# Projection of balances and benefits in life insurance with various dividend strategies

Debbie Kusch Falden

Department of Mathematical Sciences
Faculty of Science
University of Copenhagen

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PhD Thesis by:

Debbie Kusch Falden
Nordre Frihavnsgade 100A, 3. tv.
DK-2100 Copenhagen Ø
debbiefalden@hotmail.com

Assessment committee:

Jesper Lund Pedersen (chairperson)
Associate Professor, University of Copenhagen, Denmark

Jennifer Alonso García
Professor, The Université libre de Bruxelles, Belgium

Sascha Desmettre
Assistant Professor, Johannes Kepler University Linz, Austria

Supervisor:

Mogens Steffensen
Professor, University of Copenhagen, Denmark

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# Abstract

This thesis consists of four independent research projects concerning challenges and techniques within the mathematics of life insurance. They are centered around projection of balances and benefits for with-profit insurance contracts, and study various extensions of the projection model. Firstly, we consider projection of balances with the policyholder behavior options surrender and free-policy conversion, where we derive a system of differential equations of the state-wise projections based on a suitable approximation. Then, we study the computation of market reserves, when Management Actions depend on retrospective reserves and prospective reserves, causing interdependence. Next, we discuss the concept of forward transition rates in a doubly stochastic Markov setting linked with a stochastic interest rate, and propose forward transition rates, when the reserve is decomposed into sojourn payments and transition payments. Lastly, we study affine dividends as controls of linear-quadratic optimal control problems in an actuarial framework.

# Preface

This thesis has been submitted in partial fulfillment of the requirements for the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen. The work was carried out between April 2019 and May 2022 under supervision of Professor Mogens Steffensen (University of Copenhagen). The research was part of the project "ProBaBli - Projection of Balances and Benefits in Life Insurance" funded by Innovation Fund Denmark (award number 7076-00029) with investment from Edlund A/S. The company Edlund A/S implemented the theoretical results into software products that are currently being used by multiple Danish insurance companies.

The PhD thesis consists of an introduction and four manuscripts, where two of the manuscripts are published in international peer-reviewed journals at the time of writing. The introduction provides an overview of the contributions and connections between the manuscripts. The manuscripts are independent scientific contributions and appear with small notational discrepancies across the chapters. The author takes full responsibility for any typographical or mathematical errors.


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Debbie Kusch Falden
Copenhagen, May 2022

A few minor errors have been corrected and an ISBN has been provided in the final version of the thesis compared to the previous version submitted to the PhD School on 31st of May 2022.

Debbie Kusch Falden
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# List of papers

**Chapter 2:**
Falden, D.K.; Nyegaard, A.K. (2021). Retrospective reserves and bonus with policyholder behavior. *Risks* 9(1), 15. DOI: 10.3390/risks9010015

**Chapter 3:**
Falden, D.K.; Nyegaard, A.K. (2022). Reserve-dependent Management Actions in life insurance. Scandinavian Actuarial Journal. DOI: 10.1080/03461238.2022.2061868

**Chapter 4:**
Falden, D.K. (2022). Forward transition rates in multi-state life insurance. *Preprint.*

**Chapter 5:**
Avanzi, B.; Falden, D.K.; Steffensen, M. (2022). Stable dividends are optimal under linear-quadratic optimization. *Preprint.*

# Summary

This thesis consists of an introductory chapter and four manuscripts. Each manuscript constitutes a chapter, that considers challenges and techniques within the mathematics of life insurance. The introduction, Chapter 1, describes the scientific background and provides an overview of the scientific contributions of the thesis and their interconnections. The subsequent chapters are independent research projects regarding projection of balances and benefits for with-profit insurance contracts, including various extensions of the projection model.

Legislation imposes insurance companies to project their balance sheet items in various financial scenarios, where Management Actions, such as investment strategies and bonus allocation strategies, are taken into account. The retrospective nature of Management Actions causes projection models to focus on retrospective reserves, where state-wise projections can be calculated by a system of forward differential equations. The thesis considers challenges in the projection model arising from including policyholder behavior, as well as various extensions to the dividend strategy.

In Chapter 2, which contains the manuscript "*Retrospective reserves and bonus with policyholder behavior*", we consider projection of balances without and with the policyholder behavior options surrender and free-policy conversion. The inclusion results in a structure where the system of differential equations of the state-wise projections is non-trivial. We consider a case where we are able to find accurate differential equations and suggest an approximation method for projection including policyholder behavior in general.

Chapter 3, which contains the manuscript "*Reserve-dependent Management Actions in life insurance*", concerns the calculation of the market reserve, where Management Actions depend on the retrospective reserves and the market reserve itself. We study the complications that arise from the interdependence between the retrospective and prospective reserves, and characterize the market reserve by a partial differential equation (PDE). We reduce the dimension of the PDE in the case of linearity, and furthermore, suggest an approximation of the market reserve based on the forward rate.

In Chapter 4, which contains the manuscript "*Forward transition rates in multi-state*

*life insurance*", we discuss the concept of forward transition rates, inspired by the concept of the forward interest rate. In a doubly stochastic Markov setting, the forward transition rates are the deterministic substitutes for the stochastic transition intensities, which accurately compute the reserves. We consider previous suggestions for forward transition rates and propose forward transition rates in the case where the reserve is decomposed into sojourn payments and transition payments. Furthermore, we allow for a dependency between the interest rate and transition intensities.

In Chapter 5, which contains the manuscript "*Stable dividends are optimal under linear-quadratic optimization*", we study stable dividends allotted to the shareholders as a stability criterion for a risky business in the context of stochastic optimal control problems. We consider affine dividend strategies due to their property of being stable, which are optimal in so-called linear-quadratic optimization (LQ optimization). The Hamilton-Jacobi-Bellman equations characterize the objective of the LQ optimization in an actuarial framework, and we compare the objective of the LQ optimization and optimal controls to the classical objective of maximizing expected present value of future dividends.

# Resumé

Denne afhandling består af et introducerende kapitel og fire manuskripter, der omhandler udfordringer og teknikker indenfor livsforsikringsmatematik. Introduktionen, Kapitel 1, beskriver den teoretiske baggrunden og giver et overblik over afhandlingens videnskabelige bidrag og deres indbyrdes sammenhæng. De efterfølgende kapitler er enkeltstående forskningsprojekter, der relaterer til fremregning af balancen og hensættelser for gennemsnitsrente forsikringskontrakter, samt undersøger forskellige udvidelser af fremskrivningsmodellen.

Lovgivning pålægger forsikringsselskaber at fremregne deres balanceposter i forskellige økonomiske scenarier under hensyn til ledelseshandlinger (Management Actions), såsom investerings- og bonustildelingsstrategi. Fremregningsmodeller fokuserer på retrospektive reserver, da Management Actions har en retrospektiv karakter. De tilstandsvise fremregninger kan udføres ved at løse et system af fremadrettede differentialligninger. Afhandlingen omhandler de udfordringer der opstår, når forsikringstageradfærd og aftalebestemte optioner inkluderes i fremregningsmodellen, samt forskellige udvidelser af dividende strategien.

I kapitel 2, som indeholder manuskriptet "*Retrospective reserves and bonus with policyholder behavior*", betragter vi fremregning af balancen uden og med forsikringstager optionerne genkøb og overgang til fripolice. Inklusionen er ikke-triviel, da den resulterer i en struktur, som bryder antagelserne for systemet af differentialligninger til de tilstandsvise fremregninger. Vi betragter et særtilfælde, hvor antagelserne antagelserne for systemet af differentialligninger er opfyldt. Ydermere foreslår vi en approksimationsmetode, hvormed man generelt opnår differentialligninger til de tilstandsvise fremregninger inklusiv forsikringstageradfærd.

Kapitel 3, som indeholder manuskriptet "*Reserve-dependent Management Actions in life insurance*", omhandler beregning af markedsreserven, når ledelsesbeslutninger og forretningsstrategier afhænger af de retrospektive reserver og markedsreserven. Vi betragter de komplikationer, der opstår som følge af den indbyrdes afhængighed mellem de retrospektive og prospektive reserver. Vi karakteriserer markedsreserven ved en partiel differentialligning (PDE) og ved at antage linearitet, reducerer vi dimensionen af PDEen. Desuden foreslår vi en approksimation af markedsreserven baseret på

forward-renten.

I kapitel 4, som indeholder manuskriptet "*Forward transition rates in multi-state life insurance*", diskuterer vi begrebet forward-overgangsintensiteter, som er inspireret af forward-renten. Forward-overgangsintensiteter er de determiniske overgangsintensiteter, der kan benyttes til korrekt at udregne reserver ved at erstatte de stokastiske overgangsintensiteter for dobbeltstokastiske Markovmodeller. Vi betragter andre forslag for forward-overgangsintensiteter og kommer selv med et forslag baseret på en dekomposition af reserven i tilstandsvise betalinger og overgangs betalinger. Desuden tillader vi afhængighed mellem renten og overgangsintensiteterne.

I kapitel 5, som indeholder manuskriptet "*Stable dividends are optimal under linear-quadratic optimization*", studerer vi stabile dividender tildelt aktionærerne som et stabilitetskriterium for en risikofyldt virksomhed. Vi betragter det som stokastiske optimale kontrolproblemer, med særlig interesse i affine dividendestrategier, da de er stabile. Affine dividendestrategier er optimale i det såkaldte linear-quadratic optimization problem (LQ-optimering). Vi benytter Hamilton-Jacobi-Bellman ligningerne til at karakterisere værdifunktionen i LQ-optimeringen fra et aktuarmæssigt synspunkt. Ydermere sammenligner vi objektivet i LQ-optimering og de optimale kontroller med det klassiske objektiv om at maksimere den forventede nutidsværdi af fremtidige dividender.

# Contents

# Chapter 1

# Introduction

In addition to this introduction, the thesis consists of manuscripts that consider challenges and techniques within the mathematics of life insurance. Firstly, the introduction presents the background and results in the field that precedes the research behind the subsequent chapters. The introduction then provides an overview of the main contributions of the thesis and explains the interconnections between the chapters. Each manuscript in this thesis is written independently, therefore, each chapter also contains an introduction specific for the scientific contribution of the chapter.

## 1.1 Background

The life insurance setup considered in this thesis is based on multi-state models, where the state of the life, of the insured, is governed by a Markovian jump process. This was early on formalized by Hoem (1969) and further studied by Norberg (1991). Markov processes are extensively used within life insurance and are popular because of tractability, interpretation and computational ease. Recent study extends the Markov chain models to include duration effects, for instance, related to policyholder behavior as in the papers by Henriksen et al. (2014) and Buchardt, Møller, and Schmidt (2014), and extends to doubly stochastic Markov chain models with stochastic transition intensities as in the paper by Buchardt, Furrer, and Steffensen (2019). Research and models in life insurance are often influenced by legislation and actuarial concepts are embedded in financial laws. In particular the Solvency II Directive have had an impact on recently developed mathematical tools and techniques such as projection in Bruhn and Lollike (2021) and not least for the research in this thesis.

### 1.1.1 The classical multi-state life insurance setup

A life insurance contract is an agreement, where a policyholder pays premiums in order to receive coverage in terms of benefits from an insurance company, and where the payments specified in the contract are dependent on the state of the life of the insured.

The insurance company must be able to valuate the contract to determine the payments at initialization, and to ensure they can fulfil the contractual requirements. The calculations take the classical multi-state life insurance setup as starting point, where a continuous time Markov chain $Z = (Z(t))_{t\geq 0}$ in a finite state space $\mathcal{J} = \{0, 1, \ldots, J\}$ describes the biometric and behavioral state of the insured e.g. alive, disabled, dead, free-policy. The transition probabilities of $Z$ are given by

$$p_{ij}(s,t) = \mathbb{P}\big(Z(t) = j \mid Z(s) = i\big),$$

for $i, j \in \mathcal{J}$ and $s \leq t$, with assumed existence of suitably regular transition intensities

$$\mu_{ij}(t) = \lim_{h\downarrow 0} \frac{1}{h} p_{ij}(t, t+h),$$

for $i, j \in \mathcal{J}$, $i \neq j$. The transition probabilities satisfy Kolmogorov's forward and backward differential equations. The Markov property entails that the future states of life only depend on the current state of the insurer. Furthermore, the associated multivariate counting process, $N^k(t)$, counts the number of jumps of $Z$ into state $k \in \mathcal{J}$ up to and including time $t$

$$N^k(t) = \#\big\{s \in (0,t] \mid Z(s-) \neq k, Z(s) = k\big\},$$

where $Z(s-) = \lim_{h\downarrow 0} Z(s-h)$. Let $\mathcal{F}^Z = \big(\mathcal{F}_t^Z\big)_{t\geq 0}$ be the natural filtration generated by the state process $Z$.

Payments in the contract are linked with sojourns in states, $b^j(t)$ at time $t$ in state $j$, and transitions between states, $b^{jk}(t)$ upon transition from state $j$ to state $k$ at time $t$. The insurance contract can then be modeled by a payment process $B$ describing the accumulated benefits less premiums with dynamics

$$\mathrm{d}B(t) = b^{Z(t)}(t)\mathrm{d}t + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t)\mathrm{d}N^k(t),$$

where the payment functions $b^j(t)$ and $b^{jk}(t)$ are assumed to be deterministic and sufficiently regular. Lump sum payments at fixed time points during sojourn in states are disregarded.

The insurance company evaluates the future liabilities arising from the contract by considering the state-wise prospective reserve, $V^j(t)$, which is the expected present value of future payments conditional on being in state $j \in \mathcal{J}$ at time $t$

$$V^j(t) = \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\,du} dB(s) \,\middle|\, Z(t) = j\right],$$

where $n$ denotes the termination of the contract and $r(t)$ is the interest rate. In the classical life insurance setup the interest rate is deterministic and the reserve $V^j(t)$ can be computed either by Thiele's differential equation or by a cash flow approach. In order for the contract to be fair, the insurance company determines the payments at initialization of the contract based on the equivalence principle, $V^{Z(0-)}(0-) = 0$.

### 1.1.2 With-profit insurance

The unsystematic biometric risk, of sojourning different states in life insurance, is modeled by a Markov chain. This type of risk is diversifiable, i.e. by the law of large numbers, the objective converges towards the expectation as a result of the size of the insurance company's portfolio. Additionally, the insurance company also faces financial risk and systematic biometric risk. For with-profit life insurance products, these types of risks are handled by specifying the payments at initialization of the contract on the basis of prudent assumptions. The prudent assumptions are called the technical (first-order) basis and consists of the technical interest rate $r^*$ and the technical transition intensities $\mu^*_{ij}$, $i, j \in \mathcal{J}$, $i \neq j$. The technical basis is predetermined and deterministic, such that the technical state-wise reserves can be calculated by the use of Thiele's differential equations. In addition to the technical basis, the insurance company considers the market basis, which models the actual development of the interest rate and transition intensities of the insurance portfolio. The market basis is denoted by $(r, \mu_{ij})$ for $i, j \in \mathcal{J}$, $i \neq j$.

Due to the discrepancy between the prudent technical basis and reality, a surplus emerges over time, which serves as a buffer in case the technical basis is not sufficiently prudent. By legislation and product design the surplus is paid back to the policyholders in terms of bonus. The insured takes part of the profit, therefore, the name with-profit insurance. A common bonus scheme is additional benefits, where bonus is used to buy more insurance, which increases the complexity in the calculation of reserves. Modeling the surplus redistribution to the policyholders, and the profit allotment to the insurance company for undertaking the risk, have recently gained interest. This requires accurate valuation of reserves under the market basis.

### 1.1.3 Projection of balances

It is mandatory for the insurance company to calculate the reserves under the market basis, in a market consistent manner, where the interest rate is consistent with assumptions in the financial market. Therefore, the market interest rate is stochastic and not deterministic as otherwise assumed in the classical life insurance setup. Previously, this has been handled by assuming independence between the stochastic interest rate and the Markov chain, omitting interest rate dependent payments from the contract and substituting the interest rate by the deterministic forward interest rate at time $t$. In with-profit insurance these assumptions break as the surplus, and therefore most likely the bonus, depends on the market interest rate.

The concept behind the project "Projection of Balances and Benefits in Life Insurance" is to account for the financial risk by simulating a number of scenarios of the financial market, and projecting the insurance company's balance sheet along each scenario. Hence, at time 0 the insurance company considers the balance sheet items at future time $t \geq 0$, given the information $\mathcal{F}^r_t$ generated by a scenario up to time $t$. In order to project the insurance company's balance within a scenario, a specification of the bonus distribution strategy is required. In general a specification of the company's Management Actions that influence the assets and liabilities is required, which may depend on the financial scenario.

Management Actions are retrospective by nature. For instance, the company distributes bonus dependent on the surplus accumulation over time. The retrospective nature of Management Actions causes projection models to focus on retrospective reserves, which are defined as the expected present value of accumulated past premiums less benefits given some information. The retrospective reserves depend on the past interest rate, whereas prospective reserves depend on the unknown future interest. Since projections are calculated within a financial scenario, all information about the interest rate up to the considered time $t$, i.e. $\mathcal{F}_t^r$, is known, but the future biometric state is unknown. It would be excessive to project the balances with retrospective reserves given all past information, $\mathcal{F}_t^Z$, since all possible paths of the state process then have to be considered. Therefore, the projection models only condition on $\mathcal{F}_t^r$ and the state of the policyholder at time $t$. State-wise projections of retrospective reserves can be calculated by a system of forward differential equations.

### 1.1.4 Policyholder behavior and stochastic intensities

Solvency II imposed an interest in modeling policyholders' options inherent in life insurance. Two commonly considered policyholder behavior options are surrender and conversion to free-policy. Upon surrender, the policyholder receives a single payment and all future payments are cancelled. With the free-policy option, all future premiums are cancelled, and benefits are reduced by a free-policy factor, $f$, that depends on the time at which the policyholder goes from premium paying to free-policy.

Policyholder behavior is modeled by extending the state space of the Markov chain, $Z$, to include surrender and free-policy states, under the assumption that the options are exercised at random with some intensity. The surrender payment is often a deterministic function of time, and fits into the classical life insurance setup, whereas conversion to free-policy introduces a duration dependence on the time of conversion. The extension of the classical setup allow for transition intensities and payments dependent on duration in certain states, which is denoted the semi-Markov setup. However, the full semi-Markov setup is not necessary in order to include the free-policy option, since it can be accounted for by using $f$-modified transition probabilities instead of regular transition probabilities. The $f$-modified transition probabilities are defined as

$$p_{ij}^f(s,t) = \mathbb{E}\Big[\mathbb{1}_{\{Z(t)=j\}} f(\tau)^{\mathbb{1}_{\{j\in\mathcal{J}^f\}}} \;\Big|\; Z(s)=i\Big],$$

where $\tau$ is the time of conversion to free-policy, $\mathcal{J}^f$ is the set of free-policy states and $i \notin \mathcal{J}^f$. The $f$-modified transition probabilities satisfy a system of Kolmogorov-like forward differential equations, which can be used to calculate the reserves including policyholder behavior.

In the classical life insurance setup, the transition intensities are deterministic by assumption. In order to account for the systematic risk of the transition intensities, an interest is given to doubly stochastic Markov chain models, where the transition intensities are suitable regular diffusion processes and $(Z, \mu)$ is Markovian. In the survival model, i.e. with two states - alive and dead - and only one possible transition, a forward mortality rate can be defined similar to the forward interest rate. Calculations can then be performed as in the classical life insurance setup by replacing the stochastic transition intensity with the forward mortality rate. However, this technique does not generalize to the more advanced multi-state models, and stochastic transition intensities results in a significant increase in complexity. Nevertheless, we are interested in defining deterministic forward transition rates that calculate reserves accurately by replacing the stochastic transition intensity.

Policyholder behavior and stochastic transition intensities are non-trivial extensions of projection models, and these extensions constitute the subject of this thesis.

### 1.1.5 Optimal control theory

For with-profit insurance contracts, the accumulated surplus is distributed to the insured through a dividend payment stream. It is a crucial assumption within projection models, that the dividend strategies are affine in the surplus. Affine dividend strategies also receive attention within optimal control theory due to a desire to achieve stable dividends.

In actuarial risk theory there is an interest in considering stability criteria for an insurance company in connection with optimization, where certain decisions are made (the controls) to achieve certain outcomes (the objective) under certain constraints. Historically, a main stability criterion to be considered is the expected present value of dividends allotted to the shareholders. From an optimization perspective, there is a desire to find the dividend strategy (the control), which maximizes this the expected present value of the shareholder's dividends (the objective). Unfortunately, the optimal dividend strategy is very irregular, and therefore unreasonable in practice.This brings focus to affine dividend strategies with more realistic features, even though they are not the optimal strategy for the classical optimal control problem. However, it is well known that affine strategies are optimal in so-called linear-quadratic optimization, where the objective is to minimize the quadratic loss function of the deviation between the dividends and the surplus from benchmarks, respectively.

The research, in the first four Chapters of this thesis, focuses on dividends towards the policyholders, contrary to Chapter 5 about optimal control theory, which focuses on dividends to the shareholders. However, the insights concerning optimal affine dividends to the policyholder and shareholders respectively are compatible.

## 1.2 Overview of the thesis and key results

In addition to the introduction, the thesis consists of 4 chapters, based on independent manuscripts. This subsection describes the content of each chapter and their interconnections. All chapters are related to projection of balances for with-profit insurance contracts, and study how various extensions can be included into the projection model and projection techniques. It is not necessarily perspicuous, how the results in each chapter relates to projection, which is outlined in this subsection. Chapter 2 extends the projection model by incorporating the policyholder behavior options surrender and conversion to free-policy. Chapter 3 mainly deals with interdependence between retrospective and prospective reserves, when Management Actions depend on both types. In Chapter 4 the doubly stochastic Markov chain models are linked with stochastic interest rates. Chapter 5 concerns which optimal control problem results in affine optimal dividend strategies.

### 1.2.1 Retrospective reserves and bonus with policyholder behavior

Chapter 2 contains the paper Falden and Nyegaard (2021), which considers projection of balances without and with policyholder behavior for a with-profit life insurance contract. Legislation imposes insurance companies to project their balance sheet in a number of scenarios of the financial market, using a specification of Management Actions that may depend on the financial situation of the insurance company. Since Management Actions are retrospective in nature, we focus on projection of the savings account and the surplus with redistribution of bonus by a dividend strategy.

We use the bonus scheme spoken of as additional benefits, where bonus is used to buy more insurance. We decompose the accumulated payments of an insurance contract into two payment streams; one that contains the payments not regulated by bonus, $B_1$, and one that contains the profile of payments regulated by bonus, $B_2$. Let $Q(t)$ be the number of payment processes $B_2(t)$ bought over time $[0,t]$ and $Q(0-)=1$. The savings account is the technical value of future payments guaranteed at time $t \geq 0$

$$X(t) = V_1^{*Z(t)}(t) + Q(t-)V_2^{*Z(t)}(t),$$

where $V_i^{*Z(t)}$ is the technical reserve for payment stream $i=1,2$, and $X(0-)=0$. The surplus is the difference between past premiums less benefits over time $[0,t]$ accumulated with the market interest rate, $r$, and the savings account at time $t$

$$Y(t) = -\int_0^t e^{\int_s^t r(u)\mathrm{d}u}\mathrm{d}B(s) - X(t),$$

and $Y(0-)=0$.

The state-wise projections of the savings account, $X$, and the surplus, $Y$, are given by

$$\tilde{X}^j(t) = \mathbb{E}\Big[\mathbb{1}_{\{Z(t)=j\}}X(t) \;\Big|\; \mathcal{F}_t^r, Z(0)\Big],$$
$$\tilde{Y}^j(t) = \mathbb{E}\Big[\mathbb{1}_{\{Z(t)=j\}}Y(t) \;\Big|\; \mathcal{F}_t^r, Z(0)\Big].$$

Bruhn and Lollike (2021) derive differential equations for the state-wise projections of the savings account and the surplus to use for projection in a given interest rate scenario, under the assumption the dynamics of the savings account and the surplus are affine. For regular payments in a life insurance contract, this is true if the dividend strategy is affine in $X(t)$ and $Y(t)$. The assumption breaks when including the free-policyholder option, where payments are reduced by a free-policy factor

$$f(\tau, X(\tau-)) = \frac{X(\tau-)}{X(\tau-) - V_1^{*0-}(\tau)}, \tag{1.2.1}$$

which is not affine in the savings account, and it depends on the time at which the policyholder goes from premium paying to free-policy. Since we are not able to use the established projection techniques with this free-policy factor, we propose an approximation of the free-policy factor based on state-wise projections

$$\tilde{f}(\tau) = \frac{\tilde{X}^0(\tau)}{\tilde{X}^0(\tau) - p_{Z(0)0}(0,\tau)V_1^{*0-}(\tau)}. \tag{1.2.2}$$

In the case, where all benefits are regulated by bonus, we can actually find accurate differential equations for the state-wise projections of the savings account and the surplus with the free-policy factor in Equation (1.2.1). Furthermore, in that case the state-wise projections coincide with the projections based on the approximated free-policy factor in Equation (1.2.2). Therefore, we consider the approximated free-policy factor a reasonable approximation of the free-policy factor, which can be used in a general case. In order to account for duration dependence since time upon conversion, the $\tilde{f}$-modified transition probabilities from Buchardt and Møller (2015) are adopted.

### 1.2.2 Reserve-dependent Management Actions in life insurance

Chapter 3 contains the paper Falden and Nyegaard (2022), which considers calculation of the market reserve, where Management Actions depend on the market reserve itself. The market reserve is the expected present value of future guaranteed and non-guaranteed payments from the insurer to the insured under the market basis. It is influenced by the Management Actions that depend on all balance sheet items.

The inclusion of market reserve-dependent dividends and investments results in a challenging structure for the projection model, where the retrospective savings account and surplus are interdependent with the prospectively calculated market reserve. This forms a so-called forward-backward stochastic system, where the differential equations for the projection of the savings account and the surplus have initial boundary conditions, while the market reserve has a given terminal value.

The result is a characterization of the market reserve by a partial differential equation (PDE), where the market reserve can be computed backwards for all values of the savings account, the surplus and the financial market. This disentangle the forward-backward system in the calculation of the market reserve, but is very computational demanding, and the solution to the PDE systems does not necessarily fit into the affine assumptions for the dividend strategy that is needed to make the projection model.

Under the assumption the dividend strategy is in the form

$$\delta(t,x,y,v,r) = \delta_0(t,r) + \delta_1(t,r)\cdot x + \delta_2(t,r)\cdot y + \delta_3(t,r)\cdot v,$$

for deterministic functions $\delta_0, \delta_1, \delta_2$ and $\delta_3$, the market reserve is given by

$$V(t,x,y,r) = h_0(t,r) + h_1(t,r)\cdot x + h_2(t,r)\cdot y,$$

where the functions $h_0$, $h_1$, and $h_2$ satisfy a system of PDEs. Affinity of the dividend strategy reduces the dimension of the PDE for calculation of the market reserve, and expresses the market reserve as a affine function of the savings account and the surplus such that it fits into the assumptions of the projection model. Furthermore, the result applies for any choice of investment strategy in the setup, and in the Black-Scholes market of the financial market the functions $h_0$, $h_1$, and $h_2$ satisfy a system of ordinary differential equations (ODEs).

### 1.2.3 Forward transition rates in multi-state life insurance

Chapter 4 contains the manuscript Falden (2022), which considers stochastic transition intensities that are dependent with the interest rate in a multi-state life insurance setup. The projection model is based on simulating scenarios of the financial market, and projecting balance sheet items by calculating the biometric risk of reserves analytically given each scenario. Previously, the financial market and state process are assumed to be independent, but it is reasonable to assume the financial scenario have an impact on the transition intensities. In particular, transitions based on policyholder options as surrender or free-policy, since the market interest rate may make it advantageous or cause unemployment.

We are interested in being able to handle scenario-dependent transition intensities. Assume the transition intensities are given by deterministic functions of the stochastic interest rate $\mu_{ij}(t) = g_{ij}(t, r(t))$, for $t \geq 0$. As a starting point, this causes no issues in the state-wise projections of the savings account and the surplus, since we condition on the information of the interest rate up to a given time $t$, $\mathcal{F}_t^r$. Therefore, we can consider the transition intensities as known up to time $t$ and are able to calculate retrospective reserves. Complications arise when Management Actions depend on prospective reserves, since the prospective reserves are based on the interest rate and transition intensities after time $t$. The use of the deterministic forward interest rate will not lead to accurate calculations, and we need to consider doubly stochastic Markov chain models.

We consider previous suggestion for forward transition rates, in particular, the so-called forward equation rates proposed by Buchardt, Furrer, and Steffensen (2019). The forward equation rates evaluate insurance contracts correctly, if the contract consists of sojourn payments only. Similar to the forward equation rates, we propose forward transition rates based on solutions to a system of Kolmogorov forward equations. Our proposal evaluates insurance contracts correctly, if the contract consists of transition payments only. The idea is then to decompose the prospective market reserve into a reserve based on sojourn payments, which is valuated by the use of forward equation rates, and a reserve based on transition payments, which is valuated by the use of our proposed forward transition rates. Furthermore, we allow dependency between the interest rate and transition intensities. We incorporate this dependency into the model by extending the state space with one absorbing state, which is attainable from every state with the transition intensity of the interest rate $r$, under the assumption that the sample paths of $r$ are non-negative.

### 1.2.4 Stable dividends are optimal under linear-quadratic optimisation

Chapter 5 contains the manuscript Avanzi, Falden, and Steffensen (2022), which considers the optimal control problem that results in affine dividend strategies within actuarial risk theory. In projection models it is a crucial assumption that the dividend strategies are affine in the surplus. As part of the Management Actions the redistribution of bonus contains certain degrees of freedom, and the insurance company are interested in knowing, which objective is optimized by an affine dividend strategy. It is well known that affine strategies are optimal in so-called linear-quadratic (LQ) optimization, where the objective is to minimize the quadratic loss function of the deviation between the dividends and the surplus from benchmarks, respectively.

We model the surplus of an insurance company by dynamics

$$dY(t) = c(t)dt + dS(t) - dD(t), \tag{1.2.3}$$

where $c$ is the deterministic variation, $S(t)$ is a Lévy process and $D(t)$ is the dividend process. The LQ objective is to minimize the expectation of the aggregated quadratic deviation between the dividends and the surplus to benchmarks, respectively. The objective is expressed by a value function, which satisfies a system of differential equations, referred to as the Hamilton-Jacobi-Bellman equation (HJB equation). Based on a verification lemma of the HJB equation, we are able to express the optimal dividend strategy as an affine function in the surplus. Furthermore, we show the HJB equation and optimal dividend strategy coincide with the HJB equation and optimal dividend strategy in a setup, where the surplus is modeled as a diffusion process and continuous payments only, making it adequate to consider that setup for further study.

We are interested in calculating the expected present value of future dividends

$$\mathbb{E}\left[\int_t^T e^{-r(s-t)}\mathrm{d}D(s)\big|Y(t)=y\right].$$

For the with-profit insurance contract, where dividends are bonus to the policyholder, this corresponds to the future discretionary benefits. Then dividends are allotted to the shareholders, as in the case of the manuscript, the expected present value of future dividends forms a stability criterion for the insurance company. Historically, the classical optimization problem and stability criterion consists of determining the dividend strategy, which maximizes the expected present value of future dividends to the shareholders, while remaining solvent. Unfortunately, the optimal dividend strategy is very irregular and thus unreasonable in practice. Therefore, the LQ objective is advantageous as it results in affine dividends, which are more stable.

There is no obvious way to compare the LQ objective to the classical objective. In order to consider and compare the optimal dividend strategies and the expected present value of future dividends, we study suitable choices for the benchmarks in the LQ optimization problem and preform a numerical study. The manuscript i Chapter 5 seeks to maximize the expected present value of future dividends, this is not an objective in the projection model, where dividends are redistributed to the policyholder.

## Chapter 2

# Retrospective reserves and bonus with policyholder behavior


Abstract

Legislation imposes insurance companies to project their assets and liabilities in various financial scenarios. Within the setup of with-profit life insurance, we consider retrospective reserves and bonus, and we study projection of balances with and without policyholder behavior. The projection resides in a system of differential equations of the savings account and the surplus, and the policyholder behavior options surrender and conversion to free-policy are included. The inclusion results in a structure where the system of differential equations of the savings account and the surplus is non-trivial. We consider a case, where we are able to find accurate differential equations and suggest an approximation method to project the savings account and the surplus including policyholder behavior in general. To highlight the practical applications of the results in this paper, we study a numerical example.



**Keywords:** With-profit life insurance; Bonus; Surplus; Dividends; Projection of balances; Retrospective reserve; Policyholder behavior.


## 2.1 Introduction

In with-profit life insurance, prudent assumptions about the interest rate and biometric risks at initialization of an insurance contract result in a surplus emerging over time. This surplus belongs to the policyholders and must be paid back in terms of bonus. The redistribution of bonus contains certain degrees of freedom, which is part of the *Management Actions*. Furthermore, bonus must be taken into account when insurance companies determine their assets and liabilities. Legislation imposes insurance companies to project their balance sheet, and companies must be able to perform projections of assets and liabilities in a number of scenarios of the financial market. This requires a specification of the future dividend strategy and, in general, a specification of the *Future Management Actions*. Management actions may depend on the financial scenario, the present as well as the past entries of the balance sheet and their relations, and other aspects of the financial situation of the insurance company. Therefore, future management actions have a complex nature and are difficult to predict and formalize mathematically. In this paper, we model the projection of the savings account and the surplus of an insurance contract, where we assume the future dividend strategy has a simple structure. How the dividend strategy is designed in practise to fit the model is beyond the scope of this paper, but the model establishes a foundation for projecting balances in life insurance. In the projection model, biometric risks play an important role as well. We model the state of the policyholder using a Markov model, and study state-wise projections of the savings account and the surplus.

The modeling of surplus and bonus in life insurance is not new. Norberg (1999) introduces the individual surplus of a life insurance contract, and Steffensen (2006b) derives differential equations for prospective reserves in the case, where dividends are linked to the surplus. In our model, we also consider dividends linked to the surplus, but distinct from Steffensen (2006b), we derive differential equations for the projected savings account and surplus. Jensen and Schomacker (2015) study the valuation of an insurance contract with the bonus scheme spoken of as additional benefits, where dividends are used to buy more insurance, in a scenario-based model for the financial market. Our paper has some similarities with Jensen and Schomacker (2015) in the sense that we also study a scenario-based model with additional benefits. In Jensen and Schomacker (2015) the bonus allocation is discretized, while we allocate bonus continuously, resulting in difference equations in Jensen and Schomacker (2015) and ordinary differential equations in our model. Furthermore, we study state-wise projections of the savings account and the surplus, whereas Jensen and Schomacker (2015) study the expected savings account and the expected surplus.

Steffensen (2006b) considers prospective reserves, while we focus on the savings account, which is a retrospective reserve including past bonus, and the surplus of an insurance contract. The retrospective approach without bonus is studied in Norberg (1991) and studied with bonus in Asmussen and Steffensen (2020). Bruhn and Lollike (2021) also reflect on the retrospective perspective, and study retrospective reserves with and without bonus. They model the savings account and the surplus of an insurance contract, and derive differential equations for the state-wise projections. The retrospective approach is practicable when considering projection of liabilities in various financial scenarios, since the retrospective reserves depend on the past interest rate, whereas prospective reserves depend on the unknown future interest rate.

This paper serves as an extension to Bruhn and Lollike (2021). The extension resides in the incorporation of the policyholder behavior options surrender and conversion to free-policy. Upon surrender, the policyholder receives a single payment and all future payments cancel, and with the free-policy option, all future premiums cancel and benefits are reduced by a free-policy factor. We model policyholder behavior as random transitions in the Markov model from the classical life insurance setup extended with surrender and free-policy states as studied in for instance Henriksen et al. (2014). This is in contrast to modeling rational policyholder behavior as in Steffensen (2002). Buchardt and Møller (2015) study the calculation of prospective reserves without bonus including policyholder behavior using a cash flow approach, and Buchardt, Møller, and Schmidt (2014) consider the inclusion of policyholder behavior in semi-Markov models. A general extension of the concepts to non-Markovian models is studied in Christiansen and Djehiche (2020), where in addition payments are allowed to depend on prospective reserves. In our model, payments depend on the retrospective savings account. In working paper Ahmad, Buchardt, and Furrer (2021), they study a setup similar to ours with bonus and policyholder behavior, but they are included separately. We include policyholder behavior options in combination with bonus in our model of the retrospective savings account and surplus, and our approach is based on differential equations of the state-wise projections. Buchardt and Møller (2015) introduce the notion of modified probabilities to calculate prospective reserves including conversion to free-policy. The same modified probabilities appear in our system of differential equations for the state-wise projections of the savings account and the surplus.

We propose here a framework for the projection of liabilities in various financial scenarios with a general model of the future management actions, among these the redistribution of bonus. Furthermore, any policyholder response to the financial market and the savings account and the surplus can be implemented in our framework. Other papers derive or suggest specific rules for management and/or policyholder decision making. In both financial and actuarial literature, optimization of life insurance payments are discussed, typically from an individual point of view over the life cycle. Seminal works are Richard (1975) and Campbell (1980), but the area continues to attract interest, see for instance Chen et al. (2006), Chiappori et al. (2006), and Kraft and Steffensen (2008). Browne and Kim (1993) discuss life insurance demand from a macroeconomic perspective, and Nielsen (2005) considers optimal distribution of surplus on a corporate level. Modeling or derivation of optimal policyholder behavior is a recurrent topic in actuarial literature. De Giovanni (2010) models surrender risk adapted to the financial market, and the modeling and statistical examination of surrender on macroeconomic conditions are studied in for instance Loisel and Milhaud (2011) and Barsotti, Milhaud, and Salhi (2016). The modeling of free-policy behavior is most often assumed random and uncorrelated across the portfolio, see for instance Henriksen et al. (2014) and Buchardt and Møller (2015).

In Section 2.2, we present the general life insurance setup and the model of the savings account, the surplus, and the dividends. We define the projection of the savings account and the surplus without policyholder behavior and state the results from Bruhn and Lollike (2021) in Section 2.3. Section 2.4 extends the setup from Section 2.2 to include policyholder behavior. Section 2.5 consists of the key results in this paper. We consider the ideal free-policy factor in our retrospective setup including bonus, but this free-policy factor does not satisfy the simple structure of the model in Section 2.3. Therefore, the result concerning the projection of the savings account and the surplus in Section 2.3 does not apply with the ideal free-policy factor. We consider the case with all benefits regulated by bonus. In this case, we show that we actually can project the savings account and the surplus with the ideal free-policy factor. Furthermore, we suggest an approximation of the free-policy factor, for which the state-wise projections of the savings account and the surplus coincide with the state-wise projections using the ideal free-policy factor. This is one of the two main results of the paper. The second main result is a method to project the savings account and the surplus with the approximated free-policy factor in a general case. In Section 2.6, we present a numerical example to emphasize the practical applications of our results. Section 2.7 concludes the paper.

## 2.2 Life insurance setup

The classic multi-state setup in life insurance is taken as a starting point, and we extend this with policyholder behavior in Section 2.4. A Markov process, $Z = \big(Z(t)\big)_{t\geq 0}$, in a finite state space $\mathcal{J}^\circ = \{0, 1, ..., J-1\}$ describes the state of the holder of a life insurance contract, and payments in the contract link with sojourns in states and transitions between states. The transition probabilities of $Z$ are

$$p_{ij}(s,t) = \mathbb{P}\big(Z(t) = j \mid Z(s) = i\big),$$

for $i, j \in \mathcal{J}^\circ$ and $s \leq t$. We assume that the transition intensities

$$\mu_{ij}(t) = \lim_{h\downarrow 0} \frac{1}{h} p_{ij}(t, t+h),$$

exist for $i, j \in \mathcal{J}^\circ$, $i \neq j$.

The transition probabilities satisfy the Kolmogorov's differential equations (see for instance Buchardt and Møller (2015) Proposition 4).

The processes $N^k(t)$ for $k \in \mathcal{J}^\circ$ count the number of jumps of $Z$ into state $k$ up to time $t$.

$$N^k(t) = \#\{s \in (0,t] \mid Z(s-) \neq k, Z(s) = k\},$$

where $Z(s-) = \lim_{h\downarrow 0} Z(s-h)$.

We consider with-profit life insurance products, where payments specified in the contract are based on prudent assumptions about interest rate and transition intensities. These assumptions are called the technical basis, and denoted by $(r^*, \mu_{ij}^*)$ for $i, j \in \mathcal{J}^\circ$, $i \neq j$. The market basis models the actual development of the interest rate and transition intensities of the insurance portfolio. The market basis is denoted by $(r, \mu_{ij})$ for $i, j \in \mathcal{J}^\circ$, $i \neq j$. The market interest rate is stochastic, and practice is to simulate a number of scenarios of the interest rate and study the projection model in each scenario, as we do in the numerical simulation study in Section 2.6. Available information about the market interest rate is represented by the filtration $\mathcal{F}^r = \left(\mathcal{F}_t^r\right)_{t\geq 0}$, where $\mathcal{F}_t^r = \sigma\big(r(s)\big|0 \leq s \leq t\big)$. We assume the market transition intensities are deterministic.

Due to the prudent technical basis, a surplus arises, which by legislation is to be paid back to the policyholders as bonus. We use the bonus scheme spoken of as additional benefits, where bonus is used to buy more insurance. This is denoted as defined contributions since premiums are fixed and benefits are increased by bonus in contrast to defined benefits, where bonus is used to lower premiums and benefits are fixed.

The accumulated payments of an insurance contract is decomposed into two payment streams; one that contains the payments not regulated by bonus, $B_1$, and one that contains the profile of payments regulated by bonus, $B_2$, as presented in Asmussen and Steffensen (2020). An example is an insurance contract consisting of a life annuity and a term insurance. Often only the life annuity is scaled by bonus and the term insurance as well as the premiums are fixed. Then the payment stream $B_1$ consists of the term insurance and the premiums, and the payment stream $B_2$ consists of the life annuity.

The dynamics of the payment streams are in the following form for $i = 1, 2$

$$\mathrm{d}B_i(t) = b_i^{Z(t)}(t)\mathrm{d}t + \sum_{k:k\neq Z(t-)} b_i^{Z(t-)k}(t)\mathrm{d}N^k(t), \tag{2.2.1}$$

where $b_i^j(t)$ denotes the payment rate during sojourn in state $j$ and $b_i^{jk}(t)$ the single payment upon transition from state $j$ to state $k$ at time $t$. The payment functions $b_i^j(t)$ and $b_i^{jk}(t)$ are assumed to be deterministic and sufficiently regular. For notational convenience, we disregard lump sum payments at fixed time points during sojourn of states, even though it does not impose mathematical difficulties.

**Definition 2.2.1.** *The prospective technical reserve at time $t \leq n$ for payment stream $\mathrm{d}B_i(t)$, $i = 1, 2$ is given by*

$$V_i^{*Z(t)}(t) = \mathbb{E}^*\left[\int_t^n e^{-\int_t^s r^*(u)\mathrm{d}u}\mathrm{d}B_i(s)\;\middle|\; Z(t)\right],$$

*where $n$ denotes termination of the contract and $\mathbb{E}^*$ means that the technical transition intensities, $\mu_{jk}^*$, $j, k \in \mathcal{J}^\circ$, $j \neq k$, are used in the distribution of $Z$.*

Since the technical interest rate and transition intensities are determined at initialization of the insurance contract and therefore known for all $t \in [0, n]$, the prospective technical reserves are deterministic conditional on $Z(t) = j$. The principle of equivalence states that $V_1^{*Z(0)}(0) + V_2^{*Z(0)}(0) = 0$.

### 2.2.1 The savings account, the surplus and the dividends

Similar to Asmussen and Steffensen (2020), the surplus is returned to the insured through a dividend payment stream $D$. A process $Q(t)$ denotes the number of payment processes $B_2$ bought up to time $t$. Additional benefits are bought under the technical basis, and as dividends are used to buy $B_2(t)$ at the price of $V_2^*(t)$, we must have that

$$\mathrm{d}D^{Z(t)}(t) = \mathrm{d}Q(t)V_2^{*Z(t)}(t). \tag{2.2.2}$$

The policyholder experiences the total payment process with dynamics

$$\mathrm{d}B(t) = \mathrm{d}B_1(t) + Q(t-)\mathrm{d}B_2(t),$$

which is the payment process guaranteed at time $t$. A decreasing $Q$ results in decreasing guaranteed benefits, which from a practical point-of-view is unreasonable. A negative value of $Q$ results in benefit payments from the insured to the insurance company which is unrealistic. We do not require that $Q$ is non-decreasing or that $Q$ is non-negative in this setup in order to obtain a simple mathematical model.

The savings account of an insurance contract is denoted by $X(t)$, and it is the technical value of future payments guaranteed at time $t \geq 0$, i.e. the following relation between $X(t)$ and $Q(t)$ holds

$$X(t) = V_1^{*Z(t)}(t) + Q(t-)V_2^{*Z(t)}(t) \Leftrightarrow Q(t-) = \frac{X(t) - V_1^{*Z(t)}(t)}{V_2^{*Z(t)}(t)}.$$

The savings account is equal to zero at the beginning of the insurance contract, $X(0-) = 0$. Then by the principle of equivalence, $V_1^{*Z(0-)}(0-) + V_2^{*Z(0-)}(0-) = 0$, the initial condition $Q(0-) = 1$ holds.

Due to the relationship between $X$ and $Q$, the payment process, $\mathrm{d}B(t)$, is a linear function in $X$

$$\mathrm{d}B(t, X(t)) = b^{Z(t)}(t, X(t))\mathrm{d}t + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))\mathrm{d}N^k(t), \tag{2.2.3}$$

where

$$b^j(t,x) = b_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^j(t),$$
$$b^{jk}(t,x) = b_1^{jk}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t).$$

**Proposition 2.2.2.** *The savings account, $X$, has dynamics*

$$\begin{aligned}\mathrm{d}X(t) &= r^*(t)X(t)\mathrm{d}t - \mathrm{d}B(t, X(t)) + \mathrm{d}D^{Z(t)}(t) \\ &\quad + \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big),\end{aligned}$$

*where the sum-at-risk is given by*

$$R^{*jk}(t,x) = b^{jk}(t,x) + \chi^{jk}(t,x) - x,$$

*and*

$$\chi^{jk}(t,x) = V_1^{*k}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} V_2^{*k}(t),$$

*is the technical value of guaranteed payments after the transition from state $j$ to state $k$.*

*Proof.* See Asmussen and Steffensen (2020), Chapter 6.7. □

The surplus $Y(t)$ is the difference between past premiums less benefits over time $[0,t]$ accumulated with the market interest rate and the savings account at time $t$.

$$Y(t) = -\int_0^t e^{\int_s^t r(u)\mathrm{d}u}\mathrm{d}B(s, X(s)) - X(t).$$

The market interest rate over time $[0,t]$ is known at time $t$ such that $Y(t)$ only depends on the market interest rate prior to time $t$, and $Y(0-) = 0$.

**Proposition 2.2.3.** *The surplus, $Y$, has dynamics*

$$\begin{aligned}\mathrm{d}Y(t) &= r(t)Y(t)\mathrm{d}t - \mathrm{d}D^{Z(t)}(t) + c^{Z(t)}(t, X(t))\mathrm{d}t \\ &\quad - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),\end{aligned}$$

*where the surplus contribution is given by*

$$c^j(t,x) = (r(t) - r^*(t))x + \sum_{k:k\neq j} R^{*jk}(t,x)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

*Proof.* See Asmussen and Steffensen (2020), Chapter 6.7. □

We assume that the technical basis is prudent compared to the market basis such that the surplus contribution, $c^j(t,x)$, is non-negative. A prudent technical basis chosen several years ago may not be prudent today due to the current low interest rate environment and therefore the interest rate part of the surplus contribution may be negative resulting in a possibly negative surplus. In practice, a negative surplus would be covered by the equity of the insurance company, but in this setup, we allow the surplus to be negative.

The dividend payments stream, $\mathrm{d}D^{Z(t)}(t)$, describes how the surplus is returned to the insured. We assume that the dividend process is continuous and depends on the savings account and the surplus, such that the dynamics are

$$\mathrm{d}D^{Z(t)}(t) = \delta^{Z(t)}(t, X(t), Y(t))\mathrm{d}t.$$

The dynamics of the savings account and the surplus are affine if and only if the dividend process is. The main results of this paper rely on affinity in the dynamics of the savings account and the surplus, and therefore we make the assumption that the dividend process is affine in $X(t)$ and $Y(t)$

$$\delta^j(t, x, y) = \delta_0^j(t) + \delta_1^j(t) \cdot x + \delta_2^j(t) \cdot y, \tag{2.2.4}$$

for sufficiently regular and deterministic functions $\delta_0^j, \delta_1^j$ and $\delta_2^j$, $j \in \mathcal{J}^\circ$. This is a restriction in the degree of freedom in the dividend allocation strategy of the insurance companies, and therefore of the future management actions in the model. How the dividend strategy is chosen in practise to cope with our model is beyond the scope of this paper, but other papers derive specific rules for management actions and agents behavior, see for instance Nielsen (2005), Chen et al. (2006), and Kraft and Steffensen (2008). The restriction that the dividends are affine may lead to negative dividends, which results in a decreasing $Q$ and that the insurance company lowers the guaranteed benefits. From a practical point-of-view this is unreasonable, but affine dividends turn out to be mathematical tractable, and therefore we make the assumption of affine dividends in our model. The user of the model must be aware of the possibility of negative dividends.

## 2.3 State-wise projections without policyholder behavior

In order to satisfy legislation, insurance companies and present research focus on the projection of balances in life insurance using simulation methods. Both the savings account, $X$, and the surplus, $Y$, are entries of the balance sheet, and in order to project these, we simulate scenarios of the interest rate and study the projection of the savings account and the surplus in each scenario. To account for the biometric risks, one approach is to use simulation methods. In practice, it can be computational heavy to simulate the biometric history of an entire insurance portfolio, and therefore we study state-wise projections to eliminate the biometric part of the simulation.

**Definition 2.3.1.** *for $j \in \mathcal{J}^\circ$. The subscript $Z(0)$ denotes that the expectation is the conditional expectation given $Z(0)$. The expectation is taken under the market basis conditional on $Z(0)$ and the interest rate filtration at time $t$. Therefore, the market* *interest rate is known up to and including time $t$, but information about the state process $Z$ is only known at time 0.*

Bruhn and Lollike (2021) derive differential equations for the state-wise projections of the savings account and the surplus from Definition 2.3.1 to use for projection in a given interest rate scenario. The theorem below states the main result of Bruhn and Lollike (2021), and the purpose of this paper is to extend these differential equations to a setup including policyholder behavior.

**Lemma 2.3.2.** *The dynamics of the savings account, $X$, from Proposition 2.2.2 and the dynamics of the surplus, $Y$, from Proposition 2.2.3 are in the form*

$$
\begin{aligned}
\mathrm{d}X(t) &= \Big(\alpha_{0,X}^{Z(t)}(t) + \alpha_{1,X}^{Z(t)}(t)X(t) + \alpha_{2,X}^{Z(t)}(t)Y(t)\Big)\mathrm{d}t \\
&\quad + \sum_{k:k\neq Z(t-)} \Big(\lambda_{0,X}^{Z(t-)k}(t) + \lambda_{1,X}^{Z(t-)k}(t)X(t-)\Big)\mathrm{d}N^k(t), \\
\mathrm{d}Y(t) &= \Big(\alpha_{0,Y}^{Z(t)}(t) + \alpha_{1,Y}^{Z(t)}(t)X(t) + \alpha_{2,Y}^{Z(t)}(t)Y(t)\Big)\mathrm{d}t \\
&\quad + \sum_{k:k\neq Z(t-)} \Big(\lambda_{0,Y}^{Z(t-)k}(t) + \lambda_{1,Y}^{Z(t-)k}(t)X(t-)\Big)\mathrm{d}N^k(t),
\end{aligned}
$$

*for deterministic functions $\alpha_{i,H}^{j}$ and $\lambda_{i,H}^{jk}$ for $i = 0,1,2$, $H = X,Y$ and $j,k \in \mathcal{J}^\circ$, $j \neq k$.*
*See Appendix 2.A for the expressions of $\alpha$ and $\lambda$ for the savings account and the surplus.*

**Theorem 2.3.3.** *Let $X$ and $Y$ have dynamics in the form of Lemma 2.3.2. Then the state-wise projections of $X$ and $Y$ from Definition 2.3.1 satisfy the following system of ordinary differential equations*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j(t) = &\sum_{k:k\neq j} \mu_{kj}(t)\tilde{X}^k(t) - \sum_{k:k\neq j} \mu_{jk}(t)\tilde{X}^j(t) \\
&+ \alpha_{0,X}^{j}(t)p_{Z(0)j}(0,t)\tilde{X}^j(t) + \alpha_{1,X}^{j}(t) + \alpha_{2,X}^{j}(t)\tilde{Y}^j(t) \\
&+ \sum_{k:k\neq j} \mu_{kj}(t)\Big(\lambda_{0,X}^{kj}(t)p_{Z(0)k}(0,t) + \lambda_{1,X}^{kj}(t)\tilde{X}^k(t)\Big), \\
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}^j(t) = &\sum_{k:k\neq j} \mu_{kj}(t)\tilde{Y}^k(t) - \sum_{k:k\neq j} \mu_{jk}(t)\tilde{Y}^j(t) \\
&+ \alpha_{0,Y}^{j}(t)p_{Z(0)j}(0,t) + \alpha_{1,Y}^{j}(t)\tilde{X}^j(t) + \alpha_{2,Y}^{j}(t)\tilde{Y}^j(t) \\
&+ \sum_{k:k\neq j} \mu_{kj}(t)\Big(\lambda_{0,Y}^{kj}(t)p_{Z(0)k}(0,t) + \lambda_{1,Y}^{kj}(t)\tilde{X}^k(t)\Big),
\end{aligned}
$$

*and $\tilde{X}^j(0-) = \tilde{Y}^j(0-) = 0$ for $j \in \mathcal{J}^\circ$.*

*Proof.* See Bruhn and Lollike (2021). $\square$

Kolmogorov's forward differential equations can be used to calculate the transition probabilities in Theorem 2.3.3.

## 2.4 Life insurance setup including policyholder behavior

Now, we extend the setup from Section 2.2 to include policyholder behavior. We include the policyholder behavior options surrender and conversion to free-policy. Upon surrender, the policyholder receives a single payment and all future payments cancel. With the free-policy option, all future premiums cancel, and benefits are reduced by a free-policy factor, $f$, that depends on the time at which the policyholder goes from premium paying to free-policy. We study how the introduction of policyholder behavior affects the dynamics of the savings account, $X$, from Proposition 2.2.2 and the surplus, $Y$, from Proposition 2.2.3. The objective is to be able to perform state-wise projections of the savings account and the surplus including policyholder behavior.

Policyholder behavior is modelled in the classic way by extending the state space of the Markov chain, $Z$, to include surrender and free policy states as presented in Henriksen et al. (2014), and the state space of $Z$ from Section 2.2 is extended as illustrated in Figure 2.1[1]. We do not consider the modeling or derivation of the surrender rate and the free-policy rate. The modeling of optimal surrender rates is studied in for instance De Giovanni (2010), Loisel and Milhaud (2011), and Barsotti, Milhaud, and Salhi (2016), but little attention has been paid in existing literature to the choice of free-policy rate, which is often modelled as a deterministic intensity as in Henriksen et al. (2014) and Buchardt and Møller (2015). The extension of the state space in Figure 2.1 can also be obtained as a specific case of the more general state space expansion in Christiansen and Djehiche (2020).

The state $J$ corresponds to surrender, and we assume that surrender can only happen from state 0. The state space $\mathcal{J}^f$ denotes the free-policy states, and it is a copy of $\mathcal{J}$ in the sense that it holds the same number of states and that state $i \in \mathcal{J}^f$ corresponds to state $i-(J+1) \in \mathcal{J}$. We assume that conversion to free-policy can only occur from state 0 and that the transition intensities in $\mathcal{J}^f$ equal the transition intensities in $\mathcal{J}$. We assume throughout the rest of this paper that $Z(0) \in \mathcal{J}$. The classical 7-state model from for example Buchardt and Møller (2015) is contained in this setup, where state 0 in our model corresponds to the premium-paying active state.

In order to model payments including policyholder behavior, the payment streams from Equation (2.2.1) are decomposed in benefits, $\mathrm{d}B_i^+(t)$ and premiums, $\mathrm{d}B_i^-(t)$ for $i=1,2$. The sojourn payments and payments upon transition are then decomposed in $b_i^{j+}$ and $b_i^{j-}$, and $b_i^{jk+}$ and $b_i^{jk-}$ respectively. We consider defined contributions such that the payment stream increased by bonus only contains benefits i.e. $b_2^{j-}(t) = b_2^{jk-}(t) = 0$ for all $t \geq 0$ and $j,k \in \mathcal{J}$, $j \neq k$.

[1]The figure is elaborated by the authors

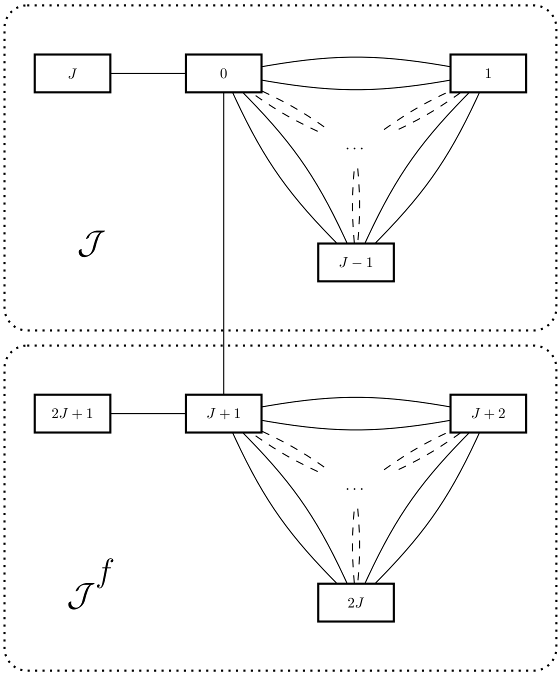


**Figure 2.1:** *Multi-state model including policyholder behavior options*

The technical benefit and premium reserves respectively in the non-free-policy states, $Z(t) \in \mathcal{J}$, are given by

$$V_i^{*Z(t)\pm}(t) = \mathbb{E}^* \left[ \int_t^n e^{-\int_t^s r^*(u)\mathrm{d}u} \mathrm{d}B_i^{\pm}(s) \,\middle|\, Z(t) \right],$$

for $i = 1, 2$, and where $n$ is termination of the insurance contract.

Defined contributions imply that

$$V_2^{*Z(t)}(t) = V_2^{*Z(t)+}(t) + V_2^{*Z(t)-}(t) = V_2^{*Z(t)+}(t), \tag{2.4.1}$$

for $Z(t) \in \mathcal{J}$.

The duration $U$ in the free-policy states is

$$U(t) = \inf\{s \in [0, t] \mid Z(t - s) \in \mathcal{J}\}.$$

Payments in the free-policy states equal a free-policy factor, $f \in [0, 1]$, times the benefits in the corresponding premium-paying state. We allow the free-policy factor to depend on the savings account, i.e. $f(t, X(t))$, and the benefits are reduced with the free-policy factor evaluated at the time of conversion to free-policy, $f(t - U(t), X(t - U(t)))$. We introduce the mapping of $Z(t)$ that returns the corresponding premium-paying state if $Z(t) \in \mathcal{J}^f$

$$g\big(Z(t)\big) = \mathbb{1}_{\{Z(t)\in\mathcal{J}^f\}}\big(Z(t) - (J+1)\big).$$

Policyholder behavior is modelled solely on the market basis, and therefore $\mu^*_{0J}(t) = \mu^*_{0(J+1)}(t) = 0$ for all $t \geq 0$. The remaining transition intensities in $\mathcal{J}^f$ equal the corresponding transition intensities in $\mathcal{J}$ on the technical basis. Hence, the technical reserve in a free-policy state equals the free-policy factor times the technical benefit reserve in the corresponding premium-paying state

$$V_i^{*Z(t)}(t) = f(t - U(t), X(t - U(t)))V_i^{*g(Z(t))+}(t),$$

for $i = 1, 2$ and $Z(t) \in \mathcal{J}^f$.

The inclusion of policyholder behavior changes the payment process from Equation (2.2.3) and the sum-at-risk from Proposition 2.2.2. Now, the payment process and the sum-at-risk depend on time, the savings account, and the duration in the free-policy states.

**Proposition 2.4.1.** *The total payment process guaranteed at time t including policyholder behavior is*

$$\begin{aligned}
&\mathrm{d}B(t, X(t),\ U(t),\ X(t - U(t))) \\
&\qquad = b^{Z(t)}(t, X(t),\ U(t),\ X(t - U(t)))\mathrm{d}t \\
&\qquad\quad + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-),\ U(t),\ X((t - U(t))-))\mathrm{d}N^k(t),
\end{aligned}$$

*where the continuous payment function during sojourns in states and the payment function upon transition between states are*

$$\begin{aligned}
&b^j(t, x, u, x^f) \\
&\quad = \mathbb{1}_{\{j\in\mathcal{J}\}}\Big(b_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j+}(t)} b_2^j(t)\Big) \\
&\qquad + \mathbb{1}_{\{j\in\mathcal{J}^f\}}\Big(f(t-u, x^f)b_1^{g(j)+}(t) + \frac{x - V_1^{*g(j)+}(t)f(t-u, x^f)}{V_2^{*g(j)+}(t)} b_2^{g(j)+}(t)\Big),
\end{aligned}$$

$$\begin{aligned}
&b^{jk}(t, x, u, x^f) \\
&\quad = \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\Big(b_1^{jk}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j+}(t)} b_2^{jk}(t)\Big) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}} f(t-u, x^f) b_1^{g(j)g(k)+}(t) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}} \frac{x - V_1^{*g(j)+}(t)f(t-u, x^f)}{V_2^{*g(j)+}(t)} b_2^{g(j)g(k)+}(t),
\end{aligned}$$

*for $j,k \in \mathcal{J} \cup \mathcal{J}^f$, $j \neq k$. We assume that there are no continuous payments in the surrender states, and that there is no payment upon transition between $\mathcal{J}$ and $\mathcal{J}^f$.*

**Proposition 2.4.2.** *Including policyholder behavior, the sum-at-risk from Proposition 2.2.2 is*

$$\begin{aligned}
R^{*jk}&(t,x,u,x^f) \\
&= b^{jk}(t,x,u,x^f) \\
&\quad + \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\left(V_1^{*k}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j+}(t)}V_2^{*k+}(t) - x\right) \\
&\quad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}V_1^{*g(k)+}(t)f(t-u,x^f) \\
&\quad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}\left(\frac{x - V_1^{*g(j)+}(t)f(t-u,x^f)}{V_2^{*g(j)+}(t)}V_2^{*g(k)+}(t) - x\right) \\
&\quad + \mathbb{1}_{\{j=0,\ k=J+1\}}\left(V_1^{*g(k)+}(t)f(t,x) + \frac{x - V_1^{*j}(t)}{V_2^{*j+}(t)}V_2^{*g(k)+}(t)f(t,x) - x\right).
\end{aligned}$$

*The last line corresponds to the sum-at-risk upon conversion to free-policy, where $u = 0$.*

*Remark* 2.4.3. In the last line of the sum-at-risk from Proposition 2.4.2, $g(k) = g(J+1) = 0 = j$, and by Equation (2.4.1), the sum-at-risk upon conversion to free-policy is

$$\big(x - V_1^{*0-}(t)\big)f(t,x) - x.$$

The dynamics of the savings account, $X$, and the surplus, $Y$, including policyholder behavior are equal to the dynamics in Proposition 2.2.2 and Proposition 2.2.3, where the payment process and the sum-at-risk are given by Proposition 2.4.1 and Proposition 2.4.2. Thus, the dynamics of the savings account are

$$\begin{aligned}
\mathrm{d}X(t) &= r^*(t)X(t)\mathrm{d}t - \mathrm{d}B(t,X(t),U(t),X(t-U(t))) + \delta^{Z(t)}(t,X(t),Y(t))\mathrm{d}t \\
&\quad + \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t,X(t-),U(t-),X((t-U(t-))-)) \\
&\qquad\qquad \times \big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big),
\end{aligned} \tag{2.4.2}$$

and the dynamics of the surplus are

$$\begin{aligned}
\mathrm{d}Y(t) &= r(t)Y(t)\mathrm{d}t - \delta^{Z(t)}(t,X(t),Y(t))\mathrm{d}t + c^{Z(t)}(t,X(t),U(t),X(t-U(t)))\mathrm{d}t \\
&\quad - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t,X(t-),U(t-),X((t-U(t-))-)) \\
&\qquad\qquad \times \big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),
\end{aligned} \tag{2.4.3}$$

where the surplus contribution is given by

$$c^j(t,x,u,x^f) = (r(t) - r^*(t))x + \sum_{k:k\neq j} R^{*jk}(t,x,u,x^f)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

The dividend strategy $\delta$ is given by Equation (2.2.4).

The above dynamics of the savings account and the surplus contain the free-policy factor, $f$, and the duration, $U(t)$, which implies that they are not in the form of Lemma 2.3.2. Therefore, Theorem 2.3.3 cannot be used to project the savings account and the surplus including policyholder behavior.

## 2.5 State-wise projections including policyholder behavior

In this section, the main results of the paper are presented by extending the result from Section 2.3 to include policyholder behavior. First, we describe the inclusion of policyholder behavior in the life insurance setup with bonus and the choice of free-policy factor. In general, the inclusion of the ideal choice of free-policy factor breaks the linearity assumption of Section 2.3. We consider a certain case where the linearity assumption is satisfied, and suggest an approximation of the ideal free-policy factor. The main results of this paper are that in the certain case, the state-wise projections of the savings account and the surplus with the ideal free-policy factor and the approximated free-policy factor respectively coincide, and that we extend Theorem 2.3.3 to include policyholder behavior in a general case.

### 2.5.1 Policyholder behavior including bonus

The extension of the classic life insurance setup without bonus to include policyholder behavior is described in existing literature. See Buchardt, Møller, and Schmidt (2014) or Buchardt and Møller (2015) for a description of this extension. Without bonus, the payment upon surrender is usually chosen to be the technical reserve in state 0, $b^{0J}(t) = V^{*0}(t)$, such that the insured receive their savings account upon surrender, the sum-at-risk upon surrender is equal to zero, and the modeling of surrender can be omitted on the technical basis. Without bonus, the technical reserve, $V^*(t)$, is the technical value of future payments guaranteed at time $t$, since all payments are guaranteed. In our setup with bonus, this corresponds to the savings account, $X(t)$. The payment upon surrender in the setup with bonus is equal to the savings account $X(t)$ such that bonus obtained prior to time $t$ is included in the payment upon surrender. Then the sum-at-risk of the savings account upon surrender is equal to zero. This complies with the assumption that payments are linear in the savings account.

Without bonus, the free-policy factor is usually chosen according to the principle of equivalence such that there is no jump in the technical reserve upon conversion to free-policy, i.e.

$$f^{\circ}(t) = \frac{V^{*0}(t)}{V^{*0+}(t)},$$

where the superscript $\circ$ refer to the setup without bonus.

To resemble the setup without bonus, the ideal free-policy factor in the setup with bonus is the free-policy factor, where the sum-at-risk of the savings account upon conversion to free-policy is equal to zero, resulting in no jump in $X$ upon conversion to free-policy. The sum-at-risk upon conversion to free-policy is given in Remark 2.4.3, and setting this equal to zero implies that

$$f(t, X(t-)) = \frac{X(t-)}{X(t-) - V_1^{*0-}(t)}. \tag{2.5.1}$$

This free-policy factor is nonlinear in the savings account, which implies that the dynamics of the savings account and the surplus from Equations (2.4.2) and (2.4.3) do not satisfy the linearity assumption in Lemma 2.3.2 with this choice of free-policy factor.

The objective when including policyholder behavior is to ensure that the savings account is unaffected when the behavior option is exercised. This is achieved when the sum-at-risk is equal to zero upon surrender and upon conversion to free-policy. In the study of prospective reserves, Christiansen and Djehiche (2020) denote this concept actuarial equivalence, and obtain adjustment factors similar to our free-policy factor, but their adjustment factors depend on the prospective reserve where our free-policy factor depends on the retrospective savings account.

Let $X_{\mathrm{id}}$ be the savings account and let $Y_{\mathrm{id}}$ be the surplus with the ideal free-policy factor from Equation (2.5.1) above. Similar to Definition 2.3.1, the state-wise projections of the savings account and the surplus are given by

$$\tilde{X}^j_{\mathrm{id}}(t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}} X_{\mathrm{id}}(t) \;\middle|\; \mathcal{F}^r_t\right], \tag{2.5.2}$$

$$\tilde{Y}^j_{\mathrm{id}}(t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}} Y_{\mathrm{id}}(t) \;\middle|\; \mathcal{F}^r_t\right], \tag{2.5.3}$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$.

### 2.5.2 The case with all benefits regulated by bonus

We consider the case, where all benefits are regulated by bonus such that the payment stream not increased by bonus, $B_1$, only contains premiums i.e. $B_1^+ = 0$. In this case, we show that the dynamics of the savings account and the surplus with the ideal free-policy factor from Equation (2.5.1), are in the form of Lemma 2.3.2 such that Theorem 2.3.3 can be used to find differential equations for the state-wise projections of the savings account and the surplus including policyholder behavior.

In the example of an insurance contract consisting of a life annuity and a term insurance, both products are regulated by bonus in the case $B_1^+ = 0$, in contrast to the case where only the life annuity is scaled by bonus.

The assumptions of defined contributions and $B_1^+ = 0$ imply that the total payment process has dynamics

$$\mathrm{d}B_1^-(t) + Q(t-)\mathrm{d}B_2^+(t),$$

where $Q(0-) = 1$ due to the principle of equivalence.

In the continuous payment functions during sojourns in states and the payment functions upon transition between states from Proposition 2.4.1, the terms including the free-policy factor are multiplied by either $b_1^{j+}$, $b_1^{jk+}$ or $V_1^{*j+}$ for $j, k \in \mathcal{J}$. In the case $B_1^+ = 0$, these are all equal to zero and therefore the free-policy factor does not appear in the payment functions.

The continuous payment functions during sojourns in states and the payment functions upon transition between states from Proposition 2.4.1 are in this case

$$\begin{aligned} b^j(t,x) = {} & \mathbb{1}_{\{j \in \mathcal{J}\}} \Big( b_1^{j-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)} b_2^{j+}(t) \Big) \\ & + \mathbb{1}_{\{j \in \mathcal{J}^f\}} \Big( \frac{x}{V_2^{*g(j)+}(t)} b_2^{g(j)+}(t) \Big), \end{aligned} \tag{2.5.4}$$

$$\begin{aligned} b^{jk}(t,x) = {} & \mathbb{1}_{\{j,k \in \mathcal{J},\ j \neq k\}} \Big( b_1^{jk-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)} b_2^{jk+}(t) \Big) \\ & + \mathbb{1}_{\{j,k \in \mathcal{J}^f,\ j \neq k\}} \Big( \frac{x}{V_2^{*g(j)+}(t)} b_2^{g(j)g(k)+}(t) \Big), \end{aligned} \tag{2.5.5}$$

for $j, k \in \mathcal{J} \cup \mathcal{J}^f$.

Similar to the payment functions, the terms including the free-policy factor in the sum-at-risk from Proposition 2.4.2 are multiplied by $V_1^{*j+}$ for $j \in \mathcal{J}$, except for the sum-at-risk upon conversion to free-policy. Thus, in the case $B_1^+ = 0$, the sum-at-risk is

$$\begin{aligned} R^{*jk}(t,x) = {} & b^{jk}(t,x) + \mathbb{1}_{\{j,k \in \mathcal{J},\ j \neq k\}} \left( V_1^{*k-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)} V_2^{*k+}(t) - x \right) \\ & + \mathbb{1}_{\{j,k \in \mathcal{J}^f,\ j \neq k\}} \left( \frac{x}{V_2^{*g(j)+}(t)} V_2^{*g(k)+}(t) - x \right) \\ & + \mathbb{1}_{\{j=0,\ k=J+1\}} \Big( \big(x - V_1^{*j-}(t)\big) f(t,x) - x \Big). \end{aligned} \tag{2.5.6}$$

With the free-policy factor from Equation (2.5.1), the last line in the sum-at-risk above is equal to zero. Therefore, in the case $B_1^+ = 0$ with the free-policy factor from

Equation (2.5.1), neither the payment functions (2.5.4) and (2.5.5) nor the sum-at-risk (2.5.6) depend on the duration in the free-policy states, and they are linear in the savings account. This implies that the dynamics of $X_{\text{id}}(t)$ and $Y_{\text{id}}(t)$ are in the form of Lemma 2.3.2, leading to the result in Theorem 2.3.3. Hence, in this case, we actually have differential equations for the projected savings account and the projected surplus with the free-policy factor from Equation (2.5.1) given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j_{\text{id}}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{X}^k_{\text{id}}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{X}^j_{\text{id}}(t)\\ &+\hat{\alpha}^j_{0,X}(t)p_{Z(0)j}(0,t)+\hat{\alpha}^j_{1,X}(t)\tilde{X}^j_{\text{id}}(t)+\hat{\alpha}^j_{2,X}(t)\tilde{Y}^j_{\text{id}}(t)\\ &+\sum_{k:k\neq j}\mu_{kj}(t)\Big(\hat{\lambda}^{kj}_{0,X}(t)p_{Z(0)k}(0,t)+\hat{\lambda}^{kj}_{1,X}(t)\tilde{X}^k_{\text{id}}(t)\Big),\end{aligned} \tag{2.5.7}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}^j_{\text{id}}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}^k_{\text{id}}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}^j_{\text{id}}(t)\\ &+\hat{\alpha}^j_{0,Y}(t)p_{Z(0)j}(0,t)+\hat{\alpha}^j_{1,Y}(t)\tilde{X}^j_{\text{id}}(t)+\hat{\alpha}^j_{2,Y}(t)\tilde{Y}^j_{\text{id}}(t)\\ &+\sum_{k:k\neq j}\mu_{kj}(t)\Big(\hat{\lambda}^{kj}_{0,Y}(t)p_{Z(0)k}(0,t)+\hat{\lambda}^{kj}_{1,Y}(t)\tilde{X}^k_{\text{id}}(t)\Big),\end{aligned} \tag{2.5.8}$$

and $\tilde{X}^j_{id}(0-) = \tilde{Y}^j_{id}(0-) = 0$ for $j \in \mathcal{J} \cup \mathcal{J}^f$. The expressions for $\hat{\alpha}^j$ and $\hat{\lambda}^{jk}$ are in Appendix 2.B.

We compare the differential equations of the projected savings account and the projected surplus in the case $B_1^+ = 0$ using the free-policy factor from Equation (2.5.1) with the differential equations without policyholder behavior. This comes down to a comparison of the coefficients $\alpha^j$ and $\lambda^{jk}$ from Appendix 2.A and $\hat{\alpha}^j$ and $\hat{\lambda}^{jk}$ from Appendix 2.B. The coefficient $\alpha^j$ and the corresponding $\hat{\alpha}^j$ consist of the same terms, but $\hat{\alpha}^j$ is decomposed in the cases $j \in \mathcal{J}$ and $j \in \mathcal{J}^f$ in the same sense as the payment functions and the sum-at-risk from Equations (2.5.4), (2.5.5) and (2.5.6), since there are only benefits in the free-policy states. This also goes for $\lambda^{jk}$ and $\hat{\lambda}^{jk}$.

*Remark* 2.5.1. The case $B_1^+ = B_2^+$ corresponds to the case $B_1^+ = 0$, since the total payment process when $B_1^+ = B_2^+$ is

$$\mathrm{d}B_1(t) + Q(t-)\mathrm{d}B_2^+(t) = \mathrm{d}B_1^-(t) + \underbrace{(1+Q(t-))}_{=\tilde{Q}(t-)}\mathrm{d}B_2^+(t),$$

which has the same form as the payment process in the case $B_1^+ = 0$, but where $\tilde{Q}(0-) = 2$ since $Q(0-) = 1$ due to the principle of equivalence. When the benefits in $B_1$ are equal to the benefits in $B_2$, all benefits are regulated equally by bonus, and therefore the case $B_1^+ = B_2^+$ can be rewritten to be in the form of $B_1^+ = 0$. Hence, the results above also apply for $B_1^+ = B_2^+$.

If benefits not regulated by bonus cancel due to conversion to free-policy, $B_1^+ = 0$ after conversion to free-policy, and the result above still applies. An example is an insurance contract consisting of a life annuity and a term insurance, where the life annuity is regulated by bonus, and the term insurance cancels upon conversion to free-policy. Throughout this paper, we assume that payments in the free-policy states equal a free-policy factor times the benefits in the corresponding premium-paying state. The example does not comply with this assumption, but we can easily extend our setup to include this case.

### 2.5.3 Approximation of the free-policy factor

In the general setup, $B_1^+(t) \geq 0$ for $t \geq 0$, we cannot project the savings account and the surplus including policyholder behavior by Theorem 2.3.3, since the assumptions are violated. The dynamics of the savings account and the surplus depend on the duration in the free-policy states, $U$. Furthermore, the derivation of Theorem 2.3.3 relies on linearity of $X$ and $Y$ in the dynamics from Lemma 2.3.2, which breaks when the free-policy factor depends on the savings account. This motivates an approximation of the ideal free-policy factor from Equation (2.5.1), which does not depend on $X$.

Just before conversion to free-policy, the policyholder must be premium paying and active, i.e. $Z(t-) = 0$. A reasonable approximation of the free-policy factor is therefore

$$\begin{aligned}
\hat{f}(t) =& \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}} f(t, X(t)) \,\middle|\, \mathcal{F}_t^r\right] \\
=& \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}} \frac{X(t-)}{X(t-) - V_1^{*Z(t-)-}(t)} \,\middle|\, \mathcal{F}_t^r\right].
\end{aligned}$$

We have not developed methods to calculate the projection of a fraction containing the savings account, $X(t)$, in both the nominator and the denominator. Therefore, we cannot continue with the approximation above. Alternatively, the nominator and denominator in the free-policy factor can be projected separately

$$\begin{aligned}
\tilde{f}(t) =& \frac{\mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}} X(t-) \,\middle|\, \mathcal{F}_t^r\right]}{\mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}} \left(X(t-) - V_1^{*Z(t-)-}(t)\right) \,\middle|\, \mathcal{F}_t^r\right]} \\
=& \frac{\tilde{X}^0(t)}{\tilde{X}^0(t) - p_{Z(0)0}(0,t) V_1^{*0-}(t)}.
\end{aligned} \tag{2.5.9}$$

The above free-policy factor does not depend on the savings account, but on the state-wise projection of the savings account. This approximation of the ideal free-policy factor motivates one of the main results of this paper presented in Corollary 2.5.2 below.

**Corollary 2.5.2.** *Let $X_{id}$ be the savings account and $Y_{id}$ be the surplus modeled with the ideal free-policy factor from Equation (2.5.1), and let $X_{ap}$ be the savings account* *and $Y_{ap}$ be the surplus modeled with the approximated free-policy factor from Equation (2.5.9). The state-wise projections are given by Equations (2.5.2) and (2.5.3), and*

$$\tilde{X}^j_{ap}(t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}} X_{ap}(t) \;\middle|\; \mathcal{F}^r_t \right],$$
$$\tilde{Y}^j_{ap}(t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}} Y_{ap}(t) \;\middle|\; \mathcal{F}^r_t \right],$$

*for $j \in \mathcal{J} \cup \mathcal{J}^f$, respectively.*

*In the case where all benefits are regulated by bonus, $B_1^+ = 0$*

$$\tilde{X}^j_{id}(t) = \tilde{X}^j_{ap}(t),$$
$$\tilde{Y}^j_{id}(t) = \tilde{Y}^j_{ap}(t),$$

*for $j \in \mathcal{J} \cup \mathcal{J}^f$.*

*Proof.* Assume all benefits are regulated by bonus, $B_1^+ = 0$. The state-wise projections of the savings account and the surplus with the ideal free-policy factor satisfy the differential equations in Equations (2.5.7) and (2.5.8).

Equations (2.5.4), (2.5.5) and (2.5.6) in Section 2.5.2 state that only the sum-at-risk depends on the free-policy factor. The sum-at-risk with the approximated free-policy factor, $\tilde{f}$, is

$$\begin{aligned} R^{*jk}(t,x) = b^{jk}(t,x) &+ \mathbb{1}_{\{j,k \in \mathcal{J},\ j \neq k\}} \left( V_1^{*k-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)} V_2^{*k+}(t) - x \right) \\ &+ \mathbb{1}_{\{j,k \in \mathcal{J}^f,\ j \neq k\}} \left( \frac{x}{V_2^{*g(j)+}(t)} V_2^{*g(k)+}(t) - x \right) \\ &+ \mathbb{1}_{\{j=0,\ k=J+1\}} \Big( \big(x - V_1^{*j-}(t)\big) \tilde{f}(t) - x \Big). \end{aligned} \tag{2.5.10}$$

The dynamics of $X_{\mathrm{ap}}$ and $Y_{\mathrm{ap}}$ are in the form of Equations (2.4.2) and (2.4.3) with the payment functions from Equations (2.5.4) and (2.5.5) and the sum-at-risk from Equation (2.5.10). This implies that the dynamics of $X_{\mathrm{ap}}$ and $Y_{\mathrm{ap}}$ are in the same form as in Lemma 2.3.2, since they do not depend on the duration, $U$, and they are linear in $X_{\mathrm{ap}}(t)$ and $Y_{\mathrm{ap}}(t)$.

Theorem 2.3.3 gives differential equations of the state-wise projections of the savings account and the surplus, $\tilde{X}^j_{\mathrm{ap}}$ and $\tilde{Y}^j_{\mathrm{ap}}$. These differential equations can be expressed in terms of $\hat{\alpha}$ and $\hat{\lambda}$ from the differential equations (2.5.7) and (2.5.8)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j_{\mathrm{ap}}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{X}^k_{\mathrm{ap}}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{X}^j_{\mathrm{ap}}(t) \\
&+ \hat{\alpha}^j_{0,X}(t)p_{Z(0)j}(0,t) + \hat{\alpha}^j_{1,X}(t)\tilde{X}^j_{\mathrm{ap}}(t) + \hat{\alpha}^j_{2,X}(t)\tilde{Y}^j_{\mathrm{ap}}(t) \\
&+ \mathbb{1}_{\{j=J+1\}}\mu^*_{0j}(t)\Big(\tilde{X}^0_{\mathrm{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\mathrm{ap}}(t)\big)\Big) \\
&+ \sum_{k:k\neq j}\mu_{kj}(t)\Bigg(\hat{\lambda}^{kj}_{0,X}(t)p_{Z(0)k}(0,t) + \hat{\lambda}^{kj}_{1,X}(t)\tilde{X}^k_{\mathrm{ap}}(t) \\
&\qquad - \mathbb{1}_{\{k=0,\ j=J+1\}}\Big(\tilde{X}^0_{\mathrm{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\mathrm{ap}}(t)\big)\Big)\Bigg),
\end{aligned} \tag{2.5.11}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}^j_{\mathrm{ap}}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}^k_{\mathrm{ap}}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}^j_{\mathrm{ap}}(t) \\
&+ \hat{\alpha}^j_{0,Y}(t)p_{Z(0)j}(0,t) + \hat{\alpha}^j_{1,Y}(t)\tilde{X}^j_{\mathrm{ap}}(t) + \hat{\alpha}^j_{2,Y}(t)\tilde{Y}^j_{\mathrm{ap}}(t) \\
&- \mathbb{1}_{\{j=J+1\}}\mu^*_{0j}(t)\Big(\tilde{X}^0_{\mathrm{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\mathrm{ap}}(t)\big)\Big) \\
&+ \sum_{k:k\neq j}\mu_{kj}(t)\Bigg(\hat{\lambda}^{kj}_{0,Y}(t)p_{Z(0)k}(0,t) + \hat{\lambda}^{kj}_{1,Y}(t)\tilde{X}^k_{\mathrm{ap}}(t) \\
&\qquad + \mathbb{1}_{\{k=0,\ j=J+1\}}\Big(\tilde{X}^0_{\mathrm{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\mathrm{ap}}(t)\big)\Big)\Bigg).
\end{aligned} \tag{2.5.12}$$

By inserting the expression for $\tilde{f}$ from Equation (2.5.9), the differential equations (2.5.11) and (2.5.12) are equal to the differential equations (2.5.7) and (2.5.8). Furthermore, the initial conditions are

$$\tilde{X}^j_{\mathrm{id}}(0) = \tilde{Y}^j_{\mathrm{id}}(0) = \tilde{X}^j_{\mathrm{ap}}(0) = \tilde{Y}^j_{\mathrm{ap}}(0) = 0,$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$. This implies that

$$\begin{aligned}
\tilde{X}^j_{\mathrm{id}}(t) &= \tilde{X}^j_{\mathrm{ap}}(t), \\
\tilde{Y}^j_{\mathrm{id}}(t) &= \tilde{Y}^j_{\mathrm{ap}}(t),
\end{aligned}$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$ as desired. □

Corollary 2.5.2 implies that in the case $B_1^+ = 0$, we can project the savings account and the surplus with the approximated free-policy factor and actually obtain the same accurate projections as with the ideal free-policy factor. Based on this result, we consider $\tilde{f}$ to be a reasonable approximation of $f$, that does not depend on the savings account, but instead on the projected savings account.

### 2.5.4 Projections with the approximated free-policy factor

In the general setup, $B_1^+(t) \geq 0$ for $t \geq 0$, with the approximated free-policy factor from Equation (2.5.9), the dynamics of the savings account and the surplus are linear, but they also depend on the duration through the payment functions from Proposition 2.4.1 and the sum-of-risk from Proposition 2.4.2. Therefore, we cannot use Theorem 2.3.3 to project the savings account and the surplus. This motivates an extension of Theorem 2.3.3 including duration dependence, where linearity in the dynamics of the savings account and the surplus is preserved.

**Lemma 2.5.3.** *The dynamics of the savings account, $X_{ap}$, from Equation (2.4.2) and the dynamics of the surplus, $Y_{ap}$, from Equation (2.4.3), with the approximated free-policy factor, $\tilde{f}$, from Equation (2.5.9), can be written in the form*

$$
\begin{aligned}
\mathrm{d}X_{ap}(t) = {} & \Big(\bar{\alpha}_{0,X}^{Z(t)}(t) + \bar{\alpha}_{1,X}^{Z(t)}(t)X_{ap}(t) + \bar{\alpha}_{2,X}^{Z(t)}(t)Y_{ap}(t) + \tilde{f}(t-U(t))\bar{\beta}_{0,X}^{Z(t)}(t)\Big)\mathrm{d}t \\
& + \sum_{k:k\neq Z(t-)} \Big(\bar{\lambda}_{0,X}^{Z(t-)k}(t) + \bar{\lambda}_{1,X}^{Z(t-)k}(t)X_{ap}(t-) \\
& \qquad + \tilde{f}(t-U(t))\bar{\gamma}_{0,X}^{Z(t-)k}(t)\Big)\mathrm{d}N^k(t), \\
\mathrm{d}Y_{ap}(t) = {} & \Big(\bar{\alpha}_{0,Y}^{Z(t)}(t) + \bar{\alpha}_{1,Y}^{Z(t)}(t)X_{ap}(t) + \bar{\alpha}_{2,Y}^{Z(t)}(t)Y_{ap}(t) + \tilde{f}(t-U(t))\bar{\beta}_{0,Y}^{Z(t)}(t)\Big)\mathrm{d}t \\
& + \sum_{k:k\neq Z(t-)} \Big(\bar{\lambda}_{0,Y}^{Z(t-)k}(t) + \bar{\lambda}_{1,Y}^{Z(t-)k}(t)X_{ap}(t-) \\
& \qquad + \tilde{f}(t-U(t))\bar{\gamma}_{0,Y}^{Z(t-)k}(t)\Big)\mathrm{d}N^k(t),
\end{aligned}
$$

*for deterministic functions $\bar{\alpha}_{i,H}^j, \bar{\beta}_{i,H}^j, \bar{\lambda}_{i,H}^{jk}, \bar{\gamma}_{i,H}^{jk}$ for $i = 0, 1, 2$, $H = X, Y$ and $j, k \in \mathcal{J} \cup \mathcal{J}^f$, $j \neq k$, where*

$$\bar{\beta}_{0,X}^j(t) = \bar{\beta}_{0,Y}^j(t) = \bar{\gamma}_{0,X}^{jk}(t) = \bar{\gamma}_{0,Y}^{jk}(t) = 0,$$

*for all $t \geq 0$ and $j \in \mathcal{J}$.*
*See Appendix 2.C for the expressions of $\bar{\alpha}, \bar{\beta}, \bar{\lambda}$ and $\bar{\gamma}$ for the savings account and the surplus.*

We consider the difference between the case with all benefits regulated by bonus, $B_1^+ = 0$, with the free-policy factor from Equation (2.5.1) from Section 2.5.2 and the general case, $B_1^+ \geq 0$, with the approximated free-policy factor. This comes down to a comparison of the coefficients $\hat{\alpha}$ and $\hat{\lambda}$ from Appendix 2.B with the coefficients $\bar{\alpha}$, $\bar{\beta}$, $\bar{\lambda}$ and $\bar{\gamma}$ from Appendix 2.C. Apart from the sum-at-risk upon conversion to free-policy and the duration dependent terms, the coefficients are equal. In the first case, the sum-at-risk upon conversion to free-policy is equal to zero, while in the second case, it is added to $\bar{\lambda}$. The duration dependent terms from Propositions 2.4.1 and 2.4.2 are equal to zero in the case with all benefits regulated by bonus, while in the general case they appear in $\bar{\beta}$ and $\bar{\gamma}$.

The dynamics of the savings account and the surplus in Lemma 2.5.3 allow for an extension of the dividend strategy from Equation (2.2.4) to be duration dependent. Dividends in form

$$\begin{aligned}\delta^j(t,x,y,u) &= \delta_0^j(t,u) + \delta_1^j(t)\cdot x + \delta_2^j(t)\cdot y,\\ \delta_0^j(t,u) &= \mathbb{1}_{\{j\in\mathcal{J}\}}\delta_0^j(t) + \mathbb{1}_{\{j\in\mathcal{J}^f\}}\tilde{f}(t-u)\delta_0^j(t),\end{aligned}$$

comply with the dynamics in Lemma 2.5.3.

Now, we extent the result of Theorem 2.3.3 to include duration dependence in the approximated free-policy factor from the dynamics of the savings account and the surplus in Lemma 2.5.3.

**Theorem 2.5.4.** *Let $X_{ap}$ and $Y_{ap}$ have dynamics in the form of Lemma 2.5.3 and $Z(0)\in\mathcal{J}$. The state-wise projections of the savings account and the surplus, $\tilde{X}_{ap}^j$ and $\tilde{Y}_{ap}^j$, satisfy the system of differential equations below*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}_{ap}^j(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{X}_{ap}^k(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{X}_{ap}^j(t)\\ &+\bar{\alpha}_{0,X}^j(t)p_{Z(0)j}^{\tilde{f}}(0,t) + \bar{\alpha}_{1,X}^j(t)\tilde{X}_{ap}^j(t) + \bar{\alpha}_{2,X}^j(t)\tilde{Y}_{ap}^j(t)\\ &+\sum_{k:k\neq j}\mu_{kj}(t)\Big(\bar{\lambda}_{0,X}^{kj}(t)p_{Z(0)k}^{\tilde{f}}(0,t) + \bar{\lambda}_{1,X}^{kj}(t)\tilde{X}_{ap}^k(t)\Big)\\ &+\bar{\beta}_{0,X}^j(t)p_{Z(0)j}^{\tilde{f}}(0,t) + \sum_{k:k\neq j}\mu_{kj}(t)\bar{\gamma}_{0,X}^{kj}(t)p_{Z(0)k}^{\tilde{f}}(0,t),\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}_{ap}^j(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}_{ap}^k(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}_{ap}^j(t)\\ &+\bar{\alpha}_{0,Y}^j(t)p_{Z(0)j}^{\tilde{f}}(0,t) + \bar{\alpha}_{1,Y}^j(t)\tilde{X}_{ap}^j(t) + \bar{\alpha}_{2,Y}^j(t)\tilde{Y}_{ap}^j(t)\\ &+\sum_{k:k\neq j}\mu_{kj}(t)\Big(\bar{\lambda}_{0,Y}^{kj}(t)p_{Z(0)k}^{\tilde{f}}(0,t) + \bar{\lambda}_{1,Y}^{kj}(t)\tilde{X}_{ap}^k(t)\Big)\\ &+\bar{\beta}_{0,Y}^j(t)p_{Z(0)j}^{\tilde{f}}(0,t) + \sum_{k:k\neq j}\mu_{kj}(t)\bar{\gamma}_{0,Y}^{kj}(t)p_{Z(0)k}^{\tilde{f}}(0,t),\end{aligned}$$

*where $\tilde{X}_{ap}^j(0-) = \tilde{Y}_{ap}^j(0-) = 0$, $\tilde{f}$ is the approximated free-policy factor from Equation (2.5.9), and $p_{Z(0)j}^{\tilde{f}}(0,t)$ are the $\tilde{f}$-modified probabilities*

$$p_{Z(0)j}^{\tilde{f}}(0,t) = \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(t)=j\}}\tilde{f}(t-U(t))^{\mathbb{1}_{\{j\in\mathcal{J}^f\}}}\Big],$$

*for $Z(0)\in\mathcal{J}$, $j\in\mathcal{J}\cup\mathcal{J}^f$, and $t\geq 0$.*

*Proof.* See Appendix 2.D. □

Buchardt and Møller (2015) derive forward differential equations for the same $\tilde{f}$-modified probabilities in the case where $j \in \mathcal{J}^f$. In the case where $j \in \mathcal{J}$, the $\tilde{f}$-modified probabilities are the ordinary transition probabilities that satisfy Kolmogorov's forward differential equations. Therefore, for a general $j \in \mathcal{J} \cup \mathcal{J}^f$, the $\tilde{f}$-modified probabilities satisfy the following forward differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} p^{\tilde{f}}_{Z(0)j}(0,t) = \mathbb{1}_{\{j=J+1\}} p^{\tilde{f}}_{Z(0)0}(0,t)\mu_{0(J+1)}(t)\tilde{f}(t) - p^{\tilde{f}}_{Z(0)j}(0,t) \sum_{k:k\neq j} \mu_{jk}(t)$$
$$+ \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{\substack{k\in\mathcal{J}^f \\ k\neq j}} p^{\tilde{f}}_{Z(0)k}(0,t)\mu_{kj}(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k\in\mathcal{J} \\ k\neq j}} p^{\tilde{f}}_{Z(0)k}(0,t)\mu_{kj}(t).$$

We consider Theorem 2.5.4 as one of the main results of the paper, since it enables us to project the savings account and the surplus in a general setup with the policyholder behavior options surrender and free-policy with the approximated free-policy factor from Equation (2.5.9). For instance in the example with an insurance contract consisting of a life annuity and a term insurance, where the life annuity is regulated by bonus and the term insurance and the premiums are fixed.

*Remark* 2.5.5. Let the savings account and the surplus have dynamics in the form of Lemma 2.5.3, but with a general free-policy factor, $\bar{f}$, that does not depend on the savings account. Then Theorem 2.5.4 holds with $\bar{f}$-modified probabilities.

In the danish life insurance business, it is common to scale all benefits (both those regulated by bonus and those not regulated by bonus) with the free-policy factor upon conversion to free-policy. We can imagine an insurance contract where only the benefits not regulated by bonus, $B_1^+$, are scaled with the free-policy factor upon conversion to free-policy and where $Q(0-) = 0$. Then the free-policy factor does not depend on the savings account, and Theorem 2.5.4 applies.

## 2.6 Numerical simulation example

In this section, we emphasize the practical applications of our results in a numerical simulation example, and study the state-wise projections of the savings account and the surplus in a survival model including free-policy.

To illustrate this example, we assume the interest rate follow a Vasicek model with dynamics

$$\mathrm{d}r(t) = (\phi + \psi\, r(t))\, \mathrm{d}t + \sqrt{\theta}\, \mathrm{d}W(t),$$

where $\big(W(t)\big)_{\{t\geq 0\}}$ is a Brownian motion, see for instance Björk (2009). Any other model of the interest rate can be chosen.

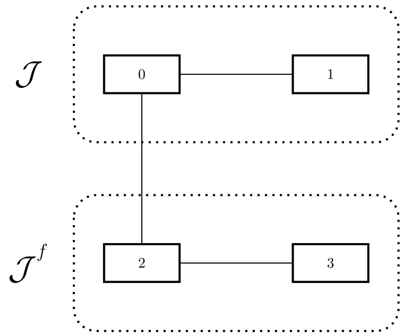


**Figure 2.2:** *Survival model in the numerical example*

The survival model including free-policy is illustrated in Figure 3.1[2], where state 0 corresponds to alive and state 1 corresponds to dead in the non-free-policy states and state 2 and state 3 corresponds to alive and dead, respectively, in the free-policy states. We consider an insured male at age $a_0$ at initialization of the insurance contract at time 0. The insurance contract consists of premiums paid continuously in state 0 until retirement age $n$, a term insurance not regulated by bonus payable upon dead before retirement age, and a life annuity regulated by bonus paid continuously when alive after retirement age. Hence, in this example, $B_1^+ \geq 0$ and we use Theorem 2.5.4 in the projection. The payment process is

$$\begin{aligned}
&\mathrm{d}B(t,X(t)) = \\
&\quad \mathbb{1}_{\{Z(t)=0\}}\left(\left(\frac{X(t)-V_1^{*0}(t)}{V_2^{*0}(t)}b_2^0(t)-\pi(t)\right)\mathrm{d}t + b_1^{01}(t)\mathrm{d}N^1(t)\right) \\
&\quad + \mathbb{1}_{\{Z(t)=2\}}\left(\frac{X(t)-\tilde{f}(t-U(t))V_1^{*0}(t)}{V_2^{*0}(t)}b_2^0(t)\mathrm{d}t + \tilde{f}(t-U(t))b_1^{01}(t)\mathrm{d}N^3(t)\right).
\end{aligned}$$

The premium rate is determined according to the principle of equivalence on the technical basis, and we use the approximated free-policy factor from Equation (2.5.9).

Inspired by Bruhn and Lollike (2021), we choose a dividend strategy equal to

$$\begin{aligned}
&\delta^{Z(t)}(t,U(t),X(t),Y(t)) = \\
&\qquad 0.5\cdot\big(r(t)-r^*(t)\big)^+X(t) + 0.01\cdot Y(t) \\
&\qquad + 0.5\cdot\sum_{k:k\neq Z(t)} R^{*Z(t)k}(t,U(t),X(t))\big(\mu^*_{Z(t)k}(t)-\mu_{Z(t)k}(t)\big),
\end{aligned}$$

where $R^{*Z(t)k}$ is the sum-at-risk from Proposition 2.4.2 with the approximated free-policy factor. The dividend strategy resembles the surplus contribution, but with

[2]The figure is elaborated by the authors

**Table 2.1:** *Components in the numerical example*

| Component | Value |
|---|---|
| Age of policyholder, $a_0$ | 30 |
| Age of retirement, $n$ | 65 |
| Termination | 80 |
| Premium, $\pi(t)$ | $0.3021694 \cdot \mathbb{1}_{\{a_0+t<n\}}$ |
| Annuity, $b_2^0(t)$ | $1 \cdot \mathbb{1}_{\{a_0+t\geq n\}}$ |
| Term insurance, $b_1^{01}(t)$ | $5 \cdot \mathbb{1}_{\{a_0+t<n\}}$ |
| $Z(0)$ | 0 |
| $\mu_{01}^*(t)$ | $0.0005 + 10^{5.88+0.038(t+a_0)-10}$ |
| $\mu_{02}(t)$ | $0.015 \cdot \mathbb{1}_{\{a_0+t<n\}}$ |
| $r^*(t)$ | 0.01 |
| $r(0)$ | 0.05 |
| $\phi$ | 0.008127 |
| $\psi$ | -0.162953 |
| $\theta$ | 0.000237 |

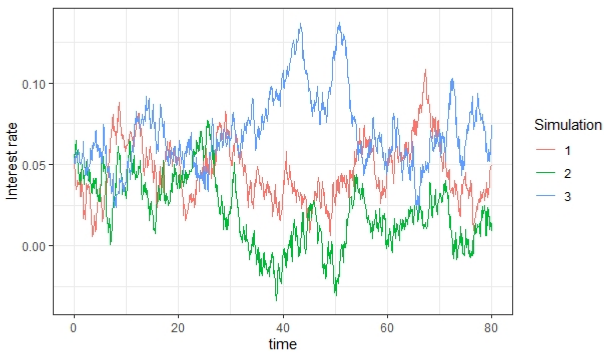


**Figure 2.3:** *Simulations of the interest rate in the numerical example*

$(r(t)-r^*(t))^+$ instead of $r(t)-r^*(t)$. This is to avoid negative dividends if $r^*(t) > r(t)$. The market death intensity is the mortality benchmark from the Danish FSA from 2019. We project the savings account and the surplus in states 0 and 2, since there are no payments in the death states. The components in the projection are stated in Table 5.1.

Figure 2.3 illustrates three simulated paths of the interest rate, simulated with an Euler scheme based on the dynamics of the interest rate. For each path of the interest rate, we project the savings account and the surplus in state 0 and 2 using Theorem 2.5.4, and illustrate the state-wise projections in Figure 2.4 (left) and Figure 2.5 (left).

The projected savings account is larger in state 0 than in the free-policy-state, since premiums cancel upon conversion to free-policy, which lowers the savings account.

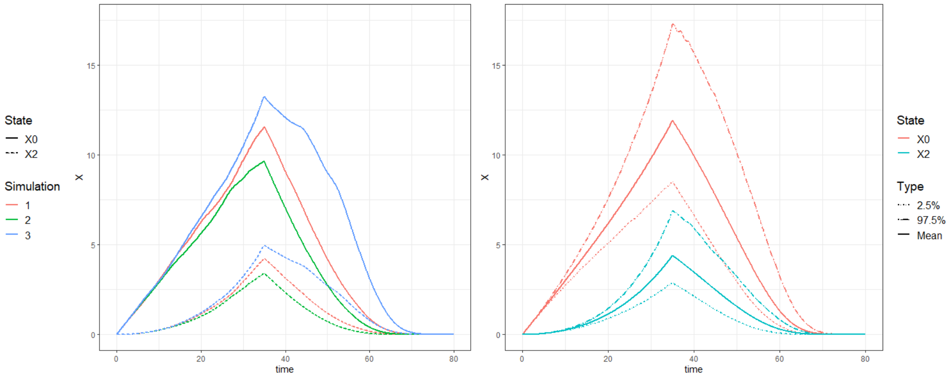


**Figure 2.4:** *Left: State-wise projections of the savings account in the three simulated scenarios of the interest rate. Right: The mean and confidence intervals of the projected savings account.*

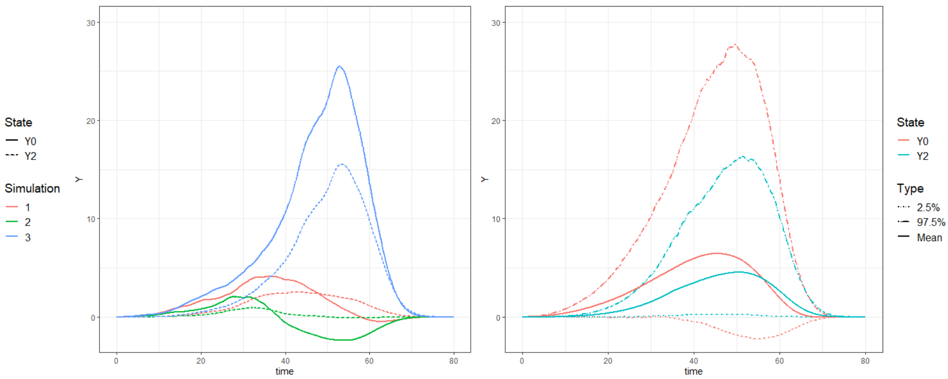


**Figure 2.5:** *Left: State-wise projections of the surplus in the three simulated scenarios of the interest rate. Right: The mean and confidence intervals of the projected surplus.*

The interest rate impacts the projected surplus in Figure 2.5 (left) significantly. A high (low) interest rate results in a high (low) surplus contribution, which effects the projected surplus as illustrated in simulation 3 (2). A high interest rate results in high dividends in our numerical example, and therefore the projected savings accounts are highest in simulation 3. For the effects of changing the dividend strategy, see Bruhn and Lollike (2021). With these calculations, the insurance company can monitor the development of the insurance contract in various scenarios of the interest rate, and for instance assess the effects of the chosen dividend strategy.

Based on 1000 simulations of the interest rate, we estimate the mean, the 2.5%-quantile, and the 97.5%-quantile of the projected savings account (see Figure 2.4 (right)) and the projected surplus (see Figure 2.5 (right)). This illustrates that within the Vasicek model with the chosen parameters and with the chosen dividend strategy, the 95%-

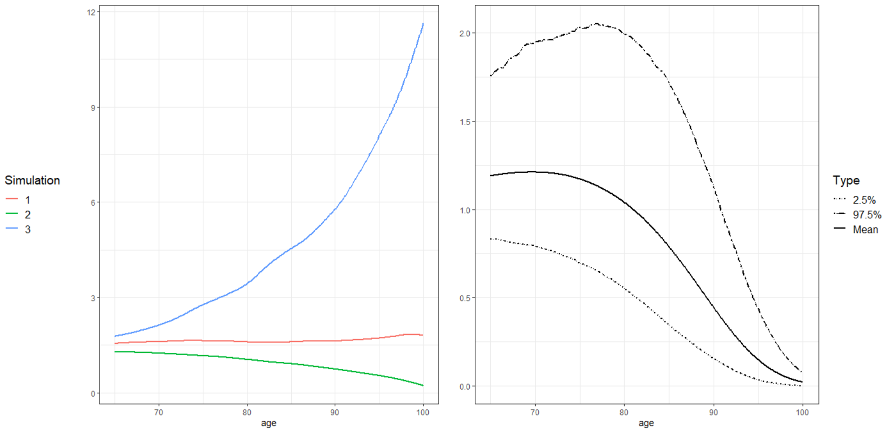


**Figure 2.6:** *Left: The expected life annuity in the three simulated scenarios of the interest conditional in the insured being alive and non-free-policy. Right: The expected life annuity and confidence intervals.*

confidence interval of the projected savings account is widest, when the insured retires at time 35, and the 95%-confidence interval of the projected surplus spans from $-2.2$ to 27.7, which indicates to the insurance company that the development of the surplus is uncertain.

The insurance company is interested in communicating the expected life annuity payment to the insured, since it is regulated by bonus, and the amount of future bonus is unknown at initialization of the insurance contract. Figure 2.6 (left) illustrates the life annuity rate in the three simulated scenarios of the interest rate conditional on the insured being alive and in the non-free-policy state at the time of the payment. In scenario 3, the savings account is higher resulting in a high life annuity. Scenario 2 has a negative surplus due to a low interest rate, which results in negative dividends with the chosen dividend strategy, and therefore the life annuity gets below 1 in this scenario. At initialization of the insurance contract, the insurance company promises the insured a life annuity of 1 given alive and non-free-policy, and hence scenario 2 is bad for the company. The projection in Figure 2.6 (left) holds information to the insurance company, that when the interest rate is low, the insurance company should react and change their dividend strategy.

Figure 2.6 (right) illustrates the expected life annuity and a 95% confidence interval of the life annuity as a function of age. The life annuity is weighted with the probability of dying and conversion to free-policy, hence it is lower than the life annuity in Figure 2.6 (left) where we condition in being alive and non-free-policy.

## 2.7 Conclusion

The paper presents a method for projecting the savings account and the surplus of a life insurance contract including policyholder behavior in various financial scenarios. We present differential equations of the projected savings account and the projected surplus without policyholder behavior, which is the result of Bruhn and Lollike (2021). When including policyholder behavior, we cannot in general project the savings account and the surplus with an ideal free-policy factor using the methods from Bruhn and Lollike (2021).

In this paper, we show that in the case, where all benefits are regulated by bonus, we can actually find accurate differential equations for the state-wise projections of the savings account and the surplus with the ideal free-policy factor. We suggest an approximation to the ideal free-policy factor, and one of the main results is that in the case, where all benefits are regulated by bonus, the projections of the savings account and the surplus based on the ideal free-policy factor coincide with the projections based on the approximated free-policy factor. Therefore, we consider the approximated free-policy factor a reasonable approximation of the ideal free-policy factor.

We are able to project the savings account and the surplus with the approximated free-policy factor in a general case, and we present differential equations of the state-wise projections of the savings account and the surplus with the approximated free-policy factor. We consider this result as a key result in the projection of balances in life insurance and a good extension of Bruhn and Lollike (2021) to include policyholder behavior outside the case, where all benefits are regulated by bonus. We illustrate a numerical simulation example in three scenarios of the interest rate to highlight the practical application of our findings. This results in a projection of the savings account and the surplus for a chosen dividend strategy, which enables the insurance company to assess the effects of their chosen management actions. Furthermore, we study distributional properties of the projections.

This paper studies a simple dividend strategy which is linear in the savings account and the surplus. In order to use this model, insurance companies must choose their future dividend strategy according to this simple setup. Future research involves extending the model to include a more complex dividend strategy and allow for dependence of for instance assets and market values. Another branch is the study of how to choose an optimal dividend strategy in this multi-state setup, see for instance Nielsen (2005).

## Acknowledgments and declarations of interest

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## 2.A Additions to Lemma 2.3.2

The coefficients in the dynamics of the savings account and surplus from Lemma 2.3.2 in the setup without policyholder behavior.

$$\begin{aligned}
\alpha^j_{0,X}(t) &= \delta^j_0(t) - b^j_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^j_2(t) \\
&\quad - \sum_{k:k\neq j} \left( b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^{jk}_2(t) + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t) \right) \mu^*_{jk}(t), \\
\alpha^j_{1,X}(t) &= r^*(t) + \delta^j_1(t) - \frac{b^j_2(t)}{V^{*j}_2(t)} - \sum_{k:k\neq j} \left( \frac{b^{jk}_2(t)}{V^{*j}_2(t)} + \frac{V^{*k}_2(t)}{V^{*j}_2(t)} - 1 \right) \mu^*_{jk}(t), \\
\alpha^j_{2,X}(t) &= \delta^j_2(t), \\
\lambda^{jk}_{0,X}(t) &= V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t), \\
\lambda^{jk}_{1,X}(t) &= \frac{V^{*k}_2(t)}{V^{*j}_2(t)} - 1
\end{aligned}$$

$$\begin{aligned}
\alpha^j_{0,Y}(t) &= \; - \delta^j_0(t) + \sum_{k:k\neq j} \left( b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^{jk}_2(t) + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t) \right) \mu^*_{jk}(t), \\
\alpha^j_{1,Y}(t) &= \; - \delta^j_1(t) + r(t) - r^*(t) + \sum_{k:k\neq j} \left( \frac{b^{jk}_2(t)}{V^{*j}_2(t)} + \frac{V^{*k}_2(t)}{V^{*j}_2(t)} - 1 \right) \mu^*_{jk}(t), \\
\alpha^j_{2,Y}(t) &= r(t) - \delta^j_2(t), \\
\lambda^{jk}_{0,Y}(t) &= \; - b^{jk}_1(t) + \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^{jk}_2(t) - V^{*k}_1(t) + \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t), \\
\lambda^{jk}_{1,Y}(t) &= \; - \frac{b^{jk}_2(t)}{V^{*j}_2(t)} - \frac{V^{*k}_2(t)}{V^{*j}_2(t)} + 1.
\end{aligned}$$

## 2.B Additions to the case $B^+_1 = 0$

The coefficients in the dynamics of the savings account and surplus from Lemma 2.3.2 in the case $B^+_1 = 0$ with the ideal free-policy factor.

$$
\begin{aligned}
\hat{\alpha}^{j}_{0,X}(t) =& -\mathbb{1}_{\{j\in\mathcal{J}\}}\Big(b^{j-}_{1}(t)-\frac{V^{*j-}_{1}(t)}{V^{*j+}_{2}(t)}b^{j+}_{2}(t)\Big)+\delta^{j}_{0}(t)\\
&-\mathbb{1}_{\{j\in\mathcal{J}\}}\sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}}\Big(b^{jk-}_{1}(t)+V^{*k-}_{1}(t)-\frac{V^{*j-}_{1}(t)}{V^{*j+}_{2}(t)}\big(b^{jk+}_{2}(t)+V^{*k+}_{2}(t)\big)\Big)\mu^{*}_{jk}(t),\\
\hat{\alpha}^{j}_{1,X}(t) =& \, r^{*}(t)-\mathbb{1}_{\{j\in\mathcal{J}\}}\frac{b^{j+}_{2}(t)}{V^{*j+}_{2}(t)}-\mathbb{1}_{\{j\in\mathcal{J}^{f}\}}\frac{b^{g(j)+}_{2}(t)}{V^{*g(j)+}_{2}(t)}+\delta^{j}_{1}(t)\\
&-\mathbb{1}_{\{j\in\mathcal{J}\}}\sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}}\Big(\frac{b^{jk+}_{2}(t)+V^{*k+}_{2}(t)}{V^{*j+}_{2}(t)}-1\Big)\mu^{*}_{jk}(t)\\
&-\mathbb{1}_{\{j\in\mathcal{J}^{f}\}}\sum_{\substack{k:k\neq j\\ k\in\mathcal{J}^{f}}}\Big(\frac{b^{g(j)g(k)+}_{2}(t)+V^{*g(k)+}_{2}(t)}{V^{*g(j)+}_{2}(t)}-1\Big)\mu^{*}_{jk}(t),\\
\hat{\alpha}^{j}_{2,X}(t) =& \, \delta^{j}_{2}(t),\\
\hat{\lambda}^{jk}_{0,X}(t) =& \, \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\Big(V^{*k-}_{1}(t)-\frac{V^{*j-}_{1}(t)}{V^{*j+}_{2}(t)}V^{*k+}_{2}(t)\Big),\\
\hat{\lambda}^{jk}_{1,X}(t) =& \, \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\left(\frac{V^{*k+}_{2}(t)}{V^{*j+}_{2}(t)}-1\right)+\mathbb{1}_{\{j,k\in\mathcal{J}^{f},j\neq k\}}\left(\frac{V^{*g(k)+}_{2}(t)}{V^{*g(j)+}_{2}(t)}-1\right).
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha}^{j}_{0,Y}(t) =& -\delta^{j}_{0}(t)+\mathbb{1}_{\{j\in\mathcal{J}\}}\sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}}\Big(b^{jk-}_{1}(t)+V^{*k-}_{1}(t)\\
&\qquad -\frac{V^{*j-}_{1}(t)}{V^{*j+}_{2}(t)}\big(b^{jk+}_{2}(t)+V^{*k+}_{2}(t)\big)\Big)\mu^{*}_{jk}(t),\\
\hat{\alpha}^{j}_{1,Y}(t) =& \, r(t)-r^{*}(t)-\delta^{j}_{1}(t)+\mathbb{1}_{\{j\in\mathcal{J}\}}\sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}}\Big(\frac{b^{jk+}_{2}(t)+V^{*k+}_{2}(t)}{V^{*j+}_{2}(t)}-1\Big)\mu^{*}_{jk}(t)\\
&+\mathbb{1}_{\{j\in\mathcal{J}^{f}\}}\sum_{\substack{k:k\neq j\\ k\in\mathcal{J}^{f}}}\Big(\frac{b^{g(j)g(k)+}_{2}(t)+V^{*g(k)+}_{2}(t)}{V^{*g(j)+}_{2}(t)}-1\Big)\mu^{*}_{jk}(t),\\
\hat{\alpha}^{j}_{2,Y}(t) =& \, r(t)-\delta^{j}_{2}(t),\\
\hat{\lambda}^{jk}_{0,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\Big(b^{jk-}_{1}(t)+V^{*k-}_{1}(t)-\frac{V^{*j-}_{1}(t)}{V^{*j+}_{2}(t)}\big(b^{jk+}_{2}(t)+V^{*k+}_{2}(t)\big)\Big),\\
\hat{\lambda}^{jk}_{1,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\Big(\frac{b^{jk+}_{2}(t)+V^{*k+}_{2}(t)}{V^{*j+}_{2}(t)}-1\Big)\\
&-\mathbb{1}_{\{j,k\in\mathcal{J}^{f},j\neq k\}}\Big(\frac{b^{g(k)g(j)+}_{2}(t)+V^{*g(k)+}_{2}(t)}{V^{*g(j)+}_{2}(t)}-1\Big).
\end{aligned}
$$

## 2.C Additions to Lemma 2.5.3

The coefficients in the dynamics of the savings account and surplus from Lemma 2.5.3.

$$
\begin{aligned}
\bar{\alpha}^j_{0,X}(t) = {} & \delta^j_0(t) - \mathbb{1}_{\{j\in\mathcal{J}\}}\left(b^j_1(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} b^j_2(t)\right) \\
& - \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\neq J+1}} \Bigg(b^{jk}_1(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} b^{jk}_2(t) \\
& \qquad + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} V_2^{*k+}(t)\Bigg)\mu^*_{jk}(t), \\
\bar{\alpha}^j_{1,X}(t) = {} & r^*(t) + \delta^j_1(t) - \mathbb{1}_{\{j\in\mathcal{J}\}}\Bigg(\frac{b^j_2(t)}{V_2^{*j+}(t)} \\
& \qquad + \sum_{\substack{k:k\neq j\\ k\neq J+1}}\left(\frac{b^{jk}_2(t)}{V_2^{*j+}(t)} + \frac{V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1\right)\mu^*_{jk}(t)\Bigg) \\
& - \mathbb{1}_{\{j\in\mathcal{J}^f\}}\left(\frac{b_2^{g(j)+}(t)}{V_2^{*g(j)+}(t)} + \sum_{k:k\neq j}\left(\frac{b_2^{g(j)g(k)+}(t)}{V_2^{*g(j)+}(t)} + \frac{V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1\right)\mu^*_{jk}(t)\right), \\
\bar{\alpha}^j_{2,X}(t) = {} & \delta^j_2(t), \\
\bar{\beta}^j_{0,X}(t) = {} & - \mathbb{1}_{\{j\in\mathcal{J}^f\}}\left(b_1^{g(j)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} b_2^{g(j)+}(t)\right) \\
& - \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j}\left(b_1^{g(j)g(k)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} b_2^{g(j)g(k)+}(t)\right)\mu^*_{jk}(t) \\
& - \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j}\left(V_1^{*g(k)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} V_2^{*g(k)+}(t)\right)\mu^*_{jk}(t), \\
\bar{\lambda}^{jk}_{0,X}(t) = {} & \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\left(V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} V_2^{*k+}(t)\right) - \mathbb{1}_{\{j=0,k=J+1\}}\tilde{f}(t)V_1^{*0-}(t), \\
\bar{\lambda}^{jk}_{1,X}(t) = {} & \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\left(\frac{V_2^{*k+}(t)}{V_2^{*j+}(t-)} - 1\right) + \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}}\left(\frac{V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t-)} - 1\right) \\
& + \mathbb{1}_{\{j=0,k=J+1\}}\left(\tilde{f}(t) - 1\right), \\
\bar{\gamma}^{jk}_{0,X}(t) = {} & \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}}\left(V_1^{*g(k)+}(t) - \frac{V_1^{*g(j)+}(t-)}{V_2^{*g(j)+}(t-)} V_2^{*g(k)+}(t)\right).
\end{aligned}
$$

$$
\begin{aligned}
\bar{\alpha}^j_{0,Y}(t) = & -\delta^j_0(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\neq J+1}} \Bigg( b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)} b^{jk}_2(t) \\
& \qquad\qquad + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)} V^{*k+}_2(t) \Bigg) \mu^*_{jk}(t), \\
\bar{\alpha}^j_{1,Y}(t) = & \, r(t) - r^*(t) - \delta^j_1(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\neq J+1}} \left( \frac{b^{jk}_2(t)}{V^{*j+}_2(t)} + \frac{V^{*k+}_2(t)}{V^{*j+}_2(t)} - 1 \right) \mu^*_{jk}(t) \\
& + \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j} \left( \frac{b^{g(j)g(k)+}_2(t)}{V^{*g(j)+}_2(t)} + \frac{V^{*g(k)+}_2(t)}{V^{*g(j)+}_2(t)} - 1 \right) \mu^*_{g(j)g(k)}(t), \\
\bar{\alpha}^j_{2,Y}(t) = & \, r(t) - \delta^j_2(t), \\
\bar{\beta}^j_{0,Y}(t) = & \, \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j} \left( b^{g(j)g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)} b^{g(j)g(k)+}_2(t) \right) \mu^*_{g(j)g(k)}(t) \\
& + \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j} \left( V^{*g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)} V^{*g(k)+}_2(t) \right) \mu^*_{g(j)g(k)}(t), \\
\bar{\lambda}^{jk}_{0,Y}(t) = & -\mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}} \left( b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)} b^{jk}_2(t) + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)} V^{*k+}_2(t) \right) \\
& + \mathbb{1}_{\{j=0,k=J+1\}} \tilde{f}(t) V^{*0-}_1(t), \\
\bar{\lambda}^{jk}_{1,Y}(t) = & -\mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}} \left( \frac{b^{jk}_2(t)}{V^{*j+}_2(t)} + \frac{V^{*k+}_2(t)}{V^{*j+}_2(t)} - 1 \right) - \mathbb{1}_{\{j=0,k=J+1\}} \left( \tilde{f}(t) - 1 \right) \\
& - \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}} \left( \frac{b^{g(j)g(k)+}_2(t)}{V^{*g(j)+}_2(t)} + \frac{V^{*g(k)+}_2(t)}{V^{*g(j)+}_2(t)} - 1 \right), \\
\bar{\gamma}^{jk}_{0,Y}(t) = & -\mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}} \left( b^{g(j)g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)} b^{g(j)g(k)+}_2(t) \right) \\
& - \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}} \left( V^{*g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)} V^{*g(k)+}_2(t) \right).
\end{aligned}
$$

## 2.D Proof of Theorem 2.5.4

We only present the proof of the differential equation for $\tilde{X}^j$, since the differential equation for $\tilde{Y}^j$ is obtained using the same calculations. All calculations are conditioned on the interest rate filtration $\mathcal{F}^r_t$.

Due to the result in Theorem 2.3.3, it suffices to prove the result for

$$\bar{\alpha}^j_{0,X} = \bar{\alpha}^j_{1,X} = \bar{\alpha}^j_{2,X} = \bar{\lambda}^{jk}_{0,X} = \bar{\lambda}^{jk}_{1,X} = 0,$$

for all $j, k$, $j \neq k$.

We consider the integral equation for $\tilde{X}^j(t)$

$$
\begin{aligned}
\tilde{X}^j(t) =& p_{Z(0)j}(0,t)X(0) \\
&+ \int_0^t \sum_{g \in \mathcal{J} \cup \mathcal{J}^f} \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(s-)=g\}} \mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}} \mathrm{d}X(s) \bigm| Z(s-)=g\big]\Big].
\end{aligned}
$$

We calculate $\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}} \mathrm{d}X(s) \bigm| Z(s-)=g\big]$ for both terms in the dynamics of $X(t)$ from Lemma 2.5.3.

$$
\begin{aligned}
&\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}} \tilde{f}(s-U(s-))\bar{\beta}_{0,X}^{Z(s-)}(s) \bigm| Z(s-)=g\big] \\
&= \mathbb{1}_{\{g \in \mathcal{J}^f\}} \bar{\beta}_{0,X}^{g}(s) p_{gj}(s,t) \mathbb{E}_{Z(0)}\big[\tilde{f}(s-U(s-)) \bigm| Z(s-)=g, Z(t)=j\big] \\
&= \mathbb{1}_{\{g \in \mathcal{J}^f\}} \bar{\beta}_{0,X}^{g}(s) p_{gj}(s,t) \mathbb{E}_{Z(0)}\big[\tilde{f}(s-U(s-)) \bigm| Z(s-)=g\big] \\
&= \mathbb{1}_{\{g \in \mathcal{J}^f\}} \bar{\beta}_{0,X}^{g}(s) p_{gj}(s,t) \frac{p^{\tilde{f}}_{Z(0)g}(0,s)}{p_{Z(0)g}(0,s)},
\end{aligned}
$$

where we use that $\bar{\beta}_{0,X}^{Z(s-)}(s) = 0$ for $Z(s-) \in \mathcal{J}$ and that $U(s-)|Z(s-)=g \perp\!\!\!\perp Z(t) = j$ for $g \in \mathcal{J}^f$ and $s \leq t$. The $\tilde{f}$-modified probabilities, $p^{\tilde{f}}_{Z(0)g}(0,s)$, are defined as

$$
p^{\tilde{f}}_{Z(0)g}(0,s) = \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(s)=g\}} \tilde{f}(s-U(s))^{\mathbb{1}_{\{g \in \mathcal{J}^f\}}}\Big],
$$

for $Z(0) \in \mathcal{J}$ and $s \geq 0$.

$$
\begin{aligned}
&\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}} \tilde{f}(s-U(s-))\bar{\gamma}_{0,X}^{Z(s-)k}(s) \mathrm{d}N^k(s) \bigm| Z(s-)=g\big] \\
&= \mathbb{1}_{\{g \in \mathcal{J}^f\}} \bar{\gamma}_{0,X}^{gk}(s) \frac{p^{\tilde{f}}_{Z(0)g}(0,s)}{p_{Z(0)g}(0,s)} \mu_{gk}(s) p_{kj}(s,t) \mathrm{d}s,
\end{aligned}
$$

where we use that $\bar{\gamma}_{0,X}^{Z(s-)k}(s) = 0$ for $Z(s-) \in \mathcal{J}$ and $U(s-)|Z(s-)=g \perp\!\!\!\perp Z(t) = j$ for $g \in \mathcal{J}^f$ and $s \leq t$ and that

$$
\begin{aligned}
\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}} \mathrm{d}N^k(s) \bigm| Z(s-)=g\big] =& \\
& p_{gj}(s,t) \mathbb{E}_{Z(0)}\big[\mathrm{d}N^k(s) \bigm| Z(s-)=g, Z(t)=j\big] \\
=& \mu_{gk}(s) p_{kj}(s,t) \mathrm{d}s,
\end{aligned}
$$

see Norberg (1991) Equation (4.12).

Inserting in the integral equation for $\tilde{X}^j(t)$

$$
\begin{aligned}
\tilde{X}^j(t) = p_{Z(0)j}(0,t)X(0) + \sum_{g \in \mathcal{J}^f} \int_0^t \Big( & p^{\tilde{f}}_{Z(0)g}(0,s) \Big( p_{gj}(s,t) \bar{\beta}_{0,X}^{g}(s) \\
& + \sum_{k: k \neq g} \mu_{gk}(s) p_{kj}(s,t) \bar{\gamma}_{0,X}^{gk}(s) \Big) \Big) \mathrm{d}s
\end{aligned}
$$

We use Leibniz's rule to differentiate $\tilde{X}^j(t)$ and use that $p_{lk}(t,t) = \mathbb{1}_{\{l=k\}}$ for $l, k \in \mathcal{J} \cup \mathcal{J}^f$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j(t) = {} & \frac{\mathrm{d}}{\mathrm{d}t} p_{Z(0)j}(0,t)X(0) \\
& + \mathbb{1}_{\{j \in \mathcal{J}^f\}} \bar{\beta}^j_{0,X}(t) p^{\tilde{f}}_{Z(0)j}(0,t) + \sum_{k: k \neq j} \mathbb{1}_{\{k \in \mathcal{J}^f\}} \mu_{kj}(t) \bar{\gamma}^{kj}_{0,X}(t) p^{\tilde{f}}_{Z(0)k}(0,t) \\
& + \sum_{g \in \mathcal{J}^f} \int_0^t \bigg( p^{\tilde{f}}_{Z(0)g}(0,s) \Big( \frac{\mathrm{d}}{\mathrm{d}t} p_{gj}(s,t) \bar{\beta}^g_{0,X}(s) \\
& \qquad + \sum_{k: k \neq g} \mu_{gk}(s) \frac{\mathrm{d}}{\mathrm{d}t} p_{kj}(s,t) \bar{\gamma}^{gk}_{0,X}(s) \Big) \bigg) \mathrm{d}s.
\end{aligned}$$

Kolmogorov's forward differential equations for the transition probabilities gives the result. □

# Chapter 3

# Reserve-dependent Management Actions in life insurance


Abstract

In a set-up of with-profit life insurance including bonus, we study the calculation of the market reserve, where Management Actions such as investment strategies and bonus allocation strategies depend on the reserve itself. Since the amount of future bonus depends on the retrospective savings account, the introduction of Management Actions that depend on the prospective market reserve results in an entanglement of retrospective and prospective reserves. We study the complications that arise due to the interdependence between retrospective and prospective reserves, and characterize the market reserve by a partial differential equation (PDE). We reduce the dimension of the PDE in the case of linearity, and furthermore, we suggest an approximation of the market reserve based on the forward rate. The quality of the approximation is studied in a numerical example.



**Keywords:** With-profit life insurance; Bonus; Prospective reserves; Management Actions


## 3.1 Introduction

In this paper, we study the calculation of the market reserve of a with-profit life insurance contract in a set-up, where the so-called Management Actions have a complex structure. The market reserve is the expected present value of future guaranteed and non-guaranteed payments from the insurer to the insured, and the Management Actions influence the payments of a life insurance contract, for instance, through the investment strategy and the bonus allocation strategy. Especially, the non-guaranteed payments are influenced by future Management Actions. The life insurance company takes many considerations into account when deciding on its Management Actions, and the decisions depend on the financial situation of the company, which is measured by the balance sheet. A fair redistribution of bonus is of great importance in with-profit life insurance, such that the policyholders who contributed to the surplus receive a reasonable amount of bonus. In order to fairly model the future bonus allocation strategy, we need a sophisticated model that takes the entire balance sheet into account. In our model, we allow the future Management Actions to depend on all balance sheet items, and the dependence on the market reserve complicates the set-up.

The modelling of bonus in with-profit life insurance is studied in Norberg (1999), Steffensen (2006b) and Asmussen and Steffensen (2020). We extend the model from Asmussen and Steffensen (2020) to allow for a broader range of investment and bonus allocation strategies, and characterize the prospective market reserve within this model. The core of the model is the surplus that arises due to prudent assumptions about the interest rate and insurance risks on which payments are specified at initialization of the life insurance contract. By legislation, the surplus is to be paid back to the policyholders as bonus. We use the bonus scheme spoken of as additional benefits, where bonus is used to buy more insurance, and therefore, the savings account of the insurance contract is influenced by bonus in terms of dividend payments. This results in a link between the savings account and the guaranteed payments, which is different from the set-up in Steffensen (2006b),where dividends only depend on the surplus, and guaranteed payments are not influenced by dividends. With the introduction of Management Actions that depend on the market reserve, the stochastic differential equation of the retrospective savings account and the retrospective surplus depend on the prospective market reserve. This paper studies the complications that arise due to the interdependence between retrospective and prospective reserves caused by the structure of the Management Actions.The result is a characterization of the market reserve by a partial differential equation (PDE) for a general model of the financial market with methods inspired by Steffensen (2000). We reduce the dimension of the PDE under the assumption of linearity of the dividend strategy with calculations similar to those in Steffensen (2006b), and suggest an approximation of the market reserve based on the forward interest rate. The quality of the approximation is studied in a numerical example.

Christiansen, Denuit, and Dhaene (2014)study reserve-dependence in benefits and costs in a life insurance set-up without bonus, and characterize the prospective market reserve by a Thiele differential equation. The inclusion of bonus in our set-up prevents us from applying the results from Christiansen, Denuit, and Dhaene (2014). The results in this paper combine the modelling of bonus in life insurance from Asmussen and Steffensen (2020) with reserve-dependence from Christiansen, Denuit, and Dhaene (2014). Djehiche and Löfdahl (2016) study nonlinear reserve-dependence in life insurance payments in a set-up without bonus and derive a backward stochastic differential equation (BSDE) of the prospective reserve. Under the Markov assumption, Djehiche and Löfdahl (2016) derive the nonlinear Thiele's equation from the BSDE. We use a similar Markov assumption to derive the PDE of the prospective market reserve in our set-up with bonus. From a simulation point of view, the entanglement of retrospective and prospective reserves is notoriously difficult to handle, and Nyegaard, Ott, and Steffensen (2021) propose a simulation method to disentangle the problem based on intrinsic values. The derivation of the PDE of the prospective market reserve under the assumption of reserve-dependent Management Actions draws parallels to the valuation of contingent claims and option pricing of financial derivatives. Especially the valuation of American options, where the decision to exercise depends on the value of the option itself. Valuation of American options is studied in, for instance, Rogers (2003) and Haugh and Kogan (2004), but our model contains an additional layer of complexity, since the underlying savings account depends on the prospective reserve. Therefore, we cannot use existing valuation methods for American options as presented in for instance Rogers (2003) and Haugh and Kogan (2004) to calculate the prospective reserve in our model. Bruhn and Lollike (2021) and Ahmad, Buchardt, and Furrer (2021) focus on a projection model of the retrospective savings account and surplus in a setting similar to ours, but where the dividend strategy is restricted to depend on the state of the insured, the savings account and the surplus. The inclusion of the prospective market reserve in the specification of the dividend strategy makes projection of the savings account and the surplus with the methods developed in Bruhn and Lollike (2021) and Ahmad, Buchardt, and Furrer (2021) impossible.

The structure of the paper is the following. In Section 4.2, we present the set-up of with-profit life insurance including bonus, introduce a model of the financial market, define the assets and liabilities of the insurance company, and link Management Actions in terms of investments and dividends to the market reserve. Calculation of the market reserve is studied in Section 3.3, where we derive the PDE, and study the case of linearity. A numerical example in Section 3.4 emphasizes the practical applications of our result.

## 3.2 Reserves in life insurance

We introduce the set-up of with-profit life insurance including bonus from Asmussen and Steffensen (2020) in a general financial market. Two decompositions of the liabilities of the insurer are presented, and we link Management Actions in terms of dividends and the investment strategy to the liabilities.

### 3.2.1 Set-up

We consider the classical model of a life insurance contract, as presented in, for instance, Norberg (1991), where a Markov process $Z = \big(Z(t)\big)_{\{t\geq 0\}}$ on a finite state space $\mathcal{J}$

describes the state of the policyholder of a life insurance contract. Payments in the contract link with sojourns in states and transitions between states.

The transition probabilities of $Z$ are given by

$$p_{ij}(s,t) = \mathbb{P}\big(Z(t) = j \mid Z(s) = i\big),$$

for $i, j \in \mathcal{J}$ and $s \leq t$. We assume that the transition intensities

$$\mu_{ij}(t) = \lim_{h \downarrow 0} \frac{1}{h} p_{ij}(t, t+h),$$

exist for $i, j \in \mathcal{J}$, $i \neq j$ and are suitably regular. The process $N^k(t)$ counts the number of jumps of $Z$ into state $k \in \mathcal{J}$ up to and including time $t$

$$N^k(t) = \#\big\{s \in (0,t] \mid Z(s-) \neq k, Z(s) = k\big\},$$

where $Z(s-) = \lim_{h \downarrow 0} Z(s-h)$. Let $\mathcal{F}^Z = \big(\mathcal{F}_t^Z\big)_{t \geq 0}$ be the natural filtration generated by the state process $Z$.

We consider a general financial market, where the insurance company invests in a money market account governed by the interest rate $r$ and $K$ traded assets. The financial market is assumed to be free of arbitrage resulting in the existence of a (not necessarily unique) martingale measure $\mathbb{Q}$. All quantities in the model of the financial market are modelled directly under the martingale measure.

The interest rate is modelled as a diffusion process with dynamics

$$\mathrm{d}r(t) = \alpha_r(t, r(t))\mathrm{d}t + \sigma_r(t, r(t))\mathrm{d}W_r(t), \tag{3.2.1}$$

where $W_r$ is a Brownian motion under the martingale $\mathbb{Q}$, and $\alpha_r : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma_r : [0,\infty) \times \mathbb{R} \to (0,\infty)$ are deterministic and sufficiently regular functions.

The general market consists of a money market account with dynamics

$$\mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t,$$

and $K$ traded assets $S(t) = \big(S_1(t), ..., S_K(t)\big)^T$ with dynamics

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \tilde{\sigma}(t, S(t), r(t))\mathrm{d}W(t), \tag{3.2.2}$$

where $W(t) = (W_1(t), ..., W_M(t))^T$ is a $M$-dimensional Brownian motion under $\mathbb{Q}$ independent of $W_r(t)$, and where

$$\tilde{\sigma}(t,s,r) = \begin{pmatrix} \sigma_{11}(t,s,r) \cdot s_1 & \sigma_{12}(t,s,r) \cdot s_1 & \dots & \sigma_{1M}(t,s,r) \cdot s_1 \\ \sigma_{21}(t,s,r) \cdot s_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{K1}(t,s,r) \cdot s_K & \sigma_{K2}(t,s,r) \cdot s_K & \dots & \sigma_{KM}(t,s,r) \cdot s_K \end{pmatrix},$$

for $s \in \mathbb{R}^K$ and sufficiently regular and deterministic functions $\sigma_{ij} : [0,\infty) \times \mathbb{R}^K \times \mathbb{R} \to (0,\infty)$. The natural filtration generated by the financial market is $\mathcal{F}^S = \left(\mathcal{F}_t^S\right)_{t\geq 0}$, and the combined information about the state process $Z$ and the financial market at time $t$ is given by $\mathcal{F}_t = \mathcal{F}_t^S \cup \mathcal{F}_t^Z$. We assume independence between the state process $Z$ and the financial market. With this specification of the financial market, the interest rate and the traded assets, $\big(r(t), S(t)\big)$, are Markov, and the ideas presented in this paper rely on the Markov property of the financial market. Our results generalize directly to any financial market, that is Markov and independent of the state process $Z$.

Furthermore, we assume the existence of a suitable regular forward interest rates $u \mapsto f(t,u)$ for $t \geq 0$, which satisfies

$$\mathbb{E}^{\mathbb{Q}}\left[ e^{-\int_t^s r(u)\mathrm{d}u} \,\middle|\, \mathcal{F}_t^S \right] = e^{-\int_t^s f(t,u)\mathrm{d}u},$$

and $f(t,t) = r(t)$ for all $0 \leq t \leq s$. The forward interest rate $u \mapsto f(t,u)$ is measurable with respect to $\mathcal{F}_t^S$.

The insurance company invests in an account $G$ that consists of investments in the money market account and in the traded assets. We assume that the proportion of $G$ invested in risky asset $k$ is given by $q_k(t)$. The account $G$ has dynamics

$$\begin{aligned} \mathrm{d}G(t) &= \Big(1 - \sum_{k=1}^K q_k(t)\Big) G(t) \frac{\mathrm{d}S_0(t)}{S_0(t)} + \sum_{k=1}^K q_k(t) G(t) \frac{\mathrm{d}S_k(t)}{S_k(t)} \\ &= r(t)G(t)\mathrm{d}t + G(t)q(t)^T \sigma(t, S(t), r(t))\mathrm{d}W(t), \end{aligned} \tag{3.2.3}$$

where $q(t) = (q_1(t), ..., q_K(t))^T$, and

$$\sigma(t,s,r) = \begin{pmatrix} \sigma_{11}(t,s,r) & \sigma_{12}(t,s,r) & \dots & \sigma_{1M}(t,s,r) \\ \sigma_{21}(t,s,r) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{K1}(t,s,r) & \sigma_{K2}(t,s,r) & \dots & \sigma_{KM}(t,s,r) \end{pmatrix}.$$

### 3.2.2 With-profit life insurance

In with-profit life insurance, payments specified in the insurance contract are based on prudent assumptions about insurance risks and the return in the financial market. We denote these assumptions the first-order (technical) basis. The first-order basis consists of the technical interest rate $r^*$ and the technical transition intensities $\mu_{ij}^*$, $i,j \in \mathcal{J}$, $i \neq j$. Assumptions about the interest rate and transition intensities on the first-order basis are prudent compared to the expectation of the actual development of the market interest rate and transition intensities. The actual future development of the market interest rate and the market transition intensities $\mu_{ij}$ is unknown and needs to be modelled. Throughout, we assume that the market transition intensities are modelled in advance, and consider $\mu_{ij}$ as externally given, which is also practise in, for instance, Danish life insurance industry. The model of the market interest rate is specified in Equation (3.2.1).

Due to the prudent first-order basis, a surplus arises which by product design is to be paid back to the policyholders in terms of bonus. The redistribution of bonus is governed by legislation (in Denmark denoted *Kontributionsbekendtgørelsen*), and life insurance companies have certain degrees of freedom in the redistribution of bonus, which is part of the Management Actions of the company. We use the bonus scheme spoken of as additional benefits where bonus is used to buy more insurance. Inspired by Asmussen and Steffensen (2020) Chapter 6, the payments of the insurance contract consist of two types of payments. The payment stream $B_1$ represents payments not regulated by bonus, and $B_2$ represents the profile of payments regulated by bonus. The payment streams contain benefits less premiums of the insurance contract

$$\mathrm{d}B_i(t) = \mathrm{d}B_i^{Z(t)}(t) + \sum_{k:k\neq Z(t-)} b_i^{Z(t-)k}(t)\mathrm{d}N^k(t),$$

and

$$\mathrm{d}B_i^j(t) = b_i^j(t)\mathrm{d}t + \Delta B_i^j(t)\mathrm{d}\epsilon_n(t),$$

for $j \in \mathcal{J}$ and where $\epsilon_n(t) = \mathbb{1}_{\{t \geq n-\}}$ is the Dirac measure, $b_i^j$ denotes continuous payments during sojourn in state $j$, and $b_i^{jk}$ denotes the single payment upon transition from state $j$ to state $k$. There is a lump sum payment of size $\Delta B_i^j(n-)$ just before the contract terminates at time $n$. Other lump sum payments at fixed time points during sojourn in states are disregarded in this set-up. We assume that the payment functions $b_i^j$, $b_i^{jk}$ and $\Delta B_i^j$ are deterministic and sufficiently regular.

The technical reserve for the payment stream $B_i$ for $i = 1, 2$ in this set-up is the present value of future payments discounted with the technical interest rate

$$V_i^{*Z(t)}(t) = \mathbb{E}^*\left[\int_t^n e^{-\int_t^s r^*(u)\mathrm{d}u}\mathrm{d}B_i(s) \,\middle|\, Z(t)\right],$$

where $\mathbb{E}^*$ implies that we use the first-order transition intensities in the distribution of $Z$. See Asmussen and Steffensen (2020) Chapter 6 Section 4 for the dynamics of $V_i^{*Z(t)}(t)$.

Bonus is distributed from the insurance company to the insured through a dividend payment stream $D$. With the bonus scheme additional benefits, bonus is used to buy more insurance, and we denote by $Q(t)$ the number of payment processes $B_2$ bought up to time $t$. Additional benefits are bought under the technical basis, and as we then use dividends to buy $B_2(t)$ at the price of $V_2^*(t)$, we must have that

$$\mathrm{d}D(t) = V_2^{*Z(t)}(t)\mathrm{d}Q(t).$$

The payment process guaranteed the policyholder at time $t$ is

$$\mathrm{d}B(s) = \mathrm{d}B_1(s) + Q(t)\mathrm{d}B_2(s).$$

### 3.2.3 Assets and liabilities

The assets, $U(t)$, of the insurance company are given by past premiums less benefits accumulated with the capital gains from investing in $G$, which consists of investments in the money market account, $S_0$, and the risky assets, $S$,

$$U(t) = -\int_0^t \frac{G(t)}{G(s)}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big), \tag{3.2.4}$$

under the assumption that $U(0) = 0$.

We consider two decompositions of the liabilities of the insurance company. One decomposition is in the savings account of the policyholder and the surplus. The savings account $X$ of an insurance contract is the technical value of future payments guaranteed at time $t$, i.e.

$$X(t) = V_1^{*Z(t)}(t) + Q(t)V_2^{*Z(t)}(t).$$

The savings account $X(t)$ depends on the process $Q(t)$, which denotes the number of payment processes $B_2$ bought up to time $t$. We can express $Q(t)$ in terms of the savings account and link the payment stream experienced by the policyholder to the savings account

$$\begin{aligned}\mathrm{d}B(t) &= \mathrm{d}B(t, X(t)) \\ &= b^{Z(t)}(t, X(t))\mathrm{d}t + \Delta B^{Z(t-)}(t, X(t-))\mathrm{d}\epsilon_n(t) \\ &\quad + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))\mathrm{d}N^k(t),\end{aligned}$$

where

$$\begin{aligned}b^j(t,x) &= b_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^j(t), \\ \Delta B^j(t,x) &= \Delta B_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} \Delta B_2^j(t), \\ b^{jk}(t,x) &= b_1^{jk}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t).\end{aligned}$$

Note that since $V_i^{*j}(n-) = \Delta B_i^j(n-)$, the lump sum payment at termination of the contract is equal to the savings account, $\Delta B^j(n-, x) = x$.

The surplus $Y$ is the difference between the assets and the savings account

$$Y(t) = U(t) - X(t). \tag{3.2.5}$$

We assume that the proportion of the account $G$ invested in the risky asset $S_k$ can be written in the form

$$q_k(t) = \frac{\tilde{\pi}_k(t)Y(t)}{U(t)},$$

where $\tilde{\pi}_k$ is a sufficiently regular process. The investment strategy of the insurance company is $\tilde{\pi}(t) = (\tilde{\pi}_1(t), ..., \tilde{\pi}_K(t))^T$. Hence, the proportion of $G$ invested in the risky asset $k$ is proportional to the surplus divided by the assets, leading to a larger investment if the surplus is large compared to the savings account.

**Proposition 3.2.1.** *The savings account, $X$, and the surplus, $Y$, have dynamics*

$$\begin{aligned}\mathrm{d}X(t) &= r^*(t)X(t)\mathrm{d}t - \mathrm{d}B(t, X(t)) + \mathrm{d}D(t)\\ &\quad + \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big),\end{aligned}$$

$$\begin{aligned}\mathrm{d}Y(t) &= r(t)Y(t)\mathrm{d}t + Y(t)\tilde{\pi}(t)^T\sigma(t, S(t), r(t))\mathrm{d}W(t) - \mathrm{d}D(t) + c^{Z(t)}(t, X(t))\mathrm{d}t\\ &\quad - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),\end{aligned}$$

*where*

$$\begin{aligned}R^{*jk}(t,x) &= b^{jk}(t,x) + \chi^{jk}(t,x) - x,\\ \chi^{jk}(t,x) &= V_1^{*k}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t),\\ c^j(t,x) &= (r(t) - r^*(t))x + \sum_{k:k\neq j}\big(\mu^*_{jk}(t) - \mu_{jk}(t)\big)R^{*jk}(t,x).\end{aligned}$$

*Proof.* See Asmussen and Steffensen (2020) Chapter 6 Section 7, for the dynamics of the savings account. For the surplus, insert the dynamics of the account $G$ from

Equation (3.2.3) and the dynamics of the savings account

$$
\begin{aligned}
\mathrm{d}Y(t) =& -\mathrm{d}G(t)\int_0^t \frac{1}{G(s)}\big(\mathrm{d}B_1(s)+Q(s)\mathrm{d}B_2(s)\big) - \big(\mathrm{d}B_1(t)+Q(t)\mathrm{d}B_2(t)\big) - \mathrm{d}X(t)\\
=& \, r(t)\Big(-\int_0^t \frac{G(t)}{G(s)}\big(\mathrm{d}B_1(s)+Q(s)\mathrm{d}B_2(s)\big) - X(t)\Big)\mathrm{d}t + r(t)X(t)\mathrm{d}t\\
&+\Big(-\int_0^t \frac{G(t)}{G(s)}\big(\mathrm{d}B_1(s)+Q(s)\mathrm{d}B_2(s)\big)\Big)\frac{\tilde{\pi}(t)^T Y(t)}{U(t)}\sigma(t,S(t),r(t))\mathrm{d}W(t)\\
&- r^*(t)X(t)\mathrm{d}t - \mathrm{d}D(t)\\
&- \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t,X(t-))\big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big)\\
=& \, r(t)Y(t)\mathrm{d}t + Y(t)\tilde{\pi}(t)^T\sigma(t,S(t),r(t))\mathrm{d}W(t) + c^{Z(t)}(t,X(t)\mathrm{d}t - \mathrm{d}D(t)\\
&- \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t,X(t-))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),
\end{aligned}
$$

which completes the proof. □

Based on the principle of equivalence on the technical basis, a natural constraint is that the savings account and the surplus are equal to zero at initialization of the contract i.e. $X(0-)=0$ and $Y(0-)=0$. This assumption implies that the savings account and the surplus are retrospective reserves. Hence, the decomposition of the liabilities into the savings account and the surplus is a decomposition based on retrospective reserves. Another decomposition of the liabilities is based on prospective reserves, and the natural constraint on the prospective reserves is that they are equal to zero at termination of the insurance contract. The prospective reserves are the market value of guaranteed payments, the market value of future bonus payments, also denoted as Future Discretionary Benefits (FDB), and future profits.

Uncertainties in future payments arise from two different types of risk. There is the risk associated with the state of the insured described by the state process $Z$, and the risk from investments in the risky assets. Inspired byAsmussen and Steffensen (2020) Chapter 6 Section 3, we evaluate the risk associated with $Z$ under the physical measure $\mathbb{P}$ due to diversification, and evaluate financial risks under the risk-neutral measure $\mathbb{Q}$ determined by the financial market. Therefore, valuation of future payments is performed under the product measure $\mathbb{P}\otimes\mathbb{Q}$.

The market value of the guaranteed payments, $V^{g,Z(t)}(t)$, is the expected present value of the future payments that are guaranteed the insured at time $t$

$$
V^{g,Z(t)}(t) = \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s)+Q(t)\mathrm{d}B_2(s)\big)\;\Big|\;\mathcal{F}_t\right].
$$

*Remark* 3.2.2. We can express the market value of the guaranteed payments in terms of the savings account

$$
\begin{aligned}
V^{g,Z(t)}(t) &= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\,du}\big(dB_1(s) + Q(t)dB_2(s)\big) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\,du} dB_1(s) \,\middle|\, Z(t), \mathcal{F}_t^S\right] \\
&\quad + Q(t)\cdot\mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\,du} dB_2(s) \,\middle|\, Z(t), \mathcal{F}_t^S\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_t^n e^{-\int_t^s f(t,u)\,du} dB_1(s) \,\middle|\, Z(t),\ r(t)\right] \\
&\quad + Q(t)\cdot\mathbb{E}^{\mathbb{P}}\left[\int_t^n e^{-\int_t^s f(t,u)\,du} dB_2(s) \,\middle|\, Z(t),\ r(t)\right],
\end{aligned}
$$

since $Q(t)$ is measurable with respect to $\mathcal{F}_t$, $Z$ is Markov, and $Q(t)$ is a function of $X(t)$. The forward interest rate can be inserted in the discount factor, since the state process $Z$ and the financial market are independent, such that the market value of the payment streams $dB_1$ and $dB_2$ consists of the valuation of risks associated with $Z$ only and can be performed under $\mathbb{P}$ independent of the financial market.

The market value of the future bonus payments (FDB) is

$$
V^{b,Z(t)}(t) = \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(Q(s) - Q(t)\big)\mathrm{d}B_2(s) \,\middle|\, \mathcal{F}_t\right].
$$

The market reserve is the expected present value under the market basis of future guaranteed and non-guaranteed payments, and therefore it is the sum of the market value of the guaranteed payments and the market value of the future bonus payments

$$
\begin{aligned}
V^{Z(t)}(t) &= V^{g,Z(t)}(t) + V^{b,Z(t)}(t) \\
&= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \,\middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

Future profit is the difference between the assets and the market reserve, $V^{p,Z(t)}(t) = U(t) - V^{Z(t)}(t)$. The market reserve is the expected present value of future payments from the insurance company to the insured, while future profit is the expected present value of payments allotted the insurance company for taking on risks.

Note that in the first decomposition of the liabilities, the sum of the retrospective savings account and surplus is equal to the assets, and in the second decomposition, the sum of the prospective market reserve and future profit is equal to the assets. Hence, $U(t) = X(t) + Y(t) = V^{p,Z(t)}(t) + V^{Z(t)}(t)$.

### 3.2.4 Reserve-dependent dividends and investments

Calculation of the balance sheet items requires a specification of the investment strategy and the dividend payment stream. These are part of the Management Actions of the insurance company, and the determination of the investment strategy and the dividend payment stream holds certain degrees of freedom.

We assume that dividends are allocated continuously such that

$$\mathrm{d}D(t) = \tilde{\delta}(t)\mathrm{d}t,$$

where $\tilde{\delta}$ is the dividend strategy of the insurance company.

When deciding the investment strategy and the dividend allocation strategy, the insurance company considers its financial situation in terms of relations between balance sheet items. Therefore, an attractable model of the Management Actions includes the possibility that dividends and investments depend on all balance sheet items.

We consider a set-up where the investment strategy of the insurance company depends on the savings account, the surplus, the market reserve, the interest rate, and the traded assets

$$\tilde{\pi}_k(t) = \pi_k\big(t, X(t), Y(t), V^{Z(t)}(t), r(t), S(t)\big), \tag{3.2.6}$$

for deterministic and sufficiently regular functions $\pi_k$, $k = 1, ..., K$. In the same way, we allow dividends to depend on the savings account, the surplus, the market reserve, the interest rate, and the traded assets

$$\tilde{\delta}(t) = \delta^{Z(t)}\big(t, X(t), Y(t), V^{Z(t)}(t), r(t), S(t)\big), \tag{3.2.7}$$

for a deterministic and sufficiently regular function $\delta$. Due to the relations between the balance sheet items, this specification of the investment strategy and the dividend strategy above also allow investments and dividends to depend on the assets, $U(t)$, the market value of guaranteed payments, $V^{g,Z(t)}(t)$, the market value of future bonus payments, $V^{b,Z(t)}(t)$, and future profits, $V^{p,Z(t)}(t)$. It is reasonable to assume that the dividend process depends on FDB, since it is likely that the amount of bonus depends on the reserve of future bonus. Hedging of interest rate risks in the market value of guaranteed payments, $V^{g,Z(t)}(t)$, is of great interest of the insurance company, and the general specification of the investment strategy above enables this. To find such an investment strategy, the insurance company must compute the interest rate sensitivity of $V^{g,Z(t)}(t)$ and then choose an investment strategy $\pi_k$ with the same interest rate sensitivity.

With this specification of the investment strategy and the dividend strategy, there is a forward-backward entanglement of the prospective market reserve in the retrospective savings account and surplus, since the investment strategy and the dividend strategy appear in the dynamics from Proposition 3.2.1.

## 3.3 Calculation of the market reserve

The set-up with investments and dividends linked to all balance sheet items is attractable, since the Management Actions of the insurance company may depend on the entire balance sheet. Calculation of the market reserve within this set-up is complicated due to the interdependence between retrospective and prospective reserves. We characterize the market reserve by a PDE and consider the case of linearity that leads to a reduction in the dimension of the PDE.

### 3.3.1 PDE of the market reserve

Informally, $\big(X(t), Y(t), r(t), S(t), Z(t)\big)$ is seen to be Markov with the specification of the investment strategy and the dividend strategy in Equations (3.2.6) and (3.2.7), since the dynamics of the savings account and the surplus depend solely on $\big(X(t), Y(t), r(t), S(t), Z(t)\big)$. Hence, with a slight misuse of notation where $V$ is now a function and not a stochastic process, we write the market reserve as

$$\begin{aligned} &V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\ &\quad = \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[ \int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \,\middle|\, X(t), Y(t), r(t), S(t), Z(t) \right]. \end{aligned} \tag{3.3.1}$$

Since the savings account and the surplus depend on the stochastic interest rate and the traded assets, the market reserve also depends on $r(t)$ and $S(t)$.

**Proposition 3.3.1.** *Assume that $V^j(t, x, y, r, s)$ is sufficiently differentiable. Then the market reserve satisfies the following PDE*

$$\begin{aligned} \frac{\partial}{\partial t}V^j(t, x, y, r, s) &= rV^j(t, x, y, r, s) - b^j(t, x) - \sum_{k:k\neq j} R^{jk}(t, x, y, r, s)\mu_{jk}(t) \\ &\qquad - \mathcal{D}_x V^j(t, x, y, r, s) - \mathcal{D}_y V^j(t, x, y, r, s) - \mathcal{D}_r V^j(t, x, y, r, s) \\ &\qquad - \mathcal{D}_s V^j(t, x, y, r, s), \\ V^j(n-, x, y, r, s) &= x, \end{aligned} \tag{3.3.2}$$

*where*

$$\begin{aligned}
\mathcal{D}_x V^j(t,x,y,r,s) &= \frac{\partial}{\partial x} V^j(t,x,y,r,s)\big(r^*(t)x - b^j(t,x) \\
&\qquad + \delta(t,x,y,V^j(t,x,y,r,s),r,s) - \sum_{k:k\neq j} R^{*jk}(t,x)\mu^*_{jk}(t)\big), \\
\mathcal{D}_y V^j(t,x,y,r,s) &= \frac{\partial}{\partial y} V^j(t,x,y,r,s)\Bigg(ry - \delta(t,x,y,V^j(t,x,y,r,s),r,s) + c^j(t,x) \\
&\qquad + \sum_{k:k\neq j} R^{*jk}(t,x)\mu_{jk}(t)\Bigg) + \frac{1}{2}\frac{\partial^2}{\partial y^2} V^j(t,x,y,r,s) y^2 \sigma_y^2, \\
\mathcal{D}_r V^j(t,x,y,r,s) &= \frac{\partial}{\partial r} V^j(t,x,y,r,s)\alpha_r(t,r) + \frac{1}{2}\frac{\partial^2}{\partial r^2} V^j(t,x,y,r,s)\sigma_r^2(t,r), \\
\mathcal{D}_s V^j(t,x,y,r,s) &= \sum_{k=1}^{K} \frac{\partial}{\partial s_k} V^j(t,x,y,r,s) r s_k \\
&\quad + \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} \frac{\partial^2}{\partial s_k \partial s_l} V^j(t,x,y,r,s) s_k s_l \sum_{m=1}^{M} \sigma_{km}(t,r,s)\sigma_{lm}(t,s,r), \\
R^{jk}(t,x,y,r,s) &= b^{jk}(t,x) + V^k(t,\chi^{jk}(t,x), y - R^{*jk}(t,x), r, s) - V^j(t,x,y,r,s),
\end{aligned}$$

*and*

$$\sigma_y^2 = \pi(t,x,y,V^j(t,x,y,r,s),r,s)^T \sigma(t,s,r)\sigma(t,s,r)^T \pi(t,x,y,V^j(t,x,y,r,s),r,s).$$

*Conversely, if a function $V^j(t,x,y,r,s)$ satisfies the PDE above, it is indeed the market reserve defined in Equation* (3.3.1)*.*

*Proof.* See Appendix 3.A. □

The boundary condition is due to the lump sum payment at time $n-$.

*Remark* 3.3.2. In the Black-Scholes model of the financial market, where the interest rate is constant and deterministic, $r(t) = r \in \mathbb{R}$, and the volatility is constant, $\sigma(t,s,r) = \sigma > 0$, the state-wise market reserve is a function of the savings account and the surplus, and is independent of the traded asset, $S$, given the savings account and the surplus, under the condition that the dividend strategy and the investment strategy do not depend on $S$. The function $V^j(t,x,y)$ satisfies a PDE equal to the PDE in Proposition 3.3.1, but where $\mathcal{D}_r V^j(t,x,y,r,s) = \mathcal{D}_s V^j(t,x,y,r,s) = 0$ and $\sigma_y = \sigma\pi(t,x,y,V^j(t,x,y))$. This result also applies in the Black-Scholes model with a deterministic and time-dependent interest rate $r(t)$.

In order to calculate the market reserve, we must solve the PDE from Proposition 3.3.1 for all values of $j, x, y, r$ and $s$, which is computationally demanding if even possible. One way to reduce the dimension of the PDE is to assume a more specific model for the financial market, which is the case in Remark 3.3.2. Another approach is to study the special case of linearity in the dividend strategy.

### 3.3.2 Linearity

The payment stream, $\mathrm{d}B(t,x)$, and the sum-at-risk, $R^{*jk}(t,x)$, are by construction linear in the savings account. Therefore, the dynamics of the savings account and the surplus from Proposition 3.2.1 are linear in the savings account, the surplus and the market reserve if and only if the investment strategy from Equation (3.2.6) and the dividend strategy from Equation (3.2.7) are linear.

**Proposition 3.3.3.** *Assume that the dividend strategy from Equation* (3.2.7) *is in the form*

$$\delta^j(t,x,y,v,r) = \delta_0^j(t,r) + \delta_1^j(t,r)\cdot x + \delta_2^j(t,r)\cdot y + \delta_3^j(t,r)\cdot v,$$

*for deterministic functions* $\delta_0^j, \delta_1^j, \delta_2^j$ *and* $\delta_3^j$*. Then the market reserve is given by*

$$V^j(t,x,y,r) = h_0^j(t,r) + h_1^j(t,r)\cdot x + h_2^j(t,r)\cdot y, \tag{3.3.3}$$

*where the functions* $h_0$*,* $h_1$*, and* $h_2$ *satisfy the system of PDEs stated in Appendix 3.B.*

*Proof.* Since the function $V^j(t,x,y,r) = h_0^j(t,r) + h_1^j(t,r)\cdot x + h_2^j(t,r)\cdot y$ satisfies the PDE in Proposition 3.3.1, when $h_0^j(t,r), h_1^j(t,r)$ and $h_2^j(t,r)$ satisfy the system of PDEs in Appendix 3.B for all $j \in \mathcal{J}$, Proposition 3.3.1 gives the result. $\square$

It is worth noticing that linearity of the dividend strategy is enough to make sure that the market reserve does not depend on the risky assets, $S$, when dividends are independent of $S$, and therefore the result applies for any choice of investment strategy. Therefore, the insurance company can choose an investment strategy that hedges interest rate risk in the market value of guaranteed payments. The existence of a solution is not certain, but if the system of PDEs has a solution, Proposition 3.3.1 gives that Equation (3.3.3) is in fact the market reserve. The linear structure of the market reserve in Equation (3.3.3) is similar to the results in Steffensen (2006b), where linearity of the surplus in the dividend strategy is inherited in the prospective reserve.

The result in Proposition 3.3.3 reduces the dimension of the PDE of the market reserve compared to the case without linearity. This simplifies the calculation of the market reserve, since it is less computational heavy to solve the system of PDEs for the $h$ functions for all values of $r$, compared to finding the solution to the PDE in Proposition 3.3.1 for all values of $x, y, r$ and $s$.

*Remark* 3.3.4. In the Black-Scholes model of the financial market, still under the assumption of linearity of the dividend strategy, the market reserve has representation

$$V^j(t,x,y) = h_0^j(t) + h_1^j(t)\cdot x + h_2^j(t)\cdot y,$$

where the functions $h_0, h_1$ and $h_2$ satisfy a system of ordinary differential equations (ODEs). Hence, despite the forward-backward entanglement of the market reserve in the savings account and the surplus, the market reserve can be calculated as the solution to a system of backward ODEs in this case. The result also apply in the case, where the interest rate is time-dependent and deterministic.

From a computational point of view, it is demanding to solve PDEs, and therefore it is a desirable result that a combination of linearity of the dividend strategy and the Black-Scholes model of the financial market, reduces the dimension of the PDE from Proposition 3.3.1 in such a way that we are able to calculate the market reserve as the solution to a system of ODEs. The ODEs in Remark 3.3.4 fit into the class of Riccati equations. It is not certain that Riccati equations have solutions, but if a solution exists it is relatively easy to solve the system of ODEs numerically. The existence of solutions highly depends on the choice of the dividend strategy. With the choice in Example 3.3.5 below, we actually have an analytical solution.

**Example 3.3.5.** When dividends are equal to the surplus contribution from Proposition 3.2.1

$$\delta^j(t,x,y,v) = c^j(t,x),$$

the dividends are linear in the savings account and the market reserve is given by

$$V^j(t,x,y,r) = x,$$

since the functions $h_0^j(t,r) = h_2^j(t,r) = 0$ and $h_1^j(t,r) = 1$ solve the PDEs from Appendix 3.B for all $j \in \mathcal{J}$. Hence, the market reserve is equal to the savings account. In this case, the technical basis become redundant since the surplus that arise due to the prudent technical basis is immediately distributed as dividends to the savings account. In this case, the surplus, $Y(t)$, is equal to zero, and the same holds for future profits.

For the majority of dividend strategies, an analytical expression for the market reserve is difficult to obtain, and the market reserve must be calculated numerically.

### 3.3.3 Approximation of the market reserve

In general, it is computationally more demanding to solve PDEs compared to solving ODEs by numerical methods, and there exist more precise methods for solving ODEs.

Under the assumption of linearity in the dividend strategy, we are able to calculate the market reserve as the solution to a system of backwards PDEs by Proposition 3.3.3. In a Black-Scholes model of the financial market, we actually obtain a system of ODEs by Remark 3.3.4.

It may be desirable to lose some accuracy in order to decrease computation time by making approximations that result in ODEs instead of PDEs. Therefore, we aim to approximate the model with a stochastic interest rate in Equation (3.2.1) by a Black-Scholes model. To do this, we replace the stochastic interest rate with the deterministic forward interest rate. Due to linearity in the dividend strategy, the market reserve does not depend on the risky assets, $S$, by Proposition 3.3.3, and therefore we only approximate the stochastic interest rate. When calculating the market reserve, this corresponds to approximating the solution of the PDEs for the $h$ functions from Proposition 3.3.3 by the solution to a system of ODEs based on the forward interest rate. We consider the approximation

$$\begin{aligned} r(t) &\approx f(0,t), \\ h_i^j(t,r) &\approx \tilde{h}_i^j(t), \end{aligned}$$

for $i = 0, 1, 2$ and $j \in \mathcal{J}$. The functions $\tilde{h}_i^j$ satisfy the system of ODEs given by the equations in Appendix 3.B, where $h_i^j(t,r)$ is replaced by $\tilde{h}_i^j(t)$, $r$ is replaced by $f(0,t)$, and it is noted that $\frac{\partial}{\partial r}\tilde{h}_i^j(t) = 0$.

In a set-up without bonus, the market reserve is the expected present value of future payments discounted by the forward interest rate. Therefore, we consider the forward interest rate an appropriate approximation of the stochastic interest rate. Due to linearity of the dividend strategy, calculation of the market reserve does not depend on the investment strategy, and therefore the quality of the approximation does not depend on the choice of investment strategy. When we approximate the interest rate with the forward interest rate, the quality of the investment strategy decreases (for instance the investment strategy, where the insurance company hedges interest rate risk in the market value of guaranteed payments), but the examination of this is out of the scope of this paper, since our focus is the calculation of the market reserve. We investigate the quality of the approximation with the forward interest rate in a numerical example.

## 3.4 Numerical study

In this section, we emphasize the practical applications of our results in a numerical example. Within a survival model with a stochastic interest rate and linearity in the dividend strategy, we solve the PDEs in Proposition 3.3.3 and compare the resulting market reserve with the solution of the ODEs obtained by approximating with the forward interest rate as described in Section 3.3.3.

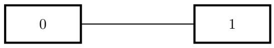


**Figure 3.1:** *Survival model in the numerical example*

**Table 3.1:** *Components in the numerical example*

| **Component** | **Value** |
|---|---|
| Age of policyholder, $a_0$ | 65 |
| Termination, $n$ | 45 |
| Premium | 15.22021 |
| Annuity, $b_2^0(t)$ | 1 |
| $Z(0)$ | 0 |
| $\mu_{01}^*(t)$ | $0.0005 + 10^{5.6+0.04(t+a_0)-10}$ |
| $\mu_{01}(t)$ | $1.1 \cdot \mu_{01}^*(t)$ |
| $r^*(t)$ | 0.01 |
| $r(0)$ | 0.05 |
| $\phi$ | 0.008127 |
| $\psi$ | -0.162953 |
| $\theta$ | 0.000237 |

The survival model is illustrated in Figure 3.1, where state 0 corresponds to alive and state 1 corresponds to dead. We consider an insured male at age $a_0$ at initialization of the insurance contract, and the insurance contract consists of a life annuity regulated by bonus, which is paid by a single premium of $V_2^{*0}(0)$ at time 0. Then the savings account at time 0 is equal to the single premium, $X(0) = V_2^{*0}(0)$, and $\mathrm{d}B_1 = 0$, since all payments are regulated by bonus. The payment process is

$$\mathrm{d}B(t, X(t)) = \mathbb{1}_{\{Z(t)=0\}} \frac{X(t)}{V_2^{*0}(t)} b_2^0(t)\mathrm{d}t.$$

We assume the interest rate from Equation (3.2.1) follows a Vasicek model with dynamics

$$\mathrm{d}r(t) = (\phi + \psi\ r(t))\ \mathrm{d}t + \sqrt{\theta}\ \mathrm{d}W(t), \tag{3.4.1}$$

where $\big(W(t)\big)_{\{t\geq 0\}}$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. Let $u \mapsto f(t,u)$ be the forward interest rate calculated at time $t \geq 0$.

The components in this example are stated in Table 5.1. The parameters in the interest rate model are inspired by Falden and Nyegaard (2021), and the technical mortality rate is the same as in Bruhn and Lollike (2021). The market mortality rate in this example is chosen to be $1.1 \cdot \mu^*(t)$, such that the technical basis is prudent compared to the market basis. The premium is determined according to the principle of equivalence.

There are only dividends in state 0, since upon death all payments cancel. We assume the dividend process is in the form

$$\begin{aligned}\delta^0(t) =& \lambda_1(t)c^0(t,X(t)) + \lambda_2(t)V^{b,0}(t)\\ =& \lambda_1(t)X(t)\Big(r(t) - r^*(t) + \mu_{01}(t) - \mu_{01}^*(t)\Big)\\ &+ \lambda_2(t)\Big(V^0(t,X(t),Y(t)) - \frac{X(t)}{V_2^{*0}(t)}V_2^{g,0}(t,r(t))\Big),\end{aligned}$$

where $V_2^{g,0}(t,r) = \mathbb{E}^{\mathbb{P}}\left[\int_t^n e^{-\int_t^s f(t,u)\mathrm{d}u}\mathrm{d}B_2(s) \;\middle|\; Z(t), r(t)\right]$ and $c^0$ is the surplus contribution from Proposition 3.2.1.

The case where $\lambda_1(t) = 1$ and $\lambda_2(t) = 0$ corresponds to Example 3.3.5, and in this case the market reserve is equal to the savings account. It is reasonable to assume that $\lambda_1(t) \in (0,1)$, since a part of the surplus contribution is then immediately distributed as dividends. We let $\lambda_1(t) = 0.5$ and $\lambda_2(t) = 0.05$, hence half of the surplus contribution and 5 % of FDB are distributed as bonus.

In this example, the PDEs from Proposition 3.3.3 result in $h_0^0(t,r) = 0$, $h_2^0(t,r) = 0$ and

$$\begin{aligned}\frac{\partial}{\partial t}h_1^0(t,r) =& -h_1^0(t,r)^2\lambda_2(t) - \frac{b_2^0(t)}{V_2^{*0}(t)}\\ &+ h_1^0(t,r)\bigg((1-\lambda_1(t))\Big(r - r^*(t) + \mu_{01}(t) - \mu_{01}^*(t)\Big)\\ &\qquad + \frac{b_2^0(t)}{V_2^{*0}(t)} + \frac{\lambda_2(t)V_2^{g,0}(t,r)}{V_2^{*0}(t)}\bigg)\\ &-(\phi + \psi r)\frac{\partial}{\partial r}h_1^0(t,r) - \frac{\theta}{2}\frac{\partial^2}{\partial r^2}h_1^0(t,r),\\ h_1^0(n,r) =& 1,\end{aligned}$$

for the model with stochastic interest rate, which reduces to an ODE when inserting the forward interest rate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}_1^0(t) =& -\tilde{h}_1^0(t)^2\lambda_2(t) - \frac{b_2^0(t)}{V_2^{*0}(t)}\\ &+ \tilde{h}_1^0(t)\bigg((1-\lambda_1(t))\Big(f(0,t) - r^*(t) + \mu_{01}(t) - \mu_{01}^*(t)\Big)\\ &\qquad + \frac{b_2^0(t)}{V_2^{*0}(t)} + \frac{\lambda_2(t)V_2^{g,0}(t)}{V_2^{*0}(t)}\bigg).\\ \tilde{h}_1^0(n) =& 1.\end{aligned}$$

The PDE for the function $h_1^0$ is solved numerically using the Explicit finite difference method, and the ODE for the function $\tilde{h}_1^0$ is solved numerically using the Runge Kutta forth-order method.

**Table 3.2:** *The market reserve at time zero*

| PDE solution | ODE solution | Relative difference |
|---|---|---|
| 13.16423 | 13.24555 | 0.00618 |

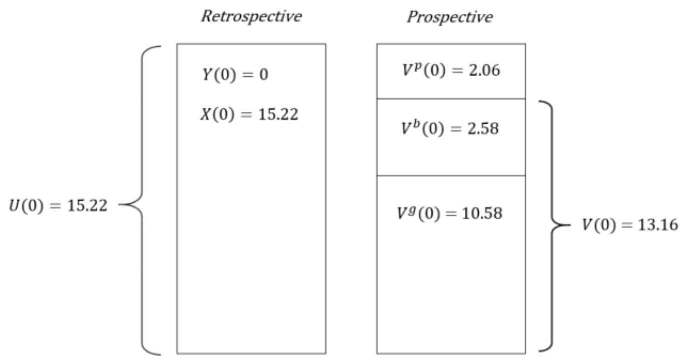


**Figure 3.2:** *The retrospective and the prospective decomposition of the liabilities at time 0 in the numerical example. The market reserve is calculated using the PDE method.*

We calculate the market reserve at time zero by computing the function $h_1^0$ as the solution to the PDE and by solving the ODE for $\tilde{h}_1^0$ based on the deterministic forward interest rate. The results are presented in Table 3.2.

In this example, there is a small difference in the value of the market reserve at time zero. When we approximate using the forward interest rate, the market reserve is larger than in the model with the stochastic interest rate. Hence, the approximation method is conservative from an accounting point-of-view.

The decomposition of the liabilities based on retrospective and prospective reserves, respectively, at time 0 for this example is illustrated in Figure 3.2. The surplus is equal to zero at initialization of the contract, and therefore, the retrospective decomposition only consists of the savings account. The market value of the guaranteed payments constitutes around two thirds of the prospective decomposition, and FDB is almost equal to future profits.

In order to get a better understanding of the difference between the two methods to calculate the market reserve, we compare the function $t \mapsto \tilde{h}_1^0(t)$, which is the solution of the ODE, to the mean, the 2.5%-quantile, and the 97.5%-quantile of the stochastic process $t \mapsto h_1^0(t, r(t))$, since the market reserve is $V^0(t,x,r) = h_1^0(t,r) \cdot x$, and the approximated market reserve is $\tilde{V}^0(t,x) = \tilde{h}_1^0(t) \cdot x$. We compute $\mathbb{E}\left[h_1^0(t,r(t))\right]$ by simulating 1000 interest rate paths, simulated with an Euler scheme based on the dynamics of the interest rate in Equation (3.4.1), interpolate the solution to the PDE

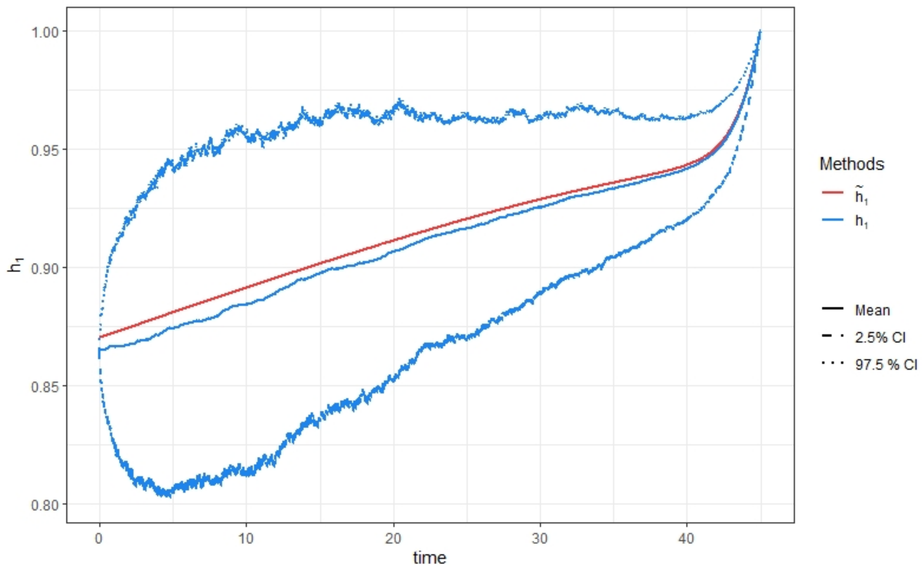


**Figure 3.3:** *Illustration of the functions $\tilde{h}_1^0$ and $h_1^0$. The red is $\tilde{h}_1^0$ based on the forward interest rate, the blue lines are $h_1^0$ based on the simulated interest rate paths, where the mean is the solid line, and the 2.5%-quantile and the 97.5%-quantile are the dashed line.*

of $h_1^0$ over $r$, consider the function for each simulated interest rate path and calculate the empirical mean.

In this example, the market reserve is decreasing since benefits are paid out immediately after the premium payment at time 0. The market reserve at time $t$ is $V(t, X(t)) = h_1^0(t, r) \cdot X(t)$, and therefore the development of the $h_1^0$ functions in Figure 3.3 does not have a one-to-one correspondence with the development of the market reserve. Based on the values of $\mathbb{E}[h_1(t, r(t))]$ and $\tilde{h}_1^0(t)$ in Figure 3.3, the development of the market reserve is similar to the development of the savings account, which is also a decreasing process in this example. When the contract terminates, the market reserve equals zero since there are no future payments. The payment of $X(n-)$ at termination of the insurance contract results in the boundary conditions $h_1^0(n-, r) = \tilde{h}_1^0(n-) = 1$, and is consistence with $V(n, X(n)) = 0$, since

$$X(n) = Q(n) V_2^{*0}(n) = 0.$$

The approximation $\tilde{h}_1^0(t)$ is in general larger than, but close to $\mathbb{E}\left[h_1^0(t, r(t))\right]$. Therefore based on this example, we consider the approximation with the forward interest rate reasonable, since $\tilde{h}_1^0$ is close to the estimated mean and within the 95% confidence interval of $h_1^0(t, r(t))$. The computation time for solving the ODE is significantly lower than for solving the PDE, and therefore the approximation is useful if one can accept the relative difference.

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## 3.A Proof of Proposition 3.3.1

Construct a martingale $m$ as

$$\begin{aligned}
m(t) &= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[ \int_0^n e^{-\int_0^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \,\middle|\, \mathcal{F}_t \right] \\
&= \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \\
&\quad + e^{-\int_0^t r(u)\mathrm{d}u} V^{Z(t)}(t, X(t), Y(t), r(t), S(t)).
\end{aligned}$$

The dynamics of $m$ are

$$\begin{aligned}
\mathrm{d}m(t) = e^{-\int_0^t r(u)\mathrm{d}u}\Big(&\mathrm{d}B_1(t) + Q(t)\mathrm{d}B_2(t) + r(t)V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&+ \mathrm{d}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\Big).
\end{aligned}$$

By the multidimensional Itô formula, we have the dynamics of the market reserve

$$\begin{aligned}
&\mathrm{d}V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&\quad = \frac{\partial}{\partial t} V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&\qquad + \mathcal{D}_x V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&\qquad + \mathcal{D}_y V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&\qquad + \frac{\partial}{\partial y} V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&\qquad \times Y(t)\pi\big(t, X(t), Y(t), V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\big)^T \\
&\qquad \times \sigma(t, S(t), r(t))\mathrm{d}W(t) \\
&\qquad + \mathcal{D}_r V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&\qquad + \frac{\partial}{\partial r} V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\sigma_r(t, r(t))\mathrm{d}W_r(t) \\
&\qquad + \mathcal{D}_s V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&\qquad + \sum_{k=1}^{K} \frac{\partial}{\partial s_k} V^{Z(t)}(t, X(t), Y(t), r(t), S(t))S_k(t) \\
&\qquad \times \sum_{m=1}^{M} \sigma_{km}(t, S(t), r(t))\mathrm{d}W_m(t) \\
&\qquad + \sum_{k:k\neq Z(t-)} \Big(V^k(t, \chi^{Z(t-)k}(t, X(t-)), Y(t-) \\
&\qquad\qquad - R^{*Z(t-)k}(t, X(t-)), r(t), S(t)) \\
&\qquad\qquad - V^{Z(t-)}(t, X(t-), Y(t-), r(t), S(t))\Big)\mathrm{d}N^k(t). \qquad (3.A.1)
\end{aligned}$$

Combining this, the dynamics of $m(t)$ are

$$
\begin{aligned}
\mathrm{d}m(t) = e^{-\int_0^t r(u)\mathrm{d}u}\Big(& b^{Z(t)}(t, X(t)) + r(t)V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&+ \frac{\partial}{\partial t}V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&+ \mathcal{D}_x V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&+ \mathcal{D}_y V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&+ \mathcal{D}_r V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&+ \mathcal{D}_s V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&+ \sum_{k:k\neq Z(t)} R^{Z(t)k}(t, X(t), Y(t), r(t), S(t))\mu_{Z(t)k}(t)\Big)\mathrm{d}t \\
&+ e^{-\int_0^t r(u)\mathrm{d}u}\Delta B^{Z(t)}(t, X(t))\mathrm{d}\epsilon_n(t) + e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}M(t),
\end{aligned}
$$

where $M$ is a martingale with dynamics

$$
\begin{aligned}
\mathrm{d}M(t) = & \frac{\partial}{\partial r}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\sigma_r(t, r(t))\mathrm{d}W_r(t) \\
&+ \frac{\partial}{\partial y}V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
&\times Y(t)\pi\big(t, X(t), Y(t), V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\big)^T \\
&\times \sigma(t, S(t), r(t))\mathrm{d}W(t) \\
&+ \sum_{k=1}^{K} \frac{\partial}{\partial s_k}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))S_k(t) \\
&\times \sum_{m=1}^{M} \sigma_{km}(t, S(t), r(t))\mathrm{d}W_m(t) \\
&+ \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t, X(t-), Y(t-), r(t), S(t))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big).
\end{aligned}
$$

Since $e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}M(t)$ also are the dynamics of a martingale and since $m(t)$ is a martingale, the term in front of $\mathrm{d}t$ in the dynamics of $m(t)$ must be equal to zero for all $t, X(t), Y(t), r(t)$, and $S(t)$ which results in the PDE for the market reserve. Due to the lump sum payment at time $n-$, $\Delta B(n-, X(n-)) = X(n-)$, the boundary condition of the PDE is $V^j(n, x, y, r, s) = x$.

Now, assume that a function $\bar{V}^j(t, x, y, r, s)$ satisfies the PDE in Equation (3.3.2). We show that this function is in fact the market reserve in Equation (3.3.1). Consider an investment strategy and dividend strategy given by

$$
\begin{aligned}
\tilde{\pi}_k(t) &= \pi_k\big(t, X(t), Y(t), \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)), r(t), S(t)\big), \\
\mathrm{d}D^{Z(t)}(t) &= \delta\big(t, X(t), Y(t), \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)), r(t), S(t)\big)\mathrm{d}t,
\end{aligned}
$$

for $k = 1, ..., K$.

The multidimensional Itô formula, the dynamics from Equation (5.D.1) with $\bar{V}$ inserted instead of $V$, and the fact that $\bar{V}$ satisfies the PDE in Equation (3.3.2) yield that

$$
\begin{aligned}
&\mathrm{d}\Big(e^{-\int_0^t r(u)\mathrm{d}u}\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))\Big) \\
=& -r(t)\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\
&+ e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
=& e^{-\int_0^t r(u)\mathrm{d}u}\Bigg(\sum_{k:k\neq Z(t-)} \bar{R}^{Z(t-)k}(t, X(t-), Y(t-), r(t), S(t)) \\
&\quad \times \big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big) \\
&\quad - b^{Z(t)}(t, X(t))\mathrm{d}t - \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))\mathrm{d}N^k(t) \\
&\quad + \frac{\partial}{\partial y}\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))Y(t) \\
&\quad \times \pi\big(t, X(t), Y(t), \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))\big)^T\sigma(t, S(t), r(t))\mathrm{d}W(t) \\
&\quad + \frac{\partial}{\partial r}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\sigma_r(t, r(t))\mathrm{d}W_r(t) \\
&\quad + \sum_{k=1}^{K}\frac{\partial}{\partial s_k}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))S_k(t)\sum_{m=1}^{M}\sigma_{km}(t, S(t), r(t))\mathrm{d}W_m(t)\Bigg).
\end{aligned}
$$

Integrating over the interval $[t, n)$ and taking the $\mathbb{P}\otimes\mathbb{Q}$ expectation conditioning on $\mathcal{F}_t$ give that

$$
\begin{aligned}
& e^{-\int_0^n r(u)\mathrm{d}u}\underbrace{\bar{V}^{Z(n-)}(n, X(n-), Y(n-), r(n-), S(n-))}_{=X(n-)} \\
& - e^{-\int_0^t r(u)\mathrm{d}u}\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
=& -\mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\Bigg[\int_t^n e^{-\int_0^s r(u)\mathrm{d}u}\big(b^{Z(s)}(s, X(s))\mathrm{d}s \\
& + \sum_{k:k\neq Z(s-)} b^{Z(s-)k}(s, X(s-))\mathrm{d}N^k(s)\big)\,\Bigg|\,\mathcal{F}_t\Bigg],
\end{aligned}
$$

since the remaining terms in the dynamics of $\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))$ are martingales with respect to the filtration $\mathcal{F}$. Multiplying by $-\exp(-\int_0^t r(u)\mathrm{d}u)$ and including the boundary condition at time $n-$ in the payment stream gives that $\bar{V}^j(t, x, y, r, s)$ is the market reserve.

## 3.B PDEs for $h$-functions

$$
\begin{aligned}
\frac{\partial}{\partial t}h_0^j(t,r) = {} & rh_0^j(t,r) - b_1^j(t) + \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^j(t) - \sum_{k:k\neq j}\mu_{jk}(t) \\
& \times \Big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + h_0^k(t,r) + h_1^k(t,r)\big(V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big) \\
& - \ h_2^k(t,r)\big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big) - h_0^j(t,r)\Big) \\
& - h_1^j(t,r)\Big( - b_1^j(t) + \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^j(t) + \delta_0^j(t) + \delta_3^j(t)h_0^j(t,r) \\
& - \sum_{k:k\neq j}\mu_{jk}^*(t)\big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big)\Big) \\
& - h_2^j(t,r)\Big( - \delta_0^j(t) - \delta_3^j(t)h_0^j(t,r) \\
& + \sum_{k:k\neq j}\mu_{jk}^*(t)\big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big)\Big) \\
& - \frac{\partial}{\partial r}h_0^j(t,r)\alpha_r(t,r) - \frac{1}{2}\frac{\partial^2}{\partial r^2}h_0^j(t,r)\sigma_r^2(t,r), \\
h_0^j(n-,r) = {} & 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}h_1^j(t,r) = {} & rh_1^j(t,r) - \frac{1}{V_2^{*j}(t)}b_2^j(t) - \sum_{k:k\neq j}\mu_{jk}(t)\Bigg(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) \\
& + h_1^k(t,r)\frac{1}{V_2^{*j}(t)}V_2^{*k}(t) \\
& - \ h_2^k(t,r)\big(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - 1\big) - h_1^j(t,r)\Bigg) \\
& - h_1^j(t,r)\Big(r^*(t) - \frac{1}{V_2^{*j}(t)}b_2^j(t) + \delta_1^j(t) + \delta_3^j(t)h_1^j(t,r) \\
& - \sum_{k:k\neq j}\mu_{jk}^*(t)\big(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - 1\big)\Big) \\
& - h_2^j(t,r)\Big( - \delta_1^j(t) - \delta_3^j(t)h_1^j(t,r) + r - r^*(t) \\
& + \sum_{k:k\neq j}\mu_{jk}^*(t)\big(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - 1\big)\Big) \\
& - \frac{\partial}{\partial r}h_1^j(t,r)\alpha_r(t,r) - \frac{1}{2}\frac{\partial^2}{\partial r^2}h_1^j(t,r)\sigma_r^2(t,r), \\
h_1^j(n-,r) = {} & 1,
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial}{\partial t} h_2^j(t,r) = & -\sum_{k:k\neq j} \mu_{jk}(t)\big(h_2^k(t,r) - h_2^j(t,r)\big) - h_1^j(t,r)\big(\delta_2^j(t) + \delta_3^j(t) h_2^j(t,r)\big) \\
& + h_2^j(t,r)\big(\delta_2^j(t) + \delta_3^j(t) h_2^j(t,r)\big) - \frac{\partial}{\partial t} h_2^j(t,r)\alpha_r(t,r) \\
& - \frac{1}{2}\frac{\partial^2}{\partial r^2} h_2^j(t,r)\sigma_r^2(t,r), \\
h_2^j(n-,r) = & \ 0.
\end{aligned}$$

# Chapter 4

# Forward transition rates in multi-state life insurance

Abstract

The concept of forward transition rates is inspired by the concept of the forward interest rate in bond market theory. Although, the forward transition rates are appropriate for the mortality rate in the survival model, the generalisation to multi-state models is non-trivial. Various definitions for forward transition rates in the multi-state model have been proposed with different properties and ambitions. We propose a definition for forward transition rates when the reserve of the insurance contract is decomposed into sojourn payments and transition payments. Furthermore, we discuss the concept of forward transition rates in a doubly stochastic Markov setting linked with a stochastic interest rate.

**Keywords:** Life insurance; Forward rates; Doubly stochastic Markov models; Kolmogorov's forward equations.

## 4.1 Introduction

The classical multi-state life insurance setup was first formalized by Hoem (1969) and further studied by Norberg (1991). In multi-state models, the state of life of the insured is governed by a Markovian jump process, and payments in the life insurance contract are linked with sojourns in states and transitions between states. In order to evaluate a contract, the insurance company computes transition probabilities, which are fairly easy to compute, when the state process is Markovian. The Markovian assumption does not comply with change in longevity as a result of pandemics or treatment of diseases, neither does it account for the financial market's influences on exercising of policyholder options. The doubly stochastic Markov setting resolves this issue, by allowing the transition rates to be stochastic, and depend on macro-demographic conditions in the population. The systematic (undiversifiable) risk life insurance is accounted for in the doubly stochastic Markov model, where the state process is assumed to be Markov, conditionally on the transition rates.

The area of stochastic transition rates in life insurance was initially considered in the survival model, i.e. with two states - alive and dead - and only one possible transition modeled by the mortality rate. The extension of the classical setup to a stochastic mortality rate is considered in for instance Dahl (2004) and Miltersen and Persson (2005). They derived the forward mortality rate inspired by the concept of the forward interest rate in bond market theory. The forward mortality rate is the deterministic transition rate, that allows calculations to be preformed as in the classical life insurance setup by substituting the stochastic mortality rate by the forward mortality rate at a fixed time point. Miltersen and Persson (2005) further addressed the issue of determining a unique forward mortality rate, when the mortality rate and the interest rate are dependent. The challenges arising from dependence between the interest rate and transition rates for multi-state models are considered in Buchardt (2014).

The concept of forward mortality rate does not unproblematically generalize to the more advanced multi-state models as discussed in Norberg (2010). It is not straightforward how to define deterministic forward transition rates that calculate reserves accurate by substituting the stochastic transition rates. Christiansen and Niemeyer (2014), Buchardt (2017) and Buchardt, Furrer, and Steffensen (2019) studied different suggestions for forward transition rates in the multi-state model. Buchardt, Furrer, and Steffensen (2019) established a theoretical framework for discussion and comparison of forward rates definitions, in which they highlighted the pros and cons of the different suggestions by Christiansen and Niemeyer (2014), Buchardt (2017) and their own. The forward transition rates proposed by Buchardt, Furrer, and Steffensen (2019), the so-called forward equation rates, are based on solutions to a system of Kolmogorov forward equations, but as a means of calculation transition rates for given transition probabilities. The forward equation rates can be used to evaluate insurance contracts correctly, if the contract only consists of sojourn payments, but not in general if the contract contains transition payments.

We use the established a theoretical framework from Buchardt, Furrer, and Steffensen (2019), and similar to the forward equation rates, we propose a forward transition rate based on solutions to a system of Kolmogorov forward equations. Our proposal intends to evaluate insurance contracts correctly, if the contract only consists of transition payments. The idea is then to decompose the expected present value of future payments, the prospective reserve, into a reserve based on sojourn payments, which is valuated by the use of forward equation rates, and a reserve based on transition payments, which is valuated by the use of our proposed forward transition rates. Furthermore, we extend the established a theoretical framework from Buchardt, Furrer, and Steffensen (2019) to allow dependency between the interest rate and transition rates. The generalization to include correlation with the interest rate are incorporated by extending the state space.

The structure of the paper is the following. In Section 4.2, we present the setup of the doubly stochastic Markov setting and life insurance model. Different suggestions for forward transition rates in the multi-state model are considered in Section 4.3, including out proposal. Section 4.4 generalize the framework to include dependency with the interest rate.

## 4.2 Reserves in Life Insurance

We introduce the setup of doubly stochastic Markov processes similar to the setup presented in Buchardt, Furrer, and Steffensen (2019) and consider the reserve for a life insurance contract.

### 4.2.1 Setup

Let $(\omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{J} = \{0, 1, .., J\}$ a finite state space, where each state represents a possible state of the life of a policyholder. For each possible transition from state $i \in \mathcal{J}$ to $j \in \mathcal{J}$, $i \neq j$, we consider a stochastic process $(\mu_{ij}(t))_{t\geq 0}$ with non-negative continuous sample paths and $E[\mu_{ij}(t)] < \infty$ for all $t \in [0, \infty)$. This enables the construction of a jump process $Z = (Z(t))_{t\geq 0}$, with values in $\mathcal{J}$ and deterministic initial state $Z(0) \in \mathcal{J}$, which conditionally on $\mu = (\mu_{ij}(t))_{t\geq 0; i,j\in\mathcal{J}}$ is a Markov process with transition rates $\mu$. The process $Z$ is called a doubly stochastic Markov process and describes the state of the policyholder. Payments in the contract link with sojourns in states and transitions between states.

Let

$$
\begin{aligned}
\mathcal{F}_t^Z =& \sigma(Z(s) : 0 \leq s \leq t), \\
\mathcal{F}_t^\mu =& \sigma((\mu_{ij}(s))_{i,j,\in\mathcal{J}} : 0 \leq s \leq t),
\end{aligned}
$$

be the natural filtrations generated by $Z$ and $\mu$ respectively. The transition probabilities of $Z$ conditionally on $\mu$ are given by

$$
P_{ij}^\mu(t,s) = P\left(Z(s) = j \middle| Z(t) = i, \mathcal{F}_\infty^\mu\right),
$$

for $i, j \in \mathcal{J}$, time $t \leq s$ and $\mathcal{F}_\infty^\mu = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t^\mu\right)$, with transition rates

$$
\lim_{h\searrow 0} \frac{1}{h} P_{ij}^\mu(t, t+h) = \mu_{ij}(t),
$$

for $i \neq j$. The counting process associated with $Z$ counts the number of jumps of $Z$ from state $i$ into state $j \in \mathcal{J}$ up to and including time $t$

$$N_{ij}(t) = \#\{s \in (0,t] \mid Z(s-) = i, Z(s) = j\},$$

where $Z(s-) = \lim_{h \downarrow 0} Z(s-h)$, and the process

$$M_{ij}(t) := N_{ij}(t) - \int_0^t \mathbb{1}_{\{Z(s-)=i\}} \mu_{ij}(s) \mathrm{d}s,$$

is a martingale with respect to the filtration $\mathbf{F}^Z \vee \mathcal{F}_\infty^\mu \vee \mathcal{F}_\infty^r$, where $\mathbf{F}^Z = \{\mathcal{F}_t^Z\}_{t \geq 0}$.

The insurance company investing in a money market account, $S_0$, governed by a interest rate, $r$, such that

$$\mathrm{d}S_0(t) = r(t) S_0(t) \mathrm{d}t.$$

We assume the interest rate is stochastic with continuous sample paths. Let

$$\mathcal{F}_t^r = \sigma(r(s) : 0 \leq s \leq t),$$

be the natural filtration generated by $r$. The combined information about the state process $Z$, the transition rates and the financial market at time $t$ be given by $\mathcal{F}_t = \mathcal{F}_t^r \cup \mathcal{F}_t^Z \cup \mathcal{F}_t^\mu$. Initially, we assume independence between the interest rate, $r$, and the transition rates, $\mu$, and therefore also independence between $r$ and the state process, $Z$. In Section 4.4, we consider a setup where $r$ and $\mu$ are not independent. Furthermore, we assume the existence of suitably regular forward interest rates $u \mapsto f(t,u)$ that satisfies

$$\mathbb{E}\left[ e^{-\int_t^s r(u)\mathrm{d}u} \,\middle|\, \mathcal{F}_t^r \right] = e^{-\int_t^s f(t,u)\mathrm{d}u}, \tag{4.2.1}$$

for all $0 \leq t \leq s$ and $f(t,t) = r(t)$. The forward interest rate $u \mapsto f(t,u)$ is measurable with respect to $\mathcal{F}_t^r$.

Note, the conditional transition probabilities satisfy Kolmogorov's forward differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s} P_{jk}^\mu(t,s) = -\sum_{\ell:\ell \neq k} \mu_{k\ell}(s) P_{jk}^\mu(t,s) + \sum_{\ell:\ell \neq k} \mu_{\ell k}(s) P_{j\ell}^\mu(t,s),$$
$$P_{jk}^\mu(t,t) = \mathbb{1}_{\{j=k\}},$$

for $j,k \in \mathcal{J}$. Kolmogorov's forward differential equation above depends solely on $(\mu_{jk}(u))_{u \in [t,s];\ j,k \in \mathcal{J}}$. Therefore,

$$P_{jk}^\mu(t,s) = P\left(Z(s) = k \middle| Z(t) = j, \mathcal{F}_s^\mu\right),$$

is $\sigma(Z(t)) \vee \mathcal{F}_s^\mu$- measurable.

### 4.2.2 Life insurance model

Let the stochastic process $B$ describe the accumulated benefits less premiums of the insurance contract, and let $B$ have dynamics

$$\mathrm{d}B(t) = \sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(t)=i\}} b_i(t)\mathrm{d}t + \sum_{j:j\neq i} \mathbb{1}_{\{Z(t-)=i\}} b_{ij}(t)\mathrm{d}N_{ij}(t),$$

where $b_i(t)$ denotes the payment rate during sojourn in state $i$, and $b_{ij}(t)$ the single payment upon transition from state $i$ to state $j$ at time $t$. We assume that the payment functions $b_i(t)$ and $b_{ij}(t)$ are deterministic and sufficiently regular. Lump sum payments at fixed time points during sojourn in states are disregarded.

Uncertainties in future payments arise from three different types of risk. The unsystematic biometric risk associated with the state of the insured described by the state process $Z$, the systematic biometric risk from the stochastic transition rates, and the systematic financial risk from the stochastic interest rate. The unsystematic risk is diversifiable and is handled by increasing the size of the portfolio. The systematic risk is undiversifiable and affects the entire portfolio no matter the size of the portfolio.

The prospective reserve of the insurance contract is the expected present value of future payments at time $t$, given by

$$\begin{aligned}
V_{Z(t)}(t) =& \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \middle| \mathcal{F}_t\right] \\
=& \int_t^n e^{-\int_t^s f(t,u)\mathrm{d}u} \sum_{j\in\mathcal{J}} \Bigg( b_j(s)\mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}} \middle| Z(t), \mathcal{F}_t^\mu\right] \\
&+ \sum_{k:k\neq j} b_{jk}(s)\mathbb{E}\left[\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s) \middle| Z(t), \mathcal{F}_t^\mu\right]\Bigg)\mathrm{d}s.
\end{aligned}$$

The forward interest rate can be inserted in the discount factor, since $r$ is independent of $\mu$ and $Z$, such that the expectation can be decomposed, and what remains, in order to calculate the market value of the payment stream $B$, is valuation of biometric risks, which is independent of the financial market. It is enough to condition on the information about the present state $Z(t)$, since

$$\mathcal{F}_{(t,\infty)}^{Z,\mu} = \sigma(Z(s) : s > t) \cup \sigma((\mu_{ij}(s))_{i,j,\in\mathcal{J}} : s > t),$$

is independent of $\mathcal{F}_t^Z$ conditional on $\sigma(Z(t)) \vee \mathcal{F}_t^\mu$.

Therefore, we are interested in calculating expectations of the form

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}} \middle| Z(t), \mathcal{F}_t^\mu\right], \\
&\mathbb{E}\left[\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s) \middle| Z(t), \mathcal{F}_t^\mu\right],
\end{aligned}$$

to calculate the reserve of an insurance contract. The insurance company calculates reserves for its entire insurance portfolio, and therefore, it uses generalized computation tools that are applicable for each policy. Inspired by the forward interest rate in Equation (4.2.1), we seek corresponding deterministic functions that replace the stochastic transition rates in calculation of the reserves, in that case, classical computation tools, based on deterministic transition rates, can be used to calculate the reserve, but where deterministic transition rates are replaced by forward transition rates.

## 4.3 Forward transition rates

In the survival model, the forward mortality rate is inspired by the concept of forward interest rate, and is defined similarly. The generalisation from the forward mortality rate, to the forward transition rates in a more general multi-state insurance model is non-trivial. Previous suggestions of forward transition rates in for instance Norberg (2010), Christiansen and Niemeyer (2014) and Buchardt, Furrer, and Steffensen (2019) have different advantages and disadvantages. Furthermore, the extraction and applicability in computation tools are not clear.

In order to calculate the reserves, we are interested in the following quantities

$$P^m_{Z(t)j}(t,s) = \mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}}\middle| Z(t), \mathcal{F}^\mu_t\right], \tag{4.3.1}$$

$$P^m_{Z(t)j}(t,s)m_{jk}(t,s) = \mathbb{E}\left[\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s)\middle| Z(t), \mathcal{F}^\mu_t\right], \tag{4.3.2}$$

which satisfy a system of differential equations similar to Kolmogorov's forward differential equations

$$\frac{\partial}{\partial s} P^m_{Z(t)j}(t,s) = \sum_{k\neq j} P^m_{Z(t)k}(t,s)m_{kj}(t,s) - P^m_{Z(t)j}(t,s)\sum_{k\neq j} m_{jk}(t,s), \tag{4.3.3}$$

for $j \neq Z(t)$, and

$$\sum_{j\in\mathcal{J}} P^m_{Z(t)j}(t,s) = 1,$$
$$P^m_{Z(t)j}(t,t) = \mathbb{1}_{\{Z(t)=j\}},$$

where $s \mapsto m_{kl}(t,s)$ is the $\sigma(Z(t)) \vee \mathcal{F}^\mu_t$-measurable candidate forward transition rate for $k, \ell\mathcal{J}, k \neq \ell$, and the functions $s \mapsto P^m_{Z_t j}(t,s)$ are differentiable and $\sigma(Z(t)) \vee \mathcal{F}^\mu_t$-measurable.

### 4.3.1 Forward transition rates in literature

The marginal forward transition rates, as considered in Christiansen and Niemeyer (2014), resemble forward interest rates and originate from forward mortality. They are given by the non-negative solution $m_{ij}(t,s)$ for $i,j \in \mathcal{J}$ and $s \in (t,n)$ to

$$\mathbb{E}\left[e^{-\int_t^n \mu_{ij}(s)\mathrm{d}s} | \mathcal{F}_t^\mu\right] = e^{-\int_t^n m_{ij}(t,s)\mathrm{d}s}. \tag{4.3.4}$$

They are $\mathcal{F}_t^\mu$- measurable and do not depend on the current state of the insured. Futhermore, do not depend on the structure or state space of the jump process $Z$. Unfortunately, they only satisfy Equations (4.3.1) and (4.3.2) under specific and often unrealistic dependency structures between the transition rates, and are only usable when the transition probabilities can be expressed in the form of Equation (4.3.4).

The state-wise forward transition rates as suggested by Norberg (2010) and studied in Buchardt (2017) is given by

$$m_{jk}(t,s) = \frac{\mathbb{E}\left[\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s) \middle| Z(t), \mathcal{F}_t^\mu\right]}{\mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}} \middle| Z(t), \mathcal{F}_t^\mu\right]},$$

for $j,k \in \mathcal{J}, j \neq k$ and $s \geq t$. By construction, Equations (4.3.1), (4.3.2) and (4.3.3) hold under some minor regularity conditions. The state-wise forward transition rates are $\sigma(Z(t)) \vee \mathcal{F}_t^\mu$-measurable and depend on the current state $Z(t)$, leading to different forward rates for different values of $Z(t)$. This is not an issue when we calculate the reserve, since we condition on the current state, but we need to take precautions when using standard computation tools.

The state-wise forward transition rates consist of the two terms

$$\mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}} \middle| Z(t), \mathcal{F}_t^\mu\right] = \mathbb{E}\left[P^\mu_{Z(t)j}(t,s) \middle| Z(t), \mathcal{F}_t^\mu\right],$$
$$\mathbb{E}\left[\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s) \middle| Z(t), \mathcal{F}_t^\mu\right] = \mathbb{E}\left[P^\mu_{Z(t)j}(t,s)\mu_{jk}(s) \middle| Z(t), \mathcal{F}_t^\mu\right],$$

which is exactly the expectations, we need in order to calculate the reserve. Therefore, if we are able to obtain these expectations, we loose the necessity of forward transition rates. The forward transition rates are only useful in a two-step procedure, where they are first calibrated, and then used in standard computation tools that fits calculations in the classical life insurance setup. The challenge lies in the calibration of these expectations for every $s \in (t,n]$ and the recalculation for different $t$. Under the assumption that the interest rate is zero, the quantities are linked to insurance contracts and should be extracted from the market.

Buchardt, Furrer, and Steffensen (2019) studies forward equation rates as a $\mathcal{F}_t^\mu$-measurable suggestion for forward transition rates, based on a concept of Kolmogorov forward equations by a means of calculating transition rates instead of transition probabilities.

### 4.3.2 Forward equation rates

The forward equation rates as suggested by Buchardt, Furrer, and Steffensen (2019) are the $\mathcal{F}_t^\mu$-measurable solutions, $m_{ij}(t,s)$ for every $i,j \in \mathcal{J}$ and $s \geq t$, to

$$\frac{\partial}{\partial s}\mathcal{P}_{jk}(t,s) = \sum_{\ell \neq k} \mathcal{P}_{j\ell}(t,s) m_{\ell k}(t,s) - \mathcal{P}_{jk}(t,s) \sum_{\ell \neq k} m_{k\ell}(t,s),$$

for $k \neq j$ and where

$$\sum_{k \in \mathcal{J}} \mathcal{P}_{jk}(t,s) = 1,$$
$$\mathcal{P}_{jk}(t,t) = \mathbb{1}_{\{j=k\}},$$

for the auxiliary $\mathcal{F}_t^\mu$-measurable functions

$$\begin{aligned} \mathcal{P}_{jk}(t,s) &= \mathbb{E}[P_{jk}^\mu(t,s)|\mathcal{F}_t^\mu] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Z(s)=k\}}\middle|Z(t)=j \vee \mathcal{F}_s^\mu\right]|\mathcal{F}_t^\mu\right], \end{aligned} \tag{4.3.5}$$

which are assumed to be differentiable. The forward equation rates do not depend on the present state, $Z(t)$, but involve all the transition probabilities, not only the transition probabilities from the current state. The system of equations for the forward equation rates consists of $J(J-1)$ unknowns and $J(J-1)$ equations. It is shown in Buchardt, Furrer, and Steffensen (2019), that if $\mathcal{P}_{jk}(t,\cdot)$ is differentiable for all $j,k \in \mathcal{J}$ and we have a decrement model ($\mathcal{P}_{jk}(t,\cdot) = 0$ for $k < j$, i.e. return to a state is not possible), then forward equations rates $m$ exist and are unique.

Norberg (2010) suggested similar forward transition rates with

$$\mathbb{E}\left[\mathbb{1}_{\{Z(s)=k\}}\middle|Z(t)=j \vee \mathcal{F}_t^\mu\right],$$

instead of $\mathcal{P}_{jk}(t,s)$, where the forward rates are $\sigma(Z(t)) \vee \mathcal{F}_t^\mu$-measurable. This results in non-unique solutions to the equations, since there is more unknown variables than equations.

Equations (4.3.1) and (4.3.3) holds for the forward equation rates, when

$$\mathcal{P}_{jk}(t,\cdot) = P_{Z(t)k}^m(t,\cdot),$$

on $\{Z(t)=j\}$, but Equation (4.3.2) does not in general hold, since $\mu_{jk} = 0$ does not imply $m_{jk}(t,\cdot) = 0$ unless $\mathcal{P}_{jk}(t,\cdot) = 0$, where there is either direct or indirect transition from $j$ to $k$. If the contract only consists of sojourn payments and no payments upon transition ($b_{jk} = 0$ for $j,k \in \mathcal{J}$), then we only require that the forward transition rates satisfy Equations (4.3.1) and (4.3.3).

The challenge lies within calibrating the quantities in Equation (4.3.5), which are linked to insurance contracts with sojourn payments, and should be extracted from the market, under the assumption that the interest rate is zero. We only need the transition probabilities in Equation (4.3.1) from the market, compared to the state-wise forward rates, which also need the transition densities in Equation (4.3.2) linked to transition payments. The forward equation rates are $\mathcal{F}_t^\mu$-measurable and do not depend on the current state $Z(t)$, therefore they fit better into standard computation tools.

### 4.3.3 Forward transition density rates

Inspired by the forward equation rates, we are interested in computing forward transition density rates, which are $\mathcal{F}_t^\mu$-measurable, based on Equations (4.3.2) and (4.3.3) and correctly evaluate contracts consisting of transition payments only.

Define the auxiliary $\mathcal{F}_t^\mu$-measurable function $s \mapsto \tilde{\mathcal{P}}_{jkl}(t,s)$ for $s \geq t$ by

$$\begin{aligned}\tilde{\mathcal{P}}_{ijk}(t,s) &= \mathbb{E}[P_{ij}^\mu(t,s)\mu_{jk}(s)|\mathcal{F}_t^\mu] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}}\mu_{jk}(s)\middle|Z(t)=i \vee \mathcal{F}_s^\mu\right]|\mathcal{F}_t^\mu\right],\end{aligned}$$

where $\tilde{\mathcal{P}}_{ijk}(t,s) = P_{Z(t)j}^m(t,s)m_{jk}(t,s)$ on the set $\{Z(t)=i\}$ for any $i \in \mathcal{J}$ and all $j,k \in \mathcal{J}$, $j \neq k$. Equation (4.3.2) is then satisfied. In order to satisfy Equation (4.3.3), define the forward rates as the $\mathcal{F}_t^\mu$-measurable solutions, $m_{kj}(t,s)$ for every $k,j \in \mathcal{J}$, to

$$\frac{\partial}{\partial s}\frac{\tilde{\mathcal{P}}_{ikj}(t,s)}{m_{kj}(t,s)} = \sum_{\ell \neq k}\tilde{\mathcal{P}}_{i\ell k}(t,s) - \sum_{\ell \neq k}\tilde{\mathcal{P}}_{ik\ell}(t,s),$$

for $k \notin \{i,j\}$ and $m_{kj}(t,s) \neq 0$. This results in $JJ(J-1)$ equations for $J(J-1)$ unknowns.

### 4.3.4 Decomposition of the market reserve

The forward equation rates based on transition probabilities evaluate sojourn payments correctly. We considered forward transition density rates that evaluate transition payments correctly. Consider the decomposition of the payments into sojourn payments and transition payments

$$\begin{aligned}\mathrm{d}B_s(t) &= \sum_{j\in\mathcal{J}}\mathbb{1}_{\{Z(t)=j\}}b_j(t)\mathrm{d}t, \\ \mathrm{d}B_t(t) &= \sum_{j\in\mathcal{J}}\sum_{k:k\neq j}\mathbb{1}_{\{Z(t-)=j\}}b_{jk}(t)\mathrm{d}N_{jk}(t).\end{aligned}$$

The reserve for each of the payment stream is given by

$$V^s_{Z(t)}(t) = \int_t^n e^{-\int_t^s f(t,u)\mathrm{d}u} \sum_{j\in\mathcal{J}} b_j(s)\mathbb{E}\left[\mathbb{1}_{\{Z(s)=j\}}\middle| Z(t), \mathcal{F}^\mu_t\right] \mathrm{d}s,$$

$$V^t_{Z(t)}(t) = \int_t^n e^{-\int_t^s f(t,u)\mathrm{d}u} \sum_{j\in\mathcal{J}} \sum_{k:k\neq j} b_{jk}(s)\mathbb{E}\left[\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s)\middle| Z(t), \mathcal{F}^\mu_t\right] \mathrm{d}s,$$

where $V_{Z(t)}(t) = V^s_{Z(t)}(t) + V^t_{Z(t)}(t)$. Calculate $V^s_{Z(t)}(t)$ by the use of the forward equation rates and calculate $V^t_{Z(t)}(t)$ by the forward transition density rates. In order to calculate $V^s_{Z(t)}(t)$, we need all the forward equation rates in the model, and in order to calculate $V^t_{Z(t)}(t)$, we need all the forward transition density rates. Therefore, this method is not advantageous, since we need to calibrate $2 \times J(J-1)$ forward rates. It is only useful if it fits into standard computation tools.

## 4.4 Dependent interest rate and transition rates

It is reasonable to assume the interest rate and transitions rates are independent, when the states represent biometric states of life of the insured such as alive, dead and disable. However, it is realistic to assume the financial market have an impact on the transition rates that are based on policyholder options, for instance, surrender or free-policy. The market interest rate may make it advantageous to exercise a policyholder option or the financial market causes unemployment, which often results in policyholder options being exercised. Therefore, we are interested in considering dependence between the interest rate and transition rates.

### 4.4.1 Interest rate deflated conditional transition probabilities

The prospective reserve of the insurance contract, when the interest rate and transition rates are dependent, is given by

$$\begin{aligned} V_{Z(t)}(t) =& \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \middle| \mathcal{F}_t\right] \\ =& \int_t^n \sum_{j\in\mathcal{J}} \Bigg( b_j(s)\mathbb{E}\left[e^{-\int_t^s r(u)\mathrm{d}u}\mathbb{1}_{\{Z(s)=j\}}\middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t\right] \\ &+ \sum_{k:k\neq j} b_{jk}(s)\mathbb{E}\left[e^{-\int_t^s r(u)\mathrm{d}u}\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s)\middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t\right]\Bigg) \mathrm{d}s. \end{aligned} \tag{4.4.1}$$

It is enough to condition on the state $Z(t)$, since

$$\mathcal{F}^{Z,\mu,r}_{(t,\infty)} = \sigma(Z(s) : s > t) \cup \sigma((\mu_{ij}(s))_{i,j,\in\mathcal{J}} : s > t) \cup \sigma(r(s) : s > t),$$

is independent of $\mathcal{F}^Z_t$ given $\sigma(Z(t)) \vee \mathcal{F}^\mu_t \vee \mathcal{F}^r_t$ by the markov property.

Therefore, we are interested in calibrating expectations of the form

$$P^{m,f}_{Z(t)j}(t,s) = \mathbb{E}\left[ e^{-\int_t^s r(u)\mathrm{d}u} \mathbb{1}_{\{Z(s)=j\}} \middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t \right], \tag{4.4.2}$$

$$P^{m,f}_{Z(t)j}(t,s) m_{jk}(t,s) = \mathbb{E}\left[ e^{-\int_t^s r(u)\mathrm{d}u} \mathbb{1}_{\{Z(s-)=j\}} \mu_{jk}(s) \middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t \right]. \tag{4.4.3}$$

We do not assume the forward transition rates satisfy a system of differential equations similar to Equation (4.3.3) with $P^{m,f}_{Z(t)j}$ instead of $P^m_{Z(t)j}$, and can no longer assume the sum of the probabilities is one,

$$\sum_{j \in \mathcal{J}} P^{m,f}_{Z(t)j}(t,s) \neq 1,$$

for the interest rate not equal to zero. However, we are able to find a differential equation for the conditional transition probabilities multiplied with the discount factor, and we show that the same quantities arise in a setup, where we extend the state space.

The Tower property implies

$$\begin{aligned}
&\mathbb{E}\left[ e^{-\int_t^s r(u)\mathrm{d}u} \mathbb{1}_{\{Z(s)=j\}} \middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t \right] \\
&= \mathbb{E}\left[ e^{-\int_t^s r(u)\mathrm{d}u} \mathbb{E}\left[ \mathbb{1}_{\{Z(s)=j\}} \middle| \sigma(Z(t)) \vee \mathcal{F}^\mu_\infty \vee \mathcal{F}^r_\infty \right] \middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t \right] \\
&= \mathbb{E}\left[ e^{-\int_t^s r(u)\mathrm{d}u} P^\mu_{Z(t)j}(t,s) \middle| Z(t), \mathcal{F}^\mu_t \vee \mathcal{F}^r_t \right].
\end{aligned}$$

Let $P^{\mu r}_{ij}(t,s) = e^{-\int_t^s r(u)\mathrm{d}u} P^\mu_{ij}(t,s)$ be the interest rate deflated conditional transition probabilities.

**Proposition 4.4.1.** *The interest rate deflated conditional transition probabilities satisfy the following differential equation*

$$\frac{\partial}{\partial s} P^{\mu r}_{jk}(t,s) = \sum_{\ell:\ell \neq k} \mu_{\ell k}(s) P^{\mu r}_{j\ell}(t,s) - \sum_{\ell:\ell \neq k} \mu_{k\ell}(s) P^{\mu r}_{jk}(t,s) - P^{\mu r}_{jk}(t,s) r(s),$$

*where*

$$P^{\mu r}_{jk}(t,t) = \mathbb{1}_{\{j=k\}}.$$

*Proof.* See Appendix 4.A. □

This differential equation is equal to Kolomogorov's forward differential equation for the conditional transition probabilities, where we extend the state space with one absorbing state, which is attainable from every state with the transition rate of the interest rate, $r$, under the assumption that the sample paths of $r$ are non-negative.

Assume the sample paths of the interest rate, $r$, are non-negative. Let $\tilde{Z} = (Z(t))_{t\geq 0}$ be a doubly stochastic Markov process on state space $\tilde{\mathcal{J}} = \mathcal{J} \cup \{J+1\}$ with transition rates

$$\begin{aligned}
\tilde{\mu}_{jk}(t) &= \mu_{jk}(t), && \text{for } j,k \in \mathcal{J}. \\
\tilde{\mu}_{j(J+1)}(t) &= r(t), && \text{for } j \in \mathcal{J}. \\
\tilde{\mu}_{(J+1)k}(t) &= 0, && \text{for } k \in \mathcal{J}.
\end{aligned}$$

**Corollary 4.4.2.** *The transition probabilities of $\tilde{Z}$ conditionally on $\tilde{\mu}$ and $r$*

$$\tilde{P}^{\tilde{\mu}r}_{jk}(t,s) = P\left(\tilde{Z}(s) = k \middle| \tilde{Z}(t) = j, \mathcal{F}^{\tilde{\mu}}_{\infty} \vee \mathcal{F}^{r}_{\infty}\right),$$

*satisfy the following differential equation*

$$\frac{\partial}{\partial s}\tilde{P}^{\tilde{\mu}r}_{jk}(t,s) = \sum_{\ell\in\mathcal{J}:\ell\neq k} \mu_{\ell k}(s)\tilde{P}^{\tilde{\mu}r}_{j\ell}(t,s) - \sum_{\ell\in\mathcal{J}:\ell\neq k} \mu_{k\ell}(s)\tilde{P}^{\tilde{\mu}r}_{jk}(t,s) - \tilde{P}^{\tilde{\mu}r}_{jk}(t,s)r(s),$$

*where*

$$\tilde{P}^{\tilde{\mu}r}_{jk}(t,t) = \mathbb{1}_{\{j=k\}}.$$

*for $j,k \in J$ and $t \leq s$.*

*Proof.* See Appendix 4.B. □

Proposition 4.4.1 and Corollary 4.4.2 state that the interest rate deflated conditional transition probabilities and the conditional transition probabilities of $\tilde{Z}$ satisfy the same system of differential equations, with the same initial conditions. Therefore they are equal

$$\tilde{P}^{\tilde{\mu}r}_{jk}(t,s) = P^{\mu r}_{jk}(t,s),$$

for $j,k \in J$ and $t \leq s$. Furthermore, the prospective reserve with dependent interest rate and transition rates, can be computed by extending the state space, under the assumption that the sample paths of $r$ are non-negative and that all payments to state $J+1$ and sojourn in state $J+1$ are equal to zero.

**Proposition 4.4.3.** *Assume the sample paths of the interest rate, $r$, are non-negative and*

$$b_{j(J+1)}(t) = b_{J+1}(t) = 0,$$

*for $j \in \mathcal{J}$ and $t \geq 0$. The state-wise prospective reserve in Equation* (4.4.1) *is equal to the expected accumulated benefits in the setup of* $(\tilde{Z}, \tilde{\mu})$

$$V_i(t) = \mathbb{E}\left[\int_t^n d\tilde{B}(s) \big| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right],$$

*for $i \in \mathcal{J}$.*

*Proof.* See Appendix 4.C. □

Hence, we are able to handle dependency between transition rates and the interest rate by considering interest rate deflated transition probabilities, or by assuming $r$ have non-negative sample paths and extending the state space.

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## 4.A Proof of Proposition 4.4.1

Define for $s \geq t$

$$X_k(s) = \mathbb{1}_{\{Z(s)=k\}} e^{-\int_t^s r(u)\mathrm{d}u}.$$

Conditional on the filtration $\sigma(Z(t)) \vee \mathcal{F}^\mu_\infty \vee \mathcal{F}^r_\infty$, integration by parts yields the dynamics of $X_k(s)$

$$\mathrm{d}X_k(s) = e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}\mathbb{1}_{\{Z(s)=k\}} - r(s)\mathbb{1}_{\{Z(s)=k\}} e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}s,$$

and the dynamics of the indicator function is

$$\mathrm{d}\mathbb{1}_{\{Z(s)=k\}} = \sum_{\ell:\ell\neq k} \mathrm{d}N_{\ell k}(s) - \mathrm{d}N_{k\ell}(s).$$

Hence the dynamics of $X_k(s)$ conditional on $\sigma(Z(t)) \vee \mathcal{F}^\mu_\infty \vee \mathcal{F}^r_\infty$ is

$$\begin{aligned}
\mathrm{d}X_k(s) &= e^{-\int_t^s r(u)\mathrm{d}u}\Bigg(\sum_{\ell:\ell\neq k} \mathrm{d}M_{\ell k}(s) - \mathrm{d}M_{k\ell}(s) \\
&\qquad + \Big(\mathbb{1}_{\{Z(s)=\ell\}}\mu_{\ell k}(s) - \mathbb{1}_{\{Z(s)=k\}}\mu_{k\ell}(s) - r(s)\mathbb{1}_{\{Z(s)=k\}}\Big)\mathrm{d}s\Bigg) \\
&= e^{-\int_t^s r(u)\mathrm{d}u}\Bigg(\sum_{\ell:\ell\neq k} \mathrm{d}M_{\ell k}(s) - \mathrm{d}M_{k\ell}(s)\Bigg) \\
&\quad + \sum_{\ell:\ell\neq k}\Big(X_\ell(s)\mu_{\ell k}(s) - X_k(s)\mu_{k\ell}(s) - r(s)X_k(s)\Big)\mathrm{d}s,
\end{aligned}$$

since

$$M_{k\ell}(s) := N_{k\ell}(s) - \int_0^s \mathbb{1}_{\{Z(v-)=k\}}\mu_{k\ell}(v)\mathrm{d}v,$$

is a martingale with respect to the filtration $\mathbf{F}^Z \vee \mathcal{F}^\mu_\infty \vee \mathcal{F}^r_\infty$, where $\mathbf{F}^Z = \{\mathcal{F}^Z_t\}_{t\geq 0}$. By definition

$$\mathbb{E}\left[X_k(s)\middle| Z(t) = j \vee \mathcal{F}^\mu_\infty \vee \mathcal{F}^r_\infty\right] = P^{\mu r}_{jk}(t,s).$$

It then follows from Fubini's theorem and the martingale properties of $M_{\ell k}(s)$ for

$\ell, k \in \mathcal{J}$ that

$$
\begin{aligned}
P_{jk}^{\mu r}(t,s) =& P_{jk}^{\mu r}(t,t) + \int_t^s \mathbb{E}\left[\mathrm{d}X_k(v) \middle| Z(t) = j \vee \mathcal{F}_\infty^\mu \vee \mathcal{F}_\infty^r\right] \\
=& P_{jk}^{\mu r}(t,t) \\
&+ \int_t^s \mathbb{E}\Bigg[\sum_{\ell:\ell\neq k}\Big(X_\ell(v)\mu_{\ell k}(v) - X_k(v)\mu_{k\ell}(v) \\
&\qquad - r(v)X_k(v)\Big)\mathrm{d}v \Bigg| Z(t) = j \vee \mathcal{F}_\infty^\mu \vee \mathcal{F}_\infty^r\Bigg] \\
=& P_{jk}^{\mu r}(t,t) + \int_t^s \sum_{\ell:\ell\neq k}\Big(P_{j\ell}^{\mu r}(t,v)\mu_{\ell k}(v) - P_{jk}^{\mu r}(t,v)\mu_{k\ell}(v) \\
&\qquad - r(v)P_{jk}^{\mu r}(t,v)\Big)\mathrm{d}v.
\end{aligned}
$$

Differentiate w.r.t. s to obtain the desired result.

## 4.B Proof of Corollary 4.4.2

The conditional transition probabilities satisfy Kolmogorov's forward differential equation

$$
\frac{\partial}{\partial s}\tilde{P}_{jk}^{\tilde{\mu}r}(t,s) = \sum_{\ell\in\tilde{\mathcal{J}}:\ell\neq k}\tilde{\mu}_{\ell k}(s)\tilde{P}_{j\ell}^{\tilde{\mu}r}(t,s) - \sum_{\ell\in\tilde{\mathcal{J}}:\ell\neq k}\tilde{\mu}_{k\ell}(s)\tilde{P}_{jk}^{\tilde{\mu}r}(t,s),
$$

for $j, k \in \mathcal{J}$, with

$$
\tilde{P}_{jk}^{\tilde{\mu}r}(t,t) = \mathbb{1}_{\{j=k\}}.
$$

We decompose the sums into $\ell \in \mathcal{J}$ and $\ell = J+1$, and use that $\tilde{\mu}_{j(J+1)}(t) = r(t)$ and $\tilde{\mu}_{(J+1)j}(t) = 0$ for $j \in \mathcal{J}$,

$$
\frac{\partial}{\partial s}\tilde{P}_{jk}^{\tilde{\mu}r}(t,s) = \sum_{\ell\in\mathcal{J}:\ell\neq k}\tilde{\mu}_{\ell k}(s)\tilde{P}_{j\ell}^{\tilde{\mu}r}(t,s) - \sum_{\ell\in\mathcal{J}:\ell\neq k}\tilde{\mu}_{k\ell}(s)\tilde{P}_{jk}^{\tilde{\mu}r}(t,s) - r(s)\tilde{P}_{jk}^{\tilde{\mu}r}(t,s).
$$

Since $\tilde{\mu}_{jk}(t) = \mu_{jk}(t)$ for $j, k \in \mathcal{J}$, we obtain the result.

## 4.C Proof of Proposition 4.4.3

The expected accumulated benefits in the setup of $(\tilde{Z}, \tilde{\mu})$ is

$$\begin{aligned}
V_i(t) &= \mathbb{E}\left[\int_t^n \mathrm{d}\tilde{B}(s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \\
&= \int_t^n \sum_{j \in \tilde{\mathcal{J}}} \left( b_j(s) \mathbb{E}\left[\mathbb{1}_{\{\tilde{Z}(s)=j\}} \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\mu} \vee \mathcal{F}_t^r\right] \right. \\
&\quad \left. + \sum_{k \in \tilde{\mathcal{J}}: k \neq j} b_{jk}(s) \mathbb{E}\left[\mathbb{1}_{\{\tilde{Z}(s)=j\}} \mu_{jk}(s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\mu} \vee \mathcal{F}_t^r\right] \right) \mathrm{d}s.
\end{aligned}$$

We can disregard the state $J+1$ in the sums, since $b_{j(J+1)}(t) = b_{J+1}(t) = 0$ and $\mu_{(J+1)j}(t) = 0$ for $t \geq 0$ and $j \in \mathcal{J}$. The Tower property and $\tilde{\mu}_{jk}(t) = \mu_{jk}(t)$ for $j, k \in \mathcal{J}$ gives that

$$\begin{aligned}
V_i(t) &= \int_t^n \sum_{j \in \mathcal{J}} \left( b_j(s) \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\tilde{Z}(s)=j\}} \middle| \tilde{Z}(t) = i \vee \mathcal{F}_\infty^{\tilde{\mu}} \vee \mathcal{F}_\infty^r\right] \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \right. \\
&\quad + \sum_{k \in \mathcal{J}: k \neq j} b_{jk}(s) \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\tilde{Z}(s-)=j\}} \middle| \tilde{Z}(t) = i \vee \mathcal{F}_\infty^{\tilde{\mu}} \vee \mathcal{F}_\infty^r\right] \right. \\
&\qquad \left. \left. \times \tilde{\mu}_{jk}(s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \right) \mathrm{d}s \\
&= \int_t^n \sum_{j \in \mathcal{J}} \left( b_j(s) \mathbb{E}\left[\tilde{P}_{ij}^{\tilde{\mu}r}(t,s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \right. \\
&\quad \left. + \sum_{k \in \mathcal{J}: k \neq j} b_{jk}(s) \mathbb{E}\left[\tilde{P}_{ij}^{\tilde{\mu}r}(t,s) \mu_{jk}(s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \right) \mathrm{d}s.
\end{aligned}$$

By Proposition 4.4.1 and Corollary 4.4.2

$$\tilde{P}_{jk}^{\tilde{\mu}r}(t,s) = P_{jk}^{\mu r}(t,s),$$

for $j, k \in J$ and $t \leq s$, such that

$$\begin{aligned}
V_i(t) &= \int_t^n \sum_{j \in \mathcal{J}} \left( b_j(s) \mathbb{E}\left[P_{Z(t)j}^{\mu,r}(t,s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \right. \\
&\quad \left. + \sum_{k: k \neq j} b_{jk}(s) \mathbb{E}\left[P_{Z(t)j}^{\mu,r}(t,s) \mu_{jk}(s) \middle| \tilde{Z}(t) = i \vee \mathcal{F}_t^{\tilde{\mu}} \vee \mathcal{F}_t^r\right] \right) \mathrm{d}s,
\end{aligned}$$

which is the reserve for $i \in \mathcal{J}$.

# Chapter 5

# Stable dividends are optimal under linear-quadratic optimization

Abstract

Within actuarial risk theory, there is an interest in considering stability criteria for a risky business in the context of stochastic optimal control problems. A desirable criterion is stable dividends (the controls) allotted to the shareholders. Unfortunately, the optimal strategies obtained in the actuarial risk literature, such as the barrier strategy, rarely have this property and are hardly acceptable in practice. Affine dividend strategies receive attention due to the property of being stable. It is well known that affine strategies are optimal in so-called linear-quadratic optimization. This is almost always formalized in diffusion models, whereas we consider when affine dividend strategies are optimal based on a surplus model following a Lévy process. We characterize the objective by the Hamilton-Jacobi-Bellman equations, and compare the objective of LQ optimization and optimal controls to the classical objective of maximizing expected present value of future dividends. In practise it is much more realistic to have stable dividends, and this frameworks opens up a point of view for control problems where stable dividends are optimal.

**Keywords:** Linear-Quadratic; Optimization; Dividends; Stability.

## 5.1 Introduction

The "stability problem" of Bühlmann (1970) in actuarial risk theory refers to stochastic modelling and optimization, whose goal is to inform decision makers about important choices they have to make, when running a risky business, according to certain stability criteria. Historically, main criteria include the probability of ruin as in Asmussen and Albrecher (2010) and the expected present value of dividends as in Albrecher and Thonhauser (2009) and Avanzi (2009). The general endeavour of the stability problem is still a contemporary one and can be seen through the lens of modern Enterprise Risk Management, described in Taylor (2013). Essentially, Enterprise Risk Management is a process by which certain decisions are made (the controls) to achieve certain outcomes (the objective) under certain constraints, where some are hard constraints, and others are chosen such as risk appetite. Such a procedure can be advantageously informed by stylised modelling as the one developed in the prolific risk theoretical literature; see also Cairns (2000, Section 1.4, on the value of simple but tractable models) and Gerber and Loisel (2012, on the value of ruin theory for risk managers, in particular for capital modelling)

Stability criteria are most often linked to the surplus of a company, for instance, the probability of ruin corresponds to the probability of a negative surplus. Historically, risk theoretical surplus models focused on insurance type dynamics where most of the risk is a downside risk with deterministic income and stochastic losses. The classical formulation is the Cramér-Lundberg model, which is a compound Poisson surplus model formalized by Lundberg (1909) and Cramér (1930). Recently, more general risky business types have been considered, for instance, where the stochastic nature is mostly on the upside as gains. The most general formulations are in terms of spectrally negative or positive Lévy processes.

The objective of minimizing the probability of ruin over infinite time implies that the surplus increases without a limit. In order to resolve this issue Finetti (1957) allowed for a surplus leakage to the shareholders of the company, referred to as dividends, and formed stability criteria assessed by the distribution of dividend payouts. The classical objective is to maximize the expected present value of future dividends until the company is ruined. Loeffen (2009) and Yin and Wen (2013) studied optimal dividend problems within the classical objective for spectrally negative Lévy processes. Realistic features in the context of the dividends criterion are increasingly being considered and introduced, insofar as tractability is not lost. Avanzi, Tu, and Wong (2016) developed a list of realistic features one might want to include, in particular based on the corporate finance literature. One important, well-known aspect, is that companies and investors like stable dividends Lintner (1956), Fama and Babiak (1968) and Avanzi, Tu, and Wong (2016). Unfortunately, the optimal strategies obtained in the actuarial risk literature rarely have this property. The typical example of an optimal strategy is the barrier strategy, under which dividend payouts are very irregular, which is hardly acceptable in practice. This was first pointed out by Gerber (1974), but received little attention until recently. In an attempt to address this issue, Avanzi and Wong (2012) introduced a linear dividend strategy, in a diffusion framework, leading to mean reversion and much improved stability. This was generalised to affine dividend strategies by Albrecher and Cani (2017), in a Cramér-Lundberg framework, who also derived a closed-form Laplace transform of the time to ruin. Importantly, both Avanzi and Wong (2012) and Albrecher and Cani (2017) illustrate that affine dividend strategies are very close to the optimal barrier strategies, but they make the process much safer. This latter point is more rigorously explored in Albrecher and Cani (2017) by their theoretical analysis of ruin.

In both Avanzi and Wong (2012) and Albrecher and Cani (2017) the dividend strategy is not the optimal solution form of an optimal control problem. Optimal parameters that maximise the expected present value of dividends are obtained, but the strategy class is specified. However, it is well known that affine strategies are optimal in related contexts such as in Cairns (2000) and Steffensen (2006a), and objectives on this form are referred to as linear-quadratic optimization (LQ optimization). Affine dividend strategies are arguably much more realistic than the usual optimal ones such as barrier strategy. In this paper, we establish a connection between the fields of risk theory and quantitative finance. In order to show that the objective of LQ optimization entails affine optimal strategies, we characterize the value function by the so-called Hamilton-Jacobi-Bellman differential equations, and express the value function as a quadratic function in the surplus. The objective of LQ optimization and optimal controls are relevant to compare to the classical objective of maximizing expected present value of future dividends, and we consider suitable choices for the benchmarks based on the comparison.

The paper is organised as follows. Section 5.2 introduces the surplus process and the LQ objective, we propose in this paper, is analysed and motivated. We derive the Hamilton-Jacobi-Bellman equation and an appropriate verification lemma in Section 5.3, along with an expression that characterize the value function. The LQ objective is compared to the classical objective in Section 5.4, where we also study choices of benchmarks in the LQ problem and the resulting optimal dividend strategy. Numerical illustrations are provided in Section 5.5.

## 5.2 The optimization problem

### 5.2.1 The surplus model

We model the surplus of a company at time $t$ after distribution of dividends by the dynamics

$$dX(t) = c(t)dt + dS(t) - dD(t), \tag{5.2.1}$$

where $c(t)$ is deterministic and represents the predictable modification component of the surplus due to income and expenses, $S(t)$ is stochastic and represents the aggregate random variations of the surplus due to, for instance, losses with $S(0) = 0$, and $D(t)$ is the aggregated net dividends with $D(0) = 0$.

If $c(t)$ is a positive constant and $S(t)$ is a compound Poisson process with negative jumps only, then (5.2.1) has the dynamics of a Cramér-Lundberg process. Conversely, if $c(t)$ is a negative constant and $S(t)$ is a compound Poisson process with positive jumps only, then (5.2.1) has the dynamics of a so-called dual model Mazza and Rullière (2004).

We assume $S(t)$ is the following Lévy process

$$S(t) = \sum_{i=1}^{N(t)} Y_i + \varsigma W(t), \quad N(0) = W(0) = 0,$$

where $(Y_i)_{i \in \mathbb{N}}$ is i.i.d. and $Y_i$ can follow any distribution on $\mathbb{R}$, with finite first two moments,

$$E[Y_i^j] = p_j, \quad j = 1, 2,$$

where $W(t)$ is a Brownian motion, and where $N(t)$ is an inhomogeneous Poisson process with intensity $\lambda(t)$, $t \geq 0$. Such two-sided formulations are rare in the actuarial literature, but they exist; see Cheung (2011) and Labbé, Sendov, and Sendova (2011, for references with negative and positive $c(t)$, respectively) or Cheung, Liu, and Willmot (2018).

The dividend process, $D(t)$, is not strictly increasing. Hence, we allow negative dividends, spoken of as capital injections. Furthermore, dividends and capital injections can be paid continuously or as lump sums upon jumps in $S(t)$, such that the dynamics of $D$ is given by

$$\mathrm{d}D(t) = l(t, X(t))\mathrm{d}t + i(t, X(t-))\mathrm{d}N(t).$$

### 5.2.2 The Linear-Quadratic (LQ) objective

We consider a finite time frame $T \geq 0$ and would like to consider a general objective of the form

$$\begin{aligned} \min E_{t,x} \Big[ & \text{discounted penalties for continuous dividends away from a benchmark} \\ & + \text{discounted penalties for lump sum (discrete) dividends} \\ & + \text{discounted penalties for the wealth process away from a benchmark} \\ & + \text{subject to a constraint on terminal wealth } X(T) \qquad \Big], \end{aligned}$$

where the subscript of the expectation refers to the expectation conditional of $X(t) = x$.

This is operationalised into the following value function,

$$
\begin{aligned}
V(t,x) = \min_{l,i} E_{t,x}\Bigg[ \quad & \tfrac{1}{2}\int_t^T e^{-\delta(s-t)}\Big(l(s,X(s)) - l_0(s) - l_1(s)X(s)\Big)^2 ds \\
+ \quad & \tfrac{1}{2}\int_t^T e^{-\delta(s-t)}\gamma^i(s) i(s,X(s-))^2 dN(s) \\
+ \quad & \tfrac{1}{2}\int_t^T e^{-\delta(s-t)}\Big(X(s) - x_0(s)\Big)^2 d\Gamma(s) \\
+ \quad & \kappa e^{-\delta(T-t)}\left(X(T) - x_T\right)^\tau \quad \Bigg],
\end{aligned} \tag{5.2.2}
$$

for $t \leq T$, where $\delta$ is a financial impatience factor. To get a better understanding of the objective behind this value function, we explain (5.2.2) line by line.

- The first line compares the continuous payout of dividends with an affine benchmark. Dividends are generally not paid continuously, but we use a continuous model that provides a tractable stylised formulation of a discrete real life situation; for comments about this see Cairns (2000). The benchmark, $l_0(s) + l_1(s)X(s)$, consist of two functions, a fixed target, $l_0(s)$, and a target which is proportional to the surplus level, $l_1(s)$. It is reasonable to assume both functions are positive, although it is not technically required.

- The second line accounts for lump sum payments. The lump sum payments are interpreted as extra dividends or capital injections paid on top of the regular dividends. Therefore, the only admissible lump sum payments are upon jumps in the surplus process, where an abrupt change of surplus level due to a jump may require a discrete adjustment of the surplus. The benchmark is zero, since we prefer not to have lump sum dividends, and we introduce a weight function $\gamma^i(s) \geq 0$, $s \geq 0$ to adjust this preference. The undesirable signals of lump sum dividends are discussed in Avanzi, Tu, and Wong (2016) and Avanzi et al. (2017). The squared function means we equally dislike lump sum dividends and capital injections, and that we prefer a series of small dividend payouts to one single large one.

- The last two lines consider the surplus process. The third line compares the surplus with a surplus benchmark. The benchmark, $x_0(s)$, could be a result of regulatory constraints or correspond to the explicit target capitalisation of the company. Companies often set and publish such targets; see, e.g. Australian Actuaries Institute (2016, for insurance companies). It is reasonable to assumes the function $x_0$ is non-negative, to not aim for the company to ruin. In order to balance this objective with the first two lines, the third line contains a mixed aggregate weight function $\Gamma(t) = \int_0^t \gamma(s)ds$, $0 \leq t < T$. It is written using the Riemann-Stietjes notation to allow for a final mass at termination $\Delta\Gamma(T) \geq 0$.

- The last line serves to control the terminal value of the surplus to a benchmark, $x_T$. The parameter $\kappa$ is a Lagrangian multiplier, and the parameter $\tau$ allows for three levels of constraints on the terminal value $X(T)$. The case $\tau = 0$ corresponds to absence of constraint. If $\tau = 1$, the expected value of the terminal value is $x_T$, where $\kappa$ is solved to satisfy this constraint. For $\tau = 2$ the constraint is stronger and the process is forced to reach $x_T$ at time $T$ by letting $\kappa$ go to infinity. See Steffensen (2006a) and Steffensen (2001) for details.

The final weight function mass $\Delta\Gamma(T)$ is not redundant with the last row for $\tau = 2$, since the third row expresses a preference and the fourth row expresses a constraint. Therefore, they are operationalised differently, where the weight at $\Delta\Gamma(T)$ remains a finite constant, while $\kappa$ is meant to diverge in the constraint, such that $X(T)$ is exactly $x_T$. Furthermore, we can have the constraint with $\tau = 1$, and use the weight $\Delta\Gamma(T)$ to express a strong preference for the terminal value of the surplus without the binding constraint of $\tau = 2$.

Except for the last line, all distances from the dividends and the surplus to the benchmarks, respectively, are penalised by a quadratic loss function. Objectives on this form are well known in the quantitative finance literature such as Wonham (1968) and Björk (2009), and optimization in this context is commonly referred to as "linear-quadratic (LQ) optimization". LQ optimazation is most commonly formalized with an underlying diffusion process without jumps and mainly considered in the context of pensions funds within actuarial risk theory Cairns (2000), Steffensen (2006a). It is also well known that LQ optimization results in affine optimal strategies, which induces the desire to understand the objective and the resulting dividend strategy in a broader actuarial context. The objective and optimal controls are relevant to compare to the classical objective of maximizing expected present value of future dividends. In order to show that the LQ objective leads to affine optimal strategies, we characterize the value function by differential equations, and express the value function as a quadratic function in the surplus.

## 5.3 HJB equation and verification lemma

### 5.3.1 HJB equation and verification lemma for the LQ objective

Under the assumption that the optimal control strategies exist, the value function satisfies a system of differential equations, referred to as the Hamilton-Jacobi-Bellman equation (HJB equation). The optimal control strategies are the functions $t \mapsto l(t, X(t))$ and $t \mapsto i(t, X(t))$ that minimize the value function and are predictable with respect to the filtration generated by the surplus process. The subscript of a function refers to the partial derivative with respect to that subscript i.e. $V_t(t,x) = \frac{\partial}{\partial t} V(t,x)$.

**Proposition 5.3.1.** *Under the assumption that the optimal control strategies exist and the value function is twice continuously differentiable, $V \in C^{1,2}$. The value function satisfies the HJB equation*

$$
\begin{aligned}
0 = V_t(t,x) - \delta V(t,x) + \inf_{l,i} \Bigg\{ & \frac{1}{2}\Big(l(t,x) - l_0(t) - l_1(t)x\Big)^2 \\
& + \frac{1}{2}\gamma^i(t) i(t,x)^2 \lambda(t) \\
& + \frac{1}{2}\gamma(t)\Big(x - x_0(t)\Big)^2 \\
& + V_x(t,x)\Big(c(t) - l(t,x)\Big) \\
& + \frac{1}{2}V_{xx}(t,x)\varsigma^2 \\
& + \lambda(t)\mathbb{E}\Big[V(t, x + Y_1 - i(t,x)) - V(t,x)\Big]\Bigg\}, \qquad (5.3.1)
\end{aligned}
$$

*with boundary condition*

$$
V(T,x) = \kappa(x - x_T)^\tau + \Delta\Gamma(T)(x - x_0(T))^2. \qquad (5.3.2)
$$

*For each $(t,x) \in [0,T] \times \mathbb{R}$ the infimum is attained by the optimal control strategies, and $Y_1$ is one of the stochastic variables in the Lévy process representing a jump size.*

*Proof.* See Appendix 5.A □

The HJB equation characterizes the value function if the conditions in Proposition 5.3.1 are satisfied, but the converse is also true. It is a sufficient condition such that if a function satisfies the HJB equation, it is the value function.

**Proposition 5.3.2.** *Assume a function $H$ satisfies the HJB equation*

$$
\begin{aligned}
0 = H_t(t,x) - \delta H(t,x) + \inf_{l,i} \Bigg\{ & \frac{1}{2}\Big(l(t,x) - l_0(t) - l_1(t)x\Big)^2 \\
& + \frac{1}{2}\gamma^i(t) i(t,x)^2 \lambda(t) \\
& + \frac{1}{2}\gamma(t)\Big(x - x_0(t)\Big)^2 \\
& + H_x(t,x)\Big(c(t) - l(t,x)\Big) \\
& + \frac{1}{2}H_{xx}(t,x)\varsigma^2 \\
& + \lambda(t)\mathbb{E}\Big[H(t, x + Y_1 - i(t,x)) - H(t,x)\Big]\Bigg\},
\end{aligned}
$$

*with boundary condition*

$$H(T,x) = \kappa(x - x_T)^\tau + \Delta\Gamma(T)(x - x_0(T))^2,$$

*and* $H_x(t,X)\varsigma \in \mathcal{L}^2$*. Furthermore, assume that the infimum is attained by admissible control strategies* $\tilde{l}$ *and* $\tilde{i}$ *for each fixed* $(t,x)$*. Then the optimal value function to the control problem,* $V$ *from Equation* (5.2.2)*, is*

$$V(t,x) = H(t,x),$$

*and the optimal control strategies are* $l^* = \tilde{l}$ *and* $i^* = \tilde{i}$*.*

*Proof.* See Appendix 5.B □

The HJB equation is given as the infimum over admissible controls of a partial differential equation (PDE). By the quadratic structure of the HJB equation, the infimum is not obtained in the limits of the admissible controls going to infinity or minus infinity. Therefore, in order to find expressions for the optimal control strategies, we consider the critical point, where the partial derivatives of the expression in the curly brackets with respect to $l$ and $i$ both equal 0

$$l^*(t) = l_0(t) + l_1(t)x + V_x(t,x), \tag{5.3.3}$$

$$i^*\gamma^i(t) - \mathbb{E}\Big[V_x(t, x + Y_1 - i^*)\Big] = 0, \tag{5.3.4}$$

for sufficiently regular $V$. Hence, the optimal continuous dividend payment is equal to the benchmark plus the derivative of the value function with respect to the surplus, and the optimal dividend payment upon jumps, is related to the expectation of the derivative of the value function with respect to the surplus after a jump. The optimal controls minimize the value function if the second derivative of the expression in the curly brackets with respect to $l$ and $i$ is positive. This is true for $l$, where the second derivative equals 1, but we need to make sure $\gamma^i(t) + \mathbb{E}\Big[V_{xx}(t, x + Y_1 - i)\Big] > 0$. Based on these expressions, it is not clear that the dividend strategies in the LQ optimazation problem have an affine structure. However, we are able next to express the value function as a quadratic function, which shows the affine optimal dividend strategies.

### 5.3.2 Quadratic value function and affine optimal dividend strategy.

The sufficient condition of satisfying the HJB equation serves as a verification lemma, such that we are able to characterize the value function by a function that satisfies the HJB equation. We guess a solution to the HJB equation based on separation of $x$ inspired by Cairns (2000) and Steffensen (2006a).

**Proposition 5.3.3.** *The value function in Equation* (5.2.2) *is given by*

$$V(t,x) = q(t)x^2 + p(t)x + r(t) \tag{5.3.5}$$

*where the functions $q$, $p$ and $r$ satisfy the system of ODEs stated in Appendix 5.C along with the stated terminal conditions.*

*Proof.* Proposition 5.3.2 gives the result, since the function $V(t,x) = q(t)x^2 + p(t)x + r(t)$ satisfies the HJB equation in Proposition 5.3.2, when the deterministic functions $q(t), p(t)$ and $r(t)$ satisfy the system of ODEs in Appendix 5.C with terminal conditions obtained by considering

$$V(T,x) = \kappa(x - x_T)^\tau + \Delta\Gamma(T)(x - x_0(T))^2$$

for all values of $x \in \mathbb{R}$, and since $V_x(t,X)\varsigma \in \mathcal{L}^2$. $\square$

The expression for the value function in Equation (5.3.5) implies that the optimal controls from Equations (5.3.3) and (5.3.4) are affine in the surplus

$$l^*(t) = l_0(t) + l_1(t)x + 2q(t)x + p(t), \tag{5.3.6}$$

$$i^* = \frac{2q(t)x + 2q(t)p_1 + p(t)}{\gamma^i(t) + 2q(t)}, \tag{5.3.7}$$

where we need the second order condition $\gamma^i(t) + 2q(t) > 0$ for the optimal controls to minimize the value function. Therefore, the objective described in Section 5.2.2 results in the desirable affine dividend strategies as the optimal strategies.

The HJB equation from Proposition 5.3.1 characterizes the value function by a PDE and expresses the dividends strategy in terms of the derivative of the value function with respect to the surplus, which is computational demanding to calculate if even possible. The quadratic representation in Proposition 5.3.3 reduces the dimension of the PDE from Proposition 5.3.1 to a system of ordinary differential equations (ODE) and expresses the optimal dividend strategy in terms of the solutions to the ODEs and as affine functions in the surplus. The ODEs in Appendix 5.C fit into the class of Riccati equations. It is not certain that Riccati equations have solutions, but if a solution exists it is relatively easy to solve the system of ODEs numerically. Hence, we are able to compute the value function for any given $(t,x) \in [0,T] \times \mathbb{R}$ by solving the system of ODEs, but we are in general not interested in the explicit value of the value function. We are interested in understanding the objective, and the resulting optimal dividend strategy, which can be expressed in terms of $q(t)$, $p(t)$ and $r(t)$.

### 5.3.3 Coincidence with a diffusion surplus model

The HJB equation in Proposition 5.3.1 is similar to the HJB equation obtained in Cairns (2000) and Steffensen (2006a), with additional terms emerging from changes in the underlying surplus model caused by jumps in the Lévy process and dividend payments upon the jumps. With the value function expressed as a quadratic function in Proposition 5.3.3, we can express the last line of the HJB equation in terms of the derivative and second derivative of the value function with respect to the surplus

$$\begin{aligned}
&\lambda(t)\mathbb{E}\Big[V(t,x+Y_1-i(t,x))-V(t,x)\Big] \\
&=\mathbb{E}\Big[\Big(2q(t)x+p(t)\Big)\lambda(t)(Y_1-i(t,x))+q(t)\lambda(t)(Y-i(t,x))^2\Big] \\
&=V_x(t,x)\lambda(t)(p_1-i(t,x))+\frac{1}{2}V_{xx}(t,x)\lambda(t)(p_2+i(t,x)^2-2p_1 i(t,x)). \qquad (5.3.8)
\end{aligned}$$

Since the objective is based on expectation and from Equation (5.3.8), we can model the surplus as a diffusion process without jumps and obtain the same HJB equation and optimal control strategies.

**Corollary 5.3.4.** *Let the surplus, $\hat{X}$, have dynamics*

$$\begin{aligned}
d\hat{X}(t)=&\Big(c(t)-l(t,\hat{X}(t))+\lambda(t)\big(p_1-i(t,\hat{X}(t))\big)\Big)dt \\
&+\Big(\sqrt{\lambda(t)\big(p_2+i(t,\hat{X}(t))^2-2p_1 i(t,\hat{X}(t))\big)+\varsigma^2}\Big)dW(t).
\end{aligned}$$

*Under the assumption that the optimal control strategies exist for value function*

$$\begin{aligned}
\hat{V}(t,x)=\min_{l,i}E_{t,x}\Bigg[&\tfrac{1}{2}\int_t^T e^{-\delta(s-t)}\Big(l(s,\hat{X}(s))-l_0(s)-l_1(s)\hat{X}(s)\Big)^2 ds \\
&+\tfrac{1}{2}\int_t^T e^{-\delta(s-t)}\gamma^i(s)i(s,\hat{X}(s))^2\lambda(s)\,ds \\
&+\tfrac{1}{2}\int_t^T e^{-\delta(s-t)}\Big(\hat{X}(s)-x_0(s)\Big)^2 d\Gamma(s) \\
&+\kappa e^{-\delta(T-t)}\Big(\hat{X}(T)-x_T\Big)^{\tau}\Bigg],
\end{aligned}$$

*for $t\le T$ and $\hat{V}\in\mathcal{C}^{1,2}$. The value function satisfies the HJB equation*

$$\begin{aligned}
0=\hat{V}_t(t,x)-\delta\hat{V}(t,x)+\inf_{l,i}\Bigg\{&\frac{1}{2}\Big(l(t,x)-l_0(t)-l_1(t)x\Big)^2 \\
&+\frac{1}{2}\gamma^i(t)i^2(t,x)\lambda(t) \\
&+\frac{1}{2}w^x(t)\Big(x-x_0(t)\Big)^2 \\
&+\hat{V}_x(t,x)\Big(c(t)-l(t,x)\Big) \\
&+\frac{1}{2}\hat{V}_{xx}(t,x)\varsigma^2 \\
&+V_x(t,x)\lambda(t)(p_1-i(t,x)) \\
&+\frac{1}{2}V_{xx}(t,x)\lambda(t)(p_2+i(t,x)^2-2p_1 i(t,x)),
\end{aligned}$$

*with boundary condition*

$$\hat{V}(T,x) = \kappa(x - x_T)^\tau + \Delta\Gamma(T)(x - x_0(T))^2.$$

*Conversely if a function satisfies the HJB equation, it is the value function $\hat{V}$.*

*Proof.* See Björk (2009, Chapter 19, Theorem 19.5 and Theorem 19.6) □

The optimal dividend strategies that gives the infimum in the value function are given by Equation (5.3.6) and (5.3.7). By Equation (5.3.8), the HJB equation and optimal dividend strategies for the original setup coincides with the setup in Corollary 5.3.4. Instead of considering the underlying surplus model with jumps, we can study a corresponding setup, where the surplus is modeled as a diffusion process. Therefore, it is adequate to model the surplus as a diffusion process with continuous payments only, and we can disregard jumps in subsequent sections. It simplifies the calculations to only consider the continuous parts, and all the results can be extended to include jumps by the setup in Corollary 5.3.4. Note that disregarding the jumps is due to the specific structure of the value function in Equation (5.2.2) and emphasized in Equation (5.3.8), and does not hold for a general objective function when the Lévy process is approximated by a diffusion process.

## 5.4 Comparison to classical objective

### 5.4.1 The objectives

In actuarial risk theory, the stability criterion of minimizing the probability of ruin, which corresponds to minimizing the probability that the surplus becomes negative, implies that the surplus increases throughout time and grow to infinity, such that the capital requirement for a company increases as the company gets older, even if the risk of the company does not change. This issue is resolved in the classical optimization of the stability problem by Finetti (1957), where the objective is to maximize the expected present value of future dividends distributed to the shareholders. For this objective a company aspires to maximize the shareholders' wealth and distribute dividends accordingly, where no more dividend are paid after ruin.

The Dividend Discount Model by Williams (1938), also known as the Gordon Model by Gordon (1962), assesses the companies financial situation by the expected present value of future dividends, and Avanzi, Tu, and Wong (2016) describe the dividends relation to the size and profitability of a company, where an increase in dividend payout impacts the share price of the company positively. This emphasizes an interest in the expected present value of future dividends until ruin

$$V^b(t,x) = \mathbb{E}_{t,x}\left[\int_t^{\tau_x} e^{-\tilde{\delta}(s-t)}\mathrm{d}D(s)\right],$$

where $\tau_x = \inf\{s \geq t : X(s) = 0 | X(t) = x\}$ is the time of ruin and $\tilde{\delta}$ is a financial impatience factor, not necessarily equal to $\delta$.

The company aims to distribute dividends such that the shareholders' dividend payouts, $V^b$, is maximized. The optimal dividend strategy that maximize $V^b$ is in general the band strategy, and reduces to the barrier strategy depending on the attainable values of $Y_i$ (Morill, 1966). The barrier strategy immediately pays dividends when the surplus exceeds a certain barrier, and the amount of dividends are the difference from the surplus to the barrier. This strategy results in very irregular dividend payouts that are unreasonable in practice as addressed in Avanzi, Tu, and Wong (2016). Even though the band strategy is optimal for the stability criterion of maximizing the expected present value of future dividends, it is not desirable in practice. Realistic features for the dividend strategy are smooth and stable payouts that increase according to the surplus, and the dividend process is preferably non-decreasing. This draws attention to an affine dividend strategy, which is shown in Avanzi and Wong (2012) and Albrecher and Cani (2017) to be close the barrier strategy, but improves stability and decreases the time to ruin. Therefore, the affine dividend strategy is attractive even though, it is not the optimal dividend strategy in the classical objective.

The value function in this paper is based on minimizing the dividends and surplus deviation from benchmarks by a quadratic loss function as described in Section 5.2.2. This results in an optimal dividend strategy that is actually affine, such that we are able to obtain the affine dividend strategy as an optimal strategy, but the objective is not to maximize the present value of future dividends. The objective for the value function in the linear-quadratic optimization is to punish deviation from a benchmark, and thereby controlling the dividends and the surplus towards a target dividend payout and target surplus respectively. The explicit value of the value function does not assesses the companies financial situation in contrary to the expected present value of future dividends. We are interested in understanding the objective behind the value function and the resulting optimal dividend strategy, not the exact value of the value function.

There is no obvious way to compare the two approaches, as the objectives behind the value functions are different, making the value functions incomparable. One is not a special case of the other. Therefore, the purpose of this paper is to understand the objective the company needs to contemplate in order to obtain affine dividends as the optimal strategy, and we consider how it relates to the classical objectives in various aspects. In order to compare the optimal dividend strategy for the LQ objective to the classical objective, and compare the present value of future dividends for each of the strategies, we need to study suitable choices for the benchmarks.

### 5.4.2 Benchmarks and the expected present value of future dividends.

The company may have a target dividend payout and target surplus, and it would be evident for the company to distribute bonus according to the optimal dividend strategy in the LQ problem with the targets as benchmarks. We consider the expected present value of future dividends and how different choices of benchmarks effect this value.

The surplus model and dividend payments are restricted to continuous payments, and we assume that the benchmarks are constants. The optimal dividend strategy is given by

$$l^*(t,x) = l_0 + p(t) + \big(l_1 + 2q(t)\big)x \tag{5.4.1}$$

where $q(t)$ and $p(t)$ solves the differential equations from Proposition 5.3.3, and the surplus has dynamics

$$\mathrm{d}X(t) = \big(c - l_0 - p(t)\big)\mathrm{d}t - \big(l_1 + 2q(t)\big)X(t)\mathrm{d}t + \varsigma \mathrm{d}W(t). \tag{5.4.2}$$

We are able to calculate the expected present value of future dividends for this dividend strategy

$$V^{LQ}(t,x) = \mathbb{E}_{t,x}\left[\int_t^T e^{-\tilde{\delta}(s-t)}\Big(l_0 + p(s) + \big(l_1 + 2q(s)\big)X(s)\Big)\mathrm{d}s\right] \tag{5.4.3}$$

by a PDE.

**Proposition 5.4.1.** *Assume $V^{LQ}(t,x) \in \mathcal{C}^{1,2}$. Then the expected present value of future dividends satisfies the following partial differential equation*

$$\begin{aligned}
V_t^{LQ}(t,x) &= \tilde{\delta}V^{LQ}(t,x) - l_0 - p(t) - \big(l_1 + 2q(s)\big)x \\
&\qquad - V_x(t,x)\Big(c - l_0 - p(t) - \big(l_1 + 2q(s)\big)x\Big) - \frac{1}{2}V_{xx}(t,x)\varsigma^2, \\
V^{LQ}(T,x) &= 0.
\end{aligned} \tag{5.4.4}$$

*Conversely, if a function satisfies the partial differential equation above, it is indeed the expected present value of future dividends defined in Equation* (5.4.3)*.*

*Proof.* See Appendix 5.D. □

Similar to Section 5.3.2, we can express the expected present value of future dividends by a function that satisfies the PDE in 5.4.1. We guess a solution

$$V^{LQ}(t,x) = f(t)x + g(t),$$

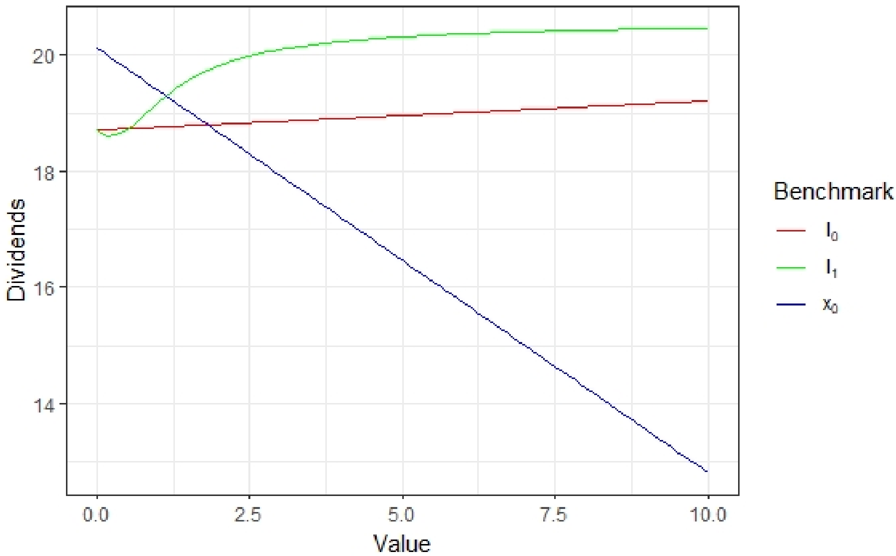


**Figure 5.1:** *The expected present value of future dividends for different values of benchmarks. The red line varies $l_0$, the green line varies $l_1$ and the blue line is for varying values of $x_0$*

where $f(t)$ and $g(t)$ satisfy the following differential equations

$$f_t(t) = f(t)\big(\tilde{\delta} + l_1 + 2q(t)\big) - l_1 - 2q(t)$$
$$g_t(t) = g(t)\tilde{\delta} + f(t)\big(l_0 + p(t) - c\big) - l_0 - p(t),$$

with terminal conditions $f(T) = 0$ and $g(T) = 0$. This function satisfies the PDE in 5.4.1 and is therefore the expected present value of future dividends. Hence, we can calculate the expected present value of future dividends for the optimal dividend strategy in the LQ problem by solving the ODEs for $f(t)$ and $g(t)$.

The expected present value of future dividends for different benchmark values is illustrated in Figure 5.5. We fix the values of the parameters that are not varied to Table 5.1. The steepest change is for the benchmark of the surplus, where a larger benchmark causes the company to distribute less dividends to achieve a higher surplus, and the expected present value of future dividends therefore decreases. There is an increase when either $l_0$ or $l_1$ increases, where larger benchmarks for the dividends cause the company to distribute more dividends, except for very small values of $l_0$.

In order to get a better understanding of the optimal dividend strategy in the LQ optimization problem, we illustrate the coefficient functions for the optimal dividend strategy $t \mapsto l_0 + p(t)$ and $t \mapsto l_1 + 2q(t)$, along with the benchmarks, for the parameters in Table 5.1.

Apart from termination, where the surplus benchmark is zero, the coefficients of the optimal control seem constant in Figure 5.2. This is in correspondence with

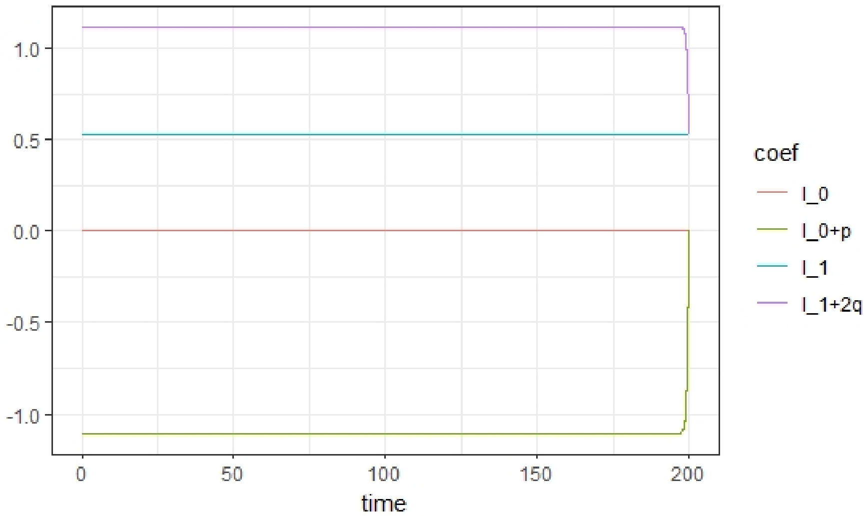


**Figure 5.2:** *The coefficient functions for the optimal dividend strategy and the benchmarks. The red and blue lines are the benchmarks $l_0$ and $l_1$, the green and turquoise lines are the coefficient functions, where the turquoise is multiplied with the surplus in the dividend strategy.*

Cairns (2000) for $T \to \infty$, because of the Markov property and the time-homogeneous objective. Based on Figure 5.2 a larger proportion of the surplus is distributed as dividends compared to the benchmark $l_1$. The function $l_1 + 2q(t)$ is even above 1. It is adjusted by the part that is not multiplied with the surplus, $l_0 + p(t)$, which is negative. A negative value of dividends corresponds to a capital injection, and increases the surplus. Therefore, if the surplus is close to zero, the dividends are negative and a capital injection increases the surplus. The LQ objective is based on minimizing quadratic deviation, and therefore, it is indifferent if the dividends and surplus are above or bellow the benchmarks respectively. It would be reasonable to constrain the dividends to be non-negative in the objective or punish capital injection with a higher weight, but this would not lead to affine dividend strategies as studied in Steffensen (2001).

Assume the company wants to choose the benchmarks such that the expected present value of future dividends, $V^{LQ}$, is maximized. Based on Figure 5.5, it is maximized by letting $x_0$ go towards minus infinity or letting either $l_0$ or $l_1$ go towards infinity. This is unreasonable in practice, since the company is ruined if the surplus is negative, and there is no constraint on the surplus being non-negative. Furthermore, if $l_0$ or $l_1$ go towards infinity, dividends are immediately payed when the surplus deviates from $x_0$, which resembles the barrier strategy. Therefore, it is undesirable to choose the benchmarks from the perspective of maximizing $V^{LQ}$. It is a drawback in the LQ optimization problem that we do not avoid ruin or stop paying dividends after ruin, and there is no restriction on non-negative dividends.

### 5.4.3 Sub-optimal control problem

In the classical objective the company aims to maximize the expected present value of future dividends until ruin, but for practical reasons the company is not interested in using the optimal dividend strategy, the barrier strategy. The absence of constrain on non-negative surplus in the LQ objective makes it undesirable to use the benchmarks that maximize the expected present value of future dividends with the LQ optimal dividend strategy. Avanzi and Wong (2012) and Albrecher and Cani (2017) study the control problem of maximizing the expected present value of future dividends until ruin, where the dividend strategy is restricted to linear and affine in the surplus respectively

$$\max_{\tilde{l_0},\tilde{l_1}} \mathbb{E}_{t,x}\Big[\int_t^{\tau_x} e^{-\tilde{\delta}(s-t)} \left(\tilde{l_0}+\tilde{l_1}X(s)\right) \,\mathrm{d}s\Big]. \tag{5.4.5}$$

This results in optimal parameters, but the dividend strategy is not the optimal solution form of the classical optimal control problem, since the strategy class is specified. Both papers give an explicit expression for Equation (5.4.5) and numerically solve for the parameters that maximize this expression. By Albrecher and Cani (2017) the optimal value of $\tilde{l_0}$ is zero in different numerical experiments. Hence, it is not desirable to pay immediate and fixed dividends when the surplus is close to zero.

The sub-optimal dividend strategies in Avanzi and Wong (2012) and Albrecher and Cani (2017) are close to the optimal barrier strategy and improve stability. We consider the optimal parameters from Avanzi and Wong (2012) and Albrecher and Cani (2017) as a suggestion for benchmarks, such that the benchmarks are the parameters that for an affine dividend strategy maximize the expected present value of future dividends until ruin.

The surplus is an Ornstein-Uhlenbeck process when the dividend strategy is a continuous rate $\tilde{l_1} > 0$ of the surplus

$$\mathrm{d}X(t) = c \,\mathrm{d}t - \tilde{l_1}X(t)\,\mathrm{d}t + \varsigma\,\mathrm{d}W(t).$$

It reverts around the level $\frac{c}{\tilde{l_1}}$, since the drift is positive, when the surplus is below the level, and negative above. Therefore, the strategy is referred to as the mean reverting dividend strategy in Avanzi and Wong (2012). With the parameters of the optimization problem in Equation (5.4.5) as benchmarks in the LQ optimization problem, a suggestion for surplus benchmark is

$$\tilde{x_0} = \frac{c}{\tilde{l_1}}.$$

The optimal dividend strategy for the LQ objective (5.4.1) implies that the surplus has a positive drift at time $t$, when the surplus is below

$$\frac{c-l_0-p(t)}{l_1+2q(t)},$$

**Table 5.1:** *Components in the numerical example*

| Component | Value |
|---|---|
| $c(t)$ | 1 |
| $\varsigma$ | 0.5 |
| $\delta$ | 0.05 |
| $\tilde{\delta}$ | 0.05 $T$ |
| 200 | |
| X(0) | 0.628 |
| $l_0$ | 0 |
| $l_1$ | 1/1.884 |
| $x_0$ | 1.884 |
| $w^x(t)$ | 1 |
| $\gamma$ | 0 |
| $k$ | 0 |
| $b^*$ | 1.256 |

and a negative drift, when the surplus is above for $l_1 + 2q(t) > 0$. Furthermore, the optimal dividend strategy increases and decreases accordingly to the surplus such that the dividend strategy reverts around $c$. We consider the optimal dividend strategy for the LQ objective with the optimal parameters from Avanzi and Wong (2012) and Albrecher and Cani (2017) as benchmarks in a numerical study.

## 5.5 Numerical study

In a numerical study we compare the barrier strategy and the mean reverting strategy to the dividend strategy in the LQ problem, where the mean reverting strategy is used as benchmark.

We use the parameters in Table 5.1 inspired by Avanzi and Wong (2012). The optimal barrier for these values is $b^* = 1.256$ and for $l_0 = 0$ the optimal level for the mean reverting strategy is $x_0 = 1.884$ with $l_1 = \frac{c}{x_0}$.

We simulate 2500 paths of the surplus for each of the three different dividend strategies using an Euler scheme with discretization step of $\frac{1}{400}$ up to time $T$, where the surplus paths are stopped for the barrier and mean reverting strategy if the surplus is negative or 0. Note the surplus for the strategy for the LQ objective is never negative, for small enough discretization steps and negative values of $l_0 + p(t)$, as discussed in Section 5.4.2.

Figure 5.3 illustrate a scatterplot of 2500 outcomes of the present value of dividends for the barrier strategy and LQ strategy, combined with a scatterplot of 2500 outcomes of the present value of dividends for the mean reverting strategy and LQ strategy. The points above the 45 degrees line are the simulations, where the LQ problem strategy

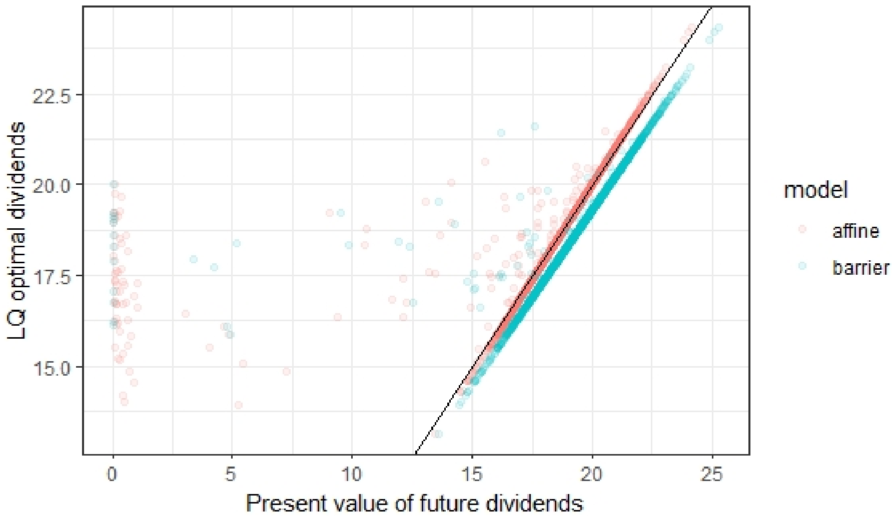


**Figure 5.3:** *The green points are a scatterplot of 2000 outcomes of the present value of dividends for the barrier strategy and LQ problem strategy, the red points are a scatterplot of 2000 outcomes of the present value of dividends for the mean reverting strategy and LQ problem strategy, and the black line is the 45 degrees line.*

outperforms the other strategies respectively. In most cases the barrier strategy results in a higher present value of dividends, which is expected as it is the optimal strategy. The band around 10-20 on the second-axis are the cases where the company is ruined before time $T$.

The violin plot in Figure 5.4 illustrate the distribution of the present value of dividends for the three different strategies based on the simulations of the surplus. The mean is highest for the barrier strategy, and the deviation is smallest for the LQ strategy. The LQ objective is based on quadratic difference, so the violin plot should be symmetric around the mean with enough simulations.

The Tables 5.2, 5.3 and 5.4 state mean and standard error of the 2500 simulations of present value of future dividends for each of the three dividend strategies for varying initial surplus, $\varsigma$ and $\tilde{\delta}$. The "mr" stands for the mean reverting strategy. We see that the present value of future dividends is in general higher for the barrier strategy, but the values for the mean reverting strategy and the LQ strategy are close. We also see that the standard error is smaller for the LQ strategy for all the different parameters, such that the deviation of the present value of future dividends is smaller for different simulation.

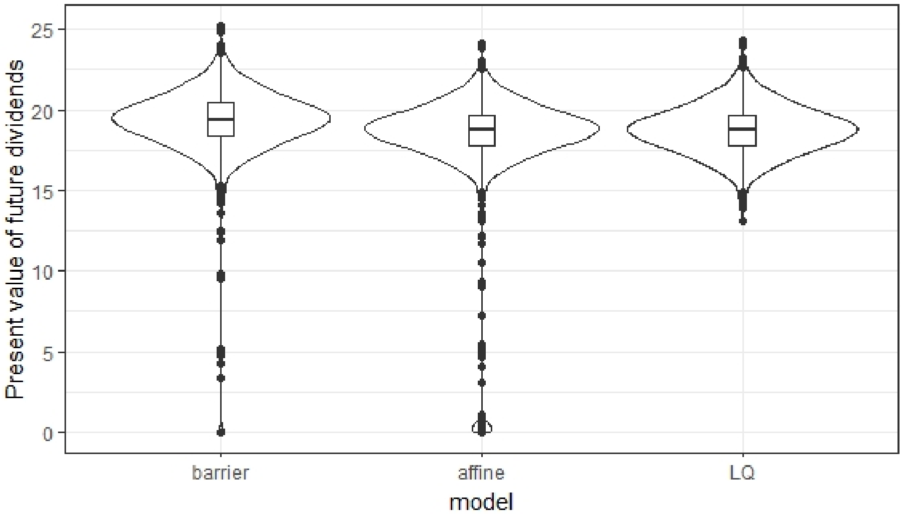


**Figure 5.4:** *Violin plot that illustrate the distributions of 2500 simulated present values of future dividends for the barrier strategy, mean reverting strategy and the LQ strategy.*

**Table 5.2:** *Present value of future dividends varying initial surplus $x$, where $c = 1$, $\tilde{\delta} = 0.05$,$\sigma = 0.5$*

| x | mean | | | sd | | | | |
|---|---|---|---|---|---|---|---|---|
| | LQ | mr | barrier | LQ | mr | barrier | $b^*$ | $g^*$ |
| 0.1 b* | 18.054 | 11.441 | 12.483 | 1.511 | 8.879 | 9.139 | 1.256 | 0.465 |
| 0.5 b* | 18.727 | 18.279 | 19.179 | 1.522 | 3.231 | 2.381 | 1.256 | 0.531 |
| 1 b* | 19.502 | 19.323 | 20.018 | 1.512 | 2.096 | 1.859 | 1.256 | 0.560 |
| 1.5 b* | 20.077 | 19.900 | 20.614 | 1.556 | 1.99 | 1.816 | 1.256 | 0.570 |
| 2 b* | 20.740 | 20.456 | 21.31 | 1.530 | 2.249 | 1.763 | 1.256 | 0.576 |

**Table 5.3:** *Present value of future dividends varying $\varsigma$, , where $c = 1$, $\tilde{\delta} = 0.05$, $x = 0.5b^*$*

| $\varsigma$ | mean | | | sd | | | | |
|---|---|---|---|---|---|---|---|---|
| | LQ | mr | barrier | LQ | mr | barrier | $b^*$ | $g^*$ |
| 0.1 | 19.857 | 19.836 | 19.908 | 0.317 | 0.705 | 0.323 | 0.083 | 7.519 |
| 0.5 | 18.709 | 18.173 | 19.162 | 1.485 | 3.440 | 2.436 | 1.256 | 0.531 |
| 1 | 16.850 | 15.801 | 18.091 | 3.039 | 5.543 | 4.563 | 3.563 | 0.192 |

**Table 5.4:** *Present value of future dividends varying $\tilde{\delta}$, where $c = 1$, $\sigma = 0.5$, $x = 0.5b^*$*

| $\tilde{\delta}$ | mean | | | sd | | | | |
|---|---|---|---|---|---|---|---|---|
| | LQ | mr | barrier | LQ | mr | barrier | $b^*$ | $g^*$ |
| 0.01 | 97.550 | 96.849 | 98.04 | 3.496 | 8.484 | 7.926 | 5.175 | 2.66 |
| 0.05 | 18.798 | 18.263 | 19.233 | 1.583 | 3.493 | 2.575 | 1.256 | 1.884 |
| 0.1 | 9.036 | 8.688 | 9.377 | 1.068 | 2.062 | 1.562 | 1.075 | 1.543 |

### 5.5.1 Initial surplus

The expected present value of future dividends until ruin with the barrier strategy, $V^b$, is in general larger than the expected present value of future dividends with the LQ strategy, $V^{LQ}$, where the benchmarks are the parameters from the mean reverting strategy. We consider how much the initial surplus differs for $V^{LQ}$ to be equal to $V^b$. Hence we want to solve the following equation for $\xi$

$$V^{LQ}(t, X(t) + \xi(t)) = V^b(t, X(t)).$$

By the quadratic representation of $V^{LQ}$ we get that

$$\xi(t) = \frac{V^b(t, X(t)) - g(t) - f(t)X(t)}{f(t)},$$

where

$$V^b(t, X(t)) = \frac{e^{rX(t)} - e^{sX(t)}}{re^{rb^*} - se^{sb^*}}$$

for the roots of $\frac{1}{2}\varsigma^2 z + cz - \tilde{\delta} = 0$

$$r = \frac{-c + \sqrt{c^2 + 2\tilde{\delta}\varsigma^2}}{\varsigma^2},$$
$$s = \frac{-c - \sqrt{c^2 + 2\tilde{\delta}\varsigma^2}}{\varsigma^2}.$$

Figure shows $\xi$ as a function of the initial surplus.

The difference $\xi$ is negative for very small values of $X(0)$, where the surplus with barrier strategy is likely to ruin. It is not a fair comparison for small values of $X(0)$, as the LQ objective does not stop paying dividends after ruin and are likely to make capital injections. Apart from values of $X(0)$, the additional amount of initial surplus needed for $V^{LQ}$ to be equal to $V^b$ increases as the initial surplus does.

It is not simple to compare the LQ objective to the classical objective, since the reasoning for the objectives are different. We therefore consider how they relate in various aspects, and discuss suitable choices for the benchmarks based on the comparison. It is a drawback in the LQ optimization problem, that we do not stop paying dividends after ruin, and there is no restriction on non-negative dividends.

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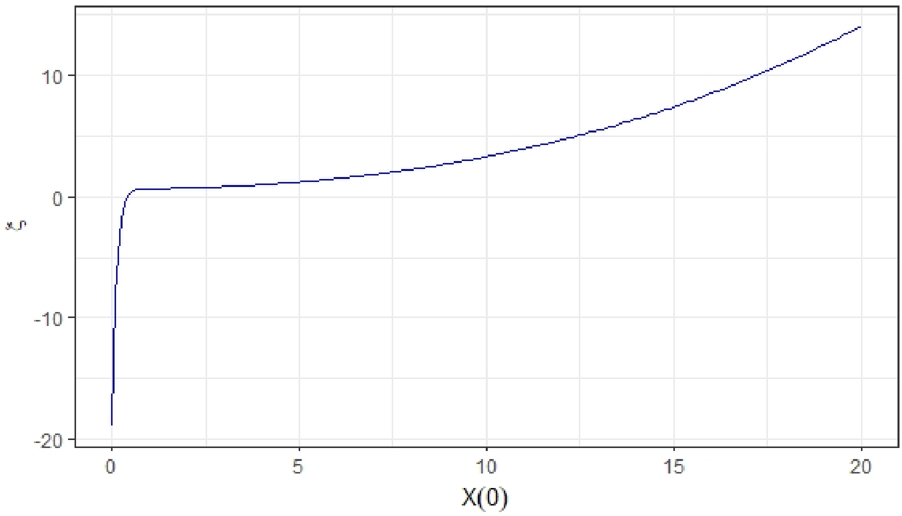


**Figure 5.5:** *The change in initial surplus for the the expected present value of future dividends to be equal, considered as a function of the initial surplus.*

## 5.A Proof of Proposition 5.3.1

The proof of the continuous parts is given in Björk (2009, Chapter 19, Theorem 19.5). Hence, it suffices to prove the result for the jumps, where

$$c(t) = l(t, X(t)) = l_0(t) = l_1(t) = \Gamma(t) = \varsigma = 0,$$

for all $t \geq 0$. Assume also initially that $\delta = 0$ and let $(t, x) \in [0, T] \times \mathbb{R}$ be fixed and define

$$\hat{i}(s, y) = \begin{cases} i(s, y), & (s, y) \in [t, t+h] \times \mathbb{R} \\ i^*(s, y), & (s, y) \in (t+h, T] \times \mathbb{R} \end{cases}$$

where $t + h < T$, $\mathbf{i}^*$ is the optimal strategy and $\mathbf{i}$ is an arbitrary fixed control strategy. Define

$$\begin{aligned} \mathcal{J}(t, x, \hat{\mathbf{i}}) =& E_{t,x}\left[\frac{1}{2}\int_t^T \gamma^i(s)\hat{i}(s, X^{\hat{\mathbf{i}}}(s-))^2 dN(s) + \kappa(X^{\hat{\mathbf{i}}}(T) - x_T)^\tau\right] \\ =& E_{t,x}\left[\frac{1}{2}\int_t^{t+h} \gamma^i(s)i(s, X^{\mathbf{i}}(s-))^2 dN(s)\right] + E_{t,x}\left[V(t+h, X^{\mathbf{i}^*}(t+h))\right] \end{aligned}$$

where the last equation comes from partitioning the time interval and that the optimal stategy is used in the interval $(t + h, T]$. We assume $V$ is sufficiently regular and use

Itôs lemma to obtain

$$V(t+h, X^{\mathbf{i}}_{t+h}) = V(t,x) + \int_t^{t+h} V_t(s, X^{\mathbf{i}}(s))\mathrm{d}s \\ + \int_t^{t+h} \Big(V(s, X^{\mathbf{i}}(s)) - V(s, X^{\mathbf{i}}(s-))\Big)\mathrm{d}N(s).$$

The last term in $\mathcal{J}(t,x,\hat{\mathbf{i}})$ is then given by

$$E_{t,x}\left[V(t+h, X^{\mathbf{i}}(t+h))\right] = V(t,x) + E_{t,x}\left[\int_t^{t+h} V_t(s, X^{\mathbf{i}}(s))\mathrm{d}s\right] \\ + E_{t,x}\Bigg[\int_t^{t+h} \lambda(s) \int_{-\infty}^{\infty} \Bigg(V(s, X^{\mathbf{i}}(s-) + y - i(s, X^{\mathbf{i}}(s-)) \\ - V(s, X^{\mathbf{i}}(s-))\Bigg)\mathrm{d}F_Y(y)\ \mathrm{d}s\Bigg]$$

for $F_Y$ the distribution of any of the jump sizes, for instance $Y_1$, where we use the following is a martingale

$$\int_t^{t+h} \Big(V(s, X^{\mathbf{i}}(s-) + \Delta X^{\mathbf{i}}(s)) - V(s, X^{\mathbf{i}}(s-))\Big)\mathrm{d}N(s) \\ - \int_t^{t+h} \lambda(s) \int_{-\infty}^{\infty} V(s, X^{\mathbf{i}}(s-) + y - i(s, X^{\mathbf{i}}(s-)) - V(s, X^{\mathbf{i}}(s-))\mathrm{d}F_Y(y)\ \mathrm{d}s.$$

Note the optimal control strategy minimize $\mathcal{J}$ therefore

$$V(t,x) = \mathcal{J}(t,x,\mathbf{i}^*) \leq \mathcal{J}(t,x,\hat{\mathbf{i}}),$$

with equality if and only if the control strategy is the optimal $\mathbf{i} = \mathbf{i}^*$. The inequality is

$$V(t,x) \leq E_{t,x}\left[\frac{1}{2}\int_t^{t+h} \gamma^i(s) i(s, X^{\mathbf{i}}(s-))^2 dN(s)\right] + V(t,x) \\ + E_{t,x}\left[\int_t^{t+h} V_t(s, X^{\mathbf{i}^*}(s))\mathrm{d}s\right] \\ + E_{t,x}\Bigg[\int_t^{t+h} \lambda(s) \int_{-\infty}^{\infty} \Bigg(V(s, X^{\mathbf{i}}(s-) + y - i^*(s, X^{\mathbf{i}^*}(s-)) \\ - V(s, X^{\mathbf{i}^*}(s-))\Bigg)\mathrm{d}F_Y(y)\ \mathrm{d}s\Bigg]$$

Dividing by $h$ and assume sufficient regularity to consider the limit $h \to 0$ within the expectation

$$0 \leq V_t(t,x) + \frac{1}{2}\lambda(t)\gamma^i(t) i^2(t,x) + \lambda(t) E\big[V(t, x + Y_1 - i(t,x)) - V(t,x)\big]$$

where we use that $X^{\mathbf{i}}(t-) = x$. Since the control strategy $\mathbf{i}$ is arbitrary, this inequality holds for all choices of control strategies, therefore also the strategy that gives the infimum of this quantity

$$0 \leq \inf_i \left\{ V_t(t,x) + \frac{1}{2}\lambda(t)\gamma^i(t)i^2(t,x) + \lambda(t)E\big[V(t, x + Y_1 - i^*(t,x)) - V(t,x)\big] \right\}.$$

We also have that infimum over an arbitrary strategy, must be smaller than any other strategy also the optimal strategy, hence, $\inf_i \mathcal{J}(t,x,\hat{\mathbf{i}}) \leq \mathcal{J}(t,x,\mathbf{i}^*)$, which gives the other inequality, and HJB equation with $\delta = 0$.

For $\delta \neq 0$

$$\begin{aligned} &V(t,x) = \\ &\qquad \min_i E_{t,x}\left[\frac{1}{2}\int_t^T {}^{-\delta(s-t)}\gamma^i(s)i(s,X(s-))^2 dN(s) + e^{-\delta(T-t)}\kappa(X^{\mathbf{i}}(T) - x_T)^\tau\right] \\ &\quad = e^{\delta(t)}\tilde{V}(t,x). \end{aligned}$$

By the previous results, the function $\tilde{V}(t,x)$ satisfy

$$0 = \tilde{V}_t(t,x) + \inf_i \left\{ e^{-\delta s}\frac{1}{2}\lambda(t)\gamma^i(t)i^2(t,x) + \lambda(t)E\big[\tilde{V}(t,x+Y_1 - i(t,x)) - \tilde{V}(t,x)\big] \right\},$$

where $\tilde{V}_t(t,x) = e^{-\delta t}V_t(t,x) - \delta e^{-\delta t}V(t,x)$. Multiply everthing by $e^{\delta t}$ to obtain the desired.

## 5.B Proof of Proposition 5.3.2

The proof of the continuous parts is given in Björk (2009, Chapter 19, Theorem 19.6). Hence, it suffices to prove the result for the jumps, where

$$c(t) = l(t, X(t)) = l_0(t) = l_1(t) = \Gamma(t) = \varsigma = 0.$$

Assume $H \in C^1$, $C^2$ except at countable many points and solves the HJB equation. Furthermore, assume for fixed $(t,x) \in [0,T] \times \mathbb{R}$ that the control strategy $\tilde{i}(t,x)$ minimizes the HJB equation of $H$.

Let $\mathbf{i}$ be an arbitrary control strategy and $X^{\mathbf{i}}(s)$ be the surplus with dynamics

$$\mathrm{d}X^{\mathbf{i}}(s) = \mathrm{d}S(s) - i(s, X^{\mathbf{i}}(s-))\mathrm{d}N(s),$$

for $t \leq s \leq T$ and $X^{\mathbf{i}}(t-) = x$.

Since $H$ solves the HJB equation we have for $t \leq s \leq T$

$$\begin{aligned} 0 \leq &H_t(s, X^{\mathbf{i}}(s-)) - \delta H(s, X^{\mathbf{i}}(s-)) \\ &+ \frac{1}{2}\gamma^i(s)i^2(s, X^{\mathbf{i}}(s-))\lambda(s) \\ &+ \lambda(s)\mathbb{E}\Big[H(s, X^{\mathbf{i}}(s-) + Y_1 - i(s, X^{\mathbf{i}}(s-))) - H(s, X^{\mathbf{i}}(s-))\Big], \end{aligned}$$

$P - a.s.$ for all possible control strategies $\mathbf{i}$. We consider the integral over $(t, T]$ for both sides of the inequality multiplied by the positive function $e^{-\delta(s-t)}$ for $t \le s \le T$.

$$\begin{aligned}
0 \le & \int_t^T e^{-\delta(s-t)} H_t(s, X^{\mathbf{i}}(s-))\mathrm{d}s - \delta \int_t^T e^{-\delta(s-t)} H(s, X^{\mathbf{i}}(s-))\mathrm{d}s \\
& + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) i^2(s, X^{\mathbf{i}}(s-)) \Big(\lambda(s)\mathrm{d}s - \mathrm{d}N(s)\Big) \\
& + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) i^2(s, X^{\mathbf{i}}(s-))\mathrm{d}N(s) \\
& + \int_t^T e^{-\delta(s-t)} \lambda(s) \mathbb{E}\Big[H\big(s, X^{\mathbf{i}}(s-) + Y_1 - i(s, X^{\mathbf{i}}(s-))\big) - H(s, X^{\mathbf{i}}(s-))\Big]\mathrm{d}s.
\end{aligned}$$

Itôs lemma implies that

$$\begin{aligned}
e^{-\delta(T-t)} H(T, X_T^{\mathbf{i}}) = & H(t, x) - \delta \int_t^T e^{-\delta(s-t)} H(s, X^{\mathbf{i}}(s))\mathrm{d}s \\
& + \int_t^T e^{-\delta(s-t)} H_t(s, X^{\mathbf{i}}(s))\mathrm{d}s \\
& + \int_t^T e^{-\delta(s-t)} \Big(H(s, X^{\mathbf{i}}(s)) - H(s, X^{\mathbf{i}}(s-))\Big)\mathrm{d}N(s),
\end{aligned}$$

such that

$$\begin{aligned}
H(t, x) \le & e^{-\delta(T-t)} H(T, X^{\mathbf{i}}(T)) \\
& + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) i^2(s, X^{\mathbf{i}}(s-)) \Big(\lambda(s)\mathrm{d}s - \mathrm{d}N(s)\Big) \\
& + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) i^2(s, X^{\mathbf{i}}(s-))\mathrm{d}N(s) \\
& + \int_t^T e^{-\delta(s-t)} \lambda(s) \mathbb{E}\Big[H\big(s, X^{\mathbf{i}}(s-) + Y_1 - i(s, X^{\mathbf{i}}(s-))\big) \\
& \qquad - H(s, X^{\mathbf{i}}(s-))\Big]\mathrm{d}s \\
& - \int_t^T e^{-\delta(s-t)} \Big(H(s, X^{\mathbf{i}}(s)) - H(s, X^{\mathbf{i}}(s-))\Big)\mathrm{d}N(s).
\end{aligned}$$

Take $\mathbb{E}_{t,x}[\cdot]$ on both sides of the inequality.

$$
\begin{aligned}
H(t,x) \leq & \mathbb{E}_{t,x}\Bigg[ e^{-\delta(T-t)}\kappa\Big(X^{\mathbf{i}}(T)-k\Big)^{\tau} \\
& + \frac{1}{2}\int_t^T e^{-\delta(s-t)}\gamma^i(s) i^2(s, X^{\mathbf{i}}(s-))\mathrm{d}N(s)\Bigg] \\
& + E_{t,x}\Bigg[\int_t^T e^{-\delta(s-t)}\lambda(s)\int_{-\infty}^{\infty} H\big(s, X^{\mathbf{i}}(s-)+y-i(s,X^{\mathbf{i}}(s-)\big) \\
& \qquad\qquad - H(s, X^{\mathbf{i}}(s-))\mathrm{d}F_Y(y) \\
& - \int_t^T e^{-\delta(s-t)}\Big(H(s,X^{\mathbf{i}}(s)) - H(s, X^{\mathbf{i}}(s-))\Big)\mathrm{d}N(s)\Bigg] \\
& = \mathcal{J}(t,x,\mathbf{i}),
\end{aligned}
$$

Where we use the compensated jump measure $\mathbb{E}\left[\mathrm{d}N(s)\right] = \lambda(s)\mathrm{d}s$ and the compensated martingale

$$
\begin{aligned}
& \int_t^{t+h} H\big(s, X^{\mathbf{i}}(s-)+\Delta X^{\mathbf{i}}(s)\big) - H(s, X^{\mathbf{i}}(s-))\mathrm{d}N(s) \\
& - \int_t^{t+h}\lambda(s)\int_{-\infty}^{\infty} H\big(s, X^{\mathbf{i}}(s-)+y-i(s,X^{\mathbf{i}}(s-)\big) - H(s,X^{\mathbf{i}}(s-))\mathrm{d}F_Y(y)\ \mathrm{d}s.
\end{aligned}
$$

Since the control strategy is arbitrary we also have that

$$
H(t,x) \leq \inf_{l,i}\ \mathcal{J}(t,x,\mathbf{i}) = V(t,x). \tag{5.B.1}
$$

For the optimal control strategy $\tilde{\mathbf{i}}$ the HJB equation implies $P$-a.s.

$$
\begin{aligned}
0 = & H_t(s, X^{\tilde{\mathbf{i}}}(s-)) - \delta H(s, X^{\tilde{\mathbf{i}}}(s-)) \\
& + \frac{1}{2}\gamma^i(s)\tilde{i}^2(s, X^{\tilde{\mathbf{i}}}(s-))\lambda(s) \\
& + \lambda(s)\mathbb{E}\Big[H\big(s, X^{\tilde{\mathbf{i}}}(s-)+Y_1-\tilde{i}(s, X^{\tilde{\mathbf{i}}}(s-))\big) - H(s, X^{\tilde{\mathbf{i}}}(s-))\Big],
\end{aligned}
$$

Consider the integral over $(t,T]$ for both sides of the equality multiplied by the positive function $e^{-\delta(s-t)}$ for $t \leq s \leq T$, and the expression of Itôs lemma

$$
\begin{aligned}
H(t, X^{\tilde{\mathbf{i}}}(t)) =& e^{-\delta(T-t)} H(T, X^{\tilde{\mathbf{i}}}) \\
& + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) \tilde{i}^2(s, X^{\tilde{\mathbf{i}}}(s-)) \Big( \lambda(s) \mathrm{d}s - \mathrm{d}N(s) \Big) \\
& + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) \tilde{i}^2(s, X^{\tilde{\mathbf{i}}}(s-)) \mathrm{d}N(s) \\
& + \int_t^T e^{-\delta(s-t)} \lambda(s) \mathbb{E} \Big[ H\big(s, X^{\tilde{\mathbf{i}}}(s-) + Y_1 - \tilde{i}(s, X^{\tilde{\mathbf{i}}}(s-))\big) \\
& \qquad - H(s, X^{\tilde{\mathbf{i}}}(s-)) \Big] \mathrm{d}s \\
& - \int_t^T e^{-\delta(s-t)} \bigg( H(s, X^{\tilde{\mathbf{i}}}(s)) - H(s, X^{\tilde{\mathbf{i}}}(s-)) \bigg) \mathrm{d}N(s).
\end{aligned}
$$

Take $\mathbb{E}_{t,x}[\cdot]$ on both sides of the equality

$$
\begin{aligned}
H(t, x) =& \mathbb{E}_{t,x} \left[ e^{-\delta(T-t)} \kappa \bigg( X^{\mathbf{i}}(T) - k \bigg)^{\tau} + \frac{1}{2} \int_t^T e^{-\delta(s-t)} \gamma^i(s) \tilde{i}^2(s, X^{\tilde{\mathbf{i}}}(s-)) \mathrm{d}N(s) \right] \\
& + E_{t,x} \bigg[ \int_t^T e^{-\delta(s-t)} \lambda(s) \int_{-\infty}^{\infty} H\big(s, X^{\tilde{\mathbf{i}}}(s-) + y - i(s, X^{\tilde{\mathbf{i}}}(s-)\big) \\
& \qquad - H(s, X^{\tilde{\mathbf{i}}}(s-)) \mathrm{d}F_Y(y) \\
& - \int_t^T e^{-\delta(s-t)} \bigg( H(s, X^{\tilde{\mathbf{i}}}(s)) - H(s, X^{\tilde{\mathbf{i}}}(s-)) \bigg) \mathrm{d}N(s) \bigg] \\
& = \mathcal{J}(t, x, \tilde{\mathbf{i}}).
\end{aligned}
$$

We must have that

$$
H(t, x) = \mathcal{J}(t, x, \tilde{\mathbf{i}}) \geq \inf_{l,i} \mathcal{J}(t, x, \mathbf{i}) = V(t, x),
$$

which together with (5.B.1) shows that

$$
H(t, x) = \mathcal{J}(t, x, \tilde{\mathbf{i}}) = V(t, x),
$$

and $\tilde{l}$ and $\tilde{i}$ are the optimal control strategies.

**Table 5.5:** *Terminal conditions*

| $\tau$ | 0 | 1 | 2 |
|---|---|---|---|
| $q(T)$ | $\Delta\Gamma(T)$ | $\Delta\Gamma(T)$ | $\Delta\Gamma(T)+\kappa$ |
| $p(T)$ | $-2\Delta\Gamma(T)x_0(T)$ | $-2\Delta\Gamma(T)x_0(T)+\kappa$ | $-2\Delta\Gamma(T)x_0(T)-2\kappa x_T$ |
| $r(T)$ | $\Delta\Gamma(T)x_0(T)^2$ | $\Delta\Gamma(T)x_0(T)^2-\kappa x_T$ | $\Delta\Gamma(T)x_0(T)^2+\kappa x_T^2$ |

## 5.C Additions to Proposition 5.3.3

$$\begin{aligned}
q_t(t) =& \delta q(t) + 2q(t)^2 - \frac{1}{2}\gamma^x(t) - \Big(\gamma^i(t)+2q(t)\Big)2\lambda(t)\frac{q(t)^2}{\big(\gamma^i(t)+2q(t)\big)^2} + 2q(t)l_1(t) \\
&+ 4\lambda(t)\frac{q(t)}{\gamma^i(t)+2q(t)} \qquad (5.C.1)\\
p_t(t) =& \delta p(t) + \gamma(t)x_0(t) - \Big(\gamma^i(t)+q(t)\Big)\lambda(t)\frac{2q(t)p_1+p(t)}{\gamma^i(t)+2q(t)}\frac{2q(t)}{\gamma^i(t)+2q(t)} \\
&- 2q(t)c(t) - 2q(t)\lambda(t)p_1 + 2q(t)l_0(t) + p(t)l_1(t) + 2q(t)p(t) \\
&- (1-2p_1+p(t))\lambda(t)\frac{2q(t)}{\gamma^i(t)+2q(t)} + 2\lambda(t)\frac{2q(t)p_1+p(t)}{\gamma^i(t)+2q(t)} \qquad (5.C.2)\\
r_t(t) =& \delta r(t) + \frac{1}{2}p(t)^2 - \frac{1}{2}\gamma(t)x_0(t)^2 - \Big(\frac{1}{2}\gamma^i(t)+q(t)\Big)\lambda(t)\Big(\frac{1-2q(t)p_1-p(t)}{\gamma^i(t)+2q(t)}\Big)^2 \\
&- p(t)c(t) - p(t)\lambda(t)p_1 + p(t)l_0(t) - 2q(t)\varsigma^2 \qquad (5.C.3)\\
&+ \Big(2p_1 - 1 - p(t)\Big)\lambda(t)\frac{2q(t)p_1+p(t)}{\gamma^i(t)+2q(t)}. \qquad (5.C.4)
\end{aligned}$$

with terminal conditions

## 5.D Proof of Proposition 5.4.1

Construct a martingale $m$ as

$$\begin{aligned}
m(t) &= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_0^T e^{-\tilde{\delta}s}\Big(l_0 + p(s) + \big(l_1+2q(s)\big)X(s)\Big)\mathrm{d}s \;\middle|\; \mathcal{F}_t\right] \\
&= \int_0^t e^{-\tilde{\delta}s}\Big(l_0 + p(s) + \big(l_1+2q(s)\big)X(s)\Big)\mathrm{d}s + e^{-\tilde{\delta}t}V^{LQ}(t, X(t)).
\end{aligned}$$

The dynamics of $m$ are

$$\begin{aligned}
\mathrm{d}m(t) = e^{-\tilde{\delta}t}\Big(l_0 + p(t) + \big(l_1+2q(t)\big)X(t) - \tilde{\delta}V^{LQ}(t,X(t))\mathrm{d}t \\
+ \mathrm{d}V^{LQ}(t,X(t))\Big).
\end{aligned}$$

By the Itô formula, we have the dynamics

$$
\begin{aligned}
&\mathrm{d}V^{LQ}(t,X(t)) \\
&\quad = V_t^{LQ}(t,X(t))\mathrm{d}t + V_x^{LQ}(t,X(t))\Big(c - l_0 - p(t) - \big(l_1 + 2q(t)\big)X(t)\Big)\mathrm{d}t \\
&\qquad + \frac{1}{2}V_{xx}^{LQ}(t,X(t))\varsigma^2\mathrm{d}t + V_x^{LQ}(t,X(t))\varsigma\mathrm{d}W(t)
\end{aligned} \tag{5.D.1}
$$

Combining this, the dynamics of $m(t)$ are

$$
\begin{aligned}
\mathrm{d}m(t) = e^{-\tilde{\delta}t}\Bigg( & l_0 + p(t) + \big(l_1 + 2q(t)\big)X(t) - \tilde{\delta}V^{LQ}(t,X(t)) \\
& + V_t^{LQ}(t,X(t)) + V_x^{LQ}(t,X(t))\Big(c - l_0 - p(t) - \big(l_1 + 2q(t)\big)X(t)\Big) \\
& + \frac{1}{2}V_{xx}^{LQ}(t,X(t))\varsigma^2\Bigg)\mathrm{d}t \\
& + e^{-\tilde{\delta}t}V_x^{LQ}(t,X(t))\varsigma\mathrm{d}W(t).
\end{aligned}
$$

Since $e^{-\tilde{\delta}t}V_x^{LQ}(t,X(t))\varsigma\mathrm{d}W(t)$ are the dynamics of a martingale and since $m(t)$ is a martingale, the term in front of $\mathrm{d}t$ in the dynamics of $m(t)$ must be equal to zero for all $t$ and $X(t)$ which results in the partial differential equation for the expected present value of future dividends. By the expression og $V^{LQ}$ the boundary condition of the partial differential equation is $V^{LQ}(T,x) = 0$.

Now, assume that a function $\bar{V}^{LQ}(t,x)$ satisfies the partial differential equation in Equation (5.4.4). We show that this function is in fact the expected present value of future dividends in Equation (5.4.3).

The Itô formula, the dynamics from Equation (5.D.1) with $\bar{V}$ inserted instead of $V$, and the fact that $\bar{V}$ satisfies the partial differential equation in Equation (5.4.4) yield that

$$
\begin{aligned}
&\mathrm{d}\Big(e^{-\tilde{\delta}t}\bar{V}^{LQ}(t,X(t))\Big) \\
&= -\tilde{\delta}\bar{V}^{LQ}(t,X(t))\mathrm{d}t + e^{-\tilde{\delta}t}\mathrm{d}\bar{V}^{LQ}(t,X(t)) \\
&= e^{-\tilde{\delta}t}\Bigg(\Big(l_0 + p(t) + \big(l_1 + 2q(t)\big)X(t)\Big)\mathrm{d}t + V_x^{LQ}(t,X(t))\varsigma\mathrm{d}W(t)\Bigg).
\end{aligned}
$$

Integrating over the interval $[t,T)$ and taking the expectation conditioning on $\mathcal{F}_t$ give that

$$
\begin{aligned}
& e^{-\tilde{\delta}T}\bar{V}^{LQ}(T,X(T)) - e^{-\tilde{\delta}t}\bar{V}^{LQ}(t,X(t)) \\
&= -\mathbb{E}\Bigg[\int_t^T e^{-\tilde{\delta}s}\Big(l_0 + p(s) + \big(l_1 + 2q(s)\big)X(s)\Big)\mathrm{d}s \;\Bigg|\; \mathcal{F}_t\Bigg],
\end{aligned}
$$

since the remaining term in the dynamics of $\bar{V}^{LQ}(t, X(t))$ is a martingale with respect to the filtration $\mathcal{F}$. Multiplying by $-\exp(-\tilde{\delta}t)$ gives that $\bar{V}^{LQ}(t, x)$ is the expected present value of future dividends.

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