Topological Hochschild homology of adic rings

Adriano Córdova Fedeli

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Adriano Córdova Fedeli
Department of Mathematical Sciences
University of Copenhagen
Universitetparken 5
2100 Copenhagen
Denmark

adrianox@gmail.com

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Advisor:	Dustin Clausen University of Copenhagen & IHES
Assesment committee:	Fabien Pazuki (chair) University of Copenhagen
	Alexander Efimov Steklov Mathematical Institute of Russian Academy of Sciences
	Arthur César le Bras University of Strasbourg

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Abstract

Let R be an $\mathbb{E}_\infty$-ring, and let $I \subset \pi_0 R$ be a finitely generated ideal such that R is complete along I. This thesis studies localizing invariants arising from pairs of the form (R, I). Precisely, the pair (R, I) gives rise to a category Nuc_R, the category of *nuclear R-modules*: this category contains the usual category of R-modules, as well as many I-complete R-modules with continuous maps between them. We then study localizing invariants applied to such categories. In this context, a localizing invariant T is said to be continuous if $T(\mathrm{Nuc}_R) = \varprojlim_n T(R/\!\!/I^n)$. Efimov proved that algebraic K-theory is continuous. The main result of this thesis builds from the continuity of K-theory to prove the same for topological cyclic and Hochschild homology.

Resumé

Lad R være en $\mathbb{E}_\infty$ ring, og lad $I \subset \pi_0 R$ være et endeligt frembragt ideal sådan at R er komplet med hensyn til I. Denne PhD afhandling studerer lokaliserende invarianter som optræder fra par på formen (R, I). Nærmere bestemt giver paret (I, R) anledning til en kategori Nuc_R, kategorien af *nuklear R-moduler*: denne kategori indeholder den sædvanlige kategori af R-moduler såvel som flere I-komplette R-moduler med kontinuerte afbildinger mellem dem. Derefter studerer vi lokaliserende invarianter anvendt på sådanne kategorier. I denne kontekst siges en lokaliserende invariant at være kontinuert, hvis $T(\mathrm{Nuc}_R) = \varprojlim_n T(R/\!\!/l^n)$. Efimov beviste at algebraisk K-teori er kontinuert. Hovedresultatet i denne af handling bruger kontinuiteten af K-teori til at bevise kontinuiteten af topologisk cyklisk kohomologi og Hochschild homologi.

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Contents

Introduction

The algebraic K-theory[1] of a given qcqs scheme X is given by the K-theory of the ∞-category $\mathrm{Perf}(X)$, the ∞-category of perfect complexes over X. This is a global definition which when X is of the form $\mathrm{Spec}(A)$ recovers the usual K-theory of the commutative ring A. Starting instead from the K-theory of commutative rings, the K-theory of X can be obtained by gluing the K-theory of an affine cover: for example, say X is separated and covered by two Zariski opens $U = \mathrm{Spec}(R_1)$ and $V = \mathrm{Spec}(R_2)$ with intersection $U \cap V = \mathrm{Spec}(R_{12})$, then we can define a spectrum

$$K(X) := K(R_1) \times_{K(R_{12})} K(R_2). \tag{1}$$

This spectrum turns out to be independent of the presentation $X = U \cup V$: more generally, a result of Thomason[TT90, Theorem 10.3] says that the functor

$$U \subset X \mapsto K(\mathrm{Perf}(U))$$

is a sheaf of spectra on the Zariski site of a qcqs scheme X. Then, as the definition of K-theory via $\mathrm{Perf}(X)$ agrees with the usual K-theory of commutative rings in the affine case, Thomason's result gives both that (1) is well defined and that the two approaches to the K-theory of the scheme X in this paragraph, via $\mathrm{Perf}(X)$ and via gluing from affines, agree. Moreover, Thomason's result holds with K-theory replaced by an arbitrary localizing invariant.

The previous has an analog in the world of rigid geometry. Let X now be a qcqs rigid analytic variety over a non-archimedean field C. When $X = \mathrm{Sp}(R)$ is affinoid, its K-theory can be defined, following [Mor16, 3.1], as the pushout of spectra

$$\begin{array}{ccc} K(R_0) & \longrightarrow & K(R) \\ \downarrow & & \downarrow \\ \varprojlim_n K(R_0/\pi^n) & \longrightarrow & K^{\mathrm{cont}}(R) \end{array}$$

where $R_0 \subset R$ is any subring of definition an $\pi \in C$ is a topologically nilpotent unit. This is well defined: it is independent of R_0 and π. Morever, defining the K-theory of a qcqs rigid analytic variety by gluing from affinoid pieces [2] gives a well defined spectrum $K^{\mathrm{cont}}(X)$, this is [Mor16, Lemma 3.4], and it uses pro-cdh descent for schemes. The definition of $K^{\mathrm{cont}}(X)$ is an analog of the second definition of the K-theory of a scheme on the previous paragraph. There is also an analog of the first definition, a sort of global definition: attached to any rigid analytic space X there is a category Nuc_X, called the *nuclear category* of X, of which it is possible to consider its K-theory[3], $K(\mathrm{Nuc}_X)$. There is a comparison map

$$K(\mathrm{Nuc}_X) \to K^{\mathrm{cont}}(X) \tag{2}$$

[1] always non-connective in this introduction.

[2] that is, as in (1) but with the analytic topology playing the role of the Zariski topology

[3] the category Nuc_X is a dualizable, presentable and stable ∞-category, and such categories are the input for Efimov K-theory, as explained in [Hoy].

between these two definitions of the K-theory of a rigid analytic space. By a result of Andreychev [And23], the functor $U \subset X \mapsto K(\mathrm{Nuc}_U)$ is a sheaf for the analytic topology on X[4]. Then, as both sides in the comparison map in (2) are determined by their values on affinoids. When $X = \mathrm{Sp}(R)$ is affinoid, we also write Nuc_R for $\mathrm{Nuc}_{\mathrm{Sp}(R)}$. The following recent result of Efimov ensures that the comparison map in (2) is an equivalence.

Theorem (Efimov)**.** *Let R be an affinoid C-algebra. Then the comparison map*

$$K(\mathrm{Nuc}_R) \to K^{\mathrm{cont}}(R) \tag{3}$$

is an equivalence of spectra.

This result also ensures that $K^{\mathrm{cont}}(R)$ is well defined and independent of the presentation of $\mathrm{Sp}(R)$, because Nuc_R turns out to be (Corollary 1.33.1), and this gives a proof of the version of pro-cdh descent for K-theory from [Mor16, Lemma 3.4]. The comparison map in (2) makes sense for an arbitrary localizing invariant, and Andreychev's descent result holds for any localizing invariant, where for a localizing invariant T we let $T^{\mathrm{cont}}(R)$ be the pushout

$$\begin{array}{ccc}
T(\mathrm{Perf}_{R_0}) & \longrightarrow & T(\mathrm{Perf}_R) \\
\downarrow & & \downarrow \\
\varprojlim_n T(\mathrm{Perf}_{R_0/\pi^n}) & \longrightarrow & T^{\mathrm{cont}}(R),
\end{array}$$

Here we use the previous theorem of Efimov to prove an analog for topological Hochschild homology

Theorem 0.1. *Let R be an affinoid C-algebra. Then the comparison map*

$$THH(\mathrm{Nuc}_R) \to THH^{\mathrm{cont}}(R) \tag{4}$$

is an equivalence.

The comparison maps in Efimov's theorem and in Theorem 0.1 also exist at the level of R_0, and are more fundamental: the equivalency of the map

$$T(\mathrm{Nuc}_{R_0}) \longrightarrow T^{\mathrm{cont}}(R_0)(= \varprojlim_n T(R_0/\pi^n))$$

at the level of R_0 implies the equivalency of the same map for $R = R_0[\pi^{-1}]$; this is because of the existence of a commutative diagram

$$\begin{array}{ccc}
\mathrm{Mod}_{R_0} & \longrightarrow & \mathrm{Mod}_R \\
\downarrow & & \downarrow \\
\mathrm{Nuc}_{R_0} & \longrightarrow & \mathrm{Nuc}_R,
\end{array}$$

[4]The category Nuc_X is a certain enlargement of the category Perf_X, but, in contrast to the case of schemes, Andreychev's result doesn't hold for Nuc_X replaced by Perf_X.

which turns out to be a Verdier square[5] (Lemma 1.33), so it is sent to a pushout by any localizing invariant. Given this, the equivalency of all the previous comparison maps is reduced to the following more general question.

Question 0.2. Given a connective $\mathbb{E}_\infty$-ring R (playing the role of R_0 above), a finitely generated ideal $I \subset \pi_0(R)$ (playing the role of $(\pi) \subset R_0$), and a localizing invariant T (such as K-theory or topological Hochschild homology), when is the map

$$T(\mathrm{Nuc}_{R_I^\wedge}) \to \varprojlim_{n\in\mathbb{N}} T(R/\!\!/I^n) \tag{5}$$

an equivalence? Here the notation stands for the following: $R_I^\wedge$ denotes the derived completion of R with respect to I taken in *condensed* spectra (Definition 1.21), for which it makes sense to consider $\mathrm{Nuc}_{R_I^\wedge}$ (Section 1.2.3), and each $R/\!\!/I^n$ is an $\mathbb{E}_\infty$-ring in spectra which stands for an appropriate derived quotient (Lemma 1.24).

Efimov's theorem then admits the following formulation.

Theorem (Efimov)**.** *Let R be as in Question 0.2. Then for every localizing invariant T there is a fiber sequence*

$$T(\mathrm{Nuc}_{R_I^\wedge}) \to T(\prod_n \mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/I^n}) \to T(\prod_n \mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/I^n}) \tag{6}$$

Where $\mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/I^n}$ denotes the category of finitely generated projective modules over $R/\!\!/I^n$, and the second map is given by the identity minus the projection of the $1{+}n$-th factor to the n-th factor.

The category $\prod_n \mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/I^n}$ on this theorem is regarded as an additive ∞-category, and the non-conective K-theory in the theorem is taken in the setting of additive ∞-categories[6] From this, to deduce Question 0.2 for K-theory it remains to commute the $\mathbb{N}$-indexed products in the last theorem with K-theory. If all the $\mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/I^n}$ are additive 1-categories (for example, if R is a Noetherian discrete ring or if $I = (x)$ and R is discrete with bounded x-torsion) then the K-theory commutes with the product of additive 1-categories by [KW20, Thm 1.2]. This commutation turns out to hold in general.

Theorem (Proposition 2.11)**.** *Let I be a small set and let $(\mathcal{A}_i)_{i\in I}$ be a family of additive ∞-categories. Then the canonical map of non-connective K-theory spectra*

$$K(\prod_{i\in I} \mathcal{A}_i) \longrightarrow \prod_{i\in I} K(\mathcal{A}_i)$$

is an equivalence.

[5]that is, taking fibers in each row gives a localization sequence, and the fibers of both rows agree.

[6]By this we mean that if $\mathcal{A}$ is an additive ∞-category and T is a localizing invariant such as non-connective K-theory, we let $T(\mathcal{A})$ denote T applied to the stable envelope of $\mathcal{A}$. See Section 2.

Versions of this result have been studied before, see [Car95], [KW19], or the introduction to Section 2 for an account. Crucially, the proof of the last theorem relies on the commutation of K-theory with products of *stable* ∞-categories, proved by Kasprowski and Winges in [KW19]. The main use of Proposition 2.11 in here is to prove the same but for topological Hochschild homology.

Theorem (Proposition 2.15)**.** *Let I be a small set and let $(\mathcal{A}_i)_{i\in I}$ be a family of additive ∞-categories. Then the canonical map*

$$THH(\prod_{i\in I}\mathcal{A}_i) \longrightarrow \prod_{i\in I} THH(\mathcal{A}_i)$$

is an equivalence.

We remark that even when the $\mathcal{A}_i$'s are all 1-categories the proof of this result needs the above K-theoretic analog for additive ∞-categories. The following is a consequence of Proposition 2.15 and (6), and it can be seen as a more fundamental formulation of Theorem 0.1 above.

Theorem (Corollary 3.26.3)**.** *The map*

$$THH(\mathrm{Nuc}_{R^\wedge_I}) \to \varprojlim_n THH(R/\!\!/I^n)$$

from Question 0.2 is an equivalence of spectra.

This theorem implies a characterization of THH of $\mathrm{Nuc}_{R^\wedge_I}$ independent of the rings $R/\!\!/I^n$.

Corollary 1 (Corollary 3.27.1)**.** *Notation as in Question 0.2. The inclusion* $\mathrm{Mod}_R \to \mathrm{Nuc}_{R^\wedge_I}$ *induces an equivalence*

$$THH(R)^\wedge_I \to THH(\mathrm{Nuc}_{R^\wedge_I}).$$

of R-modules in spectra.

The last theorem and its corollary are a refinement of the fact that there are canonical maps

$$THH(R) \to THH(\mathrm{Nuc}_R) \to \varprojlim_n THH(R/\!\!/I^n)$$

both of which are equivalences modulo the ideal I. This fact doesn't rely on any of the previous results in this introduction: the composite map is shown to be an equivalence in [CMM20, 5.2], where much more is proved about this composite and its variants for other invariants. Here we show the first map to be an equivalence modulo I in Corollary 3.11.1, the proof of this goes by unraveling the definition of the Hochschild homology of the category Nuc_R. More precisely, we show that it admits a description as a relative *solid* tensor product (Proposition 3.11)

$$THH(\mathrm{Nuc}_R) = \widetilde{R} \otimes_{R \otimes^{\blacksquare} R} R$$

where $\widetilde{R}$ is an $R \otimes^{\blacksquare} R$-module with a map $\widetilde{R} \to R$ to the diagonal $R \otimes^{\blacksquare} R$-module R obtained from the monoidal structure in Solid_R, and we show that this map is an equivalence modulo (I, I)[7]. By the affirmative answer to Question 0.2 for THH, we know that the map $\widetilde{R} \to R$ becomes an actual equivalence after base change along $R \otimes^{\blacksquare} R \to R$, but we do not know if it is an equivalence before base change. If yes, this would give another affirmative answer to Question 0.2 for THH without using the previous results on this introduction. Inspired by this, we prove that in any case an affirmative answer to Question 0.2 for THH would in turn imply an affirmative answer to the same question for K-theory[8]

Proposition (Proposition 3.13)**.** *Suppose that the canonical map*

$$THH(\mathrm{Nuc}_R) \to \varprojlim_{n \in \mathbb{N}} THH(R /\!\!/ I^n)$$

is an equivalence. Then the analog maps for TC and K-theory are also equivalences.

This is derived from the following statement, whose proof does not use the previous results on this introduction.

Proposition (Corollary 3.17.1)**.** *Let E be a truncating invariant commuting with infinite products of additive ∞-categories. Then*

$$E(\mathrm{Nuc}_R) \to \varprojlim_{n \in \mathbb{N}} E(R /\!\!/ I^n)$$

is an equivalence.

This result is proved as follows. We first prove that Nuc_R embeds into a category of 'lax-perfect complexes', denoted $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ and given by a subcategory of the Ind-completion of the lax limit of $n \in \mathbb{N} \mapsto \mathrm{Perf}_{R_n}$ (Definition 1.37). We then analyze the cofiber of this inclusion: there is a category Cof^b_R (Definition 1.71) and a map $\mathrm{laxPerf}^b_R \to \mathrm{Cof}^b_R$ which, informally, remembers the successive cofibers of the objects in a lax-limit. The Ind-extension of this map kills Nuc_R (Proposition 1.72), so there is an induced functor

$$\mathrm{Ind}(\mathrm{laxPerf}^b_R)/\mathrm{Nuc}_R \to \mathrm{Ind}(\mathrm{Cof}^b_R)$$

which we show to be induced by a *nilpotent extension* of additive ∞-categories in the sense of [ES21, Def 3.1.1] (Proposition 3.17). It is proved in [ES21, Thm 4.2.1] that nilpotent extensions agree on truncating invariants, so

$$E(\mathrm{Nuc}_R) \to E(\mathrm{laxPerf}^b_R) \to E(\mathrm{Cof}^b_R)$$

is a fiber sequence of spectra. The categories inside the middle and right terms can be shown to have the same additive motive as the product $\prod_n \mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/ I^n}$ from above

[7] this is the ideal generated by the two copies of I in $R \otimes^{\blacksquare} R$

[8] but, recall, the only current proof of an affirmative answer to Question 0.2 for THH uses Efimov's results, which also imply an affirmative answer for K-theory.

(Section 3.2.1), with the map between them inducing the identity minus the projection. As E in the last proposition commutes with the infinite product of the categories $\mathrm{Proj}^{\mathrm{fg}}_{R/\!\!/I^n}$, the map $E(\mathrm{Nuc}_R) \to \varprojlim_{n\in\mathbb{N}} E(R/\!\!/I^n)$ is an equivalence.

Notation

1. An denotes the ∞-category of anima/spaces, and Sp denotes the ∞-category of spectra.
2. We write $\hom(-,-)$ for hom spaces in ∞-categories. We write $\mathrm{map}(-,-)$ for mapping spectra and we write $\mathrm{Map}(-,-)$ for internal mapping spectra in closed symmetric monoidal stable ∞-categories.
3. We write $\mathrm{Cat}^{\mathrm{st}}_\infty$ for the ∞-category of small stable ∞-categories and exact functors between them, and we write Pr^L for the ∞-category of presentable ∞-categories and colimit preserving functors between them.
4. We write $\mathcal{C}^w \subset \mathcal{C}$ for the full subcategory of a given ∞-category $\mathcal{C}$ spanned by the compact objects.
5. We write $\mathrm{Ind}(\mathcal{C})$ for the Ind extension of an ∞-category $\mathcal{C}$. If $\mathcal{C}$ is small, then $\mathrm{Ind}(\mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ spanned by the left-exact functors. We make sense of this notation also when $\mathcal{C}$ is not necessarily small, but still *accessible*. In this case, $\mathrm{Ind}(\mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ spanned by the left-exact *accessible* functors (see [Lur09, Rmk 7.1.6.2]).
6. We write $\mathcal{M}_{add}$ and $\mathcal{M}_{loc}$ for the ∞-categories of additive and localizing motives, respectively. Similarly, we write $\mathcal{U}_{add}$ and $\mathcal{U}_{loc}$ for the universal additive and localizing invariant, respectively. We refer to [BGT13] for a study of these categories and functors.
7. I sometimes use ‘we’ without the grammatical number being clear. Apparently, when singular this is called the *royal we* and it was used by kings and monarchs to refer to themselves. We do not posses any of those titles nor aim for them; instead, I see this usage as an emphasizer of the fact that math is a collaborative effort (but all mistakes in here are mine).

1 Nuclear modules over adic rings

1.1 Abstract nuclear objects

This section is about introducing nuclear objects and proving basic properties about them. Informally, an object X in a given symmetric monoidal category $\mathcal{C}$ is called nuclear (Definition 1.2) if for every compact object P it holds that

$$P^\vee \otimes X = \mathrm{Hom}(P, X)$$

That is, an object is nuclear if compacts behave as dualizable when tested against it. As nuclearity is a property defined by testing against compact objects, it will only give information about the subcategory generated by them, so in this section we will restrict to the case where $\mathcal{C}$ is compactly generated. Moreover, if the monoidal unit $1 \in \mathcal{C}$ is compact, then objects appropietely generated under colimits by it are nuclear, and we will suppose this too. More precisely, if $\mathrm{Nuc}(\mathcal{C}) \subset \mathcal{C}$ denotes the full subcategory spanned by those objects which are nuclear in the above sense, the unit being compact implies the existence of a fully faithful functor

$$\mathrm{RMod}_{\mathrm{End}_{\mathcal{C}}(1)} \hookrightarrow \mathrm{Nuc}(\mathcal{C})$$

sending $\mathrm{End}_{\mathcal{C}}(1)$ to 1 and commuting with colimits, but we will see that this inclusion is usually not an equivalence. For example, if every compact is dualizable then every object of $\mathcal{C}$ is nuclear (and if a compact is not dualizable, it can't be nuclear). In any case, the properties of the subcategory of nuclear objects will vary with the input category. Here we fix the following generality:

Situation 1.1. For the rest of this section $(\mathcal{C}, \otimes_{\mathcal{C}})$ will denote a compactly generated closed symmetric monoidal stable ∞-category. Moreover, we suppose that $- \otimes_{\mathcal{C}} -$ commutes with colimits in each variable and that the unit $1 \in \mathcal{C}$ is compact.

The section starts by giving a definition of the category $\mathrm{Nuc}(\mathcal{C})$ (Definition 1.2) following [Scha], and recalling the properties proved in there. We then study a case where the map $\mathrm{Nuc}(\mathcal{C}) \to \mathcal{C}$ has a nice right adjoint (Lemma 1.9), which will be the case in later sections. The section then ends by discussing the case where $\mathcal{C}$ comes from an additive ∞-category $\mathcal{A}$. This will be the case in all later sections. In this case, there exists a subcategory $\mathrm{BNuc}_0(\mathcal{C}) \subset \mathrm{Nuc}(\mathcal{C})$ (Definition 1.15) that under some conditions generates the whole of $\mathrm{Nuc}(\mathcal{C})$ under colimits, and it is built from $\mathcal{A}$ under certain sequential colimits, which makes it useful for calculations, see Lemma 1.16.

Definition 1.2. An object $X \in \mathcal{C}$ is called *nuclear* if the map

$$\mathrm{map}_{\mathcal{C}}(1_{\mathcal{C}}, \mathrm{Map}(P, 1_{\mathcal{C}}) \otimes_{\mathcal{C}} X) \to \mathrm{map}_{\mathcal{C}}(P, X)$$

is an equivalence for every compact P. The full subcategory spanned by the nuclear objects will be denoted $\mathrm{Nuc}(\mathcal{C}) \subset \mathcal{C}$.

Definition 1.3. A map $f\colon P \to X$ in $\mathcal{C}$ is called *trace-class* if there exists a map

$$g\colon 1_{\mathcal{C}} \to \mathrm{Map}(P, 1_{\mathcal{C}}) \otimes_{\mathcal{C}} X$$

such that f agrees with the composite

$$P \xrightarrow{\mathrm{id}_P \otimes g} P \otimes_{\mathcal{C}} \mathrm{Map}(P, 1_{\mathcal{C}}) \otimes_{\mathcal{C}} X \xrightarrow{ev_P \otimes \mathrm{id}_X} X.$$

Equivalently, a map f is trace-class if it is in the image of the canonical map

$$\pi_0 \mathrm{map}_{\mathcal{C}}(1_{\mathcal{C}}, \mathrm{Map}(P, 1_{\mathcal{C}}) \otimes_{\mathcal{C}} X) \to \pi_0 \mathrm{map}_{\mathcal{C}}(P, X).$$

Definition 1.4. An object $N \in \mathcal{C}$ is called *basic nuclear* if it can be written as a sequential colimit

$$N = \operatorname{colim}(P_0 \to P_1 \to \cdots)$$

where each $P_i \in \mathcal{C}$ and each map $P_i \to P_{i+1}$ is trace-class.

Remark 1.5. Equivalently, an object is basic nuclear if it can be written as a sequential colimit of trace-class maps between *compact* objects. In fact, any trace-class map $X \to Y$ factors as $X \to P \to Y$, where $X \to P$ is trace class and P is compact. This is because the unit is compact, so any witness $1 \to X^\vee \otimes Y$ of the trace-class map $X \to Y$ factors through some $X^\vee \otimes P$, where P is a compact mapping to Y.

Lemma 1.6. *Basic nuclear objects are nuclear. The class of basic nuclear objects is stable under all countable colimits, and the class of nuclear objects is stable under all small colimits. Moreover, if the full subcategory spanned by the basic nuclear objects is (essentially) small then:*

1. *An object of $\mathcal{C}$ is nuclear if and only if it can be written as a filtered colimit of basic nuclear objects.*
2. *The ∞-category* $\mathrm{Nuc}(\mathcal{C})$ *(Definition 1.2) is equivalent to the* ω_1-Ind-*completion of the full subcategory spanned by the basic nuclear objects.*

Proof. This is proved in [Scha, 13.13] in the condensed setting, but the proof is the same. The smallness condition is needed when taking colimits over all basic nuclear mapping to an object (see the proof in [Scha, 13.13]). Here we prove in detail that basic nuclear objects are nuclear. Let N be a basic nuclear object, so $N = \operatorname{colim}(P_0 \to P_1 \to \cdots)$ is a sequential colimit of compacts along trace-class maps $f_i\colon P_i \to P_{i+1}$. Let Q be a compact object. We pick witnesses g_i of each f_i as in Definition 1.3, so that f_i agrees with the composite

$$P_i \xrightarrow{\mathrm{id}_{P_i}\otimes g_i} P_i \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1} \xrightarrow{ev_{P_i}\otimes \mathrm{id}_{P_{i+1}}} P_{i+1}$$

Proving that N is nuclear amounts to show that the map

$$\mathrm{map}(Q, 1) \otimes N \to \mathrm{map}(Q, N) \tag{7}$$

is an equivalence. Both sides of (7) commute with the colimit presenting N, and there are level-wise maps in the other direction

$$\mathrm{map}(Q, P_i) \to \mathrm{map}(Q, P_i) \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1} \to \mathrm{map}(Q, 1) \otimes P_{i+1} \tag{8}$$

It will suffice to show that both compositions of these maps are given by going one step in the colimit. The composition (8) with the map $\mathrm{map}(Q, 1) \otimes P_{i+1} \to \mathrm{map}(Q, P_{i+1})$ in (7) is given by going right-right-down in the following diagram

$$\begin{tikzcd}
\mathrm{map}(Q, P_i) \arrow[r] \arrow[dr] & \mathrm{map}(Q, P_i) \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1} \arrow[r] \arrow[d] & \mathrm{map}(Q, 1) \otimes P_{i+1} \arrow[d] \\
& \mathrm{map}(Q, P_i \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1}) \arrow[r] & \mathrm{map}(Q, P_{i+1})
\end{tikzcd} \tag{9}$$

and going diagonal-right gives the map $\mathrm{map}(Q, f_i)$ in the colimit, so we want the last diagram to commute. Adjoining the triangle in (9) gives a diagram

$$\begin{tikzcd}
Q \otimes \mathrm{map}(Q, P_i) \arrow[r, "\mathrm{id}\otimes g_i"] \arrow[dr, "\mathrm{ev}_{Q,P_i}\otimes g_i"'] & Q \otimes \mathrm{map}(Q, P_i) \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1} \arrow[d, "\mathrm{ev}_{Q,P_i}\otimes \mathrm{id}"] \\
 & P_i \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1}
\end{tikzcd}$$

which clearly commutes, so the triangle in (9) commutes. Adjoining the square in (9) gives

$$\begin{tikzcd}
Q \otimes \mathrm{map}(Q, P_i) \otimes \mathrm{map}(P_i, 1) \arrow[r] \arrow[d, "\mathrm{ev}_{Q,P_i}"] & Q \otimes \mathrm{map}(Q, 1) \arrow[d, "\mathrm{ev}_Q"] \\
P_i \otimes \mathrm{map}(P_i, 1) \arrow[r, "\mathrm{ev}_{P_i}"] & 1
\end{tikzcd}$$

(up to tensoring all the diagram with P_{i+1}) and this last diagram clearly commutes, so (9) commutes. The composition in the other direction is given by going right-right-down in the following diagram

$$\begin{tikzcd}
\mathrm{map}(Q, 1) \otimes P_i \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{id}\otimes g_i"] & \mathrm{map}(Q, P_i) \arrow[r, "\mathrm{id}\otimes g_i"] & \mathrm{map}(Q, P_i) \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1} \arrow[d] \\
\mathrm{map}(Q, 1) \otimes P_i \otimes \mathrm{map}(P_i, 1) \otimes P_{i+1} \arrow[urr, "\mathrm{can}\otimes\mathrm{id}"] \arrow[rr] & & \mathrm{map}(Q, 1) \otimes P_{i+1}
\end{tikzcd}$$

and going down-right gives the map $\mathrm{map}(Q, 1) \otimes f_i$ in the colimit, so we want the last diagram to commute. It is clear that the triangle on the left commutes, so it remains to show that the triangle on the right also commutes. For this we can suppose that $P_{i+1} = 1$ and then adjoin Q as before. This concludes the proof that basic nuclear objects are nuclear. □

Remark 1.7. From now on we will assume that the hypothesis of Lemma 1.6 is always satisfied. That is, we assume that basic nuclear objects always from an essentially small subcategory of the given ambient category $\mathcal{C}$. As hinted in Lemma 1.6, this has two consequences. First, this implies that the category of nuclear objects is generated under ω_1-filtered by basic nuclear objects, so many things can be proved by reducing to the case of basic nuclear objects. Second, this implies that $\mathrm{Nuc}(\mathcal{C})$ is presentable, which in turn implies that the inclusion $\mathrm{Nuc}(\mathcal{C}) \subset \mathcal{C}$ always admits a right adjoint.

Corollary 1.7.1. *Nuclear objects are closed under tensor product.*

Proof. By the above, and under Remark 1.7, it suffices to show that basic nuclear objects are closed under tensor products, for which it suffices to note that trace-class maps are closed under tensor products. □

The definition of nuclear can be packed into a functor:

Definition 1.8. Let $(-)^{\mathrm{tr}}\colon \mathcal{C} \to \mathcal{C}$ denote the filtered-colimit preserving functor defined by

$$X^{\mathrm{tr}}(P) := \mathrm{map}_{\mathcal{C}}(1_{\mathcal{C}}, \mathrm{Map}(P, 1_{\mathcal{C}}) \otimes_{\mathcal{C}} X) \in \mathrm{Sp}$$

for compact objects $P \in \mathcal{C}$. This functor comes with a natural transformation $(-)^{\mathrm{tr}} \Rightarrow \mathrm{id}_{\mathcal{C}}$, and an object $X \in \mathcal{C}$ is nuclear if and only if such natural transformation induces an equivalence $X^{\mathrm{tr}} \xrightarrow{\sim} X$.

Lemma 1.9. *Suppose that the endofunctor $(-)^{\mathrm{tr}}\colon \mathcal{C} \to \mathcal{C}$ lands in nuclear objects. Then the right adjoint to the inclusion of nuclear modules into $\mathcal{C}$ is given by the resulting functor $(-)^{\mathrm{tr}}\colon \mathcal{C} \to \mathrm{Nuc}(\mathcal{C})$ from Definition 1.8.*

Proof. We have to check that for every nuclear object N the map

$$\mathrm{map}_{\mathcal{C}}(N, X^{\mathrm{tr}}) \to \mathrm{map}_{\mathcal{C}}(N, X)$$

is an equivalence. By writing N as a colimit of basic nuclears, we can reduce to the case where N is basic nuclear. That is, N is a sequential colimit of compact objects along trace-class maps. Let $P \to Q$ be a trace-class map between compact objects, then the induced morphism $\mathrm{Map}(Q, -) \to \mathrm{Map}(P, -)$ of endofunctors of $\mathcal{C}$ factors through the endofunctor $P^\vee \otimes (-)$: this follows from the following diagram

$$\begin{tikzcd}
\mathrm{Map}(Q,-) \arrow[rr] \arrow[d] & & \mathrm{Map}(P \otimes P^\vee \otimes Q, -) \arrow[d] \\
P^\vee \otimes Q \otimes \mathrm{Map}(Q,-) \arrow[r] & P^\vee \otimes (-) \arrow[r] & \mathrm{Map}(P,-)
\end{tikzcd}$$

which commutes because both compositions are adjoint to elements in $\pi_0\mathrm{map}_{\mathrm{End}(\mathcal{C})}(P \otimes \mathrm{Map}(Q, -), \mathrm{id}_{\mathcal{C}})$ coming from the same element in $\pi_0\mathrm{map}_{\mathrm{End}(\mathcal{C})}(P \otimes P^\vee \otimes Q \otimes \mathrm{Map}(Q, -), \mathrm{id}_{\mathcal{C}})$ under the witness $1 \to P^\vee \otimes Q$. We apply this to a presentation of the basic nuclear N as a sequential colimit of compact objects along trace-class maps $f_i\colon P_i \to P_{i+1}$ to get commutative diagrams

$$\begin{tikzcd}
\mathrm{Map}(P_{i+1},-) \arrow[r] \arrow[dr] & \mathrm{Map}(P_i,-) \\
P_{i+1}^\vee \otimes (-) \arrow[r] \arrow[u] & P_i^\vee \otimes (-) \arrow[u]
\end{tikzcd} \tag{10}$$

where the diagonal arrow is the one we've just produced and the lower arrow is induced by f_i. The commutativity of the last diagram is the same calculation as the one done in the proof of the fact that basic nuclear objects are nuclear. This gives backward maps in the colimit, giving that $\mathrm{map}(N, X^{\mathrm{tr}}) = \mathrm{map}(N, X)$. $\square$

Lemma 1.10. *Suppose that compact objects in $\mathcal{C}$ are stable under tensor products, that formal duals of compact objects are nuclear, and that for any two compact objects P and Q the canonical map*

$$\mathrm{Map}(P, 1) \otimes \mathrm{Map}(Q, 1) \to \mathrm{Map}(P \otimes Q, 1)$$

is an equivalence. Then, for every nuclear N and for every compact P, the natural map

$$\mathrm{Map}(P, N) \to \mathrm{Map}(P, 1) \otimes N$$

is an equivalence of objects of $\mathcal{C}$. That is, the condition for nuclearity also holds "internally".

Proof. The statements amounts to show that for every compact Q the map

$$\mathrm{map}(P \otimes Q, N) \to \mathrm{map}(Q, \mathrm{Map}(P, 1) \otimes N)$$

is an equivalence of spectra. By nuclearity of N, the left hand side is given by the undelying spectrum of $\mathrm{Map}(P \otimes Q, 1) \otimes N$. Similarly, $\mathrm{Map}(P, 1) \otimes N$ is nuclear because of the assumptions combined with Corollary 1.7.1, so the right hand side is the underlying spectrum of $\mathrm{Map}(Q, 1) \otimes \mathrm{Map}(P, 1) \otimes N$. These two expressions agree by the assumptions. □

Lemma 1.11. *Let $A \in \mathrm{CAlg}(\mathrm{Nuc}(\mathcal{C}))$ and suppose that the conclusion of Lemma 1.10 holds for $N = A$* [9]*. Then there is an equivalence*

$$\mathrm{Mod}_A(\mathrm{Nuc}(\mathcal{C})) \xrightarrow{\sim} \mathrm{Nuc}(\mathrm{Mod}_A(\mathcal{C})).$$

Moreover, if the hypothesis of Lemma 1.9 hold for $\mathcal{C}$ then they also hold for $\mathrm{Mod}_A(\mathcal{C})$, giving an explicit right adjoint

$$(-)^{\mathrm{tr}_A} \colon \mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Nuc}(\mathrm{Mod}_A(\mathcal{C}))$$

to the inclusion of nuclear modules into all modules.

Proof. By Lemma 1.17 below, the functor $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ restricts to a functor between the respective nuclear categories. This gives the map

$$\mathrm{Mod}_A(\mathrm{Nuc}(\mathcal{C})) \to \mathrm{Nuc}(\mathrm{Mod}_A(\mathcal{C}))$$

from the statement, which is fully faithful because it is compatible with the inclusions of both source and target in $\mathrm{Mod}_A(\mathcal{C})$. It remains to show that this map is essentially surjective. For this, it suffices to show for any $M \in \mathrm{Nuc}(\mathrm{Mod}_A(\mathcal{C}))$ the underlying object of $\mathcal{C}$ lies in $\mathrm{Nuc}(\mathcal{C})$, as then M can be resolved, as an A-module, by objects of the form $A^{\otimes n} \otimes M$, which lie in $\mathrm{Mod}_A(\mathrm{Nuc}(\mathcal{C}))$, proving that the map on the statement is essentially surjective. So let M be as above and let P be a compact object in $\mathcal{C}$. Then

$$\begin{aligned}
\mathrm{map}_{\mathcal{C}}(P, M) &= \mathrm{map}_{\mathrm{Mod}_A(\mathcal{C})}(P \otimes A, M) \\
&= \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(P \otimes A, A) \otimes_A M (*) \\
&= \mathrm{Map}_{\mathcal{C}}(P, 1) \otimes A \otimes_A M (*) \\
&= \mathrm{Map}_{\mathcal{C}}(P, 1) \otimes M (*)
\end{aligned}$$

[9] that is, suppose that A is "internally nuclear"

which is the nuclearity of M. This finishes the proof of the equivalence on the statement. For the last claim, let $(-)^{\mathrm{tr}_A}$ denote the endofunctor of $\mathrm{Mod}_A(\mathcal{C})$ given by

$$M^{\mathrm{tr}_A}(Q) := \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(Q, A) \otimes_A M(*)$$

where Q is a compact object of $\mathrm{Mod}_A(\mathcal{C})$. Then what it remains to prove is that M^{tr_A} is nuclear in $\mathrm{Mod}_A(\mathcal{C})$. From the first part of the proof, it suffices to show that the underlying object of $\mathcal{C}$ is nuclear. And for this, note that the previous chain of equivalences gives that $M^{\mathrm{tr}_A} = M^{\mathrm{tr}}$, from which the claim follows by the assumption that M^{tr} is nuclear. □

Many of the examples for the category $\mathcal{C}$ arise from stabilizing categories generated by compact projective objects. The next lines specialize to this case. More precisely, we consider the following setting:

Situation 1.12. Let $\mathcal{A}$ be a symmetric monoidal additive ∞-category (thought of as the compact projective objects). In the following lines we focus on nuclear modules over the category

$$\mathcal{C} := \mathrm{Ind}(\mathrm{Stab}(\mathcal{A}))$$

where $\mathrm{Stab}(\mathcal{A})$ denotes the stable envelope of $\mathcal{A}$ (see the first lines of Section 2.1 for a definition of the stable envelope).

In order to talk about nuclear modules over $\mathcal{C}$, we need to make it fit in the setting of Situation 1.1. That is, we need a closed symmetric monoidal structure on $\mathcal{C}$ where the tensor product commutes with colimits in each variable and the unit is compact. This will follow from the following rewriting of $\mathcal{C}$:

Remark 1.13. The category $\mathrm{Stab}(\mathcal{A})$ is given by inverting Σ in the category $\mathcal{P}_{\Sigma,f}(\mathcal{A})$ (see Section 2.1). Let $\mathcal{C}_{\geq 0} := \mathrm{Ind}(\mathcal{P}_{\Sigma,f}(\mathcal{A}))$, so that objects of $\mathcal{A}$ are compact projective generators of $\mathcal{C}_{\geq 0}$. This notation makes sense: the category $\mathcal{C}$ carries a t-structure whose connective part is $\mathcal{C}_{\geq 0}$. To see this, note that there is an equivalence

$$\mathcal{C} := \mathrm{Ind}(\mathrm{Stab}(\mathcal{A})) \cong \mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathcal{A}))$$

as both categories satisfy the same universal property in the category of presentable stable ∞-categories, and the latter carries a bi-complete t-structure as in [Lurb, C.1.2.10(b)].

Remark 1.14. Using the characterization $\mathcal{C} = \mathrm{Sp}(\mathcal{P}_{\Sigma}(\mathcal{A})) = \mathcal{P}_{\Sigma}(\mathcal{A}) \otimes \mathrm{Sp}$ from the last lines, [Lura, 4.8.1.10] then gives a symmetric monoidal structure on $\mathcal{P}_{\Sigma}(\mathcal{A})$[10], and this symmetric monidal structure is unique such that the tensor product commutes with colimits in each variable and such that the inclusion $\mathcal{A} \to \mathcal{P}_{\Sigma}(\mathcal{A})$ is symmetric monoidal. This implies that the unit is compact. This then gives a symmetric monoidal structure on $\mathcal{P}_{\Sigma}(\mathcal{A}) \otimes \mathrm{Sp} \simeq \mathcal{C}$.

[10] here we use Remark [Lur09, 5.5.8.10] to identify $\mathcal{P}_{\Sigma}(\mathcal{A}) = \mathcal{P}^{sifted}(\mathcal{A})$ as the category $\mathcal{P}^{all}_{fin,disc}(\mathcal{A})$ obtained from $\mathcal{A}$ by freely adjoining all the small colimits restricted to preserving the finite discrete colimits that exist in $\mathcal{A}$

We let $\mathrm{Nuc}(\mathcal{C})$ denote the category of nuclear modules in $\mathcal{C}$ with respect to the monoidal structure of Remark 1.14.

Definition 1.15. Let $\mathrm{BNuc}_0(\mathcal{C})$ denote the full subcategory of $\mathrm{Nuc}(\mathcal{C})$ spanned by sequential colimits of elements in the essential image of $\mathcal{A} \to \mathcal{C}$ along trace-class maps.

Note that $\mathrm{BNuc}_0(\mathcal{C})$ depends not only on $\mathcal{C}$ but also on the inclusion $\mathcal{A} \to \mathcal{C}$. In what follows, instead of assuming that the full subcategory spanned by basic nuclear objects is small, as in Remark 1.7, we only suppose that $\mathrm{BNuc}_0(\mathcal{C})$ is small. The idea is that checking smallness of $\mathrm{BNuc}_0(\mathcal{C})$ is easier than to check smallness of the subcategory spanned by all basic nuclear objects (which is needed for Lemma 1.6), and under some hypotheses, all satisfied in the examples on this thesis, smallness of $\mathrm{BNuc}_0(\mathcal{C})$ implies smallness of the full subcategory spanned by basic nuclear objects:

Lemma 1.16. *Suppose that*

1. *The category* $\mathrm{BNuc}_0(\mathcal{C})$ *from Definition 1.15 is small.*
2. *The functor* $(-)^{\mathrm{tr}} \colon \mathcal{C} \to \mathcal{C}$ *from Definition 1.8 preserves connective objects and lands in nuclear modules.*

Then the category of basic nuclear objects is small, and the smallest full subcategory of $\mathcal{C}$ *containing* $\mathrm{BNuc}_0(\mathcal{C})$ *and closed under small colimits and desuspensions is* $\mathrm{Nuc}(\mathcal{C})$*.*

Proof. Let $\mathcal{B}$ denote the smallest full subcategory of $\mathcal{C}$ containing $\mathrm{BNuc}_0(\mathcal{C})$ and closed under small colimits. We start by showing that $\mathcal{B} = \mathrm{Nuc}(\mathcal{C}) \cap \mathcal{C}_{\geq 0}$. As $\mathrm{BNuc}_0(\mathcal{C}) \subset \mathrm{Nuc}(\mathcal{C}) \cap \mathcal{C}_{\geq 0}$ and the latter is stable under small colimits, $\mathcal{B} \subset \mathrm{Nuc}_{\mathcal{C}} \cap \mathcal{C}_{\geq 0}$. We now show the reverse inclusion. Let $N \in \mathcal{C}$ be nuclear and connective. Let $\mathcal{B}_f$ denote the smallest full subcategory of $\mathcal{C}$ containing $\mathrm{BNuc}_0(\mathcal{C})$ and closed under finite colimits, and let $N' \in \mathcal{B}$ denote the colimit of the filtered diagram of all objects in $\mathcal{B}_f$ mapping to N, this object exists because $\mathrm{BNuc}_0(\mathcal{C})$ is small. Let C denote the cofiber of the canonical map $N' \to N$. As $N' \in \mathcal{B}$, it suffices to show that C vanishes. By left completeness of the t-structure on $\mathcal{C}$ (Remark 1.13), it is enough to show that C is n-connective for all n. Let's induct on $n \geq 0$. If $n = 0$ then C is 0-connective (and nuclear) as a cofiber of connective (and nuclear) objects. Let $n = 1$, $P \in \mathcal{A}$ and let $P \to C$ be a map. The composition $P \to C \to \Sigma N'$ vanishes by 1-connectivity of the target, hence $P \to C$ lifts to a map $P \to N$. We now claim that this lift factors as a composition

$$P \to Q \to N$$

where $Q \in \mathcal{A}$ and both maps are trace-class. To see this, note that, as N is connective, there is a fiber sequence $B_0 \to N \to \Sigma B_1$ where $B_0 \in \mathrm{Ind}(\mathcal{A})$ and $B_1 \in \mathcal{C}$ is connective. The functor $(-)^{\mathrm{tr}}$ preserves connective objects by the assumption that duals of objects of $\mathcal{A}$ are connective, so the composite $P \to N = N^{\mathrm{tr}} \to \Sigma B_1^{\mathrm{tr}}$ vanishes (where the equality $N = N^{\mathrm{tr}}$ is the nuclearity of N), giving a lift $P \to B_0^{\mathrm{tr}}$. As B_0 is a filtered colimit of objects of $\mathcal{A}$, there exists a $Q \in \mathcal{A}$ mapping to B_0 and a lift of $P \to B_0^{\mathrm{tr}}$ to Q^{tr}. The resulting composite $P \to Q \to N$ is the desired one, where the first map is trace-class by construction and the second one because N is nuclear, so every map to N from a compact is trace-class. Iterating this argument, we see that the map $P \to N$ factors through an element of $\mathrm{BNuc}_0(\mathcal{C})$, so it lifts to N', and this shows that the map $P \to C$ we started with is zero. This shows that C is 1-connective. Let $n > 1$ and suppose that C is $(n-1)$-connective. Let $P \in \mathcal{A}$ and $\Sigma^{n-1}P \to C$ map. As above, the map $\Sigma^{n-1}P \to C$ factors as the composition of two trace-class maps $\Sigma^{n-1}P \to \Sigma^{n-1}Q \to C$. Iterating this, the map $\Sigma^{n-1}P \to C$ factors through an element $M \in \Sigma^{n-1}\mathrm{Nuc}_0(\mathcal{C}) \subset \Sigma\mathcal{B}_f$ (as $n-1 \geq 1$). Then the composition $M \to C \to \Sigma N'$ is zero, hence $M \to C$ lifts to N and, as it is in $\mathcal{B}_f$, it lifts to N', and so does the map $\Sigma^{n-1}P \to C$ (as it factors through M). Then $\Sigma^{n-1}P \to C$ must be zero too, hence the n-connectivity of C. This concludes the proof of the equality $\mathcal{B} = \mathrm{Nuc}(\mathcal{C}) \cap \mathcal{C}_{\geq 0}$. Let N be a nuclear object, not necessarily connective. As the t-structure on $\mathcal{C}$ is right complete (Remark 1.13), N can be written as the colimit of its truncations $N_{\geq -n}$. As $(-)^{\mathrm{tr}}$ commutes with colimits, we can write

$$N = N^{\mathrm{tr}} = \mathrm{colim}_{n\in\mathbb{N}}(N_{\geq -n})^{\mathrm{tr}}$$

where each $(N_{\geq -n})^{\mathrm{tr}}$ is nuclear an $(-n)$-connective by the second hypothesis on the statement. So each $N^{\mathrm{tr}}_{\geq -n}$ is a desuspension of an object of $\mathcal{B}$. As $\mathcal{B}$ is in the subcategory generated by colimits under $\mathrm{BNuc}_0(\mathcal{C})$, each $N^{\mathrm{tr}}_{\geq -n}$ is in the subcategory generated under small colimits and desuspensions $\mathrm{BNuc}_0(\mathcal{C})$, and so is N. ☐

Categories of nuclear modules enjoy certain functoriality:

Lemma 1.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories admitting nuclear modules in the sense presented above. That is, they are compactly generated closed symmetric monoidal stable ∞-categories such that $-\otimes-$ commutes with colimits in each variable and such that the monidal unit is compact. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving, symmetric monoidal functor. Then F restricts to nuclear modules.*

Proof. As every nuclear object is a colimit of basic nuclear objects, it suffices to prove that F preserves basic nuclear objects. For this, it suffices to show that F sends trace-class maps to trace-class maps. Let $f\colon P \to Q$ be a trace-class map between compact objects of $\mathcal{C}$ and let $g\colon 1 \to P^\vee \otimes Q$ be a witness of it. The map $F(f)$ can be written as the composite

$$\begin{tikzcd} & F(P\otimes P^\vee)\otimes_R F(Q) \arrow[dr, "F(\mathrm{ev}_P)\otimes\mathrm{id}"] & \\ F(P) \arrow[ur, "\mathrm{id}\otimes F(g)"] \arrow[rr, "F(f)"] & & F(Q) \end{tikzcd}$$

so to exhibit it as trace-class it suffices to construct a map between $F(\mathrm{ev}_P)$ and $\mathrm{ev}_{F(P)}$ over $F(1) \cong 1$. For this, it suffices to produce a map $\alpha\colon F(P^\vee) \to (F(P))^\vee$ such that $\mathrm{id}_{F(P)} \otimes \alpha$ commutes with the two projections $F(\mathrm{ev}_P)$ and $\mathrm{ev}_{F(P)}$ to $F(1) \cong 1$. Taking α to be the adjoint of $F(\mathrm{ev}_P)$ works. ☐

1.2 Adic rings

Classically, an adic ring is a topological ring R that admits a two sided ideal I such that every open contains a translation of a power of I. This ideal then determines the topology: the pair consisting of the ring underlying R together with the ideal I is sufficient to recover the topological ring. Here, instead of packing the data of an adic ring into a topological ring, the data will be packed into a *condensed* ring. More precisely, given a connective $\mathbb{E}_\infty$-ring R^δ[11] playing the role of the underlying ring of the topological ring R in the previous paragraph, and a finitely generated ideal $I \subset \pi_0 R^\delta$, we view R^δ as a condensed spectrum via the map $\mathrm{Sp} \to \mathrm{Cond}(\mathrm{Sp})$ from Remark 1.19 below, and we let

$$R^{\mathrm{ad}} := (R^\delta)^\wedge_I$$

where the completion is now taken in the category of R^δ-modules in condensed spectra rather than in topological R^δ-modules. In the generality presented here, an adic ring will be a condensed ring arising as R^{ad} (see 1.21). This section is organized as follows. 1.2.1 is about definitions and basic properties about condensed objects, completions and adic rings. 1.2.2 is about solid modules over adic rings, and 1.2.3 is about nuclear modules over adic rings, in the sense of Section 1.1.

1.2.1 Definitions

As an adic ring will be a special kind of condensed ring in spectra, let's start by recalling the notion of a condensed object in a category.

Definition 1.18. [Scha, Definition 11.7]. Let $\mathcal{C}$ be an ∞-category that admits finite products and all small filtered colimits. For an uncontable strong limit cardinal κ, we let $\mathrm{Cond}_\kappa(\mathcal{C})$ stand for the full subcategory of the category $\mathrm{Fun}(\mathrm{ExDisc}_\kappa^{\mathrm{op}}, \mathcal{C})$, the category of contravariant functors from the category ExDisc_κ of κ-small extremally disconnected sets to $\mathcal{C}$, spanned by those functors that preserve finite products[12]. Then

$$\mathrm{Cond}(\mathcal{C}) := \mathrm{colim}_\kappa \mathrm{Cond}_\kappa(\mathcal{C}),$$

where the colimit runs over the uncountable strong limit cardinals, and the transition maps are given by left Kan extensions.

When $\mathcal{C} = \mathrm{An}$ (resp Sp), the category $\mathrm{Cond}(\mathrm{An})$ (resp. $\mathrm{Cond}(\mathrm{Sp})$) will be referred as the category of *condensed anima* (resp *condensed spectra*). See [Schb] and [Scha] for more about the foundations of Condensed Mathematics.

Remark 1.19. The categories $\mathcal{C}$ and $\mathrm{Cond}(\mathcal{C})$ interact with each other via the inclusion $\mathcal{C} \cong \mathrm{Cond}_\omega(\mathcal{C}) \to \mathrm{Cond}(\mathcal{C})$. The resulting functor will be denoted by

$$(-)^\delta \colon \mathcal{C} \to \mathrm{Cond}(\mathcal{C}).$$

[11] in this introduction the $(-)^\delta$ stands to indicate that R^δ is just a spectrum.

[12] that is, those functors F such that $F(X \sqcup Y) = F(X) \times F(Y)$

Alternatively, the inclusion of $\mathrm{Cond}_\kappa(\mathrm{An})$ in $\mathrm{Fun}(\mathrm{ExDisc}_\kappa^{\mathrm{op}}, \mathrm{An})$ admits a left adjoint, the sheafification functor. These adjoints are compatible as κ varies and give a sheafification functor

$$\mathrm{colim}_\kappa \mathrm{Fun}(\mathrm{ExDisc}_\kappa^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Cond}(\mathrm{An})$$

which we won't name, as we will only use it in the next line. Then there exists a unique colimit preserving functor

$$(-)^\delta \colon \mathrm{An} \to \mathrm{Cond}(\mathrm{An})$$

given by sending the point to the sheafification of the constant functor on the point. Informally, for an anima X and a profinite set $\varprojlim_{i\in I} S_i$ writen as a cofiltered limit of finite sets S_i, the previous functor is given by

$$X^\delta \colon \varprojlim_{i\in I} S_i \mapsto \mathrm{colim}_{i\in I} X^{S_i}.$$

Tensoring, there exists an analogous map

$$(-)^\delta \colon \mathcal{C} \to \mathrm{Cond}(\mathcal{C})$$

for any presentable ∞-category $\mathcal{C}$.

Notation 1.20. Every connective condensed $\mathbb{E}_\infty$-ring $R \in \mathrm{CAlg}(\mathrm{Cond}(\mathrm{Sp}_{\geq 0}))$ has an underlying connective $\mathbb{E}_\infty$-ring spectrum $R(*)$, given by evaluating at a point. We write R^δ for the object $R(*)^\delta \in \mathrm{CAlg}(\mathrm{Cond}(\mathrm{Sp}_{\geq 0}))$ given by the constant condensed object obtained from the functor $\mathrm{Sp} \to \mathrm{Cond}(\mathrm{Sp})$ from Remark 1.19.

Definition 1.21. A connective condensed $\mathbb{E}_\infty$-ring R is said to be an *adic ring* if there is a constructible closed subset $Z \subset \mathrm{Spec}(\pi_0 R(*))$ such that R identifies with the completion of R^δ along Z.

Equivalently, R is an adic ring if for every finitely generated ideal I cutting out Z the natural map $R^\delta \to R$ in the $R(*)$-linear ∞-category $\mathrm{Mod}_{R^\delta}(\mathrm{Cond}(\mathrm{Sp}))$ extends to an equivalence

$$(R^\delta)^\wedge_I \xrightarrow{\sim} R.$$

Note that the completion in the previous definition is taken in the $R(*)$-linear ∞-category of R^δ-modules in condensed spectra. The notion of completion along a finitely generated ideal does indeed make sense for any $R(*)$-linear ∞-category, more about this in the following remark:

Remark 1.22. Let S be a connective $\mathbb{E}_2$-ring and let $I \subset \pi_0(S)$ be a finitely generated ideal. This remark is here to recall the notions of I-nilpotent (or I-torsion), I-local and I-complete modules in stable S-linear ∞-categories (such as the category of condensed S^δ-modules, as in the last definition) as explained in [Lurb, II.7]. Let $\mathcal{C}$ be a stable S-linear ∞-category. An object $C \in \mathcal{C}$ is called *I-nilpotent* if for every $x \in I$ the object $S[x^{-1}] \otimes_S C$ vanishes. The full subcategory spanned by I-nilpotent objects will be denoted by $\mathcal{C}^{\mathrm{Nil}(I)}$. As the condition of being I-nilpotent is stable under colimits, the inclusion $\mathcal{C}^{\mathrm{Nil}(I)} \subset \mathcal{C}$ admits a right adjoint $\Gamma_I \colon \mathcal{C}^{\mathrm{Nil}(I)} \to \mathcal{C}$. When $I = (x)$ is principal there exists a fiber sequence in $\mathcal{C}$

$$\Gamma_{(x)}C \to C \to S[x^{-1}] \otimes_S C$$

essentially by the definition of I-nilpotent, from which it follows that $\Gamma_{(x)}$ preserves colimits. Using that $\Gamma_{J+(x)} \cong \Gamma_{(x)} \circ \Gamma_J$ for any ideal J ([Lurb, 7.1.2.4]) we conclude that the functor Γ_I preserves colimits for I finitely generated and it is given by tensoring with $\Gamma_I(S)$. We let L_I be the functor sitting in the fiber sequence

$$\Gamma_I \to \mathrm{id}_{\mathcal{C}} \to L_I.$$

The functor L_I can be recovered from the iterative construction of Γ_I. We call an object $D \in \mathcal{C}$ *I-local* if for every I-nilpotent object C the mapping space $\mathrm{Map}_{\mathcal{C}}(C, D)$ is contractible. Equivalently, D is I-local if the map $D \to L_I(D)$ is an equivalence. Let $\mathcal{C}^{\mathrm{Loc}(I)}$ denote the full subcategory of $\mathcal{C}$ spanned by the I-local objects. As L_I preserves colimits, the inclusion of I-local objects into $\mathcal{C}$ admits a right adjoint G_I, giving a fiber sequence

$$G_I \to \mathrm{id}_{\mathcal{C}} \to (-)^{\wedge}_I.$$

Finally, an object $C \in \mathcal{C}$ is called *I-complete* if for every I-local object D the mapping space $\mathrm{Map}_{\mathcal{C}}(D, C)$ is contractible. Equivalently, C is I-complete if the map $C \to C^{\wedge}_I$ is an equivalence. Let $\mathcal{C}^{\mathrm{Cpl}(I)}$ denote the full subcategory spanned by I-complete objects. The functor $(-)^{\wedge}_I$ is left adjoint to the inclusion of I-complete modules into $\mathcal{C}$; as such, it will be called the *I-completion* functor.

Remark 1.23. Let R be an adic ring as in Definition 1.21 and let $Z \subset \mathrm{Spec}(\pi_0 R(*))$ be a closed defining its topology. An R-module will be called *complete* if it is I-complete in the sense of Remark 1.22 for any finitely generated ideal I cutting out Z. The notion of completeness depends only on the closed Z and not on the choice of I. As the usual I-adic completion, the current notion of completion can be realized by a specific tower:

Lemma 1.24. *Let R be an adic ring. Then there exists a tower*

$$\cdots \to R_{n+1} \to R_n \to \cdots \to R_1$$

of connective $\mathbb{E}_\infty$-ring spectra under $R(*)$ such that $R \xrightarrow{\sim} \varprojlim_n R_n^\delta$ in condensed R^δ-modules. Moreover, the tower can be chosen such that each R_n is almost perfect as an $R(*)$-module and each $R_{n+1} \to R_n$ is surjective on π_0 with nilpotent kernel.

Proof. Let $I = (x_1, \cdots x_k)$ be a presentation of the finitely generated ideal I. For each x_i, let $R(x_i)_k$ denote the pushout of the diagram

$$R(*) \xleftarrow{t \mapsto 0} R(*)\{t\} \xrightarrow{t \mapsto x_i^n} R(*)$$

in the category $\mathrm{CAlg}_{R(*)}$, where $R(*)\{t\}$ denotes the free commutative algebra over $R(*)$ in one generator. Let $R_n := R(x_1)_n \otimes_R \cdots \otimes_R R(x_k)_n$. Then [Lurb, 8.1.2.3] implies that for every connective $R(*)$-module M the natural map

$$M_I^\wedge \to \varprojlim_n M \otimes_{R(*)} R_n$$

is an equivalence. It is now easy to check that the map $R \to \varprojlim_n R_n^\delta$ is an equivalence: on S-valued points, where S is a extremally disconnected set, this map is the map

$$C(S, R(*))_I^\wedge \to \varprojlim_n C(S, R_n) = \varprojlim_n C(S, R(*)) \otimes_{R(*)} R_n$$

which is an equivalence by the above applied to the $R(*)$-module $M = C(S, R(*))$. Finally, the fact that this choice of tower satisfies the conditions of the statement is [Lurb, 8.1.2.2] $\square$

Corollary 1.24.1. *Let R be an adic ring and let $M \in \mathrm{Mod}_R(\mathrm{Cond}(\mathrm{Sp}))$. If M is connective, then the canonical map $\beta\colon M \to \varprojlim_n (R_n^\delta \otimes_{R^\delta} M)$ exhibits the target as the I-completion of the source.*

Proof. This can be checked after evaluating at an arbitrary extemally disconnected set S. Then the statement reduced to the fact recalled in the previous proof that for every connective $R(*)$-module M the natural map

$$M_I^\wedge \to \varprojlim_n M \otimes_{R(*)} R_n$$

is an equivalence. $\square$

1.2.2 Solid modules over adic rings

Let R be an adic ring. Let $(R, \mathbb{S})_\blacksquare$ denote the analytic structure on R induced from solid spectra by the map $\mathbb{S} \to R$ in the sense of [Scha, 12.8]. We let $S \mapsto (R, \mathbb{S})_\blacksquare[S]$ denote the measures with respect to this analytic structure and we let $\mathrm{Solid}_R = \mathcal{D}((R, \mathbb{S})_\blacksquare)$ denote the category of modules for this analytic ring. As the adic ring R is a solid spectrum, there is a symmetric monoidal equivalence

$$\mathrm{Mod}_R(\mathrm{Solid}) \xrightarrow{\sim} \mathrm{Solid}_R$$

so the compact objects in Solid_R are generated by base changes of compact objects in Solid. That is, the compact objects in Solid_R are generated under finite colimits, desuspensions and retracts by objects of the form $\prod_J \mathbb{S} \otimes^\blacksquare R$, where J is a small set.

Remark 1.25. The left hand side of this equivalence could also be taken as a definition of the category of solid modules over an adic ring R. Then an analytic ring structure on R is determined by its category of complete modules as a full subcategory of $\mathrm{Mod}_R(\mathrm{Cond}(\mathrm{Sp}))$: that is, more generally, the structure of an analytic associative ring $(\mathcal{A}, \mathcal{M})$ as in [Scha, 12.1] is recovered by the left adjoint to the inclusion $\mathcal{D}(\mathcal{A}, \mathcal{M}) \subset \mathcal{D}(\mathcal{A})$.

Lemma 1.26. *Let A be a Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 A$ is a finitely generated $\mathbb{Z}$-algebra (see [Lura, 7.2.4.30]) and let $I \subset \pi_0 A$ be an ideal. Let $\{A_n\}_{n\in\mathbb{N}}$ be a tower of $\mathbb{E}_\infty$-rings realizing the I-completion of A as in Lemma 1.24. Let $M, N \in \mathrm{Mod}_A(\mathrm{Solid})$.*

1. *M is I-complete if and only if each $\pi_n M$ is I-complete.*
2. *If M is connective, then M is I-complete if and only if $M \otimes_A \pi_0 A$ is I-complete.*
3. *If M and N are connective and I-complete, then the solid A-module $M \otimes_A^{\blacksquare} N$ is I-complete.*
4. *If M and N are connective and M is I-complete, then the previous point gives a canonical map*
$$\varprojlim_n (M \otimes_A^{\blacksquare} N) \otimes_A^{\blacksquare} A_n \to M \otimes_A^{\blacksquare} (\varprojlim_n N \otimes_A^{\blacksquare} A_n)$$
which is an equivalence.
5. *If M and N are connective, then the canonical map*
$$\varprojlim_n (M \otimes_A^{\blacksquare} N) \otimes_A^{\blacksquare} A_n \to \varprojlim_n (M \otimes_A^{\blacksquare} A_n) \otimes_A^{\blacksquare} (N \otimes_A^{\blacksquare} A_n)$$
is an equivalence.

Proof. Recall that for any connective $M \in \mathrm{Mod}_A(\mathrm{Solid})$ its I-completion is given by the map $M \to \varprojlim_n M \otimes_A^{\blacksquare} A_n$ (Corollary 1.24.1). It is then clear that M is I-complete if and only if for each extremally disconnected S the A-module $M(S)$ is I-complete. Then 1. follows from [Lurb, 7.3.4.1]. For 2., suppose $M \otimes_A \pi_0 A$ is I-complete. The hypotheses imply that each $\pi_n A$ is almost perfect as a $\pi_0 A$-module ([Lura, 7.2.4.17]), so each $\pi_n A$ can be written as a geometric realization of finite free $\pi_0 A$-modules ([Lura, 7.2.4.10]). Then, as tensoring with M (over A) commutes with this geometric realization (because M and the terms in the geometric realization are connective) and a geometric realization of connective objects is I-complete if each object is, we conclude that $M \otimes \pi_n A$ is I-complete. We can also conclude that $M \otimes_A A_{\leq n}$ is I-complete: the case $n = 0$ holds by assumption, and then from the previous line and an induction on the sequence $\pi_{n+1} A[n+1] \to A_{\leq n+1} \to A_{\leq n}$ it follows that each $M \otimes_A A_{\leq n}$ is I-complete. As the fibers of $M \to M \otimes_A A_{\leq n}$ are $n+1$-connective and I-completeness can be checked on homotopy groups by 1., it follows that M is I complete. For 3., as a tensor product of connective objects is connective, we can apply 2. and prove the statement after base-change to $\pi_0 A$. This is the same as replacing A by $\pi_0 A$, replacing M by $M \otimes_A \pi_0 A$ and similarly for N. As A is a finitely generated $\mathbb{Z}$-algebra, there exists a polynomial algebra B in finitely many generators and a surjection $B \twoheadrightarrow A$ that exhibits A as an almost perfect B-module. The tensor product $M \otimes_A^{\blacksquare} N$ can be written as a geometric realization of the connective modules $M \otimes_B^{\blacksquare} \otimes A^n \otimes_B^{\blacksquare} N$, so it suffices to show that each of them is complete with respect to an ideal of B going to I under the surjection $B \twoheadrightarrow A$. As each A^n is almost perfect as a B-module, each $M \otimes_B^{\blacksquare} \otimes A^n$ is complete. This reduces the statement to the case where A is a polynomial algebra over $\mathbb{Z}$ in finitely many generators. Moreover, as a module is I-complete if and only if it is (x)-complete for each $x \in I$, we can suppose that the ideal $I = (x)$ is principal. Moreover, as M is a geometric realization of connective objects of the form $(\bigoplus_{i\in I} \prod_{J_i} A)^\wedge_x$ and similarly for N, we can suppose that both M and N have this form. In this setting, a tower as in Lemma 1.24 realizing the completion is given by $A_n := A/x^n$, and as modules of the form $\bigoplus_{i\in I} \prod_{J_i} A$ have no torsion it follows that M is concentrated in degree zero and given by

$$(\bigoplus_{i\in I} \prod_{J_i} A)^\wedge_x = \varprojlim_n \bigoplus_{i\in I} \prod_{J_i} A/x^n = \operatorname{colim}_{\substack{f\colon I\to\mathbb{N} \\ f\to\infty}} \prod_{i\in I} x^{f(i)} \prod_{J_i} A$$

where the colimit runs over these functions for which for a given n there are finitely many $i \in S$ for which $f(i) \leq n$. From this rewriting of M and N it is clear that their tensor product is complete. 4. is then saying that there exists an equivalence$(M \otimes^{\blacksquare}_A N)^\wedge_I \xrightarrow{\sim} M \otimes^{\blacksquare}_A N^\wedge_I$. The existence of the map follows from 3., and the fact that it is an equivalence follows from checking modulo I. For 5., the right hand side can first be rewritten as

$$\varprojlim_n \varprojlim_m (M \otimes^{\blacksquare}_A A_n) \otimes^{\blacksquare}_A (N \otimes^{\blacksquare}_A A_m)$$

As $M \otimes A_n$ is I-complete (because $I^n = 0$ in A_n), the previous point applied to $M = M \otimes A_n$ gives that the term inside the $\varprojlim_n$ in the last expression is $(M \otimes^{\blacksquare}_A A_n) \otimes^{\blacksquare}_A N^\wedge_I$. Applying the second point again with $M = N^\wedge_I$ then gives that

$$M^\wedge_I \otimes^{\blacksquare}_A N^\wedge_I \xrightarrow{\sim} \varprojlim_n (M \otimes^{\blacksquare}_A A_n) \otimes^{\blacksquare}_A (N \otimes^{\blacksquare}_A A_n)$$

and this gives the third point by noting that $(M \otimes^{\blacksquare}_A N)^\wedge_I \xrightarrow{\sim} M^\wedge_I \otimes^{\blacksquare}_A N^\wedge_I$. ☐

Remark 1.27. Let (R_0, I_0) and (R_1, I_1) be two adic rings. Let $(I_0, I_1) \subset \pi_0(R_0 \otimes^{\blacksquare}_{\mathbb{S}} R_1)(*)$ be the ideal generated by the images of I_0 and I_1. Then $(R_0 \otimes^{\blacksquare}_{\mathbb{S}} R_1)^\wedge_{(I_0,I_1)}$ is an adic ring in the sense of Definition 1.21. In fact, saying that $(R_0 \otimes^{\blacksquare}_{\mathbb{S}} R_1)^\wedge_{(I_0,I_1)}$ is an adic ring amounts to note that it is (I_0, I_1)-complete and it is discrete[13] modulo (I_0, I_1): the latter is because R_0 is discrete modulo I_0 and R_1 is discrete modulo I_1, and the solid tensor product of discrete condensed spectra is discrete. Moreover, the underlying commutative algebra in spectra of the adic ring $(R_0 \otimes^{\blacksquare}_{\mathbb{S}} R_1)^\wedge_{(I_0,I_1)}$ is the (I_0, I_1)-completion in spectra of the commutative algebra $R_0(*) \otimes_{\mathbb{S}} R_1(*)$. In a formula, the canonical map

$$(R_0^\delta \otimes_{\mathbb{S}} R_1^\delta)^\wedge_{(I_0,I_1)} \xrightarrow{\sim} (R_0 \otimes^{\blacksquare}_{\mathbb{S}} R_1)^\wedge_{(I_0,I_1)}$$

is an equivalence of condensed algebras, where both completions happen in $\mathrm{Cond}(\mathrm{Sp})$.

Lemma 1.28. *Let (R_0, I_0) and (R_1, I_1) be two adic rings. Then the towers of $\mathbb{E}_\infty$-rings $(R_{0,n})_n$ and $(R_{1,n})_n$ realizing the completions of R_0 and R_1 in the sense of Lemma 1.24 can be chosen such that*

$$(R_0 \otimes^{\blacksquare}_{\mathbb{S}} R_1)^\wedge_{(I_0,I_1)} \xrightarrow{\sim} \varprojlim_n (R_{0,n} \otimes_{\mathbb{S}} R_{1,n})^\delta$$

[13] that is, in the image of $(-)^\delta : \mathrm{Sp} \to \mathrm{Cond}(\mathrm{Sp})$

as condensed spectra. That is, $(R_{0,n} \otimes_{\mathbb{S}} R_{1,n})_n$ is a tower realizing the completion of the adic ring $(R_0 \otimes_{\mathbb{S}}^{\blacksquare} R_1)^{\wedge}_{(I_0,I_1)}$ from Remark 1.27.

Proof. Let $(x_i)_i$ and $(y_j)_j$ be finitely many generators for I_0 and I_1, respectively. Then R_0 is a module over $\mathbb{S}\langle\langle (x_i)_i \rangle\rangle$, the completion in condensed spectra of the free $\mathbb{E}_\infty$-ring in the generators $(x_i)_i$. This is an adic ring in the sense of Definition 1.21. If $(A_{0,n})_n$ is a tower realizing the completion of $\mathbb{S}\langle\langle (x_i)_i \rangle\rangle$ (in the sense of Lemma 1.24), then

$$(R_{0,n})_n := (R_0(*) \otimes_{\mathbb{S}\langle\langle x_i, i \rangle\rangle(*)} A_{0,n})_n$$

is a tower realizing the completion of R_0. Similarly for R_1. Now $R_0 \otimes_{\mathbb{S}}^{\blacksquare} R_1$ is a module over $\mathbb{S}\langle\langle x_i, i \rangle\rangle \otimes_{\mathbb{S}}^{\blacksquare} \mathbb{S}\langle\langle (y_j)_j \rangle\rangle = \mathbb{S}\langle\langle (x_i, y_j)_{i,j} \rangle\rangle$, and the (I_0, I_1)-completion of $R_0 \otimes_{\mathbb{S}}^{\blacksquare} R_1$ agrees with its $(x_i, y_j)_{i,j}$-completion as a $\mathbb{S}\langle\langle (x_i, y_j)_{i,j} \rangle\rangle$-module. A tower realizing the completion of $\mathbb{S}\langle\langle (x_i, y_j)_{i,j} \rangle\rangle$ is given by $(A_{0,n} \otimes_{\mathbb{S}} A_{1,n})$ (see [Lurb, 8.1.2.2]), hence

$$(R_0 \otimes_{\mathbb{S}}^{\blacksquare} R_1)^{\wedge}_{(I_0,I_1)} \xrightarrow{\sim} \varprojlim_n (R_0 \otimes^{\blacksquare}_{\mathbb{S}\langle\langle (x_i)_i \rangle\rangle} A_{0,n}) \otimes_{\mathbb{S}}^{\blacksquare} (R_1 \otimes^{\blacksquare}_{\mathbb{S}\langle\langle (y_j)_j \rangle\rangle} A_{1,n}) \xrightarrow{\sim} \varprojlim_n (R_{0,n} \otimes_{\mathbb{S}} R_{1,n})^{\delta}$$

where the first equivalence is Corollary 1.24.1 and the second uses that

$$R_0(*) \otimes_{\mathbb{S}\langle\langle (x_i)_i \rangle\rangle(*)} A_{0,n} \xrightarrow{\sim} R_0 \otimes^{\blacksquare}_{\mathbb{S}\langle\langle (x_i)_i \rangle\rangle} A_{0,n},$$

and similarly for R_1. □

Lemma 1.29. *Let R be an adic ring. Then compact objects in* Solid_R *are complete.*

Proof. As any compact is a finite colimit of shifts of objects of the form $(R, \mathbb{S})_{\blacksquare}[S]$ for S an extemally disconnected set, it suffices to prove that the measure $(R, \mathbb{S})_{\blacksquare}[S]$ is complete. Let $\mathbb{S}\langle\langle x_1, \cdots, x_n \rangle\rangle$ denote the completion of the free $\mathbb{E}_\infty$-algebra $\mathbb{S}\langle x_1, \cdots, x_n \rangle$ at the ideal $(x_1, \cdots, x_n)$. Then $(R, \mathbb{S})_{\blacksquare}[S]$ can be written as a geometric realization of tensor products over $\mathbb{S}[x_1, \cdots, x_n]_{\blacksquare}$ of the modules R, $\mathbb{S}\langle\langle x_1, \cdots, x_n \rangle\rangle$ and $\mathbb{S}\langle\langle x_1, \cdots, x_n \rangle\rangle \otimes_{\mathbb{S}_{\blacksquare}} \mathbb{S}_{\blacksquare}[S]$, where the x_i's go to generators of a finitely generated ideal I in R determining its topology. Those three modules are complete, the last one by a direct computation, so Lemma 1.26 implies that $(R, \mathbb{S})_{\blacksquare}[S]$ is complete. □

1.2.3 Nuclear modules over adic rings

The category Solid_R associated to an adic ring R is cocomplete symmetric monoidal, generated under filtered colimits by compact objects and with compact unit. It then fits in the framework of Section 1.1, and it is possible to consider its subcategory $\mathrm{Nuc}(\mathrm{Solid}_R)$ of nuclear modules, as defined in Definition 1.2. In order to make the notation lighter, we will write $\mathrm{Nuc}_R := \mathrm{Nuc}(\mathrm{Solid}_R)$. The next result shows that the category Solid_R admits a relatively nice nuclearization functor.

Proposition 1.30. *Let R be an adic ring. Then*

1. *The full subcategory of* Nuc_R *spaned by the basic nuclear objects is small. That is, we are in the setting of Remark 1.7.*

2. *Lemma 1.9 holds. That is, the inclusion of* Nuc_R *into* Solid_R *fits in an adjunction*

$$\mathrm{Nuc}_R \underset{(-)^{\mathrm{tr}}}{\overset{incl}{\rightleftarrows}} \mathrm{Solid}_R$$

where the right adjoint is given by the functor $X \mapsto X^{\mathrm{tr}}$ *from Definition 1.8.*

Proof. To prove the first assertion we apply Lemma 1.16 with $\mathcal{A}$ being the full subcategory spanned by the compact projective objects in $\mathrm{Solid}_{R,\geq 0}$. We first check that the two hypotheses of Lemma 1.16 are satisfied. The first one says that The category $\mathrm{BNuc}_0(\mathcal{C})$ from Definition 1.15 is small. An object of $\mathrm{BNuc}_0(\mathcal{C})$ is a filtered colimit of objects of $\mathcal{A}$ along trace-class. An object of $\mathcal{A}$ is, up to retracts, given by R tensored with an infinite product of copies of the sphere. Any trace-class map $f\colon \prod_S \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R \to \prod_T \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R$, where S and T are any two small sets, factors through $\prod_{\mathbb{N}} \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R$ by Lemma 1.32 below, so $\mathrm{BNuc}_0(\mathcal{C})$ is equivalently given by certain subcategory of the full small subcategory of Solid_R spanned by sequential colimits by a single compact projective generator. This shows the first hypothesis of Lemma 1.16. The second says that the functor $(-)^{\mathrm{tr}}$ from Definition 1.8 preserves connective objects and lands in Nuc_R. The functor $(-)^{\mathrm{tr}}$ preserves connective objects because duals in Solid_R of compact projective objects in $\mathrm{Solid}_{R,\geq}$ are connective. To see that the functor $(-)^{\mathrm{tr}}$ lands in Nuc_R, note that the subcategory of nuclear modules is given by those modules M for which $M^{\mathrm{tr}} \xrightarrow{\sim} M$, then, as the functor $(-)^{\mathrm{tr}}$ commutes with colimits and every object is a colimit of compact objects, it suffices to see that $(-)^{\mathrm{tr}}$ sends compact objects to nuclear objects. As compact objects are complete by Lemma 1.29, it suffices to show that M^{tr} is nuclear whenever M is complete. Let M be complete and let S be an extremally disconected set. Then

$$M^{\mathrm{tr}}(S) = (\mathrm{Map}_{\mathrm{Solid}}(\mathbb{S}, R) \otimes_{\mathrm{Solid}_R} M)(*) = (C(S, R(*))^{\delta} \otimes_{R^{\delta}} M)^{\wedge}_{I}(*)$$

where the first is by definition and the second follows from the solid tensor product preserving complete objects, as in Lemma 1.26. Now both completion and tensoring with a discrete object commute with evaluating at a point, so

$$M^{\mathrm{tr}}(S) = (C(S, R(*)) \otimes_{R(*)} M(*))^{\wedge}_{I} = (M(*)^{\delta})^{\wedge}_{I}(S).$$

This shows that M^{tr} is itself complete and its S-valued points are given by the rightmost term in the last equation, from which it follows that $(M^{\mathrm{tr}})^{\mathrm{tr}} = M^{\mathrm{tr}}$, which amounts to say that M^{tr} is nuclear. This concludes the proof of the first point. For the second point, as the first point holds, we are in the setting of Remark 1.7 and it makes sense to ask if Lemma 1.9 holds, and it does as the hypothesis of Lemma 1.9 is that $(-)^{\mathrm{tr}}$ sends compact objects to nuclear objects, which was showed in the previous lines. ☐

Remark 1.31. The proof of Proposition 1.30 gives that for a complete object M, M^{tr} is given by $(M^{\delta})^{\wedge}_{I}$. By Lemma 1.29, this holds for any compact object in Solid_R

Lemma 1.32. *Every trace-class map $f\colon \prod_S \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R \to \prod_T \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R$, where S and T are any two small sets, factors through $\prod_{\mathbb{N}} \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R$.*

Proof. Each base change $f\otimes_R R_n$ is a map in

$$\mathrm{map}_{Solid_R}(\prod_S \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n, (\prod_T \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n)^{\mathrm{tr}}) = \bigoplus_S \prod_T \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n(*)$$

so it factors through a projection $\prod_S \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n \to \prod_{S_n} \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n$ for some finite subset $S_n \subset S$. Then, as compact objects in Solid_R are complete by Lemma 1.29, Corollary 1.24.1 implies that f is recovered as the limit $\varprojlim_n f\otimes_R R_n$. Then f factors through the projection

$$\prod_S \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n \to \prod_{\cup_n S_n} \mathbb{S}\otimes_{\mathbb{S}_\blacksquare} R_n$$

to the at most countable subset $\cup S_n \subset S$. □

Lemma 1.33. *Let R be a condensed adic ring. Then the following is a pullback square*

$$\begin{tikzcd} \mathrm{Mod}_{R(*)} \arrow[r] \arrow[d] & \mathrm{Mod}_{LR(*)} \arrow[d] \\ \mathrm{Nuc}_R \arrow[r] & \mathrm{Nuc}_{LR}. \end{tikzcd}$$

Where L denotes localization with respect to the topology of R, as defined in Remark 1.22 (for instance, if R is p-adic then L inverts p).

Proof. The algebra LR is nuclear as an R-module: this follows from the iterative construction of the functor L after picking finitely many generators of an ideal I defining the topology of R. Hence Lemma 1.11 gives that $\mathrm{Nuc}_{LR} = \mathrm{Mod}_{LR}(\mathrm{Nuc}_R)$. The fibers of the horizontal arrows are then given by the categories of torsion modules in $\mathrm{Mod}_{R(*)}$ and Nuc_R. To conclude the statement it suffices to show that the induced map between these two categories of torsion modules is an equivalence. In other words, it suffices to show that torsion nuclear R-modules are relatively discrete (i.e. in the essential image of $\mathrm{Mod}_{R(*)} \to \mathrm{Nuc}_R$). Let M be torsion and nuclear. Let M be written as a filtered colimit of compact objects P_i's. As M is nuclear, $M = M^{\mathrm{tr}}$ and M is the filtered colimit of the P_i^{tr}'s. By Remark 1.31, each P_i^{tr} is complete and discrete modulo I. If $B \in \mathrm{Solid}_R$ is an object that is complete and discrete modulo I, then

$$\Gamma B = \Gamma((B^\delta)^\wedge_I) = \Gamma(B^\delta) = (\Gamma B(*))^\delta \otimes_{R^\delta} R$$

is relatively discrete, so, as $M = \Gamma M$ (because it is torsion), M is the colimit of the ΓP_i's, each of which is relatively discrete by the last equation. As relatively discrete modules are stable under colimits, M is relatively discrete. □

Corollary 1.33.1. *Let $f\colon R \to R'$ be an adic map of condensed adic rings*[14]*. Suppose that the map $L_R R(*) \to L_R R'(*)$ is an equivalence. Then the following is a pullback square*

$$\begin{tikzcd}
\mathrm{Mod}_{R(*)} \arrow[r] \arrow[d] & \mathrm{Mod}_{R'(*)} \arrow[d] \\
\mathrm{Nuc}_R \arrow[r] & \mathrm{Nuc}_{R'}.
\end{tikzcd}$$

Proof. As the map $R \to R'$ is adic, there is an equivalence $L_R R'(*) = L'_R R'(*)$. Then, using Lemma 1.33 above, both the outer square and the square on the right in the following diagram are pullback squares

$$\begin{tikzcd}
\mathrm{Mod}_{R(*)} \arrow[r] \arrow[d] & \mathrm{Mod}_{R'(*)} \arrow[r] \arrow[d] & \mathrm{Mod}_{L_R R'(*)} \arrow[d] \\
\mathrm{Nuc}_R \arrow[r] & \mathrm{Nuc}_{R'} \arrow[r] & \mathrm{Mod}_{L_R R'}
\end{tikzcd}$$

so the square on the left is also a pullback. $\square$

The category Solid_R is not presentable. It is compactly generated, but compact objects do not form a small category. This can be solved by restricting the cardinality of the compact objects. This throws away some information, but by Lemma 1.35 it doesn't change the category of nuclear modules. We remark that, in the context of Remark 1.7, it would be circular to use Lemma 1.35 to show that colimits over "all basic nuclears mapping to an object" exist, as the proof of Lemma 1.35 uses that the category of nuclear objects is built from basic nuclear objects under small colimits.

Remark 1.34. Given a compact object x in a cocomplete stable ∞-category $\mathcal{C}$, there is a fully faithful functor $\mathrm{RMod}_{\mathrm{End}_{\mathcal{C}}(x)} \to \mathcal{C}$ [Lura, 7.1.2.1]. Moreover, if $\mathcal{C}$ has a symmetric monoidal structure and $x \otimes x$ is in the subcategory generated by x, then $\mathrm{RMod}_{\mathrm{End}_{\mathcal{C}}(x)}$ inherits a symmetric monoidal structure such that the inclusion into $\mathcal{C}$ is symmetric monoidal. We apply this to the element $x = \prod_{\mathbb{N}} \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R \in \mathrm{Solid}_R$. Precisely, we let

$$A := \mathrm{End}_{\mathrm{Solid}_R}\Big(\prod_{\mathbb{N}} \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R\Big)$$

so that there is a fully faithful, colimit preserving functor $\mathrm{RMod}_A \to \mathrm{Solid}_R$ sending A to the compact $\prod_{\mathbb{N}} \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R$.

Lemma 1.35. *There is an induced functor*

$$\mathrm{Nuc}(\mathrm{RMod}_A) \xrightarrow{incl} \mathrm{Nuc}(\mathrm{Solid}_R)$$

which is an equivalence. Here the symmetric monoidal structure on RMod_A is the one inherited from the one in Solid_R as in Remark 1.34.

[14]that is, a map in CAlg(Solid) between two adic rings such that the image of an ideal of definition of R under the map $R(*) \to R'(*)$ is an ideal of definition for R'.

Proof. The functor $\mathrm{RMod}_A \to \mathrm{Solid}_R$ restricts to nuclear objects by Lemma 1.17. By Lemma 1.16 it suffices to show that the functor *incl* sends $\mathrm{BNuc}_0(\mathrm{RMod}_A)$ to $\mathrm{BNuc}_0(\mathrm{Solid}_R)$ (see Lemma 1.16 for notation) in an essentially surjective way. For this, it suffices to note that every trace-class map $f\colon \prod_S \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R \to \prod_T \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R$, where S and T are any two small sets, factors through $\prod_{\mathbb{N}} \mathbb{S} \otimes_{\mathbb{S}_\blacksquare} R$. This holds by Lemma 1.32 above. $\square$

1.3 Nuclear modules inside lax-perfect modules

1.3.1 Statements

Let R be an adic ring in the sense of Definition 1.21. Let Solid_R denote the category of solid R-modules as in Section 1.2.2 and let $\mathrm{Nuc}_R := \mathrm{Nuc}(\mathrm{Solid}_R)$ denote the category of nuclear R-modules in the sense of Definition 1.2. This section is here to present another characterization of the category Nuc_R. Informally, this characterization is based on the fact that nuclear modules are built out of colimits in Solid_R from sequential colimits of infinite products of the form $\prod_{\mathbb{N}} R$ along trace-class maps (Lemma 1.62), and that trace-class maps between such objects factor through perfect modules modulo an ideal of definition, so that, by varying the ideal of definition, each trace-class map gives rise to a lax-perfect module. In more detail, the idea is the following: as said in the previous lines, any object in Nuc_R is built from small colimits and desuspensions of objects of the form

$$N = \mathrm{colim}\left(N^0 \xrightarrow{f_0} N^1 \xrightarrow{f_1} \cdots\right)$$

where each $N^i = \prod_{\mathbb{N}} R$ and the maps $f_i\colon N^i \to N^{i+1}$ are all trace-class. As explained in Lemma 1.60 below, for each $k \in \mathbb{N}$ the map $f_i \otimes_R R_k$ factors through a finite free R_k-modules N^i_k, and these modules can be chosen such that there are maps $N^i_{k+1} \to N^i_k$ compatible with the base changes of f_i. These modules and maps assemble to produce lax-perfect modules (Definition 1.37)

$$N^{i'} = \{N^i_k \in \mathrm{Perf}_{R_k} \,|\, N^i_{k+1} \otimes_{R_{k+1}} R_k \to N^i_k\}_{k \in \mathbb{N}}.$$

Then the basic nuclear N can be sent to the colimit of the $N^{i'}$'s in an appropriate category, in a functorial way. That is, there is a small category $\mathrm{laxPerf}^b_R$, which is a full subcategory of the lax limit of the functor $n \in \mathbb{N}^{\mathrm{op}} \mapsto \mathrm{Perf}_{R_n}$, where objects such as $N^{i'}$ live, these objects will be referred to as lax-perfect complexes (Definition 1.37), and under this the following result is proved in this section:

Proposition (Proposition 1.43)**.** *There is an adjunction of presentable stable ∞-categories*

$$\mathrm{Nuc}_R \underset{\mathbf{R}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathrm{Ind}(\mathrm{laxPerf}^b_R) \tag{11}$$

where the left adjoint $\mathbf{L}$ *sends* N *to the sequential colimit of the* $N^{i'}$*'s. In this adjunction, the left adjoint is fully faithful and the right adjoint preserves colimits.*

While this result says that Nuc_R can be viewed as a full subcategory of $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$, the next result says that Nuc_R can be realized as a subcategory of $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$ by intrinsic means. In order to make this more precise, recall that an object $X \in \mathrm{Solid}_R$ is nuclear if and only if the map $X^{\mathrm{tr}} \to X$ is an equivalence (Definition 1.8). That is, nuclearity in Solid_R is characterized in terms of a trace functor. Similarly, there is a colimit preserving endofunctor of the category $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$, denoted T and equipped with a natural transformation $T \Rightarrow \mathrm{Id}$ (Definition 1.47), that can be seen as an analog of the functor $(-)^{\mathrm{tr}}$. The endofunctor T is simple: given a lax-perfect complex $P \in \mathrm{laxPerf}_R^b$, we can consider the lax-perfect complex P^n which agrees with P in degrees $\geq n$ and in degrees $< n$ is base-changed from degree n. Then

$$T(P) := \varprojlim_{n \in \mathbb{N}^{\mathrm{op}}} P^n$$

where the limit is taken in the big category $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$. Under this, an object $Y \in \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ is said to be nuclear if $T(Y) \xrightarrow{\sim} Y$ (Definition 1.48). This naming is motivated by the following result:

Proposition (Proposition 1.72)**.** *The essential image of the fully faithful functor* $\mathbf{L}\colon \mathrm{Nuc}_R \to \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ *from the previous Theorem lands in the full subcategory spanned by those objects which are nuclear in the sense of the previous line. That is, those objects* Y *for which* $T(Y) \xrightarrow{\sim} Y$*.*

It is now natural to ask if the functors $\mathbf{L} \circ \mathbf{R}$ and T coincide, pretty much as for Solid_R where the trace functor $(-)^{\mathrm{tr}}$ agrees with the colocalization defining Nuc_R (see Proposition 1.30). It turns out that the functors $(-)^{\mathrm{tr}}$ and T are not completely analog: as $(-)^{\mathrm{tr}}$ is a colocalization functor, it is idempotent. In contrast, the functor T is not idempotent (see Remark 1.57). In particular, the endofunctors $\mathbf{L} \circ \mathbf{R}$ and T are different.

This section ends by showing that, even if not idempotent, the functor T has its advantages over $\mathbf{L} \circ \mathbf{R}$. Crucially, the cofiber of $T \to \mathrm{Id}$ is easier than the cofiber of $\mathbf{L} \circ \mathbf{R} \to \mathrm{Id}$. Precisely, there is a fiber sequence

$$T \to \mathrm{Id} \to \mathbf{G} \circ \mathbf{F}$$

of endofunctors of $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$, where the third term comes from a certain adjunction

$$\mathrm{Ind}(\mathrm{laxPerf}_R^b) \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \mathrm{Ind}(\mathrm{Cof}_R^b)$$

where, informally, $\mathbf{F}$ sends a lax-perfect complex $P = (P_i)_{i \in \mathbb{N}}$ to all its possible cofibers P_i/P_j (see Definition 1.67). The category Cof_R^b is nice: for example, it is generated by an additive ∞-catgory and its K theory is the product of the K theories of the rings R_n, for $n \in \mathbb{N}$. These are properties that we would like the actual cofiber of the inclusion $\mathbf{L}\colon \mathrm{Nuc}_R \hookrightarrow \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ from (11) to have. Despite $\mathbf{L} \circ \mathbf{R}$ and T being different, we will see in Proposition 3.17 that $\mathbf{L} \circ \mathbf{R}$ lies somewhere between T and $T \circ T$ (see Lemma 3.22 for a precise statement), and that this can be pushed to show that the cofiber of the inclusion $\mathbf{L}\colon \mathrm{Nuc}_R \hookrightarrow \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ is close to $\mathrm{Ind}(\mathrm{Cof}_R^b)$ in the following sense:

Proposition (Proposition 3.17). *Let E be a truncating invariant. Then*

$$E(\mathrm{cof}(\mathrm{Nuc}_R \hookrightarrow \mathrm{Ind}(\mathrm{laxPerf}^b_R))) \to E(\mathrm{Ind}(\mathrm{Cof}^b_R))$$

is an equivalence.

Remark 1.36. The definitions of nuclearity discussed in this overview, where nuclear objects are defined as those objects which are fixed by a "trace functor" (T or $(-)^{\mathrm{tr}}$), are a priori different from the definition of the nuclear objects in a closed symmetric monoidal category from Definition 1.2. In the case of the closed symmetric monoidal category Solid_R these two definitions agree by Proposition 1.30. It turns out that the same holds for $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$. That is, the category $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ carries a closed symmetric monoidal structure with compact unit, inherited from the categories Perf_{R_n} for varying $n \in \mathbb{N}$, and the subcategory

$$\mathrm{Nuc}(\mathrm{Ind}(\mathrm{laxPerf}^b_R)) \subset \mathrm{Ind}(\mathrm{laxPerf}^b_R)$$

defined categorically as in Definition 1.2 coincides with the subcategory spanned by those objects for which $T(Y) \xrightarrow{\sim} Y$. Moreover, by the results of the previous paragraphs this subcategory contains Nuc_R in such a way that modding out by the nuclear objects or moding out by Nuc_R gives cofiber categories which are nilpotent extensions of each other. This remark won't be proved nor needed in the rest of the paper.

1.3.2 Definitions and Proofs

Definition 1.37. Let $\mathrm{laxPerf}^b_R$ be defined as the following pullback in Cat_∞

$$\begin{tikzcd}
\mathrm{laxPerf}^b_R \arrow[r] \arrow[d] & \lim^{\mathrm{lax}}_{n\in\mathbb{N}}(\mathrm{Perf}_{R_n}) \arrow[d] \\
\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n}) \arrow[r] & \prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n}
\end{tikzcd}$$

where $\mathrm{Vec}_{R_n} \subset \mathrm{Perf}_{R_n}$ is the full subcategory spanned by retracts of finite free R_n-modules. Here the vertical map on the right is the canonical map, and the lower horizontal map is the fully faithful canonical inclusion of the stable envelope of a product to the product of the stable envelopes (Section 2.1).

In other words, $\mathrm{laxPerf}^b_R \subset \lim^{\mathrm{lax}}_{n\in\mathbb{N}}(\mathrm{Perf}_{R_n})$ is the full subcategory spanned by those objects $\{P_n \in \mathrm{Perf}(R_n)\}_{n\in\mathbb{N}}$ together with R_{n+1}-linear maps $P_{n+1} \to P_n$ such that there is an $N \geq 0$ for which every $\Sigma^N P_n$ is connective and there exists a $k \geq 0$ for which the Tor-amplitude of each P_n is $\leq k$.

Definition 1.38. Let $\mathrm{laxVec}^{\mathrm{s}}_R$ denote the full subcategory of $\mathrm{laxPerf}^b_R$ spanned by those objects which are degree-wise connective, of Tor-amplitude ≤ 0, and whose transition maps are surjective on π_0.

In other words, $\mathrm{laxVec}^{\mathrm{s}}_R$ is spanned by those objects $\{P_n \in \mathrm{Perf}(R_n)\}_{n\in\mathbb{N}}$ for which each P_n is a retract of a finite free R_n-module in degree zero and the transition maps $P_{n+1} \to P_n$ are surjective on π_0.

Remark 1.39. The additive ∞-category $\mathrm{laxVec}^{\mathrm{s}}_R$ will be considered as an exact ∞-category by declaring a sequence to be exact if and only if it is split exact (not only degree-wise!).

The previous remark is motivated by the following result.

Lemma 1.40. *The inclusion* $\mathrm{laxVec}^{\mathrm{s}}_R \subset \mathrm{laxPerf}^{\mathrm{b}}_R$ *is exact, where the source carries the split exact structure from Remark 1.39. This inclusion induces an equivalence*

$$\mathrm{Stab}(\mathrm{laxVec}^{\mathrm{s}}_R) \xrightarrow{\sim} \mathrm{laxPerf}^{b}_R$$

That is, the stable ∞-category $\mathrm{laxPerf}^{b}_R$ *is a stable envelope of the exact ∞-category* $\mathrm{laxVec}^{\mathrm{s}}_R$

Proof. It suffices to prove the two conditions in Lemma 2.3. Let $V = \{V_n \in \mathrm{Vec}_{R_n}\}_n$ and $W = \{W_i \in \mathrm{Perf}_{R_n}\}_n$ be two lax-vector bundles in $\mathrm{laxVec}^{\mathrm{s}}_R$. The first condition in Lemma 2.3 amounts to show that the mapping spectrum $\mathrm{Map}_{\mathrm{laxPerf}_R}(V, W)$ is connective. Writing a lax-inverse limit as an iterated lax-pullback, it is possible to see that this mapping spectrum is given by the limit of the following diagram of spectra

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Map}_{R_3}(V_3, W_3) \arrow[dl] \arrow[dr, two heads] & & \mathrm{Map}_{R_2}(V_2, W_2) \arrow[dl] \arrow[dr, two heads] & & \mathrm{Map}_{R_1}(V_1, W_1) \arrow[dl] \\
\cdots\cdots & & \mathrm{Map}_{R_3}(V_3, W_2) & & \mathrm{Map}_{R_2}(V_2, W_1) &
\end{tikzcd}$$

The limit of this diagram can be calculated iteratively. Let $P_1 := \mathrm{Map}_{R_1}(V_1, W_1)$. For each $n \geq 2$, let

$$P_n := \mathrm{Map}_{R_n}(V_n, W_n) \times_{\mathrm{Map}_{R_n}(V_n, W_{n-1})} P_{n-1}$$

The construction gives maps $P_n \to P_{n-1}$, and the limit of the previous diagram is equivalent to the inverse limit of the P_n's along these maps. The maps $P_n \to P_{n-1}$ are pullbacks of the maps

$$\mathrm{Map}_{R_n}(V_n, W_n) \to \mathrm{Map}_{R_n}(V_n, W_{n-1}),$$

which are easily seen to be surjective on π_0 (as indicated in the last diagram). From this observation it follows that each P_n is connective and that each $P_n \to P_{n-1}$ is surjective on π_0. This implies the connectivity of $\varprojlim_n P_n \simeq \mathrm{Map}_{\mathrm{laxPerf}_R}(V, W)$. It remains to show the second condition in Lemma 2.3. That is, that the smallest stable subcategory containing $\mathrm{laxVec}^{\mathrm{s}}_R$ is the whole. Let $\mathcal{C}$ denote the smallest stable subcategory of $\mathrm{laxPerf}^{\mathrm{b}}_R$ containing $\mathrm{laxVec}^{\mathrm{s}}_R$. Let $k \in \mathbb{N}$ and let $\mathcal{C}_k$ denote the full subcategory of $\mathrm{laxPerf}^{\mathrm{b}}_R$ spanned by the objects which are degree-wise connective and of Tor-amplitude $\leq k$. That is, an object $\{P_i\}_{i\in\mathbb{N}}$ is in $\mathcal{C}_k$ if for each $i \in \mathbb{N}$ the R_i-module P_i is connective and has Tor-amplitude $\leq k$. Definitions imply that every object of $\mathrm{laxPerf}^{\mathrm{b}}_R$ is in some $\mathcal{C}_k$ up to a shift. As $\mathcal{C}$ is stable under shifts, it suffices to show that $\mathcal{C}_k \subset \mathcal{C}$ for each $k \geq 0$. By an induction using Lemma 1.41 below, it suffices to show that $\mathcal{C}_0 \subset \mathcal{C}$. Let $V \in \mathcal{C}_0$ be represented by $\{V_n\}$, where each V_n is a retract of a finite free R_n-module in degree zero. It is possible to produce a $W \in \mathrm{laxVec}^{\mathrm{s}}_R$ and a degree-wise split inclusion $V \to W$: for each $i \geq j \geq 1$ pick R_i-vector bundles V_i^j such that:

$$1)\ V_i^j \text{ lifts the } R_j\text{-vector bundle } V_j; \qquad 2)\ V_j^j = V_j; \qquad 3)\ V_{i+1}^j \otimes_{R_{i+1}} R_i \cong V_i^j\ ;$$

This collection of vector bundles can be summed up to get a lax-vector bundle W, which on degree i is given by $W_i = \bigoplus_{j=1}^i V_i^j$ and whose transition maps are determined by condition 3) above. Then W has surjective transition maps and there is a split inclusion $V \to W$. Letting $\overline{W}$ denote the cokernel of $V \to W$, which again lies in $\mathrm{laxVec}^{\mathrm{s}}_R$, the sequence $V \to W \to \overline{W}$ is a cofiber sequence in $\mathrm{laxPerf}^{\mathrm{b}}_R$ where the middle term and the rightmost term are in $\mathcal{C}$ by definiton, so $V \in \mathcal{C}$ too, showing that $\mathcal{C}_0 \subset \mathcal{C}$. ☐

Lemma 1.41. *Notations as in the proof of Lemma 1.40. Let $k \geq 1$ and $Y \in \mathcal{C}_k$. Then there exists an $X \in \mathcal{C}_{k-1}$, a $Z \in \mathcal{C}_0$, and a fiber sequence*

$$X \to Y \to \Sigma^k Z$$

in the stable ∞-category $\mathrm{laxPerf}^b_R$.

Proof. Recall a version of this statement for an additive ∞-category $\mathcal{A}$: the stable envelope of $\mathcal{A}$ comes with subcategories

$$\mathcal{A}_{[m,n]} \subset \mathrm{Stab}(\mathcal{A})$$

indexed by the poset of finite intervals of $\mathbb{Z}$, ordered by inclusion (see Remark 2.1). These subcategories are defined such that $\mathcal{A}_{[0,0]}$ is the essential image of $\mathcal{A}$, and then, recursively, $\mathcal{A}_{[m,n+1]}$ is the full subcategory spanned by those Y fitting in a fiber sequence

$$X \to Y \to \Sigma^{n+1} Z$$

in $\mathrm{Stab}(\mathcal{A})$ where $X \in \mathcal{A}_{[m,n]}$ and $Z \in \mathcal{A}$. The current proof will use this decomposition for the case of $\mathcal{A} = \mathrm{Vec}_{R_n}$, for which $\mathrm{Stab}(\mathcal{A}) = \mathrm{Perf}_{R_n}$. Let $Y = \{Y_n\}_{n\in\mathbb{N}}$ be an object in $\mathcal{C}_k$ as in the statement. For each $n \geq 0$ let

$$X_n \to Y_n \to \Sigma^k Z_n$$

be a fiber sequence in Perf_{R_n} as above, so $X_n \in \mathrm{Vec}_{R_n,[0,k-1]}$ and $Z_n \in \mathrm{Vec}_{R_n}$. The next step is to assemble the X_n's into an object of $\mathcal{C}_{k-1}$. This amounts to produce maps $X_{n+1} \otimes_{R_{n+1}} R_n \to X_n$. This maps are given by the fact that the composite

$$X_{n+1} \otimes_{R_{n+1}} R_n \to Y_{n+1} \otimes_{R_{n+1}} R_n \to Y_n \to \Sigma^k Z_n$$

vanishes because it is a map from an object in $\mathrm{Vec}_{R_n,[0,k-1]}$ to an object in $\mathrm{Vec}_{R_n,[k,k]}$, and a mapping spectrum between such objects is always 1-connective. Let $X := \{X_n\} \in \mathcal{C}_{k-1}$ with the maps $X_{n+1} \to X_n$ induced by the above. It is clear that this object comes with a map $X \to Y$, and the cofiber of this map, named $\Sigma^k Z$, is degree-wise given by $\Sigma^k Z_n$, so $Z \in \mathcal{C}_0$. ☐

As mentioned at the beginning of the section, the goal is to embed Nuc_R inside $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$. Consider the following functor in the other direction.

Notation 1.42. There is a canonical inverse limit functor

$$\lim\colon \mathrm{laxPerf}^b_R \to \mathrm{Solid}_R$$

sending a lax perfect complex $(P_n)_n$ to $\varprojlim_n P_n$. Under this notation, let

$$\mathbf{R}\colon \mathrm{Ind}(\mathrm{laxPerf}^b_R) \xrightarrow{\lim_!} \mathrm{Solid}_R \xrightarrow{(-)^{\mathrm{tr}}} \mathrm{Nuc}_R \tag{12}$$

where the functor $\lim_!$ is the colimit preserving extension of the functor $\lim\colon \mathrm{laxPerf}^b_R \to \mathrm{Solid}_R$ from the previous line (the notation is the one for left Kan extensions) and the second functor is the right adjoint to the inclusion of Nuc_R into Solid_R (see Proposition 1.30).

Proposition 1.43. *Let* $\mathbf{R}$ *be the functor defined in* (12)*. Then there exists an adjunction*

$$\mathrm{Nuc}_R \underset{\mathbf{R}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathrm{Ind}(\mathrm{laxPerf}^b_R)$$

where $\mathbf{L}$ *is left adjoint to* $\mathbf{R}$*. Moreover, the functor* $\mathbf{L}$ *is fully faithful.*

The construction of the left adjoint $\mathbf{L}$ and the proof of Proposition 1.43 require some study of the category $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$. The construction goes by applying the adjoint functor theorem to $\mathbf{R}$. In order to do so, though, we have to understand the functor $\mathbf{R}$.

By Lemma 1.40, in order to understand the functor $\mathbf{R}$ it is enough to understand what it does to the subcategory $\mathrm{laxVec}^{\mathrm{s}}_R$ defined in Definition 1.37. It is possible to go one step further and consider the following objects inside $\mathrm{laxVec}^{\mathrm{s}}_R$.

Definition 1.44. An object $V \in \mathrm{laxVec}^{\mathrm{s}}_R$ is called *free* if it can be represented as $V = \{V_n | V_{n+1} \to V_n\}_{n\in\mathbb{N}}$ such each $V_n = R_n^{r_n}$ is a finite free R_n-module of rank $r_n \leq r_{n+1} \in \mathbb{N}$ and the maps

$$V_{n+1} = R_{n+1}^{r_{n+1}} \to R_n^{r_n} = V_n$$

are given by base change and projection onto the first r_n coordinates.

Lemma 1.45. *Every object in* $\mathrm{laxVec}^{\mathrm{s}}_R$ *is isomorphic to a retract of a free object.*

Proof. Let $V \in \mathrm{laxVec}^{\mathrm{s}}_R$ be represented by $\{V_n | V_{n+1} \to V_n\}_{n\in\mathbb{N}}$, where each V_n is a retract of a finite free R_n-module and each map $V_{n+1} \to V_n$ is surjective. Let $k \in \mathbb{N}$ and consider the following condition for a $V \in \mathrm{laxVec}^{\mathrm{s}}_R$:

($\bullet_k$) $V_n = 0$ if $n < k$ and the map $V_{n+1} \otimes_{R_{n+1}} R_n \xrightarrow{\sim} V_n$ is an equivalence if $n \geq k$.

If V satisfies $(\bullet_k)$ then the inverse limit $\varprojlim_n V_n \in \mathrm{Mod}_R$ is locally free of finite rank and $R_m \otimes_R \varprojlim_n V_n = V_m$ for $m \geq k$ (SAG 8.3.5.4, 8.3.5.7). Picking a complement $\overline{W} \oplus \varprojlim_n V_n \cong R^s$, for some $s \in \mathbb{N}$ and $\overline{W} \in \mathrm{Mod}_R$, we see that setting $W_n := R_n \otimes_R W$ if $n \geq k$ and $W_n = 0$ for $n < k$ gives a lax vector bundle $W = \{W_n\}_{n\in\mathbb{N}}$ such that $V \oplus W$ is free in the sense of Definition 1.44. For the general case it suffices to realize a V as in the statement as a countable product $V = \prod_{k\in\mathbb{N}} V^k$ such that $V^k \in \mathrm{laxVec}^{\mathrm{s}}_R$ satisfies $(\bullet_k)$. This is ensured by the following, which follows from idempotent lifting:

(∗) Let $R \to S$ be a surjective ring map with nilpotent kernel (such as $R_{n+1} \to R_n$). Let P be an R-module, let $Q_0, \cdots, Q_k$ be projective S-modules and let $P \to Q_0 \oplus \cdots \oplus Q_k$ be a surjective R-linear map. Then there exists a decomposition $P \cong P_0 \oplus \cdots \oplus P_k \oplus P_{k+1}$ as a direct sum of R-modules and equivalences $P_i \otimes_R S \xrightarrow{\sim} Q_i$ for $0 \leq i \leq k$. Moreover, if $g_i \colon P_i \to Q_i$ denotes the adjoint to the previous equivalence and $g_{k+1} := 0$, the induced map

$$P \cong P_0 \oplus \dots P_k \oplus P_{k+1} \xrightarrow{\oplus_i g_i} Q_0 \oplus \cdots \oplus Q_k$$

agrees with the original map.

□

Corollary 1.45.1. *The inverse limit functor*

$$\lim \colon \mathrm{laxPerf}^b_R \to \mathrm{Solid}_R$$

from Notation 1.42 sends objects in $\mathrm{laxVec}^{\mathrm{s}}_R$ *to retracts of products of copies of* R*.*

Proof. The functor on the statement on a free object in the sense of Definition 1.44 is easy: it is isomorphic to an (at most countable) product $\prod_J R \in \mathrm{Solid}_R\mathrm{Solid}_R$. Lemma 1.45 then gives that the functor on the statement sends every $V \in \mathrm{laxVec}^{\mathrm{s}}_R$ to a retract of a product of copies of R. □

Definition 1.46. Let $n \in \mathbb{N}$ and let $\mathrm{laxPerf}_{R,\geq n} \subset \mathrm{laxPerf}_R$ denote the full subcategory spanned by those objects $\{P_k \in \mathrm{Perf}(R_k)\}_{k\in\mathbb{N}}$ for which $P_k = 0$ for $k < n$. The forgetful functor $\mathrm{laxPerf}_R \to \mathrm{laxPerf}_{R,\geq n}$ is left adjoint to the inclusion. The forgetful functor has a further left adjoint, which will be denoted by

$$(-)^n \colon \mathrm{laxPerf}^b_{R,\geq n} \to \mathrm{laxPerf}^b_R$$

and it is given by sending an object on the source to the lax perfect complex that agrees with it in degrees $\geq n$ and that in degrees $< n$ is base changed from its value in degree n. Let $P \in \mathrm{laxPerf}^b_R$. The counits $P^n \to P$ for this adjunctions for varying $n \in \mathbb{N}$ assemble to give a natural map

$$\varepsilon_P \colon \varprojlim_{n\in\mathbb{N}} P^n \to P$$

in $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$. Note that the inverse limit is taken in the latter Ind-category, where it exists.

Definition 1.47. Let

$$T\colon \mathrm{Ind}(\mathrm{laxPerf}_R^b) \to \mathrm{Ind}(\mathrm{laxPerf}_R^b)$$

denote the colimit preserving endofunctor whose restriction to compact objects is given by $P \mapsto \varprojlim_{n\in\mathbb{N}} P^n$. By the previous lines, the functor T comes with a natural transformation $T \to \mathrm{id}$.

Definition 1.48. An object $X \in \mathrm{Ind}(\mathrm{laxPerf}_R)$ is *nuclear* if the map $T(X) \to X$ is an equivalence. A map $f\colon P \to Q$ between compact objects in $\mathrm{Ind}(\mathrm{laxPerf}_R)$ is *trace-class* if there exists a map $g\colon P \to \varprojlim_n Q^n$ such that f agrees with

$$P \xrightarrow{g} \varprojlim_n Q^n \xrightarrow{\varepsilon_P} Q.$$

An object $N \in \mathrm{Ind}(\mathrm{laxPerf}_R)$ is *basic nuclear* if it can be written as a sequential colimit

$$N = \mathrm{colim}(N_0 \to N_1 \to \cdots)$$

of compacts $N_i \in \mathrm{laxPerf}_R$ along trace-class maps.

Remark 1.49. It is not immediate that a basic nuclear object in the sense of Definition 1.48 is nuclear in the sense of Definition 1.48. This will be proved in Proposition 1.72.

Notation 1.50. Let $\mathrm{Solid}_R^{\mathrm{lax}}$ be the lax limit of the functor

$$n \in \mathbb{N} \mapsto \mathrm{Mod}_{R_n}(\mathrm{Solid}_R)$$

The category $\mathrm{Solid}_R^{\mathrm{lax}}$ depends not only on the adic ring R but also on the chosen tower $(R_n)_n$ from Lemma 1.24. Moreover, as we are taking the analytic structure in R to be the one for which $R^+ = \mathbb{S}$, it holds that $\mathrm{Solid}_R = \mathrm{Mod}_R(\mathrm{Solid})$, and the last functor is also just given by $n \in \mathbb{N} \mapsto \mathrm{Mod}_{R_n}(\mathrm{Solid})$. Note that there is a pair of functors

$$\begin{tikzcd} \mathrm{Solid}_R \arrow[r, shift left, "h^*"] & \mathrm{Solid}_R^{\mathrm{lax}} \arrow[l, shift left, "h_*"] \end{tikzcd}$$

where h^* is induced by base change and h_* is the functor taking the inverse limit of the underlying objects of Solid_R. As suggested by the notation, this pair forms an adjunction:

Lemma 1.51. *The functor h^* is left adjoint to h_*.*

Proof. Let $X \in \mathrm{Solid}_R$ and let $Y \in \mathrm{Solid}_R^{\mathrm{lax}}$ be represented by $\{Y_n \in \mathrm{Solid}_{R_n} | Y_{n+1} \to Y_n\}$. Then

$$\mathrm{map}_{\mathrm{Solid}_R}(X, h_*Y) = \varprojlim_n \mathrm{map}_{\mathrm{Solid}_{R_n}}(X \otimes_R R_n, Y_n)$$

and $\mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(h^*X, Y)$ is given by the limit of the diagram

$$\begin{tikzcd} & \mathrm{map}_{R_2}(X\otimes_R R_2, Y_2) \arrow[dl] \arrow[dr, "\otimes_{R_2} R_1"] & & \mathrm{map}_{R_1}(X\otimes_R R_1, Y_1) \arrow[dl] \\ \cdots & & \mathrm{map}_{R_1}(X\otimes_R R_1, Y_1) & \end{tikzcd}$$

where all maps going down-left are equivalences. It is then easy to see that both expressions agree. $\square$

Remark 1.52. There is also a canonical fully faithful map $i\colon \mathrm{laxPerf}_R^b \to \mathrm{Solid}_R^{\mathrm{lax}}$ induced by the inclusions $\mathrm{Perf}_{R_k} \subset \mathrm{Mod}_{R_k}(\mathrm{Solid})$ and functoriality of lax-limits. Under this, there is an equivalence of functors

$$\lim_!(-) = h_* \circ i(-)\colon \mathrm{laxPerf}_R^b \to \mathrm{Nuc}_R$$

where the first functor is th eone from Notation 1.42. In the following we usually omit the functor i. That is, for an object $P \in \mathrm{laxPerf}_R^b$ we write h_*P for $\lim_! P$ and vice versa.

Lemma 1.53. *Let $P \in \mathrm{laxPerf}_R^b$ and $X \in \mathrm{Solid}_R$. Let η and ϵ be the unit and counit of the adjunction $h^* \vdash h_*$. Then*

1. *If X is connective then η_X is an equivalence if and only if X is complete.*
2. *If X is connective and η_X is an equivalence then $\eta_{X^{\mathrm{tr}}}$ is an equivalence.*
3. *η_{h_*P} and $h_*(\epsilon_P)$ are inverse equivalences.*
4. *$\epsilon_{h^*h_*P}$ is an equivalence.*

Proof. 1. is just saying that when X is connective the map $\eta_X\colon X \to \varprojlim_k X \otimes_R R_k$ is a completion for X (Corollary 1.24.1). For 2., as η_X is an equivalence, X is complete. Then Proposition 1.30 gives that X^{tr} is given by $(X^{\delta})^{\wedge}_I$, so it is both complete and connective and the claim follows from 1. To check 3. it suffices to consider the case where $P \in \mathrm{laxVec}_R^{\mathrm{s}}$. Then, by Lemma 1.45, it suffices to consider the case when P is free. Then $h_*P = \prod_J R$ (where J is an at most countable set) is complete and connective, so 3. follows from 1. and the triangular relations for an adjunction. Point 4. reduces to the case where P is free. Then h_*P is a product of copies of R and the claim is clear. $\square$

Lemma 1.54. *Let $P, Q \in \mathrm{laxPerf}_R^b$ and let $X, Y \in \mathrm{Solid}_R$. Then:*

1. *There is a natural equivalence*

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}_R^b)}(P, T(Q)) = \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(P, h^*(h_*Q)^{\mathrm{tr}}) = \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(P, h^*(h_*Q))$$

   compatible with the maps to $\mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(P, Q)$.

2. *Suppose that $\eta_Y\colon Y \to h_*h^*Y$ is an equivalence. Then*

$$\mathrm{map}_{\mathrm{Solid}_R}(X, Y) \to \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(h^*X, h^*Y)$$

   is an equivalence.

Proof. For 1. it suffices to consider the case where Q (and P if you want) is free. Unraveling the definition of T from Definition 1.47, the mapping spectrum on the left of 1. can be rewritten as

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(P, T(Q)) = \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(P, \varprojlim_n Q^n) = \varprojlim_n \mathrm{map}_{\mathrm{laxPerf}^b_R}(P, Q^n)$$

so it can be computed as the inverse limit over $n \in \mathbb{N}$ of the limits of the diagrams

$$\begin{tikzcd}[column sep=small] & \mathrm{map}_{R_2}(P_2, Q_2^n) \arrow[dl] \arrow[dr, "\otimes_{R_2} R_1"] & & \mathrm{map}_{R_1}(P_1, Q_1^n) \arrow[dl] \\ \cdots & & \mathrm{map}_{R_1}(P_1 \otimes_{R_2} R_1, Q_1^n) & \end{tikzcd} \tag{13}$$

where P is represented by $\{P_i | P_{i+1} \to P_i\}$, and the limit of the last diagram calculates $\mathrm{map}_{\mathrm{laxPerf}^b_R}(P, Q^n)$. Exchanging the order in which the limits are taken, we calculate

$$\varprojlim_n \mathrm{map}_{R_k}(P_k, Q_k^n) = \mathrm{map}_{\mathrm{Solid}_{R_k}}(P_k, \varprojlim_n (Q_n \otimes_{R_n} R_k)) = \mathrm{map}_{\mathrm{Solid}_{R_k}}(P_k, h_* Q \otimes_R R_k)$$

where the first equivalence holds by the definition of $Q_k^n := Q_n \otimes_{R_n} R_k$, and the second equivalence is Lemma 1.56. Now the last term can be rewritten as

$$\mathrm{map}_{\mathrm{Solid}_{R_k}}(P_k, h_* Q \otimes_R R_k) = \mathrm{map}_{\mathrm{Solid}_{R_k}}(P_k, (h_* Q \otimes_R R_k)^{\mathrm{tr}}) = \mathrm{map}_{\mathrm{Solid}_{R_k}}(P_k, (h_* Q)^{\mathrm{tr}} \otimes_R R_k) \tag{14}$$

where the first equivalence is because P_k is nuclear (even perfect) in Solid_{R_k}, and the second one is because the object $h_* Q \in \mathrm{Solid}_R$ is complete, so $(h_* Q)^{\mathrm{tr}}$ is given by $(h_* Q^\delta)^\wedge_I$, and a similar description holds over R_k, from which it is evident that

$$(h_* Q \otimes_R R_k)^{\mathrm{tr}} = (h_* Q)^{\mathrm{tr}} \otimes_R R_k$$

(here the first $(-)^{\mathrm{tr}}$ is in Solid_R and the second one is in Solid_{R_k}). It follows that $\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(P, T(Q))$ is also given by the limit of the diagram (13) but where $\mathrm{map}_{R_k}(P_k, Q_k^n)$ is replaced by any of the expressions in (14), and replacing the first and the third precisely gives the other two mapping spectra in the statement. This concludes the proof of 1. The second point on the statement follows from the fact that the pair $h^* \vdash h_*$ from Notation 1.50 is an adjunction:

$$\mathrm{map}_{\mathrm{Solid}_R}(X, Y) = \mathrm{map}_{\mathrm{Solid}_R}(X, h_* h^* Y) = \mathrm{map}_{\mathrm{Solid}^{\mathrm{lax}}_R}(h^* X, h^* Y).$$

□

Remark 1.55. From the previous proof it follows that the presentation of T as the inverse limit from Definition 1.47 is just one way of presenting the functor T as an inverse limit. For example, given a lax-perfect complex $P = (P_n)$ and letting $P'^k = (P_{n+k} \otimes_{R_{n+k}} R_n)_n$, then

$$T(P) \simeq \varprojlim_k P'^k.$$

Lemma 1.56. *Let $P = \{P_j\}_{j\in\mathbb{N}} \in \mathrm{laxPerf}^b_R$. Then the canonical map*

$$\alpha_P \colon (\varprojlim_j P_j) \otimes_R R_k \longrightarrow \varprojlim_{j>k} (P_j \otimes_{R_j} R_k)$$

is an equivalence, where all limits are computed in Solid_R

Proof. Let $\mathcal{C} \subset \mathrm{laxPerf}^b_R$ denote the full subcategory spanned by those $P \in \mathrm{laxPerf}^b_R$ such that the map α_P of the statement is an equivalence. If $W \in \mathrm{laxPerf}^b_R$ is a free lax-vector bundle with surjective transition maps, the map α_W of the statement is given by

$$\alpha_W \colon (\varprojlim_j W_j) \otimes_R R_k \simeq (\prod_{\mathbb{N}} R) \otimes_R R_k \xrightarrow{\simeq} \prod_{\mathbb{N}} R_k \simeq \varprojlim_j (W_j \otimes_{R_j} R_k)$$

where the second map is an equivalence as R_k is almost perfect as an $R(*)$-module. Hence $W \in \mathcal{C}$ and as $\mathcal{C}$ is stable under retracts, desuspensions and cofibers. This implies that $\mathrm{laxPerf}^b_R \subset \mathcal{C}$. □

Remark 1.57. We can now show that the functor T is not idempotent, and that in particular it is not given by $\mathbf{L} \circ \mathbf{R}$. Let $R = \mathbb{Z}_p$ and let V be a free lax-vector bundle (Definition 1.44) such that $\lim V = \prod_{\mathbb{N}} \mathbb{Z}_p$. We show that there is a free vector bundle W and a map $W \to T(V)$ that does not lift to $T(T(V))$ along the canonical map

$$T(T(V)) \xrightarrow{can_{T(V)}} T(V)$$

Let $W := ((\mathbb{Z}/p^n)^n)_{n\in\mathbb{N}}$. By Lemma 1.54, giving a map $\alpha \colon W \to T(V)$ in $\mathrm{Ind}(\mathrm{laxPerf}^b_{\mathbb{Z}_p})$ is the same as giving a map $W \to h^*(\prod_{\mathbb{N}} \mathbb{Z}_p)^{\mathrm{tr}}$ in the category $\mathrm{Solid}^{\mathrm{lax}}_{\mathbb{Z}_p}$. For this, it suffices to give a trace-class endomorphism of $\prod_{\mathbb{N}} \mathbb{Z}_p$ that modulo p^n depends only on the first n variables. Thus, let α be the map induced by the trace-class endomorphism

$$\alpha' \colon \prod_{\mathbb{N}} \mathbb{Z}_p \xrightarrow{(1,p,p^2,p^3,\cdots)} \prod_{\mathbb{N}} \mathbb{Z}_p$$

Suppose that α lifts. By Lemma 3.21, $T(V)$ is in the subcategory generated under small colimits by lax-vector bundles with surjective transition maps (that is, it is in the connective part), so there is a free lax-vector bundle V' with a map to $T(V)$ such that the lift of α factors through $T(V')$. This implies that α factors as

$$W \to T(V') \to V' \to T(V)$$

Then, by the same considerations as above, the map α' can be written as a composite of two trace-class maps

$$\prod_{\mathbb{N}} \mathbb{Z}_p \xrightarrow{\beta} \prod_{\mathbb{N}} \mathbb{Z}_p \xrightarrow{\gamma} \prod_{\mathbb{N}} \mathbb{Z}_p$$

where the trace-class map β is also such that modulo p^n factors over the first n coordinates. We will see that this is not possible. Let e_n reprensent the n-th basis vector of $\prod_{\mathbb{N}} \mathbb{Z}_p$.

So $\alpha(e_n) = p^n e_n$. As β/p^n only depends on the first n coordinates, let $\beta(e_n) = p^n b_n$. Then $\gamma(b_n) = e_n$, so γ hits every basis vector on the target. As γ is trace-class, γ/p^n factors over a finite $\mathbb{Z}/p^n$-module and still hits every basis vector on the target, and this is not possible.

Definition 1.58. For a non decreasing function $j\colon \mathbb{N} \to \mathbb{N}$, let V_j denote the free lax vector bundle[15] which on degree n is given by $R_n^{j(n)}$.

Remark 1.59. As in Remark 1.55, the functor T on a free lax vector bundle V_j determined by a function $j\colon \mathbb{N} \to \mathbb{N}$ can similarly be described as

$$T(V) = \varprojlim_{f \geq j} V_f.$$

The following lemma is a formal way of saying that trace-class maps factor over a vector bundle modulo an ideal of definition and that this factorizations are such that these vector bundles assemble into a lax-vector bundle as the power of the ideal of defintion varies. Morever, the lemma says that giving a trace-class map is the same as giving the lax-vector bundle:

Lemma 1.60. *Let $M \in \mathrm{Solid}_R$ be an I-complete object. Let $V^0 := \prod_{\mathbb{N}} \mathbb{S} \otimes^{\blacksquare} R$. Then the maps*

$$\operatorname*{colim}_{\substack{j\colon \mathbb{N}\to\mathbb{N} \\ j(n+1)\geq j(n)}} \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(V_j, h^*M^{\mathrm{tr}}) \xrightarrow{\sim} \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(h^*V^0, h^*M^{\mathrm{tr}}) = \mathrm{map}_{\mathrm{Solid}_R}(V^0, M^{\mathrm{tr}})$$

are equivalences, where the first map is induced by the canonical maps $h^*V_0 \to V_j$ and the second one is the equivalence from Lemma 1.54. Here the colimit runs over the non decreasing functions from $\mathbb{N}$ to $\mathbb{N}$.

Proof. As M is complete, M^{tr} is given by the completion of $M^\delta := M(*)$ in Solid_R (see the proof of Proposition 1.30 for a proof). Then

$$\mathrm{map}_{\mathrm{Solid}_R}(\prod_{\mathbb{N}} \mathbb{S} \otimes^{\blacksquare} R, M^{\mathrm{tr}}) = \varprojlim_n \bigoplus_{\mathbb{N}} M^\delta \otimes_{R(*)} R_n$$

The right hand side of the last equation fits in the following commutative diagram

$$\begin{tikzcd}
\varprojlim_{n\in\mathbb{N}} \bigoplus_{\mathbb{N}} M^\delta \otimes_{R(*)} R_n \arrow[r] \arrow[d, dashed] & \prod_{n\in\mathbb{N}} \bigoplus_{\mathbb{N}} M^\delta \otimes_{R(*)} R_n \arrow[r] \arrow[d] & \prod_{n\in\mathbb{N}} \bigoplus_{\mathbb{N}} M^\delta \otimes_{R(*)} R_n \arrow[d] \\
\operatorname*{colim}_{\substack{j\colon \mathbb{N}\to\mathbb{N} \\ j(n+1)\geq j(n)}} \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(V_j, h^*M^{\mathrm{tr}}) \arrow[r] & \operatorname*{colim}_{\substack{j\colon \mathbb{N}\to\mathbb{N} \\ j(n+1)\geq j(n)}} \prod_{n\in\mathbb{N}} (M^\delta \otimes_{R(*)} R_n)^{j(n)} & \operatorname*{colim}_{\substack{j\colon \mathbb{N}\to\mathbb{N} \\ j(n+1)\geq j(n)}} \prod_{n\in\mathbb{N}} (M^\delta \otimes_{R(*)} R_n)^{j(n+1)} \arrow[l]
\end{tikzcd}$$

[15] see Definition 1.44 for the definition of free lax vector bundle.

where the rightmost and middle vertical arrows are equivalences, the horizontal maps on the right are given by inclusion minus base change, and the rows are fiber sequences: to see that the lower one is a fiber sequence, note that the maps in the lower row respect the colimit, so it suffices to show that for each $n \in \mathbb{N}$ the sequence

$$\mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(V_j, h^* M^{\mathrm{tr}}) \to \prod_{n \in \mathbb{N}} (M^\delta)^{j(n)} \to \prod_{n \in \mathbb{N}} (M^\delta)^{j(n+1)}$$

is a fiber sequence of spectra, which follows from the definition of $\mathrm{Solid}_R^{\mathrm{lax}}$ as the lax limit of the functor $n \mapsto \mathrm{Mod}_{R_n}(\mathrm{Solid}_R)$. It follows that the vertical arrow on the left of the last diagram exists and is an equivalence, and this is the statement. $\square$

Let $f\colon P \to Q$ be a trace-class map between compact objects in $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$. It seems natural to expect for the map $\lim_!(f)\colon \lim_! P \to \lim_! Q$ to be a trace-class map in Solid_R. In the following Lemma, which is not used in the rest of this section, we prove that this is almost the case: the map $\lim_!(f)$ lifts to $(\lim_! Q)^{\mathrm{tr}}$, but the object $\lim_! P$ is not necessarily compact (recall that compact objects are base changed from Solid, so infinite products of R are not always compact). Nevertheless, this is good enough:

Lemma 1.61. *Let $f\colon P \to Q$ be a trace-class map between compact objects in $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$. Then $\lim_!(f)$ lifts to $(\lim_! Q)^{\mathrm{tr}}$. In particular, the functor $\lim_!\colon \mathrm{Ind}(\mathrm{laxPerf}_R^b) \to \mathrm{Solid}_R$ sends basic nuclear objects on the source (in the sense of 1.48) to nuclear objects in Solid_R.*

Proof. By definition, a map $f\colon P \to Q$ as in the statement is trace-class if it can be written as a composite

$$P \to T(Q) \xrightarrow{\mathrm{can}} Q$$

It is then enough to show that this factorization ensures the existence of a dotted arrow making the following square commute:

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}_R^b)}(P, T(Q)) \arrow[r, dashed] \arrow[d, "\mathrm{can}_0"] & \mathrm{map}_{\mathrm{Solid}_R}(h_* P, (h_* Q)^{\mathrm{tr}}) \arrow[d, "\mathrm{can}_1"] \\
\mathrm{map}_{\mathrm{laxPerf}_R^b}(P, Q) \arrow[r, "\lim_!"] & \mathrm{map}_{\mathrm{Solid}_R}(h_* P, h_* Q)
\end{tikzcd} \tag{15}$$

where $\mathrm{can}_0\colon T(Q) \to Q$ and $\mathrm{can}_1\colon (h_* Q)^{\mathrm{tr}} \to h_* Q$ are the canonical maps, and we identify $\lim_!$ and h_* for objects in $\mathrm{laxPerf}_R^b$ as in 1.52. We can use Lemma 1.54 to rewrite the square (15) as

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(P, h^*(h_* Q)^{\mathrm{tr}}) \arrow[r, dashed] \arrow[d, "\epsilon_Q \circ h^*(\mathrm{can}_1) \circ"'] & \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(h^* h_* P, h^*(h_* Q)^{\mathrm{tr}}) \arrow[d, "h^*(\mathrm{can}_1) \circ"] \\
\mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(P, Q) \arrow[r, "h^* h_*"] & \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(h^* h_* P, h^* h_* Q)
\end{tikzcd} \tag{16}$$

where, precisely, we used point 1. from 1.54 to rewrite the top left corner and we used point 2. from 1.54 to rewrite both terms on the right, which applies because of parts 2. and 3. of 1.53. Let the dotted arrow be the one induced by precomposing with $\epsilon_P\colon h^*h_*P \to P$. Let's see that this choice makes the square commute. Let $f \in \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(P, h^*(h_*Q)^{\mathrm{tr}})$. Going right-down in (16) then sends f to the composite given by going down-right-right in the following commutative diagram

$$\begin{tikzcd}
h^*h_*P \arrow[r, "h^*h_*f"] \arrow[d, "\epsilon_P"'] & h^*h_*h^*(h_*Q)^{\mathrm{tr}} \arrow[r, "h^*h_*h^*(\mathrm{can}_1)"] \arrow[d, "\epsilon_{h^*(h_*Q)^{\mathrm{tr}}}"] & h^*h_*h^*h_*Q \arrow[d, "\epsilon_{h^*h_*Q}"] \\
P \arrow[r, "f"] & h^*(h_*Q)^{\mathrm{tr}} \arrow[r, "h^*(\mathrm{can}_1)"] & h^*h_*Q
\end{tikzcd}$$

where both squares commute by functoriality of ϵ. Then, by commutativity of (1.3.2), going right-down in (16) sends f to $\epsilon_{h^*h_*Q} \circ h^*h_*(h^*\mathrm{can}_1 \circ f)$. Now $\epsilon_{h^*h_*Q} = h^*h_*\epsilon_Q$ because the triangular relations give that they are both left inverses of the equivalence $h^*\eta_{h_*Q}$ (where the fact that the latter is an equivalence is 1.53 3.). Then, rewriting, going right-down in (16) sends f to

$$\epsilon_{h^*h_*Q} \circ h^*h_*(h^*\mathrm{can}_1 \circ f) = h^*h_*\epsilon_Q \circ h^*h_*(h^*\mathrm{can}_1 \circ f) = h^*h_*(\epsilon_Q \circ \mathrm{can}_1 \circ f)$$

which is the same as where f goes if going down-right in (16), so the diagram commutes. $\square$

Lemma 1.62. *Let $N \in \mathrm{BNuc}_0(\mathrm{Solid}_R)$ (see Definition 1.15 for notation). Then there exists a basic nuclear $N' \in \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ and an equivalence $N \xrightarrow{\sim} \mathbf{R}(N')$. Moreover, the resulting morphism*

$$\hom_{\mathrm{Ind}(\mathrm{laxPerf}_R^b)}(N', -) \xrightarrow{\mathbf{R}} \hom_{\mathrm{Nuc}_R}(N, \mathbf{R}(-)),$$

induced by the functor $\mathbf{R}$ and the equivalence of the first part, is an equivalence of functors from $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$ to An.

Proof. By definition, N can be written as a sequential colimit

$$N = \mathrm{colim}(P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} \cdots)$$

where $P_i = \prod_{\mathbb{N}} \mathbb{S} \otimes^{\blacksquare} R$ and each $f_i\colon P_i \to P_{i+1}$ is trace-class. Let $g_i\colon P_i \to P_{i+1}^{\mathrm{tr}}$ be witnesses for the f_i's, so that each f_i factors as a composite

$$P_i \xrightarrow{g_i} P_{i+1}^{\mathrm{tr}} \xrightarrow{h_{i+1}} P_{i+1}$$

Then Lemma 1.60 gives that each h^*g_i factors as a composite

$$h^*P_i \to h^*h_*V^i \to V^i \to h^*P_{i+1}^{\mathrm{tr}}$$

where $V^i \in \mathrm{laxVec}^s_R$. In particular there are commutative triangles

$$\begin{tikzcd}
\cdots \arrow[dr] & & V^i \arrow[dr] & & V^{i+1} \arrow[dr] & & \cdots \\
\cdots \arrow[r] & h^*P_i \arrow[ur] \arrow[rr, "h^*(f_i)"] & & h^*P_{i+1} \arrow[ur] \arrow[rr, "h^*(f_{i+1})"] & & h^*P_{i+1} \arrow[ur] \arrow[r] & \cdots
\end{tikzcd}$$

in $\mathrm{Solid}^{\mathrm{lax}}_R$. Let N' denote the sequential colimit of the V^i's in $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ taken along the maps $V^i \to V^{i+1}$ induced by the last diagram. Then

$$\lim_! N' = \mathrm{colim}_i h_* V^i = \mathrm{colim}_i h_* h^* P_i = N$$

where the first equivalence is Remark 1.52, the second one is given by the triangles above, and the last one is because each $\eta_{P_i}\colon P_i \to h_* h^* P_i$ is an equivalence by Lemma 1.53. This gives that $N = \lim_! N' = \mathbf{R}(N')$, where the last equivalence holds because N is nuclear. To conclude the statement it remains to show that N' is basic nuclear in the sense of Definition 1.48. That is, that each $V^i \to V^{i+1}$ factors through the canonical map $T(V^{i+1}) \to V^{i+1}$ as a map in $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$. To prove this, note that each $V^i \to V^{i+1}$ factors as

$$V^i \to h^* P_{i+1} \to h^* h_* V^{i+1} \to V^i$$

and the composite $V^i \to h^* h_* V^{i+1}$ of the first two maps lifts to $h^*(h_* V^{i+1})^{\mathrm{tr}}$ because V^i is level-wise perfect (see Lemma 1.54). So the maps $V^i \to V^{i+1}$ factor as

$$V^i \to h^*(h_* V^{i+1})^{\mathrm{tr}} \to V^{i+1}$$

in $\mathrm{Solid}^{\mathrm{lax}}_R$, which by Lemma 1.54 is equivalent to factoring over $T(V^{i+1})$ in $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$. This concludes the proof of the first part of the statement. It remains to prove the second equivalence on the statement. That is, it remains to prove that the composite

$$\hom_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(N', -) \xrightarrow{\lim_!} \hom_{\mathrm{Nuc}_R}(N, \lim_!(-)) = \hom_{\mathrm{Nuc}_R}(N, \mathbf{R}(-)) \tag{17}$$

is an equivalence, where the last rewriting is by nuclearity of N in Solid_R. So it suffices to show that the first map in the last equation is an equivalence of functors from $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ to An. For this, let

$$\mathrm{map}'_{\mathrm{Solid}^{\mathrm{lax}}_R}(h^* P_i, -)\colon \mathrm{Ind}(\mathrm{laxPerf}^b_R) \to \mathrm{Sp}$$

be the Ind-extension of the functor sending a $W \in \mathrm{laxPerf}^b_R$ to $\mathrm{map}_{\mathrm{Solid}^{\mathrm{lax}}_R}(h^* P_i, W)$. The maps $V^i \to h^* P_{i+1} \to V^{i+1}$ in $\mathrm{Solid}^{\mathrm{lax}}_R$ give a sequence of colimit preserving functors from $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ to spectra

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V^{i+1}, -) \to \mathrm{map}'_{\mathrm{Solid}^{\mathrm{lax}}_R}(h^* P_{i+1}, -) \to \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V^i, -),$$

which is such that the composite is induced by the map $V^i \to V^{i+1}$ in $\mathrm{laxPerf}^b_R$. Then we can rewrite the left hand side of (17) as

$$\begin{aligned}\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(N', -) &= \varprojlim_i \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V^i, -) \\ &= \varprojlim_i \mathrm{map}'_{\mathrm{Solid}^{\mathrm{lax}}_R}(h^* P_i, -) \\ &= \varprojlim_i \mathrm{map}_{\mathrm{Solid}_R}(P_i, \lim_! -) \\ &= \mathrm{map}_{\mathrm{Nuc}_R}(N, \lim_!(-))\end{aligned} \tag{18}$$

as functors from $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ to spectra. The first equivalence is by writing N' as the colimit of the V^i's. The second is by the previous lines. The third is because the two sides agree levelwise: they are both colimit preserving, by compactness of P_i in Solid_R, and they agree on compacts, for which there is the adjunction $h^* \vdash h_* = \lim_!$ (Remark 1.52). Finally, the last equivalence is by writing N as the colimit of the P_i´s. □

Proof of Proposition 1.43. The adjoint $\mathbf{L}$ of the statement exists if the colimit-preserving functor $\mathbf{R}$ preserves small limits. Since $\mathbf{R}$ is exact, it suffices to show that it preserves small products. Let's first show that $\mathbf{R}$ preserves small products of elements in $\mathrm{laxPerf}^b_R$. Let J be a set and let $\{P_j\}_{j\in J}$ be a collection of objects in $\mathrm{Ind}(\mathrm{laxPerf}^b_R)$ indexed by the set J. By Lemma 1.16, it suffices to fix an $N \in \mathrm{BNuc}_0(\mathrm{Solid}_R)$ and to compare mapping spectra against it. As $N \in \mathrm{BNuc}_0(\mathrm{Solid}_R)$, Lemma 1.62 gives an $N' \in \mathrm{Ind}(\mathrm{laxPerf}^b_R)$ which is basic nuclear in the sense of Definition 1.48 and is such that there is an equivalence $N = \mathbf{R}(N')$. Then there is the following chain of equivalences

$$\begin{aligned}\mathrm{map}_{\mathrm{Nuc}_R}(\mathbf{R}(N'), \mathbf{R}(\prod_J P_j)) &= \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(N', \prod_J P_j) \\ &= \prod_J \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(N', P_j) \\ &= \prod_J \mathrm{map}_{\mathrm{Nuc}_R}(\mathbf{R}(N'), \mathbf{R}(P_j)) \\ &= \mathrm{map}_{\mathrm{Nuc}_R}(\mathbf{R}(N'), \prod_J \mathbf{R}(P_j))\end{aligned}$$

where the first and third equivalences are given by the second part of Lemma 1.62. Replacing $\mathbf{R}(N')$ with N gives that $\mathbf{R}$ commutes with products of elements in $\mathrm{laxPerf}^b_R$. The existence of the left adjoint $\mathbf{L}$ now follows from Lemma 1.63 below. It remains to prove that $\mathbf{L}$ is fully faithful. Let N ad N' as above. The second part of Lemma 1.62 gives that the counit $\epsilon_{N'} \colon \mathbf{LR}(N') \xrightarrow{\simeq} N'$ is an equivalence, hence the unit is an equivalence at $\mathbf{R}(N') = N$. This shows that the unit is an equivalence at every element of $\mathrm{BNuc}_0(\mathrm{Solid}_R)$. As everything commutes with colimits Definition 1.15 shows that the unit is an equivalence, hence the fully faithfulness of $\mathbf{L}$. □

Lemma 1.63. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving functor between stable, presentable and dualizable categories. If $\mathcal{C}$ is compactly generated, then suppose that F preserves small products of compact objects. If $\mathcal{C}$ is ω_1-compactly generated, then suppose that F preserves products of ω_1-compact objects. Then F preserves small limits. In particular, F has a left adjoint.*

Proof. This follows from the description of products

$$\prod_{j\in J} \mathrm{colim}_{i\in I_j} X_{j,i} = \mathrm{colim}_{(g_j)_j \in \prod_{j\in J} I_j} \prod_{j\in J} X_{j,g_j}$$

valid in any dualizable category $\mathcal{E}$ because dualizability implies that the colimit functor $\mathrm{Ind}(\mathcal{E}) \to \mathcal{E}$ preserves limits, which implies the above formula. □

Remark 1.64. Let M be a basic nuclear object of $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$. We do not know if the unit $\mathbf{LR}(M) \to M$ is an equivalence. This holds when infinite products of copies of R are compact in Solid_R.

Remark 1.65. The functor $\mathbf{L}$ assembles the association from the introduction into a functor. That is, applying the corollary in the setting of Lemma 1.62 for $M = N'$ (notation as in Lemma 1.62) gives that there exists an equivalence $\mathbf{L}(N) = N'$, which is precisely what the functor $\mathbf{L}$ was supposed to do.

Recall that the right adjoint to the inclusion $\mathrm{Nuc}_R \subset \mathrm{Solid}_R$ is given by the functor $(-)^{\mathrm{tr}}$. The analog in this setting would be an equivalence $\mathbf{LR} \cong T$. We have seen that this is not true (Remark 1.57). Nevertheless, these functors compare well. The rest of this section aims to approximate the cofiber of $\mathbf{L}$ using the functor T.

Remark 1.66. As the functor $\mathbf{R}$ preserves colimits, the functor $\mathbf{L}$ in the adjunction

$$\mathrm{Nuc}_R \underset{\mathbf{R}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathrm{Ind}(\mathrm{laxPerf}_R^b)$$

of Proposition 1.43 can be viewed as a morphism in $\mathrm{Pr}_\infty^{\mathrm{dual}}$, and the fact that $\mathbf{L}$ is fully faithful ensures that it admits a cofiber in $\mathrm{Pr}_\infty^{\mathrm{dual}}$, which is necessarily compactly generated.

Definition 1.67. Let $\widetilde{\mathrm{Cof}}_R^b$ denote the full subcategory of $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{laxPerf}_R^b)$ spanned by those functors $h\colon \mathbb{N}^{\mathrm{op}} \to \mathrm{laxPerf}_R$ such that:

1. For every $j \leq i$, the perfect R_i-module $h(j)_i$ is zero.
2. For each $0 < i < j$, the sequence

$$h(j)_i \otimes_{R_i} R_{i-1} \to h(j)_{i-1} \to h(j-1)_{i-1}$$

   is a cofiber sequence in Perf_{R_i}.
3. There is an uniform bound on the Tor-amplitudes of the $h(j)_i$, $i, j \in \mathbb{N}$.

Note that the ∞-category $\widetilde{\mathrm{Cof}}^b_R$ is stable. Given this definition, there is a functor

$$F\colon \mathrm{laxPerf}^b_R \to \widetilde{\mathrm{Cof}}^b_R \tag{19}$$

that sends a lax-perfect complex P to the functor $h_P(j)_i := \mathrm{cof}(P_j \otimes_{R_j} R_i \to P_i)$. This description on objects clearly assembles into a functor.

The functor F takes the generating subcategory $\mathrm{laxVec}^{\mathrm{s}}_R \subset \mathrm{laxPerf}^b_R$ to a certain subcategory of $\widetilde{\mathrm{Cof}}^b_R$, which can be described:

Definition 1.68. Let $\mathrm{Cof}^0_R \subset \mathrm{Cof}^b_R$ denote the full subcategory of Cof^b_R spanned by those functors $h\colon \mathbb{N}^{\mathrm{op}} \to \mathrm{laxPerf}^b_R$ such that:

1. For each $j \in \mathbb{N}^{\mathrm{op}}$ and $i \in \mathbb{N}$, $h(j)_i$ is an R_i-vector bundle placed in degree zero.
2. For each $0 < i < j$, the cofiber sequence of R_{i-1}-vector bundles

$$h(j)_i \otimes_{R_i} R_{i-1} \to h(j)_{i-1} \to h(j-1)_{i-1}$$

is split.

It follows from the definitions that the functor $F\colon \mathrm{laxPerf}^b_R \to \widetilde{\mathrm{Cof}}^b_R$ takes $\mathrm{laxVec}^{\mathrm{s}}_R$ to $\Sigma\mathrm{Cof}^0_R \subset \widetilde{\mathrm{Cof}}^b_R$, giving an additive functor

$$\Sigma^{-1}F_{|\mathrm{laxVec}^{\mathrm{s}}_R} : \mathrm{laxVec}^{\mathrm{s}}_R \longrightarrow \mathrm{Cof}^0_R \tag{20}$$

between these two additive ∞-categories. Conversely, Lemma 1.70 below says that this restriction is enough to recover the full functor F. The proof of Lemma 1.70 requires the following small remark about the structure of the category Cof^0_R.

Remark 1.69. Let $t \in \mathbb{N}$ and let $i_t\colon \mathbb{N}^{\mathrm{op}}_{\leq t} \to \mathbb{N}^{\mathrm{op}}$ denote the inclusion of the natural numbers $\leq t$. Then any $h \in \mathrm{Cof}^0_R$ restricts to an $i_t^*h \in \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}_{\leq t}, \mathrm{laxPerf}^b_R)$, which satisfies the truncated version of Definition 1.68. Using this notation, the canonical map

$$h \xrightarrow{\sim} \varprojlim_t i_{t*}i_t^*h \tag{21}$$

is an equivalence in Cof^0_R, where in this claim it is implicit that each $i_{t*}i_t^*h$ lies in Cof^0_R and that the limit exists in this category. There is a relative version of this for the inclusions of the form $i_t^{t+1}\colon \mathbb{N}^{\mathrm{op}}_{\leq t} \to \mathbb{N}^{\mathrm{op}}_{\leq t+1}$, and for h as above there is a splitting

$$i_{t+1}^*h \cong h^{t+1} \oplus i^{t+1}_{t*}i_t^*h$$

where h^{t+1} is the functor in $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}_{\leq t+1}, \mathrm{laxVec}_R)$ given by $h^{t+1}(j) = 0$ for $j \leq t$ and $h^{t+1}(t+1)_n$ it is zero for $n > t$, it is $h(t+1)_t$ at level t, and is the base changes of $h(t+1)_t$ at levels below t. An induction then gives an equivalence of functors $i_{t*}i_t^*h = \prod_{j\leq t} i_{j*}h^j$. Combining this with (21) gives an equivalence

$$h \xrightarrow{\simeq} \prod_{t\in\mathbb{N}} i_{t*}h^t$$

in $\mathrm{Ind}(\mathrm{Cof}_R^b)$. These lines also give an equivalence of lax vector bundles

$$h(t) = \prod_{j\leq t} h^j(j).$$

When reading the next statement, recall that $\mathrm{laxPerf}_R^b$ is the stable envelope of $\mathrm{laxVec}_R^{\mathrm{s}}$.

Lemma 1.70. *The inclusion of* Cof_R^0 *into* $\widetilde{\mathrm{Cof}}_R^b$ *extends to a fully faithful functor*

$$\mathrm{Stab}(\mathrm{Cof}_R^0) \to \widetilde{\mathrm{Cof}}_R^b,$$

where exact sequences in the additive ∞*-category* Cof_R^0 *are the ones that split. Under this, the (shifted) functor* $\Sigma^{-1}F$ *from* (19) *is the functor induced on stable envelopes by the functor between additive* ∞*-categories*

$$\alpha : \mathrm{laxVec}_R^{\mathrm{s}} \longrightarrow \mathrm{Cof}_R^0$$

sending a V *on the source to* $\alpha(V)(j)_i := \ker(V_j \otimes_{R_j} R_i \to V_i)$*.*

Proof. For the assertion about the stable envelope it suffices to check the first condition on Lemma 2.3. This condition says that given $g, h \in \mathrm{Cof}_R^0$ the mapping spectrum $\mathrm{map}_{\mathrm{Cof}_R^b}(g, h)$ is connective. To show this it suffices to show, as in the proof of Lemma 1.40, that for each $j \in \mathbb{N}_{\geq 1}$ the map

$$\mathrm{map}_{\mathrm{laxPerf}_R^b}(g(j), f(j)) \longrightarrow \mathrm{map}_{\mathrm{laxPerf}_R^b}(g(j), f(j-1)) \tag{22}$$

is a map between connective spectra and has connective fiber. The last line of Remark 1.69 implies that $f(j-1)$ is a retract of $f(j)$, so it will suffice to show that the source of (22) is connective. Again by the last line of Remark 1.69, it suffices to show that for each $t, k \leq j$ the mapping spectrum $\mathrm{map}_{\mathrm{laxPerf}_R^b}(g^t(t), f^k(k))$ is connective. As $g^t(t)$ is zero above degree t and base changed from degree t in degrees below t, there is an equivalence

$$\mathrm{map}_{\mathrm{laxPerf}_R^b}(g^t(t), f^k(k)) = \mathrm{map}_{\mathrm{Perf}_{R_t}}(g^t(t)_t, f^k(k)_t)$$

and the latter is connective because it is a mapping spectra between R_t-vector bundles in degree zero. The second claim in the statement follows from the first. $\square$

Because of the previous lemma, the category $\mathrm{Stab}(\mathrm{Cof}_R^0)$ is enough, so from now on we drop the bigger category $\widetilde{\mathrm{Cof}}_R^b$,. Precisely, we make the following definition.

Definition 1.71. Let $\mathrm{Cof}_R^b := \mathrm{Stab}(\mathrm{Cof}_R^0)$. Under this, the functor

$$\mathrm{Ind}(F) \colon \mathrm{Ind}(\mathrm{laxPerf}_R^b) \to \mathrm{Ind}(\widetilde{\mathrm{Cof}}_R^b)$$

from (19) lands in $\mathrm{Ind}(\mathrm{Cof}_R^b)$, and we let

$$\mathbf{F} \colon \mathrm{Ind}(\mathrm{laxPerf}_R^b) \to \mathrm{Ind}(\mathrm{Cof}_R^b)$$

denote the induced functor.

Consider the adjunction

$$\operatorname{Ind}(\mathrm{laxPerf}^b_R) \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \operatorname{Ind}(\mathrm{Cof}^b_R)$$

where $\mathbf{G}$ is a right adjoint to $\mathbf{F}$. This adjunction is introduced in order to prove the following:

Proposition 1.72. *The essential image of the fully faithful functor*

$$\mathbf{L}\colon \mathrm{Nuc}_R \to \operatorname{Ind}(\mathrm{laxPerf}^b_R)$$

constructed in Proposition 1.43 land in the full subcategory of $\operatorname{Ind}(\mathrm{laxPerf}^b_R)$ *spanned by the objects which are nuclear in the sense of Definition 1.48.*

Remark 1.73. When infinite products of copies of R are compact in Solid_R, we are able to prove that the essential image of Nuc_R exhausts the full subcategory of $\operatorname{Ind}(\mathrm{laxPerf}^b_R)$ spanned by the objects which are nuclear in the sense of Definition 1.48. This is because Remark 1.64 holds. Moreover, in this case the sequence

$$\mathrm{Nuc}_R \underset{\mathbf{R}}{\overset{\mathbf{L}}{\rightleftarrows}} \operatorname{Ind}(\mathrm{laxPerf}^b_R) \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \operatorname{Ind}(\mathrm{Cof}^b_R) \tag{23}$$

is a fiber sequence in $\mathrm{Pr}^L_{\mathrm{st}}$.

The proof of Proposition 1.72 needs an explicit calculation of what the functor $\mathbf{G}$ does. This is covered by Lemma 1.74 and Remark 1.76. Precisely, they show that $\mathbf{G}$ it is the exact and colimit preserving extension of the functor that takes an $h \in \mathrm{Cof}^b_R$ and sends it to the inverse limit of its colimns $\varprojlim_{j\in\mathbb{N}^{\mathrm{op}}} h(j)$ in $\operatorname{Ind}(\mathrm{laxPerf}^b_R)$. Note that, unraveling the definitions, this fact gives a fiber sequence of endofunctors

$$T \to \mathrm{Id} \to \mathbf{GF}$$

of the ∞-category $\operatorname{Ind}(\mathrm{laxPerf}^b_R)$.

Lemma 1.74. *Notation as in Remark 1.69. Let $h \in \mathrm{Cof}^0_R$. Then for every $j \in \mathbb{N}$ there is a natural equivalence*

$$\mathbf{G}(i_{j*}h^j) = h^j(j)$$

Proof. Let h be as in the statement, $j \in \mathbb{N}$ and let $V = \{V_i\}_{i\in\mathbb{N}} \in \mathrm{laxVec}^{\mathrm{s}}_R$ be a lax vector bundle with surjective transition maps. There are equivalences

$$\begin{aligned}\mathrm{map}_{\operatorname{Ind}(\mathrm{laxPerf}^b_R)}(V, \mathbf{G}(i_{j*}h^j)) &= \mathrm{map}_{\operatorname{Ind}(\mathrm{laxPerf}^b_R)}(i_j^*\mathbf{F}(V), h^j)\\ &= \mathrm{map}_{\operatorname{Ind}(\mathrm{laxPerf}^b_R)}(\mathbf{F}(V)(j), h^j(j))\end{aligned}$$

where the first equivalence is by applying adjunctions and the second one is because $h^j(t)$ is zero for $t < j$. After this rewriting, to prove the statement it suffices to show that the canonical map $V \to F(V)(j)$ induces an equivalence after applying $\mathrm{map}(-, h^j(j))$. This follows from Lemma 1.75 below. $\square$

Lemma 1.75. *Let $W = \{W_i\}_{i\in\mathbb{N}}$ be an object of* $\mathrm{laxPerf}_R^b$ *such that $W_i \simeq 0$ for $i \geq j$ (for example, $W = h^j(j)$ from the previous proof). Then there is a natural equivalence of spectra*

$$\mathrm{map}_{\mathrm{laxPerf}_R^b}(F(V)(j), W) \xrightarrow{\simeq} \mathrm{map}_{\mathrm{laxPerf}_R^b}(V, W)$$

Proof. Let V^j be the lax vector bundle

$$V^j := \cdots \to V_{j+1} \to V_j \to V_j \otimes_{R_j} R_{j-1} \to \cdots \to V_j \otimes_{R_j} R_1$$

and consider the cofiber sequence

$$V^j \to V \to F(V)(j)$$

The statement is then equivalent to the vanishing of the spectrum $\mathrm{map}_{\mathrm{laxPerf}_R^b}(V^j, W)$. As in Lemma 1.40, this mapping spectrum is given by the limit of the following diagram[16]

$$\begin{tikzcd}
0 \arrow[dr] & & \mathrm{map}_{R_{j-1}}(V_j \otimes_{R_j} R_{j-1}, W_{j-1}) \arrow[dl, "\simeq"'] \arrow[dr] & & \cdots \arrow[dl, "\simeq"'] \arrow[dr] & & \mathrm{map}_{R_1}(V_j \otimes_{R_j} R_1, W^{(1)}) \arrow[dl, "\simeq"'] \\
& \mathrm{map}_{R_j}(V_j, W_{j-1}) & & \cdots & & \mathrm{map}_{R_2}(V_j \otimes_{R_j} R_2, W^{(1)}) &
\end{tikzcd}$$

which is seen to be zero since the diagonal arrows in the direction indicated in the diagram are equivalences. ☐

Remark 1.76. It is now possible to show that the exact and colimit-preserving functor

$$\mathbf{G}\colon \mathrm{Ind}(\mathrm{Cof}_R^b) \to \mathrm{Ind}(\mathrm{laxPerf}_R^b)$$

sends an $h \in \mathrm{Cof}_R^0$ to $\varprojlim_{j\in\mathbb{N}^{\mathrm{op}}} h(j)$. This amounts to note that there are equivalences

$$\begin{aligned}
\mathrm{map}(-, \varprojlim_j h(j)) &= \varprojlim_j \mathrm{map}(\mathbf{F}(-), i_{j*}i_j^* h) \\
&= \mathrm{map}(\mathbf{F}(-), \varprojlim_j i_{j*}i_j^* h) \\
&= \mathrm{map}(\mathbf{F}(-), h)
\end{aligned}$$

where the first equivalence follows from Lemma Lemma 1.74 and the last one from Remark 1.69. This characterization of the functor $\mathbf{G}$ implies that

$$T \longrightarrow \mathrm{id} \longrightarrow \mathbf{GF} \tag{24}$$

is a fiber sequence of colimit-preserving endofunctors of $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$.

[16]a priori only the mapping spaces are given by such diagram, but passing from the mapping space to the mapping spectrum commutes with limits, since the mapping spectrum is the unique such that composed with Ω^∞ gives the mapping space.

Proof of Proposition 1.72. The explicit description of the functor $\mathbf{G}$ in Remark 1.76 gives that the composition $\mathbf{R} \circ \mathbf{G}$ is zero. By passing to left adjoints, $\mathbf{F} \circ \mathbf{L}$ is also zero. If $X \in \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ is such that $\mathbf{F}(X) \simeq 0$, then (24) gives that $T(X) \to X$ is an equivalence: that is, X is nuclear. As $\mathbf{F} \circ \mathbf{L} = 0$, every object in the essential image of $\mathbf{L}\colon \mathrm{Nuc}_R \to \mathrm{Ind}(\mathrm{laxPerf}_R^b)$ is nuclear. □

2 Localizing invariants and infinite products

This section is about localizing invariants applied to infinite products of additive ∞-categories. Precisely, we show that K-theory and topological Hochschild homology commute with small products of additive ∞-categories (Proposition 2.10, Proposition 2.15). Versions of this question have already been studied and answered. Carlsson showed that K-theory commutes with products of exact 1-categories with a cylinder functor, see [Car95]. And Kasprowski and Winges showed in [KW20], following a characterization of Grayson ([Gra12]) that K-theory commutes with infinite products of additive 1-categories. In the direction of considering ∞-categories, Kasprowski and Winges proved that non-connective K-theory commutes with infinite products of stable ∞-categories, see [KW19]. The results on this section rely on the fact, that K-theory commutes with products of stable ∞-categories. The idea is to reduce questions about infinite products of additive ∞-categories to questions about infinite products of stable ∞-categories, to then apply their result.

This section is organized as follows. 2.1 is about *stable envelopes* of additive categories. The stable envelope of an additive category is a canonical way of passing from an additive category to a stable category. Precisely, the stable envelope is the left adjoint of the forgetful functor from stable categories to additive categories. As we want to reduce questions about additive categories to questions about stable categories, the study of the stable envelope functor will be crucial. The language of 2.1 is then used in 2.2 to prove the following version of commutation of K-theory with infinite products

Proposition (Proposition 2.10)**.** *Let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of additive ∞-categories indexed by a set I. Then the canonical map*

$$K_{\geq 0}(\prod_i \mathcal{A}_i) \to \prod_i K_{\geq 0}(\mathcal{A}_i)$$

is an equivalence.

As mentioned, variants of this result already exist when the $\mathcal{A}_i$'s are (pre)stable, and the previous proposition builds from them. Section 2.4 then builds from the previous result to show analogous results for topological Hochschild homology and topological cyclic homology. Precisely, the main result in Section 2.4 is the following.

Proposition (Proposition 2.15)**.** *Let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of additive ∞-categories* *indexed by a set I. Then the canonical map*

$$THH(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i)) \to \prod_{i\in I} THH(\mathrm{Stab}(\mathcal{A}_i))$$

is an equivalence. Here $\mathrm{Stab}(-)$ *stands for the stable envelope functor, from additive to stable categories, mentioned on the first lines of this introduction.*

2.1 Stable envelopes of additive ∞-categories

Let Ex_∞ denote the ∞-category of small exact ∞-categories and exact functors as defined in [Bar13, Definition 1.3], and let $\mathrm{Cat}^{\mathrm{st}}_\infty \subset \mathrm{Ex}_\infty$ denote the full subcategory spanned by those ∞-categories which are stable. This inclusion admits a left adjoint

$$\mathrm{Stab}(-)\colon \mathrm{Ex}_\infty \to \mathrm{Cat}^{\mathrm{st}}_\infty,$$

see [Kle20], which will be referred to as the *stable envelope* functor. For an exact ∞-category $\mathcal{E}$, the unit of the previous adjunction $\mathcal{E} \to \mathrm{Stab}(\mathcal{E})$ is a nice map to a stable ∞-category: it is fully faithful, it preserves and reflects exact sequences, and it is closed under extensions.

When $\mathcal{E}$ is an additive ∞-category endowed with the split exact structure, the stable envelope $\mathrm{Stab}(\mathcal{E})$ is given by the Spanier–Whitehead construction

$$\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) := \mathrm{colim}\left(\mathcal{P}_{\Sigma,f}(\mathcal{E}) \xrightarrow{\Sigma} \mathcal{P}_{\Sigma,f}(\mathcal{E}) \xrightarrow{\Sigma} \cdots\right),$$

where the colimit is taken in Cat_∞ and $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ is the smallest full subcategory of $\mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathrm{An})$, the category of finite-product-preserving functors from $\mathcal{E}^{\mathrm{op}}$ to An, containing $\mathcal{E}$ and closed under finite colimits. In general, the stable envelope of $\mathcal{E}$ is given by looking at the exact sequences $x \to y \to z$ in $\mathcal{E}$ and inverting $y/x \to z$ inside $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$. Alternatively, inverting the morphisms $y/x \to z$ in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ gives a prestable ∞-category, which will be denoted $\mathrm{Stab}(\mathcal{E})_{\geq 0}$. As suggested by the notation, stabilizing $\mathrm{Stab}(\mathcal{E})_{\geq 0}$ also gives a stable envelope for $\mathcal{E}$. More precisely, the canonical map

$$\mathcal{SW}(\mathrm{Stab}(\mathcal{E})_{\geq 0}) \to \mathrm{Stab}(\mathcal{E}) \tag{25}$$

is an equivalence for every exact ∞-category $\mathcal{E}$, see [Kle20, 3.7].

Remark 2.1. Let $\mathcal{A}$ be an additive ∞-category. The category $\mathcal{P}_{\Sigma,f}(\mathcal{A})$ defined in the previous paragraph comes with a filtration by exact ∞-categories

$$\mathcal{A}_{[0,0]} \subset \mathcal{A}_{[0,1]} \subset \cdots \subset \mathcal{A}_{[0,n]} \subset \cdots \subset \mathcal{P}_{\Sigma,f}$$

defined recursively as follows. The subcategory $\mathcal{A}_{[0,0]}$ is given by the essential image of the Yoneda embedding $\mathcal{A} \to \mathcal{P}_{\Sigma,f}(\mathcal{A})$. Then, inductively, for each $n \geq 0$ the ∞-category $\mathcal{A}_{[0,n]} \subset \mathcal{P}_{\Sigma,f}(\mathcal{A})$ is given by the full subcategory spanned by those objects $X \in \mathcal{P}_{\Sigma,f}(\mathcal{A})$ that fit in a cofiber sequence

$$X_{n-1} \to X \to \Sigma^n Y$$

where $X_{n-1} \in \mathcal{A}_{[0,n-1]}$ and $Y \in \mathcal{A}_{[0,0]}$. The category $\mathcal{A}_{[0,n]}$ is then regarded as an exact ∞-category by declaring a sequence exact if it is taken to an exact sequence via the canonical inclusion $\mathcal{A}_{[0,n]} \subset \mathrm{Stab}(\mathcal{A})$. More generally, for any finite interval $[a,b] \subset \mathbb{Z}$ we consider $\mathcal{A}_{[a,b]} := \Sigma^a \mathcal{A}_{[0,b-a]} \subset \mathrm{Stab}(\mathcal{A})$. These subcategories are such that $\mathcal{A}_I \subset \mathcal{A}_J$ if $I \subset J$, and

$$\mathrm{colim}_{I \subset \mathbb{Z}} \mathcal{A}_I = \mathrm{Stab}(\mathcal{A}),$$

where the colimit runs over the finite intervals, ordered by inclusion. Note that the suspension functor $\Sigma\colon \mathrm{Stab}(\mathcal{A}) \to \mathrm{Stab}(\mathcal{A})$ induces exact functors $\mathcal{A}_{[a,b]} \to \mathcal{A}_{[a,b+1]}$, which we will also denote by Σ. More generally:

Lemma 2.2. *Let $F\colon K \to \mathrm{Stab}(\mathcal{A})$ be a map from a finite simplicial set of dimension $d \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ be natural numbers such that $F(k) \in \mathcal{A}_{[n,m]}$ for every $k \in K$. Then $\varprojlim F \in \mathcal{A}_{[n-d,m]}$.*

Lemma 2.3. *Let $\mathcal{D}$ be a stable ∞-category and let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory. Suppose that*

1. *For every $x, y \in \mathcal{A}$ the mapping spectrum $\mathrm{Map}_{\mathcal{D}}(x,y)$ is connective.*
2. *The smallest stable subcategory of $\mathcal{D}$ containing $\mathcal{A}$ is $\mathcal{D}$ itself.*

Then the inclusion $\mathcal{A} \subset \mathcal{D}$ exhibits $\mathcal{D}$ as the stable envelope of the split-exact ∞-category $\mathcal{A}$.

Proof. The first condition on the statement ensures that the inclusion $\mathcal{A} \subset \mathcal{D}$ is an exact functor of exact ∞-categories. The universal property of the stable envelope then gives an exact functor $\alpha\colon \mathrm{Stab}(\mathcal{A}) \to \mathcal{D}$ between stable ∞-categories. The rest of this proof shows that this functor is fully faithful and essentially surjective. Let $X \in \mathcal{A}$ and let S_X denote the collection of objects of $\mathrm{Stab}(\mathcal{A})$ for which the transformation

$$\beta_{-,X}\colon \mathrm{Map}_{\mathrm{Stab}(\mathcal{A})}(-,X) \to \mathrm{Map}_{\mathcal{D}}(\alpha(-),\alpha(X))$$

evaluates to an equivalence. As $\mathcal{A}$ maps fully faithfully to both $\mathrm{Stab}(\mathcal{A})$ and $\mathcal{D}$, $\beta_{C,X}$ induces an equivalence on connective covers for every $C \in \mathcal{A}$. Now the first condition on the statement (and a similar condition for the stable envelope of an additive ∞-category) imply that $\beta_{C,X}$ is an equivalence of spectra for every $C \in \mathcal{A}$. Then $\mathcal{C} \subset S_X$, S_X is stable and the inclusion $S_X \subset \mathrm{Stab}(\mathcal{A})$ is exact, which forces $S_X = \mathrm{Stab}(\mathcal{A})$. A similar argument letting X vary gives that $\alpha\colon \mathrm{Stab}(\mathcal{A}) \to \mathcal{D}$ is fully faithful on mapping spectra. Fully faithfulness of α on mapping spectra implies that its essential image is stable. As the essential image of α contains $\mathcal{A}$, the second condition ensures that α is essentially surjective, hence an equivalence. □

Having discussed some basic properties of stable envelopes of additive ∞-categories, we now turn to their K-theory. We let

$$K_{\geq 0}\colon \mathrm{Ex}_\infty \to \mathrm{Sp}_{\geq 0}$$

denote the (connective) K-theory functor for exact ∞-categories given by the ∞-categorical Q-construction, see [Bar13]. This is a generalization of Quillen's Q-construction to the setting of exact ∞-categories. We let K denote non-connective K-theory of stable ∞-categories. Viewing a stable ∞-category $\mathcal{C}$ as an exact category, there is an induced map $K_{\geq 0}(\mathcal{C}) \to K(\mathcal{C})$ which exhibits the source as the connective cover of the target. The starting point is the following form of Quillen's resolution theorem.

Lemma 2.4. *[Qui73, Theorem 3]. Let $\mathcal{A}$ be an additive ∞-category and let $I \subset J$ be two finite non-empty subintervals of $\mathbb{Z}$. Then the inclusion $\mathcal{A}_I \subset \mathcal{A}_J$ from Remark 2.1 induces an equivalence*

$$K_{\geq 0}(\mathcal{A}_I) \xrightarrow{\sim} K_{\geq 0}(\mathcal{A}_J).$$

Theorem 2.5. *Let $\mathcal{A}$ be an additive ∞-category. Then the induced map*

$$K_{\geq 0}(\mathcal{A}) \xrightarrow{\sim} K_{\geq 0}(\mathrm{Stab}(\mathcal{A}))$$

is an equivalence.

Proof. This follows from the previous theorem combined with the rewriting

$$\mathrm{colim}_{I \subset \mathbb{Z}} \mathcal{A}_I = \mathrm{Stab}(\mathcal{A})$$

where I runs over finite non-empty intervals, and from the fact that the functor $K_{\geq 0}$ commutes with filtered colimits of exact ∞-categories. □

2.2 Localizing invariants of products of additive ∞-categories

Let I be a small set and let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of small additive ∞-categories indexed by I. Below we show that the map

$$K_{\geq 0}(\prod_{i \in I} \mathcal{A}_i) \longrightarrow \prod_{i \in I} K_{\geq 0}(\mathcal{A}_i)$$

is an equivalence of connective spectra, see Proposition 2.11 and Corollary 2.11.1. As mentioned, versions of this question have already been studied and answered. In [KW19] it was proved that non-connective K-theory commutes with infinite products of stable ∞-categories:

Theorem 2.6. *[KW19, Theorem 1.3] The universal additive invariant $\mathcal{U}_{add} \colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathcal{M}_{add}$ commutes with small products. Moreover, for a family of stable ∞-categories $\{\mathcal{C}_i\}_{i \in I}$ indexed by a small set I the canonical map of non-connective K-theory spectra*

$$K(\prod_{i \in I} \mathcal{C}_i) \to \prod_{i \in I} K(\mathcal{C}_i)$$

is an equivalence.

This was then used in [BKW19, 2.39] to prove that non-connective K-theory commutes with infinite products of *pre*stable ∞-categories, where the argument goes by reducing to the case of stable ∞-categories. Here we deduce the case of additive ∞-categories from the case of prestable ∞-categories. Concretely, we will need the following version of [BKW19, 2.39]:

Proposition 2.7. *Let $\mathcal{A}$ be an additive ∞-category. Let $\mathcal{A}_{[0,\infty)}$ be obtained from $\mathcal{A}$ as in Remark 2.1 and let K denote non-connective K-theory. Then the map*

$$K(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_{[0,\infty)})) \to \prod_{i\in I} K(\mathrm{Stab}(\mathcal{A}_{[0,\infty)}))$$

is an equivalence[17]*. Moreover, the map of universal additive invariants*

$$\mathcal{U}_{add}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_{[0,\infty)})) \to \prod_{i\in I} \mathcal{U}_{add}(\mathrm{Stab}(\mathcal{A}_{[0,\infty)}))$$

is also an equivalence.

Remark 2.8. Note that the two equivalences in Proposition 2.7 do not follow from each other, this is because non-connective K-theory is not corepresented as an additive invariant. For example, for the first to follow from the second it would suffice to know that the canonical functor $\mathcal{M}_{add} \to \mathcal{M}_{loc}$ from additive motives to localizing motives commutes with products. This is true, but out of the scope of this thesis.

Proof. The statement is the claim that the following composite is an equivalence

$$K(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_{[0,\infty)})) \to K(\prod_{i\in I}\mathrm{Stab}(\mathcal{A}_{[0,\infty)})) \xrightarrow{\sim} \prod_{i\in I} K(\mathrm{Stab}(\mathcal{A}_{[0,\infty)}))$$

where the second map is the equivalence given by Theorem 2.6. It is then enough to show that the first map is an equivalence. Rewriting $\mathrm{Stab}(\mathcal{A}_{[0,\infty)})$ as $\mathrm{colim}_{k\in\mathbb{N}}\mathcal{A}_{[-k,\infty)}$, it is enough to prove that the following composite

$$\prod_{i\in I}\mathcal{A}_{[0,\infty)} \longrightarrow \prod_{i\in I}\mathrm{colim}_{k\in\mathbb{N}}\mathcal{A}_{[-k,\infty)} \xrightarrow{\sim} \mathrm{colim}_{f\in\mathbb{N}^I}\prod_{i\in I}\mathcal{A}_{[-f(i),\infty)}, \tag{26}$$

in which the second map is again an equivalence, induces an equivalence upon applying the functor $F := K(\mathrm{Stab}(-))$. Note that the left hand side of (26) is the value of the diagram of the colimit on the right hand side of (26) for $f = 0$. It will then suffice to show that F sends all the transition maps in the colimit of the right hand side of (26) to equivalences. As the poset $\mathbb{N}^I$ is filtered and F commutes with filtered colimits of exact ∞-categories, it will suffice to show that for an $f \in \mathbb{N}^I$ taking values in even numbers the map $F(0 \leq f))$ is an equivalence (as even functions are cofinal). This last claim is a

[17] the arguments on the next lines work for a general prestable ∞-category $\mathcal{C}$ in the place of $\mathcal{A}_{[0,\infty)}$, but we stick to $\mathcal{A}_{[0,\infty)}$ as this is what shows up when analyzing the additive case.

particular case of Lemma 2.9 below with $g : i \mapsto g(i) = \infty$. Moreover, by how the lemma is phrased the same arguments show that

$$\mathcal{U}_{add}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_{[0,\infty)})) \to \mathcal{U}_{add}(\prod_{i\in I}\mathrm{Stab}(\mathcal{A}_{[0,\infty)})) \tag{27}$$

is an equivalence. This combined with Theorem 2.6 give the last claim on the statement. $\square$

Lemma 2.9. *Let I be a set and let $(\mathcal{A}_i)_{i\in I}$ be additive ∞-categories. Let $f\colon I \to 2\mathbb{N}$ be a function taking values on even numbers and let $g\colon I \to \mathbb{N}\cup\{\infty\}$ be such that $f(i) < g(i)$ for each $i \in I$. Then the map*

$$\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i) \xrightarrow{\mathrm{Stab}((\Sigma^{f(i)})_i)} \mathrm{Stab}(\prod_{i\in I}\mathcal{A}_{i,[-f(i),g(i))})$$

agrees with the canonical inclusion in degree zero after applying $\mathcal{U}_{add}$. The same holds for the map $\mathrm{Stab}((\Sigma^{-f(i)})_i)$.

Proof. This is [BKW19, 2.39], but in this generality. We repeat the argument here for completeness. Consider the following two sequences of exact functors from the additive ∞-category $\mathcal{A}_i$ to the exact ∞-category $\mathcal{A}_{i,[-f(i),g(i))}$

$$\bigoplus_{0\le k<f(i)}\Sigma^k \xrightarrow{\mathrm{id}\oplus 0\oplus\mathrm{id}\cdots\oplus 0} \bigoplus_{0\le k<f(i)}\Sigma^k \longrightarrow \bigoplus_{0<k\le f(i)}\Sigma^k$$

$$\Sigma^0 \oplus \bigoplus_{0\ <k<f(i)}\Sigma^k \xrightarrow{0\oplus\mathrm{id}\oplus\cdots\oplus 0} \Sigma^{f(i)} \oplus \bigoplus_{0<k<f(i)}\Sigma^k \longrightarrow \bigoplus_{0<k\le f(i)}\Sigma^k$$

which are well defined and point-wise exact, hence exact after applying $\mathrm{Stab}(\prod_{i\in I}(-))$ everywhere. Noting that, as the sequences only differ on the middle term, additivity gives

$$\mathrm{Stab}(\prod_{i\in I}\Sigma^0) \cong \mathrm{Stab}(\prod_{i\in I}\Sigma^{f(i)}).$$

$\square$

What is proved here is the next two results. The first result, Proposition 2.10, relies only on Lemma 2.9, and it doesn't need any commutation of K-theory with infinite products, but it also doesn't imply it. The second result, Proposition 2.11, shows commutation of K-theory with infinite products of additive ∞-categories, and it relies on Proposition 2.7.

Proposition 2.10. *Let I be a small set and let $(\mathcal{A}_i)_{i\in I}$ be a family of additive ∞-categories. Then the canonical map*

$$\mathcal{U}_{loc}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i)) \longrightarrow \mathcal{U}_{loc}(\prod_{i\in I}\mathrm{Stab}(\mathcal{A}_i))$$

is an equivalence.

Proof. Recall that $\mathrm{Stab}(\mathcal{A}) = \mathrm{colim}_{n\in\mathbb{N}} \mathcal{A}_{[-n,n]}$. This lets us write the infinite product appearing on the right hand side of the statement as

$$\prod_{i\in I} \mathrm{Stab}(\mathcal{A}_i) \cong \mathrm{colim}_{f\in\mathbb{N}^I} \prod_{i\in I} \mathcal{A}_{i,[-f(i),f(i)]}$$

where the colimit runs over the poset of functions $f\colon I \to \mathbb{N}$, the order relation is given by $f \leq g$ iff $f(i) \leq g(i)$ for all $i \in I$, and the maps in the colimit are the canonical inclusions. Applying $\mathrm{Stab}(-)$ on both sides of the last equation gives

$$\prod_{i\in I} \mathrm{Stab}(\mathcal{A}_i) \cong \mathrm{colim}_{f\in\mathbb{N}^I} \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[-f(i),f(i)]}\big) \tag{28}$$

and under this equivalences the map on the statement is induced by the structure map of the colimit for the function $0 \in \mathbb{N}^I$. As the poset $\mathbb{N}^I$ filtered, it suffices to prove that for any $f \in \mathbb{N}^I$ the map induced by $0 \leq f$ is an equivalence after applying $\mathcal{U}_{loc}$. Consider the exact functors $\Sigma^{f(i)}\colon \mathcal{A}_i \longrightarrow \mathcal{A}_{i,[-f(i),f(i)]}$ given by including each $\mathcal{A}_i$ in degree $f(i)$. These assemble into a functor

$$\mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_i\big) \xrightarrow{\mathrm{Stab}((\Sigma^{f(i)})_i)} \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[-f(i),f(i)]}\big)$$

which by Lemma 2.9[18] coincides with the map induced by $0 \leq f$ after applying the functor $\mathcal{U}_{loc}$ (this is immediate if f is bounded, by additivity). Then, after re-indexing, it suffices to show that the functor

$$\mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_i\big) \xrightarrow{\mathrm{Stab}((\Sigma^{f(i)})_i)} \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[0,f(i)]}\big) \tag{29}$$

is sent to an equivalence by $\mathcal{U}_{loc}$. As this functor is fully faithful, it suffices to show that its cofiber is sent to zero by $\mathcal{U}_{loc}$. Consider the two cofiber sequences in $\mathrm{Cat}^{\mathrm{st}}_\infty$

$$\mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_i\big) \xrightarrow{\mathrm{Stab}((\Sigma^{f(i)})_i)} \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[0,f(i)]}\big) \longrightarrow \mathcal{C} \tag{30}$$

$$\mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[f(i),\infty)}\big) \xrightarrow{\mathrm{can}} \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[0,\infty)}\big) \longrightarrow \mathcal{D}$$

where the functor on the upper left is $\mathrm{Stab}((\Sigma^{f(i)})_i)$ from (29), and the one on the lower left is the canonical inclusion. As by definition these sequences become Verdier sequences after idempotent completion, the functor on the upper left is sent to an equivalence by $\mathcal{U}_{loc}$ if and only if $\mathcal{C}$ is sent to zero. The rest of the proof shows that this is the case by showing that $\mathcal{C}$ is equivalent to $\mathcal{D}$ and that $\mathcal{U}_{loc}(\mathcal{D}) \cong 0$. The fact that $\mathcal{U}_{loc}(\mathcal{D}) \cong 0$ can be proved in two ways: one is to apply the second eqation in Proposition 2.7 to the source and target of the map can of which $\mathcal{D}$ is the cofiber, this gives the claim immediately,

[18]the lemma works for even f, here we can suppose that f is even by cofinality.

but it used the fact that $\mathcal{U}_{add}$ commutes with infinite products. The other way to see that $\mathcal{U}_{loc}(\mathcal{D}) \cong 0$ is by applying Lemma 2.9 to both maps from $\mathrm{Stab}(\prod_i \mathcal{A}_i)$ to the source and target of can to conclude that both maps become equivalences after applying $\mathcal{U}_{add}$, so $\mathcal{U}_{add}(\mathrm{can})$ is an equivalence too. This doesn't use the fact that $\mathcal{U}_{add}$ commutes with infinite products. Either way, this shows that $\mathcal{U}_{loc}(\mathcal{D}) = 0$. Note that there is a induced exact functor $F\colon \mathcal{C} \to \mathcal{D}$. As it remains to show that $\mathcal{U}_{loc}(\mathcal{C})$ is zero, it is enough to show that the functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence after idempotent completion. We start by showing essential surjectivity. Any object of $\mathcal{D}$ is represented by an object $x \in \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[0,\infty)}\big)$. Writing $\mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[0,\infty)}\big) = \mathrm{colim}_{k\in\mathbb{N}} \prod_{i\in I} \mathcal{A}_{i,[-k,\infty)}$, there exists a $k \in \mathbb{N}$ such that $x \in \prod_{i\in I} \mathcal{A}_{i,[-k,\infty)}$. Let $x = (x_i)_{i\in I}$, where each $x_i \in \mathcal{A}_{i,[-k,\infty)}$, and let

$$x_{i,\leq f(i)} \to x_i \to x_{i,>f(i)}$$

be exact sequences in $\mathrm{Stab}(\mathcal{A}_i)$ where $x_{i,\leq f(i)} \in \mathcal{A}_{i,[-k,f(i)]}$ and $x_{i,>f(i)} \in \mathcal{A}_{i,(f(i),\infty)}$, as in the previous proof. Then $x = \big(x_{i,\leq f(i)}\big)_{i\in I}$ in $\mathcal{D}$ and it is easy to see that $\big(x_{i,\leq f(i)}\big)_{i\in I}$ is in the essential image of $\mathcal{C}$, showing that F is essentially surjective. It remains to show that F is fully faithful. Let $x, y \in \mathcal{C}$ be represented by objects $(x_i)_i, (y_i)_i \in \mathrm{Stab}\big(\prod_{i\in I} \mathcal{A}_{i,[0,f(i)]}\big) = \mathrm{colim}_{k\in\mathbb{N}} \prod_{i\in I} \mathcal{A}_{i,[-k,f(i)+k]}$, then mapping spectra in $\mathcal{C}$ can be described as

$$\mathrm{Map}_{\mathcal{C}}(x,y) = \mathrm{colim}_{(z_i)\in\left((x_i)/\mathrm{colim}_k \prod_i \mathcal{A}_{i,[f(i)-k,f(i)+k]}\right)^{\mathrm{op}}} \prod_{i\in I} \mathrm{Map}_{\mathrm{Stab}(\mathcal{A}_i)}(\mathrm{fib}(x_i \to z_i), y_i). \tag{31}$$

Similarly, mapping spectra in $\mathcal{D}$ can be described as

$$\mathrm{Map}_{\mathcal{D}}(F(x),F(y)) = \mathrm{colim}_{(z_i)\in\left((x_i)/\mathrm{colim}_k \prod_i \mathcal{A}_{i,[f(i)-k,\infty)}\right)^{\mathrm{op}}} \prod_{i\in I} \mathrm{Map}_{\mathrm{Stab}(\mathcal{A}_i)}(\mathrm{fib}(x_i \to z_i), y_i). \tag{32}$$

and we want these mapping spectra to be isomorphic. To show this, it suffices to consider the case where x is represented by an object of $\prod_{i\in I} \mathcal{A}_{i,[0,f(i)]}$, as the general case follows by shifting x. We apply Joyal's version of Quillen's Theorem A [Lur09, 4.1.3.1]: let $(z_i) \in \big((x_i)/\mathrm{colim}_k \prod_i \mathcal{A}_{i,[f(i)-k,\infty)}\big)^{\mathrm{op}}$ and let

$$\mathcal{B} := \Big((x_i)/\mathrm{colim}_k \prod_i \mathcal{A}_{i,[f(i)-k,f(i)+k]}\Big)\backslash(z_i).$$

We want $\mathcal{B}$ to be weakly contractible. Let

$$z_{i,\leq f(i)} \to z_i \to z_{i,>f(i)}$$

be exact sequences in $\mathrm{Stab}(\mathcal{A}_i)$ for each $i \in I$, where $z_{i,\leq f(i)} \in \mathcal{A}_{i,[f(i)-k',f(i)]}$ and $z_{i,>f(i)} \in \mathcal{A}_{i,(f(i),\infty)}$, as before. As $x_i \in \mathcal{E}_{[0,f(i)]}$, the map $(x_i)_{i\in I} \to (z_i)_{i\in I}$ factors through $(z_{i,\leq f(i)})_{i\in I}$. As $(z_{i,\leq f(i)})_{i\in I} \in \prod_i \mathcal{A}_{i,[f(i)-k',f(i)+k']}$, this shows that $\mathcal{B}$ is non empty. Let $g\colon K \to \mathcal{B}$ be a map from a finite simplicial set, we want to show that g has a cocone point. The map g determines a map

$$\widetilde{g}\colon \big(K^{\triangleright}\big)^{\triangleleft} \to \prod_{i\in I} \mathrm{Stab}(\mathcal{A}_i)$$

that sends the left cone point to $(x_i) \to (z_i)$ and then the right cone point to (z_i), and sends K to $\mathrm{colim}_k \prod_i \mathcal{A}_{i,[f(i)-k,f(i)+k]}$. The composition $K \xrightarrow{g} \mathcal{B} \to \big((x_i)/\prod_{i\in I}\mathrm{Stab}(\mathcal{A}_i)\big)\backslash(z_i)$ has a limit whose underlying object $(w_i) \in \prod_{i\in I}\mathrm{Stab}(\mathcal{A}_i)$ is given by the limit of the functor $\widetilde{g}_{|K^{\triangleright}}$. As K is a finite simplicial set, there exists a $k \in \mathbb{N}$ such that $\widetilde{g}_{|K^{\triangleright}}$ lands in $\prod_i \mathcal{A}_{i,[f(i)-k,\infty)}$ (the ∞ is because of the cone point (z_i)) and now Lemma 2.2 gives that $(w_i) \in \prod_i \mathcal{A}_{i,[f(i)-k',\infty)}$ for some $k' \in \mathbb{N}$. Note that (w_i) comes with a map $(x_i) \to (w_i)$. Same as when arguing that $\mathcal{B}$ is non empty, the map $(x_i) \to (w_i)$ map factors through an element $(w_{i,\leq f(i)})$ which lies in $\prod_i \mathcal{A}_{i,[f(i)-k',f(i)+k']}$, and this gives a left cone for the original map g. This shows that $\mathcal{B}$ is cofiltered, hence weakly contractible. □

Proposition 2.11. *Let I be a small set and let $(\mathcal{A}_i)_{i\in I}$ be a family of additive ∞-categories. Then the canonical map of non-connective K-theory spectra*

$$K(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i)) \longrightarrow \prod_{i\in I} K(\mathrm{Stab}(\mathcal{A}_i))$$

is an equivalence.

Proof. This is not a corollary of Proposition 2.10, because even if non-connective K-theory becomes representable in $\mathcal{M}_{loc}$ the previous result doesn't show that $\mathcal{U}_{loc}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i))$ is an infinite product in $\mathcal{M}_{loc}$ (it actually is an infinite product, but this won't be showed here). The proof of the current statement goes instead by repeating the previous proof. Using the same notations as in the previous proof, it suffices to show that $K(\mathcal{C}) = 0$. Using that $\mathcal{C} \cong \mathcal{D}$, it suffices to show that $K(\mathcal{D}) = 0$. Equivalently, it suffices to show that $K(\mathrm{can})$ is an equivalence, which follows from Proposition 2.7. □

Corollary 2.11.1. *Let I be a small set and let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of small additive ∞-categories indexed by the set I. Then the map*

$$K_{\geq 0}(\prod_{i\in I}\mathcal{A}_i) \longrightarrow \prod_{i\in I} K_{\geq 0}(\mathcal{A}_i)$$

is an equivalence of spectra.

Proof. Follows from Proposition 2.11 and from Theorem 2.5. □

2.3 Additive invariants of split lax-limits

This section is about two Eilenberg swindle lemmas for additive invariants of certain lax inverse limits of small exact ∞-categories. These are Lemma 2.12 and Lemma 2.13. The latter implies the former, but the former has a slightly simpler proof. I learnt the trick in Lemma 2.12 in a talk by Efimov, and the trick in Lemma 2.13 is just an elaboration. This section ends with two corollaries of Lemma 2.13, which calculate the additive motives of the categories $\mathrm{laxPerf}^b_R$ and Cof^b_0 from Definition 1.37 and Definition 1.67, respectively.

Lemma 2.12. *Let $\{\mathcal{E}_i\}_{i\in\mathbb{N}}$ be a collection of small exact ∞-categories together with exact functors $\mathcal{E}_{i+1} \to \mathcal{E}_i$. Then the canonical projection induces an equivalence*

$$\mathcal{U}_{add}(\mathrm{Stab}(\mathrm{laxlim}_{i\in\mathbb{N}}\mathcal{E}_i)) \xrightarrow{\simeq} \mathcal{U}_{add}(\mathrm{Stab}(\prod_{i\in\mathbb{N}} \mathcal{E}_i)).$$

Proof. Let $G\colon \prod_i \mathcal{E}_i \to \mathrm{laxlim}_i\mathcal{E}_i$ be the functor that sends an object $(x_i \in \mathcal{E}_i)_i$ to itself with vanishing transition maps in the lax-limit. Consider the functor $F\colon \mathrm{laxlim}_i\mathcal{E}_i \to \prod_i \mathcal{E}_i$ in the other direction given by sending an object of the lax limit to its underlying object in the product. Then the composite $F\circ G$ is the identity. We now show that $\mathcal{U}_{add}(\mathrm{Stab}(G\circ F))$ is the identity. Let

$$\widetilde{\oplus}\colon \mathrm{laxlim}_i\mathcal{E}_i \to \mathrm{laxlim}_i\mathcal{E}_i$$

denote the functor given by sending

$$(\cdots x_{n+1} \to x_n \to \cdots \to x_0) \mapsto (\cdots (x_{n+1})^{n+2} \to (x_n)^{n+1} \to \cdots \to x_0),$$

where the map $(x_{n+1})^{n+2} \to (x_n)^{n+1}$ is given by the original structure map in the first $n+1$ entries and by the zero map in the last entry. Let $\eta\colon \widetilde{\oplus} \to \widetilde{\oplus}$ denote the natural transformation given on the n-th entry by the map denoted $\eta_n\colon (x_n)^{n+1} \to (x_n)^{n+1}$ that sends the first n copies to the last n copies, and the last copy to zero. Then it is easy to check that η indeed defines a natural transformation, and that there is an exact sequence of functors

$$\mathrm{Stab}(\widetilde{\oplus}) \xrightarrow{\mathrm{Stab}(\eta)} \mathrm{Stab}(\widetilde{\oplus}) \longrightarrow \mathrm{Id} \oplus \Sigma \circ \mathrm{Stab}(GF)$$

from which $\mathrm{id} = \mathrm{Stab}(GF)$ in $\pi_0\mathrm{End}(\mathcal{U}_{add}(\mathrm{Stab}(\mathrm{laxlim}_{i\in\mathbb{N}}\mathcal{E}_i)))$. $\square$

Lemma 2.13. *Let $\{\mathcal{E}_i\}_{i\in\mathbb{N}}$ be a collection of small exact ∞-categories together with exact functors $f_i\colon \mathcal{E}_i \to \mathcal{E}_{i-1}$ admitting fully faithful and exact right (or left) adjoints $r_i\colon \mathcal{E}_{i-1} \to \mathcal{E}_i$. Let $\mathcal{A}_i = \ker(f_i)$ with its induced exact structure. Suppose that the units $\mathrm{id}_{\mathcal{E}_i} \to r_i \circ f_i$ are point-wise egressive (that is, epimorphisms for the exact structure). Then there exists a canonical exact functor of exact ∞-categories*

$$\prod_{i\in\mathbb{N}} \mathcal{A}_i \to \varprojlim_{i\in\mathbb{N}} \mathcal{E}_i$$

where the exact structures are the canonical ones, which induces an equivalence after applying $\mathcal{U}_{add}(\mathrm{Stab}(-))$. If each $\mathcal{A}_i$ is additive then there is an equivalence

$$\prod_{i\in\mathbb{N}} K(\mathrm{Stab}(\mathcal{A}_i)) \xrightarrow{\simeq} K(\mathrm{Stab}(\varprojlim_{i\in\mathbb{N}} \mathcal{E}_i))$$

where K denotes non-connective K-theory.

Proof. We start by constructing the map on the statement. Let $j < i$. The adjoints $r_i, r_{i+1}, \cdots, r_j$ compose to an exact functor $r_{j,i}\colon \mathcal{E}_j \to \mathcal{E}_i$ which is fully faithful and right adjoint to a similar composition of the f_i's. These functors assemble into fully faithful functors $r_{j,\infty}\colon \mathcal{E}_j \to \varprojlim_{i\in\mathbb{N}} \mathcal{E}_i$. This lets us think in terms of full exact subcategories $\mathcal{A}_j \subset \mathcal{E}_j \overset{r_{j,k}}{\hookrightarrow} \mathcal{E}_k \overset{r_{k,\infty}}{\hookrightarrow} \varprojlim_{i\in\mathbb{N}} \mathcal{E}_i$ whenever $j \leq k$. Assembling the functors $\mathcal{A}_j \to \mathcal{E}_j \xrightarrow{r_{j,i}} \mathcal{E}_i$ for each $j \leq i$ gives a functor $\prod_{j\leq i} \mathcal{A}_j \to \mathcal{E}_i$ which is natural in $i \in \mathbb{N}$. Taking the limit over $i \in \mathbb{N}$ gives an exact functor

$$G\colon \prod_{i\in\mathbb{N}} \mathcal{A}_i \to \varprojlim_{i\in\mathbb{N}} \mathcal{E}_i.$$

as the one appearing in the statement. To give a functor in the other direction it suffices to give for each $j \in \mathbb{N}$ an exact functor from $\varprojlim_{i\in\mathbb{N}} \mathcal{E}_i$ to $\mathrm{Stab}(\varprojlim_i \mathcal{E}_i)$ that lands in $\mathcal{A}_i$, viewed as a full subcategory. Let $x = (f_{i+1}(x_{i+1}) \cong x_i)_i \in \varprojlim_{i\in\mathbb{N}} \mathcal{E}_i$ be an object, where $x_i \in \mathcal{E}_i$, and consider the functor given on objects by

$$x \mapsto \mathrm{fib}\big(r_{j,\infty}(x_j) \to r_{j-1,\infty}(x_{j-1})\big) \cong \mathrm{fib}\big(r_{j,\infty}(x_j) \to r_{j,\infty} r_{j-1,j} f_j(x_j)\big) \in \mathrm{Stab}(\varprojlim_i \mathcal{E}_i)$$

where the fiber is taken stably. This description on objects clearly promotes to a functor, and the hypotheses imply that this functor lands in $\mathcal{A}_j$. As $j \in \mathbb{N}$ varies these functors assemble into an exact functor $F\colon \varprojlim_i \mathcal{E}_i \to \prod_j \mathcal{A}_j$ which is a left inverse to G. Define $x_{>j} := \mathrm{fib}(x \to r_{j,\infty}(x_j))$ (and let $x_{>-1} := x$) and note that there are canonical maps $x_{>j} \to x_{>j-1}$ induced by the maps $r_{j,\infty}(x_j) \to r_{j-1,\infty}(x_{j-1})$, and that

$$\mathrm{fib}\big(r_{j,\infty}(x_j) \to r_{j-1,\infty}(x_{j-1})\big) \cong \mathrm{cofib}\big(x_{>j} \to x_{>j-1}\big)$$

in $\mathrm{Stab}(\varprojlim_i \mathcal{E}_i)$. Now note that the product $\prod_{j\geq i} x_{>j}$ exists in $\varprojlim_i \mathcal{E}_i$ and there are two exact sequences

$$\prod_{j\geq 0} x_{>j} \to \prod_{j\geq -1} x_{>j} \to GF(x) \quad \text{and} \quad \prod_{j\geq 0} x_{>j} \to \prod_{j\geq -1} x_{>j} \to x$$

in $\varprojlim_i \mathcal{E}_i$. Here the first map is induced by the maps $x_{>j} \to x_{>j-1}$ and the third one by the canonical inclusion. Additivity now gives that Id and GF are equivalent after applying $\mathcal{U}_{add}(\mathrm{Stab}(-))$. The last assertion on the statement now follows from the first part and Proposition 2.11. □

Remark 2.14. Let $\mathrm{laxPerf}_R$ denote the lax limit of the functor $n \in \mathbb{N}^{\mathrm{op}} \mapsto \mathrm{Perf}_{R_n}$. The previous Lemma 2.13 combined with Proposition 2.10 give that the canonical map

$$\mathcal{U}_{loc}(\mathrm{laxPerf}_R) \to \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n})$$

is an equivalence. This turns out to hold even after restricting to bounded perfect complexes, see Lemma 3.15 below.

2.4 Cyclic and Hochschild homology of products

As said in the introduction, this section is here to prove the following result.

Proposition 2.15. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of small additive ∞-categories. Then the canonical map*

$$THH(\mathrm{Stab}(\prod_I \mathcal{A}_i)) \longrightarrow \prod_I THH(\mathrm{Stab}(\mathcal{A}_i))$$

is an isomorphism of spectra.

The proof of Proposition 2.15 requires some notation:

Notation 2.16. Let $\mathcal{C}$ be a small stable ∞-category, let $\mathcal{A} \subset \mathcal{C}$ be a full subcategory and let $F\colon \mathcal{C} \to \mathcal{C}$ be an exact endofunctor. Let $(\mathcal{C}, \mathcal{A})^F$ denote the category of pairs (X, η) where $X \in \mathcal{A}$ and $\eta\colon X \to F(X)$ is a morphism in $\mathcal{C}$. More formally, the category $(\mathcal{C}, \mathcal{A})^F$ sits in the pullback

$$\begin{tikzcd}
(\mathcal{C}, \mathcal{A})^F \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "\text{(s,t)}"] \\
\mathcal{A} \arrow[r, "(\text{incl}, F)"] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

where $\mathrm{incl}\colon \mathcal{A} \to \mathcal{C}$ denotes the canonical inclusion. When $\mathcal{A} = \mathcal{C}$, we write $\mathcal{C}^F := (\mathcal{C}, \mathcal{C})^F$.

The additive ∞-category $(\mathcal{C}, \mathcal{A})^F$ will be considered as an exact ∞-category by calling a sequence exact if it is carried to a fiber sequence under the canonical functor $(\mathcal{C}, \mathcal{A})^F \to \mathcal{C}^F$, where the latter category is stable.

Lemma 2.17. *Let $\mathcal{A}$ be a small additive ∞-category and let $n \geq 1$. Then*

1. *Every exact sequence in $(\mathrm{Stab}(\mathcal{A}), \mathcal{A})^{\Sigma^n}$ splits.*
2. *The canonical inclusion $(\mathrm{Stab}(\mathcal{A}), \mathcal{A})^{\Sigma^n} \to \mathrm{Stab}(\mathcal{A})^{\Sigma^n}$ exhibits $\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$ as the stable envelope of the additive ∞-category $(\mathrm{Stab}(\mathcal{A}), \mathcal{A})^{\Sigma^n}$.*

Proof. By the definition of the exact structure of $(\mathrm{Stab}(\mathcal{A}), \mathcal{A})^{\Sigma^n}$, the first point amounts to show that the mapping spectrum between two objects $X', Y' \in (\mathrm{Stab}(\mathcal{A}), \mathcal{A})^{\Sigma^n}$ (calculated in the stable ∞-category $\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$) is connective. Let X and Y denote the underlying objects of $\mathcal{A}$ for X' and Y', so that X' is given by $X \xrightarrow{0} \Sigma^n X$ (here the map has to be the zero map as $n \geq 1$). The same holds for Y' and Y. Then the mapping spectrum $\mathrm{map}_{\mathrm{Stab}(\mathcal{A})^{\Sigma^n}}(X', Y')$ fits in the pullback

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Stab}(\mathcal{A})^{\Sigma^n}}(X', Y') \arrow[r] \arrow[d] & \mathrm{map}_{\mathrm{Fun}(\Delta^1, \mathrm{Stab}(\mathcal{A}))}(X', Y') \arrow[d] \\
\mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(X, Y) \arrow[r] & \mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(X, Y) \times \mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(\Sigma^n X, \Sigma^n Y)
\end{tikzcd}$$

As the spectrum in the lower left is connective, it will suffice to show that the fiber of the left vertical arrow is connective. As the diagram is a pullback, it suffices to show that the fiber of the right vertical arrow is connective. Writing the upper right corner as

$$\mathrm{map}_{\mathrm{Fun}(\Delta^1,\mathrm{Stab}(\mathcal{A}))}(X',Y') = \mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(X,Y)\times_{\mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(X,\Sigma^n Y)}\mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(\Sigma^n X,\Sigma^n Y)$$

we see that the fibers of the vertical arrows on the previous diagram are given by the spectrum $\Sigma^{-1}\mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(X,\Sigma^n Y)$, which is connective because $n\geq 1$. This shows the first point on the statement. For the second point, it suffices to show that both conditions in Lemma 2.3 are satisfied. The first condition is precisely the statement on the previous lines about connectivity of mapping spectra. The second condition says that the smallest stable subcategory of $\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$ containing $(\mathrm{Stab}(\mathcal{A}),\mathcal{A})^{\Sigma^n}$ is the whole of $\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$. Let $(Z\to\Sigma^n Z)\in\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$. Up to a shift, the object Z is in some $\mathcal{A}_{[0,m]}\subset\mathrm{Stab}(\mathcal{A})$ (see Remark 2.1). If $m=0$ then the claim is clear. In general, by induction, Z fits in a fiber sequence

$$Z_0\xrightarrow{f} Z\xrightarrow{g} Z_1$$

where $Z_0\in\mathcal{A}_{[0,m-1]}$ and $Z_1\in\mathcal{A}_{[m,m]}$. The composite $Z_0\xrightarrow{f} Z\to\Sigma^n Z\xrightarrow{g}\Sigma^n Z_1$ vanishes because the mapping spectrum $\mathrm{map}_{\mathrm{Stab}(\mathcal{A})}(Z_0,Z_1)$ is connective and $n\geq 0$. This gives an induced morphism $Z_0\to\Sigma^n Z_0$ in $\mathrm{Stab}(\mathcal{A})$ together with a morphism from $(Z_0\to\Sigma^n Z_0)$ to $(Z\to\Sigma^n Z)$ in $\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$. This induces a map $(Z_1\to\Sigma^n Z_1)$ fitting in a fiber sequence

$$(Z_0\to\Sigma^n Z_0)\to(Z\to\Sigma^n Z)\to(Z_1\to\Sigma^n Z_1)$$

of objects of $\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$. The object $(Z_1\to\Sigma^n Z_1)$ is a shift of an object of $(\mathrm{Stab}(\mathcal{A}),\mathcal{A})^{\Sigma^n}$, so it lies in the smallest stable subcategory generated by $(\mathrm{Stab}(\mathcal{A}),\mathcal{A})^{\Sigma^n}$, and $(Z_0\to\Sigma^n Z_0)$ lies in the smallest stable subcategory generated by $(\mathrm{Stab}(\mathcal{A}),\mathcal{A})^{\Sigma^n}$ by induction on $m\geq 0$. Then the same must hold for $(Z\to\Sigma^n Z)$, which was arbitrary. □

Corollary 2.17.1. *Let $\mathcal{A}$ be a small additive ∞-category and let $n\geq 1$. Then the canonical functor $(\mathrm{Stab}(\mathcal{A}),\mathcal{A})^{\Sigma^n}\to\mathrm{Stab}(\mathcal{A})^{\Sigma^n}$ induces an equivalence*

$$K_{\geq 0}((\mathrm{Stab}(\mathcal{A}),\mathcal{A})^{\Sigma^n})\xrightarrow{\simeq}K_{\geq 0}(\mathrm{Stab}(\mathcal{A})^{\Sigma^n}).$$

Proof. This follows from putting together Theorem 2.5 and Lemma 2.17 □

Remark 2.18. Suppose that $\mathcal{A}$ is the category of finitely generated projective modules over some $\mathbb{E}_1$-ring B. Then $\mathrm{Stab}(\mathcal{A})=\mathrm{Perf}_B$, and the proof of Lemma 2.17 shows that $(\mathrm{Perf}_B)^{\Sigma^n}$ is generated by the single compact generator $B\xrightarrow{0}\Sigma^n B$ (as $n\geq 1$). This implies, using [Lura, 7.1.2.1], that $(\mathrm{Perf}_B)^{\Sigma^n}$ can be identified with the category of perfect complexes over the endomorphism ring of the compact generator $B\xrightarrow{0}\Sigma^n B$. This endomorphism can be computed as in the proof of Lemma 2.17: it is given by the split square-zero extension of B by $\Sigma^{n-1}B$. In a formula, there is an equivalence

$$(\mathrm{Perf}_B)^{\Sigma^n}\cong\mathrm{Perf}_{B\oplus\Sigma^{n-1}B}.$$

Under Notation 2.16, for the next result we let $K_{red}(\mathcal{C}^F) := \mathrm{fib}(K_{\geq 0}(\mathcal{C}^F) \to K_{\geq 0}(\mathcal{C}^0))$, where the map is induced by the natural transformation of functors $F \to 0$.

Theorem 2.19. *Let $\mathcal{A}$ be a small additive ∞-category. There is a functorial equivalence of spectra*

$$\mathrm{colim}_{n\in\mathbb{N}}\Omega^n K_{red}(\mathrm{Stab}(\mathcal{A})^{\Sigma^n}) \xrightarrow{\sim} THH(\mathrm{Stab}(\mathcal{A}))$$

where the notation is as in 2.16. Moreover, the n-th transition map in the previous colimit is $(n+1)$-connective.

Proof. If $\mathcal{A}$ is of the form $\mathrm{Proj}(B)$ for a connective $\mathbb{E}_1$-ring B, then $\mathrm{Perf}(B)^{\Sigma^n} = \mathrm{Perf}_{B\oplus\Sigma^{n-1}B}$ by Remark 2.18. Then the equivalence on the statement is [DM94] and the connectivity bound on the statement is [Ram, 3.2]. The general case reduces to this one. Precisely, let $\mathcal{A}$ be presented as a filtered colimit of additive ∞-categories of the form $\mathrm{Proj}(B)$ for B a connective $\mathbb{E}_1$-ring. Applying the colimit preserving functor $\mathrm{Stab}(-)$ to this gives a presentation of $\mathrm{Stab}(\mathcal{A})$ as a filtered colimit of categories of the form Mod_B, for B a connective $\mathbb{E}_1$-ring. The construction of Notation 2.16 sending a stable ∞-category $\mathcal{C}$ to $\mathcal{C}^{\Sigma^n}$ commutes with filtered colmits, and so does THH, hence the statement reduces to the case $\mathcal{A} = \mathrm{Proj}(B)$. $\square$

Proof of Proposition 2.15. Using Theorem 2.19, it is enough to show that the canonical map

$$K_{\geq 0}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i)^{\Sigma^n}) \longrightarrow \prod_{i\in I} K_{\geq 0}(\mathrm{Stab}(\mathcal{A}_i)^{\Sigma^n}) \tag{33}$$

is an equivalence, as then it is possible to pass to the colimit over n and pull out the product on the right hand side out of the colimit by the connectivity bound from Theorem 2.19. The rest of the proof shows that (33) is an equivalence. Lemma 2.17 together with Theorem 2.5 give an equivalence

$$K_{\geq 0}((\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i), \prod_{i\in I}\mathcal{A}_i)^{\Sigma^n}) \xrightarrow{\sim} K_{\geq 0}((\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i))^{\Sigma^n}) \tag{34}$$

and the category on the source of this last equation can be rewritten as

$$(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i), \prod_{i\in I}\mathcal{A}_i)^{\Sigma^n} = \prod_{i\in I}(\mathrm{Stab}(\mathcal{A}_i), \mathcal{A}_i)^{\Sigma^n}. \tag{35}$$

Note that this rewriting is compatible with exact structures: by Lemma 2.17 exact structures on both sides are split when $n \geq 1$. These facts then fit in a commutative diagram:

$$\begin{tikzcd}
K_{\geq 0}((\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i),\prod_{i\in I}\mathcal{A}_i)^{\Sigma^n}) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & K_{\geq 0}((\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i))^{\Sigma^n}) \arrow[d] \\
K_{\geq 0}(\prod_{i\in I}(\mathrm{Stab}(\mathcal{A}_i),\mathcal{A}_i)^{\Sigma^n}) \arrow[r] \arrow[d, "\simeq"] & K_{\geq 0}(\prod_{i\in I}(\mathrm{Stab}(\mathcal{A}_i))^{\Sigma^n}) \arrow[d, "\simeq"] \\
\prod_{i\in I}K_{\geq 0}((\mathrm{Stab}(\mathcal{A}_i),\mathcal{A}_i)^{\Sigma^n}) \arrow[r, "\simeq"] & \prod_{i\in I}K_{\geq 0}((\mathrm{Stab}(\mathcal{A}_i))^{\Sigma^n})
\end{tikzcd}$$

where the arrows indicated as equivalences are indeed equivalences: the upper horizontal map is an equivalence by (34) prod, the vertical arrow in the upper left corner is an equivalence by (35), the vertical arrow in the lower left is an equivalence because K-theory of additive ∞-categories commutes with products of additive ∞-categories (Corollary 2.11.1), the lower horizontal arrow is an equivalence by the same reason that the upper horizontal is. Finally, the vertical arrow in the lower right is an equivalence because K-theory commutes with products of stable ∞-categories (Theorem 2.6). This shows that the composite of the two vertical arrows on the right is an equivalence, which is what was left to show. $\square$

The next couple of results are condensed versions of the previous lines. Informally, any localizing invariant has a condensed enhancement given by just applying $\mathrm{Cond}(-)$ everywhere. Formally:

Definition 2.20. Let

$$THH^{\mathrm{cd}}\colon \mathrm{Cond}(\mathcal{M}_{loc}) \xrightarrow{\mathrm{Cond}(THH)} \mathrm{Cond}(\mathrm{Cyc}(\mathrm{Sp})) \cong \mathrm{Cyc}(\mathrm{Cond}(\mathrm{Sp}))$$

be a condensed enhancement of THH, where $\mathcal{M}_{loc}$ denotes the category of localizing motives as defined in [BGT13].

For $T = TC, TC^{-}$ or TP, there is a functor

$$T^{\mathrm{cd}}\colon \mathrm{Cyc}(\mathrm{Cond}(\mathrm{Sp})) \to \mathrm{Cond}(\mathrm{Sp})$$

given by the equivalence $\mathrm{Cond}(\mathrm{Cyc}(\mathrm{Sp})) \cong \mathrm{Cyc}(\mathrm{Cond}(\mathrm{Sp}))$ and the functor $\mathrm{Cond}(T)$.

Definition 2.21. For $\mathcal{C}$ a condensed stable ∞-category, let

$$T^{\mathrm{cd}}(\mathcal{C}) := T^{\mathrm{cd}}(THH^{\mathrm{cd}}(\mathcal{C})) \in \mathrm{Cond}(\mathrm{Sp}),$$

where the condensed category $\mathcal{C}$ is confused with its condensed motive.

Corollary 2.21.1. *Let I be a set and let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of* condensed *additive small ∞-categories. Then the canonical map*

$$THH^{\mathrm{cd}}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i)) \longrightarrow \prod_{i\in I} THH^{\mathrm{cd}}(\mathrm{Stab}(\mathcal{A}_i))$$

is an equivalence of condensed *spectra. Moreover, if $I = \mathbb{N}$ and there are exact functors $\mathcal{A}_{i+1} \to \mathcal{A}_i$, then the equivalence holds with $\prod_{i\in I}\mathcal{A}_i$ replaced by* $\mathrm{laxlim}_{i\in\mathbb{N}}\mathcal{A}_i$.

Proof. The first equivalence follows from taking S-valued points, for S an extremally disconnected set, and from everything commuting with finite products. The last claim follows from Lemma 2.12. $\square$

Proposition 2.22. *Let I be a set and let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of* condensed *additive small ∞-categories. Let $T\colon \mathrm{CycSp} \to \mathrm{Sp}$ denote one of $TC, TC^-, (-)_{hS^1}$ or TP. Then the canonical map*

$$T^{\mathrm{cd}}(\mathrm{Stab}(\prod_{i\in I}\mathcal{A}_i)) \xrightarrow{\sim} \prod_{i\in I} T^{\mathrm{cd}}(\mathrm{Stab}(\mathcal{A}_i)).$$

is an equivalence. Moreover, if $I = \mathbb{N}$ and there are exact functors $\mathcal{A}_{i+1} \to \mathcal{A}_i$, then the equivalence holds with $\prod_{i\in I}\mathcal{A}_i$ replaced by $\mathrm{laxlim}_{i\in\mathbb{N}}\mathcal{A}_i$.

Proof. For $T = TC$ or $T = TC^-$ the statement follows from observing that Corollary 2.21.1 is an equivalence of cyclotomic condensed spectra, and then applying T on both sides. For $T = (-)_{hS^1}$ the statement follows as in the previous case by noting that $(-)_{hS^1}$ still commutes with products of (condensed) connective spectra. The case $T = TP$ now follows from the fiber sequence $\Sigma(-)_{hS^1} \to TC^- \to TP$. The last part follows from Lemma 2.12. $\square$

3 K-theory and Hochschild homology of nuclear modules

Let R be an adic ring and let $\cdots R_{n+1} \to R_n \to \cdots \to R_1$ be a tower of $\mathbb{E}_\infty$-rings under R realizing its completion, as in Lemma 1.24. We say that a localizing invariant T satisfies continuity for R if the canonical map

$$T(\mathrm{Nuc}_R) \to \varprojlim_n T(\mathrm{Mod}_{R_n})$$

is an equivalence. This section is about continuity of localizing invariants in the sense of the previous lines. In this direction, Efimov proved the following:

Theorem 3.1 (Efimov)**.** *The canonical map of localizing motives*

$$\mathcal{U}_{loc}(\widetilde{\mathrm{Nuc}}_R) \xrightarrow{\sim} \varprojlim_n \mathcal{U}_{loc}(\mathrm{Mod}_{R_n})$$

is an equivalence, where $\widetilde{\mathrm{Nuc}}_R$ is a certain enlargement of the usual category of nuclear modules over R (see Theorem 3.24).

We will show that $\widetilde{\mathrm{Nuc}}_R$ and Nuc_R have the same localizing motive (Corollary 3.24.1), which gives Efimov's continuity of K-theory. Efimov's results and the techniques of the previous section can be combined to prove the following result.

Theorem (Corollary 3.26.3, Corollary 3.27.1)**.** *The map*

$$THH(\mathrm{Nuc}_R) \to \varprojlim_n THH(R_n)$$

is an equivalence. Moreover, the natural inclusion $\mathrm{Mod}_{R(*)} \to \mathrm{Nuc}_R$ *induces an equivalence*

$$THH(R(*))^{\wedge}_{I} \to THH(\mathrm{Nuc}_R)$$

of $R(*)$*-modules.*

As a partial converse, we show that the continuity of THH (which uses the continuity of K-theory) implies the continuity of TC, which in turn implies the continuity of K-theory. More generally, we show the following, independent of Efimov's results:

Proposition (Proposition 3.14)**.** *Let* $T_1 \to T_2$ *be a map of localizing invariants. Suppose that* T_2/T_1 *is nilinvariant and that* T_1 *and* T_2 *commute with products of additive categories. Then* T_1 *satisfies continuity if and only if* T_2 *does.*

Corollary 2 (Proposition 3.13)**.** *Suppose that* THH *satisfies continuity for* R*. Then the same holds for* K*-theory.*

This section is organized as follows. 3.1 is about trying to compute $THH(\mathrm{Nuc}_R)$ directly, without using the two theorems from this introduction. We do not prove continuity of THH in 3.1, but we show that it is given by a certain relative tensor product, similar to the case of the topological Hoschschild homology of an ordinary ring, but here the tensor product is solid. Precisely, we show that

$$THH(\mathrm{Nuc}_R) = (\widetilde{R} \otimes_{R \otimes^{\blacksquare} R} R)(*)$$

where $\widetilde{R} \in \mathrm{Mod}_{R\otimes^{\blacksquare} R}(\mathrm{Solid})$ is a certain object which agrees with the module $R \in \mathrm{Mod}_{R\otimes^{\blacksquare} R}(\mathrm{Solid})$ (with the diagonal action) modulo an ideal of definition (Corollary 3.11.1). We believe that $\widetilde{R}$ is just R. 3.2 is about deducing continuity of K-theory from continuity of THH. In 3.3 we explain results of Efimov concerning the continuity of K-theory and how they imply the continuty of Hochschild homology. 3.3 then ends by showing that THH of nuclear modules over an adic ring R agrees with the I-completion of the topological Hochschild homology of the underlying $\mathbb{E}_\infty$-ring $R(*)$ (Corollary 3.27.1). The section ends with Section 3.4, which is a sort of example of the results of this section. It is showed that the previous permit a rephrasing of some results of [BMS18] in terms of nuclear modules. This rephrasing makes it possible to pass to the generic fiber at the level of categories without forgetting the topology: for example, given a perfectoid ring R with its p-adic topology, there is an isomorphism of graded rings

$$TP_*(\mathrm{Nuc}_{R[1/p]}) \cong B^+_{dR}(R)[\sigma^{\pm 1}]$$

where σ lives in degree two, see Lemma 3.31.

3.1 Remarks about the Hochschild homology of nuclear modules

This subsection is about giving an expression for the topological Hochschild homology of nuclear modules. We show that

$$THH(\mathrm{Nuc}_R) = M \otimes_{R\otimes^{\blacksquare} R} R\,(*)$$

where $M \in \mathrm{Mod}_{R\otimes_{\blacksquare}R}(\mathrm{Solid})$ is the module obtained by starting with $R \in \mathrm{Mod}_{R\otimes_{\blacksquare}R}(\mathrm{Solid})$ and considering the source of the counit at R of the adjunction whose left adjoint is the map

$$\mathrm{Mod}_{R\boxtimes R}(\mathrm{Solid} \otimes \mathrm{Solid}) \to \mathrm{Mod}_{R\otimes_{\blacksquare}R}(\mathrm{Solid})$$

induced by the solid tensor product (Proposition 3.11). We show that M agrees with R modulo an ideal of definition of R, which implies that the canonical map of $R(*)$-modules

$$THH(R(*)) \to THH(\mathrm{Nuc}_R)$$

is an equivalence modulo the ideal I, see Corollary 3.11.1.

Notation 3.2. In this subsection R stands for an adic ring in the sense of Definition 1.21, and Nuc_R stands for $\mathrm{Nuc}(\mathrm{Solid}_R)$, where the $\mathrm{Nuc}(-)$ construction is the one from Definition 1.2. Recall that an adic ring is always solid as a condensed spectrum.

Remark 3.3. For the duration of this subsection the category Solid will be treated as presentable compactly generated by restricting the cardinality of the profinite sets giving rise to compact objects. This does not change the category Nuc_R, see Lemma 1.35.

Remark 3.4. The category Nuc_R is rigid in the sense of [GR17, 1.9], and this implies that the spectrum $THH(\mathrm{Nuc}_R)$ is given by the image of R under the composite

$$R \in \mathrm{Nuc}_R \xrightarrow{m_r} \mathrm{Nuc}_R \otimes \mathrm{Nuc}_R \xrightarrow{m} \mathrm{Nuc}_R \xrightarrow{(-)(*)} \mathrm{Sp}$$

where the second functor is the multiplication map for Nuc_R, the first is its right adjoint, and the third functor is evaluation of a condensed object at a point. The tensor product of Nuc_R with itself can be understood as follows:

Lemma 3.5. *The square*

$$\begin{tikzcd}
\mathrm{Nuc}_R \otimes \mathrm{Nuc}_R \arrow[r] \arrow[d] & \mathrm{Nuc}_R \otimes \mathrm{Solid}_R \arrow[d] \\
\mathrm{Solid}_R \otimes \mathrm{Nuc}_R \arrow[r] & \mathrm{Solid}_R \otimes \mathrm{Solid}_R
\end{tikzcd}$$

is a pullback square, and all functors are fully faithful. In particular, an object of $\mathrm{Solid}_R \otimes \mathrm{Solid}_R$ *lies in the subcategory* $\mathrm{Nuc}_R \otimes \mathrm{Nuc}_R$ *if it lies both in* $\mathrm{Nuc}_R \otimes \mathrm{Solid}_R$ *and in* $\mathrm{Solid}_R \otimes \mathrm{Nuc}_R$.

Proof. Follows from the next lemma. □

Lemma 3.6. *Let $L\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful colimit preserving functor between dualizable stable ∞-categories. Suppose that the right adjoint of L preserves colimits. Then the diagram*

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \arrow[r, "L\otimes id"] \arrow[d, "id\otimes L"] & \mathcal{D} \otimes \mathcal{C} \arrow[d, "id\otimes L"] \\
\mathcal{C} \otimes \mathcal{D} \arrow[r, "L\otimes id"] & \mathcal{D} \otimes \mathcal{D}
\end{tikzcd}$$

is a pullback square of presentable ∞-categories, and all the functors on the diagram are fully faithful.

Proof. The left vertical arrow can be rewritten as

$$\mathrm{LFun}(\mathcal{D}^\vee, \mathcal{C}) \hookrightarrow \mathrm{LFun}(\mathcal{D}^\vee, \mathcal{C})$$

showing that it is fully faithful. The same argument shows that the lower horizontal arrow is fully faithful (after twisting the factors). Similarly, the lower horizontal arrow can be written as

$$\mathrm{LFun}(\mathcal{C}^\vee, \mathcal{D}) \hookrightarrow \mathrm{LFun}(\mathcal{D}^\vee, \mathcal{D})$$

and the pullback of the diagram on the statement is given by the full subcategory of $\mathrm{LFun}(\mathcal{C}^\vee, \mathcal{D})$ spanned by those functors $F\colon \mathcal{C}^\vee \to \mathcal{D}$ such that the composite $F \circ L^\vee \colon \mathcal{D}^\vee \to \mathcal{D}$ lands in $\mathcal{C}$. Let $R\colon \mathcal{D} \to \mathcal{C}$ be the right adjoint of L. If $F \circ L^\vee$ lands in $\mathcal{C}$, then $F \circ L^\vee \circ R^\vee \cong F$ lands in $\mathcal{C}$ too, so the pullback is given by $\mathrm{LFun}(\mathcal{C}^\vee, \mathcal{C})$. $\square$

Using the previous lemma it is now possible to show that $THH(\mathrm{Nuc}_R)$ can be computed in Solid_R:

Lemma 3.7. *Let $M\colon \mathrm{Solid}_R \otimes \mathrm{Solid}_R \to \mathrm{Solid}_R$ denote the multiplication map for the symmetric monoidal category Solid_R, and let M_r denote its right adjoint. Let m denote the analogous map for Nuc_R, as in Remark 3.4. Then*

$$m \circ m_r(R) \cong M \circ M_r(R)$$

Proof. Consider the following diagram

$$\begin{tikzcd} \mathrm{Nuc}_R \otimes \mathrm{Nuc}_R \arrow[r, "i_0"] \arrow[d, "m"] & \mathrm{Solid}_R \otimes \mathrm{Solid}_R \arrow[d, "M"] \\ \mathrm{Nuc}_R \arrow[r, "i_1"] & \mathrm{Solid}_R \end{tikzcd}$$

and let $i_{0,r}$ denote the right adjoint of the functor i_0 on the upper row. Then it follows formally from the fact that the vertical maps are fully faithful that

$$i_1 \circ m \circ m_r \cong M \circ i_0 \circ i_{0,r} \circ M_r \circ i_1,$$

and the statement reduces to show that the object $M_r(R) = M_r \circ i_1(R)$ lies in $\mathrm{Nuc}_R \otimes \mathrm{Nuc}_R$. Let $F := M_r(R) \in \mathrm{Solid}_R \otimes \mathrm{Solid}_R$. By Lemma 3.5 it suffices to show that F lies in $\mathrm{Solid}_R \otimes \mathrm{Nuc}_R$. Let $\mathcal{E} := \mathrm{Solid}_R^\omega$ denote the full subcategory spanned by compact objects inside Solid_R. There is an equivalence

$$\mathrm{Solid}_R \otimes \mathrm{Solid}_R = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Solid}_R) = \mathrm{Fun}'(\mathcal{E}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$$

where the right hand side denotes those functors that preserve finite limits separately in each variable. Under this equivalence, a functor G on the right hand side lies in $\mathrm{Solid}_R \otimes \mathrm{Nuc}_R$ on the left hand side if for every profinite set S it holds that

$$G(S,-)^{\mathrm{tr}} = G(S,-)$$

as objects of Solid_R, where $S \in \mathcal{E}$ really refers to the compact object $R \otimes^{\blacksquare} \mathbb{S}[S] \in \mathcal{E}$. This proof cares about the case $G = F$. Let $S = \varprojlim_{i\in I} S_i$ and $T = \varprojlim_{i\in I} T_i$ be two profinite sets, presented over the same cofiltered diagram (this is always possible). Unraveling the definitions, the functor F is given by

$$F(S,T) = \mathrm{Hom}_{\mathrm{Solid}_{R\otimes^{\blacksquare} R}}(R \otimes^{\blacksquare} \mathbb{S}[S] \otimes^{\blacksquare} R \otimes^{\blacksquare} \mathbb{S}[T], R) = (\mathrm{colim}_{i\in I} \bigoplus_{S_i \times T_i} R(*))^{\wedge}_{I}$$

where the $R(*) \otimes R(*)$-module structure in the last term is diagonal, so the I-completion is also a completion with respect to the ideal (I, I) of $R(*) \otimes R(*)$. In particular, the object $F(S,-) \in \mathrm{Solid}_R$ is I-complete. Similarly,

$$F(S,-)^{\mathrm{tr}}(T) = ((\mathrm{colim}_{i\in I} \bigoplus_{T_i} R(*))^{\wedge}_{I} \otimes_{\mathrm{Solid}_R} F(S,-))(*)$$

which as $F(S,-)$ is I-complete can be rewritten using Remark 1.31 as

$$F(S,-)^{\mathrm{tr}}(T) = (\mathrm{colim}_{i\in I} \bigoplus_{T_i} R(*) \otimes_{\mathrm{Solid}_R} F(S,*))^{\wedge}_{I}$$

which is the same as $F(S,T)$. This shows that F is in $\mathrm{Solid}_R \otimes \mathrm{Nuc}_R$. □

Remark 3.8. Recall that Solid_R is just notation for $\mathrm{Mod}_R(\mathrm{Solid})$. Similarly, let $\mathrm{Solid}_{R\otimes^{\blacksquare} R}$ denote $\mathrm{Mod}_{R\otimes^{\blacksquare} R}(\mathrm{Solid})$. For the next lemma, note that the multiplication functor M of the previous lemma factors as a composite

$$M\colon \mathrm{Solid}_R \otimes \mathrm{Solid}_R \xrightarrow{M'} \mathrm{Solid}_{R\otimes^{\blacksquare} R} \to \mathrm{Solid}_R$$

where the first map is induced by the two algebra maps of the form $R \to R \otimes^{\blacksquare} R$, and the second map is induced by the map $R \otimes^{\blacksquare} R \to R$ of algebras in solid spectra.

Lemma 3.9. *Let $F \in \mathrm{Solid}_R \otimes \mathrm{Solid}_R$ denote the image of the diagonal module $R \in \mathrm{Solid}_{R\otimes^{\blacksquare} R}$ along the right adjoint of the canonical functor*

$$M'\colon \mathrm{Solid}_R \otimes \mathrm{Solid}_R \to \mathrm{Solid}_{R\otimes^{\blacksquare} R}$$

of the previous remark. Then the counit map $M'(F) \to R$ is an equivalence modulo $(I, I) \subset R \otimes^{\blacksquare} R$[19].

[19] that is, the ideal generated by the two copies of I

Proof. Using that $\mathrm{Solid}_R = \mathrm{Mod}_R(\mathrm{Solid})$, the source of the functor M' can be rewritten as

$$\mathrm{Solid}_R \otimes \mathrm{Solid}_R \cong \mathrm{Mod}_{R \boxtimes R}(\mathrm{Solid} \otimes \mathrm{Solid})$$

from which it is clear that the functor M' on the statement is the functor induced by the multiplication map $m' \colon \mathrm{Solid} \otimes \mathrm{Solid} \to \mathrm{Solid}$ for the symmetric monoidal structure on Solid after passing to $R \boxtimes R$-modules on the source and $m'(R \boxtimes R) = R \otimes^{\blacksquare} R$-modules on the target. Consider the diagram

$$\begin{tikzcd}
\mathrm{Mod}_{R\boxtimes R}(\mathrm{Solid} \otimes \mathrm{Solid}) \arrow[r, "i_0"] \arrow[d, "M'"] & \mathrm{Solid} \otimes \mathrm{Solid} \arrow[d, "m'"] \\
\mathrm{Mod}_{R\otimes^{\blacksquare} R}(\mathrm{Solid}) \arrow[r, "i_1"] & \mathrm{Solid}
\end{tikzcd}$$

which commutes by the first assertion in Lemma 3.10 below. Let M'_r and m'_r denote the right adjoints of M' and m', then

$$i_1 \circ M' \circ M'_r \cong m' \circ i_0 \circ M'_r \cong m' \circ m_r \circ i_1$$

where the first equivalence is the comutativity of the previous diagram and the second equivalence is the second assertion in Lemma 3.10 below. As the module R is discrete modulo (I, I), it suffices to check that the counit $M' \circ M_r \to \mathrm{id}$ is an equivalence on discrete modules. As this can be checked after applying i_1, the previous chain of equivalences says that it is enough to check that the counit $m' \circ m'_r(N) \to N$ is an equivalence for every discrete module N (that is, for every spectrum). As both m' and m_r commute with colimits (the latter because the tensor product of Solid preserves compact objects), it suffices to check the claim for $N = \mathbb{S}$. This follows from

$$m' \circ m'_r(\mathbb{S}) \xrightarrow{\sim} THH(\mathrm{Nuc}_{\mathbb{S}}) \xleftarrow{\sim} THH(\mathbb{S}) \cong \mathbb{S}$$

where the first equivalence is Lemma 3.7 and the second one is because $\mathrm{Sp} \xrightarrow{\sim} \mathrm{Nuc}_{\mathbb{S}}$. $\square$

Lemma 3.10. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor between symmetric monoidal presentable ∞-categories. Let $A \in \mathrm{CAlg}(\mathcal{C})$ and let $F_A \colon \mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_{F(A)}(\mathcal{D})$ denote the functor induced by F. Then the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathcal{C}) \arrow[r, "res_0"] \arrow[d, "F_A"] & \mathcal{C} \arrow[d, "F"] \\
\mathrm{Mod}_{F(A)}(\mathcal{D}) \arrow[r, "res_1"] & \mathcal{D}
\end{tikzcd}$$

Moreover, if F is symmetric monoidal and G and G_A denote the right adjoints of F and F_A, then the following natural transformation is an equivalence

$$\mathrm{res}_0 \circ G_A \xrightarrow{\sim} G \circ \mathrm{res}_1 .$$

Proposition 3.11. *There is a canonical map of $R(*)$-modules*

$$THH(\mathrm{Nuc}_R) \to (R \otimes^{\blacksquare}_{R\otimes^{\blacksquare} R} R)(*)$$

which is an equivalence modulo I.

Proof. Using notations from Lemma 3.9, the map on the statement is given by the map

$$THH(\mathrm{Nuc}_R) = (M'(F) \otimes_{R\otimes^{\blacksquare} R} R)(*) \to (R \otimes_{R\otimes^{\blacksquare} R} R)(*)$$

induced by the counit $M'(F) \to R$ of Lemma 3.9. This map is an equivalence modulo I by the same lemma, and this concludes the proof of the statement. $\square$

From this, the I-adic continuity of THH follows:

Corollary 3.11.1. *The inclusion $\mathrm{Mod}_{R(*)} \to \mathrm{Nuc}_R$ induces an equivalence*

$$THH(R(*))^{\wedge}_I \to THH(\mathrm{Nuc}_R)^{\wedge}_I$$

of $R(*)$-modules.

Proof. This follows from Proposition 3.11 by noting that $THH(R(*))$ and $R \otimes^{\blacksquare}_{R\otimes^{\blacksquare} R} R$ agree modulo the ideal I. $\square$

Remark 3.12. Assuming continuity of K-theory, Corollary 3.11.1 can be improved by noting that $THH(\mathrm{Nuc}_R)$ is already I-complete. See Corollary 3.27.1.

3.2 Continuity of Hochschild homology implies continuity of K-theory

In this section we prove that continuity of topological Hochschild homology implies continuity of K-theory.

Proposition 3.13. *Let R be an adic ring in the sense of Definition 1.21, and let $(R_n)_{n\in\mathbb{N}}$ be a tower of $\mathbb{E}_\infty$-rings realizing its completion, as in Lemma 1.24. Suppose that the canonical map*

$$THH(\mathrm{Nuc}_R) \to \varprojlim_{n\in\mathbb{N}} THH(\mathrm{Mod}_{R_n})$$

is an equivalence. Then the analog maps for TC and K-theory are also equivalences.

The proof does not require more notation than the one that has already been introduced, so it is given now, but it relies on a couple of results proved in Section 3.2.1 and Proposition 3.17 below.

Proof. The proof goes by using the sequence

$$\mathrm{Nuc}_R \xrightarrow{\mathbf{L}} \mathrm{Ind}(\mathrm{laxPerf}^b_R) \xrightarrow{\mathbf{F}} \mathrm{Ind}(\mathrm{Cof}^b_R) \tag{36}$$

produced in Proposition 1.72 (which is however not a localization sequence, by Remark 1.57) and the criteria from Corollary 3.16.1, which says that a localizing invariant which commutes with products of additive ∞-categories satisfies continuity if and only if it sends (36) to a fiber sequence of spectra. As non-connective K-theory commutes with products of additive ∞-categories by Proposition 2.11, to deduce continuity for K-theory it suffices to show that both TC and the fiber of $K \to TC$ send (36) to a fiber sequence of spectra. For the fiber of $K \to TC$ this follows from Proposition 3.17 below, which says that, more generally, any truncating invariant sends (36) to a fiber sequence of spectra. For TC, it commutes with infinite products of additive categories by Proposition 2.22, so it suffices to show that TC satisfies continuity, which follows from the assumption that THH does: for example, by using Corollary 3.16.1 for THH in the forward direction to deduce that TC sends the sequence in there to a fiber sequence, and then use it for TC in the backward direction, which is possible by Proposition 2.22, to deduce that TC satisfies continuity. This concludes the proof. $\square$

The method of the previous proof actually proves:

Proposition 3.14. *Let $T_1 \to T_2$ be a map of localizing invariants. Suppose that T_2/T_1 is nilinvariant and that T_1 and T_2 commute with products of additive categories. Then T_1 satisfies continuity if and only if T_2 does.*

The rest of this section is dedicated to proving the statements used in the proof of Proposition 3.13.

3.2.1 Motives of lax-perfect complexes

Lemma 3.15. *There is a canonical equivalence*

$$\mathcal{U}_{add}(\mathrm{laxPerf}_R^b) \to \mathcal{U}_{add}(\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n}))$$

Moreover, the canonical map

$$\mathcal{U}_{loc}(\mathrm{laxPerf}_R^b) \to \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n})$$

is also an equivalence.

Proof. Let laxVec_R denote the lax limit of the functor

$$n \in \mathbb{N}^{\mathrm{op}} \mapsto \mathrm{Vec}_{R_n}$$

in Cat_∞. We consider this lax limit as an exact category by declaring a sequence to be exact if it is exact at each level. Exact sequences in laxVec_R are level-wise split, but not necessarily split. There are exact and fully faithful inclusions

$$\mathrm{laxVec}_R^s \to \mathrm{laxVec}_R \to \mathrm{laxPerf}_R^b$$

coming from the fact that they are all subcategories of the lax limit of $n \mapsto \mathrm{Perf}_{R_n}$. These inclusion are all exact, and as such they extend to fully faithful functors

$$\mathrm{Stab}(\mathrm{laxVec}^s_R) \to \mathrm{Stab}(\mathrm{laxVec}_R) \to \mathrm{laxPerf}^b_R$$

which are equivalences because the composite is an equivalence (Lemma 1.40). Let $\mathcal{E}_i$ denote the lax limit of the functor

$$n \in \mathbb{N}^{\mathrm{op}}_{\leq i} \mapsto \mathrm{Vec}_{R_n}$$

equipped with the canonical exact structure. Here $\mathbb{N}_{\leq i}$ denotes the finite poset of natural numbers less than or equal to i. Then $\mathrm{laxVec}_R = \varprojlim_i \mathcal{E}_i$ as exact ∞-categories. The collection of exact ∞-categories $(\mathcal{E}_i)_i$ together with the canonical exact functors $\mathcal{E}_i \to \mathcal{E}_{i-1}$ satisfy the hypotheses of Lemma 2.13, which then gives an equivalence

$$\mathcal{U}_{add}(\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n})) \to \mathcal{U}_{add}(\mathrm{Stab}(\varprojlim_{i\in\mathbb{N}} \mathcal{E}_i))$$

which amounts to the first equivalence on the statement. Passing to $\mathcal{U}_{loc}$ and using Proposition 2.10 from the previous section to commute the product with $\mathrm{Stab}(-)$ gives the second equivalence. $\square$

Lemma 3.16. *There is a canonical equivalence*

$$\mathcal{U}_{add}(\mathrm{Cof}^b_R) \to \mathcal{U}_{add}(\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n}))$$

Moreover, the diagonal map

$$\mathcal{U}_{loc}(\mathrm{Cof}^b_R) \to \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n})$$

sending h to $(h(n))_n$ is also an equivalence.

Proof. For each $i \in \mathbb{N}$ let $\mathcal{E}_i := \mathrm{Cof}^0_R \cap \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}_{\leq i}, \mathrm{laxPerf}^b_R)$ inside $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{laxPerf}^b_R)$. That is, $\mathcal{E}_i$ is the category obtained as in Definition 1.68 but where $\mathbb{N}$ is replaced by $\mathbb{N}_{\leq i}$. Then $\mathrm{Cof}^0_R = \varprojlim_{i\in\mathbb{N}} \mathcal{E}_i$. Viewing each $\mathcal{E}_i$ with its canonical exact structure from its inclusion into the stable category $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{laxPerf}^b_R)$, the last equivalence becomes an equivalence of exact categories. In fact, both sides carry the split exact structure (Lemma 1.70). The exact categories $\mathcal{E}_i$ satisfy the hypotheses of Lemma 2.13 (the left and right adjoints are given by Kan extensions along $\mathbb{N}_{\leq i} \to \mathbb{N}$), which then gives an equivalence

$$\mathcal{U}_{add}(\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n})) \to \mathcal{U}_{add}(\mathrm{Stab}(\varprojlim_{i\in\mathbb{N}} \mathcal{E}_i))$$

which amounts to the first equivalence on the statement. Passing to $\mathcal{U}_{loc}$ and using Proposition 2.10 from the previous section to commute the product with $\mathrm{Stab}(-)$ gives the second equivalence. $\square$

Corollary 3.16.1. *Let T be a localizing invariant commuting with products of additive ∞-categories or of small stable ∞-categories. Then T satisfies continuity in the sense of Section 3 if and only if T sends the sequence*

$$\mathrm{Nuc}_R \xrightarrow{\mathbf{L}} \mathrm{Ind}(\mathrm{laxPerf}_R^b) \xrightarrow{\mathbf{F}} \mathrm{Ind}(\mathrm{Cof}_R^b)$$

to a fiber sequence of spectra.

Proof. Using Lemma 3.15 and Lemma 3.16 above, the sequence in the statement goes to

$$\mathcal{U}_{loc}(\mathrm{Nuc}_R) \to \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n}) \to \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n})$$

after applying $\mathcal{U}_{loc}$, where the second arrow is the identity minus the projection map. Moreover, the categories on the middle and on the right of the last equation are also equal to $\mathcal{U}_{loc}(\mathrm{Stab}(\prod_{\mathbb{N}} \mathrm{Vec}_{R_n}))$, again by Lemma 3.15 and Lemma 3.16 above. Then the statement follows from the hypotheses. □

3.2.2 Truncating invariants via lax-perfect modules

This section is here to prove the following result.

Proposition 3.17. *Let E be a truncating invariant*[20]*. Then the sequence*

$$E(\mathrm{Nuc}_R) \to E(\mathrm{laxPerf}_R^b) \to E(\mathrm{Cof}_R^b)$$

is a fiber sequence of spectra.

Corollary 3.17.1. *Let E be a truncating invariant. Suppose that E commutes with infinite products of additive categories. Then the canonical map*

$$E(\mathrm{Nuc}_R) \to \varprojlim_n E(R_n)$$

is an equivalence.

Proof. By Corollary 3.16.1 and Proposition 3.17 above. □

Proposition 3.17 is based on the fact that nilpotent extensions of additive ∞-categories induce equivalences on truncating invariants. The notion of a nilpotent extension of additive ∞-categories is a generalization of the notion of a map of connective rings which is surjective on π_0 and has nilpotent kernel, see [ES21, 3.1.1]. Precisely, an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to be a nilpotent extension of additive ∞ categories if the following two conditions hold:

[20]A localizing invariant E is called *truncating* if for every connctive $\mathbb{E}_1$-ring A the canonical map $A \to \pi_0(A)$ induces an equivalence

$$E(A) \xrightarrow{\sim} E(\pi_0(A))$$

1. f is essentially surjective.
2. For all objects $V, W \in \mathcal{A}$ the map

$$\mathrm{map}_{\mathcal{A}}(V, W) \to \mathrm{map}_{\mathcal{B}}(f(V), f(W))$$

is surjective on π_0.
3. There exists an $n \geq 1$ such that, for every collection of n composable morphisms $f_1, \cdots f_n$ in $\mathcal{A}$, if each f_i vanishes in $\mathcal{B}$ then the composite $f_1 \circ \cdots \circ f_n$ vanishes in $\mathcal{A}$.

Then the following theorem holds, where for an additive ∞-category $\mathcal{A}$ we let

$$\mathrm{Mod}_{\mathcal{A}} := \mathrm{Ind}(\mathrm{Stab}(\mathcal{A})).$$

Theorem 3.18. *[ES21, 4.2.1] Let $\mathcal{A} \to \mathcal{B}$ be a nilpotent extension of additive ∞-categories and let E be a truncating invariant taking values in a stable ∞-category. Then the induced map*

$$E(\mathrm{Mod}_{\mathcal{A}}) \to E(\mathrm{Mod}_{\mathcal{B}})$$

is an equivalence.

Remark 3.19. Let $\mathcal{A}$ be an additive ∞-category. A colimit preserving colocalization $\mathcal{D} \subset \mathrm{Mod}_{\mathcal{A}}$[21] gives rise (and is determined by) an endofunctor $L_{\mathcal{D}} \colon \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{A}}$. In the following we care about colocalizations for which the associated endofuctor sends connective objects to (-1)-connective objects (or just connective), where an object of $\mathrm{Mod}_{\mathcal{A}}$ is connective if it lies in the subcategory generated under small colimits by objects in the essential image of $\mathcal{A}$ in $\mathrm{Mod}_{\mathcal{A}}$ under the canonical functor, and an object is (-1)-connective if it is a negative shift of a connective object. In this case, the cofiber of the inclusion $\mathcal{D} \subset \mathrm{Mod}_{\mathcal{A}}$ is generated by an additive ∞-category. Precisely, letting $\mathcal{B}$ denote the essential image of $\mathcal{A}$ under the projection $p \colon \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{A}}/\mathcal{D}$, regarded as an additive ∞-category, there is an equivalence

$$\mathrm{Mod}_{\mathcal{B}} \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{A}}/\mathcal{D} \tag{37}$$

In fact, the assumption on $L_{\mathcal{D}}$ implies that mapping spectra between objects of $\mathcal{B}$ are connective, and this implies that the functor in (37) is fully faithful, see Lemma 2.3. As it is also essentially surjective, it is an equivalence. Conversely, if $\mathrm{Mod}_{\mathcal{B}} \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{A}}/\mathcal{D}$, then the mapping spectra between objects in $p(\mathcal{A}) = \mathcal{B} \subset \mathrm{Mod}_{\mathcal{A}}/\mathcal{D}$ are connective and the fiber sequence of spectra

$$\mathrm{map}_{\mathrm{Mod}_{\mathcal{A}}}(V, L_{\mathcal{D}}(W)) \to \mathrm{map}_{\mathrm{Mod}_{\mathcal{A}}}(V, W) \to \mathrm{map}_{\mathrm{Mod}_{\mathcal{A}}/\mathcal{D}}(p(V), p(W))$$

where $V, W \in \mathcal{A}$, implies that $L_{\mathcal{D}}$ sends connective objects to -1-connective objects.

[21] that is, a colimit preserving fully faithful functor between presentable ∞-categories with a colimit preserving right adjoint.

The following corollary of Theorem 3.18 is the form in which this theorem will be used in here:

Corollary 3.19.1. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive ∞-categories, and let*

$$F\colon \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{B}}$$

denote the colimit preserving functor induced at the level of presentable ∞-categories. Let $\mathcal{D} \subset \mathrm{fib}(F)$ be a colimit preserving colocalization. Suppose that the following conditions hold:

1. *The endofunctor associated to the induced colocalization $\mathcal{D} \subset \mathrm{Mod}_{\mathcal{A}}$ sends connective objects to (-1)-connective objects.*
2. *f is essentially surjective.*
3. *For all objects $V, W \in \mathcal{A}$ the map*

   $$\mathrm{map}_{\mathcal{A}}(V, W) \to \mathrm{map}_{\mathcal{B}}(f(V), f(W))$$

   is surjective on π_0.
4. *Let G denote the right adjoint to F, and let $T := \mathrm{fib}(\mathrm{id} \to \mathrm{GF})$. Suppose that for every $V, W \in \mathcal{A}$ and every morphism $V \xrightarrow{\alpha} T(W)$ in $\mathrm{Ind}(\mathrm{Stab}(\mathcal{A}))$ there exists an $X \in \mathrm{fib}(F)$ and a commutative diagram*

   $$\begin{tikzcd} T(V) \arrow[r] \arrow[d, "can"] & X \arrow[d] \\ V \arrow[r, "\alpha"] & T(W) \end{tikzcd}$$

Then the induced map $\mathrm{Mod}_{\mathcal{A}}/\mathcal{D} \to \mathrm{Mod}_{\mathcal{B}}$ comes from a nilpotent extension of additive ∞-categories. In particular, for every truncating invariant E the sequence

$$E(\mathcal{D}) \to E(\mathrm{Mod}_{\mathcal{A}}) \to E(\mathrm{Mod}_{\mathcal{B}})$$

is a fiber sequence.

Proof. Let $\mathcal{C}$ denote the cofiber of the inclusion $\mathcal{D} \subset \mathrm{Mod}_{\mathcal{A}}$. As in Remark 3.19, the first condition on the statement implies that $\mathcal{C} \simeq \mathrm{Mod}_{\mathcal{B}'}$, where $\mathcal{B}'$ is the additive ∞-category generated by the image of $\mathcal{A}$ in $\mathcal{C}$. The induced map $\mathcal{C} \to \mathrm{Mod}_{\mathcal{B}}$ is induced by a functor $\mathcal{B}' \to \mathcal{B}$, which by the hypotheses on the statement is a nilpotent extension of additive ∞-categories. Then the fiber sequence

$$\mathcal{D} \to \mathrm{Mod}_{\mathcal{A}} \to \mathcal{C}$$

plus the fact that $\mathcal{C}$ and $\mathrm{Mod}_{\mathcal{B}}$ agree on truncating invariants (by Theorem 3.18) give that any truncating invariant sends

$$\mathcal{D} \to \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{B}}$$

to a fiber sequence. $\square$

Proof of Proposition 3.17. The proof goes by applying Corollary 3.19.1 to the case $\mathcal{A} = \mathrm{laxVec}^s_R$ (Definition 1.38) and $\mathcal{B} = \mathrm{Cof}^0_R$ (Definition 1.68), and to the functor $f\colon \mathcal{A} \to \mathcal{B}$ given by the functor $\Sigma^{-1}F_{|\mathrm{laxVec}^s_R}$ from (20) that remembers the successive kernels of a lax-vector bundle with surjective transition maps. Then functor

$$F\colon \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{B}}$$

from 3.19.1 is precisely the functor

$$\mathbf{F}\colon \mathrm{Ind}(\mathrm{laxPerf}^b_R) \to \mathrm{Ind}(\mathrm{Cof}^b_R)$$

from Proposition 1.72. And we take $\mathcal{D}$ to be Nuc_R, which is killed by F. So it suffices to check that the rest of the hypotheses of Corollary 3.19.1 are satisfied. The first condition says that the endofunctor classifying the colocalization $\mathrm{Nuc}_R \subset \mathrm{Ind}(\mathrm{laxPerf}^b_R)$ sends connective objects to (-1)-connective, which by Remark 3.19 is the same as saying that the cofiber is generated by an additive ∞-category. This is Lemma 3.20 below. This shows the first condition. The third condition says that for every $V, W \in \mathrm{laxVec}^s_R$ the map

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V, W) \to \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(\mathbf{F}(V), \mathbf{F}(W)) = \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V, \mathbf{G} \circ \mathbf{F}(W))$$

is surjective on π_0. As Proposition 1.72 gives a fiber sequence of functors $T \to \mathrm{Id} \to \mathbf{G} \circ \mathbf{F}$, it suffices to show that $\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V, T(W))$ is connective. This is Lemma 3.21 below. This shows the third condition. Finally, the fourth condition is Lemma 3.22. $\square$

Lemma 3.20. *The ∞-category*

$$\mathcal{C} := \mathrm{Ind}(\mathrm{laxPerf}^b_R)/\mathrm{Nuc}_R$$

is generated by an additive ∞-category. Precisely, there exists an additive ∞-category $\mathcal{C}_0 \subset \mathcal{C}^w$ such that $\mathcal{C} = \mathrm{Mod}_{\mathcal{C}_0}$.

Proof. The category $\mathcal{C}$ is compactly generated, and the projection $p\colon \mathrm{Ind}(\mathrm{laxPerf}^b_R) \to \mathcal{C}$ sends compact objects to compact objects. Let $\mathcal{C}_0 \subset \mathcal{C}^w$ denote the essential image of laxVec^s_R under the projection p. We will now prove that the inclusion $\mathcal{C}_0 \subset \mathcal{C}^w$ extends uniquely to equivalence $\mathrm{Stab}(\mathcal{C}_0) \xrightarrow{\sim} \mathcal{C}^w$, where the exact structure on $\mathcal{C}_0$ is given by the split exact sequences. This can be proved by showing the two conditions in Lemma 2.3. The first condition says that for every $V, W \in \mathrm{laxVec}^s_R$ the mapping spectra $\mathrm{map}_{\mathcal{C}}(p(V), p(W))$ is connective. This spectrum fits in a fiber sequence

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V, \mathbf{LR}(W)) \to \mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V, W) \to \mathrm{map}_{\mathcal{C}}(V, W)$$

from which it suffices to show that the leftmost and the middle term are connective, as the cofiber of two connective objects is connetive. The middle term is connective by Lemma 1.40. For the leftmost term, note that object $\mathbf{R}(W)$ is connective and nuclear, so Lemma 1.16 gives that it is a small colimit of objects in $\mathrm{BNuc}_0(\mathrm{Solid}_R)$. As the functor

$\mathbf{L}$ sends $\mathrm{BNuc}_0(\mathrm{Solid}_R)$ to connective objects (even to $\mathrm{Ind}(\mathrm{laxVec}^{\mathrm{s}}_R)$), the object $\mathbf{LR}(W)$ is a small colimit of objects in $\mathrm{laxVec}^{\mathrm{s}}_R$ in degree zero. Then, as V is compact projective, $\mathrm{map}(V, \mathbf{LR}(W))$ is a colimit of connective objects, hence the connectivity. The second condition amounts to show that the fully faithful inclusion $\mathrm{Stab}(\mathcal{C}_0) \hookrightarrow \mathcal{C}^w$ is essentially surjective. This follows because this inclusion factors the projection

$$p^w\colon \mathrm{laxPerf}^b_R = \mathrm{Stab}(\mathrm{laxVec}^{\mathrm{s}}_R) \to \mathrm{Stab}(\mathcal{C}_0) \hookrightarrow \mathcal{C}^w$$

and p^w is essentially surjective. □

Lemma 3.21. *Let $V, W \in \mathrm{laxVec}^{\mathrm{s}}_R$. Then the spectrum $\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}^b_R)}(V, T(W))$ is connective.*

Proof. As in (13), it is possible to compute $\mathrm{map}(V, T(W))$ as the limit of

$$\begin{tikzcd}[column sep=small]
 & \mathrm{map}_{\mathrm{Solid}_{R_2}}(V_2, \overline{W} \otimes_R R_2) \arrow[dl] \arrow[dr] & & \mathrm{map}_{\mathrm{Solid}_{R_1}}(V_1, \overline{W} \otimes_R R_1) \arrow[dl] \\
\cdots & & \mathrm{map}_{\mathrm{Solid}_{R_2}}(V_2, \overline{W} \otimes_R R_1) &
\end{tikzcd}$$

where $\overline{W} := \varprojlim W$ is isomorphic to a retract of $\prod_{\mathbb{N}} R$ in Solid_R. The limit of this diagram is connective because every term is and the diagonal maps from left to right are surjective. □

Lemma 3.22. *Let $V, W \in \mathrm{laxVec}^s_R$ and let $f\colon V \to T(W)$ be a map. Then there exists a map $g\colon T(V) \to \mathbf{LR}(W)$ making the following diagram commute*

$$\begin{tikzcd}
T(V) \arrow[r, "g"] \arrow[d, "\mathrm{can}"] & \mathbf{LR}(W) \arrow[d, "can"] \\
V \arrow[r, "f"] & T(W)
\end{tikzcd}$$

where the vertical arrow on the left is the canonical one from Definition 1.47, and the one on the right is the one induced from Proposition 1.72 and the fact that $\mathbf{F} \circ \mathbf{L}$ vanishes.

Proof. By Lemma 1.45, V can be written as a retract of a free object $\widetilde{V}$ in the sense of Definition 1.44, and similarly W can be written as a retract of a free object $\widetilde{W}$. The map $V \to T(W)$ then induces a map

$$\widetilde{V} \to V \to T(W) \to T(\widetilde{W})$$

Supposing for a moment that the statement holds for free objects, we get a map $T(\widetilde{V}) \to \mathbf{LR}(\widetilde{W})$ which in turn induces a map $T(V) \to \mathbf{LR}(W)$ which proves the statement for the original map $V \to T(W)$. This is a diagram chase that uses the additivity of the functors T and $\mathbf{LR}$. So we can suppose that both V and W are free in the sense of

Definition 1.44. So h_*V and h_*W are both isomorphic to $\prod_{\mathbb{N}} R$ (see Notation 1.50). Let $V^0 := \prod_{\mathbb{N}} \mathbb{S} \otimes^{\blacksquare} R$ be a compact projective generator and let's fix a map $V^0 \to h_*V$ isomorphic to the canonical map $\prod_{\mathbb{N}} \mathbb{S} \otimes^{\blacksquare} R \to \prod_{\mathbb{N}} R$. Let f' denote the image of f under the composite

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}_R^b)}(V, T(W)) \to \mathrm{map}_{\mathrm{Solid}_R}(h_*V, (h_*W)^{\mathrm{tr}}) \to \mathrm{map}_{\mathrm{Solid}_R}(V^0, (h_*W)^{\mathrm{tr}})$$

where the first map is the one produced in (15) and the second one is the one induced by the map $V^0 \to h_*V$. Then there is a factorization of f' as a composite:

$$V^0 \to B \to (h_*W)^{\mathrm{tr}}$$

where B is a basic nuclear that can be written as a sequential colimit along trace-class maps

$$B = \mathrm{colim}_{\mathbb{N}} (P_0 \to P_1 \to \cdots)$$

where each P_i is a countable product of copies of R (see Lemma 1.62). Let the $V = (R_n^{r(n)})_{n\in\mathbb{N}}$ be a representative for V. Then V is determined by the function $r\colon \mathbb{N} \to \mathbb{N}$, and we write $V = V_r$. More generally, for any non-decreasing function $s\colon \mathbb{N} \to \mathbb{N}$ we let V_s denote the free object in laxVec_R^s given by $(R_n^{s(n)})_{n\in\mathbb{N}}$. As V^0 is compact, we can suppose that the map $V^0 \to B$ factors through a trace-class map $\alpha\colon V^0 \to P_0$. Then Lemma 1.60 gives that there exists a non-decreasing function $s\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha \otimes_R R_n$ factors through the projection $V^0 \otimes_R R_n \to R_n^{s(n)}$ onto the first $s(n)$ coordinates. We then know from Remark 1.65 that $\mathbf{L}(B)$ admits a map from V_s. Up to increasing s, we can suppose that $s(n) \geq r(n)$ for each $n \in \mathbb{N}$, so that there is a canonical map $V_s \to V_r = V$. This gives a diagram

$$\begin{tikzcd}
V_s \arrow[r] \arrow[d] & \mathbf{L}(B) \arrow[r] & \mathbf{LR}(W) \arrow[d] \\
V \arrow[rr] & & T(W)
\end{tikzcd}$$

in $\mathrm{Ind}(\mathrm{laxPerf}_R^b)$. Commutativity of this diagram amounts to check that two maps $V_s \to T(W)$ are homotopic. These maps live in the mapping spectra

$$\mathrm{map}_{\mathrm{Ind}(\mathrm{laxPerf}_R^b)}(V_s, T(W)) = \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(V_s, h^*(h_*W)^{\mathrm{tr}})$$

where the equivalence is Lemma 1.54. So it suffices to see that they are homotopic as maps from V_s to $h^*(h_*W)^{\mathrm{tr}}$. The following equivalence given by Lemma 1.60 below

$$\underset{\substack{j\colon \mathbb{N}\to\mathbb{N} \\ j(n+1)\geq j(n)}}{\mathrm{colim}} \mathrm{map}_{\mathrm{Solid}_R^{\mathrm{lax}}}(V_j, h^*(h_*W)^{\mathrm{tr}}) = \mathrm{map}_{\mathrm{Solid}_R}(V^0, (h_*W)^{\mathrm{tr}})$$

says that, after possibly increasing s, the equality of the two maps can be tested in π_0 of the mapping spectra $\mathrm{map}_{\mathrm{Solid}_R}(V^0, (h_*W)^{\mathrm{tr}})$, where it holds by construction. This shows that the last diagram commutes after possibly increasing s. Now it is easy to see that $T(V) \to V$ factors through V_s (for example, by Remark 1.59), and this gives the statement. $\square$

3.3 Continuity of K-theory and continuity of Hochschild homology

Let R be an adic ring (Definition 1.21) and let $(R_n)_{n\in\mathbb{N}}$ be a tower realizing its completion, as in Lemma 1.24. Knowing that topological cyclic homology invariants commute with certain products (Corollary 2.21.1), the missing step to conclude continuity for those invariants (in the sense of Section 3) is a statement relating Nuc_R to such infinite products. This statement is the following theorem by Efimov:

Theorem 3.23 (Efimov)**.** *There exists a fiber sequence*

$$\mathcal{U}_{loc}(\mathrm{Nuc}_R) \longrightarrow \mathcal{U}_{loc}(\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n})) \xrightarrow{id-pr} \mathcal{U}_{loc}(\mathrm{Stab}(\prod_{n\in\mathbb{N}} \mathrm{Vec}_{R_n}))$$

where Vec_{R_n} denotes the additive ∞-category of R_n-vector bundles, and $\mathrm{Stab}(-)$ is the stable envelope functor from Section 2.1. Here the second arrow is the one induced by the identity minus the product of the projections $\mathrm{Vec}_{R_{n+1}} \to \mathrm{Vec}_{R_n}$ for varying $n \in \mathbb{N}$.

As mentioned in the introduction, there is also an unbounded version of this statement. To state it, let $\widetilde{\mathrm{Nuc}}_R$ denote the inverse limit of the tower

$$\cdots \to \mathrm{Mod}_{R_{n+1}} \to \mathrm{Mod}_{R_n} \to \cdots$$

taken in the category of presentable dualizable ∞-categories. This limit indeed exists. As Nuc_R is dualizable, the universal property of limits gives a map $\mathrm{Nuc}_R \to \widetilde{\mathrm{Nuc}}_R$. This map is fully faithful, and the latter can be thought of as an unbounded version of the former. This adjective is further motivated by the following analog to Theorem 3.23 above:

Theorem 3.24 (Efimov)**.** *There exists a fiber sequence*

$$\mathcal{U}_{loc}(\widetilde{\mathrm{Nuc}}_R) \longrightarrow \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n}) \xrightarrow{id-pr} \mathcal{U}_{loc}(\prod_{n\in\mathbb{N}} \mathrm{Perf}_{R_n}).$$

Combining the previous two theorems with the facts about infinite products of additive ∞-categories proved in the previous sections gives that this two versions of nuclear modules, although different, have the same localizing invariant.

Corollary 3.24.1. *The inclusion induces an equivalence*

$$\mathcal{U}_{loc}(\mathrm{Nuc}_R) \to \mathcal{U}_{loc}(\widetilde{\mathrm{Nuc}}_R)$$

of localizing invariants.

Proof. This follows from Proposition 2.10 and the previous two theorems. □

The next results in this section are phrased for *condensed* localizing invariants, and the next remark and definition are here to make sense of this. This makes no real difference escept for the last couple of results in this section (Lemma 3.27 and Corollary 3.27.1), so feel free to evaluate everything at a point and stick to the non-condensed statements.

Remark 3.25. For any profinite set S, the condensed ring $C(S,R)$ of continuous functions from S to R is again an adic ring. In fact, the ring $C(S,R)$ is just the I-completion of the spectrum of maps from S to $R(*)^\delta$. Using this observation, the category Nuc_R lifts to a condensed category

$$S \in \mathrm{ExDisc} \mapsto \mathrm{Nuc}_{C(S,R)} \in \mathrm{Pr}^{\mathrm{dual}}_{st}.$$

Definition 3.26. For a localizing invariant $E\colon \mathcal{M}_{loc} \to \mathrm{Sp}$, let

$$E^{\mathrm{cd}} := \mathrm{Cond}(E)\colon \mathrm{Cond}(\mathcal{M}_{loc}) \to \mathrm{Cond}(\mathrm{Sp})$$

so that, for example, $E^{\mathrm{cd}}(\mathrm{Nuc}_R)(S) = E(\mathrm{Nuc}_{C(S,R)})$.

Corollary 3.26.1. *Let E be a localizing invariant that commutes with countable products of stable ∞-categories. Then the maps*

$$E^{\mathrm{cd}}(\mathrm{Nuc}_R) \to E^{\mathrm{cd}}(\widetilde{\mathrm{Nuc}}_R) \to \varprojlim_{n\in\mathbb{N}} E^{\mathrm{cd}}(\mathrm{Mod}_{R_n})$$

are equivalences of condensed spectra.

Proof. For an extemally disconnected set S the condensed algebra $C(S,R)$ is again an adic ring, so it suffices to prove the statement for an arbitrary adic ring R after evaluating at a point. Then the first equivalence follows from Corollary 3.24.1 and the second from Theorem 3.24 and the hypothesis on E. □

Corollary 3.26.2. *The maps*

$$K^{\mathrm{cd}}(\mathrm{Nuc}_R) \to K^{\mathrm{cd}}(\widetilde{\mathrm{Nuc}}_R) \to \varprojlim_{n\in\mathbb{N}} K^{\mathrm{cd}}(\mathrm{Mod}_{R_n})$$

are equivalences of condensed spectra.

Proof. This follows from the previous corollary and Theorem 2.6. □

This corollary does not apply to THH or TC. In fact, as far as we know from Corollary 2.21.1, THH and TC only commute with infinite products of *additive* ∞-categories. Using Theorem 3.23 instead, it is easy to see that these infinite products are enough.

Corollary 3.26.3. *Let THH^{cd} and TC^{cd} denote the condensed invariants defined in Definition 2.20. Then the canonical maps*

$$THH^{\mathrm{cd}}(\widetilde{\mathrm{Nuc}}_R) \leftarrow THH^{\mathrm{cd}}(\mathrm{Nuc}_R) \to \varprojlim_{n\in\mathbb{N}} THH^{\mathrm{cd}}(\mathrm{Mod}_{R_n})$$

and

$$TC^{\mathrm{cd}}(\widetilde{\mathrm{Nuc}}_R) \leftarrow TC^{\mathrm{cd}}(\mathrm{Nuc}_R) \to \varprojlim_{n\in\mathbb{N}} TC^{\mathrm{cd}}(\mathrm{Mod}_{R_n})$$

are all equivalences of condensed spectra.

Knowing the previous corollary, it is now possible to give a description of the topological Hochschild homology of Nuc_R that does not depend on the chosen tower $(R_n)_{n\in\mathbb{N}}$ from Lemma 1.24:

Lemma 3.27. *There is an equivalence of condensed spectra*

$$THH^{\mathrm{cd}}(\mathrm{Nuc}_R) \xrightarrow{\sim} \mathrm{colim}_{m\in\Delta^{\mathrm{op}}}(R^{\otimes^{\blacksquare} m})^{\wedge}_{(I,\cdots,I)}$$

where the $(-)^{\wedge}_{(I,\cdots,I)}$ denotes completion of the condensed commutative algebra $R^{\otimes^{\blacksquare} m}$ with respect to the ideal generated by the m images of I inside $\pi_0(R^{\otimes^{\blacksquare} m})$.

Proof. The previous corollary says that $THH^{\mathrm{cd}}(\mathrm{Nuc}_R)$ is equivalent to the inverse limit of the $THH^{\mathrm{cd}}(\mathrm{Mod}_{R_n})$'s. This inverse limit commutes with the geometric realizations presenting the THH of the R_n's, so

$$THH^{\mathrm{cd}}(\mathrm{Nuc}_R) \xrightarrow{\sim} \mathrm{colim}_{m\in\Delta^{\mathrm{op}}}(\varprojlim_n R_n^{\otimes m})$$

and the inner term in the latter is equivalent to $(R^{\otimes^{\blacksquare} m})^{\wedge}_{(I,\cdots,I)}$ by Lemma 1.28 (and independent of the R_n's). $\square$

As R is a commutative algebra, $THH(R(*))$ is free over S^1 in commutative $R(*)$-algebras, and this gives $R(*)$-module structures to both $THH(R(*))$ and $THH(\mathrm{Nuc}_R)$. The previous lemma is close to saying that the topological Hochschild homology of Nuc_R is the I-adic completion of the topological Hochschild homology of the $\mathbb{E}_\infty$-ring $R(*)$ under this module structures. This is indeed the case, as the next result shows.

Corollary 3.27.1. *The inclusion $\mathrm{Mod}_{R(*)} \to \mathrm{Nuc}_R$ induces an equivalence*

$$THH(R(*))^{\wedge}_I \to THH(\mathrm{Nuc}_R)$$

of $R(*)$-modules.

Proof. By Corollary 3.11.1 it suffices to show that the $R(*)$-module $THH(\mathrm{Nuc}_R)$ is I-complete. By the proof of Lemma 3.27, $THH(\mathrm{Nuc}_R)$ is a geometric realization of $R(*)$-modules of the form $(R(*)^{\otimes m})^{\wedge}_{(I,\cdots,I)}$, where the $R(*)$-module structure is given by the action on the leftmost term of the tensor product. As each $(R(*)^{\otimes m})^{\wedge}_{(I,\cdots,I)}$ is connective and I-complete (for the leftmost copy of I), $THH(\mathrm{Nuc}_R)$ is also I-complete. $\square$

3.4 An example: nuclear modules over perfectoid rings

Definition 3.28. A *p-adic* ring is an adic ring (Definition 1.21) with the p-adic topology. That is, a p-adic ring is a solid connective commutative algebra which agrees with the p-completion of its evaluation at a point.

Lemma 3.29. *Let R be a p-adic ring. Then the inclusion $\mathrm{Mod}_{R(*)} \to \mathrm{Nuc}_R$ induces an equivalence*

$$(THH(R(*))^{\delta})^{\wedge}_p = THH^{\mathrm{cd}}(\mathrm{Mod}_{R(*)})^{\wedge}_p \xrightarrow{\sim} THH^{\mathrm{cd}}(\mathrm{Nuc}_R)$$

of condensed spectra, where the p-adic completion is taken in condensed spectra. That is, the topological Hochschild homology of Nuc_R is given by the p-completion of the topological Hochschild homology of the underlying $\mathbb{E}_\infty$-ring $R(*)$. The same holds for TC, TC^- and TP.

Proof. This follows from Corollary 3.27.1, but here we give an easier proof in the case where the ideal of definition is (p). The first equivalence holds because $THH(R(*))^{\delta} = THH^{\mathrm{cd}}(\mathrm{Mod}_{R(*)})$ as condensed spectra. For the second, Corollary 3.26.3 gives that

$$THH^{\mathrm{cd}}(\mathrm{Nuc}_R) \xrightarrow{\sim} \varprojlim_{n\in\mathbb{N}} THH^{\mathrm{cd}}(\mathrm{Mod}_{R_n})$$

and the latter can be rewritten as $\varprojlim_n \mathrm{colim}_{m\in\Delta^{\mathrm{op}}} R_n^{\otimes m} = \mathrm{colim}_{m\in\Delta^{\mathrm{op}}} \varprojlim_n R_n^{\otimes m}$, where limit and colimit commute because of a connectivity argument. By Lemma 1.28 the last expression is equivalent to $\mathrm{colim}_{m\in\Delta^{\mathrm{op}}}(R^{\otimes m})^{\wedge}_p$, where again it is possible to comute the colimit with the p-completion by a connectivity argument. □

Lemma 3.30. *Let R be a p-adic ring. Then the canonical map*

$$THH(\mathrm{Nuc}_R)[p^{-1}] \to THH(\mathrm{Nuc}_{R[p^{-1}]})$$

is an equivalence.

Proof. It is clear that forgetting about nuclear modules the map

$$THH(R(*))[p^{-1}] \to THH(R(*)[p^{-1}])$$

is an equivalence. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\mathrm{Nil}(p)}_{R(*)} \arrow[r] \arrow[d] & \mathrm{Mod}_{R(*)} \arrow[r] \arrow[d] & \mathrm{Mod}_{R(*)[p^{-1}]} \arrow[d] \\
\mathrm{Nuc}^{\mathrm{Nil}(p)}_{R} \arrow[r] & \mathrm{Nuc}_{R} \arrow[r] & \mathrm{Nuc}_{R[p^{-1}]}
\end{tikzcd}$$

where the rows are localization sequences and the leftmost vertical arrow is an equivalence by Lemma 1.33. Then

$$\begin{aligned} THH(\mathrm{Nuc}^{\mathrm{Nil}(p)}_R) &= THH(\mathrm{Mod}^{\mathrm{Nil}(p)}_{R(*)}) \\ &= \Gamma_{(p)} THH(R) \\ &= \Gamma_{(p)}(THH(R)^{\wedge}_p) \\ &= \Gamma_{(p)}(THH(\mathrm{Nuc}_R)) \end{aligned}$$

where the second equality holds by the first sentence of the proof, the third one because the difference is both (p)-local and (p)-nilpotent, so it is zero, and the last equality is Lemma 3.29. Now the fact that the lower row in the last diagram is sent to a fiber sequence by THH together with the fiber sequence of functors $\Gamma_{(p)} \to \mathrm{id} \to (-)[p^{-1}]$ imply the statement. $\square$

Considering perfectoid rings as (condensed) adic rings with the p-adic topology, Lemma 3.29 applied to a perfectoid ring R gives an equivalence $THH(R, \mathbb{Z}_p) \xrightarrow{\sim} THH(\mathrm{Nuc}_R)$, where the former is another way of writing the p-completion of $THH(R)$. It is then possible to rewrite the calculations of [BMS18, 6.1, 6.2, 6.3] in terms of nuclear categories by simply replacing $THH(R, \mathbb{Z}_p)$ by $THH(\mathrm{Nuc}_R)$ everywhere. For example, [BMS18, 6.1] says that the canonical map

$$\mathrm{Sym}_R(THH_2(\mathrm{Nuc}_R)) \to THH_*(\mathrm{Nuc}_R)$$

is an isomorphism of graded rings[22], and that $THH_2(\mathrm{Nuc}_R)$ is a free R-module of rank 1. Similarly, [BMS18, 6.2, 6.3] says that there exist isomorphisms of graded rings

$$TC^-_*(\mathrm{Nuc}_R) \simeq A_{\mathrm{inf}}(R)[u,v]/(uv - \xi) \quad \text{and} \quad TP_*(\mathrm{Nuc}_R) \simeq A_{\mathrm{inf}}(R)[\sigma^{\pm 1}]$$

where $|u| = |\sigma| = 2$, $|v| = -2$, and ξ has degree zero and is a generator of the kernel of the canonical map $A_{\mathrm{inf}}(R) \to R$. Moreover, the previous isomorphisms can be chosen such that the graded ring homomorphisms

$$TC^-_*(\mathrm{Nuc}_R) \underset{can}{\overset{\varphi}{\rightrightarrows}} TP_*(\mathrm{Nuc}_R)$$

are φ-linear and $A_{\mathrm{inf}}(R)$-linear, respectively, and $\varphi(u) = \sigma$, $\mathrm{can}(u) = \xi \cdot \sigma$ and $\mathrm{can}(v) = \sigma^{-1}$.

One advantage of working with nuclear modules is that it is possible to use Lemma 3.30 to pass to the generic fiber at the level of categories:

Lemma 3.31. *Let R be a perfectoid ring. There exists an isomorphism of graded rings*

$$TP_*(\mathrm{Nuc}_{R[1/p]}) \simeq B^+_{\mathrm{dR}}(R)[\sigma^{\pm 1}]$$

where σ has degree 2 and the ring B^+_{dR} is given by the (ξ)-adic completion of the ring $A_{\mathrm{inf}}(R)[1/p]$, where ξ is the kernel of the canonical map $A_{\mathrm{inf}}(R) \to R$.

Proof. Recall that the spectral sequences calculating periodic homology

$$\begin{aligned} E^2_{i,j} &= \widehat{H}^{-i}(BS^1, THH_j(\mathrm{Nuc}_R)) &&\Rightarrow TP_{i+j}(\mathrm{Nuc}_R) \\ \widetilde{E}^2_{i,j} &= \widehat{H}^{-i}(BS^1, THH_j(\mathrm{Nuc}_{R[1/p]})) &&\Rightarrow TP_{i+j}(\mathrm{Nuc}_{R[1/p]}) \end{aligned}$$

[22] here the spectrum $THH(\mathrm{Nuc}_R)$ carries a symmetric monoidal structure coming from the symmetric monoidal structure of Nuc_R and the lax-monoidaity of the functor THH.

collapse by Bökstedt periodicity (using Lemma 3.30 for the second). The descending filtration given by the spectral sequence E on $TP_0(\mathrm{Nuc}_R) \cong A_{\mathrm{inf}}(R)$ is given by $\mathrm{Fil}^n_E(A_{\mathrm{inf}}(R)) = (\xi)^n \subset A_{\mathrm{inf}}(R)$, and the element $\xi^n \in (\xi)^n$ is sent to a generator of $\mathrm{gr}^n(A_{\mathrm{inf}}(R))$. The base change map $\mathrm{Nuc}_R \to \mathrm{Nuc}_{R[1/p]}$ induces both a ring homomorphism on periodic homology

$$f\colon A_{\mathrm{inf}}(R)[1/p] \cong TP_0(\mathrm{Nuc}_R)[1/p] \longrightarrow TP_0(\mathrm{Nuc}_{R[1/p]})$$

and compatible $R[1/p]$-linear equivalences

$$E^2_{i,j}[1/p] \xrightarrow{\sim} \widetilde{E}^2_{i,j} \tag{38}$$

at the level of spectral sequences (they are equivalences by Lemma 3.30). As f respects filtrations, there is an inclusion $(f(\xi))^n \subset \mathrm{Fil}^n_{\widetilde{E}} TP_0(\mathrm{Nuc}_{R[1/p]})$, and we claim that this inclusion is an equivalence. To show this, let $x \in \mathrm{Fil}^n_{\widetilde{E}} TP_0(\mathrm{Nuc}_{R[1/p]})$ and let x_n denote its reduction to the n-th graded piece. As f induces equivalences on graded pieces ((38)), there exists a $y \in A_{\mathrm{inf}}(R)[1/p]$ such that $f(\xi^n y)$ reduces to x_n in the n-th graded piece. Then $x - f(\xi)^n f(y) \in \mathrm{Fil}^{n+1} TP_0(\mathrm{Nuc}_{R[1/p]})$. As $TP_0(\mathrm{Nuc}_{R[1/p]})$ is is separated and complete with respect to the filtration given by $\widetilde{E}$, an induction gives that $x \in (f(\xi))^n$. This shows that the filtrations agree. Now, as f is a mod-ξ isomorphism by (38) (here we use that ξ is a non-zero divisor in $A_{\mathrm{inf}}(R)[1/p]$) and $TP_0(\mathrm{Nuc}_{R[1/p]})$ is $f(\xi)$-complete, the latter must agree with the ξ-completion of $A_{\mathrm{inf}}(R)[1/p]$, which is $B^+_{dR}(R)$. ☐

Let

$$HH(-/R)\colon \mathrm{CAlg}(\mathrm{Cond}(\mathrm{Sp}))_{R/} \longrightarrow \mathrm{Cond}(\mathrm{Sp}) \tag{39}$$

denote the Hochschild homology functor from commutative condensed R-algebras to condensed spectra, and let

$$\begin{aligned} HC^-_{\blacksquare}(-/R) &:= HH^{\blacksquare}(-/R)^{hS^1} \\ HP_{\blacksquare}(-/R) &:= HH^{\blacksquare}(-/R)^{tS^1}. \end{aligned}$$

be two solid versions of cyclic and periodoc homology arising from it. These are functors taking values in solid spectra and receiving a map from the solidifications of HC^- and HP, respectively. In [BMS18, 6.3] it is explained in which sense the topological theory is a deformation of the algebraic theory. Here is a version of this, which is an analog of [BMS18, Theorem 6.7], over the generic fiber.

Proposition 3.32. *Let R be a perfectoid ring and let A be a p-complete R-algebra. Then there is a S^1-equivariant cofiber sequence*

$$THH(\mathrm{Nuc}_{A[1/p]})[2] \xrightarrow{u} THH(\mathrm{Nuc}_{A[1/p]}) \longrightarrow HH^{\blacksquare}(A[1/p]/R[1/p]) \tag{40}$$

of condensed $THH(\mathrm{Nuc}_{A[1/p]})$-modules, where A is viewed as an adic condensed R-algebra with its p-adic topology. Moreover, there is a natural equivalence of condensed spectra

$$HH^{\blacksquare}(A/R) \xrightarrow{\sim} THH(\mathrm{Nuc}_A) \otimes_{THH(\mathrm{Nuc}_R)} R \tag{41}$$

which also holds when A and R are replaced by $A[1/p]$ and $R[1/p]$. In particular, the right hand side of (41) *is solid. After applying fixed points or Tate constructions to* (40)*, there are induced cofiber sequences*

$$TC^{-}(\mathrm{Nuc}_{A[1/p]})[2] \xrightarrow{u} TC^{-}(\mathrm{Nuc}_{A[1/p]}) \longrightarrow HC^{-}_{\blacksquare}(A[1/p]/R[1/p])$$
$$TP(\mathrm{Nuc}_{A[1/p]})[2] \xrightarrow{\xi\cdot\sigma} TP(\mathrm{Nuc}_{A[1/p]}) \longrightarrow HP_{\blacksquare}(A[1/p]/R[1/p])$$

of condensed spectra.

Proof. By [BMS18, Theorem 6.7] there is a fiber sequence

$$THH(\mathrm{Nuc}_A)[2] \xrightarrow{u} THH(\mathrm{Nuc}_A) \longrightarrow HH(A/R)^{\wedge}_{p}$$

of condensed $THH(\mathrm{Nuc}_A)$-modules. Analogously, [BMS18, Theorem 6.7] for $A = R$ gives a fiber sequence

$$THH(\mathrm{Nuc}_R)[2] \xrightarrow{u} THH(\mathrm{Nuc}_R) \longrightarrow R$$

of condensed $THH(\mathrm{Nuc}_R)$-modules. Comparing these two fiber sequences after base changing the latter to to $THH(\mathrm{Nuc}_A)$-modules gives isomorphisms $THH(\mathrm{Nuc}_A)\otimes_{THH(\mathrm{Nuc}_R)} R \cong HH(A/R)^{\wedge}_{p} \cong HH^{\blacksquare}(A/R)$ of condensed spectra, where the last isomorphism follows from the solid tensor product preserving complete objects. Inverting p on top of everything and using Lemma 3.30 gives (40). $\square$

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