m}})$ and $\exp_q(\epsilon e^{i\theta_{2m+1}})$. Arguing as in the proof of Proposition 4.6, β_m also lies inside Ω. Let Δ_m be the geodesic triangle in Ω with vertices q, $\exp_q(\epsilon e^{i\theta_{2m}})$ and $\exp_q(\epsilon e^{i\theta_{2m+1}})$. Let D_ϵ be the collection of all such triangles constructed for all vertices of $\partial\Omega$. Let $\Omega_\epsilon = \Omega \setminus D_\epsilon$. Note that the interior of Ω is homeomorphic to the interior of Ω_ϵ. Hence Ω_ϵ is connected. Moreover, the boundary $\partial\Omega_\epsilon$ is parametrized by an injective curve consisting of piece-wise smooth geodesic arcs which satisfies (4.4). Thus we can apply Theorem 4.7 to conclude

$$|\partial\Omega_\epsilon| \leq \frac{2\pi}{\sqrt{\kappa}}.$$

Taking $\epsilon \to 0$, we obtain

$$|\partial\Omega| \leq \frac{2\pi}{\sqrt{\kappa}}$$

and hence

$$L_h(\sigma) \leq \sum_{i=1}^{k+1} |\partial\Omega_i| \leq (k+1)\frac{2\pi}{\sqrt{\kappa}}.$$

□

5 Entropy bounds for immersed non-spherical closed self-shrinkers

We recall that for any immersed curve $\sigma : I \to \mathbb{H}$, where $\mathbb{H}$ is the upper half space, the immersed rotationally symmetric hypersurface Σ_σ in $\mathbb{R}^{n+1}$ with profile curve $\sigma(s) = (x(s), r(s))$ is given by

$$\Sigma_\sigma := \{(x(s), r(s)\omega) : s \in I, \ \omega \in \mathbb{S}^n\}.$$

Now we prove Theorem 1.2.

Proof of Theorem 1.2. By [MMM22, Proposition 2.3], Σ_σ is a self-shrinker if and only if σ is a geodesic in $(\mathbb{H}, g_A)$, where g_A is given by

$$g_A = r^{2(n-1)} e^{-\frac{x^2+r^2}{2}} (dx^2 + dr^2). \tag{5.1}$$

Since Σ_σ is compact, either $I = \mathbb{S}^1$ and Σ_σ is an immersion from $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Or $I = (a, b)$, $\sigma(a^+), \sigma(b^-)$ lie in $\partial\mathbb{H}$ and Σ_σ is an immersion from $\mathbb{S}^n$. Since Σ_σ is non-spherical, the latter case is excluded and thus σ is a closed immersed geodesic. By Theorem 4.8, the length of σ satisfies

$$L(\sigma) \leq (k+1)\frac{2\pi}{\sqrt{\kappa_n}},$$

where κ_n is the lower bound of the Gaussian curvature of g_A computed in [MMM22, section 3]. Together with

$$\lambda(\Sigma_\sigma) = (4\pi)^{-n/2}\omega_{n-1}L_A(\sigma) \tag{5.2}$$

and the numbers E_n from [MMM22, equation (3.1)], given by

$$E_n = \frac{2\pi\omega_{n-1}}{(4\pi)^{n/2}\sqrt{\kappa_n}}$$

we obtain (1.2). □

Remark 5.1. As in the proof of Theorem 1.2, one can use Theorem 4.8 to prove an entropy upper bound for the class of f-invariant closed self-shrinkers N such that $f(N)$ is a closed immersed geodesic with k self-intersections in $(0, \infty) \times (0, \pi/g)$ with respect to the metric h given in (3.11). However, as of now no such examples are known to the authors' knowledge.

References

[AL86] Uwe Abresch and J. Langer. The normalized curve shortening flow and homothetic solutions. *J. Differ. Geom.*, 23:175–196, 1986.

[Ang92] Sigurd B. Angenent. Shrinking doughnuts. In *Nonlinear diffusion equations and their equilibrium states, 3. Proceedings from the third conference, held August 20-29, 1989 in Gregynog, Wales, United Kingdom*, pages 21–38. Boston, MA etc.: Birkhäuser, 1992.

[BBI01] D. Burago, Yu. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2001.

[BK21] Yakov Berchenko-Kogan. The entropy of the angenent torus is approximately 1.85122. *Exp. Math.*, 30(4):587–594, 2021.

[BNS21] Reto Buzano, Huy The Nguyen, and Mario B Schulz. Noncompact self-shrinkers for mean curvature flow with arbitrary genus. *arXiv preprint arXiv:2110.06027*, 2021.

[BW16] Jacob Bernstein and Lu Wang. A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. *Invent. Math.*, 206(3):601–627, 2016.

[CHH22] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits. Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness. *Acta Math.*, 228(2):217–301, 2022.

[CM12a] Tobias H. Colding and William P. II Minicozzi. Generic mean curvature flow. I: Generic singularities. *Ann. Math. (2)*, 175(2):755–833, 2012.

[CM12b] Tobias H. Colding and William P. II Minicozzi. Smooth compactness of self-shrinkers. *Comment. Math. Helv.*, 87(2):463–475, 2012.

[CR15] Thomas E. Cecil and Patrick J. Ryan. *Geometry of hypersurfaces*. Springer Monogr. Math. New York, NY: Springer, 2015.

[DK17] Gregory Drugan and Stephen J. Kleene. Immersed self-shrinkers. *Trans. Am. Math. Soc.*, 369(10):7213–7250, 2017.

[DN18] Gregory Drugan and Xuan Hien Nguyen. Shrinking doughnuts via variational methods. *J. Geom. Anal.*, 28(4):3725–3746, 2018.

[GN21] Harald Garcke and Robert Nürnberg. Numerical approximation of boundary value problems for curvature flow and elastic flow in Riemannian manifolds. *Numer. Math.*, 149(2):375–415, 2021.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differ. Geom.*, 31(1):285–299, 1990.

[Ilm94] Tom Ilmanen. *Elliptic regularization and partial regularity for motion by mean curvature*, volume 520 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 1994.

[KKM18] Nikolaos Kapouleas, Stephen James Kleene, and Niels Martin Møller. Mean curvature self-shrinkers of high genus: non-compact examples. *J. Reine Angew. Math.*, 739:1–39, 2018.

[KM23] Nikolaos Kapouleas and Peter McGrath. Generalizing the linearized doubling approach. I: General theory and new minimal surfaces and self-shrinkers. *Camb. J. Math.*, 11(2):299–439, 2023.

[MM09] Annibale Magni and Carlo Mantegazza. Some remarks on Huisken's monotonicity formula for mean curvature flow. In *Singularities in nonlinear evolution phenomena and applications. Proceedings of the workshop, Pisa, Italy, May 2008*, pages 157–169. Pisa: Edizioni della Normale, 2009.

[MMM22] John Man Shun Ma, Ali Muhammad, and Niels Martin Møller. Entropy bounds, compactness and finiteness theorems for embedded self-shrinkers with rotational symmetry. *J. Reine Angew. Math.*, 793:239–259, 2022.

[Mra21] Alexander Mramor. On self shrinkers of medium entropy in $\mathbb{R}^4$. *arXiv preprint arXiv:2106.10243, (to appear in Geometry and Topology)*, 2021.

[Mün80] Hans Friedrich Münzner. Isoparametrische Hyperflächen in Sphären. I. *Math. Ann.*, 251:57–71, 1980.

[Mün81] Hans Friedrich Münzner. Isoparametrische Hyperflächen in Sphären. II: Über die Zerlegung der Sphäre in Ballbündel. *Math. Ann.*, 256:215–232, 1981.

[Ngu14] Xuan Hien Nguyen. Construction of complete embedded self-similar surfaces under mean curvature flow. III. *Duke Math. J.*, 163(11):2023–2056, 2014.

[Nic11] Liviu I Nicolaescu. The coarea formula. In *seminar notes. Citeseer*, 2011.

[PT86] Richard S. Palais and Chuu-Lian Terng. Reduction of variables for minimal submanifolds. *Proc. Am. Math. Soc.*, 98:480–484, 1986.

[Rie23] Oskar Riedler. Closed embedded self-shrinkers of mean curvature flow. *J. Geom. Anal.*, 33(6):27, 2023. Id/No 172.

[Sto94] Andrew Stone. A density function and the structure of singularities of the mean curvature flow. *Calc. Var. Partial Differ. Equ.*, 2(4):443–480, 1994.

John Man Shun Ma
Department of Mathematics, Southern University of Science and Technology
No 1088, xueyuan Rd., Xili, Nanshan District, Shenzhen, Guangdong, China
518055
email: hunm@sustech.edu.cn

Ali Muhammad
Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
email: ajhm@math.ku.dk