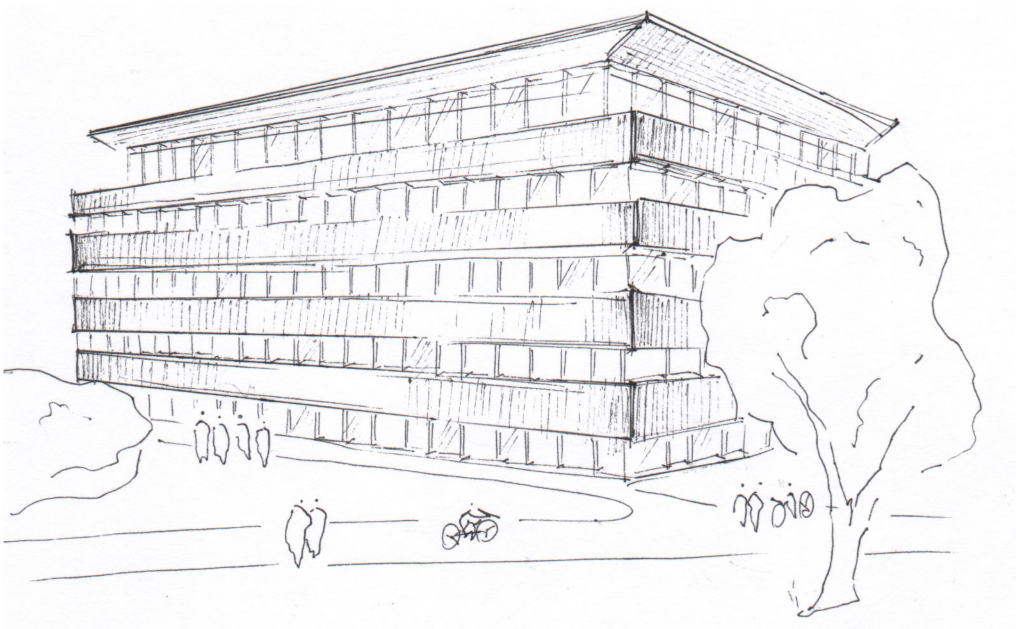
KØBENHAVNS UNIVERSITET
DET NATUR- OG BIOVIDENSKABELIGE FAKULTET

MODULI OF ALGEBRAIC HYPERSURFACES VIA HOMOTOPY PRINCIPLES



PHD THESIS

ALEXIS AUMONIER

SEPTEMBER 2023

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen.

Alexis Aumonier
Institut for Matematiske Fag
Københavns Universitet
Universitetsparken 5
DK-2100 København Ø
Danmark

alexis-aumonier@math.ku.dk
https://isomorphisme.github.io

Date of submission:	1^{st} September 2023
Date of defence:	6^{th} October 2023
Adviser:	Søren Galatius Københavns Universitet, Denmark
Assessment committee:	Nathalie Wahl (chair) Københavns Universitet, Denmark
	Michael Weiss Westfälische Wilhelms-Universität Münster, Germany
	Kai Cieliebak Universität Augsburg, Germany

This work was partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682922), the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151), and the Swedish Research Council (grant No. 2021-06594) during the research program "Higher Algebraic Structures in Algebra, Topology and Geometry" held at Institut Mittag–Leffler in the spring of 2022.

ISBN 978-87-7125-218-7

Abstract

In this thesis, I prove a general h-principle for algebraic sections of vector bundles, and use it to investigate the homology of moduli spaces of smooth algebraic hypersurfaces. The thesis consists of an introduction followed by three papers, the last of which is joint with Ronno Das.

In the first paper, I consider spaces of algebraic sections of vector bundles subject to differential relations. On smooth projective complex varieties, I prove that the homology of such a space coincides in a range with that of a space of continuous sections of an associated bundle. As an immediate consequence, I show stability of the rational cohomology for complement of discriminants in linear systems of hypersurfaces of increasing degree. This paper is the most technical and its results are used repeatedly throughout the thesis.

In the second paper, I study the locus of smooth hypersurfaces inside the Hilbert scheme of a smooth complex projective variety. Using the results of the first paper, I show how part of its cohomology can be computed via an h-principle akin to a scanning map. I also explain how to compare the rational cohomology to that of classifying spaces of diffeomorphisms groups of hypersurfaces.

In the third paper, Ronno Das and I study the cohomology of the universal smooth hypersurface bundle with marked points. We adapt the arguments of the first paper to show another h-principle. Using rational models, we deduce rational homological stability for this space.

Resumé

I denne afhandling viser jeg et h-princip vedrørende algebraiske sektionsrum af vektorbundter, og bruger det til at studere homologien af moduli rummene af glatte hyperflader. Afhandlingen består af en introduktion og tre artikler, hvoraf den sidste er et samarbejde med Ronno Das.

I den første artikel studerer jeg rum af algebraiske sektioner af vektorbundter, der opfylder differentialrelationer. For glatte projektive komplekse varieteter viser jeg, at en del af sektionsrummets homologi er lig med den af et rum af kontinuerte sektioner af et relateret bundt. Som ummidelbar korollar beviser jeg, at den rationelle kohomologi af komplementerer af diskriminanter i lineære systemer af hyperflader stabiliserer, når hyperfladers grad stiger. Denne artikel er den mest teknisk krævende i afhandlingen, og indeholder resultater, som bruges gennem hele afhandlingen.

I den anden artikel undersøger jeg locusen af glatte hyperflader i Hilbert-skemaet af en glat projektiv kompleks varietet. Jeg viser, hvordan en del af dens kohomologi kan beregnes via et h-princip, der ligner en skanningsafbildning. Jeg forklarer også, hvordan den rationelle kohomologi kan sammenlignes med den af klassificerende rum af diffeomorfisme grupper af hyperflader.

I den trejde artikel, som er et samarbejde med Ronno Das, studerer vi kohomologien af det universelle glatte hyperfladerbundt med markerede punkter. Vi tilpasser beviserne fra den første artikel til at vise et andet h-princip. Ved hjælp af rationelle modeller deducerer vi rationel homologisk stabilitet for dette rum.

Résumé

Dans cette thèse, je montre un h-principe général concernant les sections algébriques de fibrés vectoriels, et l'utilise pour étudier l'homologie d'espaces de modules d'hypersurfaces lisses. Cette thèse comprend une introduction ainsi que trois articles, le dernier étant écrit en collaboration avec Ronno Das.

Dans le premier article, je considère des espaces de sections algébriques de fibrés vectoriels sujets à des relations différentielles. Dans le cas de variétés complexes lisses et projectives, je prouve que l'homologie d'un tel espace coïncide jusqu'à un certain degré avec celle d'un espace de sections continues d'un fibré auxiliaire. En guise d'application, je montre de la stabilité homologique rationnelle pour les complémentaires des discriminants dans des systèmes linéaires d'hypersurfaces de degrés croissants. Ce papier est certainement le plus technique et contient des résultats utilisés tout au long de cette thèse.

Dans le second article, j'étudie le lieu géométrique des hypersurfaces lisses à l'intérieur du schéma de Hilbert d'une variété complexe lisse et projective. En utilisant les résultats du premier papier, je montre qu'une partie de sa cohomologie peut être calculée via un h-principe s'apparentant aux méthodes de scanographie topologique. J'en profite pour aussi expliquer comment comparer la cohomologie rationnelle à celle d'espaces classifiants de groupes de difféomorphismes d'hypersurfaces.

Dans le troisième article, Ronno Das et moi-même étudions la cohomologie du fibré universel des hypersurfaces lisses avec des points marqués. On adapte les arguments de mon premier papier pour prouver un autre h-principe. On en déduit un phénomène de stabilité homologique en utilisant des modèles rationnels.

Acknowledgements

First and foremost, I would like to heartily thank Søren for his guidance during the last four years. I have benefited immensely from his insights, suggestions, questions, encouragements… needless to say, this thesis would not exist without him. Thanks for giving me the freedom to find my own path, whilst making sure I was not completely going astray. I could not have asked for a better adviser, both mathematically and humanly. Tak Søren!

The beginning of my Danish journey can be traced back to the summer of 2017. I am grateful to Jesper for generously hosting me back then, and for his constant cheerfulness. Many thanks are also due to Oscar for introducing me to the beautiful world of manifolds in Cambridge, and for enlightening mathematical discussions during my PhD. I am thankful to Nathalie, not only for chairing my thesis committee, but also for creating such a joyful atmosphere at the GeoTop center. Thanks to Ronno for exchanging good and bad puns, and for our mathematical collaboration. I have learned a lot from our discussions.

I am pleased to have had great office mates over the years, and special thanks are due to Kaif and Jeroen for delightful office nonsense. It is a pleasure to thank Vignesh, both for being an important honorary member of the office, and also for sharing his mathematical enthusiasm. Thanks to Calista for helpful advice and being a great academic sister, and to Severin for being a friendly housemate. I would also like to extend my thanks to my fellow PhD students and everyone else in the topology group and the administration for pleasant collegiality.

I am also very grateful to have had a happy life during my years spent in Copenhagen. Special thanks to Nanna for being a good friend and PhD companion over the years, and to Nena for great culinary collaborations and being a worthy board game rival. En stor tak til alle fægtere i Frederiksberg Slots Fægteklub, især til Pia for hendes varme velkomst, et à Quentin pour de belles touches. Mange tak også til Inger for et hyggeligt liv på Vestagervej.

Last but not least[1], un immense merci à toute ma famille. Merci pour les messages, les appels, les cartes et autres colis de chocolats. Merci à grand-mère et maman pour notre escapade danoise sur l'île de Møn. Merci à grand-père pour les nouvelles échangées et ses recommandations littéraires. Merci à papa pour les plaisanteries en tout genre. Merci à mes frères qui m'ont supporté. Et enfin merci pour tout le reste. Il serait trop long de tout écrire.

[1] En anglais dans le texte.

Contents

Thesis Statement

This thesis consists of an introduction and three papers.

The first paper is reproduced, with minor modifications, from my Master Thesis at the University of Copenhagen. As part of my 4+4 PhD programme at the University of Copenhagen, I submitted my Master Thesis after two years, in August 2021, and defended it in September 2021. The paper is now publicly available at arXiv: 2112.00326. It has been submitted to a journal for publication and is currently under review.

The third paper is written jointly with Ronno Das. A co-authorship statement was signed and given to the secretariat at the Institut for Matematiske Fag.

The image on the front page was made by Anna-Julia Plichta for the department of mathematical sciences, and is reproduced here with the authorisation of Jim Høyer.

Part I

Introduction

Much of the work in this thesis can be seen as my attempt to understand linear systems of smooth divisors. These are arguably the simplest moduli spaces in algebraic geometry, and I will explain in this introduction what they are and what I can say about them. This introduction is meant to be understandable to non-experts, and not supposed to be precise in any way.

Moduli of hypersurfaces

Loosely speaking, a bundle over T consists of objects E_t for each $t \in T$ which depend continuously on t. This is often written altogether as $E \to T$ and the E_t are called *fibres*. Given a family of objects $\mathcal{F}$, we have a *moduli functor*

$$T \longmapsto \{E \to T \mid E_t \in \mathcal{F},\ \forall t \in T\}$$

which records the set of bundles with fibres in the family. The word "functor" reflects the following important construction: given a bundle $E \to T$ and a map $f\colon S \to T$, we can form the *pullback* $f^*E \to S$ by letting $(f^*E)_s = E_{f(s)}$ for each $s \in S$. A moduli functor can sometimes be representable, and in fact those appearing in this thesis will indeed be. In that case, there exists a bundle

$$\pi\colon \mathcal{U} \to \mathcal{M}$$

with the special property that each fibre is an element of the family $\mathcal{F}$, and any bundle $E \to T$ with fibres in $\mathcal{F}$ is pulled back from π along a unique map $T \to \mathcal{M}$. Then $\mathcal{M}$ is called the *moduli space* and $\mathcal{U}$ the *universal bundle* representing the moduli functor. I have been deliberately vague to encompass two situations appearing in this thesis: the topological and the algebraic geometrical. In the former, bundles are smooth and proper maps with fibres manifolds. In the later, they are flat morphisms with fibres varieties.

The objects in the family $\mathcal{F}$ of choice in this thesis are called *hypersurfaces* or *divisors*. By definition, they are the subvarieties of codimension 1 in a chosen complex variety X. More concretely, a hypersurface $Z \subset X$ is locally given by a single equation, i.e. X can be covered by open subsets U_i where

$$Z \cap U_i = \{x \in X \mid s_i(x) = 0\}$$

with $s_i\colon U_i \to \mathbb{C}$ a non-zero polynomial. In fact these polynomials can be glued to form, not a global function $X \to \mathbb{C}$, but rather a *global section* $s\colon X \to \mathcal{L}$ of a line bundle $p\colon \mathcal{L} \to X$. In other words, p is a bundle in the above sense with fibres $\mathbb{C}$, and $p \circ s = \mathrm{id}$. Let me write

$$\Gamma(X, \mathcal{L}) = \{s\colon X \to \mathcal{L} \mid p \circ s = \mathrm{id}\}$$

for the vector space of global sections of $p\colon \mathcal{L} \to X$, and given a non-zero section $s \in \Gamma(X, \mathcal{L}) \setminus 0$

$$V(s) = \{x \in X \mid s(x) = 0\}$$

for its *vanishing locus*, i.e. the hypersurface it defines. Let me also notice that the solutions to the equation $s(x) = 0$ do not change if s is multiplied by a non-zero scalar $\lambda \in \mathbb{C}^\times$. Thus any element in the projectivisation

$$\big(\Gamma(X, \mathcal{L}) \setminus 0\big)/\mathbb{C}^\times = \mathbb{P}\big(\Gamma(X, \mathcal{L})\big)$$

has a well-defined vanishing locus attached to it. Fixing $\mathcal{L}$, the hypersurfaces obtained this way are called *linearly equivalent*, and the set of those is the *complete linear system* $|\mathcal{L}|$. This is the fundamental example in this thesis: when X is smooth and projective, the moduli functor on the category of complex varieties

$$T \longmapsto \{E \to T \text{ flat morphism} \mid \forall t \in T,\ E_t \subset X \text{ is in } |\mathcal{L}|\}$$

is represented by the flat family

$$\big\{(s,x) \in \mathbb{P}\big(\Gamma(X,\mathcal{L})\big) \times X \;\big|\; s(x) = 0\big\} \longrightarrow \mathbb{P}\big(\Gamma(X,\mathcal{L})\big)$$

and $\mathbb{P}\big(\Gamma(X,\mathcal{L})\big) \cong |\mathcal{L}|$.

Let me now introduce an interesting geometric property. If $V(s)$ is a hypersurface given as the vanishing locus of a global section s, it is called *smooth* if the derivative $\mathrm{d}s(x) \neq 0$ for all $x \in V(s)$. There exists in fact a direct characterisation found by Cayley [Cay48] in the 19th century, who constructed the *discriminant*

$$\Delta \colon \Gamma(X,\mathcal{L}) \longrightarrow \mathbb{C}$$

as a homogeneous polynomial such that $\Delta(s) \neq 0$ if and only if $V(s)$ is smooth. By the discussion above, the Zariski open subset

$$\mathcal{M}_{\mathrm{hyp}}(\mathcal{L}) = \big\{s \in \mathbb{P}\big(\Gamma(X,\mathcal{L})\big) \;\big|\; \Delta(s) \neq 0\big\} \subset \mathbb{P}\big(\Gamma(X,\mathcal{L})\big)$$

is the moduli space representing the functor of bundles with fibres smooth hypersurfaces in the family $|\mathcal{L}|$. A large part of this thesis is dedicated to computing its homology.

Homology and stability

Outside of a few rare cases, computing the whole homology of a moduli of smooth hypersurfaces is out of reach. Instead, I focus in this thesis on a phenomenon known as *homological stability*. Broadly speaking, a sequence of spaces $\{Y_d\}_{d\in\mathbb{N}}$ satisfies homological stability if given an i the homology group $H_i(Y_d)$ is independent of d for $d \gg 0$ large enough. This group is then known as the ith *stable homology*, and computing it is an interesting question. In the context of moduli spaces, this is hardly a new idea. For instance, Harer showed that the moduli spaces of smooth curves of genus g exhibit homological stability when the genus increases [Har85]. An expression for the stable rational homology was a conjecture of Mumford [Mum83] resolved by Madsen and Weiss [MW07]. More recently, Galatius and Randal-Williams greatly extended these results (both stability and computation of the stable homology) to moduli spaces of higher dimensional manifolds in the series of papers [GRW14, GRW17, GRW18].

To even state homological stability for moduli spaces of smooth hypersurfaces, we first need a sequence of them. A natural candidate is given by the sequence

$$\mathcal{M}_{\mathrm{hyp}}(\mathcal{L}),\ \mathcal{M}_{\mathrm{hyp}}(\mathcal{L}^{\otimes 2}),\ \mathcal{M}_{\mathrm{hyp}}(\mathcal{L}^{\otimes 3}),\ \ldots$$

of moduli associated to tensor powers of a chosen line bundle $\mathcal{L}$. Vakil and Wood had conjectured [VW15] that the rational homology of this sequence stabilises when $\mathcal{L}$ is an ample line bundle. This was proved by Tommasi for $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ [Tom14], and I managed to prove the conjecture in full generality in [Aum22]. Although unusual, there are to my knowledge no maps between any two spaces in the sequence that could induce isomorphisms on the stable homology groups. Instead I showed stability as a by-product of the computation of the, a fortiori stable, homology. In the continuation of this work, Ronno Das and I computed in [AD23] the stable rational homology of the universal bundle above $\mathcal{M}_{\mathrm{hyp}}(\mathcal{L}^{\otimes d})$.

In their work I alluded to above, Galatius and Randal-Williams have explained how to compute the stable homology of the classifying space $B\mathrm{Diff}(M)$ of the diffeomorphism group $\mathrm{Diff}(M)$ of a manifold M. From the point of view of this introduction, $B\mathrm{Diff}(M)$ is the topological moduli space parameterising smooth manifold bundles with fibre isomorphic to M. Of course, when M happens to be a hypersurface given as the vanishing locus of a section $s \in \Gamma(X, \mathcal{L})$, there is also the algebraic moduli space $\mathcal{M}_{\mathrm{hyp}}(\mathcal{L})$ parameterising algebraic bundles of varieties with fibre linearly equivalent to M. In the paper [Aum23], I relate the stable homologies of these two moduli spaces using *tangential structures*. These are additional data on the tangent space of the manifold, and I use them to record special properties of a hypersurface deduced from the algebraic geometry.

Finally, one can wonder what happens when linear equivalence is replaced by homological equivalence. For this latter relation, two hypersurfaces given as vanishing loci of sections of line bundles $\mathcal{L}$ and $\mathcal{L}'$ are equivalent if $\mathcal{L}$ and $\mathcal{L}'$ have equal Chern classes. In my work [Aum23], I show that the homology of the larger moduli space of those hypersurfaces can be computed using a technique from algebraic topology known as *scanning*.

Homotopy principle

In this final section of the introduction, I want to highlight the main technical result of my thesis. It takes the form of a *homotopy principle*, or *h-principle* for short. As the name suggests, h-principles are part of a general heuristic to transform geometric problems into homotopical ones. They have been wonderfully exposed in general in other places, e.g. [EM02]. In this section, I want to focus on how they look like in the context of the moduli of smooth hypersurfaces.

As defined earlier, smoothness of a section s of a line bundle $\mathcal{L} \to X$ is a twofold condition expressed on both its value and its derivative. Indeed, recall that s is smooth if $(s, \mathrm{d}s)(x) \neq 0$ for all $x \in X$. Let us write

$$j^1(s) = (s, \mathrm{d}s)$$

for that couple, so that s is smooth if and only if $j^1(s)$ never vanishes. As for the notation, $j^1(s)$ is called the *first jet expansion* of s and is a global section of an auxiliary bundle $J^1\mathcal{L}$ named the *first jet bundle* of $\mathcal{L}$. With this shift of perspective, smoothness is a single condition on the value only of a global section. Although it does not seem to achieve much, this reformulation is very useful for two reasons. Firstly the space of never vanishing continuous sections of the jet bundle

$$\Gamma_{\mathcal{C}^0}\big(X, J^1\mathcal{L} \setminus 0\big)$$

is amenable to techniques from homotopy theory. Secondly the h-principle predicts that

$$\begin{aligned} \{s \in \Gamma(X, \mathcal{L}) \mid s \text{ is smooth}\} &\longrightarrow \Gamma_{\mathcal{C}^0}\big(X, J^1\mathcal{L} \setminus 0\big) \\ s &\longmapsto j^1(s) \end{aligned}$$

induces an isomorphism on homology. This was shown by Vassiliev in the context of $\mathcal{C}^\infty$ sections of bundles on manifolds [Vas94]. In [Aum22], I adapt his methods to the context of algebraic sections, at the price of only obtaining an isomorphism in a range of degrees.

Bibliography

[AD23] Alexis Aumonier and Ronno Das. Homological stability for the space of hypersurfaces with marked points. *preprint*, 2023.

[Aum22] Alexis Aumonier. An h-principle for complements of discriminants, October 2022. arXiv:2112.00326 [math].

[Aum23] Alexis Aumonier. Scanning the moduli of embedded smooth hypersurfaces. *preprint*, 2023.

[Cay48] Arthur Cayley. On the theory of elimination. *Cambridge and Dublin Math. J.*, 3:116–120, 1848.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, June 2002.

[GRW14] Søren Galatius and Oscar Randal-Williams. Stable moduli spaces of high-dimensional manifolds. *Acta Mathematica*, 212(2):257–377, January 2014. Publisher: Institut Mittag-Leffler.

[GRW17] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. II. *Annals of Mathematics*, 186(1):127–204, July 2017. Publisher: Department of Mathematics of Princeton University.

[GRW18] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. I. *Journal of the American Mathematical Society*, 31(1):215–264, January 2018.

[Har85] John L. Harer. Stability of the Homology of the Mapping Class Groups of Orientable Surfaces. *Annals of Mathematics*, 121(2):215–249, 1985. Publisher: Annals of Mathematics.

[Mum83] David Mumford. Towards an Enumerative Geometry of the Moduli Space of Curves. In Michael Artin and John Tate, editors, *Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday. Volume II: Geometry*, Progress in Mathematics, pages 271–328. Birkhäuser, Boston, MA, 1983.

[MW07] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Annals of Mathematics*, 165(3):843–941, May 2007.

[Tom14] Orsola Tommasi. Stable cohomology of spaces of non-singular hypersurfaces. *Advances in Mathematics*, 265:428–440, November 2014.

[Vas94] V A Vassiliev. *Complements of Discriminants of Smooth Maps: Topology and Applications*, volume 98 of *Translations of mathematical monographs*. American Mathematical Society, Providence, Rhode Island, 1994.

[VW15] Ravi Vakil and Melanie Matchett Wood. Discriminants in the Grothendieck ring. *Duke Mathematical Journal*, 164(6):1139–1185, April 2015. Publisher: Duke University Press.

Part II

Papers

PAPER A

An h-principle for complements of discriminants

This chapter contains the preprint version of the following paper:

Alexis Aumonier. *An h-principle for complements of discriminants.* 2022.

The paper is reproduced, with minor modifications, from my Master Thesis at the University of Copenhagen. The preprint version is publicly available at arXiv: 2112.00326.

AN H-PRINCIPLE FOR COMPLEMENTS OF DISCRIMINANTS

ALEXIS AUMONIER

Abstract. We compare spaces of non-singular algebraic sections of ample vector bundles to spaces of continuous sections of jet bundles. Under some conditions, we provide an isomorphism in homology in a range of degrees growing with the jet ampleness. As an application, when $\mathcal{L}$ is a very ample line bundle on a smooth projective complex variety, we prove that the rational cohomology of the space of non-singular algebraic sections of $\mathcal{L}^{\otimes d}$ stabilises as $d \to \infty$ and compute the stable cohomology. We also prove that the integral homology does not stabilise using tools from stable homotopy theory.

Contents

1. Introduction

The purpose of this paper is to study spaces of non-singular holomorphic sections of vector bundles by comparing them to spaces of continuous sections of appropriate jet bundles. The latter are particularly amenable to computations using tools from homotopy theory.

Given a holomorphic line bundle $\mathcal{L}$ on a smooth projective complex variety X, one may consider the vector space of all holomorphic global sections $\Gamma_{\text{hol}}(X;\mathcal{L})$. To each section $s \in \Gamma_{\text{hol}}(X;\mathcal{L})$ is associated a geometric object: its vanishing set

$$V(s) := \{x \in X \mid s(x) = 0\} \subset X,$$

and s is called non-singular whenever its derivative $ds \in \Gamma_{\text{hol}}(\Omega^1_X \otimes \mathcal{L})$ does not vanish on $V(s)$. This implies in particular that $V(s)$ is a smooth subvariety of X. It has been known for a century now that when $\mathcal{L}$ is a very ample line bundle, Bertini theorem implies that a generic section is non-singular. There is thus a Zariski open subset

$$\Gamma_{\text{hol,ns}}(X;\mathcal{L}) \subset \Gamma_{\text{hol}}(X;\mathcal{L})$$

consisting of those non-singular sections, which geometrically can be interpreted as a moduli space of equations of certain smooth hypersurfaces in X. A prime example being the space

$\Gamma_{\text{hol,ns}}(\mathbb{CP}^n; \mathcal{O}(d))$ (sometimes modded out by $\mathbb{C}^*$ or $GL_{n+1}(\mathbb{C})$) of smooth hypersurfaces of degree d in the complex projective space $\mathbb{CP}^n$.

The cohomology ring of $\Gamma_{\text{hol,ns}}(X; \mathcal{L})$, sometimes known as the ring of characteristic classes, is therefore an important object in the study of hypersurface bundles. In this article, we give a way of computing it in a range.

Before revealing our main theorem, we will extend the classical situation above in two directions. To begin, instead of limiting ourselves to line bundles, we will look at sections of bundles of possibly higher rank. Furthermore, we observe that being non-singular imposes conditions on the value and derivative of a global section. We will generalise this situation by looking at a broader class of conditions on higher order derivatives, thus encompassing various other flavours of moduli spaces: hypersurfaces with simple nodes, smooth complete intersections, etc. (Although explicit computations of cohomology rings will only appear in forthcoming work.)

Having said this, let X be a smooth projective complex variety and $\mathcal{E}$ be a holomorphic vector bundle on X. One can construct a new holomorphic vector bundle $J^r\mathcal{E}$ called the r-th jet bundle of $\mathcal{E}$ together with a map on global sections $j^r : \Gamma_{\text{hol}}(\mathcal{E}) \to \Gamma_{\text{hol}}(J^r\mathcal{E})$. Intuitively, for a section s of $\mathcal{E}$, the associated section $j^r(s)$ of the jet bundle records all derivatives of s up to order r. For $\mathfrak{T} \subset J^r\mathcal{E}$ a subset which we think of as "forbidden derivatives", we say that a section s of $\mathcal{E}$ is non-singular if $j^r(s)(x) \notin \mathfrak{T}$ for all $x \in X$. For instance, when $\mathcal{E}$ is a line bundle and $\mathfrak{T} \subset J^1\mathcal{E}$ is the zero section, we recover the classical notion of non-singular sections discussed at the beginning of this article.

Theorem 1.1 (see Theorem 2.13 for full generality). *Let X be a smooth complex projective variety and $\mathcal{E}$ be a holomorphic vector bundle on it. Let $r \geq 0$ be an integer and $\mathfrak{T} \subset J^r\mathcal{E}$ be a closed subvariety of the r-th jet bundle of $\mathcal{E}$ of codimension at least $\dim_{\mathbb{C}} X + 1$. We write*

$$\Gamma_{\text{hol,ns}}(\mathcal{E}) := \{s \in \Gamma_{\text{hol}}(\mathcal{E}) \mid \forall x \in X \; j^r(s)(x) \notin \mathfrak{T}\}$$

for the space of non-singular holomorphic sections of $\mathcal{E}$. If $\mathcal{E}$ is d-jet ample, the composition

$$\Gamma_{\text{hol,ns}}(\mathcal{E}) \xrightarrow{j^r} \Gamma_{\text{hol}}(J^r\mathcal{E} - \mathfrak{T}) \hookrightarrow \Gamma_{\mathcal{C}^0}(J^r\mathcal{E} - \mathfrak{T})$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d-r}{r+1}$.

The theorem above can be strengthened, and in Section 2 we introduce a more general class of allowed subsets $\mathfrak{T} \subset J^r\mathcal{E}$ of the jet bundle as well as give a sharper range of degrees. We also take advantage of that section to give the definition of jet ampleness and jet bundles in algebraic geometry.

1.1. **Motivations and applications.** Motivated by their stabilisation result in the Grothendieck ring of varieties [VW15], Vakil and Wood conjectured that for a very ample line bundle $\mathcal{L}$ on a smooth projective complex variety, the space of non-singular sections of $\mathcal{L}^{\otimes d}$ should exhibit cohomological stability. In the special case of the projective space, Tommasi obtained the following result.

Theorem 1.2 (Tommasi, [Tom14]). *Let $d, n \geq 1$ be integers. Let $U_{d,n}$ be the space of non-singular holomorphic sections of $\mathcal{O}(d)$ on $\mathbb{CP}^n$. The rational cohomology of $U_{d,n}$ is isomorphic to the rational cohomology of the space $\mathrm{GL}_{n+1}(\mathbb{C})$ in degrees $* < \frac{d+1}{2}$.*

In work in progress, she furthermore investigates an extension of this result to arbitrary smooth projective varieties [Tom23]. Using different techniques, O. Banerjee also confirmed the conjecture of Vakil and Wood in the case of smooth projective curves [Ban21].

The present work was strongly motivated by the result of Tommasi and the wish to understand the stable cohomology from a more homotopy theoretic point of view. At the time of writing, let us in particular mention the following result:

Theorem 1.3 (Tommasi, work in progress in [Tom23])**.** *Let X be a smooth projective complex variety of dimension n and $\mathcal{L}$ be a very ample line bundle on X. Let $d \geq 1$ be an integer and U_d be the space of non-singular holomorphic sections of $\mathcal{L}^{\otimes d}$. There is a Vassiliev spectral sequence converging to the homology of U_d. Working with rational coefficients, this spectral sequence degenerates on the E_2-page in the stable range if and only if the stable cohomology is a free commutative graded algebra on the cohomology of X shifted by one degree.*

Assuming this degeneration, the rational cohomology of U_d in degrees $* < \lfloor \frac{d+1}{2} \rfloor$ is given by the free commutative graded algebra $\Lambda\left(H^{*-1}(X;\mathbb{Q})\right)$ on the cohomology of X shifted by one degree.

In the last section (Section 8) of this paper, we apply our main theorem to spaces of smooth hypersurfaces to prove a homological stability result with rational coefficients.

Theorem 1.4 (see Theorem 8.2)**.** *Let X be a smooth projective complex variety and $\mathcal{L}$ be a very ample line bundle on X. The rational cohomology ring of the space $\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)$ of non-singular sections (in the classical sense) of the d-th tensor power of $\mathcal{L}$ is isomorphic to $\Lambda\left(H^{*-1}(X;\mathbb{Q})\right)$ in degrees $* < \frac{d-1}{2}$.*

Firstly, let us point out that this agrees with the work in progress of Tommasi. In fact, one can use our main theorem to show the degeneration of the Vassiliev spectral sequence she constructed. Secondly, in contrast to many other instances of homological stability, one should remark that there are no natural stabilisation maps from spaces of non-singular sections of $\mathcal{L}^d$ to those of $\mathcal{L}^{d+1}$. Thus, we only mean that the cohomology rings abstractly stabilise, and the answer only depends on X and not on $\mathcal{L}$. After the apparition of the first version of the present article, and using different tools, Das and Howe proved a version of the above theorem for hypersurfaces in algebraic varieties over any algebraically closed field [DH22].

On the other hand, we find it quite interesting to notice that there is in general no integral homological stability. In fact, we prove the following result about the moduli space of smooth hypersurfaces of degree d in $\mathbb{CP}^2$:

Theorem 1.5 (see Proposition 8.10)**.** *Let $d \geq 6$ be an integer. We have:*

$$H_2(\Gamma_{\mathrm{hol,ns}}(\mathbb{CP}^2, \mathcal{O}(d)); \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & d \equiv 0 \mod 2 \\ 0 & d \equiv 1 \mod 2. \end{cases}$$

Besides the phenomenon this result illustrates, its proof showcases the potential of homotopical methods allowed by our main theorem. Indeed, the computation comes down to simple manipulations of Steenrod squares where the parity of d is reflected in the Chern class of $\mathcal{O}(d)$. In contrast, a more classical approach following the work of Vassiliev [Vas99] would require knowledge of non-trivial differentials in spectral sequences that quickly grow out of hand when d increases.

For good measure, we also study the p-torsion in the homology of $\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)$ and show that it stabilises when $p \geq \dim_{\mathbb{C}} X + 2$ and $d \to \infty$. (See Proposition 8.15.)

The results of this paper are also inspired by analogies with theorems in arithmetic probabilities, such as Poonen's Bertini theorem over finite fields [Poo04], and in motivic statistics in the Grothendieck ring of varieties as in [VW15] or [BH19]. The recent results of Bilu and Howe particularly influenced the current formulation of our main theorem and we would like to recommend the introduction of their paper [BH19] to the reader interested in an overview of these analogies. Finally, we also wish to mention that I. Banerjee recently announced a result relating non-singular sections of a line bundle on an algebraic curve and smooth sections of the same line bundle [Ban20].

1.2. **Acknowledgements.** I would like to thank my PhD advisor Søren Galatius for suggesting to compare algebraic sections to continuous sections of jet bundles. It is a pleasure to thank him for his encouragement and many helpful discussions. I would also like to thank Orsola Tommasi for discussing and sharing her work with me, as well as Ronno Das for helpful discussions related to this project. I was supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151) as well as the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682922).

2. Statement of the main theorem

We begin by a few preliminary definitions before stating precisely our main theorem. Throughout this article, X is a smooth projective complex variety and $\mathcal{E}$ is a holomorphic vector bundle on X. We denote by Γ the space of sections of a vector bundle, and decorate it with subscripts "hol" or $\mathcal{C}^0$ to indicate respectively holomorphic or continuous sections. We will make extensive use of Čech (or sheaf, as they will agree in our setting) cohomology with compact support which we denote by $\check{H}^*_c$ and refer to [Bre97] for its definition and standard properties. All homology and cohomology groups will be taken with integral coefficients, unless otherwise specified. We recall the following definition of jet ampleness.

Definition 2.1 (Compare [BDRS99]). Let $k \geq 0$ be an integer. Let $x_1, \ldots, x_t$ be t distinct points in X and $(k_1, \ldots, k_t)$ be a t-uple of positive integers such that $\sum_i k_i = k+1$. Denote by $\mathcal{O}$ the structure sheaf of X and by $\mathfrak{m}_i$ the maximal ideal sheaf corresponding to x_i. We regard the tensor product $\otimes_{i=1}^t \mathfrak{m}_i^{k_i}$ as a subsheaf of $\mathcal{O}$ under the multiplication map $\otimes_{i=1}^t \mathfrak{m}_i^{k_i} \to \mathcal{O}$. We say that $\mathcal{E}$ is *k-jet ample* if the evaluation map

$$\Gamma_{\text{hol}}(\mathcal{E}) \longrightarrow \Gamma_{\text{hol}}\left(\mathcal{E} \otimes \left(\mathcal{O}/\otimes_{i=1}^t \mathfrak{m}_i^{k_i}\right)\right) \cong \bigoplus_{i=1}^t \Gamma_{\text{hol}}\left(\mathcal{E} \otimes \left(\mathcal{O}/\mathfrak{m}_i^{k_i}\right)\right)$$

is surjective for any $x_1, \ldots, x_t$ and $k_1, \ldots, k_t$ as above.

Example 2.2. A vector bundle $\mathcal{E}$ is 0-jet ample if and only if it is spanned by its global sections. In the case of a line bundle, 1-jet ampleness corresponds to the usual notion of very ampleness. On a curve, a line bundle is k-jet ample whenever it is k-very ample. However, on higher dimensional varieties, a k-jet ample line bundle is also k-very ample but the converse is not true in general. Finally, and most importantly for us, if $\mathcal{A}$ and $\mathcal{B}$ are holomorphic vector bundles which are respectively a- and b-jet ample, then their tensor product $\mathcal{A} \otimes \mathcal{B}$ is $(a+b)$-jet ample. (See [BDRS99, Proposition 2.3].)

To ease the readability of various statements throughout the paper, we will use the following notation.

Definition 2.3. For a holomorphic vector bundle $\mathcal{E}$ on X and an integer $r \in \mathbb{N}$, we define $N(\mathcal{E}, r) \geq 0$ to be the largest integer N such that $\mathcal{E}$ is $((N+1) \cdot (r+1) - 1)$-jet ample. If no such integer exists, we set $N(\mathcal{E}, r) = -1$, although we shall never consider such a case in this paper.

Let us also recall the construction of the jet bundle from [Gro67, IV.16.7] (where it is called the sheaf of principal parts). The diagonal morphism $\Delta\colon X \to X \times X$ gives a surjection of sheaves $\Delta^\sharp\colon \Delta^*\mathcal{O}_{X\times X} \to \mathcal{O}_X$. Denoting by $\mathcal{I}$ the kernel, we have $\mathcal{O}_X \cong \Delta^*\mathcal{O}_{X\times X}/\mathcal{I}$. For an integer $r \geq 0$, we define the r-th jet bundle of $\mathcal{O}_X$ to be

$$J^r\mathcal{O}_X := \Delta^*\mathcal{O}_{X\times X}/\mathcal{I}^{r+1}.$$

The projections $p_i\colon X \times X \to X$ give two $\mathcal{O}_X$-algebra structures on $J^r\mathcal{O}_X$ and, unless otherwise specified, we use the one given by the first projection p_1. The other morphism induced by p_2 is denoted by

$$d^r_X\colon \mathcal{O}_X \longrightarrow J^r\mathcal{O}_X.$$

For a holomorphic vector bundle $\mathcal{E}$ on X, we define its *r-th jet bundle* to be

$$J^r\mathcal{E} := J^r\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} \tag{1}$$

where $J^r\mathcal{O}_X$ is seen as an $\mathcal{O}_X$-module via the morphism d^r_X for the tensor product, and the result is regarded as an $\mathcal{O}_X$-module again via p_1. It comes with the morphism

$$d^r_{X,\mathcal{E}} := d^r_X \otimes \mathcal{E}\colon \mathcal{E} \longrightarrow J^r\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} = J^r\mathcal{E}.$$

Taking global sections, we obtain the *jet map*:

$$j^r = \Gamma(d^r_{X,\mathcal{E}})\colon \Gamma_{\mathrm{hol}}(\mathcal{E}) \longrightarrow \Gamma_{\mathrm{hol}}(J^r\mathcal{E}). \tag{2}$$

The most important observation for us is the following: if $x \in X$ is a point with maximal ideal sheaf $\mathfrak{m}$, the fibre $(J^r\mathcal{E})|_x$ is naturally identified with the complex vector space $\mathcal{E}_x/\mathfrak{m}_x^{r+1}\mathcal{E}_x$. Furthermore, the composition

$$\mathcal{E}_x \xrightarrow{(d^r_{X,\mathcal{E}})_x} (J^r\mathcal{E})_x \longrightarrow (J^r\mathcal{E})|_x = \mathcal{E}_x/\mathfrak{m}_x^{r+1}\mathcal{E}_x$$

is the natural quotient map. (Here, and everywhere else, we write $\mathcal{E}_x$ for the stalk of the sheaf $\mathcal{E}$ and $\mathcal{E}|_x = \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$ for the fibre of the bundle $\mathcal{E}$.) Intuitively, for a holomorphic section s of $\mathcal{E}$, one should think of the value of $j^r(s)$ at a point $x \in X$ as the tuple of all derivatives of s at x up to order r. In particular, the following lemma is a direct consequence of the definitions.

Lemma 2.4. *Let $\mathcal{E}$ be a holomorphic vector bundle on X and let $N(\mathcal{E}, r)$ be as in Definition 2.3. Let $(x_0, \ldots, x_p)$ be a tuple of $p+1$ distinct points in X. If $p \leq N(\mathcal{E}, r)$, the simultaneous evaluation of the jet map (2) at these points*

$$\begin{aligned} j^r_{(x_0,\ldots,x_p)}\colon \Gamma_{\mathrm{hol}}(\mathcal{E}) &\longrightarrow (J^r\mathcal{E})|_{x_0} \times \cdots \times (J^r\mathcal{E})|_{x_p} \\ s &\longmapsto (j^r(s)(x_0), \ldots, j^r(s)(x_p)) \end{aligned}$$

is surjective. □

We shall now explain what we precisely mean by restricting the behaviour of sections of $\mathcal{E}$. In particular, we will require certain subsets of the jet bundle to be "semi-algebraic". This is a technical condition which is quite arbitrary. We believe that a clearer and more general notion could be used, but we were unfortunately not able to make the arguments of Section 4 work without it. Our arguments rely on multiple properties of these sets: they admit cell decompositions, have a well-defined dimension, and they behave well under projections and closure. (See Section 4.2 for their single but crucial use.)

There is a well-studied concept of real semi-algebraic subsets of an Euclidean space. They are subsets defined by polynomial equations and inequalities.

Definition 2.5 (Compare [BCR98])**.** A semi-algebraic subset of $\mathbb{R}^n$ is a union of finitely many subsets of the form

$$\{x \in \mathbb{R}^n \mid P(x) = 0,\ Q_1(x) > 0, \dots,\ Q_l(x) > 0\},$$

where $l \in \mathbb{N}$ and $P, Q_1, \dots, Q_l \in \mathbb{R}[X_1, \dots, X_n]$.

We adapt the definition to families, i.e. to subsets of vector bundles, by demanding the standard definition to be satisfied locally in charts. This is well-defined because an algebraic variety X has an atlas whose transition functions are algebraic, hence respect the semi-algebraicity.

Let us be more precise. First, we briefly recall the notion of an algebraic atlas on X. To lighten the notation, we let n be the complex dimension of X and m be the complex rank of $J^r\mathcal{E}$. We denote by $V(-)$ the vanishing set of the tuple of polynomials.

The variety X can be covered by Zariski open subsets, each of the form

$$U \cong V(f_1, \dots, f_{d-n}) \subset \mathbb{C}^d$$

for some integer $d \geq 1$ and polynomials $f_1, \dots, f_{d-n}$. Furthermore, if U and W are Zariski open subsets of X with $\alpha\colon U \cong V(f_1, \dots, f_{d-n}) \subset \mathbb{C}^d$ and $\beta\colon W \cong V(g_1, \dots, g_{d'-n}) \subset \mathbb{C}^{d'}$, the homeomorphism on the intersection

$$\alpha(W \cap U) \cap V(f_1, \dots, f_{d-n}) \xrightarrow{\cong} W \cap U \xrightarrow{\cong} \beta(U \cap W) \cap V(g_1, \dots, g_{d'-n})$$

is given by a rational function whose domain is a subset of $\mathbb{C}^d$ and codomain is a subset of $\mathbb{C}^{d'}$. Recall also that the algebraic vector bundle $J^r\mathcal{E}$ is equivalently given by the data of trivialising Zariski open subsets $U_i \subset X$ (over which $J^r\mathcal{E}|_{U_i} \cong U_i \times \mathbb{C}^m$) and transition functions on overlaps $U_i \cap U_j \to \mathrm{GL}_m(\mathbb{C})$. Most importantly for us, the transition functions are regular morphisms.

Definition 2.6. Let n be the complex dimension of X and m be the complex rank of $J^r\mathcal{E}$. A subset $\mathfrak{T} \subset J^r\mathcal{E}$ is *real semi-algebraic* if there exists a cover $X = \bigcup U_i$ by Zariski open subsets such that the following conditions hold for each i:

(1) the jet bundle may be trivialised over U_i via a map $\varphi_i\colon J^r\mathcal{E}|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{C}^m$;
(2) there is a chart $\phi_i\colon U_i \xrightarrow{\cong} V(f_1^i, \dots, f_{d_i-n}^i) \subset \mathbb{C}^{d_i}$ for some polynomials $f_1^i, \dots, f_{d_i-n}^i$;
(3) and the image in $\mathbb{R}^{2(d_i+m)}$ of $\mathfrak{T}|_{U_i}$ via the map

$$J^r\mathcal{E}|_{U_i} \xrightarrow{\varphi_i} U_i \times \mathbb{C}^m \xrightarrow{\phi_i \times \mathrm{id}} V(f_1^i, \dots, f_{d_i-n}^i) \times \mathbb{C}^m \subset \mathbb{C}^{d_i+m} \cong \mathbb{R}^{2(d_i+m)}$$

is a real semi-algebraic subset. (Here $\mathfrak{T}|_{U_i}$ is the restriction of $\mathfrak{T}$ above U_i.)

We will often drop the adjective "real" as we will never consider any complex analogue. In essence, a subset $\mathfrak{T} \subset J^r\mathcal{E}$ is semi-algebraic in the sense of Definition 2.6 when it is semi-algebraic in the usual way when "read in charts". As all the change-of-coordinates maps described above are rational functions, being semi-algebraic is independent of the choice of the cover. Indeed, the image of a semi-algebraic set by a rational function is still semi-algebraic (see [BCR98, Section 2.2]).

A semi-algebraic subset has a well-defined dimension (as in [BCR98, Section 2.8]) which can be thought of as the maximal dimension in a decomposition into cells of the form $]0,1[^d$ (see [BCR98, Corollary 2.8.9]). We therefore get a well-defined dimension for a semi-algebraic subset $\mathfrak{T} \subset J^r\mathcal{E}$ by looking at the dimensions when "reading in charts":

Definition 2.7. Let $\mathfrak{T} \subset J^r\mathcal{E}$ be a semi-algebraic subset. Let $X = \bigcup U_i$ by a finite cover as in Definition 2.6 (the finiteness can always be arranged by compactness of X) and write $\mathfrak{T}_{U_i} \subset \mathbb{R}^{2(d_i+m)}$ for the semi-algebraic sets obtained using the condition (3). Each of them has a well-defined dimension and we let the *dimension of* $\mathfrak{T}$ be their maximum.

In the following definition, we denote by $\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E}$ the complex rank of $J^r\mathcal{E}$.

Definition 2.8. We say that a subset $\mathfrak{T} \subset J^r\mathcal{E}$ is an *admissible Taylor condition* if it is closed, real semi-algebraic and has dimension at most $2(\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E} - 1)$. We will use the notation $\mathfrak{T}|_x := (J^r\mathcal{E})|_x \cap \mathfrak{T}$ for the fibre above a point $x \in X$.

Remark 2.9. Although our definition is quite technical and general, the typical admissible Taylor conditions arise as subvarieties of high enough codimension. Indeed, any closed subvariety $\mathfrak{T} \subset J^r\mathcal{E}$ of the jet bundle of complex codimension at least $\dim_{\mathbb{C}} X + 1$ defines an admissible Taylor condition.

Motivated by the previous remark, and to help general bookkeeping throughout the paper, we will use the following notation.

Definition 2.10. The (real) *excess codimension* of an admissible Taylor condition $\mathfrak{T}$ is defined to be the number $e(\mathfrak{T}) = \mathrm{codim}_{\mathbb{R}}\mathfrak{T} - \dim_{\mathbb{R}} X \geq 2$, where $\mathrm{codim}_{\mathbb{R}}\mathfrak{T}$ is the real codimension of $\mathfrak{T}$ in the jet bundle $J^r\mathcal{E}$.

We are now ready to define what it means for a section to be singular with respect to an admissible Taylor condition $\mathfrak{T}$.

Definition 2.11. A holomorphic section s of the vector bundle $\mathcal{E}$ is said to be *singular* if there exists a point $x \in X$ such that $j^r(s)(x) \in \mathfrak{T}|_x$. Similarly, a (continuous) section s of the vector bundle $J^r\mathcal{E}$ is said to be *singular* if there exists a point $x \in X$ such that $s(x) \in \mathfrak{T}|_x$. A section that is not singular is said to be *non-singular*.

Example 2.12. If $\mathcal{E}$ is a line bundle, we may take $\mathfrak{T}$ to be the zero section of $J^1\mathcal{E}$. It is an admissible Taylor condition and a singular section is one that vanishes at a point on X where its derivative also vanishes. In particular, if s is a non-singular section, its zero set $Z(s) := \{x \in X \mid s(x) = 0\} \subset X$ is a smooth submanifold.

When talking about spaces of sections Γ, we will use the subscript "ns" to denote the subspace of ***n**on-**s**ingular* sections. The following is our main result.

Theorem 2.13. *Let $r \geq 0$ and $N \geq 1$ be integers. Let $\mathcal{E}$ be an $((N+1)\cdot(r+1)-1)$-jet ample vector bundle on X and let $\mathfrak{T} \subset J^r\mathcal{E}$ be an admissible Taylor condition. The composition*

$$\Gamma_{\mathrm{hol,ns}}(\mathcal{E}) \xrightarrow{j^r} \Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E}) \hookrightarrow \Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E})$$

induces an isomorphism in homology:

$$H_*(\Gamma_{\mathrm{hol,ns}}(\mathcal{E});\mathbb{Z}) \longrightarrow H_*(\Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E});\mathbb{Z})$$

in the range of degrees $* < N\cdot(e(\mathfrak{T})-1) + e(\mathfrak{T}) - 2$.

2.1. **Outline of the paper.** There are two key ingredients in the proof of the main theorem 2.13. The first one is a spectral sequence à la Vassiliev (see [Vas94, Chapter III] for an analogous statement in the case of smooth sections, and [Vas99] for more explicit computations in small degrees). The starting idea is that one should study the space of singular sections and deduce the homology of the space of non-singular sections via Alexander duality. The former has a natural filtration given by counting the number of singularities and it is used to construct a spectral sequence converging to its cohomology. Comparing spectral sequences allows us to compare sections of $\mathcal{E}$ and $J^r\mathcal{E}$. The second ingredient is a version of the classical Stone–Weierstrass theorem adapted from the work of Mostovoy [Mos06] which allows us to compare holomorphic and continuous sections of $J^r\mathcal{E}$.

We first explain how to resolve the singular subspaces and construct the Vassiliev spectral sequence in Section 3. We study its first page in Section 4. In Section 5, we explain how to go from holomorphic sections to continuous sections. Then, in Section 6, we construct a morphism of spectral sequences and use it to compare various spaces of sections. We finish proving our main theorem in Section 7. Lastly, in Section 8, we apply our results to study spaces of non-singular sections of a very ample line bundle on a projective variety.

3. Resolution of singularities

In this section, we choose an admissible Taylor condition $\mathfrak{T} \subset J^r\mathcal{E}$ inside the r-th jet bundle of a holomorphic vector bundle $\mathcal{E}$ on X, and we will write for brevity

$$\Gamma = \Gamma_{\text{hol}}(\mathcal{E}) \quad \text{and} \quad \Sigma = \Gamma_{\text{hol}}(\mathcal{E}) - \Gamma_{\text{hol,ns}}(\mathcal{E})$$

for the vector space Γ of all holomorphic sections of $\mathcal{E}$ and its subspace Σ of singular sections. We also define the *singular space* of a section $f \in \Gamma$

$$\operatorname{Sing}(f) := \{x \in X \mid j^r(f)(x) \in \mathfrak{T}\} \subset X \tag{3}$$

as the space of points where f is singular (as in Definition 2.11). Our final goal, Theorem 2.13, is to understand the homology of the space of non-singular sections $\Gamma_{\text{hol,ns}}(\mathcal{E}) = \Gamma - \Sigma$. By Alexander duality

$$\check{H}^i_c(\Sigma) \cong \widetilde{H}_{2\dim_{\mathbb{C}}\Gamma - i - 1}(\Gamma - \Sigma),$$

it is equivalent to understand the compactly supported Čech cohomology of its complement Σ. To achieve that, we want to construct a spectral sequence converging to $\check{H}^*_c(\Sigma)$. This spectral sequence arises from a resolution of the space Σ which we define in this section.

3.1. **Construction of the resolution.** We will construct a space $R\mathfrak{X} \to \Sigma$ mapping surjectively to the singular subspace Σ. The inverse image of a section $f \in \Sigma$ with $j+1$ singularities will be a j-simplex Δ^j. This will allow us to show that $R\mathfrak{X} \to \Sigma$ induces an isomorphism in cohomology with compact supports (up to some modifications). The space $R\mathfrak{X}$ will be advantageously filtered by subspaces $R^j\mathfrak{X}$ related via pushout diagrams resembling the skeletal decomposition of a simplicial space. This filtration then yields a spectral sequence computing the cohomology of $R\mathfrak{X}$, hence that of Σ.

This is inspired by the so-called truncated resolution of Mostovoy [Mos12] but written in a more functorial way as in [Vok07].

In what follows, the space Γ is given its canonical topology coming from the fact that it is a finite dimensional complex vector space. Let F be the category whose objects are the finite sets $[n] := \{0, \dots, n\}$ for $n \geq 0$ and whose morphisms are *all* maps of sets $[n] \to [m]$. Let Top be the category of topological spaces and continuous maps between them. We define the following functor

$$\begin{aligned} \mathfrak{X}\colon \mathsf{F}^{\mathrm{op}} &\longrightarrow \mathsf{Top} \\ [n] &\longmapsto \mathfrak{X}[n] := \{(f, s_0, \dots, s_n) \in \Gamma \times X^{n+1} \mid \forall i,\ s_i \in \mathrm{Sing}(f)\} \end{aligned} \tag{4}$$

where $\mathfrak{X}[n]$ is given the subspace topology from $\Gamma \times X^{n+1}$. On morphisms, for a map of sets $g\colon [n] \to [m]$, we define

$$\begin{aligned} \mathfrak{X}(g)\colon \mathfrak{X}[m] &\longrightarrow \mathfrak{X}[n] \\ (f, s_0, \dots, s_m) &\longmapsto (f, s_{g(0)}, \dots, s_{g(n)}). \end{aligned}$$

For an integer $k \geq 0$, we denote by $\mathsf{F}_{\leq k}$ the full sub-category of F on objects $[n]$ for $n \leq k$. Let us also write

$$|\Delta^n| = \{(t_0, \dots, t_n) \mid \forall i, 0 \leq t_i \leq 1 \text{ and } t_0 + \dots + t_n = 1\} \subset \mathbb{R}^{n+1}$$

for the standard topological n-simplex, and denote by $\partial|\Delta^n|$ its boundary. In particular, the assignment $[n] \mapsto |\Delta^n|$ gives a functor $\mathsf{F} \to \mathsf{Top}$. For an integer $j \geq 0$, we define the *j-th geometric realisation of $\mathfrak{X}$* by the following coend:

$$\begin{aligned} R^j\mathfrak{X} &:= \int^{[n]\in \mathsf{F}_{\leq j}} \mathfrak{X}[n] \times |\Delta^n| \\ &= \left(\bigsqcup_{0\leq n\leq j} \mathfrak{X}[n] \times |\Delta^n| \right) / \sim \end{aligned} \tag{5}$$

where the equivalence relation $\sim$ is generated by $(\mathfrak{X}(g)(z), t) \sim (z, g_*(t))$ for all maps $g\colon [n] \to [m]$ in F. (Here $g_*\colon |\Delta^n| \to |\Delta^m|$ denotes the usual map induced on the simplices by functoriality.) This is of course reminiscent of the classical geometric realisation of a simplicial space. Note however that here a cell $|\Delta^n|$ in the geometric realisation is indexed by an *unordered* set of singularities, even though the functor $\mathfrak{X}$ is defined using ordered tuples. Indeed, all the permutations $[n] \to [n]$ are valid morphisms in our category F.

Let $j \geq 1$ be an integer. We now describe how $R^j\mathfrak{X}$ may be obtained from $R^{j-1}\mathfrak{X}$ via a pushout diagram. Let L_j be the following set:

$$L_j := \{(f, s_0, \dots, s_j) \in \Gamma \times X^{j+1} \mid \exists l \neq k \text{ such that } s_l = s_k\} \subset \mathfrak{X}[j] \tag{6}$$

topologised as a subspace of $\mathfrak{X}[j]$. This should be thought of as the analogue of the "latching object" of a simplicial space. We denote by

$$L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|$$

the quotient space of $L_j \times |\Delta^j|$ by the symmetric group $\mathfrak{S}_{j+1}$ acting on L_j by permuting the singularities s_i, and on $|\Delta^j|$ by permuting the coordinates. Denote by $\widehat{\cdot}$ the omission of an element in a tuple.

Lemma 3.1. *The formula*

$$((f, s_0, \dots, s_j), (t_0, \dots, t_j)) \longmapsto \begin{cases} \big((f, s_0, \dots, \widehat{s_l}, \dots, s_j), (t_0, \dots, t_k + t_l, \dots, \widehat{t_l}, \dots, t_j)\big) \\ \text{if there exists } k \neq l \text{ such that } s_l = s_k \end{cases}$$

gives a well-defined map $L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j| \to R^{j-1}\mathfrak{X}$.

Proof. The formula appears ill-defined as we are choosing arbitrarily two indices k and l. The identifications made by the coend formula (5) show that any choice will yield the same class in the quotient. □

Recall that a point $t = (t_0, \ldots, t_j) \in |\Delta^j|$ is in the boundary $\partial|\Delta^j|$ if one of its coordinates vanishes. An argument similar to the proof of Lemma 3.1 above gives the following.

Lemma 3.2. *The formula*

$$((f, s_0, \ldots, s_j), (t_0, \ldots, t_j)) \longmapsto \left((f, s_0, \ldots, \widehat{s_l}, \ldots, s_j), (t_0, \ldots, \widehat{t_l}, \ldots, t_j)\right) \ \textit{if}\ t_l = 0$$

gives a well-defined map $\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j| \to R^{j-1}\mathfrak{X}$. □

Consider the following pushout diagram of spaces:

$$\begin{tikzcd}
L_j \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j| \arrow[r, hook] \arrow[d, hook] & \mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j| \arrow[d] \\
L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j| \arrow[r] & \left(L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|\right) \bigcup \left(\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j|\right).
\arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Equivalently, the pushout is the union of the top-right and bottom-left spaces inside $\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} |\Delta^j|$. The maps defined above in Lemma 3.1 and Lemma 3.2 glue to a continuous map

$$\alpha_{j-1} \colon \left(L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|\right) \bigcup \left(\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j|\right) \longrightarrow R^{j-1}\mathfrak{X}.$$

The natural map $\mathfrak{X}[j] \times |\Delta^j| \to R^j\mathfrak{X}$ factors through the quotient by the symmetric group action and gives a map

$$\beta_j \colon \mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} |\Delta^j| \longrightarrow R^j\mathfrak{X}.$$

From the coend formula (5) and the inclusion of the full sub-category $\mathsf{F}_{\leq j-1} \subset \mathsf{F}_{\leq j}$, we also get a natural map $R^{j-1}\mathfrak{X} \to R^j\mathfrak{X}$. We are now ready to state the

Proposition 3.3. *The following square is a pushout diagram of topological spaces:*

$$\begin{tikzcd}
\left(L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|\right) \bigcup \left(\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j|\right) \arrow[r, "\alpha_{j-1}"] \arrow[d, hook] & R^{j-1}\mathfrak{X} \arrow[d] \\
\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} |\Delta^j| \arrow[r, "\beta_j"'] & R^j\mathfrak{X}.
\arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{7}$$

Proof. We may construct the pushout P as the quotient

$$P := \left(R^{j-1}\mathfrak{X} \bigsqcup \mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} |\Delta^j|\right) / \sim .$$

One may check that the map β_j together with the natural map $R^{j-1}\mathfrak{X} \to R^j\mathfrak{X}$ gives a map from the disjoint union above which factors through the quotient. Hence we get a well-defined map $P \to R^j\mathfrak{X}$. We now construct a continuous inverse. Recall that $R^j\mathfrak{X}$ is defined in (5) as a quotient of

$$\left(\bigsqcup_{0 \leq n \leq j-1} \mathfrak{X}[n] \times |\Delta^n|\right) \bigsqcup \left(\mathfrak{X}[j] \times |\Delta^j|\right).$$

The natural map $\left(\bigsqcup_{0 \leq n \leq j-1} \mathfrak{X}[n] \times |\Delta^n|\right) \to R^{j-1}\mathfrak{X} \to P$ together with the identity of $\mathfrak{X}[j] \times |\Delta^j|$ gives a map from the disjoint union that factors through the quotient and yields a well-defined map $R^j\mathfrak{X} \to P$. One may finally verify that it is the inverse of the map $P \to R^j\mathfrak{X}$ constructed above. $\square$

We now turn to proving some topological results about our constructions.

Lemma 3.4. *For any integer $n \geq 0$, the subspace $\mathfrak{X}[n] \subset \Gamma \times X^{n+1}$ defined in (4) is closed.*

Proof. Let $\mathrm{ev}\colon \Gamma \times X^{n+1} \to (J^r\mathcal{E})^{n+1}$ be the simultaneous evaluation of the jet map j^r (defined in (2)) at $(n+1)$ points of X. We observe directly from the definitions that $\mathfrak{X}[n] = \mathrm{ev}^{-1}(\mathfrak{T}^{n+1})$, hence is closed as the inverse image of a closed set. $\square$

Lemma 3.5. *For any $n \geq 0$, the map $\rho_n\colon \mathfrak{X}[n] \to \Gamma$ given by $(f, s_0, \ldots, s_n) \mapsto f$ is a proper map.*

Proof. The projection onto the first factor $\Gamma \times X^{n+1} \to \Gamma$ is proper as X^{n+1} is compact. Hence so is its restriction ρ_n to the closed subspace $\mathfrak{X}[n]$. $\square$

In particular, the map ρ_n is closed, so $\Sigma = \rho_1(\mathfrak{X}[1])$ is closed in Γ. We have natural projections maps $\mathfrak{X}[n] \times |\Delta^n| \to \mathfrak{X}[n] \xrightarrow{\rho_n} \Gamma$ for any $n \geq 0$. They give rise to a map

$$\tau_j\colon R^j\mathfrak{X} \longrightarrow \Sigma \tag{8}$$

for every integer $j \geq 0$.

Lemma 3.6. *For any integer $j \geq 0$, the map $\tau_j\colon R^j\mathfrak{X} \to \Sigma$ is a proper map.*

Proof. We have to show that the preimage of any compact set is compact. Equivalently, because Σ is locally compact and Hausdorff, we will show that τ_j is a closed map with compact fibres. From Lemma 3.5, for any n, the map ρ_n is closed and hence so is the composition $\mathfrak{X}[n] \times |\Delta^n| \to \mathfrak{X}[n] \xrightarrow{\rho_n} \Gamma$. This implies that τ_j is closed. It remains to see that it has compact fibres. If $f \in \Sigma$, we observe that $\tau_j^{-1}(f) = \beta_j\left(\rho_j^{-1}(f)\right)$ which is compact as $\rho_j^{-1}(f)$ is, by Lemma 3.5. $\square$

A major advantage of the pushout square (7) is that it allows us to prove the following topological lemma.

Lemma 3.7. *For any integer $j \geq 0$, the space $R^j\mathfrak{X}$ is paracompact and Hausdorff. Furthermore, the natural map $R^{j-1}\mathfrak{X} \to R^j\mathfrak{X}$ is a closed embedding.*

Proof. Firstly, from Lemma 3.4, we know that $R^0\mathfrak{X} = \mathfrak{X}[0] \subset \Gamma \times X$ is a closed subset, hence is itself paracompact Hausdorff. Then the lemma is proven inductively using the pushout diagram (7) together with the fact that

$$\left(\left(L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|\right) \bigcup \left(\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j|\right)\right) \hookrightarrow \mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} |\Delta^j|$$

is a closed embedding. $\square$

In the sequel, using the closed embedding of Lemma 3.7 just above, we will simply write $R^{j-1}\mathfrak{X} \subset R^j\mathfrak{X}$. For an integer $j \geq 0$, we let

$$Y_j := \{(f, s_0, \ldots, s_j) \in \mathfrak{X}[j] \mid s_l \neq s_k \text{ if } l \neq k\} = \mathfrak{X}[j] - L_j \subset \mathfrak{X}[j] \tag{9}$$

be the subspace of $\mathfrak{X}[j]$ where the singularities are pairwise distinct. For later use, we record the following homeomorphism, which is a direct consequence of the pushout square (7) and the fact that the vertical maps therein are closed embeddings:

$$R^j\mathfrak{X} - R^{j-1}\mathfrak{X} \cong Y_j \times_{\mathfrak{S}_{j+1}} \mathrm{Interior}(|\Delta^j|). \tag{10}$$

Let us now discuss why $\tau_j \colon R^j\mathfrak{X} \to \Sigma$ needs to be slightly modified to obtain a meaningful "resolution" of Σ. The fibre $\tau_j^{-1}(f)$ above a section $f \in \Sigma$ that has at most $j+1$ singularities is by construction a j-simplex. Hence it is contractible and one might hope that τ_j induces an isomorphism in cohomology. This is unfortunately not the case. Indeed, $\tau_j^{-1}(f)$ is not contractible if f has at least $j+2$ singularities. To fix this problem, we will modify $R^j(\Sigma)$ by gluing a cone over each fibre $\tau_j^{-1}(f)$ which is not contractible. The precise construction is as follows.

Let $N \geq 0$ be an integer. We let

$$\Sigma_{\geq N+2} := \{f \in \Gamma \mid \#\mathrm{Sing}(f) \geq N+2\} \subset \Sigma \tag{11}$$

denote the subspace of those sections with at least $N+2$ singularities. We denote by $\overline{\Sigma_{\geq N+2}}$ its closure in Σ (or equivalently, in Γ). Observe that the surjectivity of the map τ_N implies the following equality:

$$\tau_N\left(\tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right)\right) = \overline{\Sigma_{\geq N+2}}.$$

We glue fibrewise a cone over each $f \in \overline{\Sigma_{\geq N+2}}$ by defining the space $R^N_{\text{cone}}(\Sigma)$ as the following homotopy pushout:

$$\begin{tikzcd} \tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \arrow[r, hook] \arrow[d, "\tau_N"'] & R^N\mathfrak{X} \arrow[d] \\ \overline{\Sigma_{\geq N+2}} \arrow[r] \arrow[ur, phantom, "\mathrm{ho}\ulcorner"] & R^N_{\text{cone}}\mathfrak{X}. \end{tikzcd} \tag{12}$$

All three defining spaces in the corners of (12) map to Σ, hence we obtain a surjective projection map

$$\pi \colon R^N_{\text{cone}}\mathfrak{X} \longrightarrow \Sigma. \tag{13}$$

We want to prove that π induces an isomorphism in Čech cohomology with compact supports. We begin with a couple of lemmas.

Lemma 3.8. *The map $\pi \colon R^N_{\mathit{cone}}\mathfrak{X} \longrightarrow \Sigma$ is proper.*

Proof. We will prove that is it closed with compact fibres, which implies the properness. By definition of the homotopy pushout, $R^N_{\text{cone}}\mathfrak{X}$ is a quotient of the following disjoint union:

$$R^N\mathfrak{X} \bigsqcup \tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \times [0,1] \bigsqcup \overline{\Sigma_{\geq N+2}}.$$

The map π is induced by the following three maps: the projection $\tau_N \colon R^N\mathfrak{X} \to \Sigma$, the projection $\tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \times [0,1] \to \tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \to \Sigma$, and the inclusion $\overline{\Sigma_{\geq N+2}} \hookrightarrow \Sigma$. The first two are closed by Lemma 3.6 and the last one is the inclusion of a closed subset, hence closed.

Finally, we prove that the fibres of π are compact. We saw in the proof of Lemma 3.6 that for any $f \in \Sigma$, the fibre $\tau_N^{-1}(f)$ was compact. Now, $\pi^{-1}(f)$ is either $\tau_N^{-1}(f)$ if $f \in \Sigma - \overline{\Sigma_{\geq N+2}}$ or a cone over it if $f \in \overline{\Sigma_{\geq N+2}}$. In any case it is compact. $\square$

Lemma 3.9. *The space $R^N_{\mathit{cone}}\mathfrak{X}$ is paracompact, locally compact, and Hausdorff.*

Proof. The paracompactness and Hausdorffness follow from the definition as a homotopy pushout and Lemma 3.7. It is locally compact as its maps properly to the locally compact space Σ. $\square$

These topological properties will justify our subsequent manipulations of compactly supported Čech cohomology, which agrees with sheaf cohomology with compact supports in this context. The most important corollary is the following

Proposition 3.10. *The map $\pi\colon R^N_{cone}\mathfrak{X} \longrightarrow \Sigma$ induces an isomorphism in Čech cohomology with compact supports.*

Proof. The properness of π proved in Lemma 3.8 implies that it induces a well-defined map in cohomology with compact supports. We also observed in the proof of that lemma that a fibre of π is either a simplex or a cone, hence contractible. The proposition then follows from the Vietoris–Begle theorem [Bre97, V.6.1]. ☐

3.2. **Construction of the spectral sequence.** Let $N \geq 1$ be an integer. Recall from Lemma 3.7 that we have closed embeddings $R^{j-1}\mathfrak{X} \subset R^j\mathfrak{X}$. We define the following filtration on $R^N_{\text{cone}}\mathfrak{X}$:

$$F_0 = R^0\mathfrak{X} \subset F_1 = R^1\mathfrak{X} \subset \cdots \subset F_N = R^N\mathfrak{X} \subset F_{N+1} = R^N_{\text{cone}}\mathfrak{X}.$$

Following standard arguments, we obtain from the filtration a spectral sequence:

$$E_1^{p,q} = \check{H}^{p+q}_c(F_p, F_{p-1}) \cong \check{H}^{p+q}_c(F_p - F_{p-1}) \implies \check{H}^{p+q}_c(R^N_{\text{cone}}\mathfrak{X})$$

where the isomorphism between the cohomology groups on the first page follows from [Bre97, II.12.3]. Using Proposition 3.10 and Alexander duality, we obtain:

$$\check{H}^{p+q}_c(R^N_{\text{cone}}\mathfrak{X}) \cong \check{H}^{p+q}_c(\Sigma) \cong \widetilde{H}_{2\dim_{\mathbb{C}}\Gamma-(p+q)-1}(\Gamma - \Sigma)$$

where $\widetilde{H}$ denotes reduced singular homology. Letting $s = -p-1$ and $t = 2\dim_{\mathbb{C}}\Gamma - q$, we regrade our spectral sequence and obtain the following

Proposition 3.11. *There is a spectral sequence on the second quadrant $s \leq -1$ and $t \geq 0$:*

$$E^1_{s,t} = \check{H}^{2\dim_{\mathbb{C}}\Gamma-1-s-t}_c(F_{-s-1} - F_{-s-2}; \mathbb{Z}) \implies \widetilde{H}_{s+t}(\Gamma - \Sigma; \mathbb{Z}).$$

The differential d^r on the r-th page of the spectral sequence has bi-degree $(-r, r-1)$, i.e. it is a morphism $d^r_{s,t}\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

4. Cohomology groups on the E^1-page

As in the last section, we choose a holomorphic vector bundle $\mathcal{E}$ on X and an admissible Taylor condition $\mathfrak{T} \subset J^r\mathcal{E}$ inside the r-th jet bundle of $\mathcal{E}$. For the remainder of this section, we also let

$$N = N(\mathcal{E}, r)$$

be the largest integer $N \geq 0$ such that $\mathcal{E}$ is $((N+1)\cdot(r+1)-1)$-jet ample as in Definition 2.3. As discussed in the introduction, we assume that such an N exists. If not, the statements in this section are either trivially false, or trivially true as they describe elements of the empty set. For brevity, we still use the following notations

$$\Gamma = \Gamma_{\text{hol}}(\mathcal{E}) \quad \text{and} \quad \Sigma = \Gamma_{\text{hol}}(\mathcal{E}) - \Gamma_{\text{hol,ns}}(\mathcal{E})$$

as well as $\mathfrak{X}$ for the associated functor $\mathsf{F}^{\text{op}} \to \mathsf{Top}$ as in (4).

We will study the first page of the spectral sequence from Proposition 3.11 converging to the cohomology of $R^N_{\text{cone}}\mathfrak{X}$:

$$E^1_{s,t} = \check{H}^{2\dim_{\mathbb{C}}\Gamma-1-s-t}_c(F_{-s-1} - F_{-s-2}; \mathbb{Z}).$$

We will first show that for $-N-1 \leq s \leq -1$ the groups $E^1_{s,t}$ can be written, via Thom isomorphisms, in terms of the cohomology of $\mathfrak{T}$. We will then study qualitatively the cohomology of $F_{N+1} - F_N$, i.e. the column $E^1_{-N-2,*}$, and show that it does not have any influence on the cohomology of the limit in a range of degrees up to around N. Later in Section 5 we will construct spectral sequences for spaces of sections of $J^1\mathcal{E} \setminus \mathfrak{T}$, and in Section 6 we will compare them. The explicit computations of the present section will show that these various spectral sequences are isomorphic in a range from the first page and onwards.

4.1. **The first steps of the filtration.** For an integer $j \geq 0$, recall from (9) the space

$$Y_j = \{(f, s_0, \ldots, s_j) \in \mathfrak{X}[j] \mid s_l \neq s_k \text{ if } l \neq k\} \subset \mathfrak{X}[j].$$

Lemma 4.1. *For $0 \leq j \leq N(\mathcal{E}, r)$, there is a fibre bundle:*

$$\text{Interior}\left(|\Delta^j|\right) \longrightarrow F_j - F_{j-1} \longrightarrow Y_j/\mathfrak{S}_{j+1}.$$

Proof. Recall from the definition of the filtration on $R^N_{\text{cone}}\mathfrak{X}$ that $F_j = R^j\mathfrak{X}$ for $0 \leq j \leq N$. As a consequence of the pushout square (7), we observed in (10) that we have the following homeomorphism:

$$R^j\mathfrak{X} - R^{j-1}\mathfrak{X} \cong Y_j \times_{\mathfrak{S}_{j+1}} \text{Interior}(|\Delta^j|).$$

Projecting down to the first factor gives the required fibre bundle. □

By an *affine bundle* we mean a torsor for a vector bundle. In the sequel, they will arise naturally from fibrewise surjective linear maps between vector bundles. For any integer $j \geq 1$, the bundle $(J^r\mathcal{E})^j$ projects down to X^j and we may consider its restriction to the open subset $\text{Conf}_j(X) \subset X^j$ of those tuples of points which are pairwise distinct. The symmetric group $\mathfrak{S}_j$ acts on these spaces by permuting the coordinates. In particular, it acts on the subspace $\mathfrak{T}^j \subset (J^r\mathcal{E})^j$ and we let

$$\mathfrak{T}^{(j)} := \left(\mathfrak{T}^j|_{\text{Conf}_j(X)}\right)/\mathfrak{S}_j \tag{14}$$

be the orbit space of the restriction of $\mathfrak{T}^j$ over the subspace $\text{Conf}_j(X) \subset X^j$.

Lemma 4.2. *Let $0 \leq j \leq N(\mathcal{E}, r)$ be an integer and recall from (9) the space Y_j of those tuples $(f, s_0, \ldots, s_j) \in \Gamma \times \text{Conf}_{j+1}(X)$ where f is singular at the s_i. We may simultaneously evaluate the jet map at these points:*

$$\begin{aligned} Y_j &\longrightarrow \mathfrak{T}^{j+1}|_{\text{Conf}_{j+1}(X)} \\ (f, s_0, \ldots, s_j) &\longmapsto (j^r(f)(s_0), \ldots, j^r(f)(s_j)). \end{aligned}$$

Taking $\mathfrak{S}_{j+1}$-orbits on the domain and codomain of this map yields an affine bundle:

$$Y_j/\mathfrak{S}_{j+1} \longrightarrow \mathfrak{T}^{(j+1)}$$

whose fibre has complex dimension $\dim_{\mathbb{C}} \Gamma - (j+1)\text{rk}_{\mathbb{C}} J^r\mathcal{E}$. (Here $\text{rk}_{\mathbb{C}} J^r\mathcal{E}$ denotes the complex rank of the vector bundle $J^r\mathcal{E}$.)

Proof. The simultaneous evaluation of the jet map gives a map

$$\begin{tikzcd} \Gamma \times \text{Conf}_{j+1}(X) \arrow[rr] \arrow[dr] & & (J^r\mathcal{E})^{j+1}|_{\text{Conf}_{j+1}(X)} \arrow[dl] \\ & \text{Conf}_{j+1}(X) & \end{tikzcd} \tag{15}$$

of vector bundles over the configuration space $\text{Conf}_{j+1}(X)$. Under the assumption $0 \leq j \leq N(\mathcal{E}, r)$, Lemma 2.4 shows that this map of bundles is fibrewise surjective. Therefore the top map of (15) is an affine bundle. Subtracting the ranks, we obtain that its fibre has complex dimension $\dim_{\mathbb{C}} \Gamma - (j+1)\text{rk}_{\mathbb{C}} J^r\mathcal{E}$.

Now, the pullback of the affine bundle (15) to the subspace $\mathfrak{T}^{j+1}|_{\mathrm{Conf}_{j+1}(X)}$ is an affine bundle with total space Y_j. Finally taking $\mathfrak{S}_{j+1}$-orbits yields the following affine bundle:

$$Y_j/\mathfrak{S}_{j+1} \longrightarrow \left(\mathfrak{T}^{j+1}|_{\mathrm{Conf}_{j+1}(X)}\right)/\mathfrak{S}_{j+1} = \mathfrak{T}^{(j+1)}$$

which still has the rank that we have computed above. □

The quotient maps $Y_j \to Y_j/\mathfrak{S}_{j+1}$ and $\mathfrak{T}^{j+1}|_{\mathrm{Conf}_{j+1}(X)} \to \mathfrak{T}^{(j+1)}$ are principal $\mathfrak{S}_{j+1}$-bundles and hence are classified by (homotopy classes of) maps to the classifying space $B\mathfrak{S}_{j+1}$. Composing with the sign representation $B\mathfrak{S}_{j+1} \xrightarrow{B\mathrm{sign}} B\mathbb{Z}/2$, we obtain two well-defined homotopy classes of maps:

$$Y_j/\mathfrak{S}_{j+1} \longrightarrow B\mathbb{Z}/2 \quad \text{and} \quad \mathfrak{T}^{(j+1)} \longrightarrow B\mathbb{Z}/2.$$

We will write $\mathbb{Z}^{\mathrm{sign}}$ for the corresponding local coefficient systems.

Proposition 4.3. *Let $-N(\mathcal{E}, r) - 1 \leq s \leq -1$. Then, we have the following isomorphism:*

$$E^1_{s,t} \cong \check{H}_c^{-t-2s\cdot\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E}}(\mathfrak{T}^{(-s)}; \mathbb{Z}^{\mathrm{sign}})$$

where $\mathfrak{T}^{(-s)}$ is the space defined in (14) and $\mathbb{Z}^{\mathrm{sign}}$ is the local coefficient system described above.

Proof. Recall from Proposition 3.11 that the first page of the spectral sequence is given by

$$E^1_{s,t} = \check{H}_c^{2\dim_{\mathbb{C}}\Gamma-1-s-t}(R^{-s-1}\mathfrak{X} - R^{-s-2}\mathfrak{X}; \mathbb{Z}).$$

Via a homeomorphism $\mathrm{Interior}\left(|\Delta^j|\right) \cong \mathbb{R}^j$, we see that the fibre bundle of Lemma 4.1 is homeomorphic to a vector bundle. Applying the Thom isomorphism to the latter, we obtain:

$$E^1_{s,t} \cong \check{H}_c^{2\dim_{\mathbb{C}}\Gamma-t}(Y_{-s-1}/\mathfrak{S}_{-s}; \mathbb{Z}^{\mathrm{sign}}).$$

Another application of the Thom isomorphism using Lemma 4.2 yields

$$E^1_{s,t} \cong \check{H}_c^{-t-2s\cdot\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E}}(\mathfrak{T}^{(-s)}; \mathbb{Z}^{\mathrm{sign}}).$$ □

4.2. **The last step of the filtration.** We study the last non-trivial part of the E^1-page, that is the column $s = -N(\mathcal{E}, r) - 2$ where:

$$E^1_{-N-2,t} = \check{H}_c^{2\dim_{\mathbb{C}}\Gamma+1+N-t}(R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}; \mathbb{Z}).$$

The methods from the last section do not apply to the space $R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}$ and we will not be able to express the cohomology groups $E^1_{-N-2,t}$ in terms of other "known" groups. However, using the technical assumptions made in Definition 2.8 about the Taylor condition $\mathfrak{T}$, we will obtain a vanishing result for $E^1_{-N-2,t}$. This will be enough for the proof of our main theorem.

Recall the projection map $\tau_N\colon R^N\mathfrak{X} \to \Sigma$ from (8). From the homotopy pushout square (12), we obtain the following homeomorphism:

$$R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X} \cong \left(\left(\tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \times]0,1]\right) \bigsqcup \overline{\Sigma_{\geq N+2}}\right)/\sim$$

where $(z, 1) \in \tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \times]0,1]$ is identified with $\tau_N(z) \in \overline{\Sigma_{\geq N+2}}$ in the quotient. Indeed, there is a natural continuous bijection from the right-hand side to the left-hand side. It is in fact a homeomorphism, as the top arrow in the homotopy pushout square (12) is the inclusion of a closed subset. In other words, this is the fibrewise (for the map τ_N) open cone over $\overline{\Sigma_{\geq N+2}}$. We stratify this space by the following locally closed subspaces (this is analogous to [Tom14, Lemma 18]):

$$\begin{aligned}
\mathrm{Str}_{-1} &:= \overline{\Sigma_{\geq N+2}}, \\
\mathrm{Str}_0 &:= \left(\tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \times]0,1[\right) \cap \left(R^0\mathfrak{X}\times]0,1[\right), \\
\mathrm{Str}_j &:= \left(\tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \times]0,1[\right) \cap \left((R^j\mathfrak{X} - R^{j-1}\mathfrak{X})\times]0,1[\right) \quad \text{for } 1 \leq j \leq N.
\end{aligned}$$

For $0 \leq j \leq N$, let

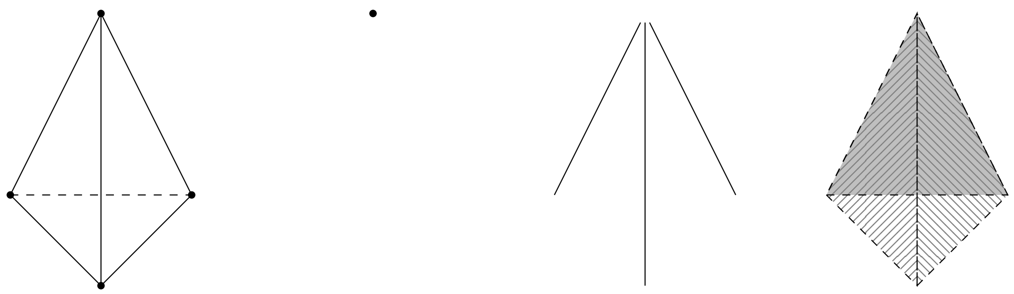
$$Y_j^{\geq N+2} := \left\{(f, s_0, \dots, s_j) \in \Gamma \times \mathrm{Conf}_{j+1}(X) \mid f \in \overline{\Sigma_{\geq N+2}} \text{ and } s_i \in \mathrm{Sing}(f)\right\} \subset Y_j. \tag{16}$$

Using the homeomorphism (10) identifying the difference between two consecutive steps of the resolution, we have a homeomorphism

$$\mathrm{Str}_j \cong \left(Y_j^{\geq N+2} \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j|\right) \times]0,1[. \tag{17}$$

for $0 \leq j \leq N$, where $|\mathring{\Delta}^j|$ denotes the interior of the simplex.

It is easier to think about this stratification by looking at one fibre $\pi^{-1}(f)$ at a time. Then, we are just decomposing an open cone over a union of simplices into the following pieces: the apex (corresponding to $\mathrm{Str}_{-1} \cap \pi^{-1}(f)$), the open segments from the 0-simplices to the apex (corresponding to $\mathrm{Str}_0 \cap \pi^{-1}(f)$), the open (filled) triangles between the 1-simplices and the apex, etc. Figure 1 below shows the strata in a single fibre $\pi^{-1}(f)$ when f has 3 singular points and $N = 1$. In this case, $\tau_N^{-1}(f)$ consists of three 1-simplices glued together (i.e. a triangle), so $\pi^{-1}(f)$ is the cone over that triangle.



$\pi^{-1}(f)$ $\quad$ $\pi^{-1}(f) \cap \mathrm{Str}_{-1}$ $\quad$ $\pi^{-1}(f) \cap \mathrm{Str}_0$ $\quad$ $\pi^{-1}(f) \cap \mathrm{Str}_1$ (the three sides only)

FIGURE 1. Decomposition of the open cone.

If we find an integer $D \geq 0$ such that $\check{H}_c^k(\mathrm{Str}_j) = 0$ for all $-1 \leq j \leq N$ and all $k > D$, then the same result will hold for the union, i.e. $\check{H}_c^k(R^N_{\text{cone}}\mathfrak{X} - R^N\mathfrak{X}) = 0$ for $k > D$. In what follows, we set out to find such a D as small as we can. With that in mind, we make the following ad hoc definition of cohomological dimension:

Definition 4.4. We say that a space Z has *cohomological dimension* D with respect to a local coefficient system $\mathcal{A}$ if D is the smallest integer such that $\check{H}_c^k(Z; \mathcal{A}) = 0$ for all $k > D$. We will denote it by $\mathrm{cohodim}(Z, \mathcal{A})$, or simply $\mathrm{cohodim}(Z)$ if $\mathcal{A} = \mathbb{Z}$.

The only non-trivial local coefficient system we will need is $\mathbb{Z}^{\text{sign}}$, which is induced on the quotient $Y_j^{\geq N+2}/\mathfrak{S}_{j+1}$ by the sign representation $\mathfrak{S}_{j+1} \to \mathbb{Z}/2$.

Lemma 4.5. *For $0 \leq j \leq N$, we have*

$$\operatorname{cohodim}(\mathrm{Str}_j) = 1 + j + \operatorname{cohodim}\left(Y_j^{\geq N+2}/\mathfrak{S}_{j+1}, \mathbb{Z}^{\text{sign}}\right).$$

Proof. From the homeomorphism (17), we have a trivial fibre bundle

$$]0,1[\longrightarrow \mathrm{Str}_j \longrightarrow Y_j^{\geq N+2} \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j|.$$

This implies that $\operatorname{cohodim}(\mathrm{Str}_j) = 1 + \operatorname{cohodim}\left(Y_j^{\geq N+2} \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j|\right)$. Now, we have another fibre bundle:

$$|\mathring{\Delta}^j| \longrightarrow Y_j^{\geq N+2} \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j| \longrightarrow Y_j^{\geq N+2}/\mathfrak{S}_{j+1}.$$

Hence, by the Thom isomorphism, we obtain:

$$\operatorname{cohodim}\left(Y_j^{\geq N+2} \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j|\right) = j + \operatorname{cohodim}\left(Y_j^{\geq N+2}/\mathfrak{S}_{j+1}, \mathbb{Z}^{\text{sign}}\right). \qquad \square$$

We thus have reduced our problem to studying the cohomology of $Y_j^{\geq N+2}/\mathfrak{S}_{j+1}$ for $0 \leq j \leq N$, as well as that of $\overline{\Sigma_{\geq N+2}}$. We shall do so by comparing these spaces to a known one, namely the space

$$Y_N = \{(f, s_0, \ldots, s_N) \in \Gamma \times \mathrm{Conf}_{N+1}(X) \mid s_i \in \mathrm{Sing}(f)\}.$$

First, let us introduce some notation. Using charts on X, we may cover Y_N by finitely many semi-algebraic sets, whose intersections are also semi-algebraic. Recall, e.g. from [BCR98, Theorem 2.3.6], that every semi-algebraic set is the disjoint union of cells, each homeomorphic to an open disc $]0,1[^d$ for some $d \geq 0$. The largest d in such a decomposition is called the dimension of the semi-algebraic set. Let $\dim Y_N$ be the largest of the dimensions of the semi-algebraic sets in a cover of Y_N. (It depends a priori of the chosen cover, but we suppress this from the notation.) The following lemma is a crucial result for controlling our spectral sequence.

Lemma 4.6. *For $0 \leq j \leq N$, we have*

$$\dim Y_N \geq \operatorname{cohodim}\left(Y_j^{\geq N+2}/\mathfrak{S}_{j+1}, \mathbb{Z}^{\text{sign}}\right).$$

Proof. Forgetting the last singularity yields a map

$$Y_{N+1} \longrightarrow Y_N, \quad (f, s_0, \ldots, s_{N+1}) \longmapsto (f, s_0, \ldots, s_N)$$

and we will write $Y_N^{\geq N+2} \subset Y_N$ for its image. As the projection map is semi-algebraic (when read in charts), its image is semi-algebraic (in charts) and $\dim Y_N^{\geq N+2} \leq \dim Y_N$. Let $0 \leq j \leq N$. Only remembering the $(j+1)^{\text{st}}$ singularities gives a map

$$\begin{aligned} Y_N^{\geq N+2} &\longrightarrow Y_j^{\geq N+2} \\ (f, s_0, \ldots, s_N) &\longmapsto (f, s_0, \ldots, s_j). \end{aligned} \tag{18}$$

Notice that this map is not surjective, as it may happen that a section $f \in \overline{\Sigma_{\geq N+2}}$ has fewer than $N+1$ singularities. We study the map (18) locally via charts. Let $U_0, \ldots, U_N \subset X$ be charts on X as in Definition 2.6. Then the subsets

$$U := \left\{(f, s_0, \ldots, s_j) \in \overline{\Sigma_{\geq N+2}} \times U_0 \times \cdots \times U_j \mid s_k \in \mathrm{Sing}(f), s_i \neq s_j \forall i \neq j\right\} \subset Y_j^{\geq N+2}$$

and

$$V := \{(f, s_0, \dots, s_N) \in \Gamma \times U_0 \times \cdots \times U_N \mid s_k \in \mathrm{Sing}(f), s_i \neq s_j \forall i \neq j\} \cap Y_N^{\geq N+2} \subset Y_N^{\geq N+2}$$

are semi-algebraic. Indeed, they are the preimages of the semi-algebraic set $\mathfrak{T}^{j+1}$ (respectively $\mathfrak{T}^{N+1}$) via the simultaneous evaluation of the jet map which is algebraic, hence semi-algebraic. (See [BCR98, Proposition 2.2.7].) The restriction of the map (18) to U and V is an algebraic map, hence semi-algebraic map, $\phi\colon V \to U$ between semi-algebraic sets. Using [BCR98, Theorem 2.8.8], we obtain the following inequality on the dimensions (as defined above using cell decompositions):

$$\dim(V) \geq \dim(\phi(V)).$$

Furthermore, the definition of $Y_j^{\geq N+2}$ implies that the semi-algebraic map $\phi\colon V \to U$ has dense image, i.e. $\overline{\phi(V)} = U$. Using that the closure has the same dimension ([BCR98, Proposition 2.8.2]) and the inequality above, we obtain:

$$\dim(V) \geq \dim(U).$$

Varying the charts $U_0, \dots, U_N \subset X$, we may cover the domain and codomain of (18) by subsets defined like U and V. If U' and V' are two other such subsets, then $U \cap U'$ and $V \cap V'$ are also semi-algebraic sets because they are intersections of semi-algebraic sets. (This follows from the Definition 2.5.) Hence the argument shows that the inequality on the dimensions holds also on intersections. Let $\dim Y_j^{\geq N+2}$ denote the maximum of the dimensions in a cover of $Y_j^{\geq N+2}$ by semi-algebraic sets. Then, an argument using the Mayer–Vietoris spectral sequence shows that the cohomological dimension of $Y_j^{\geq N+2}$ is less than its dimension $\dim Y_j^{\geq N+2}$. Therefore

$$\dim Y_N \geq \dim Y_N^{\geq N+2} \geq \dim Y_j^{\geq N+2} \geq \mathrm{cohodim}\left(Y_j^{\geq N+2}\right). \tag{19}$$

Finally, from the principal $\mathfrak{S}_{j+1}$-bundle $Y_j^{\geq N+2} \to Y_j^{\geq N+2}/\mathfrak{S}_{j+1}$, we see that the dimension of the orbit space is the same as that of $Y_j^{\geq N+2}$. Therefore the inequality (19) holds when replacing the rightmost term with $\mathrm{cohodim}(Y_j^{\geq N+2}/\mathfrak{S}_{j+1}, \mathbb{Z}^{\mathrm{sign}})$. $\square$

Repeating the proof with the map $Y_N^{\geq N+2} \to \overline{\Sigma_{\geq N+2}}$, $(f, s_0, \dots, s_N) \mapsto f$ yields the

Lemma 4.7. *The following inequality holds:*

$$\dim Y_N \geq \mathrm{cohodim}(\overline{\Sigma_{\geq N+2}}, \mathbb{Z}). \qquad \square$$

The final computation to be made is the content of the following lemma. It uses the notation $e(\mathfrak{T})$ of excess codimension established in Definition 2.10.

Lemma 4.8. *The dimension of Y_N satisfies:*

$$\dim Y_N \leq 2 \dim_{\mathbb{C}} \Gamma - (N+1) e(\mathfrak{T}).$$

Proof. The proof of Lemma 4.2 shows that the simultaneous evaluation of the jet map

$$\begin{aligned} Y_N &\longrightarrow \mathfrak{T}^{N+1}|_{\mathrm{Conf}_{N+1}(X)} \\ (f, s_0, \dots, s_N) &\longmapsto (j^r(f)(s_0), \dots, j^r(f)(s_N)) \end{aligned}$$

is an affine bundle whose fibre has complex dimension $\dim_{\mathbb{C}} \Gamma - (N+1)\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E}$. Therefore, on dimensions:

$$\dim Y_N \leq \dim(\mathfrak{T}^{N+1}|_{\mathrm{Conf}_{N+1}(X)}) + 2 \dim_{\mathbb{C}} \Gamma - 2(N+1)\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E}.$$

Now, because $\mathfrak{T}$ is a semi-algebraic subset of $J^r\mathcal{E}$ of dimension less than $2\mathrm{rk}_{\mathbb{C}}J^r\mathcal{E} - e(\mathfrak{T})$, we obtain that:

$$\dim(\mathfrak{T}^{N+1}|_{\mathrm{Conf}_{N+1}(X)}) \leq (N+1)(2\mathrm{rk}_{\mathbb{C}}J^r\mathcal{E} - e(\mathfrak{T})).$$

The lemma is then proven by combining these two inequalities. □

Assembling all the estimations we have obtained so far, we can state and prove the following.

Proposition 4.9. *The cohomology groups in the column $s = -N(\mathcal{E}, r) - 2$ on the first page of the spectral sequence:*

$$E^1_{-N-2,t} = \check{H}_c^{2\dim_{\mathbb{C}}\Gamma+1+N-t}(R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}; \mathbb{Z})$$

vanish for $t < N \cdot e(\mathfrak{T}) + e(\mathfrak{T})$.

Proof. We had set up the stratification Str_j, $-1 \leq j \leq N$, on $R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}$ so that

$$\mathrm{cohodim}\left(R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}\right) \leq \max_j \mathrm{cohodim}\left(\mathrm{Str}_j\right).$$

For $0 \leq j \leq N$, combining Lemma 4.5, Lemma 4.6, and Lemma 4.8, we get:

$$\mathrm{cohodim}\left(\mathrm{Str}_j\right) \leq 1 + j + 2\dim_{\mathbb{C}}\Gamma - (N+1)e(\mathfrak{T}) \leq 2\dim_{\mathbb{C}}\Gamma - N(e(\mathfrak{T})-1) - (e(\mathfrak{T})-1).$$

Similarly, using Lemma 4.7 and Lemma 4.8, we obtain:

$$\mathrm{cohodim}\left(\mathrm{Str}_{-1}\right) \leq 2\dim_{\mathbb{C}}\Gamma - (N+1)e(\mathfrak{T}).$$

Therefore $\mathrm{cohodim}\left(R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}\right) \leq 2\dim_{\mathbb{C}}\Gamma - N(e(\mathfrak{T})-1) - (e(\mathfrak{T})-1)$ and the result follows. □

4.3. **Differentials and summary.** Summing up all the results so far, we have the following proposition.

Proposition 4.10. *Let $\mathcal{E}$ be a holomorphic vector bundle on X and $\mathfrak{T} \subset J^r\mathcal{E}$ be an admissible Taylor condition. Let $N = N(\mathcal{E}, r)$. The resolution and its filtration described in Section 3 give rise to a spectral sequence on the second quadrant $s \leq -1$ and $t \geq 0$ converging to the homology of the space of non-singular sections $\Gamma_{\mathrm{hol,ns}}(\mathcal{E})$:*

$$E^1_{s,t} = \check{H}_c^{2\dim_{\mathbb{C}}\Gamma-1-s-t}(F_{-s-1} - F_{-s-2}; \mathbb{Z}) \implies \widetilde{H}_{s+t}(\Gamma_{\mathrm{hol,ns}}(\mathcal{E}); \mathbb{Z}).$$

The differentials on the r-th page have bi-degree $(-r, r-1)$. Furthermore, for $-N-1 \leq s \leq -1$, we have the following isomorphisms for all $t \geq 0$:

$$E^1_{s,t} \cong \check{H}_c^{-t-2s\cdot\mathrm{rk}_{\mathbb{C}}J^r\mathcal{E}}(\mathfrak{T}^{(-s)}; \mathbb{Z}^{\mathrm{sign}}).$$

(The space $\mathfrak{T}^{(-s)}$ is defined in (14).) Moreover, for $t < N \cdot e(\mathfrak{T}) + e(\mathfrak{T})$:

$$E^1_{-N-2,t} = 0.$$

We briefly describe the zones in Figure 2 below, where we have chosen to fix $e(\mathfrak{T}) = 2$ to lighten the notation. Firstly, the only possibly non-vanishing groups lie in the coloured squares. All groups $E^r_{s,t}$ with $s \leq -N-3$ are zero as the filtration finishes after $N+1$ steps. According to Proposition 4.9, the groups below the horizontal solid line in the column $s = -N-2$ vanish. The differentials coming from the groups below the upper staircase never hit groups in the column where $s = -N-2$ and $t \geq 2N+2$. Finally, the lower staircase delimits the zone of total degree $* \leq N-1$. We have also drawn some differentials d^r to the group $E^r_{-N-2,2N+2}$ for $r = 1, 2, 3$ and $N+1$.

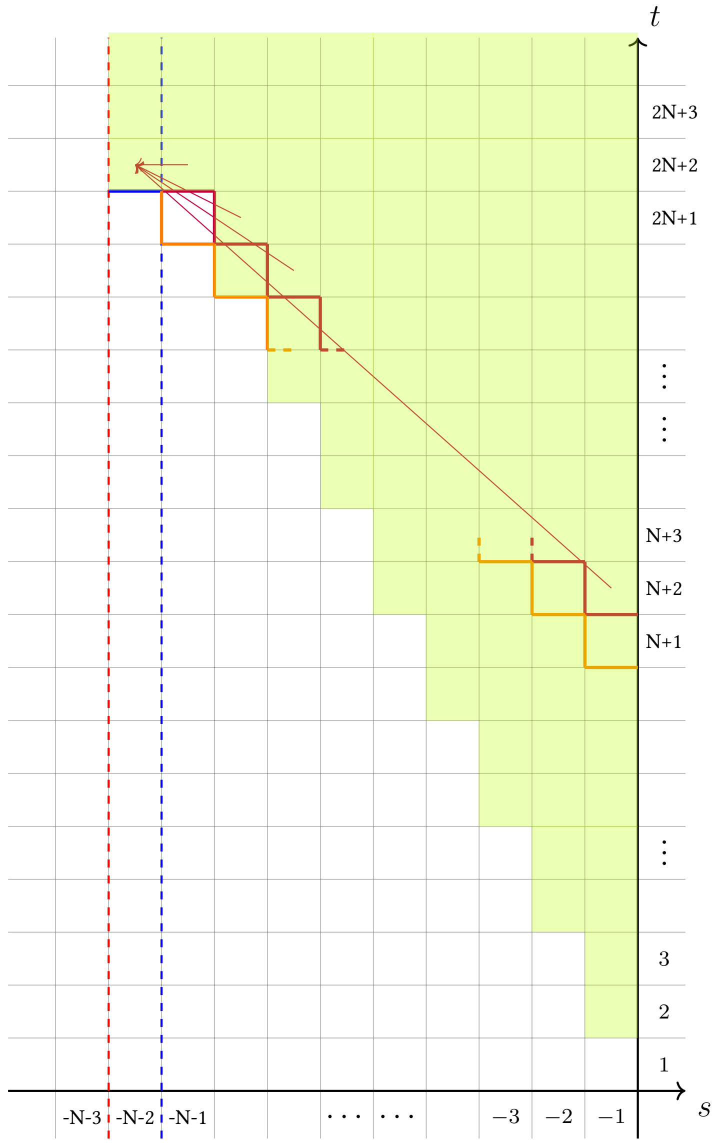


FIGURE 2. First page of the spectral sequence when $e(\mathfrak{T}) = 2$.

5. INTERPOLATING HOLOMORPHIC AND CONTINUOUS SECTIONS

In this section, we introduce and study section spaces that lie in-between holomorphic and continuous sections of the jet bundle $J^r\mathcal{E}$. They will be written as combinations of holomorphic and "anti-holomorphic" sections. We first explain how to take the complex conjugate of a holomorphic section. We then construct these spaces and finish by explaining how the resolution and the spectral sequence from the previous sections can be adapted to them.

5.1. **Complex conjugation of sections.** Using the fact that X is projective, we choose once and for all a very ample holomorphic line bundle $\mathcal{L}$ on it as well as a basis $z_0, \ldots, z_M$ of the complex vector space of holomorphic global sections $\Gamma_{\mathrm{hol}}(\mathcal{L})$.

We denote by $\overline{\mathcal{L}}$ the complex conjugate line bundle of $\mathcal{L}$. It is obtained from the underlying real vector bundle of $\mathcal{L}$ by having the complex numbers act by multiplication by their complex conjugates. We regard it as a smooth complex line bundle. We now define a complex conjugation operation $\mathcal{L} \to \overline{\mathcal{L}}$. Recall that the line bundle $\mathcal{L}$ may be constructed as a quotient

$$\mathcal{L} := \left(\bigsqcup_i U_i \times \mathbb{C} \right) \Big/ (x, v_i) \sim (x, t_{ji}(v_i))$$

from the data $(\{U_i\}_i, (t_{ij})_{i,j})$ of trivialising open sets $U_i \subset X$ and transition functions $t_{ij} \colon U_i \cap U_j \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ satisfying a cocycle condition. Similarly, $\overline{\mathcal{L}}$ may be constructed via such a quotient by replacing the transition functions by their complex conjugates $\overline{t_{ij}}$. The formula

$$\begin{aligned} \bigsqcup_i U_i \times \mathbb{C} &\longrightarrow \bigsqcup_i U_i \times \mathbb{C} \\ (x, v) &\longmapsto (x, \overline{v}) \end{aligned}$$

then gives a well defined $\mathbb{R}$-linear isomorphism $\mathcal{L} \to \overline{\mathcal{L}}$. On continuous global sections, we thus obtain an $\mathbb{R}$-linear *complex conjugation operation*:

$$\overline{\cdot} \colon \Gamma_{\mathcal{C}^0}(\mathcal{L}) \longrightarrow \Gamma_{\mathcal{C}^0}(\overline{\mathcal{L}}). \tag{20}$$

For a complex vector space V, we denote by $\overline{V}$ the $\mathbb{C}$-vector space whose underlying set is V with the $\mathbb{C}$-module structure given by multiplication by the complex conjugate. We get a $\mathbb{C}$-linear map:

$$\overline{\Gamma_{\mathrm{hol}}(\mathcal{L})} \hookrightarrow \overline{\Gamma_{\mathcal{C}^0}(\mathcal{L})} \xrightarrow{(20)} \Gamma_{\mathcal{C}^0}(\overline{\mathcal{L}}). \tag{21}$$

We let

$$\eta := \sum_{j=0}^{M} z_j \otimes \overline{z_j} \in \Gamma_{\mathrm{hol}}(\mathcal{L}) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L})}. \tag{22}$$

Its image via the composition of the the map (21) and the multiplication map $\Gamma_{\mathcal{C}^0}(\mathcal{L}) \otimes_{\mathbb{C}} \Gamma_{\mathcal{C}^0}(\overline{\mathcal{L}}) \to \Gamma_{\mathcal{C}^0}(\mathcal{L} \otimes \overline{\mathcal{L}})$ is a never vanishing section. It therefore gives an explicit trivialisation of the smooth complex line bundle $\mathcal{L} \otimes \overline{\mathcal{L}} \cong X \times \mathbb{C}$. In particular, we obtain an isomorphism on the level of continuous sections

$$\Gamma_{\mathcal{C}^0}(\mathcal{L} \otimes \overline{\mathcal{L}}) \cong \Gamma_{\mathcal{C}^0}(X \times \mathbb{C}) = \mathcal{C}^0(X, \mathbb{C}). \tag{23}$$

5.2. **Stabilisation.** For every integer $k \geq 0$, we now construct the following commutative diagram.

$$\begin{tikzcd} \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^k)} \arrow[d, "\gamma_k"'] \arrow[dr, "\varphi_k"] & \\ & \Gamma_{\mathcal{C}^0}(J^r\mathcal{E}) \\ \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^{k+1}\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^{k+1})} \arrow[ur, "\varphi_{k+1}"'] & \end{tikzcd} \tag{24}$$

The horizontal maps are given by the composition

$$\begin{aligned}\varphi_k\colon \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E})\otimes\mathcal{L}^k\right)\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}^k\right)} \\ \longrightarrow \Gamma_{\mathcal{C}^0}\left(J^r\mathcal{E}\otimes\mathcal{L}^k\right)\otimes_{\mathbb{C}}\Gamma_{\mathcal{C}^0}\left(\overline{\mathcal{L}}^k\right) \\ \longrightarrow \Gamma_{\mathcal{C}^0}\left(J^r\mathcal{E}\otimes\mathcal{L}^k\otimes\overline{\mathcal{L}}^k\right)\cong\Gamma_{\mathcal{C}^0}\left(J^r\mathcal{E}\right)\end{aligned} \tag{25}$$

where the first arrow is induced by the map (21), the second arrow is the multiplication map, and the last isomorphism is (23) applied to $\left(\mathcal{L}\otimes\overline{\mathcal{L}}\right)^k\cong\mathcal{L}^k\otimes\overline{\mathcal{L}}^k$.

We construct the vertical map in the diagram (24) as the composition:

$$\begin{aligned}\gamma_k\colon \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E})\otimes\mathcal{L}^k\right)\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}^k\right)} \\ \longrightarrow \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E})\otimes\mathcal{L}^k\right)\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}^k\right)}\otimes_{\mathbb{C}}\left(\Gamma_{\mathrm{hol}}\left(\mathcal{L}\right)\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}\right)}\right) \\ \cong \left(\Gamma_{\mathrm{hol}}\left((J^r\mathcal{E})\otimes\mathcal{L}^k\right)\otimes_{\mathbb{C}}\Gamma_{\mathrm{hol}}\left(\mathcal{L}\right)\right)\otimes_{\mathbb{C}}\left(\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}^k\right)}\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}\right)}\right) \\ \longrightarrow \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E})\otimes\mathcal{L}^{k+1}\right)\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}^{k+1}\right)}\end{aligned} \tag{26}$$

where the first arrow is given by tensoring with the element η defined in (22), the isomorphism is given by reordering the factors, and the last arrow is given by the multiplication maps.

The commutativity of the diagram (24) follows directly from the fact that η is sent to the constant function equal to 1 via the isomorphism (23). Loosely speaking, the vertical map γ_k is a "multiplication by η", which amounts to multiplying a continuous section of $J^r\mathcal{E}$ by the constant function 1 after using the chosen identification (23).

Example 5.1. It is illuminating to think about the case $X=\mathbb{CP}^n$, $\mathcal{L}=\mathcal{O}(1)$ and $\mathcal{E}=\mathcal{O}(d+1)$. In this example, $\Gamma_{\mathrm{hol}}(\mathcal{E})$ is the space of homogeneous polynomials of degree $d+1$ in $n+1$ variables. One may also prove an isomorphism $J^1(\mathcal{O}(d+1))\cong\mathcal{O}(d)^{\oplus(n+1)}$ as holomorphic vector bundles. (See [DRS00, Proposition 2.2] for a proof.)

We may then view $\Gamma_{\mathrm{hol}}\left((J^1\mathcal{E})\otimes\mathcal{L}^k\right)\otimes_{\mathbb{C}}\overline{\Gamma_{\mathrm{hol}}\left(\mathcal{L}^k\right)}$ as the space of $(n+1)$-uples of homogeneous polynomials of bi-degree $(d+k,k)$: that is of degree $d+k$ in the variables z_i and of degree k in the complex conjugate variables $\overline{z_i}$. In this case, the image of η in $\Gamma_{\mathcal{C}^0}\left(\mathcal{L}\otimes\overline{\mathcal{L}}\right)$ is $|z|^2:=z_0\overline{z_0}+\cdots+z_n\overline{z_n}$. The isomorphism $\Gamma_{\mathcal{C}^0}\left(\mathcal{L}\otimes\overline{\mathcal{L}}\right)\cong\mathcal{C}^0(X,\mathbb{C})$ corresponding to (23) sends a section s to the map

$$z=[z_0:\ldots:z_n]\in\mathbb{CP}^n\longmapsto\frac{s(z)}{|z|^2}\in\mathbb{C}.$$

Under these identifications, the map γ_k is then:

$$(f_0,\ldots,f_n)\longmapsto\left((z_0\overline{z_0}+\cdots+z_n\overline{z_n})\cdot f_0,\ldots,(z_0\overline{z_0}+\cdots+z_n\overline{z_n})\cdot f_n\right)$$

which sends a tuple of polynomials of bi-degree $(d+k,k)$ to one of bi-degree $(d+k+1,k+1)$. (Compare [Mos06] for a related situation.)

We will need the following small result, analogous to Lemma 2.4. Let $(x_0,\ldots,x_p)$ be a tuple of points in X. We may evaluate a continuous section of $J^r\mathcal{E}$ simultaneously at all these points:

$$\begin{aligned}\mathrm{ev}_{(x_0,\ldots,x_p)}\colon \Gamma_{\mathcal{C}^0}\left(J^r\mathcal{E}\right) &\longrightarrow (J^r\mathcal{E})|_{x_0}\times\cdots\times(J^r\mathcal{E})|_{x_p} \\ s &\longmapsto \left(s(x_0),\ldots,s(x_p)\right).\end{aligned} \tag{27}$$

Lemma 5.2. *Let $\mathcal{E}$ be a holomorphic vector bundle on X and $N(\mathcal{E}, r) \in \mathbb{N}$ be as in Definition 2.3. Let $(x_0, \dots, x_p)$ be a tuple of $p+1$ distinct points in X. If $p \leq N(\mathcal{E}, r)$, the composition*

$$\Gamma_{\text{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)} \xrightarrow{\varphi_k} \Gamma_{\mathcal{C}^0}\left(J^r\mathcal{E}\right) \longrightarrow (J^r\mathcal{E})|_{x_0} \times \dots \times (J^r\mathcal{E})|_{x_p}$$

of the map φ_k of (25) and the simultaneous evaluation (27) is surjective.

Proof. The case $k = 0$ is a direct consequence of Lemma 2.4. The result for $k \geq 1$ then follows from the commutativity of the diagram (24). □

5.3. **Non-singular sections.** We define

$$\mathcal{N}(k) \subset \Gamma_{\text{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)}$$

to be subspace of elements sent to non-singular sections of $J^r\mathcal{E}$ (as in Definition 2.11) under the map φ_k defined in (25). We say that an $s \in \Gamma_{\text{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)}$ is non-singular if it is in the subspace $\mathcal{N}(k)$. We define the singular subset to be the complement

$$\mathcal{S}(k) := \left(\Gamma_{\text{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)}\right) - \mathcal{N}(k).$$

Remark 5.3. When $k = 0$, $\mathcal{N}(0) \subset \Gamma_{\text{hol}}(J^r\mathcal{E})$ is the usual subspace of non-singular sections of $J^r\mathcal{E}$ as in Definition 2.11.

Example 5.4. In the case $X = \mathbb{CP}^n$, $\mathcal{L} = \mathcal{O}(1)$ and $\mathcal{E} = \mathcal{O}(d+1)$, recall from Example 5.1 that the space $\Gamma_{\text{hol}}\left((J^1\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)}$ corresponds to $(n+1)$-uples of homogeneous polynomials of degree $d+k$ in the holomorphic variables z_i and of degree k in the complex conjugate variables $\overline{z_i}$. Under this identification, if the Taylor condition $\mathfrak{T} \subset J^1(\mathcal{O}(d+1))$ is the zero section, the space of non-singular sections $\mathcal{N}(k)$ contains exactly those $(n+1)$-uples of polynomials that never vanish simultaneously.

5.4. **Resolution and spectral sequence.** We now explain how the results from Section 3 can be adapted to the case

$$\Gamma = \Gamma_{\text{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)} \quad \text{and} \quad \Sigma = \mathcal{S}(k)$$

to construct a resolution of $\mathcal{S}(k)$ and a spectral sequence converging to its cohomology, or equivalently to the homology of $\mathcal{N}(k)$ by Alexander duality. In this case, the definition of the singular space (3) of $f \in \Gamma$ has to be changed to

$$\operatorname{Sing}(f) := \{x \in X \mid \varphi_k(f)(x) \in \mathfrak{T}\} \subset X.$$

In particular, in the case $k = 0$, it agrees with Definition 2.11. The topological results about the resolution just follow from the fact that $\mathfrak{T} \subset J^r\mathcal{E}$ is closed. In particular, Lemma 3.4 still holds with its proof nearly unchanged: one has to replace the jet map j^r by φ_k. The construction of the spectral sequence is then unchanged.

The computations of cohomology groups on the E^1-page from Section 4 can also be adapted in this case. We first describe what to adapt for the first steps of the filtration. The analogue of Lemma 4.2 with the jet map j^r replaced by φ_k still holds as the key point is the surjectivity established in Lemma 5.2. The other result, Lemma 4.1, remains unchanged. Hence, Proposition 4.3 is true in our new setting.

The adaptations are similar to examine the last step $R^N_{\text{cone}}\mathfrak{X} - R^N\mathfrak{X}$. Indeed, the same stratification works, as well as the cohomological dimension estimates. In details, Lemma 4.5 is unchanged, and

Lemma 4.8 is proved similarly by just replacing the jet map by φ_k. The other two results, Lemma 4.6 and Lemma 4.7, also hold when rewriting the proof by changing the jet map j^r by φ_k. Indeed, the key ingredients were the semi-algebraicity of the Taylor condition $\mathfrak{T}$ (which remains unchanged), and the fact that the jet map was complex algebraic, hence real semi-algebraic. The map φ_k is no longer complex algebraic, but is given by a ratio of algebraic maps and complex conjugates of algebraic maps. In particular, it is real semi-algebraic. This is enough for the proof to go through.

To sum up, we have the following analogue of Proposition 4.10.

Proposition 5.5. *Let $\mathcal{E}$ be a holomorphic vector bundle on X and $\mathfrak{T} \subset J^r\mathcal{E}$ be an admissible Taylor condition. Let*

$$\Gamma = \Gamma_{\text{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\text{hol}}\left(\mathcal{L}^k\right)}$$

and $\mathcal{N}(k) \subset \Gamma$ be the subspace of non-singular sections. Let $N = N(\mathcal{E}, r)$. The resolution and its filtration described in Section 3 give rise to a spectral sequence on the second quadrant $s \leq -1$ and $t \geq 0$ converging to the homology of the space of non-singular sections:

$$E^1_{s,t} = \check{H}_c^{2\dim_{\mathbb{C}}\Gamma - 1 - s - t}(F_{-s-1} - F_{-s-2}; \mathbb{Z}) \implies \widetilde{H}_{s+t}(\mathcal{N}(k); \mathbb{Z}).$$

The differentials on the r-th page have bi-degree $(-r, r-1)$. Furthermore, for $-N-1 \leq s \leq -1$, we have the following isomorphisms for all $t \geq 0$:

$$E^1_{s,t} \cong \check{H}_c^{-t-2s\cdot\mathrm{rk}_{\mathbb{C}}J^r\mathcal{E}}(\mathfrak{T}^{(-s)}; \mathbb{Z}^{\text{sign}}).$$

Moreover, for $t < N \cdot e(\mathfrak{T}) + e(\mathfrak{T})$:

$$E^1_{-N-2,t} = 0.$$

Lastly, let us mention that in the particular example where $X = \mathbb{CP}^n$, $\mathcal{L} = \mathcal{O}(1)$, $\mathcal{E} = \mathcal{O}(d+1)$ and $\mathfrak{T} \subset J^1\mathcal{E}$ is the zero section, the spectral sequence is completely analogous to that of [Mos12].

6. Comparison of spectral sequences

From our definition of non-singularity, it follows that the jet map j^r sends a non-singular section f of $\mathcal{E}$ to a non-singular section $j^r(f)$ of $J^r\mathcal{E}$. Likewise, the stabilisation map described in (26) sends elements in $\mathcal{N}(k)$ to elements in $\mathcal{N}(k+1)$. We shall see that these maps induce isomorphisms in homology in a range of degrees up to around $N = N(\mathcal{E}, r)$. We first explain the argument for the jet map j^r and then go through the required modifications for the stabilisation map.

6.1. **The case of the jet map.** Reading Proposition 4.10 and Proposition 5.5, we may observe that we have similar looking spectral sequences, one converging to the homology of $\Gamma_{\text{hol,ns}}(\mathcal{E})$ and the other one to that of $\Gamma_{\text{hol,ns}}(J^r\mathcal{E})$. In particular, in the range $-N-1 \leq s \leq -1$, the terms $E^1_{s,t}$ are given by the same cohomology groups

$$E^1_{s,t} \cong \check{H}_c^{-t-2s\cdot\mathrm{rk}_{\mathbb{C}}J^r\mathcal{E}}(\mathfrak{T}^{(-s)}; \mathbb{Z}^{\text{sign}})$$

in both spectral sequences. If we had a morphism of spectral sequences that happened to be an isomorphism in this range, then, using the vanishing result $E^1_{-N-2,t} = 0$ for $t < N \cdot e(\mathfrak{T}) + e(\mathfrak{T})$, the morphism induced on the E^∞-page would be an isomorphism in the range of degrees $* < N(e(\mathfrak{T})-1) + e(\mathfrak{T}) - 2$. (See Figure 2 where we have drawn some differentials.) We shall construct such a morphism of spectral sequences, whilst making sure that it is compatible with the morphism induced on homology by the jet map j^r:

$$\widetilde{H}_{s+t}(\Gamma_{\text{hol,ns}}(\mathcal{E})) \longrightarrow \widetilde{H}_{s+t}(\Gamma_{\text{hol,ns}}(J^r\mathcal{E})).$$

For the sake of completeness, we recall when a morphism is compatible with a morphism of spectral sequences. (See e.g. [Wei94, Section 5.2].) If two spectral sequences $E^r_{p,q}$ and $E'^r_{p,q}$ converge respectively to H_* and H'_*, we say that a map $h\colon H_* \to H'_*$ is *compatible* with a morphism $f\colon E \to E'$ if h maps F_pH_n to $F_pH'_n$ (here F_p denotes the filtration) and the associated maps $F_pH_n/F_{p-1}H_n \to F_pH'_n/F_{p-1}H'_n$ correspond to $f^\infty_{p,q}\colon E^\infty_{p,q} \to E'^\infty_{p,q}$ (where $q = n-p$) under the isomorphisms $E^\infty_{p,q} \cong F_pH_n/F_{p-1}H_n$ and $E'^\infty_{p,q} \cong F_pH'_n/F_{p-1}H'_n$. The main point being that if f is an isomorphism in a range, then h also is an isomorphism in a range. (See [Wei94, Comparison Theorem 5.2.12].)

Let $d_1 := 2\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\mathcal{E})$ and $d_2 := 2\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(J^r\mathcal{E})$ be the real dimensions of the complex vector spaces of sections. We define the shriek morphism $j^!$ as the unique morphism making the following square commutative:

$$\begin{tikzcd}
\widetilde{H}_*(\Gamma_{\mathrm{hol,ns}}(\mathcal{E})) \arrow[r, "(j^r)_*"] \arrow[d, "\cong"'] & \widetilde{H}_*(\Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E})) \arrow[d, "\cong"] \\
\check{H}^{d_1-1-*}_c(\Gamma_{\mathrm{hol}}(\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(\mathcal{E})) \arrow[r, dashed, "j^!"'] & \check{H}^{d_2-1-*}_c(\Gamma_{\mathrm{hol}}(J^r\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E}))
\end{tikzcd} \tag{28}$$

where the vertical isomorphisms are given by Alexander duality and the top map is induced by the jet map j^r in homology. As our spectral sequences actually converge to the Čech cohomology with compact support of the singular subspaces, we will construct our morphism of spectral sequences such that it is compatible with $j^!$.

The spectral sequences arose from filtrations, so we now recall some notation from Section 3. We let $\mathfrak{X}$ be the functor $\mathsf{F}^{\mathrm{op}} \to \mathsf{Top}$ constructed there using $\Gamma = \Gamma_{\mathrm{hol}}(\mathcal{E})$ and $\Sigma = \Gamma_{\mathrm{hol}}(\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(\mathcal{E})$. As we have explained in Section 5.4, the resolution also works for $\Gamma_{\mathrm{hol}}(J^r\mathcal{E})$ and its singular subspace, and we let $\mathfrak{Y}\colon \mathsf{F}^{\mathrm{op}} \to \mathsf{Top}$ be the associated functor in this case. We denote the filtration of $R^N_{\mathrm{cone}}\mathfrak{X}$ by

$$F^1_{-1} = \varnothing \subset F^1_0 = R^0\mathfrak{X} \subset \cdots \subset F^1_N = R^N\mathfrak{X} \subset F^1_{N+1} = R^N_{\mathrm{cone}}\mathfrak{X},$$

and the analogous one of $R^N_{\mathrm{cone}}\mathfrak{Y}$ by

$$F^2_{-1} = \varnothing \subset F^2_0 = R^0\mathfrak{Y} \subset \cdots \subset F^2_N = R^N\mathfrak{Y} \subset F^2_{N+1} = R^N_{\mathrm{cone}}\mathfrak{Y}. \tag{29}$$

We will slightly abuse notation and also write

$$j^!\colon \check{H}^*_c(R^N_{\mathrm{cone}}\mathfrak{X}) \to \check{H}^{*+d_2-d_1}_c(R^N_{\mathrm{cone}}\mathfrak{Y}) \tag{30}$$

for the bottom map defined by making the following square commutative:

$$\begin{tikzcd}
\check{H}^*_c(\Gamma_{\mathrm{hol}}(\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(\mathcal{E})) \arrow[r, "j^!"] \arrow[d, "\cong"'] & \check{H}^{*+d_2-d_1}_c(\Gamma_{\mathrm{hol}}(\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E})) \arrow[d, "\cong"] \\
\check{H}^*_c(R^N_{\mathrm{cone}}\mathfrak{X}) \arrow[r, dashed, "j^!"'] & \check{H}^{*+d_2-d_1}_c(R^N_{\mathrm{cone}}\mathfrak{Y})
\end{tikzcd}$$

Recall from the general theory that the spectral associated to the filtration F^i_*, $i = 1, 2$, arises from an exact couple $(\check{H}^\bullet_c(F^i_*), \check{H}^\bullet_c(F^i_* - F^i_{*-1}))$. The map of spectral sequences that we want is then constructed via a map of exact couples as in the following lemma.

Lemma 6.1. *Let $\delta = d_2 - d_1 = 2(\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\mathcal{E}) - \dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(J^r\mathcal{E}))$. There exists a morphism of exact couples*

$$\left(j_p^!,\, j_{(p)}^!\right)_{p\geq 0} : \left(\check{H}_c^*(F_p^1), \check{H}_c^*(F_p^1 - F_{p-1}^1)\right) \longrightarrow \left(\check{H}_c^{*+\delta}(F_p^2), \check{H}_c^{*+\delta}(F_p^2 - F_{p-1}^2)\right)$$

satisfying the following two assertions:

(1) For $0 \leq p \leq N$, the map $j_{(p)}^!$ in the following diagram is an isomorphism:

$$\begin{tikzcd} \check{H}_c^*(F_p^1 - F_{p-1}^1) \arrow[r, "\cong"] \arrow[d, "j_{(p)}^!"'] & \check{H}_c^{\bullet}(\mathfrak{T}^{(p+1)}; \mathbb{Z}^{\mathrm{sign}}) \\ \check{H}_c^{*+\delta}(F_p^2 - F_{p-1}^2) \arrow[r, "\cong"'] & \check{H}_c^{\bullet}(\mathfrak{T}^{(p+1)}; \mathbb{Z}^{\mathrm{sign}}) \end{tikzcd} \tag{31}$$

where

$$\bullet = * - 2\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\mathcal{E}) - p + 2(p+1)\mathrm{rk}_{\mathbb{C}} J^r\mathcal{E}$$

and the horizontal isomorphisms are given by Thom isomorphisms as in Proposition 4.3.

(2) The map $j_{N+1}^!$ is equal to the shriek map (30).

Unpacking the definition of a morphism of exact couples, we see that it amounts to providing morphisms $j_p^!$ and $j_{(p)}^!$ for $0 \leq p \leq N+1$ such that the following diagram commutes

$$\begin{tikzcd} \check{H}_c^{*-1}(F_{p-1}^1) \arrow[r] \arrow[d, "j_{p-1}^!"] & \check{H}_c^*(F_p^1 - F_{p-1}^1) \arrow[r] \arrow[d, "j_{(p)}^!"] & \check{H}_c^*(F_p^1) \arrow[r] \arrow[d, "j_p^!"] & \check{H}_c^*(F_{p-1}^1) \arrow[d, "j_{p-1}^!"] \\ \check{H}_c^{*-1+\delta}(F_{p-1}^2) \arrow[r] & \check{H}_c^{*+\delta}(F_p^2 - F_{p-1}^2) \arrow[r] & \check{H}_c^{*+\delta}(F_p^2) \arrow[r] & \check{H}_c^{*+\delta}(F_{p-1}^2) \end{tikzcd}$$

where the horizontal morphisms in the diagram are given by the long exact sequence of the pair (F_p^i, F_{p-1}^i) for $i = 1, 2$.

This result says exactly what we need: there a morphism of spectral sequences compatible with $j^!$ (by (2)) and giving an isomorphism in the vertical strip $-N-1 \leq s \leq 1$ (by (1)). The lemma, as well as the strategy of proof, is adapted from [Vok07, Proposition 4.7]. First, let us state the most important consequence:

Proposition 6.2. *For a holomorphic vector bundle $\mathcal{E}$ on X, the jet map*

$$j^r : \Gamma_{\mathrm{hol,ns}}(\mathcal{E}) \longrightarrow \Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E})$$

induces an isomorphism in homology in the range of degrees $* < N(\mathcal{E}, r) \cdot (e(\mathfrak{T}) - 1) + e(\mathfrak{T}) - 2$. $\square$

To understand how to construct the degree-shifting morphisms of Lemma 6.1, it is helpful to give a description of the shriek map between cohomology groups arising from Alexander duality as in the diagram (28). We shall do so generally first (following [Vok07, Appendix D]) and then specialise to our situation to prove the lemma at hand.

6.1.1. *Alexander duality and shriek maps.* Let $p\colon E \to B$ be a vector bundle between oriented paracompact topological manifolds of dimension n and m respectively. Let $j\colon K \subset E$ be a closed subset, and let $i\colon B \hookrightarrow E$ be the zero section. We will see B as a submanifold of E via i. Using Alexander duality (the vertical isomorphisms in the diagram below), we may define the *shriek map*

$$i^!\colon \check{H}^*_c(B\cap K) \to \check{H}^{*+(n-m)}_c(K) \tag{32}$$

to be the unique morphism making the following diagram commute:

$$\begin{tikzcd} H_*(B, B-B\cap K) \arrow[r, "i_*"] & H_*(E, E-K) \\ \check{H}^{m-*}_c(B\cap K) \arrow[u, "\cong"] \arrow[r, dashed, "i^!"'] & \check{H}^{n-*}_c(K) \arrow[u, "\cong"'] \end{tikzcd}$$

The goal of this section is to give a more intrinsic definition of $i^!$ that will allow us to define the required morphisms in Lemma 6.1.

Firstly, Vokřínek proves in [Vok07, Proposition D.1] the following:

Lemma 6.3. *The diagram below commutes:*

$$\begin{tikzcd} H_*(B, B-B\cap K) \arrow[rr, "i_*"] & & H_*(E, E-K) \\ \check{H}^{m-*}_c(B\cap K) \arrow[u, "\cong"] & \check{H}^{m-*}_{p^{-1}c}(K) \arrow[l, "k^*"'] \arrow[r, "-\cup j^*\tau"'] & \check{H}^{n-*}_c(K) \arrow[u, "\cong"'] \end{tikzcd}$$

where the vertical isomorphisms are given by Alexander duality, $k\colon B\cap K \hookrightarrow K$ is the inclusion, $\tau \in H^\delta(D(E), S(E))$ is the Thom class of p, and $p^{-1}c$ is the family of supports defined as:

$$p^{-1}c = \left\{ F \subset K \;\middle|\; F \text{ closed and } \overline{p(F)} \subset B\cap K \text{ is compact} \right\}$$

so that $\check{H}^*_{p^{-1}c}$ denotes Čech cohomology with supports in $p^{-1}c$. (See e.g. [Bre97, Chapter II.2].)

Sketch of proof. We repeat Vokřínek's proof here for convenience. First, we explain the morphisms in Alexander duality. Recall from e.g. [Bre97, Corollary V.10.2] that we have fundamental classes $[B] \in H^{BM}_m(B)$ and $[E] \in H^{BM}_n(E)$, where H^{BM}_* denotes Borel–Moore homology (also known as homology with closed support). Using the proper inclusions $(E,\varnothing) \hookrightarrow (E, E-K)$ and $(B,\varnothing) \hookrightarrow (B, B-B\cap K)$, they give rise to classes $o_E \in H^{BM}_n(E, E-K)$ and $o_B \in H^{BM}_m(B, B-B\cap K)$. If $U \subset E$ is a closed neighbourhood of K, we get a morphism

$$\check{H}^{n-*}_c(U) \xrightarrow{-\cap o_E|U} H_*(U, U-K) \longrightarrow H_*(E, E-K)$$

where $o_E|U$ is the image of o_E via the excision isomorphism $H^{BM}_n(E, E-K) \cong H^{BM}_n(U, U-K)$. (Note that it is important for U to be closed, so that the inclusion $U \hookrightarrow E$ is proper, hence induces a morphism in Borel–Moore homology.) Likewise, we get a morphism

$$\check{H}^{m-*}_c(B\cap U) \xrightarrow{-\cap o_B|U} H_*(B\cap U, B\cap(U-K)) \longrightarrow H_*(B, B-B\cap K).$$

Now, the isomorphisms in Alexander duality are given by taking the colimit over all closed neighbourhoods U of K of the two morphisms constructed above. (This is explained in [Bre97, V.9].) Hence, to prove the lemma, it suffices to check commutativity of the following diagram:

$$\begin{tikzcd}
H_*(B\cap U, B\cap(U-K)) \arrow[rr, "g_*"] & & H_*(U, U-K) \\
\check{H}^{m-*}_c(B\cap U) \arrow[u, "-\cap o_B|U"] & \check{H}^{m-*}_{p^{-1}c}(U) \arrow[l, "g^*"'] \arrow[ur, "-\cap g_*(o_B|U)"] \arrow[r, "-\cup h^*\tau"'] & \check{H}^{n-*}_c(U) \arrow[u, "-\cap o_E"']
\end{tikzcd}$$

where $g\colon B\cap U \hookrightarrow U$ and $h\colon U \hookrightarrow E$ are the inclusions. The left part commutes by naturality of the cap products. The right part commutes by observing that the fundamental classes can be chosen to correspond under the Thom isomorphism, which implies that $h^*\tau \cap o_E|U = g_* o_B|U$, and finishes the proof. $\square$

In the statement of Lemma 6.3, if the morphism k^* were invertible, the shriek map (32) would be given by "$(k^*)^{-1}$" followed by taking the cup product with the "Thom class" $j^*\tau$. However, it is not invertible in general. There is nevertheless a way around that problem which we explain below, using ε-small neighbourhoods of $B\cap K$ in K and the continuity property of Čech cohomology.

We choose, once and for all, a bundle metric on $p\colon E\to B$. For a real number $\varepsilon>0$, denote by D_ε (resp. S_ε, $\mathring{D}_\varepsilon$) the closed disc (resp. sphere, open disc) sub-bundle of $E\to B$ of radius ε (for the chosen metric). In [Vok07, Lemma D.2], Vokřínek proves:

Lemma 6.4. *The following diagram commutes:*

$$\begin{tikzcd}
H_*(B, B-B\cap K) \arrow[rr] & & H_*(E\cap\mathring{D}_\varepsilon, (E-K)\cap\mathring{D}_\varepsilon) \arrow[r] & H_*(E, E-K) \\
\check{H}^{m-*}_c(B\cap K) \arrow[u, "\cong"] & & \check{H}^{n-*}_c(K\cap\mathring{D}_\varepsilon) \arrow[u, "\cong"'] \arrow[r] \arrow[d, "\cong"] & \check{H}^{n-*}_c(K) \arrow[u, "\cong"'] \\
& \check{H}^{m-*}_c(K\cap D_\varepsilon) \arrow[ul, "(l_\varepsilon)_*"] \arrow[r, "-\cup\tau_\varepsilon"'] & \check{H}^{n-*}_c(K\cap D_\varepsilon, K\cap S_\varepsilon) &
\end{tikzcd} \tag{33}$$

where the vertical isomorphisms on the first row are given by Alexander duality, the one on the second row follows from general results about cohomology with compact supports, $l_\varepsilon\colon B\cap K\hookrightarrow K\cap D_\varepsilon$ is the inclusion, τ_ε is the restriction of the Thom class of $E\to B$, and the rightmost horizontal arrows are induced by the inclusions. (Recall that cohomology with compact supports in covariant for open inclusions.)

Sketch of proof. The left part of the diagram can be shown to commute by a proof analogous to that of Lemma 6.3. The right-hand square is seen to commute by a direct verification. $\square$

Taking the limit $\varepsilon\to 0$, the morphisms $(l_\varepsilon)_*$ induce a morphism from the colimit

$$\operatorname*{colim}_{\varepsilon\to 0} \check{H}^{m-*}_c(K\cap D_\varepsilon) \longrightarrow \check{H}^{m-*}_c(B\cap K)$$

which is an isomorphism by the continuity property of Čech cohomology with compact supports (see, e.g. [Bre97, Theorem 14.4] where it is stated using sheaf cohomology which agrees with Čech cohomology here). We finally obtain another description of the shriek map $i^!$:

Proposition 6.5 (Compare [Vok07, Theorem D.3])**.** *The shriek map $i^!$ defined in* (32) *is equal to the composite obtained as one goes along the bottom path in the diagram* (33) *above, i.e.:*

$$\begin{aligned} i^!\colon \check{H}_c^{m-*}(B\cap K) &\xleftarrow{\cong} \operatorname*{colim}_{\varepsilon\to 0} \check{H}_c^{m-*}(K\cap D_\varepsilon) \\ &\longrightarrow \operatorname*{colim}_{\varepsilon\to 0} \check{H}_c^{n-*}(K\cap D_\varepsilon, K\cap S_\varepsilon) \cong \operatorname*{colim}_{\varepsilon\to 0} \check{H}_c^{n-*}(K\cap \mathring{D}_\varepsilon) \\ &\longrightarrow \check{H}_c^{n-*}(K). \end{aligned}$$

Furthermore, in the case where both E and B are themselves vector bundles over a same base, $K = E$, and $i\colon B \hookrightarrow E$ is the inclusion of a sub-bundle, the shriek map $i^!$ is the Thom isomorphism of the bundle $E \to B$ given by choosing a splitting of i.

Proof. The first part follows from Lemma 6.3 and Lemma 6.4. The second part is shown by direct inspection of the construction. □

6.1.2. *The proof of Lemma 6.1.* We shall apply the general theory described in the last section to our case. To lighten the notation, we write

$$\Gamma_1 := \Gamma_{\mathrm{hol}}(\mathcal{E}), \quad \Sigma_1 := \Gamma_{\mathrm{hol}}(\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(\mathcal{E})$$

and

$$\Gamma_2 := \Gamma_{\mathrm{hol}}(J^r\mathcal{E}), \quad \Sigma_2 := \Gamma_{\mathrm{hol}}(J^r\mathcal{E}) - \Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E}).$$

The jet map j^r gives a linear embedding of Γ_1 into Γ_2 such that the image of the singular subspace is precisely given by the intersection with the bigger singular subspace, i.e.

$$j^r(\Sigma_1) = j^r(\Gamma_1)\cap\Sigma_2.$$

Choosing a complementary linear subspace of $j^r(\Gamma_1)$ inside Γ_2, we obtain a projection giving a vector bundle

$$\Gamma_2 \longrightarrow j^r(\Gamma_1) \cong \Gamma_1 \tag{34}$$

of real rank $\delta = d_2 - d_1$. Below, we apply Vokřínek's results to this situation.

We first set up the notation. Let $\varepsilon > 0$ be a positive real number and denote by D_ε (resp. S_ε, $\mathring{D}_\varepsilon$) the closed disc (resp. sphere, open disc) sub-bundle of radius ε of the vector bundle (34). Recall from (29) the functor $\mathfrak{Y}$ giving rise to the resolution of Σ_2. We also define $\mathfrak{Y}_{D_\varepsilon}\colon \mathsf{F}^{\mathrm{op}} \to \mathsf{Top}$ to be the sub-functor of $\mathfrak{Y}$ given by

$$\mathfrak{Y}_{D_\varepsilon}[n] := \{(f, s_0, \dots, s_n) \in \mathfrak{Y}[n] \mid f\in D_\varepsilon\}$$

and likewise for $\mathfrak{Y}_{S_\varepsilon} \subset \mathfrak{Y}$ and $\mathfrak{Y}_{\mathring{D}_\varepsilon} \subset \mathfrak{Y}$ using only sections $f\in S_\varepsilon$ or $\mathring{D}_\varepsilon$. Let $\tau_\varepsilon \in H^\delta(\Sigma_2\cap D_\varepsilon, \Sigma_2\cap S_\varepsilon)$ be the restriction of the Thom class of the vector bundle (34) to Σ_2. (Recall that the Thom class is an element of $H^\delta(D_\varepsilon, S_\varepsilon)$.) In all what follows, we see $\Gamma_1\subset\Gamma_2$ via the embedding $j = j^r$. Let $l_\varepsilon\colon \Sigma_1 \hookrightarrow \Sigma_2\cap D_\varepsilon$ be the inclusion (which is proper, hence induces a morphism on compactly supported cohomology). We explained in Proposition 6.5 that the shriek map $j^!$ is obtained from the zigzag

$$\check{H}_c^*(\Sigma_1) \overset{(l_\varepsilon)_*}{\leftarrow} \check{H}_c^*(\Sigma_2\cap D_\varepsilon) \overset{-\cup\tau_\varepsilon}{\to} \check{H}_c^{*+\delta}(\Sigma_2\cap D_\varepsilon, \Sigma_2\cap S_\varepsilon) \cong \check{H}_c^{*+\delta}(\Sigma_2\cap\mathring{D}_\varepsilon) \to \check{H}_c^{*+\delta}(\Sigma_2)$$

by taking a colimit as $\varepsilon\to 0$.

We mimic that construction at the level of the resolutions. Let $0\le p\le N+1$ be an integer. Recall from (29) that F_p^i denoted the p-th step of the filtration of the resolution of Σ_i. We denote by

F^2_{p,D_ε}, F^2_{p,S_ε}, and $F^2_{p,\mathring{D}_\varepsilon}$ the analogous filtrations on the resolutions obtained from the subfunctors $\mathfrak{Y}_{D_\varepsilon}$, $\mathfrak{Y}_{S_\varepsilon}$ and $\mathfrak{Y}_{\mathring{D}_\varepsilon}$ respectively. Because a singular point of a section $f \in \Gamma_1$ is also a singular point of $j^r(f) \in \Gamma_2$, the jet map gives a map on resolutions

$$\begin{aligned} \mathfrak{X}[p] &\longrightarrow \mathfrak{Y}[p] \\ (f, s_0, \dots, s_p) &\longmapsto (j^r(f), s_0, \dots, s_p). \end{aligned}$$

which preserves the filtrations. Let $\tilde{l}_\varepsilon \colon F^1_p \hookrightarrow F^2_{p,D_\varepsilon}$ be the induced inclusion. Let $\gamma_\varepsilon \in H^\delta(F^2_{p,D_\varepsilon}, F^2_{p,S_\varepsilon})$ be the pullback of τ_ε along $(F^2_{p,D_\varepsilon}, F^2_{p,S_\varepsilon}) \to (\Sigma_2 \cap D_\varepsilon, \Sigma_2 \cap S_\varepsilon)$. The following diagram then commutes by naturality of all the constructions involved:

$$\begin{tikzcd}
\check{H}^*_c(F^1_p) & \check{H}^*_c(F^2_{p,D_\varepsilon}) \arrow[l, "(\tilde{l}_\varepsilon)_*"'] \arrow[r, "-\cup\gamma_\varepsilon"] & \check{H}^{*+\delta}_c(F^2_{p,D_\varepsilon}, F^2_{p,S_\varepsilon}) \cong \check{H}^{*+\delta}_c(F^2_{p,\mathring{D}_\varepsilon}) \arrow[r] & \check{H}^{*+\delta}_c(F^2_p) \\
\check{H}^*_c(\Sigma_1) \arrow[u] & \check{H}^*_c(\Sigma_2 \cap D_\varepsilon) \arrow[u] \arrow[l, "(l_\varepsilon)_*"] \arrow[r, "-\cup\tau_\varepsilon"'] & \check{H}^{*+\delta}_c(\Sigma_2 \cap D_\varepsilon, \Sigma_2 \cap S_\varepsilon) \cong \check{H}^{*+\delta}_c(\Sigma_2 \cap \mathring{D}_\varepsilon) \arrow[u] \arrow[r] & \check{H}^{*+\delta}_c(\Sigma_2) \arrow[u]
\end{tikzcd}$$

where all the vertical maps are induced by the proper projections $F^i_p \to \Sigma_i$. The morphism $j^!_p \colon \check{H}^*_c(F^1_p) \to \check{H}^{*+\delta}_c(F^2_p)$ is then defined as the colimit, when $\varepsilon \to 0$, of the top composition in the diagram above. (Recall that $(\tilde{l}_\varepsilon)_*$ is an isomorphism in the colimit, by continuity of Čech cohomology.) In particular, when $p = N+1$, the vertical map are isomorphisms (by 3.10), which proves the assertion (2) of Lemma 6.1 by noticing that the bottom composition is the shriek map $j^!$.

The morphisms $j^!_{(p)} \colon \check{H}^*_c(F^1_p - F^1_{p-1}) \to \check{H}^{*+\delta}_c(F^2_p - F^2_{p-1})$ are defined analogously, i.e. by the colimit as $\varepsilon \to 0$ of the zig-zag:

$$\begin{aligned} \check{H}^*_c(F^1_p - F^1_{p-1}) &\longleftarrow \check{H}^*_c(F^2_{p,D_\varepsilon} - F^2_{p-1,D_\varepsilon}) \\ &\longrightarrow \check{H}^{*+\delta}_c(F^2_{p,D_\varepsilon} - F^2_{p-1,D_\varepsilon}, F^2_{p,S_\varepsilon} - F^2_{p-1,S_\varepsilon}) \cong \check{H}^{*+\delta}_c(F^2_{p,\mathring{D}_\varepsilon} - F^2_{p-1,\mathring{D}_\varepsilon}) \\ &\longrightarrow \check{H}^{*+\delta}_c(F^2_p - F^2_{p-1}) \end{aligned}$$

where, as before, the first morphism is induced by the inclusion, the second morphism is the cup product with the Thom class, and the third is induced covariantly by the open inclusion.

One may check, using naturality of the various constructions involved, that the morphisms $j^!_p$ and $j^!_{(p)}$ give a morphism of exact couples. This amounts to staring at the following commutative diagram.

$$\begin{tikzcd}
\check{H}_c^{*-1}(F^1_{p-1}) \arrow[r] & \check{H}_c^{*}(F^1_p - F^1_{p-1}) \arrow[r] & \check{H}_c^{*}(F^1_p) \arrow[r] & \check{H}_c^{*}(F^1_{p-1}) \\
\check{H}_c^{*-1}(F^2_{p-1,D_\varepsilon}) \arrow[r] \arrow[u] \arrow[d] & \check{H}_c^{*}(F^2_{p,D_\varepsilon} - F^2_{p-1,D_\varepsilon}) \arrow[r] \arrow[u] \arrow[d] & \check{H}_c^{*}(F^2_{p,D_\varepsilon}) \arrow[r] \arrow[u] \arrow[d] & \check{H}_c^{*}(F^2_{p-1,D_\varepsilon}) \arrow[u] \arrow[d] \\
\check{H}_c^{*-1+\delta}(F^2_{p-1,D_\varepsilon}, F^2_{p-1,S_\varepsilon}) \arrow[r] \arrow[d, "\cong"] & \check{H}_c^{*+\delta}(F^2_{p,D_\varepsilon} - F^2_{p-1,D_\varepsilon}, F^2_{p,S_\varepsilon} - F^2_{p-1,S_\varepsilon}) \arrow[r] \arrow[d, "\cong"] & \check{H}_c^{*+\delta}(F^2_{p,D_\varepsilon}, F^2_{p,S_\varepsilon}) \arrow[r] \arrow[d, "\cong"] & \check{H}_c^{*+\delta}(F^2_{p-1,D_\varepsilon}, F^2_{p-1,S_\varepsilon}) \arrow[d, "\cong"] \\
\check{H}_c^{*-1+\delta}(F^2_{p-1,\mathring{D}_\varepsilon}) \arrow[r] \arrow[d] & \check{H}_c^{*+\delta}(F^2_{p,\mathring{D}_\varepsilon} - F^2_{p-1,\mathring{D}_\varepsilon}) \arrow[r] \arrow[d] & \check{H}_c^{*+\delta}(F^2_{p,\mathring{D}_\varepsilon}) \arrow[r] \arrow[d] & \check{H}_c^{*+\delta}(F^2_{p-1,\mathring{D}_\varepsilon}) \arrow[d] \\
\check{H}_c^{*-1+\delta}(F^2_{p-1}) \arrow[r] & \check{H}_c^{*+\delta}(F^2_p - F^2_{p-1}) \arrow[r] & \check{H}_c^{*+\delta}(F^2_p) \arrow[r] & \check{H}_c^{*+\delta}(F^2_{p-1})
\end{tikzcd}$$

To conclude the proof, we verify the assertion (1) of Lemma 6.1, i.e. that the morphism

$$j^!_{(p)} \colon \check{H}_c^{*}(F^1_p - F^1_{p-1}) \longrightarrow \check{H}_c^{*+\delta}(F^2_p - F^2_{p-1})$$

is an isomorphism. Recall from (10) that

$$F^2_p - F^2_{p-1} \cong Y_p(\mathfrak{Y}) \times_{\mathfrak{S}_{p+1}} |\mathring{\Delta}^p| \quad \text{and} \quad F^1_p - F^1_{p-1} \cong Y_p(\mathfrak{X}) \times_{\mathfrak{S}_{p+1}} |\mathring{\Delta}^p|$$

where we defined as in (9) the subspace

$$Y_p(\mathfrak{Y}) := \{(f, s_0, \dots, s_p) \in \mathfrak{Y}[p] \mid s_l \neq s_k \text{ if } l \neq k\} \subset \mathfrak{Y}[p]$$

and likewise for $Y_p(\mathfrak{X}) \subset \mathfrak{X}[p]$. Recall also that these spaces were vector bundles over $\mathfrak{T}^{(p+1)}$. (See Section 4.) Hence, we have an inclusion of vector bundles:

$$\begin{tikzcd}
F^1_p - F^1_{p-1} \arrow[rr, hook] \arrow[dr] & & F^2_p - F^2_{p-1} \arrow[dl] \\
& \mathfrak{T}^{(p+1)} &
\end{tikzcd}$$

Now, the second part of Proposition 6.5 applies and finishes the proof. □

6.2. **The case of the stabilisation map.** Choose some integer $k \geq 0$. We now describe how the argument of the previous section can be made with the stabilisation map

$$\gamma_k \colon \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^k)} \longrightarrow \Gamma_{\mathrm{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^{k+1}\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^{k+1})}$$

from (26). First of all, it is a linear embedding. Hence, by choosing a complementary subspace, we get a vector bundle

$$\Gamma_{\mathrm{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^{k+1}\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^{k+1})} \longrightarrow \gamma_k\left(\Gamma_{\mathrm{hol}}\left((J^r\mathcal{E}) \otimes \mathcal{L}^k\right) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^k)}\right)$$

analogous to the one in (34). From the commutativity of the diagram (24), we see that a singularity $x \in X$ for $f \in \mathcal{S}(k)$ is also a singularity of $\gamma_k(f) \in \mathcal{S}(k+1)$. Therefore, we also get a map induced on the respective resolutions of $\mathcal{S}(k)$ and $\mathcal{S}(k+1)$. Together with the fact that non-singular sections are sent to non-singular sections, this is enough for the argument to be repeated in that case.

Proposition 6.6. *The restriction of the stabilisation map γ_k to the non-singular subspaces*

$$\gamma_k \colon \mathcal{N}(k) \longrightarrow \mathcal{N}(k+1)$$

induces an isomorphism in homology in the range of degrees $* < N(\mathcal{E}, r) \cdot (e(\mathfrak{T}) - 1) + e(\mathfrak{T}) - 2$. □

Combining Proposition 6.2 and Proposition 6.6, we obtain the following.

Proposition 6.7. *Each map in the composition*

$$\Gamma_{\mathrm{hol,ns}}(\mathcal{E}) \longrightarrow \Gamma_{\mathrm{hol,ns}}(J^r\mathcal{E}) = \mathcal{N}(0) \longrightarrow \operatorname*{colim}_{k\to\infty} \mathcal{N}(k)$$

induces an isomorphism in homology in the range of degrees $* < N(\mathcal{E}, r) \cdot (e(\mathfrak{T}) - 1) + e(\mathfrak{T}) - 2$. □

7. Comparison of holomorphic and continuous sections

We shall relate $\operatorname{colim}_k \mathcal{N}(k)$ to the space $\Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E})$ of non-singular continuous sections of the jet bundle. Recall from the stabilisation diagram (24) that every non-singular space $\mathcal{N}(k)$ maps via φ_k to $\Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E})$. The aim of this section is to prove the following result about the map induced from the colimit.

Proposition 7.1. *The map*

$$\operatorname*{colim}_{k\to\infty} \mathcal{N}(k) \longrightarrow \Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E}) \tag{35}$$

is a weak homotopy equivalence.

Combining this result with Proposition 6.7 readily implies Theorem 2.13. Proposition 7.1 is a direct consequence of the openness of the subspace of non-singular sections, which follows from the fact that the admissible Taylor condition $\mathfrak{T} \subset J^r\mathcal{E}$ is closed (see the discussion after Lemma 3.5), and the following

Lemma 7.2. *Let F be a finite CW-complex. The map*

$$\mathcal{C}^0\big(F, \operatorname*{colim}_{k\to\infty} \mathcal{N}(k)\big) \longrightarrow \mathcal{C}^0\big(F, \Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E})\big)$$

induced by (35) has a dense image.

As in [Mos06], we will need an adaptation of the classical Stone–Weierstrass theorem for real vector bundles.

Theorem 7.3 (Stone–Weierstrass). *Let $E \to B$ be a finite rank real vector bundle over a compact Hausdorff space. Let $A \subset \mathcal{C}^0(B, \mathbb{R})$ be a subalgebra and $\{s_j\}_{j\in J}$ be a set of sections such that*

(1) the subalgebra A separates the points of B: for any $x, y \in B$, there exists $h \in A$ such that $h(x) \neq h(y)$;
(2) for any $x \in B$, there exists $h \in A$ such that $h(x) \neq 0$;
(3) for any $x \in B$, the fibre E_x is spanned by the $s_j(x)$ as an $\mathbb{R}$-vector space.

Then the A-module generated by the s_j is dense for the sup-norm (induced by the choice of any inner product on E) in the space of all continuous sections of E.

Proof of Lemma 7.2. Let F be a finite CW-complex. By adjunction, a continuous map $F \to \Gamma_{\mathcal{C}^0,\mathrm{ns}}(J^r\mathcal{E})$ corresponds to a section of the underlying real vector bundle of $J^r\mathcal{E} \times F \to X \times F$. We shall apply Theorem 7.3 to that vector bundle.

Recall that we have chosen in Section 5 a very ample line bundle $\mathcal{L}$ on X and explained how to define the complex conjugate $\overline{s}$ of a section s of $\mathcal{L}$. For any integer $k \geq 0$, define the squared norm of a holomorphic section of $\mathcal{L}$ by

$$\begin{aligned} |\cdot|^2 \colon \Gamma_{\text{hol}}(\mathcal{L}^k) &\longrightarrow \Gamma_{\mathcal{C}^0}(\mathcal{L}^k \otimes \overline{\mathcal{L}}^k) \cong \mathcal{C}^0(X, \mathbb{C}) \\ s &\longmapsto |s|^2 := s\overline{s} \end{aligned}$$

where the isomorphism with continuous maps was obtained in (23). Notice that $|s|^2$ is in fact a real valued function $X \to \mathbb{R} \subset \mathbb{C}$. We also let

$$A_k := \left\{ |g(\cdot,\cdot)|^2 \colon X \times F \to \mathbb{R} \mid g \in \mathcal{C}^0(F, \Gamma_{\text{hol}}(\mathcal{L}^k)) \right\} \subset \mathcal{C}^0(X \times F, \mathbb{R})$$

where if $g \in \mathcal{C}^0(F, \Gamma_{\text{hol}}(\mathcal{L}^k))$, we see $g(\cdot,\cdot)$ as a map from $X \times F$ to $\mathcal{L}^k$ by adjunction. Keeping the notation from Theorem 7.3, we let A to be the subalgebra of $\mathcal{C}^0(X \times F, \mathbb{R})$ generated by all the A_k for $k \geq 0$. For the set of sections as in Theorem 7.3, we take the following:

$$\left\{ (x,u) \mapsto (s(x,u), u) \colon X \times F \to J^r\mathcal{E} \times F \mid s \in \mathcal{C}^0(F, \Gamma_{\text{hol}}(J^r\mathcal{E})) \right\} \tag{36}$$

where again, for $s \in \mathcal{C}^0(F, \Gamma_{\text{hol}}(J^r\mathcal{E}))$, we see $s(\cdot,\cdot)$ as a map from $X \times F$ to $J^r\mathcal{E}$ by adjunction. We may now check the conditions of Theorem 7.3.

(1) Let $(x,u) \neq (x',u') \in X \times F$. Consider the first case where $x \neq x'$. For $k \geq 2$, $\mathcal{L}^k$ is 2-very ample (see Example 2.2). Hence there exists a section $s \in \Gamma_{\text{hol}}(\mathcal{L}^2)$ such that $s(x) \neq 0$ and $s(x') = 0$. Then the map $(x,u) \mapsto |s(x)|^2$ is in A_k and separates (x,u) and (x',u') as $|s(x)|^2 \neq 0$ and $|s(x')|^2 = 0$. In the other case where $x = x'$, we have that $u \neq u'$. By the 1-very ampleness of $\mathcal{L}$ we may choose $s \in \Gamma_{\text{hol}}(\mathcal{L})$ such that $s(x) = s(x') \neq 0$. Let $\rho : F \to \mathbb{R}_+$ be a bump function such that $\rho(u) = 0$ and $\rho(u') = 1$. Then the map $(x,u) \mapsto |\rho(u)s(x)|^2$ is in A_1 and separates the points. Indeed it is vanishing at (x,u) but non-vanishing at (x',u').
(2) The second point is exactly what we have just proved in the first case of the first point above.
(3) It suffices to prove that the fibre of $J^r\mathcal{E}$ above $x \in X$ is spanned by the sections $s(x)$ for $s \in \Gamma_{\text{hol}}(J^r\mathcal{E})$. This is implied by the 0-jet ampleness of $\mathcal{E}$ (see Example 2.2).

By construction, any element in the image of the map

$$\mathcal{C}^0\big(F,\ \operatorname*{colim}_{k\to\infty} \mathcal{N}(k)\big) \longrightarrow \mathcal{C}^0\big(F,\ \Gamma_{\mathcal{C}^0,\text{ns}}\left(J^r\mathcal{E}\right)\big)$$

is, by adjunction, in the A-module generated by the set (36). □

8. Applications

8.1. Non-singular sections of line bundles. Our first application concerns the case of non-singular sections of line bundles, which was the starting motivation for this work. Here, a direct corollary of our main theorem reads as:

Corollary 8.1. *Let X be a smooth projective complex variety and $\mathcal{L}$ be a very ample line bundle on it. Let $d \geq 1$ be an integer. The jet map*

$$j^1 \colon \Gamma_{\text{hol,ns}}\left(\mathcal{L}^d\right) \longrightarrow \Gamma_{\mathcal{C}^0,\text{ns}}\left(J^1\mathcal{L}^d\right)$$

from non-singular holomorphic sections of $\mathcal{L}^d$ to continuous never vanishing sections of $J^1\mathcal{L}^d$, induces an isomorphism in homology in the range of degrees $* < \frac{d-1}{2}$.

Proof. It is a straightforward application of Theorem 2.13 by taking the admissible Taylor condition $\mathfrak{T}$ to be the zero section of $J^1\mathcal{L}^d$ and recalling from Example 2.2 that if $\mathcal{L}$ is very ample, then the tensor power $\mathcal{L}^d$ is d-very ample. $\square$

More interestingly, we can furthermore compute the stable rational cohomology. This agrees with a computation also made by Tommasi in [Tom23].

Theorem 8.2. *Let $n = \dim_{\mathbb{C}} X$ be the complex dimension of X. For $d \geq 1$, there is a rational homotopy equivalence:*

$$\Gamma_{\mathcal{C}^0,\mathrm{ns}}\left(J^1\mathcal{L}^d\right) \xrightarrow{\simeq_{\mathbb{Q}}} \prod_{i=1}^{2n+1} K\left(H_{i-1}(X;\mathbb{Q}), i\right).$$

In particular, the rational cohomology of $\Gamma_{\mathcal{C}^0,\mathrm{ns}}\left(J^1\mathcal{L}^d\right)$ is given by the free commutative graded algebra

$$\Lambda\left(H^{*-1}(X;\mathbb{Q})\right)$$

on the cohomology of X shifted by one degree.

Remark 8.3. This result implies in particular that the rational (co)homology of $\Gamma_{\mathrm{hol},\mathrm{ns}}\left(\mathcal{L}^d\right)$ stabilises as $d \to \infty$. As we will see below, the integral cohomology does not stabilise in general.

Remark 8.4. The stable cohomology only depends on the topology of X. This is in accordance with the analogies between topology and arithmetic and motivic statistics mentioned in the introduction. In both the results of Poonen and Vakil–Wood, the limit is expressed by a zeta function which only depends on X.

Example 8.5. For $X = \mathbb{CP}^n$ and $\mathcal{L} = \mathcal{O}(1)$, we find that the stable rational cohomology is the exterior algebra

$$\Lambda_{\mathbb{Q}}(t_1, t_3, \ldots, t_{2n+1})$$

where t_i is in degree i. This agrees with the result of Tommasi in [Tom14].

Proof of Theorem 8.2. Recall that the non-singular sections of $J^1\mathcal{L}^d$ are precisely the never-vanishing ones. We choose a Riemannian metric once and for all and denote by $\mathrm{Sph}\left(J^1\mathcal{L}^d\right) \to X$ the unit sphere bundle of the vector bundle $J^1\mathcal{L}^d$. We may scale a never vanishing section to have norm equal to 1 (for the chosen metric) in each fibre. We thus obtain a homotopy equivalence:

$$\Gamma_{\mathcal{C}^0,\mathrm{ns}}\Big(J^1\mathcal{L}^d\Big) \xrightarrow{\simeq} \Gamma_{\mathcal{C}^0}\Big(\mathrm{Sph}\left(J^1\mathcal{L}^d\right)\Big).$$

We now rationalise the sphere bundle in the following sense. By [Lle85, Theorem 3.2], there is a fibration $S^{2n+1}_{\mathbb{Q}} \to \mathrm{Sph}\left(J^1\mathcal{L}^d\right)_{\mathbb{Q}} \to X$ and a morphism of fibrations:

$$\begin{tikzcd}
S^{2n+1} \arrow[rr] \arrow[d] & & S^{2n+1}_{\mathbb{Q}} \arrow[d] \\
\mathrm{Sph}\left(J^1\mathcal{L}^d\right) \arrow[rr] \arrow[dr] & & \mathrm{Sph}\left(J^1\mathcal{L}^d\right)_{\mathbb{Q}} \arrow[dl] \\
& X &
\end{tikzcd}$$

such that the map induced on the fibres $S^{2n+1} \to S^{2n+1}_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2n+1)$ is a rationalisation. As X is homotopy equivalent to a finite CW-complex and S^{2n+1} is nilpotent (it is indeed simply connected), we may use [Mø87, Theorem 5.3] that shows that the map $\mathrm{Sph}\left(J^1\mathcal{L}^d\right) \to \mathrm{Sph}\left(J^1\mathcal{L}^d\right)_{\mathbb{Q}}$ induces a map

$$\Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}\left(J^1\mathcal{L}^d\right)\right) \xrightarrow{\simeq_{\mathbb{Q}}} \Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}\left(J^1\mathcal{L}^d\right)_{\mathbb{Q}}\right)$$

which is a rationalisation. (In general, one has to restrict to some path component. However both spaces are connected in our situation.) Now, oriented rational odd sphere bundles are classified by their Euler class (see e.g. [FHT01, II.15.b]). In our situation, the orientation is induced from the canonical one on the complex vector bundle $J^1\mathcal{L}^d$ and the Euler class vanishes for dimensional reasons. It follows directly that $\mathrm{Sph}\left(J^1\mathcal{L}^d\right)_{\mathbb{Q}} \to X$ is a trivial bundle. Therefore

$$\Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}\left(J^1\mathcal{L}^d\right)_{\mathbb{Q}}\right) \cong \mathrm{map}(X, K(\mathbb{Q}, 2n+1))$$

where $\mathrm{map}(-,-)$ denotes the space of continuous functions with its compact open topology. Finally, in [Tho56] (see also [Hae82] for an accessible reference), Thom proves that this mapping space is homotopy equivalent to a product of Eilenberg–MacLane spaces

$$\mathrm{map}(X, K(\mathbb{Q}, 2n+1)) \simeq \prod_{i=0}^{2n+1} K\left(H^{2n+1-i}(X;\mathbb{Q}), i\right) \simeq \prod_{i=0}^{2n+1} K\left(H_{i-1}(X;\mathbb{Q}), i\right)$$

where the last equivalence comes from Poincaré duality. More precisely, he proves that if

$$\mathrm{ev}\colon \mathrm{map}(X, K(\mathbb{Q}, 2n+1)) \times X \longrightarrow K(\mathbb{Q}, 2n+1)$$

is the evaluation map, and $\chi \in H^{2n+1}(K(\mathbb{Q}, 2n+1);\mathbb{Q})$ is the fundamental class, we may write

$$\mathrm{ev}^*(\chi) = \sum_i \chi_i$$

where $\chi_i \in H^i(\mathrm{map}(X, K(\mathbb{Q}, 2n+1)); H^{2n+1-i}(X;\mathbb{Q}))$. Then the projection

$$\mathrm{map}(X, K(\mathbb{Q}, 2n+1)) \to K(H^{2n+1-i}(X;\mathbb{Q}), i)$$

is determined by the cohomology class χ_i. □

8.1.1. *Geometric description of the stable classes and Mixed Hodge Structures.* As a Zariski open subset of the affine space $\Gamma_{\mathrm{hol}}(\mathcal{L}^d)$, the subspace $\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)$ inherits a structure of complex variety and its cohomology thus has a natural mixed Hodge structure. On the other hand, we may endow the stable cohomology computed in Theorem 8.2 with a mixed Hodge structure defined as follows. Recall that the cohomology $H^*(X;\mathbb{Q})$ can be equipped with a mixed Hodge structure using the structure of complex variety on X, and denote by $\mathbb{Q}(-1)$ the Tate–Hodge structure of pure weight 2. By first tensoring these structures and then applying the symmetric algebra functor (see e.g. [PS08, Section 3.1]), we obtain a mixed Hodge structure on the stable cohomology. In this section, we show the following.

Proposition 8.6. *The morphism of Theorem 8.2*

$$\Lambda\left(H^{*-1}(X;\mathbb{Q}) \otimes \mathbb{Q}(-1)\right) \longrightarrow H^*(\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d);\mathbb{Q})$$

is compatible with the mixed Hodge structures.

Proof. By the universal property of the (graded) symmetric algebra, it is enough to see that the morphism

$$H^{*-1}(X;\mathbb{Q})\otimes\mathbb{Q}(-1)\longrightarrow H^*(\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d);\mathbb{Q})$$

respects the mixed Hodge structures. We will do this by giving a more geometric description of this map. Let

$$\pi\colon \Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)\times X\longrightarrow \Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)$$

be the trivial fibre bundle, and let

$$j\colon \Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)\times X\longrightarrow J^1\mathcal{L}^d-\{0\}$$

be the jet evaluation. By integrating along the fibres of π, we obtain in cohomology a morphism of mixed Hodge structures:

$$\pi_!\circ j^*\colon H^*(J^1\mathcal{L}^d-\{0\})\otimes\mathbb{Q}(n)\longrightarrow H^{*-2n}(\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)).$$

The extra Tate twist $\mathbb{Q}(n)$ comes from the definition of the Gysin map $\pi_!$ via Poincaré duality. (See [PS08, Corollary 6.25].) As the Euler class of the jet bundle vanishes for dimensional reasons, we compute that

$$H^*(J^1\mathcal{L}^d-\{0\};\mathbb{Q})\cong H^*(X;\mathbb{Q})\otimes H^*(\mathbb{C}^{n+1}-\{0\};\mathbb{Q}).$$

Now $H^{2n+1}(\mathbb{C}^{n+1}-\{0\};\mathbb{Q})\cong\mathbb{Q}(-n-1)$, so we have obtained a morphism of mixed Hodge structures:

$$\pi_!\circ j^*\colon H^*(X)\otimes\mathbb{Q}(-1)\longrightarrow H^{*+1}(\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)).$$

We claim that this coincides with the morphism given in Theorem 8.2. The proof is an exercise in algebraic topology and uses the description of the mapping space given at the end of the proof of Theorem 8.2. □

8.2. **Integral homology and stability.** In this section, we focus on the special case where $X=\mathbb{CP}^1$ and $\mathcal{L}=\mathcal{O}(1)$. That is, we study the space

$$U_d:=\Gamma_{\mathrm{hol,ns}}\left(\mathbb{CP}^1,\mathcal{O}(d)\right)$$

of non-singular homogeneous polynomials in two variables of degree d. From Corollary 8.1, we know that the jet map

$$j^1\colon U_d\longrightarrow\Gamma_{\mathcal{C}^0,\mathrm{ns}}\left(J^1\mathcal{O}(d)\right)$$

induces an isomorphism in integral homology in the range of degree $*<\frac{d-1}{2}$. We prove that the section space on the right-hand side does not depend on $d\geq 1$, hence that the integral homology of U_d stabilises.

Theorem 8.7. *For $d\geq 1$, we have a homotopy equivalence*

$$\Gamma_{\mathcal{C}^0,\mathrm{ns}}\left(J^1\mathcal{O}_{\mathbb{CP}^1}(d)\right)\simeq\mathrm{map}(S^2,S^3).$$

In particular

$$H_*(U_d;\mathbb{Z})\cong H_*(\mathrm{map}(S^2,S^3);\mathbb{Z})$$

in the range of degrees $*<\frac{d-1}{2}$.

Remark 8.8. [1] Using the Lie group structure on S^3 we obtain a homotopy equivalence

$$\mathrm{map}(S^2, S^3) \simeq S^3 \times \mathrm{map}_*(S^2, S^3) = S^3 \times \Omega^2 S^3$$

which can be used to compute the integral homology. This can be done one prime at a time. Indeed, the p-primary elements have order exactly p by [Nei10, Corollary 10.16.5]. This p-primary part can then be computed directly from the Bockstein spectral sequence and the knowledge of the $\mathbb{Z}/p$-homology, which is recalled in [Nei10, Corollary 10.16.4].

Remark 8.9. In the next section, we will show that one cannot expect integral homological stability in general. The case $X = \mathbb{CP}^1$ should be seen as a very particular phenomenon.

Proof. Recall from the proof of Theorem 8.2 that we have to study continuous sections of the sphere bundle of the jet bundle:

$$S^3 \longrightarrow \mathrm{Sph}(J^1\mathcal{O}_{\mathbb{CP}^1}(d)) \longrightarrow \mathbb{CP}^1.$$

One sees that this bundle is classified by the second Stiefel–Whitney class of the jet bundle, i.e. the reduction modulo 2 of its first Chern class. Using that $d \geq 1$ and [DRS00, Proposition 2.2], we obtain an isomorphism of vector bundles:

$$J^1\mathcal{O}_{\mathbb{CP}^1}(d) \cong \mathcal{O}_{\mathbb{CP}^1}(d-1)^{\oplus 2}.$$

We compute the first Chern class to be

$$c_1(J^1\mathcal{O}_{\mathbb{CP}^1}(d)) = c_1(\mathcal{O}_{\mathbb{CP}^1}(d-1)^{\oplus 2}) = 2c_1(\mathcal{O}_{\mathbb{CP}^1}(d-1))$$

so its reduction modulo 2 vanishes regardless of d. As the sphere bundle was classified by this class, this shows that it is trivial. Therefore:

$$\Gamma_{\mathcal{C}^0,\mathrm{ns}}\left(J^1\mathcal{O}_{\mathbb{CP}^1}(d)\right) \simeq \Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}(J^1\mathcal{O}_{\mathbb{CP}^1}(d))\right) \simeq \mathrm{map}(S^2, S^3). \qquad \square$$

8.3. **Integral homology and non-stability.** As we indicated in Remark 8.3, the rational cohomology groups of the spaces $\Gamma_{\mathrm{hol,ns}}(\mathcal{L}^d)$ stabilise. That is, for a fixed $i \geq 0$, the i-th rational cohomology group is independent of d as long as $i \leq \frac{d-1}{2}$. In this section, to contrast with the very special case of the previous one, we show that one cannot expect integral stability in general.

Let us fix some notation for the remainder of this section: $d \geq 1$ is an integer, $\mathcal{L}$ is a very ample line bundle on a smooth projective complex variety X and $n = \dim_{\mathbb{C}} X$ is the complex dimension of X. As we will only be considering spaces of continuous sections, we will use Γ as a shorthand for $\Gamma_{\mathcal{C}^0}$.

The main result of this section is Theorem 8.11 below. To show its computational potential, we will show the following:

Proposition 8.10. *Let $d \geq 6$ be an integer. We have:*

$$H_2(\Gamma_{\mathrm{hol,ns}}(\mathbb{CP}^2, \mathcal{O}(d)); \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & d \equiv 0 \mod 2 \\ 0 & d \equiv 1 \mod 2. \end{cases}$$

Furthermore, the same result holds for the quotient $\Gamma_{\mathrm{hol,ns}}(\mathbb{CP}^2, \mathcal{O}(d))/\mathbb{C}^*$.

[1] Many thanks to Antoine Touzé for explaining this computation to me.

8.3.1. *A comparison map.* As stated in Corollary 8.1, we are reduced to studying the homotopy type of the space of continuous sections of the sphere bundle $\mathrm{Sph}(J^1\mathcal{L}^d)$. Even though this is a purely homotopy theoretic problem, its resolution is quite hard. We will therefore "linearise it" in the homotopical sense using spectra. This is made precise in the following result:

Theorem 8.11. *Let TX be the tangent bundle of X, and let $X^{J^1\mathcal{L}^d - TX}$ denote the Thom spectrum of the virtual bundle $J^1\mathcal{L}^d - TX$ of rank 2. There is a $2n$-connected map:*

$$\Gamma(\mathrm{Sph}(J^1\mathcal{L}^d)) \longrightarrow \Omega^{\infty+1} X^{J^1\mathcal{L}^d - TX}.$$

Our proof uses very lightly the theory of parametrised pointed spaces/spectra and is written using ∞-categories. We feel that the latter choice helps in conveying the main ideas more clearly. The unfamiliar reader is encouraged to think of bundles of pointed spaces/spectra, whilst resting assured that there exists a theory which renders all statements made here literally true. An encyclopedic reference is [MS06]. As we shall only use basic adjunctions and Costenoble–Waner duality, we suggest to simply look at [Lan21, Appendix A] for a very readable introduction.

We denote respectively by S_* and Sp the ∞-categories of pointed spaces and spectra. Likewise, we let $\mathsf{S}_{*/X} = \mathsf{Fun}(X, \mathsf{S}_*)$ and $\mathsf{Sp}_{/X} = \mathsf{Fun}(X, \mathsf{Sp})$ be the ∞-categories of parametrised pointed spaces/spectra over X. (In the definitions, X is seen as an ∞-groupoid.) We let $r\colon X \to *$ be the unique map to the point. We will use the following three standard functors:

the restriction functor:	$r^*\colon \mathsf{S}_* \longrightarrow \mathsf{S}_{*/X}$,
its right adjoint:	$r_*\colon \mathsf{S}_{*/X} \longrightarrow \mathsf{S}_*$,
its left adjoint:	$r_!\colon \mathsf{S}_{*/X} \longrightarrow \mathsf{S}_*$.

The right and left adjoint are given respectively by right and left Kan extensions. In other words, r_* takes the limit of a functor $F \in \mathsf{S}_{*/X} = \mathsf{Fun}(X, \mathsf{S}_*)$, whilst $r_!$ takes its colimit. We will also use the analogous functors in the case of parametrised spectra with the same notation. It will be clear from the context which one we are using. The crucial fact for us is that for any bundle $Y \to X$ equipped with a section s (this gives the data of a *pointed* space over X), $r_*(Y)$ is the path component of s in the section space.

As a last piece of notation, we will use $\Sigma^\infty_{/X} \dashv \Omega^\infty_{/X}$ to denote the infinite suspension/loop space adjunction between parametrised pointed spaces and spectra, and use $\Sigma^\infty \dashv \Omega^\infty$ to denote the usual adjunction in the unparametrised setting.

On our way to the proof of Theorem 8.11, we first make some formal observations. Loosely speaking, we would like to say that the section space of a fibrewise infinite loop space is the infinite loop space of the "section spectrum". This is made precise in the lemma below.

Lemma 8.12. *Let $Y \in \mathsf{S}_{*/X}$ be a parametrised space over X. We have a natural equivalence of pointed spaces:*

$$\Omega^\infty r_*(\Sigma^\infty_{/X} Y) \simeq r_*(\Omega^\infty_{/X}\Sigma^\infty_{/X} Y).$$

Proof. We use the Yoneda lemma and the adjunction $r^* \dashv r_*$. Let $Z \in \mathsf{S}_*$ be a pointed space. We have:

$$\begin{aligned}\mathrm{map}_{\mathsf{S}_*}(Z, \Omega^\infty r_*(\Sigma^\infty_{/X} Y)) &\simeq \mathrm{map}_{\mathsf{Sp}}(\Sigma^\infty, r_*(\Sigma^\infty_{/X} Y)) \\ &\simeq \mathrm{map}_{\mathsf{Sp}_{/X}}(r^*\Sigma^\infty Z, \Sigma^\infty_{/X} Y) \\ &\simeq \mathrm{map}_{\mathsf{Sp}_{/X}}(\Sigma^\infty_{/X} r^* Z, \Sigma^\infty_{/X} Y) \\ &\simeq \mathrm{map}_{\mathsf{S}_{*/X}}(r^* Z, \Omega^\infty_{/X}\Sigma^\infty_{/X} Y) \\ &\simeq \mathrm{map}_{\mathsf{S}_*}(Z, r_*(\Omega^\infty_{/X}\Sigma^\infty_{/X} Y)).\end{aligned}$$

Almost all manipulations follow from the standard adjunctions. The third equivalence uses the fact that $r^*\Sigma^\infty Z$ is the trivial parametrised spectrum with fibre $\Sigma^\infty Z$, hence is equivalent to $\Sigma^\infty_{/X} r^* Z$. □

We will need two more facts before proving Theorem 8.11. The first one is the following simple observation. If $V \to X$ is a vector bundle such that its associated sphere bundle $\mathrm{Sph}(V) \to X$ has a section s, then we may take the fibrewise infinite suspension $\Sigma^\infty_{/X}\mathrm{Sph}(V) \in \mathsf{Sp}_{/X}$ using s to give a basepoint in each fibre. On the other hand, we could have taken the fibrewise one-point compactification and then suspend using the added point at infinity as a basepoint in each fibre. Up to a suspension, these are the same parametrised spectra.

Lemma 8.13. *Let $V \to X$ be a vector bundle with a non-vanishing section, and let $\mathrm{Sph}(V) \to X$ be its associated sphere bundle. Let $\mathbb{S}^V_X$ denote the fibrewise infinite suspension of the fibrewise one-point compactification of V (using the point at infinity as the basepoint in each fibre). Then:*

$$\Sigma^\infty_{/X}\mathrm{Sph}(V) \simeq \Omega_X \mathbb{S}^V_X$$

where Ω_X denotes the desuspension in the category $\mathsf{Sp}_{/X}$.

Proof. Let us scale a non-vanishing section s of V to that it has image in the sphere bundle. We write $\mathrm{D}(V) \subset V$ for the unit disc bundle of V which we point using s, and V^+ for the fibrewise one-point compactification. We obtain the lemma by applying the fibrewise infinite suspension $\Sigma^\infty_{/X}$ to the cofibre sequence $\mathrm{Sph}(V) \to \mathrm{D}(V) \to V^+$ of parametrised pointed spaces over X. □

Recall that the ∞-category $\mathsf{Sp}_{/X}$ is symmetric monoidal, with monoidal unit $\mathbb{S}_X := r^*(\mathbb{S})$. (Here, and everywhere else, $\mathbb{S}$ denotes the sphere spectrum.) The usefulness of the whole machinery set up so far is contained in the following result. A classical reference is [MS06, Chapter 18]. In the language of ∞-categories, one may read the second section of [Lan21, Appendix A].

Lemma 8.14 (Costenoble–Waner duality)**.** *The Costenoble–Waner dualising spectrum of X is $\mathbb{S}^{-TX}_X$, the spherical fibration associated to the stable normal bundle of X. That is, we have an equivalence of functors:*

$$r_*(-) \simeq r_!(- \otimes_{\mathbb{S}_X} \mathbb{S}^{-TX}_X).$$

We are now ready to prove the result announced at the beginning of this section.

Proof of Theorem 8.11. We start by choosing once and for all a section s of the sphere bundle $\mathrm{Sph}(J^1\mathcal{L}^d)$, which provides us with a basepoint in every fibre. We may therefore apply the free infinite loop space functor $Q = \Omega^\infty\Sigma^\infty \colon \mathsf{S}_* \to \mathsf{S}_*$ fibrewise and obtain the following diagram of fibrations:

$$\begin{tikzcd}
S^{2n+1} \arrow[r] \arrow[d] & \Omega^\infty\Sigma^\infty S^{2n+1} \arrow[d] \\
\mathrm{Sph}(J^1\mathcal{L}^d) \arrow[r] \arrow[d] & \Omega^\infty_{/X}\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d) \arrow[d] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

It is a standard fact that the map $S^{2n+1} \to \Omega^\infty\Sigma^\infty S^{2n+1}$ is $(4n+1)$-connected. Hence, on section spaces, the map

$$\Gamma(\mathrm{Sph}(J^1\mathcal{L}^d)) \to \Gamma(\Omega^\infty_{/X}\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d))$$

is $2n$-connected. (Notice that both spaces are connected, so the choice of s was immaterial.) Using Lemma 8.12, we obtain:

$$\Gamma(\Omega^\infty_{/X}\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d)) \simeq r_*(\Omega^\infty_{/X}\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d)) \simeq \Omega^\infty r_*(\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d)).$$

We now make the purely formal following computation:

$$\begin{aligned}
r_*(\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d)) &\simeq r_!(\Sigma^\infty_{/X}\mathrm{Sph}(J^1\mathcal{L}^d) \otimes_{\mathbb{S}_X} \mathbb{S}_X^{-TX}) \\
&\simeq r_!(\Omega_X\mathbb{S}_X^{J^1\mathcal{L}^d} \otimes_{\mathbb{S}_X} \mathbb{S}_X^{-TX}) \\
&\simeq r_!(\Omega_X\mathbb{S}_X^{J^1\mathcal{L}^d-TX}) \\
&\simeq \Omega r_!(\mathbb{S}_X^{J^1\mathcal{L}^d-TX}) \\
&\simeq \Omega X^{J^1\mathcal{L}^d-TX}
\end{aligned}$$

where we used Lemma 8.14 for the first equivalence, Lemma 8.13 for the second, and recognised that the value of $r_!$ on a spherical fibration is the associated Thom spectrum. Summing up, we get the result.

□

8.3.2. *An example when $X = \mathbb{CP}^2$.* To show how Theorem 8.11 can be applied in practice, we use it to prove Proposition 8.10. We hope that this will convince the reader of the computational power of homotopy theoretic methods to study spaces of algebraic sections.

Following Theorem 8.11, we should investigate $\Omega^{\infty+1}X^{J^1\mathcal{L}^d-TX}$ when $X = \mathbb{CP}^2$ and $\mathcal{L} = \mathcal{O}(1)$. Because $J^1\mathcal{L}^d - TX$ is of rank 2, the spectrum $\Omega X^{J^2\mathcal{L}^d-TX}$ is 1-connective and the bottom homotopy group is $\pi_1 \cong \mathbb{Z}$ by Hurewicz theorem. We consider the fibration

$$F \longrightarrow \Omega^{\infty+1}X^{J^1\mathcal{L}^d-TX} \longrightarrow S^1$$

where F is the homotopy fibre of the right-most map, which is taken to induce an isomorphism on π_1. A generator of $\pi_1(\Omega^{\infty+1}X^{J^1\mathcal{L}^d-TX}) \cong \mathbb{Z}$ gives a section of that fibration, and we obtain:

$$\Omega^{\infty+1}X^{J^1\mathcal{L}^d-TX} \simeq S^1 \times F.$$

In particular, F is 2-connective with $\pi_2(F) \cong \pi_2(\Omega^{\infty+1}X^{J^1\mathcal{L}^d-TX})$. By Hurewicz theorem and the universal coefficient theorem, $H_2(F;\mathbb{Z}/2) \cong H_2(F;\mathbb{Z}) \otimes \mathbb{Z}/2 \cong \pi_2(F) \otimes \mathbb{Z}/2$. We thus wish to compute $\pi_2(\Omega^{\infty+1}X^{J^1\mathcal{L}^d-TX})$, which we will do using the Adams spectral sequence at the prime 2:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}\left(H^*(X^{J^1\mathcal{L}^d-TX};\mathbb{Z}/2), \mathbb{Z}/2\right) \implies \pi_*(X^{J^1\mathcal{L}^d-TX})^\wedge_2.$$

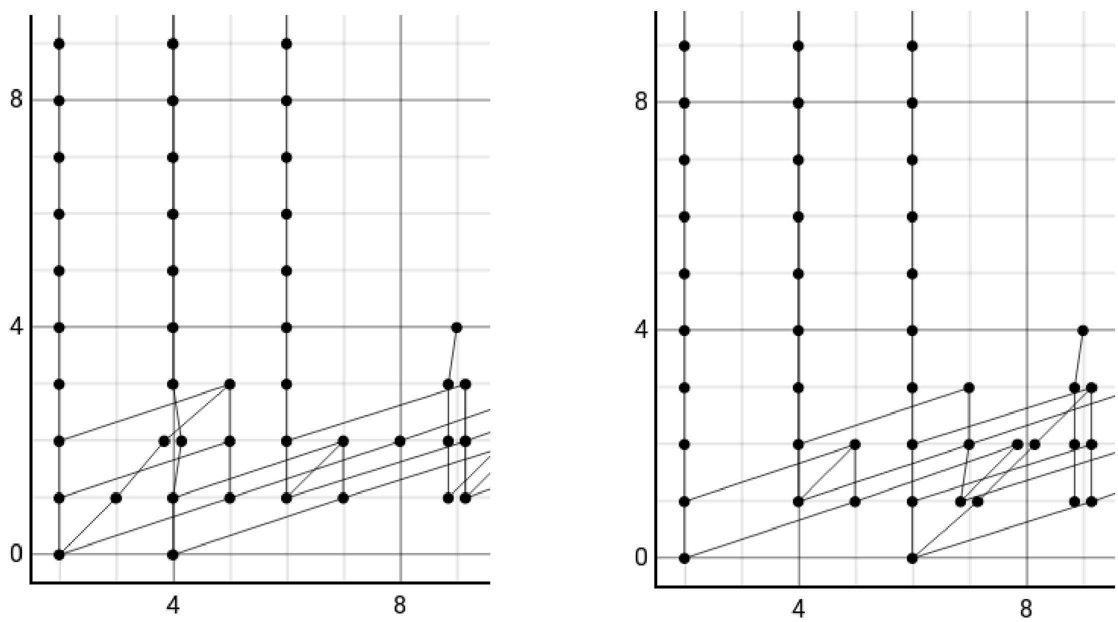
(Hence we will only compute the 2-completed group, but this will be enough for our purposes.) The E_2-page is computed by knowing the cohomology $H^*(X^{J^1\mathcal{L}^d-TX};\mathbb{Z}/2)$ as an algebra over the mod 2 Steenrod algebra $\mathcal{A}$. (See [BC18, Section 3.3] for a very readable introduction.) If U denotes the Thom class of $J^1\mathcal{L}^d - TX$, the classes in the cohomology of the Thom spectrum $X^{J^1\mathcal{L}^d-TX}$ are given via the Thom isomorphism as yU where $y \in H^*(X;\mathbb{Z}/2)$. At the prime 2, the Steenrod squares can then be computed from the formula:

$$\mathrm{Sq}^k(yU) = \sum_{i+j=k} \mathrm{Sq}^i(y)\mathrm{Sq}^j(U) = \sum_{i+j=k} \mathrm{Sq}^i(y) w_j U$$

where w_j is the j-th Stiefel–Whitney class of $J^1\mathcal{L}^d - TX$. In our case, writing $\mathbb{Z}/2[x]/(x^3)$ for the cohomology ring of $X = \mathbb{CP}^2$, the total Stiefel–Whitney class is given by:

$$w(J^1\mathcal{L}^d - TX) = \begin{cases} 1 & d \equiv 0 \mod 2 \\ 1+x & d \equiv 1 \mod 2. \end{cases}$$

We used the handy tool [CC] to compute the E_2-page for us, and obtained the following:



A. Case $d \equiv 0 \mod 2$

B. Case $d \equiv 1 \mod 2$

Following the established convention, we use the Adams grading: the horizontal axis is indexed by $t-s$, the vertical one by s. Every dot represents a copy of $\mathbb{Z}/2$. The vertical lines represent multiplication by $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{A}}(\mathbb{Z}/2,\mathbb{Z}/2)$. We suggest to the unfamiliar reader to look at [BC18, Section 4.3] for more explanations.

From this, standard arguments about differentials (see e.g. [BC18, Section 4.8]) show that

$$\pi_3(X^{J^1\mathcal{L}^d-TX})^\wedge_2 \cong \begin{cases} \mathbb{Z}/2 & d \equiv 0 \mod 2 \\ 0 & d \equiv 1 \mod 2. \end{cases}$$

Therefore

$$H_2(F;\mathbb{Z}/2) \cong \pi_2(F) \otimes \mathbb{Z}/2 \cong \pi_3(X^{J^1\mathcal{L}^d - TX}) \otimes \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z}/2 & d \equiv 0 \mod 2 \\ 0 & d \equiv 1 \mod 2. \end{cases}$$

Using Künneth theorem, we obtain:

$$H_2(\Omega^{\infty+1} X^{J^1\mathcal{L}^d - TX};\mathbb{Z}/2) \cong H_2(S^1 \times F;\mathbb{Z}/2) \cong H_2(F;\mathbb{Z}/2)$$

which finishes the proof of Proposition 8.10.

8.4. **Stability of p-torsion.** In this final section, we study the p-torsion in the homology of the space $\Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}(J^1\mathcal{L}^d)\right)$. On the one hand, we have just seen in Proposition 8.10 that it depends on d in general. On the other hand, the result below shows that when the prime p is slightly bigger the dimension of X, the p-torsion is independent of $\mathcal{L}$.

Proposition 8.15. *Let X be a smooth complex projective variety of complex dimension n and $\mathcal{L}$ be a holomorphic line bundle on it. Let p be a prime such that $p \geq n+2$. Then the fibrewise p-localisation of the sphere bundle $\mathrm{Sph}(J^1\mathcal{L}) \to X$ is trivial. In particular, we have an equivalence of p-local spaces*

$$\Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}(J^1\mathcal{L})\right)_{(p)} \simeq \mathrm{map}(X, S^{2n+1}_{(p)}).$$

As an immediate consequence, combining the proposition above with Corollary 8.1 shows that the p-torsion in the homology of $\Gamma_{\mathrm{hol,ns}}(X;\mathcal{L}^d)$ stabilises when $p \geq \dim_{\mathbb{C}} X + 2$ and $d \to \infty$.

The proof uses the following result which we learned from [BM14, Proposition 4.1].

Lemma 8.16. *For $p \geq \frac{k}{2} + \frac{3}{2}$, the space $\mathrm{map}_1(S^k_{(p)}, S^k_{(p)})$ of maps homotopic to the identity is $(k-1)$-connected.*

Proof. The proof is given in [BM14], but we sketch it here for convenience. We shall assume that k is odd, as we will only use this case in this paper. Recall the evaluation fibration

$$\Omega^k_1 S^k_{(p)} \longrightarrow \mathrm{map}_1(S^k_{(p)}, S^k_{(p)}) \longrightarrow S^k_{(p)}.$$

Using the associated long exact sequence of homotopy groups, it suffices to show that $\pi_i(\Omega^k_1 S^k_{(p)})$ vanishes for $i \leq k-1$. Using the assumption $p \geq \frac{k}{2} + \frac{3}{2}$, this follows from Serre's calculations on p-torsion in the homotopy groups of spheres. □

Proof of Proposition 8.15. Let

$$S^{2n+1}_{(p)} \longrightarrow \mathrm{Sph}(J^1\mathcal{L})_{(p)} \longrightarrow X$$

be the fibrewise p-localisation of $\mathrm{Sph}(J^1\mathcal{L}) \to X$. By [Mø87, Theorem 5.3], we have a homotopy equivalence

$$\Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}(J^1\mathcal{L})\right)_{(p)} \simeq \Gamma_{\mathcal{C}^0}\left(\mathrm{Sph}(J^1\mathcal{L})_{(p)}\right).$$

As the sphere bundle is canonically oriented (using the complex orientation of $J^1\mathcal{L}$), the fibration $\mathrm{Sph}(J^1\mathcal{L})_{(p)} \to X$ is classified by a map

$$X \longrightarrow B\mathrm{map}_1(S^{2n+1}_{(p)}, S^{2n+1}_{(p)}).$$

By Lemma 8.16, the codomain of that map is $(2n+1)$-connected. As the domain has real dimension $2n$, the classifying map must be null-homotopic thus showing that the fibration is trivial. □

References

[Ban20] Ishan Banerjee. Stable Cohomology of Discriminant Complements for an algebraic curve. *arXiv:2010.14644 [math]*, October 2020. 4

[Ban21] Oishee Banerjee. Filtration of cohomology via symmetric semisimplicial spaces. *arXiv:1909.00458 [math]*, October 2021. 3

[BC18] Agnes Beaudry and Jonathan A. Campbell. A Guide for Computing Stable Homotopy Groups. *Contemporary Mathematics*, 718:89–136, 2018. 41

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy. *Real Algebraic Geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin Heidelberg, 1998. 6, 7, 17, 18

[BDRS99] Mauro C. Beltrametti, Sandra Di Rocco, and Andrew J. Sommese. On generation of jets for vector bundles. *Revista Matemática Complutense*, 12(1):27–44, 1999. 4

[BH19] Margaret Bilu and Sean Howe. Motivic Euler products in motivic statistics. *arXiv:1910.05207 [math]*, October 2019. 4

[BM14] Martin Bendersky and Jeremy Miller. Localization and Homological Stability of Configuration Spaces. *The Quarterly Journal of Mathematics*, 65(3):807–815, September 2014. 42

[Bre97] Glen E. Bredon. *Sheaf Theory*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2 edition, 1997. 4, 13, 27, 28

[CC] Hood Chatham and Dexter Chua. The Spectral Sequences Project. 41

[DH22] Ronno Das and Sean Howe. Cohomological and motivic inclusion-exclusion. Technical Report arXiv:2204.04165, arXiv, April 2022. 3

[DRS00] Sandra Di Rocco and Andrew J. Sommese. Line bundles for which a projectivized jet bundle is a product. *Proceedings of the American Mathematical Society*, 129(6):1659–1663, December 2000. 22, 37

[FHT01] Yves Felix, Steve Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001. 35

[Gro67] Alexander Grothendieck. Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. *Publications Mathématiques de l'IHÉS*, 32:5–361, 1967. 5

[Hae82] André Haefliger. Rational Homotopy of the Space of Sections of a Nilpotent Bundle. *Transactions of the American Mathematical Society*, 273(2):609–620, 1982. 35

[Lan21] Markus Land. Reducibility of low dimensional Poincaré duality spaces. *To appear in Münster J. Math*, November 2021. 38, 39

[Lle85] Irene Llerena. Localization of fibrations with nilpotent fibre. *Mathematische Zeitschrift*, 188(3):397–410, September 1985. 34

[Mos06] Jacob Mostovoy. Spaces of rational maps and the Stone–Weierstrass theorem. *Topology*, 45(2):281 – 293, 2006. 8, 22, 32

[Mos12] Jacob Mostovy. Truncated simplicial resolutions and spaces of rational maps. *The Quarterly Journal of Mathematics*, 63(1):181–187, March 2012. 8, 24

[MS06] J. May and J. Sigurdsson. *Parametrized Homotopy Theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, December 2006. 38, 39

[Mø87] Jesper Michael Møller. Nilpotent Spaces of Sections. *Transactions of the American Mathematical Society*, 303(2):733, October 1987. 35, 42

[Nei10] Joseph Neisendorfer. *Algebraic Methods in Unstable Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, Cambridge, 2010. 37

[Poo04] Bjorn Poonen. Bertini theorems over finite fields. *Annals of Mathematics*, 160(3):1099–1127, November 2004. 4

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge Structures*. Springer Science & Business Media, February 2008. 35, 36

[Tho56] René Thom. L'homologie des espaces fonctionnels. *Colloque de topologie algébrique, Louvain*, pages 29–39, 1956. 35

[Tom14] Orsola Tommasi. Stable cohomology of spaces of non-singular hypersurfaces. *Advances in Mathematics*, 265:428 – 440, 2014. 2, 16, 34

[Tom23] Orsola Tommasi. Stable cohomology of complements of discriminants. *In preparation*, 2023. 3, 34

[Vas94] V.A. Vassiliev. *Complements of discriminants of smooth maps : topology and applications*, volume 98 of *Translations of mathematical monographs*. American Mathematical Society, Providence, Rhode Island, 1994. 8

[Vas99] V.A. Vassiliev. How to Calculate Homology Groups of Spaces of Nonsingular Algebraic Projective Hypersurfaces. *Proc. Steklov Inst. Math.*, 225:121–140, 1999. 3, 8

[Vok07] Lukáš Vokřínek. A generalization of Vassiliev's h-principle. *arXiv:0705.0333 [math]*, May 2007. 8, 26, 27, 28, 29

[VW15] Ravi Vakil and Melanie Matchett Wood. Discriminants in the Grothendieck ring. *Duke Mathematical Journal*, 164(6):1139–1185, April 2015. 2, 4

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994. 25

Department of Mathematical Sciences, University of Copenhagen,
Universitetsparken 5, 2100 København Ø, Denmark

Email address: alexis-aumonier@math.ku.dk

PAPER B

Scanning the moduli of embedded smooth hypersurfaces

This chapter contains the preprint version of the following paper:

Alexis Aumonier. *Scanning the moduli of embedded smooth hypersurfaces.* 2023.

The preprint version is not made publicly available yet.

SCANNING THE MODULI OF EMBEDDED SMOOTH HYPERSURFACES

ALEXIS AUMONIER

Abstract. We study the locus of smooth hypersurfaces inside the Hilbert scheme of a smooth projective complex variety. In the spirit of scanning, we construct a map to a continuous section space of a projective bundle, and show that it induces an isomorphism in integral homology in a range of degrees growing with the ampleness of the hypersurfaces. When the ambient variety is a curve, this recovers a result of McDuff and Segal about configuration spaces. We compute the rational cohomology of the section space and exhibit a phenomenon of homological stability for hypersurfaces with first Chern class going to infinity. For simply connected varieties, the rational cohomology is shown to agree with the stable cohomology of a moduli space of hypersurfaces, with a peculiar tangential structure, as studied by Galatius and Randal-Williams.

1. Introduction

A hypersurface in a smooth projective complex variety X is the zero locus $V(s)$ of a non-zero global section $s \in H^0(X, \mathcal{L})$ of an algebraic line bundle $\mathcal{L}$:

$$V(s) := \{x \in X \mid s(x) = 0\} \subset X.$$

Such a hypersurface is said to be smooth when the derivative $ds(x) \neq 0$ for all $x \in V(s)$. Smooth hypersurfaces obtained from sections of $\mathcal{L}$ are parameterised by the complement of the discriminant inside the complete linear system $|\mathcal{L}| = \mathbb{P}\big(H^0(X, \mathcal{L})\big)$, a classical object dating back to the early days of complex geometry. To allow variations of the line bundle, Grothendieck introduced in [Gro62] the functor of relative effective Cartier divisors. We follow his footsteps and, for a given polynomial $P \in \mathbb{Q}[x]$, consider the functor on the category of complex schemes

$$\begin{aligned} \mathfrak{M}^{\mathrm{sm},P} \colon \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ T &\longmapsto \left\{ Z \subset X \times T \;\middle|\; \begin{array}{l} Z \to T \text{ is flat and proper and for all } t \in T \\ Z_t \subset X_t \text{ is a smooth effective Cartier divisor} \\ \text{with Hilbert polynomial } \chi\big(Z_t, \mathcal{I}_{Z_t}^{-1}(n)\big) = P(n) \end{array} \right\} \end{aligned}$$

As already discovered by Grothendieck, this functor is representable by an open subscheme $\mathcal{M}^{\mathrm{sm},P} \subset \mathrm{Hilb}(X)$ of the Hilbert scheme of X. Under conditions on the polynomial P, it can in fact be explicitly constructed using projective bundles [BLR90]. Recording the isomorphism class of a Cartier divisor produces the *Abel–Jacobi morphism*

$$\mathcal{M}^{\mathrm{sm},P} \longrightarrow \mathrm{Pic}^P(X), \quad Z \longmapsto [\mathcal{I}_Z^{-1}]$$

to the Picard scheme. Given a cohomology class $\alpha \in \mathrm{NS}(X) \subset H^2(X;\mathbb{Z})$ in the *Néron–Severi group* of X, the component $\mathrm{Pic}^\alpha(X) \subset \mathrm{Pic}(X)(\mathbb{C})$ parameterises holomorphic line bundles on X with first Chern class α. Our main object of interest in this article is the *moduli space of smooth hypersurfaces* with fixed Chern class

$$\mathcal{M}_{\mathrm{hyp}}^{\alpha} \longrightarrow \mathrm{Pic}^\alpha(X)$$

obtained by restricting the Abel–Jacobi morphism after analytification. It is the parameter space

$$\mathcal{M}^{\alpha}_{\mathrm{hyp}} \cong \{Z \subset X \text{ smooth hypersurface with } c_1(\mathcal{O}_X(Z)) = \alpha\}$$

whose points are smooth complex hypersurfaces embedded inside X. In particular, the fibre of the Abel–Jacobi map above an isomorphism class of a line bundle $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$ is the complement of the discriminant inside the linear system $|\mathcal{L}|$.

In the present work, we investigate the topology of $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$ using tools from algebraic topology. To state our results, we write $d(\alpha)$ for the largest integer d such that all line bundles with first Chern class α are d-jet ample.

Theorem 1.1 (See Theorem 4.6 for a precise version)**.** *Let X be a smooth projective complex variety and let $\alpha \in \mathrm{NS}(X)$ be ample enough. Taking the first jet yields a map*

$$\mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0}\big(\mathbb{P}(J^1\mathcal{O}_X)\big)$$

which induces an isomorphism in integral homology onto the path component that it hits, in degrees $* < \frac{d(\alpha)-3}{2}$.

Remark 1.2. *In fact, $\pi_0\big(\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))\big) \cong H^2(X;\mathbb{Z})$ and the jet map hits the component corresponding to $\alpha \in \mathrm{NS}(X) \subset H^2(X;\mathbb{Z})$.*

We recall in Section 2 the notion of jet ampleness, and explain in Appendix A how to estimate the number $d(\alpha)$. Among other things, we show that given any integer $M \geq 0$ and classes $\alpha, \beta \in \mathrm{NS}(X)$ with β ample, we have that $d(\alpha + k\beta) \geq M$ for all large enough $k \gg 0$. (See Proposition A.7.) In particular, the degree range of our main theorem can be arbitrarily big. In the remainder of this introduction, we describe applications of our main theorem and connect our results to the existing literature on moduli spaces of manifolds.

1.1. Rational computations and stability.

A main advantage of our main theorem resides in the fact that the homotopy type of spaces of continuous sections can be approached by purely homotopical methods. This is particularly effective if one is willing to look at the rational information only. Using tools from rational homotopy theory, we show:

Theorem 1.3 (See Theorem 6.5 for a precise version)**.** *Let n be the complex dimension of X. Let $\alpha \in H^2(X;\mathbb{Z})$ and denote by $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$ the component hit by the jet map from $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$. The rational cohomology of that section space is computed by the cohomology of the following commutative differential graded algebra:*

$$\mathrm{Sym}^*_{\mathrm{gr}}\big(\mathbb{Q}z \oplus H_1(X;\mathbb{Q}) \oplus H_*(X;\mathbb{Q})[1]\big),$$
$$\textit{with } d(z) = 0,\ d(H_1(X;\mathbb{Q})) = 0,\ \textit{and } d(x) = \varphi(x) \textit{ for } x \in H_*(X;\mathbb{Q})[1].$$

Here $\mathrm{Sym}^*_{\mathrm{gr}}$ denotes the free graded commutative algebra on a graded vector space, $[1]$ increases the grading by one, and $\mathbb{Q}z$ is a one-dimensional vector space generated by z in degree 2. The differential is encoded by a morphism $\varphi\colon H_*(X;\mathbb{Q}) \to \mathrm{Sym}^*_{\mathrm{gr}}(\mathbb{Q}z \oplus H_1(X;\mathbb{Q}))$ which can be computed explicitly in terms of the Chern classes of $J^1\mathcal{O}_X$ and α.

Let us also mention another application of our main theorem in the form of a rational homological stability phenomenon:

Theorem 1.4 (See Corollary 6.8)**.** *Let X be a smooth projective complex variety whose tangent bundle is a topologically trivial complex vector bundle. Let $\alpha \in \mathrm{NS}(X)$ be ample and assume that $d(\alpha) \geq 1$. Then, for any integer $k \geq 1$, there is map*

$$\mathcal{M}_{\mathrm{hyp}}^{k\alpha} \longrightarrow \mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}})$$

inducing an isomorphism in rational homology in the range of degrees $* < \frac{k \cdot d(\alpha)-3}{2}$. In particular, the rational homology stabilises as $k \to \infty$.

Remark 1.5. *By a theorem of Wang [Wan54], if the tangent bundle of X is holomorphically trivial then X is an abelian variety. If we only require topological triviality as in the theorem above, other examples exist such as products of abelian varieties with curves [SA15].*

1.2. **Configuration spaces on curves.** On an algebraic curve, the first Chern class of a line bundle is simply its degree under the identification $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$. The vanishing locus of a non-zero global section of a line bundle of degree $d > 0$ is a set of d points counted with multiplicity. In fact, such a section is non-singular precisely when these d points are distinct. In that case, the moduli space of embedded hypersurfaces is the classically studied configuration space and our main theorem recovers parts of a result of McDuff and Segal [McD75, Seg79] (see also [RW13] for improved ranges):

Theorem 1.6 (See Theorem 7.2)**.** *Let X be a smooth projective complex curve of genus g. Let $\alpha \in H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ be such that $\alpha > 2g-2$. The jet map*

$$\mathrm{UConf}_{\alpha}(X) \cong \mathcal{M}_{\mathrm{hyp}}^{\alpha} \longrightarrow \Gamma_{\mathcal{C}^0}^{\alpha}(\mathbb{P}(J^1\mathcal{O}_X)) \cong \Gamma_{\mathcal{C}^0}^{\alpha}(\dot{T}X)$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{\alpha-2g-3}{2}$.

1.3. **Characteristic classes and moduli spaces of manifolds.** Let $H = V(s) \subset X$ be a hypersurface defined by a non-singular section of a line bundle $\mathcal{L}$ on X. In the series of papers [GRW14, GRW17, GRW18], Galatius and Randal-Williams have investigated the homology of moduli spaces of manifolds. In this article, we have tried to compare their results to ours in the case where the manifold under investigation is H. Deferring the technical details to Section 8.2, we describe here the main contents.

Let n be the complex dimension of X, so that H is of real dimension $2n-2$. Given a tangential structure $\ell \colon H \to B$ on H with $\theta \colon B \to BO(2n-2)$, Galatius and Randal-Williams study the moduli space $\mathcal{M}^{\theta}(H,\ell)$ classifying smooth H-bundles with θ-structure. One could wonder which tangential structure is the most natural on H. In the last section of this article, we find a reasonable candidate and show:

Theorem 1.7 (See Theorem 8.11 and Corollary 8.15 for a precise version)**.** *Let X be a simply connected smooth complex projective variety of dimension $n \geq 4$ and $\mathcal{L}$ be a very ample line bundle on it. Let*

$$\Gamma_{\mathrm{ns}}(\mathcal{L}) := \left\{ s \in H^0(X,\mathcal{L}) \mid j^1(s)(x) \neq 0 \text{ for all } x \in X \right\}$$

be the subspace of non-singular global sections. There is a map $\theta \colon B \to BO(2n-2)$ such that a hypersurface H defined by a non-singular section of $\mathcal{L}$ inherits a tangential structure $\ell \colon H \to B$. In other words, for this tangential structure there is a map classifying the universal bundle:

$$\Gamma_{\mathrm{ns}}(\mathcal{L}) \longrightarrow \mathcal{M}^{\theta}(H,\ell).$$

Furthermore, writing Δ for the discriminant, the subgroup of $(\deg\Delta)$-th roots of unity $\mu_{\deg\Delta} \subset \mathbb{C}^\times$ acts on $\mathcal{M}^\theta(H,\ell)$ and the induced map on the quotients

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow \mathcal{M}^\theta(H,\ell)/\mu_{\deg\Delta}$$

induces an isomorphism in rational cohomology in the stable ranges.

1.4. **Outline.** In Section 2 we recall known properties about the Picard scheme, jet bundles, jet ampleness, and the topology of smooth hypersurfaces. In Section 3 we define precisely the moduli space $\mathcal{M}^\alpha_{\mathrm{hyp}}$. In Sections 4 and 5 we state and prove our main theorem. The rest of the paper is dedicated to applications. We make rational computations in Section 6 and describe the relation to scanning and characteristic classes of manifold bundles in Sections 7 and 8. Finally, we have assembled in Appendix A various results concerning jet ampleness.

1.5. **The proof strategy for the main theorem.** As explained in the introduction, we have a sequence of spaces

$$|\mathcal{L}| \setminus \Delta \longrightarrow \mathcal{M}^\alpha_{\mathrm{hyp}} \longrightarrow \mathrm{Pic}^\alpha(X) \tag{1}$$

where $\Delta \subset |\mathcal{L}|$ is the discriminant hypersurface. It turns out not to be a fibration, but only a microfibration. On the other hand, we have an actual fibration

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \longrightarrow B\big(\mathrm{Map}(X,\mathbb{C}^\times)/\mathbb{C}^\times\big) \tag{2}$$

obtained by modding out by the constant functions $\mathbb{C}^\times \subset \mathrm{Map}(X,\mathbb{C}^\times)$ and delooping the $\mathrm{Map}(X,\mathbb{C}^\times)$-principal fibration

$$\mathrm{Map}(X,\mathbb{C}^\times) \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0) \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

sending a section in the total space to its projectivisation. We observe the weak homotopy equivalences

$$B\big(\mathrm{Map}(X,\mathbb{C}^\times)/\mathbb{C}^\times\big) \simeq K(H^1(X;\mathbb{Z}),1) \simeq \mathrm{Pic}^\alpha(X)$$

and have proved in the earlier work [Aum22] that the jet map

$$|\mathcal{L}| \setminus \Delta \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0)/\mathbb{C}^\times$$

induces an isomorphism in homology in a range of degrees. In essence, the proof consists in comparing with two (micro)fibrations (1) and (2): we will leverage the homotopy (resp. homology) equivalence of their bases (resp. fibres) to obtain a homology equivalence between their total spaces.

Contents

Acknowledgements. This work is part of my PhD thesis and I would like to thank my advisor Søren Galatius for many useful discussions, suggestions, and encouragements. Thanks to Ronno Das as well, for comments on this project and joint work on another where we heavily use microfibrations. I was supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151) as well as the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682922).

2. Preliminary material

2.1. **The Picard scheme.** We begin with recollections on line bundles on smooth projective complex varieties and their moduli. A more precise, and much more general, account of this standard material can be found in Kleiman's notes [Kle05]. In what follows, X is a connected smooth projective complex variety.

Definition 2.1. *The* absolute Picard group *of a complex scheme T is the set $\mathrm{Pic}_{\mathrm{abs}}(T)$ of isomorphism classes of algebraic line bundles on T equipped with the group law given by the tensor product.*

Definition 2.2. *The Picard functor of X*

$$T \longrightarrow \mathrm{Pic}_{\mathrm{abs}}(X \times T)/\mathrm{Pic}_{\mathrm{abs}}(T)$$

from complex schemes to abelian groups is represented by a scheme $\mathrm{Pic}(X)$ called the Picard scheme *of X (relative to $\mathrm{Spec}(\mathbb{C})$).*

Remark 2.3. *The analytification of $\mathrm{Pic}(X)$, sometimes called the* Picard space *in this article, is the group of isomorphism classes of holomorphic line bundles on X.*

Lemma 2.4 (Compare [Kle05, Exercise 4.3])**.** *There exists a (non-unique) algebraic line bundle $\mathcal{P}$ on $\mathrm{Pic}(X) \times X$ satisfying the following property: given any complex scheme T and a line bundle $\mathcal{L}$ on $X \times T$, there exists a unique morphism $h\colon T \to \mathrm{Pic}(X)$ such that*

$$\mathcal{L} \cong (1 \times h)^*\mathcal{P} \otimes f^*\mathcal{N}$$

for $f\colon X \times T \to T$ the projection map and $\mathcal{N}$ some line bundle on T. □

Definition 2.5. *Any choice of a line bundle $\mathcal{P}$ as in Lemma 2.4 will be called a* Poincaré line bundle *on X.*

The Picard scheme $\mathrm{Pic}(X)$ only parameterises *isomorphism classes* of line bundles on X. One should think of the choice of a Poincaré line bundle as making compatible choices of *representatives* of those isomorphisms classes.

We will need a further decomposition of the Picard scheme into components. To introduce it, we let $\mathcal{O}_X(1)$ be a very ample line bundle on X and write $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and $n \in \mathbb{Z}$.

Definition 2.6. *Let $\mathbb{C} \subset k$ be a field extension, and write $X_k = X \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(k)$ for the base change. The* Hilbert polynomial *of a closed subscheme $Z \subset X_k$ is the function*

$$P_Z\colon \mathbb{N} \longrightarrow \mathbb{Z}, \quad n \longmapsto \chi(Z, \mathcal{O}_Z(n))$$

where $\mathcal{O}_Z$ is the structure sheaf of Z.

Given a polynomial $P \in \mathbb{Q}[x]$, let $\mathrm{Pic}^P(X)(-) \subset \mathrm{Pic}(X)(-)$ be the subfunctor of the Picard functor whose T-points are represented by the line bundles $\mathcal{L}$ on $X \times T$ such that

$$\chi(X_t, \mathcal{L}_t^{-1}(n)) = P(n) \quad \text{for all } t \in T$$

where X_t and $\mathcal{L}_t$ denote the base change to t.

Proposition 2.7 (Compare [Kle05, Theorems 4.8 and 6.20])**.** *The Picard functor* $\mathrm{Pic}^P(X)(-)$ *is represented by a complex quasi-projective scheme denoted* $\mathrm{Pic}^P(X)$*. The Picard scheme is the disjoint union of the* $\mathrm{Pic}^P(X)$ *when* P *runs over all polynomials* P*.* □

Passing to complex points, the picture is vastly simplified by Hodge theory as recalled in the following two results.

Definition 2.8. *The* Néron–Severi *group of* X*, denoted* $\mathrm{NS}(X)$*, is the image of the morphism* $c_1 \colon \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ *sending an isomorphism class of a line bundle to its first Chern class.*

Proposition 2.9. *The Picard space* $\mathrm{Pic}(X)$ *a disjoint union of connected components indexed by the Néron–Severi group*

$$\mathrm{Pic}(X) = \bigsqcup_{\alpha \in \mathrm{NS}(X)} \mathrm{Pic}^\alpha(X).$$

Each space $\mathrm{Pic}^\alpha(X)$ *is a torus, non-canonically isomorphic to* $H^1(X;\mathcal{O}_X)/H^1(X;\mathbb{Z})$*, and parameterises isomorphism classes of holomorphic line bundles on* X *with Chern class* α*.* □

2.2. **Jet bundles.** We recall the definition of jet bundles in algebraic geometry and explain the construction of the jet evaluation map which will be used throughout this article. The main reference for this section is [Gro67, Section 16.7]. In this section only, the full generality offered by schemes will be convenient, so we momentarily work in this setting.

Let $f \colon Y \to S$ be a morphism of schemes, $\Delta \colon Y \to Y \times_S Y$ be the diagonal and $\mathcal{I}$ be its ideal sheaf. We let $p_i \colon Y \times_S Y \to Y$ be the two projections for $i = 1, 2$.

Definition 2.10. *Let* $\mathcal{F}$ *be an* $\mathcal{O}_Y$*-module. Its* relative r-jet bundle *is defined by*

$$J^r_{Y/S}\mathcal{F} := (p_1)_*\big(\mathcal{O}_{Y\times_S Y}/\mathcal{I}^{r+1} \otimes p_2^*\mathcal{F}\big).$$

The two projections p_i *give two morphisms of sheaves of rings* $\mathcal{O}_Y \to J^r_{Y/S}\mathcal{O}_Y$*. We choose the one given by* p_1 *to define an* $\mathcal{O}_Y$*-module structure. The other one, induced by* p_2*, is denoted by*

$$d^r_{Y/S} \colon \mathcal{O}_Y \to J^r_{Y/S}\mathcal{O}_Y$$

and is called the jet map*. In particular* $J^r_{Y/S}\mathcal{F} = J^r_{Y/S}\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$ *where* $J^r_{Y/S}\mathcal{O}_Y$ *is seen as a right* $\mathcal{O}_Y$*-module via* $d^r_{Y/S}$*. We will also write* $d^r_{Y/S} \colon \mathcal{F} \to J^r_{Y/S}\mathcal{F}$ *for the tensor of the jet map with* $\mathcal{F}$*.*

The fibre of the jet bundle at a closed point $y \in Y$ with maximal ideal sheaf $\mathfrak{m}$ is $(J^r_{Y/S}\mathcal{F})|_y \cong \mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F}$. Intuitively, the jet map should be thought of as taking the r-th Taylor expansion of a function. In particular, as the Leibniz rule for differentation shows, it is not a morphism of $\mathcal{O}_X$-modules when $r > 0$. On the contrary, taking the derivative of a function commutes with multiplication by a constant. At the level of the relative jet bundles, the functions on S play the role of the scalars, and this fact is expressed by the following:

Lemma 2.11. *The pushforward of the jet map*

$$f_* d^r_{Y/S} \colon f_* \mathcal{F} \longrightarrow f_* J^r_{Y/S} \mathcal{F}$$

is a morphism of $\mathcal{O}_S$-modules.

Proof. The claim can be checked locally on an affine cover. We can thus assume that $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism between affine schemes, $\mathcal{F} = \widetilde{M}$ is a B-module, and I is the kernel of the multiplication map $B \otimes_A B \to B$. Then $f_* J^r_{X/S} \mathcal{F}$ corresponds to

$$(B \otimes_A B)/I^{r+1} \otimes_{B \otimes_A B} ((B \otimes_A B) \otimes_B M)$$

and $f_* \mathcal{F}$ is M, both seen as A-modules via f. In these coordinates, the jet map is

$$m \longmapsto (1 \otimes 1) \otimes ((1 \otimes 1) \otimes m)$$

which is visibly A-linear. $\square$

Definition 2.12. *Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. The* fibrewise jet evaluation map *is the composition of the pushforward of the jet map followed by the counit:*

$$f^* f_* \mathcal{F} \longrightarrow f^* f_* J^r_{Y/S} \mathcal{F} \longrightarrow J^r_{Y/S} \mathcal{F}.$$

To explain the definition above, we assume that Y and S are complex varieties for the rest of this section. As we will alternate between two points of view on vector bundles (as sheaves or as spaces over the base), it will sometimes be convenient to be explicit about which viewpoint is adopted:

Definition 2.13. *Let $\mathcal{F}$ be a vector bundle (i.e. a locally free sheaf of $\mathcal{O}_Y$-modules of finite rank) on a complex variety Y. The total space of the associated* geometric vector bundle *is*

$$\mathbb{V}(\mathcal{F}) := \underline{\operatorname{Spec}}_Y \big(\operatorname{Sym}^*(\mathcal{F}^\vee) \big)^{\mathrm{an}}$$

where $(-)^{\mathrm{an}}$ denotes the analytification functor.

Suppose now that $\mathcal{F}$ is a vector bundle on Y such that $f_* \mathcal{F}$ is also a vector bundle on S. As sets, we have an identification

$$\mathbb{V}(f_* \mathcal{F}) = \{(s, \sigma) \mid s \in S,\ \sigma \in H^0(Y_s; \mathcal{F}|_{Y_s})\}$$

where $Y_s = f^{-1}(s) \subset Y$ is the fibre above s. In particular, when $S = \operatorname{Spec} \mathbb{C}$ is a point, this is the space of global sections $H^0(Y; \mathcal{F})$. In general, it should be thought of as a space of fibrewise sections.

Lemma 2.14. *Under the above assumptions, the counit map $f^* f_* \mathcal{F} \to \mathcal{F}$ induces the evaluation map*

$$\begin{aligned} \mathbb{V}(f^* f_* \mathcal{F}) \cong \mathbb{V}(f_* \mathcal{F}) \times_S Y &\longrightarrow \mathbb{V}(\mathcal{F}) \\ ((s, \sigma), y) &\longmapsto \sigma(y) \end{aligned}$$

on geometric realisation.

Proof. Recall first that $f^* f_* \mathcal{F}$ is the sheafification of $U \longmapsto \mathcal{F}(f^{-1} f(U)) \otimes_{\mathcal{O}_S(f(U))} \mathcal{O}_Y(U)$. (When $f(U)$ is not open, we mean taking the colimit over all open subsets of S containing it.) Chasing through the adjunction, the counit map is then seen to be the sheafification of the map

$$\begin{aligned} \mathcal{F}(f^{-1} f(U)) \otimes_{\mathcal{O}_S(f(U))} \mathcal{O}_Y(U) &\longrightarrow \mathcal{F}(U) \\ a \otimes r &\longmapsto r \cdot a|_U \end{aligned}$$

and the claim follows. $\square$

Summarising the situation, we see that the fibrewise jet evaluation map

$$\mathbb{V}(f_*\mathcal{F}) \times_S Y \longrightarrow \mathbb{V}(J^r_{Y/S}\mathcal{F})$$

is given above a point $s \in S$ by

$$\begin{aligned} H^0(Y_s; \mathcal{F}|_{Y_s}) \times Y &\longrightarrow \mathbb{V}(J^r\mathcal{F}|_{Y_s}) \\ (\sigma, y) &\longmapsto j^r\sigma(y) \in \mathcal{F}|_{Y_s}/\mathfrak{m}^{r+1}\mathcal{F}|_{Y_s} \end{aligned}$$

where $\mathfrak{m}$ is the maximal ideal sheaf of $y \in Y_s$. In other words, it takes the Taylor expansion of σ at y up to order r.

2.3. **Jet ampleness.** Having now defined jet bundles, we state the crucial definition of jet ampleness of a line bundle on a smooth projective complex variety X.

Definition 2.15 (Compare [BRS99]). *Let $k \geq 0$ be an integer. Let $x_1, \dots, x_t$ be t distinct points in X and $(k_1, \dots, k_t)$ be a t-uple of positive integers such that $\sum_i k_i = k+1$. Denote by $\mathcal{O}_X$ the structure sheaf of X and by $\mathfrak{m}_i$ the maximal ideal sheaf corresponding to x_i. We regard the tensor product $\otimes_{i=1}^t \mathfrak{m}_i^{k_i}$ as a subsheaf of $\mathcal{O}_X$ under the multiplication map $\otimes_{i=1}^t \mathfrak{m}_i^{k_i} \to \mathcal{O}_X$. We say that a line bundle $\mathcal{L}$ is* k-jet ample *if the evaluation map*

$$\Gamma(\mathcal{L}) \longrightarrow \Gamma\left(\mathcal{L} \otimes \left(\mathcal{O}_X / \otimes_{i=1}^t \mathfrak{m}_i^{k_i}\right)\right) \cong \bigoplus_{i=1}^t \Gamma\left(\mathcal{L} \otimes \left(\mathcal{O}_X/\mathfrak{m}_i^{k_i}\right)\right)$$

is surjective for any $x_1, \dots, x_t$ and $k_1, \dots, k_t$ as above.

Example 2.16. *Being 0-jet ample corresponds to being spanned by global sections. Furthemore, 1-jet ampleness is the usual notion of very ampleness. On a curve, a line bundle is k-jet ample whenever it is k-very ample. However, on higher dimensional varieties, a k-jet ample line bundle is also k-very ample but the converse is not true in general.*

The following proposition is the main tool to produce line bundles having a very high degree of jet ampleness.

Proposition 2.17 (See [BRS99, Proposition 2.3]). *If $\mathcal{A}$ and $\mathcal{B}$ are respectively a- and b-jet ample line bundles, then their tensor product $\mathcal{A} \otimes \mathcal{B}$ is $(a+b)$-jet ample.*

Definition 2.18. *Let X be a smooth projective complex variety and $\alpha \in H^2(X;\mathbb{Z})$. We write $d(X,\alpha)$ for the largest integer $d \geq -1$ such that all line bundles on X with first Chern class equal to α are d-jet ample. (By convention, we declare that being (-1)-jet ample is an empty condition.)*

We refer to Appendix A for how to compute $d(X,\alpha)$ in some special cases. Given an integer d, we also explain in Proposition A.7 how to find an α such that $d(X,\alpha) \geq d$.

2.4. **The topology of hypersurfaces.** It is well known that all smooth degree d complex hypersurfaces in $\mathbb{P}^n$ are diffeomorphic. As a way of justifying the study of the moduli space of hypersurfaces of a given Chern class, we observe that such hypersurfaces are also all diffeomorphic, provided the Chern class is ample enough. First, recall that ampleness is a numerical property:

Theorem 2.19 (Nakai–Moishezon criterion). *A line bundle $\mathcal{L}$ on a proper scheme over a field is ample if and only if $\int_Y c_1(\mathcal{L})^{\dim Y} > 0$ for every integral subscheme $Y \subset X$.*

Definition 2.20. *A Chern class $\alpha \in \mathrm{NS}(X)$ is called* ample *if it satisfies the Nakai–Moishezon criterion.*

We recall the following classical definition which is central in our work:

Definition 2.21. *A global section $s \in \Gamma(X, \mathcal{L})$ of a line bundle $\mathcal{L}$ on a smooth projective complex variety X is called* non-singular *if for all $x \in X$ we have $(s(x), ds(x)) \neq 0$.*

Remark 2.22. *Given a non-singular section $s \in \Gamma(X, \mathcal{L})$, its vanishing locus*

$$V(s) := \{x \in X \mid s(x) = 0\} \subset X$$

is a smooth hypersurface.

Any hypersurface H can be seen as a Weil divisor, hence a Cartier divisor (X is smooth), and therefore has an attached line bundle $\mathcal{O}_X(H)$. If $H = V(s)$ with $s \in \Gamma(X, \mathcal{L})$, then $\mathcal{O}_X(H) \cong \mathcal{L}$. The following bit of language will be convenient:

Definition 2.23. *The* Chern class of a hypersurface H *is the first Chern class of its associated line bundle $c_1(\mathcal{O}_X(H))$.*

Proposition 2.24. *Let X be a smooth projective complex variety with canonical sheaf K_X. Let $\alpha \in \mathrm{NS}(X)$ be a Chern class ample enough such that:*

1. *the class $\alpha - c_1(K_X)$ is ample;*
2. *for any line bundle $\mathcal{L}$ of Chern class α, the subspace $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ consisting of the non-singular global sections is non empty.*

Then all the smooth hypersurfaces of Chern class α are diffeomorphic to one another.

Remark 2.25. *Let us make some remarks on the two assumptions of the proposition above. Let $\mathcal{L}$ be a very ample line bundle on X. Then $K_X^{-1} \otimes \mathcal{L}^{\otimes k}$ is very ample for $k \gg 0$ big enough, and $\alpha = c_1(\mathcal{L}^{\otimes k})$ satisfies the first assumption. Furthermore, by Bertini theorem, the subspace $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ is dense. The second assumption is thus satisfied as soon as all line bundles of Chern class α are very ample. We explain how to arrange this in Appendix A.*

Proof. We first briefly recall the classical proof in the case of a single linear system. Let $\mathcal{L}$ be a line bundle on X with Chern class α and denote by $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ the subset of those global sections that are non-singular. The projection from the incidence variety

$$\{(s, x) \in \Gamma_{\mathrm{ns}}(\mathcal{L}) \times X \mid s(x) = 0\} \longrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L}).$$

is a proper surjective submersion between smooth manifolds, with fibres the smooth hypersurfaces. By Ehresmann's lemma, it is a fibre bundle and therefore all the fibres over a connected component are diffeomorphic. Finally, $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ is the complement of the discriminant which has codimension at least 1, hence it is connected.

Now we adapt the proof in families. Let $\mathrm{Pic}^\alpha(X)$ be the connected component of the Picard space classifying isomorphism classes of line bundles of Chern class α, and let $\mathcal{P}$ be a Poincaré line bundle on $\mathrm{Pic}^\alpha(X) \times X$. For $[\mathcal{L}] \in \mathrm{Pic}^\alpha(X)$, we write $\mathcal{P}_{[\mathcal{L}]}$ for the line bundle on X which represents the isomorphism class $[\mathcal{L}]$. Let $p\colon \mathrm{Pic}^\alpha(X) \times X \to \mathrm{Pic}^\alpha(X)$ be the projection. By cohomology and base change [Har77, Theorem III.12.11], the sheaf $p_*\mathcal{P}$ is a vector bundle provided that $H^1(X, \mathcal{P}_{[\mathcal{L}]}) = 0$ for all $[\mathcal{L}] \in \mathrm{Pic}^\alpha(X)$. This follows by the Kodaira vanishing theorem and the assumption that $\alpha - c_1(K_X)$ is ample. Let

$$\mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}} \subset \mathbb{V}(p_*\mathcal{P}) = \{([\mathcal{L}], s) \mid [\mathcal{L}] \in \mathrm{Pic}^\alpha(X),\ s \in \Gamma(\mathcal{P}_{[\mathcal{L}]})\}$$

be the subset of those sections that are non-singular. The incidence variety

$$\{([\mathcal{L}], s, x) \in \mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}} \times X \mid s(x) = 0\} \longrightarrow \mathbb{V}(p_*\mathcal{P})^{\mathrm{ns}}$$

is a smooth fibre bundle by Ehresmann's lemma. The base is connected by our second assumption on the ampleness of α, therefore all the fibres are diffeomorphic. □

3. The moduli of hypersurfaces

In this section, we precisely define our main object of interest in this paper: the moduli of smooth hypersurfaces. From now on, we adopt the following conventions:

- X is a connected smooth complex projective variety;
- $\mathrm{Pic}(X)$ is its associated Picard scheme or space (see Definition 2.2);
- $p\colon \mathrm{Pic}(X) \times X \to \mathrm{Pic}(X)$ is the first projection;
- $\mathcal{P}$ is a choice, once and for all, of a Poincaré line bundle (see Definition 2.5);
- if $\mathcal{L}$ is a line bundle on X, we write $\Gamma_{\mathrm{ns}}(\mathcal{L}) \subset \Gamma(\mathcal{L})$ for the subspace of non-singular sections.

3.1. **Moduli functors and the Hilbert scheme.** In this subsection, we define the moduli functor of smooth hypersurfaces in X and show that it is representable by an open subscheme of the Hilbert scheme of X. We give an explicit description of the representing moduli scheme using the theory of linear systems of divisors. Although providing motivation and context, this subsection is logically independent of the rest of the article. We must also say that the results presented here are well known to algebraic geometers, but we have chosen to recall them in detail to be self-contained. The reader unfamiliar with the algebro-geometric language used here is invited to jump to the following subsection where we provide a point-set model for the analytification of the moduli scheme, which will be thereafter used throughout the article.

Given a polynomial $P \in \mathbb{Q}[x]$, we may consider the Hilbert functor $\mathrm{Hilb}^P(X)(-)$ parameterising flat proper families of closed subschemes in X with given Hilbert polynomial P (recall Definition 2.6). In other words

$$\begin{aligned} \mathrm{Hilb}^P(X)(-)\colon \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ T &\longmapsto \left\{ Z \subset X \times T \;\middle|\; \begin{array}{l} Z \to T \text{ is flat and proper} \\ \text{and } \forall t \in T,\ P_{Z_t}(n) = P(n) \end{array} \right\} \end{aligned}$$

where $\mathsf{Sch}_{\mathbb{C}}$ is the category of schemes over $\mathrm{Spec}(\mathbb{C})$ and $Z_t \subset X_t$ is the fibre[1] of $Z \to T$ above $t \in T$. More generally, the Hilbert functor $\mathrm{Hilb}(X)(-)$ of X is the disjoint union of the $\mathrm{Hilb}^P(X)(-)$ as P runs over all polynomials.

Theorem 3.1 (Grothendieck [Gro61]). *The Hilbert functor* $\mathrm{Hilb}^P(X)(-)$ *is represented by the* Hilbert scheme $\mathrm{Hilb}^P(X)$ *which is projective over* $\mathrm{Spec}(\mathbb{C})$.

[1] If k is the residue field of $t \in T$, then $Z_t = Z \times_T \mathrm{Spec}(k) \subset X_t = X_k$.

Families of hypersurfaces are more commonly known as *relative effective Cartier divisors.* (See [The23, Tag 056P].) Following [Kle05, Part 3], we recall the definition of their moduli functor:

Definition 3.2. *Let $P \in \mathbb{Q}[x]$ be a polynomial. The* moduli functor of effective divisors *with Hilbert polynomial P is the functor*

$$\mathfrak{M}^P : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

$$T \longmapsto \left\{ Z \subset X \times T \;\middle|\; \begin{array}{l} Z \to T \text{ is flat and proper and for all } t \in T \\ \text{the ideal sheaf } \mathcal{I}_{Z_t} \text{ is a line bundle and is} \\ \text{such that } \chi(X_t, \mathcal{I}_{Z_t}^{-1}(n)) = P(n) \end{array} \right\}$$

We define $\mathfrak{M}^{\mathrm{sm},P} \subset \mathfrak{M}^P$ to be the subfunctor[2] *where Z_t is furthermore required to be smooth over the residue field $\mathrm{Spec}(\kappa(t))$ at $t \in T$.*

Remark 3.3. *The moduli functor $\mathfrak{M}^P$ is visibly a subfunctor of the Hilbert functor. Though, because of our conventions, the indexing Hilbert polynomials are different. If $Z \subset X$ is an effective Cartier divisor and $P(n) = \chi(X, \mathcal{I}_Z^{-1}(n))$, we let P' be the associated polynomial $P'(n) = \chi(Z, \mathcal{O}_Z(n))$. The subfunctor inclusion then reads $\mathfrak{M}^P \subset \mathrm{Hilb}^{P'}(X)$.*

We are here working with a general projective variety X with non-necessarily discrete Picard space, which can be confusing. To counteract that feeling, we remind the reader of the more classical, and enlightening, situation of linear systems of divisors:

Example 3.4 (Compare [Kle05, Definition 3.12 and Theorem 3.13]). *Let $X = \mathbb{P}^n$ and let $P' \in \mathbb{Q}[x]$ be the Hilbert polynomial of a hypersurface of degree $d \geq 1$. One can show that the Hilbert scheme is in this case the complete linear system*

$$\mathrm{Hilb}^{P'}(\mathbb{P}^n) = |\mathcal{O}(d)| = \mathbb{P}\big(H^0(\mathbb{P}^n, \mathcal{O}(d))\big).$$

Therefore $\mathrm{Hilb}^{P'}(\mathbb{P}^n)(-) = \mathfrak{M}^P(-)$ is represented by a projective space of complex dimension $\dim_{\mathbb{C}} H^0(\mathbb{P}^n, \mathcal{O}(d)) - 1$. The subfunctor $\mathfrak{M}^{\mathrm{sm},P}$ is represented by the complement of the discriminant hypersurface.

In general, we have the following general representability result:

Theorem 3.5. *The subfunctor $\mathfrak{M}^P(-) \subset \mathrm{Hilb}^{P'}(X)(-)$ is represented by a union of connected components of the Hilbert scheme $\mathrm{Hilb}^{P'}(X)$.*

Proof. [3] By [Kol96, Theorem 1.13], $\mathfrak{M}^P(-)$ is represented by an open subscheme U of the Hilbert scheme such that the inclusion $U \subset \mathrm{Hilb}(X)$ is universally closed (this uses that X is smooth over $\mathrm{Spec}(\mathbb{C})$). As the Hilbert scheme is separated (because projective), U must be a union of connected components. ☐

Our goal in now to give an explicit description of the scheme representing $\mathfrak{M}^P(-)$. For the remainder of this section, we fix a polynomial $P \in \mathbb{Q}[x]$, denote by $\mathrm{Pic}^P(X) \subset \mathrm{Pic}(X)$ the subscheme defined in Proposition 2.7, and still write $\mathcal{P}$ for the restriction of the chosen Poincaré line bundle to it. Recall that $p\colon \mathrm{Pic}^P(X) \times X \to \mathrm{Pic}^P(X)$ denotes the first projection. Let

$$\mathcal{K} := \ker\left(p^* p_* \mathcal{P} \to \mathcal{P}\right)$$

[2] Recall that smoothness is preserved under base change.

[3] I have learned this result from Bejleri's lecture notes, and I thank him for pointing me to Kollár's book.

be the kernel sheaf of the counit of $p^* \dashv p_*$. From the monoidality of p^* and standard properties of the relative Proj construction, we have an isomorphism

$$\underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\, p^*(p_*\mathcal{P})^\vee\big) \cong \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) \times X.$$

Using the surjection of sheaves $p^*(p_*\mathcal{P})^\vee \twoheadrightarrow \mathcal{K}^\vee$ we thus obtain a closed immersion

$$\underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\,\mathcal{K}^\vee\big) \subset \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) \times X.$$

Definition 3.6. *We define the* universal family *to be the morphism*

$$\mathcal{U} := \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\big(\mathrm{Sym}\,\mathcal{K}^\vee\big) \longrightarrow \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) =: \mathcal{M}$$

obtained by projecting onto the first coordinate.

The following is the main theorem of this section:

Theorem 3.7. *Assume that $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$. Then the universal family $\mathcal{U} \to \mathcal{M}$ represents the moduli functor $\mathfrak{M}^P(-)$.*

Proof. This is explained in [BLR90, Proposition 8.2.7]. To translate to the notation in that book: take f to be the structure morphism $X \to \mathrm{Spec}(\mathbb{C})$, T to be $\mathrm{Pic}^P(X)$, and $\mathcal{L}$ to be $\mathcal{P}$. The flatness assumption on $\mathcal{L}$ is implies by our assumption using cohomology and base change [Har77, Theorem III.12.11]. ☐

Remark 3.8. *The analytification of the universal family is simply the incidence variety*

$$\begin{gathered}\big\{(x, [\mathcal{L}], [s]) \text{ with } x \in X,\ [\mathcal{L}] \in \mathrm{Pic}^P(X),\ [s] \in \mathbb{P}\Gamma(\mathcal{P}_{[\mathcal{L}]}) \text{ such that } s(x) = 0\big\} \\ \big\downarrow \\ \big\{([\mathcal{L}], [s]) \text{ with } [\mathcal{L}] \in \mathrm{Pic}^P(X),\ [s] \in \mathbb{P}\Gamma(\mathcal{P}_{[\mathcal{L}]})\big\}\end{gathered}$$

above the projectivisation of the vector bundle $\mathbb{V}(p_*\mathcal{P}) \to \mathrm{Pic}^P(X)$.

Definition 3.9. *The morphism of functors*

$$\mathfrak{M}^P(T) \longrightarrow \mathrm{Pic}^P(X)(T), \quad (Z \subset X \times T) \longmapsto [\mathcal{I}_Z^{-1}]$$

is represented by the projection morphism

$$\mathcal{M} = \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)}\big(\mathrm{Sym}(p_*\mathcal{P})^\vee\big) \longrightarrow \mathrm{Pic}^P(X)$$

which is usually called the Abel–Jacobi *morphism.*

We now explain how to obtain a scheme representing $\mathfrak{M}^{\mathrm{sm},P}(-)$. We assume that the evaluation morphism $p^*p_*\mathcal{P} \to \mathcal{P}$ is surjective. Recall the jet evaluation morphism

$$p^*p_*\mathcal{P} \longrightarrow J^1_{\mathrm{Pic}^P(X)\times X/\mathrm{Pic}^P(X)}\mathcal{P}$$

from Definition 2.12. We have a commutative diagram with middle row a short exact sequence:

$$\begin{tikzcd}
 & & \left(J^1_{\mathrm{Pic}^P(X)\times X/\mathrm{Pic}^P(X)}\mathcal{P}\right)^\vee \arrow[d] & & \\
0 \arrow[r] & \mathcal{P}^\vee \arrow[ur] \arrow[r] \arrow[dr, "0"'] & p^*(p_*\mathcal{P})^\vee \arrow[r] \arrow[d] & \mathcal{K}^\vee \arrow[r] \arrow[dl, dashed, two heads] & 0 \\
 & & \mathcal{Q} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $\mathcal{Q}$ is defined to be the cokernel of the dual of the jet evaluation, and

$$\mathcal{P}^\vee \longrightarrow \left(J^1_{\mathrm{Pic}^P(X)\times X/\mathrm{Pic}^P(X)}\mathcal{P}\right)^\vee$$

is the dual of the projection morphism from the first jet bundle to the zeroth jet bundle $J^0\mathcal{P} = \mathcal{P}$. The composition $\mathcal{P}^\vee \to p^*(p_*\mathcal{P})^\vee \to \mathcal{Q}$ is the zero morphism, and we thus obtain a surjective morphism $\mathcal{K}^\vee \twoheadrightarrow \mathcal{Q}$.

Corollary 3.10. *Assume that the counit morphism $p^*p_*\mathcal{P} \to \mathcal{P}$ is surjective, and that $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$. Then the moduli functor $\mathfrak{M}^{\mathrm{sm},P}(-)$ is represented by an open subscheme of $\mathcal{M}$, hence of the Hilbert scheme of X. More precisely, let $\pi : \mathcal{U} \to \mathcal{M}$ be the universal family and*

$$\mathcal{Z} := \underline{\mathrm{Proj}}_{\mathrm{Pic}^P(X)\times X}\left(\mathrm{Sym}\,\mathcal{Q}\right) \hookrightarrow \mathcal{U}$$

be the closed subscheme determined by the surjection $\mathcal{K}^\vee \twoheadrightarrow \mathcal{Q}$. Then $\mathfrak{M}^{\mathrm{sm},P}(-)$ is represented by $\mathcal{M} \setminus \pi(\mathcal{Z})$.

Remark 3.11. *The sheaf $\mathcal{Q}$ is only a coherent sheaf of $\mathcal{O}_{\mathrm{Pic}^P(X)\times X}$-modules and $\mathcal{Z}$ is therefore not a vector bundle in general. Nonetheless, if we furthermore assume that the jet evaluation morphism is surjective (e.g. if all line bundles parameterised by $\mathrm{Pic}^P(X)$ are very ample), then $\mathcal{Q}$ is the dual of the kernel of a surjective morphism of locally free sheaves, hence itself locally free.*

Remark 3.12. *In the notation of Remark 3.8, $\mathcal{Z}^{\mathrm{an}}$ consists of those points $(x, [\mathcal{L}], [s])$ such that $j^1(s)(x) = 0$, i.e. s is singular at x.*

Proof. Smoothness is an open condition: the projection π is flat and of finite presentation, so [The23, Tag 01V9] applies and $\mathfrak{M}^{\mathrm{sm},P}(-)$ is seen to be represented by an open subscheme of $\mathcal{M}$. The subscheme $\mathcal{M} \setminus \pi(\mathcal{Z}) \subset \mathcal{M}$ is open because π is proper. It represents the moduli functor as smoothness can be checked locally using the Jacobian criterion. □

We close this section with some general remarks about our assumptions in Corollary 3.10. We show that, although stated scheme-theoretically, they can be checked after analytification.

Lemma 3.13. *Let $\mathcal{P}^{\mathrm{an}}$ denote the analytic sheaf associated to $\mathcal{P}$. If the counit morphism $p^*p_*\mathcal{P}^{\mathrm{an}} \to \mathcal{P}^{\mathrm{an}}$ is surjective, then the same is true before analytification.*

Proof. This follows from exactness of the analytification of sheaves functor. □

Lemma 3.14. *Let $P \in \mathbb{Q}[x]$ be a polynomial and suppose that $H^1(X, \mathcal{L}) = 0$ for all $[\mathcal{L}] \in \mathrm{Pic}^P(X)(\mathbb{C})$. Then $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \mathrm{Pic}^P(X)$.*

Proof. By upper semicontinuity of cohomology, the subscheme

$$\left\{t \in \operatorname{Pic}^P(X) \mid H^1(X_t, \mathcal{P}_t) = 0\right\} \subset \operatorname{Pic}^P(X)$$

is open. Assume that its complement is non empty. Then it contains a complex point: it is of locally of finite type over $\operatorname{Spec}(\mathbb{C})$ and Hilbert's Nullstellensatz applies. This cannot be the case by assumption. □

We finally comment on the relation between Hilbert polynomials and Chern classes. For a holomorphic line bundle $\mathcal{L}$ on X, recall the Hirzebruch–Riemann–Roch theorem giving an equality

$$\chi(X, \mathcal{L}) = \int_X \operatorname{ch}(\mathcal{L}) \operatorname{td}(X)$$

where $\operatorname{ch}(-)$ is the Chern character and $\operatorname{td}(X)$ is the Todd class of X. In particular, if $Z \subset X$ is an effective Cartier divisor, its Hilbert polynomial only depends on the first Chern class of its associated line bundle $\mathcal{O}_X(Z)$. We thus obtain a numerical criterion:

Lemma 3.15. *Let $P \in \mathbb{Q}[x]$ and let C be the collection of Chern classes*

$$C := \left\{c_1(\mathcal{L}) \mid [\mathcal{L}] \in \operatorname{Pic}^P(X)(\mathbb{C})\right\}.$$

If $\alpha - c_1(K_X)$ is ample for all $\alpha \in C$, then $H^1(X_t, \mathcal{P}_t) = 0$ for all $t \in \operatorname{Pic}^P(X)$.

Proof. This follows from Lemma 3.14 whose assumption is verified by the Kodaira vanishing theorem. □

3.2. **A convenient point-set model.** In this section, we unravel the result of Corollary 3.10 and give an explicit point set model for the moduli space of smooth hypersurfaces.

We begin with notations which we will use throughout the rest of the article. If $[\mathcal{L}] \in \operatorname{Pic}(X)$ is an isomorphism class of a line bundle, we write $\mathcal{P}_{[\mathcal{L}]}$ for the representative of that isomorphism class given by the restriction of $\mathcal{P}$ to $X \cong \{[\mathcal{L}]\} \times X \subset \operatorname{Pic}(X) \times X$. If $\alpha \in \operatorname{NS}(X)$ we recall from Proposition 2.9 that $\operatorname{Pic}^\alpha(X) \subset \operatorname{Pic}(X)$ denotes the connected component parameterising line bundles of Chern class α, and we will write $\mathcal{P}_\alpha$ for the restriction of the Poincaré line bundle to that component.

Definition 3.16. *The first jet bundle of $\mathcal{P}$ relative to the projection p (see Definition 2.10) is denoted*

$$J^1_p\mathcal{P} := J^1_{\operatorname{Pic}(X)\times X/\operatorname{Pic}(X)}\mathcal{P}.$$

When restricted to $\operatorname{Pic}^\alpha(X)$ for some $\alpha \in \operatorname{NS}(X)$, we will write $J^1_p\mathcal{P}_\alpha := J^1_{\operatorname{Pic}^\alpha(X)\times X/\operatorname{Pic}^\alpha(X)}\mathcal{P}_\alpha$.

Lemma 3.17. *Let K_X be the canonical sheaf of X. Let $\alpha \in \operatorname{NS}(X)$ be such that $\alpha - c_1(K_X)$ is ample. Then $p_*\mathcal{P}_\alpha$ is a vector bundle and the fibrewise jet evaluation map gives a map of vector bundles*

$$p^*p_*\mathcal{P}_\alpha \longrightarrow J^1_p\mathcal{P}_\alpha.$$

As sets, the geometric realisations are given by

$$\mathbb{V}(p^*p_*\mathcal{P}_\alpha) = \mathbb{V}(p_*\mathcal{P}_\alpha) \times X = \left\{([\mathcal{L}], x, s) \mid [\mathcal{L}] \in \operatorname{Pic}^\alpha(X),\ x \in X,\ s \in \Gamma(\mathcal{P}_{[\mathcal{L}]})\right\}$$

and

$$\mathbb{V}(J^1_p\mathcal{P}_\alpha) = \left\{([\mathcal{L}], x, v) \mid [\mathcal{L}] \in \operatorname{Pic}^\alpha(X),\ x \in X,\ v \in J^1\mathcal{P}_{[\mathcal{L}]}|_x\right\}.$$

Under these identifications, the jet evaluation map is given by

$$\operatorname{jev}\colon ([\mathcal{L}], x, s) \longmapsto ([\mathcal{L}], x, j^1s(x)).$$

Proof. The fact that $p_*\mathcal{P}_\alpha$ is a vector bundle follows directly from cohomology and base change and the Kodaira vanishing theorem under the assumption that $\alpha - c_1(K_X)$ is ample. The rest of the lemma follows from the results recalled in Section 2.2. □

Recall from Corollary 3.10 the scheme $\mathcal{M} \setminus \pi(\mathcal{Z})$ representing the moduli functor of smooth hypersurfaces. After analytification, we may restrict the Abel–Jacobi map

$$\big(\mathcal{M} \setminus \pi(\mathcal{Z})\big)^{\mathrm{an}} \longrightarrow \mathrm{Pic}(X)^{\mathrm{an}} \tag{3}$$

to the connected component $\mathrm{Pic}^\alpha(X) \subset \mathrm{Pic}(X)$ (we will from now on drop the superscript "an"), recalled in Proposition 2.9, provided that α is ample enough:

Definition 3.18. *Let $\alpha \in \mathrm{NS}(X)$ be such that $\alpha - c_1(K_X)$ is ample. The* moduli of embedded smooth hypersurfaces in X of Chern class α *is defined to be the preimage of $\mathrm{Pic}^\alpha(X)$ by the Abel–Jacobi map* (3). *In other terms, we have a homeomorphism:*

$$\mathcal{M}^\alpha_{\mathrm{hyp}} \cong \big(\mathbb{V}(p_*\mathcal{P}_\alpha) \setminus \mathrm{proj}(\mathrm{jev}^{-1}(0))\big)/\mathbb{C}^\times$$

where the scalars act fibrewise over $\mathrm{Pic}(X)$, and $\mathrm{proj}\colon \mathbb{V}(p^*p_*\mathcal{P}) \to \mathbb{V}(p_*\mathcal{P})$ is the projection induced by p.

Remark 3.19. *As sets, we have an identification*

$$\mathcal{M}^\alpha_{\mathrm{hyp}} = \{([\mathcal{L}],[s]) \mid [\mathcal{L}] \in \mathrm{Pic}^\alpha(X),\ [s] \in \Gamma_{\mathrm{ns}}(\mathcal{P}_{[\mathcal{L}]})/\mathbb{C}^\times\}.$$

That is, it will be technically convenient to think of a smooth hypersurface as a tuple consisting of a line bundle $\mathcal{L}$ and a non-singular global section of it (up to isomorphism and scaling action). However, the name of moduli space is justified by the previous section: we have a homeomorphism

$$\mathcal{M}^\alpha_{\mathrm{hyp}} \cong \{Z \subset X \textit{ smooth hypersurface with } c_1(\mathcal{O}_X(Z)) = \alpha\} \subset \mathrm{Hilb}(X)^{\mathrm{an}}.$$

4. Statement of the main theorem

In this section, we construct a topological counterpart to the moduli of smooth hypersurfaces described in Definition 3.18. We then state our main theorem comparing the two objects.

4.1. **A topological counterpart.** We begin with some generalities about the topology of continuous section spaces. Let $E \to A \times B$ be a fibre bundle on a topological space $A \times B$. We denote a point of E as a tuple (a,b,e) where $a \in A$, $b \in B$ and $e \in E|_{(a,b)}$ is in the fibre. All mapping spaces are given the compact open topology.

Definition 4.1. *The space of* fibrewise sections *of E over A is defined to be the subspace*

$$\begin{aligned}\Gamma_{\mathcal{C}^0,\mathrm{fib}}(E \to A) := \{(a,s) \mid a \in A,\ s \in \Gamma_{\mathcal{C}^0}(E|_{a\times B})\} &\hookrightarrow \mathrm{Map}(B,E)\\ (a,s) &\mapsto [b \mapsto (a,b,s(b))].\end{aligned}$$

Post-composition with the projection maps $E \to A \times B \to A$ gives a continuous map $\mathrm{Map}(B,E) \to \mathrm{Map}(B,A)$ which, when restricted to fibrewise sections, yields the projection map

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(E \to A) \longrightarrow A, \quad (a,s) \longmapsto a.$$

In particular, this projection map is continuous.

Remark 4.2. *Let Z be a topological space. A continuous map $Z \to \Gamma_{\mathcal{C}^0,\text{fib}}(E \to A)$ is the same datum as a continuous map $f\colon Z \times B \to E$ over B such that $\operatorname{proj} \circ f(z,-)\colon B \to E \to A$ is constant for any $z \in Z$.*

Remark 4.3. *When A and B are smooth projective complex varieties, one can modify the definition above by using the spaces of holomorphic maps instead of the whole mapping spaces. In fact, assume that $E = \mathbb{V}(\mathcal{E})$ is also a vector bundle and that $\pi_*\mathcal{E}$ is a vector bundle, where $\pi\colon A \times B \to A$ is the first projection. Then the holomorphic fibrewise section space is exactly $\mathbb{V}(\pi_*\mathcal{E})$. Let us notice however that the holomorphic mapping spaces are more naturally topologised using the analytic topology of the Hom scheme. Fortunately, Douady shows in [Dou66] that the inclusion inside the whole mapping space is continuous.*

To lighten the notation, and as no confusion can arise, we will from now on drop the symbol $\mathbb{V}(-)$ when considering continuous sections of a vector bundle.

Definition 4.4. *Taking fibrewise, over $\operatorname{Pic}(X)$, continuous global sections of $J^1_p\mathcal{P}$ which are never vanishing, we obtain the space*

$$\begin{aligned}\Gamma_{\mathcal{C}^0,\text{fib}}(J^1_p\mathcal{P} \setminus 0) &:= \Gamma_{\mathcal{C}^0,\text{fib}}(J^1_p\mathcal{P} \setminus 0 \to \operatorname{Pic}(X)) \\ &= \left\{([\mathcal{L}], s) \mid [\mathcal{L}] \in \operatorname{Pic}(X),\ s \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}]} \setminus 0)\right\}.\end{aligned}$$

The group $\mathbb{C}^\times$ acts by multiplying the sections by scalars and we let $\Gamma_{\mathcal{C}^0,\text{fib}}(J^1_p\mathcal{P} \setminus 0)/\mathbb{C}^\times$ be the quotient for that action.

4.2. **The main theorem.**

By Remark 4.3, the fibrewise jet map followed by the inclusion of the space of holomorphic sections inside continuous sections gives rise to a continuous map

$$j^1\colon \mathcal{M}^\alpha_{\text{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0,\text{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times.$$

Denote by $\mathbb{P}(J^1\mathcal{O}_X)$ the projectivisation of the first jet bundle of $\mathcal{O}_X$ on X. We will make use of the following:

Proposition 4.5 (Compare [CS84, Lemma 2.5])**.** *The connected components of the section space $\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$ are in one-to-one correspondence with $H^2(X;\mathbb{Z})$. For a given Chern class α, the associated connected component $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$ consists of those sections s such that the pullback $s^*\mathcal{O}(1)$ of the tautological bundle has Chern class α.*

It follows from the proposition that if $\mathcal{L}$ is a line bundle with Chern class α, the quotient map

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0) \longrightarrow \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{L})) \cong \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

has image inside the connected component $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$. Here we have used that $J^1\mathcal{L} \cong J^1\mathcal{O}_X \otimes \mathcal{L}$ and the fact that the projectivisation of a vector bundle is invariant under tensoring with a line bundle.

Now, let $\mathcal{L}_0$ be a chosen line bundle with Chern class α. By choosing an isomorphism of topological line bundles $\mathcal{L}_0 \cong \mathcal{P}_{[\mathcal{L}]}$ for each $[\mathcal{L}] \in \operatorname{Pic}^\alpha(X)$, we obtain a map

$$\Gamma_{\mathcal{C}^0,\text{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{L}_0)) \cong \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \tag{4}$$

which factors through $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$. As any two choices of isomorphisms $\mathcal{L}_0 \cong \mathcal{P}_{[\mathcal{L}]}$ differ by a non-zero constant, we see that the map is indeed uniquely well-defined and continuous. The following is our main result:

Theorem 4.6. *Let X be a smooth projective complex variety. Let $\alpha \in \mathrm{NS}(X)$ be such that $\alpha - c_1(K_X)$ is ample. The jet map*

$$j^1 \colon \mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d(X,\alpha)-3}{2}$. (See Definition 2.18.)

5. Proof of the main theorem

The proof of the main theorem is executed in two steps. In Section 5.1, we first prove:

Proposition 5.1. *Let X and α be as in Theorem 4.6. The jet map*

$$j^1 \colon \mathcal{M}^{\alpha}_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d(X,\alpha)-3}{2}$.

Then, in Section 5.2, we show the following:

Proposition 5.2. *The map defined in (4)*

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$$

is a weak homotopy equivalence.

5.1. The homology isomorphism.

The jet map fits in the following diagram where the top row is its restriction to a fibre above an $[\mathcal{L}] \in \mathrm{Pic}^{\alpha}(X)$:

$$\begin{tikzcd}
\Gamma_{\mathrm{ns}}(\mathcal{P}_{[\mathcal{L}]})/\mathbb{C}^\times \arrow[r] \arrow[d] & \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}]} \setminus 0)/\mathbb{C}^\times \arrow[d] \\
\mathcal{M}^{\alpha}_{\mathrm{hyp}} \arrow[r] \arrow[d] & \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \arrow[d] \\
\mathrm{Pic}^{\alpha}(X) \arrow[r, equal] & \mathrm{Pic}^{\alpha}(X)
\end{tikzcd}$$

The uppermost map was studied in [Aum22] where the following result was proved:

Theorem 5.3 (Compare [Aum22, Corollary 8.1])**.** *Let $\mathcal{L}$ be a d-jet ample line bundle on a smooth projective complex variety X. Then the jet map*

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} \setminus 0)/\mathbb{C}^\times$$

induces an isomorphism in homology in the range of degrees $* < \frac{d-1}{2}$.

Now, if both lower vertical maps were fibrations, a comparison of the associated Serre spectral sequence would prove the main theorem of the present paper. This is indeed the case for the map on the right-hand side. The other map is only a microfibration, which turns out to be sufficient for the argument to go through. We start by reviewing this technical notion popularised by Weiss in [Wei05].

Definition 5.4. *A map $\pi \colon E \to B$ is called a Serre microfibration if for any $k \geq 0$ and any commutative diagram*

$$\begin{tikzcd} \{0\} \times D^k \arrow[r, "u"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ {[0,1] \times D^k} \arrow[r, "v"'] & B \end{tikzcd}$$

there exists an $\varepsilon > 0$ and a map $h\colon [0,\varepsilon] \times D^k \to E$ such that $h(0,x) = u(x)$ and $\pi \circ h(t,x) = v(t,x)$ for all $x \in D^k$ and $t \in [0,\varepsilon]$.

Remark 5.5. *Any Serre fibration is a microfibration. More generally, the restriction of a Serre fibration to an open subspace of the total space is a microfibration.*

Contrary to the case of fibrations, the homotopy types of the fibres of a microfibration can vary. Nonetheless, we have the very useful comparison theorem of Raptis generalising a result of Weiss:

Theorem 5.6 (Compare [Rap17, Theorem 1.3])**.** *Let $p\colon E \to B$ be a Serre microfibration, $q\colon V \to B$ be a Serre fibration, and $f\colon E \to V$ a map over B. Suppose that $f_b\colon p^{-1}(b) \to q^{-1}(b)$ is n-connected for some $n \geq 1$ and for all $b \in B$. Then the map $f\colon E \to V$ is n-connected.*

In the present situation, we only have access to Theorem 5.3 which provides an isomorphism in homology, rather than on homotopy groups. The remedy chosen here is to suspend to work with simply connected spaces and apply the homology Whitehead theorem.

Definition 5.7. *For a map $p\colon E \to B$, its fibrewise (unreduced) k^{th} suspension is defined to be*

$$\Sigma^k_B E = \left(E \times [0,1] \times S^{k-1}\right) \Big/ \left((e,0,s) \sim (e,0,s') \text{ and } (e,1,s) \sim (e',1,s) \text{ if } p(e) = p(e')\right).$$

The fibre of the natural map $\Sigma^k_B p\colon \Sigma^k_B E \to B$ induced by p is the unreduced k^{th} suspension of the fibre of p (here modelled as the join with the sphere S^{k-1}):

$$(\Sigma^k_B p)^{-1}(b) = \Sigma^k p^{-1}(b), \quad \forall b \in B.$$

Lemma 5.8. *The map*

$$\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p \mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \longrightarrow \mathrm{Pic}^\alpha(X)$$

is a fibre bundle.

Proof. Let $U \subset \mathrm{Pic}^\alpha(X)$ be a small contractible open subset. A topological vector bundle being trivial over a contractible base, we obtain an isomorphism of vector bundles

$$\psi\colon J^1_p \mathcal{P}_\alpha|_{U \times X} \xrightarrow{\cong} U \times J^1 \mathcal{P}_{[\mathcal{L}_0]}$$

over $U \times X$, with $[\mathcal{L}_0] \in U$ a chosen basepoint. The map

$$\begin{aligned} \left(\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p \mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times\right)|_U &\longrightarrow U \times \Gamma_{\mathcal{C}^0}(J^1 \mathcal{P}_{[\mathcal{L}_0]} \setminus 0) \\ ([\mathcal{L}], s) &\longmapsto ([\mathcal{L}], \psi \circ s) \end{aligned}$$

is then a homeomorphism over U exhibiting the local triviality of the fibre bundle. ☐

We will say that a map $A \to B$ is homology m-connected if it induces an isomorphism on homology groups $H_i(A) \to H_i(B)$ for $i < m$ and a surjection when $i = m$.

Lemma 5.9. *Let $q\colon V \to B$ be a fibre bundle, and $p\colon U \to B$ be the restriction of a fibre bundle $E \to B$ to an open subset $U \subset E$. Let $f\colon U \to V$ be a map over B and suppose that for every $b \in B$, the restriction to the fibre*

$$f_b\colon p^{-1}(b) \longrightarrow q^{-1}(b)$$

is homology m-connected. Then $f\colon U \to V$ is homology m-connected.

Proof. For any $b \in B$, the suspension of f on the fibre

$$\Sigma^2 f_b\colon \Sigma^2 p^{-1}(b) \longrightarrow \Sigma^2 q^{-1}(b)$$

induces an isomorphism in homology in degrees $* \leq m+1$ and a surjective morphism in degree $* = m+2$. As both spaces are simply connected, the homology Whitehead theorem implies that this map is $(m+2)$-connected. We would like to apply Theorem 5.6 to $\Sigma^2_B f$, but $\Sigma^2_B U \subset \Sigma^2_B E$ is not open and it is unclear if $\Sigma^2_B U \to B$ is a microfibration. We resolve the issue by enlarging slightly the space to a homotopy equivalent one. More precisely, let

$$W = \left(\Sigma^2_B U\right) \cup \left((E \times (0.5, 1] \times S^1))/\sim\right) \subset \Sigma^2_B E,$$

and denote by E_b, W_b, U_b the fibres of the respective spaces above a point $b \in B$. Using in each fibre the homotopy equivalence $\left((E_b \times (0.5, 1] \times S^1)/\sim\right) \simeq S^1$ given by collapsing gives a homotopy equivalence

$$(W, W_b) \xrightarrow{\simeq} (\Sigma^2_B U, \Sigma^2 U_b)$$

for all $b \in B$. Now, the fibrewise suspension of the fibre bundles $E \to B$ and $V \to B$ are fibre bundles. As $W \subset \Sigma^2_B E$ is open, the restriction $W \to B$ is a microfibration. Applying Theorem 5.6 to the composite

$$W \xrightarrow{\simeq} \Sigma^2_B U \xrightarrow{\Sigma^2_B f} \Sigma^2_B V$$

and using that the first map is a homotopy equivalence, we obtain that $\Sigma^2_B f\colon \Sigma^2_B U \to \Sigma^2_B V$ is $(m+2)$-connected. Hence it is homology $(m+2)$-connected. Comparing the Mayer–Vietoris sequences of the fibrewise suspensions finally shows that $f\colon U \to B$ is homology m-connected. □

Proof of Proposition 5.1. By Definition 3.18, the map $\mathcal{M}^\alpha_{\mathrm{hyp}} \to \mathrm{Pic}^\alpha(X)$ is the restriction of the projective bundle $\mathbb{P}(p_*\mathcal{P}_\alpha) \to \mathrm{Pic}^\alpha(X)$ to the open subset $\mathcal{M}^\alpha_{\mathrm{hyp}}$. Using Theorem 5.3 and Lemma 5.8, we can apply Lemma 5.9 to conclude. □

5.2. **The homotopy type of the space of fibrewise sections.** We begin by making explicit some basic results in algebraic topology about homotopy fibres. The specific point-set models chosen will be useful for the proof of Proposition 5.2. For a pointed space (A, a), we let $P(A, a) = \mathrm{Map}_*(([0,1], 0), (A, a))$ be the space of paths in A starting at a. We will write cte_* for the constant loop based at a point $*$.

5.2.1. *The homotopy fibre of a homotopy fibre.* Let $\pi\colon (E, e_0) \to (B, b_0)$ be a fibration between pointed spaces, $F = \pi^{-1}(b_0)$ be the fibre, and $\Omega_{b_0} B$ be the loop space of B based at b_0. Writing $i\colon F \hookrightarrow E$ for the inclusion, the space

$$Hi := \{(e, \alpha) \in F \times P(E, e_0) \mid \alpha(1) = e\}$$

is a model of its homotopy fibre. It is well known that Hi and $\Omega_{b_0} B$ are homotopy equivalent, and the goal of this small section is to give an explicit description of the induced bijection on connected components.

We write

$$H\pi := \{(e, \gamma) \in E \times P(B, b_0) \mid \gamma(1) = \pi(e)\}$$

for the homotopy fibre of π. Let $j\colon H\pi \to E$ be the map $(e, \gamma) \mapsto e$, whose homotopy fibre is given by

$$\begin{aligned} Hj &:= \{(e, \gamma, \alpha) \in H\pi \times P(E, e_0) \mid \alpha(1) = e\} \\ &\cong \{(\gamma, \alpha) \in P(B, b_0) \times P(E, e_0) \mid \gamma(1) = \pi \circ \alpha(1)\}. \end{aligned}$$

The map $\Omega_{b_0} B \to Hj$ given by $\gamma \longmapsto (\gamma, \mathrm{cte}_{e_0})$ is a homotopy equivalence (see [Die08, Note 4.7.1]). The situation is summarised in the following diagram:

$$\begin{tikzcd} Hi \arrow[r] \arrow[d, "\simeq"'] & F \arrow[d, "\simeq"', hook] \arrow[dr, hook, "i"] & & \\ Hj \arrow[r] & H\pi \arrow[r, "j"'] & E \arrow[r, "\pi"'] & B \\ \Omega_{b_0} B \arrow[u, "\simeq"] & & & \end{tikzcd}$$

Lemma 5.10. *Let $\gamma\colon [0,1] \to B$ be a loop based at b_0. Let $\alpha\colon [0,1] \to E$ be a lift of that loop starting at e_0. The map on connected components*

$$\begin{aligned} \pi_0(\Omega_{b_0} B) &\longrightarrow \pi_0(Hi) \\ [\gamma] &\longmapsto [(\alpha(1), \alpha)] \end{aligned}$$

is well-defined and is a bijection.

Proof. The natural map $F \to H\pi$ is a homotopy equivalence as π is a fibration. Hence the induced map $Hi \to Hj$ given by $(e, \alpha) \mapsto (\mathrm{cte}_{b_0}, \alpha)$ is a homotopy equivalence. Therefore it suffices to show that $(\gamma, \mathrm{cte}_{e_0})$ and $(\alpha(1), \alpha)$ are in the same connected component of Hj. Both are in the same component as (γ, α), as seen by deforming either the first or the second path. $\square$

5.2.2. *Homotopy fibre of a principal bundle.* In this subsection, $\pi\colon (E, e_0) \to (B, b_0)$ is now a principal $G = \pi^{-1}(b_0)$-bundle. We let $\alpha\colon [0,1] \to B$ be a path from b_0 to a point b_1, and we choose a point $e_1 \in \pi^{-1}(b_1)$. As before, recall that a model for the homotopy fibre of π is given by

$$H\pi := \{(e, \gamma) \in E \times P(B, b_0) \mid \gamma(1) = \pi(e)\}.$$

We may choose a lift of the path α to a path $\beta\colon [0,1] \to E$ such that $\pi \circ \beta = \alpha$. We define $e_1 = \beta(1)$. As the action of G on $\pi^{-1}(b_1)$ is free and transitive, there exists a unique $g_1 \in G$ such that $g_1 \cdot e_1 = e_1'$ (where $\cdot$ denotes the action).

Lemma 5.11. *We keep the notation as above. Then the points (e_1', α) and $(g_1 \cdot e_0, \mathrm{cte}_{e_0})$ are in the same connected component of the homotopy fibre $H\pi$.*

Proof. The map $g_1 \cdot \beta\colon [0,1] \to E$ is a path from $g_1 \cdot e_0$ to $g_1 \cdot e_1 = e_1'$, and is such that $\pi \circ (g_1 \cdot \beta) = \pi \circ \beta = \alpha$. Thus the map

$$\begin{aligned} [0,1] &\longrightarrow H\pi \\ t &\longmapsto \big((g_1 \cdot \beta)(t),\ \alpha(t \cdot -)\big) \end{aligned}$$

is a path from (e_1', α) to $(g_1 \cdot e_0, \mathrm{cte}_{e_0})$ in $H\pi$. $\square$

5.2.3. *The proof of Proposition 5.2.* For concreteness, we start by fixing basepoints. Let $[\mathcal{L}_0] \in \mathrm{Pic}^\alpha(X)$, and let $s_0 \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0)$. We will use these as basepoints, as well as the images $[s_0] \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0)/\mathbb{C}^\times$ and $\mathbb{P}s_0 \in \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$.

Pointwise multiplication of maps gives $\mathrm{Map}(X, \mathbb{C}^\times)$ the structure of a topological group. By [CS84, Proposition 2.6], there is a principal $\mathrm{Map}(X, \mathbb{C}^\times)$-bundle:

$$\mathrm{Map}(X, \mathbb{C}^\times) \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0) \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)). \tag{5}$$

There is also the subgroup $\mathbb{C}^\times \subset \mathrm{Map}(X, \mathbb{C}^\times)$ of the constant functions, and modding out fibrewise gives a principal bundle:

$$\mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times \longrightarrow \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0)/\mathbb{C}^\times \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)). \tag{6}$$

We obtain a commutative diagram where each row is a fibration sequence

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] & \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0)/\mathbb{C}^\times \arrow[r] \arrow[d, equal] & \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times \arrow[d] \\
F_2 \arrow[r] & \Gamma_{\mathcal{C}^0}(J^1\mathcal{P}_{[\mathcal{L}_0]} \setminus 0)/\mathbb{C}^\times \arrow[r] & \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))
\end{tikzcd}$$

and the spaces F_1 and F_2 are defined as the respective homotopy fibres (at the basepoints chosen above). Using the 5-lemma and the long exact sequence of homotopy groups associated to a fibration, Proposition 5.2 follows directly from the next lemma.

Lemma 5.12. *Using the notation as above, the map induced on the homotopy fibres $F_1 \to F_2$ is a homotopy equivalence.*

Proof. We already know that $F_1 \simeq \Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)$ and $F_2 \simeq \mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times$, which are both homotopy equivalent to the discrete space $H^1(X; \mathbb{Z})$. Therefore we only need to verify that the map $F_1 \to F_2$ induces a bijection on the set of connected components. We have a diagram of sets

$$\begin{tikzcd}
\pi_0\big(\Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)\big) \arrow[r, "\cong"] \arrow[d, dashed] & \pi_0(F_1) \arrow[d] \\
\pi_0\big(\mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times\big) \arrow[r, "\cong"'] & \pi_0(F_2)
\end{tikzcd}$$

where the right vertical map is induced from $F_1 \to F_2$, the top horizontal map is explained in Lemma 5.10, the bottom horizontal map is induced by the inclusion of the fibre inside the homotopy fibre, and the dotted arrow is defined by composition. It suffices to show that this last arrow is a bijection.

Let $\gamma \in \Omega_{[\mathcal{L}_0]}\mathrm{Pic}^\alpha(X)$ be a loop. We choose a lift to a path $\alpha\colon [0,1] \to \Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times$ starting at $[s_0]$ and ending at some $[s_1']$, and write $\mathbb{P}\alpha$ for its image in $\Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X))$. We may furthermore lift that path to a path β in $\Gamma_{\mathcal{C}^0,\mathrm{fib}}(J^1_p\mathcal{P}_\alpha \setminus 0)/\mathbb{C}^\times$ starting at $[s_0]$ and ending at some point $[s_1]$. Using the principal bundle (6), there is a unique class of a map $[\varphi_1] \in \mathrm{Map}(X, \mathbb{C}^\times)/\mathbb{C}^\times$ such that $[\varphi_1 \cdot s_1] = [s_1']$. By Lemmas 5.10 and 5.11 the dotted arrow is given by

$$[\gamma] \longmapsto [\varphi_1].$$

In particular, it follows directly that it is surjective. One may furthermore check that it is compatible with the group structures: on the source given by composition of loops, and on the target given by multiplication of maps. As both groups are isomorphic to $H^1(X;\mathbb{Z})$ this shows that the surjective morphism is fact an isomorphism. □

6. Rational computations and stability

In this part, we show how Theorem 4.6 can be used to make explicit computations of the rational cohomology of $\mathcal{M}^{\alpha}_{\mathrm{hyp}}$. Assuming that the underlying variety X is topologically parallelisable, we will also exhibit a phenomenon of homological stability.

We will first recall a general strategy, dating back to Haefliger [Hae82], to compute the cohomology of continuous section spaces. In Theorem 6.5 below, we provide a commutative differential graded algebra (CDGA) computing the rational cohomology of the section space of the projective bundle. We hope that this will convince the reader that the homotopical approach taken in this paper may be useful in practical computations. We will freely use the notations and results from rational homotopy theory. A textbook account can be found in [FHT01]. In particular, we write $\Lambda(-)$ for the free commutative graded algebra. We let n be the complex dimension of X.

6.1. Haefliger's tower of section spaces.

Although Theorem 4.6 provides an integral homology isomorphism, we will mainly be interested in the rational cohomology groups for computational reasons. Fibrewise rationalisation (denoted $(-)_{f\mathbb{Q}}$) yields a fibration

$$\mathbb{P}^n_{\mathbb{Q}} \longrightarrow \mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}} \longrightarrow X. \tag{7}$$

By [Mø87, Theorem 5.3], the natural map $\mathbb{P}(J^1\mathcal{O}_X) \to \mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}}$ induces a map on section spaces

$$\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \longrightarrow \Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$$

which is a rationalisation when restricted to a connected component on the source and target. (Beware the fact that the source has $H^2(X;\mathbb{Z})$ many connected components, while the target has $H^2(X;\mathbb{Q})$ many of them.) We apply the general strategy described in [Hae82, Section 1.3] to compute the rational homotopy type of the section space $\Gamma_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$. The fibration (7) admits a Moore–Postnikov decomposition of the form

$$\begin{tikzcd}
K(\mathbb{Q},2n+1) \arrow[r] & Y_2 \simeq \mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}} \arrow[d, "p_2"] & \\
K(\mathbb{Q},2) \arrow[r] & Y_1 \arrow[r, "k_1"] \arrow[d, "p_1"] & K(\mathbb{Q},2n+2) \\
& Y_0 = X \arrow[r, "k_0"] & K(\mathbb{Q},3)
\end{tikzcd}$$

where each $p_i\colon Y_i \to Y_{i-1}$ is a principal fibration classified by the k-invariant k_{i-1}. The latter were computed by Møller:

Lemma 6.1 (Compare [Mø85, Lemma 2.1]). *The k-invariant k_0 is trivial. In particular $Y_1 \simeq X \times K(\mathbb{Q},2)$. Writing $z \in H^2(K(\mathbb{Q},2);\mathbb{Q})$ for the generator, k_1 corresponds to the cohomology class*

$$\sum_{i=0}^{n+1} (-1)^i c_i(J^1\mathcal{O}_X) \otimes z^{n+1-i} \in H^*(X;\mathbb{Q}) \otimes H^*(K(\mathbb{Q},2);\mathbb{Q}). \qquad \square$$

Let $s \in \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$ with $\alpha \in H^2(X;\mathbb{Q})$. The map $p_2 \circ s$ is a section of $Y_1 \to X$, and we denote by $\Gamma_1 \subset \Gamma_{\mathcal{C}^0}(Y_1 \to X)$ its connected component. As k_0 is trivial by Lemma 6.1, there is a homotopy equivalence

$$\Gamma_{\mathcal{C}^0}(Y_1 \to X) \simeq \mathrm{Map}(X, K(\mathbb{Q},2)) \simeq K(\mathbb{Q},2) \times K(H^1(X;\mathbb{Q}),1) \times H^2(X;\mathbb{Q}).$$

and Γ_1 corresponds to the connected component indexed by α.

Lemma 6.2 (Compare [Mø85, Lemma 2.2])**.** *Let Ψ be the composite*

$$\Psi\colon K(\mathbb{Q},2) \times K(H^1(X;\mathbb{Q}),1) \times X \simeq \Gamma_1 \times X \xrightarrow{\mathrm{ev}} Y_1 \xrightarrow{k_1} K(\mathbb{Q},2n+2). \tag{8}$$

Let $z \in H^2(K(\mathbb{Q},2);\mathbb{Q})$ be the generator. Let $\{x_j\}$ be a basis of $H^1(X;\mathbb{Z})$ and let $\{x'_j\}$ be the dual basis of $H^1(K(H^1(X;\mathbb{Z}),1);\mathbb{Z}) \cong H^1(X;\mathbb{Z})^\vee$. The morphism induced in cohomology Ψ^* sends the generator $\chi \in H^{2n+2}(K(\mathbb{Q},2n+2);\mathbb{Q})$ to the class:

$$\Psi^*(\chi) = \sum_{i=0}^{n+1}(-1)^i\Big(1 \otimes 1 \otimes c_i(J^1\mathcal{O}_X)\Big) \cup \Big(z \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha + \sum_j 1 \otimes x'_j \otimes x_j\Big)^{n+1-i}. \qquad \square$$

Let $\overline{k_1}\colon \Gamma_1 \to K(\mathbb{Q},2n+2)^X$ be the adjoint of the map (8). There is a homotopy equivalence (see [Hae82, Section 1])

$$K(\mathbb{Q},2n+2)^X \simeq \prod_{i=2}^{2n+2} K(H^{2n+2-i}(X;\mathbb{Q}),i). \tag{9}$$

Lemma 6.3 (Compare [Hae82])**.** *Let φ_i be the map to the i-th factor of the product:*

$$\varphi_i\colon \Gamma_1 \xrightarrow{\overline{k_1}} K(\mathbb{Q},2n+2)^X \longrightarrow K(H^{2n+2-i}(X;\mathbb{Q}),i).$$

The morphism induced in cohomology is given explicitly by:

$$\begin{aligned}\varphi_i^*\colon H^{2n+2-i}(X;\mathbb{Q})^\vee \cong H^i(K(H^{2n+2-i}(X;\mathbb{Q});\mathbb{Q}) &\longrightarrow H^i(\Gamma_1;\mathbb{Q})\\ y' &\longmapsto y' \cap \Psi^*(\chi).\end{aligned}$$

Here, for $w \otimes y \in H^*(\Gamma_1) \otimes H^*(X)$ and $y' \in H^*(X)^\vee$, we write $y' \cap (w \otimes y) = y'(y)w$. $\square$

Proposition 6.4 (Compare [Hae82])**.** *There is a fibration*

$$K(\mathbb{Q},2n+1)^X \longrightarrow \Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}}) \longrightarrow \Gamma_1$$

pulled back from the path space fibration over $K(\mathbb{Q},2n+2)^X$ via the map $\overline{k_1}$. $\square$

Theorem 6.5. *Let z and $\{x'_j\}$ be as in Lemma 6.2. Let $\{y'_{ik}\}$ be a basis of the rational cohomology of (9) where $y'_{ik} \in H^{2n+2-i}(X;\mathbb{Q})^\vee$ is in degree i. The rational cohomology of $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$ is given by the cohomology of the following commutative differential graded algebra:*

$$\Lambda\big(z,\ x'_j,\ s^{-1}y'_{ik}\big), \quad d(z) = 0,\ d(x'_j) = 0,\ d(s^{-1}y'_{ik}) = \varphi_i^*(y'_{ik})$$

where z is in degree 2, each x'_j is in degree 1, each $s^{-1}y'_{ik}$ is in degree $i-1$, and φ_i^* is given as in Lemma 6.3.

Proof. By Proposition 6.4, there is a homotopy pullback square:

$$\begin{tikzcd}
\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}}) \arrow[r] \arrow[d] & * \arrow[d] \\
\Gamma_1 \arrow[r, "\overline{k_1}"] & K(\mathbb{Q}, 2n+2)^X
\end{tikzcd}$$

By the Eilenberg–Moore theorem, the cohomology of the pullback is given by the derived tensor product

$$\Lambda(z, x'_j) \otimes^{\mathbb{L}}_{\Lambda(y'_{ik})} \mathbb{Q}$$

which can be computed by choosing $\Lambda(y'_{ik}) \to \big(\Lambda(s^{-1}y'_{ik}, y'_{ik}),\ d(s^{-1}y'_{ik}) = y'_{ik}\big) \simeq \mathbb{Q}$ as a cofibrant replacement. $\square$

Example 6.6. *Let X be a smooth curve ($n = 1$) of genus 1 (i.e. a torus). It is a framed manifold, hence its jet bundle has trivial Chern classes. Write a, b for the standard basis of $H^1(X;\mathbb{Z})$ such that $a^2 = b^2 = 0$ and $u = ab$ generates $H^2(X;\mathbb{Z})$. Let a', b' be the dual basis. Let $\alpha = d \cdot u$ for some $d \in \mathbb{Z}$. With the notations of Lemma 6.2 we have*

$$\begin{aligned}\Psi^*(\chi) &= \big(z \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha + 1 \otimes a' \otimes a + 1 \otimes b \otimes b'\big)^2 \\ &= (2d(z \otimes 1) - 2(1 \otimes a'b')) \otimes u + 2(z \otimes a' \otimes a) + 2(z \otimes b' \otimes b) + z^2 \otimes 1 \otimes 1.\end{aligned}$$

The morphisms φ_i^* of Lemma 6.3 are given by

$$\begin{aligned}\varphi_2^*\colon u' &\longmapsto u' \cap \Psi^*(\chi) = 2d(z \otimes 1) - 2(1 \otimes a'b') \\ \varphi_3^*\colon a' &\longmapsto a' \cap \Psi^*(\chi) = 2(z \otimes a') \\ b' &\longmapsto b' \cap \Psi^*(\chi) = 2(z \otimes b') \\ \varphi_4^*\colon 1 &\longmapsto 1 \cap \Psi^*(\chi) = z^2 \otimes 1.\end{aligned}$$

Therefore the cohomology of $\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \simeq \mathrm{Map}_{\alpha}(X, \mathbb{P}^2)$ is given by the cohomology of the CDGA:

$$\Lambda(z, a', b', y_1, y_2, y'_2, y_3), \quad d(y_1) = 2dz - 2a'b',\ d(y_2) = 2za',\ d(y'_2) = 2zb',\ d(y_3) = z^2$$

where the indices on the last four variables indicate their degrees. (See [Mø85, Section 3] for related computations.)

6.2. **Homological stability.** Despite the formula given in Theorem 6.5, it is unclear to us how the cohomology varies when α does. Nonetheless, when X is topologically parallelisable, we can make the following qualitative remark:

Proposition 6.7. *Let X be a smooth projective complex variety such that Ω^1_X is a topologically trivial vector bundle, and let $\alpha \in H^2(X;\mathbb{Q})$. Then there is a homotopy equivalence*

$$\Gamma^{\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}}) \simeq \Gamma^{k\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}})$$

for any non-zero rational number $k \in \mathbb{Q}^{\times}$.

Proof. As X is topologically parallelisable, the jet bundle $J^1\mathcal{O}_X$ is topologically trivial. Hence the section space is the mapping space into the fibre:

$$\Gamma^{k\alpha}_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)_{f\mathbb{Q}}) \simeq \mathrm{Map}_{k\alpha}(X, \mathbb{P}^n_{\mathbb{Q}})$$

where the subscript $k\alpha$ on the right-hand side indicates the connected component of the maps which pullback the generator in cohomology to $k\alpha$. Post-composing with a self map of $\mathbb{P}^n_{\mathbb{Q}}$ of degree $1/k$ gives a homotopy equivalence

$$\mathrm{Map}_{k\alpha}(X, \mathbb{P}^n_{\mathbb{Q}}) \simeq \mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}}). \qquad \square$$

Corollary 6.8. *Let X be a smooth projective complex variety which is topologically parallelisable. Let $\alpha \in \mathrm{NS}(X)$ be ample, and assume that $d(X,\alpha) \geq 1$ (see Definition 2.18). Then, for any integer $k \geq 1$, there is map*

$$\mathcal{M}^{k\alpha}_{\mathrm{hyp}} \longrightarrow \mathrm{Map}_{\alpha}(X, \mathbb{P}^n_{\mathbb{Q}})$$

inducing an isomorphism in rational cohomology in the range of degrees $* < \frac{k \cdot d(X,\alpha)-3}{2}$. In particular, the rational cohomology stabilises as $k \to \infty$. $\square$

Remark 6.9. *The rational homotopy type of the mapping space $\mathrm{Map}(X, \mathbb{P}^n_{\mathbb{Q}})$ can be easily computed without the results of the last section. In [Ber15], Berglund gives an explicit L_∞-algebra model whose underlying graded $\mathbb{Q}$-vector space is given by*

$$H^*(X;\mathbb{Q}) \otimes \mathbb{Q} \cdot \{u, w\}$$

where $H^i(X)$ sits in degree $-i$, and u, w are respectively in degrees 1 and $2n$. (This uses that X is a formal space.) For the connected component corresponding to $\alpha \in H^2(X;\mathbb{Q})$, the associated Maurer–Cartan element is $\tau = \alpha \otimes u$. In particular, the only possibly non-vanishing brackets are given by

$$[x_1 \otimes u, \dots, x_r \otimes u]_\tau = \pm \frac{(n+1)!}{(n+1-r)!}(\alpha^{n+1-r} x_1 x_2 \cdots x_r) \otimes w.$$

In fact, in the case of the torus, the Chevalley–Eilenberg complex associated to this L_∞-algebra is the CGDA given in Example 6.6.

7. Scanning and configuration spaces on curves

In this last section, we explain how the present article fits into the general philosophy of scanning maps in topology. In Theorem 7.2, we recover a special case of a result of McDuff about the homology of configuration spaces of points on a curve. We then turn our attention to characteristic classes. We explain in Theorem 8.11 a relation between the stable cohomology of $\Gamma_{\mathrm{ns}}(\mathcal{L})$ and that of moduli spaces of manifolds as studied by Galatius and Randal-Williams.

7.1. **Scanning.** We begin with a brief and intuitive exposition of the general idea behind scanning. Suppose given $M \subset N$, a d-dimensional submanifold of an n-dimensional manifold. We can try to see what M looks like by looking locally at each point of N. One can imagine looking through a magnifying glass: either we are far from M and see nothing, or close to M and see a first-order approximation of M, i.e. a tangent space, together with a small vector from the center of the lens to M. To formalise this intuition, recall the tautological quotient bundle over the Grassmannian of d-dimensional planes in $\mathbb{R}^n$:

$$\mathbb{R}^{n-d} \longrightarrow \gamma^\perp_{d,n} := \{(H,v) \mid H \in \mathrm{Gr}(d,\mathbb{R}^n),\ v \in \mathbb{R}^n/H\} \longrightarrow \mathrm{Gr}(d,\mathbb{R}^n).$$

One thinks of a point $(H,v) \in \gamma^\perp_{d,n}$ as a d-dimensional plane together with a normal vector. The Thom space $\mathrm{Gr}(d,\mathbb{R}^n)^{\gamma^\perp_{d,n}}$ is obtained by one-point compactifying the total space. This construction can be done fibrewise to the tangent bundle TN of N, and we denote by $\mathrm{Gr}(d,TN)^{\gamma^\perp_{d,n}}$ the resulting bundle over N. The submanifold M then gives a section

$$N \longrightarrow \mathrm{Gr}(d,TN)^{\gamma^\perp_{d,n}}$$

obtained by sending a point far away from M to the point at infinity (in the Thom space), and sending a point $x \in N$ close to a point $y \in M$ to the tangent space $T_yM \subset T_yN$ together with the vector pointing from y to x. Of course this requires to be made precise, e.g. by choosing a tubular neighbourhood of M inside N. In many cases, this idea can be implemented in families to obtain a map from a parameter space of submanifolds to a section space.

We are now ready to give an interpretation of the jet map of Theorem 4.6 in the spirit of scanning. We take X and α as in the assumptions of that theorem. We also denote by $n = \dim_{\mathbb{C}} X$ the complex dimension of X. The main observation is the following:

Lemma 7.1. *For integers d, m, let $\mathrm{Gr}(d,\mathbb{C}^m)$ be the Grassmannian of complex d-dimensional planes in $\mathbb{C}^m$ and $\gamma^\perp_{d,m}$ be the tautological quotient bundle. There is a homeomorphism*

$$\begin{aligned} \gamma^\perp_{d,m} &\xrightarrow{\cong} \mathrm{Gr}(d+1,\mathbb{C}^m\oplus\mathbb{C}) \setminus \mathrm{Gr}(d+1,\mathbb{C}^m) \\ (H,v) &\longmapsto (H,0)\oplus(v,1) \end{aligned}$$

where $\mathrm{Gr}(d+1,\mathbb{C}^m)$ is embedded inside $\mathrm{Gr}(d+1,\mathbb{C}^m\oplus\mathbb{C})$ via $P \mapsto (P,0)$. □

When V is an n-dimensional complex vector space, the tautological quotient bundle over $\mathrm{Gr}(n-1,V) = \mathbb{P}(V)$ is homeomorphic to $\mathbb{P}(V\oplus\mathbb{C}) \setminus \{*\}$. Hence its Thom space is $\mathbb{P}(V\oplus\mathbb{C})$. From the isomorphism $J^1\mathcal{O}_X \cong \Omega^1_X \oplus \mathcal{O}_X$ as smooth complex vector bundles, we see that

$$\mathbb{P}(J^1\mathcal{O}_X) \cong \mathrm{Gr}(n-1,\Omega^1_X)^{\gamma^\perp_{n-1,n}} \cong \mathrm{Gr}(n-1,TX)^{\gamma^\perp_{n-1,n}}.$$

Under these identifications, the jet map

$$\mathcal{M}^\alpha_{\mathrm{hyp}} \longrightarrow \Gamma_{\mathcal{C}^0}\left(\mathrm{Gr}(n-1,TX)^{\gamma^\perp_{n-1,n}}\right)$$

is very close to the general idea of scanning described above. Given a hypersurface $V(s) \subset X$, the derivative $x \mapsto ds(x)$ records the tangent space when non-zero, i.e. near the hypersurface, and $x \mapsto s(x)$ records in some sense the distance to the hypersurface, an analogue of the normal vector.

7.2. **Configuration spaces on curves.** Let us now describe the case $n = 1$ in more details. The variety X is then a curve and we think of $\alpha \in \mathrm{NS}(X)$ as an integer under the isomorphism $\mathrm{NS}(X) \subset H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ given by the complex orientation. A hypersurface of Chern class α is simply an unordered configuration of α points and we have a homeomorphism

$$\begin{aligned} \mathcal{M}^\alpha_{\mathrm{hyp}} &\xrightarrow{\cong} \mathrm{UConf}_\alpha(X) \\ ([\mathcal{L}] \in \mathrm{Pic}^\alpha(X),\ [s] \in \Gamma_{\mathrm{ns}}(\mathcal{P}_{[\mathcal{L}]})/\mathbb{C}^\times) &\longrightarrow V([s]). \end{aligned}$$

There is also an identification

$$\mathbb{P}(J^1\mathcal{O}_X) \cong \mathrm{Gr}(0,TX)^{\gamma^\perp_{0,1}} \cong \dot{TX}$$

with the fibrewise one-point compactification of the tangent bundle. In [McD75], McDuff studied a scanning map on configuration spaces of points on a manifold, i.e. spaces of 0-dimensional submanifolds. In the present work, we instead study (complex) codimension 1 submanifolds. On a curve these agree and we recover a special case of McDuff's theorem, although our scanning map is now more algebraic in nature:

Theorem 7.2. *Let X be a smooth projective complex curve of genus g. Let $\alpha \in H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ be such that $\alpha > 2g-2$. The jet map*

$$\mathrm{UConf}_\alpha(X) \cong \mathcal{M}^\alpha_{\mathrm{hyp}} \longrightarrow \Gamma^\alpha_{\mathcal{C}^0}(\mathbb{P}(J^1\mathcal{O}_X)) \cong \Gamma^\alpha_{\mathcal{C}^0}(\dot{TX})$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{\alpha-2g-3}{2}$.

Proof. This is a direct consequence of Theorem 4.6. To verify the assumption on the ampleness of $\alpha - c_1(K_X)$, recall that the canonical divisor has degree $2g-2$ and that a line bundle of positive degree is ample. The final bound is obtained by computing $d(X,\alpha) = \alpha - 2g$ using Riemann–Roch as explained in Lemma A.1. □

Remark 7.3. *The observations above lead us to ask about subvarieties of greater codimension: could the spaces $\mathrm{Gr}(n-c,\Omega^1_X)^{\gamma^\perp_{n-c,n}}$ be related to Hilbert schemes of codimension c smooth subvarieties?*

8. Characteristic classes and manifold bundles

In this section, we comment on the stable rational cohomology of $\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$. Our main motivation is trying to relate it to the stable cohomology of moduli spaces of manifolds as investigated by Galatius and Randal-Williams [GRW14, GRW17, GRW18]. None of this section uses the new results of this article, and will in fact be deduced entirely from [Aum22]. Nonetheless, we think that its fits naturally with the "moduli space point of view" adopted in this paper.

We will assume that $\dim_{\mathbb{C}} X = n \geq 4$ and that the fundamental group of X is virtually polycyclic to apply the results of [GRW19, Fri17]. We also choose a very ample line bundle $\mathcal{L}$ on X.

8.1. Recollections on stable classes.

We begin by recalling from [Aum22] the geometric interpretation of the stable classes in the rational cohomology of $\Gamma_{\mathrm{ns}}(\mathcal{L})$. As we are here mostly interested in the quotient by the scalars $\mathbb{C}^\times$, we point out the following observation that we learned from [Das21, Lemma 2.7]:

Lemma 8.1. *There is an isomorphism of $H^*(\Gamma_{\mathrm{ns}}(\mathcal{L});\mathbb{Q})$-modules:*

$$H^*(\Gamma_{\mathrm{ns}}(\mathcal{L});\mathbb{Q}) \cong H^*(\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times;\mathbb{Q}) \otimes H^*(\mathbb{C}^\times;\mathbb{Q}).$$

Proof. Let $\Delta\colon \Gamma(\mathcal{L}) \to \mathbb{C}$ be the discriminant (see [GKZ94]) so that $\Gamma_{\mathrm{ns}}(\mathcal{L}) = \Gamma(\mathcal{L}) \setminus \Delta^{-1}(0)$. There is a fibre bundle

$$\mathbb{C}^\times \longrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L}) \longrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$$

and for any fibre the map $\mathbb{C}^\times \hookrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L}) \xrightarrow{\Delta} \mathbb{C}^\times$ is of degree $\deg(\Delta) \neq 0$, hence an isomorphism on rational cohomology. The lemma then follows by the Leray–Hirsch theorem. □

Consider the universal bundle of hypersurfaces:

$$V(s) \longrightarrow U(\mathcal{L}) := \{(s,x) \in \Gamma_{\mathrm{ns}}(\mathcal{L}) \times X \mid s(x) = 0\} \xrightarrow{\pi} \Gamma_{\mathrm{ns}}(\mathcal{L}).$$

At each point $(s,x) \in U(\mathcal{L})$, the derivative $ds(x)$ is non-zero, thus giving a map

$$j\colon U(\mathcal{L}) \longrightarrow \Omega^1_X \otimes \mathcal{L} \setminus 0.$$

For $\mathcal{L}$ ample enough such that the Euler class of $\Omega^1_X \otimes \mathcal{L}$ is non zero, the cohomology of the target of j in degrees $* \geq 2n$ is

$$H^{*\geq 2n}(\Omega^1_X \otimes \mathcal{L} \setminus 0; \mathbb{Z}) \cong H^{*\geq 1}(X; \mathbb{Z})[2n-1],$$

where $[2n-1]$ indicates a shift of degrees.

Proposition 8.2 (Compare [Aum22, Proposition 8.6])**.** *Let $\mathcal{L}$ be a d-jet ample line bundle on a smooth projective complex variety X. Suppose furthermore that the Euler class of $\Omega^1_X \otimes \mathcal{L}$ is non-zero. Then the morphism given by pulling back along j and integrating along the fibres of π*

$$\pi_! \circ j^* \colon \Lambda(H^{*\geq 2n}(\Omega^1_X \otimes \mathcal{L} \setminus 0; \mathbb{Q})) \longrightarrow H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}); \mathbb{Q}) \longrightarrow H^*(\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times; \mathbb{Q})$$

is an isomorphism in the range of degrees $* < \frac{d-1}{2}$. (The second arrow is obtained from Lemma 8.1 by projecting away from the second tensor factor.) □

8.2. **Comparison with diffeomorphisms.** Let us fix a non-singular section $s \in \Gamma_{\mathrm{ns}}(\mathcal{L})$ and write $H := V(s)$ for the associated hypersurface. From the point of view of the Hilbert scheme developed in Section 3, the subspace $\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$ classifies algebraic bundles with fibres equivalent to H (as divisors) and embedded in X. Seeing H only as a smooth oriented real manifold, one can consider the classifying space $B\mathrm{Diff}^{\mathrm{or}}(H)$ of its group of orientation preserving diffeomorphisms. By definition, this latter space classifies fibre bundles with fibre H and structure group $\mathrm{Diff}^{\mathrm{or}}(H)$. In particular, there is a map

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow B\mathrm{Diff}^{\mathrm{or}}(H) \tag{10}$$

classifying the universal bundle

$$U(\mathcal{L})/\mathbb{C}^\times := \{([s], x) \in \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \times X \mid s(x) = 0\} \xrightarrow{\pi} \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times. \tag{11}$$

One could wonder if the map (10) induces an isomorphism in rational cohomology in a range of degrees. This was shown to be false when $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(d)$ by Randal-Williams [RW19]. On the other hand, one could alter the situation by considering diffeomorphisms preserving other kinds of tangential structures: we have picked orientation, but could have chosen a spin structure in some cases, or a map to a background space, etc. We describe below a peculiar tangential structure θ such that the map classifying the universal bundle

$$\Gamma_{\mathrm{ns}}(\mathcal{L}) \longrightarrow B\mathrm{Diff}^\theta(H)$$

is "very close" to being a rational homology isomorphism in a range of degrees.

Remark 8.3. *Although the space $\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$ is more geometrically natural, we will only be able to produce a map to $B\mathrm{Diff}^\theta(H)$ from $\Gamma_{\mathrm{ns}}(\mathcal{L})$. Nevertheless, by Lemma 8.1 the quotient map induces an injection in rational cohomology*

$$H^*(\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times; \mathbb{Q}) \hookrightarrow H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}); \mathbb{Q})$$

which will be good enough for our arguments.

Choose maps $[TX] \colon X \to BU(n)$ and $[\mathcal{L}] \colon X \to BU(1)$ classifying respectively the tangent bundle of X and $\mathcal{L}$ as topological complex vector bundles. Let B be the space defined by the following homotopy pullback square:

$$\begin{tikzcd} B \arrow[rr] \arrow[d] & & X \arrow[d, "{([TX],\,[\mathcal{L}])}"] \\ BU(n-1)\times BU(1) \arrow[rr, "{(``\oplus\text{''},\, \mathrm{pr}_2)}"'] & & BU(n)\times BU(1) \end{tikzcd} \tag{12}$$

where "$\oplus$": $BU(n-1)\times BU(1) \to BU(n)$ classifies taking the direct sum of vector bundles. We will adopt the point of view of spaces over $BO(2n-2)$ to describe tangential structures. (See [GRW19, Section 4.5] for a discussion.) In this language our tangential structure is the map:

$$\theta\colon B \longrightarrow BU(n-1)\times BU(1) \xrightarrow{\mathrm{pr}_1} BU(n-1) \longrightarrow BO(2n-2).$$

Remark 8.4. *Informally, a θ-structure on a $(2n-2)$-manifold M is the datum of a lift of the map classifying the tangent bundle:*

$$\begin{tikzcd} & B \arrow[d, "\theta"] \\ M \arrow[r, "{[TM]}"'] \arrow[ur, dashed] & BO(2n-2) \end{tikzcd}$$

up to homotopy. By the universal property of the homotopy pullback, this amounts to providing two maps

$$M \longrightarrow BU(n-1)\times BU(1) \quad \textit{and} \quad M \longrightarrow X$$

which become homotopic after further composing to $BU(n)\times BU(1)$ and such that $M\to BU(n-1)$ classifies the tangent bundles of M. In other words, this is the data of a map $\iota\colon M\to X$ and a complex line bundle $\mathcal{L}'$ (corresponding to $M\to BU(1)$) such that $TH\oplus\mathcal{L}'\cong\iota^*TX$ and $\iota^*\mathcal{L}\cong\mathcal{L}'$. Therefore, a θ-structure on M should be intuitively interpreted as a choice of an immersion $\iota\colon M\to X$ with normal bundle $\iota^*\mathcal{L}$.

We have chosen to construct B via the homotopy pullback (12) as it allowed us to informally understand the geometric meaning of a θ-structure. But it turns out that we can give more familiar expressions for B and the bundle classified by θ as explained in the following two lemmas.

Lemma 8.5. *There is a homotopy equivalence above X*

$$B \simeq \Omega^1_X\otimes\mathcal{L}\setminus 0.$$

Proof. We will use the following explicit point set models:

$$\begin{aligned} BU(j) &:= \{P\subset\mathbb{C}^\infty \mid P \text{ is a } j\text{-dimensional plane}\},\\ \gamma_j &:= \{(P,v)\mid P\in BU(j),\ v\in P\},\\ \gamma_j^\vee &:= \{(P,\varphi)\mid P\in BU(j),\ \varphi\colon P\to\mathbb{C} \text{ linear map}\}, \end{aligned}$$

for the classifying space, and its tautological vector bundle and the dual of it. Recall that the classical fibration sequence

$$\mathbb{C}^n\setminus 0 \longrightarrow BU(n-1) \longrightarrow BU(n)$$

can be modelled by the sphere bundle of the dual tautological bundle using the homeomorphism

$$\gamma_n^\vee\setminus 0 \cong BU(n-1), \quad (P,\varphi\colon P\twoheadrightarrow\mathbb{C})\mapsto\ker(\varphi).$$

From the pullback square

$$\begin{tikzcd} BU(n-1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BU(n) \arrow[d] \\ BU(n-1)\times BU(1) \arrow[r, "(``\oplus", \mathrm{pr}_2)"'] & BU(n)\times BU(1) \end{tikzcd}$$

we see that the homotopy fibre of the bottom map is $\mathbb{C}^n \setminus 0$. In fact, we can likewise model the fibre sequence

$$\mathbb{C}^n \setminus 0 \longrightarrow BU(n-1)\times BU(1) \overset{(``\oplus", \mathrm{pr}_2)}{\longrightarrow} BU(n)\times BU(1)$$

by the sphere bundle $\gamma_n^\vee \boxtimes \gamma_1 \setminus 0$. Indeed, we may write

$$\gamma_n^\vee \boxtimes \gamma_1 \setminus 0 \cong \{(P, L, \varphi\otimes v) \mid P \in BU(n),\ L \in BU(1),\ \varphi \otimes v \in P^\vee \otimes L \setminus 0\}$$

and use the homeomorphism

$$\gamma_n^\vee \boxtimes \gamma_1 \setminus 0 \cong BU(n-1)\times BU(1), \quad (P, L, \varphi\otimes v) \mapsto (\ker(\varphi\otimes v), L).$$

Under these identifications, one can check that the map "$\oplus$" can be modelled by

$$(P, L, \varphi \otimes v) \longmapsto \ker(\varphi\otimes v) \oplus L \subset \mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty.$$

Hence, by pulling back along $([TX], [\mathcal{L}])$, one sees that $B \simeq \Omega^1_X \otimes \mathcal{L} \setminus 0$. $\square$

Lemma 8.6. *Let $q\colon \Omega^1_X \otimes \mathcal{L} \setminus 0 \to X$ be the projection. The virtual vector bundle $q^*(TX - \mathcal{L})$ is in fact the genuine vector bundle $\theta^*\gamma$ classified by the map $\theta\colon \Omega^1_X \otimes \mathcal{L} \setminus 0 \to BO(2n-2)$.*

Proof. We will use the homeomorphism

$$\Omega^1_X \otimes \mathcal{L} \setminus 0 \cong \{(x, \varphi) \mid x \in X,\ \varphi\colon TX|_x \twoheadrightarrow \mathcal{L}|_x \text{ surjective linear map}\}$$

given by identifying a non-zero vector of $(\Omega^1_X \otimes \mathcal{L})|_x$ with a surjective linear map. As one sees from the point set models described in the proof of Lemma 8.5, the pullback vector bundle $\theta^*\gamma$ classified by θ is equivalent to the one whose fibre above a point (x, φ) is given by the kernel of φ. Writing out the vector bundles

$$q^*TX = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1 \otimes \mathcal{L} \setminus 0,\ v \in TX|_x\},$$
$$q^*\mathcal{L} = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1 \otimes \mathcal{L} \setminus 0,\ v \in \mathcal{L}|_x\},$$

we identify $\theta^*\gamma$ as the kernel of the morphism of vector bundles

$$q^*TX \longrightarrow q^*\mathcal{L},\ (x, \varphi, v) \longmapsto (x, \varphi, \varphi(v)).$$

We thus obtain the short exact sequence of vector bundles

$$0 \longrightarrow \theta^*\gamma \longrightarrow q^*TX \longrightarrow q^*\mathcal{L} \longrightarrow 0$$

which proves the lemma. $\square$

Let $H = V(s)$ be a smooth hypersurface with $s \in \Gamma_{\mathrm{ns}}(\mathcal{L})$. Using non-singularity, we obtain a map $\ell\colon H \to \Omega^1_X \otimes \mathcal{L} \setminus 0$ given by $\ell(x) = ds(x)$.

Lemma 8.7. *The map $\ell\colon H \to \Omega^1_X \otimes \mathcal{L} \setminus 0$ is $(n-1)$-connected.*

Proof. The inclusion $\iota\colon H \hookrightarrow X$ factors as

$$H \xrightarrow{\ell} \Omega^1_X \otimes \mathcal{L} \setminus 0 \longrightarrow X$$

where the second map is the projection map of the bundle, hence $(2n-1)$-connected. Therefore it suffices to show that $\iota\colon H \to X$ is $(n-1)$-connected. But this is precisely the Lefschetz hyperplane theorem. □

The maps ℓ for all hypersurfaces assemble to equip the universal bundle with our tangential structure:

Proposition 8.8. *The universal bundle $U(\mathcal{L}) \to \Gamma_{\mathrm{ns}}(\mathcal{L})$ admits the structure of a smooth fibre bundle with θ-structure given by ℓ in each fibre.*

Proof. Let $T_vU(\mathcal{L})$ be the vertical tangent bundle of the universal bundle. We have to provide the horizontal maps in the following diagram to construct a vector bundle map

$$\begin{tikzcd} T_vU(\mathcal{L}) \arrow[r] \arrow[d] & \theta^*\gamma \arrow[d] \\ U(\mathcal{L}) \arrow[r] & \Omega^1_X \otimes \mathcal{L} \setminus 0 \end{tikzcd}$$

which restricts to a linear isomorphism in each fibre. Using the notation from the proof of Lemma 8.6, we write

$$\Omega^1_X \otimes \mathcal{L} \setminus 0 = \{(x, \varphi) \mid x \in X,\ \varphi\colon TX|_x \twoheadrightarrow \mathcal{L}|_x \text{ surjective linear map}\}$$

and

$$\theta^*\gamma = \{(x, \varphi, v) \mid (x, \varphi) \in \Omega^1_X \otimes \mathcal{L} \setminus 0,\ v \in \ker(\varphi)\}.$$

Differentiating a non-singular section $s\colon X \to \mathcal{L}$ yields a short exact sequence of vector bundles

$$0 \longrightarrow TV(s) \longrightarrow TX \xrightarrow{ds} \mathcal{L} \longrightarrow 0$$

which shows that $\ker(ds(x)) = TV(s)|_x$. Hence, taking

$$U(\mathcal{L}) \longrightarrow \Omega^1_X \otimes \mathcal{L} \setminus 0, \quad (s, x) \longmapsto ds(x)$$

and its fibrewise vertical differential gives the wanted square. □

Let us now look at hypersurfaces of higher degree. For every integer $d \geq 1$, we pick a section $s_d \in \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})$ and write $H_d = V(s_d) \subset X$ for the associated hypersurface. Replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes d}$ in the diagram (12), we obtain spaces $B_d \simeq \Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0$. We write $\theta_d\colon B_d \to BO(2n-2)$ for the tangential structure and $\ell_d\colon H_d \to B_d$ for the tangential structure on H_d induced from the inclusion inside X. Let

$$\mathcal{M}^{\theta_d}(H_d, \ell_d)$$

be the connected component of (H_d, ℓ_d) in the classifying space of H_d-bundles with θ_d-structure. (See [GRW19, Section 2.2] and the references therein for precise definitions.) Work of Galatius and Randal-Williams provides the following:

Theorem 8.9 (Compare [GRW19, Theorem 4.5])**.** *Using the notations as above, let $\alpha = c_1(\mathcal{L})$ and $N := \int_X \alpha^{n+1} \neq 0$. There is a map*

$$\mathcal{M}^{\theta_d}(H_d, \ell_d) \longrightarrow \Omega^\infty MT\theta_d \simeq \Omega^\infty\big(\Omega^1_X \otimes \mathcal{L}^{\otimes d} \setminus 0\big)^{q^*\mathcal{L}^{\otimes d} - q^*TX}$$

which, when regarded as a map onto the path component that it hits, induces an isomorphism in integral homology in degrees $* \leq \frac{1}{3}Cd^{n+1} + O(d^n)$, for some constant C depending on n and satisfying $\frac{1}{2} \cdot \frac{13}{15}N \leq C$.

Proof. The connectivity assumption of [GRW19, Theorem 4.5] is verified in Lemma 8.7. The identification of the Thom spectrum is given by Lemma 8.6. Finally, the range given is explained in [GRW19, Remark 5.6] □

Lemma 8.10. *Suppose that the Euler class of $\Omega^1_X \otimes \mathcal{L}^{\otimes d}$ does not vanish. Then there is an isomorphism of commutative rings*

$$\begin{aligned} H^*\big(\mathcal{M}^{\theta_d}(H_d, \ell_d); \mathbb{Q}\big) &\cong \Lambda\big(H^{2n-1}(X)[1] \oplus H^1(X)[2] \oplus H^2(X)[3] \oplus \cdots \oplus H^{2n}(X)[2n+1]\big) \\ &=: \Lambda\big(H^{2n-1}(X)[1]\big) \otimes \Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big) \end{aligned}$$

where $H^i(X)[j]$ denotes the graded $\mathbb{Q}$-vector space $H^i(X;\mathbb{Q})$ placed in degree j.

Proof. This is a well-known computation in rational homotopy theory using the Thom isomorphism. See for example [GRW19, Remark 4.2]. □

By Proposition 8.8, the universal bundle is pulled back along a map

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \ell_d).$$

Our work describes the stable rational cohomology of the domain, whereas Galatius and Randal-Williams compute the one of the codomain. The relation between the two rings of characteristic classes is summarised in the following result:

Theorem 8.11. *Let $i \geq 0$ be an integer and let $d \gg 0$ be big enough so that*

$$H^*(\mathcal{M}^{\theta_d}(H_d, \ell_d); \mathbb{Q}) \cong \Lambda\big(H^{2n-1}(X)[1]\big) \otimes \Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big)$$

and

$$H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times; \mathbb{Q}) \cong \Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big)$$

in degrees $* \leq i$. The map classifying the universal bundle

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \ell_d)$$

induces a ring morphism in rational cohomology with the following properties in cohomological degrees $* \leq i$:

(i) Its restriction to $\Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big) \subset H^*(\mathcal{M}^{\theta_d}(H_d, \ell_d); \mathbb{Q})$ is injective.
(ii) Its restriction to $\Lambda\big(H^{2n-1}(X)[1]\big) \subset H^*(\mathcal{M}^{\theta_d}(H_d, \ell_d); \mathbb{Q})$ is zero.

In particular, its image in degrees $* \leq i$ is the subring $H^*(\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times; \mathbb{Q}) \subset H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}); \mathbb{Q})$.

Remark 8.12. *In other words, under the identifications of the theorem, the morphism induced by the classifying map in cohomology in degrees $* \leq i$ coincides with the projection away from the first tensor factor*

$$\Lambda\big(H^{2n-1}(X)[1]\big) \otimes \Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big) \longrightarrow \Lambda\big(H^{\bullet>0}(X)[\bullet+1]\big)$$

up to an automorphism of the target (depending on the exact representatives chosen for the stable classes).

Proof. Recall from [GRW19, Section 3.1] that the characteristic classes of θ_d-bundles are given by integration along the fibres. From the similar description given in Proposition 8.2, we see that, in degrees $* \leq i$, a basis of

$$H^{\bullet>0}(X)[\bullet+1] \subset H^*(\mathcal{M}^{\theta_d}(H_d,\ell_d);\mathbb{Q})$$

is sent to a basis of

$$H^{\bullet>0}(X)[\bullet+1] \subset H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d});\mathbb{Q})$$

under the morphism induced by the map classifying the universal bundle. This proves the first point. To prove the second, we recall that the image of a element $w \in H^{2n-1}(X)$ is the fibre integration

$$\pi_!(i^*w) \in H^*(\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d});\mathbb{Q})$$

where π is the universal bundle, and $i\colon U(\mathcal{L}^{\otimes d}) \to X \times \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \to X$ is the map $(f,x) \mapsto x$. From the commutative diagram

$$\begin{tikzcd} U(\mathcal{L}^{\otimes d}) \arrow[r] \arrow[d, "\pi"'] & \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \times X \arrow[d, "\mathrm{pr}_2"] \\ \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \arrow[r, equal] & \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \end{tikzcd}$$

we compute that

$$\pi_!(i^*w) = (\mathrm{pr}_2)_!(1 \otimes w) = w \cap [X] = 0.$$

The final remark about the image follows from Lemma 8.1. □

Corollary 8.13. *Suppose that $H^{2n-1}(X;\mathbb{Q})$ vanishes (e.g. X is simply connected) and let i, d be as in Theorem 8.11. Then the maps in the zigzag*

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times \longleftarrow \Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d}) \longrightarrow \mathcal{M}^{\theta_d}(H_d,\ell_d)$$

induce inclusions in rational cohomology in degrees $* \leq i$ with the same image. In particular, the outer spaces have isomorphic rational cohomology in degrees $* \leq i$. □

8.3. **The $\mathbb{C}^\times$ action and the classifying map on the quotient.** It might be unsatisfying to only have a zigzag of maps in Corollary 8.13. This situation can be improved if one is willing to map to a certain quotient of $\mathcal{M}^{\theta_d}(H_d,\ell_d)$ as we now explain. To lighten the notation, we drop the subscript d everywhere and first work with a hypersurface $H = V(s)$ given by a non-singular section $s \in \Gamma_{\mathrm{ns}}(\mathcal{L})$ as in the beginning of the previous part. Recall the following point-set model (see e.g. [GRW14, Definition 1.5]):

$$\mathcal{M}^\theta(H,\ell) = \big(E\mathrm{Diff}(H) \times \mathrm{Bun}(TH,\theta^*\gamma;\ell)\big)/\mathrm{Diff}(H)$$

where $\mathrm{Bun}(TH,\theta^*\gamma;\ell)$ denotes the connected component of ℓ in the space of bundle maps[4]. Recall also from the proof of Proposition 8.8 the point-set model:

$$\theta^*\gamma = \{(x,\varphi,v) \mid (x,\varphi) \in \Omega^1_X \otimes \mathcal{L} \setminus 0,\ v \in \ker(\varphi)\}.$$

[4]As in the previous part, we slightly abuse notation by writing ℓ for both the map $H \to B$ and the map $TH \to \theta^*\gamma$ covering it. We started doing this in Proposition 8.8.

The group $\mathbb{C}^\times$ acts fibrewise on the vector bundle $\Omega^1_X \otimes \mathcal{L}$, thus on $\theta^*\gamma$ via

$$\lambda \cdot (x, \varphi, v) = (x, \lambda \cdot \varphi, v) \quad \text{for } \lambda \in \mathbb{C}^\times,$$

and therefore acts on $\mathcal{M}^\theta(H, \ell)$ by post-composition on bundle maps.

Let $\Delta\colon \Gamma_{\mathrm{ns}}(\mathcal{L}) \to \mathbb{C}^\times$ be the discriminant (see [GKZ94] for a reference) and let

$$\mu_{\deg\Delta} \subset \mathbb{C}^\times$$

be the cyclic subgroup of the $(\deg\Delta)^{\text{th}}$ roots of unity. There is a commutative diagram whose top two rows and leftmost two columns are fibre bundles:

$$\begin{tikzcd}
\mu_{\deg\Delta} \arrow[r, hook] \arrow[d, hook] & \Delta^{-1}(1) \arrow[r] \arrow[d, hook] & \Delta^{-1}(1)/\mu_{\deg\Delta} \arrow[d, "\cong"] \\
\mathbb{C}^\times \arrow[r, hook] \arrow[d] & \Gamma_{\mathrm{ns}}(\mathcal{L}) \arrow[r] \arrow[d, "\Delta"] & \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \\
\mathbb{C}^\times/\mu_{\deg\Delta} \arrow[r, "\cong"] & \mathbb{C}^\times &
\end{tikzcd}$$

The inverse of the homeomorphism

$$\Delta^{-1}(1)/\mu_{\deg\Delta} \xrightarrow{\cong} \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$$

is explicitly given by $s \mapsto \Delta(s)^{-1/\deg\Delta} \cdot s$. This readily implies:

Proposition 8.14. *The zigzag of Corollary 8.13 can be extended to a commutative diagram:*

$$\begin{tikzcd}
\Gamma_{\mathrm{ns}}(\mathcal{L}) \arrow[r] \arrow[d] & \mathcal{M}^\theta(H, \ell) \arrow[d] \\
\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \arrow[r] & \mathcal{M}^\theta(H, \ell)/\mu_{\deg\Delta}
\end{tikzcd}$$

where both vertical arrows are quotient maps and the bottom arrow is the induced map

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \cong \Delta^{-1}(1)/\mu_{\deg\Delta} \longrightarrow \mathcal{M}^\theta(H, \ell)/\mu_{\deg\Delta}. \qquad \square$$

Corollary 8.15. *Let i, d and X be as in Corollary 8.13. Then the map*[5]

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})/\mathbb{C}^\times \longrightarrow \mathcal{M}^{\theta_d}(H_d, \ell_d)/\mu_{\deg\Delta}$$

induces an isomorphism in rational cohomology in the range of degrees $* \leq i$.

Proof. It suffices to see that the quotient map

$$\mathcal{M}^{\theta_d}(H_d, \ell_d) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \ell_d)/\mu_{\deg\Delta}$$

induces an isomorphism in rational cohomology. As the action is free, the quotient is the homotopy orbit and we thus have a fibration:

$$\mathcal{M}^{\theta_d}(H_d, \ell_d) \longrightarrow \mathcal{M}^{\theta_d}(H_d, \ell_d)/\mu_{\deg\Delta} \longrightarrow B\mu_{\deg\Delta}.$$

The monodromy action of $\pi_1(B\mu_{\deg\Delta}) = \mu_{\deg\Delta}$ on the cohomology of the fibre is trivial as it can be extended to the connected group $\mathbb{C}^\times$. A finite group has trivial rational cohomology, here $H^*(B\mu_{\deg\Delta}; \mathbb{Q}) = \mathbb{Q}$, and the result follows. $\square$

[5] The discriminant Δ depends on d but we suppress this in the notation.

Remark 8.16. *By general theory, a map from a space T to the homotopy orbit space $\mathcal{M}^\theta(H,\ell)/\!\!/\mu_{\deg\Delta}$ is given by a principal $\mu_{\deg\Delta}$-bundle $P\to T$ and an equivariant map $P\to\mathcal{M}^\theta(H,\ell)$. From that point of view, the map*

$$\Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times \longrightarrow \mathcal{M}^\theta(H,\ell)/\mu_{\deg\Delta}$$

is given by the datum of a θ-structure on the pullback of the universal bundle along the étale cover

$$\Delta^{-1}(1)\longrightarrow \Gamma_{\mathrm{ns}}(\mathcal{L})/\mathbb{C}^\times$$

with Galois group $\mu_{\deg\Delta}$.

8.4. **En rød sild.** We finish this section by a remark on the infinite loop space from Theorem 8.9. As explained in [Aum22, Theorem 8.11], there is a map

$$\Gamma_{\mathrm{ns}}(\mathcal{L}^{\otimes d})\longrightarrow \Omega^{\infty+1}X^{J^1\mathcal{L}^{\otimes d}-TX}$$

which induces an isomorphism in rational cohomology in the range of degrees $*<\frac{d-1}{2}$. [6] To compare this infinite loop space with the one appearing above, we will use the following well-known lemma:

Lemma 8.17. *Let V,W be two vector bundles on a space Z. We may assume that W is a virtual vector bundle. Then there is a cofibre sequence of Thom spectra:*

$$S(V)^W\longrightarrow Z^W\longrightarrow Z^{V\oplus W}.$$

Proof. The Thom space Z^V is defined by collapsing the sphere bundle $S(V)$ inside the disc bundle $D(V)$. Thus there is a cofibre sequence of spaces

$$S(V)\hookrightarrow D(V)\longrightarrow Z^V.$$

The lemma follows by passing to Thom spectra with respect to the virtual bundle W and using the equivalence $D(V)\simeq Z$. ☐

Applying the lemma to $Z=X$, $V=\Omega^1_X\otimes\mathcal{L}^{\otimes d}$ and $W=\mathcal{L}^{\otimes d}-TX$ yields the fibre sequence of spaces

$$\Omega^{\infty+1}X^{J^1\mathcal{L}^{\otimes d}-TX}\longrightarrow \Omega^\infty S(\Omega^1_X\otimes\mathcal{L}^{\otimes d})^{\mathcal{L}^{\otimes d}-TX}\longrightarrow \Omega^\infty X^{\mathcal{L}^{\otimes d}-TX},$$

in which we recognise the two infinite loop spaces appearing earlier in this section. Finally, let us remark that the rational homotopy type of the rightmost space

$$(\Omega^\infty X^{\mathcal{L}^{\otimes d}-TX})_{\mathbb{Q}}\simeq H^2(X;\mathbb{Q})\times K(H^1(X;\mathbb{Q}),1)$$

is not far from that of $\mathrm{Pic}(X)$. This was one of the starting observations for the present article, although it now seems to me to be red herring.

[6] To be precise, [Aum22, Theorem 8.11] only states that the map is $2n$-connected. However the connectivity solely appears as the connectivity of the map $S^{2n+1}\to\Omega^\infty\Sigma^\infty S^{2n+1}$. The latter is a rational equivalence, and the proof can be repeated after rationalisation.

Appendix A. Range estimates for jet ampleness

A.1. **The case of curves.** When X is a curve, that is of dimension 1, one can give an explicit formula for the jet ampleness of a line bundle depending only on its degree.

Lemma A.1. *Let X be a smooth projective complex curve of genus g. Let $\mathcal{L}$ be a line bundle on X and denote by $c_1(\mathcal{L}) \in H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ its degree, i.e. its first Chern class. Let $d \geq 1$ be an integer. If $c_1(\mathcal{L}) > 2g - 1 + d$ then $\mathcal{L}$ is d-jet ample.*

Proof. As there is only tangent direction at each point on a curve, to show that $\mathcal{L}$ is d-jet ample it suffices to show that the restriction map

$$H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L} \otimes \mathcal{O}_Z)$$

is surjective for all subschemes $Z \subset X$ of length $d+1$. Recall the short exact sequences of sheaves

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where $\mathcal{I}_Z$ denotes the ideal sheaf of Z. From the long exact sequence in cohomology, we see that it suffices to show that

$$H^1(\mathcal{L} \otimes \mathcal{I}_Z) = 0.$$

By Serre duality, this group is isomorphic to $H^0(K_X \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_Z^{-1})$ where K_X is the canonical sheaf. It is now enough to show that $K_X \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_Z^{-1}$ has negative degree under the assumptions of the lemma. This follows by computing that $\deg \mathcal{I}_Z^{-1} = d+1$, $\deg \mathcal{L}^{-1} = -c_1(\mathcal{L})$ and $\deg K_X = 2g-2$ by Riemann–Roch. □

A.2. **The case of toric varieties.** When X is a smooth projective toric variety, its fundamental group is trivial, hence the Picard scheme is discrete. In that case the results of this paper are simply obtained from [Aum22]. We nevertheless comment on how to compute the jet ampleness of a line bundle to give a sense of the difficulty of the problem.

The basic idea is as follows: if $\mathcal{L}$ is a d-jet ample line bundle on X, then so is its restriction to any rational curve on X. On such a rational curve $C \cong \mathbb{P}^1$, a line bundle is of the form $\mathcal{O}_{\mathbb{P}^1}(a)$ and is d-jet ample if and only if $a \geq d$. Now there are some distinguished curves on X, namely the ones invariant under the torus action, and it turns out to be enough to check jet ampleness on them:

Theorem A.2 (Compare [Roc99])**.** *Let $\mathcal{L}$ be a line bundle on a smooth projective toric variety. Then $\mathcal{L}$ is d-jet ample if and only if $\mathcal{L} \cdot C \geq d$ for any torus invariant curve $C \subset X$.*

In [Roc99], Di Rocco also proves two more equivalent criteria for jet ampleness in terms of convexity of the support function of $\mathcal{L}$ and Seshadri constants at each point of X. We refer to that paper for the full details. Importantly for us, the criterion shows that d-jet ampleness can be checked by a finite number of inequations.

A.3. **Fujita's conjecture and jet ampleness on surfaces.** Whereas Kleiman's criterion shows that ampleness is a numerical property, jet ampleness, or even just very ampleness or global generation, is a trickier question to settle. In 1985, Fujita proposed the following conjecture which remains unsolved in general:

Conjecture A.3 (Compare [Fuj87])**.** *Let X be a smooth projective complex variety of dimension n. Let A be an ample line bundle on X. Then $K_X + (n+1)A$ is globally generated, and $K_X + (n+2)A$ is very ample.*

In dimension 1, the conjecture follows from the Riemann–Roch theorem. In higher dimension, the approach taken for curves would require proving a Kodaira-type vanishing theorem for non-invertible sheaves. However, in dimension 2, the conjecture was solved by Reider by different means:

Theorem A.4 (Compare [Rei88]). *Fujita's conjecture is true for $n = 2$.*

We recommend the lecture notes of Lazarsfeld [Laz94] for a beautiful introduction to Fujita's conjecture and Reider's theorem.

A.4. **A few general remarks.** Although effective vanishing theorems like Riemann–Roch do not exist in higher dimension, there exist alternatives that can be used to provide qualitative statements about jet ampleness. The starting observation is the following cohomological criterion:

Lemma A.5. *Let $\mathcal{L}$ be a line bundle on a smooth projective variety X of dimension n. Let $d \geq 1$ be an integer. Then $\mathcal{L}$ is d-jet ample if the cohomology groups*

$$H^1(\mathcal{L} \otimes \mathcal{I}_Z)$$

vanish for all 0-dimensional subschemes $Z \subset X$ of length $\sum\limits_{j=0}^{d} \binom{n+j-1}{j}$, and ideal sheaf $\mathcal{I}_Z$.

Proof. By definition, $\mathcal{L}$ is d-jet ample if the evaluation map

$$H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}/\mathfrak{m}_1^{k_1} \cdots \mathfrak{m}_l^{k_l}) \cong \bigoplus_{i=1}^{l} H^0(\mathcal{L}/\mathfrak{m}_i^{k_i})$$

is surjective for all distinct closed points $x_1, \ldots, x_l$ with associated maximal ideal sheaves $\mathfrak{m}_1, \ldots, \mathfrak{m}_l$, and all integers $k_i \geq 1$ such that $\sum k_i = d + 1$. For a closed point with ideal sheaf $\mathfrak{m}$ and $k \geq 1$ an integer, the subscheme given by the ideal $\mathfrak{m}^k$ has length $\sum_{j=0}^{k-1} \binom{n+j-1}{j}$. Therefore the subscheme Z given by the ideal sheaf $\mathfrak{m}_1^{k_1} \cdots \mathfrak{m}_l^{k_l}$ has length

$$l(Z) = \sum_{i=1}^{l} \sum_{j=0}^{k_i-1} \binom{n+j-1}{j} \leq \sum_{j=0}^{d} \binom{n+j-1}{j}.$$

Now, if Z is a subscheme with ideal sheaf $\mathcal{I}_Z$, then surjectivity of $H^0(\mathcal{L}) \to H^0(\mathcal{L} \otimes \mathcal{O}_Z)$ is implied by vanishing of the cohomology group $H^1(\mathcal{L} \otimes \mathcal{I}_Z)$ as one sees from the long exact sequence in cohomology associated to the short exact sequence of sheaves $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. ☐

Remark A.6. *The results of [Aum22] are stated in terms of the jet ampleness of $\mathcal{L}$, which is why we use the same phrasing in this paper. But in fact, as we are only concerned with conditions on the first order derivatives of sections, we could settle for the following ad hoc weaker notion: a line bundle $\mathcal{L}$ is d-good if the evaluation map*

$$H^0(\mathcal{L}) \longrightarrow \bigoplus_{i=1}^{l} H^0(\mathcal{L}/\mathfrak{m}_i^2)$$

is surjective for all distinct closed points $x_1, \ldots, x_l$ with associated maximal ideal sheaves $\mathfrak{m}_1, \ldots, \mathfrak{m}_l$, and $2l \leq d + 1$. We claim that the proofs of [Aum22] go through to study $\Gamma_{\mathrm{ns}}(\mathcal{L})$ with this weaker assumption. However, as the bounds we obtain to estimate d-goodness or jet ampleness are not very explicit, we have opted for the stronger assumption of jet ampleness which is more commonly studied.

Proposition A.7. *Let X be smooth projective complex variety. For any integer $d \geq 1$, there exists a class $\alpha \in \mathrm{NS}(X)$ such that all line bundles of first Chern class equal to α are d-jet ample.*

Proof. The plan of attack is to start from any component of the Picard scheme and tensor all line bundles on it by a chosen very ample line bundle. This yields an isomorphism with another Picard component, where one hopes that the locus of those line bundles that are not d-jet ample has been shrunk. By repetitively doing this procedure one should arrive at a component that only contains d-jet ample line bundles. Let $M = \sum_{j=0}^{d} \binom{n+j-1}{j}$ and define

$$E_\alpha := \{(Z, \mathcal{L}) \in \mathrm{Hilb}_M(X) \times \mathrm{Pic}^\alpha(X) \mid H^1(\mathcal{L} \otimes \mathcal{I}_Z) \neq 0\} \subset \mathrm{Hilb}_M(X) \times \mathrm{Pic}^\alpha(X)$$

for any Chern class $\alpha \in \mathrm{NS}(X)$. Here $\mathrm{Hilb}_M(X)$ denotes the Hilbert scheme of length M 0-dimensional subschemes of X. By upper semicontinuity of cohomology, E_α is a closed subscheme. Let $p\colon \mathrm{Hilb}_M(X) \times \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ be the second projection. By properness of the Hilbert scheme, the image $p(E_\alpha) \subset \mathrm{Pic}^\alpha(X)$ is closed. This image is exactly those line bundles which do not satisfy the cohomological criterion of Lemma A.5. By properness of $\mathrm{Pic}^\alpha(X)$, it has only finitely many irreducible components. We will want to reduce their number, up to changing the Chern class α.

Let $\mathcal{L}_0 \in p(E_\alpha)$ and consider the subscheme

$$E(\mathcal{L}_0) := \{Z \in \mathrm{Hilb}_M(X) \mid H^1(\mathcal{L}_0 \otimes \mathcal{I}_Z) \neq 0\} \subset \mathrm{Hilb}_M(X).$$

Again, upper semicontinuity of cohomology shows that it is a closed subscheme. Let $Z \in E(\mathcal{L}_0)$. Its ideal sheaf $\mathcal{I}_Z$ is a coherent sheaf on X, so by Serre vanishing theorem there exists a very ample line bundle $\mathcal{A}$ on X such that $H^1(\mathcal{L}_0 \otimes \mathcal{A} \otimes \mathcal{I}_Z) = 0$. Furthermore, if Z' was another 0-dimensional subscheme such that $H^1(\mathcal{L}_0 \otimes \mathcal{I}_{Z'}) = 0$, then $H^1(\mathcal{L}_0 \otimes \mathcal{A} \otimes \mathcal{I}_{Z'}) = 0$ also. Therefore, we have a strict inclusion of closed subschemes

$$E(\mathcal{L}_0 \otimes \mathcal{A}) \subsetneq E(\mathcal{L}_0).$$

By properness of $\mathrm{Hilb}_M(X)$, the same argument can be repeated a finite amount of times to obtain a very ample line bundle $\mathcal{A}'$ such that $E(\mathcal{L}_0 \otimes \mathcal{A}')$ is empty.

Now consider the isomorphism given by tensoring with $\mathcal{A}'$:

$$- \otimes \mathcal{A}' \colon \mathrm{Pic}^\alpha(X) \xrightarrow{\cong} \mathrm{Pic}^{\alpha+\alpha'}(X)$$

where $\alpha' = c_1(\mathcal{A}')$. By what we have seen

$$p(E_{\alpha+\alpha'}) \subsetneq p(E_\alpha) \otimes \mathcal{A}',$$

so that $p(E_{\alpha+\alpha'})$ has strictly fewer irreducible components than $p(E_\alpha)$ (if non empty). Repeating this argument reduces the number of irreducible components until we find a $\beta \in \mathrm{NS}(X)$ such $p(E_\beta)$ is empty. For this Chern class, all line bundles are d-jet ample by Lemma A.5. $\square$

Remark A.8. *For surfaces, there is a simpler and more explicit proof using Fujita's conjecture. Indeed, the image of the ample cone under the map $A \mapsto K_X + (n+2)A$ is a cone, and any line bundle having Chern class in that cone is very ample by Reider's theorem. If $\mathcal{L}$ is a very ample line bundle, the component of the Picard scheme corresponding to $K_X + (n+2)A + (d-1)\mathcal{L}$ only contains line bundles that are d-jet ample.*

References

[Aum22] Alexis Aumonier. An h-principle for complements of discriminants, October 2022. arXiv:2112.00326 [math]. 4, 17, 27, 28, 35, 36, 37

[Ber15] Alexander Berglund. Rational homotopy theory of mapping spaces via lie theory for L_∞-algebras. *Homology, Homotopy and Applications*, 17(2):343–369, 2015. 25

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*. Springer, Berlin, Heidelberg, 1990. 1, 12

[BRS99] Mauro C. Beltrametti, Sandra Di Rocco, and Andrew J. Sommese. On generation of jets for vector bundles. *Revista Matemática Complutense*, 12(1):27–44, 1999. 8

[CS84] M. C. Crabb and W. A. Sutherland. Function Spaces and Hurwitz-Radon Numbers. *Mathematica Scandinavica*, 55(1):67–90, 1984. Publisher: Mathematica Scandinavica. 16, 21

[Das21] Ronno Das. The space of cubic surfaces equipped with a line. *Mathematische Zeitschrift*, 298(1):653–670, June 2021. 27

[Die08] Tammo tom Dieck. *Algebraic topology*. EMS textbooks in mathematics. European Mathematical Society, Zürich, 2008. OCLC: ocn261176011. 20

[Dou66] Adrien Douady. Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. *Annales de l'Institut Fourier*, 16(1):1–95, 1966. 16

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory*, volume 205 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2001. 22

[Fri17] Nina Friedrich. Homological Stability of Automorphism Groups of Quadratic Modules and Manifolds. *Documenta Mathematica*, 22:1729–1774, 2017. 27

[Fuj87] Takao Fujita. On Polarized Manifolds Whose Adjoint Bundles Are Not Semipositive. In *Algebraic Geometry, Sendai, 1985*, volume 10, pages 167–179. Mathematical Society of Japan, January 1987. 36

[GKZ94] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Birkhäuser, Boston, MA, 1994. 27, 34

[Gro61] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert. *Société mathématique de France*, 1961. 10

[Gro62] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard : théorèmes d'existence. *Société mathématique de France*, 1962. 1

[Gro67] Alexander Grothendieck. Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. *Publications Mathématiques de l'IHÉS*, 32:5–361, 1967. 6

[GRW14] Søren Galatius and Oscar Randal-Williams. Stable moduli spaces of high-dimensional manifolds. *Acta Mathematica*, 212(2):257–377, January 2014. Publisher: Institut Mittag-Leffler. 3, 27, 33

[GRW17] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. II. *Annals of Mathematics*, 186(1):127–204, July 2017. Publisher: Department of Mathematics of Princeton University. 3, 27

[GRW18] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. I. *Journal of the American Mathematical Society*, 31(1):215–264, January 2018. 3, 27

[GRW19] Søren Galatius and Oscar Randal-Williams. Moduli spaces of manifolds: a user's guide. In *Handbook of Homotopy Theory*. Chapman and Hall/CRC, 2019. Num Pages: 43. 27, 29, 31, 32, 33

[Hae82] André Haefliger. Rational homotopy of the space of sections of a nilpotent bundle. *Transactions of the American Mathematical Society*, 273:609–620, 1982. Publisher: American Mathematical Society. 22, 23

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, New York, NY, 1977. 9, 12

[Kle05] Steven L. Kleiman. The Picard scheme, April 2005. arXiv:math/0504020. 5, 6, 11

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. Springer, Berlin, Heidelberg, 1996. 11

[Laz94] Robert Lazarsfeld. Lectures on Linear Series, August 1994. arXiv:alg-geom/9408011. 37

[McD75] Dusa McDuff. Configuration spaces of positive and negative particles. *Topology*, 14(1):91–107, March 1975. 3, 26

[Mø85] Jesper Michael Møller. On the homology of spaces of sections of complex projective bundles. *Pacific Journal of Mathematics*, 116(1):143–154, January 1985. Publisher: Pacific Journal of Mathematics, A Non-profit Corporation. 22, 23, 24

[Mø87] Jesper Michael Møller. Nilpotent Spaces of Sections. *Transactions of the American Mathematical Society*, 303(2):733, October 1987. 22

[Rap17] George Raptis. On Serre microfibrations and a lemma of M. Weiss. *Glasgow Mathematical Journal*, 59(3):649–657, September 2017. Publisher: Cambridge University Press. 18

[Rei88] Igor Reider. Vector Bundles of Rank 2 and Linear Systems on Algebraic Surfaces. *Annals of Mathematics*, 127(2):309–316, 1988. Publisher: Annals of Mathematics. 37

[Roc99] Sandra Di Rocco. Generation of k-jets on toric varieties. *Mathematische Zeitschrift*, 231(1):169–188, May 1999. 36

[RW13] Oscar Randal-Williams. Homological stability for unordered configuration spaces. *The Quarterly Journal of Mathematics*, 64(1):303–326, March 2013. 3

[RW19] Oscar Randal-Williams. Miller–Morita–Mumford classes vanish on the moduli space of hypersurfaces. *Available on the author's webpage: https://www.dpmms.cam.ac.uk/˜or257/publications.htm*, 2019. 28

[SA15] Schemer and Albanese. What are examples of parallelizable complex projective varieties? *Mathematics Stack Exchange: https://math.stackexchange.com/q/1567516*, December 2015. 3

[Seg79] Graeme Segal. The topology of spaces of rational functions. *Acta Mathematica*, 143(none):39–72, January 1979. Publisher: Institut Mittag-Leffler. 3

[The23] The Stacks Project Authors. The Stacks project. *https://stacks.math.columbia.edu/*, 2023. 11, 13

[Wan54] Hsien-Chung Wang. Complex parallisable manifolds. *Proceedings of the American Mathematical Society*, 5(5):771–776, 1954. 3

[Wei05] Michael Weiss. What does the classifying space of a category classify? *Homology, Homotopy and Applications*, 7(1):185–195, January 2005. Publisher: International Press of Boston. 17

Department of Mathematical Sciences, University of Copenhagen,
Universitetsparken 5, 2100 København Ø, Denmark

Email address: alexis-aumonier@math.ku.dk

PAPER C

Homological stability for the space of hypersurfaces with marked points

This chapter contains the preprint version of the following paper:

Alexis Aumonier and Ronno Das. *Homological stability for the space of hypersurfaces with marked points.* 2023.

The preprint version is not made publicly available yet.

HOMOLOGICAL STABILITY FOR THE SPACE OF HYPERSURFACES WITH MARKED POINTS

ALEXIS AUMONIER AND RONNO DAS

Abstract. We study the space of smooth marked hypersurfaces in a given linear system. More specifically, we prove an h-principle relating its homology to that of a space of sections of an appropriate bundle. Using rational models, we explain how to compute its rational cohomology in a range of degrees, and deduce a homological stability result for hypersurfaces of increasing degree.

Contents

1. Introduction

Consider a connected smooth complex projective variety X with a line bundle $\mathcal{L}$. Let $U(\mathcal{L}) \subset \Gamma_{\mathrm{hol}}(\mathcal{L})$ be the open subset of *non-singular* holomorphic global sections of $\mathcal{L}$ and consider the incidence variety:

$$Z(\mathcal{L}) = \left\{(f,x) \in U(\mathcal{L}) \times X \;\middle|\; f(x) = 0\right\} \subset \Gamma_{\mathrm{hol}}(\mathcal{L}) \times X\,.$$

If we take the sections modulo scalar multiples, i.e. quotient by the action of $\mathbb{C}^\times$, we get the space of *$\mathcal{L}$-hypersurfaces* given by $U_d/\mathbb{C}^\times$ and the *universal non-singular $\mathcal{L}$-hypersurface* given by $Z(\mathcal{L})/\mathbb{C}^\times$.[1] In rational homology the $\mathbb{C}^\times$ behaves like a direct factor, see Lemma 3.2, and we will not worry about this quotient in the rest of the paper.

More generally, let $\mathrm{Z}^r(\mathcal{L})$ be the bundle over $U(\mathcal{L})$ whose fiber over f is $\mathrm{F}^r(\mathcal{V}(f))$, the configuration space of r distinct points on the vanishing locus $\mathcal{V}(f)$ of f, topologized as a subspace of $U \times X^r$. Explicitly,

$$\mathrm{Z}^r(\mathcal{L}) = \left\{(f, x_1, \ldots, x_r) \in \Gamma_{\mathrm{hol}}(\mathcal{L}) \times X^r \;\middle|\; f \in U(\mathcal{L}),\ f(x_i) = 0,\ x_i \neq x_j \text{ for } i \neq j\right\}$$

is the *space of $\mathcal{L}$-hypersurface sections with r distinct marked points.*

Theorem A. *Assume that X is simply connected. Suppose $\mathcal{L}$ is ample and fix $r \geq 1$. Then for each i, there exists*[2] *a d_0 such that for all $d \geq d_0$, we have an isomorphism*

$$H^i(\mathrm{Z}^r(\mathcal{L}^{\otimes d}); \mathbb{Q}) \cong H^i(A_r(\mathcal{L}))$$

where $A_r(\mathcal{L})$ is the commutative differential graded algebra explicitly described in Construction 5.5. This isomorphism is $\mathfrak{S}_r$ equivariant and preserves cup products (when d is appropriately large). In particular, $H^i(\mathrm{Z}^r(\mathcal{L}^{\otimes d}); \mathbb{Q})$ stabilizes with d, as an $\mathfrak{S}_r$-representation.

We conjecture that the classes in $H^i(\mathrm{Z}^r(\mathcal{L}^{\otimes d}); \mathbb{Q})$ for different d that are identified by these isomorphisms should also have the same Hodge weights (see Section 5.5). As a supporting fact, the consequent stabilization of Hodge–Euler characteristics was observed by Howe [How19]. The analogous statement for $r = 0$ was proved for $X = \mathbb{P}^n$ by Tommasi [Tom14] and for general X in [Aum22] and [DH22].

There are no obvious maps between $\mathrm{Z}^r(\mathcal{L}^{\otimes d})$ for varying d and the stability in Theorem A is (to our knowledge) not induced by such maps of spaces. Nevertheless, the isomorphism in Theorem A is not just an abstract isomorphism once we ascribe more meaning to $A_r(\mathcal{L})$. As an algebra it is generated by a CDGA model for $\mathrm{F}^r(X)$, a shifted copy of $H^*(X; \mathbb{Q})$, and $2r$ additional classes, r each in degrees $2n$ and $2n-1$, where $n = \dim X$. These extra classes correspond to the fundamental classes of the tangent spaces and jet spaces of X at the r marked points.

Remark 1.1. The dependence of the CDGA $A_r(\mathcal{L})$ on X and $\mathcal{L}$ is only via the graded ring $H^*(X)$, its Poincaré pairing, and the Chern class of $\mathcal{L}$. Therefore the same is true for the stable cohomology in Theorem A.

In fact, in Theorem 5.9 $A_r(\mathcal{L})$ is identified[3] as a rational model for a continuous analog of $\mathrm{Z}^r(\mathcal{L}^{\otimes d})$. Explicitly, let $J^1\mathcal{L}$ be the first-order jet bundle of $\mathcal{L}$ and define $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$ by imitating the construction above while replacing $U(\mathcal{L})$ by an analogous open subset $U_{\mathcal{C}^0}(\mathcal{L})$ of the space $\Gamma_{\mathcal{C}^0}(J^1\mathcal{L})$ of continuous sections; see Section 3 for a precise definition. There is a *jet-expansion map*

$$j^1 : \Gamma_{\mathrm{hol}}(\mathcal{L}) \to \Gamma_{\mathcal{C}^0}(J^1\mathcal{L})$$

which lies below an $\mathfrak{S}_r$-equivariant map $\mathrm{Z}^r(\mathcal{L}) \to \mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$, which we will also call a jet-expansion map and denote by j^1.

[1] In a different language, these spaces are respectively the open locus of smooth divisors inside the complete linear system $|\mathcal{L}|$ and the restriction of the universal flat family above it.

[2] It suffices to take $d_0 = 1 + \max(|\chi(X)|, k(2i + 2r + 3))$, with k such that $\mathcal{L}^{\otimes k}$ is very ample, by combining the bounds in Theorem B and Remark 5.11.

[3] Even though $A_r(\mathcal{L})$ does not depend on d, the identification of the generators *does* and also crucially requires rational coefficients, cf. the proof of Theorem 5.9.

Recall that if $\mathcal{L}$ is very ample then $\mathcal{L}^{\otimes d}$ is d-jet ample (see Definition 2.7 for the definition of jet ampleness). Then the following theorem, combined with the computation of $H^*(\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L});\mathbb{Q})$ from Theorem 5.9 and Proposition 5.12, refines Theorem A:

Theorem B. *Let $r \geq 1$ and $\mathcal{L}$ be d-jet ample for some $d > 2i + 2r + 3$. Then the jet expansion map j^1 induces an isomorphism*

$$H_i(\mathrm{Z}^r(\mathcal{L});\mathbb{Z}) \xrightarrow{\simeq} H_i(\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L});\mathbb{Z})\,.$$

For the rest of the introduction we will use cohomology with rational coefficients but will suppress it from the notation for the sake of brevity and readability.

Theorem 1.1 in [Aum22] (see also [DH22; Tom14]) should be thought of as the $r = 0$ case of Theorem B, comparing just the space $U(\mathcal{L})$ of nonsingular sections, without any markings, with the analogous continuous section space $U_{\mathcal{C}^0}(\mathcal{L})$. Given this result and the definition of $\mathrm{Z}^r(\mathcal{L})$, it seems reasonable to use the Serre spectral sequence of the fiber bundle $\mathrm{Z}^r(\mathcal{L}) \to U(\mathcal{L})$, which for $r = 1$ in fact degenerates on the E_2 page (by the Deligne degeneration theorem). The terms of this spectral sequence are given by the cohomology of $U(\mathcal{L})$ with certain *local coefficients*, with stalk $H^*(\mathcal{V}(f))$ at $f \in U(\mathcal{L})$. However, the usual technique for controlling the cohomology of $U(\mathcal{L})$ (in [Aum22] and elsewhere) is to pass to the complement in $\Gamma_{\mathrm{hol}}(\mathcal{L})$ by Alexander duality and any non-trivially twisted coefficients on $U(\mathcal{L})$ cannot possibly extend to this contractible space.

In contrast, what actually lets us make progress is the other projection $\mathrm{Z}^r(\mathcal{L}) \to \mathrm{F}^r(X)$. While we do get a Serre spectral sequence for the fibration $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L}) \to \mathrm{F}^r(X)$, on the algebraic side the map $\mathrm{Z}^r(\mathcal{L}) \to \mathrm{F}^r(X)$ is not[4] a fibration even when $\mathcal{L}$ is highly jet ample. It is however a *microfibration* (see Definition 4.3). In Section 4.1 we use this fact to reduce the proof of Theorem B to establishing a homology isomorphism (in an appropriate range) on each fiber. To be more precise, we establish such a fiberwise homology isomorphism in Theorem 6.1, after replacing the map $\mathrm{Z}^r(\mathcal{L}) \to \mathrm{F}^r(X)$ by the map $\mathrm{Z}^r(\mathcal{L}) \to (\Omega^1_X \otimes \mathcal{L})^r|_{\mathrm{F}^r(X)}$,

$$(f, x_1, \ldots, x_r) \mapsto (\mathrm{d}f(x_1), \ldots, \mathrm{d}f(x_r))\,,$$

that records not just the marked points but also the (necessarily non-zero) derivatives of the section at each of these points.

The special case of $r = 1$, $X = \mathbb{P}^n$ (and $\mathcal{L} = \mathcal{O}(d)$ for large d) was treated in [Ban21]. It general the $r = 1$ case has a slightly different flavor than $r > 1$, partly because the fiber $\mathrm{F}^1(\mathcal{L}) = Z(\mathcal{L}) \to U(\mathcal{L})$ is projective. We discuss this case in more detail in Section 1.1.2.

1.1. **Applications and computations.**

1.1.1. *Unordered marked points.* While the configuration spaces in the rest of the paper are ordered, let us deal with the unordered case here. The symmetric group $\mathfrak{S}_r$ acts on $\mathrm{Z}^r(\mathcal{L}) \subset \Gamma_{\mathrm{hol}}(\mathcal{L}) \times \mathrm{F}^r(X)$ by permuting the coordinates of $\mathrm{F}^r(X)$. The map $\mathrm{Z}^r(\mathcal{L}) \to U(\mathcal{L})$ descends to the quotient $\mathrm{Z}^r(\mathcal{L})/\mathfrak{S}_r \to U(\mathcal{L})$, and this map is also a fiber bundle, now with fiber $\mathrm{F}^r(\mathcal{V}(f))/\mathfrak{S}_r$, the *unordered* configuration space of $\mathcal{V}(f)$, over $f \in U(\mathcal{L})$. The analog of Theorem B holds with the same proof, or with rational coefficients we can just use the transfer isomorphisms

$$H^*(\mathrm{Z}^r(\mathcal{L})/\mathfrak{S}_r;\mathbb{Q}) \cong H^*(\mathrm{Z}^r(\mathcal{L});\mathbb{Q})^{\mathfrak{S}_r}\,.$$

In particular we have the following analog of Theorem A:

[4]At least not in general, excepting when $\mathrm{Aut}(X)$ acts transitively on $\mathrm{F}^r(X)$.

Corollary 1.2. *If $\mathcal{L}$ is very ample and d is sufficiently large then*

$$H^i(\mathrm{Z}^r(\mathcal{L}^{\otimes d})/\mathfrak{S}^r;\mathbb{Q}) \cong H^i(A_r(\mathcal{L})^{\mathfrak{S}_r}).$$

In particular, $\dim H^i(\mathrm{Z}^r(\mathcal{L}^{\otimes d})/\mathfrak{S}_r;\mathbb{Q})$ *stabilizes for large* d. □

1.1.2. *Marking a single point.* For $r=1$, the space $\mathrm{Z}^1(\mathcal{L}) = Z(\mathcal{L})$ is the "universal $\mathcal{L}$-hypersurface". In this case the computation of $H^*(A_1(\mathcal{L}))$ is relatively simple and we recover the following explicit description:

Corollary 1.3. *Assume that X is simply connected. Suppose $\mathcal{L}$ is d-jet ample. Writing $\Omega^1_X \otimes \mathcal{L} - 0$ for the complement of the zero section of $\Omega^1_X \otimes \mathcal{L}$, we have the following isomorphism in the range $* < \frac{d-3}{2}$:*

$$H^*(Z(\mathcal{L});\mathbb{Q}) \cong H^*\left(\Omega^1_X \otimes \mathcal{L} - 0;\mathbb{Q}\right) \otimes \mathrm{Sym}_{\mathrm{gr}}\left(H^{*-1}(X-\mathrm{pt})\right).$$

In the range $* < 2n-2$ (and for d sufficiently large) this follows from Nori's connectivity theorem [Nor93]. Taking $X = \mathbb{P}^n$, we recover the stabilization (and stable cohomology) from [Ban21, Theorem 1.1].

Remark 1.4. Corollary 1.3 gives us an exact criterion for when the stable cohomology of $Z(\mathcal{L})$ is finite dimensional: $H^i(Z(\mathcal{L}))$ vanishes for i large and $\mathcal{L}$ sufficiently jet ample (depending on i) if and only if $H^*(X)$ is concentrated in even degrees. For instance, this holds if X is $\mathbb{P}^n$ or a Grassmannian but fails if X is a curve of positive genus. In contrast, for $r > 1$ the stable cohomology is necessarily non-zero in infinitely many degrees for any X, this is already visible in the weightwise Euler characteristic described in Section 5.5 and was noted in [How19] as a required feature.

Acknowledgements. We are grateful to Oscar Randal-Williams for pointing out the relevance of microfibrations, which are a crucial ingredient in our arguments. We thank Søren Galatius, Florian Naef and Dan Petersen for helpful conversations throughout various stages of the project.

AA was supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151) as well as the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682922). RD was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 772960), by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151), as well as by Dan Petersen's Wallenberg Academy fellowship during various parts of the project.

2. Configuration spaces and jet bundles

In this section, we provide a reminder on configuration spaces and jet bundles. As in the introduction, X is a connected smooth complex projective variety of complex dimension n. We write $\Gamma_{\mathrm{hol}}(-)$ for the space of holomorphic global sections of a bundle on X.

2.1. **Configuration spaces.** For any topological space S, let $\mathrm{F}^r(S)$ be the configuration space of r ordered points in S

$$\mathrm{F}^r(S) := \left\{(x_1,\dots,x_r) \in S^r \;\middle|\; x_i \neq x_j \text{ for } i \neq j\right\}.$$

The symmetric group $\mathfrak{S}_r$ acts on $\mathrm{F}^r(S)$ by permuting coordinates. More generally, for $Z \to S$, define the fiberwise configuration space

$$\mathrm{F}^r_S(Z) := \mathrm{F}^r(Z) \cap Z^{\times_S r},$$

where $Z^{\times_S r}$ denotes the r-fold fiber product $Z \times_S Z \times_S \cdots \times_S Z$. Denoting the fiber of Z over $s \in S$ by Z_s, we have a natural identification

$$\mathrm{F}^r_S(Z) = \left\{(s, z_1, \ldots, z_r) \;\middle|\; (z_1, \ldots, z_r) \in \mathrm{F}^r(Z_s)\right\} \subset S \times \mathrm{F}^r(Z) \subset S \times Z^r .$$

In particular, the projection map $\mathrm{F}^r_S(Z) \to S$ has fiber $\mathrm{F}^r(Z_s)$ over $s \in S$. For convenience, we define $\mathrm{F}^1_S(Z) = Z$ and $\mathrm{F}^0_S(Z) = S$.

2.2. **Jet bundles.** In this work, we use jet bundles to suitably talk about derivatives of sections of vector bundles. For the unfamiliar reader, we offer a minimal overview of the general theory developed in [Gro67].

Let $\mathcal{L}$ be a holomorphic line bundle on X. Its bundle of first order jets, defined in [Gro67, Section 16.7], will be denoted by $J^1\mathcal{L}$. It is a holomorphic vector bundle on X of complex rank $\dim_{\mathbb{C}} X + 1$ which fits in an exact sequence of holomorphic vector bundles

$$0 \longrightarrow \Omega^1_X \otimes \mathcal{L} \longrightarrow J^1\mathcal{L} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{2.1}$$

where Ω^1_X is the cotangent bundle of X. Although this short exact sequence does not split in general (in the category of holomorphic vector bundles), it informally indicates that the jet bundle records the value (in $\mathcal{L}$) and the first derivative (in $\Omega^1_X \otimes \mathcal{L}$) of sections of $\mathcal{L}$. More precisely:

Definition 2.2. The short exact sequence above splits after taking holomorphic global sections. Writing $\mathrm{d}f$ for the derivative of a global section f of $\mathcal{L}$, the morphism

$$\begin{aligned} j^1 \colon \Gamma_{\mathrm{hol}}(\mathcal{L}) &\longrightarrow \Gamma_{\mathrm{hol}}(J^1\mathcal{L}) = \Gamma_{\mathrm{hol}}(\mathcal{L}) \oplus \Gamma_{\mathrm{hol}}(\Omega^1_X \otimes \mathcal{L}) \\ f &\longmapsto (f, \mathrm{d}f) \end{aligned}$$

is called the *(first order) jet expansion.*

Example 2.3. Let us consider in more details the case where $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(k)$ with $k \geq 1$. The space of global sections $\Gamma_{\mathrm{hol}}(\mathcal{O}(k))$ can be identified with the complex vector space of homogeneous polynomials of degree k in $n+1$ variables. Writing $z_0, \ldots, z_n$ for the variables, Euler's identity

$$\sum_i z_i \frac{\partial f}{\partial z_i} = k \cdot f$$

shows that knowing the $n+1$ partial derivatives of a section f in the homogeneous coordinates amounts to knowing the section. This fact can be leveraged to an isomorphism of holomorphic vector bundles $J^1\mathcal{O}(k) \cong \mathcal{O}(k-1)^{\oplus n+1}$. (For a proof, see [DS98]. Note however that such a splitting is very peculiar to projective spaces.) In that case, the jet expansion is given by

$$f \in \Gamma_{\mathrm{hol}}(\mathcal{O}(k)) \longmapsto \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \in \Gamma_{\mathrm{hol}}(\mathcal{O}(k-1))^{\oplus n+1}.$$

Definition 2.4. A global section $f \in \Gamma_{\mathrm{hol}}(\mathcal{L})$ is said to be *singular* at $x \in X$ if the first jet $j^1(s)(x) = 0$. It is called *non-singular* if it does not admit any singular point.

Remark 2.5. The vanishing locus of any non-zero global section $f \in \Gamma_{\mathrm{hol}}(\mathcal{L})$ is the subvariety given by $V(f) := \left\{x \in X \;\middle|\; f(x) = 0\right\} \subset X$. When f is non-singular, $V(f)$ is a smooth subvariety.

Example 2.6. In the situation of Example 2.3, a global section f is singular at a point $x \in \mathbb{P}^n$ precisely when $\frac{\partial f}{\partial z_i}(x) = 0$ for all $i = 0, \ldots, n$. This is indeed the more classical Jacobian criterion.

The following property, jet ampleness, is the technical key to many arguments in this work:

Definition 2.7 (Compare [BRS99])**.** Let $\mathcal{L}$ be a holomorphic line bundle on X. Let $k \geq 0$ be an integer. Let $x_1, \dots, x_t$ be t distinct points in X and $(k_1, \dots, k_t)$ be a t-uple of positive integers such that $\sum_i k_i = k+1$. Write $\mathfrak{m}_i$ for the maximal ideal sheaf corresponding to x_i, and $\mathcal{L}_{x_i}$ for the stalk of $\mathcal{L}$ at x_i. We say that $\mathcal{L}$ is *k-jet ample* if the evaluation map

$$\Gamma_{\mathrm{hol}}(\mathcal{L}) \longrightarrow \bigoplus_{i=1}^{t} \mathcal{L}_{x_i}/\mathfrak{m}_i^{k_i}\mathcal{L}_{x_i}$$

is surjective for any $x_1, \dots, x_t$ and $k_1, \dots, k_t$ as above.

Remark 2.8. In the definition above, we wrote $\mathcal{L}$ for the sheaf of sections of the line bundle to be able to talk about its stalk. We shall be guilty of this slight abuse of notation throughout this article. The back-and-forth between the two viewpoints is explained by Serre's GAGA theorems [Ser56].

Remark 2.9. Let $x \in X$, denote by $\mathfrak{m}_x$ the corresponding maximal ideal sheaf, and by $\mathcal{L}_x$ the stalk of $\mathcal{L}$ at x. The fiber of $J^1\mathcal{L}$ at x is naturally identified with $\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$ and the jet map is the natural quotient map.

3. Marked hypersurfaces and section spaces

We continue with the notation from the previous section. For a holomorphic bundle $E \to X$, recall that we denote the space of its holomorphic global sections by $\Gamma_{\mathrm{hol}}(E)$. We will write $\Gamma_{\mathcal{C}^0}(E)$ for the larger space of continuous global sections. Both section spaces are topologized as subspaces of the continuous mapping space $\mathrm{map}(X, E)$, the latter being endowed with the compact-open topology.

Let $U(\mathcal{L}) \subset \Gamma_{\mathrm{hol}}(\mathcal{L})$ be the subspace of non-singular sections. The incidence variety

$$Z(\mathcal{L}) := \big\{(f, x) \bigm| f(x) = 0\big\} \subset U(\mathcal{L}) \times X$$

is equipped with projections to $U(\mathcal{L})$ and X; the fiber over $f \in U(\mathcal{L})$ is the smooth "$\mathcal{L}$-hypersurface" $V(f) \subset X$.

Definition 3.1. For $r \geq 0$, the space of *$\mathcal{L}$-hypersurfaces with r (ordered) marked points* is

$$\mathrm{Z}^r(\mathcal{L}) := \mathrm{F}^r_{U(\mathcal{L})}(Z(\mathcal{L})) \subset \Gamma_{\mathrm{hol}}(\mathcal{L}) \times X^r\,.$$

The space $\mathrm{Z}^r(\mathcal{L})$ comes with two projection maps: to $U(\mathcal{L})$ and to $\mathrm{F}^r(X)$. The fiber of the first one over $f \in U(\mathcal{L})$ is the configuration space $\mathrm{F}^r(V(f))$. As $V(f) = V(\lambda f)$ for any $\lambda \in \mathbb{C}^\times$, it is perhaps more geometrically meaningful to consider a quotient of $\mathrm{Z}^r(\mathcal{L})$ by $\mathbb{C}^\times$. However, our constructions are easier to be made before taking any quotient. Fortunately, we will mostly be interested in the rational cohomology of $\mathrm{Z}^r(\mathcal{L})$ and the following lemma shows that it is mostly a matter of convenience:

Lemma 3.2. *For $r \geq 0$, let $\mathbb{C}^\times$ act on $\mathrm{Z}^r(\mathcal{L}) \subset U(\mathcal{L}) \times \mathrm{F}^r(X)$ by acting by scalar multiplication on $U(\mathcal{L})$ and trivially on $\mathrm{F}^r(X)$. Then this action is free and we have an isomorphism:*

$$H^*(\mathrm{Z}^r(\mathcal{L}); \mathbb{Q}) \cong H^*(\mathrm{Z}^r(\mathcal{L})/\mathbb{C}^\times \times \mathbb{C}^\times; \mathbb{Q})\,.$$

Proof. From the general theory of discriminants (see e.g. [GKZ94]) we get that $U(\mathcal{L}) \subset \Gamma_{\mathrm{hol}}(\mathcal{L})$ is the complement of a hypersurface defined by a polynomial Δ. Then for any $(f, \vec{x}) \in \mathrm{Z}^r(\mathcal{L})$, the composite

$$\mathbb{C}^\times \xrightarrow{\lambda \mapsto \lambda \cdot (f,\vec{x})} \mathrm{Z}^r(\mathcal{L}) \to U(\mathcal{L}) \xrightarrow{\Delta} \mathbb{C}^\times$$

is a map of degree $\deg(\Delta) \neq 0$ and hence induces an isomorphism on rational cohomology. The claim follows by applying the Leray–Hirsch theorem. □

The constructions above can be adapted and repeated in the setting of continuous sections. More precisely, we write

$$U_{\mathcal{C}^0}(\mathcal{L}) := \left\{ s \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}) \;\middle|\; s(x) \neq 0,\ \forall x \in X \right\} = \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0).$$

To define the analogue of the incidence variety $Z(\mathcal{L})$, we have to find a corresponding notion of vanishing at a point for sections of the jet bundle. Remark that the evaluation of a section $f \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{L})$ at a point $x \in X$ is an element

$$f(x) = (f_1(x), f_2(x)) \in (J^1\mathcal{L})|_x = (\Omega^1_X \otimes \mathcal{L})|_x \oplus \mathcal{L}|_x$$

where $(-)|_x$ denotes the fiber at x. If $f = j^1(g)$, then $f_2(x) = g(x)$ is simply the value of g at x. We take a cue from this situation and define

$$Z_{\mathcal{C}^0}(\mathcal{L}) := \left\{ (s,x) \in U_{\mathcal{C}^0}(\mathcal{L}) \times X \;\middle|\; s(x) \in (\Omega^1_X \otimes \mathcal{L})|_x \oplus 0 \subset (\Omega^1_X \otimes \mathcal{L})|_x \oplus \mathcal{L}|_x \right\}.$$

The following pullback square of topological spaces is then a direct consequence of the definitions:

$$\begin{tikzcd} Z(\mathcal{L}) \arrow[r, "j^1 \times \mathrm{id}"] \arrow[d] & Z_{\mathcal{C}^0}(\mathcal{L}) \arrow[d] \\ U(\mathcal{L}) \arrow[r, "j^1"] & U_{\mathcal{C}^0}(\mathcal{L}) \end{tikzcd}$$

More generally, denoting $\mathrm{F}^r_{U_{\mathcal{C}^0}(\mathcal{L})}(Z_{\mathcal{C}^0}(\mathcal{L}))$ by $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$, we get a pullback square

$$\begin{tikzcd} \mathrm{Z}^r(\mathcal{L}) \arrow[r, "j^1 \times \mathrm{id}"] \arrow[d] & \mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L}) \arrow[d] \\ U(\mathcal{L}) \arrow[r, "j^1"] & U_{\mathcal{C}^0}(\mathcal{L}) \end{tikzcd}$$

4. The main theorem

Having defined the spaces $\mathrm{Z}^r(\mathcal{L})$ and $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$ above, we are now ready to state our main theorem. Its proof will occupy the rest of this section.

Theorem 4.1. *Let X be a connected smooth complex projective variety, $r \geq 1$ an integer, and $\mathcal{L}$ a d-jet ample line bundle on X. Then the map*

$$j^1 \times \mathrm{id} \colon\ \mathrm{Z}^r(\mathcal{L}) \longrightarrow \mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d-3}{2} - r$.

We now proceed towards the proof of the theorem, beginning with some notation. Given a fiber bundle $E \to X$, we denote by $E^r|_{\mathrm{F}^r(X)}$ the restriction to $\mathrm{F}^r(X)$ of the product bundle E^r on X^r. We will mostly be concerned with

$$(\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \quad \text{and} \quad (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}$$

which are respectively $(\mathbb{C}^n - 0)^r$ and $(\mathbb{C}^{n+1} - 0)^r$ bundles over $\mathrm{F}^r(X)$. We will also make use of the evaluation map

$$\begin{aligned} \mathrm{ev}\colon U_{\mathcal{C}^0}(\mathcal{L}) \times \mathrm{F}^r(X) &\longrightarrow (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \\ (f, x_1, \dots, x_r) &\longmapsto (f(x_1), \dots, f(x_r)). \end{aligned}$$

It follows directly from the definitions that we have two pullback squares:

$$\begin{tikzcd} \mathrm{Z}^r(\mathcal{L}) \arrow[r] \arrow[d] & U(\mathcal{L}) \times \mathrm{F}^r(X) \arrow[d, "j^1 \times \mathrm{id}"] \\ \mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L}) \arrow[r] \arrow[d] & U_{\mathcal{C}^0}(\mathcal{L}) \times \mathrm{F}^r(X) \arrow[d, "\mathrm{ev}"] \\ (\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \arrow[r, hook] & (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \end{tikzcd}$$

Over a point $\vec{v} = \big((x_1, v_1), \dots, (x_r, v_r)\big) \in (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}$, denote the fiber of the composition $\mathrm{ev} \circ (j^1 \times \mathrm{id})$ by

$$U(\vec{v}) := \left\{ f \in U(\mathcal{L}) \;\middle|\; j^1(f)(x_i) = (x_i, v_i) \text{ for each } i = 1, \dots, r \right\}$$

and the fiber of ev by

$$U_{\mathcal{C}^0}(\vec{v}) := \left\{ s \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0) \;\middle|\; s(x_i) = (x_i, v_i) \text{ for each } i = 1, \dots, r \right\}.$$

We summarize the situation in the following commutative diagram:

$$\begin{tikzcd} U(\vec{v}) \arrow[rr] \arrow[d, hook] & & U_{\mathcal{C}^0}(\vec{v}) \arrow[d, hook] \\ \mathrm{Z}^r(\mathcal{L}) \arrow[rr, "j^1 \times \mathrm{id}"] \arrow[dr, "\mathrm{ev} \circ (j^1 \times \mathrm{id})"'] & & \mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L}) \arrow[dl, "\mathrm{ev}"] \\ & (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} & \end{tikzcd} \tag{4.2}$$

Our strategy to prove the main theorem rests on the diagram (4.2). Suppose for a moment that ev and $\mathrm{ev} \circ (j^1 \times \mathrm{id})$ are Serre fibrations. Then Theorem 4.1 follows by a Serre spectral sequence argument from proving that the map between the fibers induces a homology isomorphism in a range of degrees. We will indeed show the latter in Section 6, specifically Theorem 6.1. However, although we will see shortly that ev is a fiber bundle (Lemma 4.9), the map $\mathrm{ev} \circ (j^1 \times \mathrm{id})$ is only a Serre microfibration (Lemma 4.10). We recall this notion below, and explain why it is sufficient to carry out the outlined strategy.

4.1. **A micro review of microfibrations.** We start by recalling the definition of a microfibration.

Definition 4.3. A map $E \to B$ of topological spaces is a *Serre microfibration* if for any $k \geq 0$, given a commutative diagram

$$\begin{tikzcd} D^k \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ D^k \times [0,1] \arrow[r] & B \end{tikzcd}$$

there is an $\varepsilon > 0$ and a map $D^k \times [0, \varepsilon] \to E$ making the following diagram commute:

$$\begin{tikzcd} D^k \times \{0\} \arrow[r] \arrow[d] & E \arrow[dd] \\ D^k \times [0,\varepsilon] \arrow[ur] \arrow[d] & \\ D^k \times [0,1] \arrow[r] & B \end{tikzcd}$$

An abundant source of examples comes from the following direct consequence of the definition:

Lemma 4.4. *If $p : E \to B$ is a Serre fibration and $U \subset E$ is open, then $p|_U : U \to B$ is a Serre microfibration.* □

In this paper, wherever we use the terms fibration and microfibration, we mean Serre fibration and Serre microfibration respectively. Contrary to the case of fibrations, the homotopy types of the fibers of a general microfibration can vary. Nonetheless, comparing the total spaces of a microfibration and a fibration can done via a result originally due to Weiss and further generalized by Raptis.

Theorem 4.5 ([Rap17, Theorem 1.3])**.** *Let $p\colon E \to B$ be a Serre microfibration, $q\colon V \to B$ be a Serre fibration, and $f\colon E \to V$ a map over B. Suppose that $f_b\colon p^{-1}(b) \to q^{-1}(b)$ is n-connected for some $n \geq 1$ and for all $b \in B$. Then the map $f\colon E \to V$ is n-connected.*

As we are interested in homology rather than homotopy groups, we will need to slightly adapt Raptis' theorem to our needs. We first introduce some notation.

Definition 4.6. For a map $p\colon E \to B$, its fiberwise (unreduced) kth suspension is defined to be

$$\Sigma^k_B E = \left(E \times [0,1] \times S^{k-1}\right) \Big/ \left((e,0,s) \sim (e,0,s') \text{ and } (e,1,s) \sim (e',1,s) \text{ when } p(e) = p(e')\right).$$

The fiber of the natural map $\Sigma^k_B p : \Sigma^k_B E \to B$ induced by p is the unreduced kth suspension of the fiber of p (here modeled as the join with the sphere S^{k-1}):

$$(\Sigma^k_B p)^{-1}(b) = \Sigma^k p^{-1}(b) \qquad \forall b \in B.$$

Definition 4.7. For a natural number m, a map of topological spaces $A \to B$ is called homology m-connected if it induces an isomorphism on homology groups $H_i(A) \to H_i(B)$ for $i < m$ and a surjection when $i = m$.

Lemma 4.8. *Let $q\colon V \to B$ be a fiber bundle, and $p\colon U \to B$ be the restriction of a fiber bundle $E \to B$ to an open subset $U \subset E$. Let $f : U \to V$ be a map over B and suppose that for every $b \in B$, the restriction to the fiber*

$$f_b\colon p^{-1}(b) \longrightarrow q^{-1}(b)$$

is homology m-connected. Then $f : U \to V$ is homology m-connected.

Proof. See [Aum23, Lemma 5.9]. □

4.2. **Finishing the proof of the main theorem.** As promised, we show that the maps in the diagram (4.2) are respectively a microfibration and a fibration.

Lemma 4.9. *The evaluation map*

$$\begin{aligned} \mathrm{ev}\colon U_{\mathcal{C}^0}(\mathcal{L}) \times \mathrm{F}^r(X) &\longrightarrow (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \\ (f, x_1, \dots, x_r) &\longmapsto (f(x_1), \dots, f(x_r)) \end{aligned}$$

is a fiber bundle. Therefore, so is the pullback $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L}) \to (\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}$.

Proof. We first treat the case $r = 1$ to lighten the notation. Let $(x, v) \in J^1\mathcal{L} - 0$, with $x \in X$ and $0 \neq v \in (J^1\mathcal{L})_x$. Using charts on the manifold X, we choose a small open ball $B(x, 1) \subset \mathbb{R}^{2n} \subset X$ centered at x and of radius 1. Using the local triviality of the jet bundle, we obtain a homeomorphism $(J^1\mathcal{L} - 0)|_U \cong U \times (\mathbb{R}^{2n+2} - 0)$. We choose a small $\varepsilon > 0$ and let $B(v, \varepsilon) \subset \mathbb{R}^{2n+2} - 0$ be a small open ball neighborhood of v. Pick continuous maps

$$\varphi\colon B(v, \varepsilon/4) \longrightarrow \mathrm{Homeo}(B(v, \varepsilon))$$

and

$$\phi\colon B(x, 1/4) \longrightarrow \mathrm{Homeo}(B(x, 1))$$

such that $\varphi(w)$ is a homeomorphism sending v to w and is the identity outside $B(v, \varepsilon/2)$, and $\phi(y)$ is a homeomorphism sending x to y and is the identity outside $B(x, 1/2)$. Slightly abusing notation, we still denote by $\varphi(w)$ and $\phi(y)$ the homeomorphisms of $J^1\mathcal{L} - 0$ and X respectively obtained by extending by the identity. We use them to construct a local trivialization of the evaluation map above the subset $A = B(x, 1/4) \times B(v, \varepsilon/4) \subset J^1\mathcal{L} - 0$ as follows:

$$\begin{aligned} A \times \{s \in U_{\mathcal{C}^0}(\mathcal{L}) \mid s(x) = v\} &\overset{\cong}{\longleftrightarrow} \mathrm{ev}^{-1}(A) = \{(y, s) \mid y \in B(x, 1/4),\ s(y) \in B(v, \varepsilon/4)\} \\ \big((y, w), s\big) &\longmapsto \big(y, \varphi(w) \circ s \circ \phi(y)^{-1}\big) \\ \big(y, s(y), \varphi(s(y))^{-1} \circ s \circ \phi(y)\big) &\longleftarrow \big(y, s\big). \end{aligned}$$

One directly checks that the two given maps are inverse to each other.

We now return to the general case where $r \geq 2$. To construct a local trivialization above a neighborhood of a point $\big((x_1, v_1), \dots, (x_r, v_r)\big) \in (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}$, it suffices to pick neighborhoods of the x_i and apply the argument above at each of them. By choosing the neighborhoods small enough and disjoint, the homeomorphisms constructed as above may be composed to obtain a local trivialisation. □

On the algebraic side, for the jet evaluation map to even be surjective, we need the line bundle $\mathcal{L}$ to have enough sections. As a direct consequence of the definition of jet ampleness (see Definition 2.7), we have the following refinement of [VW15, Lemma 3.2].

Lemma 4.10. *Suppose that $\mathcal{L}$ is $(2r-1)$-jet ample. Then the map*

$$\begin{aligned}\left\{(f, x_1, \ldots, x_r) \in \Gamma_{\mathrm{hol}}(\mathcal{L}) \times \mathrm{F}^r(X) \;\middle|\; \forall i,\ f(x_i) = 0\right\} &\longrightarrow (\Omega^1_X \otimes \mathcal{L})^r|_{\mathrm{F}^r(X)} \\ (f, x_1, \ldots, x_r) &\longmapsto (\mathrm{d}f(x_1), \ldots, \mathrm{d}f(x_r))\end{aligned}$$

is a fiber bundle. The subset $\mathrm{Z}^r(\mathcal{L})$ is open in the domain, hence the restriction

$$\mathrm{Z}^r(\mathcal{L}) \longrightarrow (\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}$$

is a microfibration.

Proof. The map of the lemma is the pullback of

$$\begin{aligned}\Gamma_{\mathrm{hol}}(\mathcal{L}) \times \mathrm{F}^r(X) &\longrightarrow (J^1\mathcal{L})^r|_{\mathrm{F}^r(X)} \\ (f, x_1, \ldots, x_r) &\longmapsto (j^1(f)(x_1), \ldots, j^1(f)(x_r))\end{aligned}$$

along the inclusion $(\Omega^1_X \otimes \mathcal{L})^r|_{\mathrm{F}^r(X)} \hookrightarrow (J^1\mathcal{L})^r|_{\mathrm{F}^r(X)}$. It is then enough to show that this latter map is a fiber bundle. Both the domain and codomain are vector bundles on $\mathrm{F}^r(X)$ and the map is linear in each fiber. By the assumption on the jet ampleness of $\mathcal{L}$ it is also fiberwise surjective, and is therefore an affine bundle. The second part of the lemma follows directly from Lemma 4.4. ☐

Proof of Theorem 4.1. It suffices to apply Lemma 4.8 to the diagram (4.2). Its assumptions are fulfilled by virtue of Lemmas 4.9 and 4.10, and Theorem 6.1. ☐

5. Stability with rational coefficients

In this section, we construct a commutative differential graded algebra (CDGA) modelling the rational homotopy type of $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$. Our construction only depends on known models for configuration spaces and mapping spaces, as well as basic methods from rational homotopy. We first recall the former, and use the latter to deduce Theorem A.

Assumption 5.1. *As we will use rational homotopy theory throughout this section, we furthermore assume that X is a simply connected space. We believe that many arguments could be carried out with X only assumed nilpotent, and that the cohomological results could even hold without any restriction. However, we report such careful work to the future.*

5.1. **Recollections on the rational homotopy of configuration spaces.** Fulton and MacPherson first gave a rational model in the sense of Sullivan for the configuration spaces of points on a smooth projective complex variety in [FM94]. This model was later improved by Kříž [Kri94] and Totaro [Tot96] and we recall its construction here.

Construction 5.2. *Let $r \geq 1$ be a natural number. For any integers $1 \leq a, b \leq r$, $a \neq b$, denote by $\pi_a \colon X^r \to X$ and $\pi_{ab} \colon X^r \to X^2$ the obvious projections. Let C_r be the quotient of the graded commutative algebra*

$$H^*(X^r; \mathbb{Q})[G_{ab}]$$

where the $G_{a,b}$ are generators in degree $2n-1$ for $1 \leq a, b \leq r$, $a \neq b$, modulo the following relations:

$$\begin{aligned} G_{ab} &= G_{ba} \\ (G_{ab})^2 &= 0 \quad \textit{(automatic from graded commutativity)} \\ G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} &= 0 \quad \textit{for } a, b, c \textit{ distinct} \\ \pi_a^*(x)G_{ab} &= \pi_b^*(x)G_{ab} \quad \textit{for } a \neq b, x \in H^*(X; \mathbb{Q}). \end{aligned}$$

Define a differential d on C_r by

$$d(G_{ab}) = \pi_{ab}^*(\Delta)$$

where $\Delta \in H^{2n}(X^2;\mathbb{Q})$ is the class of the diagonal. Then the CDGA (C_r, d) is a rational model for the configuration space $\mathrm{F}^r(X)$.

5.2. **Recollections on the rational homotopy of section spaces.** We recall the results from [Aum22, Section 8.1] concerning the rational cohomology of the continuous section space $\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0)$. The rational homotopy equivalences

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0) \simeq_{\mathbb{Q}} \operatorname{map}(X, K(\mathbb{Q}, 2n+1)) \simeq \prod_{i=0}^{2n+1} K(H^{2n+1-i}(X;\mathbb{Q}), i)$$

imply that

$$H^*(\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0);\mathbb{Q}) \cong \operatorname{Sym}_{\mathrm{gr}}\left(\bigoplus_i H^{2n+1-i}(X;\mathbb{Q})^\vee[i]\right)$$

where $H^*(X;\mathbb{Q})^\vee = \operatorname{Hom}(H^*(X;\mathbb{Q}),\mathbb{Q})$ denotes the dual vector space, and $[i]$ indicates that it is placed in degree i. Using that homology is linearly dual to cohomology and Poincaré duality, we have isomorphisms:

$$H^{*-1}(X) \cong H_{2n+1-*}(X) \cong H^{2n+1-*}(X)^\vee$$

which can be used to rewrite

$$H^*(\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0);\mathbb{Q}) \cong \operatorname{Sym}_{\mathrm{gr}}\left(\bigoplus_i H^{i-1}(X;\mathbb{Q})[i]\right). \tag{5.3}$$

We will need to understand the morphism induced in cohomology by the evaluation map

$$\mathrm{ev}\colon \operatorname{map}(X, K(\mathbb{Q}, 2n+1)) \times X \longrightarrow K(\mathbb{Q}, 2n+1).$$

This is explained by Haefliger in [Hae82, Section 1.2] and we transcribe here his words in our notation. Let $\{b_j\}$ be a homogeneous basis of the graded vector space $H^*(X;\mathbb{Q})$. Let $\{b_j^\vee\}$ be the dual basis under the Poincaré pairing, so that $|b_j^\vee| = 2n - |b_j|$ and $b_i \smile b_j^\vee = \delta_{ij}[X]$ if $|b_i| = |b_j|$. Here δ_{ij} is the Kronecker delta, and $[X] \in H^{2n}(X;\mathbb{Q})$ denotes the top generator in cohomology. Under the isomorphism (5.3), each b_j corresponds to a class of degree $|b_j|+1$ in the cohomology of $\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0)$ which we denote by sb_j (the shifted class).

Lemma 5.4. *Let $\chi \in H^{2n+1}(K(\mathbb{Q}, 2n+1);\mathbb{Q})$ be the canonical generator. The morphism induced in cohomology by the evaluation map*

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0) \times X \simeq_{\mathbb{Q}} \operatorname{map}(X, K(\mathbb{Q}, 2n+1)) \times X \xrightarrow{\mathrm{ev}} K(\mathbb{Q}, 2n+1)$$

sends χ to

$$\sum_j sb_j \otimes b_j^\vee \in H^*(\Gamma_{\mathcal{C}^0}(J^1\mathcal{L}-0);\mathbb{Q}) \otimes H^*(X;\mathbb{Q}).$$

Proof. To avoid cluttering the argument with too much notation, we first explain the general case of a map $f\colon Z \times X \to K(\mathbb{Q}, 2n+1)$ by closely following Haefliger's argument. (We will later take f to be the evaluation map.) By adjunction, f is the same datum as a map $g\colon Z \to \mathrm{map}(X, K(\mathbb{Q}, 2n+1))$. Let g_i be the composition of g with the projection onto the i-th factor:

$$Z \longrightarrow \mathrm{map}(X, K(\mathbb{Q}, 2n+1)) \simeq \prod_{j=0}^{2n+1} K(H^{2n+1-j}(X;\mathbb{Q}), j) \longrightarrow K(H^{2n+1-i}(X;\mathbb{Q}), i).$$

Haefliger then explains that the morphism induced in cohomology is given by

$$\begin{aligned} g_i^*\colon H^i(K(H^{2n+1-i}(X;\mathbb{Q}), i);\mathbb{Q}) \cong H^{2n+1-i}(X;\mathbb{Q})^\vee &\longrightarrow H^i(Z;\mathbb{Q}) \\ a &\longmapsto a \cap f^*(\chi) \end{aligned}$$

where $a \cap (z \otimes x) = a(x)z$ for $a \in H^*(X)^\vee$, $x \in H^*(X)$ and $z \in H^*(Z)$. We now take f to be the evaluation map, hence g to be the identity. Decomposing in the chosen bases, we may a priori write

$$\mathrm{ev}^*(\chi) = \sum_{|b_j|=|b_k|} \lambda_{jk} \cdot (sb_j) \otimes b_k^\vee$$

for some constants $\lambda_{jk} \in \mathbb{Q}$. Using that g is the identity, we have for all $a \in H^{2n+1-i}(X)^\vee \cong H^{i-1}(X)$:

$$a \cap \mathrm{ev}^*(\chi) = sa\,.$$

Varying a through $\{b_j\}$ finishes the proof. □

5.3. **A rational model for the continuous section space with marked points.** We are now ready to construct a CDGA which we will shortly show models the rational homotopy type of the space $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$.

Construction 5.5. *Let $r \geq 1$ be a natural number. We define $A_r(\mathcal{L})$ to be the commutative graded algebra*

$$A_r(\mathcal{L}) = C_r \otimes \mathrm{Sym}_{\mathrm{gr}}\left(\bigoplus_k H^{k-1}(X;\mathbb{Q})[k] \oplus \mathbb{Q}\langle \alpha_1, \ldots, \alpha_r\rangle \oplus \mathbb{Q}\langle \eta_1, \ldots, \eta_r\rangle\right), \tag{5.6}$$

where each α_i is in degree $2n-1$, each η_i is in degree $2n$, and C_r is the rational model of $\mathrm{F}^r(X)$ recalled in Construction 5.2. Let $\pi_i : \mathrm{F}^r(X) \to X$ be the i-th projection. Define

$$\varepsilon_i := \sum_j \pi_i^*(b_j) \otimes sb_j^\vee \in C_r \otimes \mathrm{Sym}_{\mathrm{gr}}\left(\bigoplus_k H^{k-1}(X;\mathbb{Q})[k]\right).$$

which is an element in degree $2n+1$. In fact, by Lemma 5.4, it is the class represented by the composite

$$\Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0) \times \mathrm{F}^r(X) \xrightarrow{\mathrm{id}\times\pi_i} \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0) \times X \simeq_{\mathbb{Q}} \mathrm{map}(X, K(\mathbb{Q}, 2n+1)) \times X \xrightarrow{\mathrm{ev}} K(\mathbb{Q}, 2n+1).$$

We define a differential on $A_r(\mathcal{L})$ by the tensor product of the differential on C_r and the differential on the second tensor factor given by

$$d(\alpha_i) = \pi_i^*(e(\Omega^1_X \otimes \mathcal{L}))$$

and

$$d(\eta_i) = \varepsilon_i - \pi_i^*(c_1(\mathcal{L}))\alpha_i$$

(and the other generators to 0).

Remark 5.7. To lighten the notation, we will often write $H^{*-1}(X;\mathbb{Q}) := \bigoplus_k H^{k-1}(X;\mathbb{Q})[k]$.

Remark 5.8. We can define a $\mathfrak{S}_r$ action on $A_r(\mathcal{L})$ by acting on C_r by permuting coordinates of X^r, trivially on $H^{*-1}(X)$ and by permuting the α_i and η_i. It is clear that the differential defined above is $\mathfrak{S}_r$-equivariant.

Theorem 5.9. *Suppose that X is simply connected. The commutative differential graded algebra $A_r(\mathcal{L})$ of Construction 5.5 is a rational model of $\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L})$. In particular, there is an $\mathfrak{S}_r$-equivariant isomorphism*

$$H^*(\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L});\mathbb{Q}) \cong H^*(A_r(\mathcal{L}))\,.$$

We will prove this theorem in the next section. But first, we collect a few direct computational consequences.

Proposition 5.10. *Let $\mathcal{L}$ be an ample line bundle. Then there exists a $d_0 \geq 1$ such that $e(\Omega^1_X \otimes \mathcal{L}^d) \neq 0$ for all $d \geq d_0$.*

Proof. For an integer d, we compute the Euler class:

$$e(\Omega^1_X \otimes \mathcal{L}^d) = c_n(\Omega^1_X \otimes \mathcal{L}^d) = \sum_{i=0}^{n} c_i(\Omega^1_X)c_1(\mathcal{L}^d)^{n-i} = \sum_{i=0}^{n} c_i(\Omega^1_X)c_1(\mathcal{L})^{n-i}d^{n-i}.$$

Recall, e.g. from the Nakai–Moishezon criterion, that ampleness of $\mathcal{L}$ implies that $c_1(\mathcal{L})^n[X] > 0$. In particular

$$e(\Omega^1_X \otimes \mathcal{L}^d)[X] = (c_1(\mathcal{L})^n[X])d^n + o(d^n)$$

is a polynomial in d of degree n. Thus, there exists a $d_0 \geq 1$ such that this polynomial does not vanish when evaluated at all $d \geq d_0$. □

Remark 5.11. The rational root theorem implies that $d_0 = 1 + |\chi(X)|$ suffices in the proposition. For curves the polynomial is $ad - \chi(X)$ with $a = c_1(\mathcal{L})[X] \geq 1$, so $d_0 = 3$ is sufficient. We are not aware of a bound that is uniform in all X of a given dimension n for general $n > 1$.

We can now state and prove our main stability result:

Proposition 5.12. *Suppose that X is simply connected. Let $\mathcal{L}$ be an ample line bundle, and let d_0 be as in Proposition 5.10. Then for all $d \geq d_0$, the CDGA $A_r(\mathcal{L}^d)$ is isomorphic to $A_r(\mathcal{L}^{d_0})$.*

Proof. The Euler class $e(\Omega^1_X \otimes \mathcal{L}^d)$ is in cohomological degree $2n$ and $H^{2n}(X;\mathbb{Z}) \cong \mathbb{Z}$. Let us write $f(d) = e(\Omega^1_X \otimes \mathcal{L}^d)[X] \in \mathbb{Z}$ for that number. By assumption, we have $f(d) \neq 0$ for all $d \geq d_0$. We construct an explicit morphism:

$$A_r(\mathcal{L}^{d_0}) \longrightarrow A_r(\mathcal{L}^d)$$

given by the identity on the C_r tensor factor and sending the generators accordingly as follows:

$$sb_j \mapsto \frac{d_0 f(d_0)}{df(d)} sb_j, \quad \alpha_i \mapsto \frac{f(d_0)}{f(d)}\alpha_i, \quad \eta_i \mapsto \frac{d_0 f(d_0)}{df(d)}\eta_i.$$

One directly checks that this defines a morphism of CDGAs. Furthermore it is visibly an isomorphism, whose inverse is given by swapping d_0 and d in the formulas above. □

5.4. **Proof of Theorem 5.9.** The proof of Theorem 5.9 relies on recognizing $Z^r_{\mathcal{C}^0}(\mathcal{L})$ as the following pullback:

$$\begin{tikzcd}
Z^r_{\mathcal{C}^0}(\mathcal{L}) \arrow[r] \arrow[d] & F^r(X) \times \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0) \arrow[d, "\mathrm{ev}"] \\
(\Omega^1_X \otimes \mathcal{L} - 0)^{\boxtimes r}|_{F^r(X)} \arrow[r, hook] & (J^1\mathcal{L} - 0)^{\boxtimes r}|_{F^r(X)}
\end{tikzcd}$$

which is also a homotopy pullback, as Lemma 4.9 shows that the right-hand vertical arrow is a fibration. Our strategy is then to apply the Eilenberg–Moore theorem. In order to make explicit computations of rational models, we will need to model the bottom arrow as a cofibration between CDGAs. This in turn will follow from making explicit its Moore–Postnikov tower.

Let us introduce some notation. Write

$$\iota\colon \Omega^1_X \otimes \mathcal{L} - 0 \hookrightarrow J^1\mathcal{L} - 0$$

for the inclusion. We define the product of the two bundles above X as the pullback:

$$\begin{tikzcd}
(\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0) \arrow[r, "p_1"] \arrow[d, "p_2"'] & J^1\mathcal{L} - 0 \arrow[d, "q_1"] \\
\Omega^1_X \otimes \mathcal{L} - 0 \arrow[r, "q_2"'] & X
\end{tikzcd} \tag{5.13}$$

with p_i and q_i the projections. Notice that we also have a map

$$(\mathrm{id}, \iota)\colon \Omega^1_X \otimes \mathcal{L} - 0 \hookrightarrow (\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0)$$

given by the identity on the first factor and the inclusion on the second. Writing $a \in H^{2n-1}(\mathbb{C}^n - 0)$ for the generator, the Serre spectral sequence of the bundle $q_2\colon \Omega^1_X \otimes \mathcal{L} - 0 \to X$ shows that

$$H^{2n+1}(\Omega^1_X \otimes \mathcal{L} - 0; \mathbb{Q}) \cong H^2(X; \mathbb{Q}) \otimes \mathbb{Q}a.$$

Although a does not survive in the spectral sequence if $e(\Omega^1_X \otimes L) \neq 0$, we will write

$$x \cdot a \in H^{2n+1}(\Omega^1_X \otimes \mathcal{L} - 0; \mathbb{Q})$$

for $x \in H^2(X)$ using the isomorphism above. We will also write

$$b \in H^{2n+1}(\mathbb{C}^{n+1} - 0) = H^{2n+1}(J^1\mathcal{L} - 0)$$

for the generator. We will need the following computation:

Lemma 5.14. *In the cohomology group $H^{2n+1}(\Omega^1_X \otimes \mathcal{L} - 0)$, we have the equality $\iota^*(b) = c_1(\mathcal{L}) \cdot a$.*

Proof. The integration along the fibres of q_2, or Gysin map, gives an isomorphism

$$(q_2)_!\colon H^{2n+1}(\Omega^1_X \otimes \mathcal{L} - 0) \xrightarrow{\cong} H^2(X)$$

such that $(q_2)_!(x \cdot a) = x$ in our notation. It thus suffices to check that $(q_2)_!(\iota^*(b)) = c_1(\mathcal{L})$. This follows by functoriality of the Gysin maps and the push-pull formula:

$$\begin{aligned}
(q_2)_!(\iota^*(b)) &= (q_1)_! \circ \iota_!(\iota^*(b)) \\
&= (q_1)_!(b \cup \iota_!(1)) \\
&= (q_1)_!(b \cup (q_1)^*(c_1(\mathcal{L}))) \\
&= c_1(\mathcal{L}),
\end{aligned}$$

where we have used the standard fact that $\iota_!(1)$ is the first Chern class of the normal line bundle of the inclusion $\Omega^1_X \otimes \mathcal{L} - 0 \subset J^1\mathcal{L} - 0$. □

We are now ready to describe explicitly the Moore–Postnikov factorization of the rationalization of ι. Concerning notation, we write $(-)_{\mathbb{Q}}$ for the rationalization of a space.

Lemma 5.15. *There is a tower:*

$$\begin{tikzcd}
K(\mathbb{Q},2n) \arrow[r] & (\Omega^1_X \otimes \mathcal{L} - 0)_{\mathbb{Q}} \arrow[d, "(\mathrm{id},\iota)"] & \\
K(\mathbb{Q},2n-1) \arrow[r] & \big((\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0)\big)_{\mathbb{Q}} \arrow[r, "k_2"] \arrow[d, "p_1"] & K(\mathbb{Q},2n+1) \\
& (J^1\mathcal{L} - 0)_{\mathbb{Q}} \arrow[r, "k_1"] & K(\mathbb{Q},2n)
\end{tikzcd}$$

where the vertical maps are principal fibrations classified by the maps k_1 and k_2. These are given, as cohomology classes, as follows:

$$k_1 = q_1^*(e(\Omega^1_X \otimes \mathcal{L})) \in H^{2n}(J^1\mathcal{L} - 0; \mathbb{Q})$$

and

$$k_2 = p_1^*(b) - p_2^*(c_1(\mathcal{L}) \cdot a) \in H^{2n+1}((\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0); \mathbb{Q}).$$

Proof. Let us first show that the bottom vertical arrow p_1 is a principal fibration classified by k_1. For this, we have two squares:

$$\begin{tikzcd}
(\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0) \arrow[r] \arrow[d, "p_1"] & \Omega^1_X \otimes \mathcal{L} - 0 \arrow[r] \arrow[d] & * \arrow[d] \\
J^1\mathcal{L} - 0 \arrow[r, "q_1"'] & X \arrow[r, "e(\Omega^1_X \otimes \mathcal{L})"'] & K(\mathbb{Q},2n)
\end{tikzcd}$$

The square on the left-hand is a pullback by definition. The square on the right-hand is a pullback after rationalisation. So the outer square is a pullback after rationalisation.

Let us now deal with the second vertical map. First, observe that the composition

$$\Omega^1_X \mathcal{L} - 0 \xrightarrow{(\mathrm{id},\iota)} (\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0) \xrightarrow{k_2} K(\mathbb{Q},2n+1)$$

is null-homotopic. Indeed, this is the content of Lemma 5.14. Thus, if we consider the homotopy pullback square:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & * \arrow[d] \\
\big((\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0)\big)_{\mathbb{Q}} \arrow[r, "k_2"'] & K(\mathbb{Q},2n+1)
\end{tikzcd}$$

we obtain a map $\Omega^1_X \otimes \mathcal{L} - 0 \to P$ by universal property. We claim that this map is a rational cohomology equivalence, hence a rational equivalence. (Here we have used that X is simply connected, hence nilpotent.) By the Eilenberg–Moore theorem applied to the principal fibration classified by k_1, a rational model of $(\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0)$ is given by

$$H^*(X) \otimes \Lambda(x_{2n+1}, y_{2n-1}),\ d(x) = 0,\ d(y) = e(\Omega^1_X \otimes \mathcal{L}).$$

Here the indices on the variables indicate the cohomological degree, and $\Lambda(-)$ is the free graded commutative algebra functor. We have also used that X is formal to use its cohomology as a model. Again using the Eilenberg–Moore theorem, we obtain a model for P of the form

$$H^*(X) \otimes \Lambda(x_{2n+1}, y_{2n-1}, z_{2n}),\ d(x) = 0,\ d(y) = e(\Omega^1_X \otimes \mathcal{L}),\ d(z) = x - c_1(\mathcal{L})y.$$

By Lemma 5.17 below (in the notation of that lemma, z is $x - c_1(\mathcal{L})$ for us and the assumption is verified using Lemma 5.18 below), the cohomology of this CDGA is isomorphic to the cohomology of the following CDGA:

$$\big(H^*(X) \otimes \Lambda(x_{2n+1}, y_{2n-1})\big)/\big(x - c_1(\mathcal{L})\big),\ d(x) = 0,\ d(y) = e(\Omega^1_X \otimes \mathcal{L}),$$

which is also isomorphic to

$$\big(H^*(X) \otimes \Lambda(y_{2n-1})\big),\ d(y) = e(\Omega^1_X \otimes \mathcal{L}),$$

and whose cohomology is $H^*(\Omega^1_X \otimes \mathcal{L} - 0; \mathbb{Q})$. The morphism induced in rational cohomology by the map

$$(\mathrm{id}, \iota)\colon \Omega^1_X \otimes \mathcal{L} - 0 \hookrightarrow (\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0)$$

is surjective. Indeed this can be checked using the Eilenberg–Moore theorem applied to the pullback square (5.13). Therefore, using commutativity of the triangle

$$\begin{tikzcd}
\Omega^1_X \otimes \mathcal{L} - 0 \arrow[rr] \arrow[dr] & & P \arrow[d] \\
& \big((\Omega^1_X \otimes \mathcal{L} - 0) \times_X (J^1\mathcal{L} - 0)\big)_{\mathbb{Q}} &
\end{tikzcd}$$

we see that the morphism on cohomology

$$H^*(P; \mathbb{Q}) \longrightarrow H^*(\Omega^1_X \otimes \mathcal{L} - 0; \mathbb{Q})$$

is surjective. As shown above, both sides of this morphism are abstractly isomorphic rational vector spaces of finite dimension. Hence the morphism must be an isomorphism. ☐

We are now ready to give a rational model of ι in the form of a cofibration of CDGAs (see [FHT01, Chapter 14] for more details on the model structure):

Lemma 5.16. *The map induced by the inclusion*

$$(\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \longrightarrow (J^1\mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}$$

is modelled rationally by the inclusion

$$C_r \otimes \Lambda(x_1, \dots, x_r) \hookrightarrow C_r \otimes \Lambda(x_1, \dots, x_r, y_1, \dots, y_r, z_1, \dots, z_r)$$

where C_r is the model of the configuration space $\mathrm{F}^r(X)$ recalled in Construction 5.2, the classes have degrees $|x_i| = 2n+1$, $|y_i| = 2n-1$, $|z_i| = 2n$, and the differentials are given by $d(x_i) = 0$, $d(y_i) = \pi_i^*(e(\Omega^1_X \otimes \mathcal{L}))$ and $d(z_i) = x_i - \pi_i^*(c_1(\mathcal{L}))y_i$.

Proof. In the case of $r = 1$, this follows directly by applying the Eilenberg–Moore theorem twice on the tower provided by Lemma 5.15. For a general $r \geq 1$, it suffices to take the r-fold product of the case $r = 1$ and restrict to the configuration space $\mathrm{F}^r(X) \subset X^r$. ☐

The proof of our main computational theorem now follows:

Proof of Theorem 5.9. It suffices to apply the Eilenberg–Moore theorem to the pullback diagram

$$\begin{tikzcd}
\mathrm{Z}^r_{\mathcal{C}^0}(\mathcal{L}) \arrow[r] \arrow[d] & F^r(X) \times \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0) \arrow[d, "\mathrm{ev}"] \\
(\Omega^1_X \otimes \mathcal{L} - 0)^{\boxtimes r}|_{F^r(X)} \arrow[r, hook] & (J^1\mathcal{L} - 0)^{\boxtimes r}|_{F^r(X)}
\end{tikzcd}$$

and to note that the model of Lemma 5.16 is a cofibration of CDGAs. The computation of the derived tensor product then follows. □

5.4.1. *Two lemmas in homological algebra.* In the proof above, we have made use of the following two lemmas in homological algebra. These are surely well-known, but we could not track a reference.

For a CDGA (A, d_A), a homogeneous cocycle $z \in A^{i+1}$ (i.e. with $d_A z = 0$) for some $i > 0$ and a free variable x, denote by $(A[x], dx = z)$ (or just $A[x]$ if z is understood) the CDGA $A \otimes \Lambda(x)$ with $|x| = i$ and with differential d determined by $dx = z$ and $da = d_A(a)$ for $a \in A$. Note that d_A descends to a differential on A/zA since $dz = 0$; we will denote this CDGA by simply A/zA. There is a map of CDGAs $A[x] \to A/zA$ taking $x \mapsto 0$. The parity assumptions in the lemmas below are not crucial but are the only cases we need and these assumptions simplify both the statements and the proofs.

Lemma 5.17. *Suppose i is even (so that z is in odd degree) and z is such that $zA = \{a \in A \mid az = 0\}$. Then the map $A[x] \to A/zA$ induces an isomorphism on cohomology.*

Proof. Consider $A[x]$ as a double complex

$$(A[x])^{p,q} = A^{p+(i+1)q}x^{-q}$$

(supported on $q \leq 0$) with p-differential $d_A\colon A^{p+(i+1)q}x^{-q} \to A^{p+1+(i+1)q}x^{-q}$ and q-differential

$$d_z^{p,q}\colon A^{p+(i+1)q}x^{-q} \to A^{p+(i+1)q+(i+1)}x^{-q-1} = A^{p+(i+1)(q+1)}x^{-(q+1)}$$

defined by $d_z^{p,q}(ax^{-q}) = (-1)^{p+q+1}q \cdot azx^{-q-1}$. Then we have the spectral sequence

$$E_1^{p,q} = H^q(A[x]^{p,\bullet}, d_z) \implies H^{p+q}(A[x]).$$

But at any $q < 0$, the differential $d_z^{p,q}$ is, up to a non-zero scalar, multiplication by z. So by the assumption on z we have the exactness $\operatorname{Ker} d_z^{p,q} = \operatorname{Im} d_z^{p,q-1}$ for $q < 0$ and arbitrary p. Therefore E_1 is supported on $q = 0$ with

$$E_1^{p,0} = A^p/zA^p$$

and differential d_1 induced by d_A. So the spectral sequence collapses on E_2 and we get the claimed isomorphism. □

In practical cases, to verify the assumptions of Lemma 5.17 above it is useful to use the following result:

Lemma 5.18. *Suppose i is odd, A is of the form $B[t]$ with $|t| = i$ and $z \in \lambda t + B^i$ for some $\lambda \in \mathbb{Q}^\times$. Then $zA = \{a \in A \mid az = 0\}$.*

Proof. Reduce to the case $\lambda = 1$ without loss of generality, so $z = t + b$ for some $b \in B^i$. Since $|z| = i$ is odd, $z^2 = 0$, so $a \in zA \implies az = 0$. So suppose $za = 0$, and assume a is homogeneous of degree j. Then $a = b_1 t + b_0$ for some $b_0 \in B^j$, $b_1 \in B^{j-i}$. Now

$$0 = az = (b_1 t + b_0)(t + b) = b_1 tb + b_0 t + b_0 b = (b_0 - b_1 b)t + b_0 b \implies b_0 = b_1 b,$$

since $t^2 = 0$ and $tb = (-1)^{i^2} bt = -bt$. But now $a = (b_1 t + b_1 b) = b_1 z \in zA$. □

5.5. **Hodge weights.** Since $\mathrm{Z}^r(\mathcal{L})$ and $U(\mathcal{L})$ are quasiprojective varieties, their rational cohomology are canonically equipped with mixed Hodge structures. In fact, the mixed Hodge structure on the stable cohomology of $H^*(U(\mathcal{L});\mathbb{Q})$ is compatible with its description as $\mathrm{Sym}_{\mathrm{gr}}(H^{*-1}(X)(-1))$. This was shown in [Aum22, Proposition 8.6], and in [DH22, Theorem 1] after passing to the associated weight graded. Concretely, the vector space map

$$\mathrm{Sym}_{\mathrm{gr}}(H^{*-1}(X;\mathbb{Q})(-1)) \longrightarrow \mathrm{Gr}_W H^*(U(\mathcal{L});\mathbb{Q})$$

is an isomorphism of mixed Hodge structures. Here the (-1) denotes a Tate twist, in particular if b_j has weight w then sb_j in the notation of Section 5.2 has weight $w+2$. Similarly, the Kříž–Totaro CDGA, due to its identification with the Leray spectral sequence of the algebraic inclusion $\mathrm{F}^r(X) \hookrightarrow X^r$, has a mixed Hodge structure which is compatible with that on $H^*(\mathrm{F}^r(X))$ after passing to the weight graded. If we identify α_i and η_i with the fundamental classes (with a degree shift in the case of η_i) that they correspond to (i.e. identify $\mathbb{Q}\langle\alpha_i\rangle \cong \mathbb{Q}(-n)$ and $\mathbb{Q}\langle\eta_i\rangle \cong \mathbb{Q}(-n-1)$), we can carry out Construction 5.5 of $A_r(\mathcal{L})$ in the category of mixed Hodge structures.

Conjecture 5.19. *With the mixed Hodge structure defined above, the isomorphisms in Theorem A are isomorphisms of mixed Hodge structure after passing to the weight graded.*

Proving Conjecture 5.19 plausibly involves carrying out the computation in the previous section in an appropriate category of CDGAs equipped with mixed Hodge structures. However, such a lift of the computation seems far from straightforward using available technology and is at least beyond the scope of this project. Instead, we strongly believe that if we assign weights to the generators of A_r as predicted by Conjecture 5.19 then we obtain the correct weights on $H^*(\mathrm{Z}^r(\mathcal{L}))$. The following is stated as a conjecture, with an idea of a proof, because a fully detailed proof seemed to involve too much bookkeeping.

Conjecture 5.20. *With the above weight grading on $A_r(\mathcal{L})$, which makes the differentials weight preserving, the isomorphisms in Theorem A are weight preserving.*

Idea of a proof. Consider the Leray spectral sequence of the composition of the jet map and the projection

$$\pi : \mathrm{Z}^r(\mathcal{L}) \longrightarrow (\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}.$$

Since this map is algebraic, the spectral sequence is of mixed Hodge structures, and computes the correct weights in cohomology. But π and any restriction $\pi|\pi^{-1}(V)$ for V open in the codomain is also a microfibration. By Lemma 4.8 and Theorem 6.1, we may compare the Leray sheaves of π and π_{C^0} in the diagram

$$\begin{tikzcd}
\mathrm{Z}^r(\mathcal{L}) \arrow[r, "j^1"] \arrow[d, "\pi"] & \mathrm{F}^r_{C^0}(\mathcal{L}) \arrow[d, "\pi_{C^0}"] \\
(\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)} \arrow[r, equal] & (\Omega^1_X \otimes \mathcal{L} - 0)^r|_{\mathrm{F}^r(X)}
\end{tikzcd}$$

and obtain that $R^q\pi_*\mathbb{Q}$ is locally constant in the stable range. Therefore the Leray spectral sequence for π agrees with the Leray–Serre spectral sequence for π_{C^0}, starting from the E_2 page. It should now be possible to compute the differentials in this Serre spectral sequence using rational models, following Grivel [Gri79]. In essence, he shows how the Serre spectral sequence arises from a relative Sullivan extension modelling the fibration by filtering the base CDGA by the degree. It remains however to relate $A_r(\mathcal{L})$ to such a relative Sullivan extension. □

If Z is a complex quasiprojective variety, its Euler characteristic $\chi(Z)$ has a lift

$$\chi_{\mathrm{HS}}(Z) := \sum_i (-1)^i [H^i(Z)]\,,$$

taking values in the Grothendieck ring $K_0(\mathrm{MHS})$ of mixed Hodge structures. If Z further has an algebraic action by a group G then for any (finite-dimensional) G-representation V over $\mathbb{Q}$ we can define

$$\chi_{\mathrm{HS}}(Z;V) := \sum_i (-1)^i [H^i(Z) \otimes_G V]\,.$$

By Deligne's bounds on weights, cohomology classes of a given weight only occur in a fixed range of degrees. Thus Conjecture 5.19 would imply:

Theorem 5.21 ([How19, Theorem A]). *If $\mathcal{L}$ is very ample and V is a finite-dimensional $\mathfrak{S}_r$ representation, then $\chi_{HS}(F_r(\mathcal{L}^{\otimes d}); V)$ stabilizes weightwise to $\chi_{HS}(A_r(\mathcal{L}) \otimes_{\mathfrak{S}_r} V)$ as $d \to \infty$.*

Taking the weightwise Euler characteristic of Conjecture 5.20, which is well-defined since a given weight w can only appear in cohomology in degrees $* \leq w$, we recover a version of Theorem 5.21. In [How19] Howe also shows analogous stabilization results for arithmetic statistics of $\mathrm{F}^r(\mathcal{L}^{\otimes d})$ and related spaces. However these quantities a priori have contributions from cohomology in all degrees simultaneously (just like usual Euler characteristic) so our results do not concretely imply anything about them.

6. The h-principle

Let $(x_1, \ldots, x_r) \in \mathrm{F}^r(X)$ be r distinct points and $v_i \in (J^1\mathcal{L} - 0)|_{x_i}$ be non-zero vectors at these points. To lighten the notation, we will abbreviate the whole tuple by

$$\vec{v} := \big((x_1, v_1), \ldots, (x_r, v_r)\big) \in (J^1\mathcal{L} - 0)^r\,.$$

The space of holomorphic sections with derivatives prescribed by $\vec{v}$ is defined to be:

$$\Gamma_{\mathrm{hol}}(\vec{v}) := \big\{ f \in \Gamma_{\mathrm{hol}}(\mathcal{L}) \;\big|\; j^1(f)(x_i) = (x_i, v_i) \text{ for each } i = 1, \ldots, r \big\}\,.$$

This is an affine subspace of the vector space of global sections $\Gamma_{\mathrm{hol}}(\mathcal{L})$ and contains as an open subspace the previously defined

$$U(\vec{v}) := \Gamma_{\mathrm{hol}}(\vec{x}) \cap U(\mathcal{L})\,.$$

Similarly, we have an analogous space defined by continuous sections of $J^1\mathcal{L}$ with prescribed values at the x_i:

$$U_{\mathcal{C}^0}(\vec{v}) := \big\{ s \in \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}) \;\big|\; s(x_i) = (x_i, v_i) \text{ for each } i = 1, \ldots, r \big\} \subset \Gamma_{\mathcal{C}^0}(J^1\mathcal{L} - 0)\,.$$

The jet expansion j^1 restricts to these subspaces and the goal of this section is to prove the following result:

Theorem 6.1. *For a d-jet ample line bundle $\mathcal{L}$, the jet map*

$$j^1 : U(\vec{v}) \longrightarrow U_{\mathcal{C}^0}(\vec{v})$$

induces an isomorphism in integral homology in the range of degrees $* < \frac{d-1}{2} - r$.

The proof follows that of [Aum22] but we could not find a direct way of applying the theorem therein. Indeed, there are two main differences. Firstly, we apply a Vassiliev-style argument to an *affine subspace* $\Gamma_{\mathrm{hol}}(\vec{v}) \subset \Gamma_{\mathrm{hol}}(\mathcal{L})$ of the vector space of global sections. This is in contrast with [Aum22] where the full space of global sections is considered. Secondly, a section $f \in \Gamma_{\mathrm{hol}}(\vec{v})$ can only be singular at points in $X - \{x_1, \ldots, x_r\}$ because it has non-vanishing derivatives at the special chosen points x_i. As the proofs of [Aum22, Section 3] relied on compactness of X which we lose when removing points, we will need to adapt them to our case.

6.1. **Constructing the Vassiliev spectral sequence.** The space $U(\vec{v})$ is an open subset in the complex affine space $\Gamma_{\mathrm{hol}}(\vec{v})$ of complex dimension $\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\vec{v})$, whose complement we denote by $\Sigma(\vec{v})$. We topologize them using the canonical topology on the ambient complex affine space. By Alexander duality, there is an isomorphism

$$\check{H}^i_c(\Sigma(\vec{v})) \cong \widetilde{H}_{2\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\vec{v})-1-i}(U(\vec{v})).$$

We want to compute the homology of $U(\vec{v})$, but we will equivalently study the compactly supported Čech cohomology of its complement. This is technically advantageous: the complement admits a filtration by the number of singularities of a section $f \in \Sigma(\vec{v})$, which allows the construction of a spectral sequence à la Vassiliev. In practice, it is easier to work with an auxiliary filtered space mapping properly down to $\Sigma(\vec{v})$ with acyclic fibers. We construct this space and its associated spectral sequence below.

We write $X^\circ := X - \{x_1, \ldots, x_r\}$ to denote the punctured space. Define

$$\mathrm{Sing}(f) := \big\{y \in X^\circ \mid f \text{ is singular at } y\big\} \subset X^\circ$$

to be the singular subspace of a section $f \in \Sigma(\vec{v})$. Let F be the category whose objects are the finite sets $[n] := \{0, \ldots, n\}$ for $n \geq 0$ and whose morphisms are *all* maps of sets $[n] \to [m]$. Let Top be the category of topological spaces and continuous maps between them. On objects, define the following functor:

$$\begin{aligned} \mathfrak{X} \colon \mathsf{F}^{\mathrm{op}} &\longrightarrow \mathsf{Top} \\ [n] &\longmapsto \mathfrak{X}[n] := \{(f, y_0, \ldots, y_n) \in \Gamma_{\mathrm{hol}}(\vec{v}) \times (X^\circ)^{n+1} \mid \forall i,\ y_i \in \mathrm{Sing}(f)\} \end{aligned}$$

where $\mathfrak{X}[n]$ is given the subspace topology from $\Gamma_{\mathrm{hol}}(\vec{v}) \times (X^\circ)^{n+1}$. On morphisms, for a map of sets $g \colon [n] \to [m]$, we define it by:

$$\begin{aligned} \mathfrak{X}(g) \colon \mathfrak{X}[m] &\longrightarrow \mathfrak{X}[n] \\ (f, y_0, \ldots, y_m) &\longmapsto (f, y_{g(0)}, \ldots, y_{g(n)}). \end{aligned}$$

For an integer $k \geq 0$, let $\mathsf{F}_{\leq k}$ be the full sub-category of F on objects $[n]$ for $n \leq k$. Write

$$|\Delta^n| = \{(t_0, \ldots, t_n) \mid \forall i, 0 \leq t_i \leq 1 \text{ and } t_0 + \cdots + t_n = 1\} \subset \mathbb{R}^{n+1}$$

for the standard topological n-simplex, and denote by $\partial|\Delta^n|$ its boundary. In particular, the assignment $[n] \mapsto |\Delta^n|$ gives a functor $\mathsf{F} \to \mathsf{Top}$. For an integer $j \geq 0$, we define the *j-th geometric realization of* $\mathfrak{X}$ by the following coend:

$$R^j\mathfrak{X} := \int^{[n]\in \mathsf{F}_{\leq j}} \mathfrak{X}[n] \times |\Delta^n| = \left(\bigsqcup_{0\leq n\leq j} \mathfrak{X}[n] \times |\Delta^n| \right) / \sim$$

where the equivalence relation $\sim$ is generated by $(\mathfrak{X}(g)(z), t) \sim (z, g_*(t))$ for all maps $g\colon [n] \to [m]$ in F. Here $g_*\colon |\Delta^n| \to |\Delta^m|$ denotes the usual map induced on the simplices by functoriality. Note that the main difference between our construction and the geometric realisation of a simplicial space resides in the fact that we allow all maps of sets, in particular the permutations $[n] \to [n]$ are morphisms in F.

As for simplicial spaces, $R^j\mathfrak{X}$ is obtained from $R^{j-1}\mathfrak{X}$ via a pushout diagram along a subspace, which can be thought of as a kind of latching object. More precisely, define

$$L_j := \{(f, y_0, \ldots, y_j) \in \Gamma_{\text{hol}}(\vec{v}) \times (X^\circ)^{j+1} \mid \exists l \neq k \text{ such that } y_l = y_k\} \subset \mathfrak{X}[j]$$

topologized as a subspace of $\mathfrak{X}[j]$, and write $L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|$ for the quotient space of $L_j \times |\Delta^j|$ by the symmetric group $\mathfrak{S}_{j+1}$ acting on L_j by permuting the singularities y_i, and on $|\Delta^j|$ by permuting the coordinates. The following result is immediate from the definitions.

Lemma 6.2 ([Aum22, Proposition 3.3]). *There is a pushout square of topological spaces:*

$$\begin{tikzcd} \left(L_j \times_{\mathfrak{S}_{j+1}} |\Delta^j|\right) \bigcup \left(\mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} \partial|\Delta^j|\right) \arrow[r] \arrow[d, hook] & R^{j-1}\mathfrak{X} \arrow[d] \\ \mathfrak{X}[j] \times_{\mathfrak{S}_{j+1}} |\Delta^j| \arrow[r] & R^j\mathfrak{X} \end{tikzcd}$$

where the left vertical map is a closed embedding. ☐

Using the lemma inductively, one sees that the spaces $R^j\mathfrak{X}$ are paracompact and Hausdorff, and that the natural map $R^{j-1}\mathfrak{X} \to R^j\mathfrak{X}$ is a closed embedding. Another direct consequence is the following homeomorphism:

$$R^j\mathfrak{X} - R^{j-1}\mathfrak{X} \cong Y_j \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j| \tag{6.3}$$

where

$$Y_j := \left\{(f, y_0, \ldots, y_j) \in \mathfrak{X}[j] \mid y_l \neq y_k \text{ if } l \neq k\right\} = \mathfrak{X}[j] - L_j \subset \mathfrak{X}[j]$$

is the subspace of $\mathfrak{X}[j]$ where the singularities are pairwise distinct, and $|\mathring{\Delta}^j|$ is the interior of the simplex. The next lemma is stated in [Aum22, Lemma 3.5] but its proof needs to be adapted here to account for the fact that X° is not compact.

Lemma 6.4. *For any $n \geq 0$, the map $\rho_n\colon \mathfrak{X}[n] \to \Gamma_{\text{hol}}(\vec{v})$ given by $(f, y_0, \ldots, y_n) \mapsto f$ is a proper map.*

Proof. Let $K \subset \Gamma_{\text{hol}}(\vec{v})$ be compact. Then

$$\left\{(f, x) \in K \times X \;\middle|\; x \notin \operatorname{Sing}(f)\right\}$$

is open in $K \times X$ and contains $K \times \{x_i\}$ for $i = 1, \ldots, r$. By the tube lemma, it must contain some $K \times V$, where $V \subset X$ is a neighborhood of $\{x_1, \ldots, x_r\}$. Then $\rho_n^{-1}(K)$ is a closed subset of $K \times (X - V)^{n+1}$ and hence is compact. ☐

The natural projections maps $\mathfrak{X}[n] \times |\Delta^n| \to \mathfrak{X}[n] \xrightarrow{\rho_n} \Gamma_{\text{hol}}(\vec{v})$ give rise to a map from the geometric realization $\tau_j\colon R^j\mathfrak{X} \to \Sigma(\vec{v})$. The proof of [Aum22, Lemma 3.6] using the adapted Lemma 6.4 above then shows that:

Lemma 6.5. *For any integer $j \geq 0$, the map $\tau_j\colon R^j\mathfrak{X} \to \Sigma(\vec{v})$ is proper.* ☐

We are now half-way through the construction of a replacement of $\Sigma(\vec{v})$: the space $R^j\mathfrak{X}$ maps properly down to it, but the fibers are not all acyclic. Indeed, the fiber $\tau_j^{-1}(f)$ above a section $f \in \Sigma(\vec{v})$ that has at most $j+1$ singularities is a (possibly degenerate) j-simplex whose vertices are indexed by the singular points of f. Hence its homology vanishes in positive degrees. On the contrary, if f has at least $j+2$ singularities the fiber is not contractible nor acyclic in general. This issue is fixed by the following construction. Let $N \geq 0$ be an integer and let

$$\Sigma(\vec{v})_{\geq N+2} := \left\{ f \in \Gamma_{\mathrm{hol}}(\vec{v}) \mid \#\mathrm{Sing}(f) \geq N+2 \right\} \subset \Sigma(\vec{v})$$

be the subspace of those sections with at least $N+2$ singular points. We denote by $\overline{\Sigma(\vec{v})_{\geq N+2}}$ its closure in $\Sigma(\vec{v})$. Define $R^N_{\mathrm{cone}}\mathfrak{X}$ by the following homotopy pushout:

$$\begin{tikzcd} \tau_N^{-1}\left(\overline{\Sigma_{\geq N+2}}\right) \arrow[r, hook] \arrow[d, "\tau_N"'] & R^N\mathfrak{X} \arrow[d] \\ \overline{\Sigma_{\geq N+2}} \arrow[r] \arrow[ur, phantom, "\mathrm{ho}\ulcorner"] & R^N_{\mathrm{cone}}\mathfrak{X}. \end{tikzcd}$$

The three other spaces map to $\Sigma(\vec{v})$, thus yielding a map from the homotopy pushout $\pi\colon R^N_{\mathrm{cone}}\mathfrak{X} \to \Sigma(\vec{v})$.

Proposition 6.6 ([Aum22, Lemma 3.8, Lemma 3.9, Proposition 3.10])**.** *The space $R^N_{cone}\mathfrak{X}$ is paracompact, locally compact and Hausdorff. The map $\pi\colon R^N_{cone}\mathfrak{X} \to \Sigma(\vec{v})$ is proper and induces an isomorphism in cohomology with compact supports.* □

Using the closed embeddings $R^{j-1}\mathfrak{X} \subset R^j\mathfrak{X}$ obtained in Lemma 6.2, we define the following filtration on $R^N_{\mathrm{cone}}\mathfrak{X}$:

$$F_0 = R^0\mathfrak{X} \subset F_1 = R^1\mathfrak{X} \subset \cdots \subset F_N = R^N\mathfrak{X} \subset F_{N+1} = R^N_{\mathrm{cone}}\mathfrak{X}.$$

By standard arguments on spectral sequences associated to filtered complexes, we obtain:

Proposition 6.7 ([Aum22, Proposition 3.11])**.** *There is a spectral sequence on the first quadrant $s, t \geq 0$:*

$$E_1^{s,t} = \check{H}_c^{s+t}(F_s - F_{s-1}; \mathbb{Z}) \implies \check{H}_c^{s+t}(R^N_{cone}\mathfrak{X}; \mathbb{Z}) \cong \widetilde{H}_{2\dim_{\mathbb{C}}\Gamma_{\mathrm{hol}}(\vec{v})-1-s-t}(U(\vec{v}); \mathbb{Z}).$$

The differential d^r on the r-th page of the spectral sequence has bi-degree $(r, 1-r)$, i.e. it is a morphism $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$. □

6.2. **Analyzing the spectral sequence.** In this section, we describe the terms on the E_1-page of the spectral sequence given in Proposition 6.7. The constructions of the previous section and the spectral sequence depend on an integer N that we are a priori free to choose. We will follow the following convention for the remainder of this article:

Convention 6.8. *Let N be the largest integer such that $\mathcal{L}$ is $(2(r+N+1)-1)$-jet ample.*

We will consider two cases separately:

(1) When $0 \leq s \leq N$, we have $F_s - F_{s-1} = R^s\mathfrak{X} - R^{s-1}\mathfrak{X}$ whose cohomology can be understood using the homeomorphism (6.3).
(2) When $s = N+1$, we have $F_s - F_{s-1} = R^N_{\mathrm{cone}}\mathfrak{X} - R^N\mathfrak{X}$ whose cohomology we will bound.

Note that outside the band $0 \leq s \leq N+1$, all the terms $E_1^{s,t}$ vanish as the filtration giving rise to the spectral sequence is indexed from 0 to $N+1$.

6.2.1. *Cohomology in the columns* $0 \leq s \leq N$. We first treat the cases where $0 \leq s \leq N$. Let us recall the homeomorphism (6.3):

$$F_s - F_{s-1} = R^s\mathfrak{X} - R^{s-1}\mathfrak{X} \cong \left\{(f, y_0, \ldots, y_s) \in \mathfrak{X}[s] \mid y_l \neq y_k \text{ if } l \neq k\right\} \times_{\mathfrak{S}_{s+1}} |\mathring{\Delta}^s|$$

where the symmetric group $\mathfrak{S}_{s+1}$ acts on the left by permuting the singular points y_i and on the right by permuting the vertices of the simplex. There is a natural map down to the configuration space $\mathrm{F}^{s+1}(X^\circ)$ which forgets the simplex and the section f. An argument similar to that of [Aum22, Section 4.1] then shows that this map is a fiber bundle.

Lemma 6.9. *For* $0 \leq s \leq N$*, the natural map*

$$\begin{aligned} F_s - F_{s-1} \cong \left\{(f, y_0, \ldots, y_s) \in \mathfrak{X}[s] \mid y_l \neq y_k \textit{ if } l \neq k\right\} \times_{\mathfrak{S}_{s+1}} |\mathring{\Delta}^s| &\longrightarrow \mathrm{F}^{s+1}(X^\circ) \\ [(f, y_0, \ldots, y_s), \lambda] &\longmapsto \{y_0, \ldots, y_s\} \end{aligned}$$

is a fiber bundle. The fiber above a point $\vec{y} = \{y_0, \ldots, y_s\} \in \mathrm{F}^{s+1}(X^\circ)$ *is the product*

$$|\mathring{\Delta}^s| \times \left\{f \in \Gamma_{\mathrm{hol}}(\vec{v}) \mid \vec{y} \subset \mathrm{Sing}(f)\right\},$$

where the right-hand term is the complex affine subspace of $\Gamma_{\mathrm{hol}}(\vec{v})$ *of those sections having all points in* $\vec{y}$ *as singularities. It is of complex dimension* $\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\vec{v}) - (s+1)(\dim_{\mathbb{C}} X + 1)$.

Proof sketch. The main point is to see that the affine spaces $\left\{f \in \Gamma_{\mathrm{hol}}(\vec{v}) \mid \vec{y} \subset \mathrm{Sing}(f)\right\}$ all have the same dimension regardless of $\vec{y} \in \mathrm{F}^{s+1}(X^\circ)$. They are given by $(s+1)(\dim_{\mathbb{C}} X + 1)$ equations: a section f is singular at y_i when both the value $f(y_i)$ and all the partial derivatives of f at y_i vanish. In total, this imposes $1 + \dim_{\mathbb{C}} X$ equations on f. The lemma is then proven if these equations are linearly independent. The line bundle $\mathcal{L}$ is $(2(r+s+1)-1)$-jet ample by Convention 6.8, so the evaluation map

$$\Gamma_{\mathrm{hol}}(\mathcal{L}) \longrightarrow \bigoplus_{i=1}^{r} (J^1\mathcal{L})|_{x_i} \oplus \bigoplus_{j=0}^{s} (J^1\mathcal{L})|_{y_i}$$

is surjective. This directly implies that the considered equations are linearly independent. □

Remark 6.10. The proof above follows very closely the one given for [Aum22, Lemma 4.2], except for the fact that we also impose derivatives at the fixed x_i to get the existence of sections in $\Gamma_{\mathrm{hol}}(\vec{v})$. This is reflected by the appearance of the constant r in the required jet ampleness of $\mathcal{L}$.

Applying the Thom isomorphism for cohomology with compact supports yields:

Lemma 6.11 (Compare [Aum22, Proposition 4.3])**.** *For* $0 \leq s \leq N$*, we have an isomorphism*

$$E_1^{s,t} \cong \check{H}_c^{t-2\dim_{\mathbb{C}} \Gamma_{\mathrm{hol}}(\vec{v}) + 2(s+1)(\dim_{\mathbb{C}} X + 1)}(\mathrm{F}^{s+1}(X^\circ); \mathbb{Z}^{\mathrm{sign}})$$

where $\mathbb{Z}^{\mathrm{sign}}$ *is the local coefficients system on the configuration space given by the sign representation.* □

6.2.2. *Cohomology in the column $s = N + 1$.* We now turn our attention to the last column on the first page of the spectral sequence. In this case $s = N + 1$ and the groups are

$$E_1^{N+1,t} = \check{H}_c^{N+1+t}(R^N_{\text{cone}}\mathfrak{X} - R^N\mathfrak{X}; \mathbb{Z}).$$

We shall show that these group vanish for t big enough. More precisely, following [Aum22, Section 4.2] we obtain:

Lemma 6.12 (Compare [Aum22, Proposition 4.9]). *For $t > 2\dim_{\mathbb{C}} \Gamma_{\text{hol}}(\vec{v}) - 2N - 2$ we have*

$$E_1^{N+1,t} = \check{H}_c^{N+1+t}(R^N_{\mathit{cone}}\mathfrak{X} - R^N\mathfrak{X}; \mathbb{Z}) = 0.$$

Proof sketch. The proof of [Aum22, Proposition 4.9] actually applies to this situation, but we sketch the main ideas in the particular case at hand. By construction, the space $R^N_{\text{cone}}\mathfrak{X} - R^N\mathfrak{X}$ is the fiberwise (for the map $\tau_N \colon R^N\mathfrak{X} \to \Sigma(\vec{v})$) open cone over $\overline{\Sigma(\vec{v})_{\geq N+2}}$. It can be stratified by

$$\begin{aligned}
\mathrm{Str}_{-1} &:= \overline{\Sigma(\vec{v})_{\geq N+2}},\\
\mathrm{Str}_0 &:= \left(\tau_N^{-1}\left(\overline{\Sigma(\vec{v})_{\geq N+2}}\right) \times]0,1[\right) \cap \left(R^0\mathfrak{X} \times]0,1[\right),\\
\mathrm{Str}_j &:= \left(\tau_N^{-1}\left(\overline{\Sigma(\vec{v})_{\geq N+2}}\right) \times]0,1[\right) \cap \left((R^j\mathfrak{X} - R^{j-1}\mathfrak{X}) \times]0,1[\right) \qquad \text{for } 1 \leq j \leq N.
\end{aligned}$$

Furthermore, the homeomorphism (6.3) shows that for $0 \leq j \leq N$ we have a homeomorphism

$$\mathrm{Str}_j \cong \left(Y_j^{\geq N+2} \times_{\mathfrak{S}_{j+1}} |\mathring{\Delta}^j|\right) \times]0,1[$$

where

$$Y_j^{\geq N+2} := \left\{(f, y_0, \dots, y_j) \in \Gamma_{\text{hol}}(\vec{v}) \times \mathrm{F}^{j+1}(X^\circ) \mid f \in \overline{\Sigma(\vec{v})_{\geq N+2}} \text{ and } y_i \in \mathrm{Sing}(f)\right\}.$$

One sees that this latter space $Y_j^{\geq N+2}$ is a real semi-algebraic set and that the natural forgetful map

$$\left\{(f, y_0, \dots, y_N) \in \Gamma_{\text{hol}}(\vec{v}) \times \mathrm{F}^{N+1}(X^\circ) \mid y_i \in \mathrm{Sing}(f)\right\} \longrightarrow Y_j^{\geq N+2}$$

is algebraic and has dense image. This implies that the dimension of $Y_j^{\geq N+2}$ is at most that of the space on the left-hand side. One computes it to be at most $2\dim_{\mathbb{C}} \Gamma_{\text{hol}}(\vec{v}) - 2(N+1)$ (see [Aum22, Lemma 4.8]), implying that all the strata have dimension at most $2\dim_{\mathbb{C}} \Gamma_{\text{hol}}(\vec{v}) - N - 1$. Therefore the compactly supported cohomology of their union vanishes above this dimension, i.e. $\check{H}_c^{N+1+t}(R^N_{\text{cone}}\mathfrak{X} - R^N\mathfrak{X}; \mathbb{Z}) = 0$ whenever $N + 1 + t > 2\dim_{\mathbb{C}} \Gamma_{\text{hol}}(\vec{v}) - N - 1$. □

6.3. **From holomorphic to continuous sections.** In the previous sections, we have constructed a spectral sequence converging to the homology of $U(\vec{v})$ and have described some features of its first page. We would like to do the same for $U_{\mathcal{C}^0}(\vec{v})$ as well as provide a morphism of spectral sequences that is an isomorphism in a range on the first page. But the space of continuous sections of $J^1\mathcal{L}$ is not finite dimensional, hence Alexander duality cannot be applied directly. This problem can be remedied by introducing a growing filtration

$$U(\vec{v}) \xrightarrow{j^1} U_0(\vec{v}) := \Gamma_{\text{hol}}(J^1\mathcal{L} - 0) \longrightarrow U_1(\vec{v}) \longrightarrow \cdots \longrightarrow \operatorname*{colim}_{k\to\infty} U_k(\vec{v}) \xrightarrow{\simeq} U_{\mathcal{C}^0}(\vec{v})$$

where every map $U_k(\vec{v}) \to U_{k+1}(\vec{v})$ is shown to be a homology isomorphism in a range using a spectral sequence similar to the one above, and the colimit of the $U_k(\vec{v})$ is homotopy equivalent to $U_{\mathcal{C}^0}(\vec{v})$.

6.3.1. *Definition of the filtration.* We follow [Aum22, Section 5] to describe roughly how the spaces $U_k(\vec{v})$ are constructed, but refer to that article for the full details. The main idea is to consider the complex conjugate (or equivalently the dual) line bundle $\overline{\mathcal{L}}$ of $\mathcal{L}$. Taking the complex conjugates of the values of a section gives an $\mathbb{R}$-linear morphism:

$$\overline{\cdot}\colon \Gamma_{\mathcal{C}^0}(\mathcal{L}) \longrightarrow \Gamma_{\mathcal{C}^0}(\overline{\mathcal{L}}).$$

For a complex vector space V, we will denote by $\overline{V}$ the complex vector space with the same underlying abelian group, but where the $\mathbb{C}$-module structure is given by multiplication by the complex conjugate. Complex conjugation thus gives a $\mathbb{C}$-linear morphism $\overline{\Gamma_{\mathrm{hol}}(\mathcal{L})} \to \Gamma_{\mathcal{C}^0}(\overline{\mathcal{L}})$. As the tensor product $\mathcal{L} \otimes \overline{\mathcal{L}}$ is the trivial line bundle $X \times \mathbb{C}$ one can consider the multiplication map:

$$\Gamma_{\mathrm{hol}}(\mathcal{L}) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L})} \subset \Gamma_{\mathcal{C}^0}(\mathcal{L}) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathcal{C}^0}(\mathcal{L})} \longrightarrow \Gamma_{\mathcal{C}^0}(\mathcal{L} \otimes \overline{\mathcal{L}}) \cong \Gamma_{\mathcal{C}^0}(X \times \mathbb{C}). \tag{6.13}$$

Likewise, for any integer $k \geq 0$, one gets a multiplication map

$$\mu_k\colon \Gamma_{\mathrm{hol}}((J^1\mathcal{L}) \otimes \mathcal{L}^k) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^k)} \longrightarrow \Gamma_{\mathcal{C}^0}((J^1\mathcal{L}) \otimes \mathcal{L}^k \otimes \overline{\mathcal{L}^k}) \cong \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}).$$

Definition 6.14. For any integer $k \geq 0$, we define

$$\Gamma_k(\vec{v}) := \mu_k^{-1}(\Gamma_{\mathcal{C}^0}(\vec{v}))$$

and

$$U_k(\vec{v}) := \mu_k^{-1}(U_{\mathcal{C}^0}(\vec{v})).$$

Importantly for us, the space $\Gamma_k(\vec{v})$ is an affine subspace of the finite dimensional complex vector space $\Gamma_{\mathrm{hol}}((J^1\mathcal{L}) \otimes \mathcal{L}^k) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^k)}$, hence is itself finite dimensional. We now describe the maps $U_k(\vec{v}) \to U_{k+1}(\vec{v})$. Using the triviality of $\mathcal{L} \otimes \overline{\mathcal{L}}$, we can choose an element

$$\eta \in \Gamma_{\mathrm{hol}}(\mathcal{L}) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L})}$$

corresponding to the constant function with value 1 under the multiplication map (6.13). Multiplying sections by this element gives a commutative square:

$$\begin{tikzcd}
\Gamma_{\mathrm{hol}}((J^1\mathcal{L}) \otimes \mathcal{L}^k) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^k)} \arrow[r, "\mu_k"] \arrow[d, "\cdot\eta"'] & \Gamma_{\mathcal{C}^0}(J^1\mathcal{L}) \arrow[d, equal] \\
\Gamma_{\mathrm{hol}}((J^1\mathcal{L}) \otimes \mathcal{L}^{k+1}) \otimes_{\mathbb{C}} \overline{\Gamma_{\mathrm{hol}}(\mathcal{L}^{k+1})} \arrow[r, "\mu_{k+1}"'] & \Gamma_{\mathcal{C}^0}(J^1\mathcal{L})
\end{tikzcd}$$

This commutativity readily implies that the left vertical map restricts to a map

$$\cdot\eta\colon U_k(\vec{v}) \longrightarrow U_{k+1}(\vec{v}).$$

6.3.2. *Comparing spectral sequences.* As explained in [Aum22, Section 5.4], the construction of the spectral sequence and the analysis of its first page can be carried out for the spaces $U_k(\vec{v}) \subset \Gamma_k(\vec{v})$. We summarize the results here:

Proposition 6.15 (Compare [Aum22, Proposition 5.5])**.** *Let N be as in Convention 6.8. For any integer $k \geq 0$, there is a cohomologically-indexed spectral sequence supported on the strip $0 \leq s \leq N+1$ and $t \geq 0$:*

$$E_1^{s,t} \implies \widetilde{H}_{2\dim_{\mathbb{C}} \Gamma_k(\vec{v})-1-s-t}(U_k(\vec{v}); \mathbb{Z}).$$

When $0 \leq s \leq N$, there is an isomorphism

$$E_1^{s,t} \cong \check{H}_c^{t-2\dim_{\mathbb{C}} \Gamma_k(\vec{v})+2(s+1)(\dim_{\mathbb{C}} X+1)}(\mathrm{F}^{s+1}(X^\circ); \mathbb{Z}^{\mathrm{sign}})$$

where $\mathbb{Z}^{\mathrm{sign}}$ is the local coefficients system on the configuration space given by the sign representation. And when $s = N+1$ and $t > 2\dim_{\mathbb{C}} \Gamma_k(\vec{v}) - 2N - 2$ we have

$$E_1^{N+1,t} = 0. \qquad \square$$

For any $k \geq 0$ the groups on the first page in the range $0 \leq s \leq N$ are all given in terms of the cohomology of configuration spaces of points in X°. The only subtle difference is that these groups are indexed differently, as the degree shift depends on the dimension of $\Gamma_k(\vec{v})$ which varies with k. This degree shift is also apparent on the abutment of the spectral sequence. Thus, if we could construct a morphism of spectral sequences shifting the total degree by $\dim_{\mathbb{C}} \Gamma_{k+1}(\vec{v}) - \dim_{\mathbb{C}} \Gamma_k(\vec{v})$, we would obtain on the abutment a morphism

$$\widetilde{H}_*(U_k(\vec{v}); \mathbb{Z}) \longrightarrow \widetilde{H}_*(U_{k+1}(\vec{v}); \mathbb{Z}).$$

Suppose furthermore that we could construct this morphism of spectral sequences such that it were an isomorphism on the $E_1^{s,t}$ groups when $0 \leq s \leq N$ (which we recall are equal up to this degree shift). Then the vanishing result in the column $s = N+1$ would imply that the morphism on the abutment would be an isomorphism in the range of degrees $* < N$.

In [Aum22, Section 6], it is explained how to construct this morphism such that the induced morphism $\widetilde{H}_*(U_k(\vec{v}); \mathbb{Z}) \to \widetilde{H}_*(U_{k+1}(\vec{v}); \mathbb{Z})$ is the one induced by $\cdot\eta\colon U_k(\vec{v}) \to U_{k+1}(\vec{v})$ in homology. Likewise, the argument works for the jet map $j^1\colon U(\vec{v}) \to U_0(\vec{v})$. To sum up, we have:

Proposition 6.16 (Compare [Aum22, Proposition 6.6])**.** *Let N be as in Convention 6.8. Let $k \geq 0$ be an integer. The map $\cdot\eta\colon U_k(\vec{v}) \to U_{k+1}(\vec{v})$ induces an isomorphism in homology in the range of degrees $* < N$. Similarly, the jet map $j^1\colon U(\vec{v}) \to U_0(\vec{v})$ induces an isomorphism in homology in the same range.* $\square$

6.3.3. *The Stone–Weierstrass theorem.* In view of Proposition 6.16, it suffices to show that

$$\operatorname*{colim}_{k\to\infty} U_k(\vec{v}) \longrightarrow U_{\mathcal{C}^0}(\vec{v})$$

is a weak homotopy equivalence to finish the proof of Theorem 6.1. The proof is analogous to that given in [Aum22, Section 7], but using the following version of the Stone–Weierstrass theorem with interpolation.

Theorem 6.17 (Stone–Weierstrass)**.** *Let $E \to B$ be a finite rank real vector bundle over a compact Hausdorff space. Let $A \subset \mathcal{C}^0(B, \mathbb{R})$ be a subalgebra, $\{s_j\}_{j\in J}$ be a set of sections, and $\mathcal{A}$ be the A-module generated by the s_j. Let $P = \{b_1, b_2, \ldots, b_k\} \subset B$ be a finite (possibly empty) set of distinct points, and $V = \{v_1, v_2, \ldots, v_k\} \subset E$ be vectors $v_i \in E|_{b_i}$ in the fibers above the b_i. Define*

$$\mathcal{A}^{P,V} := \{f \in \mathcal{A} \mid \forall i,\ f(b_i) = v_i\} \subset \mathcal{A}$$

and

$$\Gamma_{\mathcal{C}^0}^{P,V}(E) := \{f \in \Gamma_{\mathcal{C}^0}(E) \mid \forall i,\ f(b_i) = v_i\} \subset \Gamma_{\mathcal{C}^0}^{P,V}(E)$$

to be the subsets of sections with prescribed values at the b_i. Suppose that

(1) the subalgebra A separates the points of B: for any $x, y \in B$, there exists $h \in A$ such that $h(x) \neq h(y)$;
(2) for any $x \in B$, there exists $h \in A$ such that $h(x) \neq 0$;
(3) for any $x \in B$, the fiber E_x is spanned by the $s_j(x)$ as an $\mathbb{R}$-vector space.

Then $\mathcal{A}^{P,V}$ is dense for the sup-norm (induced by the choice of any inner product on E) in the space $\Gamma_{\mathcal{C}^0}^{P,V}(E)$.

Sketch of a proof. The theorem follows from the original Stone–Weierstrass theorem for functions from a compact space to the real line when the set P is empty, and its variations allowing interpolation in general (see e.g. [Deu66, Theorem 1]). Indeed, by compactness, we may find a finite number of sections $s_1, \ldots, s_n$ such that $s_1(x), \ldots, s_n(x)$ span the fiber at each $x \in B$. Then $\mathcal{A}$ contains all sections of the form $a_1 s_1 + \cdots + a_n s_n$ for $a_i \in A$, and every continuous section of E can be written as $f_1 s_1 + \cdots + f_n s_n$ with $f_i \in \mathcal{C}^0(B, \mathbb{R})$. We may finally use the usual Stone–Weierstrass theorem, or its adaptation with interpolation, for the functions f_i. □

References

[Aum22] A. Aumonier, *An h-principle for complements of discriminants*, arXiv:2112.00326 [math], Oct. 2022 (cit. on pp. 2, 3, 12, 19, 21–27).

[Aum23] A. Aumonier, "Scanning the moduli of embedded hypersurfaces," 2023 (cit. on p. 10).

[Ban21] I. Banerjee, *Stable cohomology of the universal degree d hypersurface in $\mathbb{P}^n$*, arXiv:2007.05425 [math], May 2021 (cit. on pp. 3, 4).

[BRS99] M. C. Beltrametti, S. D. Rocco, and A. J. Sommese, "On generation of jets for vector bundles.," eng, *Revista Matemática Complutense*, vol. 12, no. 1, pp. 27–44, 1999, ISSN: 1139-1138 (cit. on p. 6).

[Deu66] F. Deutsch, "Simultaneous Interpolation and Approximation in Topological Linear Spaces," *SIAM Journal on Applied Mathematics*, vol. 14, no. 5, pp. 1180–1190, 1966, Publisher: Society for Industrial and Applied Mathematics, ISSN: 0036-1399 (cit. on p. 28).

[DH22] R. Das and S. Howe, *Cohomological and motivic inclusion-exclusion*, arXiv:2204.04165 [math], Apr. 2022 (cit. on pp. 2, 3, 19).

[DS98] S. Di Rocco and A. J. Sommese, *Line bundles for which a projectivized jet bundle is a product*, arXiv:math/9806144, Jun. 1998 (cit. on p. 5).

[FHT01] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory* (Graduate Texts in Mathematics). New York, NY: Springer, 2001, vol. 205, ISBN: 978-1-4612-6516-0 (cit. on p. 17).

[FM94] W. Fulton and R. MacPherson, "A Compactification of Configuration Spaces," *Annals of Mathematics*, vol. 139, no. 1, pp. 183–225, 1994, Publisher: Annals of Mathematics, ISSN: 0003-486X (cit. on p. 11).

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*. Boston, MA: Birkhäuser, 1994, ISBN: 978-0-8176-4770-4 (cit. on p. 7).

[Gri79] P.-P. Grivel, "Formes différentielles et suites spectrales," fra, *Annales de l'institut Fourier*, vol. 29, no. 3, pp. 17–37, 1979, Publisher: Association des Annales de l'Institut Fourier, ISSN: 0373-0956 (cit. on p. 19).

[Gro67] A. Grothendieck, "Éléments de géométrie algébrique : IV. étude locale des schémas et des morphismes de schémas, Quatrième partie," en, *Publications Mathématiques de l'IHÉS*, vol. 32, pp. 5–361, 1967, ISSN: 1618-1913 (cit. on p. 5).

[Hae82] A. Haefliger, "Rational homotopy of the space of sections of a nilpotent bundle," en, *Transactions of the American Mathematical Society*, vol. 273, pp. 609–620, 1982, Publisher: American Mathematical Society, ISSN: 0002-9947 (cit. on p. 12).

[How19] S. Howe, "Motivic random variables and representation stability II: Hypersurface sections," en, *Advances in Mathematics*, vol. 350, pp. 1267–1313, Jul. 2019, ISSN: 0001-8708 (cit. on pp. 2, 4, 20).

[Kri94] I. Kriz, "On the Rational Homotopy Type of Configuration Spaces," *Annals of Mathematics*, vol. 139, no. 2, pp. 227–237, 1994, Publisher: Annals of Mathematics, ISSN: 0003-486X (cit. on p. 11).

[Nor93] M. V. Nori, "Algebraic cycles and Hodge theoretic connectivity," en, *Inventiones Mathematicae*, vol. 111, no. 1, pp. 349–373, Dec. 1993, ISSN: 0020-9910, 1432-1297 (cit. on p. 4).

[Rap17] G. Raptis, "On Serre microfibrations and a lemma of M. Weiss," en, *Glasgow Mathematical Journal*, vol. 59, no. 3, pp. 649–657, Sep. 2017, Publisher: Cambridge University Press, ISSN: 0017-0895, 1469-509X (cit. on p. 9).

[Ser56] J.-P. Serre, "Géométrie algébrique et géométrie analytique," en, *Annales de l'Institut Fourier*, vol. 6, pp. 1–42, 1956, ISSN: 1777-5310 (cit. on p. 6).

[Tom14] O. Tommasi, "Stable cohomology of spaces of non-singular hypersurfaces," en, *Advances in Mathematics*, vol. 265, pp. 428–440, Nov. 2014, ISSN: 0001-8708 (cit. on pp. 2, 3).

[Tot96] B. Totaro, "Configuration spaces of algebraic varieties," en, *Topology*, vol. 35, no. 4, pp. 1057–1067, Oct. 1996, ISSN: 0040-9383 (cit. on p. 11).

[VW15] R. Vakil and M. M. Wood, "Discriminants in the Grothendieck ring," *Duke Mathematical Journal*, vol. 164, no. 6, pp. 1139–1185, Apr. 2015, Publisher: Duke University Press, ISSN: 0012-7094, 1547-7398 (cit. on p. 10).