Aharonov–Casher theorem for manifolds with boundary

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Abstract. In this project we extend the famous Aharonov–Casher result on the number of zero modes of the Pauli (or Dirac) operator on $\mathbb{R}^2$ with a compactly supported smooth magnetic field to the case of a planar connected, but not simply connected, region M. More specifically we consider the Dirac operator on M with a smooth magnetic field compactly supported in the bulk of M and arbitrary magnetic field supported inside the "holes" of M. The domain is given by the famous Atyiah–Patodi–Singer boundary condition. First we prove that the problem is unitarily equivalent to the case when each of the fluxes inside the holes is normalized to a value inside the interval $[-\pi, \pi)$ by adding an integer multiple of 2π to the original flux. Denoting by Φ the sum of the flux in the bulk and the normalized fluxes we then show that if M is a disc with holes the number of zero modes is given by $\left|\left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor\right|$. For M being $\mathbb{R}^2$ with circular holes the number is $\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$ provided that $|\Phi| > 1$. By means of stereographic projection we show a similar result for domain on a sphere with holes.

The index of the Dirac operator is the difference of the number of its zero modes with spin up and spin down and can be expressed by the famous index formula by Atiyah, Patodi and Singer. The Aharonov–Casher theorem extends this result (in a very particular setting) telling us that all the zero modes have the same spin, that depends on the sign of the total magnetic flux. Our result in this sense agrees with and extends the index theorem, or more precisely its generalization by Grubb to manifolds that do not have a product structure close to the boundary.

Resumé. I dette projekt udvider vi det berømte Aharonov–Casher resultat om dimensionen af kernen for Pauli (eller Dirac) operatoren på $\mathbb{R}^2$ med et kompakt støttet glat magnetfelt til tilfældet hvor operatoren er defineret på et sammenhængende, men ikke enkelt-sammenhængende område M i planen. Mere specifikt betragter vi Dirac-operatoren på M med et glat magnetfelt, der er kompakt støttet i M og et vilkårligt magnetfelt støttet i "hullern". Operatores domæne er givet ved den berømte Atyiah–Patodi–Singer-randbetingelse. Først beviser vi, at problemet er unitært ækvivalent med tilfældet hvor, hver af fluxene inde i hullerne normaliseres til en værdi i intervallet $[-\pi, \pi)$ ved at tilføje et heltalsmultiplum af 2π til den oprindelige flux. Betegner Φ summen af fluxen i M og de normaliserede fluxer viser vi derefter, at hvis M er en disk med huller, er antallet af nul-tilstande givet ved $\left|\left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor\right|$. For M er $\mathbb{R}^2$ med cirkulære huller er antallet $\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$ forudsat at $|\Phi| > 1$. Ved hjælp af stereografisk projektion viser vi et tilsvarende resultat for et domæne på kuglefladen med huller. Indekset for Dirac-operatoren er forskellen mellem antallet af dens nultilstande med spin op og spin ned og kan udtrykkes ved den berømte indeksformel af Atiyah, Patodi og Singer. Aharonov–Casher-sætningen udvider dette resultat (i en meget speciel situation) ved at fortæller os, at alle nul-tilstande har det samme spin. Det afhænger af fortegnet på den totale magnetiske flux. I samme forstand stemmer vores resultat overens med og udvider indekssætningen og dens generalisering af Grubb til mangfoldigheder der ikke har en produktstruktur tæt på randen.

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Contents

Notation

Ω_k	Open ball in $\mathbb{C}$ with centre at $w_k \in \mathbb{C}$ and radius R_k, $z_{0k} := w_k + R_k \in \mathbb{C}$
$\mathcal{A}(\Omega)$	Annulus whose inner radius is the radius of a ball Ω
$a_{(k)}$	Vector potential of the Aharonov–Bohm field $B_k = \Phi_k \delta_{w_k}$ with flux Φ_k
(r_k, φ_k)	Polar coordinates around the point w_k, we set $\varphi_k = 0$ to be the axis parallel with the Cartesian positive x-axis
∂X	Boundary of a region X
X^C	Complement of the set $X \subset \mathbb{C}$ in $\mathbb{C}$
$\overline{X}$	Closure of a subset $X \subset \mathbb{C}$
χ_X	Indicator function of a set X
$\lfloor y \rfloor$	The biggest integer strictly less than y
$\{y\}$	The biggest integer less or equal than $y \geq 0$
TM	Tangent space of a manifold M
T^*M	Cotangent space of a manifold M
$Cl(V)$	Clifford algebra on a vector space V
$C^\infty(X)$	Smooth functions on X
$C_0^\infty(X)$	Smooth functions with compact support in X
M°	Interior of a manifold M
$int\, \gamma$	Interior of a curve γ
$\pi : M \to E$	or $M \to E$ Fibre bundle E over a manifold smooth M
$(\cdot, \cdot)_E$	Inner product on fibres of a bundle E
$\Gamma(M, E)$	Smooth sections of a bundle E over a manifold M
$\mathrm{d}vol_M$	Volume form induced on a Riemannian manifold M by its metric
$L^2(M, g; E)$	or $L^2(M, E)$ Square integrable sections of the bundle E over a Riemannian manifold M with metric g.
$GL(W)$	Invertible matrices on a vector space W
$const$	Constant in general which can be of different value from one equality sign to another

Chapter 1

Introduction

In 1979 Aharonov and Casher (AC) presented a simple proof [1], of a special case of the Atiyah–Singer (AS) index theorem for closed manifold from 1963 [6]. In particular, they found the zero modes, i.e. the functions in the kernel, of the Pauli (or equivalently Dirac) operator on $\mathbb{R}^2$ with a magnetic field. In this thesis we prove a generalisation of their result for some regions in $\mathbb{R}^2$ with boundary. To compare these two results we note, that both AS theorem and AC theorem can be employed to compute the index of the magnetic Dirac operator on the sphere, since the index can be interpreted as the difference of the number of zero modes with spin up and spin down. The AC theorem then moreover asserts that one of these contributions is always zero. Such a theorem is sometimes also referred to as a vanishing theorem.

The proof of the AC result relies on the fact that the Pauli operator acts as a square of the Dirac operator. The zero modes are then found as the solutions of two decoupled homogeneous partial differential equations of the first order. The authors do not comment on the domain of the operator. However, for a smooth compactly supported magnetic field the Kato–Rellich theorem ensures that such an operator (in their suitably chosen gauge) is self-adjoint on the domain of self-adjointness of the free Pauli operator, which is simply formed by the $\mathbb{C}^2$-valued distributions in the second Sobolev space $H^2(\mathbb{R}^2, \mathbb{C}^2)$. The zero modes turn out to be analytic functions multiplied by a certain exponential factor. Aharonov and Casher then conclude that whether or not the system can host zero modes depends only on the flux of the magnetic field. The absolute value of the flux further determines the number of these modes and the sign of the flux determines their spin. This topic is quite well represented in the literature, as briefly discussed at the end of Section 2.3. Many generalisations of the result can be found.

In the 1970s, Atiyah, Patodi and Singer (APS) published a series of papers [3, 4, 5] on the celebrated *index theorem* on compact manifolds with boundary, which is a result combining the analytical index of an elliptic operator and the topological index of the underlying manifold in one formula. This generalizes the previously mentioned AS theorem which holds on closed manifolds. In order to obtain a finite index, Atiyah, Patodi and Singer introduced a global boundary condition, known as the APS boundary condition.

Even though the index theorem holds for elliptic operators of arbitrary orders provided the manifold is even dimensional, the index formula for an elliptic first order differential operator presented in [3, Thm. 3.10.] is of particular relevance to our work. The index in such a case consists of two parts: the “bulk” contribution computed from the heat kernel expansion for small times, which is equal to the index in the boundary-less case, and the boundary contribution given by the η-invariant introduced by APS. The manifolds considered in the APS paper were required to have a cylindrical end. Grubb [22] proves a result for more general manifolds by introducing yet another boundary contribution, which vanishes for cylindrical ends. These results were further extended by Gilkey in [21], where one can find more concrete formulas for this new contribution.

Results. In our generalisation we consider (planar) manifolds with boundary: first, a plane with circular holes and second a disc with circular holes. We consider a smooth magnetic field with a compact support contained in the interior of such a manifold and an additional magnetic field inside the holes that can be modelled by Aharonov–Bohm solenoids with a particular flux. Our focus is purely on a self-adjoint Dirac operator and an important part of the analysis is the determination of the domain. Since we have a boundary, the self-adjoint extensions are distinguished by different boundary conditions. Apart from self-adjointness another requirement we pose is to select a boundary condition that is elliptic. This will ensure that the realisation of the Dirac operator is a Fredholm operator, i.e. an operator with a closed range and a finite dimensional kernel and co-kernel. We remark that this is insured only in the case that the manifold is compact. A detailed classification of elliptic boundary conditions can be found in [8]. We will use the earlier mentioned APS boundary condition which in particular fulfils both of these requirements.

Standardly we work with the vector potential a of the magnetic field rather than with its strength B satisfying $\operatorname{rot} a = B$. It is clear that the vector potential is not given uniquely by this relation. A particular choice of a is called the gauge of the magnetic field. A well known fact is that the Dirac operators corresponding to different gauge choices are unitarily equivalent. In our case when the magnetic field appears also inside the holes which are not part of the manifold we can prove even more. We show that for this part of the magnetic field we can to some extent modify even the flux of the field and that the Dirac operators with the APS boundary condition with fluxes inside the holes differing by a 2π multiple of an integer are again unitarily equivalent. This allows us to consider the flux inside each hole to be in the interval $[-\pi, \pi)$ and we refer to these as normalised fluxes.

Let us denote by Φ the sum of the flux of magnetic field supported inside the interior of our manifold and the normalised fluxes in the holes. Our main results then state that there are

$$\left| \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor \right| \tag{1.1}$$

zero modes of the Dirac operator on a disc with holes whose domain is given by the APS boundary condition, and, provided that $|\Phi| > 2\pi$, there are

$$\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$$

zero modes of that operator when considered on a plane with holes. Here $\lfloor y \rfloor$ denotes the biggest integer strictly smaller than y. In both cases if $\Phi > 0$ the zero modes have spin up and if $\Phi < 0$ they have spin down. More precisely the results are stated in Theorems 40 and 39.

The formula for the unbounded region exactly agrees with the original formula of Aharonov and Casher who considered empty boundary. This is in no way surprising as in the boundary-less case our proof simply follows theirs. Moreover, in the same way the AC theorem strengthens the AS index theorem, our result on the bounded region strengthens the APS index theorem, which in this case states that the index is $\left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor$.

We proved an adaptation of the result for the Dirac operator on the sphere with holes in which case we consider only magnetic fields whose overall flux sums to zero. Then again the formula (1.1) for the number of zero modes holds with Φ denoting the sum of the flux through the bulk and the normalised fluxes through the holes except for exactly one. The result is independent of the choice of the omitted flux.

We expect that the same results should hold even for the case of arbitrarily shaped holes with smooth boundary. However, our attempts to prove this generalisation have so far been unsuccessful, despite the APS theorem holding in that case. The complication stems from the difficulty of describing the boundary values of analytic functions in the case of a boundary given by a general curve. We hope to address this problem again in the future.

Importance of the zero modes in studying the stability of matter. Apart from the mathematical interest, there is a greater motivation, for investigating the zero modes, due to the results on stability of matter in the series of papers [19, 27, 29] which led to the proof of stability of matter in [28, 18]. In the first paper the authors present a proof of stability of a single electron atom with a magnetic fields, which amounts to the fact that the Pauli operator with the Coulomb potential $V(x) = \frac{z}{|x|}$ plus the self energy of the magnetic field B

$$H(a, V, 1) + \epsilon \int B^2\,, \quad \text{where}$$
$$H(a, V, h) = \left[\sum_{j=1}^{3} \sigma^j \left(h\frac{\partial}{\partial x_j} - \mathrm{i}a_j\right)\right]^2 - V(x)\,, \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3\,,$$

is bounded below for all fields $B = \operatorname{rot} a$, $a = (a_1, a_2, a_3) \in L^6(\mathbb{R}^3; \mathbb{R}^3)$, $\operatorname{div} a = 0$. The Pauli matrices σ^j, $j \leq 3$ are defined in (2.5). (Here ϵ is a constant inserted in order for all the terms to be expressed in the same units.) The stability is established for nuclei not acceding the charge $z|e|$, where e is the elementary charge, when $z < z_c$. The bound z_c is then given as the infimum of the expression

$$\frac{\epsilon \int B^2}{(\psi, |x|^{-1}\psi)_{L^2(\mathbb{R}^3;\mathbb{C}^2)}}\,,$$

over the zero modes ψ of the magnetic Dirac operator. A lower bound on the number z_c was found in [19].

Furthermore in the paper [13], Erdős–Fournais–Solovej consider the semiclassical limit of the infimum over all magnetic fields $a \in H^1(\mathbb{R}^3; \mathbb{R}^3)$ of a similar problem

$$h^3 \left(\operatorname{Tr}[H(a, V, h)]_- + b \int B^2\right)\,, \quad V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)\,,$$

where $\operatorname{Tr}[H(a, V, h)]_-$ denotes the sum of the negative eigenvalues of the Pauli operator $H(a, V, h)$, and $b > 0$ is a parameter introduced to represent the coupling of the particle to the magnetic field. More concretely they investigate the limit $h \to 0$, simultaneously with $hb \to \infty$. They also find an upper and lower bounds for the asymptotics in the case of a greater influence of the magnetic field, i.e. for small hb. In the proof of the upper bound the zero modes of the magnetic Dirac operator in three dimensions found in [14] play an important role. Even though these bounds are very close they do not match in the highest power of hb.

Let us point out that these are problems in $\mathbb{R}^3$, however Erdős and Solovej in [14] found a way to analyse three dimensional fields by lifting from two dimensions.

Organisation of the text. Let us mention that an overview of some notation used in this thesis can be found before this introduction on page viii. The second chapter of this work is dedicated to the geometrical background material leading to the Dirac operator on *spin* and *spin*^{c} manifolds. Since connections on the *spin*^{c} structure can represent a magnetic field, the relation between the curvature of the corresponding connection on *spin*^{c} bundles and magnetism is outlined. We further discuss elliptic operators and elliptic boundary conditions following the paper [8]. We provide the standard proof of the AC theorem, since the main idea is also needed for our generalisation. Further we find the APS boundary condition for our particular setting and establish gauge invariance of the Dirac operator with the domain determined by this boundary condition.

In the third chapter we state our main results and present the proofs. The chapter is concluded by restating of the main result for a certain modification of the APS boundary condition and evaluation of the index formula from [22] and [21].

The stereographic projection is a conformal map from a sphere onto a plane. Analysis of the Dirac operator with the APS boundary condition under a conformal change of metric allows for a generalization of our Aharonov–Casher type result for a disc with holes to a sphere with holes in Chapter 4. Note that in particular the problem is, however, not conformally invariant which would otherwise lead to a direct proof of our results with a relaxed condition on the shape of the boundary; instead of all its components being circular we could for example assume them to be smooth closed curves. This generalisation is therefore still an open question and we briefly discuss some aspects of the problem in our concluding remarks in the last chapter.

In Appendix A we included some computational details concerning the Möbius transform which is used in the proof for the sphere case. In Appendix B we comment on existence of zero modes on an annulus with local boundary conditions introduced in [10] by Berry and Mondragon.

Our original methods in the proofs were insufficient to show our result Theorem 40 for the bounded region. They, however, provided an inspiration for the current proof, so we include them in Appendix C. The last appendix is dedicated to computation of the η-invariant in the cases that are relevant to our problem.

Chapter 2

Prerequisites

2.1 Principal bundles, their morphisms and connection

Even though we will present basic definitions and claims that are essential for defining the spinor bundles we still expect the reader to be familiar with some standard notions in differential geometry. We refer to e.g. [35, 38, 17] for more details on the subject.

We first introduce principal G-bundles, their associated vector bundles and a connection on a vector bundle, as these terms will be used for the definition of spinor bundles in the next section.

Recall that a **fibre bundle** E over a manifold M is a collection

$$(E, M, \pi, F)\,,$$

of smooth manifolds E, M, F called the total space, the base manifold and a typical fibre and a surjective mapping $\pi : E \to M$ called the canonical projection, such that there exists an open covering $(U_j)_{j\in J}$ of M (J being some index set) and smooth diffeomorphisms

$$t_j : \pi^{-1}(U_j) \to U_j \times F\,,$$

satisfying $pr_1 \circ t_j = \pi$, for all $j \in J$ with pr_1 denoting the projection on the first component of a Cartesian product. The mappings t_j are called local trivializations.

We further define the transition functions $t_k \circ t_j^{-1} : U_j \cap U_k \times F \to U_j \cap U_k \times F$ which induce a smooth mapping $m \mapsto g_{jk}(m)$ by

$$t_k \circ t_j^{-1}(m, u) = (m, g_{jk}(m)u)\,, \quad u \in F\,, m \in U_j \cap U_k\,. \tag{2.1}$$

By definition, $g_{jk}(m)$ belong to the group of diffeomorphisms of F for all $m \in M$ and $j, k \in J$. If moreover, they are from a certain subgroup G of the diffeomorphisms of F we call E a G-fibre bundle and refer to G as the **structure group** of E.

We will often write $\pi : E \to M$ instead of (E, M, π, F). In a standard manner we set $\Gamma(M, E)$ to be the smooth sections $u : M \to E$ of E. If M is a Riemannian manifold and the fibres of E are vector spaces equipped with a smooth inner product $(\cdot, \cdot)_E$, i.e., E is a vector bundle, we further define the square integrable sections of E by

$$L^2(M, E) := \{u : M \to E \mid \|u\|^2 := \int_M (u, u)_E \, \mathrm{d}vol_M < \infty\} ,$$

where $\mathrm{d}vol_M$ is the volume form generated by the Riemannian metric on M.

A principal G-bundle is a special case of a G-fibre bundle.

Definition 1. *Let G be a Lie group and $\pi : P \to M$ be a fibre bundle over a smooth manifold M with typical fibre G. Assume there is a right action $R : P \times G \to P$ of G on P, written $(p, g) \mapsto R(g)p$ and satisfying*

i) R acts along the fibres of P, i.e. for all $p \in P$ and $g \in G$

$$[\pi \circ R(g)](p) = \pi(p) ,$$

ii) R acts freely *on the fibres of P, i.e. if $R(g)p = R(h)p$ for some $p \in \pi^{-1}(m) \subset P$, $m \in M$ and $g, h \in G$ then $g = h$,*

iii) and transitively*, i.e. for all $p, p' \in \pi^{-1}(m) \subset P$, $m \in M$ there exists $g \in G$ such that $R(g)p = p'$,*

iv) the local trivializations $t : \pi^{-1}(V) \to V \times G$, $V \subset M$ being an open subset, are equivariant*, i.e.*

$$t^{-1}(m, g \cdot h) = R(h)t^{-1}(m, g) ,$$

for all $m \in V$ and $g, h \in G$. The dot stands for the multiplication on G. We call P a **principal G-bundle***.*

Further we define a morphism of principal bundles. The idea is depicted in Figure 2.1.

Definition 2. *Let $P_{1,2} \to M_{1,2}$ be principal $G_{1,2}$- bundles. Denote $R_{1,2}$ the corresponding right action of $G_{1,2}$ on $P_{1,2}$ and consider two mappings $\Lambda : P_1 \to P_2$ and $\lambda : G_1 \to G_2$. The pair (Λ, λ) is called a* **morphism of the principle bundles** *P_1 and P_2 if on P_1 it holds*

$$\Lambda \circ R_1(h) = R_2(\lambda(h)) \circ \Lambda ,$$

for any $h \in G_1$.

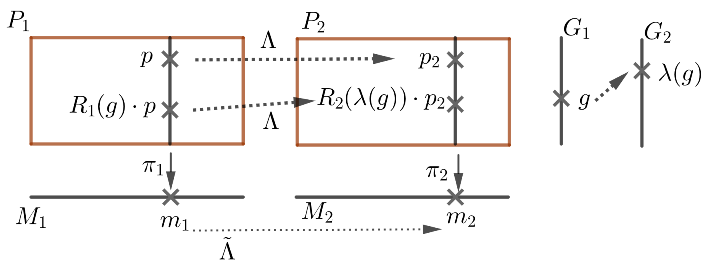

Figure 2.1: Morphism (Λ, λ) of principal G_1-bundle $P_1 \to M_1$ with the right action R_1 and the principal G_2-bundle $P_2 \to M_2$ with the right action R_2, and, the induced map $\widetilde{\Lambda} : M_1 \to M_2$.

In other words Λ maps fibres over a point in M_1 to fibres over a point in M_2. Consequently this induces a map $\widetilde{\Lambda} : M_1 \to M_2$. If $M_1 = M_2 = M$ and this induced mapping is an identity we call the morphism **vertical**. Notice that a vertical morphism carries a fibre over point $m \in M$ to a fibre over the same point m. If, moreover, λ is an embedding we say that G_1 is a **reduction of the structure group** G_2.

An example of a principle bundle is the bundle of frames on the tangent bundle TM over a smooth manifold M, $\dim M = n$. We recall that in general this is a $GL(n)$-principal bundle and if M is oriented the corresponding structure group can be reduced to $GL(n)_+$ consisting of regular $n \times n$ matrices with positive determinant. Further, existence of a Riemannian structure (metric) allows for a possibility of reduction to the $SO(n)$-principal bundle. In this work we will be particularly interested in the *spin* and *spin*^{c} structures, which will be introduced later.

The term associated vector bundle allows us to pass from a principal G-bundle to a vector bundle whose fibres are endowed with an action of the Lie group G. Recall that a representation of G on a vector space V is a group homomorphism $\rho : G \to \mathrm{End}(V)$ and can be also viewed as a mapping from $G \times V$ to V.

Definition 3. *Let $P \to M$ be a principal G-bundle with a right action R and let $L : G \times V \to V$ be a representation of G on a vector space V. Then we define an equivalence relation $(p', v') \sim (p, v)$ on $P \times V$ by*

$$(p', v') = (R(g)p, L(g^{-1})v)\,,$$

for some $g \in G$. The equivalence classes are also known as orbits, and, we denote the orbit space by $P \times_G V := (P \times V)/\sim$. Defining the projection $\pi_V[(p, v)] = \pi(p)$ we thus endow $P \times_G V$ with a natural bundle structure and call $\pi_V : P \times_G V \to M$ an **associated vector bundle** *to the principle bundle P.*

Remark 4. *The representation $L : G \times V \to V$ is a left action of the group G on V. Note, that the equivalence classes are the elements of $P \times V$ that are related by the right free product group action $R \times L : g \mapsto R(g) \times L(g^{-1})$.*

Definition 5. *Let TM be the tangent bundle over M. A* **connection on a vector bundle** *E over M is a mapping*

$$\nabla : \Gamma(M, E) \otimes TM \to \Gamma(M, E)\,,$$

such that

$$\begin{aligned} \nabla_X(u+v) &= \nabla_X u + \nabla_X v \\ \nabla_{fX+Y} u &= f\nabla_X u + \nabla_Y u \\ \nabla_X(fu) &= X(f)u + f\nabla_X u\,, \end{aligned}$$

for any $X \in TM$, $u \in \Gamma(M, E)$ and $f \in C^\infty(M)$.

Let $\gamma(s)$ for $s \in [0, 1]$ be a smooth parametrisation of a curve $\gamma \subset M$ and denote $\dot\gamma$ the corresponding velocity vector field tangent to γ. We say that a section $u \in \Gamma(M, E)$ is parallel transported along γ if $\nabla_{\dot\gamma} u = 0$.

By definition any two connections $\nabla^{(1)}$ and $\nabla^{(2)}$ differ by a one form ω with values in $\mathrm{End}(E)$ (the endomorphisms of E), i.e.

$$\nabla^{(1)}_X u = \nabla^{(2)}_X u + \omega(X) \cdot u\,, \quad u \in \Gamma(M, E)\,, X \in TM\,.$$

Note further, that if E is a trivial vector bundle, the exterior derivative d satisfies the defining properties of a connection and thus any connection on a trivial vector bundle can always be written as

$$\nabla = \mathrm{d} + \omega\,. \tag{2.2}$$

Since vector bundles are locally trivial we can find ω such that (2.2) is satisfied locally for a general (non-trivial) bundle E. We call this $\mathrm{End}(E)$ valued one form ω, which depends on the local trivialization, the **connection one from**. Finally we recall the definition of curvature which describes how a spinor changes when parallel transported along a loop on the base manifold.

Definition 6. *The* **curvature** *of a connection ∇ on a vector bundle E is the* $\mathrm{End}(E)$ *valued two form R defined by*

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\,,$$

for any vector fields $X, Y \in TM$.

Remark 7. 1. *Note that considering fields* $X, Y \in TM$ *and* $\zeta \in \Gamma(M, E)$*, a direct computation yields*

$$R(f_1 X, f_2 Y) f_3 \zeta = f_1 f_2 f_3 R(X, Y)\zeta\,,$$

for any functions $f_{1,2,3} \in C^\infty(M)$*. Therefore,* $R(X, Y)$ *depends only on the local values of* X, Y *and it defines a tensor.*

2. *Consider a smooth parametrisation* $\gamma_\epsilon(s)$*,* $s \in [0, 1]$ *of an infinitesimal parallelogram* γ_ϵ *along the integral curves of two vector fields* $\epsilon X, \epsilon Y \in TM$ *for some* $\epsilon > 0$*. Then a section* $\zeta \in \Gamma(M, E)$ *parallel transported along such a loop does not need to come back to itself and more explicitly it holds that if* $\nabla_{\dot{\gamma}_\epsilon}\zeta(\gamma_\epsilon(s)) = 0$*, for all* $s \in [0, 1]$ *then (see e.g. [17, Section 15.5])*

$$\lim_{\epsilon \to 0} \frac{\zeta(\gamma_\epsilon(1)) - \zeta(\gamma_\epsilon(0))}{\epsilon^2} = -R(X, Y)\zeta(\gamma_\epsilon(0))\,.$$

In this sense curvature measures failure of a section parallel transported along a loop to come back to itself.

2.2 Dirac operator on spin and spin^c manifolds

Our goal here is to introduce the Dirac operator on a $spin^c$ manifold, which is useful to model a relativistic charged particle in presence of a magnetic field. Since the geometrical background is not the main topic of this text we will restrict only to a concise presentation of the topic. For further details see e.g. Chapter II and Appendix D in [26], Chapter 10. in [38] or Chapter 5 in [35].

As a starting point we define the Clifford algebra. Let V be a real vector space of a finite dimension $\dim V = n$ with a real quadratic form g. Let further

$$\bigotimes V := \bigoplus_{k \geq 0} V^{k\otimes} = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots\,,$$

be the corresponding tensor algebra, and $\mathcal{I}$ the ideal in $\bigotimes V$ generated by

$$\{xy + yx - 2g(x, y)I \mid x, y \in V\}\,, \tag{2.3}$$

where I is the identity in $\bigotimes V$.

Definition 8. *The* **Clifford algebra** $Cl(V)$ *is the quotient space*

$$Cl(V) = \bigotimes V/\mathcal{I}\,.$$

Note that V is naturally embedded in $Cl(V)$ and that the elements satisfy the so-called **Clifford relations**

$$xy + yx = 2g(x,y)I\,.$$

We will also use the standard notations $\{x,y\} = xy + yx$ for the anti-commutator and $Cl(n)$ for $Cl(\mathbb{R}^n)$ with the standard metric.

Remark 9. *We address the question of representations of the complexifications of Clifford algebras following [26, Section I.5]. By a representation of $Cl(n) \otimes \mathbb{C}$ we mean an algebra homomorphism $Cl(n) \otimes \mathbb{C} \to \mathrm{End}(W)$ where W is a certain complex vector space which we refer to as a* **Clifford module***. If the dimension n is odd there exist two irreducible representations. For n even there is only one irreducible representation.*

Note that similarly one can also consider real or complex representations of $Cl(n)$. Then the algebra homomorphism maps from $Cl(n)$ and W is a real or complex vector space. Moreover any complex representation of $Cl(n)$ extends to a representation for $Cl(n) \otimes \mathbb{C}$.

Definition 10. *Consider an oriented Riemannian manifold M of dimension n with metric g and let $E \to M$ be a vector bundle with a hermitian metric which is fibrewise a $Cl(T^*_m M)$ module for all $m \in M$, in a smooth fashion such that the restriction of the linear map*

$$\sigma : Cl(T^*_m M) \to \mathrm{End}(E_m)\,,$$

to $T^*_m M$ satisfies $\sigma(x) = \sigma(x)^*$ for all $x \in T^*_m M$. Here by E_m we denoted the fibre of E over $m \in M$. The map σ is called the **Clifford multiplication***. The vector bundle E equipped with the smooth Clifford multiplication is referred to as a* **Clifford module bundle***.*

Let us point out that by definition if g is non-degenerate then for each $x \in T^*M$ the Clifford multiplication $\sigma(x)$ is a hermitian isomorphism of the fibres of E and satisfies the Clifford relations

$$\sigma(x)\sigma(y) + \sigma(y)\sigma(x) = 2g(x,y)I\,, \tag{2.4}$$

for all $x, y \in T^*M$.

Example 11. *Recall the Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.5}$$

which satisfy the relations $\{\sigma^j, \sigma^k\} = 2\delta^{jk} I$ with the Kronecker delta

$$\delta^{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} .$$

The irreducible representation of $Cl(2) \otimes \mathbb{C} \simeq \mathbb{C}(2)$ (the 2×2 complex matrices) is generated by the first two Pauli matrices; $\rho(e^1) = \sigma^1$ and $\rho(e^2) = \sigma^2$.

There are some important groups arising in connection with the Clifford algebra $Cl(V)$:

Definition 12. *The* **spin group** *$Spin(V)$ is the group generated by the elements $xy \in Cl(V)$, such that $x, y \in V$ and $g(x,x) = g(y,y) = 1$, with the induced multiplication. The* **spin^c group** *$Spin^c(V)$ is then defined by the quotient*

$$Spin^c(V) = Spin(V) \times U(1) \big/ \{\pm(1,1)\} ,$$

where $U(1)$ denotes the group of unitaries on $\mathbb{C}$.

Notice that by the Clifford relations $x^2 = 1 \in Spin(V)$ and $-x \cdot x = -1 \in Spin(V)$. Thus the right hand side in the definition of $Spin^c$ has a good meaning. We use the usual notation $Spin(n)$ and $Spin^c(n)$ in the case $V = \mathbb{R}^n$. In what follows let us restrict to the case $n \geq 2$. Observe that for $x, y, v \in V$ such that $g(x,x) = g(y,y) = 1$ the Clifford relations imply that $p(xy)(v) := yxvxy \in V$ is the reflection of v across the line x followed by the reflection across the line y, i.e. a rotation. It turns out that there is the following short exact sequence

$$1 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{p} SO(n) \to 1 , \tag{2.6}$$

where p is a double covering map (*cf.* [38, Section 10, Proposition 3.1.]) .

As a side note we remark that for $n > 2$ the group $Spin(n)$ is simply connected and p is the universal cover. Similarly there is an exact sequence

$$1 \to \mathbb{Z}_2 \to Spin^c(n) \xrightarrow{p^c} SO(n) \times U(1) \to 1 , \tag{2.7}$$

where p^c extends the double cover map p to $Spin^c(n)$ by $p^c(g,z) := (p(g), z^2)$ for any $g \in Spin(n)$ and $z \in U(1)$. Let pr_1 stand for the projection on the first component of a Cartesian product. We will further denote

$$p' := pr_1 \circ p^c : Spin^c(n) \to SO(n) . \tag{2.8}$$

Note, that by restricting a $Cl(n) \otimes \mathbb{C}$ representation (discussed in Remark 9) to the group $Spin(n) \subset Cl(n) \otimes \mathbb{C}$ we obtain a representation of $Spin(n)$. Similarly, we get a $Spin^c(n)$ representation when restricting to $Spin^c(n) \subset$

$Cl(n) \otimes \mathbb{C}$. Moreover, if n is odd the two irreducible representations of $Cl(n) \otimes \mathbb{C}$ are equivalent when restricted to either $Spin(n)$ or $Spin^c(n)$. Let us also remark that $Spin(2) \simeq U(1) \simeq SO(2)$ and $Spin(3) \simeq SU(2)$.

In what follows we consider M to be an orientable Riemannian manifold of dimension $n \geq 2$ and we denote by $SO(TM)$ the $SO(n)$-principal bundle of oriented frames on the tangent bundle TM over M.

Definition 13. *A* **spin structure on TM** *is a pair $(S(TM), \Lambda)$ of a $Spin(n)$-principal bundle $S(TM)$ over M and a mapping $\Lambda : S(TM) \to SO(TM)$ such that (Λ, p), where p is the double covering from (2.6) is a vertical morphism of the corresponding principal bundles. A manifold M admitting a spin structure on TM is called a* **spin manifold**.

In other words on a spin manifold the structure group of its frame bundle can be reduced to $Spin(n)$. Correspondingly we define a $spin^c$ structure:

Definition 14. *A* **spin^c structure** *is a pair $(S^c(TM), \Lambda)$ of a $Spin^c(n)$-principal bundle $S^c(TM)$ over M and a mapping $\Lambda : S^c(TM) \to SO(TM)$ such that (Λ, p'), where p' is given by (2.8), is a vertical morphism of the corresponding principal bundles. A manifold M admitting $spin^c$ structure on TM is called a* **spin^c manifold**.

Definition 15. *Let $(S(TM), \Lambda)$ be a spin structure on TM and $(S^c(TM), \Lambda)$ be a $spin^c$ structure on TM.*

- *A complex* **spin spinor bundle** *of TM is the associated vector bundle*

$$\mathcal{S}(TM) = S(TM) \times_\rho W$$

  where W is a complex left module over $Cl(n)$ such that $\rho : Spin(n) \to GL(W)$ is the irreducible representation of $Cl(n) \otimes \mathbb{C}$ restricted to $Spin(n)$.

- *A* **spin^c spinor bundle** *of TM is an associated vector bundle*

$$\mathcal{S}^c(TM) = S^c(TM) \times_\rho W'$$

  where W' is a complex left module over $Cl(n)$ such that $\rho : Spin^c(n) \to GL(W')$ is the irreducible representation of $Cl(n) \otimes \mathbb{C}$ restricted $Spin^c(n)$.

Remark 16. 1. *One can also define a real spin bundle by requiring W in the previous definition for spin spinor bundle to be a real left $Cl(n)$-module and ρ to be the representation given by left multiplication by elements of $Spin(n) \subset Cl(n)$.*

2. *We will refer to the complex spin or $spin^c$ spinor bundles by the common term spinor bundle.*

3. *We can choose a convenient inner product on the fibres of a spinor bundle E so that the representation ρ of $Cl(n)$ maps T^*M to hermitian endomorphisms of E. Therefore ρ is the Clifford multiplication on E which thus equipped has structure of a Clifford module bundle.*

4. *There may be several $spin^c$ structures on a $spin^c$ manifold. E.g. for an explicit construction of $spin^c$ spinor bundles over the two-sphere see Appendix A.1 in [14].*

The goal of this section is to define the Dirac operator on a $spin^c$ manifold. To that end we still need to introduce one more definition. In the following let E be a spinor bundle over an oriented Riemannian manifold M with an inner product $(\cdot,\cdot)_E$.

Definition 17. *A connection $\nabla : \Gamma(M,E) \otimes TM \to \Gamma(M,E)$ on E is called a* **Clifford connection** *if it is*

1. *metric:*

$$X(\eta,\xi)_E = (\nabla_X \eta, \xi)_E + (\eta, \nabla_X \xi)_E \,,$$

   for any sections $\eta, \xi \in \Gamma(M,E)$ and any vector field $X \in TM$, and,

2. *compatible with the Clifford multiplication σ:*

$$[\nabla_X, \sigma(\tau)] = \sigma(\nabla^{LC}_X \tau) \,,$$

   for all vector fields X and one-forms τ on M. Here, ∇^{LC} is the Levi-Civita connection on the cotangent space T^*M of M.

Definition 18. *Let ∇ be a Clifford connection on E. The Dirac operator $D : \Gamma(M,E) \to \Gamma(M,E)$ is the following composition*

$$D = -\mathrm{i} \sum_{j \leq n} \sigma(e^j) \nabla_{e_j} \,,$$

where $(e_j)_{j\leq n}$ is an orthonormal basis on TM and $(e^j)_{j\leq n}$ the corresponding dual basis on T^*M.

Note that the definition is independent of a particular choice of $(e_j)_{j\leq n}$, so D is well defined. The Dirac operator D extends as a bounded linear map from the **maximal domain** of D

$$\mathrm{dom}(D_{max}) := \{u \in L^2(M,E) \mid Du \in L^2(M,E)\}$$

to $L^2(M,E)$.

In the case of a $spin^c$ spinor bundle E we further introduce the magnetic two form β as the curvature of the Clifford connection ∇. If $\xi \in \Gamma(M, E)$ is a normalized section we have by Definition 17

$$(\nabla_Z \xi, \xi)_E = -(\xi, \nabla_Z \xi)_E \,, \text{ and}$$
$$(\nabla_X \xi, \nabla_Y \xi)_E + (\xi, \nabla_X \nabla_Y \xi)_E = -(\nabla_X \nabla_Y \xi, \xi)_E - (\nabla_Y \xi, \nabla_X \xi)_E \,,$$

for all $X, Y, Z \in TM$. This implies with Definition 6 that the $\mathrm{End}(E)$-valued curvature two form $R_E(X, Y)$ of the Clifford connection on E is anti-hermitian for all $X, Y \in TM$. Let (ξ_j) form an orthonormal basis of the fibre E_m over the point $m \in M$ and consider the pointwise trace

$$Tr[R_E(X, Y)] = \sum_{\xi_j \in E_m} (\xi_j, R_E(X, Y)\xi_j)_E \,, \quad X, Y \in TM \,,$$

on M. Since we consider everything smooth, this relation yields a two form on M.

Definition 19. *Let M be a $spin^c$ manifold of even dimension n. Let further E be a $spin^c$ spinor bundle and ∇ a Clifford connection on E with curvature R_E. The* **magnetic two form** *β is defined by*

$$\beta(X, Y) = \frac{\mathrm{i}}{2^{n/2}} Tr[R_E(X, Y)] \,.$$

We note that β is a closed form and therefore locally $\beta = \mathrm{d}\alpha$, where α is the connection one form of ∇. For more details in the two (and three) dimensional case we refer to [14].

Index of the Dirac operator

Here we assume that the dimension of the manifold n is even. In that case setting $\chi = e^1 \otimes \cdots \otimes e^n \in Cl(n)$ for some orthogonal basis $(e^j)_{j \le n}$ of T^*M we have $\chi \otimes x = -x \otimes \chi$ for all $x \in T^*M \subset Cl(n)$. Thus for any Clifford module bundle E the Clifford multiplication $\sigma(\chi)$ (called also the **chirality operator**) induces a $\mathbb{Z}^2$ grading of the bundle E, i.e. we can write $E = E_+ \oplus E_-$ where $E_\pm$ are the ± 1 eigensubspaces of $\sigma(\chi)$. If E is now a spinor bundle, the Dirac operator on E can be then written in the following form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \,,$$

where $D_\pm$ are mutual formal adjoints.

Definition 20. *We define the* **analytical index** *(or* **index***) of the Dirac operator D by*

$$\mathrm{ind}(D) = \dim \ker(D_+) - \dim \ker(D_-) \,. \tag{2.9}$$

Atiyah and Singer showed in [6] that if the manifold M is compact and has no boundary, then the analytical index is equal to the topological index

$$\operatorname{ind}(D) = \int_M AS\,.$$

The integrand AS depends both on the Riemannian curvature R_M and the magnetic two form β on E. (For the spin spinor bundle we simply set $\beta = 0$.) It is given by

$$AS = (\hat{A}(M)\mathrm{Ch}(E))_{[n]}\,, \quad \mathrm{Ch}(E) = \exp\frac{\beta}{2\pi}\,, \tag{2.10}$$

where the subscript $[n]$ refers to the $[n]$-th degree part of the form. We define the **A-roof-genus** by

$$\hat{A}(M) := \det{}^{1/2}\left(\frac{R_M/4\pi\mathrm{i}}{\sinh(R_M/4\pi\mathrm{i})}\right)\,,$$

where R_M is the Riemannian curvature of M. The expressions $\det^{1/2}$, sinh and exp are to be understood as the series expansions. Recall that if M is flat $R_M = 0$, hence $\hat{A}(M) = 1$. In such case only the two form β contributes.

Remark 21. *The index formula also holds for more general Clifford module bundles E called twisted spinor bundles. In such case* $\mathrm{Ch}(E)$ *is defined by*

$$2^{-n/2}Tr\left\{\exp\left(\frac{-R_{twist}}{2\pi\mathrm{i}}\right)\right\}\,,$$

for an $\mathrm{End}(E)$*-valued two form R_{twist} which commutes with the Clifford multiplication on E and satisfies* (c.f. *[9, Proposition 3.43, Theorem 4.3])*

$$R_E(X,Y) = \frac{1}{4}\sum_{j,k}(e_j, R_M(X,Y)e_k)\sigma(e^j)\sigma(e^k) + R_{twist}(X,Y)\,,$$

for any vectors $X, Y \in TM$. Here R_M is the Riemannian curvature of M and $(e_j)_{j\leq n}$ form a local orthonormal basis for vector fields on M with $(e^j)_{j\leq n}$ denoting the dual frame. The form R_{twist} can be found in the literature under the name twisting curvature of the bundle E. In this thesis only spinor bundles are considered which in particular means that R_{twist} is a $\mathbb{C}$ valued two form and therefore the traces in the index formula can be omitted. Our Definition 19 of the magnetic two form then corresponds exactly to an i *multiple of the twisting curvature.*

Magnetic field

In this section we will comment on the connection of the magnetic field defined in the terms of a differential two form as introduced in the previous section and the vector formalism, commonly used throughout the literature, representing the field strength by a vector $\vec{B}$ with a vector potential $\vec{a}$ satisfying $\operatorname{rot}\vec{a} = \vec{B}$.

We will restrict the following analysis to the case of a two dimensional manifold $M \subset \mathbb{R}^2 \simeq \mathbb{C}$ which is relevant to the problems considered in this thesis. A magnetic field is described by the magnetic two form $\beta = B(z)\frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$, where $B(z)$ is a real-valued function on $\mathbb{C}$ which we identify with the lift by the Riemannian metric g of its Hodge dual, i.e.

$$(\vec{B})_j = g_{jk}(*\beta)^k\,, j,k \in \{x,y,z\}\,.$$

Since β is proportional to the volume form on M we have $\vec{B} = (0,0,B)^T$ if $g_{jk} = \delta_{jk}$. By the Gauss law the real magnetic two form is a closed differential form which means that $\mathrm{d}\beta = 0$. The Poincaré lemma further implies that there is a one form α such that locally we can write $\beta = \mathrm{d}\alpha$. Moreover, for open star-shaped domains in $\mathbb{R}^n$ we can choose α globally. Therefore we associate with β a magnetic one form that is expressed by $\alpha = (a_x, a_y)$ in the coordinate basis $(\mathrm{d}x, \mathrm{d}y)$. This is then identified with the vector potential $\vec{a} = (a^x, a^y)$ in the basis (∂_x, ∂_y) directly by means of the Riemannian metric, setting $a_j = g_{jk}a^k$ with $j,k \in \{x,y\}$. In the language of differential forms we obtain by the Stokes theorem the following integral relation

$$\int_X \beta = \oint_{\partial X} \alpha\,, \tag{2.11}$$

for any domain $X \subset M$. In the case the metric $g_{jk} = \delta_{jk}$ the components of the one form α and the corresponding vector field $\vec{a}$ coincide. Then we can rewrite the integral on the right-hand side as $\oint_{\partial X}\alpha = \oint_{\partial X}\vec{a}\,\mathrm{d}\vec{s}$ with $\vec{a}\,\mathrm{d}\vec{s} = a^x\,\mathrm{d}x + a^y\,\mathrm{d}y$.

From the theory of differential geometry we know that in three dimensions the operator $\operatorname{rot} : TM \to TM$ can be written as $\operatorname{rot} = g^{-1} \circ (*\,\mathrm{d} \circ g)$. Hence, using the local relation $\beta = \mathrm{d}\alpha$ we retrieve the well-known relation of the vector quantities

$$\operatorname{rot}(a^x, a^y, 0) = (0, 0, \partial_x a^y - \partial_y a^x) = (0, 0, B)\,.$$

Further in the text we also use the complex notation

$$a = a_x + \mathrm{i}a_y\,.$$

Example 22. *Let us consider the connection $\nabla = \mathrm{d} - \mathrm{i}\alpha$ on $\mathbb{R}^2$ given by the canonical momentum obtained by the principle of correspondence and the minimal coupling. We denote by (a_x, a_y) the components of the connection one form α in the basis $(\mathrm{d}x, \mathrm{d}y)$ and by β the corresponding magnetic two form. Recall that the solution of $(-\mathrm{i}\partial_x - a_x)f = 0$, at a point (x, y_0) reads $f(x, y_0) = f(x_0, y_0)\mathrm{e}^{\mathrm{i}\int_{x_0}^{x} a_x\,\mathrm{d}x'}$,* *for some $(x_0, y_0) \in \mathbb{R}^2$. Therefore, if we consider a loop γ_ϵ for some $\epsilon > 0$, formed by integral curves of the fields $\epsilon\partial_x, \epsilon\partial_y, -\epsilon\partial_x, -\epsilon\partial_y$ we obtain the equality*

$$f_1 = f\mathrm{e}^{\mathrm{i}\oint_{\gamma_\epsilon} \vec{a}\,\mathrm{d}\vec{s}} = f\mathrm{e}^{\mathrm{i}\oint_{\gamma_\epsilon} \alpha} = f\mathrm{e}^{\mathrm{i}\int_{int\gamma_\epsilon} \beta},$$

at (x_0, y_0), where f_1 denotes the parallel transported function f along γ_ϵ. Assuming that ϵ is small we have $\int_{int\gamma_\epsilon} \beta = \epsilon^2 \beta(\partial_x, \partial_y)$ and thus for an infinitesimal loop it holds

$$f_1 - f = \mathrm{i} f \epsilon^2 \beta(\partial_x, \partial_y) + \mathcal{O}(\epsilon^4).$$

Recalling Remark 7, this example motivates Definition 19 of the magnetic two form as the curvature of a vector bundle in the particular case of the flat manifold $\mathbb{R}^2$.

Note that the vector potential is not given uniquely by the magnetic field B but we have a freedom of the gauge choice. Throughout this text we will use the divergence free gauge, i.e.

$$\partial_x a_x + \partial_y a_y = 0.$$

We will introduce the scalar potential $h(z)$, which will be very useful in the proofs of the Aharonov–Casher type theorems, so that the following holds

$$\partial_z h(z) = -\frac{\mathrm{i}\bar{a}}{2}, \tag{2.12}$$

with notation $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $a = a_x + \mathrm{i}a_y$. For a divergence free a this implies

$$\begin{aligned} -\frac{1}{4}\Delta h = -\partial_{\bar{z}}\partial_z h &= \frac{1}{4}(\partial_x + \mathrm{i}\partial_y)(\mathrm{i}a_x + a_y) \\ &= \frac{\mathrm{i}}{4}(\partial_x a_x + \partial_y a_y) + \frac{1}{4}(\partial_x a_y - \partial_y a_x) = \frac{1}{4}B. \end{aligned} \tag{2.13}$$

Now, recall that a solution of the problem

$$-\Delta h = B, \tag{2.14}$$

on $\mathbb{C}$, is for sufficiently fast decaying B given by the real-valued function

$$h(z) = -\frac{1}{2\pi}\int_{\mathbb{C}} \log|z - z'| B(z') \frac{\mathrm{i}}{2}\,\mathrm{d}z' \wedge \overline{\mathrm{d}z'}. \tag{2.15}$$

Another quantity that describes the magnetic field is called the **magnetic flux**

$$\Phi := \int_{\mathbb{C}} B \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Notice that in particular for a compactly supported B we have the asymptotic behaviour

$$h(z) = -\frac{\Phi}{2\pi} \log|z| + \mathcal{O}(|z|^{-1}), \tag{2.16}$$

as z tends to infinity. Moreover in the case of a spherically symmetric B there is no error term and for z outside of support of B

$$h(z) = -\frac{\Phi}{2\pi} \log|z|,$$

by the Newton's law.

Remark 23. *Let us now consider a smooth magnetic field B with compact support. By elliptic regularity (see e.g. [16, Section 6.3, Theorem 3]) the potential h is then also a smooth function. Note that the Poisson equation (2.14) determines h up to an addition of a harmonic function and our particular choice corresponds to the unique gauge choice via the relation (2.12), yielding a divergence free a bounded at infinity. We will refer to this as the* **Aharonov–Casher gauge**. *For a smooth magnetic field B this gauge gives a smooth vector potential which decays to zero as $|z| \to \infty$, hence, is bounded. To see this, let us assume that the support of B is contained inside a ball of radius R'' with centre at the origin and consider $|z| > R' > 2R''$ for some R'. Using the bound*

$$\left|\frac{B(z')}{z - z'}\right| \leq \frac{2}{R'}|B(z')| \in L^1(\mathbb{C}), \quad z' \in \operatorname{supp} B,$$

we can apply the dominated convergence theorem to obtain

$$\partial_z h(z) = -\frac{1}{4\pi}\partial_z \int_{\mathbb{C}} B(z') \log|z - z'|^2 \frac{\mathrm{i}}{2}\, \mathrm{d}z' \wedge \overline{\mathrm{d}z'} = -\frac{1}{4\pi}\int_{\mathbb{C}} \frac{B(z')}{z - z'} \frac{\mathrm{i}}{2}\, \mathrm{d}z' \wedge \overline{\mathrm{d}z'}.$$

Consequently we have the following estimate for large $|z|$

$$|\partial_z h(z)| \leq \frac{const}{|z|}\int_{\mathbb{C}} \frac{|B(z')|}{1 - \frac{|z'|}{|z|}} \frac{\mathrm{i}}{2}\, \mathrm{d}z' \wedge \overline{\mathrm{d}z'} \leq \frac{const}{|z|}\int_{\mathbb{C}} |B(z')| \mathrm{i}\, \mathrm{d}z' \wedge \overline{\mathrm{d}z'} \leq \frac{const}{|z|}.$$

Remark 24. 1. *In the flat space $\mathbb{R}^2$ we clearly have $a_x = a^x$ and $a_y = a^y$. Notice that if we consider the polar coordinates*

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\frac{y}{x},$$

and write $\alpha = (a_r, a_\varphi)$ for the components in the normalized basis $(\mathrm{d}r, r\,\mathrm{d}\varphi)$ and $\vec{a} = (a^r, a^\varphi)$ for the components in the dual basis $\left(\partial_r, \frac{\partial_\varphi}{r}\right)$ we also have $a_r = a^r$ and $a_\varphi = a^\varphi$.

2. *We can write the divergence in the complex notation* $\partial_x a_x + \partial_y a_y = \operatorname{Re}(2\partial_z a)$. *Consequently we see that if we define the magnetic field by the magnetic strength* B *then the vector potential defined by* (2.12) *with* (2.14) *is indeed divergence free*

$$\operatorname{Re}(2\partial_{\bar z}\bar a) = \operatorname{Re}(4\mathrm{i}\partial_{\bar z}\partial_z h) = \operatorname{Re}(-\mathrm{i}B) = 0\,.$$

3. *Let us find explicitly the vector potential* (a_r, a_φ) *for a radially symmetric field* B. *We first recall that* $a_x + \mathrm{i}a_y = \mathrm{e}^{\mathrm{i}\varphi}(a_r + \mathrm{i}a_\varphi)$ *and that for the partial differentials it holds* $\partial_x - \mathrm{i}\partial_y = \mathrm{e}^{-\mathrm{i}\varphi}\left(\partial_r - \mathrm{i}\frac{\partial_\varphi}{r}\right)$. *Then the computation*

$$2\partial_z h(z) = \mathrm{e}^{-\mathrm{i}\varphi}\left(\partial_r - \mathrm{i}\frac{\partial_\varphi}{r}\right)\frac{-\Phi}{2\pi}\log r = \mathrm{e}^{-\mathrm{i}\varphi}\frac{-\Phi}{2\pi r} = -\mathrm{i}\bar a\,,$$

yields

$$a_r = 0\,,\quad a_\varphi = \frac{\Phi}{2\pi r}\,.$$

The Dirac operator on the plane

In the first part of this thesis we consider only the mass-less Dirac operator with magnetic field $a = (a_1, a_2)$ on $\mathbb{R}^2$ or its subsets. The Dirac operator is an elliptic operator of the first order and in this particular case takes the form

$$D_a = -\mathrm{i}\sum_{j=1,2}\sigma^j(\partial_j - \mathrm{i}a_j)\,, \tag{2.17}$$

where σ^j are the first two Pauli matrices (2.5). By ∂_j we denote the partial derivative $\frac{\partial}{\partial_{x_j}}$ and a_j are the components of the magnetic one form in the basis $(\,\mathrm{d}x^1, \mathrm{d}x^2)$. Let us remark that we use the mathematical notation here. In the usual physics notation the Dirac operator describing a negatively charged particle is obtained by the principle of correspondence from the minimal coupling and has the opposite sign in front of the vector potential term a_j. Using the convention $a = a_1 + \mathrm{i}a_2$, we can write the Dirac operator in the complex notation $\partial_z = \frac{1}{2}(\partial_{x_1} - \mathrm{i}\partial_{x_2})$, $\partial_{\bar z} = \frac{1}{2}(\partial_{x_1} + \mathrm{i}\partial_{x_2})$

$$D_a = -2\mathrm{i}\begin{pmatrix} 0 & \partial_z - \frac{\mathrm{i}\bar a}{2} \\ \partial_{\bar z} - \frac{\mathrm{i}a}{2} & 0 \end{pmatrix}\,. \tag{2.18}$$

For later use we present also the Dirac operator in polar coordinates, which reads

$$\begin{aligned} D_a &= -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(\partial_r - \mathrm{i}\frac{\partial_\varphi}{r}) \\ \mathrm{e}^{\mathrm{i}\varphi}(\partial_r + \mathrm{i}\frac{\partial_\varphi}{r}) & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(a_r - \mathrm{i}a_\varphi) \\ \mathrm{e}^{\mathrm{i}\varphi}(a_r + \mathrm{i}a_\varphi) & 0 \end{pmatrix} \\ &= -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(\partial_r - \mathrm{i}a_r - \mathrm{i}\frac{\partial_\varphi}{r} - a_\varphi) \\ \mathrm{e}^{\mathrm{i}\varphi}(\partial_r - \mathrm{i}a_r + \mathrm{i}\frac{\partial_\varphi}{r} + a_\varphi) & 0 \end{pmatrix}\,, \end{aligned} \tag{2.19}$$

where (a_r, a_φ) are the components of a in the basis of one-forms $(\,\mathrm{d}r, r\,\mathrm{d}\varphi)$.

Remark 25. *Let us comment on the domain of the maximal extension of D_a in the case of $B \in C_0^\infty(\mathbb{R}^2)$,*

$$\mathrm{dom}(D_{a,max}) := \{u \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid D_a u \in L^2(\mathbb{R}^2, \mathbb{C}^2)\}\,.$$

Using the Fourier transform $\hat{u}$ it is straightforward to show that for the free Dirac operator on $\mathbb{R}^2$ this set is the first Sobolev space

$$\begin{aligned} H^1(\mathbb{R}^2, \mathbb{C}^2) &= \{u \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid (1 + |\xi|^2)^{1/2}\hat{u} \in L^2(\mathbb{R}^2, \mathbb{C}^2)\} \\ &= \{u \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid |\xi|\hat{u} \in L^2(\mathbb{R}^2, \mathbb{C}^2)\} = D_{0,max}\,. \end{aligned}$$

If $a \neq 0$ then using the Aharonov–Casher gauge (2.12) we know that a is bounded (see Remark 23) and thus any $u \in H^1(\mathbb{R}^2, \mathbb{C}^2)$ is also in the domain $\mathrm{dom}(D_{a,max})$. Conversely, for any function $u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \in \mathrm{dom}(D_{a,max})$

$$\|(\partial_{x_1} \pm \mathrm{i}\partial_{x_2})u_\mp\|_{L^2} \leq \|(\partial_{x_1} \pm \mathrm{i}\partial_{x_2} - \mathrm{i}(a_1 \mp \mathrm{i}a_2))u_\mp\|_{L^2} + \|(a_1 \mp \mathrm{i}a_2))u_\mp\|_{L^2} < \infty\,.$$

Hence $\mathrm{dom}(D_{a,max}) = H^1(\mathbb{R}^2, \mathbb{C}^2)$. Moreover, recall that we define the minimal extension D_{min} as the closure of $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ in the operator graph norm

$$\|u\|_{D_a} := \|u\|_{L^2} + \|D_a u\|_{L^2}\,.$$

For a bounded vector potential this norm is equivalent to $\|\cdot\|_{D_0}$ in which case the closure is known ([23, Corollary 4.11]) to be again $H^1(\mathbb{R}^2, \mathbb{C}^2)$. We conclude that D_a is in fact self-adjoint on the maximal domain.

2.3 The Aharonov–Casher theorem: The zero modes

We call a **zero mode** of an elliptic operator T a solution $u \in \mathrm{dom}(T)$ of the problem $Tu = 0$. If T acts on sections of a spinor bundle E over an even-dimensional manifold and $\sigma(\chi)$ is the chirality operator (*cf.* page 16) on E we will further say that the zero mode has spin up if $\sigma(\chi)u = u$ and spin down if $\sigma(\chi)u = -u$. In two dimensions we have $\sigma(\chi) = \sigma^3$ with σ^3 the third Pauli matrix (2.5). The Aharonov–Casher theorem proved in 1979 in [1] is a result on the number of zero modes of the Pauli operator on $\mathbb{R}^2$ with the magnetic field B:

$$H_a = -\sum_{j=1,2} \begin{pmatrix} (\partial_j - \mathrm{i}a_j)^2 + B & 0 \\ 0 & (\partial_j - \mathrm{i}a_j)^2 - B \end{pmatrix}\,.$$

The authors omit any detailed discussion of the domain of this operator. Their proof, which we will follow here, works e.g for the case of a magnetic field $B \in C_0^\infty(\mathbb{R}^2)$. Noting that on the level of formal expressions it holds $H_a = D_a^2$ (the Dirac operator D_a was defined in (2.17)) we conclude by Remark 25 that, in the Aharonov–Casher gauge, H_a is self-adjoint on the second Sobolev space $H^2(\mathbb{R}^2, \mathbb{C}^2)$. Let us mention that the relation $H_a = D_a^2$ is a particular case of the Lichnerowicz formula, see e.g. [9, Theorem 3.52] or [14, Theorem 3.4].

Theorem 26. *Let B be a smooth magnetic field with compact support on $\mathbb{R}^2$ and $a = (a_1, a_2)$ be vector potential in the Aharonov–Casher gauge associated with B. Let $\Phi_0 := \int B(x)\,\mathrm{d}x$ be the flux of B. Then the operator H_a has precisely $\left\lfloor \frac{|\Phi_0|}{2\pi} \right\rfloor$ zero modes, provided $|\Phi_0| > 2\pi$. If $\Phi_0 > 0$ they all have spin up. If $\Phi_0 < 0$ they have spin down. Here $\lfloor y \rfloor$ is the biggest integer strictly smaller then y. In the case $|\Phi_0| \le 2\pi$ we have* $\ker H_a = \{0\}$.

For the completeness we include the proof here.

Proof. We will view $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and denote correspondingly $a = a_1 + \mathrm{i}a_2$. First we note that clearly $D_a u = 0$ on $H^2(\mathbb{C}; \mathbb{C}^2)$ implies $H_a u = D_a^2 u = 0$. Conversely, by self-adjointness of D_a if $H_a u = 0$, then $(H_a u, u) = \|D_a u\|^2 = 0$ and thus $D_a u = 0$. Hence, the solutions of $H_a u = 0$ and $D_a u = 0$ on $H^2(\mathbb{C}, \mathbb{C}^2)$ coincide. We denote $u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in H^2(\mathbb{C}, \mathbb{C}^2)$. To find the zero modes of D_a on $\mathbb{C}$ we then need to solve the following set of equations

$$\left[\partial_{\bar z} - \frac{\mathrm{i}a}{2}\right] u^+ = 0\,, \quad \left[\partial_z - \frac{\mathrm{i}\bar a}{2}\right] u^- = 0\,. \tag{2.20}$$

Using our potential function satisfying (2.12) it is easy to check that using Aharonov–Casher gauge

$$\mathrm{e}^{-h(z)} \left(\partial_{\bar z} - \frac{\mathrm{i}a}{2}\right) u^+ = \partial_{\bar z} \mathrm{e}^{-h(z)} u^+\,, \quad \mathrm{e}^{h(z)} \left(\partial_z - \frac{\mathrm{i}\bar a}{2}\right) u^- = \partial_z \mathrm{e}^{h(z)} u^-\,.$$

Hence $u \in \mathrm{dom}(D_a)$ is a zero mode only if the functions g^+ and g^-

$$g^+ := \mathrm{e}^{-h} u^+\,, \quad g^- := \mathrm{e}^{h} u^-$$

are analytic and anti-analytic on $\mathbb{C}$, respectively, i.e.

$$g^+(z) = \sum_{k \ge 0} d_k z^k\,, \quad g^-(z) = \sum_{k \ge 0} b_k \bar z^k\,,$$

for some $d_k, b_k \in \mathbb{C}$. Since for a function $u \in \mathrm{dom}(D_a)$ we require square integrability at infinity, employing the expansion (2.16) we obtain the linearly independent zero modes

$$u = \mathrm{e}^h \begin{pmatrix} z^k \\ 0 \end{pmatrix} \in L^2(\mathbb{C}, \mathbb{C}^2)\,,$$

for $0 \leq k < \frac{\Phi_0}{2\pi} - 1$, provided that $\Phi_0 > 2\pi$. If $\Phi_0 < 2\pi$, the zero modes are

$$u = \mathrm{e}^{-h} \begin{pmatrix} 0 \\ \bar{z}^k \end{pmatrix} \in L^2(\mathbb{C}, \mathbb{C}^2)\,,$$

with $0 \geq k > \frac{\Phi_0}{2\pi} + 1$. □

Since we will later need to refer to the general solutions of the problem (2.20) we present their form again in the following remark.

Remark 27. *The zero modes of the Dirac operator* (2.18) *on an open subset $M \subset \mathbb{C}$ are of the form*

$$\begin{pmatrix} u^+ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u^- \end{pmatrix}, \qquad \textit{with } u^\pm = \mathrm{e}^{\pm h(z)} g^\pm\,,$$

where g^+ and g^- are analytic and anti-analytic on M, respectively. The function h is given by the relation (2.15).

The literature on zero modes is vast and we will mention only a couple of works generalizing the Aharonov–Casher theorem. A proof of the result on a two sphere is due to Avron and Tomaras (but was not published) and it can be found e.g. in [25] or [14, Appendix A.3]. For generalization to measure-valued magnetic fields see [15]. Singular Aharonov–Bohm type fields were considered by Hirokawa and Ogurisu in [24], by Person in [30] and by Geyler and Šťovíček in [20]. Rozenblum and Shirokov, [34], showed that for certain singular magnetic fields there could be possibly infinite dimensional space of zero modes with having possibly both spin up and spin down modes. Results for the case of even dimensional Euclidean spaces were discussed by Person in [31]. Bony, Espinoza and Raikov invesitgate almost periodic potentials in [11]. On a bounded domain with Dirichlet boundary condition the related result was studied in [12].

2.4 Elliptic boundary conditions

In this section we are following the formalism for elliptic boundary conditions introduced in [8, 7]. We, however, diverted with the convention for the Clifford multiplication which is in the cited papers considered to be anti-hermitian and satisfying the Clifford relations (2.4) with an extra minus sign on the right-hand side. For a background overview on elliptic differential operators see e.g. [37] (in particular Section 5.11.).

We start with some notation. Let M be a Riemannian manifold with a compact boundary ∂M and metric g, and, let E be a hermitian vector bundle over M. Consider a differential operator $D : \Gamma(M, E) \to \Gamma(M, E)$ of order ℓ.

Definition 28. *The* **principle symbol** *of D is the mapping $\hat{\sigma} : (T^*M)^\ell \to \mathrm{End}(E)$ defined by*

$$\hat{\sigma}(\xi_1, \xi_2, \ldots, \xi_\ell) = \frac{\mathrm{i}^\ell}{\ell!}[\ldots[D, f_1], f_2, \ldots, f_\ell]$$

for all $f_1, f_2, \ldots, f_\ell \in C^\infty(M)$ such that $\xi_j = \mathrm{d}f_j(m)$ for some $m \in M$ and all $j \leq \ell$.

We say that a differential operator D is **elliptic** if its principal symbol $\hat{\sigma}(\xi_1, \xi_2, \ldots, \xi_\ell)$ is invertible for all $(\xi_1, \xi_2, \ldots, \xi_\ell) \in (T^*M \setminus \{0\})^\ell$. The **formal adjoint** of D is the operator D^* satisfying

$$\int_M (D\varphi, \psi)_E \,\mathrm{d}vol_M = \int_M (\varphi, D^*\psi)_E \,\mathrm{d}vol_M ,$$

for all $\varphi \in \Gamma(M, E)$ with compact support contained in the interior of M and all $\psi \in \Gamma(M, E)$. Here $\mathrm{d}vol_M$ denotes the volume form on M and $(\cdot, \cdot)_E$ denotes the inner product on E. We say that D is **formally self-adjoint** if $D = D^*$.

Example 29. *1. The Dirac operator D (see Definition 18) on a spinor bundle E over oriented Riemannian manifold M of dimension n is a formally self-adjoint elliptic operator and its principal symbol coincides with the Clifford multiplication. Indeed, let $(e_j)_{j\leq n}$ be an orthonormal basis on TM and $(e^j)_{j\leq n}$ be the dual basis on T^*M. Then for any $\varphi, \psi \in \Gamma(M, E)$ where φ has compact support contained in M°*

$$\begin{aligned}
\int_M (D\varphi, \psi)_E \,\mathrm{d}vol_M &= \int_M \sum_{j\leq n} \left(-\mathrm{i}\nabla_{e_j}\varphi, \sigma(e^j)\psi\right)_E \mathrm{d}vol_M \\
&= \int_M \sum_{j\leq n} e_j \left(-\mathrm{i}\varphi, \sigma(e^j)\psi\right)_E + \left(\varphi, -\mathrm{i}\nabla_{e_j}\sigma(e^j)\psi\right)_E \mathrm{d}vol_M \\
&= \int_M \sum_{j\leq n} -e_j \left(\varphi, -\mathrm{i}\sigma(e^j)\psi\right)_E + \left(\varphi, -\mathrm{i}[\nabla_{e_j}, \sigma(e^j)]\psi\right)_E \\
&\quad + \left(\varphi, -\mathrm{i}\sigma(e^j)\nabla_{e_j}\psi\right)_E \mathrm{d}vol_M \\
&= \int_M (\varphi, D\psi)_E \,\mathrm{d}vol_M ,
\end{aligned}$$

where we used in the last equality that the sum

$$\begin{aligned}
&\sum_{j\leq n} e_j \left(\varphi, \sigma(e^j)\psi\right)_E - \left(\varphi, \sigma(\nabla^{LC}_{e_j}(e^j)\psi\right)_E \\
&\qquad = \sum_{k,j\leq n} \left(\nabla^{LC}_{e_k}\left[\left(\varphi, \sigma(e^j)\psi\right)_E e_j\right], e_k\right)
\end{aligned}$$

is a divergence term and since φ is compactly supported on M the integral of this term over M vanishes. Here $(\cdot,\cdot)$ is the complexification of the inner product on the tangent space of M.

Let $f, g \in C^\infty(M)$. For the principal symbol we have

$$\mathrm{i}[-\mathrm{i}\sum_{j\leq n}\sigma(e^j)\nabla_{e_j}, f] = \sum_{j\leq n}\sigma(e^j)\partial_j f = \sigma(\mathrm{d}f)\,.$$

2. *As another example we find the principal symbol of the Laplacian operator on $\mathbb{R}^n$ acting as $-\Delta f = -\sum_{j\leq n}\partial_j\partial_j f$ on $f \in C^\infty(\mathbb{R}^n)$. Computing*

$$[\partial_j\partial_j, f] = \partial_j[\partial_j, f] + [\partial_j, f]\partial_j = \partial_j\partial_j f + 2\partial_j f\partial_j\,,$$
$$-[[\partial_j\partial_j, f], g] = -2\partial_j f[\partial_j, g] = -2\partial_j f\partial_j g\,,$$

for any $f, g \in C^\infty(M)$ we see that denoting $\xi_1 = \mathrm{d}f(m)$ and $\xi_2 = \mathrm{d}g(m)$ for $m \in M$, the principal symbol of $-\Delta$ at a point $m \in M$ satisfies $\hat{\sigma}(\xi_1, \xi_2) = (\xi_1, \xi_2)$.

Remark 30. *Since in this thesis we will consider only Dirac operators we will, motivated by the previous example, from now on denote by $\sigma(\cdot)$ both the Clifford multiplication and the principal symbol. For the Dirac operators the difference is then that the Clifford multiplication is defined on the algebra $Cl(T^*M)$ while the principal symbol is a map from co-vectors. A mild justification for this abuse of notation is that we will use only action of the Clifford multiplication by elements in T^*M.*

We will denote by $\nu \in T^*M$ the co-vector field on the boundary ∂M dual to the inner normal vector field on ∂M. The space of co-vectors tangent to the boundary is defined by

$$T^*\partial M := \{\xi \in T^*M \mid (\xi, \nu) = 0\}\,,$$

where $(\cdot,\cdot)$ is the inner product T^*M naturally induced by the Riemannian metric g. In what follows we consider D to be the Dirac operator, though the formalism in [8] is introduced for a broader class of the so-called Dirac type operators. Using the principal symbol of D we introduce a boundary operator. Abusing the notation, by $E \to \partial M$ we mean the restriction of the bundle $E \to M$ to ∂M.

Definition 31. *Let D be a Dirac operator on a spinor bundle $E \to M$ with principal symbol σ and let ν be the one form dual to the inward normal vector on ∂M. A* **boundary operator adapted to D** *is a formally self-adjoint operator $D^\partial : \Gamma(\partial M, E) \to \Gamma(\partial M, E)$ of first order whose principal symbol is $\sigma_{D^\partial}(\xi) := \mathrm{i}\sigma(\nu)^{-1}\sigma(\xi)$ for any $\xi \in T^*\partial M$.*

The importance of boundary operators is that one can use them for a construction of elliptic boundary conditions which give rise to domains that are subsets of $H^1_{loc}(M,E) = \{u \in L^2_{loc}(M,E) \mid \nabla u \in L^2_{loc}(M,E)\}$ and on which D is a Fredholm operator, therefore the name elliptic. Here $L^2_{loc}(M,E)$ denotes sections of E that are square integrable over each compact subset $K \subset M$ and, in particular, we may have $K \cap \partial M \neq \emptyset$. We also recall that Fredholm operator is an operator with closed range and finite dimensional kernel and cokernel.

Since the Dirac operator D is formally self-adjoint, we can (see Lemma 2.2 in [8]) choose a boundary operator D^∂ so that the anti-commutation condition $\{\sigma(\nu), D^\partial\} = 0$ holds. With such boundary operators we can construct domains on which D is self-adjoint. Note, however, that D^∂ is not uniquely defined (not even by this additional anticommutation condition) as we are allowed to add any field of hermitian zero order endomorphisms on E that anti-commute with $\sigma(\nu)$. We will consider a unique choice of the boundary operator constructed as follows (for more details see the Appendices in [8]).

Definition 32. *Let E be a spinor bundle with Clifford connection ∇ and Clifford multiplication σ. Setting $A_0 = \mathrm{i}\sigma(\nu)^{-1}D - \nabla_n$, where n is the vector field dual to ν, we define the* **canonical boundary operator adapted to** *D by*

$$A = \frac{A_0 - \sigma(\nu)A_0\sigma(\nu)^{-1}}{2}\,.$$

Using $\sigma(\nu)^2 = I$, it is straightforward to check that $\{\sigma(\nu), A\} = 0$. Since A is an elliptic operator on the compact manifold ∂M, it has purely discrete spectrum. Consequently, denoting v_j, $j \in \mathbb{Z}$, an eigenvector of A corresponding to an eigenvalue λ_j, the eigenbasis $(v_j)_{j\in\mathbb{Z}}$ forms a basis of $L^2(\partial M, E)$. Since A has both positive and negative eigenvalues, we use $\mathbb{Z}$ as our index set. For some $t \in \mathbb{R}$, we denote

$$\check{H}(A) = H^{1/2}_{(-\infty,t)}(\partial M, E) \oplus H^{-1/2}_{[t,\infty)}(\partial M, E)\,,$$

where $H^s_{\mathcal{J}}(\partial M, E)$, $s \in \mathbb{R}$ is a subset of the Sobolev space on the boundary defined for $\mathcal{J} \subset \mathbb{R}$ by

$$H^s_{\mathcal{J}}(\partial M, E) = \{v = \sum_{\{j|\lambda_j \in \mathcal{J}\}} c_j v_j \mid c_j \in \mathbb{C}\,,\ \|v\|_{H^s(\partial M,E)} < \infty\}\,,$$

where we set $\|\sum_{j\in\mathbb{Z}} c_j v_j\|^2_{H^s(\partial M,E)} = \sum_{j\in\mathbb{Z}} |c_j|^2(1+\lambda_j^2)^s$. We further define norm on $\check{H}(A)$ by

$$\Big\|\sum_{j\in\mathbb{Z}} c_j v_j\Big\|^2_{\check{H}(A)} = \Big\|\sum_{\{j|\lambda_j<t\}} c_j v_j\Big\|^2_{H^{1/2}(\partial M,E)} + \Big\|\sum_{\{j|\lambda_j\geq t\}} c_j v_j\Big\|^2_{H^{-1/2}(\partial M,E)}\,, \quad (2.21)$$

for all $\sum_{j\in\mathbb{Z}} c_j v_j \in \check{H}(A)$. Fixing a finite interval $\mathcal{J}$, we remark that $H^s_{\mathcal{J}}(\partial M, E)$ is a finite dimensional space of smooth sections on ∂M, hence, $\check{H}(A)$ and its topology is independent of a particular choice of $t \in \mathbb{R}$.

Definition 33. *A* **boundary condition** *for the Dirac operator D is a closed subspace BC of $\check{H}(D^\partial)$ for some boundary operator D^∂ adapted to D. The boundary condition BC is further called* **elliptic** *if the domain*

$$\{u \in \mathrm{dom}(D_{max}) \mid u|_{\partial M} \in BC\}$$

is a subset of $H^1_{loc}(M, E)$.

Theorem 1.12 in [7] (alternatively Definition 3.7 and Theorem 3.12 in [8]) describes a wide range of elliptic boundary conditions. We are, however, interested in a particular choice called the **APS (Atiyah–Patodi–Singer) boundary condition** which is characterised as follows

$$BC_{APS} = H^{1/2}_{(-\infty,0)}(A) \oplus N(A)\,, \tag{2.22}$$

where $N(A)$ is a subspace of the kernel $\ker(A)$ such that $\ker(A) = N(A) \oplus \sigma(\nu)N(A)$. Here we consider A to be the canonical boundary operator. We then call the realisation of D on the domain

$$\mathrm{dom}(D) = \{u \in \mathrm{dom}(D_{max}) \mid u\big|_{\partial M} \in H^{1/2}_{(-\infty,0)}(A) \oplus N(A)\}\,, \tag{2.23}$$

the **Dirac operator with APS boundary condition**. By [8, Theorem 3.12.] this is a self-adjoint realisation. Let us remark that there are also other choices of more general APS boundary conditions corresponding to different (non-canonical) choices of boundary operators or replacing $H^{1/2}_{(-\infty,0)}(A)$ in (2.22) by $H^{1/2}_{(-\infty,t)}(A)$ for some $t \in \mathbb{R}$.

Remark 34. *Recall that if the dimension of M is even we can write the Dirac operator on a spinor bundle E over M in the form $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$. We say that D is coercive at infinity if there is a compact subset $K \subset M$ such that for some constant $C > 0$ and all $u \in \Gamma(M, E)$ with a compact support contained in $M \setminus K$ it holds*

$$C\|u\| \leq \|Du\|\,,$$

with $\|\cdot\|$ being the norm on $L^2(M, E)$. For M compact D is automatically coercive at infinity. In such case Theorem 5.3 in [8] implies that the Dirac operator D with the APS boundary condition is Fredholm, i.e. it has a closed range and $\dim\ker(D_\pm) < \infty$. In this work we are also interested in M being a plane with holes. Our result Theorem 39 states that in such case there are zero modes which further by Remark 42 are smooth on M but not compactly supported, and hence, the condition for coercivity cannot be satisfied. In fact zero is an eigenvalue embedded in the essential spectrum and D does not have a closed range.

2.5 The APS boundary condition for the two-dimensional Dirac operator

Example: APS boundary condition for one inner hole

In this example we will work out the APS boundary condition for the Dirac operator (2.17) on the manifold $M = \mathbb{C} \setminus \Omega$ with magnetic field B of flux Φ which is supported inside an open ball $\Omega \subset \mathbb{C}$ with radius 1 and centre at the origin. It is convenient to work in polar coordinates (r, φ). We denote by

$$(e_r, e_\varphi) = \left(\partial_r, \frac{\partial_\varphi}{r}\right), \text{ and } (e^r, e^\varphi) = (\,\mathrm{d}r, r\,\mathrm{d}\varphi)\,, \tag{2.24}$$

the orthonormal (in the standard metric on $\mathbb{R}^2$) basis of TM and the dual basis on T^*M, respectively. The inward normal one-form on the boundary is then simply $\nu = \mathrm{d}r$. In accordance we will denote $\sigma(\nu) = \sigma^r$. The Clifford connection is given by $\nabla_{e_j} = e_j - \mathrm{i}a_j$, where a_j are the components of the vector potential in basis e^j. To find the APS boundary condition we rewrite the operator D_a, (2.19), in the form

$$D_a = -\mathrm{i}\sigma^r(\nabla_{\partial_r} + A_0) = -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(\nabla_{e_r} - \mathrm{i}\nabla_{e_\varphi}) \\ \mathrm{e}^{\mathrm{i}\varphi}(\nabla_{e_r} + \mathrm{i}\nabla_{e_\varphi}) & 0 \end{pmatrix}.$$

Then we have

$$\sigma^r = \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} = (\sigma^r)^* = (\sigma^r)^{-1},$$

and using the notation from Definition 32 we obtain

$$A_0 = \mathrm{i}\sigma^3\nabla_{e_\varphi} = \sigma^3\left(\frac{\mathrm{i}\partial_\varphi}{r} + a_\varphi\right),$$

and the canonical boundary operator thus reads

$$A = \sigma^3\left(\frac{\mathrm{i}\partial_\varphi}{r} + a_\varphi\right) - \frac{1}{2r}.$$

Remark 35. *We can perform a consistency check and find the anti-commutators*

$$\begin{aligned}
\{\sigma^3, \sigma^r\} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi} \\ -\mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} = 0, \\
\{\mathrm{i}\sigma^3\partial_\varphi, \sigma^r\} &= \mathrm{i}\{\sigma^3, \sigma^r\}\partial\varphi + \mathrm{i}\sigma^3\partial\varphi(\sigma^r) = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{i}\mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} \\
&= \mathrm{i}\begin{pmatrix} 0 & -\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi} \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} = \sigma^r.
\end{aligned}$$

Then noting $\{\frac{1}{2r}, \sigma^r\} = \frac{1}{r}\sigma^r$ *we see immediately that* $\{A, \sigma^r\} = 0$.

Later we prove Lemma 37 by which we can choose $B = \Phi\delta_0$. In the case of one hole our choice of the divergence free gauge corresponds by the third point of Remark 24 to $a_r = 0$ and $a_\varphi = \frac{\Phi}{2\pi r}$ and therefore, in particular, a_φ is a constant on the boundary. This allows us to decompose an eigenfunction $u \in L^2(\partial M, \mathbb{C}^2)$ of A into the Fourier modes

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \quad u_\pm = \sum_{k\in\mathbb{Z}} u_\pm^k \mathrm{e}^{\mathrm{i}\varphi k}. \tag{2.25}$$

The eigenvalue problem $Au = \lambda u$ on the boundary $r = 1$ is equivalent to the set of the two decoupled equations

$$\begin{aligned} \left(-k + \frac{\Phi}{2\pi} - \frac{1}{2} - \lambda\right) u_+^k &= 0 \\ \left(k - \frac{\Phi}{2\pi} - \frac{1}{2} - \lambda\right) u_-^k &= 0, \end{aligned}$$

whose solution leads to the following explicit decomposition of the space $\check{H}(A)$ on $\partial\Omega_j$

$$\begin{aligned} H^{-1/2}_{(0,\infty)}(A) &= \overline{\operatorname{span}}^{\check{H}(A)} \left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi k} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni k < \frac{\Phi}{2\pi} - \frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi k} \end{pmatrix}\right]_{\mathbb{Z}\ni k > \frac{\Phi}{2\pi} + \frac{1}{2}} \right\} \\ H^{1/2}_{(-\infty,0)}(A) &= \overline{\operatorname{span}}^{\check{H}(A)} \left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi k} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni k > \frac{\Phi}{2\pi} - \frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi k} \end{pmatrix}\right]_{\mathbb{Z}\ni k < \frac{\Phi}{2\pi} + \frac{1}{2}} \right\} \\ \ker(A) &= \operatorname{span} \left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi k} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni k = \frac{\Phi}{2\pi} - \frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi k} \end{pmatrix}\right]_{\mathbb{Z}\ni k = \frac{\Phi}{2\pi} + \frac{1}{2}} \right\}, \end{aligned}$$

where $\overline{\operatorname{span}}^{\check{H}(A)}$ denotes the closure of the span in the norm $\|\cdot\|_{\check{H}(A)}$ defined by (2.21). The APS boundary condition (2.22) is therefore given by the following subspace

$$H^{1/2}_{(-\infty,0)}(A) \oplus N(A) = \overline{\operatorname{span}}^{\check{H}(A)} \left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi n} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni n > \frac{\Phi}{2\pi} - \frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi n} \end{pmatrix}\right]_{\mathbb{Z}\ni n \leq \frac{\Phi}{2\pi} + \frac{1}{2}} \right\},$$

and, the Dirac operator with this boundary condition has the domain (2.23)

$$\begin{aligned} \operatorname{dom}(D_a) = \Big\{ u \in \operatorname{dom}(D_{a,max}) \mid u^+|_{\partial\Omega} &= \sum_{\mathbb{Z}\ni k > \frac{\Phi}{2\pi} - \frac{1}{2}} u_+^k(1)\mathrm{e}^{\mathrm{i}\varphi k}, \\ u^-|_{\partial\Omega} &= \sum_{\mathbb{Z}\ni k \leq \frac{\Phi}{2\pi} + \frac{1}{2}} u_-^k(1)\mathrm{e}^{\mathrm{i}\varphi k} \Big\}. \end{aligned}$$

Let us remark that in the APS boundary condition (2.22) the "half of the kernel" $N(A)$ of A is not given uniquely and we made a particular choice here. See Remark 36 for comments on the other choice corresponding to $\sigma(\nu)N(A)$.

The APS boundary condition for N inner holes

Let us first introduce the setting. Denote by Ω_j the open ball with centre at $w_j \in \mathbb{C}$ and radius R_j, and assume $\overline{\Omega_j} \cap \overline{\Omega_k} = \emptyset$ for all $k \neq j$. We put $M = \mathbb{C} \setminus \cup_{k \leq N} \Omega_k$ for some (finite) $N \in \mathbb{N}$. Further we fix an index $j \leq N$ and denote by (r_j, φ_j) the polar coordinates centred at w_j in order to find the boundary condition on the component $\partial\Omega_j$ of the boundary. We use the corresponding adaptation of notation (2.24).

The magnetic field we treat here consists of two parts. The smooth part supported in the bulk is denoted by $B_0 \in C_0^\infty(M)$ and the magnetic field inside the holes by $B_j \in \mathscr{D}'(\Omega_j)$, $j \leq N$, where $\mathscr{D}'(\Omega_j)$ denotes the distributions (continuous functionals on the space of $C_0^\infty(\Omega_j)$) with a compact support inside Ω_j. We also denote by $B_{sing} = \sum_{j \leq N} B_j$ and the total magnetic field by

$$B = B_0 + \sum_{j \leq N} B_j \,.$$

Finally we use the notation $a, a_0, a_{sing}, h, \Phi$ etc. for the corresponding vector and scalar potentials and fluxes. To find the canonical APS boundary condition in the case of multiple holes we cannot apply the explicit gauge as before and therefore it is not in general possible to perform the decomposition (2.25) into the Fourier modes.

We can again write the Dirac operator in polar coordinates (2.19), though only locally on the neighbourhood of $\partial\Omega_j$,

$$D_a = -\mathrm{i}\sigma^{r_j}\left(\nabla_{e_{r_j}} + A_{0,j}\right) = -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi_j}\left(\nabla_{e_{r_j}} - \mathrm{i}\nabla_{e_{\varphi_j}}\right) \\ \mathrm{e}^{\mathrm{i}\varphi_j}\left(\nabla_{e_{r_j}} + \mathrm{i}\nabla_{e_{\varphi_j}}\right) & 0 \end{pmatrix} .$$

The operator $A_{0,j}$ (the restriction of A_0 from Definition 32 to the boundary component $\partial\Omega_j$) then explicitly reads

$$A_{0,j} = \mathrm{i}\sigma^3 \nabla_{e_{\varphi_j}} = \sigma^3\left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j}\right) ,$$

and the canonical boundary operator A on $\partial\Omega_j$, which we similarly denote by A_j, is given by

$$A_j = \sigma^3\left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j}\right) - \frac{1}{2R_j} \,. \tag{2.26}$$

The eigenvalue problem $A_j u = \lambda u$, $u = (u_+, u_-)^T$, then corresponds to the following pair of equations

$$\begin{aligned} \mathrm{i}\partial_{\varphi_j} u_+ &= \left(R_j\lambda + \frac{1}{2} - R_j a_{\varphi_j}\right) u_+ \\ -\mathrm{i}\partial_{\varphi_j} u_- &= \left(R_j\lambda + \frac{1}{2} + R_j a_{\varphi_j}\right) u_- \,, \end{aligned}$$

from which we infer that

$$\begin{aligned} u_{\pm} &= \exp\left[\mp \mathrm{i} \int_{\gamma_j} R_j \lambda + \frac{1}{2} \mp R_j a_{\varphi_j}(q^j)\, \mathrm{d}q^j\right] \\ &= \exp\left[\mp \mathrm{i}\left(R_j\lambda + \frac{1}{2}\right)\varphi_j + \mathrm{i}\int_{\gamma_j} \vec{a}\, \mathrm{d}\vec{s}\right], \end{aligned}$$

where $\gamma_j \subset \partial\Omega_j$ denotes the curve connecting $z_{0j} = w_j + R_j$ and the point $z \in \partial\Omega_j$ (for illustration see Figure 2.3). In the second equality we used the fact that $R_j\, \mathrm{d}q^j$ is the line element on the boundary $\partial\Omega_j$ and $(0, a_{\varphi_j})$ is cotangent vector on the boundary $\partial\Omega_j$ (*c.f.* also the first point of Remark 24). By periodicity in φ_j we require $u_{\pm}(\varphi_j = 0) = u_{\pm}(\varphi_j = 2\pi)$ which takes us to the following condition for λ

$$R_j\lambda = \mp\left(n - \frac{\Phi_j}{2\pi}\right) - \frac{1}{2},$$

for some $n \in \mathbb{Z}$. Thus we have eigenspaces corresponding to eigenvalues $\{R_j^{-1}(-n + \frac{\Phi_j}{2\pi} - \frac{1}{2})\}_{n\in\mathbb{Z}}$ and $\{R_j^{-1}(n - \frac{\Phi_j}{2\pi} - \frac{1}{2})\}_{n\in\mathbb{Z}}$ spanned by

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_j n} \\ 0 \end{pmatrix} \cdot \exp\left[\int_{\gamma_j} \mathrm{i}\vec{a}\, \mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi_j}{2\pi}\varphi_j\right] \quad \text{and} \quad \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi_j n} \end{pmatrix} \cdot \exp\left[\int_{\gamma_j} \mathrm{i}\vec{a}\, \mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi_j}{2\pi}\varphi_j\right],$$

respectively. Finally we can write the APS boundary condition on the boundary component $\partial\Omega_j$ as described in (2.23)

$$\begin{aligned} &H^{1/2}_{(-\infty,0)}(A_j) \oplus N(A_j) \\ &\quad = \overline{\mathrm{span}}^{\check{H}(A)}\left\{\left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_j n} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni n > \frac{\Phi_j}{2\pi} - \frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi_j n} \end{pmatrix}\right]_{\mathbb{Z}\ni n \le \frac{\Phi_j}{2\pi} + \frac{1}{2}}\right\} \\ &\qquad\qquad \times \exp\left[\mathrm{i}\int_{\gamma_j} \vec{a}(s_j)\, \mathrm{d}\vec{s}_j - \mathrm{i}\frac{\Phi_j}{2\pi}\varphi_j\right], \end{aligned} \tag{2.27}$$

with $\overline{\mathrm{span}}^{\check{H}(A)}$ denoting the closure of the span in the norm $\|\cdot\|_{\check{H}(A)}$ defined by (2.21).

The APS boundary condition on the outer boundary

Another setting which will be of interest to us will be the previous scene when we consider a circular outer boundary. That is, we have a bounded region $M = \Omega_{out} \setminus \cup_{k\le N}\Omega_k$, where Ω_{out} is a disc of radius R_{out} with centre at the origin. Using the corresponding polar coordinates (r_{out}, φ_{out}), we compute the boundary condition on the component $\partial\Omega_{out}$ similarly as before.

The corresponding setting is sketched in Figure 2.2. The only difference in the process of finding the boundary operator on $\partial\Omega_j$ and $\partial\Omega_{out}$, is that the inner normal vector now corresponds to $-\partial_{r_{out}}$ and thus changes the sign of the principal symbol $\sigma^{r_{out}}$. Therefore we obtain up to a sign the same canonical boundary operator as in the case of the inner holes

$$A_{out} = -\sigma^3 \left(\frac{\mathrm{i}\partial_{\varphi_{out}}}{R_{out}} + a^{\varphi_{out}} \right) + \frac{1}{2R_{out}} . \tag{2.28}$$

The same computations as before yield that the solutions of the eigenproblem $A_{out}u = \lambda u$ are

$$u^{\pm} = \exp\left[\mp \mathrm{i} \int_{\gamma_{out}} \left(-\lambda R_{out} + \frac{1}{2} \mp R_{out} a^{\varphi_{out}} \right) \, \mathrm{d}q \right] ,$$

where $\gamma_{out} \subset \partial\Omega_{out}$ goes from $z_{0,out} = R_{out}$ to z counter-clockwise (see Figure 2.4). By the 2π periodicity the corresponding eigenvalues satisfy

$$\mp \left(\lambda R_{out} - \frac{1}{2} \right) = -n + \frac{\Phi}{2\pi} .$$

Putting these two relations together we obtain the eigensolutions

$$u^{\pm} = \exp\left[\mathrm{i} \left(-\frac{\Phi}{2\pi} + n \right) \varphi_{out} + \mathrm{i} \int_0^{\varphi_{out}} \vec{a}\, \mathrm{d}\vec{s} \right] , \quad \lambda R_{out} = \mp \left(\frac{\Phi}{2\pi} - n \right) + \frac{1}{2} .$$

Thus in addition to the boundary condition (2.27) on the components $\partial\Omega_j$, $j \leq N$ we now have the APS boundary condition on $\partial\Omega_{out}$

$$\begin{aligned} &H^{1/2}_{(-\infty,0)}(A_{out}) \oplus N(A_{out}) \\ &\qquad = \overline{\mathrm{span}}^{\check{H}(A)} \left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_{out} n} \\ 0 \end{pmatrix} \right]_{\mathbb{Z} \ni n < \frac{\Phi}{2\pi} - \frac{1}{2}} , \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi_{out} n} \end{pmatrix} \right]_{\mathbb{Z} \ni n \geq \frac{\Phi}{2\pi} + \frac{1}{2}} \right\} \\ &\qquad\qquad \times \mathrm{e}^{\mathrm{i} \int_{\gamma_{out}} \vec{a}\, \mathrm{d}\vec{s} - \mathrm{i} \frac{\Phi}{2\pi} \varphi_{out}} . \end{aligned} \tag{2.29}$$

Notice that the function $\mathrm{i} \int_{\gamma_{out}} \vec{a}\, \mathrm{d}\vec{s} - \mathrm{i} \frac{\Phi}{2\pi} \varphi_{out}$ is a well defined continuous function since at $\varphi_{out} = 2\pi$ we have $\mathrm{i} \oint_{\gamma_{out}} \vec{a}\, \mathrm{d}\vec{s} = \mathrm{i}\Phi$.

Remark 36. *As pointed out previously, $N(A)$ is not unique and we simply made a choice. The other choice corresponding to APS boundary condition $H^{1/2}_{(-\infty,0)}(A) \oplus \sigma(\nu)N(A)$ corresponds to moving the equality sign for indices n in the boundary conditions (2.27) and (2.29) from the spin down components to the spin up.*

Gauge invariance of the Dirac operator with APS boundary condition

The canonical APS boundary condition is gauge invariant in the sense of the following lemma.

Lemma 37. *Let D_a and $D_{\widetilde{a}}$ be two Dirac operators with the APS boundary condition on M corresponding to magnetic fields with fluxes Φ and $\widetilde{\Phi}$, respectively, such that*

$$\Phi = \sum_{j\leq N} \Phi_j + \Phi_0$$
$$\widetilde{\Phi} = \sum_{j\leq N} \widetilde{\Phi}_j + \Phi_0 \,,$$

where Φ_j and $\widetilde{\Phi}_j$ are the fluxes through the hole Ω_j, $j \leq N$ and Φ_0 is the flux of a smooth magnetic field supported inside the interior of M. If for all $j \leq N$

$$\widetilde{\Phi}_j = \Phi_j + m_j 2\pi \,,$$

for some $m_j \in \mathbb{Z}$, then D_a and $D_{\widetilde{a}}$ are unitarily equivalent

$$\mathscr{U}^* D_a \mathscr{U} = D_{\widetilde{a}} \,,$$

with the unitary operator

$$\mathscr{U} : L^2(M, \mathbb{C}^2) \to L^2(M, \mathbb{C}^2) \,,$$
$$\mathscr{U} : u \mapsto \exp\left[\mathrm{i} \int_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right] u \,,$$

where γ connects a fixed point $z_0 \in M$ and the point $z \in M$.

Remark 38. *This Lemma allows us to choose the the fluxes inside the holes so that $\Phi_j \in [-\frac{1}{2}, \frac{1}{2}) \cdot 2\pi$ for all $j \leq N$ for the purposes of spectral analysis. Moreover, any magnetic field inside a hole can be substituted by the corresponding flux multiple of the delta function at the centre of the hole, which further justifies the notation B_{sing} for the sum of the fluxes inside the holes.*

Proof. First we show that $\mathscr{U}$ is independent of a particular choice of the path γ in its definition. Since two different paths c and d with the same end points form a loop $\gamma = c - d$ it is sufficient to prove, that for an arbitrary loop $\gamma \subset M$

$$\exp\left[\mathrm{i} \oint_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right] = 1 \,.$$

This, however, follows immediately from the equalities

$$\oint_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s} = \int_{int\,\gamma} B - \widetilde{B} = -2\pi \sum_{\{j | \Omega_j \subset int\,\gamma\}} m_j \,.$$

Notice also that the dependence on the choice of the point z_0 is only a multiplication by a constant K such that $\overline{K} = K^{-1}$ so it leaves the map $\mathscr{U}$ unitary and the relation $\mathscr{U}^* D_a \mathscr{U} = D_{\widetilde{a}}$ untouched. Let z_1 be another choice of the starting point of γ. Then since $\mathscr{U}$ is independent of our choice of the path we have

$$\mathscr{U} = \mathrm{e}^{\mathrm{i}\int_{z_0}^{z}(\vec{a}-\widetilde{\vec{a}})\,\mathrm{d}\vec{s}} = \mathrm{e}^{\mathrm{i}\int_{z_1}^{z}(\vec{a}-\widetilde{\vec{a}})\,\mathrm{d}\vec{s}-\mathrm{i}\int_{z_1}^{z_0}(\vec{a}-\widetilde{\vec{a}})\,\mathrm{d}\vec{s}} = K\mathrm{e}^{\mathrm{i}\int_{z_1}^{z}(\vec{a}-\widetilde{\vec{a}})\,\mathrm{d}\vec{s}},$$

with the previously mentioned constant $K = \mathrm{e}^{-\mathrm{i}\int_{z_1}^{z_0}(\vec{a}-\widetilde{\vec{a}})\,\mathrm{d}\vec{s}}$.

Further we find the derivative $\partial_z \mathscr{U}$. Using the fundamental theorem of calculus we compute

$$\begin{aligned}\partial_\epsilon \,|_{\epsilon=0} \int_{\gamma(0)}^{\gamma(1)+\epsilon} (a-\widetilde{a})_x(q_1,q_2)\,\mathrm{d}q_1 &= \partial_\epsilon \,|_{\epsilon=0} \int_{\gamma_x(0)}^{\gamma_x(1)+\epsilon} (a-\widetilde{a})_x(q_1,q_2(q_1))\,\mathrm{d}q_1\\ &= (a-\widetilde{a})_x(\gamma_x(1)+\epsilon, q_2(\gamma_x(1)+\epsilon)\,|_{\epsilon=0}\\ &= (a-\widetilde{a})_x(\gamma_x(1),\gamma_y(1))\\ &= (a-\widetilde{a})_x(z)\,,\end{aligned}$$

and similarly we get

$$\begin{aligned}\partial_\epsilon \,|_{\epsilon=0} \int_{\gamma(0)}^{\gamma(1)+\mathrm{i}\epsilon} (a-\widetilde{a})_y(q_1(q_2),q_2)\,\mathrm{d}q_2 &= (a-\widetilde{a})_y(z)\\ \partial_\epsilon \,|_{\epsilon=0} \int_{\gamma(0)}^{\gamma(1)+\mathrm{i}\epsilon} (a-\widetilde{a})_x(q_1,q_2(q_1))\,\mathrm{d}q_1 &= 0\\ \partial_\epsilon \,|_{\epsilon=0} \int_{\gamma(0)}^{\gamma(1)+\epsilon} (a-\widetilde{a})_y(q_1(q_2),q_2)\,\mathrm{d}q_2 &= 0\,.\end{aligned}$$

Hence

$$\begin{aligned}\partial_z \mathscr{U} &= \partial_z \mathrm{e}^{\mathrm{i}\int_\gamma (a-\widetilde{a})_x\,\mathrm{d}x + (a-\widetilde{a})_y\,\mathrm{d}y}\\ &= \mathscr{U}\frac{1}{2}(\partial_x - \mathrm{i}\partial_y)\mathrm{i}\int_{\gamma(0)}^{\gamma(1)+x+\mathrm{i}y} (a-\widetilde{a})_x\,\mathrm{d}x + (a-\widetilde{a})_y\,\mathrm{d}y\\ &= \frac{\mathrm{i}}{2}\overline{(a-\widetilde{a})}(z)\mathscr{U}\,, \quad \text{and}\\ \partial_{\bar{z}}\mathscr{U} &= \frac{\mathrm{i}}{2}(a-\widetilde{a})(z)\mathscr{U}\,.\end{aligned}$$

Thus the unitarily transformed Dirac operator is indeed the one with the potential $\widetilde{a}$, as

$$\mathscr{U}^* D_a \mathscr{U} = D_a - 2\mathrm{i}\mathscr{U}^* \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \mathscr{U} = D_a - \mathrm{i}\begin{pmatrix} 0 & \mathrm{i}\overline{(a-\widetilde{a})} \\ \mathrm{i}(a-\widetilde{a}) & 0 \end{pmatrix}.$$

Finally we need to check that the boundary condition is preserved by $\mathscr{U}$. To do so, we show that the boundary operators are unitarily equivalent

$$A(\widetilde{a}) = \mathscr{U}^* A(a) \mathscr{U}\,, \tag{2.30}$$

where $A(a)$ denotes the canonical boundary operator adapted to D_a. From this we see that the restriction to the boundary of a spinor $u\big|_{\partial\Omega_j}$ is in the negative spectral subspace of $A(\widetilde{a})$ if and only if $(\mathscr{U} u)\big|_{\partial\Omega_j}$ is in the negative spectral subspace of $A(a)$. To see that (2.30) holds, we recall that in the previous section we found the expression (2.26) for $A(a)$ on $\partial\Omega_j$ in the local polar coordinates

$$A(a) = \sigma^3 \left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j} \right) - \frac{1}{2R_j}\,. \tag{2.31}$$

Hence on the boundary $z = (R_j, \varphi_j) \in \partial\Omega_j$ the commutator with the unitary operator reads

$$\begin{aligned}
[A(a), \mathscr{U}] &= \sigma^3 \frac{\mathrm{i}}{R_j} \partial_{\varphi_j}(\mathscr{U}) \\
&= \sigma^3 \frac{\mathrm{i}}{R_j} \partial_{\varphi_j} \exp\left[\mathrm{i} \int_0^{\varphi_j} (a_{\varphi_j}(q) - \widetilde{a}_{\varphi_j}(q)) R_j \,\mathrm{d}q \right] \\
&= -\sigma^3 (a_{\varphi_j}(z) - \widetilde{a}_{\varphi_j}(z)) \mathscr{U}(z)\,,
\end{aligned}$$

which yields

$$\begin{aligned}
A(a)\mathscr{U} u &= \mathscr{U} A(a) u + [A(a), \mathscr{U}] u \\
&= \mathscr{U} \left[\sigma^3 \left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j} \right) - \frac{1}{2R_j} - \sigma^3 (a_{\varphi_j} - \widetilde{a}_{\varphi_j}) \right] u \\
&= \mathscr{U} A(\widetilde{a}) u\,.
\end{aligned}$$

□

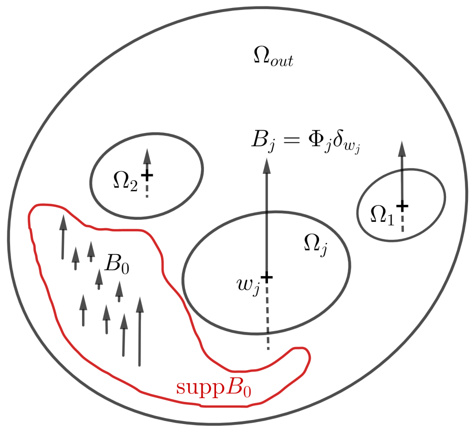

Figure 2.2: Setting of the bounded region with magnetic field.

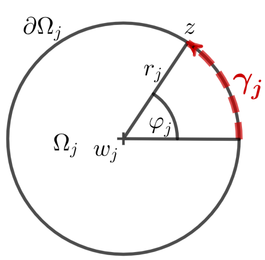

Figure 2.3: Used notation for the coordinate system of the hole Ω_j.

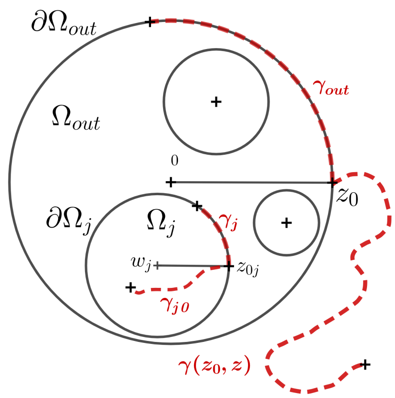

Figure 2.4: The paths used in the proofs of theorems 40 and 39.

Chapter 3

The main theorems

We start with the set up of our problem. Let $\Omega_k \subset \mathbb{C}$ denote a ball with a centre at w_k and a radius R_k. We consider the two dimensional manifold $M = \mathscr{M} \setminus \cup_{k \leq N} \Omega_k$, $N \in \mathbb{N}$, where $\mathscr{M}$ is either the whole complex plane or a bounded ball Ω_{out} with the centre at the origin and a radius R_{out}, and the magnetic field

$$B = B_{sing} + B_0 \,, \tag{3.1}$$

where B_0 is a smooth function with a compact support $\operatorname{supp} B_0 \subset M^\circ$ and $\operatorname{supp} B_{sing} \subset \cup_{k \leq N} \Omega_k$. In view of Lemma 37 we can without loss of generality assume $B_{sing} = \sum_{k \leq N} \Phi_k \delta_{w_k}$ (w_k being the centres of the circular holes Ω_k), where Φ_k is the flux of B through the k-th hole

$$\Phi_k := \int_{\Omega_k} B(z) \frac{\mathrm{i}}{2} \, \mathrm{d}z \wedge \, \mathrm{d}\bar{z} \,.$$

Moreover, Lemma 37 asserts that we can assume that the fluxes are normalised to take values in an interval of length 2π. We will therefore use the notation Φ'_j for the unique number in the interval $[-\pi, \pi)$ that differs by an integer multiple of 2π from the flux in the j-th hole, and, refer to the sum of the normalised fluxes as

$$\Phi_{sing} = \sum_{k \leq N} \Phi'_k \,.$$

For the magnetic field in the bulk we have the flux

$$\Phi_0 = \int_M B_0(z) \frac{\mathrm{i}}{2} \, \mathrm{d}z \wedge \, \mathrm{d}\bar{z} \,,$$

and the overall flux is then the sum of the bulk contribution and the normalised fluxes $\Phi := \Phi_0 + \Phi_{sing}$. In this chapter we will prove the following theorems.

Theorem 39. *Let $M = \mathbb{C} \setminus \cup_{k\leq N}\Omega_k$ and D_a the Dirac operator with the magnetic field* (3.1)*. If $|\frac{\Phi}{2\pi}| > 1$ then there are*

$$\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$$

zero modes of the operator D_a in Aharonov–Casher gauge corresponding to B_{sing} being delta functions at w_j with the APS boundary conditions (2.27) *on the inner components of the boundary. These states have spin up if $\Phi > 0$ and spin down if $\Phi < 0$. If $|\Phi| \leq 2\pi$ the system hosts no zero modes. We denoted by $\lfloor y \rfloor$ the biggest integer strictly smaller then y.*

The next theorem is the alternative to the previous one when we consider a bounded domain with holes and a circular outer boundary.

Theorem 40. *Let $M = \Omega_{out} \setminus \cup_{k\leq N}\Omega_k$ and let D_a be the Dirac operator with the magnetic field* (3.1)*. Then there are*

$$\left| \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor \right|$$

zero modes of the operator D_a with the APS boundary conditions (2.27) *on the inner components and* (2.29) *on the outer component of the boundary. In particular, there are no zero modes in the case $\Phi \in (-\pi, \pi]$. If $\Phi > 0$ then all the zero modes have spin up. If $\Phi < 0$ then they have spin down. As before we denoted by $\lfloor y \rfloor$ the biggest integer strictly smaller then y.*

As a direct consequence of this theorem we obtain the index formula for the particular setting.

Corollary 41. *Under the assumptions of Theorem 40 we obtain the index for D (defined by* (2.9)*),*

$$\operatorname{ind}(D) = \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor .$$

Remark 42. *The particular form of the (non-normalised) zero modes of the Dirac operator is also known from the proof. Depending on the sign of the total flux Φ, they are purely spin up or purely spin down*

$$\begin{pmatrix} u^+ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u^- \end{pmatrix},$$

where

$$u^+(z) = \mathrm{e}^{h(z)} \sum_{0\leq n<\frac{\Phi}{2\pi}-1} a_n z^n , \quad u^-(z) = \mathrm{e}^{-h(z)} \sum_{0\leq n<-\frac{\Phi}{2\pi}-1} b_n \bar{z}^n ,$$

if $\mathscr{M} = \mathbb{C}$, and,

$$u^+(z) = \mathrm{e}^{h(z)} \sum_{0 \le n \le \frac{\Phi}{2\pi} - \frac{1}{2}} a'_n z^n\,, \quad u^-(z) = \mathrm{e}^{-h(z)} \sum_{0 \le n \le -\frac{\Phi}{2\pi} - \frac{1}{2}} b'_n \bar{z}^n\,,$$

if $\mathscr{M} = \Omega_{out}$, with some coefficients $a_n, b_n, a'_n, b'_n \in \mathbb{C}$.

Remark 43. *In Remark 36 we commented on the other choice of the "half kernel" of A in the APS boundary conditions (2.27) and (2.29) being $\sigma(\nu)N(A)$. In this alternative we can choose by Lemma 37 the normalized fluxes inside the holes $\Phi'_j \in (-\pi, \pi]$ for all $j \le N$. An adaptation of the proofs then shows the following. The result of Theorem 39 remains the same. The content of Theorem 40 states that there are*

$$\left| \left\lfloor -\frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor \right|$$

zero modes with spin up if $\Phi > 0$ and spin down if $\Phi < 0$ and, in particular, there are no zero modes if $\Phi \in [-\pi, \pi)$.

3.1 Proof for unbounded region with holes

The main idea is to show that if the zero modes $u^\pm$ satisfy the APS boundary condition (2.27) then the functions $g^\pm$ from Remark 27 can be extended analytically in z and $\bar{z}$ inside the holes. While this is a straightforward process in the case of one hole it requires a new approach if we have several holes. Recall that g^+ and g^- are analytical in z and $\bar{z}$ on M. Thus on an annulus $\mathcal{A}(\Omega_j)$ around the hole Ω_j, such that $\mathcal{A}(\Omega_j) \cap \Omega_k = \emptyset$ for all $k \ne j$, they have the Laurent series

$$g^+(z) = \sum_{n \in \mathbb{Z}} a_n (z - w_j)^n \tag{3.2}$$

$$g^-(z) = \sum_{n \in \mathbb{Z}} b_n \overline{(z - w_j)}^n\,, \tag{3.3}$$

for some $a_n, b_n \in \mathbb{C}$. Let us denote by $\widetilde{\Omega}_j \subset \mathbb{R}^2$ an open ball such that $\widetilde{\Omega}_j \supsetneq \Omega_j$ and $\widetilde{\Omega}_j \cap \operatorname{supp}(B_0 + \sum_{k \ne j} B_k) = \emptyset$. To check that $u^\pm$ satisfy the APS boundary condition (2.27) we multiply $u^\pm$ by a function $\mathrm{e}^{G_j^\pm(z)}$ where $G_j^\pm(z)$ have the following properties

1. F_j^+ and F_j^- are analytic functions in z and $\bar{z}$ on $\widetilde{\Omega}_j$, respectively, where

$$F_j^\pm(z) := \pm h(z) + G_j^\pm(z)\,, \tag{3.4}$$

   and h was defined in (2.15).

2. The restriction of $G_j^{\pm}(z)$ to the boundary $\partial\Omega_j$ satisfies

$$G_j^{\pm}(z)\,|_{z\in\partial\Omega_j} = -\mathrm{i}\int_{\gamma_j} \vec{a}\,\mathrm{d}\vec{s} + \mathrm{i}\frac{\Phi_j'}{2\pi}\varphi_j\,,$$

where $\gamma_j \subset \partial\Omega_j$ is the curve connecting the points $z_{0j} = w_j + R_j$ and z counter-clockwise as shown in Figure 2.3.

Recall again that due to Lemma 37 and Remark 38 we are assuming that $B\big|_{\Omega_j} = B_j = \Phi_j'\delta_{w_j}$, with the normalised flux $\Phi_j' \in [-\pi,\pi)$. This, further, allows us to extend the definition of the vector potential a that is given by (2.12) inside the region $\Omega_j \setminus \{w_j\}$. For $z \in \widetilde{\Omega}_j \setminus \{w_j\}$ we then define the following pair of functions

$$G_j^{+}(z) = -\mathrm{i}\int_{\gamma(z_{0j},z)} \vec{a}\,\mathrm{d}\vec{s} + \int_{\gamma(z_{0j},z)} \frac{\Phi_j'}{2\pi(z'-w_j)}\,\mathrm{d}z'\,, \tag{3.5a}$$

$$G_j^{-}(z) = -\mathrm{i}\int_{\gamma(z_{0j},z)} \vec{a}\,\mathrm{d}\vec{s} - \int_{\gamma(z_{0j},z)} \frac{\Phi_j'}{2\pi\overline{(z'-w_j)}}\,\mathrm{d}\bar{z}'\,, \tag{3.5b}$$

where by $\gamma(z_{0j},z) \subset \widetilde{\Omega}_j \setminus \{w_j\}$ we denoted the path of integration with the endpoints $z_{0j}, z \in \widetilde{\Omega}_j \setminus \{w_j\}$ (see Figure 2.4). Note that such $G_j^{\pm}$ clearly satisfy the condition on the restriction to the boundary. The following lemma shows that $G_j^{\pm}(z)$ are well defined on $\widetilde{\Omega}_j \setminus \{w_j\}$.

Lemma 44. *$G_j^{\pm}(z)$ are independent of the choice of the path $\gamma(z_{0j},z)$ contained in $\widetilde{\Omega}_j \setminus \{w_j\}$.*

Proof. We show the equivalent statement that $G_j^{\pm}(z) = 0$ for any loop $\gamma = \gamma(z_{0j}, z = z_{0j}) \subset \widetilde{\Omega}_j \setminus \{w_j\}$. Straightforwardly, we obtain the values for the first summand of (3.5a)

$$\int_{\gamma} \vec{a}\,\mathrm{d}\vec{s} = \ell\Phi_j'\,,$$

where $\ell \in \mathbb{Z}$ is the winding number of the loop γ around the point w_j. For the other term in (3.5a), recall that the winding number can be defined by

$$\ell = \frac{1}{2\pi\mathrm{i}}\int_{\gamma}\frac{1}{z'-w_j}\,\mathrm{d}z' = \frac{-1}{2\pi\mathrm{i}}\int_{\gamma}\frac{1}{\overline{z'-w_j}}\,\mathrm{d}\bar{z}'\,.$$

Multiplying by Φ_j' then shows that indeed the right hand sides of (3.5) vanish for any closed loop $\gamma \subset \widetilde{\Omega}_j \setminus \{w_j\}$. □

Now we show that with our choice of $G_j^{\pm}$ the first requirement regarding $F_j^{\pm}$ defined by (3.4) is satisfied.

Proposition 45. *The functions $F_j^+(z)$ and $F_j^-(z)$ defined by (3.4) are analytic on $\widetilde{\Omega}_j$ in z and $\bar{z}$, respectively. And, in particular, there are series of the exponentials*

$$\mathrm{e}^{F_j^+} = \sum_{k\geq 0} c_k^+ (z-w_j)^k\,, \quad \mathrm{e}^{F_j^-} = \sum_{k\geq 0} c_k^- \overline{(z-w_j)}^k\,,$$

for some $c_k^\pm \in \mathbb{C}$ with $c_0^\pm \neq 0$.

Proof. The analyticity follows from the fact, which will be proved below, stating that $F_j^\pm$ have the following forms on $\widetilde{\Omega}_j \setminus \{w_j\}$

$$\begin{aligned} F_j^+(z) &= h(z_{0j}) + \int_{\gamma(z_{0j},z)} \sum_{\substack{k\leq N\\ k\neq j}} (2\partial_{z'} h_k)\ \mathrm{d}z'\,,\\ F_j^-(z) &= -h(z_{0j}) - \int_{\gamma(z_{0j},z)} \sum_{\substack{k\leq N\\ k\neq j}} (2\partial_{\bar{z}'} h_k)\ \mathrm{d}\bar{z}'\,, \end{aligned} \tag{3.6}$$

where $h_k = \frac{-\Phi_k'}{2\pi} \log|z-z_k|$ for $z \neq z_k$ is the scalar potential of the field B_k in the hole Ω_k. A direct computation yields that the integrands $\sum_{k\neq j} 2\partial_z h_k$ and $\sum_{k\neq j} 2\partial_{\bar{z}} h_k$ are analytic on $\widetilde{\Omega}_j$ in z and $\bar{z}$, respectively. Therefore the expressions on the right hand sides have no singularity at w_j. It follows from the next remark, that this indeed implies analyticity of $F_j^\pm$.

Remark 46. *Let g be defined on a domain in $\mathbb{C}$. If either*

$$g_{an}(z) := \int_\gamma g(w)\,\mathrm{d}w = \int_\gamma (g_1 + \mathrm{i}g_2)(\,\mathrm{d}t_1 + \mathrm{i}\,\mathrm{d}t_2)\,,$$

or

$$g_{anti}(z) := \int_\gamma g(w)\,\mathrm{d}\bar{w} = \int_\gamma (g_1 + \mathrm{i}g_2)(\,\mathrm{d}t_1 - \mathrm{i}\,\mathrm{d}t_2)\,,$$

are independent of the path γ connecting a fixed point $z_0 \in \mathbb{C}$ and a point $z \in \mathbb{C}$, then g_{an} or $g_{anti}(z)$ are analytic in z or $\bar{z}$, respectively.

Indeed, we have

$$\begin{aligned} \partial_x \int_\gamma g(w)\,\mathrm{d}w &= \partial_x \int_\gamma (g_1 + \mathrm{i}g_2)\,\mathrm{d}t_1 = g(z)\\ \partial_y \int_\gamma g(w)\,\mathrm{d}w &= \partial_y \int_\gamma \mathrm{i}(g_1 + \mathrm{i}g_2)\,\mathrm{d}t_2 = \mathrm{i}g(z)\,, \end{aligned}$$

and hence $\frac{1}{2}(\partial_x \pm \mathrm{i}\partial_y) \int_\gamma g(w)(\,\mathrm{d}t_1 \pm \mathrm{i}\,\mathrm{d}t_2) = 0$.

Now we will show that the equalities (3.6) hold. To that end we use the relation (2.12), i.e.

$$a_x = \partial_y h\,, \quad a_y = -\partial_x h\,,$$

and write

$$h(z) = h(z_{0j}) + \int_{\gamma(z_{0j},z)} \partial_x h \, \mathrm{d}x + \partial_y h \, \mathrm{d}y \,,$$

where $\gamma(z_{0j}, z) \subset \widetilde{\Omega}_j \setminus \{w_j\}$ is an arbitrary path connecting z_{0j} and z. Thus we get

$$\begin{aligned} h - \mathrm{i} \int_{\gamma(z_{0j},z)} \vec{a} \, \mathrm{d}\vec{s} &= h(z_{0j}) + \int_{\gamma(z_{0j},z)} \partial_x h \, \mathrm{d}x + \partial_y h \, \mathrm{d}y - \mathrm{i}\partial_y h \, \mathrm{d}x + \mathrm{i}\partial_x h \, \mathrm{d}y \\ &= h(z_{0j}) + 2 \int_{\gamma(z_{0j},z)} \partial_{z'} h \, \mathrm{d}z' \,, \end{aligned}$$

and similarly,

$$-h - \mathrm{i} \int_{\gamma(z_{0j},z)} \vec{a} \, \mathrm{d}\vec{s} = -h(z_{0j}) - 2 \int_{\gamma(z_{0j},z)} \partial_{\bar{z}'} h \, \mathrm{d}\bar{z}' \,.$$

Finally, we recall that the concrete form (2.15) of the potential function for $B_j = \Phi'_j \delta_{w_j}$ is $h_j = \frac{-\Phi'_j}{2\pi} \log |z - w_j|$ and compute

$$\begin{aligned} \partial_z h_j(z) &= \frac{-\Phi'_j}{4\pi} \partial_z \log |z - w_j|^2 = -\frac{1}{4\pi} \frac{\Phi'_j}{z - w_j} \,, \\ \partial_{\bar{z}} h_j(z) &= -\frac{1}{4\pi} \frac{\Phi'_j}{\overline{z - w_j}} \,, \end{aligned}$$

which together with the definitions (3.4) and (3.5) gives (3.6). □

Proposition 47. *Let $u^{\pm}$ be the zero modes of the Dirac operator that satisfy the APS boundary condition (2.27) on $\partial\Omega_j$. Then the functions g^+ and g^- from Remark 27 can be analytically extended inside the region Ω_j.*

Proof. Since the function $G_j^+(z)$ is defined on $\widetilde{\Omega}_j \setminus \{w_j\}$ and u^+ is defined on the interior M° of M, we can make sense of the product $\mathrm{e}^{G_j^+(z)} u^+(z)$ on the annulus $\widetilde{\Omega}_j \cap M^\circ$. Below, in Lemma 48, we show that the map $\tau_\varepsilon^+ : L^2(\Omega_j^C) \to L^2(\Omega_j^C)$ defined by (3.7) is continuous at zero. In the limit $\varepsilon \to 0$ this mapping applied to $\mathrm{e}^{G_j^+(z)} u^+(z)$ yields the boundary values which we compare to the boundary condition (2.27) and obtain for $z \in \partial\Omega_j$

$$\lim_{\varepsilon \to 0} \tau_\varepsilon^+ \left(\mathrm{e}^{G_j^+(z)} u^+ \right)(z) = \sum_{n \geq 0} \beta_n \mathrm{e}^{\mathrm{i}\varphi_j n} \,,$$

for some $\beta_n \in \mathbb{C}$. Here we used the fact that the normalized fluxes (see Lemma 37) satisfy $\frac{\Phi'_j}{2\pi} - \frac{1}{2} \in [-1, 0)$. Recalling the Laurent series 3.2 for g^+

we also have by Proposition 45

$$\lim_{\varepsilon\to 0}\tau_\varepsilon^+\left(e^{G_j^+}u^+\right)(z) = \lim_{\varepsilon\to 0}\tau_\varepsilon^+\left(e^{F_j^+}g^+\right)(z) = \sum_{k\geq 0}(c_k^+ e^{i\varphi_j k})\sum_{n\in\mathbb{Z}}(a_n e^{i\varphi_j n}),$$

for $z\in\partial\Omega_j$. Hence

$$\sum_{n\geq 0}\beta_n e^{i\varphi_j n} = \sum_{k\geq 0}c_k^+ e^{i\varphi_j k}\sum_{n\in\mathbb{Z}}a_n e^{i\varphi_j n},$$

and since $c_0^+\neq 0$ we conclude that $a_n = 0$ for $n<0$. Therefore $g^+ = \sum_{n\geq 0}a_n(z-w_j)^n$ which means that g^+ can be analytically extended inside Ω_j.

Analogously for u^- and g^- with Laurent series (3.2) on an annulus $\mathcal{A}(\Omega_j)$ we arrive at the condition

$$\sum_{n\geq 0}\beta_n' e^{-i\varphi_j n} = \sum_{k\geq 0}c_k^- e^{-i\varphi_j k}\sum_{n\in\mathbb{Z}}b_n e^{-i\varphi_j n},$$

for some $\beta_n'\in\mathbb{C}$, concluding that $b_n = 0$ for all $n<0$ and thus g^- can be extended anti-analytically inside Ω_j. □

Now we define the map τ_ε^+ used in the proof and show that it has a limit as ε tends to zero.

Lemma 48. *Let $\Omega\subset\mathbb{C}$ be an open ball of radius R centred at zero and Ω^C its complement in $\mathbb{C}$. For $R>0$ define the following maps*

$$\begin{aligned}\tau_\varepsilon^+ &: L^2(\Omega^C)\to L^2(\Omega^C)\\ \tau_\varepsilon^+ &: u(z)\mapsto \chi_{|z|>R}u((1+\varepsilon)z),\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\tau_\varepsilon^- &: L^2(\Omega)\to L^2(\Omega)\\ \tau_\varepsilon^- &: u(z)\mapsto \chi_{|z|<R}u((1-\varepsilon)z).\end{aligned}\tag{3.8}$$

where χ_X denotes the indicator function of the set $X\subset\mathbb{C}$. Then τ_ε^+ and τ_ε^- are continuous at $\varepsilon=0$.

Let us comment that in particular this result can be applied to functions that are defined and square integrable on an annulus with inner or outer radius R that we extend by zero to either Ω^C or Ω.

Proof. We will prove the statement only for the map τ_ε^+ since the proof for τ_ε^- runs along the same lines. First, recall that the compactly supported continuous functions $C_0(\Omega^C)$ are dense in $L^2(\Omega^C)$. Thus for any function $u\in L^2(\Omega^C)$

and any $\delta > 0$ we can find a function $v \in C_0(\Omega^C)$ such that $\|u - v\| < \frac{\delta}{4}$. Hence for an ε small enough we have

$$\begin{aligned}\|(\tau_0^+ - \tau_\varepsilon^+)u\|_{L^2(\Omega^C)} &\leq \|u - v\| + \|\tau_\varepsilon^+ v - \tau_\varepsilon^+ u\| + \|v - \tau_\varepsilon^+ v\| \\ &\leq \left(1 + \frac{1}{1+\varepsilon}\right) \|u - v\| + \|v - \tau_\varepsilon^+ v\|\,,\end{aligned}$$

where we used the following estimate

$$\begin{aligned}\int_{|z|>R} |f((1+\varepsilon)z)|^2 \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} &\leq \int_{|z|>\frac{R}{1+\varepsilon}} |f((1+\varepsilon)z)|^2 \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \frac{1}{(1+\varepsilon)^2} \int_{|z|>R} |f(z)|^2 \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,,\end{aligned}$$

for $f = v - u \in L^2(\Omega^C)$. Then by continuity of v on Ω^C we know that for all $\delta' > 0$ and ε sufficiently small we have $|v(z) - v(z + \varepsilon z)| < \delta'$ so for the choice $\delta' = \frac{\delta}{2\mu(\operatorname{supp} v)}$ with $\mu(X)$ denoting the measure of a set X we conclude

$$\|(\tau_0^+ - \tau_\varepsilon^+)u\|^2_{L^2(\Omega^C)} < \delta\,.$$

□

Applying now the L^2 integrability condition at infinity to the zero modes $u^\pm$ we can prove Theorem 39.

Proof of Theorem 39. By Proposition 47 the zero modes are of the form

$$u^+ = \mathrm{e}^h \sum_{n=0}^{n^+} a_n z^n\,, \quad u^- = \mathrm{e}^{-h} \sum_{n=0}^{n^-} b_n \bar{z}^n\,,$$

with $a_n, b_n \in \mathbb{C}$ and some integers $n^\pm$. Since the requirement $u \in \operatorname{dom}(D_a)$ in particular implies square integrability at infinity, we use the asymptotics (2.16) of the potential function h and obtain the condition $n^+ - \frac{\Phi}{2\pi} < -1$, where $\Phi = \Phi_0 + \sum_{k \leq N} \Phi'_j$. Thus

$$n^+ < \frac{\Phi}{2\pi} - 1\,,$$

from which we infer that if $\frac{\Phi}{2\pi} > 1$ there are $\lfloor \frac{\Phi}{2\pi} \rfloor$ zero modes of spin up in this system. Here $\lfloor y \rfloor$ is the biggest integer strictly less than y. In the same manner we obtain a condition for the spin down zero modes: $n^- + \frac{\Phi}{2\pi} < -1$, i.e.

$$n^- < -\frac{\Phi}{2\pi} - 1\,,$$

and we have $\lfloor -\frac{\Phi}{2\pi} \rfloor$ zero modes of spin down provided that $\frac{\Phi}{2\pi} < -1$. This concludes the proof of Theorem 39. □

3.2 Proof for the bounded region with holes

In the case of the bounded domain the condition of the square integrability, responsible for cutting off the infinite series in the final step of proof of Theorem 39, is substituted by the APS boundary condition (2.29) on the outer boundary $\partial\Omega_{out}$. To apply this boundary condition we follow a similar process as in the case of checking the boundary conditions on the inner components of the boundary, i.e. we will multiply $u^{\pm}$ by a function $\mathrm{e}^{G^{\pm}(z)}$, where $G^{\pm}(z)$ will be defined on region $\widetilde{\Omega}_{out} := \mathbb{C} \setminus \overline{\Omega_{in}}$, where $\Omega_{in} \subsetneq \Omega_{out}$ is an open ball centred at the origin satisfying $\operatorname{supp} B \subset \Omega_{in}$, so that it has the following properties:

1. The functions F^{+} and F^{-} defined by

$$F^{\pm}(z) := \pm h(z) + G^{\pm}(z)\,,$$

   are analytic in z and $\bar{z}$ on $\widetilde{\Omega}_{out}$, respectively, and they are bounded at infinity.

2. The restrictions of $G^{\pm}(z)$ to the boundary $\partial\Omega_{out}$ satisfy

$$G^{\pm}(z)\,|_{z\in\partial\Omega_{out}} = -\mathrm{i}\int_{\gamma_{out}} \vec{a}\,\mathrm{d}\vec{s} + \mathrm{i}\frac{\Phi}{2\pi}\varphi\,,$$

   where $\gamma_{out} \subset \partial\Omega_{out}$ connects the points $z_0 = R_{out}$ and $z \in \partial\Omega_{out}$.

Let $\gamma(z_0, z)$ be a path connecting z_0 and a point $z \in \widetilde{\Omega}_{out}$ (see Figure 2.4). We define

$$G^{+}(z) = -\mathrm{i}\int_{\gamma(z_0,z)} \vec{a}\,\mathrm{d}\vec{s} + \int_{\gamma(z_0,z)} \frac{\Phi}{2\pi z'}\,\mathrm{d}z'\,,$$
$$G^{-}(z) = -\mathrm{i}\int_{\gamma(z_0,z)} \vec{a}\,\mathrm{d}\vec{s} - \int_{\gamma(z_0,z)} \frac{\Phi}{2\pi \bar{z}'}\,\mathrm{d}\bar{z}'\,.$$

Note that these particular choices indeed satisfy our requirement 2. The following lemma ensures that $G^{\pm}(z)$ are well defined.

Lemma 49. *$G^{\pm}(z)$ are independent of the choice of path $\gamma(z_0, z)$ contained in $\widetilde{\Omega}_{out}$.*

Proof. We will show the equivalent statement that $G^{\pm}(z) = 0$ for any loop $\gamma = \gamma(z_0, z = z_0) \subset \widetilde{\Omega}_{out}$. Let us compute the values of the two summands separately. First we have

$$\int_{\gamma} \vec{a}\,\mathrm{d}\vec{s} = \ell\Phi\,,$$

where ℓ is the winding number of the loop γ around the origin.

Further, using the formulas for the winding number as in the proof of Lemma 44 we obtain

$$\int_\gamma \frac{\Phi}{2\pi z'}\,\mathrm{d}z' = \mathrm{i}\ell\Phi\,, \quad \text{and} \quad \int_\gamma \frac{\Phi}{2\pi \bar{z}'}\,\mathrm{d}\bar{z}' = -\mathrm{i}\ell\Phi\,,$$

which concludes the proof. □

Now we show the required analyticity of $F^{\pm}$ and their boundedness at infinity.

Lemma 50. *The functions $F^+(z)$ and $F^-(z)$ are analytic in z and $\bar{z}$, respectively, on $\widetilde{\Omega}_{out}$. Moreover, $F^{\pm}(z) \to const$ as $|z| \to \infty$.*

Proof. Similarly as in the proof of Proposition 45, it can be shown that it holds

$$\begin{aligned} F^+(z) &= h(z_0) + \int_{\gamma(z_0,z)} \left(2\partial_{z'}h + \frac{\Phi}{2\pi z'}\right)\mathrm{d}z' \\ F^-(z) &= -h(z_0) - \int_{\gamma(z_0,z)} \left(2\partial_{\bar{z}'}h + \frac{\Phi}{2\pi \bar{z}'}\right)\mathrm{d}\bar{z}'\,, \end{aligned} \tag{3.9}$$

where $\gamma(z_0, z) \subset \widetilde{\Omega}_{out}$ is an arbitrary path connecting z_0 and z. Then F^+ is analytic on $\widetilde{\Omega}_{out}$ as $2\partial_z h + \frac{\Phi}{2\pi z}$ is analytic, and F^- is anti-analytic as $2\partial_{\bar{z}} h + \frac{\Phi}{2\pi\bar{z}}$ is anti-analytic on that region (recall the relation (2.14) and Remark 46).

Since we are further interested in the limit $|z| \to \infty$, let us assume that $|z| > R'$ for some $R' > 2R_{out}$. We will show that the absolute value of the integrand in (3.9) decays like $|z|^{-2}$ when $|z|$ tends to infinity. First for the singular parts of the magnetic field $B_j = \Phi_j'\delta_{w_j}$ we have

$$2\partial_z h_j = -\frac{\Phi_j'}{2\pi}\partial_z \log|z - w_j|^2 = -\frac{\Phi_j'}{2\pi}\frac{1}{z - w_j}\,,$$

and hence for any $z \in \widetilde{\Omega}_{out}$

$$2\partial_z h_j + \frac{\Phi_j'}{2\pi z} = \frac{-\Phi_j'}{2\pi}\frac{w_j}{z(z - w_j)}\,. \tag{3.10}$$

In particular the absolute value of the right hand side is indeed bounded by a constant multiple of $|z|^{-2}$ for $|z| > R'$. For the bulk part of the magnetic field $B_0 \in C_0^\infty(M)$ with scalar potential h_0 we computed the derivative $\partial_z h_0$ in Remark 23. Hence using the definition of the flux Φ_0 we obtain the following estimate

$$\begin{aligned} \left|2\partial_z h_0 + \frac{\Phi_0}{2\pi z}\right| &= \left|\frac{1}{2\pi}\int_{\mathbb{C}}\left(\frac{B_0(z')}{z - z'} - \frac{B_0(z')}{z}\right)\frac{\mathrm{i}}{2}\,\mathrm{d}z' \wedge \overline{\mathrm{d}z'}\right| \\ &\le \frac{1}{2\pi}\int_{\mathbb{C}}\left|\frac{B_0(z')z'}{z(z - z')}\right|\frac{\mathrm{i}}{2}\,\mathrm{d}z' \wedge \overline{\mathrm{d}z'} \le const|z|^{-2}\,, \end{aligned} \tag{3.11}$$

where in the last inequality we used that

$$\left|\frac{B_0(z')z'}{z(z-z')/|z|^2}\right| \leq 2|B_0(z')z'| \in L^1(\mathbb{C})\,.$$

Let us define

$$C_0 := \int_0^\infty 2\partial_z h(z_0+t) + \frac{\Phi}{2\pi(z_0+t)}\,\mathrm{d}t\,.$$

Then this is indeed a well defined constant since an integral of analytic function along a bounded interval is bounded and therefore with use of (3.11) and (3.10)

$$|C_0| \leq C_1 + \int_{R'}^\infty \frac{C_2}{t^2}\,\mathrm{d}t < \infty\,,$$

with some constants $C_{1,2} > 0$. Further by path independence and again (3.11), (3.10) we estimate

$$\left|\int_\gamma 2\partial'_z h + \frac{\Phi}{2\pi z'}\,\mathrm{d}z' - C_0\right| \leq const\left|-\int_{|z|}^\infty \frac{\mathrm{d}t}{t^2} + \frac{1}{|z|}\int_0^{\arg(z)}\mathrm{d}\varphi\right|\,,$$

which is arbitrarily small as $|z| \to \infty$ and hence concludes the proof for F^+. The proof of asymptotics for F^- at infinity is analogous. □

Corollary 51. *The exponentials of $F^\pm$ have the following series on $\widetilde{\Omega}_{out}$*

$$\mathrm{e}^{F^+(z)} = \sum_{n\leq 0} d_n^+ z^n \quad \textit{and} \quad \mathrm{e}^{F^-(z)} = \sum_{n\leq 0} d_n^- \bar{z}^n\,,$$

for some $d_n^\pm \in \mathbb{C}$ with $d_0^\pm \neq 0$.

Proof. By the previous lemma and by analyticity of $\exp(z)$ on $\mathbb{C}$ the function $\mathrm{e}^{F^+(w^{-1})}$ is analytic and $\mathrm{e}^{F^-(w^{-1})}$ is anti-analytic on the interior of $\mathbb{C}\setminus\widetilde{\Omega}_{out}$ and converge to a non-zero constant as $w \to 0$. This implies existence of the Taylor series $\mathrm{e}^{F^+(w^{-1})} = \sum_{k\geq 0} d_k^+ w^k$ and $\mathrm{e}^{F^-(w^{-1})} = \sum_{k\geq 0} d_k^- \overline{w}^k$ with $d_0^\pm \neq 0$. Thus on the complement $\widetilde{\Omega}_{out}$ we have

$$\mathrm{e}^{F^+(z)} = \sum_{n\leq 0} d_n^+ z^n \quad \text{and} \quad \mathrm{e}^{F^-(z)} = \sum_{n\leq 0} d_n^- \bar{z}^n\,.$$

□

The proof of the Aharonov–Casher result in the case of the bounded domain now follows along the lines of the proof of Proposition 47.

Proof of Theorem 40. Since the zero modes need to satisfy the APS boundary condition on the inner components of the boundary $\partial\Omega_j$, $j \leq N$, we have by Proposition 47

$$g^+(z) = \sum_{n\geq 0} a_n z^n \,, \quad g^-(z) = \sum_{n\geq 0} b_n \bar{z}^n \,,$$

with some $a_n, b_n \in \mathbb{C}$, on the interior of Ω_{out}. Note that the product $\mathrm{e}^{G^+}u^+$ is well defined on $\widetilde{\Omega}_{out} \cap \Omega_{out}$ and so are, by Lemma 48, its boundary values on $\partial\Omega_{out}$. These are obtained by applying the map $\tau_\varepsilon^- : L^2(\Omega_{out}) \to L^2(\Omega_{out})$, defined by (3.8), and then $\lim_{\varepsilon\to 0}$. Since we require the outer boundary condition (2.29) to be satisfied by u^+, we have for $z \in \partial\Omega_{out}$

$$\lim_{\varepsilon\to 0} \tau_\varepsilon^- \left(\mathrm{e}^{G^+}u^+\right)(z) = \sum_{n<\frac{\Phi}{2\pi}-\frac{1}{2}} \beta_n \mathrm{e}^{\mathrm{i}\varphi n} \,,$$

for some $\beta_n \in \mathbb{C}$. Further by Corollary 51

$$\lim_{\varepsilon\to 0} \tau_\varepsilon^- \left(\mathrm{e}^{G^+}u^+\right)(z) \quad = \quad \lim_{\varepsilon\to 0} \tau_\varepsilon^- \left(\mathrm{e}^{F^+}g^+\right)(z) \quad = \quad \sum_{k\leq 0} d_k^+ \mathrm{e}^{\mathrm{i}\varphi k} \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}\varphi n} \,,$$

which leads to

$$\sum_{n<\frac{\Phi}{2\pi}-\frac{1}{2}} \beta_n \mathrm{e}^{\mathrm{i}\varphi n} = \sum_{k\leq 0} d_k^+ \mathrm{e}^{\mathrm{i}\varphi k} \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}\varphi n} \,.$$

Since $d_0^+ \neq 0$, this further implies $a_n = 0$ for all $n \geq \frac{\Phi}{2\pi} - \frac{1}{2}$.

Therefore we conclude that there are $\left\lfloor \frac{\Phi}{2\pi} - \frac{1}{2} \right\rfloor + 1 = \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor$ zero spin up modes.

Adaptation of these steps to $u^- = \mathrm{e}^{-h}g^-$ leads to the condition

$$\sum_{n\leq -\frac{\Phi}{2\pi}-\frac{1}{2}} \beta_n' \mathrm{e}^{-\mathrm{i}\varphi n} = \sum_{k\leq 0} d_k^- \mathrm{e}^{-\mathrm{i}\varphi k} \sum_{n\geq 0} b_n \mathrm{e}^{-\mathrm{i}\varphi n} \,,$$

for some $\beta_n' \in \mathbb{C}$, by which $b_n = 0$ if $n > -\frac{\Phi}{2\pi} - \frac{1}{2}$. Consequently if $\frac{\Phi}{2\pi} \leq -\frac{1}{2}$ there are $\left\{ -\frac{\Phi}{2\pi} - \frac{1}{2} \right\} + 1 = \left\{ \frac{|\Phi|}{2\pi} + \frac{1}{2} \right\}$ zero spin down modes, where $\{y\}$ denotes the biggest integer smaller or equal to y. The proof is now concluded by noticing that the equality $\lfloor y + \frac{1}{2} \rfloor = -\{-y + \frac{1}{2}\}$ holds [1] for any $y \in \mathbb{R}$. □

3.3 Modified boundary condition and evaluation of the index theorem

In greater generality one can add a field of hermitian endomorphisms on $\mathbb{C}^2$ to the canonical boundary operator A. In particular, we will restrict ourselves

[1] this can be easily seen by writing the real number y explicitly either as $y = k + \epsilon$ with $\epsilon \in [0, 1/2]$ or $y = k - \epsilon$ with $\epsilon \in (0, 1/2)$, where k is a suitable integer.

to operators A^q, $q \in \mathbb{R}$, such that on $\partial\Omega_j$ they are given by $A_j^q = A_j + \frac{q}{R_j}\sigma^3$, and, (in the case of bounded region) on the outer boundary we set it as $A_{out}^q = A_{out} - \frac{q}{R_{out}}\sigma^3$. Here A_j and A_{out} are the canonical boundary operators (2.26) and (2.28) on $\partial\Omega_j$ and $\partial\Omega_{out}$, respectively. With this choice we will be able to compare our result to the index formula by Gilkey [21], whose assumptions on the boundary operator require also that it commutes with the chirality operator (introduced on page 16). In the two dimensional case the chirality operator is the third Pauli matrix σ^3. In a greater detail we have

$$\begin{aligned}
A_j^q &= \sigma^3\left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j} + \frac{q}{R_j}\right) - \frac{1}{2R_j} \\
&= \begin{pmatrix} \left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j} + \frac{q}{R_j}\right) - \frac{1}{2R_j} & 0 \\ 0 & -\left(\mathrm{i}\frac{\partial_{\varphi_j}}{R_j} + a_{\varphi_j} + \frac{q}{R_j}\right) - \frac{1}{2R_j} \end{pmatrix}, \text{ and} \\
A_{out}^q &= -\sigma^3\left(\frac{\mathrm{i}\partial_{\varphi_{out}}}{R_{out}} + a^{\varphi_{out}}\right) + \frac{1}{2R_{out}} - \frac{q}{R_{out}}\sigma^3 \\
&= \begin{pmatrix} -\left(\frac{\mathrm{i}\partial_{\varphi_{out}}}{R_{out}} + a^{\varphi_{out}} + \frac{q}{R_{out}}\right) + \frac{1}{2R_{out}} & 0 \\ 0 & \left(\frac{\mathrm{i}\partial_{\varphi_{out}}}{R_{out}} + a^{\varphi_{out}} + \frac{q}{R_{out}}\right) + \frac{1}{2R_{out}} \end{pmatrix}.
\end{aligned}$$

A computation yields that the corresponding boundary condition on $\partial\Omega_j$ reads

$$\begin{aligned}
&H^{1/2}_{(-\infty,0)}(A_j^q) \oplus N(A_j^q) = \\
&\qquad \overline{\operatorname{span}}^{\check{H}(A)}\left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_j n} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni n > \frac{\Phi_j}{2\pi} - \frac{1}{2} + q}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi_j n} \end{pmatrix}\right]_{\mathbb{Z}\ni n \le \frac{\Phi_j}{2\pi} + \frac{1}{2} + q} \right\} \\
&\qquad\qquad \times \exp\left[\mathrm{i}\int_{\gamma_j} \vec{a}(s_j)\,\mathrm{d}\vec{s}_j - \mathrm{i}\frac{\Phi_j}{2\pi}\varphi_j\right], \quad (3.12)
\end{aligned}$$

and similarly on $\partial\Omega_{out}$

$$\begin{aligned}
&H^{1/2}_{(-\infty,0)}(A_{out}) \oplus N(A_{out}) = \\
&\qquad \overline{\operatorname{span}}^{\check{H}(A)}\left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_{out} n} \\ 0 \end{pmatrix}\right]_{\mathbb{Z}\ni n < \frac{\Phi}{2\pi} + q - \frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\varphi_{out} n} \end{pmatrix}\right]_{\mathbb{Z}\ni n \ge \frac{\Phi}{2\pi} + q + \frac{1}{2}} \right\} \\
&\qquad\qquad \times \exp\left[\mathrm{i}\int_{\gamma_{out}} \vec{a}\,\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi}{2\pi}\varphi_{out}\right], \quad (3.13)
\end{aligned}$$

where again $\overline{\operatorname{span}}^{\check{H}(A)}$ denotes the closure of the span in the norm $\|\cdot\|_{\check{H}(A)}$ defined by (2.21).

One can easily check that this boundary condition is preserved by the unitary $\mathscr{U}$ defined in Lemma 37. We can thus follow the steps in our proofs of theorems 39 and 40 with the choice of the normalized fluxes inside the holes

$$\frac{\Phi_j'}{2\pi} \in \left[-q-\frac{1}{2}, -q+\frac{1}{2}\right), \tag{3.14}$$

for all $j \leq N$ and conclude the following results.

Theorem 52. *Let $M = \mathbb{C} \setminus \cup_{k\leq N}\Omega_k$ and D_a the Dirac operator with the magnetic field $B = B_{sing} + B_0$ as in (3.1). Let Φ_j' be the unique number in the interval $[-q-\frac{1}{2}, -q+\frac{1}{2}) \times 2\pi$ that differs by an integer multiple of 2π from the flux in the j-th hole and $\Phi_{sing} = \sum_{j\leq N}\Phi_j'$. Let further Φ_0 be the flux of the smooth field B_0 compactly supported on M° and denote $\Phi = \Phi_{sing} + \Phi_0$. If $|\frac{\Phi}{2\pi}| > 1$ then there are*

$$\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$$

zero modes of the operator D_a with the APS boundary conditions (3.12) on the inner components of the boundary. These states have spin corresponding to the sign of Φ. If $\left|\frac{\Phi}{2\pi}\right| \leq 1$ the system hosts no zero modes. We denoted by $\lfloor y \rfloor$ the biggest integer strictly smaller then y.

Similarly in the case of the bounded domain we find

Theorem 53. *Let $M = \Omega_{out} \setminus \cup_{k\leq N}\Omega_k$ and D_a the Dirac operator with the magnetic field $B = B_{sing} + B_0$ as in (3.1). Let Φ_j' be the unique number in the interval $[-q-\frac{1}{2}, -q+\frac{1}{2}) \times 2\pi$ that differs by an integer multiple of 2π from the flux in the j-th hole and $\Phi_{sing} = \sum_{j\leq N}\Phi_j'$. The flux of the smooth compactly supported field B_0 is denoted by Φ_0 and $\Phi = \Phi_{sing} + \Phi_0$. Then there are*

$$\left| \left\lfloor \frac{\Phi}{2\pi} + q + \frac{1}{2} \right\rfloor \right|$$

zero modes of the operator D_a with the APS boundary condition (3.12) on the inner components and (3.13) on the outer component of the boundary. In particular, there are no zero modes in the case $\frac{\Phi}{2\pi} \in (-\frac{1}{2} - q, \frac{1}{2} - q]$. If $\Phi > \frac{1}{2} + q$ then all the zero modes have spin up. If $\Phi \leq -\frac{1}{2} - q$ then they have spin down. As before we denoted by $\lfloor y \rfloor$ the biggest integer strictly smaller then y.

We can then infer the index formula:

Corollary 54. *Under the assumptions of Theorem 53 we obtain the index for D (defined by (2.9)),*

$$\operatorname{ind}(D) = \left\lfloor \frac{\Phi}{2\pi} + q + \frac{1}{2} \right\rfloor.$$

Let us now compute the index $\mathrm{ind}(D)$ defined by (2.9) using the index formula. In the original work [3] proving the formula for the index of the Dirac operator on a manifold M with boundary, Atiyah, Patodi and Singer assume that M has a product structure near the boundary. Neglecting this assumption one obtains an additional boundary term that in the case of a product structure vanishes. The extended formula was proven by Grubb in [22, Corollary 5.3.]. More explicit expression of the boundary term was given by Gilkey in [21]. In particular in our two dimensional case we obtain by Theorem 8.4.d and Theorem 1.4 in [21]

$$\mathrm{ind}D = \int_M AS - \frac{1}{2}(\eta([\widetilde{A}]_{11}) + \dim\ker[\widetilde{A}]_{11}) + \frac{1}{4\pi}\int_{\partial M} Tr(\sigma^3\Psi) \qquad (3.15)$$

where σ^3 is the third Pauli matrix and $\widetilde{A}$ is the corresponding boundary operator related to the canonical boundary operator A by

$$\widetilde{A} = A - \Psi\,,$$

for some endomorphism Ψ on $\mathbb{C}^2$, and $[\widetilde{A}]_{11}$ is its top left component. We consider $\widetilde{A} = A^q$ for which $\Psi = -\frac{q}{R_j}\sigma^3$ on $\partial\Omega_j$ and $\Psi = \frac{q}{R_j}\sigma^3$ on $\partial\Omega_{out}$. The last term in the index formula then reads

$$\frac{1}{4\pi}\int_{\partial M} Tr(\sigma^3\Psi) = (1-N)q\,.$$

The integrand in the first term is the bulk contribution as in (2.10). Since in our case M is flat, i.e. the Riemannian curvature vanishes, we have that $\hat{A}(M)$ is the identity on the space of forms. Hence we have $\int_M AS = \int \frac{\beta}{2\pi} = \frac{\Phi}{2\pi}$.

The η-invariant is defined (see Appendix D) as the analytic extension of the function

$$\eta_s(A) = \sum_{\lambda\in spec(A)\setminus\{0\}} |\lambda|^{-s} sgn(\lambda)\,,$$

at the value $s = 0$ and is well defined for Dirac operators as was shown in [3]. The sum runs over the non-zero eigenvalues of the boundary operator A. For the simple case $T = -\mathrm{i}\partial_t - c$, $c \in \mathbb{R}$ we show in Proposition 74 that the analytic continuation yields

$$\eta(-\mathrm{i}\partial_t - c) = \begin{cases} -1 + 2\langle c\rangle & \text{if } c \in \mathbb{R}\setminus\mathbb{Z} \\ 0 & \text{if } c \in \mathbb{Z} \end{cases},$$

where $\langle c\rangle$ is the unique number $\widetilde{c} \in (0,1)$ such that $c - \widetilde{c} \in \mathbb{Z}$. Note that the eta-invariant $\eta(T)$ depends only on the eigenvalues of T and hence we have $\eta(T) = \eta(n-c)$. An adaptation of computations in Section 2.5 of the spectrum of the canonical boundary operator on the inner and outer components of the boundary reveals that the spectra of $[A^q_j]_{11}$ and $[A^q_{out}]_{11}$ are the following sets

$$spec\left([A^q_j]_{11}\right) = \{-R_j^{-1}\left(n - \frac{\Phi'_j}{2\pi} + \frac{1}{2} - q\right) \mid n \in \mathbb{Z}\}$$
$$spec\left([A_{out}]_{11}\right) = \{R_{out}^{-1}\left(n - \frac{\Phi}{2\pi} + \frac{1}{2} - q\right) \mid n \in \mathbb{Z}\}.$$

Employing then the properties $\eta(A) = -\eta(-A)$ and $\eta(\kappa A) = \eta(A)$ for a constant $\kappa > 0$ one obtains

$$\begin{aligned}\eta\left([A^q_j]_{11}\right) &= -\eta\left(n - \frac{\Phi'_j}{2\pi} + \frac{1}{2} - q\right)\\ &= 1 - 2\left\langle \frac{\Phi'_j}{2\pi} - \frac{1}{2} + q\right\rangle,\end{aligned}$$

for all $j \leq N$, and

$$\begin{aligned}\eta\left([A^q_{out}]_{11}\right) &= \eta\left(n - \frac{\Phi}{2\pi} + \frac{1}{2} - q\right)\\ &= -1 + 2\left\langle \frac{\Phi}{2\pi} - \frac{1}{2} + q\right\rangle.\end{aligned}$$

Let us denote by I_1 the set of indices j such that $1 = \dim \ker([A_j]_{11}) \in \{0, 1\}$, by $|I_1|$ the number of elements in I_1 and let $I_0 = \dim([A_{out}]_{11}) \in \{0, 1\}$. We make the following observations

1. By (3.14) if $j \notin I_1$ we have

$$\left\langle \frac{\Phi'_j}{2\pi} - \frac{1}{2} + q\right\rangle = \frac{\Phi'_j}{2\pi} + \frac{1}{2} + q.$$

2. For $j \in I_1$ it holds $\frac{\Phi'_j}{2\pi} - \frac{1}{2} + q = -1$ and thus ,

$$\sum_{j\in I_1}\left(\frac{\Phi'_j}{2\pi} + q\right) = -\frac{|I_1|}{2}.$$

3.

$$\eta([A_{out}]_{11}) + I_0 = \begin{cases} 1 & \text{if } I_0 = 1 \\ -1 + 2\left\langle \frac{\Phi}{2\pi} - \frac{1}{2} + q\right\rangle & \text{if } I_0 = 0 \end{cases}.$$

Omitting the last term and the outer boundary contribution in the index formula for now, we straightforwardly arrive at the expression

$$\int_M AS - \frac{1}{2}\sum_{j\leq N}(\eta([\widetilde{A}_j]_{11}) + \dim \ker[\widetilde{A}_j]_{11}) = \frac{\Phi_0}{2\pi} - \frac{1}{2}\sum_{j\notin I_1} -2\left(\frac{\Phi_j'}{2\pi}+q\right) - \frac{|I_0|}{2}$$
$$= \frac{\Phi_0}{2\pi} + \sum_{j\leq N}\left(\frac{\Phi_j'}{2\pi}+q\right) = \frac{\Phi}{2\pi} + Nq\,.$$

Finally for the index of the Dirac operator with the boundary conditions (3.12) and (3.13) we have

$$\begin{aligned}\mathrm{ind}D &= \frac{\Phi}{2\pi} + q - \left\{\begin{matrix}\frac{1}{2} & \text{if } I_0 = 1\\ -\frac{1}{2} + \left\langle \frac{\Phi}{2\pi} - \frac{1}{2} + q\right\rangle & \text{if } I_0 = 0\end{matrix}\right\} \\ &= \left\{\begin{matrix}\frac{\Phi}{2\pi} + q - \frac{1}{2} & \text{if } I_0 = 1\\ \left\lfloor \frac{\Phi}{2\pi} - \frac{1}{2} + q\right\rfloor + 1 & \text{if } I_0 = 0\end{matrix}\right\} = \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} + q \right\rfloor\,,\end{aligned}$$

where in the last equality we used that $\frac{\Phi}{2\pi} + \frac{1}{2} + q \in \mathbb{Z}$ if $I_0 = 1$. Note that this formula is in agreement with our result in Corollary 54.

The Pauli operator

We would like to point out the relation of our result and the zero modes of the corresponding Pauli operator, which are physically more relevant since by the positivity of the square $H_a = D_a^2$ the zero energy states are the ground states of the system. It is remarkable that even though the particle is moving in the region M its number of the possible lowest energy states is influenced by the magnetic field inside the holes that are not part of M. The operator H_a acts as

$$H_a = -\sum_{j=1,2}\begin{pmatrix}(\partial_j - \mathrm{i}a_j)^2 + B & 0\\ 0 & (\partial_j - \mathrm{i}a_j)^2 - B\end{pmatrix}\,,$$

on the domain

$$\mathrm{dom}(H_a) = \{u \in \mathrm{dom}(D_a) \mid D_a u \in \mathrm{dom}(D_a)\}\,.$$

As noted in the proof of the standard Aharonov–Casher result, Theorem 26, the number of the zero modes of the Dirac operator is the same as the number of its square, the Pauli operator.

Chapter 4

Aharonov–Casher on a sphere with holes

In this chapter we will prove a version of the Aharonov– Casher theorem for the magnetic Dirac operator on a sphere with holes whose boundaries are equipped with APS boundary conditions. In particular, let us consider the manifold $M = \mathbb{S}^2 \setminus \cup_{k \leq N} \Omega_k$, where $\cup_{k \leq N} \Omega_k$ is a union of mutually disjoint open discs on $\mathbb{S}^2$. We again consider the magnetic field (3.1) on M for which we moreover pose requirement that the overall flux on the sphere sums to zero

$$\int_{\mathbb{S}^2} B_0 + B_{sing} = 0 \,. \tag{4.1}$$

To motivate the condition (4.1), recall that the vector potential one-form α is globally defined and therefore the flux through the N-th hole is $\Phi_N = -\Psi$, where Ψ is the total flux minus Φ_N. This is so, since $\int_{\partial \Omega_N} \alpha$ can be integrated either as $-\Psi$ or as Φ_N as $\partial \Omega_N$ is boundary of both Ω_N and Ω_N^C which are both bounded regions. We will consider only a semi-total flux which we define as the bulk contribution Φ_0 plus the normalised fluxes through all the holes but one and we choose to omit the flux of the N-th hole

$$\widehat{\Phi} = \Phi_0 + \sum_{j \leq N-1} \Phi_j' \,.$$

The reasoning behind this comes from Lemma 64 establishing the gauge invariance of this problem which we prove later. It turns out that the problem of finding the zero modes is again gauge invariant and one can gauge away integer multiples of 2π inside each of the holes apart from exactly one. The degeneracy of the zero eigenvalue then depends on the sum of these normalised fluxes. Moreover the result does not depend on which hole was left out with non-normalised flux. The precise statement is the content of the following theorem.

Theorem 55. *Let D be the Dirac operator on M with magnetic field (3.1) that satisfies the condition (4.1). Then there are*

$$\left|\left\lfloor \frac{\widehat{\Phi}}{2\pi} + \frac{1}{2} \right\rfloor\right|$$

zero modes of the operator D with the domain given by the APS boundary conditions on $\cup_{j\leq N}\partial\Omega_j$. If $\widehat{\Phi} > 0$ then all the zero modes have spin up. If $\widehat{\Phi} < 0$ then they have spin down. Here, spin up and spin down is relative to the chirality operator $\sigma(\mu)$ where μ is dual to the normal of the sphere.

Remark 56.

1. *Notice that in particular, there are no zero modes in the case $\widehat{\Phi} \in (-\frac{1}{2}, \frac{1}{2}] \times 2\pi$.*

2. *Let us point out that the number $\left|\left\lfloor \frac{\widehat{\Phi}}{2\pi} + \frac{1}{2} \right\rfloor\right|$, where $\widehat{\Phi} = \sum_{j\leq N-1} \Phi'_j$ does not depend on the numbering of the holes. This is because we sum only over the normalised values of the fluxes and the condition that the global flux is zero expressed by (4.1). Hence if we fix an index $j_0 \leq N-1$ and put*

$$\Phi^I = \Phi'_{j_0} + \Phi_{rest}\,,$$

where $\Phi_{rest} = \sum_{j\leq N-1, j\neq j_0} \Phi'_j$, we have by (4.1) the flux $-\Phi^I$ through the hole Ω_N. To normalise this value we note that for any $y \in \mathbb{R}$ it holds $y - \lfloor y + \frac{1}{2} \rfloor \in (-\frac{1}{2}, \frac{1}{2}]$. Thus

$$\frac{\Phi'_N}{2\pi} = -\left(\frac{\Phi^I}{2\pi} - \left\lfloor \frac{\Phi^I}{2\pi} + \frac{1}{2} \right\rfloor\right) \in \left[-\frac{1}{2}, \frac{1}{2}\right)\,,$$

is the normalised flux through the N-th hole. The total flux $\Phi^{II} = \Phi_{rest} + \Phi'_N$, i.e. omitting the contribution from j_0, then satisfies

$$\begin{aligned}
\left\lfloor \frac{\Phi^{II}}{2\pi} + \frac{1}{2} \right\rfloor &= \left\lfloor \frac{\Phi_{rest}}{2\pi} + \frac{1}{2} - \left(\frac{\Phi^I}{2\pi} - \left\lfloor \frac{\Phi^I}{2\pi} + \frac{1}{2} \right\rfloor\right) \right\rfloor \\
&= \left\lfloor \frac{\Phi_{rest}}{2\pi} + \frac{1}{2} - \frac{\Phi^I}{2\pi} \right\rfloor + \left\lfloor \frac{\Phi^I}{2\pi} + \frac{1}{2} \right\rfloor \\
&= \left\lfloor -\frac{\Phi'_{j_0}}{2\pi} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{\Phi^I}{2\pi} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\Phi^I}{2\pi} + \frac{1}{2} \right\rfloor\,,
\end{aligned}$$

where in the last equality we used that $\frac{\Phi'_{j_0}}{2\pi} \in [-\frac{1}{2}, \frac{1}{2})$.

The Dirac operator with APS boundary condition in the conformal metric g^W

Let M be a two dimensional *spin*^{c} manifold with metric g. In [14, Propositions 4.1 and 4.2] the authors showed how the Levi-Civita connection ∇^{LC}, the Clifford connection ∇ and the Clifford multiplication σ on a $spin^c$ spinor bundle over M are modified under a general conformal transformation taking the metric g to a metric $g^W = W^2 g$ for some $W : M \to \mathbb{R} \setminus \{0\}$. We summarise their results in the following proposition.

Proposition 57. *In the conformal metric $g^W = W^2 g$ we have*

$$\begin{aligned} \sigma^W(\xi) &= W^{-1}\sigma(\xi)\,, \\ \nabla^W_X u &= \nabla_X u + \frac{1}{4}W^{-1}[\sigma(\xi), \sigma(\,dW)]u\,, \\ \nabla^{LC,W}_X(\alpha) &= \nabla^{LC}_X \alpha - W^{-1}X(W)\alpha + W^{-1}(\alpha,\, dW)\xi - W^{-1}\alpha(X)\, dW\,, \end{aligned}$$

for any spinor u, vector field X and a one form α. We denoted by ξ the one-form dual to X with respect to the metric g.

We point out that for any $\alpha \in T^*M$ it holds $\sigma^W(W\alpha) = \sigma(\alpha)$ and that if α is normalized in the metric g then $W\alpha$ is normalized in the conformal metric g^W. We denote by X the tangential vector field on the boundary ∂M and ξ the dual one-form. Similarly, n denotes the inner normal vector field on ∂M and ν its dual. We assume X, n to be normalised in the metric g. As a consequence of Proposition 57 we obtain the relations of the Dirac operators and their boundary operators under a conformal transform.

Corollary 58. *Consider a two dimensional $spin^c$ manifold M with the metric g which is conformally equivalent to a manifold M^W with metric $g^W = W^2 g$. The Dirac operators D on M and D^W on M^W and their respective adapted boundary operators are related by*

$$\begin{aligned} D^W &= W^{-3/2} D W^{1/2} \text{ and} \\ A^W &= W^{-1} A\,. \end{aligned}$$

In particular we see that the APS boundary condition is not conformally invariant.

Proof. The proof for D^W is presented in [14, Theorem 4.3] so we show only the relation for A^W. Writing locally on the boundary $D = \sigma(\nu)\nabla_n + \sigma(\xi)\nabla_X$ and using $\sigma(\nu)^2 = 1$ recall that by Definition 32 the canonical boundary operator A adapted to D in the metric g reads

$$2A = \sigma(\nu)\sigma(\xi)\nabla_X - \sigma(\xi)\nabla_X \sigma(\nu)\,.$$

Similarly, changing the metric from g to $g^W = W^2 g$ we obtain

$$\begin{aligned}
2A^W &= \sigma(\nu)\sigma(\xi)W^{-1}\nabla_X^W - \sigma(\xi)W^{-1}\nabla_X^W\sigma(\nu) \\
&= W^{-1}\Big(\sigma(\nu)\sigma(\xi)\big(\nabla_X + \frac{1}{4}W^{-1}[\sigma(\xi),\sigma(\,\mathrm{d}W)]\big) \\
&\quad - \sigma(\xi)\big(\nabla_X + \frac{1}{4}W^{-1}[\sigma(\xi),\sigma(\,\mathrm{d}W)]\big)\sigma(\nu)\Big) \\
&= W^{-1}\left(\sigma(\nu)\sigma(\xi)\nabla_X - \sigma(\xi)\nabla_X\sigma(\nu)\right) + \frac{W^{-2}}{4}R = W^{-1}2A + \frac{W^{-2}}{4}R\,,
\end{aligned}$$

where

$$\begin{aligned}
R &:= \sigma(\nu)\sigma(\xi)[\sigma(\xi),\sigma(\,\mathrm{d}W)] - \sigma(\xi)[\sigma(\xi),\sigma(\,\mathrm{d}W)]\sigma(\nu) \\
&= -\sigma(\xi)\{[\sigma(\xi),\sigma(\,\mathrm{d}W)],\sigma(\nu)\}\,.
\end{aligned}$$

Since the pair (ν,ξ) forms a local orthonormal basis of the one forms it holds $\sigma(\,\mathrm{d}W) = (\,\mathrm{d}W,\xi)\sigma(\xi) + (\,\mathrm{d}W,\nu)\sigma(\nu)$ and hence

$$[\sigma(\xi),\sigma(\,\mathrm{d}W)] = (\,\mathrm{d}W,\nu)\left(\{\sigma(\xi),\sigma(\nu)\} - 2\sigma(\nu)\sigma(\xi)\right) = -2(\,\mathrm{d}W,\nu)\sigma(\nu)\sigma(\xi)\,.$$

Therefore using the following (anti)-commutation identities

$$\begin{aligned}
[EF,G] &= E[F,G] + [E,G]F \\
\{EF,G\} &= E\{F,G\} - [E,G]F\,,
\end{aligned}$$

for any operators E, F, G, we infer

$$R = 2(\,\mathrm{d}W,\nu)\sigma(\xi)\sigma(\nu)\{\sigma(\xi),\sigma(\nu)\} = 0\,,$$

which concludes the proof of $A^W = W^{-1}A$. □

The Dirac operator under the stereographic projection

It is convenient to map the Dirac operator from the sphere to the plane by the stereographic projection. Here we will argue that due to this mapping we can perform the analysis for finding the zero modes of a Dirac operator on $\mathbb{S}^2$ by investigating the problem on $\mathbb{C}$ with a metric that is conformal to the standard metric on $\mathbb{C}$. We will denote by $P_\omega : \mathbb{S}^2 \setminus \{\omega\} \to \mathbb{C}$ the stereographic projection from a point $\omega \in \mathbb{S}^2$ composed with reflection across the x axis and shorten the notation as $P := P_N$ for the north pole $N = (0,0,1)^T \in \mathbb{S}^2$.

Lemma 59. *The tangent map* $P_* : (T(\mathbb{S}^2 \setminus \{N\}), g^{\mathbb{S}^2}) \to (T\mathbb{R}^2, g^W)$, *where* $g^{\mathbb{S}^2}$ *is the standard metric on* $\mathbb{S}^2$ *and* $g^W = \left(1 + \frac{x^2+y^2}{4}\right)^{-2}(\,\mathrm{d}x^2 + \,\mathrm{d}y^2)$, *is an isometry.*

Proof. A point on a unit sphere can be described as

$$\omega = \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}, \quad \vartheta \in [0,\pi]\,, \varphi \in (0, 2\pi]\,, \tag{4.2}$$

and it is mapped by P to the point $P(\omega) = 2\cot\frac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \in \mathbb{C}$, i.e.

$$x := (P(\omega))_x = 2\cot(\vartheta/2)\cos\varphi\,, \quad y := (P(\omega))_y = -2\cot(\vartheta/2)\sin\varphi\,. \tag{4.3}$$

This yields the following relation for the basis vectors $(\partial_\vartheta, \sin^{-1}\vartheta\partial_\varphi)$ and (∂_x, ∂_y) which are normalised in the standard metric on $\mathbb{S}^2$ and $\mathbb{R}^2$, respectively,

$$\begin{pmatrix} \partial_\vartheta \\ \partial_\varphi / \sin\vartheta \end{pmatrix} = (\sin(\vartheta/2))^{-2} \begin{pmatrix} -\cos\varphi & \sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$

Thus we have

$$\begin{aligned} g^W(P_*\partial_\vartheta, P_*\partial_\vartheta) &= W^2 (\sin(\vartheta/2))^{-4} g^{\mathbb{R}^2}\left(\begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix}, \begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix} \right) \\ &= W^2 (\sin(\vartheta/2))^{-4} = 1 = g^{\mathbb{S}^2}(\partial_\vartheta, \partial_\vartheta)\,, \end{aligned}$$

provided we chose the conformal factor W satisfying

$$\begin{aligned} W^{-1} = (\sin(\vartheta/2))^{-2} &= \left(\frac{\sin^2(\vartheta/2) + \cos^2(\vartheta/2)}{\sin^2(\vartheta/2)} \right) \\ &= \left(1 + \frac{(2\cot(\vartheta/2))^2}{4} \right) = \left(1 + \frac{x^2+y^2}{4} \right). \end{aligned} \tag{4.4}$$

Similarly, we get the relations

$$\begin{aligned} g^W\left(P_*\frac{\partial_\varphi}{\sin\vartheta}, P_*\frac{\partial_\varphi}{\sin\vartheta} \right) &= 1 = g^{\mathbb{S}^2}\left(\frac{\partial_\varphi}{\sin\vartheta}, \frac{\partial_\varphi}{\sin\vartheta} \right), \text{ and} \\ g^W\left(P_*\frac{\partial_\varphi}{\sin\vartheta}, P_*\partial_\vartheta \right) &= 0 = g^{\mathbb{S}^2}\left(\frac{\partial_\varphi}{\sin\vartheta}, \partial_\vartheta \right). \end{aligned}$$

□

Lemma 60. *The pullback of the stereographic projection composed with reflection across the x axis* $P^* : L^2(\mathbb{C}, g^W; \mathbb{C}^2) \to L^2(P^{-1}(\mathbb{C}), g^{\mathbb{S}^2}; \mathbb{C}^2)$ *acting as* $(P^*u)(\omega) := u(P(\omega))$ *is a unitary operator.*

Proof. A direct computation yields that for the mapping P (4.3) we have

$$\begin{pmatrix} \mathrm{d}\vartheta \\ \sin\vartheta\,\mathrm{d}\varphi \end{pmatrix} = \sin^2(\vartheta/2) \begin{pmatrix} -\cos\varphi & \sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}.$$

Therefore the volume form changes as

$$\mathrm{d}\vartheta \wedge \sin\vartheta\,\mathrm{d}\varphi = \sin^4(\vartheta/2)\,\mathrm{d}x \wedge \mathrm{d}y = \left(1 + \frac{x^2+y^2}{4}\right)^{-2} \mathrm{d}x \wedge \mathrm{d}y\,,$$

and, with the notation $(\cdot,\cdot)_{\mathbb{S}^2}$ for the inner product on $L^2(\mathbb{S}^2, g^{\mathbb{S}^2}; \mathbb{C}^2)$ and $(\cdot,\cdot)_W$ for the inner product on $L^2(\mathbb{C}, g^W; \mathbb{C}^2)$, we obtain

$$\begin{aligned} (P^* f_1, P^* f_2)_{\mathbb{S}^2} &= \int_0^\pi \int_0^{2\pi} f_1 \circ P(\vartheta,\varphi)\, \overline{f_2 \circ P(\vartheta,\varphi)}\, \mathrm{d}\vartheta \wedge \sin\vartheta\,\mathrm{d}\varphi \\ &= \int_{\mathbb{R}^2} f_1(x,y)\overline{f_2(x,y)} \left(1 + \frac{x^2+y^2}{4}\right)^{-2} \mathrm{d}x \wedge \mathrm{d}y = (f_1, f_2)_W\,, \end{aligned}$$

for any $f_{1,2} \in L^2(\mathbb{C}, g^W; \mathbb{C}^2)$ and for W given by (4.4). □

Definition 61. *Let us define the spinor bundle over $M = \mathbb{S}^2 \setminus \{N\}$ as the pull-back of the spinor bundle $\mathcal{S}(T\mathbb{R}^2)$ by the stereographic projection composed with reflection P*

$$P^*\mathcal{S}(T\mathbb{R}^2) = \{(\omega, u) \in \mathbb{S}^2 \times \mathcal{S}(T\mathbb{R}^2) \mid \pi(u) = P(\omega)\}\,,$$

where π is the bundle projection of $\mathcal{S}(T\mathbb{R}^2)$. We have as in Lemma 60 the map P^* : $\Gamma(\mathbb{R}^2, \mathcal{S}(T\mathbb{R}^2)) \to \Gamma(M, P^*(\mathcal{S}(T\mathbb{R}^2)))$ given by $P^*u(\omega) = (\omega, u \circ P(\omega))$. The corresponding Clifford multiplication and the Clifford connection on such bundle are given by

$$\begin{aligned} \sigma^M(P^*\alpha) &:= P^*\sigma^W(\alpha)\,, & \alpha &\in T^*\mathbb{R}^2 \\ \nabla^M_X &:= P^*\nabla^W_{P_*X}\,, & X &\in TM\,. \end{aligned}$$

Remark 62. *If we also define the relation between the Levi-Civita connections by $P^*\nabla^{LC,W}_{P_*X}\alpha = \nabla^{LC,M}_X(P^*\alpha)$, then the relations above indeed define a proper Clifford multiplication and connection on the spinor bundle. Therefore,*

$$[\nabla^M_X, \sigma^M(P^*\alpha)] = P^*[\nabla^W_{P_*X}, \sigma^W(\alpha)]\,,$$

and in particular, by definition of Clifford connection, it holds

$$\nabla^M_X \sigma^M(P^*\alpha) P^*u = P^*(\nabla^W_{P_*X}\sigma^W(\alpha)u)\,,$$

for any $u \in \Gamma(\mathbb{R}^2, \mathcal{S}(T\mathbb{R}^2))$, $X \in TM$ and $\alpha \in T^*\mathbb{R}^2$.

Now we are ready to state a corollary which will reduce our analysis of the Dirac operator on the spinor bundle over the sphere to the investigation of the corresponding object on the spinor bundle over the plane in a metric conformal to the standard metric on $\mathbb{R}^2 \simeq \mathbb{C}$.

Corollary 63. *The Dirac operator D^M on M is unitarily equivalent to the Dirac operator D^W on $(P(M) \subset \mathbb{C}, g^W)$,*

$$D^M P^* = P^* D^W .$$

Proof. We denote by s^j, $j = 1, 2$ an orthonormal basis on T^*M, s_j the dual basis and e^j its counterpart on $T^*\mathbb{R}^2$ such that $P^* e^j = s^j$. Note, that by Lemma 59 the last relation defines e^j that form an orthonormal frame on $T^*\mathbb{R}^2$ in the metric g^W. Following the definitions above we obtain for any section u on $\mathbb{R}^2$

$$\begin{aligned} D^M P^* u &= \sum_{j\le 2} \sigma^M(s^j) \nabla^M_{s_j}(u \circ P) = \sum_{j\le 2} P^* \sigma^W(e^j) P^*(\nabla^W_{P_* e_j}) P^* u \\ &= \sum_{j\le 2} P^*(\sigma^W(e^j) \nabla^W_{P_* e_j}) P^* u = P^*(D^W u) . \end{aligned}$$

By Remark 62 for the canonical boundary operators A^M and A^W adapted to D^M and D^W, respectively, we have

$$\begin{aligned} 2A^M &= \sigma^M(P^*\nu) \sigma^M(P^*\xi) \nabla^M_X - \sigma^M(P^*\xi) \nabla^M_X \sigma^M(P^*\nu) \\ &= P^*(\sigma^W(\nu) \sigma^W(\xi) \nabla^W_{P_* X}) - P^*(\sigma^W(\xi) \nabla^W_{P_* X} \sigma^W(\nu)) = 2P^* A^W . \end{aligned}$$

Moreover λ is an eigenvalue of A^W with eigenfunction v if and only if it is an eigenvalue of $P^* A^W$ with an eigenfunction $P^* v$. Hence $\mathrm{dom}(D^M) = P^* \mathrm{dom}(D^W)$. □

In our setting of the sphere with holes we can, due to the above, rotate the sphere so that the centre of the hole Ω_N becomes the north pole N. Then perform stereographic projection from N with reflection across the x axis to obtain a bounded region $P(M) \subset \mathbb{C}$ whose all components of the boundary are circles. We call D^W the Dirac operator on this region in the metric g^W which is unitarily equivalent to the Dirac operator on M. Therefore from now on we will focus on the analysis of the zero modes of D^W.

Similarly as in the case of the standard metric we can use the arguments for gauge invariance from Lemma 37 for the holes Ω_j, $j \le N - 1$. This will allow us, in some sense, to deform the magnetic fields inside these holes and change the values of their fluxes by an integer multiple of 2π. More precisely, the following holds.

Lemma 64. *Let a and $\widetilde{a}$ be two magnetic vector potentials whose fluxes differ by an integer multiple m_j of 2π on the inner hole $P(\Omega_j)$, for all $j \leq N-1$. Then we have the unitary equivalence*

$$\mathscr{U}^* D_a^W \mathscr{U} = D_{\widetilde{a}}^W \,,$$

with the unitary

$$\begin{aligned} \mathscr{U} &: L^2(\mathbb{C}, g^W; \mathbb{C}^2) \to L^2(\mathbb{C}, g^W; \mathbb{C}^2) \\ \mathscr{U} &: u \mapsto \exp\left[\mathrm{i} \int_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right] u \,, \end{aligned}$$

where $\gamma \subset P(M)$ is a curve connecting a fixed point $z_0 \in P(M)$ and the point z.

As we showed in the proof of Lemma 37, replacing the starting point z_0 of γ by a different point $z_1 \in P(M)$ amounts to multiplication by the constant $K = \exp\left[\mathrm{i} \int_{z_0}^{z_1} (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right]$ satisfying $\overline{K} = K^{-1}$.

Proof. By Corollary 58 the boundary operator adapted to D_a^W is $A^W(a) = W^{-1} A(a)$ if $A(a)$ is adapted to D_a. Since the unitary operator $\mathscr{U}$ commutes with W, we have, similarly as in (2.30), a unitary equivalence between the boundary operators $A^W(a)$ and $A^W(\widetilde{a})$ adapted to D_a^W and $D_{\widetilde{a}}^W$

$$\mathscr{U}^* A^W(a) \mathscr{U} = A^W(\widetilde{a}) \,.$$

□

By this lemma the Remark 38 extends to the conformal case. Harvesting all this preparation we are able to find the zero modes of the conformal Dirac operator on $\mathbb{C}$ and prove the following proposition whose immediate consequence is Theorem 55.

Proposition 65. *The zero modes of the Dirac operator D^W on $P(M)$ in the metric g^W with the APS boundary condition are of the form*

$$\begin{pmatrix} u^+ \\ 0 \end{pmatrix}, \quad u^+(z) = W^{-1/2}(z) \mathrm{e}^{h(z)} \sum_{0 \leq n < \frac{\widehat{\Phi}}{2\pi} - \frac{1}{2}} c_n z^n \,,$$

$$\begin{pmatrix} 0 \\ u^- \end{pmatrix}, \quad u^-(z) = W^{-1/2}(z) \mathrm{e}^{-h(z)} \sum_{0 \leq n \leq -\frac{\widehat{\Phi}}{2\pi} - \frac{1}{2}} b_n \overline{z}^n$$

for some coefficients $c_n, b_n \in \mathbb{C}$.

Proof. Consider a zero mode $u \in \ker(D^W)$. Then we know that for $v(z) = W(z)^{1/2}u$ it holds $Dv(z) = 0$ on $P(M)$ with $D = W^{3/2}D^W W^{-1/2}$ being the Dirac operator on $P(M)$ in the standard metric on $\mathbb{C}$. We choose coordinates $\tilde{z}$ on $P(\mathbb{S}^2 \setminus \{N\})$ with origin at $P\big((0,0,-1)^T\big)$ and mark with tilde functions on $P(M)$ expressed in these coordinates. Let us fix an arbitrary index $j \leq N-1$. We write similarly $f_j(z_j)$ for a function f on $P(M)$ in the coordinates z_j obtained by the Möbius transform $Y_{t_j} : \tilde{z} \mapsto z_j$ (see Appendix A and Lemma 67) with t_j being the antipodal point of the centre w_j of the hole $\Omega_j \subset \mathbb{S}^2$. By Remark 27 the spin up component v^+ takes the form

$$v_j^+(z) = \mathrm{e}^{h_j(z)} g_j^+(z)\,, \quad j \leq N-1\,,$$

where $g_j^+(z)$ is analytic on $Y_{t_j} \circ P(M)$ for all j and can be analytically extended to the hole $Y_{t_j} \circ P(\Omega_j)$ by Proposition 47 (or rather by the steps of its proof) applied to u_j^+. This is due to the fact that $|W_j(z_j)|$ is constant on $Y_{t_j} \circ P(\partial\Omega_j)$ and therefore the APS boundary condition on $Y_{t_j} \circ P(\partial\Omega_j)$ for D^W in coordinates z_j agrees with the boundary condition (2.27).

In Appendix A we argue that under the change of coordinates given by the Möbius transform

$$Y_{t_j} : \tilde{z} \mapsto z_j = \frac{a\tilde{z}+b}{c\tilde{z}+d}\,,$$

for some a, b, c, d complex numbers dependent on t_j, the spinor u needs to satisfy the relation (A.10), and therefore

$$W_j^{-1/2}(Y_{t_j}(\tilde{z}))v_j^+(Y_{t_j}(\tilde{z})) = \tilde{W}^{-1/2}(\tilde{z})\tilde{\mathscr{G}}(\tilde{z})\tilde{v}^+(\tilde{z})\,, \quad \mathscr{G}(z) = \frac{cz+d}{|cz+d|}\,,$$

for all $j \leq N-1$. Employing (A.6) this now leads to analyticity of $\tilde{g}^+(\tilde{z})$ on $P(\Omega_j)$ as

$$g_j^+(Y_{t_j}(\tilde{z})) = \mathrm{e}^{\tilde{h}(\tilde{z}) - h_j(Y_{t_j}(\tilde{z}))}|c\tilde{z}+d|\frac{c\tilde{z}+d}{|c\tilde{z}+d|}\tilde{g}^+(\tilde{z}) = (c\tilde{z}+d)\tilde{g}^+(\tilde{z})\,,$$

where we used that the functions $h_j(Y_{t_j}(\tilde{z}))$ and $\tilde{h}(\tilde{z})$ are in fact the same function h expressed in different sets of coordinates. Hence using that $(c\tilde{z}+d)^{-1}$ is analytic on $P(\Omega_j)$ [1] and that $j \leq N-1$ was arbitrary we conclude that g^+ is analytic on $P(\mathbb{S}^2 \setminus \Omega_N)$.

The outer boundary condition on $P(\partial\Omega_N)$ is now given by the negative subspace of $A_{out}^W = W^{-1}A_{out}$ which is the negative subspace of A_{out} since W

[1] Note that the point $\tilde{z} = -d/c \notin P(\Omega_j)$ is in fact the image of the antipodal point t_j of w_j under the mapping P.

restricted to $P(\partial\Omega_N)$ is equal to a positive constant. Therefore we may apply the same steps as in the proof of Theorem 40 (page 50) and obtain

$$u^+(z) = W^{-1/2}(z)\mathrm{e}^{h(z)} \sum_{n<\frac{\hat{\Phi}}{2\pi}-\frac{1}{2}} a_n z^n$$

on $P(M)$. The form of the modes u^- on $P(M)$ is shown by an adaptation of the previous to be

$$u^-(z) = W^{-1/2}(z)\mathrm{e}^{-h(z)} \sum_{n\leq-\frac{\hat{\Phi}}{2\pi}-\frac{1}{2}} b_n \bar{z}^n \,.$$

□

Chapter 5

Concluding remarks

In this project we computed the number of the zero modes for the magnetic Dirac operator on flat connected two dimensional manifolds with boundary whose components are circles. The domain of the operator was given by the celebrated Atiyah–Patodi–Singer global boundary condition.

For an unbounded domain we found the same formula as is stated by the Aharonov–Casher theorem which is the special case of our Theorem 39 when the domain is $\mathbb{R}^2$ and hence, simply connected. In the bounded case we can compare the index formula following from our result with an adaptation of the Atiah–Patodi–Singer index formula to manifolds without a product structure near the boundary which was studied in work of Grubb, [22], and Gilkey, [21]. The case of a disc with holes further generalizes to a domain on a sphere with holes.

So far we have assumed all the boundaries to be circular. Note, that the Atiyah–Patodi–Singer's formula for the index does not depend on a particular shape of the boundary. Therefore, a natural question is whether our results (Theorems 39, 40 and 55) still hold if we relax this requirement and consider a smooth deformation of the boundary curve. In our attempts to address this problem we intended to use the Riemannian mapping which conformally transforms the boundary back to a circle. As we saw, however, in Theorem 58 the zero modes of the Dirac operator with the APS boundary condition are not conformally invariant. Since the conformal factor along the boundary is in general not a constant we cannot use the same method as we did for the proof of our result on the sphere.

Recall that in our proofs we follow the original idea of Aharonov and Casher that the zero modes are proportional to a function that is analytic on the domain in question. On $\mathbb{R}^2$ then these functions have Taylor series. In our case they still had Laurent series on an annulus around the boundary and due to the APS boundary condition these, in fact, had no principal part. The generalization therefore requires a deeper understanding of the Hardy space on a region Ω formed by functions that are holomorphic on Ω and whose boundary values are square integrable functions on $\partial\Omega$. Knowing that such boundary values would necessarily need to satisfy the APS boundary condition would lead to the proof.

We conclude this thesis by reviewing the APS boundary condition for a smooth non-circular boundary.

5.1 The APS boundary condition for non-circular holes

As the first step we set a convenient coordinate system in which we will write the Dirac operator. To find the boundary operator and the APS boundary condition is then the same routine as in the case of circular boundaries. Since this will be a local analysis we can as well consider only one hole $\Omega \subset \mathbb{R}^2$.

We denote by $\gamma : [0, L) \to \mathbb{R}^2$ the parametrisation of the boundary $\partial\Omega$ by the arc length s and by $\dot{\gamma}(s)$ the corresponding tangent vector orthogonal to the inward normal vector n. Following the steps in Subsection 2.1. of [36] we can find $\delta > 0$ such that on the band around the boundary

$$\{x \in \mathbb{R}^2 \mid \operatorname{dist}(x, \partial\Omega) < \delta\},$$

we can use γ and n to define a local orthogonal system of coordinates as follows

$$\begin{aligned} \rho : (-\delta, \delta) \times [0, L) &\to \mathbb{R}^2 \\ \rho(r, s) &= \gamma(s) + r n(s)\,. \end{aligned}$$

In these coordinates the boundary is described by condition $r = 0$ and the normal vector corresponds to ∂_r and its dual one form is $\mathrm{d}r$.

We will denote by τ the angle between n and the positive x axis known as the turning angle, whose derivative

$$\tau(s)' = \kappa\,,$$

is the curvature of the boundary. The Jacobian of this coordinate change then reads $J = (1 + r\kappa(s))$.

Remark 66. 1. *The turning angle along a closed curve γ of length L with a winding number $\ell \in \mathbb{Z}$ satisfies*

$$\int_0^L \kappa(s)\,\mathrm{d}s = \tau(L) - \tau(0) = 2\pi\ell\,.$$

2. *Observe that by definition of τ and orthogonality of $(n, \dot{\gamma}(s))$ we have*

$$n = \begin{pmatrix} \cos\tau \\ \sin\tau \end{pmatrix}, \quad \dot{\gamma} = \begin{pmatrix} -\sin\tau \\ \cos\tau \end{pmatrix},$$

so in particular $\frac{\mathrm{d}}{\mathrm{d}s}n = \kappa\dot{\gamma}(s)$ *and* $\mathrm{d}\rho(s,r) = \dot{\gamma}J\,\mathrm{d}s + n\,\mathrm{d}r$. *This yields the following relation between the local coordinate bases of the one forms*

$$\begin{pmatrix} \mathrm{d}r \\ J\,\mathrm{d}s \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix},$$

the coordinate bases of the vectors

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \partial_r \\ J^{-1}\partial_s \end{pmatrix},$$

and, consequently, the relations

$$\begin{aligned} \partial_x \pm \mathrm{i}\partial_y &= \mathrm{e}^{\pm\mathrm{i}\tau}(\partial_r \pm \mathrm{i}J^{-1}\partial_s)\,, \\ a_x \pm \mathrm{i}a_y &= \mathrm{e}^{\pm\mathrm{i}\tau}(a_r \pm \mathrm{i}a_s)\,, \end{aligned} \tag{5.1}$$

where (a_x, a_y) *and* (a_r, a_s) *are the components of the magnetic one form in the basis* $(\mathrm{d}x, \mathrm{d}y)$ *and in the basis* $(\mathrm{d}r, (1+\kappa r)\,\mathrm{d}s)$*, respectively.*

Using (5.1) it is straightforward to rewrite the Dirac operator (2.18) in these coordinates

$$D_a = -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\tau} \\ \mathrm{e}^{\mathrm{i}\tau} & 0 \end{pmatrix} \Big((\partial_r - \mathrm{i}a_r) + \mathrm{i}\sigma^3(J^{-1}\partial_s - \mathrm{i}a_s) \Big)\,. \tag{5.2}$$

On the boundary the Jacobian $J = (1 + r\kappa(s)) = 1$ and thus, we can directly read off from (5.2) the boundary operator A_0 adapted to D_a from Definition 32

$$A_0 = \sigma^3(\mathrm{i}\partial_s + a_s)\,,$$

and, the Clifford multiplication

$$\sigma^r := \sigma(\mathrm{d}r) = -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\tau} \\ \mathrm{e}^{\mathrm{i}\tau} & 0 \end{pmatrix}.$$

We compute the relevant anti-commutation relations in order to obtain the canonical boundary operator A

$$\begin{aligned} \{\sigma^3, \sigma^r\} &= 0 \\ \mathrm{i}\partial_s\sigma^r &= \begin{pmatrix} 0 & -\mathrm{i}\kappa\mathrm{e}^{-\mathrm{i}\tau} \\ \mathrm{i}\kappa\mathrm{e}^{\mathrm{i}\tau} & 0 \end{pmatrix} + \mathrm{i}\sigma^r\partial_s \\ \mathrm{i}\{\partial_s, \sigma^r\} &= \mathrm{i}\kappa\begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\tau} \\ \mathrm{e}^{\mathrm{i}\tau} & 0 \end{pmatrix} + 2\mathrm{i}\sigma^r\partial_s \\ \{\mathrm{i}\sigma^3\partial_s, \sigma^r\} &= \mathrm{i}\sigma^3\{\partial_s, \sigma^r\} - \mathrm{i}[\sigma^3, \sigma^r]\partial_s \\ &= \mathrm{i}\kappa\sigma^3\begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\tau} \\ \mathrm{e}^{\mathrm{i}\tau} & 0 \end{pmatrix} + 2\mathrm{i}\sigma^3\sigma^r\partial_s - 2\mathrm{i}\sigma^3\sigma^r\partial_s = \kappa\sigma^r\,. \end{aligned}$$

This leads to

$$2A = A_0 - \sigma^r A_0 \sigma^r = A_0 - \sigma^r(\{A_0, \sigma^r\} - \sigma^r A_0) = 2A_0 - \kappa\sigma^r ,$$

i.e.

$$A = \sigma^3(\mathrm{i}\partial_s + a_s) - \frac{\kappa}{2} .$$

The eigenproblem $Au = \lambda u$ then corresponds to the equations

$$\begin{aligned} \partial_s u_+ &= -\mathrm{i}\left(-a_s + \frac{\kappa}{2} + \lambda\right) u_+ \\ \partial_s u_- &= \mathrm{i}\left(a_s + \frac{\kappa}{2} + \lambda\right) u_- , \end{aligned}$$

with solutions

$$u_\pm = \exp\left[\mp\mathrm{i}\int_0^s \left(\mp a_s + \frac{\kappa}{2} + \lambda\right) \mathrm{d}\tilde{s}\right] .$$

The eigenvalues are obtained by periodicity in s and satisfy

$$2\pi n = \mp\int_0^L \left(\mp a_s + \frac{\tau'}{2} + \lambda\right) \mathrm{d}\tilde{s} = \mp\left(\lambda L \mp \Phi + \frac{2\pi}{2}\right) ,$$

where $n \in \mathbb{Z}$ and Φ is the magnetic flux through the hole Ω. Inserting this back into $u^\pm$ the eigenfunctions with their corresponding eigenvalues read

$$\begin{aligned} u^\pm &= \mathrm{e}^{\mathrm{i}s2\pi L^{-1}} \mathrm{e}^{\mathrm{i}\int_0^s a_s \,\mathrm{d}\tilde{s} - \mathrm{i}\Phi s L^{-1}} \mathrm{e}^{\pm\mathrm{i}\left(\pi s L^{-1} - \frac{\tau(s)-\tau(0)}{2}\right)} \\ \lambda &= \mp L^{-1}\left(2\pi n - \Phi\right) - \pi L^{-1} . \end{aligned}$$

In the general case of arbitrary curvature we thus obtain the APS boundary condition of the form

$$\begin{aligned} &H^{1/2}_{(-\infty,0)} \oplus N(A) = \\ &\overline{\mathrm{span}}^{\check{H}(A)}\left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}ns\frac{2\pi}{L}} \\ 0 \end{pmatrix}\right]_{n>\frac{\Phi}{2\pi}-\frac{1}{2}} \cdot \mathrm{e}^{\mathrm{i}\left(\frac{\pi s}{L} - \frac{\tau(s)}{2}\right)} , \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}ns\frac{2\pi}{L}} \end{pmatrix}\right]_{n\le\frac{\Phi}{2\pi}+\frac{1}{2}} \cdot \mathrm{e}^{-\mathrm{i}\left(\frac{\pi s}{L} - \frac{\tau(s)}{2}\right)} \right\} \\ &\qquad\qquad \times \mathrm{e}^{\mathrm{i}\int_0^s a_s \,\mathrm{d}\tilde{s} - \mathrm{i}\Phi\frac{s}{L}} , \end{aligned}$$

with $\overline{\mathrm{span}}^{\check{H}(A)}$ denoting the closure of the span in the norm $\|\cdot\|_{\check{H}(A)}$ defined by (2.21).

Appendix A

Remarks on Möbius transform

Möbius transform is a mapping $Y : \mathbb{C} \to \mathbb{C}$ of the form $Y(z) = \frac{az+b}{cz+d}$ such that $ad - bc = 1$. Notice that it is an analytic mapping on $\mathbb{C} \setminus \{-\frac{d}{c}\}$ and the z derivative is

$$\partial_z Y(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}\,. \tag{A.1}$$

Such transforms can be obtained by the composition of the inverse stereographic projection from the plane to a sphere, a rotation on the sphere and stereographic projecting back to the plane.

Lemma 67. *The Möbius transform $Y_\omega = PRP^{-1}$, where P is the stereographic projection from the north pole N followed by the reflection across the x axis and R is the rotation on $\mathbb{S}^2$ along $\varphi = const$ (i.e. along a certain meridian) which maps a point $\omega \in \mathbb{S}^2 \setminus \{N\}$ to the north pole*

$$\omega = \begin{pmatrix} \cos\varphi_0 \sin\vartheta_0 \\ \sin\varphi_0 \sin\vartheta_0 \\ \cos\vartheta_0 \end{pmatrix} \mapsto R(\omega) = N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vartheta_0 \in (0,\pi]\,, \varphi_0 \in (0,2\pi]\,,$$

has the form $Y_\omega(z) = \frac{az+b}{cz+d}$ with the matrix of coefficients

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\frac{\vartheta_0}{2} & 2\mathrm{e}^{-\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2} \\ -\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2} & \cos\frac{\vartheta_0}{2} \end{pmatrix}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1\,.$$

Moreover, for the composition $Y_{\omega_1} \circ Y_{\omega_2}$ for any $\omega_{1,2} \in \mathbb{S}^2 \setminus \{N\}$, the coefficients satisfy the following relations

$$a = \overline{d}\,, \quad b = -4\overline{c}\,, \quad |a|^2 + 4|c|^2 = |d|^2 + \frac{1}{4}|b|^2 = 1\,. \tag{A.2}$$

Proof. One can easily check that $\pm 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi_0}$ are the two fixed points of Y_ω, which with the additional conditions

$$ad - bc = 1 \tag{A.3}$$
$$Y_\omega : P(\omega) = 2\mathrm{e}^{-\mathrm{i}\varphi_0}\cot\frac{\vartheta_0}{2} \mapsto \infty\,,$$

leads to the result

$$Y_\omega(z) = \frac{\cos\frac{\vartheta_0}{2}z + 2\mathrm{e}^{-\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2}}{-\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2}z + \cos\frac{\vartheta_0}{2}}\,. \tag{A.4}$$

In particular, let us point out that the relations between the coefficients of the Möbius transform (A.4) satisfy

$$a = \cos\frac{\vartheta_0}{2} = d = \bar{d}\,,\ b = 2\mathrm{e}^{-\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2} = -4\bar{c}\,,\ |a|^2 + 4|c|^2 = |d|^2 + \frac{1}{4}|b|^2 = 1\,.$$

The last relation follows from the condition (A.3).

For a composition of two such Möbius transforms $Y_{\omega_1} \circ Y_{\omega_2}$ for

$$\omega_j = \begin{pmatrix} \cos\varphi_j \sin\vartheta_j \\ \sin\varphi_j \sin\vartheta_j \\ \cos\vartheta_j \end{pmatrix}\,, j = 1, 2\,,$$

a direct computation yields

$$\begin{aligned} Y_{\omega_1} \circ Y_{\omega_2}(z) &= \frac{az+b}{cz+d}\,, \quad \text{with} \\ a &= \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - \mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)}\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \\ d &= \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - \mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} = \bar{a} \\ b &= 2\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\mathrm{e}^{-\mathrm{i}\varphi_2} + 2\mathrm{e}^{-\mathrm{i}\varphi_1}\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \\ c &= -\frac{1}{2}\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\mathrm{e}^{\mathrm{i}\varphi_2} - \frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_1}\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} = -\frac{\bar{b}}{4}\,. \end{aligned}$$

□

In what follows the particular choice of the point ω is not important so we will generally use the notation Y instead of Y_ω. Let us show that the tangent mapping Y_*, "pushforwarding" vectors at a point z to vectors at $Y(z)$, is an isometry on the tangent space on $\mathbb{C}$ in the metric $g^W = W^2\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ with $W^{-1} = 1 + \frac{|z|^2}{4}$.

Lemma 68. *Consider a conformal metric* $g^W = W^2\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ *on* $\mathbb{C}$*. The tangent map* Y_* *acts as*

$$\begin{aligned} Y_*(z)\begin{pmatrix}\hat{\partial_z}\\ \hat{\partial_{\bar z}}\end{pmatrix} &= \frac{W(Y(z))}{W(z)}\begin{pmatrix}\partial_z Y(z) & 0\\ 0 & \overline{\partial_z Y(z)}\end{pmatrix}\begin{pmatrix}\hat{\partial_z}\\ \hat{\partial_{\bar z}}\end{pmatrix}\Bigg|_{Y(z)} \\ &= |cz+d|^2\begin{pmatrix}(cz+d)^{-2} & 0\\ 0 & \overline{(cz+d)}^{-2}\end{pmatrix}\begin{pmatrix}\hat{\partial_z}\\ \hat{\partial_{\bar z}}\end{pmatrix}\Bigg|_{Y(z)} \end{aligned} \tag{A.5}$$

on the orthonormal (in the metric g^W*) basis of* $T\mathbb{R}^2 \simeq \mathbb{R}^2 \simeq \mathbb{C}$

$$\begin{pmatrix}\hat{\partial_z}\\ \hat{\partial_{\bar z}}\end{pmatrix} = W(z)^{-1}\begin{pmatrix}\partial_z\\ \partial_{\bar z}\end{pmatrix}.$$

Moreover, it is an isometry on $(\mathbb{C}, g^W)$ *for* $W = \left(1+\frac{|z|^2}{4}\right)^{-1}$*.*

Proof. The first equality in (A.5) follows by a direct computation from the definition $(Y_*X)f = X(f\circ Y)$, for any $f \in C^\infty(\mathbb{C})$ and vector X at z, and the fact that Y is an analytic function. The second equality is then a result of (A.1) and the relations (A.2) for the coefficients of a Möbius transform as

$$\begin{aligned} \frac{W(z)}{W(Y(z))} &= \frac{4|cz+d|^2+|az+b|^2}{4+|z|^2}|cz+d|^{-2} = |cz+d|^{-2}, \quad \text{since} \\ |az+b|^2 &= |az|^2+|b|^2+2\,\mathrm{Re}\,(az\bar{b}) \\ &= |z|^2 - 4|cz|^2 + 4 - 4|d|^2 - 8\,\mathrm{Re}\,(cz\bar{d}) \\ &= 4+|z|^2-4|cz+d|^2. \end{aligned} \tag{A.6}$$

Now recalling from (A.1) that $|\partial_z Y| = |cz+d|^{-2}$ we have by this computation $\frac{W(z)}{W(Y(z))} = |\partial_z Y|$ and can easily show

$$\begin{aligned} g^W(Y_*\partial_z, Y_*\partial_z) &= W(Y(z))^2|\partial_z Y(z)|^2 g(\partial_z,\partial_z) \\ &= \frac{W(Y(z))^2}{W(z)^2}|\partial_z Y(z)|^2 g^W(\partial_z,\partial_z) = g^W(\partial_z,\partial_z), \end{aligned}$$

where we used that $Y_*\partial_z = (\partial_z Y)\partial_z$ and denoted by g the standard metric on $\mathbb{C}$ and $g^W = W^2 g$. Similarly it follows that

$$\begin{aligned} g^W(Y_*\partial_{\bar z}, Y_*\partial_{\bar z}) &= g^W(\partial_{\bar z},\partial_{\bar z}), \\ g^W(Y_*\partial_{\bar z}, Y_*\partial_z) &= g^W(\partial_{\bar z},\partial_z) = 0. \end{aligned}$$

□

Consider a spinor u on the trivial spinor bundle over $\mathbb{C}$ (which is a certain complex vector bundle over $\mathbb{C}$ of dimension two), and denote by $u_j(z_j)$ this spinor in coordinates z_j, $j \in \{1,2\}$. Then we have the relation

$$u_1(z_1) = \mathcal{G}(z_2)u_2(z_2)\,, \tag{A.7}$$

for some $\mathcal{G} \in Spin^c(2)$. Assume further that the coordinates are related by the Möbius transform $Y : z_2 \mapsto z_1 = \frac{az_2+b}{cz_2+d}$. Since we know how the one forms on $\mathbb{C}$ transform under a change of coordinates, we can find $\mathcal{G}$ by applying relation (A.7) on a spinor $\sigma^W(\mathcal{T})u$. Here $\sigma^W(\mathcal{T})$ is the Clifford multiplication in metric g^W (see Proposition 57) by a real one form

$$\mathcal{T} = \frac{1}{2}(\overline{\tau}\,\widehat{\mathrm{d}z} + \tau\,\widehat{\mathrm{d}\bar{z}})\,,$$

where $\widehat{\mathrm{d}z} = W(z)\,\mathrm{d}z$ and similarly $\widehat{\mathrm{d}\bar{z}} = W(z)\,\mathrm{d}\bar{z}$ denote the orthonormal basis of one forms on $\mathbb{C}$ in metric g^W. We denote by $\mathcal{T}_j = \operatorname{Re}(\overline{\tau_j}\,\widehat{\mathrm{d}z_j})$ the one form $\mathcal{T}$ in the bases $(\widehat{\mathrm{d}z_j}, \widehat{\mathrm{d}\bar{z}_j})$, $j \in \{1,2\}$ and note that by (A.6)

$$\tau_1(Y(z)) = \frac{W(z)}{W(Y(z))\overline{\partial_z Y(z)}}\tau_2(z) = \frac{|cz+d|^2}{(cz+d)^2}\tau_2(z)\,. \tag{A.8}$$

By (A.7) we now obtain

$$\sigma^W(\mathcal{T}_1)u_1(Y(z)) = \mathcal{G}(z)\sigma^W(\mathcal{T}_2)u_2(z) = \mathcal{G}(z)\sigma^W(\mathcal{T}_2)\mathcal{G}^{-1}(z)u_1(Y(z))\,.$$

Therefore we require

$$\mathcal{G}(z)^{-1}\sigma^W(\mathcal{T}_1)\mathcal{G}(z) = \sigma^W(\mathcal{T}_2)\,. \tag{A.9}$$

Since $\sigma^W(\widehat{\mathrm{d}z}) = \sigma(\mathrm{d}z)\,, \sigma^W(\widehat{\mathrm{d}\bar{z}}) = \sigma(\mathrm{d}\bar{z})$ we have (*c.f.* Example 11)

$$\sigma^W(\mathcal{T}) = \begin{pmatrix} 0 & \overline{\tau} \\ \tau & 0 \end{pmatrix}.$$

We can easily check that setting

$$\mathcal{G}(z) = |cz+d|^{-1}\begin{pmatrix} (cz+d) & 0 \\ 0 & \overline{(cz+d)} \end{pmatrix} \in SO(2)\,,$$

indeed solves (A.9), as

$$\mathcal{G}(z)^*\sigma^W(\mathcal{T}_1)\mathcal{G}(z) = \begin{pmatrix} 0 & \frac{\overline{(cz+d)}^2}{|cz+d|^2}\overline{\tau_1} \\ \frac{cz+d^2}{|cz+d|^2}\tau_1 & 0 \end{pmatrix}$$

corresponds to the correct transformation (A.8) of the components of the one form $\mathcal{T}$ and hence the right hand side is $\sigma^W(\mathcal{T}_2)$.

We write the transformation relation for spinors on $\mathbb{C}$ under the Möbius transform once more with the particular form of $\mathcal{G}(z)$

$$u_1(z_1) = |cz_2+d|^{-1}\begin{pmatrix} (cz_2+d) & 0 \\ 0 & \overline{(cz_2+d)} \end{pmatrix} u_2(z_2)\,. \tag{A.10}$$

Appendix B

Local boundary conditions of Berry–Mondragon type

Let us make a couple of remarks on the Aharonov–Casher type result Theorem 40 when instead of the APS boundary condition we consider the following local boundary condition, introduced in [10],

$$u^- = -\mathrm{i}(n_1 + \mathrm{i}n_2)S\, u^+ \,, \tag{B.1}$$

where (n_1, n_2) are the components of the inward normal on the boundary ∂M and $S : \partial M \to \mathbb{R} \setminus \{0\}$. For simplicity, we will only consider S being a constant on each component of the boundary. The most famous example being when $S = 1$ is called the infinite mass boundary condition. This case was studied e.g. in [36].

Remark 69. *1. The usual convention is to consider the right-hand side with a plus sign and use the outward normal.*

2. This condition is automatically gauge invariant since it is preserved by any $U(1)$ transform. Therefore Lemma 37 holds also when instead of the APS boundary condition we consider the condition (B.1).

We will analyse only a special case from the setting in Theorem 40 when M is an annulus $\Omega_{out} \setminus \Omega_1$, with inner radius R_1 and outer radius R_{out} and magnetic field of flux Φ is only inside the hole, i.e. $\operatorname{supp} B \subset \Omega_1$. Let us use polar coordinates to rewrite the condition (B.1) as follows

$$\begin{aligned} u^- &= -S(R_1)\mathrm{i}e^{\mathrm{i}\varphi}u^+ \,, &&\text{on } \partial\Omega_1 \\ u^- &= S(R_{out})\mathrm{i}e^{\mathrm{i}\varphi}u^+ \,, &&\text{on } \partial\Omega_{out} \,. \end{aligned}$$

The zero modes (see Remark 27) $u^\pm = \mathrm{e}^{\pm h(z)} g^\pm$ then have to satisfy

$$R_1^{\frac{\Phi}{2\pi}} \sum_{n\in\mathbb{Z}} b_n R_1^n \mathrm{e}^{-\mathrm{i}\varphi n} = -\mathrm{i}S(R_1) R_1^{-\frac{\Phi}{2\pi}} \sum_{n\in\mathbb{Z}} a_n R_1^n \mathrm{e}^{\mathrm{i}\varphi(n+1)}$$
$$R_{out}^{\frac{\Phi}{2\pi}} \sum_{n\in\mathbb{Z}} b_n R_{out}^n \mathrm{e}^{-\mathrm{i}\varphi n} = \mathrm{i}S(R_{out}) R_{out}^{-\frac{\Phi}{2\pi}} \sum_{n\in\mathbb{Z}} a_n R_{out}^n \mathrm{e}^{\mathrm{i}\varphi(n+1)},$$

for some $a_n, b_n \in \mathbb{C}$, where we used the fact that by gauge invariance we can assume that $B = \Phi\delta_0$ and thus $h = -\frac{\Phi}{2\pi}\log|z|$. Then we obtain the conditions

$$R_1^{\frac{\Phi}{\pi}} b_{-n} R_1^{-n} = -\mathrm{i}S(R_1) a_{n-1} R_1^{n-1}$$
$$R_{out}^{\frac{\Phi}{\pi}} b_{-n} R_{out}^{-n} = \mathrm{i}S(R_{out}) a_{n-1} R_{out}^{n-1},$$

leading to

$$R_1^{\frac{\Phi}{\pi}-2n+1} b_{-n} = -\mathrm{i}S(R_1) a_{n-1} = \frac{-S(R_1)}{S(R_{out})} R_{out}^{\frac{\Phi}{\pi}-2n+1} b_{-n}.$$

Hence $b_{-n} \neq 0$ only if the ratio $-\frac{S(R_1)}{S(R_{out})} =: K$ is a positive constant and $R_1^{\frac{\Phi}{\pi}-2n+1} = K R_{out}^{\frac{\Phi}{\pi}-2n+1}$. In the particular case when $S : \partial M \to \{\pm 1\}$ we have the conditions $S(R_1) = -S(R_{out})$ and $\frac{\Phi}{\pi} - 2n + 1 = 0$. Therefore if the value of the flux is an odd integer multiple of π we have one zero mode with components

$$u^- = |z|^{\frac{\Phi}{2\pi}} \bar{z}^{-n}, u^+ = \frac{\mathrm{i}|z|^{-\frac{\Phi}{2\pi}}}{S(R_1)} z^{n-1}, \quad n = \frac{\Phi}{2\pi} + \frac{1}{2} \in \mathbb{Z}.$$

In accordance with results of Prokhorova, [33], we can form a family of operators $D_{a(t)}$ where the flux of $a(t) = \frac{2\pi t}{r}$ goes from zero to 2π as t varies from 0 to 1. Then by Lemma 37 the endpoints of such operator family are unitarily equivalent

$$\mathrm{e}^{\mathrm{i}\varphi} D_{a(0)} \mathrm{e}^{-\mathrm{i}\varphi} = \mathrm{e}^{\mathrm{i}\varphi} D_0 \mathrm{e}^{-\mathrm{i}\varphi} = D_{a(1)}.$$

We can consider the number counting signed crossings of the eigenvalues of $D_{a(t)}$ through the t-axis, known as the spectral flow $SF(D_{a(t)})$. A rigorous definition of the term for self-adjoint Fredholm operators can be found in [32]. Our computation above yields that the spectral flow of the family $D_{a(t)}$ has to be 1 or -1, which agrees with the statement of Theorem 1 in [33] that, in fact, $SF(D_{a(t)}) = 1$.

Remark 70. *1. Note that in the multiple holes setting we cannot follow the same procedure as on the annulus to conclude anything about the number of zero modes.*

2. *In the case of unbounded region with one hole there are no zero modes. Indeed, as before we have the relation of the coefficients on the boundary*

$$R_1^{\frac{\Phi}{\pi}-2n+1} b_{-n} = -\mathrm{i}S(R_1) a_{n-1}\,. \tag{B.2}$$

Moreover, the L^2 integrability at infinity of $u^\pm$ implies that $a_n \neq 0$ or $b_m \neq 0$ only if

$$n - \frac{\Phi}{2\pi} < -1 \quad \textit{or} \quad m + \frac{\Phi}{2\pi} < -1\,.$$

Therefore the left-hand side of (B.2) *is zero for $n \leq 1 + \frac{\Phi}{2\pi}$ and the right hand side is zero for $n \geq \frac{\Phi}{2\pi}$ from which we conclude that all the coefficients are zero and we have no zero modes in this case.*

Appendix C

Computation of the number of the zero modes on N holes, different method

In this section we would like to illustrate another way of obtaining the result of Theorem 39. This method, however, works only in the case when we have no outer boundary. We used this computation in the first place and it motivated the currently used approach containing also the setting with an outer boundary. Hence, using the notation from the introduction of Chapter 3, we set $M = \mathbb{C} \setminus \cup_{k \leq N} \Omega_k$. For simplicity, we consider only magnetic field supported inside the holes, i.e. no flux through the bulk. The generalisation to the case where there is flux in the bulk is, however, straightforward. Recall that due to the gauge invariance (Lemma 37) we can assume that the magnetic field inside the hole Ω_j (with centre at w_j and radius R_j) is a normalised flux $\Phi'_j \in [-\pi, \pi)$ multiple of the Dirac delta distribution $B_j = \Phi'_j \delta_{w_j}$ and therefore the scalar potential (2.15) can be written as

$$h = \sum_{k \leq N} -\frac{\Phi'_k}{2\pi} \log |z - w_k| \,. \tag{C.1}$$

We will denote by $\vec{a}_{(j)}$ the vector potential coming from the field inside the hole Ω_j, which is now by the third point of Remark 24 equal to $\left(0, \frac{\Phi'_j}{2\pi r_j}\right)$ in the basis $(\,\mathrm{d}r_j, r_j \,\mathrm{d}\varphi_j)$. Here and further in the text (r_j, φ_j) are the polar coordinates based at the centre w_j of Ω_j. Hence, we can rewrite the exponential in the boundary condition (2.27) as follows

$$\exp\left(\mathrm{i} \int_{\gamma_j} \vec{a} \,\mathrm{d}\vec{s} - \mathrm{i} \frac{\Phi'_j}{2\pi} \varphi_j\right) = \exp\left(\mathrm{i} \sum_{k \neq j} \int_{\gamma_j} \vec{a}_{(k)} \,\mathrm{d}\vec{s}\right), \tag{C.2}$$

where $\gamma_j \subset \partial\Omega_j$ denotes the curve connecting $z_{0j} = w_j + R_j$ and the point $z \in \partial\Omega_j$ (for illustration see Figure 2.3). Using a branch cut of $\log(z - w_k)$ so that it is well defined and analytic for $z \in \partial\Omega_j$ we compute

$$\begin{aligned}\int_{\gamma_j} \partial_{z'}(\log|z'-w_k|)\,\mathrm{d}z' &= \int_{\gamma_j} \partial_{z'}(\log(z'-w_k) - \mathrm{i}\arg(z'-w_k))\,\mathrm{d}z' \\ &= \log(z-w_k) - \log(z_{0j}-w_k) - \mathrm{i}\int_{\gamma_j} \partial_{z'}\arg(z'-w_k))\,\mathrm{d}z'\,.\end{aligned}$$

The real part of the path integral of the derivative of the argument can be further evaluated

$$\begin{aligned}&\mathrm{Re}\int_{\gamma_j} \partial_{z'}\arg(z'-w_k)\,\mathrm{d}z' \\ &\quad= \mathrm{Re}\int_{\gamma_j} \frac{1}{2}(\partial_{x'} - \mathrm{i}\partial_{y'})\arg(x'+\mathrm{i}y'-w_k)(\,\mathrm{d}x' + \mathrm{i}\,\mathrm{d}y') \\ &\quad= \frac{1}{2}\int_{\gamma_j} \partial_{x'}[\arg(x'+\mathrm{i}y'-w_k)]\,\mathrm{d}x' + \partial_{y'}[\arg(x'+\mathrm{i}y'-w_k)]\,\mathrm{d}y' \\ &\quad= \frac{1}{2}\int_{\gamma_j} \vec{\nabla}[\arg(z-w_k)]\cdot\,\mathrm{d}\vec{s} = \frac{1}{2}[\arg(z-w_k) - \arg(z_{0j}-w_k)]\,,\end{aligned}$$

and yields

$$\mathrm{Im}\int_{\gamma_j} \partial_{z'}h_k\,\mathrm{d}z' = -\frac{1}{2}\frac{\Phi'_k}{2\pi}[\arg(z-w_k) - \arg(z_{0j}-w_k)]\,.$$

Using this and (2.12) we obtain the following expression for the summands in the exponential (C.2)

$$\begin{aligned}\mathrm{i}\int_{\gamma_j} \vec{a}_{(k)}(s_j)\cdot\,\mathrm{d}\vec{s}_j &= \frac{\mathrm{i}}{2}\int_{\gamma_j} \overline{a_{(k)}}\,\mathrm{d}z' + a_{(k)}\,\mathrm{d}\bar{z}' \\ &= \int_{\gamma_j}(-\partial_{z'}h_k\,\mathrm{d}z' + \partial_{\bar{z}'}h_k\,\mathrm{d}\bar{z}') = -2\mathrm{i}\,\mathrm{Im}\left[\int_{\gamma_j}\partial_{z'}h_k\,\mathrm{d}z'\right] \\ &\qquad= \mathrm{i}\frac{\Phi'_k}{2\pi}(\arg(z-w_k) - \arg(z_{0j}-w_k))\,.\end{aligned}$$

We have just shown that in the particular gauge when $B_j = \Phi'_j\delta_{w_j}$ for all $j \le N$, the boundary condition (2.27) on the boundary $\partial\Omega_j$ can be rewritten as

$$\overline{\mathrm{span}}^{\check{H}(A)}\left[\left\{\begin{pmatrix}\mathrm{e}^{\mathrm{i}\varphi_j n}\\ 0\end{pmatrix}\right\}_{\mathbb{Z}\ni n\ge 0}, \left\{\begin{pmatrix}0\\ \mathrm{e}^{\mathrm{i}\varphi_j n}\end{pmatrix}\right\}_{\mathbb{Z}\ni n\le 0}\right] \times \exp\left(\mathrm{i}\sum_{k\neq j}\frac{\Phi'_k}{2\pi}\arg(z-w_k)\right). \quad (\mathrm{C.3})$$

Zero modes

Using this newly found form of the APS boundary condition we will determine the number of zero modes in the system. For conciseness we will find only the zero modes with spin up as the case of spin down is analogous. Using (C.1) and Remark 27 we can write

$$u^+ = \mathrm{e}^h f = \Pi_{k\le N}|z-w_k|^{\frac{-\Phi_k'}{2\pi}} f\,,$$

where f is analytic on the interior of M, and fixing an index $j \le N$ it is in particular analytic on an open annulus $\mathcal{A}(\Omega_j)$ around the hole Ω_j such that $\mathcal{A}(\Omega_j)\cap\Omega_k = \emptyset$ for all $k \neq j$.

Let us consider the following modification of f and u^+

$$\widetilde{f} := \Pi_{k\neq j}(z-w_k)^{-\frac{\Phi_k'}{2\pi}} f\,,\quad \widetilde{u}^+ := \mathrm{e}^{-\mathrm{i}\sum_{k\neq j}\frac{\Phi_k'}{2\pi}\arg(z-w_k)}u^+ = |z-w_j|^{-\frac{\Phi_j'}{2\pi}}\widetilde{f}\,.$$

Since $\widetilde{f}$ (using suitable branch cuts for $(z-w_k)^{-\frac{\Phi_k'}{2\pi}}$) is analytic on $\mathcal{A}(\Omega_j)$ we can further write its Laurent series

$$\widetilde{f} = \sum_{n\in\mathbb{Z}} c_n (z-w_j)^n\,,$$

with some coefficients $c_n \in \mathbb{C}$, in order to get

$$\widetilde{u}^+ = |z-w_j|^{-\frac{\Phi_j'}{2\pi}} \sum_{n\in\mathbb{Z}} c_n (z-w_j)^n\,,$$

on that annulus. Moreover, the boundary condition (C.3) and Lemma 48 yield that on $\partial\Omega_j$ this function satisfies the modified boundary condition

$$\widetilde{u}^+(z) = \sum_{n\ge 0}\beta_n \mathrm{e}^{\mathrm{i}n\varphi_j}\,,$$

for some $\beta_n \in \mathbb{C}$. Since $|z-w_j|$ is constant on $\partial\Omega_j$, we conclude that $c_n = 0$ for $n < 0$, i.e.

$$\widetilde{f} = \sum_{n\ge 0} c_n (z-w_j)^n\,,$$

on $\mathcal{A}(\Omega_j)$. Hence further by analyticity of $(z-w_k)^{\frac{\Phi_k'}{2\pi}}$ on $\mathcal{A}(\Omega_j)$ for all $k \neq j$

$$f = \Pi_{k\neq j}(z-w_k)^{\frac{\Phi_k}{2\pi}} \sum_{n\ge 0} c_n (z-w_j)^n = \sum_{n\ge 0}\widetilde{c}_n (z-w_j)^n\,,\quad \widetilde{c}_n \in \mathbb{C}\,,$$

on $\mathcal{A}(\Omega_j)$. Note that the expansion has no principal part and thus the function f can be analytically extended inside the hole Ω_j. Since the index $j \le N$

was arbitrary this implies that f is analytic on $\mathbb{C}$ and has Taylor series. Coming back to the zero mode itself we thus have

$$u^+(z) = \Pi_{k\leq N}|z - w_k|^{\frac{-\Phi'_k}{2\pi}} \sum_{n\geq 0} u_n^+ z^n \, ,$$

for $z \in M$ and some coefficients $u_n^+ \in \mathbb{C}$. Implementing L^2 integrability condition at infinity we conclude that the number of the spin up zero modes is then the number of the integers $n \geq 0$ satisfying

$$n - \sum_{k\leq N} \frac{\Phi'_k}{2\pi} < -1 \, .$$

In other words there are

$$\left\lfloor \sum_{k\leq N} \frac{\Phi'_k}{2\pi} \right\rfloor$$

zero modes with spin up. Here $\lfloor y \rfloor$ denotes the biggest integer strictly smaller then y.

Appendix D

Computation of the η- invariant of the boundary operator

The **eta-invariant** of an elliptic differential operator A on a compact manifold is the analytic extension to $s = 0$ of the **eta function**

$$\eta_s(A) = \sum_{\lambda \neq 0} |\lambda|^{-s} sgn(\lambda)\,,$$

where the sum runs over non-zero eigenvalues λ of the operator A. It is not straightforwardly seen that such analytic extension would exist, but clearly the sum is convergent for $\operatorname{Re} s$ sufficiently large. Atiyah, Patodi and Singer who introduced this notion in their series of papers [3, 4, 5] also showed (*c.f.* [3, Theorem 4.2]) that for the Dirac operator on a Riemannian manifold with a boundary the eta function of its boundary operator is holomorphic for $\operatorname{Re} s > -\frac{1}{2}$, so in particular it has a finite value at $s = 0$. Their arguments are rather complicated and involve manipulations with the corresponding heat kernel. In the context of the problems considered in this thesis we are interested only in the eta invariant of the operator $-\mathrm{i}\partial_t - c$ for $t \in [0, 2\pi]$ with a periodic boundary condition and c being a real constant parameter. We will present an elementary proof of existence of the analytic extension to $\operatorname{Re} s > -1$ for this simple case and will find its value at $s = 0$. Recall that the spectrum of the operator $-\mathrm{i}\partial_t - c$ is $n - c$, $n \in \mathbb{Z}$. Since by definition the η-invariant depends only on the spectrum we will also write $\eta(n - c)$ instead of $\eta(-\mathrm{i}\partial_t - c)$.

Remark 71. *Let us list a couple of elementary observations following directly from the definition.*

1. *If the parameter $c = 0$ we have $\eta(-\mathrm{i}\partial_t) = 0$, since*

$$\sum_{n>0} n^{-s} - \sum_{n<0} |n|^{-s} = 0\,.$$

2. *We have invariance under a shift of the parameter $c \in \mathbb{R} \setminus \mathbb{Z}$ by an integer. To see this let us write $c = m + \langle c \rangle$ where $\langle c \rangle$ is the unique number in the interval $(0,1)$ that differs from c by an integer m. Then*

$$\begin{aligned}\sum_{n>c}(n-c)^{-s} - \sum_{n<c}|n-c|^{-s} &= \sum_{n>c}(n-c)^{-s} - \sum_{n>-c}(n+c)^{-s} \\ &= \sum_{n>m+\langle c\rangle}(n-m-\langle c\rangle)^{-s} - \sum_{n>-m-\langle c\rangle}(n+m+\langle c\rangle)^{-s} \\ &= \sum_{k>\langle c\rangle}(k-\langle c\rangle)^{-s} \sum_{k>-\langle c\rangle}(k+\langle c\rangle)^{-s} \\ &= \sum_{n>\langle c\rangle}(n-\langle c\rangle)^{-s} - \sum_{n<\langle c\rangle}|n-\langle c\rangle|^{-s}\,.\end{aligned}$$

3. *In the case $c \in \mathbb{Z}$ the first remark and the computation above with $\langle c \rangle$ substituted by zero give us directly the result $\eta(n-c) = 0$.*

For $c \in (0,1)$ we consider the following form of the eta-function

$$\eta_s(n-c) = -c^{-s} + \sum_{n\geq 1}(n-c)^{-s} - (n+c)^{-s}\,. \tag{D.1}$$

To prove that this can be analytically extended to $s = 0$ we will approximate the sum by an integral expression which will turn out to be analytic at $s = 0$ and whose value at zero can be explicitly computed.

Let us define for $n \geq 1$ the following function

$$\rho(s,c,n) := (n-c)^{-s} - (n+c)^{-s} - \left(\int_n^{n+1}(x-c)^{-s}\,\mathrm{d}x - \int_n^{n+1}(x+c)^{-s}\,\mathrm{d}x\right),$$

where we can compute the integral on the right-hand side

$$\int_n^{n+1}(x\pm c)^{-s}\,\mathrm{d}x = \frac{(n+1\pm c)^{-s+1} - (n\pm c)^{-s+1}}{1-s},$$

to obtain

$$\rho(s,c,n) = n^{-s}\left(\left(1-\frac{c}{n}\right)^{-s} - \left(1+\frac{c}{n}\right)^{-s}\right) \tag{D.2}$$

$$-\frac{n^{-s+1}}{1-s}\left(\left(1+\frac{c}{n}\right)^{-s+1} - \left(1-\frac{c}{n}\right)^{-s+1}\right) \tag{D.3}$$

$$+\frac{n^{-s+1}}{1-s}\left(\left(1+\frac{1+c}{n}\right)^{-s+1} - \left(1+\frac{1-c}{n}\right)^{-s+1}\right).$$

Observe that evaluating at $s = 0$ we have $\rho(0,c,n) = 0$. Moreover, for large n we can show that this function decays like n^{-s-2}.

Lemma 72. *There exists n_0 and $R(s)$ such that for all $n > n_0$ it holds*

$$\rho(s,c,n) = s(s+1)cn^{-s-2} + R(s)n^{-s-3},$$

where $R(s)$ is given by (D.4) and in the case $n > 4$ and $s \in (-1,2)$ is satisfies the bound $|R(s)| \leq 11|s(s+1)(s+2)|$.

Proof. Let $x \in \mathbb{R}$. Expanding the function $(1+x)^{-s}$ in $|x| \in (0,1)$, around zero for a parameter $s \in \mathbb{C}$ we obtain

$$\begin{aligned} e^{-s\log(1+x)} = 1 - sx + \frac{x^2}{2}s(s+1) - \frac{x^3}{3!}s(s+1)(s+2) \\ + \frac{x^4}{4!}\frac{s(s+1)(s+2)(s+3)}{(1+\xi)^{s+4}}, \xi \in (0,x), \end{aligned}$$

where the last term is the Lagrange form of the remainder in Taylor series. Note that applying this to $\rho(s,c,n)$ the terms in the expansion that are even in $x = \pm c/n$ will cancel each other in the right-hand side expressions (D.2), (D.3). Using

$$\begin{aligned} (1+c) - (1-c) &= 2c, \\ (1+c)^2 - (1-c)^2 &= 4c, \\ (1+c)^3 - (1-c)^3 &= 2c(c^2+3), \end{aligned}$$

and considering error terms up to order n^{-s-3} we obtain the following simplification of the expansion of ρ for large n

$$\begin{aligned} \rho(s,c,n) &= n^{-s} \cdot 2scn^{-1} \\ &\quad + \frac{n^{-s+1}}{1-s}\left(2(s-1)cn^{-1} + 2\frac{(s-1)s(s+1)}{3!}c^3n^{-3}\right) \\ &\quad + \frac{n^{-s+1}}{1-s}\Bigg(-2(s-1)cn^{-1} + \frac{(s-1)s}{2}4cn^{-2} \\ &\quad - \frac{(s-1)s(s+1)}{3!}2c(c^2+3)n^{-3}\Bigg) + R(s)n^{-s-3} \\ &= n^{-s-2}\left(-\frac{s(s+1)}{3}c^3 + \frac{c(c^2+3)s(s+1)}{3}\right) + R(s)n^{-s-3} \\ &= s(s+1)cn^{-s-2} + Rn^{-s-3}. \end{aligned}$$

Here, the remainder reads

$$\begin{aligned} R(s) = \frac{s(s+1)(s+2)}{4!}\Big(4c^3(1+\xi_1)^{-s-3} + 4c^3(1-\xi_2)^{-s-3} \\ + c^4(1+\xi_3)^{-s-3} - c^4(1-\xi_4)^{-s-3} - (1+c)^4(1+\xi_5)^{-s-3} \\ + (1-c)^4(1+\xi_6)^{-s-3}\Big), \quad \text{(D.4)} \end{aligned}$$

with

$$\xi_j \in (0, c/n)\,, \text{ for } j \in \{1,2,3,4\}$$
$$\xi_5 \in (0, (1+c)/n)\,, \text{ and } \xi_6 \in (0, (1-c)/n)\,.$$

In particular for $\operatorname{Re} s \in (-1,2)$ and $n > 4$ we can use $(1+\xi_j)^{-\operatorname{Re} s-3} \leq 1$ and $(1-\xi_j)^{-\operatorname{Re} s-3} \leq 2^5$, $j \leq 6$ to obtain a rough bound $|R(s)| \leq 11|s(s+1)(s+2)|$. □

The following statement asserts the analyticity of the eta function $\eta(n+c)$.

Proposition 73. *The sum of the differences*

$$\sum_{n\geq 1} (n-c)^{-s} - (n+c)^{-s} \tag{D.5}$$

is an analytic function in $\operatorname{Re} s > -1$.

Proof. First we show, that $\sum_{n\geq 1} \rho(s,c,n)$ is analytic in s for $\operatorname{Re} s \in (-1,2)$. Let us denote by

$$f_m := \sum_{n=1}^{m} \rho(s,c,n)$$

the partial sums. This is clearly a sequence of functions that are analytic for $\operatorname{Re} s \in (-1,2)$. Lemma 72 moreover implies that f_m converges uniformly on any compact subset of $\operatorname{Re} s \in (-1,2)$ as $m \to \infty$. By Weierstrass's theorem (see Theorem 1 in [2, Section V.1.1]) we conclude that this limit is an analytic function of s on $\operatorname{Re} s \in (-1,2)$.

Now, we compute the integral

$$\begin{aligned}\int_1^\infty (x-c)^{-s} - (x+c)^{-s}\,\mathrm{d}x \\ = \lim_{x\to\infty} \frac{(x-c)^{-s+1} - (x+c)^{-s+1}}{1-s} - \left.\frac{(x-c)^{-s+1} - (x+c)^{-s+1}}{1-s}\right|_{x=1} \\ = \frac{(1+c)^{-s+1} - (1-c)^{-s+1}}{1-s}\,,\end{aligned} \tag{D.6}$$

which is clearly analytic for all $s \in \mathbb{C} \setminus \{1\}$. Furthermore, the limit for $s = 1$ exists and can be computed by L'Hospital's rule. Hence the sum (D.5) is a difference of two analytic functions which concludes the proof for $\operatorname{Re} s \in (-1,2)$. The statement is clear for $\operatorname{Re} s \geq 2$. □

Proposition 74. *If $c \in \mathbb{R} \setminus \mathbb{Z}$ it holds*

$$\eta(-\mathrm{i}\partial_t - c) = -1 + 2\langle c \rangle \in (-1, 1)\,,$$

where $\langle c \rangle$ is the unique number in $(0,1)$ which differs by an integer from c. If $c \in \mathbb{Z}$, then $\eta(-\mathrm{i}\partial_t - c) = 0$.

Note, that since $\langle c \rangle + \langle -c \rangle = 1$ Proposition 74 also asserts that

$$\eta(-\mathrm{i}\partial_t + c) = 1 - 2\langle c \rangle \in (-1, 1)\,, \quad c \in \mathbb{R} \setminus \mathbb{Z}\,.$$

Proof. The analytic extension to $s = 0$ of the finite sum

$$\sum_{n=1}^{n_0} \rho(s, c, n) = \sum_{n=1}^{n_0} (n - c)^{-s} - (n + c)^{-s} - \int_n^{n+1} (x - c)^{-s}\,\mathrm{d}x - (x + c)^{-s}\,\mathrm{d}x\,,$$

vanishes for any $n_0 \in \mathbb{N}$. Moreover, the sum $\sum_{n \geq 1} n^{-2-s}$ is convergent for $\operatorname{Re} s > -1$ and therefore $\sum_{n=n_0}^{m} n^{-2-s}$ can be made arbitrarily small for any $m > n_0$ by choosing n_0 large enough. Hence employing further Lemma 72 we see that the analytic extension of the limiting function from the previous proof $\lim_{m \to \infty} \sum_{n=1}^{m} \rho(s, c, n)$ vanishes at $s = 0$. It follows that we can compute the value of the analytic extension to $s = 0$ of (D.5) by computing the limit $s \to 0$ of the integral (D.6), which is $2c$. Thus considering formula (D.1) we obtain the result for the η-invariant

$$\eta(-\mathrm{i}\partial_t - c) = -1 + 2c \in (-1, 1)\,,$$

if $c \in (0, 1)$. For $c \notin (0, 1)$ the claim follows from this result and Remark 71.

□

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