

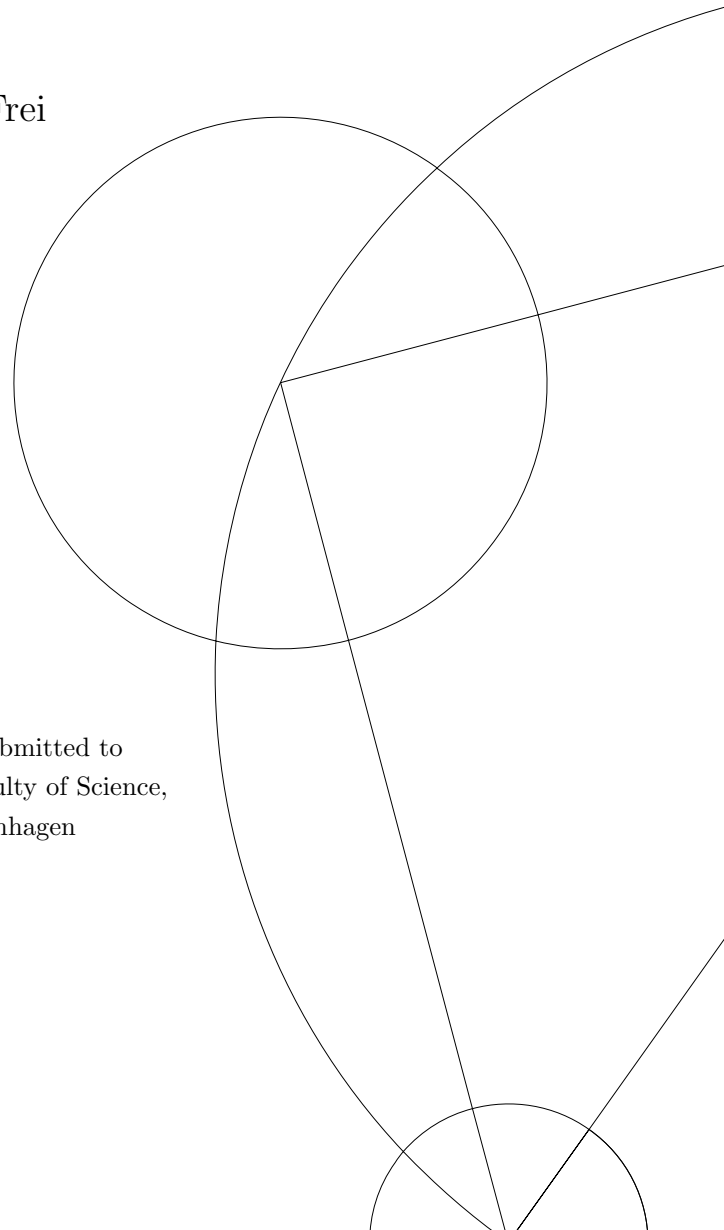


---

# Operator algebras and nonlocal games: The quantum commuting model: Optimal states and self-testing

Alexander Frei

This thesis has been submitted to  
the PhD School of the Faculty of Science,  
University of Copenhagen



**Alexander Frei**

Department of Mathematical Sciences

University of Copenhagen

frei.shahiri@gmail.com

PhD supervisor:

**Søren Eilers**

University of Copenhagen

Submission date:

January 15, 2023

Defence date:

February 20, 2023

Assessment Committee:

**Laura Mancinska** (chair)

University of Copenhagen

**Vern I. Paulsen**

University of Waterloo

**Evgenios T.A. Kakariadis**

Newcastle University

ISBN 978-87-7125-065-7

©Alexander Frei

# Preface

## English abstract

This thesis consists of two main parts: Part I on relative Cuntz–Pimsner algebras, and part II on operator algebras and nonlocal games in quantum information.

Part I centers around the classification of relative Cuntz–Pimsner algebras from preprint [A](#): More precisely, the attached preprint gives a systematic classification of gauge-equivariant representations (and whence in turn also of gauge-invariant invariant ideals) and further unravels Katsura’s construction as a canonical dilation given by the maximal covariance.

We give a short summary on its main results in the first section.

As a first application we then classify pullbacks for relative Cuntz–Pimsner algebras and use these to provide examples for the failure of pullbacks for absolute Cuntz–Pimsner algebras. This section constitutes a summary of upcoming work with Piotr M. Hajac and Mariusz Tobolski.

The second application concerns Morita equivalence for relative Cuntz–Pimsner algebras: Using our classification we swiftly recover a classical result by Fowler–Muhly–Raeburn and reveal the failure for short exact sequences by relative Cuntz–Pimsner algebras. We then outline Morita equivalences arising from higher tensor powers of correspondences and those Morita equivalences for relative Cuntz–Pimsner algebras not arising by any finite tensor power at all. This section serves as an outlook on upcoming work by the author.

Part II centers around operator algebras and nonlocal games:

The first section constitutes a summary on “Connes implies Tsirelson” from preprint [B](#).

The following sections concentrate on computing quantum values using operator algebraic techniques and the classification of optimal states and their correlations. We begin for this with a summary on uniqueness of optimal states from preprint [C](#).

We then elaborate on genuine self-testing for their correlations based on order-two moments and follow with an outlook on robust self-testing for optimal states in the quantum commuting model. Both of these constitute ongoing joint work with Azin Shahiri.

We finish the second part with a representation–theoretic classification of optimal states and their quantum value for the tilted CHSH games based on an upcoming preprint with Azin Shahiri (currently under preparation).

This further serves as a primer on ongoing work around the I3322 inequality (from a new perspective) as well as on nonlocal games exhibiting a separation between finite dimensional and quantum spatial correlations using operator algebraic techniques.

## Danish resumé

Denne afhandling består af to hoveddele: Del I om relative Cuntz-Pimsner-algebraer og del II om operatoralgebraer og ikke-lokale spil i kvanteinformation.

Del I er centreret omkring klassifikationen af relative Cuntz-Pimsner-algebraer fra preprint [A](#): Mere præcist giver det vedlagte preprint en systematisk klassifikation af gauge-ækvivariante repræsentationer (og dermed også af gauge-invariante idealer) og yderligere optrævler Katsuras konstruktion som en kanonisk dilatation givet af den maksimale kovarians.

Vi giver et kort resumé af de vigtigste resultater i det første afsnit.

Som en første anvendelse klassificerer vi derefter pullbacks for relative Cuntz-Pimsner-algebraer og bruger dette til at opstille eksempler på, at pullbacks for absolutte Cuntz-Pimsner-algebraer ikke virker. Dette afsnit er et resumé af et kommende arbejde med Piotr M. Hajac og Mariusz Tobolski.

Den anden anvendelse vedrører Morita-ækvivalens for relative Cuntz-Pimsner-algebraer:

Ved hjælp af vores klassifikation kan vi hurtigt genfinde et klassisk resultat af Fowler–Muhly–Raeburn og afsløre, at kort-eksakte følger ikke er mulige ved relative Cuntz–Pimsner-algebraer. Derefter skitserer vi Morita-ækvivalenser, der opstår ved højere tensor-potenser af korrespondancer, og de Morita-ækvivalenser for relative Cuntz-Pimsner-algebraer, der ikke opstår ved nogen endelig tensor-potenser overhovedet. Dette afsnit tjener som en oversigt over forfatterens kommende arbejde.

Del II er centreret om operatoralgebraer og ikke-lokale spil:

Det første afsnit er et resumé af “Connes implies Tsirelson” fra preprint [B](#).

De følgende afsnit koncentrerer sig om beregning af kvanteværdier ved hjælp af operatoralgebraiske teknikker og klassificering af optimale tilstande og deres korrelationer. Vi begynder med et resumé om entydighed af optimale tilstande fra preprint [C](#).

Derefter uddyber vi ægte selvtestning af deres korrelationer baseret på ordre to-momenter og følger op med en oversigt over robust selvtestning af optimale tilstande i kvantekommuteringsmodellen. Begge dele udgør et igangværende fælles arbejde med Azin Shahiri.

Vi afslutter anden del med en repræsentationsteoretisk klassifikation af optimale tilstande og deres kvanteværdi for de vippede CHSH-spil baseret på et kommende preprint med Azin Shahiri (under forberedelse).

Dette tjener desuden som en grundbog om igangværende arbejde omkring I3322-uligheden (fra et nyt perspektiv) samt om ikke-lokale spil, der udviser en adskillelse mellem endelig-dimensionale og kvante rumlige korrelationer ved hjælp af operatoralgebraiske teknikker.

## Disclaimer

The main results of this thesis are given by [preprint A](#), [preprint B](#) and [preprint C](#).

The preceding sections serve as an overview and summary of these results, together with an outlook on upcoming and ongoing work.

Each of the attached preprints is **single authored by the PhD student**.

## Acknowledgements

First and foremost I would like to thank my supervisor Søren Eilers for his kind support and encouragements throughout the PhD as well as friendly assistance in so many situations in particular and for him being such a nice person overall. :) Similarly I would like to thank Ryszard Nest for being always there mathematically and personally listening and helping out always being there. For me I see both of you Søren Eilers and Ryszard Nest as a friend.

In the same way I would like to thank Mikael Rørdam and Magdalena Musat for so many very interesting conversations about group algebras and Kazhdan property (T) as well as most importantly for their introduction to the Connes embedding problem and the Kirchberg conjectures and to quantum information from this highly interesting perspective of operator algebras.

Further I would like to thank my buddy Jeroen van der Meer for such a nice and fun time together and in the same way I would of course also like to thank all my friends and colleagues Nanna Havn Aamand, Alexis Aumonier, David Bauer, Francesco Campagna and Luigi Pagano, Kaif Hilman, Severin Mejak, Henning Olaj Milhøj, Vignesh Subramanian and Jingxuan Zhang.

Further I would like to thank each person for some deep insightful piece I've learned from:

Jamie Gabe and Chris Schafhauser, Piotr M. Hajac and Mariusz Tobolski, Ralf Meyer, Bartosz Kwasniewski and Adam Skalski, Evgenios Kakariadis and Elias Katsoulis, Laura Mancinska, Jurij Volcic and Thor Gabelgaard Nielsen, Sam Harris, Mehrdad Kalantar and David Blecher, Vern Paulsen and Tobias Fritz, Roy Araiza and Travis Russell, William Slofstra and Connor Paddock, Mario Klisse and Mikkel Munkholm, Lara Ismert and Mitch Hamidi.

Finally and most importantly I would like to thank my lovely wife Azin Shahiri for her always smiling encouragements — I kiss you and I love you. :)

*To my lovely and always smiling wife  
Azin Shahiri*

# Contents

<b>Preface</b>	<b>3</b>
English abstract . . . . .	3
Danish resumé . . . . .	4
Acknowledgements . . . . .	5
Dedication . . . . .	6
<b>1 Relative Cuntz–Pimsner algebras: Classification and Applications</b>	<b>9</b>
1.1 Classification . . . . .	9
1.2 Pullbacks . . . . .	13
1.3 Morita equivalence . . . . .	16
<b>2 Operator algebras and quantum information: Quantum values and optimal states</b>	<b>19</b>
2.1 Connes implies Tsirelson: a simple proof . . . . .	19
2.2 Quantum commuting model: states vs. correlations . . . . .	23
2.3 Optimal states: uniqueness (including all higher moments) . . . . .	23
2.4 Correlations: genuine self-testing (order-two moments) . . . . .	27
2.5 Optimal states: robustness behavior . . . . .	28
2.6 Tilted CHSH games: a representation–theoretic classification . . . . .	29
2.7 I3322 inequality: an operator–algebraic approach . . . . .	32
<b>A Relative Cuntz–Pimsner algebras: Classification of gauge-equivariant representations: a simple and complete picture</b>	<b>33</b>
A.1 Correspondences . . . . .	34
A.2 Representations . . . . .	41
A.3 Kernel and Covariance . . . . .	45
A.4 Relative Cuntz–Pimsner algebras . . . . .	66
A.5 Gauge actions: Fourier spaces . . . . .	71

## Contents

---

A.6 Uniqueness theorem . . . . .	77
A.7 Fock representation . . . . .	83
A.8 Lattice structure . . . . .	92
A.9 Pimsner dilations . . . . .	105
<b>B Connes implies Tsirelson: a simple proof</b>	<b>118</b>
B.1 Player algebra . . . . .	118
B.2 Connes algebra . . . . .	119
B.3 Ingredients . . . . .	122
B.4 Connes $\implies$ Tsirelson . . . . .	126
<b>C The quantum commuting model (Ia): Uniqueness of optimal states: The CHSH game and other examples</b>	<b>128</b>
C.1 Stinespring dilation . . . . .	128
C.2 Two-player algebra . . . . .	132
C.3 CHSH game . . . . .	133
C.4 Mermin–Peres games . . . . .	139
C.5 Pauli algebra . . . . .	145
C.6 XOR-games . . . . .	146
<b>Bibliography</b>	<b>150</b>



## Relative Cuntz–Pimsner algebras: Classification and Applications

### 1.1 Classification

We begin with an **overview on the main results** from preprint [A](#):

The main investigation on the classification begins with the characterisation of kernel and cokernel morphisms. We find that each of such are determined precisely by invariant ideals whence every short exact sequence reads

$$0 \longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow 0.$$

Based on those we suggest a parametrisation by kernel–covariance pairs describing relative Cuntz–Pimsner algebras given by 1) some generic invariant ideal (as possible kernel ideal) and 2) another bounded ideal (as possible covariance)

$$\left( X^*KX \subseteq K \mid I \subseteq \max(X) \right).$$

As such we obtain an entire lattice of kernel–covariance pairs of the form

$$\begin{array}{ccccc}
 X & \longrightarrow & \frac{X}{XK} & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow \\
 \frac{X}{XL} & \longrightarrow & \dots & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \dots & \longrightarrow & X = 0
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccccc}
 \mathcal{T}(K) = \mathcal{O}(K, 0) & \longrightarrow & \mathcal{O}(K, I) & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{O}(K, J) & \longrightarrow & \dots & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(K, \max) = \mathcal{O}(K).
 \end{array}$$

We then exploit these in the following sections to narrow down the question of representations to those correspondences faithfully embedded within ambient operator algebras. As a first instance we derive the general version for the gauge-invariant uniqueness theorem:

**Theorem 1.1.1** (Gauge-invariant uniqueness theorem: The general version):

*Every gauge-equivariant representation arises as a relative Cuntz–Pimsner algebra*

$$(K, I) = \left( \ker(A \rightarrow B) \mid \operatorname{cov}\left(\frac{X}{XK} \rightarrow B\right) \right) : \quad (X, A) \longrightarrow \mathcal{O}(K, I) \longlongequal{\quad} C^*(X \cup A) \subseteq B$$

*and as such arises as the range of kernel–covariance pairs.*

We then move on to the realisation of relative Cuntz–Pimsner algebras. This arises as a classical construction using the Fock representation and its quotients. We then seek to recover kernel–covariance pairs using their concrete representation which culminates in the following result:

**Theorem** (Relative Cuntz–Pimsner algebras: Kernel and Covariance):

## 1.1. Classification

---

For relative Cuntz–Pimsner algebra it holds

$$\begin{aligned} \ker \left( A \rightarrow \mathcal{T} \rightarrow \mathcal{O}(X; I) \right) &= 0 \\ \text{cov} \left( X \rightarrow \mathcal{T} \rightarrow \mathcal{O}(X; I) \right) &= I \end{aligned}$$

As a consequence the kernel–covariance pairs are classifying.

As such we found as desired the parametrisation in terms of kernel–covariance for the entire class of gauge-equivariant representations (and so also of gauge-invariant ideals). We then verify in the following section that our parametrisation by kernel–covariance pairs further defines an order isomorphism with the lattice of gauge-equivariant representations with its natural order:

**Theorem** (Kernel–covariance pairs: Order isomorphism):

The lattice kernel–covariance pairs defines an order isomorphism

$$\left( K \subseteq L \mid I + K \subseteq J + L \right) \iff \mathcal{O}(K, I) \leq \mathcal{O}(L, J)$$

and as such the lattice of **kernel–covariance pairs** with its natural order by inclusion describes the entire **lattice of gauge-equivariant** representations (and so also of gauge-invariant ideals).

Meanwhile we discuss the urging question of possible connecting morphisms between relative Cuntz–Pimsner algebras from different kernel strands (using our found parametrisation):

$$\begin{array}{ccccccc} (X, A) & \longrightarrow & \left( \frac{X}{XK_1}, \frac{A}{K_1} \right) & \longrightarrow & \left( \frac{X}{XK_2}, \frac{A}{K_2} \right) & \longrightarrow & \left( \frac{X}{XK}, \frac{A}{K} \right) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & & \mathcal{O}(K_1, I_1) & \overset{\times}{\dashrightarrow} & \mathcal{O}(K_2, I_2) & \longrightarrow & \mathcal{O}(K, I) \\ \downarrow & & \downarrow & \searrow & \downarrow & \nearrow & \downarrow \\ \dots & & \dots & & \dots & & \dots \end{array}$$

In particular we investigate the question on the minimal quotient

$$\begin{array}{ccccccc} \mathcal{T}\left(\frac{X}{XK}\right) = \mathcal{O}(K, 0) & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(K, I = ?) & \longrightarrow & \dots \\ \downarrow & \searrow & & \searrow & \downarrow & \nearrow & \\ \mathcal{T}\left(\frac{X}{XL}\right) = \mathcal{O}(L, 0) & \longrightarrow & & \longrightarrow & \mathcal{O}(L, J) & \longrightarrow & \mathcal{O}(L, \max) \end{array}$$

as well as the more interesting one on the existence

$$\begin{array}{ccccc} \mathcal{O}(K, 0) & \longrightarrow & \mathcal{O}(K, I) & \longrightarrow & \mathcal{O}(K, \max) \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{O}(L = ?, 0) & \longrightarrow & \mathcal{O}(L = ?, J = ?) & \longrightarrow & \mathcal{O}(L = ?, \max). \end{array}$$

We round off this section with plenty of examples describing various scenarios on possible lattices arising in nature and the existence and non-existence of connecting morphisms between gauge-

## 1.1. Classification

---

equivariant representations using our found parametrisation such as for instance

$$\begin{array}{ccccc}
 \mathcal{T}\left(X = \ell^2(a \rightarrow b)\right) & \longrightarrow & \mathcal{T}\left(X = \ell^2(b)\right) & \longrightarrow & \mathcal{T}(X = 0) \\
 \downarrow & & \parallel & \nearrow & \parallel \\
 \mathcal{O}\left(X = \ell^2(a \rightarrow b)\right) & \dashrightarrow & \mathcal{O}\left(X = \ell^2(b)\right) & & \mathcal{O}(X = 0).
 \end{array}$$

All of these examples serve to illustrate the ease of working with kernel–covariance pairs.

In particular this work on kernel–covariance pairs forms the basis on upcoming work.

We then move on to the final section as an application on Pimsner dilations:

We showcase here the ease with which we obtain various technical constructions as trivial consequences using kernel–covariance pairs. As a first such we verify the maximal dilation:

**Theorem** (Maximal dilation: absolute Cuntz–Pimsner algebra):

*The maximal dilation realises relative Cuntz–Pimsner algebras as an absolute one*

$$\mathcal{O}(K, I) = \mathcal{O}\left(Y = \mathcal{O}(K, I)(1) \mid B = \mathcal{O}(K, I)(0)\right).$$

*and further defines the maximal Hilbert bimodule.*

Following from the maximal dilation (as an object by abstract existence) we move on to seek dilations as an object small enough to allow for some combinatorial description based on the original correspondence. We provide for this an algorithm to generally produce combinatorial dilations (which may however never halt). As a positive example we apply this procedure on the maximal covariance for which the algorithm halts after its first cycle. As such the algorithm produces a suitable dilation (which we term Katsura dilation for reason further below):

**Theorem** (Katsura dilation: absolute Cuntz–Pimsner algebra):

*The canonical dilation given by the maximal covariance realises a relative Cuntz–Pimsner algebra as an absolute Cuntz–Pimsner algebra*

$$\mathcal{O}(K = 0, I) = \mathcal{O}\left(Y = X + X\left(\begin{smallmatrix} \max(X, A) \\ 0 \end{smallmatrix}\right) \mid B = A + \left(\begin{smallmatrix} \max(X, A) \\ 0 \end{smallmatrix}\right)\right)$$

*and the analogous dilation for kernel–covariance pairs with kernel ideal.*

*This dilation may well fail to define a minimal dilation (detecting covariance) and even if minimal, it generally fails to be the only minimal dilation.*

We then provide an intrinsic description for above dilation which further reveals Katsura’s construction as nothing but above dilation given by the maximal covariance in disguise:

**Corollary** (Katsura dilation: intrinsic description):

*The canonical dilation given by the maximal covariance allows the intrinsic description as the operator algebra freely generated by their abstract copies*

$$\left(A = A\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right) \cup \left(\max(X, A)/I = \left(\begin{smallmatrix} \max(X, A) \\ 0 \end{smallmatrix}\right)\right)$$

## 1.1. Classification

---

with multiplication given by

$$A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} A \max(X, A) & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix}$$

and similarly for the correspondence itself.

The analogous expression holds for kernel–covariance pairs with kernel ideal.

This further recovers the particular description from [Kat07, definition 6.1].

We then identify our dilation given by the maximal covariance as a familiar construction realising any gauge-equivariant quotient as a graph algebra itself (by outsplitting):

**Corollary** (Katsura dilation: graph correspondences):

*The canonical dilation given by the maximal covariance arises as the canonical graph:*

*Its coefficient algebra arises as the orthogonal sum of vertices*

$$W = \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} \cup \begin{pmatrix} \text{regular} \setminus R & \\ & 0 \end{pmatrix}$$

together with the correspondence given by the graph

$$EW = E \left( \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} \cup \begin{pmatrix} \text{regular} \setminus R & \\ & 0 \end{pmatrix} \right)$$

and its left action given by (whence defining the range of edges)

$$\begin{pmatrix} a & \\ & a \end{pmatrix} EW = (aE)W \quad \text{and} \quad \begin{pmatrix} 0 & \\ & b \end{pmatrix} EW = \begin{pmatrix} b & \\ & b \end{pmatrix} EW = (bE)W$$

while trivially acting for the left over last summand. As such this recovers the familiar construction for graph algebras: Any gauge-equivariant quotient arises as a graph algebra itself.

As an example we find the following **realisation for the Toeplitz algebra**

$$\mathcal{T} \left( \begin{array}{c} \curvearrowright \\ \bullet \end{array} \right) = \mathcal{O} \left( \begin{array}{c} \curvearrowright \\ \bullet \leftarrow \bullet \end{array} \right)$$

which recovers a familiar realisation from algebra theory. In particular, our realisation for the canonical dilation defines a systematic procedure for the realisation of graph algebras.

We finish the entire preprint with the question of minimal dilations:

For this we begin with a positive result for graph algebras which leads to the following characterisation for correspondences over spaces:

**Corollary** (Katsura dilation: correspondences over spaces):

*It holds for correspondences over spaces: The canonical dilation given by the maximal covariance defines a minimal dilation if and only if the maximal covariance arises as a discrete subspace.*

*In particular maximal covariances with totally disconnected spectra are not sufficient.*

We use the found characterisation to produce a plethora of examples for relative Cuntz–Pimsner algebras whose canonical dilation given by the maximal covariance do not define minimal dilations. More intriguingly, these examples infer that **the axiom of choice fails to apply**.

## 1.2 Pullbacks

As a first application of our classification we derive **the general pullback result** for arbitrary relative Cuntz–Pimsner algebras: More precisely one has the following pullback for relative Cuntz–Pimsner algebras (over some given correspondence):

$$\mathcal{O}\left((K|I) \wedge (L, J)\right) = \mathcal{O}(K|I) \oplus_{\mathcal{O}((K|I) \vee (L, J))} \mathcal{O}(L|J)$$

We will explain later below precisely to what extend these define **every possible pullback**. This is upcoming joint work with **Piotr M. Hajac and Mariusz Tobolski**.

Our work arises from a special pullback result by **Robertson and Szymanski** from [RS11]: Their special pullback result however concerns absolute Cuntz–Pimsner algebras only and runs as a **verification by hand** using the following concrete description for pullbacks of operator algebras (basically a special instance for constructing categorical limits as equalizers):

$$A_1 \oplus_A A_2 = \left\{ a_1 \oplus a_2 \mid (A_1 \rightarrow A)a_1 = (A_1 \rightarrow A)a_2 \right\} \subseteq A_1 \oplus A_2.$$

For this Robertson and Szymanski require **additional technical assumptions** in order to verify that the desired morphism into the given pullback above defines an isomorphism (basically proving injectivity and surjectivity by hand).

**Instead the author notes** that the pullback problem defines an **entirely categorical task**:

As such we may solve the entire pullback problem in an entirely abstract fashion by means of our classification using the following **key principles for pullbacks**:

For this we note first that every operator algebra defines a commutative diagram by short exact sequences (where we skipped the zeroes at each row and column) and as a special instance those that describe each pullback:

$$\begin{array}{ccccc} I \cap J & \longrightarrow & J & \longrightarrow & (\dots) \\ \downarrow & & \downarrow & & \downarrow \\ I & \longrightarrow & A & \longrightarrow & A/I \\ \downarrow & & \downarrow & & \downarrow \\ (\dots) & \longrightarrow & A/J & \longrightarrow & \frac{A}{I+J} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} (I \cap J = 0) & \longrightarrow & J & \xlongequal{\quad} & J \\ \downarrow & & \downarrow & & \downarrow \\ I & \longrightarrow & A & \longrightarrow & A/I \\ \parallel & & \downarrow & \text{pullback} & \downarrow \\ I & \longrightarrow & A/J & \longrightarrow & \frac{A}{I+J} \end{array}$$

That is the desired pullback defines **the optimal solution** within both quotients:

$$A \longrightarrow \left( \text{Pullback} = \frac{A}{I \cap J} \right) \longrightarrow \left( \frac{A}{I} \text{ and } \frac{A}{J} \right).$$

Note that a given quotient representation is gauge-equivariant precisely for gauge-invariant ideals.

## 1.2. Pullbacks

---

The optimal quotient however arises as their intersection and thus remains gauge-equivariant:

$$\left(\mathbb{T} \curvearrowright I\right) \subseteq I \quad \text{and} \quad \left(\mathbb{T} \curvearrowright J\right) \subseteq J \quad \implies \quad \left(\mathbb{T} \curvearrowright (I \cap J)\right) \subseteq (I \cap J).$$

As such the pullback problem boils down to the **optimal gauge-equivariant quotient**.

In the same vein their common quotient remains gauge-equivariant since

$$\left(\mathbb{T} \curvearrowright I\right) \subseteq I \quad \text{and} \quad \left(\mathbb{T} \curvearrowright J\right) \subseteq J \quad \implies \quad \left(\mathbb{T} \curvearrowright (I + J)\right) \subseteq (I + J).$$

and as such one may express the latter as the **optimal gauge-equivariant quotient**:

$$\left(\mathbb{T} \curvearrowright \frac{A}{I}\right) \longrightarrow \left(\mathbb{T} \curvearrowright \frac{A}{I+J}\right) \quad \text{and} \quad \left(\mathbb{T} \curvearrowright \frac{A}{J}\right) \longrightarrow \left(\mathbb{T} \curvearrowright \frac{A}{I+J}\right).$$

Since our classification induces a lattice isomorphism between each gauge-equivariant quotient (beginning at the Toeplitz algebra) and our kernel–covariance pairs we obtain as desired

$$\mathcal{O}\left((K|I) \wedge (L, J)\right) = \mathcal{O}(K|I) \oplus_{\mathcal{O}\left((K|I) \vee (L, J)\right)} \mathcal{O}(L|J).$$

This simple arguments already define **the entire proof** for our pullback result above. Note also how the entire result followed using categorical considerations (no hard analysis required).

We now address precisely to what extend these define **every possible pullback**:

Consider for this those pullbacks for possibly non-optimal quotients

$$(A/I_1) \oplus_{A/J} (A/I_2) : \quad (I_1 + I_2) \subseteq J.$$

By some abstract catorical argument our previous pullbacks **embed as a subalgebra**

$$A / \left(I_1 \cap I_2\right) \hookrightarrow (A/I_1) \oplus_{A/J} (A/I_2) : \quad (a + I_1 \cap I_2) \mapsto (a + I_1) \oplus (a + I_2).$$

The abstract categorical argument may be also written out as

$$(a/I_1) \oplus (a/I_2) = 0 \quad \implies \quad a \in I_1 \cap I_2 \quad \implies \quad a / (I_1 \cap I_2) = 0.$$

On the other hand, these pullbacks cannot agree entirely since otherwise their defining diagram as above coincide except the very right-hand bottom corner and so also in that corner, whence

$$A / \left(I_1 \cap I_2\right) = (A/I_1) \oplus_{A/J} (A/I_2) \quad \implies \quad J = (I_1 + I_2).$$

As such any other pullback **does not arise** as a relative Cuntz–Pimsner algebra:

$$(K_1|I_1) \vee (K_2|I_2) < (K|I) : \quad \mathcal{O}\left((K_1|I_1) \wedge (K_2|I_2)\right) \subsetneq \mathcal{O}(K_1|I_1) \oplus_{\mathcal{O}(K|I)} \mathcal{O}(K_2|I_2).$$

We have therefore found every pullback possibly arising as a relative Cuntz–Pimsner algebra:

## 1.2. Pullbacks

---

Summarising these are precisely the pullbacks within the lattice of kernel–covariance pairs.

We finish with a negative result on pullbacks for **absolute Cuntz–Pimsner algebras**:

For this it suffices to give a pair of quotient correspondences such that either of

$$\mathcal{O}(K|\max) \wedge \mathcal{O}(L|\max) \neq \mathcal{O}(K \cap L | \max),$$

$$\mathcal{O}(K|\max) \vee \mathcal{O}(L|\max) \neq \mathcal{O}(\dots | \max).$$

For the former we may consider the following scenario

$$\begin{array}{ccc} \mathcal{T}(X) & \longrightarrow & \mathcal{T}\left(\frac{X}{XL}\right) = \mathcal{O}\left(\frac{X}{XL}\right) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \mathcal{T}(\dots) = \mathcal{O}(\dots) \longrightarrow \dots \end{array}$$

from which we obtain as pullback of absolute Cuntz–Pimsner algebras

$$\mathcal{T}(K \cap L = 0) = \mathcal{O}(K = 0) \oplus_{\mathcal{O}(\dots)} \mathcal{O}(L) \neq \mathcal{O}(K \cap L = 0).$$

The minimal example for such is given by the following graph algebras:

$$\left( \begin{array}{ccc} \mathcal{T}(a \rightarrow b) & \longrightarrow & \mathcal{T}(b) = \mathcal{O}(b) \\ \downarrow & & \downarrow \\ \mathcal{O}(a \rightarrow b) & \longrightarrow & \mathcal{T}(\emptyset) = \mathcal{O}(\emptyset) \end{array} \right) = \left( \begin{array}{ccc} M_2 \oplus \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & 0 \end{array} \right).$$

So their pullback fails to arise as an absolute Cuntz–Pimsner algebra.

For the latter we may consider the following degenerate scenario

$$\begin{array}{ccc} \mathcal{T}(K \cap L) = \mathcal{O}(K \cap L) & \longrightarrow & \mathcal{T}(K) = \mathcal{O}(K) \\ \downarrow & & \downarrow \\ \mathcal{T}(L) = \mathcal{O}(L) & \longrightarrow & \mathcal{T}(K + L) \longrightarrow \dots \longrightarrow \mathcal{O}(K + L) \end{array}$$

from which the only pullback possibly arising as a relative Cuntz–Pimsner algebra reads

$$\begin{aligned} \left( \mathcal{T}(K \cap L) = \mathcal{O}(K \cap L) \right) &= \left( \mathcal{T}(K) = \mathcal{O}(K) \right) \oplus_{\mathcal{T}(K+L)} \left( \mathcal{T}(L) = \mathcal{O}(L) \right) \\ &\subseteq \left( \mathcal{T}(K) = \mathcal{O}(K) \right) \oplus_{\mathcal{O}(K+L)} \left( \mathcal{T}(L) = \mathcal{O}(L) \right) \neq \mathcal{O}(\text{any pair}). \end{aligned}$$

So the pullback of absolute Cuntz–Pimsner algebras does not even arise as **any relative one!**

### 1.3. Morita equivalence

---

The simplest such is given by the following graph with gauge-equivariant quotient lattice:

$$\begin{array}{ccc}
 \mathcal{T}\left(\bullet \rightrightarrows \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \leftarrow \bullet\right) = \mathcal{O}\left(\bullet \rightrightarrows \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \leftarrow \bullet\right) & \longrightarrow & \mathcal{T}\left(\bullet \rightrightarrows \begin{array}{c} \circlearrowleft \\ \bullet \end{array}\right) = \mathcal{O}\left(\bullet \rightrightarrows \begin{array}{c} \circlearrowleft \\ \bullet \end{array}\right) \\
 \downarrow & & \downarrow \\
 \mathcal{T}\left(\begin{array}{c} \circlearrowleft \\ \bullet \end{array} \leftarrow \bullet\right) = \mathcal{O}\left(\begin{array}{c} \circlearrowleft \\ \bullet \end{array} \leftarrow \bullet\right) & \longrightarrow & \mathcal{T}\left(\begin{array}{c} \circlearrowleft \\ \bullet \end{array}\right) \rightarrow \mathcal{O}\left(\begin{array}{c} \circlearrowleft \\ \bullet \end{array}\right) \rightarrow \left(\mathcal{T}(\emptyset) = 0 = \mathcal{O}(\emptyset)\right)
 \end{array}$$

Indeed each of those is singular except the last one which is fully regular. So far about pullbacks for relative Cuntz–Pimsner algebras as an **application of our classification**.

### 1.3 Morita equivalence

**The following arose as joint work** with Menevse Eryuzlu, Steven Kaliszewski and John Quigg: Recall that we began the classification preprint with the characterisation of short exact sequences in the correspondence category which we found all as (confer [Fre23, theorem 3.3])

$$0 \longrightarrow XK \longrightarrow X \longrightarrow \frac{X}{XK} \longrightarrow 0.$$

For quotient correspondence we then basically constructed each relative Cuntz–Pimsner algebras as a gauge-equivariant quotient for the original correspondence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & XK & \longrightarrow & X & \longrightarrow & \frac{X}{XK} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{O}(X \mid J \cap \max) & \longrightarrow & \mathcal{O}\left(\frac{X}{XK} \mid J\right) \longrightarrow 0.
 \end{array}$$

As such the question arose to what extend these complete as a short exact sequence by their corresponding relative Cuntz–Pimsner algebras (if possible at all !!)

$$0 \longrightarrow \mathcal{O}(XK \mid \text{some?}) \longrightarrow \mathcal{O}(X \mid J \cap \max) \longrightarrow \mathcal{O}\left(\frac{X}{XK} \mid J\right) \longrightarrow 0 \quad ??$$

For this we began with the observation that such necessarily arise as the operator algebra generated by the kernel correspondence within the relative Cuntz–Pimsner algebra. Indeed for any a commuting square (with any covariance!) possibly completing above diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & XK & \longrightarrow & X & \longrightarrow & \frac{X}{XK} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}(XK \mid \text{any}) & \longrightarrow & \mathcal{O}(X \mid I = J \cap \max) & & 
 \end{array}$$



### 1.3. Morita equivalence

---

its relative Cuntz–Pimsner algebra cannot contain additional kernel

$$XK \hookrightarrow X \hookrightarrow \mathcal{O}(X | I) \quad \implies \quad XK \hookrightarrow \mathcal{O}(XK | \text{any}).$$

The commuting square further infers that the operator algebra generated by the kernel correspondence (within the original Cuntz–Pimsner algebra) defines a quotient for any of those

$$\mathcal{T}(XK) \longrightarrow \mathcal{O}(XK | \text{any}) \longrightarrow C^*(XK \cup K) \subseteq \mathcal{O}(X | J \cap \max).$$

As such the optimal solution defines the only possible relative Cuntz–Pimsner algebra arising as an **embedded subalgebra** and as such we ruled out any other relative Cuntz–Pimsner algebra.

Using our classification it suffices now to compute its covariance (as its kernel vanishes anyways) to fully determine this optimal solution (as our only possible candidate) for which we find:

$$0 \longrightarrow \mathcal{O}(XK | I \cap K) \hookrightarrow \mathcal{O}(X | I = J \cap \max).$$

On the other hand, we found each kernel correspondence as an invariant ideal in our classification of kernel and cokernel morphisms [Fre23, theorem 3.3]. As such the operator algebra generated by the kernel correspondence defines a **hereditary subalgebra** with ideal given by the kernel:

$$\begin{aligned} C^*(XK \cup K) &= C^*(XK \cup K)C^*(X \cup A)C^*(XK \cup K), \\ C^*(XK \cup K)K &= C^*(X \cup A)C^*(XK \cup K)C^*(X \cup A). \end{aligned}$$

The latter defines the kernel for the desired short exact sequence above,

$$0 \longrightarrow \mathcal{O}(X|I)K \hookrightarrow \mathcal{O}(X|I) \longrightarrow \mathcal{O}\left(\frac{X}{XK} | J\right) \longrightarrow 0.$$

As such we recovered a classical result by Fowler–Muhly–Raeburn as in [FMR03, theorem 3.1]: The optimal solution resides as an **embedded Morita equivalence** within the kernel:

$$0 \longrightarrow \left( \mathcal{O}(XK | I \cap K) \sim \dots \right) \longrightarrow \mathcal{O}(X|I) \longrightarrow \mathcal{O}\left(\frac{X}{XK} | J\right) \longrightarrow 0.$$

As such we have narrowed down our question to the problem whether the only choice as relative Cuntz–Pimsner algebra for the kernel correspondence (as a hereditary subalgebra) coincides with the desired ideal. We provide for this **the following negative example**: For this it suffices to provide some invariant ideal with the particular property

$$X^*KX \subseteq K : \quad KX = 0 \quad \text{whereas} \quad XXK \neq 0.$$

As such the relative Cuntz–Pimsner algebra for the kernel correspondence reads

$$C^*(XK \cup K) = K + XK + KX^* + XKX^* = \mathcal{O}(XK | I \cap K)$$

### 1.3. Morita equivalence

---

which is missing above component and whence also the desired ideal

$$\mathcal{O}(XK|I \cap K) \neq \mathcal{O}(X|I)K\mathcal{O}(X|I) : \quad XXX \notin \mathcal{O}(XK|I \cap K).$$

As an example one may consider the following sequence of graph algebras

$$0 \longrightarrow \left( \mathcal{O}(a \implies b) \neq \dots \right) \longrightarrow \mathcal{O}(a \implies b \implies c) \longrightarrow \mathcal{O}(b \implies c) \longrightarrow 0.$$

We chose this example as it provides one entirely made of **absolute Cuntz–Pimsner algebras**. The minimal example when allowing also for relative ones is the same as above with single edges. Concluding that short exact sequences generally do not induce one by Cuntz–Pimsner algebras.

Moving on the author further examined Morita equivalences based on **higher tensor powers**: More precisely the author expands a result on Morita equivalence for graph algebras by Crisp and Gow from [CG06] and further uses those to construct examples (as a correspondence) with embedded Morita equivalence not arising by any of its finite tensor powers. This constitutes upcoming work under preparation.

**Operator algebras and quantum information:  
Quantum values and optimal states**

## 2.1 Connes implies Tsirelson: a simple proof

**In its most simple formulation**, the Connes embedding problem (in short the CEP) asks whether every tracial state arises via the induced trace given on the tracial ultrapower for the hyperfinite  $\text{II}_1$ -factor as a factorisation over some representation (which would then hold the entire information):

$$A \overset{\text{repr}}{\dashrightarrow} \mathcal{R}^\omega \xrightarrow{\text{tr}_\omega} \mathbb{C}.$$

The representation will be automatically unital as the induced trace is faithful by construction.

**On the other hand**, the synchronous Tsirelson conjecture asks whether every synchronous quantum commuting correlation given by some tracial state

$$C^*(\mathbb{Z}/m * \dots * \mathbb{Z}/m) \xrightarrow{\tau} \mathbb{C} : \quad p(ab|xy) = \tau(E(a|x)E(b|y))$$

arises approximately via traces on finite dimensional quotients

$$C^*(\mathbb{Z}/m * \dots * \mathbb{Z}/m) \longrightarrow Q = M(d_1) \oplus \dots \oplus M(d_r) \xrightarrow{\text{some trace}} \mathbb{C}.$$

The contribution of the preprint is to simplify the obtained

**The main contribution from our preprint** [Fre22a] is the observation that the central lifting result by Kim–Paulsen–Schafhauser from [KPS18] (more precisely an improved version from our preprint) directly entails the CEP implies the synchronous Tsirelson conjecture. More precisely their lifting result states that any representation, which admits an approximation by ucp-maps via matrix algebras, arises as an honest approximation by representations:

$$\begin{array}{ccc}
 & \overset{\text{ucp}}{\dashrightarrow} \prod_n M_n & \\
 C^*(E(1) + \dots + E(A) = 1) & \xrightarrow{\text{repr}} \mathcal{R}^\omega & \implies C^*(E(1) + \dots + E(A) = 1) \xrightarrow{\text{repr}} \mathcal{R}^\omega. \\
 & \downarrow & \downarrow \\
 & \prod_n M_n & \prod_n M_n
 \end{array}$$

Kim–Paulsen–Schafhauser then combine this with the local lifting property (as a blackbox existence result) applied on the finite dimensional operator subsystem

$$\begin{array}{ccc}
 & \overset{\text{ucp}}{\dashrightarrow} \prod_n M_n & \\
 CE(1) + \dots + CE(A) & \longrightarrow C^*(E(1) + \dots + E(A) = 1) \xrightarrow{\text{repr}} \mathcal{R}^\omega & \\
 & & \downarrow \\
 & & \prod_n M_n
 \end{array}$$

## 2.1. Connes implies Tsirelson: a simple proof

---

Our contribution in [Fre22a] is the simplified and improved version of their lifting result with some fairly elementary lifting (as an existence result) suitable for quantum information:

$$\begin{array}{ccc}
 & \xrightarrow{\text{repr}} & \prod_n (M_2 \otimes \dots \otimes M_2) \text{ } d(n)\text{-times} \\
 & \nearrow \text{dashed} & \downarrow \\
 C^*(E(1) + \dots + E(A) = 1) & \xrightarrow{\text{repr}} & \prod_n M(2^\infty) / (\text{tr}_\omega = \lim_n \text{tr}_n) = \mathcal{R}^\omega
 \end{array}$$

This result involves (1) a simplified version of the tracial ultrapower in the lower right and (2) an elementary lifting based on the construction from Kim–Paulsen–Schafhauser. More importantly however, the author realized that the proof of Connes implies the synchronous Tsirelson conjecture is already entirely contained (as our humble contribution):

**Theorem 2.1.1** ([Fre22a]). *The CEP implies the synchronous Tsirelson conjecture: More precisely, suppose every tracial state on operator algebras arises as the induced trace on the tracial ultrapower for the hyperfinite  $\text{II}_1$ -factor,*

$$A \dashrightarrow \mathcal{R}^\omega \xrightarrow{\text{tr}_\omega} \mathbb{C}.$$

Then every synchronous quantum commuting correlation arises as a limit of finite dimensional synchronous correlations using the lifting result (as above)

$$\begin{array}{ccc}
 & \xrightarrow{\text{repr}} & \prod_n (M_2 \otimes \dots \otimes M_2) \\
 & \nearrow \text{dashed} & \downarrow \\
 C^*(\mathbb{Z}/m * \dots * \mathbb{Z}/m) & \xrightarrow{\text{repr}} & \prod_n M(2^\infty) / (\text{tr}_\omega = \lim_n \text{tr}_n) = \mathcal{R}^\omega \xrightarrow{\text{tr}_\omega} \mathbb{C}
 \end{array}$$

from which the synchronous quantum commuting correlation may be read off as

$$\tau(E(a|x)E(b|y)) = \lim_{n \rightarrow \omega} \text{tr}(E_n(a|x)E_n(b|y))$$

for their corresponding projections within the sequence algebra  $E_n(-|-) \subseteq M_2 \otimes \dots \otimes M_2$  and their image within the tracial ultrapower  $E(a|x) \in \prod_n M(2^\infty) / (\text{tr}_\omega = \lim_n \text{tr}_n)$ .

This simple lifting diagram **already defines the entire proof**. Furthermore this bypasses each of the Kirchberg conjectures as well as every reformulation using microstates and the like.

On the other hand this implication entails the full version of the general Tsirelson conjecture following from the synchronous version (which contains the hard technical core): This equivalence arises as a culmination of results from [Rus20] together with [DPP19] and [KPS18].

We give some additional details about our technical contributions from above:

For this we first clarify (special thanks goes here to Mikkel Munkholm) that tracial ultrapowers do not require completion to von Neumann algebras beforehand and as such we obtain a simplified version for the tracial ultrapower of the hyperfinite  $\text{II}_1$ -factor which is more manageable (and easier to comprehend) **for quantum information purposes:**

## 2.1. Connes implies Tsirelson: a simple proof

---

**Lemma 2.1.2** ([Fre22a], details unpublished). *For tracial ultrapowers it holds  $(A, \tau)^\omega = \text{vN}(A, \tau)^\omega$ . In particular for the tracial ultrapower of the hyperfinite  $\text{II}_1$ -factor*

$$\mathcal{R}^\omega = \left( M(2^\infty) = M_2 \otimes M_2 \otimes \dots \mid \text{tr}(2^\infty) = \text{tr} \otimes \text{tr} \otimes \dots \right)^\omega$$

where the right-hand side is more manageable for quantum information purpose.

The analogous statement does not hold for ultrapowers along states merely.

This result goes hand in hand with the following observation (verbatim from some familiar version) with the crux that tracial ultrapowers do not require von Neumann algebras:

**Lemma 2.1.3** (proof verbatim from [BO08, Lemma A.9]). *Tracial ultrapowers are complete in their trace 2-norm and so already define tracial von Neumann algebras*

$$A^\omega = \prod_n A_n / \left\{ \lim_{n \rightarrow \omega} \tau_n(a_n^* a_n) \right\} : \quad \left( A^\omega, \lim_{n \rightarrow \omega} \tau_n \right) = \text{vN} \left( A^\omega, \lim_{n \rightarrow \omega} \tau_n \right)$$

for any sequence of possibly non-separable operator algebras  $A_n$ .

The second part is **our elementary lifting result** from above (details unpublished) based on the construction from Kim–Paulsen–Schafhauser. More precisely the lifting goes as follows: One begins with a single projection which may be lifted as a positive contraction:

$$\begin{aligned} q \in \prod_n M(2^\infty) / \left( \text{tr}_\omega = \lim_n \text{tr}_n \right) : \quad q^2 = q = q^* \\ \implies (q_1, q_2, \dots) \in \prod_n M(2^\infty) : \quad 0 \leq (q_1, q_2, \dots) \leq 1. \end{aligned}$$

One may then further approximate those by finite dimensional matrices as one has:

$$M(2^\infty) = \overline{M_2 \subseteq M_2 \otimes M_2 \subseteq \dots} : \quad q_n \approx a_n \in (M_2 \otimes \dots \otimes M_2) \subseteq M(2^\infty).$$

At this point we invoke the construction by **Kim–Paulsen–Schafhauser**: That is one may cut those by their spectral projection without affecting their outcome in the tracial ultrapower:

$$a_n \approx 1_{[0.5, 1]} a_n \in (M_2 \otimes \dots \otimes M_2) : \quad 1_{[0.5, 1]} a = a = q \in \prod_n M(2^\infty) / \left( \text{tr}_\omega = \lim_n \text{tr}_n \right).$$

This is cutting procedure is legitimate due the following dimension–uniform trace bound due to Kim–Paulsen–Schafhauser (for which we provide a systematic proof and its optimal bound):

**Lemma 2.1.4** ([Fre22a] optimised from [KPS18]). *Consider for any normal matrix  $n \in M(d)$  its spectral projection  $1_S(n) \in M(d)$  onto some complex region  $S \subseteq \mathbb{C}$ . Then it holds the dimension–independent optimal trace 2-bound*

$$r \|n - 1_S(n)\|_2 \leq \|n^2 - n\|_2 \quad \text{where } r := d(S \cap \sigma(n), 0) \wedge d(S^c \cap \sigma(n), 1).$$

In particular it holds for positive contractions  $a \in M(d)$

$$\|a - 1_{[0.5, 1]} a\|_2 \leq 2 \|a^2 - a\|_2 (< 2\sqrt{2} \|a^2 - a\|_2)$$

with improved optimal bound as indicated from [KPS18].

## 2.1. Connes implies Tsirelson: a simple proof

---

The following projections may be lifted the same way employing a simple cutting procedure (also due to Kim–Paulsen–Schafhauser!) and as such our lifting result reads altogether:

**Lemma 2.1.5** ([Fre22a] based on [KPS18]). *Any unital representation into the tracial ultrapower of the hyperfinite  $\text{II}_1$ -factor admits a unital lift into matrices (of possibly increasing size):*

$$\begin{array}{ccc}
 & \text{unital repr} & \rightarrow \prod_n (M_2 \otimes \dots \otimes M_2) \\
 & \text{-----} & \downarrow \\
 C^*(E(1) + \dots + E(A) = 1) & \xrightarrow{\text{unital repr}} & \prod_n M(2^\infty) / (\text{tr}_\omega = \lim_n \text{tr}_n) = \mathcal{R}^\omega
 \end{array}$$

*That is every unital representation into the tracial ultrapower arises as a representation factorising by the sequence algebra (for some certain growth in dimension).*

We finally note that variations of this lifting result are partially known (as folklore).

## 2.2 The quantum commuting model:

### States vs. correlations

In order to handle the problem of robust self-testing we advocate the systematic distinction between uniqueness of states vs. correlations. Indeed we note that most of the traditional nonlocal games come with uniqueness of the entire optimal state (as opposed to uniqueness of only its correlation). More precisely, any quantum commuting correlation arises as an operator algebraic state restricted on **order–two moments of the form**

$$\varphi : C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow \mathbb{C} : \quad p(ab|xy) = \varphi\left(E(a|x) \otimes E(b|y)\right)$$

where each players algebra arises as the **free product of cyclic groups**

$$C^*\left(n = \#\text{questions} \mid m = \#\text{answers}\right) = C^*\left(\mathbb{Z}/m * \dots * \mathbb{Z}/m - n\text{-many}\right).$$

Further most of our investigations focus on binary nonlocal games (that is each player answers with classical bits) for which the operator algebra reads for instance

$$C^*(3|2) \otimes C^*(3|2) = C^*\left(\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2\right) \otimes C^*\left(\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2\right).$$

In this case the information of order 2-moments in projection-valued measures is already entirely contained in order 2-moments by their generating unitaries (for each cyclic group)

$$\left\langle E(a \in \mathbb{Z}/2|x) \otimes E(b \in \mathbb{Z}/2|y) \right\rangle = \left\langle u(x) \otimes 1, u(x) \otimes u(y), 1 \otimes u(y) \right\rangle$$

with translation given by (due to Pontryagin duality)

$$u(x) = E(a = 1|x) - E(a = -1|x) \quad \text{and} \quad 2E(a|x) = au(x) + 1.$$

As such we consider the equivalent correlation table given by

$$\varphi\left(u(x) \otimes 1\right), \quad \varphi\left(u(x) \otimes u(y)\right), \quad \varphi\left(1 \otimes u(y)\right), \quad (1 \leq x, y \leq m).$$

This restriction on order-two moments as above defines the difference between uniqueness of optimal states vs. self-testing of correlations merely, which plays a pivotal role in upcoming joint work with Azin Shahiri on **genuine self-testing** vs. uniqueness of optimal states. We get to elaborate this problem further below in section 2.4.

## 2.3 Optimal states: uniqueness (including all higher moments)

As a first part we however elaborate uniqueness of entire optimal states, that is most traditional nonlocal games admit unique optimal states as opposed to uniqueness up to self-testing merely: For this we note in our work that the quantum value (more precisely the quantum commuting

### 2.3. Optimal states: uniqueness (including all higher moments)

---

value) arises as the operator norm

$$\omega(\text{game}) = \sup_{\varphi} \varphi(\text{game}) = \|\text{game}\|$$

with nonlocal game given as a positive polynomial

$$\text{game} = \frac{1}{|X \times Y|} \sum E(\text{winning pairs}) \in C^*(\text{Alice}) \otimes C^*(\text{Bob}).$$

where we invoke the suggestive notation  $E(ab|xy) = E(a|x) \otimes E(b|y)$ .

As such we generally structure our problem **into the subproblem:** (1) compute the operator norm for the polynomial and (2) determine the region of state space realising the operator norm:

$$(1) \quad \|\text{game}\| = ? \quad \implies \quad (2) \quad \varphi(\text{game}) = \|\text{game}\| : \quad \varphi = ?$$

Generally these are **hard subproblems** (from a complexity point of view) and they are in some sense **not independent either:** Indeed we note that it is known that the quantum approximate value may be efficiently approximated from below as observed in

[FNT14]: Can you compute the operator norm?

while the quantum commuting value may be approximated using the NPA hierarchy [NPA08].

However by the recent **breakthrough article** [JNVWY21] the class of recursively enumerable languages embeds into the class of nonlocal games in such a way that the quantum value has constant gap and gives a mean to decide the runtime for each algorithm in the sense:

$$\begin{aligned} \text{algorithm halts} & \iff \omega^*(\text{game}) \geq 2/3 \\ \text{algorithm does not halt} & \iff \omega^*(\text{game}) \leq 1/3 \end{aligned}$$

As such if the quantum and quantum commuting value would coincide, this would entail an (efficient) algorithm for deciding the runtime. The class of recursively enumerable languages however **includes the Halting problem** and as such the quantum and quantum commuting value cannot coincide for the entire class of nonlocal games.

On the other hand, the maximal tensor product of full group algebras incorporates the operator norm in each of its representations and equivalently also the value along the entire state space. As such the above subproblems are, in some sense, **not independent either.**

For particular nonlocal games one may however still hope to solve either of the subproblems: Indeed one may hope to solve the first subproblem **by means of algebraic constraints** such as for instance **Landau's trick for the CHSH game:**

$$\begin{aligned} \text{CHSH} &= u \otimes u + u \otimes v + v \otimes u - v \otimes v : \\ \text{CHSH}^* \text{CHSH} &= 4 - [u, v] \otimes [u, v] \leq 4 + 2 \otimes 2 = 8. \end{aligned}$$

Note that Landau's trick from [Lan87] is sometimes mistakenly attributed to Tsirelson.

For the second subproblem we first note that the description of correlations in terms of states on



### 2.3. Optimal states: uniqueness (including all higher moments)

---

the maximal tensor product defines a representation-free and algebraic description

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow \cancel{B(H)} \longrightarrow \mathbb{C} : \quad \varphi(E \otimes F) \quad \text{or} \quad \varphi(u \otimes v).$$

As such a pair of correlations given as seemingly different representations (think of reference frames) may define simply one and the same state after all (just from different perspectives):

$$\begin{array}{ccc} & & B(H) \\ & \nearrow & \searrow \\ C^*(\text{Alice}) \otimes C^*(\text{Bob}) & & \mathbb{C} \\ & \searrow & \nearrow \\ & & B(H') \end{array}$$

And indeed, most of the traditional self-testing results arise simply from this perspective as a unique optimal state (but seen from different reference frames as above)

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \xrightarrow{\varphi(-)} \mathbb{C} : \quad \varphi(\text{game}) = \|\text{game}\| \implies \varphi(-) \text{ unique!}$$

The problem is now to determine this region of state space maximising the quantum value: For this we **employ an additional technique** which allows us to further decompose the problem into **quotient representations** and the eigenspace/trace space on the other hand. For this we note that the minimal Stinespring dilation gives a unique decomposition of states into a minimal quotient representation followed by its vector state

$$A \xrightarrow{\text{quotient}} Q \subseteq B(A|\varphi) \xrightarrow{\langle \varphi | - | \varphi \rangle} \mathbb{C}$$

while any non-minimal Stinespring dilation arises simply as a dilation

$$A \longrightarrow \left( \begin{array}{cc} B(A|\varphi) & 0 \\ 0 & B(\text{rem}) \end{array} \right) \xrightarrow{(\langle \varphi | 0 \rangle - \begin{pmatrix} |\varphi\rangle \\ 0 \end{pmatrix})} \mathbb{C}.$$

As such the bottom right-hand corner defines entirely artificial data that does not represent any information of the original state. Indeed most ambiguity encountered in known self-testing results have their origin precisely here (which summarises the issue of reference frames above).

**More importantly however**, the minimal Stinespring dilations defines the minimal quotient algebra for the original state (and so the quotient on which the state naturally lives):

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow Q \longrightarrow \dots \longrightarrow Q(\varphi) \xrightarrow{\varphi(-)} \mathbb{C}$$

As such the task to determine the region of state space (maximising the quantum value) becomes (2a) determine the operator algebraic relations the nonlocal game enforces on all optimal states

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow Q(\text{game}) = \frac{C^*(\text{Alice}) \otimes C^*(\text{Bob})}{I(\text{game})} \xrightarrow{\varphi(-)} \mathbb{C} :$$

$$I(\text{game}) = \bigcap I(\varphi | \varphi(\text{game}) = \|\text{game}\|) \subseteq C^*(\text{Alice}) \otimes C^*(\text{Bob})$$

### 2.3. Optimal states: uniqueness (including all higher moments)

---

with maximal ideal for each state given by (for more details see the second preprint [Fre22b])

$$I(\varphi) = \{aA | \varphi = 0\} = \{\langle \varphi | Aa^*aA | \varphi \rangle = 0\} \subseteq \ker(\varphi)$$

which equivalently resembles the kernel of its representation in the minimal Stinespring dilation, and (2b) determine the maximal eigenspace/trace space within minimal quotient algebras

$$\begin{aligned} \text{ES}(\max(\text{game})) &= \{\text{game} | \varphi = \max | \varphi \rangle\} \\ T(Q(\text{game})) &= \{\tau : Q(\text{game}) \rightarrow \mathbb{C} : \tau(ab) = \tau(ba)\} \end{aligned}$$

where the latter denotes the game algebra as introduced in [OP16] and [HMPS19].

We note that in general the majority of nonlocal games however enforces no relation at all: This is the main content of upcoming work which we elaborate in more detail in section 2.4. But before we get to explain this we give some examples with unique optimal states:

**Theorem 2.3.1** ([Fre22b]). *The CHSH game given as an abstract polynomial*

$$\text{CHSH} \in C^*(\text{Alice}) \otimes C^*(\text{Bob}) : \quad C^*(\text{Alice}) = C^*(u^2 = 1 = v^2) = C^*(\text{Bob})$$

*has not only a unique correlation but in fact a unique optimal state:*

*More precisely, any optimal state maximising the CHSH polynomial*

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \xrightarrow{\varphi(-)} \mathbb{C} : \quad \varphi(\text{CHSH}) = \|\text{CHSH}\|$$

*necessarily arises as a state on the simple quotient algebra*

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow \left( Q(\text{CHSH}) = \frac{C^*(\text{Alice})}{\{u, v\} = 0} \otimes \frac{C^*(\text{Bob})}{\{u, v\} = 0} \right) \dashrightarrow \mathbb{C}$$

*with braces denoting the anticommutation  $\{u, v\} = uv + vu = 0$  for each player, and is uniquely determined on all moments such as for (using the anticommutation)*

$$\varphi\left( uvvu \otimes vuv = -u \otimes u \right) = -\frac{1}{\sqrt{2}}$$

*which follows entirely from its values:*

$\varphi(-)$	$- \otimes 1$	$- \otimes u$	$- \otimes v$	$- \otimes uv$
$1 \otimes -$	1	0	0	0
$u \otimes -$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$v \otimes -$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$uv \otimes -$	0	0	0	-1

*Note that the entire result is in abstract algebraic form and in particular representation-free. Also as the state is uniquely determined there is absolutely **no ambiguity left anymore**.*

In the preprint [Fre22b] we further note the following examples (which are partially known):

## 2.4. Correlations: genuine self-testing (order-two moments)

---

The Mermin–Peres magic square and magic pentagram games have as game algebra

$$Q(\text{magic square}) = \frac{C^*(X^2=1=Z^2)}{\{X,Z\}=0} \otimes \frac{C^*(X^2=1=Z^2)}{\{X,Z\}=0} = M_2 \otimes M_2$$

$$Q(\text{magic pentagram}) = \frac{C^*(X^2=1=Z^2)}{\{X,Z\}=0} \otimes \frac{C^*(X^2=1=Z^2)}{\{X,Z\}=0} \otimes \frac{C^*(X^2=1=Z^2)}{\{X,Z\}=0} = M_2 \otimes M_2 \otimes M_2$$

which solves problem (2a) and so also problem (2b): Both come with a unique tracial state and as such both of the Mermin–Peres games come with a unique optimal state given by

$$\varphi\left(E(a|x) \otimes E(b|y)\right) = \tau\left(E(a|x)E(b|y)\right).$$

Note that the above result is representation-free and further involves **no transpose operators!** Instead the latter is merely an artefact of rendering the optimal tracial state as vector state using a concrete representation for the opposite group algebra (within the right regular representation).

In the same vein one has for **sums of projections** as in [KRS02] which inspired [MPS21]:

The universal algebra is a simple matrix algebra and as such already comes with a unique optimal state given by the tracial state (which already entails the entire self-testing result)

$$C^*\left(p_1 + \dots + p_n = \frac{a}{b}\right) = M(b) : \quad \tau(-) = \text{tr}(-)/b$$

with irreducible expression from the continued fraction  $a/b \in \{0, 1 + \frac{1}{n-1}, \dots\}$  as in [KRS02].

In ongoing joint work with Azin Shahiri we further provide an operator algebraic approach **on robust self-testing** in the quantum commuting model which applies on above examples.

## 2.4 Correlations: genuine self-testing (accounting order-two moments)

Up until now the previous section **merely gave a prelude to highlight** how most traditional games and their self-testing results merely stem from the arbitrariness of non-minimal Stinespring dilations along with the ambiguity of the chosen concrete representation (as a reference frame) while the underlying optimal state had been entirely unique (including all moments).

With this being said, we now get to address their induced correlations on order-two moments and the procedure with which we establish an abundance of **genuine self-testing results** that do not arise as a simple dilation for any unique underlying optimal state.

This is where proper operator algebraic techniques kick-in in order to navigate also along exotic representations (think of interesting reference frames). For this we recall that synchronous correlations arise as tracial states on the full group algebra

$$\tau : C^*\left(\mathbb{Z}/m * \dots * \mathbb{Z}/m\right) \rightarrow \mathbb{C} : \quad p(ab|xy) = \tau\left(E(a|x)E(b|y)\right).$$

Apart from the low-degree degenerate cases, these groups are generally far from amenable as they even contain free groups (which includes for instance the I3322 inequality)

$$F(n-1) = \left(\mathbb{Z} * \dots * \mathbb{Z}\right) \hookrightarrow \left(\mathbb{Z}/m * \dots * \mathbb{Z}/m\right) * \mathbb{Z}/m$$

## 2.5. Optimal states: robustness behavior

---

while for binary games they even feature free groups as a quotient

$$\left(\mathbb{Z}/2 * \dots * \mathbb{Z}/2\right) * \mathbb{Z}/2 \longrightarrow F(n-1) \rtimes \mathbb{Z}/2.$$

As such we obtain an **abundance of inequivalent tracial states** when accounting all moments. In fact these are hard to handle entirely from a complexity point of view as these incorporate (before taking a semidirect product) every finitely generated operator algebra as a quotient and as such also their tracial states. On the other hand, there exists a **large body of research** on discrete groups and their quotients from an operator algebraic perspective including the intriguing **simplicity question** for discrete groups and its accompanying **unique trace property**.

**In order to handle the problem** we now plan to first exploit a particular class of fairly well-understood quotient algebras (up to the free factor problem): For this we view the underlying group algebra as a right-angled Coxeter group whose full group algebra admits the class of Hecke algebras as concrete quotient algebras (as a particular instance of quantum deformations).

For those however the simplicity question as well as the unique trace property has been answered to an extend well enough for our purposes. As such we exploit their simplicity and unique trace property to rule out **the existence of any dilations** between their tracial states.

At the same time each Hecke algebra comes with a concrete canonical tracial state (parametrising a family of tracial states for the full group algebra) whose moments have a very simple form:

$$C^*(W = \mathbb{Z}/2 * \dots * \mathbb{Z}/2) \xrightarrow{\text{Hecke}} B(\ell^2 W) \xrightarrow{\langle e| \cdot | e \rangle} \mathbb{C} : \quad \langle e | \text{word} | e \rangle = (-1 \leq s \leq 1)^{|\text{word}|}$$

As such we obtain a **pool of inequivalent traces** which may be handled sufficiently well.

On the other hand, a recent remarkable **observation by Travis Russell** however recovers their correlations by finite dimensional tracial states. As such we aim to establish in ongoing joint work a genuine self-testing result for their induced correlation based on these two observations.

This is ongoing joint work with Azin Shahiri after a **key remark by Vern Paulsen**.

## 2.5 Optimal states: robustness behavior

In recent work from 2021 [MPS21], Mancinska, Prakash and Schafhauser provide a robust self-testing result based on the algebra underlying an optimal state. Their method applies to finite dimensional measurements using a suitable version of Gowers–Hatami.

**In ongoing joint work** we pick up the approach by Mancinska–Prakash–Schafhauser at the level of optimal states and complete this as an entire robust self-testing result including the quantum commuting model using **purely operator algebraic methods altogether**:

More precisely, our result relies on the optimal state arising as some maximally entangled state on some finite dimensional quotient from which we derive the robust self-testing result using methods from the previous section on optimal states together with a few additional tricks. In particular this entails the result by Mancinska–Prakash–Schafhauser as a special instance.

Our method however does not apply for optimal states arising on infinite dimensional quotients. For such we would instead require the use of **Hilbert–Schmidt stability** which currently arises as a **hot topic in operator algebras** of independent interest.

## 2.6 The tilted CHSH games: a representation–theoretic classification

In an upcoming preprint [FS23] we take a closer look on the **tilted CHSH games**:

More precisely, we give a systematic representation–theoretic approach on the their quantum value and corresponding unique optimal states as these appear in the separation between finite dimensional and spatial strategies as recently obtained by **Coladangelo and Stark** in [CS20].

The longterm purpose is then to derive an operator algebraic argument for their separation and in turn to construct **nonlocal games exhibiting the separation** for which we first seek a better understanding (and simpler description) on optimal states for the tilted CHSH games.

We focus on this separation of correlations as their better understanding may become relevant for self-testing in practice since near-term realisations of practical quantum computing usually involves **infinite dimensional systems** which reduce to finite dimensional ones only in first order.

For this we first provide a systematic approach on the representation theory of binary–input binary–output games and in particular give a canonical description for its full group algebra. The resulting description gives an algebraic characterisation of basic building blocks appearing in a variety of articles such as for instance on the **I3322-inequality** as in [PV10].

**Proposition 2.6.1** ([FS23] under preparation; see also [RS89]). *The group algebra for binary–input binary–output games has the canonical representation as matrix valued functions*

$$C^*(\mathbb{Z}/2 * \mathbb{Z}/2) = C^*\left(u^2 = 1 = v^2 \mid \{u(z), v(z)\} = 2 \operatorname{Re}(z \in \mathbb{T})\right) \subseteq C(\mathbb{T} \rightarrow M_2)$$

with the value for the anticommutator running over the torus  $z \in \mathbb{T}$ .

As such its minimal quotients arise by anticommutation  $\{u, v\} = 2s$  running over  $-1 \leq s \leq 1$ .

In particular **any finite dimensional representation** arises as direct sum over such.

We use this canonical representation to systematically compute quantum values for nonlocal games: More precisely, the canonical representation defines a parametrisation of quantum values along minimal quotients (given by the range of anticommutation) that resemble the operator norm

$$\text{game} \in C([-2, 2] \times [-2, 2] \rightarrow M_2 \otimes M_2) : \quad \|\text{game}\| = \sup_{s,t} \|\text{game}(s, t)\|$$

and as such we may systematically compute quantum values for binary–input binary–output games as an **optimisation problem** along matrix algebras. We further note that the quantum value may not decrease within the quotient for some given optimal state since

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \xrightarrow{\pi^{(-)}} Q(\varphi) \xrightarrow{\varphi^{(-)}} \mathbb{C} \quad \varphi(\text{game}) \leq \|\pi(\text{game})\| \leq \|\text{game}\|$$

and as such the above optimisation problem further returns a range of anticommutations (as a

## 2.6. Tilted CHSH games: a representation–theoretic classification

---

quotient by restriction) that carries each optimal state

$$\text{best}(\text{game}) := \left\{ \|\text{game}(s, t)\| = \|\text{game}\| \right\} \subseteq [-2, 2] \times [-2, 2] :$$

$$C([-2, 2] \times [-2, 2] \rightarrow M_2 \otimes M_2) \longrightarrow C(\text{best}(\text{game}) \rightarrow M_2 \otimes M_2).$$

As a note asides, this additionally gives another proof that optimal states for binary–input binary–output games may be always realised by some alternative finite dimensional state.

We use this technique in the upcoming preprint for the tilted CHSH games:

**Theorem 2.6.2** ([FS23] under preparation). *The tilted CHSH games as a polynomial*

$$\text{CHSH}(\alpha \in \mathbb{R}, \beta \in \mathbb{R}) \in C^*(\mathbb{Z}/2 * \mathbb{Z}/2) \otimes C^*(\mathbb{Z}/2 * \mathbb{Z}/2)$$

admit a quantum advantage precisely some region specified in the preprint, and which further uniquely determines the anticommutation for both players as

$$Q(\text{CHSH}(\alpha, \beta)) = \frac{C^*(\text{Alice})}{\{u, v\} = 2s(\alpha, \beta)} \otimes \frac{C^*(\text{Bob})}{\{u, v\} = 2t(\alpha, \beta)}.$$

For the precise anticommutation we also refer to the upcoming preprint [FS23].

In particular we derive a ***much simpler description*** for these parameters.

The quotient for the tilted CHSH games allows us then to further derive uniqueness for the entire optimal state (including all higher moments): For this one may use a random choice of Pauli matrices to compute the maximal eigenspace. We however invoke instead a simple algebraic argument to derive the entire table of moments in a representation–free form:

**Theorem 2.6.3** ([FS23] under preparation). *The tilted CHSH games admit a unique optimal state  $\varphi(-) = \langle \varphi | - | \varphi \rangle$  given by the one dimensional subspace (as a projection)*

$$|\varphi\rangle\langle\varphi| = \lambda_+(\alpha, \beta) E(u = +1) \otimes E\left(\frac{w}{|w|} = +1\right) + \lambda_-(\alpha, \beta) E(u = -1) \otimes E\left(\frac{w}{|w|} = -1\right)$$

for  $1 \otimes w = 1 \otimes (u + v)$  and corresponding spectral projections  $E(\dots = \pm 1)$ .

For the precise form of its Schmidt coefficients we refer to the upcoming preprint.

In particular the optimal state is maximally entangled along the horizontal axis:

$$\lambda_+(\alpha, \beta) = \lambda_-(\alpha, \beta) \iff (\alpha \neq 0, \beta = 0).$$

Similarly we refer to its resulting table of moments to the upcoming preprint.

Note that the entire result is in abstract algebraic form and in particular representation–free.

Also as the state is uniquely determined there is absolutely ***no ambiguity left anymore***.

We finish this section with a simplified translation for real polynomials as binary games:

More precisely we note that the previous tilted CHSH games are given as a polynomial

$$\text{CHSH}(\alpha, \beta) = \alpha u \otimes (u + v) + v \otimes (u - v) + 2\beta u \otimes 1 \in C^*(u^2 = 1 = v^2) \otimes C^*(u^2 = 1 = v^2)$$

which we have silently viewen as a bias polynomial mimicking those for XOR-games.

## 2.6. Tilted CHSH games: a representation–theoretic classification

---

Indeed beginning from any XOR-game one may describe its game polynomial (as in section 2.3) equivalently also in terms of its bias polynomial (as for instance in [CHTW04, section 5.3])

$$\text{game} = \frac{1}{|X \times Y|} \sum E(\text{winning}) \implies \text{bias} = \sum (-1)^{V(x,y)} u(x) \otimes u(y).$$

While this defines a pretty much standard procedure, we now describe a simplified procedure for associating nonlocal games to any real bias polynomial (without increasing any question sets): For this we begin with the observation that tensor products of order-two unitaries remain order-two and whose spectral decomposition may be easily read off as

$$u \otimes v = \left( E(+1) \otimes F(+1) + E(-1) \otimes F(-1) \right) - \left( E(+1) \otimes F(-1) + E(-1) \otimes F(+1) \right)$$

from which the relations for recovering its spectral projections read

$$\begin{aligned} 1 \otimes 1 + u \otimes v &= 2 \left( E(+1) \otimes F(+1) + E(-1) \otimes F(-1) \right), \\ 1 \otimes 1 - u \otimes v &= 2 \left( E(+1) \otimes F(-1) + E(-1) \otimes F(+1) \right). \end{aligned}$$

On the other hand the spectrum satisfies the obvious relations

$$\sigma(a+1) = \sigma(a) + 1 \quad \text{and} \quad \sigma(\lambda a) = \lambda(a).$$

We may thus combine **these two observations** to relate any real bias polynomial to some game polynomial (with however possibly non-uniform distribution over questions) such as for instance

$$\begin{aligned} \text{bias} &= 1/3 u(x=1) \otimes u(y=2) - \sqrt{2} u(x=3) \otimes 1 \quad \rightsquigarrow \\ \text{bias} + \left( \frac{1}{3} + \sqrt{2} \right) &= 1/3 \left( E(a=1|x=1) \otimes E(b=1|y=2) + E(a=-1|x=1) \otimes E(b=-1|y=2) \right) \\ &\quad + \sqrt{2} \left( E(a=1|x=1) + E(a=-1|x=1) \right) \otimes \left( E(b=1|\text{any}) + E(b=-1|\text{any}) \right) = \text{game} \end{aligned}$$

with game polynomial up to some normalizing scalar.

As such any real bias polynomial defines a binary game whence also above tilted CHSH games. As another example we have any signed game as for example those exhibiting a separation between the quantum and quantum commuting value from [DPP19].

### 2.7 The I3322 inequality: an operator–algebraic approach

**In ongoing work with Azin Shahiri** we use the representation-theoretic classification from the previous section to approach the prominent I3322 inequality from an operator algebraic perspective:

We begin for this on the norm and spectrum for the underlying bias polynomial:

Using a culmination of operator algebraic relations in combination with a systematic description for the affine space of non-signalling correlations we first derive its precise operator norm and thus an antisymmetry for the spectrum.

We then exploit our found description for its finite dimensional representations (together with an additional reduction) to bound its maximal quantum violation as an optimisation problem within the numerical range for a particular 16-dimensional representation.

On the other hand we invoke an algorithmic description for the operator norm in group algebras using random walks. Embedding the I3322 inequality in the spatial tensor product we use this description to derive lower bounds for the quantum spatial value.

We ultimately wish to derive a strict bound for finite dimensional correlations, and hope to derive this way a separation from their quantum spatial counterpart.

On the other hand we attempt to flip perspectives:

More precisely, beginning from a given spatial correlation we wish to find systematic methods for constructing nonlocal games (as a polynomial in the operator algebra for the underlying group) exhibiting a separation from finite dimensional correlations and the given state.

This idea is based on discussions with Tobias Fritz and Remigiusz Augusiak.

The importance of this question stems from device independent certification in the context of real-world quantum computing: More precisely, while current models for quantum computing rest on finite dimensional spaces (up to first order) their physical realisations involve infinite dimensional spaces nonetheless. As such the separation between finite dimensional and quantum spatial correlations may become **utmost important in the near-term era** of quantum computing.



## Preprint A

# Relative Cuntz–Pimsner algebras: Classification of gauge-equivariant representations: a simple and complete picture

We give a simple and complete picture on the classification of relative Cuntz–Pimsner algebras (and so also of gauge–equivariant representations) using their intuitive **parametrisation by kernel–covariance pairs**.

For this we first present a **classification of kernel and cokernel** morphisms (in the general category of correspondences) which builds on the concept of invariant ideals as originating from work by Pimsner and as coined and further investigated by Kajiwara–Pinzari–Watatani. The existence of all such kernel and cokernel morphisms then enable us to reduce the general classification problem to the faithful case of correspondences within ambient operator algebras.

The second component arises from an **observation made by Katsura**: We unravel Katsura’s observation as an obstruction on the range of covariance ideals for correspondences embedded in ambient operator algebras, which comprises the second component of kernel–covariance pairs: As such our parametrisation runs over **invariant ideals** (as a discrete range of kernel ideals) and on the other hand over **bounded ideals** below some maximal covariance (as an upper bound on the range of covariance ideals).

We then illustrate the **lattice of relative Cuntz–Pimsner algebras** (and so also of every gauge–equivariant representation) along the range of kernel–covariance pairs. Following, we provide the general version of the gauge-invariant uniqueness theorem by its reduction to the faithful case, for which we further recall a **simplified proof by Evgenios Kakariadis**.

This establishes the first half in our classification: Every gauge–equivariant representation arises as a relative Cuntz–Pimsner algebra (for its own kernel–covariance pair) and whence the class of gauge–equivariant representations coincides with the class of relative Cuntz–Pimsner algebras. As such the kernel–covariance pairs **exhaust the gauge–equivariant representations**.

For the second half in our classification we aim to uniquely determine the relative

## A.1. Correspondences

---

Cuntz–Pimsner algebras by their parametrising kernel–covariance pairs: More precisely, we will recover every abstract kernel–covariance pair as the actual kernel and covariance from its relative Cuntz–Pimsner algebra, and as such **our kernel–covariance pairs are also classifying**.

With our found classification by kernel–covariance pairs we then further investigate the lattice structure of gauge–equivariant representations. In particular, we elaborate the existence of **connecting morphisms** between cokernel strands (given by the kernel component of kernel–covariance pairs) and illustrate our results on examples of graph algebras.

Along this discussion we further clarify Katsura’s description (using T-pairs) as a simple translation of kernel–covariance pairs (which had been already covered by Katsura himself) with the second component however not taken and further pursued as describing the range of covariance ideals.

Altogether **our classification is a simple reduction** of the gauge–invariant uniqueness theorem (along cokernel morphisms) together with the identification of kernel and covariance for relative Cuntz–Pimsner algebras.

Finally we provide a realisation of relative Cuntz–Pimsner algebras as absolute Cuntz–Pimsner algebra by the process of maximal dilation and reveal Katsura’s construction as the canonical dilation given by the maximal covariance. We further reveal the canonical dilation as a familiar construction from graph algebras and provide an example to illustrate **the lack of minimal dilations**.

As an application we provide a systematic approach for an earlier pullback result by Robertson–Szymanski which we extend to the general context in upcoming work **with Piotr M. Hajac and Mariusz Tobolski**.

## A.1 Correspondences

We begin with an introduction to correspondences and their representations. In particular, we provide a less formal and more abstract angle. This seeks to help to understand their gauge–equivariant representations from an abstract perspective — and so also to classify the entire lattice of gauge–invariant ideals. Further, this allows one to better understand dilations and in further work the shift equivalence problem from an abstract angle. A Hilbert module is a right module over an operator algebra (seen as coefficient algebra) that comes equipped with a pairing (compatible with the coefficient algebra) which renders the right module complete

## A.1. Correspondences

---

with respect to the induced norm:

$$\langle -|- \rangle : X \times X \rightarrow A : \quad \|x\|^2 := \|\langle x|x \rangle\|.$$

Given a pair of Hilbert modules over a common coefficient algebra one may introduce the notion of adjointable operators as those which admit an adjoint:

$$T : X \rightarrow Y, \quad T^* : Y \rightarrow X : \quad \langle T-|- \rangle = \langle -|T^*- \rangle.$$

We note that such adjointable operators are automatically continuous which may be seen most easily via the closed graph theorem. Moreover the class of adjointable operators over pairs of Hilbert modules defines a “consistent system” of Banach spaces with composition and involution

$$\mathcal{L}(Y|Z) \circ \mathcal{L}(X|Y) \subseteq \mathcal{L}(X|Z), \quad \mathcal{L}(X|Y)^* = \mathcal{L}(Y|X)$$

satisfying the generalized  $C^*$ -identity

$$T \in \mathcal{L}(X|Y) : \quad \|T\|^2 = \|T^*T\|.$$

The vertical separators hereby seek to convey the so-called Dirac bracket notation, that is we have the following identification which we term Dirac calculus:

**Proposition A.1.1.** *The identification (and its conjugate)*

$$\begin{aligned} X = |X \rangle &\subseteq \mathcal{L}(A|X), & X^* = \langle X| &\subseteq \mathcal{L}(X|A) : \\ |x \rangle a &:= |xa \rangle, & \langle x|y &:= \langle x|y \rangle : & \langle x|^* &= |x \rangle \end{aligned}$$

*define an isometric embedding (and its conjugate), which renders the pairing as*

$$\langle x|y \rangle = \langle x| \circ |y \rangle \in \mathcal{L}(X|A) \circ \mathcal{L}(A|X).$$

*This further renders the notion of compact operators as*

$$\mathbb{K}(X, Y) := \overline{\text{span}} |Y \rangle \langle X| \subseteq \mathcal{L}(X|Y)$$

*and the notion of adjointable operators as*

$$Tx = T \circ |x \rangle, \quad \langle Tx|y \rangle = \langle x| \circ T^* \circ |y \rangle = \langle x|T^*y \rangle.$$

*All of the above enables one to split expressions as composition of operators.*

## A.1. Correspondences

---

*Proof.* We first note that the coefficient algebra itself defines a Hilbert module:

$$\langle - | - \rangle : A \times A \rightarrow A : \quad \langle x | y \rangle = x^* y.$$

One may now verify that the assignments define mutual adjoints

$$\langle x |^* = |x \rangle \in \mathcal{L}(A|X) : \quad \langle \langle x | y \rangle | a \rangle = \dots = \langle y | x a \rangle = \langle y | |x \rangle a \rangle$$

and that the identification defines an isometric embedding

$$X = |X \rangle \subseteq \mathcal{L}(A|X) : \quad \| |x \rangle \|^2 = \| \langle x | \circ |x \rangle \| = \dots = \| \langle x | x \rangle \|.$$

We leave these as an instructive exercise for the reader.  $\square$

With this identification at hand, a Hilbert module conveniently reads as nothing but a right module together with an abstract pairing given by involution

$$(X \curvearrowright A) \quad XA \subseteq X, \quad (X \times X \rightarrow A) \quad X^*X \subseteq A$$

which we from now on simply indicate as such formal inclusions.

We may now turn our attention to the notion of  $C^*$ -correspondence: Formally these are given as Hilbert modules together with a representation of the coefficient algebra as adjointable operators. With the viewpoint from above we however obtain the alternative description as a bimodule over the coefficient algebra together with some compatible inner product pairing

$$AX \subseteq X, \quad XA \subseteq X, \quad X^*X \subseteq A \tag{A.1}$$

satisfying some relations such as (now evident in Dirac formalism)

$$(ax)^* y = x^* (a^* y), \quad x^* (ya) = (x^* y) a, \quad \text{etc.}$$

We meanwhile note that the notion of correspondences has an intrinsic asymmetry by the pairing. More precisely, the more traditional notion of Hilbert modules comes equipped with a dual pairing which renders the notion symmetric:

$$\begin{aligned} X^*X &\subseteq A, & XA &\subseteq X, \\ XX^* &\subseteq A, & AX &\subseteq X. \end{aligned}$$

In fact, this covers the main objective of covariant representations. We return to this aspect in the following section. We continue with a swift introduction to the (internal) tensor product of correspondences. Using our Dirac formalism from

## A.1. Correspondences

---

above we may now simply introduce those as formal powers such as

$$A(XY \dots Z) \subseteq (AX)Y \dots Z, \quad (XY \dots Z)A \subseteq XY \dots (ZA)$$

and where the inner product pairing now simply reads

$$\begin{aligned} (XY \dots Z)^*(XY \dots Z) &= Z^* \dots Y^*(X^*X \subseteq A)Y \dots Z \\ &\subseteq Z^* \dots (Y^*Y \subseteq B) \dots Z \subseteq \dots \subseteq (Z^*Z \subseteq A). \end{aligned}$$

Moreover, this automatically entails the balanced relation as for example

$$\begin{aligned} &\left( (xa)y - x(ay) \right)^* \left( (xa)y - x(ay) \right) = \\ &= y^*(a^*x^*)(xa)y - \dots + (y^*a^*)(x^*x)(ay) = 0. \end{aligned}$$

Let us give an illustrative example for such tensor products:

**Example A.1.2** (Graph correspondences: Tensor powers):

Consider a directed graph and regard its graph correspondence

$$X = \ell^2 \left( E = \text{edges} \right), \quad A = c_0 \left( V = \text{vertices} \right)$$

with action and pairing given by range and source say

$$a \xrightarrow{z} b \begin{matrix} \curvearrowright \\ \downarrow \end{matrix} \quad : \quad \begin{aligned} z^*z &= a, & za &= z = bz, \\ w^*w &= b, & wb &= w = bw. \end{aligned}$$

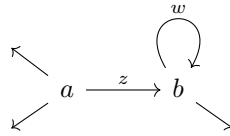
On the other hand we know by [KPQ12] that every nondegenerate correspondence over a “direct sum over a discrete set as vertices” arises as a graph correspondence:

$$A = c_0 \left( \text{vertices} = \text{some set} \right) \implies X = \ell^2(\text{edges}).$$

Indeed this follows by some simple counting argument:

$$a, b \in \text{vertices} : \quad bXa = \ell^2 \left( \text{edges} : b \leftarrow a \right) = \ell^2(bEa).$$

From this argument it furthermore follows that degenerate correspondences arise the same way when allowing for edges with heads pointing into the void such as



## A.1. Correspondences

---

and furthermore also any power of a graph correspondence arises as a graph correspondence itself and indeed is given by its paths of according length:

$$XX \dots X = \ell^2(EE \dots E), \quad A = c_0(\text{vertices}).$$

In other words, that is by concatenation of edges.

As such the tensor product may be seen as nothing but a formal power in words and one may equivalently consider also mixed powers with the dual space and its dual pairing (though formally only in the context of operator spaces).

We finish this section with an intrinsic characterization of Hilbert bimodules. For this we first require the following well-known observation:

**Proposition A.1.3** (based on [Kat04, definition 3.2]):

*Consider a correspondence (in Dirac bracket notation)*

$$\langle X|X \rangle \subseteq A, \quad XA \subseteq X, \quad AX \subseteq A$$

and regard the orthogonal complement for the kernel

$$\ker(A \curvearrowright X) = \{aX = 0\} \subseteq A :$$

$$\ker(A \curvearrowright X)^\perp = \{a \ker(A \curvearrowright X) = 0\} \subseteq A.$$

*This defines the largest ideal that renders the left action faithful.*

In particular we may in the following result simultaneously identify the orthogonal complement with its isometric image within the space of adjointable operators.

*Proof.* Before we begin with the actual argument let us make the useful observation that the kernel defines an ideal (closed and two-sided) since

$$A \ker(A \curvearrowright X)X = 0, \quad \ker(A \curvearrowright X)AX = 0,$$

$$\overline{\ker(A \curvearrowright X)X} \subseteq \overline{\ker(A \curvearrowright X)X} = 0$$

and so also a selfadjoint one (which one may also easily observe by hand)

$$\left\langle X \left| \ker(A \curvearrowright X)^* X \right. \right\rangle = \left\langle \ker(A \curvearrowright X)X \left| X \right. \right\rangle = 0 \implies \ker(A \curvearrowright X)^* X = 0.$$

This kernel determines now precisely those ideals that render the action faithful:

$$K \ker(A \curvearrowright X) = K \cap \ker(A \curvearrowright X) = 0 \iff K \upharpoonright X = 0.$$

## A.1. Correspondences

---

As such its orthogonal complement defines the largest such ideal:

$$K \ker(A \curvearrowright X) = 0 \iff K \subseteq \ker(A \curvearrowright X)^\perp.$$

Note that the orthogonal complement (as we have defined from the left only) defines itself an ideal (closed and two-sided) and so also a selfadjoint one.  $\square$

Building on this, we may now recover an **intrinsic characterization** of Hilbert bimodules (the author would thereby like to thank Adam Skalski and Bartosz Kwasniewski for unraveling the key detail for this result):

**Proposition A.1.4** (see also [Kak13, lemma 2.1]). *Consider a correspondence as seen in (A.1) and regard the orthogonal complement from Proposition A.1.3*

$$\ker(A \curvearrowright X)^\perp = \{a \mid \ker(A \curvearrowright X) = 0\} \subseteq A$$

and simultaneously identified as in that proposition by their action

$$A \supseteq \ker(A \curvearrowright X)^\perp = \left( \ker(A \curvearrowright X)^\perp \curvearrowright X \right) \subseteq \mathcal{L}(X|X).$$

Then the correspondence defines a Hilbert bimodule if and only if the compact operators (see Proposition A.1.1) all lie within the orthogonal complement

$$|X\rangle\langle X| \subseteq \ker(A \curvearrowright X)^\perp \subseteq \mathcal{L}(X|X)$$

while the dual pairing is always given by

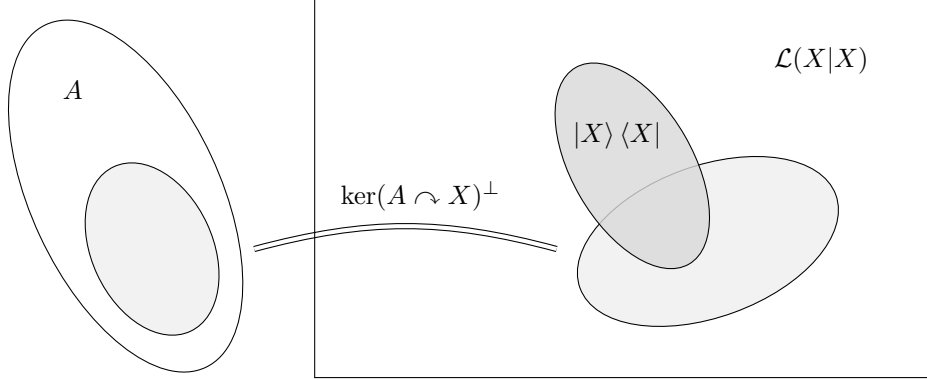
$$|-\rangle\langle -| : X \times X \rightarrow A : \quad |X\rangle\langle X| \subseteq \ker(A \curvearrowright X)^\perp \subseteq A$$

whence there is no ambiguity left anymore. As such the notion of a Hilbert bimodule **defines a pure property without any additional structure.**

Before we begin with the proof, we note that the situation above may be easiest pictured in mind (and remembered) by the following illustration:

## A.1. Correspondences

---



As such a correspondence in general may be seen as a Hilbert bimodule with a partial dual pairing. With this picture in mind let us get to the proof.

*Proof.* Instead of Hilbert bimodules, it suffices to consider the case of Hilbert modules (meaning the case of a single action and pairing) say given by

$$\langle X|X \rangle \subseteq A, \quad XA \subseteq X.$$

The kernel for the single action (in this case from the right) agrees with the orthogonal complement for the pairing (and so its support ideal):

$$\ker(X \curvearrowright A) = \langle X|X \rangle^\perp (= \text{supp}(X)^\perp).$$

Indeed using Blanchard factorization [Bla96, Lemma 1.3] this immediately follows from

$$\langle X|X \rangle a = 0 \iff \left( |X \rangle = |X \rangle \langle X|X \right) a = 0.$$

In turn this observation tells us where to search for the pairing:

$$\langle X|X \rangle \subseteq \langle X|X \rangle^{\perp\perp} = \ker(X \curvearrowright A)^\perp$$

Replacing the right action from our consideration above by the left action from the proposition we thus just revealed the condition from the proposition. Conversely, given the condition from the proposition we may simply retrieve the dual pairing using the isometric image of the orthogonal complement. Using Dirac calculus from Proposition A.1.1 we finally obtain (simply as composition)

$$|x \rangle \circ \left( \langle y| \circ |z \rangle \right) = \left( |x \rangle \circ \langle y| \right) \circ |z \rangle$$

and so the traditional compatibility (between pairings) holds trivially.  $\square$

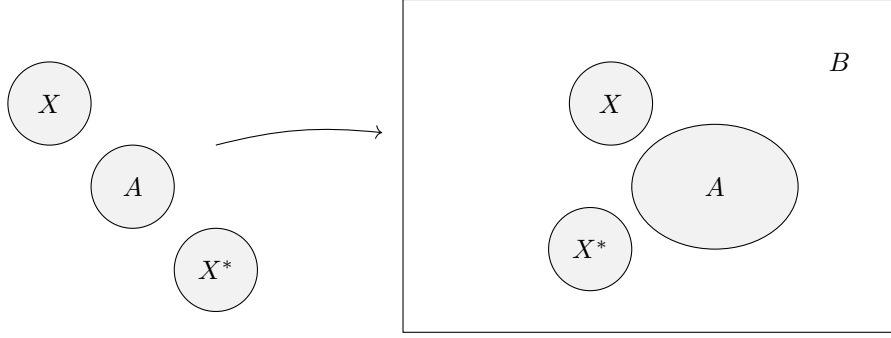


## A.2 Representations

Consider an abstract correspondence as introduced in the first section:

$$X^*X \subseteq A, \quad XA \subseteq X, \quad AX \subseteq X.$$

This structure is in some sense freely floating, and so we wish to embed this structure as a whole into an ambient operator algebra as illustrated:



A good analogy here is the embedding of Fell bundles into any crossed product. This is what we understand as a representation. More precisely, that is a representation of both the correspondence and the coefficient algebra into some ambient operator algebra say

$$(X, A) \longrightarrow B : \quad \tau : X \rightarrow B, \quad \varphi : A \rightarrow B$$

so the former being a morphism of vector spaces and the latter a morphism of operator algebras and such that the pair is coherent with the structure in between, which now reads in Dirac formalism (see Proposition A.1.1):

$$\varphi(x^*y) = \tau(x)^*\tau(y), \quad \tau(xa) = \tau(x)\varphi(a), \quad \tau(ax) = \varphi(a)\tau(x).$$

It is well-known that the latter follows automatically from the former two: Indeed using the  $C^*$ -identity we find (written in Dirac formalism)

$$\begin{aligned} & \left[ \varphi(a)\tau(x) - \tau(ax) \right]^* \left[ \varphi(a)\tau(x) - \tau(ax) \right] = \\ &= \varphi(a)^*\tau(x)^*\tau(x)\varphi(a) - \varphi(a)^*\tau(x)^*\tau(ax) - \tau(ax)\varphi(a)\tau(x) + \tau(ax)^*\tau(ax) \\ &= \varphi(x^*a^*ax) - \varphi(x^*a^*ax) - \varphi(x^*a^*ax) + \varphi(x^*a^*ax) = 0. \end{aligned}$$

## A.2. Representations

---

Consider now the structure as embedded within the ambient operator algebra

$$X = \tau(X) \subseteq B \quad \text{and} \quad A = \varphi(A) \subseteq B.$$

In particular, viewing them as a subspace and a subalgebra render the abstract inclusions from the previous section into actual corresponding inclusions

$$X^*X \subseteq A \subseteq B, \quad XA \subseteq X \subseteq B, \quad AX \subseteq X \subseteq B.$$

We note that while these representations are evidently not faithful in general, one may always pass to its quotient correspondence which renders the representation faithful. This will define the first parameter for the classification.

We now wish to extend a representation to higher and mixed tensor powers. For this we recall the following result by Kajiwara–Pinzari–Watatani:

**Proposition A.2.1** ([KPW98, Lemma 2.1]). *Representations canonically extend to the tensor product and compact operators (denoted in Dirac formalism):*

$$\left( \begin{array}{l} \tau : X \rightarrow Y \\ \varphi : A \rightarrow B \end{array} \right) \implies \left( \begin{array}{ll} XX \rightarrow B : & \tau(xy) := \tau(x)\tau(y) \\ XX^* \rightarrow B : & \tau(xy^*) := \tau(x)\tau(y)^* \end{array} \right)$$

The latter further satisfies the relations

$$\begin{aligned} KL \in (XX^*)(XX^*) \subseteq (XX^*) : & \quad \tau(K)\tau(L) = \tau(KL) \\ aKb \in A(XX^*)A \subseteq (XX^*) : & \quad \varphi(a)\tau(K)\varphi(b) = \tau(aKb) \end{aligned}$$

and so defines in particular a morphism of operator algebras.

Furthermore, suppose the morphism is isometric on the coefficient algebra.

Then it is so also on the correspondence and on compact operators:

$$\begin{aligned} \|(A \rightarrow B)\text{-}\| = \|\text{-}\| & \implies \|(X \rightarrow B)\text{-}\| = \|\text{-}\|, \\ \|(A \rightarrow B)\text{-}\| = \|\text{-}\| & \implies \|(XX^* \rightarrow B)\text{-}\| = \|\text{-}\|. \end{aligned}$$

By iteration the proposition includes all higher and any other mixed powers.

*Proof.* We provide the extension to compact operators since it demonstrates the use of Dirac calculus with a **neat trick by Kajiwara–Pinzari–Watatani**: We need to verify that the formal linear assignment on elementary compact operators remains bounded

$$\left\| \tau \left( \sum_n x_n y_n^* \right) \right\| \leq \left\| \sum_n x_n y_n^* \right\|, \quad \forall x_n, y_n \in X,$$

## A.2. Representations

---

whence the assignment allows an (a posteriori well-defined) extension to the completion of all compact operators. For this we invoke matrix calculus to reformulate the linear sum as a product of matrices

$$(x)(y^*) := \sum_n x_n y_n^* = \begin{pmatrix} x_1 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1^* \\ \vdots \\ y_N^* \end{pmatrix}.$$

Reformulated, the quite clever trick by Kajiwara Pinzari and Watatani is now to use the  $C^*$ -identity (generalized to matrices of adjointable operators):

$$\|xy^*\|^2 = \|x(y^*y)x^*\| = \|x\sqrt{y^*y}\| = \|\sqrt{y^*y}(x^*x)\sqrt{y^*y}\| = \|\sqrt{x^*x}\sqrt{y^*y}\|.$$

Note this automatically invokes matrix inflations since for example

$$x^*x = \begin{pmatrix} \langle x_1 | \\ \vdots \\ \langle x_N | \end{pmatrix} \begin{pmatrix} |x_1\rangle & \cdots & |x_N\rangle \end{pmatrix} = \begin{pmatrix} \langle x_1 | x_1 \rangle & \cdots & \langle x_1 | x_N \rangle \\ \vdots & \ddots & \vdots \\ \langle x_N | x_1 \rangle & \cdots & \langle x_N | x_N \rangle \end{pmatrix}.$$

On the other hand, this equally applies when invoking the representation and so we obtain the desired bound

$$\|\tau(xy^*)\| = \|\varphi(\sqrt{x^*x}\sqrt{y^*y})\| \leq \|\sqrt{x^*x}\sqrt{y^*y}\| = \|xy^*\|.$$

In turn this trick also applies for the correspondence and tensor product

$$X \rightarrow B : \quad \|\tau(x)\| = \|\varphi(x^*x)\| \leq \|x^*x\| = \|x\|,$$

$$XX \rightarrow B : \quad \|\tau(xy)\| = \|\varphi(y^*x^*xy)\| \leq \|y^*x^*xy\| = \|xy^*\|$$

as well as on every other type of tensor product such as of operator algebras.

Meanwhile, this moreover infers that isometricity passes from the coefficient algebra to the correspondence (and tensor product) and to compact operators. Moreover, the morphism satisfies the relations on compact operators since one easily verifies on elementary compact operators:

$$\varphi(a)\tau(xy^*)\varphi(b) = \varphi(a)\tau(x)\tau(y^*)\varphi(b) = \tau(axy^*b),$$

$$\tau(xy^*)\tau(zw^*) = \tau(x)\varphi(y^*z)\tau(w^*) = \tau(xy^*zw^*).$$

We therefore found the desired relations and the proof is complete.  $\square$

With this at hand we may now introduce the notion of covariances. We begin for

## A.2. Representations

---

this with the observation that (as in the previous proposition) we could have also defined the morphism (somewhat senseless)

$$A \supseteq \langle X|X \rangle \rightarrow B : \quad \tau(x^*y) := \tau(x)^*\tau(y)$$

which agrees with the original one on the coefficient algebra (somewhat trivially):

$$\begin{array}{ccc} \langle X|X \rangle & \xrightarrow{\quad\quad\quad} & B \\ \downarrow & \tau(-) = \varphi(-) & \parallel \\ A & \xrightarrow{\quad\quad\quad} & B. \end{array}$$

Suppose on the other hand that some elements also act as compact operators:

$$A \cap |X\rangle\langle X| := \left\{ a \in A \mid a \in |X\rangle\langle X| \right\} \subseteq A.$$

Then there is no evidence to believe that the induced morphism from the previous proposition would agree with the morphism for the algebra:

$$\begin{array}{ccc} A \cap |X\rangle\langle X| & \xrightarrow{\quad\quad\quad} & B \\ \downarrow & \tau(-) \neq \varphi(-) & \parallel \\ |X\rangle\langle X| & \xrightarrow{\quad\quad\quad} & B. \end{array}$$

Nevertheless the difference of morphisms surprisingly defines a morphism of operator algebras since (using the property from Proposition A.2.1)

$$\begin{aligned} (\varphi - \tau)a(\varphi - \tau)b &= \varphi(a)\varphi(b) - \varphi(a)\tau(b) - \tau(a)\varphi(b) + \tau(a)\tau(b) \\ &= \varphi(ab) - \tau(ab) - \tau(ab) + \tau(ab) = \varphi(ab) - \tau(ab). \end{aligned}$$

and so our representation decomposes into the sum of morphisms

$$\begin{aligned} &\left( A \cap |X\rangle\langle X| \longrightarrow B \right) = \tag{A.2} \\ &= \left( \begin{array}{ccc} A \cap |X\rangle\langle X| & & \\ \downarrow & & \\ |X\rangle\langle X| & \longrightarrow & B \end{array} \right) + \left( \begin{array}{ccc} A \cap |X\rangle\langle X| & \longrightarrow & B \\ \parallel & & \\ B & & \\ A \cap |X\rangle\langle X| & & \\ \downarrow & & \\ |X\rangle\langle X| & \longrightarrow & B \end{array} \right). \end{aligned}$$

As such we may capture the domain of equality by the covariance ideal:

$$\text{cov} \left( (X, A) \rightarrow B \right) := \ker \left( \text{---} \text{''} \text{---} \right) \trianglelefteq A. \tag{A.3}$$

### A.3. Kernel and Covariance

---

We will often drop the dependency on the coefficient algebra for simplicity. The covariance and kernel will classify the gauge-equivariant representations. Let us thus take a closer look at kernel morphisms and possible covariances:

### A.3 Kernel and Covariance

We begin this section with a characterization of kernel morphisms in the category of correspondences. For this let us introduce the general notion of morphisms between correspondences. As for representations, these are given by a pair of morphisms on the correspondence and the coefficient algebra

$$(X, A) \longrightarrow (Y, B) : \quad \tau : X \rightarrow Y, \quad \varphi : A \rightarrow B$$

where the former defines a linear morphism and the latter a morphism of operator algebras and such that the pair is coherent with the structure in between, which conveniently reads in Dirac formalism:

$$\varphi(x^*y) = \tau(x)^*\tau(y), \quad \tau(ax) = \varphi(a)\tau(x), \quad \tau(xa) = \tau(x)\varphi(a).$$

While the resulting category fails to be abelian (basically due to the algebraic morphism on the coefficient algebra) it still possesses all kernels and cokernels. As the notion of kernels and cokernels is nonstandard however in categories beyond abelian ones, let us give a quick introduction. Our category of correspondences and their morphisms has zero morphisms in the sense:

$$\begin{aligned} (X, A) \longrightarrow (Y, B) \xrightarrow{0} (Z, C) &= (X, A) \xrightarrow{0} (Z, C), \\ (X, A) \xrightarrow{0} (Y, B) \longrightarrow (Z, C) &= (X, A) \xrightarrow{0} (Z, C). \end{aligned}$$

A kernel for a morphism is the universal annihilating morphism:

$$(? , ?) \xrightarrow{\text{ker}} (X, A) \longrightarrow (Y, B) = 0$$

That is any other annihilating morphism factors uniquely over the kernel:

$$\begin{aligned} (X', A') \longrightarrow (X, A) \longrightarrow (Y, B) &= 0 \\ \implies (X', A') \dashrightarrow (? , ?) \xrightarrow{\text{ker}} (X, A). \end{aligned}$$

### A.3. Kernel and Covariance

---

Dually one may define the cokernel of morphisms. Moreover, we define a short exact sequence denoted by

$$0 \longrightarrow (?, ?) \longrightarrow (X, A) \longrightarrow (?, ?) \longrightarrow 0$$

whenever each side is the kernel (respectively cokernel) of the other:

$$\begin{aligned} (?, ?) &\longrightarrow (X, A) = \ker \left( (X, A) \longrightarrow (?, ?) \right) \\ (X, A) &\longrightarrow (?, ?) = \operatorname{coker} \left( (?, ?) \longrightarrow (X, A) \right) \end{aligned}$$

We will fill the question marks in the proposition below.

But before, we note the following equivalent notions of invariant and hereditary ideals which date all the way back to Pimsner and as explicitly coined by Kajiwara–Pinzari–Watatani (we refrain from the notion of negatively invariant ideals as we will find those from a different perspective later on):

**Lemma A.3.1** ([Pim97, Lemma 3.5 following] and [KPW98, section 4];

see further also [FMR03, section 2] and similarly also [Kat07, section 1]):

*The notion of invariant and hereditary ideals coincide. More precisely, one has the characterization for ideals in the coefficient algebra  $K \trianglelefteq A$ ,*

$$XK = \{x \in X \mid X^*x \in K\} = \{x \in X \mid x^*x \in K\}, \quad (\text{A.4})$$

and so it furthermore holds the equivalence

$$X^*KX \subseteq K \iff KX \subseteq XK. \quad (\text{A.5})$$

*The former is generally referred to as invariant and the latter as hereditary ideal.*

*Proof.* Katsura gives a neat proof based on some factorization result by Lance, more precisely [Lan95, Lemma 4.4], which itself however still requires some rather technical approximation. Instead we may verify the equivalence with the following fairly elementary observations: Let us first note that both of the right-hand spaces are automatically linear and closed by Cohen–Hewitt factorisation:

$$KX = \overline{\operatorname{span}} KX \subseteq X, \quad XK = \overline{\operatorname{span}} XK \subseteq X.$$

Now we clearly have the forward inclusions since

$$X^*(XK) = (X^*X)K \subseteq AK \subseteq K.$$

### A.3. Kernel and Covariance

---

Conversely we have using any approximate identity for the ideal

$$x^*x \in K \implies (1-e)x^*x(1-e) \rightarrow 0 \implies x = \lim_e(xe) \in XK.$$

With the characterization at hand we further obtain the forward direction

$$X^*KX \subseteq K \implies KX \subseteq XK : (X^*K^*)(KX) = X^*KX \subseteq K.$$

Alternatively, one may verify the forward direction using Blanchard factorization (whose original proof is very neat and elementary, see [Bla96, Lemma 1.3]):

$$KX = (KX)(KX)^*(KX) = (KX)(X^*KX) \subseteq (KX)K \subseteq XK$$

This however yet implicitly invokes the rather technical Cohen–Hewitt factorisation. If one wishes to refrain from using Cohen–Hewitt altogether, one may also argue in the following elementwise way (formulated in Dirac notation):

$$\begin{aligned} K|X\rangle \ni k|x\rangle = |y\rangle\langle y|y\rangle &\implies \langle y|y\rangle^3 = \langle x|k^*k|x\rangle \in K \\ \implies \langle y|y\rangle = \sqrt[3]{\langle x|k^*k|x\rangle} \in K &\implies k|x\rangle = |y\rangle\langle y|y\rangle \in XK. \end{aligned}$$

All of these variants for the forward direction have their advantage. For the converse direction we may simply argue

$$KX \subseteq XK \implies X^*KX \subseteq X^*XK \subseteq AK \subseteq K.$$

So the notion of invariant and hereditary ideals coincide. □

Let us give an example to illuminate the notion of hereditary ideals:

**Example A.3.2** (Graph correspondences: hereditary ideals):

Consider a graph correspondence as in Example A.1.2:

$$X = \ell^2(E = \text{edges}), \quad A = c_0(V = \text{vertices}).$$

Then every ideal corresponds to some collection of vertices

$$K = c_0(S = \text{some vertices}) \trianglelefteq c_0(V) = A$$

and hereditary ideals become the hereditary collection of vertices

$$c_0(S)\ell^2(E) \subseteq \ell^2(E)c_0(S) \iff SE \subseteq ES$$

### A.3. Kernel and Covariance

---

which reads written out in words

$$\text{range}(\text{given edge}) \in S \implies \text{source}(\text{given edge}) \in S$$

and whence **their name: hereditary ideals.**

In order to understand kernel morphisms (and so in turn also cokernel morphisms) we make the following observation: Suppose a morphism vanishes on the coefficient algebra, then it does so also on the entire correspondence:

$$(A \rightarrow B) = 0 \implies (X \rightarrow Y) = 0 \quad (\text{A.6})$$

Indeed one easily verifies (in a way using the  $C^*$ -identity)

$$\tau(x) = 0 \implies \varphi(x^*x) = \tau(x)^*\tau(x) = 0 \implies x^*x = 0 \implies x = 0$$

which is equivalently the commutative diagram (see the previous section)

$$\begin{array}{ccc} \langle X|X \rangle & \xrightarrow{\quad\quad\quad} & 0 \\ \downarrow & \tau(-) = \varphi(-) & \parallel \\ A & \xrightarrow{\quad\quad\quad} & 0 \subseteq B. \end{array}$$

As such one may already expect the kernel of morphisms to involve the kernel on the coefficient algebra in some crucial way. With this in mind, we may now give a new intrinsic characterization of kernel and cokernel morphism.

Meanwhile the author would like to note that the idea to consider concrete kernel correspondences (by invariant ideals) dates back to [Pim97] and their quotient correspondence to [KPW98] with their representations already appearing in the proof of [KPW98, theorem 4.3] and as explicitly in [FMR03].

The author identified those **as partial results on** the intrinsic characterisation of categorical **kernel and cokernel morphisms**, which further expanded and completed provide the **following entire classification** of kernel and cokernel morphisms in the category of correspondences, which we promote as theorem due to its usefulness (also in upcoming work).

**Theorem A.3.3** (Classification: kernel and cokernel morphisms):

*The category of correspondences has all kernel and all cokernel morphisms.*

*More precisely, they are all of the special form*

$$0 \longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0$$



### A.3. Kernel and Covariance

---

precisely for the invariant (and equivalently hereditary) ideals

$$K \trianglelefteq A : \quad X^* K X \subseteq K \quad (\iff K X \subseteq X K).$$

Conversely, given a morphism its kernel is given by

$$(XK, K) \xrightarrow{\ker} (X, A) \longrightarrow (Y, B) : \quad K = \ker(A \rightarrow B) \trianglelefteq A.$$

As a consequence, its cokernel is given as (within the image)

$$(X, A) \longrightarrow (Y, B) \xrightarrow{\text{coker}} \left( \frac{Y}{YL}, \frac{B}{L} \right) : \quad L = \sum_{n \geq 0} \langle Y^n | BAB | Y^n \rangle \trianglelefteq B.$$

Furthermore, the cokernel morphisms are precisely **the surjective morphisms**.

Before we begin with the proof for the above proposition let us give a quick clarification on the quotient norm (and a rather quite simplified proof thereof):

**Lemma A.3.4** (compare with [FMR03, Lemma 2.1] and [Kat07, Lemma 1.5]):

*The norm for a quotient correspondence seen as a Hilbert module agrees with the norm for the quotient seen as a Banach space,*

$$\|x + XK\|_{\text{Hilbert}}^2 = \inf \| \langle x|x \rangle + K \| = \inf \|x + XK\|^2 = \|x + XK\|_{\text{Banach}}^2.$$

*In particular, the quotient correspondence is complete (by Cohen–Hewitt).*

*Proof.* We give a simplified proof as compared to both references:

On the one hand we have the obvious inclusion

$$\|x + XK\|^2 \subseteq \| \langle x + XK | x + XK \rangle \| \subseteq \| \langle x|x \rangle + K \|$$

and as such the bound (when applying infima)

$$\|x + XK\|_{\text{Banach}}^2 = \inf \| \langle x|x \rangle + K \| \leq \inf \|x + XK\|^2 = \|x + XK\|_{\text{Hilbert}}^2.$$

For the converse implication we may invoke the well-known identification for the quotient norm in case of an operator algebra and ideal (**the simplifying trick**)

$$\| \langle x|x \rangle + K \| = \lim_{e \rightarrow 1} \| (1 - e) \langle x|x \rangle (1 - e) \| = \lim_{e \rightarrow 1} \|x - xe\|$$

where the limit runs over some or any approximate identity for the ideal.

As such we obtain in particular the converse inclusion

$$\left\{ \|x - xe\|^2 \mid e \rightarrow 1 \right\} \subseteq \|x + XK\|^2$$

### A.3. Kernel and Covariance

---

and so also the desired converse bound

$$\|x + XK\|_{\text{Hilbert}}^2 = \inf \|x + XK\|^2 \leq \lim_{e \rightarrow 1} \|(1-e)\langle x|x\rangle(1-e)\| = \|x + XK\|_{\text{Banach}}^2.$$

So the Hilbert module norm and the Banach space norm coincide.  $\square$

Having clarified the completeness for the quotient correspondence, we may now savely get to the desired kernel and cokernel morphisms.

*Proof of theorem A.3.3.* We begin with the statements about kernel morphisms. We first verify that each morphism admits a (categorical) kernel given by

$$(XK, K) = \ker( (X, A) \longrightarrow (Y, B) ) : \quad K = \ker(A \rightarrow B).$$

Clearly the kernel annihilates the morphism due to our observation (A.6):

$$\left( \ker(A \rightarrow B) \rightarrow A \rightarrow B \right) = 0 \quad \implies \quad \left( X \ker(A \rightarrow B) \rightarrow X \rightarrow Y \right) = 0$$

Meanwhile, let us also note that any such ideal is invariant:

$$\begin{aligned} \langle X | \ker(A \rightarrow B) | X \rangle \subseteq \ker(A \rightarrow B) : \quad (A \rightarrow B) \left( \langle X | \ker(A \rightarrow B) | X \rangle \right) &= \\ = (Y \leftarrow X) \langle X | \cdot (A \rightarrow B) \ker(A \rightarrow B) \cdot (Y \leftarrow X) | X \rangle &= 0. \end{aligned}$$

Conversely, suppose another morphism annihilates from the left

$$(W, D) \longrightarrow (X, A) : \quad (W \rightarrow X \rightarrow Y) = 0, \quad (D \rightarrow A \rightarrow B) = 0.$$

(The annihilation suffices on coefficient algebras due to our observation above.) Trivially, we have the desired inclusion at the level of coefficient algebras:

$$(D \rightarrow A \rightarrow B) = 0 \quad \implies \quad D \rightarrow \ker(A \rightarrow B)$$

On the other hand, recall that the induced morphism on the support ideal coincides with the morphism at the level of coefficient algebras. As such we necessarily also have an inclusion into the kernel,

$$\begin{array}{ccccc} \langle W|W \rangle & \dashrightarrow & \ker(A \rightarrow B) & \longrightarrow & A \\ \downarrow & & \parallel & & \parallel \\ D & \longrightarrow & \ker(A \rightarrow B) & \longrightarrow & A. \end{array}$$

### A.3. Kernel and Covariance

---

We however have the equivalence (using the characterization (A.4)):

$$W \rightarrow X \ker(A \rightarrow B) \iff \langle W|W \rangle \rightarrow \ker(A \rightarrow B).$$

As such we obtain also the desired inclusion at the level of correspondences,

$$D \rightarrow \ker(A \rightarrow B) \implies W \rightarrow X \ker(A \rightarrow B).$$

So there exists also a unique factorization over the kernel correspondence above. Every morphism thus admits a kernel given by the above kernel morphism.

We next verify that each such morphism is indeed some kernel, namely the kernel from the short exact sequence in the proposition:

$$\begin{aligned} 0 &\longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0 : \\ \ker\left( (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \right) &= (XK, K) \longrightarrow (X, A). \end{aligned}$$

For this let us first verify that the quotient gives a well-defined correspondence

$$\left(\frac{X}{XK}\right)^* \left(\frac{X}{XK}\right) \subseteq \left(\frac{A}{K}\right), \quad \left(\frac{X}{XK}\right)\left(\frac{A}{K}\right) \subseteq \left(\frac{X}{XK}\right), \quad \left(\frac{A}{K}\right)\left(\frac{X}{XK}\right) \subseteq \left(\frac{X}{XK}\right).$$

In other words we need to verify the relations

$$(XK)^*X \subseteq K, \quad X^*(XK) \subseteq K, \quad \dots, \quad (K \subseteq A)X \subseteq (XK).$$

The only nontrivial one here (which is not guaranteed automatically) is the last one which precisely calls for hereditary ideals (equivalently invariant ideals)

$$(K \subseteq A)X = KX \subseteq XK \quad (\iff KX \subseteq XK)$$

and so the quotient gives a well-defined correspondence.

Let us now get to its kernel. We already found the unique (categorical) kernel of morphisms above. As such the desired equality now easily follows from

$$\ker\left( (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \right) = \left( X \ker(A \rightarrow \frac{A}{K}), \ker(A \rightarrow \frac{A}{K}) \right) = (XK, K).$$

So far about kernel morphisms for correspondences. Let us now get to cokernel morphisms. We first verify that each cokernel is also of the special form as above

$$\begin{aligned} 0 &\longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0 : \\ \text{coker}\left( (XK, K) \longrightarrow (X, A) \right) &= (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right). \end{aligned}$$

### A.3. Kernel and Covariance

---

Clearly the quotient annihilates the kernel (using our observation (A.6)):

$$(K \rightarrow A \rightarrow \frac{A}{K}) = 0 \implies (XK \rightarrow X \rightarrow \frac{X}{XK}) = 0$$

Conversely, given any other annihilating morphism say

$$(X, A) \xrightarrow{(\tau, \varphi)} (Y, B) : (XK \rightarrow X \rightarrow Y) = 0, \quad (K \rightarrow A \rightarrow B) = 0.$$

Then both morphisms factor uniquely over the quotient (as linear maps):

$$\begin{array}{ccccccc} 0 & \longrightarrow & (XK, K) & \longrightarrow & (X, A) & \longrightarrow & (\frac{X}{XK}, \frac{A}{K}) \longrightarrow 0 \\ & & \downarrow & & \downarrow (\tau, \varphi) & & \downarrow \text{dashed} \\ 0 & \longrightarrow & 0 & \longrightarrow & (Y, B) & \xlongequal{\quad} & (Y, B) \longrightarrow 0. \end{array}$$

So it remains to verify that the factorization defines a morphism of correspondences. That however basically follows from the original morphism (using congruence classes):

$$\begin{aligned} \varphi\left((x + XK)^*(y + XK)\right) &= \varphi(x^*y) = \tau(x)^*\tau(y) = \tau(x + XK)^*\tau(y + XK), \\ \tau\left((x + XK)(a + K)\right) &= \tau(xa) = \tau(x)\varphi(a) = \tau(x + XK)\varphi(a + K). \end{aligned}$$

Recall here that the coherence with the left action follows automatically,

$$\implies \tau\left((a + K)(x + XK)\right) = \varphi(a + K)\tau(x + XK).$$

So we have found the desired cokernel:

$$\text{coker}\left( (XK, K) \longrightarrow (X, A) \right) = \left( \frac{X}{XK}, \frac{A}{K} \right).$$

We now wish to find the cokernel for general morphisms:

$$\text{coker}\left( (X, A) \longrightarrow (Y, B) \right) = (?, ?)$$

For this we may now make use of the following relation to our advantage: The kernel and cokernel operator satisfy the Galois connection (verbatim from [Lan78, section VIII.1] and confer further [Fre64, chapter 1 and 2]):

$$\ker \text{coker} \ker = \ker, \quad \ker \ker = 0 = \text{coker} \text{coker}, \quad \text{coker} \ker \text{coker} = \text{coker}.$$

### A.3. Kernel and Covariance

---

On the other hand, we have already found the form of kernel morphisms:

$$\ker \left( (Y, B) \rightarrow (?, ?) \right) = (YL, L) \rightarrow (Y, B).$$

as well as the cokernel for kernel morphisms

$$\text{coker} \left( (YL, L) \rightarrow (Y, B) \right) = (Y, B) \rightarrow \left( \frac{Y}{YL}, \frac{B}{L} \right).$$

We may thus combine these with the Galois connection between the kernel and cokernel operator to find the necessary shape of cokernel morphisms:

$$\begin{aligned} (?, ?) &= \text{coker} \left( (X, A) \rightarrow (Y, B) \right) = \\ &= \text{coker} \ker \text{coker} \left( (X, A) \rightarrow (Y, B) \right) = \left( \frac{Y}{YL}, \frac{B}{L} \right). \end{aligned}$$

So we are left with finding the invariant ideal which determines the quotient. Consider for this the factorization over the image (kernel of cokernel)

$$\begin{array}{ccccccc} 0 & \longrightarrow & (X, A) & \xlongequal{\quad} & (X, A) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow \text{---} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (YL, L) & \longrightarrow & (Y, B) & \longrightarrow & \left( \frac{Y}{YL}, \frac{B}{L} \right) & \longrightarrow & 0. \end{array}$$

As such the invariant ideal (which determines the kernel) necessarily contains the image of the coefficient algebra

$$(X, A) \rightarrow (YL, L) \quad \Longrightarrow \quad (A \rightarrow L \rightarrow B)$$

and is necessarily also the smallest such ideal (generated by the image)

$$L = BAB + \langle Y | BAB | Y \rangle + \dots = \sum_{n \geq 0} \langle Y^n | BAB | Y^n \rangle \trianglelefteq B.$$

Indeed any larger invariant ideal defines nothing but a quotient beyond:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (YL, L) & \longrightarrow & (Y, B) & \longrightarrow & \left( \frac{Y}{YL}, \frac{B}{L} \right) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & (YL', L') & \longrightarrow & (Y, B) & \longrightarrow & \left( \frac{Y}{YL'}, \frac{B}{L'} \right) & \longrightarrow & 0 \end{array}$$

So we have also found the cokernel for general morphisms. For the final assertion we note that every cokernel is onto as easily seen from their special form

$$\left( X \rightarrow \frac{X}{XK} \rightarrow 0 \right) \quad \text{and} \quad \left( A \rightarrow \frac{A}{K} \rightarrow 0 \right).$$

### A.3. Kernel and Covariance

---

For the converse consider a surjective morphism  $(X, A) \rightarrow (Y, B)$  :

$$\left( X \rightarrow Y \rightarrow 0 \right) \quad \text{and} \quad \left( A \rightarrow B \rightarrow 0 \right).$$

Recall from above that the kernel and cokernel operator define a Galois connection and so there is no choice for our morphism (to define a cokernel) than to arise as its own coimage (cokernel of kernel)

$$\begin{aligned} (X, A) \rightarrow (Y, B) &= \text{coker} \left( \text{some morphism} \right) \\ &= \text{coker ker coker} \left( \text{some morphism} \right) = \text{coker ker} \left( (X, A) \rightarrow (Y, B) \right). \end{aligned}$$

Recall for this the factorisation over its coimage (as outlined before):

$$\begin{array}{ccccccc} 0 & \longrightarrow & (XK, K) & \xrightarrow{\text{ker}} & (X, A) & \xrightarrow{\text{coker ker}} & \left( \frac{X}{XK}, \frac{A}{K} \right) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & (Y, B) & \xlongequal{\quad} & (Y, B) & \longrightarrow & 0 \end{array}$$

So we need to verify that the factorisation defines an isomorphism.

For this we obtain as our morphism is onto (on the coefficient algebra):

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & A/K & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

So the factorisation defines an isomorphism on the level of coefficient algebras.

As a consequence, the factorisation is also faithful on the correspondence (recall for instance proposition [A.2.1](#)):

$$\ker \left( \frac{A}{K} \rightarrow B \right) = 0 \quad \implies \quad \ker \left( \frac{X}{XK} \rightarrow Y \right) = 0.$$

On the other hand, the factorisation is also onto (for the correspondence)

$$\text{im} \left( \frac{X}{XK} \rightarrow Y \right) = \text{im} \left( X \rightarrow \frac{X}{XK} \rightarrow Y \right) = \text{im} \left( X \rightarrow Y \right) = Y$$

and as such the factorisation defines an isomorphism as desired.

That is every surjective morphism defines a cokernel morphism as well.  $\square$

Let us identify such quotients in the context of directed graphs:

**Example A.3.5** (Graph correspondences: quotient graphs):

### A.3. Kernel and Covariance

---

Consider a graph correspondence as in Example A.1.2

$$X = \ell^2(E = \text{edges}), \quad A = c_0(\text{ver} = \text{vertices})$$

and its hereditary ideals given by hereditary collections (as in Example A.3.2)

$$K = c_0(S \subseteq \text{vertices}) \trianglelefteq A : \quad SE \subseteq ES.$$

It is already clear that its quotients themselves arise as a graph, simply since their coefficient algebras define discrete direct sums of vertices as in A.1.2:

$$0 \longrightarrow K = c_0(S) \longrightarrow A = c_0(V) \longrightarrow \frac{A}{K} = c_0(T := V \setminus S) \longrightarrow 0.$$

And indeed we may now simply reveal the quotient as

$$0 \longrightarrow XK = \ell^2(ES) \longrightarrow X = \ell^2(E) \longrightarrow \frac{X}{XK} = \ell^2(TE) \longrightarrow 0.$$

Thus its quotients simply arise as **the complementary graphs**.

We have now found everything about kernel and cokernel morphisms.

With this in our toolbox, we now proceed to covariances. For this we note that we may render any representation faithful: Indeed we may simply factor any representation over the quotient correspondence

$$\begin{array}{ccccccc} 0 & \longrightarrow & (XK, K) & \xrightarrow{\ker} & (X, A) & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \text{---} \\ 0 & \longrightarrow & 0 & \longrightarrow & (B, B) & \xlongequal{\quad} & (B, B) \longrightarrow 0. \end{array}$$

The resulting factorization is faithful (isometric) on the coefficient algebra and as such also on the entire correspondence (see proposition A.2.1):

$$K = \ker(A \rightarrow B) : \quad \ker\left(\frac{A}{K} \rightarrow B\right) = 0 \implies \ker\left(\frac{X}{XK} \rightarrow Y\right) = 0.$$

So we may always first pass to the (unique) quotient correspondence to render any representation faithful. This is the first step in the classification of gauge-equivariant representations. The second step is to understand the possibly occurring covariances in nature. This is captured as an important observation by Katsura below. For this let us recall Katsura's ideal

$$\max(X, A) = \ker(A \curvearrowright X)^\perp \cap XX^* \trianglelefteq A \tag{A.7}$$

### A.3. Kernel and Covariance

---

which we denoted as maximal ideal for reasons which will become clear in the result below, and we will often drop the dependence on the coefficient algebra for simplicity. Note also that we have already encountered this ideal in different context on Hilbert bimodules (see Proposition A.1.3 and A.1.4). With this ideal in mind let us get to Katsura's observation (which we split as a result on faithful representations and further as one on faithful morphisms in general):

**Proposition A.3.6** (First part of [Kat04, Proposition 3.3]):

*Consider an embedding into some ambient operator algebra*

$$(X, A) \subseteq B : \quad A \subseteq B \quad (\implies X \subseteq B).$$

*Then its covariance (A.3) lies perpendicular to the kernel*

$$\text{cov}( (X, A) \subseteq B ) \perp \ker(A \curvearrowright X).$$

*In particular it holds for general representations (and factored as above)*

$$(X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \dashrightarrow B :$$

$$0 \subseteq \text{cov} \left( \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow B \right) \subseteq \max \left( \frac{X}{XK}, \frac{A}{K} \right).$$

*So the range of possible covariances is bounded from above by Katsura's ideal, whence its name and notation as maximal ideal in (A.7).*

*Proof.* We recast the arguments from Katsura in our language. For this let us first recall the commutative diagram for the covariance ideal,

$$\begin{array}{ccc} \text{cov}(X \rightarrow B) \subseteq A & \longrightarrow & B \\ \downarrow & & \parallel \\ |X \rangle \langle X| & \longrightarrow & B. \end{array}$$

In our case the representation is also faithful (all the horizontal paths):

$$A \subseteq B, \quad X \subseteq B, \quad XX^* \subseteq B, \quad \dots$$

On the other hand, the covariance ideal defines an ideal in the coefficient algebra (as we have observed in the previous section). As such we also have

$$\ker(A \curvearrowright X) \text{cov}(X \rightarrow B) \subseteq \text{cov}(X \rightarrow B).$$



### A.3. Kernel and Covariance

---

So one may place this expression in the commutative diagram above to obtain

$$\begin{array}{ccc} \ker(A \curvearrowright X) \operatorname{cov}(X \rightarrow B) & \longrightarrow & B \\ \downarrow & & \parallel \\ \ker(A \curvearrowright X) |X\rangle\langle X| = 0 & \longrightarrow & B \end{array}$$

and simply trace back the desired orthogonality (using the top path).

The remaining points follow from the discussion preceding the proposition.  $\square$

Let us reveal the maximal covariance in the case of graph algebras:

**Example A.3.7** (Graph correspondences: maximal covariance):

*Consider a graph correspondence as in Example A.1.2:*

$$X = \ell^2(E = \text{edges}), \quad A = c_0(\text{vertices}).$$

*For these the trivially acting portion (the kernel for the left action) and the compactly acting portion correspond to sources and finite receivers:*

$$\ker(A \curvearrowright X) = c_0(a : |a \text{ edges}| = 0) = c_0(\text{sources}),$$

$$A \cap XX^* = c_0(a : |a \text{ edges}| < \infty) = c_0(\text{fin receivers}).$$

*As such the orthogonal complement reads together*

$$\begin{aligned} \max(X, A) &= \ker(A \curvearrowright X)^\perp \cap XX^* = \\ &= c_0(a : 0 < |a \text{ edges}| < \infty) = c_0(\text{regular}). \end{aligned}$$

*That is the maximal covariance corresponds to **the regular vertices** and as such any other covariance ideal corresponds to simply some collection of such.*

We finish this section with an investigation of covariances for morphisms between correspondences (as opposed to representations into operator algebras). We begin with faithful morphisms between correspondences. As for representations we have (compare Proposition A.2.1): Being faithful passes from the coefficient algebra to the correspondence (and to any other power):

$$(X, A) \rightarrow (Y, B) : \quad A \subseteq B \quad \implies \quad X \subseteq Y.$$

So the faithful morphisms may be seen as nothing but a subcorrespondence: That is those whose inner product already lies in the subalgebra and similar for the

### A.3. Kernel and Covariance

---

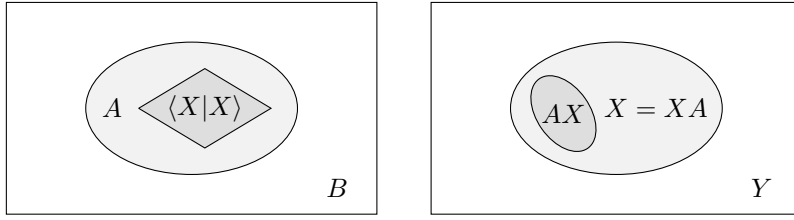
action from either side,

$$\begin{aligned} \langle Y|Y \rangle \subseteq B, \quad BY \subseteq Y, \quad YB \subseteq Y : \\ X \subseteq Y, \quad A \subseteq B : \quad \langle X|X \rangle \subseteq A, \quad AX \subseteq X, \quad XA \subseteq X \end{aligned} \tag{A.8}$$

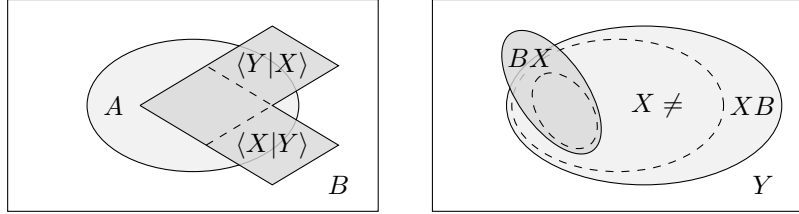
while for comparison mixed expressions only satisfy

$$\langle X|Y \rangle \subseteq B, \quad BX \subseteq Y, \quad XB \subseteq Y.$$

Schematically the inclusions for subcorrespondences may look like



which compare to mixed expressions as possibly only



So one may think of a subcorrespondence as some sort of coherent restriction: Think of global actions and their restrictions to possibly partial actions only. The action of compact operators on the ambient correspondence further reads

$$(XX^*)Y \subseteq X(Y^*Y) \subseteq XB \subseteq Y$$

Altogher, we may thus really think of a subcorrespondence simply as a subspace and subalgebra, which will be quite convenient also further on. In particular one has (for rather trivial reason)

$$(T - T')Y = 0 \quad \Longrightarrow \quad (T - T')X = 0$$

and as such also for  $a \in A$  and  $k \in XX^*$ ,

$$(a - k)Y = 0 \quad \Longrightarrow \quad (a - k)X = 0. \tag{A.9}$$

This basically trivial implication already verifies Katsura's second observation, and

### A.3. Kernel and Covariance

---

note how **thinking in terms of subspaces** and subalgebras paid off:

**Proposition A.3.8** (Second part of [Kat04, Proposition 3.3]):

For a subcorrespondence as in (A.8) the covariance arises as pullback

$$\begin{aligned} \text{cov} \left( X \subseteq Y \right) &= \left\{ a \in A \mid \text{im}(a \curvearrowright Y) \in \text{im}(XX^* \curvearrowright Y) \right\} \\ &= (A \curvearrowright Y)^{-1} \left( \text{im}(A \curvearrowright Y) \cap \text{im}(XX^* \curvearrowright Y) \right) \end{aligned}$$

which we usually abbreviate simply as their common intersection.

As a consequence it follows the characterization for trivial covariance

$$\text{cov} \left( (X, A) \subseteq B \right) = 0 \iff \text{im}(A \rightarrow B) \cap \text{im}(XX^* \rightarrow B) = 0$$

which reveals the familiar slogan **for the Toeplitz representation**:

Those whose coefficient algebra has trivial intersection with compact operators.

*Proof.* This is nothing but the trivial implication (A.9). □

We continue with the covariance for kernel and cokernel morphisms. We begin for this with the following fairly standard result due to Kajiwara–Pinzari–Watatani which will become particularly useful in later context:

**Proposition A.3.9** (see [KPW98, Proposition 4.2]):

A short exact sequence of correspondences (see theorem A.3.3)

$$0 \longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow 0$$

induces a morphism (of operator algebras)

$$\mathcal{L}(X) \longrightarrow \mathcal{L}\left(\frac{X}{XK}\right) : \quad T(x + XK) := Tx + XK$$

whose kernel admits the equivalent characterization

$$X^{-1}(K) := \left\{ X^*TX \subseteq K \right\} = \left\{ TX \subseteq XK \right\} = \ker \left( \mathcal{L}(X) \rightarrow \mathcal{L}\left(\frac{X}{XK}\right) \right). \quad (\text{A.10})$$

Further the morphism commutes with the left action by the coefficient algebra,

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & \frac{A}{AK} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{L}(X) & \longrightarrow & \mathcal{L}\left(\frac{X}{XK}\right) & & \end{array}$$

*Proof.* In order to verify the induced morphism let us first note the following:

### A.3. Kernel and Covariance

---

Adjointable operators restrict to kernel correspondences:

$$T \in \mathcal{L}(X) \implies T \in \mathcal{L}(XK) : \quad T(XK) \subseteq (TX)K \subseteq XK.$$

As such we obtain a commutative diagram and whence the adjointable operator descends also to the quotient,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & XK & \longrightarrow & X & \longrightarrow & \frac{X}{XK} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & T & & & & \\
 & & & & & & & & \\
 0 & \longrightarrow & XK & \longrightarrow & X & \longrightarrow & \frac{X}{XK} & \longrightarrow & 0.
 \end{array}$$

The resulting operator is adjointable as one easily verifies

$$\langle T(x + XK) \mid y + XK \rangle = \langle TX \mid y \rangle = \langle x \mid T^*y \rangle = \langle x + XK \mid T^*(y + XK) \rangle.$$

As such we got the desired morphism (of operator algebras)

$$\mathcal{L}(X) \longrightarrow \mathcal{L}\left(\frac{X}{XK}, \frac{A}{K}\right) : \quad T(x + XK) := Tx + XK.$$

We meanwhile note that the morphism is generally not onto!

Moreover, the morphism clearly commutes with the left action by the coefficient algebra as one easily verifies (think in terms of subspaces)

$$(a + K)(x + XK) = ax + XK = a(x + XK).$$

The remaining claim about the kernel immediately follows from (A.4).  $\square$

We continue with the analogous result for compact operators as for example in Fowler–Muhly–Raeburn (for which we provide a slightly simplified proof):

**Proposition A.3.10** (see [FMR03, Lemma 2.6]):

*A short exact sequence of correspondences (see theorem A.3.3)*

$$0 \longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0$$

*induces a short exact sequence at the level of compact operators*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (XK)(XK)^* = XKX^* & \longrightarrow & XX^* & \longrightarrow & \left(\frac{X}{XK}\right)\left(\frac{X}{XK}\right)^* & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{L}(X) & \longrightarrow & \mathcal{L}\left(\frac{X}{XK}\right) & & 
 \end{array}$$

### A.3. Kernel and Covariance

---

which restricts from the morphism of adjointable operators as indicated.

*Proof.* Clearly, the induced morphism on compact operators commutes with the morphism of adjointable operators since (think in terms of subspaces)

$$(x + XK)(y + XK)^*(z + XK) = xy^*z + XK = xy^*(z + XK).$$

Next we already know that the kernel correspondence (as a subcorrespondence) defines an embedding at the level of compact operators:

$$(XK, K) \subseteq (X, A) \implies (XK)(XK)^* \subseteq XX^*.$$

For kernel correspondences these now further define an ideal as for example

$$\left( (XK)(XK)^* = XKX^* \right) XX^* = XK(X^*XX^*) = XKX^*.$$

On the other hand, the right-hand morphism is also clearly onto since

$$X \longrightarrow \frac{X}{XK} \longrightarrow 0 \implies XX^* \longrightarrow \left( \frac{X}{XK} \right) \left( \frac{X}{XK} \right)^* \longrightarrow 0.$$

Regarding exactness in the middle, we may now invoke the characterization of the kernel from the previous proposition, which reads for compact operators

$$X^{-1}(K) \cap XX^* = \ker \left( XX^* \longrightarrow \left( \frac{X}{XK} \right) \left( \frac{X}{XK} \right)^* \right).$$

While the ideal clearly lies inside the kernel (left as an exercise for the reader) the converse inclusion now easily follows from the above description:

$$\begin{aligned} X^{-1}(K) \cap XX^* &= XX^* \left( X^{-1}(K) \cap XX^* \right) XX^* \\ &\subseteq X \left( X^* X^{-1}(K) X \right) X^* \subseteq XKX^*. \end{aligned}$$

So we have found the desired exactness for compact operators.  $\square$

We may now easily derive the desired covariance for kernel and cokernel morphisms (with the previous results in mind)

**Proposition A.3.11.** *Kernel and cokernel morphisms have full covariance,*

$$0 \longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow 0 :$$

$$\operatorname{cov} \left( (XK, K) \rightarrow (X, A) \right) = K \cap XKX^*,$$

$$\operatorname{cov} \left( (X, A) \rightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \right) = A \cap XX^*.$$

### A.3. Kernel and Covariance

---

*This stands in contrast to the covariance for representations:*

*The covariance for representations is bounded by Katsura's ideal (A.7).*

*Proof.* Note that kernel morphisms define a particular type of subcorrespondences and so we find ourselves in the situation of proposition A.3.8: The nontrivial converse of implication (A.9) holds for kernel correspondences,

$$a \in K, \quad k \in XKX^* : \quad (a - k)XKX^* = 0 \quad \implies \quad (a - k)X = 0.$$

Note that we may equivalently verify the implication

$$KX^*(a - k)X^*K = 0 \quad \implies \quad X^*(a - k)X = 0.$$

For this we note the inclusion (due to invariance):

$$X^*(a - k)X \subseteq X^*(K + XKX^*)X = (X^*KX) + (X^*X)K(X^*X) \subseteq K.$$

As such the above implication holds true since for

$$K \left( X^*(a - k)X \subseteq K \right) K = 0 \quad \implies \quad X^*(a - k)X = 0$$

so kernel morphisms have full covariance as desired.

Regarding the cokernel morphism, we may combine proposition A.3.9 and A.3.10 to obtain the desired full covariance

$$\begin{array}{ccc}
 A \cap XKX^* & \xrightarrow{\quad\quad\quad} & \frac{A}{K} \\
 \downarrow & & \downarrow \\
 \mathcal{L}(X) & \xrightarrow{\quad\quad\quad} & \mathcal{L}\left(\frac{X}{XK}\right) \\
 \downarrow & \nearrow \text{dashed} & \downarrow \\
 XX^* & \xrightarrow{\quad\quad\quad} & \left(\frac{X}{XK}\right)\left(\frac{X}{XK}\right)^*
 \end{array}$$

Concluding that kernel and cokernel morphisms have full covariance. □

Recall next that we may always render representations faithful by passing to the

### A.3. Kernel and Covariance

---

induced representation on the quotient

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (XK, K) & \xrightarrow{\ker} & (X, A) & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & (B, B) & \xlongequal{\quad} & (B, B) \longrightarrow 0.
 \end{array}$$

The previous result allows us now to further clarify the relation between the covariance on the quotient correspondence (the relevant one) and the covariance for the original representation (the author would like to note that he found this special instance as an observation made by Katsura in [Kat07]):

**Proposition A.3.12.** *Consider a cokernel morphism (simply some onto morphism as observed in proposition A.3.3) followed by an arbitrary morphism*

$$(X, A) \longrightarrow (Y, B) \longrightarrow 0 \quad \text{and} \quad (Y, B) \longrightarrow (Z, C).$$

Then the covariance for the composition arises as a pullback

$$\text{cov}(X \rightarrow Y \rightarrow Z) = (A \rightarrow B)^{-1} \text{cov}(Y \rightarrow Z) \cap XX^*.$$

In particular the covariance for a representation may be recovered from the induced representation on the quotient correspondence

$$\begin{array}{ccc}
 (X, A) & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) \dashrightarrow B : \\
 \text{cov}(X \rightarrow B) & = & (A \rightarrow \frac{A}{K})^{-1} \text{cov}\left(\frac{X}{XK} \rightarrow B\right) \cap XX^*
 \end{array}$$

which resembles Katsura's observation [Kat07, lemma 5.10 statement (v)].

*Proof.* The result now easily follows from our previous proposition: Indeed as cokernel morphisms have full covariance we obtain a covariance diagram for our cokernel and one for the arbitrary morphism

$$\left( \begin{array}{ccc} A \cap XX^* & \longrightarrow & B \cap YY^* \\ \downarrow & & \downarrow \\ XX^* & \longrightarrow & YY^* \end{array} \right) \quad \text{and} \quad \left( \begin{array}{ccc} \text{cov}(Y \rightarrow Z) & \longrightarrow & C \cap ZZ^* \\ \downarrow & & \downarrow \\ YY^* & \longrightarrow & ZZ^* \end{array} \right)$$

### A.3. Kernel and Covariance

---

and so also a commuting diagram for their composition

$$\begin{array}{ccccc}
 (A \rightarrow B)^{-1}(\dots) \cap XX^* & \longrightarrow & \text{cov}(Y \rightarrow Z) & \longrightarrow & C \cap YY^* \\
 \downarrow & & \downarrow & & \downarrow \\
 XX^* & \longrightarrow & YY^* & \longrightarrow & ZZ^*.
 \end{array}$$

That is the pullback lies within the covariance

$$(A \rightarrow B)^{-1} \text{cov}(Y \rightarrow Z) \cap XX^* \subseteq \text{cov}(X \rightarrow Z).$$

For the converse inclusion it suffices to establish

$$\begin{aligned}
 \text{cov}(X \rightarrow Z) &\subseteq (A \rightarrow B)^{-1} \text{cov}(Y \rightarrow Z) \\
 \iff (A \rightarrow B) \text{cov}(X \rightarrow Z) &\subseteq \text{cov}(Y \rightarrow Z)
 \end{aligned}$$

which is to verify the covariance diagram:

$$\begin{array}{ccccc}
 \text{cov}(X \rightarrow Z) & \longrightarrow & (A \rightarrow B) \text{cov}(X \rightarrow Z) & \longrightarrow & C \cap ZZ^* \\
 & & \downarrow & & \downarrow \\
 & & YY^* & \longrightarrow & ZZ^*
 \end{array} \quad ?$$

Indeed this may be easily seen by following the covariance diagram

$$\begin{array}{ccccc}
 \text{cov}(X \rightarrow Z) & \longrightarrow & (\dots) & \longrightarrow & C \cap ZZ^* \\
 \downarrow & & & & \downarrow \\
 XX^* & \longrightarrow & YY^* & \longrightarrow & ZZ^*
 \end{array}$$

and the full covariance for our cokernel morphisms (once more)

$$\begin{array}{ccccc}
 A \cap XX^* & \longrightarrow & B \cap YY^* & & \\
 \downarrow & & \downarrow & & \\
 XX^* & \longrightarrow & YY^* & \longrightarrow & ZZ^*.
 \end{array}$$

So the covariance for the composition arises from the pullback as desired.

The remaining statement arises now as special instance from above.  $\square$

We finish this section with the following negative result about the covariance for subcorrespondences: While we have found that kernel and cokernel morphisms have full covariance, this is not the case for subcorrespondences in general. That



### A.3. Kernel and Covariance

---

is simply the converse of implication (A.9) fails in general:

$$a \in A, \quad k \in XX^* : \quad (a - k)X = 0 \quad \not\Rightarrow \quad (a - k)Y = 0.$$

For this we consider the following somewhat minimal example:

**Example A.3.13** (Subcorrespondence with zero covariance):

Consider the direct sum of an operator algebra as both the coefficient algebra and the Hilbert module (which we depict as diagonal operators)

$$Y = \begin{pmatrix} D & \\ & D \end{pmatrix} = B \quad \Longrightarrow \quad Y^*Y \subseteq B, \quad YB \subseteq Y$$

but with left action given by the flip automorphism on  $B = D \oplus D$ :

$$\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \curvearrowright \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left[ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Regard the subcorrespondence given by the subalgebra (non-invariant ideal!)

$$\left( X = \begin{pmatrix} D & \\ & 0 \end{pmatrix} = A \right) \subseteq \left( Y = \begin{pmatrix} D & \\ & D \end{pmatrix} = B \right).$$

Then its left action vanishes identically (and in particular by compact operators):

$$A \curvearrowright X = \left[ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} D & \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right] \begin{pmatrix} D & \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \\ & D \end{pmatrix} \begin{pmatrix} D & \\ & 0 \end{pmatrix} = 0$$

On the other hand it never vanishes on the ambient correspondence

$$A \curvearrowright Y = \left[ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} D & \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right] \begin{pmatrix} D & \\ & D \end{pmatrix} = \begin{pmatrix} 0 & \\ & D \end{pmatrix} \begin{pmatrix} D & \\ & D \end{pmatrix} = \begin{pmatrix} 0 & \\ & D \end{pmatrix}.$$

As such the covariance diagram is **maximally noncommuting**:

$$\begin{array}{ccc} A = A \cap XX^* & \longrightarrow & B \\ \downarrow & \neq & \downarrow \\ XX^* & \longrightarrow & YY^* \end{array} : \quad \begin{array}{l} \ker \left( A \subseteq B \subseteq YY^* \right) = 0, \\ \ker \left( A \rightarrow XX^* \subseteq YY^* \right) = A. \end{array}$$

So one needs to stay cautious about the covariance **for subcorrespondences**:

In worst case one needs to verify a particular covariance by hand.

This finishes our section on kernel and cokernel morphisms on one hand, and the possible covariances on the other hand. With this at hand we may now proceed to the gauge-equivariant representations and their classification — in other words the classification of relative Cuntz–Pimsner algebras.

## A.4 Relative Cuntz–Pimsner algebras

We introduce in this section the relative Cuntz–Pimsner algebras and elaborate why and how these serve to classify the gauge-equivariant representations.

We begin with the Toeplitz algebra: That is the universal representation (more precisely the initial representation) as such that any other representation uniquely factors via the universal one,

$$\begin{array}{ccc} (X, A) & \xlongequal{\quad\quad\quad} & (X, A) \\ \downarrow & & \downarrow \\ \mathcal{T}(X, A) & \dashrightarrow & (B, B). \end{array}$$

Next recall from the previous section that we may render any representation faithful by passing to the quotient correspondence

$$\begin{array}{ccccccc} 0 & \longrightarrow & (XK, K) & \xrightarrow{\ker} & (X, A) & \longrightarrow & (\frac{X}{XK}, \frac{A}{K}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & (B, B) & \xlongequal{\quad\quad\quad} & (B, B) & \longrightarrow & 0 \end{array}$$

and recall from theorem A.3.3 that any such quotient arises precisely for some invariant ideal (equivalently hereditary ideal)

$$K = \ker(A \rightarrow B) \trianglelefteq A : \quad X^*KX \subseteq K \quad (\iff KX \subseteq XK).$$

On the other hand, we found in proposition A.3.6 that faithful representations have their covariance bounded from above by the maximal covariance,

$$0 \subseteq \text{cov}\left(\left(\frac{X}{XK}, \frac{A}{K}\right) \rightarrow B\right) \subseteq \max\left(\frac{X}{XK}, \frac{A}{K}\right).$$

As such we may aim to classify representations by pairs of some invariant ideal (as possible covariance) and another bounded ideal (as possible covariance):

$$\left( K \trianglelefteq A : X^*KX \subseteq K \mid I \trianglelefteq \frac{A}{K} : I \subseteq \max\left(\frac{X}{XK}\right) \right) \quad (\text{A.11})$$

To handle this task, we may consider the class of representations with kernel and covariance at least a given pair of invariant and bounded ideal,

$$(X, A) \longrightarrow (B, B) : \quad K \subseteq \ker(A \rightarrow B), \quad I \subseteq \text{cov}\left(\frac{X}{XK} \rightarrow B\right).$$

#### A.4. Relative Cuntz–Pimsner algebras

---

Note that any such class contains at least the trivial representations and furthermore that representations may well have larger kernel and covariance only (which we cannot exclude at this moment as we will explain below). The universal such representation defines now the relative Cuntz–Pimsner algebra for a given kernel–covariance pair as in (A.11):

$$\begin{array}{ccc}
 (X, A) & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) \equiv \left(\frac{X}{XK}, \frac{A}{K}\right) \\
 & & \downarrow \qquad \qquad \qquad \downarrow \\
 & & \mathcal{O}(K, I) \dashrightarrow (B, B).
 \end{array}$$

$K \subseteq \ker(A \rightarrow B)$   
 $I \subseteq \text{cov}\left(\frac{X}{XK} \rightarrow B\right)$

As such we obtain an entire “2-dimensional lattice” of relative Cuntz–Pimsner algebras (as class of universal representations) by first following along the lattice of cokernel morphisms (given by invariant ideals)

$$\begin{array}{ccccccc}
 & & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & (0, 0) \\
 & & \nearrow & & \nearrow & & \nearrow & & \nearrow \\
 & & \dots & \longrightarrow & \left(\frac{X}{XL}, \frac{A}{L}\right) & \longrightarrow & \left(\frac{X}{X(K+L)}, \frac{A}{K+L}\right) & \longrightarrow & \dots \\
 & & \nearrow & & \nearrow & & \nearrow & & \nearrow \\
 & & \dots & \longrightarrow & \left(\frac{X}{X(K \cap L)}, \frac{A}{K \cap L}\right) & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) & \longrightarrow & \dots \\
 & & \nearrow & & \nearrow & & \nearrow & & \nearrow \\
 (X, A) & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots
 \end{array}$$

#### A.4. Relative Cuntz–Pimsner algebras

---

followed by the lattice of covariance ideals (on the chosen cokernel)

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) & \longrightarrow & \dots & & \\
 & & \downarrow & & & & \\
 \mathcal{T}\left(\frac{X}{XK}, \frac{A}{K}\right) = \mathcal{O}(K, 0) & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \mathcal{O}(K, I \cap J) & \longrightarrow & \mathcal{O}(K, I) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \mathcal{O}(K, J) & \longrightarrow & \mathcal{O}(K, I + J) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(K, \max) = \mathcal{O}\left(\frac{X}{XK}, \frac{A}{K}\right)
 \end{array}$$

with Toeplitz algebras and absolute Cuntz–Pimsner algebras as extreme points.

We now explain the precise goal for our classification: We first note for this that a relative Cuntz–Pimsner algebra comes equipped with gauge-action rendering the representation gauge-equivariant (which we explain in the following section). As such the possible range of relative Cuntz–Pimsner algebras are at most the gauge-equivariant representations. The surprising first goal of their classification then states that every gauge-equivariant representation indeed arises itself as a relative Cuntz–Pimsner algebra. More precisely, consider first the induced representation rendering the representation faithful

$$(X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \dashrightarrow (B, B) : \quad K = \ker(A \rightarrow B)$$

followed by the covariance for the resulting representation

$$\left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow (B, B) : \quad I := \text{cov}\left(\left(\frac{X}{XK}, \frac{A}{K}\right) \subseteq B\right).$$

To better illuminate the problem let us restrict the representation to its range, that is the operator algebra generated by (the image of) the correspondence,

$$(X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow C^*(X \cup A) \subseteq B.$$

With this description the first problem states that

$$(X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow \mathcal{O}(K, I) = C^*(X \cup A).$$

#### A.4. Relative Cuntz–Pimsner algebras

---

Or put in other words, the pairs of invariant ideals as kernel and bounded ideals covariance exhaust the gauge-equivariant representations. We will solve this problem via the familiar gauge-invariant uniqueness theorem.

The second goal is to determine that in fact every possible kernel and covariance arises itself as an actual kernel and covariance. More precisely, consider any pair of invariant ideal as kernel and bounded ideal as covariance:

$$\left( \begin{array}{l} K \trianglelefteq A : \quad X^* K X \subseteq K \\ I \trianglelefteq A/K : \quad I \subseteq \max(X/XK) \end{array} \right)$$

Then there simply may be no such representation with precisely the given kernel and covariance ideal (so that not every such pair would actually arise):

$$(X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow (B, B) \quad :$$

$$\left( \ker(A \rightarrow B), \operatorname{cov}\left(\frac{X}{XK} \rightarrow B\right) \right) = (K, I) \quad ?$$

Or put in other words, two possibly different pairs of kernel and covariance ideal could in principle lead to one and the same relative Cuntz–Pimsner algebra:

$$\mathcal{O}(K, I) = \mathcal{O}(K', I') \quad \implies \quad (K, I) = (K', I') \quad ?$$

We will however find the following relations (which are both nontrivial):

$$(X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow \mathcal{O}(K, I) \quad :$$

$$\left( \ker\left(\frac{A}{K} \rightarrow \mathcal{O}(K, I)\right) = 0 \mid \operatorname{cov}\left(\frac{X}{XK} \rightarrow \mathcal{O}(K, I)\right) = I \right)$$

and as such also the desired kernel–covariance pair

$$\left( \ker(A \rightarrow \mathcal{O}(K, I)) = K \mid \operatorname{cov}\left(\frac{X}{XK} \rightarrow \mathcal{O}(K, I)\right) = I \right). \quad (\text{A.12})$$

For this the Fock representation will come into play: Its concrete representation allows us to actually compute its kernel and covariance and so to verify the desired relations. As such each possible pair of invariant ideal (as kernel) and bounded ideal (as covariance) arises itself as actual kernel and covariance. Summarizing these goals, the kernel–covariance pairs completely parametrize the entire lattice of gauge-equivariant representations. In other words, the lattice of relative Cuntz–Pimsner algebras (as schematically given above) classifies the entire lattice of gauge-equivariant representations.

With this at hand, we then further investigate the connecting morphisms

#### A.4. Relative Cuntz–Pimsner algebras

---

between relative Cuntz–Pimsner algebras. More precisely, we will find that our parametrisation given by kernel–covariance pairs defines a lattice isomorphism in the sense that

$$(K \subseteq L \mid “I \subseteq J”) \iff \mathcal{O}(K, I) \leq \mathcal{O}(L, J)$$

where the latter denotes the factorization (as common notation):

$$\begin{array}{ccc} (X, A) & \xlongequal{\quad\quad\quad} & (X, A) \\ \downarrow & & \downarrow \\ \mathcal{O}(K, I) & \dashrightarrow & \mathcal{O}(L, J). \end{array}$$

In [Kat07] this has been already observed using however so-called T-pairs instead: We will unravel these as nothing but transformed versions of our kernel–covariance pairs and as such give a natural interpretation for such pairs in general.

Following, we further give a precise description for when connecting morphisms exist between cokernel strands (as from where and to where)

$$\mathcal{O}(K, I = ??) \dashrightarrow \mathcal{O}(L, J) \quad \text{and} \quad \mathcal{O}(K, I) \dashrightarrow \mathcal{O}(L, \max) ??$$

which generalizes the results from Katsura’s work [Kat07]. Together we thus obtain a plethora of connecting morphisms between cokernel strands such as

$$\begin{array}{ccccc} \longrightarrow & (\frac{X}{XK}, \frac{A}{K}) & \longrightarrow & (\frac{X}{XL}, \frac{A}{L}) & \longrightarrow \\ & \downarrow & & \downarrow & \\ \longrightarrow & \mathcal{T}(\frac{X}{XK}, \frac{A}{K}) & \longrightarrow & \mathcal{T}(\frac{X}{XL}, \frac{A}{L}) & \longrightarrow \\ & \downarrow & & \downarrow & \\ \dashrightarrow & \mathcal{O}(K, I) & \dashrightarrow & \mathcal{O}(L, J) & \dashrightarrow \\ & \downarrow & & \downarrow & \\ & \mathcal{O}(\frac{X}{XK}, \frac{A}{K}) & & \mathcal{O}(\frac{X}{XL}, \frac{A}{L}) & \\ & \swarrow \quad \searrow & & \swarrow \quad \searrow & \end{array}$$

and note that plenty of connecting morphisms also will be missing. For this we will give some further examples from graph algebras to illuminate the lack of connecting morphism. With this in mind and without further ado, we now get to the class of equivariant representations.

## A.5 Gauge actions: Fourier spaces

We begin with the following observation to motivate gauge-equivariant representations: For every fixed complex number on the torus we may consider the automorphism which rotates the correspondence

$$z \in \mathbb{T} : (X, A) \longrightarrow (X, A) : z \curvearrowright x = zx, \quad z \curvearrowright a = a$$

which together define a circle action  $\mathbb{T} \curvearrowright (X, A)$ .

Clearly these do not affect the kernel of representations as for

$$(X, A) \longrightarrow B : \ker( A \xrightarrow{1} A \longrightarrow B ) = \ker( A \longrightarrow B ).$$

Similarly they do not affect the covariance as they act trivially on compacts,

$$zz^* = 1 : XX^* \longrightarrow XX^* : x(zz^*)y^* = xy^* :$$

$$\begin{array}{ccccc} A \cap XX^* & \xlongequal{\quad} & A \cap XX^* \subseteq A & \longrightarrow & B \\ \downarrow & & \downarrow & \xlongequal{=?} & \parallel \\ XX^* & \xlongequal{\quad} & XX^* & \longrightarrow & B. \end{array}$$

As such every relative Cuntz–Pimsner algebra (as a universal representation for some relation) admits a unique gauge action which renders its representation gauge-equivariant (think in terms of generators and relations):

$$\exists \mathbb{T} \curvearrowright \mathcal{O}(K, J) : \left( \mathbb{T} \curvearrowright (X, A) \right) \longrightarrow \left( \mathbb{T} \curvearrowright \mathcal{O}(K, I) \right).$$

We may thus hope at best to classify the relative Cuntz–Pimsner algebras amongst those representations which come along with some gauge-action rendering the representation equivariant. More precisely, that is written out

$$\begin{aligned} (\varphi, \tau) : \left( \mathbb{T} \curvearrowright (X, A) \right) &\longrightarrow \left( \mathbb{T} \curvearrowright B \right) : \\ z \curvearrowright \tau(x) &= \tau(z \curvearrowright x) = \tau(zx), \\ z \curvearrowright \varphi(a) &= \varphi(z \curvearrowright a) = \varphi(a). \end{aligned}$$

## A.5. Gauge actions: Fourier spaces

---

Equivalently these are the conventional gauge-equivariant representations of operator algebras (from any of the preceding relative Cuntz–Pimsner algebras)

$$\pi : \left( \mathbb{T} \curvearrowright \mathcal{O}(K, I) \right) \longrightarrow \left( \mathbb{T} \curvearrowright B \right) : \quad z \curvearrowright \pi(-) = \pi(z \curvearrowright -).$$

Consider now the relative Cuntz–Pimsner algebras (as universal representations) which allow factorizations for our gauge-equivariant representation:

$$\begin{aligned} (X, A) &\longrightarrow \mathcal{O}(K, I) \dashrightarrow B \\ \iff &\left( K \subseteq \ker(A \rightarrow B) \mid I \subseteq \text{cov}\left(\frac{X}{XK} \rightarrow B\right) \right) \end{aligned}$$

As such — if the gauge-equivariant representation has a chance of being a universal representation for some kernel–covariance pair — then certainly the best chance is given by the kernel–covariance pair for the representation itself:

$$\begin{aligned} (K, I) &= \left( \ker(A \rightarrow B) \mid \text{cov}\left(\frac{X}{XK} \rightarrow B\right) \right) : \\ (X, A) &\longrightarrow \mathcal{O}(K, I) \iff C^*(X \cup A) \subseteq B ? \end{aligned}$$

This is what the gauge-equivariant uniqueness theorem will establish in the next section and so also the first goal in the classification of relative Cuntz–Pimsner algebras. So let us prepare ourselves a bit more for this by taking a closer look at gauge-equivariant representations and their Fourier spaces.

It is well known that every operator algebra equipped with a circle action (such as our gauge-equivariant representations) comes along with Fourier spaces

$$B(n \in \mathbb{Z}) = \left\{ b \mid \left( b(z) := z \curvearrowright b \right) = z^n b \right\} \subseteq B$$

and in particular its fixed point algebra

$$B(n = 0) = \left\{ b \mid \left( b(z) := z \curvearrowright b \right) \equiv b \right\} \subseteq B$$

which define a Fell bundle over the integers

$$B(m)B(n) \subseteq B(m+n), \quad B(n)^* = B(-n)$$

together with a conditional expectation onto its fixed point algebra

$$E : B \longrightarrow B(0) : \quad E(b) = \int_{\mathbb{T}} b(z) dz$$



## A.5. Gauge actions: Fourier spaces

---

and more generally with projections onto any of its Fourier spaces

$$E_n : B \longrightarrow B(n) : E_n(b) = \int_{\mathbb{T}} z^{-n} b(z) dz.$$

Recall that conditional expectations given by averaging are automatically faithful since the state space separates the positive elements (and since averaging runs as Bochner integral):

$$\begin{aligned} b \geq 0 &\implies (b(z) = z \curvearrowright b) \geq 0 \implies SB(b(z) = z \curvearrowright b) \geq 0 : \\ E(b \geq 0) = 0 &\implies SB(E(b)) = SB\left(\int b(z) dz\right) = \int SB(b(z)) dz = 0 \\ \implies SB(b(z) = z \curvearrowright b) &\equiv 0 \implies SB(b = 1 \curvearrowright b) = 0 \implies b = 0. \end{aligned}$$

We meanwhile note that the Fourier spaces densely span the operator algebra by [Exe94, proposition 2.5] (which seems a rather intuitive yet nontrivial relation)

$$B = \overline{(\dots + B(-1) + B(0) + B(1) + \dots)} = \sum_n B(n)$$

and we encourage the reader to have a look into the beautiful proof by Exel: It does not require any Cesaro approximations and instead only invokes the elementary isomorphism (as the basic version of Coburn)

$$\begin{aligned} C^*(\mathbb{Z}) = C^*(u^*u = 1 = uu^*) &= C(\sigma u \mid u^*u = 1 = uu^*) = C(\mathbb{T}) \\ \implies C^*(\mathbb{Z}) &= \overline{(\dots + \mathbb{C}u^* + \mathbb{C}e + \mathbb{C}u + \dots)} = C(\mathbb{T}). \end{aligned}$$

**A short digression:** One may meanwhile wonder how it may be possible that any element may be approximated by Fourier sums while the series of its Fourier coefficients only converges in Cesaro mean (and generally diverges in norm). The answer to this question becomes evident when replacing for instance the torus with an open disk and the trigonometric by polynomial approximations:

$$\{\dots, 1/z, 1, z, z^2, \dots\} \subseteq C(\mathbb{T}) \rightsquigarrow \{1, z, z^2, \dots\} \subseteq C(\mathbb{D})$$

It is well-known here that any continuous function may be approximated in norm by polynomials (by Stone–Weierstrass). On the other hand, the convergent power series correspond to the class of Taylor analytic functions:

$$f(z) = \sum_n a_n z^n \iff f \in C^\infty(\mathbb{D})$$

## A.5. Gauge actions: Fourier spaces

---

The difference lies in the fact that a polynomial approximation of continuous functions generally changes each of the coefficient in the sequence — also the previously already set coefficients! As such one may think of the convergent Fourier series as a **generalization of analytic functions**.

We now further restrict our attention on the range of the representation (also since the ambient rest does not reflect the correspondence): More precisely, that is the operator algebra generated by (the image of) the correspondence,

$$(X, A) \longrightarrow C^*(X \cup A) \subseteq B \quad \rightsquigarrow \quad B = C^*(X \cup A).$$

Recall that a correspondence (and so its image under the representation) satisfies its defining inclusions as explained in more detail in section [A.1](#):

$$X^*X \subseteq A, \quad XA \subseteq X, \quad AX \subseteq X.$$

As such the range (as a generated operator algebra) admits “a fine structure”

$$\begin{aligned} C^*(X \cup A) = \overline{\text{span}} \left( \dots + (X^* + XX^*X^* + \dots) + \right. \\ \left. + (A + XX^* + XX^*X^* + \dots) + (X + XXX^* + \dots) + \dots \right) \end{aligned}$$

which we organised in groups according to the Fourier spaces they belong to. As such we also obtain “a fine structure” for the fixed point algebra

$$B(n=0) = E\left(B = C^*(X \cup A)\right) = \overline{\text{span}}\left(A + XX^* + XX^*X^* + \dots\right)$$

which allows us to deduce the gauge-invariant uniqueness theorem by induction. Note further that for any correspondence (including possibly degenerate ones!)

$$\begin{aligned} X^*X \subseteq A, \quad XA \subseteq X, \quad AX \subseteq X \\ \implies XA = X, \quad XXX^* = XX, \quad \dots \end{aligned}$$

As such the Fourier spaces further satisfy

$$\begin{aligned} B(n \geq 1) = E_n\left(B = C^*(X \cup A)\right) &= \overline{\text{span}}\left(X^n + X^nXX^* + \dots\right) \\ &= X^n\left(A + XX^* + XX^*X^* + \dots\right) = X^nB(0) \end{aligned}$$

where we left the linear spans and closures implicit for more pointy statements, and by involution also for negative Fourier spaces and as such altogether

$$B(n \geq 1) = X^nB(0), \quad B(n \leq -1) = B(0)X^{-n}.$$

## A.5. Gauge actions: Fourier spaces

---

As a consequence, the Fell bundle (restricted on the range) further satisfies

$$B(n \geq 1) = B(1)^n : \quad B(n) = X^n B(0) \subseteq B(1)^n B(0) \subseteq B(1)^n \subseteq B(n)$$

and similarly for its negative Fourier spaces, and as such defines altogether a semi-saturated Fell bundle (see for instance [Exe94, proposition 4.8]):

$$\begin{aligned} B(m \geq 0)B(n \geq 0) &= B(m+n), \\ B(m \leq 0)B(n \leq 0) &= B(m+n). \end{aligned}$$

These relations are quite useful for bipartite inflations as introduced in [MPT08]. With the knowledge about Fourier spaces at hand, we may now get back to the classification problem from above:

$$\begin{aligned} (K, I) &= \left( \ker(A \rightarrow B) \mid \text{cov}\left(\frac{X}{XK} \rightarrow B\right) \right) : \\ (X, A) &\longrightarrow \mathcal{O}(K, I) \iff C^*(X \cup A) \subseteq B ? \end{aligned}$$

At first we may pass to the quotient correspondence

$$(X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right) \dashrightarrow \mathcal{O}(K, I) \longrightarrow B$$

to obtain an honest embedding since

$$K = \ker(A \rightarrow B) \implies A/K \subseteq B \implies X/XK \subseteq B.$$

As such we may assume that our correspondence lies faithfully in the ambient operator algebra (upon replacing the original with the quotient correspondence)

$$\left(\frac{X}{XK}, \frac{A}{K}\right) \rightsquigarrow (X, A) \implies \mathcal{O}(K, I) \rightsquigarrow \mathcal{O}(0, I) \implies X \subseteq B, A \subseteq B \quad (\text{A.13})$$

and at the same time restrict our attention to its range:

$$C^*(X \cup A) \subseteq B \rightsquigarrow C^*(X \cup A) = B.$$

We may thus think of both ambient algebras as each some faithful completion for the correspondence, but under a priori possibly different norm topologies:

$$\mathcal{O}(X; I) = C^*(X \cup A) \longleftarrow X \cup A \longrightarrow C^*(X \cup A) = B$$

As another valuable perspective, one may also think of the above problem as gauge-equivariant envelopes (comparable to the maximal and minimal  $C^*$ -cover

## A.5. Gauge actions: Fourier spaces

---

for non-selfadjoint operator algebras):

$$\mathcal{T}(X) \longrightarrow \dots \longrightarrow \mathcal{O}(X; I) \longrightarrow Q \longrightarrow \dots : \quad (X, A) \subseteq (\mathbb{T} \curvearrowright Q).$$

The author would like to thank Elias Katsoulis for bringing closer this idea.

Next as our representation is gauge-equivariant it restricts in particular to fixed point algebras and further commutes with conditional expectations:

$$\begin{array}{ccc} \mathcal{O}(X; I) & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathcal{O}(X; I)(0) & \longrightarrow & B(0) \end{array} \quad \begin{array}{ccc} \mathcal{O}(X; I) & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathcal{O}(X; I)(0) & \longrightarrow & B(0) \end{array}$$

As such it suffices to verify the above equality on fixed point algebras:

$$\pi : \mathcal{O}(X; I)(0) \equiv B(0) \quad \implies \quad \pi : \mathcal{O}(X; I) \equiv B \quad (\text{A.14})$$

Indeed this follows as standard argument from the faithful conditional expectation (sufficiently for the relative Cuntz–Pimsner algebra):

$$\begin{aligned} \pi(a) = 0 &\implies \pi(a^*a) = 0 \implies \pi(E(a^*a)) = E(\pi(a^*a)) = 0 \\ &\implies E(a^*a) = 0 \implies a^*a = 0 \implies a = 0. \end{aligned}$$

Another interesting less commonly known **argument due to Exel** basically exploits that continuous functions, whose Fourier coefficients all vanish, already vanish identically themselves (see [Exe94, proposition 2.5 and 2.9]):

$$\begin{aligned} E_n(a)^* E_n(a) &\in \mathcal{O}(X; I)(n - n = 0) : \\ \pi(a) = 0 &\implies \pi(E_n(a)^* E_n(a)) = E_n(\pi(a))^* E_n(\pi(a)) \equiv 0 \\ &\implies E_n(a)^* E_n(a) \equiv 0 \implies E_n(a) \equiv 0 \implies a = 0. \end{aligned}$$

For the fixed point algebra we however found the “fine structure”

$$\mathcal{O}(X; I)(n = 0) = \overline{\text{span}} \left( A + XX^* + XXX^*X^* + \dots \right)$$

so the fixed point algebras arises as an increasing union (inductive limit)

$$\mathcal{O}(X; I)(n = 0) = \text{closure} \left( \bigcup_N \overline{\text{span}} \left( A + XX^* + \dots + X^N X^{-N} \right) \right).$$

## A.6. Uniqueness theorem

---

As such we may verify the faithfulness by induction along  $n \in \mathbb{N}$ :

$$\mathcal{O}(X; I) \supseteq \overline{\text{span}}(A + XX^* + \dots + X^n X^{-n}) \hookrightarrow B ? \quad (\text{A.15})$$

The point here is not that each summand embeds separately (which is trivial)

$$A \hookrightarrow B, \quad XX^* \hookrightarrow B, \quad XXX^*X^* \hookrightarrow B, \quad \dots$$

but instead that the summands will have plenty of correlation among each other (namely precisely the amount of covariance) which causes lots of sums to collapse within the representation such as the covariance itself,

$$\begin{aligned} \text{cov}(X \rightarrow B) &= \ker \left[ \left( \begin{array}{ccc} \text{cov}(X \rightarrow B) & \longrightarrow & B \\ & & \parallel \\ & & B \end{array} \right) - \left( \begin{array}{ccc} \text{cov}(X \rightarrow B) & & \\ & \downarrow & \\ & XX^* & \longrightarrow B \end{array} \right) \right] \\ \implies \left\{ (a) + (-a) \in A + XX^* \mid a \in \text{cov}(X \rightarrow B) \right\} &\subseteq \ker \left( A + XX^* \longrightarrow B \right). \end{aligned}$$

In fact, proposition [A.3.8](#) tells us that the covariance precisely captures this kernel. As an interesting question to the encouraged reader: can you figure out why?

Before we proceed, we would like to note that the idea for the above induction and its proof are due to Evgenios Kakariadis as in [\[Kak16\]](#) and note that this simplifies the original proof by Katsura substantially! We will not go further into how these approaches compare (since this would not benefit our current work) but we would like to note that Katsura's approach analysing cores may be quite valuable after all in the more general context of product systems and so we refer the curious reader to [\[Kat04, section 5\]](#) for comparison. With this we may now proceed to the gauge-invariant uniqueness theorem.

## A.6 Uniqueness theorem

We may now state and proof our version of the gauge-invariant uniqueness theorem for arbitrary gauge-equivariant representations which arise as relative Cuntz–Pimsner algebra along kernel–covariance pairs:

**Theorem A.6.1** (Gauge-invariant uniqueness theorem: The general version). *Consider a gauge-equivariant representation (as in the previous section)*

$$\left( \mathbb{T} \curvearrowright (X, A) \right) \longrightarrow \left( \mathbb{T} \curvearrowright B \right)$$

## A.6. Uniqueness theorem

---

and choose the kernel-covariance pair for the representation

$$(K, I) = \left( \ker(A \rightarrow B) \mid \operatorname{cov}\left(\frac{X}{XK} \rightarrow B\right) \right).$$

Then the quotient from the relative Cuntz-Pimsner algebra is faithful:

$$(X, A) \longrightarrow \mathcal{O}(K, I) \longlongequal{\quad} C^*(X \cup A) \subseteq B.$$

Thus each gauge-equivariant defines a relative Cuntz-Pimsner algebra.

On the other hand, each relative Cuntz-Pimsner algebra defines itself a gauge-equivariant representations. As such the gauge-equivariant representations agree with relative Cuntz-Pimsner algebras (the range of kernel-covariance pairs).

We have basically already proven the theorem: Indeed we have already reduced the general version to the case of faithful representations (see the previous section) and the faithful case is a classical result due to Katsura from [Kat04] resp. the simplified version due to Kakariadis in [Kak16]. For convenience of the reader we review the simplified version:

We begin for this with a couple of observations and useful relations. At first consider the increasing subalgebras from the induction problem (A.15) (which exhaust the fixed point algebra):

$$\overline{\operatorname{span}}(A + XX^* + \dots + X^n X^{-n}) \subseteq \mathcal{O}(X; I)$$

In order to simplify their induction we need to **address a technical detail**: Note that while the sum of ideals is already closed, this fails in general for the sum of subalgebras,

$$A \subseteq B, \quad A' \subseteq B : \quad (A + A') \subseteq \overline{(A + A')}.$$

However the sum of an ideal and a subalgebra is closed nevertheless:

$$A \subseteq B, \quad J \trianglelefteq B : \quad (A + J) = \overline{(A + J)}.$$

**Quick proof for an algebra and ideal (as for pairs of ideals):**

Consider the short exact sequences of possibly incomplete algebras,

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & (A + J) & \longrightarrow & (A + J)/J \longrightarrow 0 \\ & & \parallel & & \downarrow & \pi & \downarrow \\ 0 & \longrightarrow & J & \longrightarrow & \overline{(A + J)} & \longrightarrow & \overline{(A + J)}/J \longrightarrow 0. \end{array}$$

## A.6. Uniqueness theorem

---

Note that the morphism between quotients defines an embedding since

$$J \cap (A + J) = J \implies (A + J)/J \subseteq \overline{(A + J)}/J.$$

Since the range of \*-homomorphisms is always closed we obtain also

$$\pi(A + J) = \pi(A) = \overline{\pi(A)} = \overline{\pi(A + J)} = \pi(\overline{A + J}).$$

As such the quotients agree and so (by basic homological algebra)

$$(A + J)/J = \overline{(A + J)}/J \implies (A + J) = \overline{(A + J)}. \quad (\text{A.16})$$

So the sum of an algebra and ideal **defines a closed subalgebra**.

With this at hand we find in our case (using the obvious inclusion)

$$AXX^* \subseteq XX^* \implies \overline{\text{span}}(A + XX^*) = A + \overline{\text{span}}(XX^*)$$

and similarly on larger sums

$$\overline{\text{span}}\left(A + XX^* + \dots + X^n X^{-n}\right) = A + \overline{\text{span}}\left(XX^* + \dots + X^n X^{-n}\right).$$

This finishes our **discussion on the technical detail**.

With this the induction problem asks for the kernel

$$\ker\left(\mathcal{O}(X; I) \supseteq A + \overline{\text{span}}\left(XX^* + \dots + X^n X^{-n}\right) \longrightarrow B\right)$$

and so for the coefficient algebra that intersects compact operators

$$A \cap \left(XX^* + \dots + X^n X^{-n}\right) \subseteq B$$

where we drop from now on the closed linear span for more pointy statements.

We would like to better understand this intersection. For this we first discuss the following result, which gives an interesting algebraic description for non-commutative Cartan subalgebras from [Exe11] as the nondegenerate ones:

**Lemma A.6.2.** *A subalgebra contains some approximate identity for the ambient algebra if and only if the **subalgebra is nondegenerate**:*

$$A \subseteq B : \quad AB = B \iff (1 - e)B \rightarrow 0 \text{ for } A \ni e \rightarrow 1.$$

*On the other hand, the nondegeneracy holds equivalently if the subalgebra reaches*

## A.6. Uniqueness theorem

---

the entire ambient algebra hereditarily (with implicit Cohen–Hewitt)

$$AB = B \iff \text{her}(A) = ABA = B.$$

Further, the subalgebra remains nondegenerate on intermediate subalgebras:

$$A \subseteq B_0 \subseteq B : \quad AB = B \implies AB_0 = B_0.$$

The analogous statements also hold when replacing the subalgebra and the ambient algebra by any pair of operator algebras (after suitable modifications).

*Proof.* The forward direction is obvious since for any approximate unit

$$A \ni e \rightarrow 1 : \quad (1 - e)B = (1 - e)AB \rightarrow 0$$

where we implicitly use Cohen–Hewitt as usual.

The converse is also obvious as evidently

$$(1 - e)B \rightarrow 0 \implies B \subseteq AB.$$

Regarding the hereditary subalgebra we have:

$$\begin{aligned} AB = B \quad (\implies B = B^* = (AB)^* = B^*A^* = BA) \\ \implies B = (AB) = A(BA) = ABA \implies B = ABA \subseteq AB \end{aligned}$$

Regarding intermediate algebras we have:

$$\begin{array}{ccc} AB = B & \iff & (1 - e)B \rightarrow 0 \text{ for } A \ni e \rightarrow 1 \\ & & \Downarrow \\ AB_0 = B_0 & \iff & (1 - e)B_0 \rightarrow 0 \text{ for } A \ni e \rightarrow 1 \end{array}$$

For arbitrary pairs of operator algebras (instead of a subalgebra):

One needs to replace the equality by an inclusion and an additional closure.  $\square$

With the nondegeneracy in mind we may now unravel the intersection above: For this we first note that the compact operators are nondegenerate in the sum of higher order compact operators,

$$\left( XX^* + \dots + X^n X^{-n} \right) = XX^* \left( XX^* + \dots + X^n X^{-n} \right).$$



## A.6. Uniqueness theorem

---

Indeed we may simply pull out a rabbit from each summand (using Blanchard):

$$XX^* = (XX^*)XX^*, \quad XXX^*X^* = (XX^*)XXX^*X^*, \quad \dots$$

As a consequence we obtain the inclusion (similarly as in lemma A.6.2):

$$A \cap (XX^* + \dots + X^n X^{-n}) \subseteq (A \cap (XX^* + \dots + X^n X^{-n}))XX^*.$$

The latter however lies inside the algebra of compact operators since

$$(A \cap (XX^* + \dots + X^n X^{-n}))XX^* \subseteq AXX^* \subseteq XX^*$$

and as such inside the covariance for the representation (see proposition A.3.8).

As the covariance cannot decrease we obtain the combined relation:

$$A \cap (XX^* + \dots + X^n X^{-n}) = \text{cov}(X \rightarrow B) = \text{cov}(X \rightarrow \mathcal{O}(X; I)).$$

In particular the kernel arises as sum of compact operators:

$$\begin{aligned} \ker \left( A + (XX^* + \dots + XX^n X^{-n} X^*) \longrightarrow B \right) \\ \subseteq XX^* + (XX^* + \dots + XX^n X^{-n} X^*) \subseteq \mathcal{O}(X; I). \end{aligned}$$

Havin these introductory observations in mind, we may now get to the proof of the gauge-invariant uniqueness theorem.

*Proof of theorem A.6.1 (Gauge-invariant uniqueness theorem):* We have already reduced the general version to the case of faithful representations in (A.13):

$$\left( \frac{X}{\overline{XK}}, \frac{A}{\overline{K}} \right) \rightsquigarrow (X, A) \implies \mathcal{O}(K, I) \rightsquigarrow \mathcal{O}(0, I) \implies X \subseteq B, \quad A \subseteq B$$

Furthermore the problem reduces to fixed point algebras as in (A.14)

$$\mathcal{O}(X; I)(0) = B(0) \implies \mathcal{O}(X; I) = B$$

which may be solved as an induction along its fine structure as in (A.15):

$$\mathcal{O}(X; I) \supseteq A + (XX^* + \dots + X^n X^{-n}) \longleftarrow B ?$$

With this at hand we may now begin the **proof by induction from [Kak16]**: The base case simply states the inclusion that we are already well aware of,

$$\mathcal{O}(X; I) = C^*(X \cup A) \longleftarrow A \longrightarrow C^*(X \cup A) = B.$$

## A.6. Uniqueness theorem

---

Consider now the induction step (while assuming the induction hypothesis):

$$\ker\left( A + \left( XX^* + \dots + X^{n+1}X^{-n-1} \right) \longrightarrow B \right) = 0 ?$$

We may reduce this problem to the induction hypothesis via compression

$$\begin{aligned} & X^* \ker\left( A + \left( XX^* + \dots + X^{n+1}X^{-n-1} \right) \longrightarrow B \right) X \\ & \subseteq \ker\left( X^*X + X^*\left( XX^* + \dots + X^{n+1}X^{-n-1} \right)X \longrightarrow B \right) \\ & \subseteq \ker\left( A + \left( XX^* + \dots + X^nX^{-n} \right) \longrightarrow B \right) = 0 \end{aligned}$$

from which we infer that the kernel vanishes for the compression

$$XX^* \ker\left( A + \left( XX^* + \dots + X(X^nX^{-n})X^* \right) \longrightarrow B \right) XX^* = 0.$$

We however found that the kernel lies inside the sum of compact operators:

$$\begin{aligned} & \ker\left( A + \left( XX^* + \dots + XX^nX^{-n}X^* \right) \longrightarrow B \right) \\ & \subseteq XX^* + \left( XX^* + \dots + XX^nX^{-n}X^* \right) \subseteq \mathcal{O}(X; I). \end{aligned}$$

At the same time, the compact operators define a nondegenerate subalgebra within higher order compact operators and as such have trivial annihilator:

$$\begin{aligned} & \left( XX^* + \dots + XX^nX^{-n}X^* \right) = XX^* \left( XX^* + \dots + XX^nX^{-n}X^* \right) \\ & \implies (XX^*)^\perp \cap \left( XX^* + \dots + XX^nX^{-n}X^* \right) = 0. \end{aligned}$$

So the kernel above necessarily also vanishes without compression.  $\square$

Concluding the current and previous section, we have achieved the first half in the classification of relative Cuntz–Pimsner algebras: that is we have found that every gauge-equivariant representation arises itself as relative Cuntz–Pimsner algebra for some kernel–covariance pair. In short, that is the kernel–covariance pairs exhaust the gauge-equivariant representations.

We may now proceed with the second half on the classification of relative Cuntz–Pimsner algebras: The kernel–covariance pairs classify the relative Cuntz–Pimsner algebra and equivalently every possible kernel–covariance pair arises itself in nature. As such we need to construct sufficiently many representations (as well as any

## A.7. Fock representation

---

nontrivial one whatsoever). This is where the Fock representation and the quotients thereof will come into play.

## A.7 Fock representation

We begin with the following problem: Note that up until now we have not come across any representation whatsoever (put aside sufficiently many) except the trivial representation

$$B = 0 : \quad X \longrightarrow 0, \quad A \longrightarrow 0$$

and as such it **could be in principle** that the relative Cuntz–Pimsner algebras (for some correspondences) all coincide as the only trivial representation

$$(X, A) \longrightarrow \mathcal{T}(X, A) = \dots = \mathcal{O}(K, I) = \mathcal{O}(K', I') = \dots = 0 ??$$

As such the question arises whether a correspondence admits always a nontrivial representation (and further also sufficiently many faithful ones). For this we consider a failing attempt which leads us in turn to the Fock representation: As a first guess we may consider the linking algebra associated to our correspondence (seen as Hilbert module only)

$$B = \begin{pmatrix} A \\ X \end{pmatrix} \begin{pmatrix} A^* & X^* \end{pmatrix} = \begin{pmatrix} A & X^* \\ X & XX^* \end{pmatrix} \subseteq \mathcal{L} \left[ \begin{pmatrix} A \\ X \end{pmatrix} \right]$$

together with the representation given by the canonical embedding

$$\begin{pmatrix} A \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} A & X^* \\ X & XX^* \end{pmatrix} = B, \quad \begin{pmatrix} 0 \\ X \end{pmatrix} \subseteq \begin{pmatrix} A & X^* \\ X & XX^* \end{pmatrix} = B.$$

Now while this representation respects the Hilbert module structure (basically by construction) it does not respect the left module structure:

$$\begin{aligned} \begin{pmatrix} 0 & x^* \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix} &= \begin{pmatrix} x^*y & 0 \end{pmatrix}, \\ \begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} &= \begin{pmatrix} xa & 0 \end{pmatrix}, \\ \begin{pmatrix} a \\ 0 \end{pmatrix} \begin{pmatrix} x & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \end{pmatrix} \neq \begin{pmatrix} ax & 0 \end{pmatrix}. \end{aligned}$$

The solution is to simply extend these down the diagonal to infinity which brings us straight to the Fock space representation: More precisely, the Fock space is the

## A.7. Fock representation

---

infinite direct sum of increasing tensor powers (as in section A.1)

$$\mathcal{F}(X) := A \oplus X \oplus XX \oplus XXX \oplus \dots = \begin{pmatrix} A \\ X \\ XX \\ \vdots \end{pmatrix}$$

and the representation as diagonal action and as right shift respectively:

$$A \rightarrow \mathcal{L}(\mathcal{F}X) : \quad a(A \oplus X \oplus XX \oplus \dots) := aA \oplus aX \oplus aXX \oplus \dots$$

$$X \rightarrow \mathcal{L}(\mathcal{F}X) : \quad x(A \oplus X \oplus XX \oplus \dots) := 0 \oplus xA \oplus xX \oplus xXX \oplus \dots$$

We visualize them in matrix notation (using a formal left and right shift):

$$X \otimes R = X \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \dots & \end{pmatrix}, \quad A \otimes 1 = A \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \dots \end{pmatrix}, \quad X^* \otimes L = X^* \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \dots \end{pmatrix}.$$

Further the Fock representation comes with the (inner) circle action

$$\mathbb{T} \curvearrowright \mathcal{L}(\mathcal{F}X) : \quad (z \curvearrowright -) := \begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \dots \end{pmatrix} - \begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \dots \end{pmatrix}^*$$

which renders the representation gauge-equivariant:

$$\begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \dots \end{pmatrix} A \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \dots \end{pmatrix} \begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \dots \end{pmatrix}^* = 1A \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \dots \end{pmatrix},$$

$$\begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \dots \end{pmatrix} X \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \dots & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & z & & \\ & & z^2 & \\ & & & \dots \end{pmatrix}^* = zX \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \dots & \end{pmatrix}.$$

Furthermore the representation of compact operators reads

$$XX^* \mapsto X \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \dots & \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \dots \end{pmatrix} X^* = XX^* \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \dots \end{pmatrix}$$

With these preliminary computations ready we may now reveal the Fock representation as the universal representation (i.e. the Toeplitz algebra):

**Proposition A.7.1** (Fock representation = Toeplitz algebra):

## A.7. Fock representation

---

The Fock representation has trivial kernel and covariance

$$\ker\left( A \longrightarrow \mathcal{L}(\mathcal{F}X) \right) = 0, \quad \text{cov}\left( X \longrightarrow \mathcal{L}(\mathcal{F}X) \right) = 0$$

and as such defines the universal representation (by theorem A.6.1).

*Proof.* Clearly the representation is faithful (and so has trivial kernel) as

$$\begin{pmatrix} a & & \\ & a & \\ & & \dots \end{pmatrix} \begin{pmatrix} A \\ \dots \\ \dots \end{pmatrix} = \begin{pmatrix} aA \\ \dots \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \end{pmatrix} \implies a = 0.$$

As a consequence, we may further compute the covariance as the intersection within the representation (as the special instance from proposition A.3.8)

$$\text{cov}\left( X \longrightarrow \mathcal{L}(\mathcal{F}X) \right) = A \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix} \cap XX^* \begin{pmatrix} 0 & & \\ & 1 & \\ & & \dots \end{pmatrix} = 0.$$

So the covariance is trivial as well and the proposition follows.  $\square$

Note that more importantly we established the existence of any nontrivial representation whatsoever (equivalently the universal representation is different from the trivial representation) and further also the existence of faithful representations (equivalently the universal representation is faithful). Put in other words, we may now distinguish Toeplitz algebras along the lattice of quotient correspondences (the first dimension of kernel–covariance pairs):

$$\mathcal{T}\left(\frac{X}{XK}\right) = \mathcal{O}(K, 0) = \mathcal{O}(K', 0) = \mathcal{T}\left(\frac{X}{XK'}\right) \implies K = K'$$

Indeed we have even found the stronger kernel relation from (A.12):

$$\ker\left( A \longrightarrow \mathcal{T}(X) \right) = 0 \implies \ker\left( A \longrightarrow \frac{A}{K} \longrightarrow \mathcal{T}\left(\frac{X}{XK}\right) \right) = K$$

We are still left with the question (which we get to next)

$$\mathcal{O}(K, I) = \mathcal{O}(K', I') \implies (K = K' \mid I = I') \text{ ?}$$

For this we may now consider the Fock representation as a concrete realization for the Toeplitz algebra and as such further construct every relative Cuntz–Pimsner algebra (along kernel–covariance pairs) as a concrete quotient thereof: For example given first a quotient correspondence (for some invariant ideal)

$$K \triangleleft A : \quad X^* K X \subseteq K : \quad (X, A) \longrightarrow \left(\frac{X}{XK}, \frac{A}{K}\right)$$

## A.7. Fock representation

---

we may construct the corresponding Toeplitz algebra as quotient by the kernel (more precisely its ideal generated within the original Toeplitz representation)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (XK, K) & \longrightarrow & (X, A) & \longrightarrow & \left(\frac{X}{XK}, \frac{A}{K}\right) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{T}X(K \subseteq A)\mathcal{T}X & \longrightarrow & \mathcal{T}X & \longrightarrow & \mathcal{T}\left(\frac{X}{XK}\right) \longrightarrow 0
 \end{array}$$

where we omit as usual the closed linear span for more pointy statements.

Before continuing we replace for convenience the original correspondence by the quotient correspondence,

$$\left(\frac{X}{XK}, \frac{A}{K}\right) \rightsquigarrow (X, A) \quad \Longrightarrow \quad \mathcal{T}\left(\frac{X}{XK}\right) \rightsquigarrow \mathcal{T}(X).$$

Consider next an ideal (as possible covariance) bounded from above by the maximal covariance (as in proposition A.3.6),

$$I \trianglelefteq A : \quad I \subseteq \max(X, A).$$

The bound from above is necessary as larger covariances force an additional kernel and as such would factor over some further quotient correspondence. Similarly as for relative Toeplitz algebras, the relative Cuntz–Pimsner algebra arises now as coequalizer for the chosen covariance

$$\mathcal{O}(K, I) = \text{coeq} \left( \begin{array}{ccc} I \subseteq A \cap XX^* & \longrightarrow & \mathcal{T}X \\ \downarrow & & \parallel \\ XX^* & \longrightarrow & \mathcal{T}X \end{array} \right)$$

and as such also as quotient by their difference as in (A.2):

$$\begin{aligned}
 0 & \longrightarrow \mathcal{T}X \left( (\varphi - \tau)I \right) \mathcal{T}X \longrightarrow \mathcal{T}X \longrightarrow \mathcal{O}(X; I) \longrightarrow 0 : \\
 (\varphi - \tau) & = \left( \begin{array}{ccc} A \cap XX^* & \longrightarrow & \mathcal{T}X \\ & \parallel & \\ & \mathcal{T}X & \end{array} \right) - \left( \begin{array}{ccc} A \cap XX^* & & \\ \downarrow & & \\ XX^* & \longrightarrow & \mathcal{T}X \end{array} \right).
 \end{aligned}$$

We now invoke the Fock representation as Toeplitz algebra: Recall that we have already found here the concrete embedding for the coefficient algebra as well as

## A.7. Fock representation

---

the concrete embedding of compact operators (see above)

$$\mathcal{TX} \subseteq \mathcal{L} \left[ \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} \right] : \quad A \mapsto A \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix}, \quad XX^* \mapsto XX^* \begin{pmatrix} 0 & & \\ & 1 & \\ & & \dots \end{pmatrix}$$

and as such their difference reads

$$A \cap XX^* \mapsto A \cap XX^* \begin{pmatrix} 1 & & \\ & 0 & \\ & & \dots \end{pmatrix}.$$

We therefore found the relative Cuntz–Pimsner algebra as quotient

$$0 \longrightarrow \mathcal{TX} \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{TX} \longrightarrow \mathcal{TX} \longrightarrow \mathcal{O}(X; I) \longrightarrow 0. \quad (\text{A.17})$$

This was originally established by Muhly and Solel in [MS98, theorem 2.19].

We now wish to verify that the induced quotient representation remains faithful: That is we note that the quotient **could in principle** introduce new kernel,

$$I \subseteq A \cap XX^* : \quad \ker \left( A \longrightarrow \mathcal{TX} \longrightarrow \mathcal{O}(X; I) \right) = 0 ?$$

Indeed we note that the quotient **does introduce new kernel** as soon as the covariance **exceeds the maximal covariance** (as we have noted also above). As such we have to make sure this does not happen as long as the covariance lies below the maximal covariance from proposition A.3.6. For this we note that the trivial kernel above may be equivalently verified now as the trivial intersection

$$I \subseteq \max(X, A) : \quad A \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix} \cap \mathcal{TX} \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{TX} = 0 ? \quad (\text{A.18})$$

On the other hand, we wish to also verify the covariance relation from (A.12):

$$\text{cov} \left( A \longrightarrow \mathcal{TX} \longrightarrow \mathcal{O}(X; I) \right) = I ?$$

The problem here is that the covariance could in principle increase as well. Indeed the construction (and even our very definition) of relative Cuntz–Pimsner algebras guarantees just a least covariance for the provided covariance ideal. This becomes more evident as follows: Consider for this the difference morphism which

## A.7. Fock representation

---

factors by the universal property via the Toeplitz algebra:

$$\begin{array}{ccccccc}
 & & & & A \cap XX^* & \xlongequal{\quad} & A \cap XX^* \\
 & & & & \downarrow & & \downarrow \\
 & & & & & \varphi - \tau & \\
 0 & \longrightarrow & \mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X & \longrightarrow & \mathcal{T}X & \longrightarrow & \mathcal{O}(X; I) \longrightarrow 0.
 \end{array}$$

As such the covariance may be read off from the common intersection as

$$\left( \begin{matrix} A \cap XX^* & & \\ & 0 & \\ & & \dots \end{matrix} \right) \cap \mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X = \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} ? \quad (\text{A.19})$$

This meanwhile also highlights how the covariance **could in principle increase**. Let us begin to verify that the representation remains faithful along the quotient. For this we begin with the following well-known relation (which goes all the way back to an observation by Joachim Cuntz made in [Cun77]):

**Proposition A.7.2.** *The ideal generated by the covariance as in (A.17) coincides with the ideal of compact operators of the form*

$$\mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X = \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} I (A \ X^* \ \dots) = \mathbb{K} \left[ \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} I \right]$$

with implicit closed linear spans for more pointy statements.

*Sketch of proof:* The result follows most easily using the formal right and left shift operators (as further above) from which the covariance ideal reads

$$\begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} = I \left[ \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & \dots \end{pmatrix} \right] = I \otimes (1 - RL).$$

Recall however that these satisfy the well-known relation

$$LR = 1 \quad \implies \quad L(1 - RL) = 0 = (1 - RL)R.$$

As such we obtain as the only contributions for the ideal

$$\mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X = \sum_{mn} \left( X \begin{pmatrix} 0 & & \\ & 1 & \\ & & \dots \end{pmatrix} \right)^m \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \left( X^* \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & \dots \end{pmatrix} \right)^n.$$



## A.7. Fock representation

---

On the other hand they generate the system of matrix units such as

$$\begin{pmatrix} 0 & & \\ 1 & 0 & \\ & \dots & \end{pmatrix}^m \begin{pmatrix} 1 & & \\ 0 & \dots & \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 0 & & \dots \end{pmatrix}^n = \begin{pmatrix} 0 & & \\ 0 & 1 & 0 \\ 0 & & \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0 \ 1 \ 0).$$

As such we obtain for the above ideal

$$\mathcal{T}X \begin{pmatrix} I & & \\ 0 & \dots & \end{pmatrix} \mathcal{T}X = \dots = \sum_{mn} \begin{pmatrix} 0 \\ X^n \\ 0 \end{pmatrix} I (0 \ X^{-n} \ 0) = \mathcal{F}(X) \mathcal{F}(X)^*$$

which is the desired relation for the covariance ideal.  $\square$

In order to handle the kernel for the induced representation on the quotient we need to take a closer look into the ideal of compact operators from [A.7.2](#): We begin for this with the well-known approximation by “finite rank matrices”. More precisely, recall that compact operators on Fock space may be approximated by their finite matrix blocks (along finite  $S \subseteq \mathbb{N}$ )

$$\begin{pmatrix} 0 & & \\ \mathbb{K}[\dots] & & \\ & & 0 \end{pmatrix} \ni \begin{pmatrix} 0 & & \\ k(S) & & \\ & & 0 \end{pmatrix} \xrightarrow{S \rightarrow \mathbb{N}} \begin{pmatrix} k_{00} & k_{01} & \\ k_{10} & k_{11} & \\ & & \dots \end{pmatrix} \in \mathbb{K} \left[ \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} \right].$$

Indeed one easily verifies this (using Dirac calculus from [A.1.1](#)):

$$\begin{pmatrix} \dots & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} (A^* \ X^* \ \dots) \rightarrow 0 \leftarrow \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} (A^* \ X^* \ \dots) \begin{pmatrix} \dots & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

In particular we obtain for the diagonal compact operators as in [A.7.2](#):

$$\mathbb{K} \left[ \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} I \right] \cap \begin{pmatrix} \mathcal{L}(A) & & \\ \mathcal{L}(X) & & \\ & & \dots \end{pmatrix} = \begin{pmatrix} I & & \\ X I X^* & & \\ & & \rightarrow 0 \end{pmatrix}.$$

Meanwhile the author would like to take a moment to thank his previous tutor Dominic Enders for highlighting this perspective during personal discussions.

The idea is now to use the previous relation in contrast to the following **observation by Katsura** which we reformulate in our language: Consider for this the diagonal operators on Fock space

$$\begin{pmatrix} \mathcal{L}(A) & & \\ \mathcal{L}(X) & & \\ & & \dots \end{pmatrix} \subseteq \mathcal{L} \left[ \mathcal{F}X = \begin{pmatrix} A \\ X \\ \dots \end{pmatrix} \right]$$

## A.7. Fock representation

---

and the representation between such diagonal operators:

$$\begin{pmatrix} 0 & & & \\ & \mathcal{L}(X^n) & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \mathcal{L}(X^n) \otimes 1 & \\ & & & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \mathcal{L}(X^n \otimes X) & \\ & & & 0 \end{pmatrix}$$

This generally fails to define a faithful representation: one may for instance consider the graph correspondence for any finite acyclic graph such as

$$X = \ell^2(E = \bullet \longrightarrow \bullet) \quad \implies \quad XX = \ell^2(EE = \emptyset) = 0.$$

Katsura's crucial observation tells us now that this becomes faithful when restricted to the subspace of compact operators by our covariance ideal:

**Proposition A.7.3** ([Kat04, Lemma 4.7]). *The representation above defines an embedding when restricted to the maximal covariance as in proposition A.7.2,*

$$\begin{pmatrix} 0 & & & \\ & |X^n\rangle \max(X, A) \langle X^n| & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & |X^n\rangle \max(X, A) \langle X^n| \otimes 1 & \\ & & & 0 \end{pmatrix}.$$

As such the representation defines also an embedding for any covariance ideal below the maximal covariance.

*Proof from [Kat04]:* We note for the kernel (in Dirac bracket notation)

$$\begin{aligned} & \langle X^n | \ker \left[ |X^n\rangle \max(X, A) \langle X^n| \curvearrowright |X^n\rangle \otimes |X\rangle \right] |X^n\rangle \\ & \subseteq \ker \left[ \langle X^n | X^n \rangle \max(X, A) \langle X^n | X^n \rangle \curvearrowright |X\rangle \right] \\ & \subseteq \ker(A \curvearrowright X) \cap \ker(A \curvearrowright X)^\perp = 0 \end{aligned}$$

where we have used the obvious inclusion

$$\langle X^n | X^n \rangle \max(X, A) \langle X^n | X^n \rangle \subseteq \max(X, A) \subseteq \ker(A \curvearrowright X)^\perp.$$

(Note the intersection reflects also the first level as in proposition A.1.3.)

For an ideal such as the kernel above it holds however

$$\langle X^n | \left( \ker[\dots] = \ker[\dots]^* \ker[\dots] \right) |X^n\rangle = 0 \quad \implies \quad \ker[\dots] |X^n\rangle = 0.$$

As such we found the inclusion

$$\ker \left[ |X^n\rangle \max(X, A) \langle X^n| \curvearrowright |X^n\rangle \otimes |X\rangle \right] \subseteq \ker \left[ |X^n\rangle \langle X^n| \curvearrowright |X^n\rangle \right] = 0.$$

## A.7. Fock representation

---

In particular, the same holds true for any covariance below the maximal.  $\square$

With Katsura's observation at hand we may now verify the desired kernel and covariance relation as in (A.12) which will resolve the second half of our classification of relative Cuntz–Pimsner algebras. We note here that the kernel relation is already due to Katsura as established in [Kat04]:

**Theorem A.7.4** (Relative Cuntz–Pimsner algebras: Kernel and Covariance):  
*For the relative Cuntz–Pimsner algebra as above it holds*

$$\begin{aligned} \ker\left( A \rightarrow \mathcal{T} \rightarrow \mathcal{O}(X; I) \right) &= 0 \\ \text{cov}\left( X \rightarrow \mathcal{T} \rightarrow \mathcal{O}(X; I) \right) &= I \end{aligned}$$

*As a consequence, the kernel–covariance pairs are also classifying.*

*Proof.* We begin with **the kernel relation** due to Katsura from [Kat04]:  
As in the beginning discussion we need to verify the relation (A.18):

$$I \subseteq \max(X, A) : \quad A \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix} \cap \mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X = 0 ?$$

For this we first revealed in proposition A.7.2 that the ideal generated by our covariance (within the Toeplitz algebra) agrees with compact operators. As such the intersection with the coefficient algebra reads

$$A \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix} \cap \mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X = A \begin{pmatrix} 1 & & \\ & 1 & \\ & & \dots \end{pmatrix} \cap \begin{pmatrix} I & & \\ & |X\rangle I \langle X| & \\ & & \rightarrow 0 \end{pmatrix}.$$

In contrast, Katsura's observation from proposition A.7.3 tells us that these embed along the diagonal and as such the norm remains also constant throughout,

$$\begin{pmatrix} a & & \\ & a & \\ & & \dots \end{pmatrix} \in \begin{pmatrix} I & & \\ & |X\rangle I \langle X| & \\ & & \dots \end{pmatrix} : \quad \|a\| = \|a \curvearrowright X\| = \dots = \|a \curvearrowright X^n\|.$$

Both the vanishing of compact operators along the diagonal and the constant norm are only possible for the trivial intersection and as such the trivial kernel.

We continue with **the covariance relation** from (A.12): For this we may now

## A.8. Lattice structure

---

simply verify the common intersection as in (A.19) also using proposition A.7.2:

$$\begin{aligned} & \begin{pmatrix} A \cap XX^* & & \\ & 0 & \\ & & \dots \end{pmatrix} \cap \mathcal{T}X \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix} \mathcal{T}X = \\ & = \begin{pmatrix} A \cap XX^* & & \\ & 0 & \\ & & \dots \end{pmatrix} \cap \begin{pmatrix} I & IX^* & \\ XI & XIX^* & \\ & & \dots \end{pmatrix} = \begin{pmatrix} I & & \\ & 0 & \\ & & \dots \end{pmatrix}. \end{aligned}$$

As such the covariance **does not increase** and the theorem is proven.  $\square$

We have thus established also the **second half in our classification**:

More precisely, we have first found that the class of relative Cuntz–Pimsner algebras exhausts the gauge-equivariant representations (which was the content of the gauge-invariant uniqueness theorem). On the other hand we now found that the parametrisation via kernel–covariance pairs **is also classifying**:

$$\mathcal{O}(K, I) = \mathcal{O}(K', I') \quad \implies \quad (K, I) = (K', I')$$

**Altogether we have thus found:** the lattice of kernel–covariance pairs parametrises the entire lattice of gauge-equivariant representations (as points).

## A.8 Lattice structure

While our discussion (so far) captured the lattice of gauge-equivariant representations **as individual points** along the lattice, this still leaves open how the lattice structure of kernel–covariance pairs **reflects the lattice structure** of gauge-equivariant representations (among each other) to which we now get: Recall for this that our kernel–covariance pairs encode the covariance for the quotient correspondence (which rendered the representation faithful)

$$(X, A) \longrightarrow B : \quad K = \ker(A \rightarrow B) \quad \implies \quad I = \text{cov}\left(\frac{X}{XK} \rightarrow B\right)$$

and its intrinsic characterisation on the quotient (as bounded ideal)

$$\left( K \trianglelefteq A \mid X^*KX \subseteq K \right) \quad \implies \quad \left( I \trianglelefteq A/K \mid I \subseteq \max\left(\frac{X}{XK}\right) \right).$$

We therefore begin with a translation of our kernel–covariance pairs which lives on the original correspondence. This allows us to give an intrinsic order on kernel–covariance pairs reflecting the lattice structure of representations. Along this translation we further reveal Katsura’s T-pair version as nothing but our formulation using kernel–covariance pairs (with maximal covariance in disguise).

## A.8. Lattice structure

---

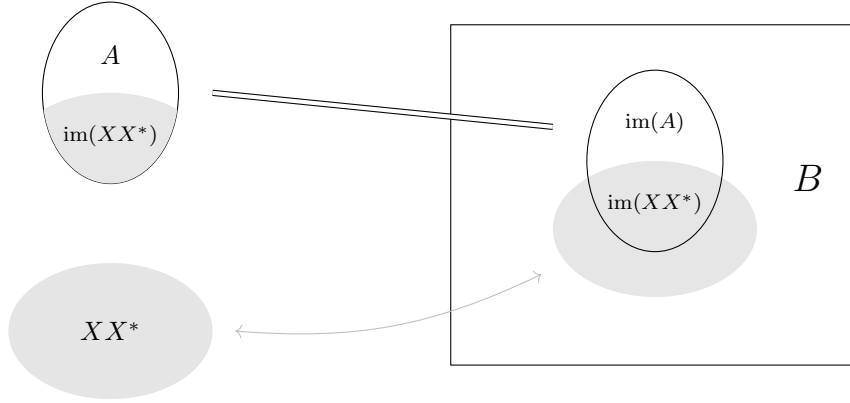
For our translation we first recall that the covariance for an embedding (faithful representation) may be simply read off as common intersection within the ambient algebra (as in proposition A.3.8)

$$(X, A) \subseteq B : \quad \text{cov}(X \rightarrow B) = \text{im}(A \rightarrow B) \cap \text{im}(XX^* \rightarrow B).$$

More precisely, one may take the portion of the coefficient algebra

$$\text{cov}(X \rightarrow B) = \left\{ a \in A \mid a \in \text{im}(XX^* \rightarrow B) \right\} = A \cap \text{im}(XX^*).$$

That is however the same amount of information as the actual intersection, as long as one keeps track of the embedding for the coefficient algebra:



With this picture in mind, we continue on some general representation

$$(X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \dashrightarrow B : \quad K = \ker(A \rightarrow B).$$

We first note that for a quotient (i.e. surjective mapping) there is absolutely no loss of generality when pulling back any ideal along the quotient since for

$$I \trianglelefteq A/K \quad \rightsquigarrow \quad (A \rightarrow \frac{A}{K})^{-1} I \trianglelefteq A : \quad I = (A \rightarrow \frac{A}{K})(A \rightarrow \frac{A}{K})^{-1} I$$

and so we may use the equivalent intrinsic definition of kernel–covariance pairs as those with covariance below the maximal covariance within the quotient:

$$(I + K) \trianglelefteq A : \quad I/K \subseteq \max\left(\frac{X}{XK}\right) \tag{A.20}$$

We wrote the covariance ideal (here and below) as sum with the kernel ideal simply to guarantee that the ideal arises indeed as pullback from the quotient.

On the other hand, note that the amount of covariance (as described above) does not change either in the sense of how much common intersection the coefficient

## A.8. Lattice structure

---

algebra has with compact operators since (see also proposition A.3.10):

$$\begin{aligned} \operatorname{im}\left( A \longrightarrow \frac{A}{K} \longrightarrow B \right) &= \operatorname{im}\left( \frac{A}{K} \longrightarrow B \right), \\ \operatorname{im}\left( XX^* \longrightarrow \left(\frac{X}{XK}\right)\left(\frac{X}{XK}\right)^* \longrightarrow B \right) &= \operatorname{im}\left( \left(\frac{X}{XK}\right)\left(\frac{X}{XK}\right)^* \longrightarrow B \right). \end{aligned}$$

So there is really no loss of generality from this perspective either.

As such we obtain another equivalent **extrinsic definition** of kernel–covariance pairs as those with covariance ideal describing the amount of common intersection between the coefficient algebra and compact operators:

$$(X, A) \longrightarrow B : \quad (I + K) = \operatorname{im}(A \rightarrow B) \cap \operatorname{im}(XX^* \rightarrow B).$$

For comparison between kernel–covariance pairs we however take from now on the portion within the coefficient algebra as above, that is

$$(I + K) = \left\{ a \in A \mid a \in \operatorname{im}(XX^* \rightarrow B) \right\} = A \cap \operatorname{im}(XX^* \rightarrow B)$$

and we note this agrees with our intrinsic definition (somewhat obvious now). Meanwhile we also keep in mind the viewpoint on the covariance as amount of common intersection as it provides an interesting perspective on representations.

With both these definitions at hand (the intrinsic and the extrinsic) we may now get to the lattice of gauge-equivariant representations. Given a pair of representations we define the usual order of representations as

$$\left( (X, A) \rightarrow B \right) \leq \left( (X, A) \rightarrow B' \right) : \quad (X, A) \longrightarrow B \dashrightarrow B'.$$

More precisely, that is the representation factors over the other and note that the sole existence of such a factorisation entails a unique such as the representations are all generated as an operator algebra by (the image of) the correspondence:

$$B = C^*(A \cup X) \implies B \dashrightarrow B' \text{ uniquely}$$

Given a factorisation we now easily infer for their kernel and covariance

$$\begin{aligned} \ker(A \rightarrow B) &\subseteq \ker(A \rightarrow B \rightarrow B'), \\ A \cap \operatorname{im}(XX^* \rightarrow B) &\subseteq A \cap \operatorname{im}(XX^* \rightarrow B \rightarrow B'). \end{aligned}$$

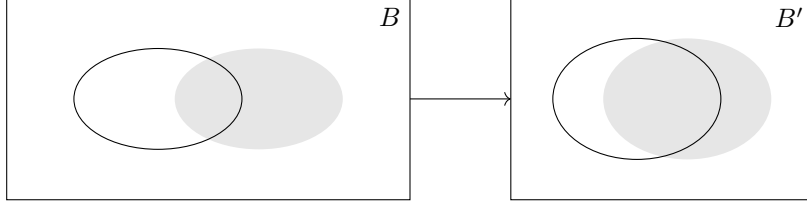
Indeed the latter may be easily seen as (somewhat trivially)

$$\operatorname{im}(a \rightarrow B) \in \operatorname{im}(XX^* \rightarrow B) \implies \operatorname{im}(a \rightarrow B \rightarrow B') \in \operatorname{im}(XX^* \rightarrow B \rightarrow B').$$

## A.8. Lattice structure

---

Schematically the amount of intersection could look something like:



So we have found the following converse direction (using theorem A.7.4):

$$\left( K \subseteq L \mid I + K \subseteq J + L \right) \iff \mathcal{O}(K, I) \leq \mathcal{O}(L, J)$$

What about the forward direction? That is assume we have an inclusion of kernel–covariance pairs as above. As we have an inclusion of kernel ideals we obtain in particular for their quotient correspondence

$$\begin{array}{ccccc} (X, A) & \longrightarrow & \left( \frac{X}{XK}, \frac{A}{K} \right) & \longrightarrow & \left( \frac{X}{XL}, \frac{A}{L} \right) \\ & & \downarrow & & \downarrow \\ & & \mathcal{O}(K, I) & & \mathcal{O}(L, J) \end{array}$$

and so we may replace our correspondence as usual by the quotient

$$\left( \frac{X}{XK}, \frac{A}{K} \right) \rightsquigarrow (X, A) \implies \mathcal{O}(K, I) \rightsquigarrow \mathcal{O}(0, I).$$

Recall that the relative Cuntz–Pimsner algebra satisfies (by definition)

$$(X, A) \longrightarrow \mathcal{O}(X; I) \dashrightarrow B \iff I \subseteq \text{cov}(X \rightarrow B)$$

and as such we need to verify the least amount of covariance

$$I \subseteq \text{cov} \left( (X, A) \longrightarrow \left( \frac{X}{XL}, \frac{A}{L} \right) \longrightarrow \mathcal{O}(L, J) \right) ?$$

This basically follows now from our study of kernel morphisms and covariance ideals from section A.3, which we recall now for more clarity in our context: At first we found that kernel and cokernel morphisms have full covariance. That is in our context the fully commutative diagram for the quotient morphism and on

## A.8. Lattice structure

---

the other hand the covariance diagram for the quotient representation:

$$\begin{array}{ccc}
 A \cap XX^* & \longrightarrow & A/L \\
 \downarrow & & \downarrow \\
 XX^* & \longrightarrow & (\frac{X}{XL})(\frac{X}{XL})^*
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 J/L & \longrightarrow & \mathcal{O}(L, J) \\
 \downarrow & & \parallel \\
 (\frac{X}{XL})(\frac{X}{XL})^* & \longrightarrow & \mathcal{O}(L, J).
 \end{array}$$

Combing these with our assumption on covariance ideals we obtain

$$\begin{array}{ccccc}
 (I + 0) & \longrightarrow & J/L & \longrightarrow & \mathcal{O}(L, J) \\
 \downarrow & & \downarrow & & \parallel \\
 XX^* & \longrightarrow & (\frac{X}{XL})(\frac{X}{XL})^* & \longrightarrow & \mathcal{O}(L, J)
 \end{array}$$

and as such the desired amount of covariance,

$$(I + 0) \subseteq (J + L) \quad \implies \quad I \subseteq \text{cov} \left( (X, A) \rightarrow \mathcal{O}(L, J) \right).$$

As such we also found the forward direction and so the order isomorphism, which is the **main conclusion of this article**:

**Theorem A.8.1** (Kernel–covariance pairs: Order isomorphism):

The kernel–covariance pairs as in (A.20) define the order isomorphism

$$\left( K \subseteq L \mid I + K \subseteq J + L \right) \iff \mathcal{O}(K, I) \leq \mathcal{O}(L, J) \quad (\text{A.21})$$

and as such the lattice of **kernel–covariance pairs** with its natural order by inclusion describes the entire **lattice of gauge-equivariant** representations, equivalently the entire **lattice of gauge-invariant ideals**.

Let us give an example of our result for some graph algebra, also to illustrate the ease of working with such kernel–covariance pairs:

**Example A.8.2** (Graph correspondences: gauge-invariant ideals):

Consider a graph correspondence as in example A.1.2,

$$X = \ell^2(E = \text{edges}), \quad A = c_0(\text{vertices})$$

and recall its quotient graphs as in A.3.5 (given by hereditary ideals as in A.3.2) as well as their covariance ideals given by their sets of regular vertices as in A.3.7.



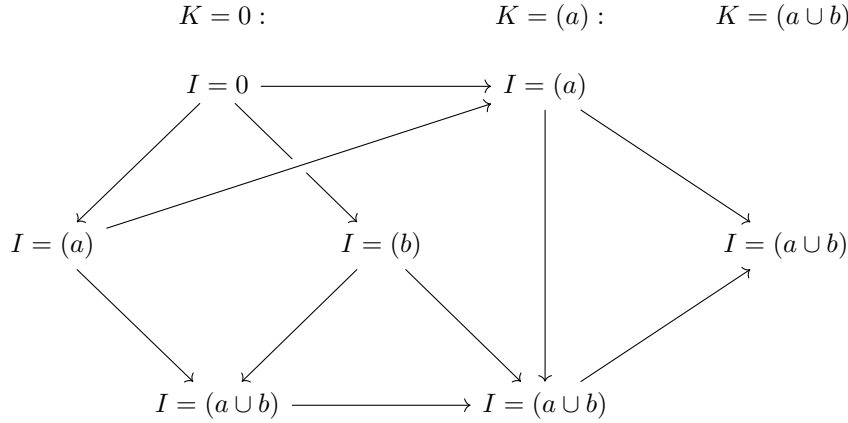
## A.8. Lattice structure

---

As an example consider the following graph and its quotient graphs

$$\begin{array}{ccc}
 K = 0 : & K = (a) : & K = (a \cup b) : \\
 \left( \begin{array}{c} \textcirclearrowleft \\ a \rightarrow b \\ \textcirclearrowleft \end{array} \right) & \longrightarrow & \left( \begin{array}{c} \textcirclearrowleft \\ b \end{array} \right) & \longrightarrow & \left( \emptyset \right).
 \end{array}$$

Then its lattice of gauge-equivariant representations reads (as Hasse diagram)



and so equivalently also the lattice of gauge-invariant ideals.

Note how we now easily read off the order via kernel-covariance pairs.

Before we continue let us make a few remarks on the order isomorphism:

In particular the following discussion on connecting morphisms will basically cover the notion of suprema and infima as addressed in the following remark:

**Remark A.8.3** (Lattice isomorphism: suprema and infima):

Note that as both lattices are order isomorphic they will be also lattice isomorphic as unions and intersections (finite or arbitrary) as well as top and bottom elements are determined as suprema and infima respectively

$$\text{order iso} \left( \sup_s a_s \right) = \sup_s \left( \text{order iso} a_s \right)$$

which aside their existence depend only on the given order.

Put in other words the notion of a lattice is really just a pure property and defines no additional structure so there is really no difference between the lattice of gauge-equivariant representations and the lattice of kernel-covariance pairs. We will however later discover that arbitrary suprema and infima of kernel-covariance pairs do not necessarily always arise as intersections and sums of their kernel

## A.8. Lattice structure

---

and covariance ideals, that is we only have

$$\inf_s (K_s | I_s) \leq \left( \bigcap_s K_s \mid \bigcap_s I_s \right) \quad \text{and} \quad \sup_s (K_s | I_s) \geq \left( \sum_s K_s \mid \sum_s I_s \right).$$

Indeed while the intersection und sum of invariant ideals remain invariant

$$\langle X | \left( \bigcap_s K_s \right) | X \rangle \subseteq \bigcap_s K_s \quad \text{and} \quad \langle X | \left( \sum_s K_s \right) | X \rangle \subseteq \sum_s K_s$$

the intersection and sum of covariance ideals may not always end up below the maximal covariance,

$$\bigcap_s I_s \subseteq \max \left( \frac{X}{X(\bigcap_s K_s)} \right) \quad \text{and} \quad \sum_s I_s \subseteq \max \left( \frac{X}{X(\sum_s K_s)} \right) ?$$

Schematically that is the next possible kernel–covariance pair may lie just further beyond (as seen within the lattice of any pairs of ideals) such as

$$\begin{array}{c} \left( K : \dots \mid I \subseteq \max \left( \frac{X}{XK} \right) \right) \\ \uparrow \\ \left( (K_1 + K_2) : \dots \mid (I_1 + I_2) \not\subseteq \max \left( \frac{X}{X(K_1 + K_2)} \right) \right) \\ \uparrow \\ \left( K_s : \dots \mid I_s \subseteq \max \left( \frac{X}{XK_s} \right) \right). \end{array}$$

For example the requirement to have at least as much covariance as all the given covariance ideals can force larger kernel than just the sum of given kernel ideals, or put more drastically there may exist no connecting morphism from each relative Cuntz–Pimsner algebra to the cokernel strand over the sum of kernel ideals. We will find such an example in [A.8.5](#) between already even just a pair of kernel–covariance pairs of the form:

$$\begin{array}{ccccccc} (X, A) & \longrightarrow & \left( \frac{X}{XK_1}, \frac{A}{K_1} \right) & \longrightarrow & \left( \frac{X}{XK_2}, \frac{A}{K_2} \right) & \longrightarrow & \left( \frac{X}{XK}, \frac{A}{K} \right) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & & \mathcal{O}(K_1, I_1) & \overset{\times}{\dashrightarrow} & \mathcal{O}(K_2, I_2) & \longrightarrow & \mathcal{O}(K, I) \\ \downarrow & & \downarrow & \searrow & \downarrow & \nearrow & \downarrow \\ \dots & & \dots & \dashrightarrow & \dots & & \dots \end{array}$$

Furthermore we note that from either description the order defines a partial order

## A.8. Lattice structure

---

(as opposed to just a preorder) as easily seen from

$$\left( (X, A) \longrightarrow C^*(X \cup A) = B \right) \quad \text{and} \quad \left( (X, A) \longrightarrow C^*(X \cup A) = B' \right) :$$

$$\left( \begin{array}{ccccc} (X, A) & \longleftarrow & (X, A) & \longleftarrow & (X, A) \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B' & \longrightarrow & B \end{array} \right) = \left( \begin{array}{ccc} (X, A) & \longleftarrow & (X, A) \\ \downarrow & & \downarrow \\ B & \longleftarrow & B \end{array} \right)$$

so one is a retract of the other and similarly the other way around, or one may equivalently also argue using their kernel from the Toeplitz algebra

$$\left( \mathcal{T}X \longrightarrow B \longrightarrow B' \right) \implies \ker(\mathcal{T}X \rightarrow B) \subseteq \ker(\mathcal{T}X \rightarrow B')$$

from which they had been already the same quotient

$$\ker(\mathcal{T}X \rightarrow B) = \ker(\mathcal{T}X \rightarrow B') \implies \left( \mathcal{T}X \longrightarrow B = B' \right).$$

Alternatively one may now also argue using kernel–covariance pairs,

$$\left( K \subseteq L \subseteq K \mid I \subseteq J \subseteq I \right) \implies \left( K = L \mid I = J \right).$$

Altogether we remark that the lattice of gauge-equivariant representations coincides entirely with the lattice of kernel–covariance pairs, while suprema and infima of kernel–covariance pairs may lay only beyond of just the intersection and sum of their kernel and covariance ideals.

With this in mind we continue on the remaining questions from section A.4: More precisely, we wish to find suitable characterisations when morphisms exists between different quotient strands (based on kernel–covariance pairs)

$$\begin{array}{ccccccc} \left( \frac{X}{XK}, \frac{A}{K} \right) & \longrightarrow & \mathcal{O}(K, 0) & \longrightarrow & \mathcal{O}(K, I) & \longrightarrow & \mathcal{O}(K, \max) \\ \downarrow & & \downarrow & & \downarrow \text{---} & & \\ \left( \frac{X}{XL}, \frac{A}{L} \right) & \longrightarrow & \mathcal{O}(L, 0) & \longrightarrow & \mathcal{O}(L, J) & \longrightarrow & \mathcal{O}(L, \max) \end{array}$$

and we warn ahead that these **won't always exist**. Now at first we note that clearly the existence infers the inclusion of kernel ideals and we may for simplicity replace the original correspondence with the quotient

$$(X, A) := \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow \mathcal{O}(K = 0, 0) \longrightarrow \mathcal{O}(K = 0, I) \longrightarrow \dots$$

In particular there always exist connecting morphism from at least the Toeplitz

### A.8. Lattice structure

---

algebra and so also some further relative Cuntz-Pimsner algebras

$$\begin{array}{ccccccc}
 \mathcal{T}\left(\frac{X}{XK}\right) = \mathcal{O}(K, 0) & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(K, I = ?) & \longrightarrow & \dots \\
 \downarrow & & & \swarrow \checkmark & \downarrow \checkmark & & \swarrow \times \\
 \mathcal{T}\left(\frac{X}{XL}\right) = \mathcal{O}(L, 0) & \longrightarrow & & \longrightarrow & \mathcal{O}(L, J) & \longrightarrow & \mathcal{O}(L, \max).
 \end{array}$$

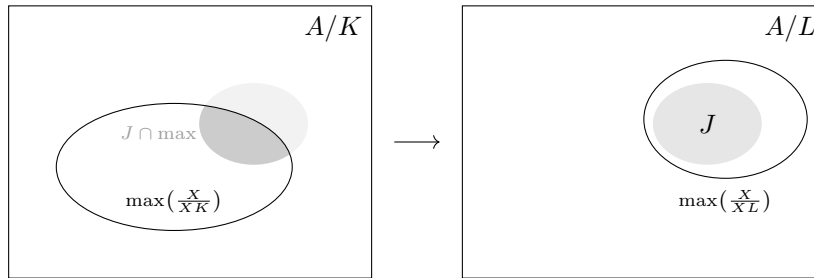
As such the first question is to find the smallest relative Cuntz-Pimsner algebra from which connecting morphisms exist. This may be now easily solved as

$$\mathcal{O}(K, I) \dashrightarrow \mathcal{O}(L, J) \iff (I + K) \subseteq (J + L)$$

and as such the largest covariance ideal (i.e. within the maximal covariance) simply arises as intersection with the given covariance from the quotient,

$$I = \max\left(\frac{X}{XK}\right) \cap (A/K \rightarrow A/L)^{-1} J = \max\left(\frac{X}{XK}\right) \cap (J + K).$$

Schematically the intersection can look something like this:



The reader may easily find some examples with (using graph algebras as above):

$$\max\left(\frac{X}{XK}\right) \neq 0 : (J \cap \max) = 0 \ / \ 0 \neq (J \cap \max) \neq \max \ / \ (J \cap \max) = \max \ ?$$

Moreover one may now easily guess the **meet of kernel-covariance pairs**:

$$\bigwedge_s (K_s \mid I_s) = \left( K = \bigcap_s K_s \mid I = \bigcap_s I_s \cap \max\left(\frac{X}{XK}\right) \right)$$

So far about connecting morphisms **from preceding cokernel strands**.

The other direction however is more interesting: That is a relative Cuntz-Pimsner does not necessarily connect to every following quotient correspondence and in

## A.8. Lattice structure

---

there not even beginning at every relative Cuntz-Pimsner algebra either,

$$\begin{array}{ccccc}
 \mathcal{O}(K, 0) & \longrightarrow & \mathcal{O}(K, I) & \longrightarrow & \mathcal{O}(K, \max) \\
 \downarrow & & \downarrow & \searrow & \\
 \mathcal{O}(L =?, 0) & \longrightarrow & \mathcal{O}(L =?, J =?) & \longrightarrow & \mathcal{O}(L =?, \max).
 \end{array}$$

Note this also describes the lattice of gauge-invariant ideals for the given relative Cuntz-Pimsner algebra (simply as each such defines a quotient). We note however that as soon as it connects to another relative Cuntz-Pimsner algebra (in some following quotient correspondence) then it certainly also does so to the absolute Cuntz-Pimsner algebra for that quotient correspondence. As such this introduces an obstruction which may be now handle using (A.21):

$$\mathcal{O}(K, I) \leq \mathcal{O}(L, \max) \iff J_{\min} := \left( A/K \rightarrow A/L \right) I \subseteq \max\left( \frac{X}{XL} \right)$$

This condition fails from time to time (we provide a simple example below). In case this condition is met we obtain **as smallest solution**

$$\begin{array}{ccccccc}
 \mathcal{O}(K, 0) & \longrightarrow & \mathcal{O}(K, I) & \longrightarrow & \dots & & \\
 \downarrow & & \downarrow & \searrow & \searrow & \searrow & \\
 \mathcal{O}(L, 0) & \longrightarrow & \dots & \longleftarrow & \mathcal{O}(L, J_{\min}) & \longrightarrow & \dots \longrightarrow \mathcal{O}(L, \max)
 \end{array}$$

while in case the condition fails then there simply is **no connecting morphism**. As such we also obtain the lattice for any relative Cuntz-Pimsner algebra which is really **just our main result restated** (while this also generalizes O-pairs):

**Corollary A.8.4** (Relative Cuntz-Pimsner algebra: gauge-invariant ideals):  
*Consider a relative Cuntz-Pimsner algebra (as described in section A.4)*

$$(X, A) \longrightarrow \left( \frac{X}{XK}, \frac{A}{K} \right) \longrightarrow \mathcal{O}(K, 0) \longrightarrow \mathcal{O}(K, I)$$

for some kernel-covariance pair as in (A.20) above.

Then its lattice of gauge-invariant ideals simply runs over pairs as in (A.21)

$$\left( K \subseteq L \mid I + K \subseteq J + L \right) \iff \left( K \subseteq L \mid J_{\min} \subseteq J \subseteq \max\left( \frac{X}{XL} \right) \right)$$

that is in words simply over all kernel-covariance pairs above.

Let us give an example for when there is no connecting morphism:

**Example A.8.5** (Graph correspondences: no connecting morphism):

## A.8. Lattice structure

---

Consider as an example the following graph and its quotient graphs

$$\begin{array}{ccccc}
 K = 0 : & & K = (a) : & & K = (a \cup b) : \\
 \left( \begin{array}{c} a \rightarrow b \end{array} \right) & \longrightarrow & \left( \begin{array}{c} b \end{array} \right) & \longrightarrow & \left( \begin{array}{c} \emptyset \end{array} \right).
 \end{array}$$

Then there is no connecting morphism for the absolute Cuntz-Pimsner algebras between the first and second quotient (by simply reading off covariance ideals):

$$\begin{array}{ccccc}
 \mathcal{T}\left( X = \ell^2(a \rightarrow b) \right) & \longrightarrow & \mathcal{T}\left( X = \ell^2(b) \right) & \longrightarrow & \mathcal{T}(X = 0) \\
 \downarrow & & \parallel & \nearrow & \parallel \\
 \mathcal{O}\left( X = \ell^2(a \rightarrow b) \right) & \dashrightarrow \times \dashrightarrow & \mathcal{O}\left( X = \ell^2(b) \right) & & \mathcal{O}(X = 0).
 \end{array}$$

Indeed the obstruction fails for the covariance ideal:

$$\begin{array}{l}
 I_1 = (b) = \text{reg}(a \rightarrow b) \quad \text{and} \quad K_2 = (a) \subseteq \text{her}(a \rightarrow b) : \\
 (A \rightarrow A/K_2)I_1 = (b) \not\subseteq \text{reg}(\text{quotient graph} = b).
 \end{array}$$

The issue here is that the hereditary ideal is simply not saturated. Alternatively one may note that the first defines the simple algebra of  $2 \times 2$  matrices.

In particular, we obtain for the **join of kernel-covariance pairs**

$$\left( K_1 = 0 \mid I_1 = (b) \right) \vee \left( K_2 = (a) \mid I_2 = (a) \right) = \left( K = (a \cup b) \mid \dots \right).$$

In other words, the join as **next possible kernel-covariance pair** lies only beyond of just the sum of kernel and covariance ideals. So we found an example for the issue (mentioned further above) that suprema and infima will be generally **beyond just intersections and sums of ideals**.

We finish this section with a widely missed identification of Katsura's work:

That is we clarify how **Katsura's T-pairs (and O-pairs)** are nothing but the pullback version of our kernel-covariance pairs from above. More precisely we expand on the requirement given in [Kat07, definition 5.1]

$$J(K) := \left\{ a \in A \mid \dots \text{ and } aX^{-1}(K) \subseteq K \right\} : \quad K \subseteq I \subseteq J(K)$$

and how this defines a **translation of the constraint** from proposition A.3.6:

That is any covariance for an embedding into an operator algebra is necessarily

## A.8. Lattice structure

---

orthogonal to the kernel (for its left action) which read in our case

$$\text{cov}\left(\frac{X}{XK} \rightarrow B\right) \perp \ker\left(\frac{A}{K} \curvearrowright \frac{X}{XK}\right)$$

and as such these cannot exceed the maximal covariance (a.k.a. Katsura's ideal)

$$\text{cov}\left(\frac{X}{XK} \rightarrow B\right) \subseteq \left[\left(\frac{X}{XK}\right)\left(\frac{X}{XK}\right)^* \cap \ker\left(\frac{A}{K} \curvearrowright \frac{X}{XK}\right)^\perp\right] = \max\left(\frac{X}{XK}\right).$$

As such we found instead our kernel-covariance pairs as given **by invariant ideals as kernel** (which defines some sort of discrete range for kernel ideals)

$$K \trianglelefteq A : \quad X^* K X \subseteq K$$

together with **ideals bounded from above** as covariance (which defines an upper bound on the range of covariance ideals)

$$I \trianglelefteq A/K : \quad 0 \subseteq I \subseteq \max\left(\frac{X}{XK}\right)$$

while we found in the **second half of our classification** that each such kernel-covariance pair indeed arises itself (more precisely theorem [A.7.4](#)). In order to establish Katsura's requirement we first note the obvious

$$I = \left(A \rightarrow A/K\right)^{-1} J \iff K \subseteq I$$

and on the other hand the inclusion (for quotient maps)

$$J \subseteq \max\left(\frac{X}{XK}\right) \iff \left(A \rightarrow A/K\right)^{-1} J \subseteq \left(A \rightarrow A/K\right)^{-1} \max\left(\frac{X}{XK}\right).$$

As such **Katsura's condition** simply states (see [\[Kat07, lemma 5.2\]](#))

$$J(K) = \left(A \rightarrow A/K\right)^{-1} \max\left(\frac{X}{XK}\right)$$

for which it further suffices to verify (see also [\[Kat07, lemma 5.2\]](#))

$$\left\{ aX^{-1}(K) \subseteq K \right\} = \left(A \rightarrow A/K\right)^{-1} \ker\left(\frac{A}{K} \curvearrowright \frac{X}{XK}\right)^\perp$$

since the dotted condition represents nothing but compactly acting coefficients. This is however now easily verified: Consider for this the pullback (which contains

## A.8. Lattice structure

---

the same information)

$$\ker(A \curvearrowright \frac{X}{XK}) = \left( A \rightarrow A/K \right)^{-1} \ker\left(\frac{A}{K} \curvearrowright \frac{X}{XK}\right) :$$

$$\left( A \rightarrow A/K \right) \left( A \rightarrow A/K \right)^{-1} (\dots) = (\dots)$$

and which further reads (as in proposition A.3.9)

$$\ker(A \curvearrowright \frac{X}{XK}) = \left\{ a \frac{X}{XK} = 0 \right\} = \{aX \subseteq XK\} = \{X^*aX \subseteq K\} = X^{-1}(K).$$

Put together we obtain the desired relation for the orthogonal complement.

We note that the relation has been worked out by Katsura in [Kat07, lemma 5.2] which however has been not continued further on: Katsura chose to work with the cryptic requirement instead of pursuing their kernel–covariance counterpart. Possibly because they got **only partially recognized** as covariance ideals.

Finally the author notes that the results here arose from a more detailed study of [Kat07] which builds on [FMR03] and further [KPW98] and [Pim97]. More precisely, the author realized the relations drawn in [Kat07, lemma 5.10] (which extend [FMR03, lemma 2.9]) as a partial result on **categorical kernel** and cokernel morphisms which led to **their intrinsic characterization** (in theorem A.3.3) and so also on the **range of possible kernel ideals**.

On the other hand the author realized the first observation made in [Kat04, proposition 3.3] as an intrinsic characterisation on the range of **possible covariance ideals** for the induced representation on the quotient. These let the author to **systematically employ** such kernel–covariance pairs, which allowed on one hand to handle **the general version** of the gauge-invariant uniqueness-theorem **by reduction to the faithful case** which follows from the **sleek and simplifying proof** by Evgenios Kakariadis in [Kak16] (which draws from the second observation made in [Kat04, proposition 3.3]) and on the other hand the **critical observation** made by Takeshi Katsura in [Kat04, lemma 4.7] in his seminal paper from 2004, which led the author to **retrieve kernel–covariance pairs** from their relative Cuntz-Pimsner algebra (in theorem A.7.4). **The main difference** however is that we didn't need to build any ad-hoc semi-kind-of categorical pushout for correspondences as was handled in [Kat07]. Instead it is all based on **the simple idea of reduction** to faithful representations using **kernel and cokernel morphisms**.



## A.9 Pimsner dilations

This final section introduces the notion of dilations and verifies the existence of the maximal dilation as **Hilbert bimodule**. We further reveal Katsura's construction as a **particular nonmaximal dilation** and illustrate the lack of minimal dilations. Meanwhile, the author would like to take this opportunity **to thank Ralf Meyer** for sharing his enlightening perspective on the Pimsner dilation as **maximal dilation**.

We begin with the concept of dilations: More precisely that is any **gauge-equivariant** factorisation over some **intermediate correspondence** such as

$$\begin{array}{ccc} & (Y, B) & \\ \text{---} & \nearrow & \searrow \text{---} \\ (X, A) & \xrightarrow{\quad\quad\quad} & \mathcal{O}(K, I) \end{array}$$

where the gauge-equivariance boils down to simply

$$Y \longrightarrow \mathcal{O}(K, I)(1) \quad \text{and} \quad B \longrightarrow \mathcal{O}(K, I)(0).$$

As the original correspondence generates the relative Cuntz–Pimsner algebra, so does also the intermediate one

$$C^*(X \cup A) = \mathcal{O}(K, I) \quad \implies \quad C^*(Y \cup B) = \mathcal{O}(K, I)$$

whence the factorisation defines a relative Cuntz–Pimsner algebra itself:

$$(Y, B) \longrightarrow \mathcal{O}(K, I) = \mathcal{O}\left(Y, B \mid L = ? \ J = ?\right)$$

As such the task is now to find dilations which generate the relative Cuntz–Pimsner algebra as an **absolute Cuntz–Pimsner algebra**:

$$(X, A) \dashrightarrow \left( Y = ?, B = ? \right) \dashrightarrow \mathcal{O}(K, I) = \mathcal{O}\left( Y, B \mid L = 0, J = \max \right) ?$$

As the kernel should be trivial we have no choice than to look within the relative Cuntz–Pimsner algebra itself. Also we may as usual assume that our original correspondence embeds itself (simply by replacing our original correspondence by its quotient correspondence). As such our intermediate correspondence necessarily

## A.9. Pimsner dilations

---

arises as an intermediate subspace

$$X \subseteq Y \subseteq \mathcal{O}(K = 0, I)(1) \quad \text{and} \quad A \subseteq B \subseteq \mathcal{O}(K = 0, I)(0).$$

On the other hand we found (as a well-known description) that the absolute Cuntz–Pimsner algebra arises as the smallest gauge-equivariant quotient for which the coefficient algebra faithfully embeds into (and so also the correspondence) or in other words the coefficient algebra detects the gauge-invariant ideals within the absolute Cuntz–Pimsner algebra. So we aim to find an **intermediate subspace which detects** the remaining gauge-invariant ideals

$$J \trianglelefteq \mathcal{O}(K = 0, I) : \quad B \cap J = 0 \implies J = 0 ?$$

By our main result (theorem A.7.4) we found a parametrisation for the entire lattice of gauge-equivariant representations

$$\begin{array}{ccccc} \mathcal{O}(K = 0, I = 0) & \longrightarrow & \mathcal{O}(K \neq 0, I = 0) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}(K = 0, I = \text{some}) & \dashrightarrow & \dots & \dashrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}(K = 0, I = \text{max}) & \dashrightarrow & \dots & \dashrightarrow & \dots \end{array}$$

given by kernel–covariance pairs

$$\left( K \trianglelefteq A : X^* K X \subseteq K \mid I \trianglelefteq \frac{A}{K} : I \subseteq \max\left(\frac{X}{XK}\right) \right).$$

and so also of gauge-invariant ideals (within the Toeplitz algebra) whence also for the relative Cuntz–Pimsner algebra. Our original coefficient algebra however already detects the kernel component:

$$A \cap \mathcal{T}(X, A) \left( K \subseteq A \right) \mathcal{T}(X, A) = 0 \implies K = 0$$

As such the only gauge-invariant ideals which our original coefficient algebra cannot detect are precisely the covariance ideals (with trivial kernel component)

$$I \subseteq \max(X, A) : \quad A \cap \mathcal{T}(X, A) \begin{pmatrix} I \\ 0 \end{pmatrix} \mathcal{T}(X, A) = 0.$$

Furthermore we have already taken a quotient for some covariance (which brought us to our relative Cuntz–Pimsner algebra) and so the remaining gauge-invariant

## A.9. Pimsner dilations

---

ideals arise as remaining quotients

$$\mathcal{O}(K = 0; I = 0) \longrightarrow \mathcal{O}(K = 0, I) \longrightarrow \dots \longrightarrow \mathcal{O}(K = 0, \max).$$

As such we need to find an intermediate coefficient algebra which detects all of the remaining covariance ideals beyond the given one  $I \subseteq J \subseteq \max(X, A)$ :

$$A \subseteq B \subseteq \mathcal{O}(K, I)(0) : \quad B \cap \mathcal{O}(K, I) \begin{pmatrix} J \\ 0 \end{pmatrix} \mathcal{O}(K, I) \neq 0 \quad (\text{A.22})$$

with an intermediate subspace as correspondence (as described in section A.1)

$$X \subseteq Y \subseteq \mathcal{O}(K, I)(1) : \quad Y^*Y \subseteq B, \quad BY \subseteq Y, \quad YB \subseteq Y.$$

As such one may first choose an intermediate subalgebra which detects the remaining covariance ideals (and if desired also a chosen subspace) and from there simply enlarge the chosen subalgebra to form a correspondence, for instance as the smallest correspondence above:

$$\bigcap \left\{ (Y_0 \subseteq Y | B_0 \subseteq B) \mid Y^*Y \subseteq B, \quad BY \subseteq Y, \quad YB \subseteq Y \right\}.$$

Indeed the intersection of any class of correspondences forms a correspondence:

$$\left( Y = \bigcap Y_n \mid B = \bigcap B_n \right) : \quad Y_n^*Y_n \subseteq B_n, \dots \implies Y^*Y \subseteq B, \dots$$

We meanwhile need to verify that there always exists one above: For this we simply consider the maximal dilation (**also known as Pimsner dilation**)

$$\left( Y = \mathcal{O}(K, I)(1) \mid B = \mathcal{O}(K, I)(0) \right).$$

Indeed the maximal dilation defines **even a Hilbert bimodule** and so also a correspondence simply as Fourier spaces define Fell bundles (confer section A.5):

$$\begin{aligned} \mathcal{O}(K, I)(-1)\mathcal{O}(K, I)(1) &\subseteq \mathcal{O}(K, I)(-1 + 1 = 0), \\ \mathcal{O}(K, I)(0)\mathcal{O}(K, I)(1) &\subseteq \mathcal{O}(K, I)(0 + 1 = 1), \quad \dots \end{aligned}$$

On the other hand the maximal dilation is also easily seen to detect all of the remaining covariance simply as each is generated from the fixed point algebra:

$$\begin{aligned} \begin{pmatrix} J \\ 0 \end{pmatrix} &\subseteq A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + XX^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \subseteq \mathcal{O}(K, I)(0) \\ \implies \begin{pmatrix} J \\ 0 \end{pmatrix} &\subseteq \mathcal{O}(K, I)(0) \cap \left[ \mathcal{O}(K, I) \begin{pmatrix} J \\ 0 \end{pmatrix} \mathcal{O}(K, I) \right] \end{aligned} \quad (\text{A.23})$$

## A.9. Pimsner dilations

---

**In fact one may argue more generally:** Given any operator algebra with a given circle action and consider its fixed point algebra (as in section A.5)

$$\mathbb{T} \curvearrowright B : \quad B(n=0) = \left\{ b \mid (b(z) := z \curvearrowright b) \equiv b \right\}.$$

Then its fixed point algebra detects every gauge-invariant subalgebra:

$$\left( A = \mathbb{T} \curvearrowright A \right) \subseteq B : \quad A \cap B(0) = 0 \implies A = 0.$$

Indeed this follows using the conditional expectation (confer section A.5)

$$\mathbb{T} \curvearrowright B : \quad E(b) = \int_{\mathbb{T}} (b(z) = z \curvearrowright b) dz.$$

As the subalgebra is gauge-invariant the conditional expectation does not leave the subalgebra (using its construction as Bochner integral)

$$\left( A = \mathbb{T} \curvearrowright A \right) \implies E(A) = \int_{\mathbb{T}} (z \curvearrowright A \subseteq A) dz \subseteq A.$$

On the other hand, every operator algebra is spanned by its positive portion,

$$A = \text{pos}(A) - \text{pos}(A) + i \text{pos}(A) - i \text{pos}(A), \quad \text{pos}(A) := \{0 \leq a \in A\}.$$

The conditional expectation (given by averaging) is however faithful on the positive portion and as such we have have found the detection

$$E(\text{pos } A) \subseteq A \cap B(0) = 0 \implies \text{pos}(A) = 0 \implies A = 0.$$

This well-known technique is **quite worthwhile in other context.**

As such we have found the following familiar result with ease (and in fact there was basically nothing left to prove anymore):

**Theorem A.9.1** (Maximal dilation: absolute Cuntz–Pimsner algebra):

*The maximal dilation realises relative Cuntz–Pimsner algebras as absolute one*

$$\mathcal{O}(K, I) = \mathcal{O} \left( Y = \mathcal{O}(K, I)(1) \mid B = \mathcal{O}(K, I)(0) \right).$$

*and further defines the maximal Hilbert bimodule.*

*Proof.* This is now an immediate consequence from (A.23) satisfying (A.22).  $\square$

This dilation however is rather large in the sense that there is not much control over its behavior (besides its universal description). For instance one may think

## A.9. Pimsner dilations

---

of the maximal dilation similar to maximal Furstenberg boundary. Instead we therefore seek for some dilation small enough **to be tractable combinatorially** while large enough to detect covariance. As explained above, one may for this simply begin with a small subalgebra which detects covariance and simply enlarge the chosen subalgebra to form a correspondence. In practice one may for instance **attempt to run the algorithm**

$$\begin{aligned} Y = BX + X + XB + BXB &\implies B' = B + Y^*Y \\ \implies Y' = B'Y + Y + YB' &\implies \dots \end{aligned} \tag{A.24}$$

with implicit closed linear span as usual.

While there always exist a smallest correspondence above (as we found above) this process may never halt and whence leave us clueless about its combinatorial behavior. In good cases however, the algorithm halts and thus allows for its combinatorial description. This happens in particular for the canonical subalgebra **given by the maximal covariance itself:**

$$A \subseteq \left[ B = A + \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \right] \subseteq \mathcal{O}(K, I)(0) \tag{A.25}$$

whose sum defines a subalgebra since

$$\begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \subseteq \begin{pmatrix} \max(X, A)A & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix}$$

which is nothin but the relation (from proposition [A.2.1](#))

$$(\varphi - \tau) \max(X, A) \cdot \varphi(A) = (\varphi - \tau) \begin{pmatrix} \max(X, A)A & \\ & 0 \end{pmatrix} \subseteq (\varphi - \tau) \max(X, A).$$

On the other hand its left action **keeps the space invariant**

$$\begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} X \subseteq (A + XX^*)X \subseteq X \tag{A.26}$$

and as such the algorithm halts right after the first round,

$$Y = X + X \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} = XB : \quad Y^*Y = BX^*XB \subseteq B$$

and as such we got the **canonical dilation as a combinatorial object.**

Using the left and right shift (as in section [A.7](#)) we further even note

$$\begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \curvearrowright X = \max(X, A)(1 - RL) \cdot XR = 0 \tag{A.27}$$

## A.9. Pimsner dilations

---

which is nothing but the obvious relation (see also proposition [A.2.1](#))

$$\tau(XX^*)\tau(X) = \tau(XX^*X) \implies (\varphi - \tau)\left(A \cap XX^*\right)\tau(X) = 0.$$

This resembles Katsura's construction from [[Kat04](#)] and so we refer to the canonical dilation given by the maximal covariance as **Katsura dilation** (and note also here that there was basically nothing left to prove anymore):

**Theorem A.9.2** (Katsura dilation: absolute Cuntz–Pimsner algebra):

*The canonical dilation given by the maximal covariance realises a relative Cuntz–Pimsner algebra as an absolute Cuntz–Pimsner algebra*

$$\mathcal{O}(K = 0, I) = \mathcal{O}\left( Y = X + X \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \mid B = A + \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \right)$$

and the analogous dilation for kernel–covariance pairs with kernel ideal.

*Proof.* This is now an immediate consequence from ([A.25](#)) satisfying ([A.22](#)).  $\square$

We meanwhile note that this dilation (given by the maximal covariance) may well fail to define a minimal dilation (detecting covariance) while even if minimal it may possibly also fail to be the only minimal dilation (which is yet open). We will provide a class of examples for the first failure in [A.9.8](#).

**Corollary A.9.3** (Katsura dilation: intrinsic description):

*The canonical dilation given by the maximal covariance allows the intrinsic description as the operator algebra freely generated by their abstract copies*

$$\left( A = A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \cup \left( \max(X, A)/I = \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \right)$$

with multiplication given by

$$A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \subseteq \left( A \max(X, A) \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \subseteq \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix}$$

and similarly for the correspondence itself.

*The analogous expression holds for kernel–covariance pairs with kernel ideal.*

*This further recovers the particular description from [[Kat07](#), definition 6.1].*

*Proof.* By part of our main result (the nontrivial part of theorem [A.7.4](#)) we found that the relative Cuntz–Pimsner algebra does not introduce additional kernel which is the faithful copy of the coefficient algebra (as a familiar result):

$$(X, A) \longrightarrow \mathcal{O}(K = 0, I) : \quad A = A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

## A.9. Pimsner dilations

---

On the other hand the relative Cuntz–Pimsner algebra also does not introduce additional covariance (asides the already given covariance) which reads

$$\left( \begin{array}{c} A \cap XX^* \\ 0 \end{array} \right) \cap \mathcal{T}(X, A) \left( \begin{array}{c} I \\ 0 \end{array} \right) \mathcal{T}(X, A) = \left( \begin{array}{c} I \\ 0 \end{array} \right).$$

and as such the added maximal covariance defines a faithful copy up to

$$\max(X, A)/I = \left( \begin{array}{c} \max(X, A) \\ 0 \end{array} \right) \subseteq \mathcal{O}(K = 0, I).$$

Note that the maximal covariance absorbs the coefficient algebra and as such their sum already defines a closed and thus complete operator algebra (see the quick proof (A.16) on the sum of an algebra and ideal from section A.6):

$$A \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} \max(X, A) \\ 0 \end{array} \right) = \overline{A \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} \max(X, A) \\ 0 \end{array} \right)} \subseteq \mathcal{O}(K = 0, I).$$

In fact this holds in any representation as also their universal:

$$\begin{aligned} C^* \left( A \cup \max(X, A)/I \mid A \max(X, A)/I \subseteq \max(X, A)/I \right) \\ = A + \max(X, A)/I = \overline{A + \max(X, A)/I}. \end{aligned}$$

As such any concrete representation which provides faithful disjoint copies for the coefficient algebra and the maximal covariance (mod covariance ideal) defines a faithful representation for their freely generated operator algebra:

$$A \cap \max(X, A)/I = 0 \subseteq B \quad \implies \quad C^*(A \cup \max(X, A)/I) \subseteq B$$

Indeed this simply follows by some basic linear algebra. This holds in particular for their copy in the relative Cuntz–Pimsner algebra

$$A \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \cap \left( \begin{array}{c} \max(X, A) \\ 0 \end{array} \right) = \left( \begin{array}{c} \ker(A \curvearrowright X) \\ 0 \end{array} \right) \cap \left( \begin{array}{c} \max(X, A) \\ 0 \end{array} \right) = 0$$

where we have used their trivial intersection

$$\ker(A \curvearrowright X) \cap \max(X, A) \subseteq \ker(A \curvearrowright X) \cap \ker(A \curvearrowright X)^\perp = 0$$

and as such the dilation arises as universal representation

$$\begin{aligned} C^* \left( A \cup \max(X, A)/I \mid A \max(X, A)/I \subseteq \max(X, A)/I \right) \\ = A \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} \max(X, A) \\ 0 \end{array} \right) \subseteq \mathcal{O}(K = 0, I). \end{aligned}$$

Note this defines a quite general argument which applies also in other context.

## A.9. Pimsner dilations

---

Finally Katsura's description is nothing but the isomorphism

$$C^*\left( A \cup M \mid AM \subseteq M \right) = \left\{ a \oplus (\text{im } a + m) \in A \oplus (\text{im } A + M \subseteq B) \right\}$$

which simply enforces faithful disjoint copies for any  $A \rightarrow B$ :

$$A = A(1 \oplus 1), \quad A(1 \oplus 1) \cap (0 \oplus M) = 0, \quad M = 0 \oplus (M \subseteq B).$$

There is however nothing special about this choice of formal description. Instead it is the universal description as freely generated copies with one absorbing which captures its properties:

$$\begin{aligned} C^*\left( A \cup \max(X, A)/I \mid A \max(X, A)/I \subseteq \max(X, A)/I \right) \\ = A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \subseteq \mathcal{O}(K = 0, I). \end{aligned}$$

The reader may now similarly argue for the correspondence.  $\square$

We may now easily recover the classical result that any gauge-equivariant quotient for some (possibly relative) graph algebra **remains a graph algebra**. For this we first recall that any quotient correspondence (as kernel component) arises as a quotient graph (confer example [A.3.5](#))

$$\begin{aligned} A/K = c_0(V)/c_0\left( H = \text{hereditary} \right) = c_0\left( W = V \setminus H \right) \\ X/XK = \ell^2(E)/\ell^2(EH) = \ell^2\left( F := WE \right) \end{aligned}$$

and as such we may replace the original graph by the quotient graph

$$X = \ell^2(E := F), \quad A = c_0(V := W) \quad \implies \quad \mathcal{O}(K = 0, I).$$

On the other hand recall that any covariance ideal for a graph (in our case the quotient graph) arises simply as a regular set of vertices (confer example [A.3.7](#)):

$$\max(X, A) = c_0\left( \text{regular} \right) \quad \implies \quad I = c_0\left( R \subseteq \text{regular} \right).$$

With this notation in mind we may now find the canonical dilation as a graph. We note for this that as it was given **by the algorithm above** we may recover the canonical dilation as a combinatorial object **from the original data**, which boils down in our case to the canonical dilation arising as a graph:

**Corollary A.9.4** (Katsura dilation: graph correspondences):



## A.9. Pimsner dilations

---

The canonical dilation given by the maximal covariance as in (A.25)

$$\left( XB = X + X \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \mid B = A + \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \right)$$

arises as the following canonical graph (with notation from above):

Its coefficient algebra arises as the orthogonal sum of vertices

$$W = \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} \cup \left( \text{regular} \setminus R \right) \begin{pmatrix} \\ & 0 \end{pmatrix}$$

together with the correspondence given by the graph

$$EW = E \left( \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} \cup \left( \text{regular} \setminus R \right) \begin{pmatrix} \\ & 0 \end{pmatrix} \right)$$

and its left action given by (whence defining the range of edges)

$$\begin{pmatrix} a & \\ & a \end{pmatrix} EW = (aE)W \quad \text{and} \quad \begin{pmatrix} 0 & \\ & b \end{pmatrix} EW = \begin{pmatrix} b & \\ & b \end{pmatrix} EW = (bE)W$$

while trivially acting for the left over last summand.

As such the graph reads in more classical terms

$$W = \left( \text{all vertices} = \text{singular} \cup \text{regular} \right) \amalg \left( \text{regular} \setminus R \right)$$

$$EW = E \left( \text{all vertices} \right) \amalg E \left( \text{regular} \setminus R \right)$$

with range and source map given by

$$s(- \amalg \emptyset) = s(-) \amalg \emptyset \quad \text{and} \quad s(\emptyset \amalg -) = \emptyset \amalg s(-),$$

$$r(- \amalg \emptyset) = r(-) \amalg \emptyset = r(\emptyset \amalg -).$$

Note that the combined summands basically recover the original graph whereas the last provides an additional copy to make up for the maximal covariance. On the other hand the edges **all point into the original copy of vertices**. As such this recovers the familiar construction for graph algebras: Any gauge-equivariant quotient arises as a graph algebra itself.

*Proof.* In order to find the canonical dilation as a graph it suffices to recover its coefficient algebra as an **orthogonal sum of vertices** (see example A.1.2). In our case the coefficient algebra already reads as a sum of vertices

$$A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} = c_0(\text{vertices}) \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \left( c_0(\text{regular}) \begin{pmatrix} \\ & 0 \end{pmatrix} \right)$$

## A.9. Pimsner dilations

---

which we may decompose now further into an orthogonal sum:

First the singular vertices (that is the nonregular ones) are trivially disjoint from the regular ones and as such define an orthogonal summand,

$$\text{vertices} = \text{singular} \cup \text{regular} : \quad \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \perp \begin{pmatrix} \text{regular} & \\ & \text{regular} \end{pmatrix}.$$

On the other hand the sum on regular vertices may be also taken as

$$\text{regular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \text{regular} & \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} + \begin{pmatrix} \text{regular} & \\ & 0 \end{pmatrix}$$

whose summands belong to the relative Cuntz–Pimsner algebra since

$$\tau(XX^*) = \begin{pmatrix} 0 & \\ & XX^* \end{pmatrix} \quad \text{and} \quad (\varphi - \tau)(A \cap XX^*) = \begin{pmatrix} A \cap XX^* & \\ & 0 \end{pmatrix}$$

which is available only for the compactly acting portion! The latter summand vanishes precisely for the given covariance (see theorem A.7.4 or corollary A.9.3)

$$\begin{pmatrix} \text{regular} \setminus R & \\ & 0 \end{pmatrix} \neq 0 \quad \text{and} \quad \begin{pmatrix} R & \\ & 0 \end{pmatrix} = 0$$

and as such the sum reduces to the **non-zero vertices**

$$\begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} + \begin{pmatrix} \text{regular} & \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} + \begin{pmatrix} \text{regular} \setminus R & \\ & 0 \end{pmatrix}.$$

As such the coefficient algebra for our dilation decomposes into

$$\text{vertices} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \text{regular} & \\ & 0 \end{pmatrix} = \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} + \begin{pmatrix} \text{regular} \setminus R & \\ & 0 \end{pmatrix}$$

and so we have found the **vertices for our graph correspondence**.

We may now simply read off the edges from our correspondence as

$$X + X \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} = \ell^2(E) \left( \text{singular} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 0 & \\ & \text{regular} \end{pmatrix} + \begin{pmatrix} \text{regular} \setminus R & \\ & 0 \end{pmatrix} \right)$$

with the source of edges as evident. On the other hand its left action reads (using the induced morphism on compact operators from A.2.1)

$$\begin{pmatrix} a & \\ & a \end{pmatrix} EW = \varphi(a)\tau(E)W = \tau(aE)W = (aE)W$$

$$\begin{pmatrix} 0 & \\ & b \end{pmatrix} EW = \tau(b)\tau(E)W = \tau(bE)W = (bE)W$$

$$\begin{pmatrix} c & \\ & 0 \end{pmatrix} EW = (\varphi - \tau)c\tau(E)W = 0$$

and so we have found the desired graph. □

## A.9. Pimsner dilations

---

We illustrate the canonical dilation for the following prominent graph:

**Example A.9.5** (Katsura dilation: Toeplitz graph):

Consider the correspondence given by the single loop

$$\left( X = \ell^2 \left( \begin{array}{c} x \\ \curvearrowright \\ a \end{array} \right) = \mathbb{C}x \mid A = c_0(\text{vertices}) = \mathbb{C}a \right).$$

Its Toeplitz algebra recovers the traditional Toeplitz algebra since

$$\mathcal{T}(X, A) = C^* \left( x \cup a \mid x^*x = a, ax = x \right) = C^* \left( x^*x = 1 \right) = \mathcal{T}$$

and as such its **suggestive name as Toeplitz graph**. On the other hand its absolute Cuntz–Pimsner algebra recovers the traditional circle algebra

$$\mathcal{O}(X, A) = C^* \left( x \cup a \mid x^*x = a = xx^* \right) = C^* \left( x^*x = 1 = xx^* \right) = C(\mathbb{T}).$$

These already define all its gauge-equivariant representations:

Indeed there is only a single covariance ideal given by the single vertex

$$\max(X, A) = c_0 \left( \text{regular} = a = \text{vertices} \right) = A$$

and no further quotient graph (except the trivial one).

As such we found the entire lattice of gauge-equivariant representations.

Further we may now compute the graph for the canonical dilation:

For this we may now simply read off the graph as (confer corollary A.9.4)

$$\left( W = (a) \amalg (a \setminus \emptyset) = a \amalg a \mid EW = (xa) \amalg (xa) = x \amalg x \right) :$$

$$\left( \begin{array}{l} s(x \amalg \emptyset) = a \amalg \emptyset, \quad s(\emptyset \amalg x) = \emptyset \amalg a \\ r(x \amalg \emptyset) = a \amalg \emptyset = s(\emptyset \amalg x) \end{array} \right) \Rightarrow \left( \begin{array}{c} x \amalg \emptyset \\ \curvearrowright \\ a \amalg \emptyset \longleftarrow \emptyset \amalg x \longleftarrow \emptyset \amalg a \end{array} \right).$$

Put together we found the realisation for the Toeplitz algebra:

$$\mathcal{T} \left( \begin{array}{c} \curvearrowright \\ \bullet \end{array} \right) = \mathcal{O} \left( \begin{array}{c} \curvearrowright \\ \bullet \longleftarrow \bullet \end{array} \right)$$

The latter appears sometimes as well under the name Toeplitz graph.

We now continue with the issue about the existence of minimal dilations. For this we begin with the following positive result for relative graph algebras:

**Corollary A.9.6** (Katsura dilation: minimal dilation). For graph correspondences

## A.9. Pimsner dilations

---

the canonical dilation is minimal. That is roughly speaking, there is no smaller graph which realises the relative graph algebra as an absolute one.

*Proof.* Consider a subalgebra detecting each covariance  $B \subseteq c_0(\text{regular})$ :

$$B \cap c_0(S \subseteq \text{regular}) = 0 \implies c_0(S \subseteq \text{regular}) = 0.$$

Then the subalgebra necessarily contains each summand and so also

$$c_0(S = \{r\}) = \mathbb{C}(r) \subseteq B \implies B = \bigoplus_r \mathbb{C}(r) = c_0(\text{regular}).$$

As such the maximal covariance defines a minimal dilation.  $\square$

The previous observation suggests now the following result for correspondences over spaces (as coefficient algebra) and in particular for integer actions:

**Corollary A.9.7** (Katsura dilation: correspondences over spaces):

*It holds for correspondences with abelian maximal covariance (and so also for correspondences over spaces): The canonical dilation given by the maximal covariance defines a minimal dilation if and only if the maximal covariance has discrete spectrum (resp. the maximal covariance defines a discrete subspace). Maximal covariances with totally disconnected spectra are not sufficient.*

*Proof.* Note that the entire problem deals with an abelian operator algebra. By Stone–Weierstrass it thus suffices to focus on any ideal over some point in the spectrum and any subalgebra over some pair of points of the form

$$\ker(\omega : B \rightarrow \mathbb{C}) = \{b(\omega) = 0\} \quad \text{and} \quad \text{eq}(\omega_i : B \rightrightarrows \mathbb{C}) = \{b(\omega_1) = b(\omega_2)\}.$$

For the ideal we first observe for a pair of closed subspaces  $E, F \subseteq X = \Gamma B$ :

$$\ker(E) \cap \ker(F) = \ker(E \cup F) = 0 \iff E \cup F = X.$$

As such the only ideals some non-isolated point cannot detect are

$$\ker(\omega) \cap \ker(F) = 0 : \quad \ker(F = \overline{X - \omega} = X) = 0$$

whence the non-isolated point would already detect every ideal.

Suppose on the other hand that the spectrum defines a discrete space.

Then every subalgebra as above fails detect even two ideals:

$$\text{eq}(\omega_1, \omega_2) \cap \ker(X - \omega_i) = \{b(X - \omega_i) = 0 = b(\omega_i)\} = 0$$

## A.9. Pimsner dilations

---

The corollary now follows from combining these two observations.

We leave the final statement about totally disconnected spaces to the reader.  $\square$

With the previous corollary in mind, we may now illustrate some negative examples on the existence of minimal dilations altogether:

**Example A.9.8** (Minimal dilations: failures of existence). *For this we may now simply consider any integer action as our correspondence,*

$$\mathbb{Z} \curvearrowright C_0(\text{space}) \implies X = C_0(\text{space}) = A.$$

*Note that any integer action defines a regular correspondence and as such the coefficient algebra defines the maximal covariance. Consider now some continuous space such as for instance the real line for which we thus obtain:*

*a) The canonical dilation **fails to be minimal** as for instance the following ideals already detect each covariance*

$$C_0(\mathbb{R} - r) = \ker(r) \subseteq C_0(\mathbb{R}) : \quad C_0(\mathbb{R} - r) \cap I = 0 \implies I = 0.$$

*b) On the other hand none of those is minimal either since furthermore detection of covariance already happens for those with discrete complement such as for*

$$C_0(\mathbb{R} - \{r_1, \dots, r_n\}) = \ker(r_1 \cup \dots \cup r_n) \subseteq C_0(\mathbb{R}).$$

*Consider now some enumeration of rational numbers  $\{q_1, q_2, \dots\} = \mathbb{Q}$ .*

*As such we obtain some decreasing sequence of ideals with detection:*

$$\ker(q_1) \supseteq \ker(q_1 \cup q_2) \supseteq \dots \supseteq \ker(q_1 \cup \dots \cup q_n) \supseteq \dots$$

*Their intersection however fails to detect any covariance since*

$$\ker(\mathbb{Q} = \{q_1, q_2, \dots\}) = \ker(\overline{\{q_1, q_2, \dots\}} = \mathbb{R}) = 0.$$

*As such the given sequence of ideals admits no minimal dilation.*

*In fact we have even found that **the axiom of choice fails to apply!***

## Acknowledgements

The author would like to thank his supervisor Søren Eilers for his kind support and encouragements, as well as Evgenios Kakariadis for pointing out to [Kat07] as a valuable follow-up article by Takeshi Katsura, which inspired the author on finding the missing components. Moreover, the author acknowledges the support under the Marie–Curie Doctoral Fellowship No. 801199.

## Preprint B

# Connes implies Tsirelson: a simple proof

More precisely, we give a quick and very simple proof of “the Connes embedding problem implies the synchronous Tsirelson conjecture” that relies on only *two elementary ingredients*: 1) the well-known description of synchronous correlations as traces on the algebra per player  $C^*(\text{player}) = C^*(\text{in}|\text{out})$  and 2) an elementary lifting result by Kim, Paulsen and Schafhauser.

Moreover, this bypasses every of the deep results by Kirchberg as well as any other implicit reformulation as the microstates conjecture and the like.

Meanwhile, we also give a different construction of Connes’ algebra  $\mathcal{R}^\omega$  appearing in the Connes embedding problem, which is more suitable for the purposes of quantum information theory and much easier to comprehend for the reader without any prior knowledge in operator algebras.

Most importantly, however, we present this proof for the following reason: Since the recent refutation of the synchronous Tsirelson conjecture by  $\text{MIP}^*=\text{RE}$ , there exists a nonlocal game which violates the synchronous Tsirelson conjecture, and by the proof of  $\text{MIP}^*=\text{RE}$  even a synchronous such game. The approach however is based on contradiction with the undecidability of the Halting problem, and so remains implicit. As such the quest now has started to give an *explicit example of a synchronous game* violating the synchronous Tsirelson conjecture together with a direct argument for the failure, and the current article serves as a direct translation to the corresponding operator algebra and its tracial state violating the Connes embedding problem.

Meanwhile the author would like to stress that he does not take any credit for any of the results as already available. Our only contribution lies in combining the well-known results from quantum information theory with the lifting result as established by Kim Paulsen and Schafhauser in [KPS18].

### B.1 Algebra per player: $C^*(\text{in}|\text{out})$

Let us swiftly introduce the algebra per player, which defines one of the two relevant algebras appearing in our main result below. It is well-known (by Gelfand and

## B.2. Connes algebra

---

(Pontryagin duality) that the universal  $C^*$ -algebra generated by a projection valued measure agrees with the full group  $C^*$ -algebra

$$C^*(\mathbb{Z}/A) = C^*(e_1, \dots, e_A \in \text{proj} \mid e_1 + \dots + e_A = 1)$$

and as such we obtain as the algebra per player their amalgamated free product

$$C^*(X = \text{in} \mid A = \text{out}) := \overbrace{C^*(\mathbb{Z}/A) * \dots * C^*(\mathbb{Z}/A)}^{X\text{-many}} = C^*(\mathbb{Z}/A * \dots * \mathbb{Z}/A)$$

for some finite sets of questions and answers as cardinal numbers  $A, X \in \mathbb{N}$ .

Their labeling as input and output thereby serve to assist the reader to identify their respective question set and answer set, while we will later on omit their actual answer and question sets for more pointy statements.

From here one may (we won't be needing it though) define the algebra for two-player games as the maximal tensor product of each player's algebra (with corresponding question and answer sets)

$$C^*(\text{two player}) = C^*(\text{Alice}) \otimes C^*(\text{Bob}) = C^*(X|A) \otimes C^*(Y|B)$$

and any quantum commuting strategy arises simply as a state

$$\varphi : C^*(\text{two player}) \rightarrow \mathbb{C} : p(ab|xy) = \varphi(e(a|x) \otimes e(b|y)).$$

For more details on its relevance we refer the reader to [Fre22b, section 2].

For us the above algebra per player however will be sufficient since we will be dealing with synchronous strategies exclusively, and whence tracial states on the algebra per player. Having introduced the first algebra relevant for our main result, let us proceed to the second one.

## B.2 Connes' algebra: $\mathcal{R}^\omega$

We introduce in this section the tracial ultrapower of the hyperfinite  $\text{II}_1$ -factor, in short also referred to as Connes' algebra, which defines the main player in the Connes embedding problem. For this we will pursue another construction for the Connes algebra, which defines a more suitable approach when working in quantum information theory (and which may be also much easier to comprehend for the reader without any prior knowledge in operator algebras).

But before we do so, let us first describe how the usual construction of the hyperfinite  $\text{II}_1$ -factor goes along, just for having both constructions available. Pick

## B.2. Connes algebra

---

your favorite UHF-algebra of infinite-type like

$$\begin{aligned} \text{UHF} &= M_2 \otimes M_2 \otimes \dots = \bigotimes_{\infty} M_2, \quad \text{or} \quad \text{UHF} = \left( \bigotimes_{\infty} M_3 \right) \otimes \left( \bigotimes_{\infty} M_7 \right), \\ \text{or} \quad \text{UHF} &= \left( \bigotimes_{\infty} M_2 \right) \otimes \left( \bigotimes_{\infty} M_3 \right) \otimes \dots = \bigotimes_{p: \text{ all primes}} \left( \bigotimes_{\infty} M_p \right). \end{aligned}$$

and consider its unique tracial state, for example

$$\text{tr} : \text{UHF} = \bigotimes_{\infty} M_2 \rightarrow \mathbb{C} : \quad \text{tr}(a_1 \otimes a_2 \otimes 1 \otimes \dots) = \text{tr}(a_1) \text{tr}(a_2).$$

Just as an intermediate result — which may however also be omitted!<sup>1</sup> — one may embed the chosen UHF-algebra in the GNS-representation for the tracial state. Taking the von-Neumann algebraic completion of the UHF-algebra within the GNS-representation produces then the hyperfinite  $\text{II}_1$ -factor:

$$\mathcal{R} = \overline{\text{UHF}}^{\text{w*}} \subseteq B(\overline{\text{UHF}}^2)$$

The story however wouldn't be over here: From here we would still need to proceed to the ultrapower of the hyperfinite  $\text{II}_1$ -factor as follows: Take the infinite repeated power of either the UHF-algebra from above (or alternatively the hyperfinite  $\text{II}_1$ -factor that arose from the completion), which we denote for shorthand by

$$\ell^\infty(\text{UHF}) := \left\{ a \in \prod_n \text{UHF} \mid \sup_n \|a_n\| < \infty \right\} = \prod_n \text{UHF}.$$

The resulting operator algebra comes together with the induced trace

$$\text{tr}_\omega : \ell^\infty(\text{UHF}) \rightarrow \mathbb{C} : \quad \text{tr}_\omega(a) = \lim_{n \rightarrow \omega} \text{tr}(a_n)$$

where the limit is taken along any free ultrafilter. In order to render the trace faithful one passes, together with the trace, to the following quotient which defines the so-called tracial ultrapower

$$\text{UHF}^\omega := \frac{\ell^\infty(\text{UHF})}{c_\omega(\text{UHF}; \text{tr}) := \{a \in \ell^\infty(\text{UHF}) \mid \text{tr}_\omega(a^*a) = 0\}}.$$

This would be the final object appearing in the Connes embedding problem:

$$\mathcal{R}^\omega = \frac{\ell^\infty(\mathcal{R})}{c_\omega(\mathcal{R}; \text{tr})} = \frac{\ell^\infty(\text{UHF})}{c_\omega(\text{UHF}; \text{tr})} = \text{UHF}^\omega$$

---

<sup>1</sup>Due to [BO08, lemma A.9], which following the proof also works for  $C^*$ -algebras, and a Kaplansky density argument.



## B.2. Connes algebra

---

Now, while this construction has its merits in the classification of factors due to the hyperfinite  $\text{II}_1$ -factor appearing in the intermediate construction, it would be way too overloaded for applications in quantum information theory, and in particular for the relation between the Connes embedding problem and the Tsirelson conjecture. So we refrain from this construction!

Instead we introduce now a different construction which, as promised above, defines a more convenient approach when working in quantum information theory. For this we shortcut the construction above and basically start at its very end. More precisely, we take the infinite product on matrices of arbitrary sizes, which we denote suggestively as above by

$$\ell^\infty(M) := \left\{ a \in \prod_n M_n \mid \sup_n \|a_n\| < \infty \right\} = \prod_n M_n.$$

As above, this comes together with the induced trace (along any free ultrafilter)

$$\text{tr}_\omega : \ell^\infty(M) \rightarrow \mathbb{C} : \quad \text{tr}_\omega(a) = \lim_{n \rightarrow \omega} \text{tr}_n(a_n)$$

which we render faithful as above by passing to the quotient by

$$c_\omega(M; \text{tr}) := \left\{ a \in \prod_n M_n \mid \text{tr}_\omega(a^*a) = \lim_n \text{tr}_n(a_n^*a_n) = 0 \right\}.$$

Summarizing the construction: considering sequences of matrices of arbitrary sizes allows us to pickup all finite-dimensional representations, and passing to the quotient allows us to also do so approximately.

On the other hand, the surprising feature (surprising just to some extent) is that this defines also the same tracial ultrapower as before,

$$M^\omega = \frac{\ell^\infty(M)}{c_\omega(M; \text{tr})} = \frac{\ell^\infty(\mathcal{R})}{c_\omega(\mathcal{R}; \text{tr})} = \mathcal{R}^\omega.$$

This follows basically by some Kaplansky density type argument and a careful diagonal reindexing — just with many more steps (see also the footnote above). Since this would however escape the scope of the current article, we refrain from presenting a proof in here and instead leave it to the experienced reader.

Now the left-hand side construction we have just provided is much closer to correlations, and so we will from now on use our construction in what follows. So far on the operator algebra appearing in the Connes embedding problem.

### B.3 Two ingredients

In this section we prepare two simple ingredients, proposition B.3.1 and proposition B.3.2 below, which are all what is needed for “the Connes embedding problem implies the synchronous Tsirelson conjecture”. We recall for this the trace space for an operator algebra  $A$  as those states which possess the trace property

$$TA = \{ \tau : A \rightarrow \mathbb{C} \mid \tau(ab) = \tau(ba) \}.$$

With this notation in mind let us begin with the first ingredient:

**Proposition B.3.1** ([PSSTW16, corollary 5.6]). *The set of synchronous quantum-commuting correlations is realized by the trace space restricted on two-moments:*

$$\tau \in TC^*(X = \text{in} \mid A = \text{out}) : \quad p(ab \mid xy) = \tau(e(a \mid x)e(b \mid y)).$$

*Similarly the set of finite dimensional synchronous correlations is realized by those of such traces which live on finite dimensional quotients thereof*

$$C^*(X = \text{in} \mid A = \text{out}) \longrightarrow \text{some fin dim quotient} \xrightarrow{\text{some trace}} \mathbb{C}$$

*and then restricted on two-moments as above.*

We skip the proof since it is fairly elementary and well-known. The second ingredient is the following lifting result found by Kim, Paulsen and Schafhauser, which we recall together with its proof for convenience of the reader.

**Proposition B.3.2** ([KPS18, lemma 3.5]). *The following lifting problem has a solution: Every representation into the hyperfinite  $\text{II}_1$ -factor lifts to matrices*

$$\begin{array}{ccc}
 & & \ell^\infty(M) \\
 & \nearrow \text{dashed} & \downarrow \\
 C^*(\mathbb{Z}/m) = \underbrace{\mathbb{C} \oplus \dots \oplus \mathbb{C}}_{m\text{-many}} & \longrightarrow & \ell^\infty(M)/c(M, \text{tr}) = \mathcal{R}^\omega
 \end{array}$$

*and unital representations may be lifted unitaly. As a consequence, the lifting problem also has a solution for any number of inputs and outputs*

$$C^*\left(\underbrace{\mathbb{Z}/m * \dots * \mathbb{Z}/m}_{n\text{-many}}\right) = C^*(n = \text{in} \mid m = \text{out}).$$

Before we begin with the proof itself, let us note that the lifting problem easily

### B.3. Ingredients

---

admits a solution by positive maps: Any such representation is determined by some tuple of mutually orthogonal projections in the quotient

$$\mathbb{C} \oplus \dots \oplus \mathbb{C} \rightarrow \ell^\infty(M)/c(M, \text{tr}) : \quad e_1 \mapsto q_1, \quad \dots \quad e_m \mapsto q_m$$

and so in particular by some tuple of positive elements. Any single positive element in a quotient may however be easily lifted itself

$$\pi : A \rightarrow B \rightarrow 0 : \quad \forall b \in \text{pos}(B) \exists a \in \text{pos}(A) : \quad \pi(a) = b$$

and so also an entire tuple of positive elements — each element one-by-one. Put together this defines a solution by some positive map

$$\mathbb{C} \oplus \dots \oplus \mathbb{C} \rightarrow \ell^\infty(M) : \quad e_1 \mapsto a_1, \quad \dots \quad e_m \mapsto a_m.$$

At the same time one may always arrange for such a lift without increasing the norm of each lift and so arrange for contractions

$$\|a_1\| = \|q_1\| \leq 1, \quad \dots \quad , \|a_m\| = \|q_m\| \leq 1.$$

On the other hand, in case of some unital representation one may moreover arrange for a unital lift by adding the remainder on the last,

$$\text{rem} = 1 - (a_1 + \dots + a_m) : \quad a'_m := a_m + \text{rem}.$$

The gist is now to also arrange for some projection valued lift: this is one of the main accomplishments by Kim, Paulsen and Schafhauser in their article on synchronous games. At the heart of this problem lies the following technical result, which we formulate in its slightly improved, optimal version.

Either such bound (be it optimal or just some upper bound) then allows us to deform the tuple of positive elements into an actual set of projections.

**Lemma B.3.3** (compare [KPS18, lemma 3.4]). *Consider for any positive matrix contraction  $a \in M_n$  the spectral projection onto the interval  $[1/2, 1] \subseteq \mathbb{R}$ :*

$$p(a) := 1_{[1/2, 1]}(a) \in M_n : \quad p(a)^2 = p(a) = p(a)^*.$$

*Then it holds the upper bound for the distance*

$$\|a - p(a)\|_\varphi \leq 2\|a^2 - a\|_\varphi$$

### B.3. Ingredients

---

uniformly in every positive matrix contraction and any state 2-norm

$$\|x\|_\varphi^2 = \langle x|x \rangle_\varphi = \varphi(x^*x) = \varphi(|x|^2)$$

and the bound above is optimal for faithful states.

*Proof.* We give a slightly different proof than in [KPS18]: Instead the optimal bound can be easily read off from the spectrum as follows. For this denote for shorthand the left-hand and right-hand side as

$$f(x) := x - h(x), \quad g(x) := x^2 - x = x(1 - x).$$

Since states are positive, we have as a sufficient condition

$$f(a)^2 \leq g(a)^2 \implies \varphi(f(a)^2) \leq \varphi(g(a)^2).$$

This however can be now read off from the spectrum as

$$\forall x \in \sigma(a) : |f(x)| \leq |g(x)|$$

which in our case boils down to the condition

$$\begin{cases} |x| \leq 2|x| \cdot |1 - x| & \text{for } x \in [0, 1/2] \cap \sigma(a), \\ |1 - x| \leq 2|x| \cdot |1 - x| & \text{for } x \in [1/2, 1] \cap \sigma(a). \end{cases}$$

Finally note that the bound is optimal for faithful states: simply use some matrix whos spectrum contains the eigenvalue  $1/2$  — for instance half the identity.  $\square$

With the previous bound at hand one may now derive the desired solution to the lifting problem, which we sketch for completeness.

*Sketch of proposition B.3.2 (based on the construction by [KPS18]):* Say we have already found a lift to some tuple of positive contractions

$$a_1, \dots, a_m \in \ell^\infty(M) = \prod_n M_n.$$

While each element of the tuple is a matrix sequence itself like

$$a = (a_1, a_2, \dots) \in \ell^\infty(M)$$

we will keep viewing each matrix sequence as a single element. The reader new to operator-algebraic techniques may however also savelly run the following procedure

### B.3. Ingredients

---

indexwise for each sequence in the tuple. Replace the first one in the tuple by the spectral projection from lemma B.3.3, then cut-off the resulting projection from the next one and apply the lemma again to that,

$$p_1 := p(a_1), \quad a'_2 := (1 - p_1)a_2(1 - p_1), \quad p_2 := p(a'_2).$$

Continuing this way one needs to cut-off all the previous projections, for example

$$a'_3 = (1 - p_1 - p_2)a_3(1 - p_1 - p_2).$$

This way we guarantee their orthogonality since for each next step

$$1 \leq k + 1 \leq m : \quad a'_{k+1} \perp p_1, \dots, p_k \implies p(a'_{k+1}) \perp p_1, \dots, p_k.$$

The upper bound in lemma B.3.3 now guarantees that the deformation procedure remains a lift for the original tuple in the quotient: Indeed recall that the quotient is given by the ideal

$$c(M, \text{tr}) = \{a \in \ell^\infty(M) \mid \|a\|_2 = 0\}$$

which reads when written out as a sequence  $a = (a_1, \dots) \in \prod_n M_n$ :

$$\|a\|_2^2 = \text{tr}_\omega(a^*a) = \lim_{n \rightarrow \omega} \text{tr}(a_n^*a_n) = 0.$$

The original tuple in the quotient however consists of mutually orthogonal projections and so the deformation procedure remains a lift since: Taking the spectral projection does not alter the equivalence class since by lemma B.3.3

$$\|p(a) - a\|_2 \leq 2\|a^2 - a\|_2 = 0$$

(it helps to view them as a sequence of matrices and one as a sequence of projection matrices, which indexwise get closer and closer in trace 2-norm), nor does the cutting-off procedure since this cannot be seen by any orthogonal pair in a quotient anyways:

$$\pi : A \rightarrow B \rightarrow 0 : \quad \pi(a) \perp \pi(a') \implies \pi(1 - a)\pi(a')\pi(1 - a) = \pi(a').$$

This completes the sketch of the construction for proposition B.3.2.  $\square$

We have now successfully established our ingredients, and so we may now proceed to the main result of the article:

## B.4 Connes $\implies$ Tsirelson

With our two simple ingredients at hand, namely proposition B.3.1 and B.3.2, we may now verify that “Connes implies the synchronous Tsirelson conjecture”.

**Theorem B.4.1.** *The Connes embedding problem implies the synchronous Tsirelson conjecture (and so further also the general Tsirelson conjecture).*

*More precisely, suppose the Connes embedding problem was to hold true: That is every tracial state arises as a factorisation by some representation*

$$A \xrightarrow{\text{repr}} \mathcal{R}^\omega \xrightarrow{\text{tr}_\omega} \mathbb{C}$$

*with unique tracial state on the hyperfinite  $\text{II}_1$ -factor  $\text{tr} : \mathcal{R} \rightarrow \mathbb{C}$ .*

*Then the synchronous Tsirelson conjecture would also hold true:*

$$C_{qc}^s(n|m) \subseteq \overline{C_q(n|m) \cap C^s(n|m)} \left( \subseteq C_{qa}^s(n|m) \subseteq C_{qc}^s(n|m) \right).$$

*As consequence, the recent refutation of the synchronous version of the Tsirelson conjecture by [JNVWY21] implies the failure of the Connes embedding problem.*

Before we begin with the proof we need to make the following remark:

For comparison it still holds true by [KPS18] (regardless of [JNVWY21])

$$\overline{C_q(n|m) \cap C^s(n|m)} = C_{qa}^s(n|m)$$

which requires yet a difficult result by Kirchberg on amenable traces.

With this remark in mind, let us us begin with the proof for above main result:

*Proof.* The result now basically follows from the previous ingredients:

Step 1: Consider a synchronous quantum-commuting strategy, which by proposition

B.3.1 is given by some trace restricted on two-moments,

$$\tau \in TC^*(\text{in}|\text{out}) : p(ab|xy) = \tau(e_{ax}e_{by}).$$

Step 2: Assuming the Connes embedding problem, all traces would approximately arise as the unique trace on the hyperfinite  $\text{II}_1$ -factor and so also our trace:

$$C^*(\text{in}|\text{out}) \dashrightarrow \ell^\infty(M)/c_\omega(M, \text{tr}) = \mathcal{R}^\omega \xrightarrow{\text{tr}_\omega} \mathbb{C}.$$

#### B.4. Connes $\implies$ Tsirelson

---

Step 3: By proposition B.3.2 any such representation admits a lift

$$\begin{array}{ccc}
 & & \ell^\infty(M) \\
 & \nearrow \text{dashed} & \downarrow \\
 C^*(\text{in|out}) & \longrightarrow & \ell^\infty(M)/c_\omega(M, \text{tr}) = \mathcal{R}^\omega.
 \end{array}$$

Putting all together, any such resulting lift however gives a desired approximation by finite dimensional synchronous correlations:

$$\begin{array}{ccc}
 & & \ell^\infty(M) \\
 & \nearrow & \downarrow \\
 C^*(\text{in|out}) & & \ell^\infty(M)/c_\omega(M, \text{tr}) = \mathcal{R}^\omega \\
 & & \xrightarrow{\text{tr}_\omega} \mathbb{C}
 \end{array}$$

More precisely, simply recall the trace as it was originally defined:

$$\prod_n M_n = \ell^\infty(M) \ni a = (a_1, a_2, \dots) : \quad \text{tr}_\omega(a) = \lim_{n \rightarrow \omega} \text{tr}_n(a_n).$$

This completes the proof and so we conclude as desired: the Connes embedding problem implies the synchronous Tsirelson conjecture.  $\square$

## Acknowledgements

The author would like to thank his supervisor Søren Eilers for his kind support and encouragements, as well as Ryszard Nest and Mikkel Munkholm for helpful discussions on the hyperfinite  $\text{II}_1$ -factor. Moreover, the author acknowledges the support under the Marie–Curie Doctoral Fellowship No. 801199.

## Preprint C

# The quantum commuting model (Ia): Uniqueness of optimal states: The CHSH game and other examples

We present in this paper that the CHSH game admits one and only one optimal state and so remove all ambiguity of representations. More precisely, we use the well-known universal description of quantum commuting correlations as state space on the universal algebra for two player games, and so allows us to unambiguously compare quantum strategies as states on this common algebra. As such we find that the CHSH game leaves a single optimal state on this common algebra.

In turn passing to any non-minimal Stinespring dilation for this unique optimal state is the only source of ambiguity (including self-testing): More precisely, any state on some operator algebra may be uniquely broken up into its minimal Stinespring dilation as an honest representation for the operator algebra followed by its vector state. Any other Stinespring dilation however arises simply as an extension of the minimal Stinespring dilation (i.e. as an embedding of the minimal Hilbert space into some random ambient one). As such this manifests the only source of ambiguity appearing in most (but not all!) traditional self-testing results such as for the CHSH game as well as in plenty of similar examples.

We then further demonstrate the simplicity of our arguments on the Mermin–Peres magic square and magic pentagram game.

Most importantly however, we present this article as an illustration of operator algebraic techniques on optimal states and their quotients, and we further pick up the results of the current article in another following one (currently under preparation) to derive a first robust self-testing result in the quantum commuting model.

### C.1 Stinespring dilation

The Stinespring dilation for states on operator algebras will play the major role of the current article — more precisely it provides a convenient approach for verifying uniqueness of optimal states. As such we begin with an introduction



### C.1. Stinespring dilation

---

for convenience of the reader new to operator algebras.

Meanwhile we moreover explain the issue of randomly chosen non-minimal Stinespring dilations (in random ambient spaces) which do not reflect any of the properties of the original state. As such this accounts for all of the ambiguity arising in most (but not all!) self-testing results in literature.

Consider a state on a unital operator algebra

$$\varphi : A \rightarrow \mathbb{C} : \quad \varphi \geq 0, \quad \varphi(1) = 1.$$

From this we may define an inner product pairing

$$A \times A \rightarrow \mathbb{C} : \quad \langle x|y \rangle := \varphi(x^*y)$$

which defines a Hilbert space. We formally denote the Hilbert space in bracket notation such that the inner product conveniently reads as composition,

$$H = "A|\varphi" : \quad \varphi(x^*y) = \langle \varphi|x^*y|\varphi \rangle.$$

This allows us to later split such expressions to our desire.<sup>1</sup> Based on the Hilbert space we obtain an induced representation by left multiplication

$$A \rightarrow B(\overline{A|\varphi}) : \quad a(x|\varphi) := ax|\varphi$$

together with an induced vector state by  $|\varphi\rangle = 1|\varphi$ :

$$B(\overline{A|\varphi}) \rightarrow \mathbb{C} : \quad T \mapsto \langle \varphi|T|\varphi \rangle.$$

Both combined however retrieve the original state  $\langle \varphi|a|\varphi \rangle = \varphi(a)$ .

As such our state allows for a dilation to the vector state:

$$\begin{array}{ccc} A & \xrightarrow{\varphi(-)} & \mathbb{C} : \\ & \searrow & \nearrow \langle \varphi|-|\varphi \rangle \\ & B(\overline{A|\varphi} = \text{minimal space}) & \end{array}$$

This defines the (so-called) minimal Stinespring dilation in the following sense: Each dilation arises simply as an embedding of this minimal dilation into some

---

<sup>1</sup>This can be made rigorous in the setting of Hilbert modules.

### C.1. Stinespring dilation

---

random ambient space. More precisely, consider another dilation say

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi(-)} & \mathbb{C} \\
 & \searrow & \nearrow \langle \varphi' | - \rangle \\
 & B(H = \text{some space}) & 
 \end{array}$$

Then the space defines an ambient space for the minimal one above: More precisely, the embedding is formally given by the isometry

$$A|\varphi\rangle \subseteq A|\varphi'\rangle \subseteq H : \quad x|\varphi\rangle = x|\varphi'\rangle, \quad |\varphi\rangle = |\varphi'\rangle.$$

Indeed one easily verifies using the above dilation

$$\langle \varphi | x^* y | \varphi \rangle = \varphi(x^* y) = \langle \varphi' | x^* y | \varphi' \rangle.$$

Using the orthogonal decomposition we may invoke matrix formalism

$$B(H = \overline{A|\varphi\rangle} \oplus \text{other}) = \begin{pmatrix} B(\overline{A|\varphi\rangle}) & B(\text{other} \rightarrow \overline{A|\varphi\rangle}) \\ B(\overline{A|\varphi\rangle} \rightarrow \text{other}) & B(\text{other}) \end{pmatrix}$$

and so the representation into the random ambient dilation reads:

$$\begin{aligned}
 A &\rightarrow B(A|\varphi\rangle \oplus \text{other}) : \quad a \mapsto \begin{pmatrix} a \\ * \\ * \end{pmatrix}, \\
 B(A|\varphi\rangle \oplus \text{other}) &\rightarrow \mathbb{C} : \quad \begin{pmatrix} a \\ * \\ * \end{pmatrix} \mapsto (\langle \varphi | 0 \rangle) \begin{pmatrix} a \\ * \\ * \end{pmatrix} \begin{pmatrix} |\varphi\rangle \\ 0 \end{pmatrix}.
 \end{aligned}$$

As such we have found that each dilation simply arises as an embedding into some random ambient space, and the description in matrix formalism illustrates how these ambient spaces do not reflect any properties of the original state.

On the other hand, the Stinespring dilation defines also the minimal quotient to which the original state descends. More precisely, the minimal quotient is given by the kernel (see [Kwa17, subsection 2.1] and [KM21, subsection 3.1])

$$\begin{aligned}
 0 &\longrightarrow I(\varphi) \longrightarrow A \longrightarrow B(\overline{A|\varphi\rangle}) : \\
 A &\longrightarrow A/I(\varphi) \subseteq B(\overline{A|\varphi\rangle}) \xrightarrow{\varphi(-)} \mathbb{C}
 \end{aligned}$$

### C.1. Stinespring dilation

---

while any other quotient lies in between:

$$I \trianglelefteq A : I \subseteq \ker(\varphi) \implies I \subseteq I(\varphi) :$$

$$A \longrightarrow A/I \longrightarrow A/I(\varphi) \xrightarrow{\varphi(-)} \mathbb{C}.$$

Summarizing, every state decomposes uniquely as a representation onto its minimal quotient followed by its dilation as vector state (a good mnemonic here is the analogous decomposition of morphisms into epi plus mono):

$$A \xrightarrow{\text{quotient}} A/I(\varphi) \xrightarrow{\langle \varphi | - | \varphi \rangle} \mathbb{C} : \varphi = \text{“mono} \circ \text{epi”}$$

In turn, one may use the unique decomposition for comparison of states as we will illustrate in this article for the CHSH and the Mermin–Peres games.

Finally let us make another remark regarding the canonical decomposition: Suppose one chooses some factorization via some unital completely positive map (in short ucp-map) instead of an honest representation (which accounts for a choice of POVM instead of some honest PVM):

$$\begin{array}{ccc} A & \xrightarrow{\varphi(-)} & \mathbb{C} \\ & \searrow \text{ucp} & \nearrow \langle \varphi | - | \varphi \rangle \\ & B(S = \text{small space}) & \end{array}$$

Using a minimal Stinespring dilation for the ucp-map we obtain an embedding into another ambient space

$$\begin{array}{ccc} A & \longrightarrow & B(H = \text{ambient space}) \\ \parallel & & \downarrow S \subseteq H \\ A & \xrightarrow{\text{ucp}} & B(S = \text{small space}) \xrightarrow{\langle \varphi | - | \varphi \rangle} \mathbb{C}. \end{array}$$

In other words, this may be understood as a mistaken decomposition:

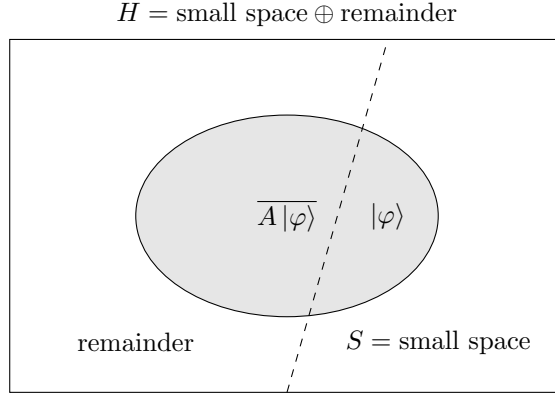
$$\varphi = (\text{mono} \circ \text{mono}) \circ \text{epi} = \text{mono} \circ (\text{mono} \circ \text{epi}) = \langle \varphi | - | \varphi \rangle \circ \text{ucp}.$$

Moreover, our previous discussion reveals this also as an ambient space for the

## C.2. Two-player algebra

---

original state, and as such reveals our “small space” as an incompatible cut:



That is any choice of POVM (instead of an honest PVM) simply arises as the minimal Stinespring dilation embedded in some ambient space and then **cut incompatibly onto some small portion**. As such this accounts for an additional, much worse source of ambiguity for self-testing.

This concludes our introduction and discussion on the Stinespring dilation from an operator algebraic perspective. With this at hand, we may now proceed to nonlocal games and their optimal states — and their uniqueness.

## C.2 The two-player algebra

We continue with an introduction on the two-player algebra, which provides the universal setting for describing quantum correlations all as states on this common algebra — and so allows us to compare them on a common ground.

It is well-known that the universal  $C^*$ -algebra generated by a single projection valued measure agrees with the full group  $C^*$ -algebra

$$C^*(\mathbb{Z}/A) = C^*(e_1, \dots, e_A \in \text{proj} \mid e_1 + \dots + e_A = 1)$$

and as such we obtain as the universal  $C^*$ -algebra per player for a given number of questions and answers the free product of groups

$$C^*(X = \text{in} \mid A = \text{out}) := C^*(\mathbb{Z}/A * \dots * \mathbb{Z}/A : X\text{-many}).$$

Finally a commuting pair of families of observables may be gathered algebraically by the maximal tensor product of their operator algebras and as such the algebra for two-player games has the universal description (with corresponding question

### C.3. CHSH game

---

and answer sets)

$$C^*(\text{two player}) := C^*(\text{Alice}) \otimes C^*(\text{Bob}) = C^*(X|A) \otimes C^*(Y|B)$$

and any quantum commuting strategy arises simply as a state

$$\varphi : C^*(\text{two player}) \rightarrow \mathbb{C} : \quad p(ab|xy) = \varphi\left(e(a|x) \otimes e(b|y)\right).$$

Indeed, applying any Stinespring dilation (see section C.1) one obtains the classical description of correlations as a commuting representation of projection valued measures together with a vector state,

$$\begin{array}{ccc} C^*(\text{Alice}) \otimes C^*(\text{Bob}) & \xrightarrow{\varphi(-)} & \mathbb{C} \\ \searrow \pi = \alpha \otimes \beta & & \nearrow \langle \varphi | - | \varphi \rangle \\ & B\left( \overline{C^*(\text{two player})} | \varphi \rangle \subseteq H \right) & \end{array}$$

This classical description however has the disadvantage that a Stinespring dilation introduces a level of ambiguity due to the chosen representation, which is basically the cause for the usual self-testing results.

Instead the description in terms of states on the two-player algebra allows now for a direct comparison of such on a common universal algebra, and so with this at hand we may now approach uniqueness of optimal states as follows.

### C.3 CHSH game: single optimal state

We prove in this section that the CHSH game has one and only one optimal state (compared to merely unique optimal correlation). For this we begin with a quick review on the CHSH game phrased in the language of operator algebras.

Recall from [CHSH69] that the CHSH game is a binary-input binary-output game, and as such the two-player algebra is generated by a pair of order-2 unitaries for every player (compare section C.2):

$$\begin{aligned} C^*(\text{two player}) &= C^*(\text{Alice}) \otimes C^*(\text{Bob}) : \\ C^*(\text{Alice}) &= C^*(u^2 = 1 = v^2) = C^*(\text{Bob}). \end{aligned}$$

Note that the CHSH game is an XOR-game as introduced in [CHTW04].

### C.3. CHSH game

---

As such we may equivalently describe the CHSH game by its bias polynomial

$$\text{win} = \{a \oplus b = xy\} : \quad \text{CHSH} := u \otimes u + u \otimes v + v \otimes u - v \otimes v$$

and an optimal state for the corresponding nonlocal game translates to maximizing the bias (see appendix C.6 for a summary on XOR games):

$$\varphi : C^*(\text{two player}) \rightarrow \mathbb{C} : \quad \varphi(\text{CHSH}) = \|\text{CHSH}\|.$$

This concludes our quick review on the CHSH game. With this at hand, we may now prove the desired uniqueness of the optimal state for the CHSH game.

**Theorem C.3.1.** *The CHSH game has one and only one optimal state. More precisely, consider the bias polynomial for the CHSH game*

$$\text{CHSH} = u \otimes u + u \otimes v + v \otimes u - v \otimes v$$

and suppose a state realises the maximal bias (see proposition C.6.1)

$$\varphi : C^*(\text{two player}) \rightarrow \mathbb{C} : \quad \varphi(\text{CHSH}) = \|\text{CHSH}\|.$$

Then the state necessarily factors over the matrix algebra

$$\begin{array}{ccc} C^*(\text{Alice}) \otimes C^*(\text{Bob}) & \xrightarrow{\quad} & \mathbb{C} \\ & \searrow \scriptstyle - \otimes - & \nearrow \scriptstyle \varphi(-) \\ & & M(2) \otimes M(2) \end{array}$$

given by the canonical quotient (see proposition C.5.1)

$$M(2) = C^*(u^2 = 1 = v^2 | \{u, v\} = 0).$$

Note that the collection of words in generators reduces, by the anticommutation in the quotient, to the finite list of words (up to linear span)

$$\text{words}(u, v) = \{1, u, v, uv = -vu, uvu = -v, \dots\} \implies \{1, u, v, uv\}$$

whence every state is already linearly determined on those:

$$\varphi : M(2) \otimes M(2) \rightarrow \mathbb{C} : \quad \varphi(\{1, u, v, uv\} \otimes \{1, u, v, uv\}) = ?$$

With this observation in mind, the optimal state is uniquely and entirely determined

### C.3. CHSH game

---

(including all higher and mixed moments) by the correlation table

$\varphi(\dots) =$	$- \otimes 1$	$- \otimes u$	$- \otimes v$	$- \otimes uv$
$1 \otimes -$	1	0	0	0
$u \otimes -$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$v \otimes -$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$uv \otimes -$	0	0	0	-1

from which there is no ambiguity left anymore! (Recall section C.1.)

In short, there is one and only one optimal state for the CHSH game.

Before we begin with the proof, let us note how our notion of uniqueness on states compares with the traditional uniqueness on correlations:

**Remark C.3.2.** The entire correlation table for the optimal state above entails all quantum commuting correlations, since these are given by the state space restricted on two-moments of the form

$$\begin{array}{cccccc}
 \varphi(1 \otimes 1), & \varphi(1 \otimes u), & \varphi(1 \otimes v), & \cancel{\varphi(1 \otimes uv)}, & \cancel{\varphi(1 \otimes vu)}, & \cancel{\varphi(1 \otimes uvu)} \quad \cancel{\cdot} \\
 \varphi(u \otimes 1), & \varphi(u \otimes u), & \varphi(u \otimes v), & \cancel{\varphi(u \otimes uv)}, & \cancel{\varphi(u \otimes vu)}, & \cancel{\varphi(u \otimes uvu)} \quad \cancel{\cdot} \\
 \varphi(v \otimes 1), & \varphi(v \otimes u), & \varphi(v \otimes v), & \cancel{\varphi(v \otimes uv)}, & \cancel{\varphi(v \otimes vu)}, & \cancel{\varphi(v \otimes uvu)} \quad \cancel{\cdot} \\
 \cancel{\varphi(uv \otimes 1)}, & \cancel{\varphi(uv \otimes u)}, & \cancel{\varphi(uv \otimes v)}, & \cancel{\cdot} & & 
 \end{array}$$

These values are equivalently determined by the traditional correlation table

$$p(a, b = 0, 1 | x, y = 0, 1) = \varphi(E(a|x) \otimes E(b|y))$$

given by the corresponding spectral projections

$$u = E(0|0) - E(1|0), \quad v = E(0|1) - E(1|1)$$

which reduces to the familiar values for the CHSH game

$$p(a \oplus b = xy) = \frac{1 + 1/\sqrt{2}}{4}, \quad p(a \oplus b \neq xy) = \frac{1 - 1/\sqrt{2}}{4}.$$

In particular, the traditional uniqueness of correlations — given for example by self-testing — excludes all higher and mixed moments of order-2 unitaries.

*Proof.* Our approach follows the original one by Landau from [?] and its sequel [?], which we however extend to uniqueness of states using some additional arguments from operator algebras. We begin with the following observation: Suppose a state

### C.3. CHSH game

---

is optimal for the bias polynomial, then it is necessarily also optimal for the squared bias

$$\varphi(\text{CHSH}) = \|\text{CHSH}\| \implies \varphi(\text{CHSH}^2) = \|\text{CHSH}\|^2$$

which indeed follows by Cauchy–Schwarz:<sup>2</sup>

$$\begin{aligned} \|\text{CHSH}\|^2 &= \langle \varphi | \text{CHSH}^* | \varphi \rangle \langle \varphi | \text{CHSH} | \varphi \rangle \\ &\leq \langle \varphi | \text{CHSH}^* \text{CHSH} | \varphi \rangle \leq \|\text{CHSH}\|^2 \end{aligned}$$

In our case the squared bias however reduces to

$$\text{CHSH}^2 = 4 - [u, v] \otimes [u, v] : \quad \|\text{CHSH}\|^2 = 8$$

and so an optimal state necessarily satisfies

$$\varphi(\text{CHSH}^2) = 8 \implies \varphi\left([u, v] \otimes [u, v]\right) = -4.$$

We note that this defines only a necessary condition.

Arguing once more by Cauchy–Schwarz we obtain as above

$$\varphi\left([u, v]^* \otimes [u, v]^* [u, v] \otimes [u, v]\right) = 16$$

and this once more is only a necessary but not sufficient condition.

Noting that  $u^2 = 1 = v^2$ , this expression reads written out

$$\begin{aligned} 16 &= \varphi\left((2 - uvv - vvv) \otimes (2 - uvv - vvv)\right) \\ &= \varphi(2 \otimes 2) - \varphi(2 \otimes uvv) - \dots + \varphi(vvv \otimes vvv). \end{aligned}$$

Note that each summand involves a unitary:

$$1 \otimes 1, \quad 1 \otimes uvv, \quad \dots, \quad vvv \otimes vvv.$$

Thus by a simple counting argument this is only possible if the state achieves the value plus/minus one with correct sign on each of the unitaries. In particular

$$\varphi(uvv \otimes 1) = -1 = \varphi(1 \otimes uvv).$$

---

<sup>2</sup>One may either use the bound  $|\varphi\rangle\langle\varphi| \leq \|\varphi\|^2$  together with  $a \leq b \implies c^*ac \leq c^*bc$ , or equivalently the general Cauchy–Schwarz for ucp-maps  $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$  as originally proven by Choi in [Cho74]. For a neat proof of the latter see also [Pau03, proposition 3.2].



### C.3. CHSH game

---

Within the Stinespring dilation we may however cut such relations into

$$\langle x|y \rangle := \langle \varphi | uv \otimes 1 | uv \otimes 1 | \varphi \rangle = \varphi(uvw \otimes 1) = -1$$

with both pieces of norm-one

$$\langle x|x \rangle = \langle \varphi | uv \otimes 1 | vu \otimes 1 | \varphi \rangle = 1$$

$$\langle y|y \rangle = \langle \varphi | vu \otimes 1 | uv \otimes 1 | \varphi \rangle = 1$$

and similarly for the second tensor factor  $\varphi(1 \otimes uvw)$ .

That is however only possible if these are colinear with

$$uv \otimes 1 | \varphi \rangle = -vu \otimes 1 | \varphi \rangle, \quad 1 \otimes uv | \varphi \rangle = -1 \otimes vu | \varphi \rangle$$

and so we have the anticommutation

$$\{u, v\} \otimes 1 | \varphi \rangle = 0 = 1 \otimes \{u, v\} | \varphi \rangle.$$

Finally note that these anticommutators lie in the center

$$\{u, v\} \otimes 1, 1 \otimes \{u, v\} \in \mathcal{Z}C^*(\text{two player})$$

as one easily verifies on generators:

$$u\{u, v\} = v + uvu = \{u, v\}u, \quad v\{u, v\} = vuv + v = \{u, v\}v.$$

The anticommutator therefore also vanishes on the entire subspace

$$H := \overline{C^*(\text{two player}) | \varphi \rangle} = \overline{C^*(\text{Alice}) \otimes C^*(\text{Bob}) | \varphi \rangle} :$$

$$\{u, v\} \otimes 1 C^*(\text{Alice}) \otimes C^*(\text{Bob}) | \varphi \rangle = 0$$

$$1 \otimes \{u, v\} C^*(\text{Alice}) \otimes C^*(\text{Bob}) | \varphi \rangle = 0.$$

that is within the minimal Stinespring dilation (see section C.1)

$$C^*(\text{two player}) \rightarrow B(H = \min) : \quad \{u, v\} \otimes 1 = 0 = 1 \otimes \{u, v\}.$$

In other words, the state factors as desired via (see proposition C.5.1)

$$M(2) \otimes M(2) = C^*(u^2 = 1 = v^2 | \{u, v\} = 0) \otimes C^*(u^2 = 1 = v^2 | \{u, v\} = 0)$$

### C.3. CHSH game

---

which gives us the diagram for the optimal state:

$$\begin{array}{ccc}
 C^*(\text{Alice}) \otimes C^*(\text{Bob}) & \xrightarrow{\hspace{10em}} & \mathbb{C} \\
 \searrow^{-\otimes-} & & \nearrow_{\langle \varphi | - | \varphi \rangle} \\
 & M(2) \otimes M(2) \longrightarrow B(H = \min) & 
 \end{array}$$

This finishes the first part of the proposition and so we may freely restrict our attention from now on to the optimal state already acting on the quotient

$$C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow M(2) \otimes M(2) \xrightarrow{\varphi(-)} \mathbb{C}$$

and so also the bias polynomial for the CHSH game

$$\text{CHSH} = u \otimes (u + v) + v \otimes (u - v) \in M(2) \otimes M(2).$$

From here the rest of the theorem basically follows from traditional results on the CHSH game. We however present a new more algebraic approach below. But at first we untwist the CHSH polynomial. For this we note that the combined expressions (in brackets above) are easily seen to anticommute

$$\{u + v, u - v\} = 1 - \{u, v\} + \{u, v\} - 1 = 0.$$

Due to the anticommutation of our generators, these define also unitaries of order-2 (up to normalization)

$$(u \pm v)^2 = u^2 \pm (uv + vu) + v^2 = 1 \pm \{u, v\} + 1 = 2.$$

In other words, these define just another abstract pair of Pauli matrices and so they give rise to the automorphism (see proposition C.5.1)

$$M(2) \rightarrow M(2) : \quad u \mapsto \frac{u + v}{\sqrt{2}}, \quad v \mapsto \frac{u - v}{\sqrt{2}}$$

which is at the same time its own inverse. Using this automorphism we obtain the untwisted bias polynomial (basically in the new basis)

$$\text{CHSH}/\sqrt{2} = u \otimes u + v \otimes v.$$

From here we give a new algebraic approach by verifying the trace property from which we deduce the final uniqueness, and from which we may moreover easily read off the values for the correlation table: Suppose our state maximises the untwisted

#### C.4. Mermin–Peres games

---

bias polynomial and so also the unitary summands

$$\varphi(u \otimes u + v \otimes v) = 2 \implies \varphi(u \otimes u) = 1 = \varphi(v \otimes v).$$

From this it follows within the Stinespring dilation (similar as above):

$$u \otimes 1 |\varphi\rangle = 1 \otimes u |\varphi\rangle, \quad v \otimes 1 |\varphi\rangle = 1 \otimes v |\varphi\rangle.$$

Indeed one easily verifies (and similarly for the other generator)

$$\langle \varphi | (u \otimes 1 - 1 \otimes u)^* (u \otimes 1 - 1 \otimes u) | \varphi \rangle = \langle \varphi | 2(1 \otimes 1 - u \otimes u) | \varphi \rangle = 0.$$

The entire trace property follows now by iteration on words of generators

$$(u_1 \dots u_n u) \otimes 1 |\varphi\rangle = (u_1 \dots u_n) \otimes u |\varphi\rangle = \dots = 1 \otimes (u u_n \dots u_1) |\varphi\rangle$$

and we note that the order gets flipped (which is a cause for the transpose). Full matrix algebras however carry a unique tracial state and so we obtain as desired the unique optimal state for the CHSH game:

$$\varphi(u_1 \dots u_m \otimes v_1 \dots v_n) = \tau(u_1 \dots u_m v_n \dots v_1).$$

The particular values for the correlation table may now be easily derived from this relation: Indeed we obtain for example

$$\varphi(u \otimes v) = \tau\left(u \cdot \frac{u+v}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tau(1) + \frac{1}{\sqrt{2}}\tau(uv) = \frac{1}{\sqrt{2}}.$$

This concludes the proof whence the optimal state is unique.  $\square$

### C.4 Mermin–Peres games: single perfect state

We continue in this section with another example of nonlocal games: the Mermin–Peres magic square and magic pentagram games. Both admit a unique winning state for rather obvious reasons (see the theorem below). In order to understand this uniqueness argument, we will however need a more thorough introduction first to linear constraint system games and their game algebra (and we note that the following arguments also apply to nonlinear constraint system games with any complex variables).

As such we begin with an overview on linear constraint system games summarising the considerations from [CM14] and [CLS17], as well as [KPS18] and [Gol21]: More precisely, a linear constraint system game is a nonlocal game given by a

#### C.4. Mermin–Peres games

---

system of linear equations (which we formulate in multiplicative form) such as

$$u, v, \dots \in \exp\{2\pi i \mathbb{Z}/2\} : \quad uvw = 1, \quad vz = -1, \quad \dots$$

The referee asks each player an equation (

$$X = \{uvw = 1, vz = -1, \dots\} = Y$$

and each player is asked to fill out the variables of their equation say

$$x = (vz = -1) \quad \Longrightarrow \quad a = (v = -1, z = 1).$$

As a first rule the players are asked to answer in a synchronous way:

$$x = \text{some equation} = y \quad \Longrightarrow \quad a = b.$$

As such a winning state (being synchronous) defines a tracial state on the algebra for a single player (see [PSSTW16, corollary 5.6]). More precisely, consider the two-player algebra (from section C.2) for identical algebras per player

$$C^*(\text{two player}) = C^*(\text{player}) \otimes C^*(\text{player})$$

and consider a winning state

$$\varphi : C^*(\text{player}) \otimes C^*(\text{player}) \rightarrow \mathbb{C} : \quad \varphi(E(\text{losing})) = 0.$$

Then the synchronicity rule enforces on the Stinespring dilation:

$$\varphi(E(a \neq b | x = y)) = 0 \quad \Longrightarrow \quad E(a|x) \otimes 1 |\varphi = 1 \otimes E(a|x) |\varphi.$$

By iteration such states enjoy the special property (note the flip in order)

$$\begin{aligned} & \varphi(E(a_1|x_1) \dots E(a_m|x_m) \otimes E(b_1|y_1) \dots E(b_n|y_n)) \\ &= \varphi(E(a_1|x_1) \dots E(a_m|x_m) E(b_n|y_n) \dots E(b_1|y_1) \otimes 1) \end{aligned}$$

and similar for flipping words in other ways from left to right.

Thus such states define a tracial state on either algebra per player

$$\varphi(- \otimes 1) : C^*(\text{player}) \otimes 1 \rightarrow \mathbb{C}$$

$$\varphi(1 \otimes -) : 1 \otimes C^*(\text{player}) \rightarrow \mathbb{C}$$

which both allow us to recover the original state from above relation.

#### C.4. Mermin–Peres games

---

So we may from now on restrict our attention to a tracial state

$$\tau : C^*(\text{player}) \rightarrow \mathbb{C} : \quad \tau(ab) = \tau(ba).$$

For tracial states however every additional rule enforces the corresponding 2-moments of losing pairs to vanish in the minimal Stinespring dilation (see [OP16] and [HMPS19]). As such we are left to determine the game algebra for our synchronous game

$$C^*(\text{sync game}) := C^*\left(\text{player} \mid E(\text{losing}) = 0\right) \quad (\text{C.1})$$

where we denote for shorthand (similar as in appendix C.6)

$$E(ab|xy) := E(a|x)E(b|y) : \quad E(S) = \{E(s) \mid s \in S\}$$

and any tracial state on the quotient defines a winning correlation:

$$\tau : C^*(\text{sync game}) \rightarrow \mathbb{C} : \quad p(ab|xy) = \tau\left(E(a|x)E(b|y)\right). \quad (\text{C.2})$$

With this at hand we now get to the first particular rule for linear constraint system games. For this we first note that for each equation as input we obtain a projection valued measure whence commuting projections for each variable. For the equation  $x = (uvw = 1)$  we obtain for example the triple

$$E(u \in \mathbb{Z}/2 \mid x), \quad E(v \in \mathbb{Z}/2 \mid x), \quad E(w \in \mathbb{Z}/2 \mid x)$$

which we retrieve from the projection valued measure as a disjoint union like

$$E(v = -1 \mid x) := \sum E\left(u \in \mathbb{Z}/2, v = -1, w \in \mathbb{Z}/2 \mid x\right)$$

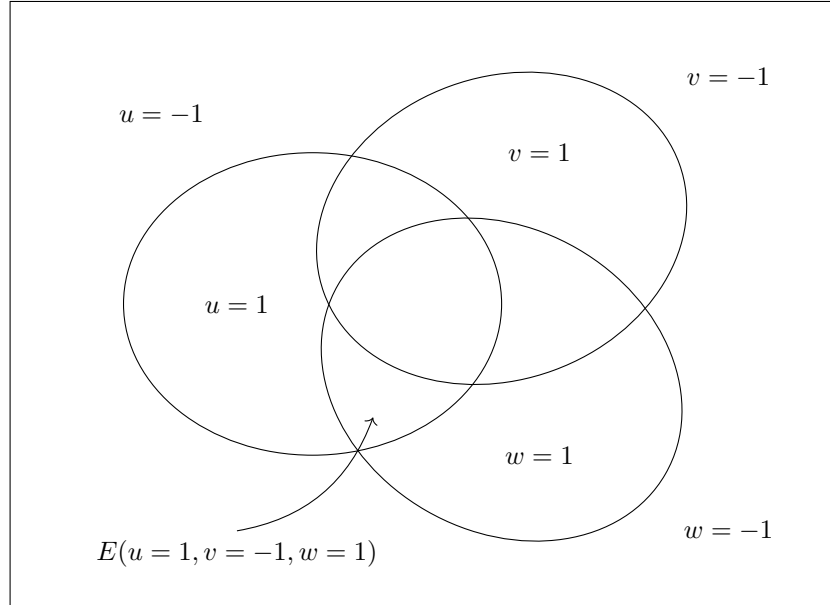
and we recover the projection valued measure simply as intersection

$$E(u, v, w \mid x) = E(u|x)E(v|x)E(w|x).$$

This is all easiest remembered as a Venn diagram (for commuting projections):

#### C.4. Mermin–Peres games

---



The first particular rule for linear constraint system games asks now the players to answer consistently for shared variables, for example

$$x = (uvw = 1), \quad y = (vz = -1) \quad \Longrightarrow \quad "a(v) = b(v)"$$

which boils down to the equality for our projections (inside the game algebra).

$$E(v \mid x = (uvw = 1)) = E(v \mid y = (vz = -1)).$$

Indeed this follows very quickly from (think in terms of Venn-diagrams!)

$$E(v = 1 \mid x)E(v = -1 \mid y) = 0 = E(v = -1 \mid x)E(v = 1 \mid y).$$

That is the projections are all independent of the chosen equation (cf. [CM14]).

With this at hand we may now get to the second particular rule for linear constraint system games. For this we first pass to their generating unitaries

$$U := E(u = 1) - E(u = -1), \quad V := E(v = 1) - E(v = -1), \quad \dots$$

We now ask these unitaries to satisfy each equation, for example

$$f(u, v, w) = uvw = 1 \quad \Longrightarrow \quad f(U, V, W) = UVW = 1.$$

Note that the unitaries appearing in a common equation arose as commuting

#### C.4. Mermin–Peres games

---

projections. As such they (isomorphically) generate the abelian subalgebra

$$C(\sigma U) \otimes C(\sigma V) \otimes C(\sigma W) \subseteq C^*(\text{player})$$

and we may have instead also asked for any continuous function such as

$$g(U, V, W) = \cos(UV^3) + W^2 - 1 = 0.$$

The generated subalgebra however agrees with the isomorphic copy

$$C^*(g) = C(\text{supp } g)E(\text{supp } g) \subseteq C^*(\text{player})$$

and so modding out (the ideal generated) by either side results in the same quotient, that is the requirement above reduces to its support projection:

$$g(U, V, W) = 0 \quad \iff \quad E(\text{supp } g) = 0.$$

That is we obtain for our example (read backwards)

$$E(u, v, w) = E(u)E(v)E(w) = 0, \quad \forall uvw \neq 1 \quad \iff \quad UVW = 1.$$

This is nothing but our second particular rule for linear constraint system games:

The players need to respond with variables satisfying their equation like

$$x = (uvw = 1) \quad \implies \quad a = (u, v, w = \pm 1) : \quad a(u)a(v)a(w) = 1.$$

Thus the game algebra (C.1) for linear constraint systems reads (see also [CM14]):

$$\begin{aligned} & C^*\left(\text{“}uvw = 1, vz = -1, \dots\text{”}\right) \\ &= C^*\left(U^2 = 1, V^2 = 1, \dots \mid UVW = 1 : [U, V] = 0, \dots; VZ = -1 : \dots\right). \end{aligned}$$

While allowing also for continuous functions, note that there was also nothing special about unitaries of order–two. Instead all the above also holds perfectly fine for variables with values in any (possibly different) list of complex numbers

$$u \in \{\alpha_1, \dots, \alpha_u\}, \quad v \in \{\beta_1, \dots, \beta_v\}, \quad \dots$$

upon replacing the generators with

$$U = \alpha_1 E(u = \alpha_1) + \dots + \alpha_u E(u = \alpha_u).$$

This concludes the introduction to linear constraint system games.

For the following statement let us denote the trace space for an operator algebra

#### C.4. Mermin–Peres games

---

$A$  as those states which possess the trace property

$$TA = \left\{ \tau : A \rightarrow \mathbb{C} \mid \tau(ab) = \tau(ba) \right\}.$$

With this in mind one may now easily settle uniqueness of winning states:

**Theorem C.4.1.** *The game algebra (C.1) for the Mermin–Peres magic square and magic pentagram game read respectively*

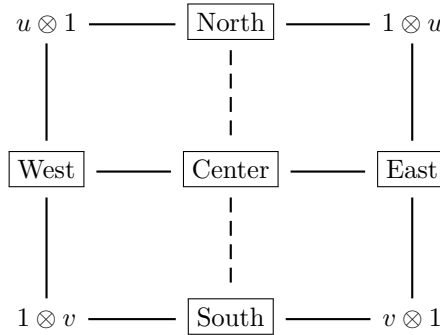
$$\begin{aligned} C^*(\text{magic square}) &= M_2 \otimes M_2 \\ C^*(\text{magic pentagram}) &= M_2 \otimes M_2 \otimes M_2 \end{aligned}$$

and as such both admit one and only one winning state given by the unique tracial state on either matrix algebra:

$$\begin{aligned} T(M_2 \otimes M_2) &= \{\text{tr}/2 \otimes \text{tr}/2\} \\ T(M_2 \otimes M_2 \otimes M_2) &= \{\text{tr}/2 \otimes \text{tr}/2 \otimes \text{tr}/2\} \end{aligned}$$

As for the CHSH game, this entails all quantum commuting strategies.

*Proof.* The algebra for the Mermin–Peres magic square is generated by



where we have commuting rows and columns automatically built in as tensor product while using abstract copies of order two unitaries:

$$u^2 = 1 = v^2 : \quad u \otimes 1, \quad v \otimes 1, \quad 1 \otimes u, \quad 1 \otimes v.$$

Also all the boxes on outer edges are already determined:

$$\boxed{\text{North}} = u \otimes u, \quad \boxed{\text{West}} = u \otimes v, \quad \boxed{\text{East}} = v \otimes u, \quad \boxed{\text{South}} = v \otimes v.$$



## C.5. Pauli algebra

---

Using the center box one now easily derives their anticommutation:

$$\boxed{\text{West}} \cdot \boxed{\text{East}} = \boxed{\text{Center}} = -\boxed{\text{North}} \cdot \boxed{\text{South}} \implies 1 \otimes \{u, v\} = 0,$$

$$\boxed{\text{West}} \cdot \boxed{\text{East}} = \boxed{\text{Center}} = -\boxed{\text{South}} \cdot \boxed{\text{North}} \implies \{u, v\} \otimes 1 = 0.$$

But the universal algebra generated by a pair of anticommuting order-2 unitaries is nothing but the abstract Pauli algebra from proposition C.5.1 and so we found

$$C^*(\text{magic square}) = M_2 \otimes M_2.$$

Similarly one argues for the magic pentagram and the proof is complete.  $\square$

## Acknowledgements

The author would like to thank his supervisor Søren Eilers for his kind support and encouragements, as well as Connor Paddock for suggesting the quite valuable survey article [SB20]. Moreover, the author acknowledges the support under the Marie–Curie Doctoral Fellowship No. 801199.

## C.5 Pauli algebra

**Proposition C.5.1.** *The algebra of  $2 \times 2$ -matrices (equivalently the algebra generated by its system of matrix units) has the equivalent description as the universal algebra generated by anticommuting order-2 unitaries*

$$M(2) = C^*(u^2 = 1 = v^2 \mid \{u, v\} = 0)$$

which identifies the generating order-2 unitaries as Pauli matrices

$$u = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = Z, \quad v = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = X$$

or any other (possibly rotated) pair of anticommuting Pauli matrices.

We thus suggestively denote this as **the abstract algebra of Pauli matrices**.

*Proof.* Note that the system of matrix units is equivalently generated by the partial isometry with the orthogonality relation for its range and source:

$$w = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}: \quad \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \perp \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}.$$

The algebra generated by such a partial isometry is now easily seen to be equivalent

## C.6. XOR-games

---

to the algebra generated by anticommuting order-2 unitaries. Indeed with the suggested identification from the proposition one has

$$2w := v(1 + u) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

with inverse translation given by

$$\begin{aligned} u &:= w^*w - ww^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \\ v &:= w + w^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

and one easily verifies the orthogonality relation for the range and source, as well as the anticommutation relation, respectively.  $\square$

## C.6 Overview on XOR-games

We provide an overview of XOR-games for convenience of the reader, and we do so from an abstract, operator algebraic perspective. Meanwhile, the author would like to acknowledge that the considerations of the current section arose as joint work with Azin Shahiri from a forthcoming work on the systematic (operator algebraic and representation-theoretic) classification of optimal states for the tilted CHSH games from [LLP10] and further [AMP12] with longterm goal for a better understanding of the I3322-inequality from [Fro81] and [CG04].

An XOR-game is a binary game (that is each player answers with classical bits) whose winning condition depends only on the resulting parity for their answers. More precisely, that is the winning pairs arise as

$$\text{parity} \subseteq \{0, 1\} \times X \times Y : \quad \text{winning} = \left\{ (a \oplus b | x, y) \in \text{parity} \right\}$$

which equivalently reads written out

$$(a, b | x, y) \in \text{winning} \iff (a \oplus b | x, y) \in \text{parity}.$$

The most prominent example is of course the CHSH game as in section C.3.

To give another less obvious example however one may also consider the graph 2-coloring game, whose winning condition may be equivalently formulated as

$$\begin{aligned} x \sim y &\implies a \oplus b = 1, \\ x = y &\implies a \oplus b = 0. \end{aligned}$$

## C.6. XOR-games

---

For convenience we consider bits from now on as  $\pm 1$ -valued:

$$A = \{1, -1\} = B : \quad a \oplus b = 0, 1 \quad \rightsquigarrow \quad ab = \pm 1.$$

The point of such games is that they allow to verify the winning probability based on the bias instead, at which we take a closer look now. Recall for this the two-player algebra with generators (see section C.2):

$$\begin{aligned} C^*(\text{two player}) &= C^*(X|A=2) \otimes C^*(Y|B=2) : \\ u(x) &= E(a=0|x) - E(a=1|x). \end{aligned}$$

Now for nonlocal games in general the winning probability may be described by the operator norm for the game polynomial (which we take unnormalized)

$$\text{game} := \sum E(\text{win}) : \quad \omega_{qc}(\text{game}) = \frac{\|\text{game}\|}{|X \times Y|}$$

and where we used the shorthand notation

$$E(ab|xy) := E(a|x) \otimes E(b|y) : \quad E(S) = \{E(s) | s \in S\}.$$

For XOR-games we may now introduce the bias polynomial (also unnormalized)

$$\text{bias} := \sum_{xy} \beta(x, y) u(x) \otimes u(y)$$

with bias function given by the sum over the possible winning parities

$$\beta(x, y) := \sum \left\{ \text{par} = \pm 1 \mid (\text{par}|xy) \in \text{win} \right\} = 1, 0, -1.$$

For the 2-coloring example above this reads for example

$$\beta(x \sim y) = -1, \quad \beta(x = y) = 1, \quad \beta(x \not\sim y, x \neq y) = 0.$$

The point is now that the bias polynomial satisfies

$$2\text{game} = |\text{win}| + \text{bias}.$$

Recall that we take here the winning set as  $\text{win} \subseteq \{0, 1\} \times X \times Y$ . Since the game polynomial is positive we obtain for the spectrum

$$0 \leq \sigma(\text{game}) \leq \|\text{game}\| : \quad \max \sigma(\text{game}) = \|\text{game}\|$$

## C.6. XOR-games

---

and by the relation above

$$2 \max \sigma(\text{game}) = \max \sigma(\text{bias}) + |\text{win}|.$$

We may therefore relate the winning probability to the maximal upper bound for the bias. We remark that the above procedure — that is passing from the norm to the spectrum — may be seen as a linearization process since

$$\|a + 1\| \neq \|a\| + 1 \quad \rightsquigarrow \quad \sigma(a + 1) = \sigma(a) + 1, \quad \sigma(\lambda a) = \lambda \sigma(a).$$

The goal is now to undo the linearization process — that is passing from the spectrum for the bias polynomial back to its norm. We achieve this with the following observation:

**Proposition C.6.1.** *Bias polynomials for XOR-games have symmetric spectrum, from which the maximal bias agrees with the operator norm:*

$$\sigma(\text{bias}) = -\sigma(\text{bias}) : \quad \max \sigma(\text{bias}) = \|\text{bias}\|.$$

As a consequence for any XOR-game

$$2\|\text{game}\| = |\text{win}| + \|\text{bias}\|$$

whence the winning probability relates to the problem (see [FNT14]):

“Can you compute the operator norm” for the bias polynomial?

*Proof.* The automorphism which swaps the measurement outcomes for Alice and leaves the measurements for Bob untouched (or vice versa)

$$\begin{aligned} \Phi : C^*(X|2) \otimes C^*(Y|2) &\rightarrow C^*(X|2) \otimes C^*(Y|2) : \\ u(x) \otimes 1 &\mapsto -u(x) \otimes 1, \quad 1 \otimes u(y) \mapsto 1 \otimes u(y) \end{aligned}$$

sends any bias polynomial to its negative

$$\Phi(\text{bias}) = -\sum \beta(xy)u(x) \otimes u(y) = -\text{bias}.$$

The spectrum however is invariant under automorphism whence

$$\sigma(\text{bias}) = \sigma(\Phi \text{bias}) = \sigma(-\text{bias}) = -\sigma(\text{bias}).$$

The remaining statements follow now from the symmetric spectrum.  $\square$

**Remark C.6.2.** *One needs to stay cautious since polynomials (which do not strictly arise as bias polynomial) generally fail to have symmetric spectrum:*

## C.6. XOR-games

---

for example the I3322 inequality from [Fro81] and as reinvented in [CG04] might even have distinct values for the minimal and maximal bias.

We finish this overview with an interesting negative result from [CHTW04], and for which we note that it easily extends to include also quantum commuting strategies simply by using a Stinespring dilation as discussed in section C.1:

**Theorem C.6.3** ([CHTW04, theorem 3]). *Nonlocal games with binary outputs are never pseudo-telepathy games — neither for quantum commuting strategies.*

## Bibliography

- [AMP12] Antonio Acín, Serge Massar, and Stefano Pironio. Randomness versus Nonlocality and Entanglement. *Physical Review Letters*, 108:100402, 2012.
- [Bla96] Etienne Blanchard. Déformations de  $C^*$ -algèbres de Hopf. *Bulletin de la Société Mathématique de France*, 124(1):141–215, 1996.
- [BO08] Nathaniel Brown and Narutaka Ozawa.  $C^*$ -Algebras and Finite-Dimensional Approximations, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.
- [CG04] Daniel Collins and Nicolas Gisin. A relevant two qubit Bell inequality inequivalent to the CHSH inequality. *Journal Math Physics*, 37(5):1775–1787, 2004.
- [CG06] Tyrone Crisp and Daniel Gow. Contractible subgraphs and Morita equivalence of graph  $C^*$ -algebras. *Proceedings of the American Mathematical Society*, 134(7):2003–2013, 2006.
- [Cho74] Man-Duen Choi. A Schwarz inequality for positive linear maps on  $C^*$ -algebras. *Illinois Journal of Mathematics*, 18(4):565–574, 1974.
- [CHSH69] John Clauser, Michael Horne, Abner Shimony, and Richard Holt. Proposed Experiment to Test Local Hidden-Variable Theories. *Physical Review Letters*, 23(15):880–884, 1969.
- [CHTW04] Richard Cleve, Peter Høyer, Ben Toner, and John Watrous. Consequences and limits of nonlocal strategies. In *19th IEEE Annual Conference on Computational Complexity*, pages 236–249, 2004.
- [CLS17] Richard Cleve, Li Liu, and William Slofstra. Perfect commuting-operator strategies for linear system games. *Journal of Mathematical Physics*, 58(1):012202, 2017.
- [CM14] Richard Cleve and Rajat Mittal. Characterization of Binary Constraint System Games. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming*, pages 320–331. Springer Berlin Heidelberg, 2014.
- [CS20] Andrea Coladangelo and Jalex Stark. An inherently infinite-dimensional quantum correlation. *Nat Commun*, 11:3335, 2020.

- [Cun77] Joachim Cuntz. Simple  $C^*$ -algebras generated by isometries. *Communications in Mathematical Physics*, 57(2):173–185, 1977.
- [DPP19] Ken Dykema, Vern Paulsen, and Jitendra Prakash. Non-closure of the set of quantum correlations via graphs. *Communications in Mathematical Physics*, 365(3):1125–1142, 2019.
- [Exe94] Ruy Exel. Circle Actions on  $C^*$ -Algebras, Partial Automorphisms, and a Generalized Pimsner-Voiculescu Exact Sequence. *Journal of Functional Analysis*, 122(2):361–401, 1994.
- [Exe11] Ruy Exel. Noncommutative Cartan subalgebras of  $C^*$ -algebras. *New York Journal of Mathematics*, 17:331–382, 2011.
- [FMR03] Neal Fowler, Paul Muhly, and Iain Raeburn. Representations of Cuntz-Pimsner algebras. *Indiana University Mathematics Journal*, 52(3):569–605, 2003.
- [FNT14] Tobias Fritz, Tim Netzer, and Andreas Thom. Can you compute the operator norm? *Proceedings of the AMS*, 142(12):4265–4276, 2014.
- [Fre64] Peter Freyd. *Abelian Categories*. Harper and Row and John Weatherhill, 1964.
- [Fre22a] Alexander Frei. Connes implies Tsirelson: a simple proof. 2022.
- [Fre22b] Alexander Frei. The quantum commuting model (Ia): uniqueness of optimal states. 2022.
- [Fre23] Alexander Frei. Relative Cuntz–Pimsner algebras: Classification of gauge-equivariant representations: a simple and complete picture. 2023.
- [Fro81] Marcel Froissart. Constructive generalization of Bell’s inequalities. *II Nuovo Cimento B*, 64(2):241–251, 1981.
- [FS23] Alexander Frei and Azin Shahiri. The tilted CHSH games: an operator algebraic classification. 2023.
- [Gol21] Adina Goldberg. Synchronous linear constraint system games. *Journal of Mathematical Physics*, 62(3):032201, 2021.
- [HMPS19] William Helton, Kyle Meyer, Vern Paulsen, and Matthew Satriano. Algebras, synchronous games, and chromatic numbers of graphs. *New York Journal of Mathematics*, 25:328–361, 2019.
- [JNVWY21] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen.  $MIP^* = RE$ . *Communications of the ACM*, 64(11):155–181, November 2021.
- [Kak13] Evgenios Kakariadis. The Dirichlet property for tensor algebras. *Bulletin of the London Mathematical Society*, 45(6):1119–1130, 2013.

- [Kak16] Evgenios Kakariadis. A note on the gauge invariant uniqueness theorem for  $C^*$ -correspondences. *Israel Journal of Mathematics*, 215(2):513—521, Sep 2016.
- [Kat04] Takeshi Katsura. On  $C^*$ -algebras associated with  $C^*$ -correspondences. *Journal of Functional Analysis*, 217(2):366–401, 2004.
- [Kat07] Takeshi Katsura. Ideal structure of  $C^*$ -algebras associated with  $C^*$ -correspondences. *Pacific Journal of Mathematics*, 230(1):107–145, 2007.
- [KM21] Bartosz Kosma Kwasniewski and Ralf Meyer. Essential crossed products for inverse semigroup actions: simplicity and pure infiniteness. *Documenta Mathematica*, 26:271–335, 2021.
- [KPQ12] Steven Kaliszewski, Nura Patani, and John Quigg. Characterizing graph  $C^*$ -correspondences. *Houston Journal of Mathematics*, 38(3):751–759, 2012.
- [KPS18] Se-Jin Kim, Vern Paulsen, and Christopher Schafhauser. A synchronous game for binary constraint systems. *Journal Math Physics*, 59(3):032201, 2018.
- [KPW98] Tsuyoshi Kajiwara, Claudia Pinzari, and Yasuo Watatani. Ideal Structure and Simplicity of the  $C^*$ -Algebras Generated by Hilbert Bimodules. *Journal of Functional Analysis*, 159(2):295–322, 1998.
- [KRS02] Stanislav Kruglyak, Slavik Rabanovich, and Yuri Samoilenko. On Sums of Projections. *Functional Analysis and Its Applications*, 36(3):182–195, 2002.
- [Kwa17] Bartosz Kosma Kwasniewski. Exel’s crossed product and crossed products by completely positive maps. *Houston Journal of Mathematics*, 43(2):509–567, 2017.
- [Lan78] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer New York, second edition edition, 1978.
- [Lan87] Lawrence Landau. Experimental tests of general quantum theories. *Lett Math Phys*, 14(1):33–40, 1987.
- [Lan95] Christopher Lance. *Hilbert  $C^*$ -Modules: A Toolkit for Operator Algebrists*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995.
- [LLP10] Thomas Lawson, Noah Linden, and Sandu Popescu. Biased nonlocal quantum games. 2010.



- [MPS21] Laura Mancinska, Jitendra Prakash, and Christopher Schafhauser. Constant-sized robust self-tests for states and measurements of unbounded dimension. 2021.
- [MPT08] Paul Muhly, James Pask, and Mark Tomforde. Strong shift equivalence of  $C^*$ -correspondences. *Israel Journal of Mathematics*, 167:315–346, 2008.
- [MS98] Paul Muhly and Baruch Solel. Tensor Algebras over  $C^*$ -Correspondences: Representations, Dilations, and  $C^*$ -Envelopes. *Journal of Functional Analysis*, 158(2):389–457, 1998.
- [NPA08] Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, 2008.
- [OP16] Carlos M. Ortiz and Vern I. Paulsen. Quantum graph homomorphisms via operator systems. *Linear Algebra and its Applications*, 497:23–43, 2016.
- [Pau03] Vern Paulsen. *Completely Bounded Maps and Operator Algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003.
- [Pim97] Michael Pimsner. A class of  $C^*$ -algebras generalizing both Cuntz–Krieger algebras and crossed products by  $\mathbb{Z}$ . In Dan-Virgil Voiculescu, editor, *Free Probability Theory*, volume 12 of *Fields Institute Communications*, pages 189–212, 1997.
- [PSSTW16] Vern Paulsen, Simone Severini, Daniel Stahlke, Ivan Todorov, and Andreas Winter. Estimating quantum chromatic numbers. *Journal of Functional Analysis*, 270(6):2188–2222, 2016.
- [PV10] Károly Pál and Tamás Vértesi. Maximal violation of a bipartite three-setting, two-outcome Bell inequality using infinite-dimensional quantum systems. *Phys. Rev. A*, 82(2):022116, 2010.
- [RS89] Iain Raeburn and Allan Sinclair. The  $C^*$ -algebra generated by two projections. *Mathematica Scandinavica*, 65(2):278–290, 1989.
- [RS11] David Robertson and Wojciech Szymański.  $C^*$ -algebras associated to  $C^*$ -correspondences and applications to mirror quantum spheres. *Illinois Journal of Mathematics*, 55(3):845–870, 2011.
- [Rus20] Travis Russell. Two-outcome synchronous correlation sets and Connes’ embedding problem. *Quantum Information and Computation*, 20(5 and 6):361–374, 2020.
- [SB20] Ivan Supic and Joseph Bowles. Self-testing of quantum systems: a review. *Quantum*, 4:337, 2020.

