PATTERNS IN THE HOMOLOGY OF ALGEBRAS:

VANISHING, STABILITY, AND HIGHER STRUCTURES

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Abstract: This thesis studies patterns in the homology and cohomology of algebras. We investigate the vanishing of homology and cohomology groups, homological stability questions, and homology operations arising from E_k-structures. In Chapter 1, we introduce the notion of algebraic coset poset. This construction is inspired by work of Boyd, Hepworth and Patzt. It generalizes the notion of coset poset for groups considered in the literature and allows us to associate "geometrically flavored" semi-simplicial A-modules to certain algebras A. These "spaces with A-action" play an important role in the two subsequent chapters in which we use associated "isotropy" spectral sequences to prove theorems about the homology of A. In Chapter 2, we prove that the homology of any Temperley–Lieb algebra on an odd number of strands vanishes in all positive homological degrees. This improves a result of Boyd–Hepworth. In Chapter 3, we derive an explicit formula for the second homology of certain Iwahori–Hecke algebras. This generalizes a result of Boyd for the second homology of Coxeter groups and is the Iwahori–Hecke analogue of a theorem of Howlett. In Chapter 4, which is based on joint work with Richard Hepworth and Jeremy Miller, we specify conditions for the existence of an E_k-algebra structure on the "classifying space" of a family of abstract algebras, building on work of Berger, Fiedorowicz and Smith. We then describe an E_2-algebra structure on the "classifying space" of certain families of Iwahori–Hecke algebras and show that it does not extend to an E_3-structure in general. Chapter 5, which is based on joint work with Benjamin Brück and Peter Patzt, studies the top-dimensional rational cohomology of the integral symplectic groups. It follows from a theorem of Gunnells that this unstable cohomology group is trivial. We implement an idea of Putman for a new proof of Gunnells' theorem and explain how the vanishing result follows.

Resumé: I denne afhandling studeres mønstre i homologien og cohomologien af algebraer. Vi undersøger forsvinden af homologi- og cohomologigrupper, spørgsmål vedrørende homologisk stabilitet, og homologioperationer som opstår fra E_k-strukturer. I kapitel 1 introducerer vi begrebet algebraisk coset poset. Denne konstruktion er inspireret af arbejde af Boyd, Hepworth og Patzt. Den generaliserer begrebet coset poset for grupper studeret i litteraturen og gør det muligt at associere "geometrisk farvede" semi-simplicial A-moduler med visse algebraer A. Disse "rum med A-virkning" spiller en vigtig rolle i de to efterfølgende kapitler, hvor vi bruger associerede "isotropi"-spectralsekvenser til at bevise sætninger vedrørende homologien af A. I kapitel 2 beviser vi at homologien af enhver Temperley–Lieb algebra med et ulige antal strenge forsvinder i alle positive homologigrader. Dette forbedrer et resultat af Boyd–Hepworth. I kapitel 3 udleder vi en eksplicit formel for anden homologi af visse Iwahori–Hecke algebraer. Dette generaliserer et resultat af Boyd for anden homologi af Coxeter grupper og er Iwahori–Hecke analogen af en sætning af Howlett. I kapitel 4, som er baseret på et samarbejde med Richard Hepworth og Jeremy Miller, specificerer vi betingelser for eksistensen af en E_k-algebra struktur på "det klassificerende rum" af en familie af abstrakte algebraer, som bygge på arbejdet af Berger, Fiedorowicz og Smith. Vi beskriver så en E_2-algebra struktur på "det klassificerende rum" af visse familier af Iwahori–Hecke algebraer og beviser at den ikke generelt kan udvides til en E_3-struktur. I kapitel 5, som er baseret på et samarbejde med Benjamin Brück og Peter Patzt, studeres den topdimensionelle rationelle cohomologi af de symplektiske grupper over de hele tal. Det følger af en sætning af Gunnells, at denne ustabile cohomologigruppe er triviel. Vi implementerer en idé af Putman til et nyt bevis for Gunnells sætning og forklarer hvordan forsvindingsresultatet følger.

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Contents

Introduction

From spaces and groups to abstract algebras

Homological stability techniques have been key in advancing our understanding of the homology of many families of spaces and groups (e.g. [**Har85**, **Dwy80**, **Gal11**, **GRW18**, **GRW17**, **Wah08**, **vdK80**, **HV04**, **HVW06**, **Vog79**, **Cha87**, **Hep16**]). Given a sequential diagram of topological spaces or groups

$$X_0 \to X_1 \to \dots \to X_n \to X_{n+1} \to \dots$$

we say that the family $\{X_n\}_{n\in\mathbb{N}}$ satisfies homological stability, if for any $i \in \mathbb{N}$ the diagram obtained by passing to the i-th homology group

$$H_i(X_0) \to H_i(X_1) \to \dots \to H_i(X_n) \to H_i(X_{n+1}) \to \dots$$

has the property that all but finitely many of the induced morphisms are isomorphisms. This property splits the sequence of homology groups $\{H_i(X_n)\}_{n\in\mathbb{N}}$ into two pieces; an infinite set of "stable" homology groups that are all connected via isomorphisms and a finite set of "unstable" homology groups.

The classical argument to detect homological stability originates from Quillen's work on algebraic K-theory [**Qui**]. This approach has been formalized by Randal-Williams–Wahl for families of groups [**RWW17**]. Their work highlights an extra structure that is only visible if one considers families of objects. Let $\mathcal{G}$ be the groupoid associated to a family of groups $\{G_n\}_{n\in\mathbb{N}}$, which has the natural numbers $\mathbb{N}$ as its set of objects and whose automorphisms are given by $\mathcal{G}(n,n) = G_n$. Randal-Williams–Wahl [**RWW17**] showed that many of the groupoids $\mathcal{G}$, which are constructed from groups that satisfy homological stability, have the structure of a braided or symmetric monoidal category. This means that there is a certain natural isomorphism

$$b_{m,n} : m+n \xrightarrow{\cong} n+m$$

for any pair of objects (m,n) that encodes a "commutativity property". Geometrically, this gives the classifying space

$$N_\bullet\mathcal{G} = \bigsqcup_{n\in\mathbb{N}} B_\bullet G_n$$

the structure of an E_2-algebra [**Fie**]. This perspective has been used by Krannich [**Kra19**] to formulate a homological stability framework for E_1-modules over E_2-algebras, generalizing the framework for groups [**RWW17**] to topological moduli spaces. Galatius, Kupers and Randal-Williams [**GKRW19a**] gave a conceptual explanation for the occurrence of stability patterns using E_k-cellular techniques. This new approach to homological stability lead to the discovery of "secondary" stability patterns in the homology of mapping class groups [**GKRW19b**], new homological stability results for general linear groups over finite fields [**GKRW18**] and novel information about the unstable homology of general linear groups over infinite fields [**GKRW20**].

Homology stability techniques found useful generalizations in other areas of mathematics (e.g. [**CF13**]). Recently, Hepworth [**Hep20**], Boyd–Hepworth [**BH20**] and Boyd–Hepworth–Patzt [**BHP20**] proposed and demonstrated that, in analogy to groups and spaces, Quillen's approach to prove homological stability results can also be applied to detect patterns in the homology of sequences of abstract algebras $\{A_n\}_{n\in\mathbb{N}}$. In their work, they proved stability results for Iwahori–Hecke [**Hep20**], Temperley–Lieb [**BH20**] and Brauer algebras [**BHP20**], respectively. The main part of this thesis, occupying the first four chapters, extends and builds on these ideas.

We will now explain the general setting in which the work presented in this thesis takes place. Thereafter, we state and explain the main results obtained in each chapter using the intuition from the perhaps more familiar situation in the category of spaces and groups.

General mathematical setting

Let R be a unital commutative ring.

Definition 1. *An augmented R-algebra (A, ϵ) is a R-algebra A together with an algebra morphism*

$$\epsilon : A \to R.$$

The examples of augmented R-algebras that we will study in this thesis are the following:

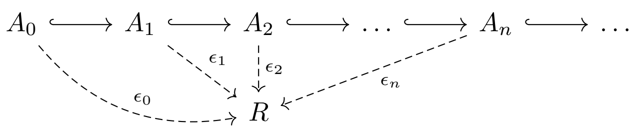
- the Temperley–Lieb algebras $(\mathcal{TL}_n(a), \epsilon)$ with parameter $a \in R$,
- the group ring $(R[G], \epsilon)$, equipped with the trivial augmentation $\epsilon : g \mapsto 1$, of the symplectic group $G = \mathrm{Sp}_{2n}(\mathbb{Z})$,
- the Iwahori–Hecke algebras $(\mathcal{H}^q(W, S), \epsilon)$ of certain Coxeter groups (W, S) with deformation parameter $q \in R$.

The augmentation map of an augmented R-algebra gives R the structure of a one-dimensional representation, which we denote by $\mathbb{1}$. The homology and cohomology of an augmented R-algebra (A, ϵ) with coefficients in some A-module M can then be defined (e.g. [**Ben91**], Chapter 2) as

$$H_\star(A, M) = \mathrm{Tor}^A_\star(\mathbb{1}, M) \text{ and } H^\star(A, M) = \mathrm{Ext}^\star_A(\mathbb{1}, M).$$

For group algebras $R[G]$ with trivial augmentation, this is exactly the definition of group homology.

By a sequence or family of augmented R-algebras (A_n, ϵ_n) we will usually mean a commutative diagram of the following shape in the category of R-algebras.



Such a diagram induces a sequence of morphisms between the i-th homology groups for any $i \in \mathbb{N}$ (similarly for cohomology)

$$H_i(A_n, \mathbb{1}) \longrightarrow H_i(A_{n+1}, \mathbb{1}).$$

We say that the sequence $\{(A_n, \epsilon_n)\}$ satisfies homological stability, if these maps are isomorphisms whenever n is sufficiently large compared to i. For families of group algebras with trivial augmentations and inclusion maps $R[G_n] \hookrightarrow R[G_{n+1}]$ induced by the inclusion of subgroups $G_n \subset G_{n+1}$, this recovers the notion of homological stability for groups discussed in the first part of this introduction. For abstract R-algebras, this is the type of stability result proved by Hepworth [**Hep20**] for Iwahori–Hecke algebras and Boyd–Hepworth–Patzt

[**BHP20**] for the homology of Brauer algebras. In [**BH20**], Boyd–Hepworth proved a vanishing line for the homology of certain Temperley–Lieb algebras which implies that the "stable" homology of these algebras is trivial. In particular, Temperley–Lieb algebras satisfy homological stability as well.

Overview

We will now survey the individual chapters of this thesis.

A note on algebraic coset posets. Let A be an algebra, $\mathfrak{A}$ a set of subalgebras A_i of A and M an A-module. In the first chapter, we describe how one can construct a "geometrically flavored" semi-simplicial A-module

$$CP(A, \mathfrak{A}, M)$$

from this data, which we call the algebraic coset poset of $(A, \mathfrak{A}, M)$. This is analogous to the notion of a coset poset for groups considered in the literature. Let G be a group and $\mathfrak{G}$ a set of subgroups of G. The coset poset $CP(G, \mathfrak{G})$ is the poset with the set of cosets in G of the subgroups contained in $\mathfrak{G}$ as underlying set and order relation given by inclusion of cosets. Topological properties of this G-space have been studied by many authors [**AH93**, **BW20**, **Bro00**, **Brü20b**, **Brü20a**, **Ram05**, **SW16**, **Wel18**]. The most important instance of a coset poset for groups in the context of this thesis is the Davis complex of a Coxeter group W [**Dav08**]. This is a contractible W-space [**Dav08**] that plays an important role in the low-dimensional homology calculations for Coxeter groups of Boyd [**Boy20**]. Our interest in algebraic analogues of coset posets stems from the work of Boyd, Hepworth and Patzt on homological stability for families of algebras [**Hep20**, **BH20**, **BHP20**]. Quillen's classical approach to homological stability for sequences of groups $\{G_n\}_{n\in\mathbb{N}}$ uses the existence of a highly connected G_n-space for any $n \in \mathbb{N}$. The stability result is then derived by studying an isotropy spectral sequence attached to these complexes. In their work on homological stability for families of algebras, Boyd, Hepworth and Patzt use algebraic analogues of such G_n-spaces to derive their results. In the two subsequent chapters, we will similarly use algebraic analogues of the Davis complex of a Coxeter group to study patterns in the homology of algebras. We will derive our results by studying "isotropy" spectral sequences attached to this "space with A-action", that are analogous to the isotropy spectral sequences of the classical Davis complex that Boyd studied in [**Boy20**].

Algebraic coset posets and vanishing theorems for the homology of Temperley–Lieb algebras. In Chapter 2, we study the homology $H_\star(\mathcal{TL}_n(a), \mathbb{1})$ of the family of augmented Temperley–Lieb algebras $\{(\mathcal{TL}_n(a), \epsilon_n)\}_{n\in\mathbb{N}}$. The main result of this chapter is the following improvement of a vanishing theorem of Boyd–Hepworth [**BH20**].

Theorem. *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$ be odd. Consider a Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands with parameter a. Then,*

$$H_0(\mathcal{TL}_n(a), \mathbb{1}) = \mathbb{1} \text{ and } H_\star(\mathcal{TL}_n(a), \mathbb{1}) = 0 \text{ for } \star > 0.$$

Boyd–Hepworth proved this for fields R of a certain characteristic and parameter $a = 0$ ([**BH20**], Theorem D). Building on their work, Randal-Williams [**RW21**] removed both assumptions, but an invertibility condition on certain elements in R remained. We show that none of these assumptions are necessary and formulate a different argument for the results of Boyd–Hepworth, and the strengthening obtained by Randal-Williams. Our strategy is similar to Boyd's work [**Boy20**]. We define an algebraic Davis poset for Temperley–Lieb algebras in the sense of Chapter 1, prove that it is contractible and study the "isotropy" spectral sequence of this "space with $\mathcal{TL}_n(a)$-action".

Algebraic coset posets and the low-dimensional homology of Iwahori–Hecke algebras. In Chapter 3, we study the homology groups $H_\star(\mathcal{H}^q_n, \mathbb{1})$ of the family of augmented Iwahori–Hecke algebras $\{(\mathcal{H}^q_n, \epsilon_n)\}_{n\in\mathbb{N}}$ associated to a Coxeter group (W, S) of finite rank $|S| < \infty$. These homology groups have been studied by Benson–Erdmann–Mikaelian in [**BEM10**] and by Hepworth in [**Hep20**]. We prove the Iwahori–Hecke analogue of a formula obtained by Boyd for the second homology of the Coxeter group (W, S) [**Boy20**]. This can also be seen as the Iwahori–Hecke analogue of a theorem of Howlett [**How88**].

THEOREM. *Let $q \neq -1$ be a unit in an integral domain R and $\mathcal{H} = \mathcal{H}^q(W, S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$, then we have the following natural identification:*

$$H_2(\mathcal{H}, \mathbb{1}) \cong H_0(\mathcal{D}^X_{A_3}, R_{(1+q,2)}) \oplus H_0(\mathcal{D}^X_{\neg A_3}, \mathscr{L}) \oplus R_{(1+q)}\{E(\mathcal{D}_{even})\} \oplus H_1(\mathcal{D}_{odd}, R_{(1+q)})$$

Here, $\mathcal{D}_{odd}, \mathcal{D}_{even}, \mathcal{D}^X_{\neg A_3}$ and $\mathcal{D}^X_{A_3}$ are certain graphs attached to the Coxeter system (W, S), $\mathscr{L}$ is a local coefficient system, and $1 + q \in R$ is the second q-integer.

Our strategy for proving this theorem is a direct generalization of the approach that Boyd employed in [**Boy20**]. We will define an algebraic Davis poset for Iwahori–Hecke algebras in the sense of Chapter 1, prove that it is contractible and study the "isotropy" spectral sequence of this "space with $\mathcal{H}^q$-action".

Deformations of the free E_∞-algebra on a point. Chapter 4 is based on joint work with Richard Hepworth and Jeremy Miller. We specify conditions under which the "classifying space"

$$N_\bullet\mathcal{A} = \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1}, A_n, \mathbb{1})$$

of a family of augmented algebras $\{(A_n, \epsilon_n)\}_{n\in\mathbb{N}}$ admits the structure of an E_k-algebra. Here, $B_\bullet(\mathbb{1}, A_n, \mathbb{1})$ denotes the two sided bar-construction. In particular, if A_n is R-projective (see [**CE99**], Ch. II Proposition 5.3 and Ch. IX §6), then

$$H_\star(B_\bullet(\mathbb{1}, A_n, \mathbb{1})) \cong H_\star(A_n, \mathbb{1}).$$

In direct analogy with the group case discussed in the first part of the introduction, this requires a "braided structure". In order to apply the ideas of [**Fie, BFSV03, Ber99, May74, BE74, Smi89**], we need additional conditions that ensure the compatibility of the braiding with the augmentations $\{\epsilon_n\}_{n\in\mathbb{N}}$. We then apply our criterion to the family of augmented Iwahori–Hecke algebras $\{(\mathcal{H}^q(\Sigma_n), \epsilon_n)\}_{n\in\mathbb{N}}$ and prove the following theorem.

THEOREM. *Let R be a commutative unital ring and let $q \in R$ be a deformation parameter. Consider the associated sequence of Iwahori–Hecke algebras $\{(\mathcal{H}^q(\Sigma_n), \epsilon_n)\}_{n\in\mathbb{N}}$, where Σ_n denotes the symmetric group permuting n letters. If $q \in R$ is a unit, then there exists an E_2-algebra*

$$N_\bullet\mathcal{H}^q = \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1}, \mathcal{H}^q(\Sigma_n), \mathbb{1}) \in Alg_{E_2}(sMod_R)$$

in the category of simplicial R-modules, whose homology groups in charge n are exactly $H_\star(\mathcal{H}^q(\Sigma_n), \mathbb{1})$ i.e. the homology groups of $\mathcal{H}^q(\Sigma_n)$ with trivial coefficients. Furthermore,

i) if $q = 1$, then the E_2-structure extends to an E_∞-structure and $N_\bullet\mathcal{H}^1 = R[E_\infty(*)]$ is the R-linearization of the free E_∞-algebra on a point $*$.
ii) if $q = -1 \neq 1 \in R$, then the E_2-structure of $N_\bullet\mathcal{A}$ does not extend to an E_3-structure.

An alternative proof of a theorem of Gunnells. Chapter 5 is based on joint work with Benjamin Brück and Peter Patzt. In this last chapter, we implement an idea of Putman for a new proof of a theorem of Gunnells [**Gun00**] for the symplectic group $\mathrm{Sp_{2n}}(\mathbb{Z})$. The theorem implies that the rational cohomology group of the symplectic group $\mathrm{Sp_{2n}}(\mathbb{Z})$ vanishes in its virtual cohomological dimension, $vcd(\mathrm{Sp_{2n}}(\mathbb{Z})) = n^2$.

Theorem (cf. [**Gun00**], Corollary 4.12.). *Let* $n \geq 1$*, then* $H^{n^2}(\mathrm{Sp}_{2n}(\mathbb{Z});\mathbb{Q}) = 0$.

We combine connectivity results obtained by Putman [**Put09**] and Church–Putman [**CP17**] to formulate an alternative proof of the main technical result in [**Gun00**] and explain how the vanishing result follows. The strategy of proof that we employ is inspired by recent work of Church, Farb and Putman [**CFP19, CFP14**].

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CHAPTER 1

A note on algebraic coset posets

Summary: Inspired by recent work of Boyd and Hepworth, we introduce a notion of coset posets for abstract algebras. This generalizes the notion of coset poset for groups considered in the literature and gives a way to associate geometrically flavored semi-simplicial A-modules to certain algebras A. We use this construction to define algebraic analogues of the classical Coxeter and Davis complex of a Coxeter system. These semi-simplicial modules come equipped with a filtration that is reminiscent of the Coxeter cell structure on the classical Davis complex. In the two subsequent chapters, we will use algebraic coset posets to study the homology of Temperley–Lieb algebras and the homology of Iwahori–Hecke algebras following ideas of Boyd, Hepworth and Patzt.

1. Introduction

Let G be a group and let $\mathfrak{H} = \{H_i\}$ be a finite set of subgroups $H_i \leq G$. We start by explaining two well-known constructions, the coset poset and the coset complex, that associate a G-space to this datum.

Definition 2 (Coset poset). *The coset poset $CP(G, \mathfrak{H})$ is the poset with underlying set*

$$\{gH_i : g \in G \text{ and } H_i \in \mathfrak{H}\}$$

the set of cosets in G of the subgroups contained in $\mathfrak{H}$ and order relation given by inclusion of cosets $g_iH_i \subseteq g_jH_j$.

Taking the nerve of the coset poset $CP(G, \mathfrak{H})$ and geometric realization, we obtain a space that we also denote by $CP(G, \mathfrak{H})$. A different way of constructing a G-space from $(G, \mathfrak{H})$ is the following.

Definition 3 (Coset complex). *View the set of cosets $\{gH_i : g \in G \text{ and } H_i \in \mathfrak{H}\}$ as a set-cover of the group G. The coset complex $CC(G, \mathfrak{H})$ is the simplicial complex that arises as the nerve of this set-cover i.e. a k-simplex of $CC(G, \mathfrak{H})$ is a set of $k+1$ cosets $\{g_iH_i : 0 \leq i \leq k\}$ whose intersection is nonempty*

$$\bigcap_{i=0}^{k} g_iH_i \neq \emptyset.$$

Taking geometric realization of the abstract simplicial complex $CC(G, \mathfrak{H})$, we obtain a space that we also denote by $CC(G, \mathfrak{H})$. Topologicial properties of coset posets and coset complexes for groups have been studied by many authors [**AH93**, **BW20**, **Bro00**, **Brü20b**, **Brü20a**, **Ram05**, **SW16**, **Wel18**]. Important complexes related to Coxeter groups and Artin groups arise via these constructions [**Dav08**, **CD95**]. It follows from work of Abels–Holz [**AH93**], Theorem 1.4 (b), that the homotopy type of any coset complex can be encoded in a coset poset. If $\tilde{\mathfrak{H}}$ denotes the set of subgroups consisting of all finite intersections of subgroups contained in $\mathfrak{H}$, then there is a homotopy equivalence[1]

$$CP(G, \tilde{\mathfrak{H}}) \simeq CC(G, \mathfrak{H}).$$

Let R be a commutative ring. The goal of this chapter is to generalize the notion of coset poset for groups and define a similar notion for abstract R-algebras. An algebraic coset poset will be a semi-simplicial A-module that is constructed from an algebra A, a finite set of subalgebras $\mathfrak{A} = \{S_i\}$ and an A-module M. This is motivated by work of Hepworth [**Hep20**], Boyd–Hepworth [**BH20**] and Boyd–Hepworth–Patzt [**BHP20**], who defined algebraic analogues of the complex of injective words - a classical object within the area of homological stability studied in [**Far79**, **Maa79**, **BW83**, **Ker05**, **RW13**] - for the families of Iwahori–Hecke, Temperley–Lieb and Brauer algebras, respectively. They then used the algebraic analogues to study the homology and cohomology of these algebras.

The method described in this chapter can be used to define algebraic analogues of coset posets arising in geometric group theory and algebraic topology. In the two subsequent chapters, we will encounter algebraic versions of Coxeter and Davis complexes that arise this way. Our interest in algebraic analogues of these complexes stems from work of Boyd [**Boy20**], where the classical Davis complex and its "Coxeter CW structure" is used to calculate homology groups of Coxeter groups. We will use the algebraic analogues and techniques developed by Boyd [**Boy20**] and Davis [**Dav08**] to study the homology of Temperley–Lieb and Iwahori–Hecke algebras in a similar spirit as Hepworth [**Hep20**] and Boyd–Hepworth [**BH20**].

[1] See [**Brü20a**], Remark 3.5, for an explanation of how this follows from [**AH93**], Theorem 1.4 (b).

Future work. Using the notion of algebraic coset poset defined in this chapter it might be possible ask and investigate questions for families of R-algebras that have been studied for coset posets and coset complexes in the literature. Boyd–Hepworth demonstrated that the complex of injective words for Temperley–Lieb algebras constructed in [**BH20**] has interesting combinatorial properties [**BH21**]. Their work is inspired by results of Reiner–Webb for group algebras [**RW04**]. In a similar fashion, it might be interesting to study algebraic coset posets from a purely combinatorial perspective.

2. Algebraic analogues of coset posets

After giving an alternative definition of coset posets for groups from a categorical perspective, we introduce algebraic coset posets for R-algebras.

2.1. Coset posets for groups from a categorical perspective. Let G be a group and $\mathfrak{G}$ a finite set of subgroups. We consider $\mathfrak{G}$ as a small category with morphisms given by inclusions.

Definition 4. *Let* **G-set** *denote the category of G-sets. The coset system associated to $(G, \mathfrak{G})$ is the functor $CS(\mathfrak{G}) : \mathfrak{G} \to$* **G-set** *that assigns $H \mapsto G/H$ for all $H \in \mathfrak{G}$ and maps the inclusion $H \hookrightarrow H'$ to the induced G-equivariant map $G/H \to G/H'$.*

We will now explain how to extract the coset poset of $(G, \mathfrak{G})$ from the associated coset system.

Definition 5 (compare with 9.1 [**Hir14**] and Ch. XII, §5.1 [**BK72**]). *Let $\mathcal{P}$ be a poset (viewed as a small category) and let $\mathcal{D}$ be a category with coproducts. We write $\mathrm{ss}(\mathcal{D})$ for the category of semi-simplicial objects in $\mathcal{D}$. Consider a diagram $F : \mathcal{P} \to \mathcal{D}$. We call the semi-simplicial object $\sqcup_\star(F) \in \mathrm{ss}(\mathcal{D})$ with k-simplices*

$$(\sqcup_\star F)_k = \bigsqcup_{y_\bullet \in N_k(\mathcal{P})} F(y_0)$$

the semi-simplicial replacement of F. Here, $N_k(\mathcal{P}) = \{y_0 \lneq \cdots \lneq y_k : y_i \in \mathcal{P}\}$ is the set of k-simplices of the order complex of $\mathcal{P}$. The i-th face map of $\sqcup_\star F$ is induced by the i-th face map of the order complex of $\mathcal{P}$. Recall that these are given by omitting the i-th entry of a flag $y_0 \lneq \cdots \lneq y_k$. For the 0-th face map one composes with the arrow $F(y_0) \to F(y_1)$ coming from $y_0 \leq y_1$.

Definition 6. *The coset poset $CP(G, \mathfrak{G})$ of $(G, \mathfrak{G})$ is to the semi-simplicial replacement $\sqcup_\star CS(\mathfrak{G})$ of the coset system $CS(\mathfrak{G}) : \mathfrak{G} \to$* **G-set** *of $(G, \mathfrak{G})$.*

Unraveling Definition 6 shows that the coset poset defined there is isomorphic to the order complex, viewed as a semi-simplicial set, of the coset poset introduced in Definition 2. The advantage of this categorical perspective is that it readily generalizes to other settings as we will see now.

2.2. Coset posets for families of R-algebras. Let R be a commutative ring. Given an R-algebra A and a finite set of subalgebras $\mathfrak{A} = \{S_i\}$, one can construct a similar geometrically flavored object if one additionally fixes an A-module M. In the following, we consider the family of subalgebras $\mathfrak{A}$ as a category with morphisms given by the inclusions.

Definition 7. *Let* **A-mod** *denote the category of A-modules. The coset system associated to $(A, \mathfrak{A}, M)$ is the functor $CS(A, \mathfrak{A}, M) : \mathfrak{A} \to$* **A-mod** *that assigns $S \mapsto A \otimes_S M$ and maps the inclusion $S \hookrightarrow S'$ to the induced map $A \otimes_S M \to A \otimes_{S'} M$.*

If it is clear from the context what A is, we will sometimes use the notation $CS(\mathfrak{A}, M) = CS(A, \mathfrak{A}, M)$.

Definition 8. *The coset poset* $CP(A, \mathfrak{A}, M) \in$ **ss(A-mod)** *of* $(A, \mathfrak{A}, M)$ *is the semi-simplicial* A*-module*

$$CP(A, \mathfrak{A}, M) = \sqcup_\star CS(A, \mathfrak{A}, M)$$

that is construct as the semi-simplicial replacement of the coset system

$$CS(A, \mathfrak{A}, M) : \mathfrak{A} \to \textbf{A-mod}.$$

If it is clear from the context what A is, we will sometimes use the notation $CP(\mathfrak{A}, M) = CP(A, \mathfrak{A}, M)$.

Definition 9. *The homology* $H_\star(CP(A, \mathfrak{A}, M))$ *of the coset poset* $CP(A, \mathfrak{A}, M)$ *is the homology of the following chain complex in the category of* A*-modules. The* k*-th chain module of* $(CP(A, \mathfrak{A}, M), \delta)$ *is the module of* k*-simplices of the underlying coset poset*

$$CP(A, \mathfrak{A}, M)_k = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ \in N_k(\mathfrak{A})}} A \otimes_{y_0} M.$$

The differential δ *is the alternating sum of the face maps of the coset poset*

$$\delta = \sum_{i=0}^{k} (-1)^i d_i.$$

Remark 1. *The homology of a coset poset* $CP(\mathfrak{A}, M)$ *is essentially an instance of functor homology. We invite the interested reader to compare our definitions with page 155 et seqq. of* [**Qui78**] *and section 16.2.2 of* [**Ric20**].

We end this subsection by recording the relationship between the coset poset for families of groups and families of algebras.

Lemma 1. *Let* R *be a commutative ring,* G *be a group and* $\mathfrak{G}$ *a set of subgroups. Let* RG *be the group ring,* $R\mathfrak{G} = \{RH\}_{H \in \mathfrak{G}}$ *and* $\mathbb{1}$ *the trivial* G*-module. Then,*

$$CP(RG, R\mathfrak{G}, \mathbb{1}) \cong R[-] \circ CP(G, \mathfrak{G}) \text{ as semi-simplicial } G\text{-modules,}$$

where $R[-]$: **G-set** $\to$ **RG-mod** *is the* R*-linearization functor.*

Proof: This follows by unraveling the definition. □

3. Algebraic analogues of Coxeter and Davis complexes

In this section, we explain the construction of the two main examples of algebraic coset posets that occur in this thesis. These algebraic coset posets are abstractions of two CW-complexes associated to a Coxeter system (W, S) (see Definition 32). The algebraic Davis poset will be a generalization of the complex Σ, called the Davis complex, defined in [**Dav08**], Chapter 7. The algebraic Coxeter poset generalizes the classical notion of the Coxeter complex of (W, S). A definition of the "classical" Davis and Coxeter poset for groups is given in the following example. Figure 1 displays both constructions for the symmetric group Σ_3 permuting three elements. In this case, $(W, S) = (\Sigma_3, \{s_0, s_1\})$, where s_0 and s_1 denotes the two standard generators.

Example 1. *Let* (W, S) *be a Coxeter system with a finite generating set* $|S| < \infty$ *(see Definition 32). A subset of* $F \subset S$ *is called spherical if the subgroup* W_F *generated by* F *is finite.*

i) *Let* $Y = \mathfrak{D}$ *be the poset of spherical subsets of* S*. Equivalently,* $\mathfrak{D}$ *is the poset of spherical subgroups* $\{W_F\}_{F \in \mathfrak{D}}$ *of* W*. The classical Davis complex of* (W, S) *is the coset poset associated to* $(W, \mathfrak{D})$ *in the sense of Definition 2 or Definition 6 (see* [**Dav08**]*, Chapter 7).*

*ii) Let $Y = \mathfrak{C}$ be the poset of all proper (possibly empty) subsets of S. Equivalently, $\mathfrak{C}$ is the poset of standard subgroups $\{W_F\}_{F \in Y}$. The barycentric subdivision of the classical Coxeter complex of (W, S) is the coset poset $CP(W, \mathfrak{C})$ associated to $(W, \mathfrak{C})$ in the sense of Definition 2 or Definition 6 (e.g. [**AB08**], Definition 3.1. or [**BB05**], §3 Exercise 16). We will call $CP(W, \mathfrak{C})$ the Coxeter poset in the following.*

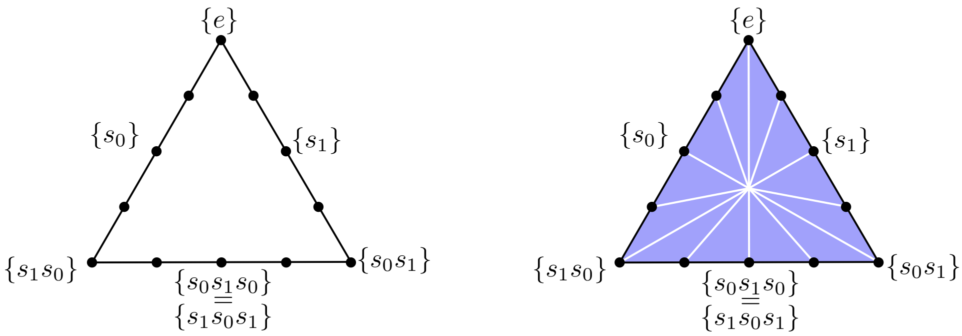

FIGURE 1. Coxeter poset (left) and Davis complex (right) of $(\Sigma_3, \{s_0, s_1\})$.

The "classical" Davis complex comes equipped with "Coxeter CW-filtration" (see [**Dav08**], Chapter 7). For the "classical" Davis complex of $(\Sigma_3, \{s_0, s_1\})$, the CW-filtration is illustrated in Figure 2. The algebraic coset posets that we define in this section will be equipped with a filtration of semi-simplicial A-modules, which is reminiscent of this geometric CW-structure. The algebraic "Coxeter CW-filtration" will play an important role in the next two chapters of this thesis.

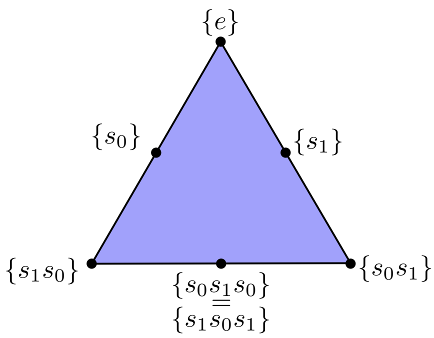

FIGURE 2. Cellular Davis complex of $(\Sigma_3, \{s_0, s_1\})$.

3.1. Construction. Let S be a set with n elements and 2^S its power set, which we consider as a poset using the inclusion of subsets. Let Y be a subposet of 2^S, A be a R-algebra and $\hat{\mathfrak{A}} = \{A_i : i \in S\}$ be a set of subalgebras.

DEFINITION 10. *For any set $y \in Y$, we write $A_y = \langle A_i : i \in y \rangle$ for the smallest subalgebra of A containing all A_i with $i \in y$. Let $\mathfrak{A} = \{A_y : y \in Y\}$ for the set of subalgebras indexed by Y. We assume that $\mathfrak{A}$ has the property that if $y \neq y' \in Y$, then $A_y \neq A_{y'}$. In particular, we will identify Y and $\mathfrak{A}$.*

EXAMPLE 2. *We explain the R-linear analogue of Example 1. Let $A = R[W]$ be the group algebra and $\hat{\mathfrak{A}} = \{A_{\{s\}} = R[W_{\{s\}}] : s \in S\}$ be the set of subalgebras corresponding to the group algebras of the subgroups of W generated by each individual generator $s \in S$.*

i) Let $Y = \mathfrak{D}$ be the poset of spherical subsets of S. Then, $\mathfrak{D} = \{R[W_F] : F \in \mathfrak{D}\}$ can be identified with the poset of group algebras of spherical subgroups of W.
ii) Let $Y = \mathfrak{C}$ be the poset of proper subsets of S. Then, $\mathfrak{C} = \{R[W_F] : F \in \mathfrak{C}\}$ can be identified with the poset of group algebras of standard subgroups of W.

DEFINITION 11. *Let M be an A-module. Then, we can define a coset system*

$$CS(Y, M) : Y \to \mathbf{A\text{-}mod} : y \mapsto A \otimes_{A_y} M$$

from which we obtain an algebraic coset poset $CP(Y, M) \in \mathbf{ss(A\text{-}mod)}$.

EXAMPLE 3. *We continue in the setting of Example 2 and pick $M = \mathbb{1}$ the trivial $R[W]$-module. Then,*

i) the algebraic coset poset $CP(R[W], \mathfrak{D}, \mathbb{1})$ arising from Definition 11 is exactly the R-linearization of the classical Davis complex (see Example 1 and Lemma 1).
ii) the algebraic coset poset $CP(R[W], \mathfrak{C}, \mathbb{1})$ arising from Definition 11 is exactly the R-linearization of (the subdivision of) the Coxeter complex (see Example 1 and Lemma 1).

The poset Y is naturally ranked via the poset map given by the cardinality function

$$| - | : Y \to \mathbb{N} : y \mapsto |y|$$

and therefore admits a natural filtration.

DEFINITION 12. *Let $\alpha \in \mathbb{N}$ be a natural number and let $F^\alpha Y$ be the subposet of elements $y \in Y$ of cardinality $|y| \leq \alpha$. Then, Y is filtered by $(F^\alpha Y)_{\alpha \in \mathbb{N}}$ and $F^\alpha Y = Y$, if $\alpha \geq n$.*

It follows that any coset system and algebraic coset poset associated to Y also admits such a filtration. This filtration is the algebraic analogue of the "Coxeter CW-filtration" on the classical Davis complex (see [**Dav08**], Chapter 7).

DEFINITION 13. *The coset system $CS(Y, M)$ is filtered by coset systems*

$$C(F^\alpha Y, M) : F^\alpha Y \to \mathbf{A\text{-}mod}$$

and, by functoriality of semi-simplicial replacements, the coset poset $CP(Y, M) \in \mathbf{ss(A\text{-}mod)}$ is filtered by semi-simplical A-modules $CP(F^\alpha Y, M) \in \mathbf{ss(A\text{-}mod)}$.

Associated to the filtration of the coset poset $CP(Y, M)$ is a spectral sequence, which we call the recellulation spectral sequence because it implements the passage to the "Coxeter CW-structure" if one inputs the classical Davis complex (see Example 4).

PROPOSITION 1 (Recellulation spectral sequence). *The filtration of the algebraic coset poset $CP(Y, M)$ by semi-simplical A-modules $CP(F^\alpha Y, M) \in \mathbf{ss(A\text{-}mod)}$ gives rise to a spectral sequence converging to the homology of $CP(Y, M)$ with E^0-page given by*

$$E^0_{\alpha,\star} = CP(F^\alpha Y, M)_{\alpha+\star} / CP(F^{\alpha-1} Y, M)_{\alpha+\star} \cong \bigoplus_{|F|=\alpha} E(Y, M)_{F,\star}$$

where $E(Y, M)_{F,\star}$ is the chain complex defined in Definition 14.

DEFINITION 14. *Let $F \in Y$ be a set of rank $|F| = \alpha$. Let $(Y/F)_k$ denote the subset of k-simplices $y_\bullet = y_0 \lneq \cdots \lneq y_k$ of $N_k(Y)$ satisfying $y_k = F$, then*

$$E(Y, M)_{F,\beta} = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_{\alpha+\beta} = F \\ \in (Y/F)_{\alpha+\beta}}} A \otimes_{A_{y_0}} M$$

with differential given by

$$\delta = \sum_{i=0}^{(\alpha+\beta)-1} (-1)^i d_i$$

where d_i denotes the i-th face map of the algebraic coset poset $CP(Y, M)$.

Proof of Proposition 1: The filtration of the algebraic coset poset $CP(Y, M)$ gives rise to a filtered differential graded module CP in the sense of Definition 2.5. in [**McC01**] with

i) $CP = \bigoplus_{k=0}^{\infty} CP(Y, M)_k$
ii) differential $d : CP \to CP$ of degree (-1) given by $\delta : CP(Y, M)_k \to CP(Y, M)_{k-1}$, where
$$\delta = \sum_{i=0}^{k} (-1)^k d_i,$$
iii) and a filtration $F^\alpha(CP) = \bigoplus_{k=0}^{\infty} CP(F^\alpha Y, M)_k$ that the differential respects.

Applying Theorem 2.6. of [**McC01**] yields a spectral sequence that, because the filtration is bounded, converges to the homology of the algebraic coset poset $CP(Y, M)$. The E^0-page of this spectral sequence is described by

$$E^0_{\alpha,\beta} = F^\alpha(CP)_{\alpha+\beta}/F^{\alpha-1}(CP)_{\alpha+\beta} = CP(F^\alpha Y, M)_{\alpha+\beta}/CP(F^{\alpha-1}Y, M)_{\alpha+\beta}.$$

By Definition 8,

$$CP(F^\alpha Y, M)_{\alpha+\beta} = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_{\alpha+\beta} \\ \in F^\alpha Y}} A \otimes_{A_{y_0}} M.$$

Note that this term is trivial, if $\beta > 0$. Observe that $CP(F^{\alpha-1}Y, M)_{\alpha+\beta}$ is the submodule indexed by flags $y_\bullet \in F^\alpha Y$ in which each subset satisfies $|y_i| \leq \alpha - 1$. It follows that any summand not contained in $CP(F^{\alpha-1}Y, M)_{\alpha+\beta}$ is indexed by a flag $y_\bullet$ that contains a subset F of cardinality α and that we must have $y_{\alpha+\beta} = F$. We hence have shown that

$$E^0_{\alpha,\beta} = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_{\alpha+\beta} \\ |y_{\alpha+\beta}| = \alpha}} A \otimes_{A_{y_0}} M.$$

The E^0-differential is induced by the differential $d = \sum_{i=0}^{\alpha+\beta} (-1)^i d_i$ defined in item ii) above. Observe that the contribution of the differential $d_{\alpha+\beta}$ is always zero, because its image is in $CP(F^{\alpha-1}Y, M)$, and that the top-element in the flag $y_0 \lneq \cdots \lneq y_{\alpha+\beta}$ is preserved by the remaining partial differentials. It follows that the chain complexes on the E^0-page split

$$E^0_{\alpha,\star} = \bigoplus_{|F|=\alpha} E(Y, M)_{F,\star}$$

where the chain complex $E(Y, M)_{F,\star}$ is as in Definition 14. □

Example 4. *We continue in the setting of Example 3 and consider the recellulation spectral sequence for the R-linearization of the classical Davis complex $CP(R[W], \mathfrak{D}, \mathbb{1})$ of the Coxeter group W. One can show that the homology of $E(\mathfrak{D}, \mathbb{1})_{F,\star}$ is concentrated in degree zero and that the chain complex $E^1_{\star,0}$ appearing on the E^1-page is exactly the cellular chain complex of the "Coxeter CW-structure" on the classical Davis complex.*

We close this chapter by defining the Coxeter, Davis and cellular Davis complex of an R-algebra to which the constructions above apply. We do not know whether this definition is useful in this generality, but it is a convenient way to package what we mean by these terms in the following two chapters.

Definition 15. *Let A be a R-algebra, S be a finite set, $\hat{\mathfrak{A}} = \{A_i : i \in S\}$ be a set of subalgebras and M be an A-module.*

i) *An algebraic Davis poset of A with coefficients in M is an algebraic coset poset $CP(Y, M)$, where Y is a subposet of 2^S as above.*

ii) *If $Y = \mathfrak{C}$ is the poset of all proper subsets of S, we call $CP(\mathfrak{C}, M)$ the Coxeter poset of A with coefficients in M.*

iii) *Assume that the E^1-page of the recellulation spectral sequence of an algebraic Davis complex of A with coefficients in M is concentrated in the zero-th row $E^1_{\star,0}$, then the associated cellular Davis complex with coefficients in M is the chain complex $E^1_{\star,0}$.*

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CHAPTER 2

Algebraic coset posets and vanishing theorems for the homology of Temperley–Lieb algebras

Summary: We prove that the homology of any Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on an odd number of strands vanishes in positive degrees. This improves a result obtained by Boyd–Hepworth. We also give alternative arguments for the following two vanishing results of Boyd–Hepworth. (1) The stable homology of Temperley–Lieb algebras is trivial. (2) If the parameter $a \in R$ is a unit, then the homology of any Temperley–Lieb algebra is concentrated in degree zero. In the appendix of this chapter, we show that the $\mathcal{TL}_n(a)$-complex, which we use to prove these theorems, is the cellular Davis complex of Temperley–Lieb algebras in the sense of Chapter 1, Definition 15, and that the algebraic Davis and Coxeter complex of a Temperley–Lieb algebra are contractible.

1. Introduction

Let R be a commutative unital ring. Intuitively, the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands is the R-algebra, whose underlying R-module has a basis given by isotopy classes of planar diagrams. The multiplication of two planar diagrams is explained by gluing them together. Any circles appearing in the resulting diagram correspond to multiplication by $a \in R$. This is illustrated in Figure 1, a precise definition is given in the next section.

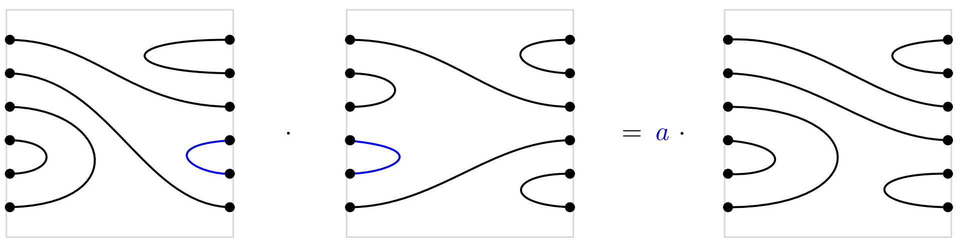

FIGURE 1. Multiplication of two planar diagrams in $\mathcal{TL}_6(a)$

The Temperley–Lieb algebra $\mathcal{TL}_n(a)$ has a natural augmentation $\epsilon_n : \mathcal{TL}_n(a) \to R$ that maps all non-identity planar diagrams to zero. The 1-dimensional representation ϵ_n gives rise to the trivial module $\mathbb{1}$. In this chapter, we examine the homology of Temperley–Lieb algebras with coefficients in this module

$$H_\star(\mathcal{TL}_n(a), \mathbb{1}) = \mathrm{Tor}^{\mathcal{TL}_n(a)}(\mathbb{1}, \mathbb{1}).$$

These homology groups have been studied by Boyd–Hepworth in [BH20]. The main finding of this chapter is the following theorem, which removes all conditions in the vanishing results for the homology of Temperley–Lieb algebras on an odd number of strands obtained by Boyd–Hepworth.

THEOREM A. *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$ be odd. Consider the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands. Then*

$$H_0(\mathcal{TL}_n(a), \mathbb{1}) = \mathbb{1} \text{ and } H_\star(\mathcal{TL}_n(a), \mathbb{1}) = 0 \text{ for } \star > 0.$$

Previously known vanishing results: Boyd–Hepworth proved that the homology of a Temperley–Lieb algebra on an odd number of strands is trivial in homological degrees $1 \leq d \leq n-1$ as part of their Theorem B, [BH20], and that the homology is zero for all positive degrees if the underlying ring R is a field of a certain characteristic ([BH20], Theorem D). In both cases, they assume that the parameter $a \in R$ used to define $\mathcal{TL}_n(a)$ is of a specific form and, in ([BH20], Theorem D), that certain binomials $\binom{k}{r}$ are units in the field R [1]. In the document [RW21], Randal–Williams builds on the work Boyd–Hepworth and shows that the assumptions on the parameter $a \in R$ can be removed using base change techniques (see Theorem B', [RW21]). An invertibility condition on the binomials $\binom{k}{r}$ remains (see item (ii) before Corollary 3.2, [RW21]).

[1]Theorem B and D, [BH20], assume that $a = v + v^{-1}$ for a unit $v \in R$. Theorem D, [BH20], additionally assumes that $a = 0$ and that R is a field, whose characteristic does not divide $\binom{k}{r}$ for $1 \leq r \leq k$ and $n = 2k + 1$.

The proof of Theorem A presented here removes all of these conditions and gives an alternative argument for the vanishing results due to Boyd–Hepworth, as well as the strengthening obtained by Randal–Williams.

In addition to Theorem A, our methods allow us to prove a vanishing line for the homology of Temperley–Lieb algebras on an even number of strands. This vanishing line is weaker than the one obtained by Boyd–Hepworth ([**BH20**], Theorem B). Boyd–Hepworth prove a slope 1 vanishing line, the one we prove in this chapter is of slope $\frac{1}{2}$.

Theorem B. *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$ be even. Consider the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands. Then, $H_0(\mathcal{TL}_n(a), \mathbb{1}) = \mathbb{1}$,*

$$H_\star(\mathcal{TL}_n(a), \mathbb{1}) = 0 \text{ for } 0 < \star < \frac{n}{2}$$

and

$$H_{\star+\frac{n}{2}}(\mathcal{TL}_n(a), \mathbb{1}) \cong H_\star(\mathcal{TL}_n(a), Cup(M))$$

for $\star \geq 0$, where $Cup(M)$ is a certain $\mathcal{TL}_n(a)$-module (see Definition 21 and Definition 23).

The coefficient $\mathcal{TL}_n(a)$-module $Cup(M)$ in Theorem B is, in general, not isomorphic to the Fineberg module appearing in the description of the higher homology groups obtained by Boyd–Hepworth [**BH20**], Theorem 5.1.

Together, Theorem A and B give an alternative proof of the fact that the homology of Temperley–Lieb algebras satisfies homological stability and that the stable homology is trivial. This also follows from the work of Boyd–Hepworth and is discussed in their paper [**BH20**]. Homological stability questions for groups play a fundamental role in algebraic topology and algebraic K-theory. This chapter can be seen as a contribution to the set of ideas formulated by Hepworth [**Hep20**], Boyd–Hepworth [**BH20**] and Boyd–Patzt–Hepworth [**BHP20**] that aim to extend techniques used to study the homology of families of groups or spaces to families of abstract algebras.

We also obtain an alternative proof for ([**BH20**], Theorem A), which shows that the homology of any Temperley–Lieb algebra is trivial in positive degrees if the parameter $a \in R$ is a unit.

Theorem C ([**BH20**], Boyd–Hepworth). *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$. Consider the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands. If $a \in R$ is a unit, then $H_0(\mathcal{TL}_n(a), \mathbb{1}) = \mathbb{1}$ and $H_\star(\mathcal{TL}_n(a), \mathbb{1}) = 0$ for $\star > 0$.*

Comparison with work of Boyd–Hepworth [**BH20**] and Boyd [**Boy20**]: The general strategy of proof that we employ in this chapter is similar to the one used by Boyd–Hepworth (i.e. we construct a certain highly connected $\mathcal{TL}_n(a)$-chain complex and study spectral sequences attached to it). The chain complex that we use is different from the complex of planar injective words studied by Hepworth–Boyd [**BH20**]. Indeed, we will use the "cellular Davis complex" $\mathcal{TLD}$ of the Temperley–Lieb algebra $\mathcal{TL}_n(a)$. The structure of the complex $\mathcal{TLD}$ is "sensitive" to the question whether the number of strings of a Temperley–Lieb algebra is even or odd. This is what enables us to prove Theorem A above. In the first part of this chapter, we will give an ad hoc definition of the complex $\mathcal{TLD}$ and formulate a direct proof that it is contractible. This makes the first part self-contained. In the appendix, we give a more conceptual description of $\mathcal{TLD}$ by proving that it is isomorphic to the cellular Davis complex of an algebraic coset poset in the sense of Chapter 1, Definition 15. This leads to a second, admittedly much more involved, proof that $\mathcal{TLD}$ is contractible and explains how the complex was discovered. From this perspective, the content of this chapter is inspired by and in direct analogy with the approach that Boyd employed in [**Boy20**] to derive formulas for the low-dimensional homology of Coxeter groups. Boyd's work [**Boy20**] was also a key source of inspiration for the content of the next chapter.

THEOREM D. *The complex $\mathcal{TL}\mathfrak{D}$ defined in Definition 22 is contractible and isomorphic to the cellular Davis complex of a family of Temperley–Lieb algebras with coefficients in $\mathbb{1}$ in the sense of Chapter 1, Definition 15.*

In the appendix and in preparation for the proof of Theorem D, we use algebraic discrete Morse theory [**Skö06, Skö18, Koz05**] to study the connectivity properties of the Coxeter and Davis poset of Temperley–Lieb algebras (see Chapter 1, Definition 15). We show that both are contractible.

THEOREM E. *Let $\mathcal{TL}_n(a)$ be an arbitrary Temperley–Lieb algebra. Then,*

i) the Coxeter poset $CP(\mathfrak{C}, \mathbb{1})$ of the Temperley–Lieb algebra is contractible.
ii) the Davis poset $CP(\mathfrak{D}, \mathbb{1})$ of the Temperley–Lieb algebra is contractible.

Future work. In this chapter, we only proved vanishing theorems. However, Boyd–Hepworth showed that for Temperley–Lieb algebras $\mathcal{TL}_n(a)$ on an even number of strings the homology group $H_{n-1}(\mathcal{TL}_n(a), \mathbb{1})$ is in general non-zero, if the parameter a is not a unit (Theorem C, [**BH20**]). The author would be interested to determine whether the homology of a Temperley–Lieb algebras $\mathcal{TL}_n(a)$ on an even number of strings has a concise description and whether one can say more about these homology groups using the methods described in this thesis.

Acknowledgments. It is a great pleasure to thank Rachael Boyd and Richard Hepworth for their interest, constructive feedback and for fruitful conversations.

2. Temperley–Lieb algebras

This section contains necessary background knowledge on Temperley–Lieb algebras that we mainly learned from Kassel–Turaev [**KT08**] and the exposition in Boyd–Hepworth [**BH20**]. In 1971, Temperley–Lieb introduced these algebras in their article [**TL71**].

DEFINITION 16 ([**KT08**], Definition 5.24; [**BH20**], Definition 2.1). *Let R be a commutative ring with 1, $a \in R$ and $n \in \mathbb{N}$. The Temperley–Lieb algebra $\mathcal{TL}_n = \mathcal{TL}_n(a)$ with parameter $a \in R$ is the R-algebra with generators $U_0, \dots, U_{n-2}$ and the following relations*

i) $U_iU_j = U_jU_i$, if $|i-j| \geq 2$.
ii) $U_iU_jU_i = U_i$, if $j = i \pm 1$.
iii) $U_i^2 = aU_i$ for all i.

The unit 1 corresponds to the empty product of the generators U_i. Note that $TL_0(a) = TL_1(a) = R$.

Jones [**Jon85**] used Temperley–Lieb algebras to define the polynomial invariant for knots, which we now call the Jones polynomial. The following diagrammatic interpretation is due to Kauffman [**Kau87, Kau90**]. We follow [**KT08**], Section 5.7.4, in our exposition.

DEFINITION 17. *A planar diagram of $n \geq 1$ arcs $D = \{\gamma_1, \dots, \gamma_n\}$ in $[0,1] \times \mathbb{R}$ is a disjoint union of n smoothly embedded arcs $\gamma_i : [0,1] \to [0,1] \times \mathbb{R}$ such that:*

i) The images of any two arcs γ_i and γ_j are pairwise disjoint.
ii) The points $\gamma_i(0)$ and $\gamma_i(1)$ are a subset of the points

$$\{(0,0), \dots, (0, n-1), (1,0), \dots, (1, n-1)\}.$$

iii) The tangent vector at $\gamma_i(0)$ and $\gamma_j(1)$ are parallel to the x-axis $\mathbb{R} \times 0$.

Let $a \in R$ and $P_n(a)$ be the free R-module spanned by the set of isotopy classes $[D]$ of planar diagrams D. We will now explain how the module $P_n(a)$ can be equipped with the structure of an associative R-algebra. Given two isotopy classes of planar diagrams $[D]$ and $[D']$, we obtain a diagram in $[0,1] \times \mathbb{R}$ by pasting D into $[0, \frac{1}{2}] \times \mathbb{R}$ and D' into $[\frac{1}{2}, 1] \times \mathbb{R}$. We can choose representatives $D \in [D]$ and $D' \in [D']$, such that the resulting diagram consist of a planar diagram $D \circ D'$ and $k(D, D') \geq 0$ circles. The product of $[D]$ and $[D']$ is defined as

$$[D] \cdot [D'] = a^{k(D,D')}[D \circ D'].$$

The reader is invited to revisit Figure 1.

Theorem 2 ([**KT08**], Theorem 5.34; [**BH20**], 2.1 ff.). *The Temperley–Lieb algebra $\mathcal{TL}_n$ is isomorphic to the R-algebra of planar diagrams $P_n(a)$. The isomorphism is given by mapping a generator $U_i \in \mathcal{TL}_n$ to the isotopy class of the following planar diagram.*

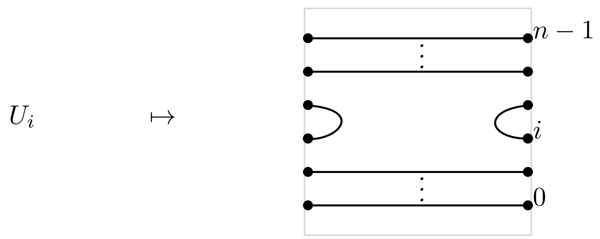

Figure 2. Translating between algebraic and diagrammatic definition of $\mathcal{TL}_n$

Definition 18. *An element $A \in \mathcal{TL}_n$ is called a planar diagram, if the isomorphism in Theorem 2 identifies A with an isotopy class $[D] \in P_n(a)$ of some planar diagram D.*

Let $\widehat{\mathcal{TL}}_n$ be the left $\mathcal{TL}_n$-submodule of $\mathcal{TL}_n$ spanned by the generators $\{U_i : 0 \leq i \leq n-2\}$. In the diagrammatic interpretation (see Theorem 2), $\widehat{\mathcal{TL}}_n$ is the submodule spanned by all planar diagrams except the identity diagram.

Definition 19. *The trivial module $\mathbb{1}$ for the Temperley–Lieb algebra $\mathcal{TL}_n$ is defined via the following exact sequence*

$$0 \to \widehat{\mathcal{TL}}_n \hookrightarrow \mathcal{TL}_n \twoheadrightarrow \mathbb{1} \to 0$$

Equivalently, the trivial module $\mathbb{1}$ is the one-dimensional $\mathcal{TL}_n$-module coming from the augmentation $\epsilon : \mathcal{TL}_n \to R$ that sends every generator U_i to zero.

3. Vanishing theorems for the homology of Temperley–Lieb algebras

This section builds on ideas contained in [**BH20**] and [**Boy20**]. We introduce the cellular Davis complex for Temperley–Lieb algebras $\mathcal{TLD}$ and give a direct proof that it is contractible. We then use it to derive the vanishing results for the homology of Temperley–Lieb algebras, which we stated as Theorem A, B and C in the introduction.

Let $\mathcal{TL} = \mathcal{TL}_{n+1}$ be the Temperley–Lieb algebra on $n+1$ strands with parameter $a \in R$ and let $U_0, \ldots, U_{n-1}$ be the standard generators. We sometimes identify $\{U_0, \ldots, U_{n-1}\}$ with the set $\langle n \rangle = \{0, \ldots, n-1\}$.

Definition 20.

i) A (possibly empty) subset $F \subseteq \langle n \rangle$ is called innermost, if for any

$$i \neq j \in F : |i - j| \geq 2.$$

ii) *Let $A \in \mathcal{TL}$ be a planar diagram in the sense of Definition 18 with corresponding isotopy class $[A] \in P_n(a)$. $A \in \mathcal{TL}_n$ is represented by certain monomials in the generating set $\{U_0, \dots, U_{n-1}\}$. We write*

$$F(A) = \{U_{i_k} : A = U_{i_1} \cdot \dots \cdot U_{i_k}\}$$

for the set of possible last letters in a monomial representing A. Using the identification in Theorem 2, the set $F(A)$ has the following diagrammatic interpretation:

$$F(A) = \{U_i : [A] \text{ has an arc connecting } (1, i) \text{ and } (1, i+1)\}$$

We call an arc connecting $(1, i)$ and $(1, i+1)$ a right cup at position i.

iii) *We call $F(A)$ the set of innermost right cups of the planar diagram $A \in \mathcal{TL}$. Note that this is an innermost set.*

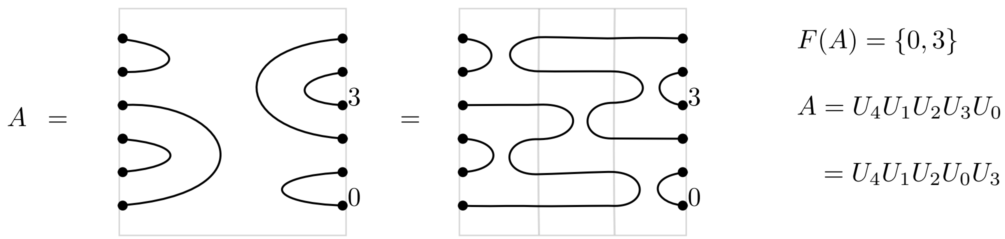

FIGURE 3. The set of innermost right cups of an element in $\mathcal{TL}_6$.

Note that by relation *i)* in Definition 16, all generators in an innermost set commute with each other. The following modules are therefore well-defined.

DEFINITION 21. *Let $F \subseteq \langle n \rangle$ be an innermost set. Then, we write $Cup(F)$ for the left submodule of $\mathcal{TL}$ generated by $\prod_{i \in F} U_i$. Using Theorem 2, this is the $\mathcal{TL}_n$-submodule of $\mathcal{TL}_n \cong P_n(a)$ spanned by all isotopy classes of diagrams $[D] \in P_n(a)$ that for any $i \in F$ have a right cup at position i.*

3.1. The cellular Davis complex of Temperley–Lieb algebras. Using the definitions above, we will now introduce and study the $\mathcal{TL}_n$-chain complex that enabled us to formulate our proofs for Theorem A, B and C.

DEFINITION 22. *The cellular Davis complex $(\mathcal{TL}\mathfrak{D}, \delta)$ of $\mathcal{TL}$ is the chain complex with*

$$\mathcal{TL}\mathfrak{D}_\alpha = \bigoplus_{\substack{F \subseteq \langle n \rangle \text{ innermost}, \\ |F| = \alpha}} Cup(F)$$

and differential

$$\delta_\alpha : \mathcal{TL}\mathfrak{D}_\alpha \to \mathcal{TL}\mathfrak{D}_{\alpha-1}$$

factorizes summand-wise as

$$Cup(F) \to \oplus_{s \in F} Cup(F_s) \hookrightarrow \mathcal{TL}\mathfrak{D}_{\alpha-1},$$

where $F_s = F - \{s\}$ for $s \in F$. The first arrow in this factorization is the map

$$A \mapsto \sum_{s \in F} (-1)^{\gamma(s)} \iota_F^{F_s}(A),$$

where $\iota_F^{F_s} : Cup(F) \hookrightarrow Cup(F_s)$ is the inclusion and $\gamma(s) = \gamma_F(s) = |\{s' \in F_s : s' > s\}|$.

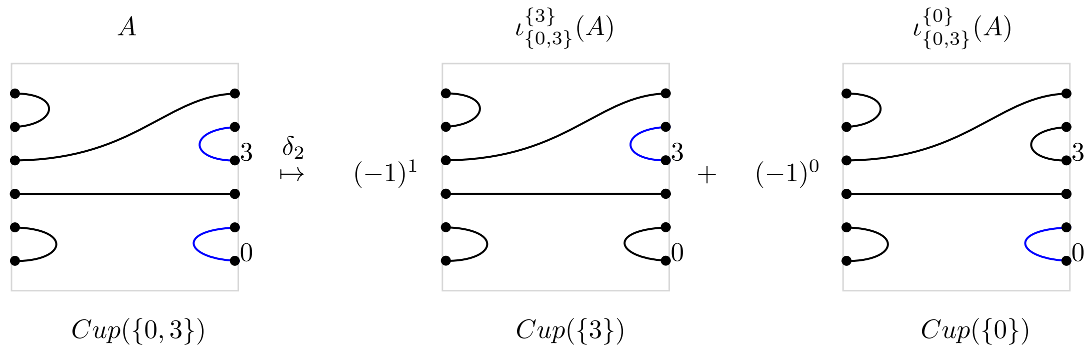

FIGURE 4. The differential of $(\mathcal{TL}_6\mathfrak{D}, \delta)$ evaluated on an element in $Cup(\{0,3\})$.

REMARK 2. *Theorem D, proved in the appendix, explains how the chain complex in Definition 22 was discovered and why we named it the "cellular Davis complex" of $\mathcal{TL}$.*

We start by verifying that this really is a chain complex. This argument is standard.

Lemma 3. *The cellular Davis complex $(\mathcal{TL}\mathfrak{D}, \delta)$ of $\mathcal{TL}$ is a $\mathcal{TL}$-chain complex.*

Proof: Let $A \in Cup(F) \subset \mathcal{TL}\mathfrak{D}_\alpha$ be a planar diagram. We need to argue that

$$\delta^2(A) = 0 \in \mathcal{TL}\mathfrak{D}_{\alpha-2}.$$

Observe that

$$\begin{aligned}\delta^2(A) &= \delta\Big(\sum_{s\in F}(-1)^{\gamma(s)}\iota_F^{F_s}(A)\Big)\\ &= \sum_{s\in F}(-1)^{\gamma_F(s)}\Big(\sum_{t\in F_s}(-1)^{\gamma_{F_s}(t)}\iota_F^{F_{s,t}}(A)\Big)\\ &= \sum_{\substack{(s,t)\in F\times F\\ s\neq t}}(-1)^{\gamma_F(s)+\gamma_{F_s}(t)}\iota_F^{F_{s,t}}(A),\end{aligned}$$

where $\iota_F^{F_{s,t}} : Cup(F) \hookrightarrow Cup(F_{s,t})$ denotes the inclusion. Clearly, $F_{s,t} = F_{t,s}$. It therefore suffices to show that if $s < t$, then

$$(-1)^{\gamma_F(s)+\gamma_{F_s}(t)} + (-1)^{\gamma_F(t)+\gamma_{F_t}(s)} = 0.$$

This holds because $s < t$ implies that

$$\gamma_F(t) = \gamma_{F_s}(t) \text{ and } \gamma_F(s) = \gamma_{F_t}(s) - 1.$$

The $\mathcal{TL}$-equivariance of the differential δ follows from the $\mathcal{TL}$-equivariance of the inclusion maps. □

THEOREM 4. *The cellular Davis complex $(\mathcal{TL}\mathfrak{D}, \delta)$ of $\mathcal{TL}$ is contractible with*

$$H_0(\mathcal{TL}\mathfrak{D}, \delta) = \mathbb{1}.$$

Proof: For a planar diagram $A \in \mathcal{TL}_n$, let $R\{[A]_{Cup(F)}\} \subseteq Cup(F)$ denote the R-linear summand spanned by the associated isotopy class $[A]$ in the diagrammatic picture (see Definition 21), using the convention that $R\{[A]_{Cup(F)}\} = 0$ if $A \notin Cup(F)$. Then, the R-module $Cup(F)$ admits the following decomposition

$$Cup(F) = \bigoplus_{\substack{A\in\mathcal{TL},\\ \text{planar diagram}}} R\{[A]_{Cup(F)}\}.$$

It follows that, as a chain complex in R-modules, the cellular Davis complex is a coproduct

$$\mathcal{TL}\mathfrak{D} = \bigoplus_{\substack{A\in\mathcal{TL},\\ \text{planar diagram}}} \mathcal{TL}\mathfrak{D}(A),$$

where $\mathcal{TL}\mathfrak{D}(A)$ is the following subcomplex associated to a planar diagram $A \in \mathcal{TL}$:

$$\mathcal{TL}\mathfrak{D}(A)_\alpha = \bigoplus_{F\subseteq F(A), |F|=\alpha} R\{[A]_{Cup(F)}\} \subseteq \bigoplus_{\substack{F\subseteq\langle n\rangle \text{ innermost},\\ |F|=\alpha}} Cup(F)$$

This splitting of the chain complex $\mathcal{TL}\mathfrak{D}$ follows from the fact that for any planar diagram $A \in \mathcal{TL}$

$$A \in Cup(F) \iff F \subseteq F(A)$$

and because the partial differentials in $\mathcal{TL}\mathfrak{D}$ are inclusion maps $Cup(F) \hookrightarrow Cup(F_s)$ with $F_s \subset F$.[2]

We will calculate the homology of each subcomplex $\mathcal{TL}\mathfrak{D}(A)$. If $A = id$, then $F(A) = \emptyset$ and $\mathcal{TL}\mathfrak{D}(A) = R\{[id]_{Cup(\emptyset)}\}[0]$ is concentrated in degree 0. If $A \neq id$, then $\mathcal{TL}\mathfrak{D}(A)$ is exactly the augmented chain complex $\tilde{C}(\Delta)[+1]$ of the simplex Δ on the vertex set $F(A)$, where the vertices are ordered using the opposite order of $F(A) \subset \langle n\rangle$ and the chain complex is shifted up by one degree. Therefore, the homology of the complex $\mathcal{TL}\mathfrak{D}(A)$ is zero in every degree, if $A \neq id$. The identification $H_0(\mathcal{TL}\mathfrak{D}, \delta) = \mathbb{1}$ as a $\mathcal{TL}$-module holds because $im(\delta_1) = \widehat{\mathcal{TL}}$, where the right side is as in Definition 19, and $\mathcal{TL}\mathfrak{D}_0 = Cup(\emptyset) = \mathcal{TL}$. □

3.2. Homology with coefficients in $Cup(F)$. Let $F \subset \langle n\rangle$ be an innermost set. The homology groups $H_\star(\mathcal{TL}, Cup(F))$ of the Temperley–Lieb algebra $\mathcal{TL}$ with coefficients in $Cup(F)$ will occur on the E_1-page of the spectral sequences, which we will use to derive Theorem A, B and C. For this reason, we will now study the modules $Cup(F)$ and the homology groups $H_\star(\mathcal{TL}, Cup(F))$. We begin by collecting several important observations about innermost sets.

The following notion explains why the cellular Davis complex is "sensitive" to the question whether the underlying Temperley–Lieb algebra is defined on an even or an odd number of arcs.

DEFINITION 23. *If $n+1$ is even, we call $M = \{0, 2, \dots, n-1\} \subset \langle n\rangle$ the unique maximal innermost set.*

Note that if $n+1$ is even, then $M = \{0, 2, \dots, n-1\} \subset \langle n\rangle$ really is the unique innermost set of maximal cardinality. If the number of arcs $n+1 \geq 3$ is odd and in contrast to the even case, there exist multiple different innermost sets of maximal cardinality (see Figure 5).

In particular the next observation always applies, if the Temperley–Lieb algebra is defined on an odd number of $n+1 \geq 3$ of arcs.

OBSERVATION 1. *If $F \subset \langle n\rangle$ is nonempty and innermost, but not the unique maximal innermost, then there exists an index $0 \leq i \leq n$ such that either $i-1 \notin F$ and $s = i+1 \in F$ or $i \notin F$ and $s = i-2 \in F$.*[3]

Proof: Because F is nonempty, there exists a smallest $z \in F$. If $z \neq 0$, we set $0 \leq i := z-1 \leq n$. Then $i-1 \notin F$ by minimality of z and $i+1 = z \in F$ per definition, so the first case applies. If $z = 0$, then there exists a maximal even number $2 \leq 2k \leq n$ such that $2l \in F$ for all $l < k$ and $2k \notin F$. Indeed, if $n+1$ is even, then $2k \leq n-1$ by the assumption that F is not the unique maximal innermost and setting $i := 2k$, the second case applies.

[2]A similar splitting has been used by Boyd–Hepworth–Patzt in their work on the homology of Brauer algebras [**BHP20**], Definition 5.5.

[3]It might happen that $i-1, i-2 < 0$. This will not cause any problems.

If $n+1$ is odd, then n is even and $n \notin \langle n \rangle$. Hence, $2k \leq n$ exists and setting $i := 2k$, the second case applies. □

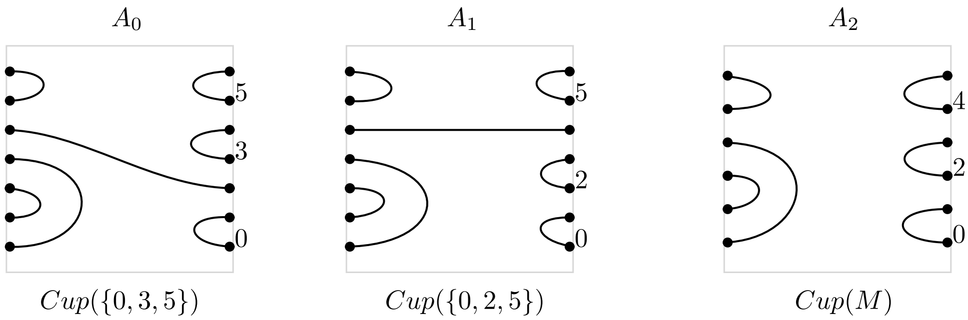

FIGURE 5. Innermost sets of maximal cardinality for in $\langle 5 \rangle$ (left) and $\langle 4 \rangle$ (right).

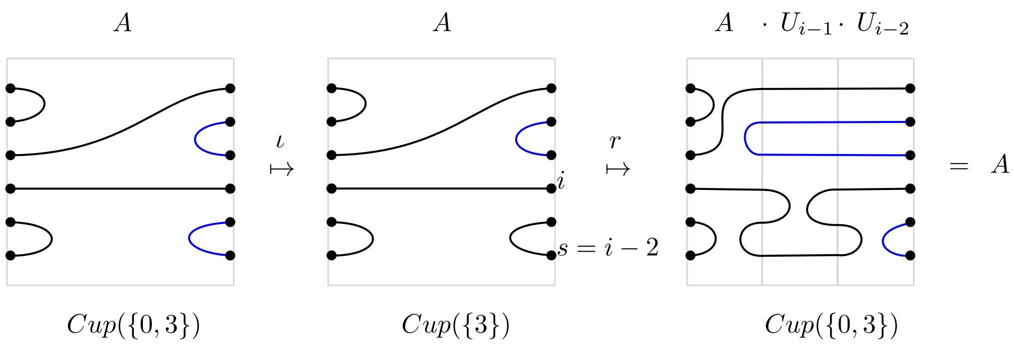

FIGURE 6. Illustration of the retraction constructed in Proposition 2 using Observation 1.

The following is closely related to Section 3 of [**BH20**] and replaces the "inductive resolutions", which Boyd–Hepworth introduced, in our setting.

PROPOSITION 2. *Let $F \subset \langle n \rangle$ be nonempty and innermost, but not the unique maximal innermost, and choose $i \in \{0, \dots, n\}$ and $s \in \{i-2, i+1\}$ as in Observation 1. Let $F_s = F - \{s\}$. Then,*

$$Cup(F) \hookrightarrow Cup(F_s)$$

is a retract of left $\mathcal{TL}$-modules.

Proof: Assume that $i \notin F$ and $s = i-2 \in F$. Consider the map of left modules (see Figure 6)

$$Cup(F_s) \to Cup(F) : c \mapsto c \cdot (U_{i-1}U_{i-2}).$$

This is well-defined: If $c \in Cup(F_s)$, then $c = A \cdot \prod_{j \in F_s} U_j$ for some $A \in \mathcal{TL}$. Therefore,

$$c \cdot (U_{i-1}U_{i-2}) = (A \cdot \prod_{j \in F_s} U_j)(U_{i-1}U_{i-2}) = (A \cdot U_{i-1})(\prod_{j \in F} U_j),$$

where we used that U_{i-1} commutes with all $\{U_z\}_{z \in F_s}$ and that $s = i-2 \in F$. The commutativity of U_{i-1} with $\{U_z\}_{z \in F_s}$ follows from Observation 1 because

$$|(i-1) - z| \geq \min\{|(i-1) - (i-4)|, |(i-1) - (i+1)|\} \geq 2 \text{ for any } z \in F_s.$$

We therefore proved that $c \cdot (U_{i-1}U_{i-2}) \in Cup(F)$, hence the map is well-defined. To see that this defines a retraction, one observes: If $c \in Cup(F)$, then $c = c' \cdot U_{i-2}$. Therefore,

$$c \cdot (U_{i-1}U_{i-2}) = c' \cdot (U_{i-2}U_{i-1}U_{i-2}) = c' \cdot U_{i-2} = c.$$

For the case that $i-1 \notin F$ and $s = i+1 \in F$, we consider the map of left modules

$$Cup(F_s) \to Cup(F) : c \mapsto c \cdot (U_iU_{i+1})$$

and observe that $|i-z| \geq \min\{|i-(i-2)|, |i-(i+3)|\} \geq 2$ for $z \in F_s$. This implies that the map is well-defined. To see that this defines a retraction, one uses that if $c \in Cup(F)$, then $c = c' \cdot U_{i+1}$. Therefore,

$$c \cdot (U_iU_{i+1}) = c' \cdot (U_{i+1}U_iU_{i+1}) = c' \cdot U_{i+1} = c.$$

□

Remark 3. *The elements that we use to define the retractions in Proposition 2 are exactly the same elements that Boyd–Hepworth use in Section 3 of* [**BH20**] *to define "inductive resolutions".*

The role of the following corollary in the arguments presented in this chapter is similar to the role of Theorem F in [**BH20**].

Corollary 1. *Let $F \subseteq \langle n \rangle$ be innermost, but not the unique maximal innermost. Then*

$$Cup(F) \hookrightarrow \mathcal{TL}$$

is a retract of left $\mathcal{TL}$-modules and $H_\star(\mathcal{TL}, Cup(F)) = 0$ for $\star > 0$

Proof: If $F = \emptyset$, the retraction statement is trivial because $Cup(\emptyset) = \mathcal{TL}$. If $F \neq \emptyset$, it follows from Proposition 2 by induction. The existence of a retraction implies that the map

$$H_\star(\mathcal{TL}, Cup(F)) \to H_\star(\mathcal{TL}, \mathcal{TL})$$

is injective. The homology $H_\star(\mathcal{TL}, \mathcal{TL})$ is nontrivial only for $\star = 0$. Hence, the claim follows. □

In the next lemma, we compute the degree zero homology of the Temperley–Lieb algebras with coefficients in $Cup(F)$.

Lemma 5. *Assume that $F \subseteq \langle n \rangle$ is an innermost set. Then,*

$$H_0(\mathcal{TL}, Cup(F)) \cong \mathbb{1} \otimes_{\mathcal{TL}} Cup(F) = \begin{cases} R, & \text{if } F = \emptyset \\ R/a, & \text{if } n = 1 \text{ and } F = \{0\} \\ 0, & \text{if } n > 1 \text{ and } F \neq \emptyset \end{cases}$$

Proof: We compute each case individually. If $F = \emptyset$, then $Cup(F) = \mathcal{TL}$ and we find that:

$$\mathbb{1} \otimes_{\mathcal{TL}} Cup(F) = \mathbb{1} \otimes_{\mathcal{TL}} \mathcal{TL} \cong \mathbb{1} = R$$

If $n = 1$ and $F = \{0\}$, then $Cup(F) = \widehat{\mathcal{TL}}$. As in Definition 19, we have a short exact sequence

$$0 \leftarrow \mathbb{1} \twoheadleftarrow \mathcal{TL} \hookleftarrow Cup(F) = \widehat{\mathcal{TL}} \leftarrow 0$$

and therefore, $H_0(\mathcal{TL}, Cup(F)) \cong H_1(\mathcal{TL}, \mathbb{1})$. The homology of the Temperley–Lieb algebra $\mathcal{TL}$ on one generator with trivial coefficients has been completely computed by Boyd–Hepworth in [**BH20**], Proposition 7.1, and it follows from their computation that $H_1(\mathcal{TL}, \mathbb{1}) = R/a$. For the last case, we assume that $n > 1$ and $F \neq \emptyset$. We fix a generator U_i, whose index i has the property that $i \in F$. By definition of $Cup(F)$, we have that $A \in Cup(F)$ if and only if $A = A' \cdot (\prod_{j \in F} U_j)$ for some $A' \in \mathcal{TL}$. Because $\mathcal{TL}$ has $n > 1$

generators, it follows that $i-1$ or $i+1$ indexes a generator U_{i-1} or U_{i+1}. By relation $ii)$ of Definition 16, it follows that

$$A' \cdot (U_i U_{i-1}) \cdot \prod_{j\in F} U_j = A' \cdot \prod_{j\in F} U_j \text{ or } A' \cdot (U_i U_{i+1}) \cdot \prod_{j\in F} U_j = A' \cdot \prod_{j\in F} U_j$$

Note in either case, $B := A' \cdot (U_i U_{i-1})$ or $B := A' \cdot (U_i U_{i+1})$, we have that $B \in \widehat{\mathcal{TL}}$, where $\widehat{\mathcal{TL}}$ is as in Definition 19. We can therefore conclude that any elementary tensor $1 \otimes_{\mathcal{TL}} A \in \mathbb{1} \otimes_{\mathcal{TL}} Cup(F)$ is equal to zero because

$$1 \otimes_{\mathcal{TL}} A = 1 \cdot B \otimes_{\mathcal{TL}} (\prod_{j\in F} U_j) = 0 \otimes_{\mathcal{TL}} (\prod_{j\in F} U_j) = 0.$$

This completes the proof. □

3.3. Proof of Theorem A, B and C. We will now prove the first three main theorems. The following result, stated as Theorem A in the introduction, generalizes ([**BH20**], Theorem D) and the "odd"-part of ([**BH20**], Theorem B).

THEOREM 6. *If $n+1$ is odd, then $H_0(\mathcal{TL}, \mathbb{1}) = \mathbb{1}$ and $H_\star(\mathcal{TL}, \mathbb{1}) = 0$ for $\star > 0$.*

Proof: Let $P_\star$ be a free resolution of the trivial $\mathcal{TL}$-module $\mathbb{1}$ and consider the double complex $P_\star \otimes_{\mathcal{TL}} \mathcal{TL}\mathfrak{D}$, where $\mathcal{TL}\mathfrak{D}$ is the cellular Davis complex for Temperley–Lieb algebras (see Definition 22). The horizontal and vertical filtration of $P_\star \otimes_{\mathcal{TL}} \mathcal{TL}\mathfrak{D}$ give rise to two spectral sequences. The vE^1-page of the vertical spectral sequence is given by:

$$vE^1_{a,b} = P_a \otimes_{\mathcal{TL}} H_b(\mathcal{TL}\mathfrak{D})$$

It follows from Theorem 4 that the vE^1-page is concentrated in degree $b = 0$, where it is equal to the complex $vE^1_{a,0} = P_a \otimes_{\mathcal{TL}} \mathbb{1}$. By definition, we therefore have that:

$$vE^2_{a,0} = H_a(\mathcal{TL}, \mathbb{1})$$

The collapsing of the vertical spectral sequence on the vE^2-page implies that the horizontal spectral sequence converges to $H_{a+b}(\mathcal{TL}, \mathbb{1})$. The horizontal spectral sequence has hE^1-page given by:

$$\begin{aligned} hE^1_{a,b} &= H_a(P_\star \otimes_{\mathcal{TL}} \mathcal{TL}\mathfrak{D}_b) \\ &\cong \bigoplus_{\substack{F \subseteq \langle n\rangle \text{ innermost}, \\ |F| = b}} H_a(P_\star \otimes_{\mathcal{TL}} Cup(F)) \\ &\cong \bigoplus_{\substack{F \subseteq \langle n\rangle \text{ innermost}, \\ |F| = b}} H_a(\mathcal{TL}, Cup(F)) \end{aligned}$$

Because $n+1$ is odd, Corollary 1 applies to any innermost subset $F \subset \langle n \rangle$. Together with Lemma 5, this implies that:

$$hE^1_{a,b} = \begin{cases} \mathbb{1}, & \text{if } (a,b) = (0,0) \\ 0, & \text{if } (a,b) \neq (0,0) \end{cases}$$

The theorem follows. □

The next vanishing result, stated as Theorem B in the introduction, is similar to the "even"-part of ([**BH20**], Theorem B). The vanishing line, which Boyd–Hepworth obtain, is stronger than ours (slope 1 versus slope $\frac{1}{2}$) and in fact, optimal (see [**BH20**], Theorem C). The description of the high-dimensional homology in terms of homology with coefficients is new in the sense that the coefficient system is different from the one Boyd–Hepworth obtain in [**BH20**].

THEOREM 7. *If $n+1$ is even, then $H_0(\mathcal{TL}, \mathbb{1}) = \mathbb{1}$, $H_\star(\mathcal{TL}, \mathbb{1}) = 0$ for $0 < \star < \frac{n+1}{2}$ and*

$$H_{\star + \frac{n+1}{2}}(\mathcal{TL}, \mathbb{1}) \cong H_\star(\mathcal{TL}, Cup(M))$$

for $\star \geq 0$ and where $M = \{0, 2, \ldots, n-3, n-1\} \subseteq \langle n \rangle$ is the unique maximal innermost set.

Proof: By the same argument as in the proof of Theorem 6, we obtain a spectral sequence converging to $H_{a+b}(\mathcal{TL}, \mathbb{1})$ with hE^1-page:

$$hE^1_{a,b} \cong \bigoplus_{\substack{F \subseteq \langle n \rangle \text{ innermost,} \\ |F| = b}} H_a(\mathcal{TL}, Cup(F))$$

Because $n+1$ is even, there exists a unique innermost set $M = \{0, 2, \ldots, n-1\} \subset \langle n \rangle$ of maximal cardinality $|M| = \frac{n+1}{2}$. Corollary 1 applies to all innermost subsets $F \subset \langle n \rangle$ except M. Together with Lemma 5, this implies that

$$hE^1_{a,b} = \begin{cases} \mathbb{1}, & \text{if } (a,b) = (0,0) \\ H_a(\mathcal{TL}, Cup(M)), & \text{if } (a,b) = (a, \frac{n+1}{2}) \\ 0, & \text{else.} \end{cases}$$

The spectral sequence collapses on the hE^1-page and the theorem follows. □

REMARK 4. *Lemma 5 implies that in the setting of Theorem 7 the following holds: For $n+1 = 2$, $H_{\frac{n+1}{2}}(\mathcal{TL}, \mathbb{1}) = R/a$, and for $n+1 > 2$, $H_{\frac{n+1}{2}}(\mathcal{TL}, \mathbb{1}) = 0$. This is consistent with Theorem B and C of* [**BH20**] *mentioned above.*

We close this section by presenting an alternative proof of ([**BH20**], Theorem A) i.e. Theorem C stated in the introduction.

THEOREM 8 (Theorem A, [**BH20**]). *If the parameter $a \in R$ is a unit, then $H_0(\mathcal{TL}, \mathbb{1}) = \mathbb{1}$ and $H_\star(\mathcal{TL}, \mathbb{1}) = 0$ for $\star > 0$*

Proof: By Theorem 6 and Theorem 7, it suffices to prove that for $n+1$ even and $M \subset \langle n \rangle$ the unique maximal innermost set, we have that $H_\star(\mathcal{TL}, Cup(M)) = 0$ for $\star \geq 0$. To see this, we consider the innermost set $M_0 = M - \{0\}$. There is a map of left $\mathcal{TL}$-modules

$$Cup(M_0) \to Cup(M) : c \mapsto c \cdot U_0$$

Observe that the map obtained by precomposition with the inclusion

$$Cup(M) \hookrightarrow Cup(M_0) \to Cup(M)$$

is multiplication by a. If a is a unit, the inclusion induced map

$$H_\star(\mathcal{TL}, Cup(M)) \to H_\star(\mathcal{TL}, Cup(M_0))$$

must therefore be an injection. By Corollary 1, the target of this map is zero if $\star > 0$. For $\star = 0$, we invoke Lemma 5. □

4. Appendix: Davis and Coxeter posets for Temperley–Lieb algebras

In this appendix, we explain how the cellular Davis complex $\mathcal{TL}\mathfrak{D}$ introduced in Definition 22 was discovered. More precisely, after introducing the necessary notation, we will use the notion of algebraic coset poset (see Chapter 1, Definition 8) to define an algebraic Davis and Coxeter poset for Temperley–Lieb algebras. We then prove that both complexes are contractible using an algebraic version of discrete Morse theory (Theorem E). We continue by studying the recellulation spectral sequence of the algebraic Davis poset introduced in Chapter 1, Proposition 1. The spectral sequence will collapse on the E^2-page and the E^1-page will contain a single contractible chain complex, which we prove to be isomorphic to the cellular Davis complex $\mathcal{TL}\mathfrak{D}$ describe in Definition 22. This shows that $\mathcal{TL}\mathfrak{D}$ can be seen as the Temperley–Lieb analogue of the Davis complex of a Coxeter group equipped with the "Coxeter cell" CW-structure (see Chapter 1, Example 4, and [**Dav08**]). In [**Boy20**], Boyd used a spectral sequence associated to the cellular Davis complex of a Coxeter group (W, S) to derive formulas for the low-dimensional homology groups of W. This appendix shows that the approach to homology computations for Temperley–Lieb algebras described in the first part of the chapter is in analogy to Boyd's work [**Boy20**]. In the next chapter, we will use a similar idea to study the homology of Iwahori–Hecke algebras. This appendix builds on ideas contained in [**Boy20, Dav08, BH20**] and [**BHP20**].

4.1. Notation and induced modules. In this subsection we gather some definitions and observations that we need to state and prove the main results, Theorem D and E, of this appendix. Let R be a commutative unital ring. Let $\mathcal{TL} = \mathcal{TL}_{n+1}$ be the Temperley–Lieb algebra on $n+1$ strands with parameter $a \in R$ and let $U_0, \ldots, U_{n-1}$ be the standard generators. We often identify $\{U_0, \ldots, U_{n-1}\}$ with the set $\langle n \rangle = \{0, \ldots, n-1\}$.

Definition 24.

i) Let $\mathfrak{D} = 2^{\langle n \rangle}$ be the poset of all subsets of $\langle n \rangle$.
ii) Let $\mathfrak{C}$ be the poset of all (possibly empty) proper subsets of $\langle n \rangle$.

Let $\mathbb{1}$ be the trivial module of $\mathcal{TL}$ (see Definition 19). We will use the following notation throughout this section.

Notation 1. *Let $F \in \mathfrak{D}$.*

- *We write $\mathcal{TL}_F$ for the subalgebra of $\mathcal{TL}$ generated by $\{U_i : i \in F\}$.*
- *I_F will denote the left ideal of $\mathcal{TL}$ generated by the set $\{U_i : i \in F\}$. This is analogous to [**BH20**], Definition 2.11.*
- *We denote the tensor product $\mathcal{TL} \otimes_{\mathcal{TL}_F} \mathbb{1}$ by $\mathcal{TL} \otimes_F \mathbb{1}$.*

The following lemma is a slight generalization of [**BH20**], Lemma 2.12. The proof is exactly the same as the one given by Boyd–Hepworth.

Lemma 9 ([**BH20**], Lemma 2.12). *Let $F \in \mathfrak{D}$. Then, $\mathcal{TL} \otimes_F \mathbb{1}$ and $\mathcal{TL}/I_F$ are isomorphic as left $\mathcal{TL}$-modules via the maps*

$$\phi : \mathcal{TL} \otimes_F \mathbb{1} \to \mathcal{TL}/I_F : B \otimes_F r \mapsto B \cdot r + I_F$$

and

$$\psi : \mathcal{TL}/I_F \to \mathcal{TL} \otimes_F \mathbb{1} : B + I_F \mapsto B \otimes_F 1.$$

Proof: The map ϕ is well defined because if $i \in F$ so that $B \cdot U_i \otimes_F r = B \otimes_F U_i \cdot r = 0$, then $B \cdot U_i \in I_F$ so that $B \cdot U_i \cdot r + I_F = I_F$. The map ψ is well defined because if $B \in I_F$, then $B = \sum_{i \in F} B'_i \cdot U_i$ so that $B \otimes_F 1 = \sum_{i \in F} B'_i \cdot U_i \otimes_F 1 = 0$. One then verifies that the two maps are inverses. □

The previous lemma and the diagrammatic interpretation of Temperley–Lieb algebras given in Theorem 2 imply the following.

Corollary 2. *The left $\mathcal{TL}$-module $\mathcal{TL} \otimes_F \mathbb{1}$ has a R-module basis given by elementary tensors $A \otimes_F 1$, where A is a planar diagram with the property that $F(A) \cap F = \emptyset$.*

Let $A \in \mathcal{TL}$ be a planar diagram. Let $R\{A \otimes_F 1\} \subseteq \mathcal{TL} \otimes_F \mathbb{1}$ denote the R-summand described in Corollary 2 i.e. we have that $R\{A \otimes_F 1\} = 0$, if $F(A) \cap F \neq \emptyset$, and $R\{A \otimes_F 1\} \cong R$, if $F(A) \cap F = \emptyset$. In particular, the R-module $\mathcal{TL} \otimes_F \mathbb{1}$ admits the following decomposition

$$\mathcal{TL} \otimes_F \mathbb{1} = \bigoplus_{\substack{A \in \mathcal{TL}, \\ \text{planar diagram}}} R\{A \otimes_F 1\}.$$

This decomposition will be important for the rest of this chapter.

4.2. The algebraic Davis and Coxeter poset of Temperley–Lieb algebras. In this subsection, we define the algebraic Davis and Coxeter poset of Temperley–Lieb algebras. The construction of algebraic coset posets for general families of R-algebras has been explained in Chapter 1, Definition 8. We will now apply this to Temperley–Lieb algebras.

DEFINITION 25.

i) The Davis poset $CP(\mathfrak{D}, \mathbb{1})$ of $\mathcal{TL}$ is the algebraic coset poset associated to

$$(\mathcal{TL}, \{\mathcal{TL}_F\}_{F\in\mathfrak{D}}, \mathbb{1})$$

i.e. $CP(\mathfrak{D}, \mathbb{1})$ is a semi-simplicial $\mathcal{TL}$-module with $\mathcal{TL}$-module of k-simplices given by

$$CP(\mathfrak{D}, \mathbb{1})_k = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ y_i \in \mathfrak{D}}} \mathcal{TL} \otimes_{y_0} \mathbb{1}.$$

ii) The Coxeter poset $CP(\mathfrak{C}, \mathbb{1})$ of $\mathcal{TL}$ is the algebraic coset poset associated to

$$(\mathcal{TL}, \{\mathcal{TL}_F\}_{F\in\mathfrak{C}}, \mathbb{1})$$

i.e. $CP(\mathfrak{C}, \mathbb{1})$ is a semi-simplicial $\mathcal{TL}$-module with $\mathcal{TL}$-module of k-simplices given by

$$CP(\mathfrak{C}, \mathbb{1})_k = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ y_i \in \mathfrak{C}}} \mathcal{TL} \otimes_{y_0} \mathbb{1}.$$

Let $A \otimes_{y_0} 1$ be in the summand $\mathcal{TL} \otimes_{y_0} \mathbb{1}$ indexed by $y_\bullet = (y_0 \lneq \cdots \lneq y_k)$. We recall the definition of the face maps for $k \geq 1$:

$$d_0(A \otimes_{y_0} 1) = A \otimes_{y_1} 1 \text{ in the summand indexed by } d_0(y_\bullet)$$

and for all $i \neq 0$

$$d_i(A \otimes_{y_0} 1) = A \otimes_{y_0} 1 \text{ in the summand indexed by } d_i(y_\bullet).$$

4.3. Overview: Statement and Proof of Theorem E. Our first goal is to prove the following theorem, which is essentially Theorem E stated in the introduction.

THEOREM 10. *The Davis poset $CP(\mathfrak{D}, \mathbb{1})$ and the Coxeter poset $CP(\mathfrak{C}, \mathbb{1})$ of the Temperley–Lieb algebra $\mathcal{TL}$ are contractible and the zero-th homology group is the trivial $\mathcal{TL}$-module $\mathbb{1}$ in both cases.*

The proof of this theorem is distributed over the next subsections. After observing some structural properties of the Coxeter and Davis poset, we will introduce an algebraic version of discrete Morse theory for chain complex based on [**Skö06, Skö18, Koz05**]. We will then use algebraic Morse theory to prove that the Davis complex is contractible. After this, we formulate a second algebraic Morse theory argument, which is more involved, to show that the Coxeter poset is contractible as well.

4.4. A splitting of the Davis and Coxeter poset. We start the proof of Theorem 10 by collecting observations about the structure of the Davis and Coxeter poset of Temperley–Lieb algebras. The following notion will be key throughout this appendix.

DEFINITION 26. *A set $y \in \mathfrak{D}$ is called good for a planar diagram A, if*

$$y \cap F(A) = \emptyset,$$

and bad for A, if

$$y \cap F(A) \neq \emptyset.$$

Equivalently, a set $y \in \mathfrak{D}$ is good for A, if $A \otimes 1$ is an R-basis element of $\mathcal{TL} \otimes_y \mathbb{1}$, and bad for A, if $A \otimes 1 = 0 \in \mathcal{TL} \otimes_y \mathbb{1}$ (see Corollary 2 et seqq.).

By Definition 25 and Corollary 2 et seqq., the R-module $CP(\mathfrak{D}, \mathbb{1})_k$ of k-simplices (similar for $CP(\mathfrak{C}, \mathbb{1})_k$) admits the following decomposition

$$\begin{aligned} CP(\mathfrak{D}, \mathbb{1})_k &= \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ y_i \in \mathfrak{D}}} \mathcal{TL} \otimes_{y_0} \mathbb{1} \\ &= \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ y_i \in \mathfrak{D}}} \bigoplus_{\substack{A \in \mathcal{TL}, \\ \text{planar diagram}}} R\{A \otimes_{y_0} 1\} \\ &= \bigoplus_{\substack{A \in \mathcal{TL}, \\ \text{planar diagram}}} \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ y_i \in \mathfrak{D} \\ y_0 \text{ good for } A}} R\{A \otimes_{y_0} 1\}. \end{aligned}$$

This leads to the following observation.

Observation 2. *Let $A \in \mathcal{TL}$ be a fixed planar diagram. The module of k-simplices in $CP(\mathfrak{D}, \mathbb{1})$ (and similar $CP(\mathfrak{C}, \mathbb{1})$) contains a unique R-module summand*

$$CP(\mathfrak{D}, \mathbb{1})_k^A := \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_k \\ y_i \in \mathfrak{D} \\ y_0 \text{ good for } A}} R\{A \otimes_{y_0} 1\}.$$

It follows from the definition of the face maps that the R-modules $\{CP(\mathfrak{D}, \mathbb{1})_k^A\}$ form a semi-simplicial R-submodule of $CP(\mathfrak{D}, \mathbb{1})$ (and similar $CP(\mathfrak{C}, \mathbb{1})$). Therefore, we obtain the following splitting.

Observation 3. *There are splittings of semi-simplicial R-modules:*

i)

$$CP(\mathfrak{D}, \mathbb{1}) = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} CP(\mathfrak{D}, \mathbb{1})^A.$$

ii)

$$CP(\mathfrak{C}, \mathbb{1}) = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} CP(\mathfrak{C}, \mathbb{1})^A.$$

This is analogous to the splitting observed by [**BHP20**] for Brauer algebras. We will study the individual pieces $CP(\mathfrak{D}, \mathbb{1})^A$ (and similar $CP(\mathfrak{C}, \mathbb{1})^A$) of the splitting in Observation 3 using an algebraic version of discrete Morse theory. The necessary background on algebraic Morse theory is explained in the next subsection.

4.5. Algebraic Morse theory. This subsection introduces discrete Morse theory for based chain complexes as described by Sköldberg [**Skö06, Skö18**] and Kozlov [**Koz05**]. We will use algebraic discrete Morse theory in the subsequent sections to prove Theorem D and E stated in the introduction. Let R be a commutative unital ring.

Definition 27. *Let $(C_\star, \partial_n)$ be a chain complex such that C_n is a free R-module with basis Ω_n and define $\Omega = \sqcup \Omega_n$. We call a pair $(C_\star, \Omega)$ a based chain complex. Given $a \in \Omega_n$, there is a unique element $\omega(a, b) \in R$ for any $b \in \Omega_{n-1}$ defined by the property that*

$$\partial_n(a) = \sum_{b \in \Omega_{n-1}} \omega(a, b) \cdot b \ .$$

We call $\omega(a, b)$ the weight of a to b.

From a based chain complex we can construct directed graph as follows.

Definition 28. *Let $C_\star$ be a based chain complex with R-basis Ω. Let $\Gamma(C_\star)$ be the directed graph with vertex set Ω and set of directed edges $\{a \to b : \omega(a, b) \neq 0\}$.*

For algebraic Morse theory, the analogue of the gradient vector field of a Morse function is encoded as a matching on the directed graph $\Gamma(C_\star)$.

Definition 29 ([**Skö06**]). *A partial matching on a directed graph $\Gamma = (V, E)$ is a set of edges $M \subset E$ such that no vertex is incident to more than one edge in M. We write $\Gamma^M = (V, E^M)$ for the directed graph obtained by reversing the direction of all edges in M, $E^M = (E - M) \cup \{b \to a : a \to b \in M\}$.*

We will only be interested in based chain complexes $(C_\star, \Omega)$ with the property that Ω is a finite set. In this setting, the notion of Morse matching has the following particularly concise description.

Definition 30 ([**Skö06**], Lemma 1). *Let $(C_\star, \Omega)$ be a based chain complex and assume that Ω is a finite set. A Morse matching on $(C_\star, \Omega)$ is a partial matching M on the directed graph $\Gamma(C_\star)$ with the following two properties:*

- *i) If $a \to b \in M$, then $\omega(a, b)$ is a unit in R.*
- *ii) The graph $\Gamma(C_\star)^M$ has no directed cycles*

A vertex in $c \in \Gamma(C_\star)$ is called M-critical, if c is not incident to an edge in M. We write Ω_n^c for the set of critical basis elements in Ω_n.

The next lemma will make it easier to check second condition in Definition 30 and recovers the acyclicity condition in Definition 1.2 of [**Koz05**].

Lemma 11. *Let $(C_\star, \Omega)$ be a based chain complex and let M be a partial matching on $\Gamma(C_\star)$. For $a \to b \in M$, we write $a \to d(a)$. Then, any directed cycle in the graph $\Gamma(C_\star)^M$ is of the form:*

$$d(b_0) \to b_0 \to d(b_1) \to b_1 \to d(b_2) \to b_2 \to \cdots \to d(b_k) \to b_k \to d(b_0)$$

with $k \geq 1$ and all $b_i \in \Omega_n - \Omega_n^c$ distinct.

Proof: Observe that $r : \Omega \to \mathbb{N}$ with $b \mapsto n$ if $b \in \Omega_n$ defines a height function on $\Gamma(C_\star)^M$ with the property that the full subgraphs $r^{-1}(n)$ are discrete. An edge $a \to b$ in $\Gamma(C_\star)^M$ is increasing, i.e. $r(b) = r(a) + 1$, if and only if $a \to b \in M$. We call the edges that are not in M decreasing. It follows that any directed cycle in $\Gamma(C_\star)^M$ contains equally many increasing and decreasing edges. By the definition of matching, any increasing edge of a directed cycle $\Gamma(C_\star)^M$ needs to be preceded and followed by a decreasing edge, otherwise some vertex would be incident to two edges in M. This implies that in any directed cycle is an alternating sequence of increasing and decreasing edges as claimed. □

The main theorem of algebraic Morse theory is the following.

Theorem 12 ([**Koz05**], Theorem 2.1, [**Skö06**] Theorem 1, [**Skö18**] Theorem 2). *Let $(C_\star, \Omega)$ be a based chain complex that is equipped with a Morse matching $\mathcal{M}$. Then $C_\star$ is homotopy equivalent to a chain complex $C_\star^{\mathcal{M}}$, whose chain module $C_n^{\mathcal{M}}$ have a free R-basis given by the critical elements Ω_n^c in Ω_n.*

Sköldberg deduces the following corollary that we will frequently use.

Corollary 3 ([**Skö06**], Corollary 2). *In the setting of Theorem 12: If the submodules $C_n^{\mathcal{M}}$ of C_n form a subcomplex $C_\star^{\mathcal{M}}$ of $C_\star$, then $C_\star^{\mathcal{M}} \hookrightarrow C_\star$ is a deformation retract.*

4.6. Contractibility of the Davis poset. We will now use algebraic Morse theory to show that the Davis complex $CP(\mathfrak{D}, \mathbb{1})$ of the Temperley–Lieb algebra $\mathcal{TL}$ is contractible. This is the first part of Theorem 10. Recall the splitting described in Observation 3

$$CP(\mathfrak{D}, \mathbb{1}) = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} CP(\mathfrak{D}, \mathbb{1})^A.$$

We our first goal is to compute the homology of each summand $CP(\mathfrak{D},\mathbb{1})^A$. Note that the poset $\mathfrak{D}$ has a terminal object, $\langle n\rangle$. We will use this to define a Morse matching on $CP(\mathfrak{D},\mathbb{1})^A$.

Let $A \in \mathcal{TL}$ be a planar diagram and consider the summand $CP(\mathfrak{D},\mathbb{1})^A$ of the Davis poset. By definition (see Observation 2) the R-module of k-simplices of $CP(\mathfrak{D},\mathbb{1})^A$ has a R-basis $\Omega_k(A)$ indexed by the set of flags $y_\bullet = (y_0 \lneq \cdots \lneq y_k) \in N_k(\mathfrak{D})$ with the property that y_0 is good for A. In the following, we view $(CP(\mathfrak{D},\mathbb{1})^A_\star, \Omega(A) = \sqcup_k \Omega_k(A))$ as a based chain complex in the sense of Definition 27.

The associated directed graph $\Gamma(CP(\mathfrak{D},\mathbb{1})^A_\star)$ (see Definition 28) has vertex set $\Omega(A)$. For any basis element $y_\bullet = (y_0 \leq \cdots \leq y_k) \in \Omega_k(A)$ there is a directed edge $y_\bullet \to d_i(y_\bullet)$ to the i-th face of $y_\bullet$ if and only if either $i \neq 0$ or $i = 0$ and y_1 is good for A. We remark that the weight $\omega(y_\bullet, d_i(y_\bullet)) = (-1)^i$ of any edge $y_\bullet \to d_i(y_\bullet)$ is a unit in R (compare with Definition 27).

With this notation in place, we can define a Morse matching (see Definition 30).

Lemma 13. *Let $Q(A) = \{y_\bullet = (y_0 \lneq \cdots \lneq y_k) \in \Omega(A) : y_k = \langle n\rangle$ and $k > 0\}$. Then*

$$\mathcal{M}(A) = \{y_\bullet \to d_k(y_\bullet) : y_\bullet = (y_0 \lneq \cdots \lneq y_k = \langle n\rangle) \in Q(A)\}$$

defines a Morse matching on $(CP(\mathfrak{D},\mathbb{1})^A_\star, \Omega(A))$ with set of critical basis elements:

$$\Omega(A)^{\mathcal{M}(A)} = \begin{cases} \{y_\bullet = \langle n\rangle\}, & \text{if } A = id \\ \emptyset & \text{if } A \neq id \end{cases}$$

Proof: We will check all conditions in Definition 30.

$\mathcal{M}(A)$ is well defined: Let $y_\bullet = (y_0 \lneq \cdots \lneq y_k = \langle n\rangle) \in Q(A)$. Because $k > 0$, the condition that y_0 is good for A also holds for $d_k(y_\bullet)$. Hence, $d_k(y_\bullet) \in \Omega(A)$ and the edge $y_\bullet \to d_k(y_\bullet)$ exists.

$\mathcal{M}(A)$ is a partial matching: If $y_\bullet \in Q(A)$, then $d_k(y_\bullet) \notin Q(A)$. Hence any element of $Q(A)$ is contained in exactly one edge of $\mathcal{M}(A)$. If $y_\bullet \notin Q(A)$ and incident to an edge in $\mathcal{M}(A)$, then it must be the edge $y_\bullet \lneq \langle n\rangle \to y_\bullet \in \mathcal{M}(A)$.

$\mathcal{M}(A)$ is a Morse matching: Because $\omega(y_\bullet, d_k(y_\bullet)) = (-1)^k$ is a unit in R and using Lemma 11, it suffices to verify that $\Gamma(CP(\mathfrak{D},\mathbb{1})^A_\star)^{\mathcal{M}(A)}$ contains no directed cycles of the form

$$d_k(b_0) \to b_0 \to d_k(b_1) \to b_1 \to d_k(b_2) \to b_3 \to \cdots \to d_k(b_m) \to b_m \to d_k(b_0)$$

for $m \geq 1$ and all $b_i \in Q(A)_k$ distinct. Such a cycle cannot exist because $d_k(b_0) \to b_0 \to d_k(b_1) \to b_1$ implies that $b_0 \in Q(A)_k$ has two distinct codimension-1 faces that do not contain $\langle n\rangle$, which is absurd.

Conclusion: Observe that $\langle n\rangle$ is good for A if and only if $A = id$. Given any flag $\langle n\rangle \neq y_\bullet \in \Omega(A)$, then either $y_\bullet \in Q(A)$ or $y_\bullet \lneq \langle n\rangle \in Q(A)$. Therefore, the set of critical basis elements is $\{y_\bullet = \langle n\rangle\}$, if $A = id$, and $\emptyset$, otherwise. □

Proof of Theorem 10 for the Davis poset: The contractibility follows from Observation 3, Lemma 13 and the main theorem of algebraic discrete Morse theory Theorem 12. The module structure on the zero-th homology group follows from Observation 3. □

4.7. Contractibility of the Coxeter poset - step 1. We will now show that the Coxeter poset $CP(\mathfrak{C}, \mathbb{1})$ of $\mathcal{TL}$ is contractibility. This is the second part of Theorem 10. Recall that by Observation 3, we have that

$$CP(\mathfrak{C}, \mathbb{1}) = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} CP(\mathfrak{C}, \mathbb{1})^A.$$

We will, again, study each summand $CP(\mathfrak{C}, \mathbb{1})^A$. For each summand, the homology computation has two steps and involves two Morse matchings. In the first step of the proof we will use Corollary 3 to reduce the homology calculation to a smaller subcomplex. This subcomplex is then analyzed using a second Morse matching, which we explain in the next subsection. The set up is similar to the above, we repeat it here for the convenience of the reader.

Let $A \in \mathcal{TL}$ be a planar diagram and consider the summand $CP(\mathfrak{C}, \mathbb{1})^A$ of the Coxeter poset. By definition (see Observation 2), the R-module of k-simplices of $CP(\mathfrak{C}, \mathbb{1})^A$ has a R-basis $\Omega_k(A)$ indexed by the set of flags $y_\bullet \in \mathfrak{C}$ with the property that y_0 is good for A. In the following, we view $(CP(\mathfrak{C}, \mathbb{1})^A_\star, \Omega(A) = \sqcup_k \Omega_k(A))$ as a based chain complex in the sense of Definition 27.

The associated directed graph $\Gamma(CP(\mathfrak{C}, \mathbb{1})^A_\star)$ (see Definition 28 in the appendix) has vertex set $\Omega(A)$. For any basis element $y_\bullet = (y_0 \leq \cdots \leq y_k) \in \Omega_k(A)$ there is a directed edge $y_\bullet \to d_i(y_\bullet)$ to the i-th face of $y_\bullet$ if and only if either $i \neq 0$ or $i = 0$ and y_1 is good for A. We remark that the weight $\omega(y_\bullet, d_i(y_\bullet)) = (-1)^i$ of any edge $y_\bullet \to d_i(y_\bullet)$ is a unit in R (compare with Definition 27).

With this notation, in place we can define the first Morse matching (see Definition 30).

Lemma 14. *Let $Q_0(A) = \{y_\bullet \in \Omega(A) : y_0 = \emptyset$ and y_1 is good for $A\}$. Then,*

$$\mathcal{M}_0(A) = \{y_\bullet \to d_0(y_\bullet) : y_\bullet \in Q_0(A)\}$$

defines a Morse matching on $(CP(\mathfrak{C}, \mathbb{1})^A_\star, \Omega(A))$ with set of critical basis elements:

$$\Omega(A)^{\mathcal{M}_0(A)} = \begin{cases} \{y_\bullet = \emptyset\}, & \text{if } A = id \\ \{y_\bullet = \emptyset\} \sqcup \{y_\bullet \in \Omega(A) : y_0 = \emptyset \text{ and } y_1 \text{ is bad for } A\}, & \text{if } A \neq id \end{cases}$$

Proof: We will check all conditions in Definition 30.

$\mathcal{M}_0(A)$ is well defined: Let $y_\bullet = (\emptyset \lneq y_1 \cdots \lneq y_k) \in Q_0(A)$. Because y_1 is good for A, the condition that the zero-th entry of the flag $d_0(y_\bullet)$ is good for A holds. Hence, $d_0(y_\bullet) \in \Omega(A)$ and the edge $y_\bullet \to d_k(y_\bullet)$ exists.

$\mathcal{M}_0(A)$ is a partial matching: If $y_\bullet = \emptyset \lneq y_1 \lneq \cdots \lneq y_k \in Q_0(A)$, then $d_0(y_\bullet) \notin Q_0(A)$. Hence any element of $Q_0(A)$ is contained in exactly one edge of $\mathcal{M}_0(A)$. If $y_\bullet \notin Q_0(A)$ and incident to an edge in $\mathcal{M}_0(A)$, then it must be the edge $\emptyset \lneq y_\bullet \to y_\bullet \in \mathcal{M}_0(A)$.

$\mathcal{M}_0(A)$ is a Morse matching: Because $\omega(y_\bullet, d_0(y_\bullet)) = 1$ is a unit in R and using Lemma 11, it suffices to verify that $\Gamma(CP(\mathfrak{C}, \mathbb{1})^A_\star)^{\mathcal{M}(A)}$ contains no directed cycles of the form

$$d_0(b_0) \to b_0 \to d_0(b_1) \to b_1 \to d_0(b_2) \to b_3 \to \cdots \to d_0(b_m) \to b_m \to d_0(b_0)$$

for $m \geq 1$ and all $b_i \in Q_0(A)_k$ distinct. Such a cycle cannot exist, because $d_0(b_0) \to b_0 \to d_0(b_1) \to b_1$ implies that $b_0 = \emptyset \lneq y_1 \lneq \cdots \lneq y_k \in Q(A)_k$ has two distinct codimension-1 faces that do not contain $\emptyset$, which is absurd.

Conclusion: Observe that $\emptyset$ is good for all planar diagrams $A \in \mathcal{TL}$. Given any flag $\emptyset \neq y_\bullet \notin \Omega(A)$. If $y_0 \neq \emptyset$, then $\emptyset \subsetneq y_\bullet \in Q_0(A)$ and $d_0(\emptyset \subsetneq y_\bullet) = y_\bullet$. If $y_0 = \emptyset$, then y_1 is bad for A. Hence, the set of critical basis elements is $\{y_\bullet = \emptyset\}$, if $A = id$, and $\{y_\bullet = \emptyset\} \sqcup \{y_\bullet \in \Omega(A) : y_0 = \emptyset$ and y_1 is bad for $A\}$, otherwise. □

Observation 4. *In the situation of Lemma 14:*
If $y_\bullet \in \Omega(A)^{\mathcal{M}_0(A)}$ is critical, then $d_i(y_\bullet) \in \Omega(A)^{\mathcal{M}_0(A)}$ for any $i \neq 0$.

Corollary 4. *The subcomplex $CP(\mathfrak{C}, \mathbb{1})_\star^{\mathcal{M}_0(A)} \hookrightarrow CP(\mathfrak{C}, \mathbb{1})_\star^A$ spanned by the critical basis elements $\Omega(A)^{\mathcal{M}_0(A)}$ of the Morse matching in Lemma 14 is a deformation retract of $CP(\mathfrak{C}, \mathbb{1})_\star^A$.*

Proof: Observation 4 implies that $CP(\mathfrak{C}, \mathbb{1})_\star^{\mathcal{M}_0(A)}$ is a subcomplex of $CP(\mathfrak{C}, \mathbb{1})_\star^A$. Then, the claim follows from Lemma 14 and Corollary 3. □

Corollary 5. *If $A = id$, then $CP(\mathfrak{C}, \mathbb{1})_\star^A$ is contractible.*

Proof: In this case, the only critical basis element in Lemma 14 is $y_\bullet = \emptyset$, which has degree 0. Therefore the claim follows from Corollary 4. □

4.8. Contractibility of the Coxeter poset - step 2. We now study the complexes $CP(\mathfrak{C}, \mathbb{1})_\star^{\mathcal{M}_0(A)}$ for $A \neq id$ (see Corollary 4). In this case, $F(A) \neq \emptyset$ and for any flag $y_\bullet \in \Omega_k^{\mathcal{M}_0(A)}$ there exist a maximal index $\rho = \rho(y_\bullet) \geq 0$ such that $y_\rho \subseteq F(A)$. We use this to define a Morse matching (see Definition 30) on $CP(\mathfrak{C}, \mathbb{1})_\star^{\mathcal{M}_0(A)}$.

Lemma 15. *Let $Q_1(A) \subseteq \Omega^{\mathcal{M}_0(A)}$ be the subset of flags $y_\bullet = (\emptyset \subsetneq y_1 \subsetneq \cdots \subsetneq y_k)$ such that $y_\bullet \neq \emptyset$, $\rho = \rho(y_\bullet) > 0$ and either*

i) $\rho < k$ and $y_\rho = F(A) \cap y_{\rho+1}$ or
ii) $\rho = k$ and $y_\rho = F(A)$.

Then

$$\mathcal{M}_1(A) = \{y_\bullet \to d_\rho(y_\bullet) : y_\bullet \in Q_1(A)\}$$

defines a Morse matching on $(CP(\mathfrak{C}, \mathbb{1})_\star^{\mathcal{M}_0(A)}, \Omega(A)^{\mathcal{M}_0(A)})$ without any critical basis elements.

Proof: We will check all conditions in Definition 30.

$\mathcal{M}_1(A)$ is well defined: Let $y_\bullet = (\emptyset \subsetneq y_1 \cdots \subsetneq y_k) \in Q_1(A)$. Because $\rho = \rho(y_\bullet) > 0$, the zero-th entry of the flag $d_\rho(y_\bullet)$ is the empty set. Because y_1 is bad for A, any y_i with $1 \leq i \leq k$ is bad for A. If $k > 1$, it follows that the first entry of the flag $d_\rho(y_\bullet)$ is bad for A. If $k = \rho = 1$, then $d_\rho(y_\bullet) = \emptyset \in \Omega(A)^{\mathcal{M}_0(A)}$. Hence, $d_\rho(y_\bullet) \in \Omega(A)^{\mathcal{M}_0(A)}$ and the edge $y_\bullet \to d_\rho(y_\bullet)$ exists.

$\mathcal{M}_1(A)$ is a partial matching: If $y_\bullet = (\emptyset \subsetneq y_1 \subsetneq \cdots \subsetneq y_k) \in Q_1(A)$, then $d_\rho(y_\bullet) \notin Q_1(A)$. Indeed, if $\rho = \rho(y_\bullet) = k$, then $\rho(d_k(y_\bullet)) = k - 1$. If $1 < \rho = \rho(y_\bullet) < k$, then condition *ii)* can never apply to $d_\rho(y_\bullet)$ and, because of the proper containment $y_{\rho-1} \subsetneq y_\rho = F(A) \cap y_{\rho+1}$, condition *i)* can not apply either. Hence, any element of $Q_1(A)$ is contained in exactly one edge of $\mathcal{M}_1(A)$. If $y_\bullet \notin Q_1(A)$ and incident to an edge in $\mathcal{M}_1(A)$, then this edge must have the form $z_\bullet \to y_\bullet$ with $z_\bullet \in Q_1(A)$. The flag $z_\bullet$ is given by one of the following two cases:

i) If $\rho(y_\bullet) < k$, then $z_\bullet = (y_0 \subsetneq \cdots \subsetneq y_{\rho(y_\bullet)} \subsetneq F(A) \cap y_{\rho(y_\bullet)+1} \subsetneq y_{\rho(y_\bullet)+1} \subsetneq \cdots \subsetneq y_k)$.
ii) If $\rho(y_\bullet) = k$, then $z_\bullet = (y_0 \subsetneq \cdots \subsetneq y_k \subsetneq F(A))$.

It is impossible that both candidates for $z_\bullet$ exist simultaneously. Hence any $y_\bullet$ is contained in exactly one edge of $\mathcal{M}_1(A)$.

$\mathcal{M}_1(A)$ is a Morse matching: Since $\omega(y_\bullet, d_\rho(y_\bullet)) = (-1)^\rho$ is a unit in R and using Lemma 11 it suffices to verify that $\Gamma(CP(\mathfrak{C},\mathbb{1})_\star^{\mathcal{M}_0(A)})^{\mathcal{M}_1(A)}$ contains no directed cycles

$$d_\rho(b_0) \to b_0 \to d_\rho(b_1) \to b_1 \to d_\rho(b_2) \to b_3 \to \cdots \to d_\rho(b_m) \to b_m \to d_\rho(b_0)$$

with $m \geq 1$, $d_\rho = d_{\rho(b_i)}$ and all $b_i \in Q_1(A)_k$ distinct. Observe that $d_\rho(b_i) \to b_i \to d_\rho(b_{i+1})$ implies that $(b_i)_{\rho(b_i)}$ is a proper subset of $(b_{i+1})_{\rho(b_{i+1})}$. The existence of such a cycle would therefore imply that $(b_0)_\rho$ is a proper subset of itself, which is absurd.

Conclusion: Given any $y_\bullet = (\emptyset \lneq \cdots \lneq y_k) \notin Q_1(A)$. If $\rho(y_\bullet) = k$, then case $ii)$ in "partial matching" applies. If $\rho(y_\bullet) < k$, then $y_{\rho+1} \nsubseteq F(A)$ and case $i)$ in "partial matching" applies. It follows that there are no critical basis elements. □

COROLLARY 6. *If $A \neq id$, then $CP(\mathfrak{C},\mathbb{1})_\star^A$ is acyclic.*

Proof: Lemma 15 and Theorem 12 imply that $CP(\mathfrak{C},\mathbb{1})_\star^{\mathcal{M}_0(A)}$ is acyclic. The claim then follows from Corollary 4. □

Proof of Theorem 10 for the Coxeter poset: The contractibility of the Coxeter poset follows from Corollary 5, Corollary 6 and Observation 3. The module structure of the zero-th homology group follows from Observation 3. □

4.9. Overview: Statement and Proof of Theorem D. We now turn to the second goal of this appendix, the proof of Theorem D stated in the introduction. This theorem relates the cellular Davis complex of Temperley–Lieb algebras $\mathcal{TLD}$ introduced in Definition 22 to the Davis poset $CP(\mathfrak{D},\mathbb{1})$ of $\mathcal{TL}$, that we defined and studied in this appendix. We will prove the following theorem, which implies Theorem D.

THEOREM 16. *Let R be a commutative unital ring. Let $a \in R$ and $\mathcal{TL} = \mathcal{TL}_{n+1}(a)$.*

i) The recellulation spectral sequence of the algebraic Davis poset $CP(\mathfrak{D},\mathbb{1})$ described in Chapter 1, Proposition 1, collapses on the E^2-page and the E^1-page is concentrated in the zero-th row.

ii) The $\mathcal{TL}$-chain complex in the zero-th row $(E^1_{\star,0}, \partial^1)$ is isomorphic to the cellular Davis complex $\mathcal{TLD}$ introduce in Definition 22.

In the next subsection, we will describe the recellulation spectral sequence for Temperley–Lieb algebras and use algebraic discrete Morse theory to prove part $i)$ of Theorem 16. In the last subsection of the appendix, we explain the proof of part $ii)$.

4.10. The recellulation spectral sequence for Temperley–Lieb algebras. In this subsection, we study the spectral sequence constructed in Chapter 1, Proposition 1, specialized to the Davis poset $CP(\mathfrak{D},\mathbb{1})$ of the Temperley–Lieb algebra $\mathcal{TL} = \mathcal{TL}_{n+1}$ and prove part $i)$ of Theorem 16. We start by describing the recellulation spectral sequence for $\mathcal{TL}$.

The chain complexes $E(\mathfrak{D},\mathbb{1})_{F,\star}$ in the following definition occur as summands on the E^0-page of the recelluation spectral sequence,

$$E^0_{\alpha,\star} \cong \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=\alpha}} E(\mathfrak{D},\mathbb{1})_{F,\star}.$$

DEFINITION 31. *Let $F \in \mathfrak{D}$ such that $|F| = \alpha$. Let $(\mathfrak{D}/F)_k$ be the set of k-simplices $y_\bullet = y_0 \lneq \cdots \lneq y_k$ of $N(\mathfrak{D})$ satisfying $y_k = F$, then*

$$E(\mathfrak{D},\mathbb{1})_{F,\beta} = \bigoplus_{\substack{y_0\lneq\cdots\lneq y_{\alpha+\beta}=F\\ \in(\mathfrak{D}/F)_{\alpha+\beta}}} \mathcal{TL} \otimes_{y_0} \mathbb{1} \ .$$

The differential of $E(\mathfrak{D},\mathbb{1})_{F,\star}$ is given by

$$\delta = \sum_{i=0}^{(\alpha+\beta)-1} (-1)^i d_i \ ,$$

where d_i denotes the i-th face map of the Davis poset $CP(\mathfrak{D},\mathbb{1})$. This chain complex is concentrated in non positive degrees $\beta \leq 0$.

We will use algebraic Morse theory to show that the homology of these complex is concentrated in degree $\beta = 0$. We also identify the $\mathcal{TL}$-module structure on the nontrivial homology groups. These intermediate results are summarized in the following proposition.

PROPOSITION 3. *For any $F \in \mathfrak{D}$, the homology of $E(\mathfrak{D},\mathbb{1})_{F,\star}$ is concentrated in degree 0.*

- *i) If F is innermost, then $H_0(E(\mathfrak{D},\mathbb{1})_{F,\star}) \cong Cup(F)$.*
- *ii) If F is not innermost, then $H_0(E(\mathfrak{D},\mathbb{1})_{F,\star}) = 0$ and $E(\mathfrak{D},\mathbb{1})_{F,\star}$ is acyclic.*

We will spend the rest of this subsection proving Proposition 3. Note that part $i)$ of Theorem 16 follows from it.

Proof of part $i)$ of Theorem 16: This follows from Proposition 3. □

REMARK 5. *For Iwahori–Hecke algebras, which we study in the next chapter, it is possible to derive the connectivity properties of the analogues of the complexes $E(\mathfrak{D},\mathbb{1})_{F,\star}$ from connectivity properties of the Coxeter and Davis poset by relating $E(\mathfrak{D},\mathbb{1})_{F,\star}$ to the cofibers of the inclusion of the Coxeter poset into the Davis poset. Because $\mathcal{TL}$ is in general not free as a module over $\mathcal{TL}_F$ for $F \in \mathfrak{D}$, we cannot apply this strategy in the present situation. In fact, the previous connectivity calculations imply that the cofiber is acyclic but the homology of $E(\mathfrak{D},\mathbb{1})_{F,\beta}$ turns out to be nontrivial.*

We start the proof of Proposition 3 by observing what happens if $F = \emptyset$ is the empty set.

OBSERVATION 5. *If $F = \emptyset$, then $H_0(E(\mathfrak{D},\mathbb{1})_{F,\star}) = E(\mathfrak{D},\mathbb{1})_{F,0} = \mathcal{TL} \otimes_F \mathbb{1} \cong \mathcal{TL} = Cup(F)$, because $Cup(F)$ is the left submodule of $\mathcal{TL}$ generated by $\prod_{i\in\emptyset} U_i = 1$.*

From now on we assume that $F \in \mathfrak{D}$ with $|F| = \alpha > 0$.

A splitting for $E(\mathfrak{D},\mathbb{1})_{F,\star}$. Let $F \in \mathfrak{D}$ with $|F| = \alpha > 0$. In this paragraph we will show that the complex $E(\mathfrak{D},\mathbb{1})_{F,\star}$ admits a direct sum decomposition. We will then study each of the summands, using algebraic Morse theory.

By Definition 31 and Corollary 2 et seqq., the R-module $E(\mathfrak{D},\mathbb{1})_{F,\beta}$ admits the following decomposition

$$\begin{aligned} E(\mathfrak{D},\mathbb{1})_{F,\beta} &= \bigoplus_{\substack{y_0 \subsetneq \cdots \subsetneq y_{\alpha+\beta} = F \\ \in (\mathfrak{D}/F)_{\alpha+\beta}}} \mathcal{TL} \otimes_{y_0} \mathbb{1} \\ &= \bigoplus_{\substack{y_0 \subsetneq \cdots \subsetneq y_{\alpha+\beta} = F \\ \in (\mathfrak{D}/F)_{\alpha+\beta}}} \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} R\{A \otimes_{y_0} 1\} \\ &= \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} \bigoplus_{\substack{y_0 \subsetneq \cdots \subsetneq y_{\alpha+\beta} = F \\ \in (\mathfrak{D}/F)_{\alpha+\beta} \\ y_0 \text{ good for } A}} R\{A \otimes_{y_0} 1\} \ . \end{aligned}$$

Hence, we obtain the following analogue of Observation 2 for the chain complex $E(\mathfrak{D},\mathbb{1})_{F,\star}$.

OBSERVATION 6. *Let $A \in \mathcal{TL}$ be a fixed planar diagram. The chain module $E(\mathfrak{D}, \mathbb{1})_{F,\beta}$ contains a unique R-module summand*

$$E(\mathfrak{D}, \mathbb{1})^A_{F,\beta} = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_{\alpha+\beta} = F \\ \in (\mathfrak{D}/F)_{\alpha+\beta} \\ y_0 \text{ good for } A}} R\{A \otimes_{y_0} 1\}.$$

The differential of $E(\mathfrak{D}, \mathbb{1})_{F,\star}$ is induced by the differential of $CP(\mathfrak{D}, \mathbb{1})$. It follows that the R-modules $\{E(\mathfrak{D}, \mathbb{1})^A_{F,\beta}\}$ form a subcomplex and that we have the following analogue of Observation 3.

OBSERVATION 7. *There is a splitting*

$$E(\mathfrak{D}, \mathbb{1})_{F,\star} = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} E(\mathfrak{D}, \mathbb{1})^A_{F,\star}.$$

Algebraic Morse theory - step 1. We will now compute the homology of each summand $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ using algebraic discrete Morse theory. The following proposition summarizes the results.

PROPOSITION 4. *For any $F \in \mathfrak{D}$ and $A \in \mathcal{TL}$ a planar diagram. The homology of $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ is concentrated in degree 0.*

i) If $F \not\subseteq F(A)$, then $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ is acyclic.
ii) If $F \subseteq F(A)$, then the zero-th homology of $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ is isomorphic to R.

We will prove this proposition in two steps. In the first step, we reduce the homology calculation to a small subcomplex using Corollary 3 and is explained in this paragraph. In the second step, which will be described in the next paragraph, we compute the homology of the small subcomplexes.

Fix a planar diagram $A \in \mathcal{TL}$ and let $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]$ be subcomplex as in Observation 7 shifted up α-many degrees. The module of k-simplices of $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]$ is a free R-module with basis $\Omega_k(A, F)$ indexed by flags $y_\bullet \in (\mathfrak{D}/F)_k$ satisfying with $y_k = F$ and y_0 is good for A. In the following we view $(E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha], \Omega(A, F) = \sqcup_k \Omega_k(A, F))$ as a based chain complex in the sense of Definition 27.

We obtain a Morse matching $\mathcal{M}_0(A, F)$ on $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]$ analogous to the one in Lemma 14.

Lemma 17. *Let $Q_0(A, F) = \{y_\bullet \in \Omega(A, F) : y_0 = \emptyset$ and y_1 good for $A\}$. Then*

$$\mathcal{M}_0(A, F) = \{b \to d_0(b) : b \in Q_0(A, F)\}$$

defines a Morse matching on $(E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha], \Omega(A, F))$ with set of critical basis elements:

$$\Omega(A, F)^{\mathcal{M}_0(A)} = \begin{cases} \emptyset, & \text{if } F \text{ is good for } A \\ \{y_\bullet \in \Omega(A, F) : y_0 = \emptyset \text{ and } y_1 \text{ is bad for } A\}, & \text{if } F \text{ is bad for } A \end{cases}$$

Proof: The argument that $\mathcal{M}_0(A, F)$ is well defined, a partial matching and a Morse matching are exactly as in Lemma 14. However, the set of critical basis elements is different. We determine possible candidates for critical basis elements: Given any flag $y_\bullet = (y_0 \lneq \cdots \lneq y_k) \notin Q_0(A, F)$. If $y_0 \neq \emptyset$, then $\emptyset \lneq y_\bullet \in Q_0(A, F)$ and we have a matching $d_0(\emptyset \lneq y_\bullet) = y_\bullet$. If $y_0 = \emptyset$, then either $y_\bullet = \emptyset$ or y_1 is bad for F. We need to determine which of these flags are not part of the matching. Assume first that F is good for A. Then the assumption that $F \neq \emptyset$ and the condition that flags in $\Omega(A, F)$ need to terminate with F imply that it can neither be the case that $y_\bullet = \emptyset$, because this flag is not in $\Omega(A, F)$, nor the case that $y_0 = \emptyset$ and y_1 is bad for A, because this would imply that F

is bad for A. Hence there are no critical basis elements, if F is good for A. If one assumes that F is bad for A, then $y_0 = \emptyset$ and y_1 bad for A describes the critical basis elements. □

Corollary 7. *If $F \neq \emptyset$ is good for A, then $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ is acyclic.*

Proof: This follows from the previous lemma and Theorem 12. □

Corollary 8. *If $F \neq \emptyset$ is bad for A, then $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]^{\mathcal{M}_0}$ is a deformation retract of $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]$*

Proof: This follows from the previous lemma. Observe that the critical basis elements span a subcomplex of $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]$ and invoke Corollary 3. □

Algebraic Morse theory - step 2. We will now compute the homology of the complexes $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]^{\mathcal{M}_0}$ constructed in the last paragraph (see Corollary 8).

Case I. Assume that $F \neq \emptyset$ is bad for A and that $F \not\subseteq F(A)$. Recall that we write $\rho = \rho(y_\bullet)$ for the maximal index $i \geq 0$ in $y_\bullet$ such that $y_i \subseteq F(A)$.

Observation 8. *Note that the condition that a flag $y_\bullet = (y_0 \subsetneq \cdots \subsetneq y_k = F) \in \Omega_k(A, F)$ terminates with F implies that $\rho(y_\bullet) < k$, if $F \not\subseteq F(A)$.*

The following is analogous to Lemma 15.

Lemma 18. *Let $Q_1(A, F) \subseteq \Omega(A, F)^{\mathcal{M}_0(A,F)}$ be the subset of flags $y_\bullet = (\emptyset \subsetneq y_1 \subsetneq \cdots \subsetneq y_k)$ such that $0 < \rho = \rho(y_\bullet)$ and $y_\rho = F(A) \cap y_{\rho+1}$. Then*

$$\mathcal{M}_1(A, F) = \{y_\bullet \to d_\rho(y_\bullet) : y_\bullet \in Q_1(A, F)\}$$

defines a Morse matching on $(E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]^{\mathcal{M}_0(A,F)}, \Omega(A, F)^{\mathcal{M}_0(A,F)})$ without any critical basis elements.

Proof: Using Observation 8 one can check that $\mathcal{M}_1(A, F)$ is a well defined Morse matching without any critical basis elements as in Lemma 15. □

Corollary 9. *Assume that $F \neq \emptyset$ is bad for A and that $F \not\subseteq F(A)$, then $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ is acyclic.*

Proof: This follows from Corollary 8, Lemma 18 and Theorem 12. □

Case II. Assume that $F \neq \emptyset$ is bad for A and that $F \subseteq F(A)$.

Lemma 19. *The complex $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}[+\alpha]^{\mathcal{M}_0(A,F)}$ can be identified with the augmented chain complex of the poset $\partial\Delta(F)$ of proper nonempty subsets of F.*

Proof: The identification is given by sending the basis element $y_\bullet = (\emptyset \subsetneq y_1 \subsetneq \cdots \subsetneq y_k \subsetneq F)$ in $E(\mathfrak{D}, \mathbb{1})^A_{F,k+1}[+\alpha]^{\mathcal{M}_0(A,F)}$ to the basis element $y_1 \subsetneq \cdots \subsetneq y_k$ of $C_{k-1}(\partial\Delta(F))$ and the unique basis element $y_\bullet = \emptyset \subsetneq F$ in $E(\mathfrak{D}, \mathbb{1})^A_{F,1}[+\alpha]^{\mathcal{M}_0}$ to the generator of $1 \in R = C_{-1}(\partial\Delta(F))$. □

The following lemma gives a detailed description of the homology of the complex in Lemma 19 and is well known. We will use it to finish our computation.

Lemma 20. *Let $F = \{0, \ldots, \alpha - 1\}$ be an ordered set of cardinality $|F| = \alpha$. Let $\partial\Delta(F)$ be the poset of proper nonempty subsets of F ordered by inclusion, then there is a homeomorphism*

$$|\partial\Delta(F)| \cong S^{\alpha-2}$$

and, if $\partial\Delta(F)_{\alpha-2}$ *denotes the set of maximal flags* $y_0 \subsetneq \cdots \subsetneq y_{\alpha-2}$ *of the poset, then*

$$[\sum_{\substack{y_\bullet=(y_0\subsetneq\cdots\subsetneq y_{\alpha-2})\\ \in\partial\Delta(F)_{\alpha-2}}} \operatorname{sgn}_F(y_\bullet)\cdot y_\bullet] \in \tilde{H}_{\alpha-2}(\partial\Delta(F);R)$$

is a generator. Here $\operatorname{sgn}_F(y_\bullet)$ *is the signum of the permutation of* F *obtained by mapping* $i \in F$ *to the unique element in* $y_i - y_{i-1}$*, where* $y_{-1} := \emptyset$ *and* $y_{\alpha-1} := F$*.*

Proof: The first part is standard and follows from the fact that $\partial\Delta(F)$ is the poset of simplices of the boundary of a simplex. Because $\partial\Delta(F)$ is a triangulation of a $(\alpha-2)$-sphere, two maximal simplices are, if at all, glued together along exactly one common codimension-one face. The sum in the definition of the homology class in the second claim is over all maximal simplicies. To verify the second part, it therefore suffices to check that the signs of two codimension-one faces $d_i(A_\bullet) = d_j(B_\bullet)$ of simplices $A_\bullet \neq B_\bullet$ cancel. For cardinality reasons, we must have $i = j$. We have $A_t = B_t$ for $t \neq i$, $A_{i+1}-A_{i-1} = B_{i+1}-B_{i-1} = \{x, y\}$ and, without loss of generality, $A_i = A_{i-1} \cup \{x\}$ and $B_i = B_{i-1} \cup \{y\}$. It follows that the two signs are unequal, because $\operatorname{sgn}_F(A_\bullet) \neq \operatorname{sgn}_F(B_\bullet)$. □

The lemma has the following immediate consequence.

Corollary 10. *Assume that* $\emptyset \neq F \subseteq F(A)$*. Then the homology of* $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ *is concentrated in degree* $\star = 0$*, where it is equal to* R*.*

Proof: This follows from Corollary 8, Lemma 19 and Lemma 20. Because $|F| = \alpha$, the homology of the complex $C_\star(\partial\Delta(F))$ concentrated in degree $\star = \alpha-2$, where it is equal to R. □

This finished our analysis of the complexes $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$. We obtain the following proof of Proposition 4.

Proof of Proposition 4: The $F \not\subseteq F(A)$ part follows from Corollary 7 and Corollary 9. The $F \subseteq F(A)$ part follows from Observation 5 and Corollary 10. □

Recall that

$$E(\mathfrak{D}, \mathbb{1})_{F,\star} = \bigoplus_{\substack{A\in\mathcal{TL}\\ \text{planar diagram}}} E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$$

We will now use Proposition 4 to compute the homology of $E(\mathfrak{D}, \mathbb{1})_{F,\star}$ i.e. prove Proposition 3.

Proof of part ii) of Proposition 3: If F is not innermost, then $F \neq \emptyset$ and F cannot be a subset of any innermost set. It follows that for any planar diagram $A \in \mathcal{TL}$, the equation $F \not\subseteq F(A)$ holds. Therefore part $i)$ of Proposition 4 implies that

$$E(\mathfrak{D}, \mathbb{1})_{F,\star} = \bigoplus_{\substack{A\in\mathcal{TL}\\ \text{planar diagram}}} E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$$

is acyclic. □

Proof of part i) of Proposition 3: For $F = \emptyset$, this is Observation 5. Let $A \in \mathcal{TL}$ be a planar diagram and assume that F is innermost with $|F| > 0$. Part $i)$ of Proposition 4 implies that $E(\mathfrak{D}, \mathbb{1})^A_{F,\star}$ is acyclic, if $F \not\subseteq F(A)$. Hence part $ii)$ of Proposition 4 and Observation 7 imply that the homology of $E(\mathfrak{D}, \mathbb{1})_{F,\star}$ is concentrated in degree $\star = 0$ and that as an

R-module it is given by

$$H_0(E(\mathfrak{D},\mathbb{1})_{F,\star}) = \bigoplus_{\substack{A\in\mathcal{TL}\\ \text{such that}\\ F\subseteq F(A)}} R.$$

Observe that $F \subseteq F(A)$ if and only if $A \in Cup(F)$ (compare with Definition 21). Let $|F| = \alpha$. Lemma 19 and Lemma 20 imply that the homology classes of the elements

$$\Delta_F(A) = \sum_{y_\bullet\in\Omega(A,F)^{\mathcal{M}_0}_\alpha} \mathrm{sgn}_F(y_\bullet)(A\otimes_\emptyset 1) \in E(\mathfrak{D},\mathbb{1})^A_{F,0},$$

where $A \in Cup(F)$ is a planar diagram, form a R-module basis of $H_0(E(\mathfrak{D},\mathbb{1})_{F,\star})$. Let $A' \in \mathcal{TL}$ be planar, $A'A = a^{k(A',A)}(A'\circ A)$ where $(A'\circ A)$ is a planar diagram (see Definition 18 et seqq.) and observe that

$$F \subseteq F(A) \subseteq F(A'\circ A)$$

implies $\Omega(A,F)^{\mathcal{M}_0}_\alpha = \Omega(A'\circ A,F)^{\mathcal{M}_0}_\alpha$. It follows that

$$A'\cdot\Delta_F(A) = a^{k(A',A)}\Delta_F(A'\circ A)\ \ ,$$

which implies that $H_0(E(\mathfrak{D},\mathbb{1})_{F,\star}) \cong Cup(F)$. □

4.11. The cellular Davis complex of Temperley–Lieb algebras revisited. This subsection finishes the proof of Theorem D, and contains the proof of part *ii*) of Theorem 16.

Proposition 3 yields the following description of the E^1-page of the recellulation spectral sequence of any Temperley–Lieb algebra $\mathcal{TL} = \mathcal{TL}_{n+1}$.

COROLLARY 11. *The E^1-page of the recellulation spectral sequence of Davis poset $CP(\mathfrak{D},\mathbb{1})$ of $\mathcal{TL}$ contains a single chain complex $E^1_{\star,0}$, i.e. if $\beta\neq 0$, then $E^1_{\alpha,\beta} = 0$. Furthermore, there is an isomorphism of $\mathcal{TL}$-modules*

$$E^1_{\alpha,0} \cong \bigoplus_{\substack{F\subseteq\langle n\rangle\ \textit{innermost}\ ,\\ |F|=\alpha}} Cup(F).$$

We close this chapter by identifying the E^1-differentials of the recellulation spectral sequence, the final step in the proof of part *ii*) of Theorem 16 (Theorem D).

PROPOSITION 5. *Let $\mathcal{TL}\mathfrak{D}$ be the cellular Davis complex of the Temperley–Lieb algebra $\mathcal{TL}$ introduced in Definition 22 and let $(E^1_{\star,0},\partial^1)$ be the chain complex on the E^1-page of the recellulation spectral sequence of the Davis poset $CP(\mathfrak{D},\mathbb{1})$ of $\mathcal{TL}$. There is an isomorphism of chain complexes*

$$E^1_{\star,0} \to \mathcal{TL}\mathfrak{D}_\star$$

given by multiplication with $(-1)^\alpha$ on the α-th chain module i.e.

$$E^1_{\alpha,0} \to \mathcal{TL}\mathfrak{D}_\alpha : [\Delta_F(A)] \mapsto (-1)^\alpha[A]_{Cup(F)},$$

where the elements $[\Delta_F(A)]$ are the generators defined in the proof of part i) of Proposition 3 and the elements $[A]_{Cup(F)}$ are the generators defined in the proof of Theorem 4.

Proof: Recall that the E^0-differential ∂^0 of the recellulation spectral sequence (see Proposition 1) is induced by the differential of the Davis poset $CP(\mathfrak{D},\mathbb{1})$ of the Temperley–Lieb algebra $\mathcal{TL}$ (see Definition 25). By definition $E^1_{\alpha,0} \cong \ker(\partial^0 : E^0_{\alpha,0} \to E^0_{\alpha,-1})$, where $(E^0_{\star,\star},\partial^0)$ denotes the E^0-page of the recellulation spectral sequence. If we write

$$\ker(\alpha) := \ker(\delta : F^\alpha(CP(\mathfrak{D},\mathbb{1}))_\alpha \to F^\alpha(CP(\mathfrak{D},\mathbb{1}))_{\alpha-1}/F^{\alpha-1}(CP(\mathfrak{D},\mathbb{1}))_{\alpha-1})\ \ ,$$

where δ is the restricted differential of $CP(\mathfrak{D}, \mathbb{1})$, then $E^1_{\alpha,0} \cong \ker(\alpha)/F^{\alpha-1}(CP(\mathfrak{D}, \mathbb{1}))_\alpha$. An element $\theta \in \ker(\alpha)$ gives rise to an element $\delta(\theta) \in F^{\alpha-1}(CP(\mathfrak{D}, \mathbb{1}^\epsilon))_{\alpha-1}$ and by definition the differential of the E^1-page is given by

$$\partial^1([\theta]) = [\delta(\theta)] \in E^1_{\alpha-1,0} \ .$$

We will now calculate the effect of ∂^1 on the generators $[\Delta_F(A)]$ described in the proof of part $i)$ of Proposition 3. Let $A \in Cup(F)$ be a planar diagram such that $F \subseteq F(A)$. For any $s \in F$, let $F_s = F - s$ and $\gamma(s) = \gamma_F(s) = |\{s' \in F_s : s' > s\}|$. Observe that $F_s \subseteq F(A)$ and $A \in Cup(F_s)$ for any $s \in F$. Then, we have that

$$\begin{aligned}
\delta(\Delta_F(A)) &= \delta^{E(\mathfrak{D},\mathbb{1})}(\Delta_F(A)) + (-1)^\alpha d_\alpha(\Delta_F(A)) \\
&= 0 + (-1)^\alpha d_\alpha\Big(\sum_{y_\bullet \in \Omega(A,\mathfrak{D}/F)^{\mathcal{M}_0}_\alpha} \operatorname{sgn}_F(y_\bullet)(A \otimes_\emptyset 1)\Big) \\
&= (-1)^\alpha \sum_{s \in F} \sum_{y_\bullet \in \Omega(A,\mathfrak{D}/F_s)^{\mathcal{M}_0}_{\alpha-1}} \operatorname{sgn}_F(y_\bullet \lneq F)(A \otimes_\emptyset 1) \\
&= (-1)^\alpha \sum_{s \in F} (-1)^{\gamma(s)} \sum_{y_\bullet \in \Omega(A,\mathfrak{D}/F_s)^{\mathcal{M}_0}_{\alpha-1}} \operatorname{sgn}_{F_s}(y_\bullet)(A \otimes_\emptyset 1) \\
&= (-1)^\alpha \sum_{s \in F} (-1)^{\gamma(s)} \Delta_{F_s}(A).
\end{aligned}$$

Here $\delta^{E(\mathfrak{D},\mathbb{1})}$ denotes the differential of the complex $E(\mathfrak{D}, \mathbb{1})_{F,\star}$ (see Definition 31), we used Lemma 19 and Lemma 20 in the second step and that $\operatorname{sgn}_F(y_\bullet \lneq F) = \gamma(s) \cdot \operatorname{sgn}_{F_s}(y_\bullet)$ for $y_\bullet \in \Omega(A, \mathfrak{D}/F_s)^{\mathcal{M}_0}_{\alpha-1}$ in the fourth step. It follows that

$$\partial^1[\Delta_F(A)] = (-1)^\alpha \sum_{s \in F} (-1)^{\gamma(s)} [\Delta_{F_s}(A)]$$

and comparison with the definition of the differential of $\mathcal{TLD}$ in Definition 22 completes the proof. $\square$

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CHAPTER 3

Algebraic coset posets and the low-dimensional homology of Iwahori–Hecke algebras

Summary: We derive a formula for the second homology group of certain Iwahori–Hecke algebras. This generalizes a result of Boyd for the second homology of Coxeter groups and is the Iwahori–Hecke analogue of a theorem of Howlett [**How88**]. Our strategy is a direct generalization of Boyd's approach to low-dimensional homology computations for Coxeter groups [**Boy20**]. The two key components of the proof are the following. (1) An isotropy spectral sequence that is associated to an algebraic Davis poset for Iwahori–Hecke algebras, in the sense of Chapter 1, Proposition 1. (2) Understanding transfer maps between the homology groups of Iwahori–Hecke algebras.

1. Introduction

Iwahori–Hecke algebras $\mathcal{H} = \mathcal{H}^q(W,S)$ are a class of R-algebras associated to the choice of a Coxeter group (W,S) and a parameter $q \in R$. The family of algebras $\{\mathcal{H}^q(W,S)\}_{q\in R}$ is a deformation of the group algebra $R[W]$ of the Coxeter group W, where the involution relation $s^2 - 1 = (s+1)(s-1) = 0$ of the generators $s \in S$ is deformed as follows using a parameter $q \in R$ (see Definition 37).

$$(s+1)\cdot(s-1) = 0 \rightsquigarrow (s+1)\cdot(s-q) = 0 \text{ for any } s \in S$$

Note that if $q = 1$, the deformation is trivial. In this case, the Iwahori–Hecke algebra $\mathcal{H} = \mathcal{H}^q(W,S)$ is the group algebra $R[W]$.

Iwahori–Hecke algebras admit a one-dimensional representation $\mathbb{1}$ on which the generators $s \in S$ act by multiplication by $q \in R$. In the group algebra case, $q = 1$, this is the trivial $R[W]$-representation. In this chapter, we will study the low-dimensional homology groups of Iwahori–Hecke algebras with coefficients in this representation.

$$H_\star(\mathcal{H},\mathbb{1}) = \operatorname{Tor}^{\mathcal{H}}_\star(\mathbb{1},\mathbb{1})$$

The homology and cohomology groups of Iwahori–Hecke algebras have been considered in the literature. The cohomology ring for $R = \mathbb{C}$ and finite Coxeter groups of type A_n, B_n and D_n has been computed by Benson–Erdmann–Mikaelian in [**BEM10**] for many interesting choices of $q \in \mathbb{C}$. This has found applications in Nakano–Xiang's work on support varieties [**NX19**]. More recently, Hepworth [**Hep20**] used Quillen's method and an algebraic analogue of the complex of injective words to show that the homology of Iwahori–Hecke algebras associated to symmetric groups satisfy homological stability, if $q \in R$ is a unit. Hepworth's result and method is closely related to the stability theorems proved for the homology of Temperley–Lieb and Brauer algebras by Boyd–Hepworth [**BH20**] and Boyd–Hepworth–Patzt [**BHP20**], respectively.

In this chapter, we investigate the low-dimensional homology of Iwahori–Hecke algebras whose underlying Coxeter group is not necessarily the symmetric group and whose ground ring R is not necessarily $\mathbb{C}$. We will do this by generalizing techniques used by Boyd [**Boy18, Boy20**] to study the low-dimensional homology of Coxeter groups to the algebraic setting. Our main theorem, which is in direct analogy to ([**Boy20**], Theorem A), gives a formula for the first and second homology of an Iwahori–Hecke algebra in terms of graphs obtained from the Coxeter diagram of the Coxeter system (W,S). Given a commutative ring R and elements $x, y \in R$, we write $R_{(x,y)}$ for the quotient $R/(x,y)$.

THEOREM A. *Let q be a unit in an integral domain R and $\mathcal{H} = \mathcal{H}^q(W,S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$, then we have the following natural identifications:*

i) $H_0(\mathcal{H},\mathbb{1}) \cong \mathbb{1}$.
ii) $H_1(\mathcal{H},\mathbb{1}) \cong H_0(\mathcal{D}_{odd}; R_{(1+q)})$.
iii) If $q \neq -1$, then

$$H_2(\mathcal{H},\mathbb{1}) \cong H_0(\mathcal{D}^X_{A_3}, R_{(1+q,2)}) \oplus H_0(\mathcal{D}^X_{\neg A_3}, \mathscr{L}) \oplus R_{(1+q)}\{E(\mathcal{D}_{even})\} \oplus H_1(\mathcal{D}_{odd}, R_{(1+q)})$$

Here, $\mathcal{D}_{odd}, \mathcal{D}_{even}, \mathcal{D}^X_{\neg A_3}$ and $\mathcal{D}^X_{A_3}$ are certain graphs attached to the Coxeter system (W,S), $\mathscr{L}$ is a local coefficient system, and $1 + q \in R$ is the second q-integer (see Definition 46).

Comparison with work of Boyd [**Boy20**] and Howlett [**How88**]. For $q = 1$ and $R = \mathbb{Z}$, then $\mathcal{H} = \mathbb{Z}[W]$ is the group ring and $H_i(\mathcal{H},\mathbb{1}) = H_i(W,\mathbb{Z})$ is the integral group homology of W. We have that $(1+q) = 2$, $R_{(1+q)} = R_{(1+q,2)} = \mathbb{Z}_{(2)}$ and one can show that the coefficient system $\mathscr{L} = \mathbb{Z}_{(2)}$ is constant. In this case, the formula is a theorem of Boyd [**Boy20**] and recovers Howletts' computation [**How88**] of the rank of the second integral homology of a

Coxeter group. For $q \neq 1$, the theorem is new, but the strategy of proof is a direct generalization of the approach developed by Boyd in [**Boy20**].

The effect of the deformation $1 \rightsquigarrow q$. Note the passage from 2-torsion in the group case as decribed in the previous paragraph, to $(1+q)$-torsion in the case of Iwahori–Hecke algebras as described in Theorem A. Furthermore, a "splitting" occurs in the second homology group that we will explain now. The graphs $\mathcal{D}^X_{\neg A_3}$ and $\mathcal{D}^X_{A_3}$ in the theorem are different connected components of a larger graph $\mathcal{D}^X$. For the group algebra case, $q = 1$, both components have the same coefficient system and we could replace the first two terms in part *iii)* by $H_0(\mathcal{D}^X; R_{(2)})$. This is exactly the term appearing in Boyd's formula for $H_2(W;\mathbb{Z})$ [**Boy20**]. For $q \neq 1$, the coeffient system on the two connected components is different in general and therefore the term is "split" into two pieces.

We will now outline Boyd's strategy of proof for the group case and then describe how the argument changes in the deformed case.

The undeformed case, $q = 1$: In [**Boy20**], Boyd studies the isotropy spectral sequence of the classical Davis complex equipped with the "Coxeter CW structure" (compare with examples in Chapter 1, Example 1) to derive formulas for the second and third homology of Coxeter groups with integral coefficients. The classical Davis complex is a contractible coset poset [**Dav08**] associated to (W,S) and hence a W-complex. The "Coxeter CW-structure" has the property that stabilizers of k-cells are finite Coxeter groups (W_F, F) of rank $|F| = k$. The isotropy spectral sequence gives a way to compute the homology of the Coxeter group W from the homology of the stabilizers with coefficients in the sign representation. The differentials in this spectral sequence can be described as transfer maps in group homology.

The deformed case, $q \neq 1$: In this chapter we replace the classical Davis complex with an algebraic Davis poset for Iwahori–Hecke algebras, that is an algebraic coset poset associated to the R-algebra $\mathcal{H}$ (see Chapter 1, Definition 8). Similarly, one can define an algebraic Coxeter complex for Iwahori–Hecke algebras. We prove that these complexes have the same connectivity properties as their classical analogues if $q \in R$ is a unit. In fact, the next theorem is proved by reduction to the undeformed case $q = 1$ and follows from the connectivity properties of the classical analogues (see Corollary 15).

THEOREM B. *Let R be a communitative unital ring. Let q be a unit in R and $\mathcal{H} = \mathcal{H}^q(W,S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$. Then,*

i) the algebraic Davis complex $CP(\mathfrak{D}, \mathbb{1})$ of $\mathcal{H}$ is contractible.
ii) the algebraic Coxeter complex $CP(\mathfrak{C}, \mathbb{1})$ of $\mathcal{H}$ is $(|S|-1)$-spherical, if W is finite, and contractible, if W is infinite.

We then study the recellulation spectral sequence of the algebraic Davis complex described in Chapter 1, Proposition 1. This spectral sequence implements the passage to the "Coxeter CW structure" and we use it to construct the cellular Davis complex $\mathcal{H}\mathfrak{D}$ of the Iwahori–Hecke algebra $\mathcal{H}$. Before we can use the isotropy spectral sequence to derive Theorem A, we need to understand the differentials that occur in it. For this reason, we define and study transfer maps for the homology of Iwahori–Hecke algebras. Such transfer maps already play an important role in the work of Benson–Erdmann–Mikaelian [**BEM10**]. Our exposition will be parallel to the discussion of transfer for the homology of groups in Brown's book [**Bro82**].

Outline of the chapter. Sections 2, 3 and 4 contain the necessary mathematical background that is largely contained in the literature. The sections 2 and 3 concern Coxeter groups and Iwahori–Hecke algebras, respectively. Section 4 is a detailed treatment of transfer maps for Iwahori–Hecke algebras. In section 5, we define and study the algebraic Davis and Coxeter complex. This section contains the proof of Theorem B, the recellulation spectral sequence argument and a description of the cellular Davis complex for Iwahori–Hecke algebras. In section 6, we study the isotropy spectral sequence associated to the cellular Davis complex and prove Theorem A.

Future work. In principle, it should be possible to use the methods described in this chapter to derive an Iwahori–Hecke analogue of the formula for the third homology group of a Coxeter group obtained by Boyd in [**Boy20**]. One difficulty of this project would be the description of higher differentials in the isotropy spectral sequence. Understanding such differentials is already necessary if one wants to derive a formula for the second homology with deformation parameter $q = -1$. Another possible application of the techniques and results described in this chapter are homological stability questions. In an on-going project with Richard Hepworth and Jeremy Miller, we study (higher) stability patterns in the homology of Iwahori–Hecke algebras.

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2. On Coxeter groups

This section introduces background on Coxeter groups that is necessary for our purposes and that we learned from [**GP00, Mat99**].

Definition 32 ([**GP00**], § 1.2.8.). *Let S be a finite set and $M = (m_{s,t})_{s,t\in S}$ be a symmetric matrix whose entries are positive integers (or ∞) such that $m_{s,s} = 1$ and $m_{s,t} > 1$ for all $s, t \in S, s \neq t$. Define a group W by the presentation:*

$$W = \langle s \in S | s^2 = 1 \text{ for } s \in S \text{ and } (st)^{m_{s,t}} = 1 \text{ for } s \neq t, m_{s,t} < \infty\rangle$$

The pair (W, S) is called a Coxeter system and the group W a Coxeter group.

Figure 1 describes the finite Coxeter that will be of particular importance in this chapter in terms of their Coxeter graph $\mathcal{D}(W, S)$. The vertex set of $\mathcal{D}(W, S)$ is the set of Coxeter generators S. There is no edge between two generators s and t, if they commute i.e. $m_{s,t} = 2$. There is an edge labeled by $m_{s,t}$ between s and t, if $m_{s,t} \geq 3$.

It follows from the classification of finite Coxeter groups [**Cox35**], that any finite Coxeter group (W, S) whose generating set satisfies $|S| \leq 3$ is of the form A_3, B_3, H_3 or $I_2(p) \times A_1$ (see Figure 1 for their Coxeter graphs). We will use this fact in the last part of the chapter.

Let (W, S) denote a Coxeter system for the rest of this section.

Definition 33. *Let $w \in W$. We write $|w|$ for the word length of w with regard to the generating set S i.e.*

$$|w| = \inf\{n \in \mathbb{N} : w = s_{i_1} \dots s_{i_n} \text{ for } s_i \in S\}.$$

A word $s_{i_1} \dots s_{i_n}$ is called a reduced spelling of w, if

$$w = s_{i_1} \dots s_{i_n} \text{ and } |w| = n.$$

The following "standard" subgroups of W will play an important role in this chapter.

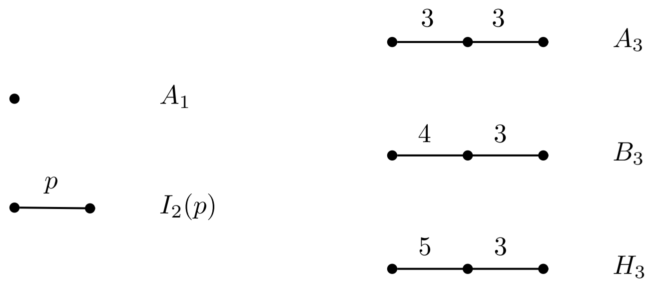


FIGURE 1. Coxeter graphs of Coxeter groups of type $A_1, I_2(p), A_3, B_3$ and H_3.

DEFINITION 34. *For a subset $J \subseteq S$, we write W_J for the subgroup of W generated by J. Such subgroups are called parabolic subgroups of W.*

We will be interested in cosets $wW_J \in W/W_J$ of W with respect to parabolic subgroups W_J. The cosets wW_J have the (perhaps surprising) property that each contains a unique element of minimal word length. This is the content of the next proposition.

PROPOSITION 6 ([**GP00**], 2.1.1). *Given $J \subseteq S$ and $w \in W$, there is a unique element $w_{min} \in wW_J$ characterized by the following (equivalent) properties:*

- *$|w_{min}| \leq |w'|$ for all $w' \in wW_J$.*
- *$|w_{min}w_J| = |w_{min}| + |w_J|$ for all $w_J \in W_J$.*

We call w_{min} the minimal (word length) representative of the left coset wW_J. Similarly, there is a unique minimal (word length) representative for any right coset W_Jw.

The set of all minimal representatives of the set of cosets W/W_J will often occur as an indexing set.

DEFINITION 35. *We write $M_J = M_J(W)$ for the set of minimal representatives of left cosets W/W_J and ${}_JM = {}_JM(W)$ for the set of minimal representatives of right cosets $W_J \backslash W$.*

We record the following facts, which we will use later.

OBSERVATION 9. *The map $w \mapsto w^{-1}$ is a bijection $M_J \to {}_JM$.*

Lemma 21 ([**Mat99**], 1.7 Corollary)**.** *For any $s \in J$ and $w \in W_J$:*

i) $|ws| < |w|$ if and only if there exists a reduced spelling of w using J ending with s.
ii) $|sw| < |w|$ if and only if there exists a reduced spelling of w using J starting with s.

Proof: The reference gives the argument for symmetric groups. The same argument applies for general Coxeter systems (W, S) using the exchange condition described in [**GP00**], Exercise 1.6.. □

DEFINITION 36. *Let $w, w_{pre}, w_{suf} \in W$. We call w_{pre} a prefix and w_{suf} a suffix of w, if $w = w_{pre}w_{suf}$ and $|w| = |w_{pre}| + |w_{suf}|$.*

Lemma 22 ([**GP00**], Deodhar's lemma, Lemma 2.1.2.)**.** *Let $J \subseteq S$. Given minimal right coset representative $x \in {}_JM$ and $s \in S$, then either $xs \in {}_JM$ or $xs = ux$ for some $u \in J$.*

COROLLARY 12 ([**GP00**], cf. after Lemma 2.1.2.).

i) If x_{pre} is a prefix of $x \in {}_JM$, then $x_{pre} \in {}_JM$ is minimal right coset representative as well.

ii) If x_{suf} is a suffix of $x \in M_J$, then $x_{suf} \in M_J$ is minimal left coset representative as well.

Proof: By Observation 9 we only need to argue for part i). It suffices to show that if $x \in {}_JM$ and $s \in S$ such that $|xs| < |x|$, then $xs \in {}_JM$. The minimality of x implies that for any $u \in J$ the word length of ux is larger than $|x|$, i.e. $|ux| = 1 + |x|$ (see Proposition 6). Hence, we cannot have $xs = ux$ for any $u \in J$ and the first case of Lemma 22 must apply. $\square$

3. On Iwahori–Hecke algebras

This section introduces background on Iwahori–Hecke algebras that is necessary for our purposes and is mainly based on [**GP00**, **Mat99**, **DJ86**]. Throughout this chapter, R will denote a commutative unital ring.

Definition 37 ([**GP00**], 4.4.5). *The Iwahori-Hecke Algebra $\mathcal{H} = \mathcal{H}(q, (W, S))$ associated to a Coxeter system (W, S) and the parameter $q \in R$ is the associative R-algebra with 1 defined by the following presentation:*

- *generators: symbols T_w for $w \in W$.*
- *relation (i): $T_w = T_{s_0} \dots T_{s_l}$ for $w = s_0 \dots s_l$ a reduced spelling of $w \in W$ and $s_i \in S$.*
- *relation (ii): $T_s^2 = q + (q-1)T_s$ for $s \in S$.*

Note that $T_e = 1 \in \mathcal{H}$, where e is the neutral element of W.

If $q = 1$, then relation (ii) states $T_s^2 = T_e = 1$. In particular, $\mathcal{H} \cong R[W]$ is the group algebra of the Coxeter group W in this case. Note that relation (ii) in Definition 37 is exactly the "deformed relation" mentioned in the introduction and can be rewritten as $T_s^2 - qT_s + T_s - q = (T_s + 1)(T_s - q) = 0$. If $q \neq 1$, we record that $T_s \neq (T_s)^{-1}$ in general.

The following lemma describes how to evaluate products in $\mathcal{H}$.

Lemma 23 ([**GP00**], 4.4.3. (b))**.** *For $s \in S$ and $w \in W$, we have*

$$T_s \cdot T_w = \begin{cases} T_{sw} & \text{if } |sw| > |w|, \\ qT_{sw} + (q-1)T_w & \text{if } |sw| < |w| \end{cases}$$

An analogous relation also holds for the product $T_w \cdot T_s$.

In analogy with the parabolic subgroups W_J of the Coxeter group W, an Iwahori–Hecke algebra contains the following important subalgebras.

Definition 38. *For $J \subseteq S$, we write $\mathcal{H}_J$ for the parabolic subalgebra of $\mathcal{H} = \mathcal{H}(q, (W, S))$ spanned by $\{T_w : w \in W_J\}$.*

Recall that the group algebra $R[G]$ of a group G is free as a module over the group ring $R[H]$ of any subgroup $H \leq G$ of G. The following theorem shows that a similar property holds for Iwahori–Hecke algebras with respect to parabolic subalgebras.

Theorem 24 (Bourbaki. See [**GP00**], 4.4.6. and §4.4.7.). *Let $J \subseteq S$. If R is a commutative ring, then the Iwahori–Hecke algebra $\mathcal{H} = \mathcal{H}(q, (W, S))$ is a free (left) $\mathcal{H}_J$-module with basis $\{T_x : x \in {}_JM\}$. In particular, $\mathcal{H}$ is a free R-module with basis $\{T_w : w \in W\}$.*

We will be interested in the homology of Iwahori–Hecke algebras with coefficients in the following two one-dimensional representations.

Corollary 13 ([**Mat99**], 1.14). *The following R-linear maps define one-dimensional representations of the Iwahori–Hecke algebra $\mathcal{H}$:*

$$\hat{\epsilon} : \mathcal{H} \to R : T_w \mapsto q^{|w|} \text{ and } \epsilon^{\pm} : \mathcal{H} \to R : T_w \mapsto (-1)^{|w|}$$

Definition 39. *We call R equipped with*

i) the left $\mathcal{H}$-action $T_w \mapsto q^{|w|}$ the trivial module and denote it by $\mathbb{1}^{(q)}$.
ii) the left $\mathcal{H}$-action $T_w \mapsto (-1)^{|w|}$ the sign module and denote it by $\mathbb{1}^{(-1)}$.

Remark 6. *If $q = -1$, then the two representations are the same.*

Remark 7. *Let $W = \Sigma_n$ be the symmetric group and R be a field. If $q \neq 0$, then the trivial and sign module are the only one-dimensional representations up to isomorphism (Lemma 3.1., [**DJ86**]). If $q = 0$, there are $2^{|S|}$ distinct one-dimensional representations (Exercise 8.9 (d), [**GP00**]).*

We close this subsection by presenting an argument of the following proposition. A similar fact plays an important role in Dipper–James' proof that the induction is a right adjoint to restriction for Iwahori–Hecke algebras associated to finite Coxeter groups (see [**DJ86**], Theorem 2.6). It will be useful in our discussion of transfer for Iwahori–Hecke algebras following [**BEM10**] in the next subsection. The statement and proof presented here is a slight refinement of ([**GP00**], Proposition 8.1.1), where the case $J = \emptyset$ is discussed in detail.

Definition 40. *Given $h \in \mathcal{H}$ and some element $z \in W$, we define $h_z \in R$ to be the coefficient of T_z in the unique sum-decomposition $h = \sum_{z\in W} h_z T_z$ obtained by Theorem 24.*

Proposition 7. *Let $J \subseteq S$ and consider two minimal left coset representatives $x, y \in M_J$. Recall that $x^{-1} \in {}_JM$ is a minimal right coset representative. Let $T_{x^{-1}}$ and T_y denote the corresponding generators of $\mathcal{H}$, then the coefficients of their product $(T_{x^{-1}} \cdot T_y)_z$ have the following values for $z \in W_J$*

$$(T_{x^{-1}} \cdot T_y)_z = \begin{cases} q^{|x|} & \text{if } x = y \text{ and } z = e \\ 0 & \text{otherwise} \end{cases}$$

Proof: We perform an induction over word length of y. If $|y| = 0$, then $y = e$ and $T_{x^{-1}} \cdot T_y = T_{x^{-1}}$ because $T_e = 1$. The claim holds in this case since either $x = e = y$ and $q^{|x|} = 1$ or $x^{-1} \notin W_J$. If $|y| > 0$, then $y = s\hat{y}$ for some $s \in S$ such that $|y| = 1 + |\hat{y}|$. Corollary 12 implies that $\hat{y} \in M_J$ is minimal as a suffix. By Lemma 23, we therefore need to inspect two cases:

i) Assume that $|x^{-1}s| > |x^{-1}|$: Then $T_{x^{-1}} \cdot T_y = T_{x^{-1}s} \cdot T_{\hat{y}}$ with $|x^{-1}s| = |x| + 1$. If $x^{-1}s \in {}_JM$, the claim holds by the induction hypothesis. Assume that $x^{-1}s \notin {}_JM$, then by Lemma 22: $T_{x^{-1}s} = T_{ux^{-1}} = T_u \cdot T_{x^{-1}}$ for some $u \in J$. Let $z \in W_J$. The induction hypothesis applies to $(T_{x^{-1}} \cdot T_{\hat{y}})$. Hence, $(T_{x^{-1}} \cdot T_{\hat{y}})_z = 0$, if $z \neq e$. Assume that $(T_{x^{-1}} \cdot T_{\hat{y}})_e$ is nonzero. By the induction hypothesis this can only happen for $x = \hat{y}$. Thus, $sx = y \in M_J$ and hence, $x^{-1}s \in {}_JM$, but this is not the case by our previous assumption. Therefore, $(T_{x^{-1}} \cdot T_{\hat{y}})_e$ must be zero as well. To finish the argument for this case, set $h := T_{x^{-1}} \cdot T_{\hat{y}}$ and let $h = \sum_{w\in W} h_w T_w$ be the unique decomposition obtained from Theorem 24. We have argued that $h_w = 0$ if $w \in W_J$ and thus know that $T_{x^{-1}} \cdot T_y = T_u \cdot h = \sum_{w\notin W_J} h_w T_u T_w$ for some $u \in J$. Observe that if $w \notin W_J$, then $uw \notin W_J$ so that as a consequence of Lemma 23: $(T_u T_w)_z = 0$ for all $z \in W_J$ and any $w \notin W_J$. But this implies $(T_{x^{-1}} \cdot T_y)_z = 0$ for all $z \in W_J$.

ii) Assume that $|x^{-1}s| < |x^{-1}|$: Then by Lemma 23

$$\begin{aligned} T_{x^{-1}} \cdot T_y &= (T_{x^{-1}} \cdot T_s) \cdot T_{\hat{y}} \\ &= (qT_{x^{-1}s} + (q-1)T_{x^{-1}}) \cdot T_{\hat{y}} = q(T_{x^{-1}s} \cdot T_{\hat{y}}) + (q-1)(T_{x^{-1}} \cdot T_{\hat{y}}). \end{aligned}$$

By Corollary 12, we have $x^{-1}s \in {}_JM$ as a prefix. Therefore, the induction hypothesis applies to both $(T_{x^{-1}s} \cdot T_{\hat{y}})$ and $(T_{x^{-1}} \cdot T_{\hat{y}})$. This implies that $(T_{x^{-1}s} \cdot T_{\hat{y}})_z = q^{|x|-1}$, if $z = e$ and $sx = \hat{y}$, as well as $(T_{x^{-1}} \cdot T_{\hat{y}})_z = q^{|x|}$, if $z = e$ and $x = \hat{y}$. For any other $z \neq e \in W_J$ both terms are zero. Note that our initial assumption implies that $x \neq \hat{y}$: If $x = \hat{y}$, then $|x^{-1}| = |y| - 1$ and therefore $|x^{-1}| = |y| - 1 < |y| = |x^{-1}s|$, a contradiction. Hence, the second term $(T_{x^{-1}} \cdot T_{\hat{y}})_z = 0$ is zero for all $z \in W_J$. On the other hand, $sx = \hat{y}$ is equivalent to $x = y$ and the claim follows because $(T_{x^{-1}} \cdot T_y)_z = q(T_{x^{-1}s} \cdot T_{\hat{y}})_z$ for $z \in W_J$. □

4. Transfer for Iwahori–Hecke algebras

This section expands the discussion of transfer maps for Iwahori–Hecke algebras contained [**BEM10**] and builds on [**BEM10, DJ86, DD93, Bro82**]. Let (W, S) be a Coxeter system and let $J \subseteq S$ be a set of Coxeter generators. Assume that the parabolic subgroup W_J is a finite index subgroup of W, then there exists a "wrong way map"

$$tr : H_\star(W, M) \to H_\star(W_J, M)$$

in group homology with coefficients in any W-module M called transfer. [**Bro82**], III.9 (A), explains that the underlying reason for the existence of the transfer map is the following. If H is a finite index subgroup of G and M is any left G-module, then there exists an isomorphism of left $\mathbb{Z}G$-modules, [**Bro82**] III.5 (5.9),

$$\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} M.$$

The following proposition is the Iwahori–Hecke analogue of this fact. For Iwahori–Hecke algebras associated to finite Coxeter groups this statement is discussed in [**DJ86**], p. 26, and, as stated there, a consequence of the uniqueness of adjoint functors[1]. We will give a hands-on proof of this isomorphism in the slightly more general setting, where W is allowed to be infinite and W_J is a finite index subgroup. The explicit formula obtained this way will later be used to relate the differentials of the cellular Davis complex $\mathcal{H}\mathfrak{D}$, constructed in the next section (see Definition 44), to the transfer maps, which we will describe in this section.

Proposition 8 ([**DJ86**], p. 26). *Assume that the parameter $q \in R$ is a unit and let M be a left $\mathcal{H}$-module. If $J \subseteq S$ is such that W_J is a finite index subgroup of W, then there is an isomorphism of left $\mathcal{H}$-modules*

$$\mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M) \cong \mathcal{H} \otimes_{\mathcal{H}_J} M$$

given by

$$\psi : Hom_{\mathcal{H}_J}(\mathcal{H}, M) \to \mathcal{H} \otimes_{\mathcal{H}_J} M : f \mapsto \sum_{x \in M_J} q^{-|x|} T_x \otimes f(T_{x^{-1}}).$$

Proof: The following argument is analogous to the one in the group case presented by Brown in [**Bro82**], III.5 (5.9). We will construct a map

$$\phi : \mathcal{H} \otimes_{\mathcal{H}_J} M \to \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$$

and then verify that it is inverse to ψ. To construct ϕ, we observe that there is a map of left $\mathcal{H}_J$-modules

$$\phi_0 : M \to \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$$

defined by sending $m \in M$ to the morphism $\phi_0(m)$, which is given by $\phi_0(m)(T_w) = T_w \cdot m$, if $w \in W_J$, and $\phi_0(m)(T_w) = 0$ otherwise. To see that this is well-defined and $\mathcal{H}_J$-equivariant, we recall that the right module structure of $\mathcal{H}$ is used to turn $\mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$ into a left $\mathcal{H}$-module i.e., $(h \cdot f)(-) = f(- \cdot h) \in \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$ for $h \in \mathcal{H}$ and $f \in \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$. Whereas a morphism in $Hom_{\mathcal{H}_J}(\mathcal{H}, M)$ is equivariant with regard to the left $\mathcal{H}_J$-module structures on $\mathcal{H}$ and M, respectively. The observation then amounts to using Lemma 23 to check that for any $m \in M, s \in J$ and $w \in W$ the following two equations hold.

[1] See also [**DJ86**], Remark, at the end of Section 2.

i) $\phi_0(m)$ is $\mathcal{H}_J$-equivariant and ϕ_0 is well-defined, because

$$\phi_0(m)(T_s \cdot T_w) = T_s \cdot \phi_0(m)(T_w).$$

ii) ϕ_0 is $\mathcal{H}_J$-equivariant, because

$$\phi_0(T_s \cdot m)(T_w) = (T_s \cdot \phi_0(m))(T_w).$$

By extension of scalars ϕ_0 induces to a map of $\mathcal{H}$-modules

$$\phi : \mathcal{H} \otimes_{\mathcal{H}_J} M \to Hom_{\mathcal{H}_J}(\mathcal{H}, M) : h \otimes m \mapsto h \cdot \phi_0(m)$$

We will verify that

$$\psi : Hom_{\mathcal{H}_J}(\mathcal{H}, M) \to \mathcal{H} \otimes_{\mathcal{H}_J} M : f \mapsto \sum_{x \in M_J} q^{-|x|} T_x \otimes f(T_{x^{-1}})$$

is the inverse of ϕ. We will use the following two implications of Theorem 24 and Observation 9.

- As a left $\mathcal{H}_J$-module $\mathcal{H}$ is free with basis $\{T_{x^{-1}} : x \in M_J\}$.
- As a right $\mathcal{H}_J$-module $\mathcal{H}$ is free with basis $\{T_x : x \in M_J\}$.

One readily checks that ψ is $\mathcal{H}$-equivariant. Before computing $\phi \circ \psi$ and $\psi \circ \phi$, we make the following preparatory observation. A map of left $\mathcal{H}_J$-modules

$$f \in \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$$

is completely determined by its values on $\{T_{x^{-1}} : x \in M_J\}$ because this set is a left $\mathcal{H}_J$-module basis of $\mathcal{H}$. Furthermore, if $m \in M$ and $y \in M_J$, we can use Proposition 7 to determine the values of the left $\mathcal{H}_J$-module map $(T_y \cdot \phi_0(m)) \in \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$ on the basis elements $\{T_{x^{-1}} : x \in M_J\}$ explicitly. They are given by

$$(T_y \cdot \phi_0(m))(T_{x^{-1}}) = \phi_0(m)(T_{x^{-1}} \cdot T_y) = \begin{cases} q^{|x|} m & \text{if } x = y \in M_J \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to show that $\phi \circ \psi = id$. Let $f \in \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$ be a map of left $\mathcal{H}_J$-modules, then

$$(\phi \circ \psi)(f) = \sum_{y \in M_J} q^{-|y|}(T_y \cdot \phi_0(f(T_{y^{-1}}))) \in \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$$

using the definitions of ψ and ϕ. By the second part of the preparatory observation the value of this function on the basis element $T_{x^{-1}}$ is exactly $f(T_{x^{-1}})$. Hence, by the first part of the preparatory observation the function is equal to f. This establishes that $\phi \circ \psi = id$. To show $\psi \circ \phi = id$, we use that any element in $\mathcal{H} \otimes_{\mathcal{H}_J} M$ may be written as a finite R-linear combination of elements of the form $T_y \otimes m$ for $y \in M_J$. To show that $\psi \circ \phi = id$, we will verify that $\psi \circ \phi$ maps $T_y \otimes m$ onto itself. Using the definitions of ϕ and ψ, we obtain:

$$(\psi \circ \phi)(T_y \otimes m) = \psi((T_y \cdot \phi_0(m))) = \sum_{x \in M_J} q^{-|x|} T_x \otimes (T_y \cdot \phi_0(m))(T_{x^{-1}})$$

The second part of the preparatory observation implies that $q^{-|x|}(T_y \cdot \phi_0(m))(T_{x^{-1}}) = m$, if $x = y \in M_J$, and zero otherwise. Therefore, the right sum-term in the last equation is equal to $T_y \otimes m$ and as a consequence $\psi \circ \phi = id$. This completes the proof. □

Observe that there is an injection of $\mathcal{H}$-modules $\iota : M \to \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$ defined by $m \mapsto (h \mapsto h \cdot m)$. This is the Iwahori–Hecke analogue of the map [**Bro82**], III.9 (9.2). Together with the above lemma, we arrive at the following definition of homology transfer for Iwahori–Hecke algebras. The analogue for the cohomology of Iwahori–Hecke algebras plays an important role in [**BEM10**] (see Lemma 2.10 and the discussion before Lemma 3.2). The definition below is parallel to [**Bro82**], III.9 (A).

DEFINITION 41. *Assume that the parameter $q \in R$ is a unit and let $J \subseteq S$ be a subset such that W_J is a finite index subgroup of W. Let M be an $\mathcal{H}$-module, let $\iota : M \to \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M)$ as in the previous paragraph and $\psi : \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M) \to \mathcal{H} \otimes_{\mathcal{H}_J} M$ be the isomorphism described in Proposition 8. The transfer from $\mathcal{H}$ to $\mathcal{H}_J$ is the morphism*

$$tr_\star = tr^{\mathcal{H}}_{\mathcal{H}_T,\star} : H_\star(\mathcal{H}, M) \to H_\star(\mathcal{H}_J, M)$$

obtained as the composition

$$H_\star(\mathcal{H}, M) \to H_\star(\mathcal{H}, \mathcal{H} \otimes_{\mathcal{H}_J} M) \cong H_\star(\mathcal{H}_J, M)$$

where the first map is induced by

$$\psi \circ \iota : M \to \mathcal{H} \otimes_{\mathcal{H}_J} M : m \mapsto \sum_{x \in M_J} q^{-|x|} T_x \otimes T_{x^{-1}} m$$

and the second-map is the change-of-rings isomorphism (Shapiro's lemma), which is available because of Theorem 24.

REMARK 8. *The cohomology transfer map defined in* [**BEM10**] *(compare with Lemma 2.10 and the discussion before Lemma 3.2) can essentially be recovered from the discussion above as follows. There is a canonical surjection $\omega : \mathcal{H} \otimes_{\mathcal{H}_J} M \to M : h \otimes m \mapsto h \cdot m$, which is the Iwahori–Hecke analogue of* [**Bro82**]*, III.9 (9.1). Then the composition*

$$\omega \circ \psi : \mathrm{Hom}_{\mathcal{H}_J}(\mathcal{H}, M) \to M$$

is given by:

$$f \mapsto \sum_{x \in M_J} q^{-|x|} T_x \cdot f(T_{x^{-1}})$$

We remark that in the proof of Lemma 2.10, [**BEM10**] *refers to* [**DD93**]*, where this map appears in a slightly different context.*

4.1. The q-Index of the transfer map. The following is well-known for group homology. If H is a finite index subgroup of G, then the inclusion $H \hookrightarrow G$ induces a morphism on group homology

$$\omega_\star : H_\star(H, M) \to H_\star(G, M)$$

for any G-module M, whose composition with the transfer map

$$\omega_\star \circ tr_\star : H_\star(G, R) \to H_\star(G, R)$$

is multiplication by the index $[G : H]$ of H in G (see [**Bro82**], Proposition (9.5), for example). We close this section by discussing the Iwahori–Hecke analogue building on [**BEM10**], where the q-index for cohomology with coefficients in the trivial $\mathcal{H}$-module plays an important role.

Let $J \subseteq S$ be a subset such that W_J is a finite index subgroup of W. Let

$$\omega_\star : H_\star(\mathcal{H}_J, M) \to H_\star(\mathcal{H}, M)$$

be the composition

$$H_\star(\mathcal{H}_J, M) \cong H_\star(\mathcal{H}, \mathcal{H} \otimes_{\mathcal{H}_J} M) \to H_\star(\mathcal{H}, M)$$

where the first map is the change-of-rings isomorphism (Shapiro's lemma) and the second map is induced by the surjective $\mathcal{H}$-module map

$$\omega : \mathcal{H} \otimes_{\mathcal{H}_J} M \to M : h \otimes m \mapsto h \cdot m$$

PROPOSITION 9. *Assume that the parameter $q \in R$ is a unit and let $J \subseteq S$ be a subset such that W_J is a finite index subgroup of W. Let M be any $\mathcal{H}$-module, then the composition*

$$\omega_\star \circ tr_\star : H_\star(\mathcal{H}, M) \to H_\star(\mathcal{H}, M)$$

is induced by the $\mathcal{H}$-module endomorphism $[W : W_J]_q^M \in End_{\mathcal{H}}(M)$ of M corresponding to multiplication with

$$\sum_{x\in M_J} q^{-|x|}(T_x \cdot T_{x^{-1}}) \in \mathcal{H}$$

We call the endomorphism $[W, W_J]_q^M \in End_{\mathcal{H}}(M)$ the q-index of W_J in W with coefficients in M.

Proof: By construction $\omega_\star \circ tr_\star : H_\star(\mathcal{H}, M) \to H_\star(\mathcal{H}, M)$ is induced by the composition

$$\omega \circ \psi \circ \iota : M \to M$$

which is given by

$$m \mapsto (\sum_{x\in M_J} q^{-|x|}(T_x \cdot T_{x^{-1}})) \cdot m$$

□

Corollary 14. *In the setting of Proposition 9:*

- *If $M = \mathbb{1}^{(q)}$ is the trivial module, then*
$$[W : W_J]_q := [W : W_J]_q^{\mathbb{1}^{(q)}} = \sum_{x\in M_J} q^{|x|} \in R.$$
*This recovers ([**BEM10**], Definition 2.4.) and should be compared with ([**BEM10**], Lemma 3.2.).*
- *If $M = \mathbb{1}^{(-1)}$ is the sign module, then*
$$[W : W_J]_q^{\pm} := [W : W_J]_q^{\mathbb{1}^{(-1)}} = \sum_{x\in M_J} q^{-|x|} \in R.$$

Note that in contrast with group homology, the q-index a priori depends on the module structure of M because $T_{x^{-1}} \neq T_x^{-1} \in \mathcal{H}$ in general. For example, if $s \in S$, then $T_s^{-1} = q^{-1}T_s - (1 - q^{-1})$. However, if $q = 1$ and $\mathcal{H} = R[W]$ is the group ring, then $T_{x^{-1}} = T_x^{-1}$. In this case, the formula of the q-index simplifies and yields the endomorphism of M given by multiplication with $[W : W_T]_1^M = [W : W_T]$ in agreement with the well-known result for group-homology [**Bro82**], III. (9.5).

5. Davis and Coxeter posets for Iwahori–Hecke algebras

In this section we work in the setting of Chapter 1. We define an algebraic Davis and Coxeter poset for the Iwahori–Hecke algebra $\mathcal{H}$ (see Chapter 1, Definition 15) and study their connectivity properties. In particular, we will prove Theorem B. In the second part, we describe and analyze the recellulation spectral sequence (see Chapter 1, Proposition 1) associated to the Davis poset of $\mathcal{H}$. We will show that the spectral sequence collapses on the E_2-page and that the E_1-page contains a single contractible chain complex. This chain complex is the Iwahori–Hecke analogue of the classical Davis complex equipped with the "Coxeter Cell CW-structure" described by Davis in [**Dav08**], Chapter 7, and Boyd [**Boy20**], Definition 2.24. We call it the cellular Davis complex of the Iwahori–Hecke algebra $\mathcal{H}$ and denote it by $\mathcal{H}\mathfrak{D}^{\epsilon}$. In preparation for the proof of Theorem A (i.e. the calculation of the low-dimensional homology groups of $\mathcal{H}$), we will give an explicit description of $\mathcal{H}\mathfrak{D}^{\epsilon}$. The content of this section mainly builds on ideas contained in [**Dav08, Boy20**].

Let (W, S) be a Coxeter system of finite rank $|S| < \infty$ and fix an identification $S \cong \langle n\rangle = \{0, \ldots, n-1\}$ once and for all.

Definition 42.

i) A subset $y \subseteq \langle n\rangle$ is called spherical, if the subgroup $W_y \subseteq W$ generated y is finite.

ii) Let $\mathfrak{D}$ be the poset of spherical subsets of $S \cong \langle n \rangle$.
iii) Let $\mathfrak{C}$ be the poset of all (possibly empty) proper subsets of $S \cong \langle n \rangle$.

Let $\mathcal{H}$ be the Iwahori–Hecke algebra associated to (W, S) with parameter $q \in R$ and $\mathbb{1}^\epsilon$ be the trivial ($\epsilon = q$) or the sign module ($\epsilon = -1$) of $\mathcal{H}$.

DEFINITION 43. *For $y \subseteq \langle n \rangle$, we set $\mathcal{H} \otimes_y \mathbb{1}^\epsilon := \mathcal{H} \otimes_{\mathcal{H}_y} \mathbb{1}^\epsilon$ in the following.*

i) Let $Y \leq 2^{\langle n \rangle}$ be a subposet of the power set of $\langle n \rangle$.
(a) We write $CS(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ for the algebraic coset system associated to

$$(\mathcal{H}, \{\mathcal{H}_y\}_{y \in Y}, \mathbb{1}^\epsilon)$$

in the sense of Chapter 1, Definition 7. Recall that this is the functor

$$CS(\mathcal{H}, Y, \mathbb{1}^\epsilon) : Y \to \mathcal{H}\text{-Mod} : y \mapsto \mathcal{H} \otimes_y \mathbb{1}^\epsilon.$$

(b) We write $CP(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ for the algebraic coset poset associated to

$$(\mathcal{H}, \{\mathcal{H}_y\}_{y \in Y}, \mathbb{1}^\epsilon)$$

in the sense of Chapter 1, Definition 8. Recall that $CP(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ is the semi-simplicial $\mathcal{H}$-module obtained from the coset system $CS(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ by the semi-simplicial replacement. Its $\mathcal{H}$-module of k-simplices is given by

$$CP(\mathcal{H}, Y, \mathbb{1}^\epsilon)_k = \bigoplus_{\substack{y_0 \subsetneq \cdots \subsetneq y_k \\ y_i \in Y}} \mathcal{H} \otimes_{y_0} \mathbb{1}^\epsilon$$

ii) The Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)$ of $\mathcal{H}$ is the algebraic coset poset associated to

$$(\mathcal{H}, \{\mathcal{H}_y\}_{y \in \mathfrak{D}}, \mathbb{1}^\epsilon).$$

iii) The Coxeter poset $CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)$ of $\mathcal{H}$ is the coset poset associated to

$$(\mathcal{H}, \{\mathcal{H}_y\}_{y \in \mathfrak{C}}, \mathbb{1}^\epsilon).$$

Before we start studying the connectivity properties of these complexes, we briefly describe the face maps of the coset poset $CP(\mathcal{H}, Y, \mathbb{1}^\epsilon)$. Let $T_x \otimes_{y_0} 1 \in \mathcal{H} \otimes_{y_0} \mathbb{1}^\epsilon$ be an elementary tensor in the summand of the algebraic coset poset $CP(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ indexed by

$$y_\bullet = y_0 \subsetneq \cdots \subsetneq y_k.$$

Let $k \geq 1$ and recall that the face maps have the following effects:

$$d_0(T_x \otimes_{y_0} 1) = T_x \otimes_{y_1} 1 \text{ in the summand indexed by } d_0(y_\bullet)$$

and for all $i \neq 0$,

$$d_i(T_x \otimes_{y_0} 1) = T_x \otimes_{y_0} 1 \text{ in the summand indexed by } d_i(y_\bullet).$$

5.1. Connectivity properties of the Davis and Coxeter poset. The goal of this subsection is the proof of the following result, which implies Theorem B stated in the introduction.

THEOREM 25. *Let R be a commutative unital ring. Let q be a unit in R and $\mathcal{H} = \mathcal{H}^q(W, S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$.*

i) The algebraic Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)$ of $\mathcal{H}$ is contractible and

$$H_0(CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)) \cong \mathbb{1}^\epsilon.$$

ii) If the underlying Coxeter group W is finite, then the algebraic Coxeter poset $CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)$ of $\mathcal{H}$ is $(|S| - 1)$-spherical. In this case, it holds that
(a) $H_0(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)) \cong \mathcal{H}$, if $|S| = 1$.
(b) $H_0(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)) \cong \mathbb{1}^\epsilon$, if $|S| \geq 2$.
(c) $\tilde{H}_{|S|-1}(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)) \cong \mathbb{1}^{\hat{\epsilon}}$, where $\hat{\epsilon} = (-1)$, if $\epsilon = q$, and $\hat{\epsilon} = q$, if $\epsilon = (-1)$.

iii) If the underlying Coxeter group W is infinite, then the algebraic Coxeter poset $CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)$ is contractible and $H_0(CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)) \cong \mathbb{1}^\epsilon$.

We will spend the rest of this subsection proving this theorem. We start with the following proposition, which relates the coset system $CS(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ of the Iwahori–Hecke algebra $\mathcal{H}$ to the coset systems $CS(W, Y)$ of the Coxeter group W equipped with the family of subgroups $\{W_y\}_{y \in Y}$ (see Chapter 1, Definition 4, for the definition of the coset systems for groups).

Proposition 10. *Let Y be as in Definition 43, item i). Assume that the parameter $q \in R$ of $\mathcal{H}$ is a unit, then the coset systems*

$$CS(\mathcal{H}, Y, \mathbb{1}^\epsilon)$$

and

$$R[-] \circ CS(W, Y)$$

are isomorphic in the functor category $\mathbf{R\text{-}mod}^{\mathbf{Y}}$. *Here, $R[-] : \mathbf{G\text{-}set} \to \mathbf{RG\text{-}mod}$ denotes is the R-linearization functor and $CS(W, Y)$ is defined as in Chapter 1, Definition 4.*

Proof: We construct a natural isomorphism $\eta : CS(\mathcal{H}, Y, \mathbb{1}^\epsilon) \to R[-] \circ CS(W, Y)$. Let $y \in Y$. Recall that $M_y(W)$ denotes the set of minimal left coset representatives of W/W_y (see Definition 35). For any $y \in Y$, $\mathcal{H}$ is a free right $\mathcal{H}_y$-module with basis $\{T_x : x \in M_y(W)\}$ by Theorem 24 and Observation 9. It follows that $\{T_x \otimes_y 1 : x \in M_y(W)\}$ is a R-module basis of $\mathcal{H} \otimes_y \mathbb{1}^\epsilon$. Furthermore, $R[W]$ is a free right W_y-module with basis $M_y(W)$ and therefore $\{x \otimes_y 1 : x \in M_y(W)\}$ is a R-module basis of $R[W] \otimes_y \mathbb{1}$. Hence, we obtain a R-linear isomorphism $\eta_y : \mathcal{H} \otimes_y \mathbb{1}^\epsilon \to R[W] \otimes_y \mathbb{1}$ using basis elements of the domain and codomain as follows

$$T_x \otimes_y 1 \mapsto \epsilon^{|x|}(x \otimes_y 1) \text{ for } x \in M_y(W).$$

Here $|x|$ denotes the word length of x in the generating set S and $\epsilon \in \{q, -1\}$ depending on whether we consider the trivial or sign representation of $\mathcal{H}$. The inverse is given by multiplying the R-basis of $R[W] \otimes_y \mathbb{1}$ by $\epsilon^{-|x|}$. It remains to check that the maps $\{\eta_y\}$ are compatible with the inclusion induced maps. We write $\iota^{\mathcal{H}} : \mathcal{H} \otimes_y \mathbb{1}^\epsilon \to \mathcal{H} \otimes_{y'} \mathbb{1}^\epsilon$ and $\iota^W : R[W] \otimes_y \mathbb{1} \to R[W] \otimes_{y'} \mathbb{1}$ for the image of the inclusion $y \hookrightarrow y'$ of subsets under $CS(\mathcal{H}, Y, \mathbb{1}^\epsilon)$ and $R[-] \circ CS(W, Y)$, respectively. Let $T_x \otimes_y 1 \in \mathcal{H} \otimes_y \mathbb{1}^\epsilon$ be a R-basis element, let $\bar{x}$ denote the minimal word length representative of the coset $xW_{y'}$ and let $w \in W_{y'}$ be the unique element with $x = \bar{x} \cdot w$ and $|x| = |\bar{x}| + |w|$ (see Proposition 6). Then

$$\begin{aligned}(\iota^W \circ \eta_y)(T_x \otimes_y 1) &= \epsilon^{|x|}(x \otimes_{y'} 1) = \epsilon^{|\bar{x} \cdot w|}(\bar{x} \otimes_{y'} w \cdot 1) = \epsilon^{|\bar{x}|}\epsilon^{|w|}(\bar{x} \otimes_{y'} 1) \\ &= \eta_{y'}(T_{\bar{x}} \otimes_{y'} \epsilon^{|w|}) = \eta_{y'}(T_{\bar{x}} \otimes_{y'} T_w \cdot 1) \\ &= \eta_{y'}(T_{\bar{x}} T_w \otimes_{y'} 1) = (\eta_{y'} \circ \iota^{\mathcal{H}})(T_x \otimes_y 1).\end{aligned}$$

□

In the group setting, the topology of the Davis and Coxeter poset are well understood. The previous proposition implies that we can use this knowledge to compute the homology of the Davis and Coxeter poset for Iwahori–Hecke algebras.

Corollary 15. *Let $\mathcal{H} = \mathcal{H}(W, S)$ be an Iwahori–Hecke algebra with parameter $q \in R$ a unit and $|S| = n$.*

i) The Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)$ of $\mathcal{H}$ is contractible.
ii) The Coxeter poset $CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)$ of $\mathcal{H}$ is $(n-1)$-spherical, if W is finite, and contractible otherwise.

Proof: Proposition 10 and the functoriality of semi-simplicial replacements imply that we have the following isomorphisms of semi-simplicial R-modules. For the algebraic Davis poset of the Iwahori–Hecke algebra $\mathcal{H}$, we have that

$$CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon) \cong R[-] \circ CP(W, \mathfrak{D}),$$

and for the algebraic Coxeter poset of the Iwahori–Hecke algebra $\mathcal{H}$, we have that

$$CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon) \cong R[-] \circ CP(W, \mathfrak{C}).$$

The classical Davis complex $CP(W, \mathfrak{D})$ is contractible (see [**Dav08**], Equation (7.1), the paragraph after Lemma 7.2.2., and Theorem 8.2.13.). If W is finite, then the classical Coxeter complex $CP(W, \mathfrak{C})$ is homeomorphic to a $(|S|-1)$-sphere (see [**AB08**], Definition 3.1. and Proposition 1.108, or [**BB05**], §3 Exercise 16.(h), for example). If W is infinite, then the classical Coxeter complex $CP(W, \mathfrak{C})$ is contractible (see [**Ser71**], Lemma 4, or [**BB05**], §3 Exercise 16.(i), for example). Hence, the same holds for the algebraic analogues. □

We continue by determining the $\mathcal{H}$-module structure of the nontrivial homology groups.

Proposition 11. *Let $\mathcal{H} = \mathcal{H}(W, S)$ be the Iwahori–Hecke algebra associated to a unit $q \in R$ and let $|S| = n$. Then,*

i) $H_0(CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)) = \mathbb{1}^\epsilon$.
If $n = 1$, then $H_0(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)) = \mathcal{H}$.
If $n \geq 2$, then $H_0(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon)) = \mathbb{1}^\epsilon$.
ii) If W is finite, then

$$\tilde{H}_{n-1}(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^{(q)})) = \mathbb{1}^{(-1)} \text{ and } \tilde{H}_{n-1}(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^{(-1)})) = \mathbb{1}^{(q)}.$$

Proof of part i) of Proposition 11: It follows from Proposition 10, that the homology group $H_0(CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon))$ is a free R-modules of rank one and that a generator is represented by the classes of $T_e \otimes_{\{i\}} 1$ for any $i \in \langle n \rangle$, respectively. The same applies to $H_0(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon))$, if $n \geq 2$. To determine the $\mathcal{H}$-module structure in these two cases, we observe that the action of $T_{s_i} \in \mathcal{H}$ for $s_i \in S$ is given by $T_{s_i} \cdot T_e \otimes_{\{i\}} 1 = T_e \otimes_{\{i\}} T_{s_i} \cdot 1 = \epsilon \cdot (T_e \otimes_{\{i\}} 1)$. Hence, the zero-th homology is given by $\mathbb{1}^\epsilon$ in both cases. If $n = 1$, then $\mathfrak{C} = \{\emptyset\}$ and the only nontrivial simplex module of $CP(\mathfrak{C}, \mathbb{1}^\epsilon)$ is $CP(\mathfrak{C}, \mathbb{1}^\epsilon)_0 = \mathcal{H} \otimes_\emptyset \mathbb{1}^\epsilon \cong \mathcal{H}$. This completes the proof of the part i). □

The proof of part ii) relies on the following lemma, which describes an explicit generator of the top-homology of the classical Coxeter complex $CP(W, \mathfrak{C})$ of a finite Coxeter group W.

Lemma 26. *Let (W, F) be a Coxeter system with W finite and $S \cong \langle n \rangle = \{0, \ldots, n-1\}$ for $n \in \mathbb{N}$. Let $CP(W, \mathfrak{C}) \in$ **ss(G-set)** be the Coxeter poset of (W, S) in the sense of Chapter 1, Example 1. Then*

$$|CP(W, \mathfrak{C})| \cong S^{n-1}.$$

The class of the cycle $\Delta_{\langle n \rangle}$, defined below, is a generator of $H_{n-1}(CP(W, \mathfrak{C}); R)$.

$$\Delta_{\langle n \rangle} = \sum_{(w, y_\bullet) \in W \times N_{n-1}(\mathfrak{C})} (-1)^{-|w|} \operatorname{sgn}_{\langle n \rangle}(y_\bullet)(w \otimes_{y_0} 1)$$

where $N_{n-1}(\mathfrak{C})$ is the set of flags $y_\bullet = (\emptyset = y_0 \lneq \cdots \lneq y_{n-1})$ in $\mathfrak{C}$ and $\operatorname{sgn}_{\langle n \rangle}(y_\bullet)$ is the signum of the permutation of $\langle n \rangle$ that sends $i \in \langle n \rangle$ to the unique element in $y_{i+1} - y_i$, where $y_n := \langle n \rangle$.

Proof: The proof is given in the appendix (see Section 7). □

Proof of part ii) of Proposition 11: It follows from Proposition 10 that

$$\Delta^\epsilon_{\langle n \rangle} = \sum_{(w, y_\bullet) \in W \times N_{n-1}(\mathfrak{C})} (-\epsilon)^{-|w|} \operatorname{sgn}_{\langle n \rangle}(y_\bullet)(T_w \otimes_{y_0} 1)$$

is a cycle that represents a generator of the top-homology $H_{n-1}(CP(\mathcal{H}, \mathfrak{C}, \mathbb{1}^\epsilon))$. It therefore suffices to compute the $\mathcal{H}$-action of the generators $\{T_{s_i}\}_{i \in \langle n \rangle}$ of $\mathcal{H}$ on the class of $\Delta^\epsilon_{\langle n \rangle}$. As a consequence of Lemma 21, any choice of $s \in S$ partitions W into two sets

$$A = \{w \in W : w \text{ doesn't have a reduced spelling starting with } s\}$$
$$sA = \{w \in W : w \text{ does have a reduced spelling starting with } s\}$$

which have the property that left multiplication by s is a bijection $A \to sA$. For $a \in A$ and a fixed flag $y_\bullet \in N_{n-1}(\mathfrak{C})$, the sum-terms in $\Delta^{\epsilon}_{\langle n \rangle}$ associated to the pairs $\{a, sa\} \times y_\bullet$ have the following property. We start using Lemma 23 to calculate:

$$T_s \cdot [(-\epsilon)^{-|a|}(T_a \otimes 1) + (-\epsilon)^{-|sa|}(T_{sa} \otimes 1)]$$
$$= (-\epsilon)^{-|a|}(T_{sa} \otimes 1) + (-\epsilon)^{-|sa|}q(T_a \otimes 1) + (-\epsilon)^{-|sa|}q(T_{sa} \otimes 1) - (-\epsilon)^{-|sa|}(T_{sa} \otimes 1)$$

If we assume that $\epsilon = q$, then $(-\epsilon)^{-|sa|}q = -(-\epsilon)^{-|a|}$. Hence, the first and the third term cancel and we obtain the following equality in this case:

$$T_s \cdot [(-\epsilon)^{-|a|}(T_a \otimes 1) + (-\epsilon)^{-|sa|}(T_{sa} \otimes 1)]$$
$$= (-1) \cdot [(-\epsilon)^{-|a|}(T_a \otimes 1) + (-\epsilon)^{-|sa|}(T_{sa} \otimes 1)]$$

It follows that for T_s acts by multiplication with (-1), if $\epsilon = q$, so the first equation of Proposition 11, part ii), holds.

If we assume that $\epsilon = (-1)$, then $(-\epsilon) = 1$. Hence, the first and the last term above cancel and we obtain the following equality in this case:

$$T_s \cdot [(-\epsilon)^{-|a|}(T_a \otimes 1) + (-\epsilon)^{-|sa|}(T_{sa} \otimes 1)]$$
$$= q \cdot [(-\epsilon)^{-|a|}(T_a \otimes 1) + (-\epsilon)^{-|sa|}(T_{sa} \otimes 1)]$$

It follows that T_s acts by multiplication with q, if $\epsilon = (-1)$. This completes the proof. □

We can now use the intermediate results of this subsection to formulate the proof of Theorem 25.

Proof of Theorem 25. The theorem follows from Corollary 15 and Proposition 11. □

5.2. The cellular Davis complex of Iwahori–Hecke algebras. The goal of this subsection is to define the celluar Davis complex $\mathcal{HD}^\epsilon$ with coefficients in $\mathbb{1}^\epsilon$ of the Iwahori–Hecke algebra $\mathcal{H}$ and prove that it is contractible. Its contractibility will be a consequence of its relation to the algebraic Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)$, which is contractible by Theorem 25. Indeed, we will study the recellulation spectral sequence constructed in Chapter 1, Proposition 1, for the Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^\epsilon)$. This spectral sequence converges to the homology of the algebraic Davis complex and we will show that it collapses on the E^2-page. The E^1-page contains a single chain complex that hence must be contractible. We will verify that this complex is isomorphic to the complex $\mathcal{HD}^\epsilon$, which we will now explicitly define. Before giving the definition, we would like to highlight that the formula for the differential of $\mathcal{HD}^\epsilon$ can be conceptualized by comparing it with the definition of the transfer map for Iwahori–Hecke algebras (see Definition 41). It is essentially given by an alternating sum of maps that induce a transfer map on homology.

Definition 44. *Let R be a commutative unital ring, $q \in R$ be a unit and let $\mathcal{H}(W, S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$. The cellular Davis complex $(\mathcal{HD}^\epsilon, \delta)$ of $\mathcal{H}$ with coefficients in $\mathbb{1}^\epsilon$ is the $\mathcal{H}$-chain complex with α-th chain module given by*

$$\mathcal{HD}^\epsilon_\alpha = \bigoplus_{F \in \mathfrak{D} : |F| = \alpha} \mathcal{H} \otimes_F \mathbb{1}^{\hat{\epsilon}},$$

where $\hat{\epsilon} = (-1)$, if $\epsilon = q$, and $\hat{\epsilon} = q$, if $\epsilon = (-1)$. The differential of $\mathcal{HD}^\epsilon$ in degree α

$$\delta_\alpha : \mathcal{HD}^\epsilon_\alpha \to \mathcal{HD}^\epsilon_{\alpha-1}$$

factorizes summand-wise as

$$\mathcal{H} \otimes_F \mathbb{1}^{\hat{\epsilon}} \to \oplus_{s\in F} \mathcal{H} \otimes_{F_s} \mathbb{1}^{\hat{\epsilon}} \hookrightarrow \mathcal{H}\mathfrak{D}^{\epsilon}_{\alpha-1}$$

where $F_s = F - \{s\}$ *for* $s \in F$. *The first arrow in this factorization can be identified with the extension of scalars by* $\mathcal{H}$ *of the map*

$$\mathbb{1}^{\hat{\epsilon}} \to \oplus_{s\in F} \mathcal{H} \otimes_{F_s} \mathbb{1}^{\hat{\epsilon}} : [\blacktriangle^{\epsilon}] \mapsto \sum_{s\in F} (-1)^{\gamma(s)} \sum_{x\in M_{F_s}(W_F)} q^{-|x|}(T_x \otimes T_{x^{-1}} \cdot [\blacktriangle^{\epsilon}])$$

where $[\blacktriangle^{\epsilon}] \in \mathbb{1}^{\hat{\epsilon}}$ *is a generator,* $\gamma(s) = |\{s' \in F_s : s' > s\}|$ *and* $M_{F_s}(W_F)$ *is the set of minimal word length representatives of left cosets* W_F/W_{F_s}. *Note that the last morphism is an alternating sum of maps that induce transfer morphisms between homology groups of Iwahori–Hecke algebras (see Definition 41).*

We will prove the following theorem in this subsection.

THEOREM 27. *Let* R *be a commutative unital ring. Let* q *be a unit in* R *and* $\mathcal{H} = \mathcal{H}^q(W, S)$ *be an associated Iwahori–Hecke algebra with* $|S| < \infty$.

i) The recellulation spectral sequence (see Chapter 1, Proposition 1) of the algebraic Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^{\epsilon})$ *collapses on the* E^2*-page and the* E^1*-page is concentrated in the zero-th row.*

ii) The $\mathcal{H}$*-chain complex in the zero-th row* $(E^1_{\star,0}, \partial^1)$ *is isomorphic to the cellular Davis complex* $\mathcal{H}\mathfrak{D}^{\epsilon}$ *with coefficients in* $\mathbb{1}^{\epsilon}$ *of the Iwahori–Hecke algebra* $\mathcal{H}$ *(see Definition 44.) In particular,* $\mathcal{H}\mathfrak{D}^{\epsilon}$ *is contractible and* $H_0(\mathcal{H}\mathfrak{D}^{\epsilon}) \cong \mathbb{1}^{\epsilon}$.

We start working towards the proof of part $i)$ of this theorem by explaining the structure of recellulation spectral sequence for the algebraic Davis poset $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^{\epsilon})$ of $\mathcal{H}$. The chain complexes $E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}$ in the following definition occur as summands of $E^0_{\alpha,\star}$ on the E^0-page of this spectral sequence,

$$E^0_{\alpha,\star} \cong \bigoplus_{\substack{F\in\mathfrak{D} \\ |F|=\alpha}} E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}.$$

DEFINITION 45. *Let* $F \in \mathfrak{D}$ *such that* $|F| = \alpha$. *Let* $(\mathfrak{D}/F)_k$ *be the set of* k*-simplices* $y_\bullet = y_0 \lneq \cdots \lneq y_k$ *of* $N(\mathfrak{D})$ *satisfying* $y_k = F$, *then*

$$E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\beta} = \bigoplus_{\substack{y_0 \lneq \cdots \lneq y_{\alpha+\beta} = F \\ \in (\mathfrak{D}/F)_{\alpha+\beta}}} \mathcal{H} \otimes_{y_0} \mathbb{1}^{\epsilon}$$

with differential given by

$$\delta = \sum_{i=0}^{(\alpha+\beta)-1} (-1)^i d_i \ ,$$

where d_i *denotes the* i*-th face map of* $CP(\mathcal{H}, \mathfrak{D}, \mathbb{1}^{\epsilon})$. *Note that this chain complex is concentrated in non positive degrees* $\beta \le 0$.

To prove part $i)$ of Theorem 27, we need to compute the homology of $E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}$. We will do this by relating the chain complex $E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}$ to the Coxeter and Davis poset of the parabolic subalgebra $\mathcal{H}_F$.

Let $F \in \mathfrak{D}$ be a spherical subset of size $|F| = \alpha$. Let $CP(\mathcal{H}_F, \mathfrak{D}_F, \mathbb{1}^{\epsilon})$ and $CP(\mathcal{H}_F, \mathfrak{C}_F, \mathbb{1}^{\epsilon})$ denote the algebraic Davis and Coxeter poset of the Iwahori–Hecke algebra $\mathcal{H}_F$, respectively. The Coxeter poset is naturally a subcomplex of the Davis poset:

$$CP(\mathcal{H}_F, \mathfrak{C}_F, \mathbb{1}^{\epsilon}) \hookrightarrow CP(\mathcal{H}_F, \mathfrak{D}_F, \mathbb{1}^{\epsilon})$$

We consider the cofiber $cofib_{F,\star}$ of this inclusion in the category of chain complexes of $\mathcal{H}_F$-modules.

Lemma 28. *There is an isomorphism of chain complexes*

$$E(\mathcal{H}, \mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star} \cong (\mathcal{H} \otimes_F cofib_{F,\star})[-\alpha],$$

where the complex on the right side is shifted down α degrees.

Proof: It follows from the definition of the algebraic Coxeter and Davis poset (see Definition 43) that

$$cofib_{F,k} = \bigoplus_{y_0 \lneq \cdots \lneq y_k = F} \mathcal{H}_F \otimes_{y_0} \mathbb{1}^{\epsilon},$$

which is concentrated in degrees $0 \leq k \leq \alpha$. The differential of $cofib^F_\bullet$ is induced by the differential of the algebraic Davis poset $CP(\mathcal{H}_F, \mathfrak{D}_F, \mathbb{1}^{\epsilon})$ and therefore given by

$$\delta_k = \sum_0^{k-1} (-1)^i d_i.$$

The claim then follows via the identification $\mathcal{H} \otimes_F (\mathcal{H}_F \otimes_{y_0} \mathbb{1}^{\epsilon}) \cong \mathcal{H} \otimes_{y_0} \mathbb{1}^{\epsilon}$. □

Recall that $\mathcal{H}$ is a free right $\mathcal{H}_F$-module by Theorem 24 and note that this implies that the functor $\mathcal{H} \otimes_F -$ is exact. We will use this observation, the identification in Lemma 28 and the connectivity results for the Coxeter and Davis poset derived in the last section to compute the homology of $E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}$.

PROPOSITION 12. *For any $F \in \mathfrak{D}$, the homology of $E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}$ is concentrated in degree 0. Furthermore,*

$$H_0(E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}) \cong \mathcal{H} \otimes_F \mathbb{1}^{\hat{\epsilon}},$$

where $\hat{\epsilon} = -1$, if $\epsilon = q$, and $\hat{\epsilon} = q$, if $\epsilon = -1$.

Proof: We first calculate the homology of $cofib_{F,\star}$ and then invoke Lemma 28. The long exact homology sequence of $\mathcal{H}_F$-modules associated to the cofiber sequence

$$CP(\mathcal{H}_F, \mathfrak{C}_F, \mathbb{1}^{\epsilon}) \hookrightarrow CP(\mathcal{H}_F, \mathfrak{D}_F, \mathbb{1}^{\epsilon}) \twoheadrightarrow cofib_{F,\star}$$

the fact that $CP(\mathcal{H}_F, \mathfrak{D}_F, \mathbb{1}^{\epsilon})$ is contractible and that $CP(\mathcal{H}_F, \mathfrak{D}, \mathbb{1}^{\epsilon})$ is $(\alpha - 1)$-spherical by Corollary 15 imply that the homology of the cofiber $cofib_{F,\star}$ is concentrated in homological degree α. For $\alpha > 0$, the connecting morphism of the long exact sequence gives an isomorphism of $\mathcal{H}_F$-modules:

$$H_\alpha(cofib_{F,\star}) \cong \tilde{H}_{\alpha-1}(CP(\mathcal{H}_F, \mathfrak{C}, \mathbb{1}^{\epsilon}))$$

In the case $F = \emptyset$, the chain complex $cofib_{F,\star}$ is concentrated in degree zero, where its chain module is a single copy of R. It follows from the freeness of $\mathcal{H}$ as a right $\mathcal{H}_F$-module (see Theorem 24), that $\mathcal{H} \otimes_F -$ is exact. Hence, Lemma 28 implies that:

$$H_\star(E(\mathfrak{D}, \mathbb{1}^{\epsilon})_{F,\star}) \cong \mathcal{H} \otimes_F H_\star(cofib_{F,\star})[-\alpha]$$

Using the description of $H_\star(cofib_{F,\star})$ above, the claim then follows from Proposition 11. □

It follows that for Iwahori–Hecke algebras with parameter $q \in R$ a unit, the E^1-page of the recellulation spectral sequence consists of a single contractible chain complex $(E^1_{\star,0}, \partial^1)$ that is concentrated in the zero-th row. This means that we completed the proof of part $i)$ of Theorem 27.

Proof of part $i)$ of Theorem 27: This follows from Proposition 12. □

Proposition 12 also provides a description of the chain modules of $(E^1_{\star,0}, \partial^1)$.

Corollary 16. *The chain modules of* $(E^1_{\star,0},\partial^1)$ *are given by*

$$E^1_{\alpha,0} = H_0(\bigoplus_{\substack{F\in\mathfrak{D}\\|F|=\alpha}} E(\mathfrak{D},\mathbb{1}^{\epsilon})_{F,\star}) \cong \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=\alpha}} \mathcal{H}\otimes_F \mathbb{1}^{\hat{\epsilon}}.$$

To prove part *ii*) of Theorem 27 and identify $(E^1_{\star,0},\partial^1)$ with $\mathcal{H}\mathfrak{D}^{\epsilon}$, we need to understand the effect of the differential of $(E^1_{\star,0},\partial^1)$ on the generators obtained under the identifications made in Corollary 16. We can explicitly describe the basis elements in the right most term. The following lemma identifies their images in the left term.

Lemma 29. *Let* $F\in\mathfrak{D}$, $|F|=\alpha$ *and recall that* $M_F(W)$ *denotes the set of minimal left coset representatives of* W/W_F. *The homology classes of the elements in the following set*

$$\{T_x\cdot\blacktriangle^{\epsilon}_{\alpha,F}\in E(\mathfrak{D},\mathbb{1}^{\epsilon})_{F,0} : x\in M_F(W)\}$$

form a R-module basis of $H_0(E(\mathfrak{D},\mathbb{1}^{\epsilon})_{F,\star}))$. *Here,*

$$\blacktriangle^{\epsilon}_{\alpha,F} = \sum_{(w,y_\bullet)\in W_F\times N_\alpha(\mathfrak{D}_F)} (-\epsilon)^{-|w|}\operatorname{sgn}(y_\bullet)(T_w\otimes 1).$$

Proof: By Lemma 28 and Theorem 24, it suffices to show that $\blacktriangle^{\epsilon}_{\alpha,F}$ represents a generator of $H_\alpha(cofib_{F,\star})$. Recall from the proof of Proposition 12 that the connecting morphism

$$H_\alpha(cofib_{F,\star})\to \tilde{H}_{\alpha-1}(CP(\mathcal{H}_F,\mathfrak{C},\mathbb{1}^{\epsilon}))$$

is an isomorphism. It therefore suffices to show that the boundary of $\blacktriangle^{\epsilon}_{\alpha,F}$ in $CP(\mathcal{H}_F,\mathfrak{D},\mathbb{1}^{\epsilon})$ represents a generator of $\tilde{H}_{\alpha-1}(CP(\mathcal{H}_F,\mathfrak{C},\mathbb{1}^{\epsilon}))$. For the group case, $q=\epsilon=1$, the boundary of $(-1)^\alpha\blacktriangle_{\alpha,F}$ in $CP(W_F,\mathfrak{D},\mathbb{1})$ is visibly Δ_F (see Lemma 26). Thus, Proposition 10 implies that the same relationship holds for all $q\in R^\times$ and coefficients in the trivial module or the sign representation:

$$\delta((-1)^\alpha\blacktriangle^{\epsilon}_{\alpha,F}) = \Delta^{\epsilon}_{\alpha,F}$$

□

We can now formulate the proof of the second part of Theorem 27. Our strategy is to evaluate the differential of $(E^1_{\star,0},\partial^1)$ on the generators described in Lemma 29 and compare the result with the differentials of the cellular Davis complex (see Definition 44).

Proof of part *ii*) of Theorem 27: We start by describing the differential of the E^1-page of recellulation spectral sequence i.e. the differential of $(E^1_{\star,0},\partial^1)$. By definition $E^1_{\alpha,0} = \ker(\partial^0 : E^0_{\alpha,0}\to E^0_{\alpha,-1})$, where $(E^0_{\star,\star},\partial^0)$ denotes the E^0-page of the recellulation spectral sequence (see Chapter 1, Proposition 1). More precisely, let

$$\ker(\alpha) := \ker(\delta : F^\alpha(CP(\mathfrak{D},\mathbb{1}^{\epsilon}))_\alpha \to F^\alpha(CP(\mathfrak{D},\mathbb{1}^{\epsilon}))_{\alpha-1}/F^{\alpha-1}(CP(\mathfrak{D},\mathbb{1}^{\epsilon}))_{\alpha-1}),$$

where δ is the restricted differential of $CP(\mathcal{H},\mathfrak{D},\mathbb{1}^{\epsilon})$, then $E^1_{\alpha,0} = \ker(\alpha)/F^{\alpha-1}(CP(\mathcal{H},\mathfrak{D})^{\epsilon})_\alpha$. An element $\theta\in\ker(\alpha)$ gives rise to an element $\delta(\theta)\in F^{\alpha-1}CP(\mathcal{H},\mathfrak{D})^{\epsilon}$ and per definition the differential of the E^1-page is given by

$$\partial^1([\theta]) = [\delta(\theta)]\in E^1_{\alpha-1,0}.$$

By Lemma 29, the elements $T_x\cdot\blacktriangle^{\epsilon}_{\alpha,F}$ for $x\in M_F(W)$ represent generators of $E^1_{\alpha,0}$. Because ∂^1 is a $\mathcal{H}$-module map, it suffices to consider $\theta = 1\cdot\blacktriangle^{\epsilon}_{\alpha,F}$. The proof of Lemma 29 shows that

$$\partial^1([(-1)^\alpha(1\cdot\blacktriangle^{\epsilon}_{\alpha,F})) = [(1\cdot\Delta^{\epsilon}_{\alpha,F})]$$

where $\Delta^{\epsilon}_{\alpha,F}$ is as in the proof of part *ii*) of Proposition 11 (compare with Lemma 26). We will now express $[(1\cdot\Delta^{\epsilon}_{\alpha,F})]$ in terms of the generators of $E^1_{\alpha-1,0}$ provided by Lemma 29. After this, comparing the resulting expression with the description of the differentials of the cellular Davis complex $\mathcal{H}\mathfrak{D}^{\epsilon}$ in Definition 44 will finish the proof.

Let $M_{F_s}(W_F)$ be the set of minimal word length representatives of left cosets W_F/W_{F_s}. Note that

$$N_{\alpha-1}(\mathfrak{D}_{F_s}) = \{y_\bullet \in N_{\alpha-1}(\mathfrak{C}_F) : y_{\alpha-1} = F_s\}.$$

For $y_\bullet \in N_{\alpha-1}(\mathfrak{D}_{F_s})$, we have that $\operatorname{sgn}_F(y_\bullet) = (-1)^{\gamma(s)} \operatorname{sgn}_{F_s}(y_\bullet)$, where $\gamma(s)$ the number of inversions caused by ordering s last i.e. $\gamma(s) = |\{s' \in F_s : s' > s\}|$. It follows that:

$$\begin{aligned}
1 \cdot \Delta^\epsilon_{\alpha,F} &= 1 \cdot \sum_{(w,y_\bullet)\in W_F\times N_{\alpha-1}(\mathfrak{C}_F)} (-\epsilon)^{-|w|} \operatorname{sgn}_F(y_\bullet)(T_w \otimes_{y_0} 1)\\
&= 1 \cdot \sum_{s\in F}\Big(\sum_{y_\bullet\in N_{\alpha-1}(\mathfrak{D}_{F_s})} \Big(\sum_{m\in M_{F_s}(W_F)} \Big(\sum_{\hat{w}\in W_{F_s}} (-\epsilon)^{-|m\hat{w}|} \operatorname{sgn}_F(y_\bullet)(T_{m\hat{w}} \otimes_{y_0} 1)\Big)\Big)\Big)\\
&= 1 \cdot \sum_{s\in F}(-1)^{\gamma(s)}\Big(\sum_{m\in M_{F_s}(W_F)} (-\epsilon)^{-|m|} T_m \Big(\sum_{(\hat{w},y_\bullet)\in W_{F_s}\times N_{\alpha-1}(\mathfrak{D}_{F_s})} (-\epsilon)^{-|\hat{w}|} \operatorname{sgn}_{F_s}(y_\bullet)(T_{\hat{w}} \otimes_{y_0} 1)\Big)\Big)\\
&= 1 \cdot \sum_{s\in F}(-1)^{\gamma(s)}\Big(\sum_{m\in M_{F_s}(W_F)} (-\epsilon)^{-|m|}(T_m \cdot \blacktriangle^\epsilon_{\alpha-1,F_s})\Big)
\end{aligned}$$

The calculation shows that under the identifications made in Corollary 16, the differential ∂^1 factors as

$$\partial^1 : \mathcal{H} \otimes_F \mathbb{1}^{\hat{\epsilon}} \to \oplus_{s\in F}\mathcal{H} \otimes_{F_s} \mathbb{1}^{\hat{\epsilon}}$$

and that ∂^1 is obtained by extension of scalars from a map

$$\mathbb{1}^{\hat{\epsilon}} \to \oplus_{s\in F}\mathcal{H} \otimes_{F_s} \mathbb{1}^{\hat{\epsilon}}.$$

Denoting the generator of $\mathbb{1}^{\hat{\epsilon}}$ by $[\blacktriangle^\epsilon]$, this map is given by

$$\begin{aligned}
[\blacktriangle^\epsilon] \mapsto &\sum_{s\in F}(-1)^{\gamma(s)} \sum_{m\in M_{F_s}(W_F)} (-\epsilon)^{-|m|}(T_m \otimes_{F_s} [\blacktriangle^\epsilon])\\
&= \sum_{s\in F}(-1)^{\gamma(s)} \sum_{m\in M_{F_s}(W_F)} q^{-|m|}(T_m \otimes_{F_s} T_{m^{-1}} \cdot [\blacktriangle^\epsilon])
\end{aligned}$$

where we used the following two observations for the last equality.

- If $\epsilon = q$, then $\hat{\epsilon} = (-1)$ and therefore $T_{m^{-1}} \cdot [\blacktriangle^\epsilon] = (-1)^{|m|}[\blacktriangle^\epsilon]$.
- If $\epsilon = (-1)$, then $(-\epsilon)^{|m|} = 1$, $\hat{\epsilon} = q$ and $T_{m^{-1}} \cdot [\blacktriangle^\epsilon] = q^{|m|}[\blacktriangle^\epsilon]$.

This completes the proof. □

6. The low dimensional homology of Iwahori–Hecke algebras

In this section, we compute the low dimensional homology of Iwahori–Hecke algebras i.e. we prove Theorem A. Our strategy is in analogy with the isotropy spectral sequence argument Boyd employs in [**Boy20**], Section 4. We start by describing the Iwahori–Hecke analogue of the isotropy spectral spectral studied by Boyd in full generality. This spectral sequence will converge to $H_\star(\mathcal{H}, \mathbb{1}^\epsilon)$ and its E^1-page is given by

$$E^1_{b,a} = \bigoplus_{\{F\in\mathfrak{D}:|F|=b\}} H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}).$$

If the underlying finite Coxeter group (W_F, F) of the Iwahori–Hecke algebra $\mathcal{H}_F$ for $F \in \mathfrak{D}$ has small rank $|F|$, then its homology with coefficients in $\mathbb{1}^{\hat{\epsilon}}$ is accessible. We will completely describe the entries written in the first two columns of the E^1-page. In the final part of this chapter, we will specialize to the case $\epsilon = q$ and use the spectral sequence to derive formulas for the first and second dimensional homology of $\mathcal{H}$ with trivial coefficients stated in Theorem A.

Remark 9. *In* [**Boy20**]*, Boyd also proves a formula for the third homology of a Coxeter group.*

6.1. The q-isotropy spectral sequence. The goal of this first subsection is the construction of the Iwahori–Hecke analogue of the isotropy spectral sequence that Boyd studied in [**Boy20**]. We call it the q-isotropy spectral sequence. We will now prove the following theorem.

THEOREM 30. *Let R be a commutative unital ring, q be a unit in R and $\mathcal{H} = \mathcal{H}^q(W,S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$. Denote by $\mathfrak{D}$ the set of spherical subsets of S (see Definition 42). Fix a total order on S. Let $\mathbb{1}^\epsilon$ be the trivial module of $\mathcal{H}$, if $\epsilon = q$, and the sign module of $\mathcal{H}$, if $\epsilon = (-1)$.*

i) There exists a spectral sequence

$$E^1_{b,a} = \bigoplus_{F\in\mathfrak{D}:|F|=b} H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \Longrightarrow H_{a+b}(\mathcal{H}, \mathbb{1}^\epsilon)$$

converging to the homology of $\mathcal{H}$ with coefficients in $\mathbb{1}^\epsilon$. Here, $\mathcal{H}_F$ denotes the parabolic subalgebra of $\mathcal{H}$ associated to $F \in \mathfrak{D}$ (see Definition 38). Furthermore, $\hat{\epsilon} = (-1)$, if $\epsilon = q$, and $\hat{\epsilon} = q$, if $\epsilon = (-1)$.

ii) Let $b > 0$. The E^1-differential $\partial^1 : E^1_{b,a} \to E^1_{b-1,a}$ of this spectral sequence restricted to the summand $H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}})$ indexed by $F \in \mathfrak{D}$ admits the following factorization

$$H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \to \oplus_{s\in F} H_a(\mathcal{H}_{F_s}, \mathbb{1}^{\hat{\epsilon}}) \hookrightarrow E^1_{b-1,a},$$

where $F_s = F - \{s\}$. The first map of this factorization is given by

$$\sum_{s\in F} (-1)^{\gamma(s)} tr^F_{F_s},$$

where $\gamma(s) = |\{s' \in F_s : s' > s\}|$ and

$$tr^F_{F_s} : H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \to H_a(\mathcal{H}_{F_s}, \mathbb{1}^{\hat{\epsilon}})$$

denotes the transfer map defined in Definition 41.

Proof: Let $P_\star$ be a free resolution of the $\mathcal{H}$-module $\mathbb{1}^\epsilon$. We consider the two spectral sequences associated to the horizontal respectively vertical filtration of the double complex $P_\star \otimes_{\mathcal{H}} \mathcal{H}\mathfrak{D}^\epsilon_\star$, where $\mathcal{H}\mathfrak{D}^\epsilon_\star$ is the cellular Davis complex of $\mathcal{H}$ introduced in Definition 44.

The exactness of $P_a \otimes_{\mathcal{H}} -$ implies that the vertical spectral sequence has vE^1-terms given by

$$vE^1_{a,b} \cong P_a \otimes_{\mathcal{H}} H_b(\mathcal{H}\mathfrak{D}^\epsilon_\star).$$

The contractibility of the cellular Davis complex and the fact that $H_0(\mathcal{H}\mathfrak{D}^\epsilon) \cong \mathbb{1}^\epsilon$ (see Theorem 27, part *ii*)) imply that the vertical spectral sequences converges to the homology of the chain complex

$$vE^2_{\star,0} = P_\star \otimes_{\mathcal{H}} \mathbb{1}^\epsilon.$$

This is by definition $H_\star(\mathcal{H}, \mathbb{1}^\epsilon)$, the homology of $\mathcal{H}$ with coefficients in $\mathbb{1}^\epsilon$. It follows that the spectral sequence constructed from the horizontal filtration of $P_\star \otimes_{\mathcal{H}} \mathcal{H}\mathfrak{D}^\epsilon_\star$ converges to $H_\star(\mathcal{H}, \mathbb{1}^\epsilon)$ as well.

The structure of the cellular Davis complex (see Definition 44) implies that the hE^1-terms of the horizontal spectral sequence can be identified as:

$$hE^1_{b,a} = H_a(P_\bullet \otimes_{\mathcal{H}} \mathcal{H}\mathfrak{D}^\epsilon_b) \cong \bigoplus_{F\in\mathfrak{D}:|F|=b} H_a(\mathcal{H}, \mathcal{H} \otimes_F \mathbb{1}^{\hat{\epsilon}}) \cong \bigoplus_{F\in\mathfrak{D}:|F|=b} H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}})$$

where the last isomorphism is the change-of-rings isomorphism (Shapiro's lemma), which is available because of Theorem 24. This finishes the proof of part *i*) of Theorem 30. The hE^1-differential of the horizontal spectral sequence $h\partial^1$ is induced by the differential of the cellular Davis complex $\mathcal{H}\mathfrak{D}^\epsilon$. Part *ii*) of Theorem 30 therefore follows from Definition 44 and Definition 41. □

6.2. The first two columns of the E^1-page. The goal of this subsection is the proof of the following proposition, which describes homology groups that appear in the first two columns of the q-spectral sequence constructed in Theorem 30. Before stating the result, we recall some notation.

- If $\epsilon = q$, then $\hat{\epsilon} = (-1)$, and if $\epsilon = (-1)$, then $\hat{\epsilon} = q$.
- Let R be a commutative unital ring. Let $x, y \in R$. We write $R_{(x,y)}$ for the quotient $R/(x, y)$.

Lemma 31. *Let $F \in \mathfrak{D}$.*

i) If $|F| = 0$, then $\mathcal{H}_F = R$. Therefore,

$$H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \cong \begin{cases} R, & \text{for } a = 0, \\ 0, & \text{for } a > 0. \end{cases}$$

ii) If $|F| = 1$, then $F = \{s\}$ for $s \in S$. In this case, $H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}})$ is given by:

(a) For $\hat{\epsilon} = -1$:

$$H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \cong \begin{cases} R_{(1+q)}, & \text{if } a \text{ is even}, \\ T_{(1+q)}(R) = \{r \in R : r(1+q) = 0\}, & \text{if } a \text{ is odd}. \end{cases}$$

(b) For $\hat{\epsilon} = q$:

$$H_a(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \cong \begin{cases} R, & \text{if } a = 0, \\ T_{(1+q)}(R), & \text{if } a \neq 0 \text{ and even}, \\ R_{(1+q)}, & \text{if } a \text{ is odd}. \end{cases}$$

iii) Moreover, $H_0(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) = R_{(\hat{\epsilon}-q)}$ for $|F| > 0$.

Proof: Part i) of the lemma is trivial. We now give an argument for part ii). Let $F = \{s\}$. Using Lemma 23, it is easy to check that

$$P'_\star = \mathcal{H}_F \xleftarrow{(T_s - q)} \mathcal{H}_F \xleftarrow{(T_s + 1)} \mathcal{H}_F \xleftarrow{(T_s - q)} \dots$$

is a free resolution[2] of the trivial $\mathcal{H}_F$-module $\mathbb{1}^{(q)}$. Therefore $P'_\star \otimes_{\mathcal{H}_F} \mathbb{1}^{\hat{\epsilon}}$ computes $H_\star(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}})$. This complex has the following form:

$$P'_\bullet \otimes_{\mathcal{H}_F} \mathbb{1}^{\hat{\epsilon}} \cong R \xleftarrow{(\hat{\epsilon} - q)} R \xleftarrow{(\hat{\epsilon} + 1)} R \xleftarrow{(\hat{\epsilon} - q)} \dots$$

Part ii) of the lemma follows. Let $|F| > 0$ and $s \in F$. To prove part iii), one can use the retraction $\mathcal{H}_F \to \mathcal{H}_s$ to show that there is an isomorphism

$$\mathbb{1}^{(q)} \otimes_{\mathcal{H}_F} \mathbb{1}^{\hat{\epsilon}} \cong \mathbb{1}^{(q)} \otimes_{\mathcal{H}_{\{s\}}} \mathbb{1}^{\hat{\epsilon}}.$$

Using that $H_0(\mathcal{H}_F, \mathbb{1}^{\hat{\epsilon}}) \cong \mathbb{1}^{(q)} \otimes_{\mathcal{H}_F} \mathbb{1}^{\hat{\epsilon}}$, the last claim follows from part ii). □

6.3. Proof of Theorem A. In this last subsection, we will use the q-isotropy spectral sequence to prove Theorem A. In particular, we will from now on assume that $\epsilon = q$ and $\hat{\epsilon} = (-1)$. The results of the previous two subsections have the following consequence.

Corollary 17. *Let $C_\star = E^1_{0 \leq \star \leq 3, 0}$ be the truncated chain complex appearing in the zero-th row of the E^1-page of the q-isotropy spectral sequence (see Theorem 30), then there are natural isomorphisms*

$$H_0(\mathcal{H}, \mathbb{1}^q) \cong H_0(C_\star)$$
$$H_1(\mathcal{H}, \mathbb{1}^q) \cong H_1(C_\star)$$

Under the additional assumption that $T_{(1+q)}(R) = 0$:

$$H_2(\mathcal{H}, \mathbb{1}^q) \cong H_2(C_\star)$$

[2] Note that for $q = 1$, this is the standard periodic resolution of the trivial $\mathbb{Z}_2$-module $\mathbb{1}$.

Proof: By item $i)$ in Lemma 31, we have $E^1_{0,k} = 0$ for $k > 0$. Hence, for $k \in \{0, 1\}$

$$E^2_{0,k} = E^\infty_{0,k} \cong H_k(\mathcal{H}, \mathbb{1}^q).$$

Under the assumption that $T_{(1+q)}(R) = 0$, it follows from item $ii.a)$ in Lemma 31 that $E^1_{1,1} = 0$ as well. Hence, in this case

$$E^2_{0,2} = E^\infty_{0,2} \cong H_2(\mathcal{H}, \mathbb{1}^q)$$

. □

Corollary 17 shows that the computation of the zero-th, first and second homology group of an Iwahori–Hecke algebra often amounts to the computation of the homology of the chain-complex $C_\star$:

$$R \leftarrow \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=1}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}) \leftarrow \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=2}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}) \leftarrow \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=3}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)})$$

By part $ii)$ of Theorem 30, the differentials of $C_\star$ are summand-wise given by an alternating sum of transfer maps. The following lemma identifies these transfer maps as multiplication by a q-index.

Lemma 32. *The transfer map $tr^F_{F_s} : H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}) \to H_0(\mathcal{H}_{F_s}, \mathbb{1}^{(-1)})$ is multiplication by the q-index*

$$[W_F, W_{F_s}]^{(-1)}_q = \sum_{x\in M_{F_s}(W_F)} (-1)^{|x|}$$

Proof: The transfer map is given by the composition

$$H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}) \cong \mathbb{1}^{(q)} \otimes_{\mathcal{H}_F} \mathbb{1}^{(-1)} \to \mathbb{1}^{(q)} \otimes_{\mathcal{H}_F} (\mathcal{H}_F \otimes_{\mathcal{H}_{F_s}} \mathbb{1}^{(-1)}) \cong \mathbb{1}^{(q)} \otimes_{\mathcal{H}_{F_s}} \mathbb{1}^{(-1)} \cong H_0(\mathcal{H}_{F_s}, \mathbb{1}^{(-1)})$$

and can be computed using Definition 41 as

$$a \otimes b \mapsto a \otimes \sum_{x\in M_{F_s}(W_F)} q^{-|x|}(T_x \otimes T_{x^{-1}} b) \mapsto \sum_{x\in M_{F_s}(W_F)} q^{-|x|}(aT_x \otimes T_{x^{-1}} b) = \sum_{x\in M_{F_s}(W_F)} (-1)^{|x|}(a \otimes b)$$

□

6.3.1. *The zero-th homology group.* We will now derive the formula of for the zero-th homology group of $\mathcal{H}$ with coefficients in $\mathbb{1}^q$ stated in Theorem A.

Lemma 33. *The zero-th differential*

$$R \xleftarrow{\partial^1} \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=1}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)})$$

of $C_\star$ is the zero map.

Proof: By Theorem 30, part $ii)$, the restriction of the differential to the summand indexed by $F = \{s\}$ is given by

$$\partial^1|_{\{s\}} = tr^{\{s\}}_\emptyset.$$

Lemma 32 implies that $tr^{\{s\}}_\emptyset$ is multiplication by $(-1)^{|e|} + (-1)^{|s|} = 0$. □

Corollary 18.

$$H_0(\mathcal{H}, \mathbb{1}^{(q)}) \cong R$$

6.3.2. *The first homology group.* We will now derive the formula for the first homology of $\mathcal{H}$ with coefficients in $\mathbb{1}^q$ stated in Theorem A. For this, we will need to understand the next differential of $C_\star$,

$$R \overset{0}{\leftarrow} \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=1}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}) \overset{\partial^1}{\longleftarrow} \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=2}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}).$$

The following lemma calculates the transfer maps that occur in the definition of this differential.

Lemma 34. *For $|F| = |\{s,t\}| = 2$ and $s \in F$, the transfer-map*

$$tr_s^F : H_0(\mathcal{H}_F, \mathbb{1}^{(q)}) \to H_0(\mathcal{H}_{F_s}, \mathbb{1}^{(q)})$$

is multiplication by one, if $m_{s,t}$ is odd, and zero otherwise. Here, $m_{s,t} \geq 2$ is the number encoding the relation of the generators s and t of the underlying Coxeter group W (see Definition 32).

Proof: Lemma 32 implies that the transfer map is multiplication by:

$$\sum_{x\in M_{F_s}(W_F)} (-1)^{|x|}$$

Calculating the transfer map therefore amounts to understanding the number of minimal coset representatives of $W_{\{s,t\}}/W_{\{t\}}$ with odd and even length, respectively. Any finite Coxeter group contains a unique element w_0 of maximal word length (see [**Dav08**], Lemma 4.6.1). W_F is a dihedral group and its element of maximal word length is $w_0 = (st)^{\frac{m}{2}}$, if $m = m(s,t)$ is even, and $w_0 = (ts)^{\frac{m-1}{2}}t$, if m is odd. It follows that the minimal word-length representative of the coset $w_0 W_{\{t\}}$ has length $m-1$ and that this is the longest element in $M_{\{t\}}(W_{\{s,t\}})$. By Corollary 12, any suffix of the longest element in $M_{\{t\}}(W_{\{s,t\}})$ is a minimal left-coset representative as well. There are m many different suffixes in each case with word-length $m-1, m-2, \ldots, 1$ and 0, respectively. These form a list of all minimal coset representatives because the cardinality of $W_{\{s,t\}}/W_{\{s\}}$ is m. The claim follows. □

In the following theorem, we derive the formula for the first homology group of $\mathcal{H}$. The proof is analogous to the argument of ([**Boy20**], Theorem A).

THEOREM 35. *Let (W,S) be a Coxeter system with $|S| < \infty$ and $q \in R$ be a unit in a commutative unital ring. Let $\mathcal{D}_{odd}$ be the graph with vertex set S and edges given by subsets $\{s,t\}$ that satisfy that $m_{s,t} \neq \infty$ is odd (see Definition 32). Then:*

$$H_1(\mathcal{H}, \mathbb{1}^{(q)}) \cong H_0(\mathcal{D}_{odd}, R_{(1+q)})$$

Proof: By Corollary 17 and Lemma 32 we only need to compute the zero-th homology of the chain complex

$$\bigoplus_{s\in S} H_0(\mathcal{H}_{\{s\}}, \mathbb{1}^{(-1)}) \overset{\partial}{\leftarrow} \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=2}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}).$$

Lemma 34 implies that the summands indexed by $F = \{s,t\} \subset S$ with $m_{s,t}$ even do not contribute to the image of the differential. Thus the problem is equivalent to computing the zero-th homology of the subcomplex

$$\bigoplus_{s\in S} H_0(\mathcal{H}_{\{s\}}, \mathbb{1}^{(-1)}) \overset{\partial}{\leftarrow} \bigoplus_{\substack{F\in\mathfrak{D}:\\|F|=2\\m_{s,t}\text{ odd}}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}).$$

By Lemma 31, part *iii*), the chain modules are free $R_{(1+q)}$-modules. We compute the differential using Theorem 30, part *ii*). For $F = \{s < t\} \subseteq S$ with $|F| = 2$ and $m_{s,t}$ odd, we find that $\gamma(s) = 1$ and $\gamma(t) = 0$. Therefore Lemma 34 implies that the differential is summand-wise given by:

$$1_{s<t} \overset{\partial}{\mapsto} (-1) \cdot 1_t + 1_s$$

If follows that the subcomplex is the chain complex of the simplicial graph $\mathcal{D}_{odd}$ with trivial coefficients in $R_{(1+q)}$ and order on the vertex set chosen as the opposite order on S. □

This computation has the following immediate consequence.

Corollary 19. *In the setting of Theorem 35 the following holds.*

i) If $(1+q) = 0 \in R$, then the first homology group $H_1(\mathcal{H}, \mathbb{1}^{(q)})$ of $\mathcal{H}$ is nontrivial.
ii) If $(1+q) \in R$ is invertible, then first homology group $H_1(\mathcal{H}, \mathbb{1}^{(q)})$ of $\mathcal{H}$ is trivial.

6.3.3. *The second homology group.* We will from now on assume that $T_{(1+q)}(R) = 0$ and $q \neq -1$. We will now derive the formula for the second homology given in Theorem A. The assumption $T_{(1+q)}(R) = 0$ is satisfied in Theorem A because of the following observation.

Observation 10. *If R is an integral domain (in particular $0 \neq 1$), then*

$$T_{(1+q)}(R) \neq 0 \text{ if and only if } q = -1$$

The description of the zero-th and first homology group of Iwahori–Hecke algebras is essentially the same as for Coxeter groups (with 2 replaced by quantum two $(1+q)$). This changes for the second homology group. We will see that the deformation of the group algebra $R[W]$ splits one of the sum terms that occurs in Boyd's description of the second homology of Coxeter groups ([**Boy20**], Theorem A) into two pieces. The reason for this is that the q-index (see Proposition 9) depends on the word length of minimal representatives.

To illustrate this, we consider the symmetric group on three generators $\{s_0, s_1, s_2\}$. This group is often denoted by A_3 in the literature and it appears in Figure 1 after the definition of a Coxeter system (see Definition 32). Let $A = \langle s_0, s_1\rangle, B = \langle s_0, s_2\rangle$ and $C = \langle s_1, s_2\rangle$ denote the three parabolic subgroups on two generators. The following is a list of all minimal coset representatives of A_3 with regard to A, B and C.

i) A_3/A contains 4 cosets with minimal representatives
$$e, s_2, s_1s_2 \text{ and } s_0s_1s_2.$$
ii) A_3/B contains 6 cosets with minimal representatives
$$e, s_1, s_0s_1, s_2s_1, s_2s_0s_1 \text{ and } s_1s_2s_0s_1.$$
iii) A_3/C contains 4 cosets with minimal representatives
$$e, s_0, s_1s_0 \text{ and } s_2s_1s_0.$$

Lemma 32 implies that the two maps $tr_A^{A_3}$ and $tr_C^{A_3}$ are trivial, but $tr_B^{A_3}$ is multiplication by 2. Note that in the group algebra case $q = 1$, this map is always zero because $2 = 0$ in $R_{(1+1)}$ in agreement with [**Boy20**], Lemma 4.3.. If $q \neq 1$, the map might be nonzero. These transfer maps occur in the final differential of the complex $C_\star$

$$\bigoplus_{\substack{F\in\mathfrak{D}\\|F|=2}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}) \overset{\partial}{\leftarrow} \bigoplus_{\substack{F\in\mathfrak{D}\\|F|=3}} H_0(\mathcal{H}_F, \mathbb{1}^{(-1)}).$$

We will need to understand this differential to derive the formula for the second homology group of $\mathcal{H}$. The next lemma describes the transfer maps that can occur.

Lemma 36. *Let $F = \{s, t, u\} \subseteq S$ with $|F| = 3$ be a spherical set of Coxeter generators of (W, S). Using the notation for Coxeter groups introduced in the discussion after Definition 32 (see Figure 1), the following holds.*

i) (W_F, F) *is a Coxeter group of one of the following types*

$$A_3, B_3, H_3 \text{ or } I_2(p) \times A_1 \text{ with } p \geq 2.$$

ii) *Let* $F_u = \{s,t\}, F_t = \{s,u\}, F_s = \{t,u\}$. *Then*

$$\partial|_F = \gamma(u) \cdot tr^F_{F_u} + \gamma(t) \cdot tr^F_{F_t} + \gamma(s) \cdot tr^F_{F_s}$$

and the following table describes the transfer maps in each case (using Lemma 32).

Type:	$tr^F_{F_u}$	$tr^F_{F_t}$	$tr^F_{F_s}$
A_3	0	2	0
B_3	0	0	0
H_3	0	0	0
$I_2(p) \times A_1$ (for p even)	0	0	0
$I_2(p) \times A_1$ (for p odd)	0	1	1

Proof: The classification of finite Coxeter groups [**Cox35**] implies that the list in part *i)* contains all finite Coxeter groups (W,S) with $|S| = 3$. Using Lemma 32, one can verify part *ii)* for the three groups A_3, B_3 and H_3 using the GAP3 [**S**$^+$**97**] code contained in the appendix (see Section 7). For Iwahori–Hecke algebras with underlying Coxeter group $W = I_2(p) \times A_1$ i.e. $m_{s,t} = p$, $m_{s,u} = 2$ and $m_{t,u} = 2$, we observe that $W/W_{\{s,t\}} \cong \langle u \rangle$, $W/W_{\{s,u\}} \cong W_{\{s,t\}}/W_{\{s\}}$ and $W/W_{\{t,u\}} \cong W_{\{s,t\}}/W_{\{t\}}$. The remaining two cases of part *ii)* therefore follow from Lemma 34. □

In preparation for the computation of the second homology group of $\mathcal{H}$, we will now introduce the graphs that appear in the formula in Theorem A. We will use the same notation as Boyd in [**Boy20**]. There will be two extra cases that amount to the splitting coming from the 2-torsion phenomenon associated with Coxeter groups (W_F, F) of type A_3. After giving the definition of these graphs, we introduce the local coefficient system appearing in the formula in Theorem A.

Definition 46. *Let* (W,S) *be a Coxeter system with* $|S| < \infty$.

i) *Let* $\mathcal{D}_{even}$ *be the graph with vertex set* S *and edges* $\{s,t\} \in E(\mathcal{D}_{even})$ *spanned by subsets* $\{s,t\}$ *with* $m_{s,t} \neq 2, \infty$ *even.*
ii) *Let* $\mathcal{D}_{odd}$ *be the graph with vertex set* S *and edges spanned by subsets* $\{s,t\}$ *with* $m_{s,t} \neq \infty$ *odd.*
iii) *Let* $\mathcal{D}^X$ *be the graph with vertices* $\{\{s,t\} \subset S : m_{s,t} = 2\}$ *and edges spanned by pairs* $\{s_1,t_1\}, \{s_2,t_2\}$ *with* $s_1 = s_2$ *and* $m_{t_1,t_2} \neq \infty$ *odd.*
iv) *Let* $\mathcal{D}^X_{A_3}$ *be the subgraph of* $\mathcal{D}^X$ *consisting of all connected components that contain a vertex* $\{s,t\}$ *for which there exists some element* $u \in S$ *with* $m_{s,u} = m_{u,t} = 3$.
v) *Let* $\mathcal{D}^X_{\neg A_3} := \mathcal{D}^X - \mathcal{D}^X_{A_3}$ *be the subgraph of* $\mathcal{D}^X$ *consisting of all connected components that do not contain a vertex as in item iv).*

We additionally define a local system of coefficients $\mathscr{L}$ on $\mathcal{D}^X$. Fix a vertex v and let $p : v \rightsquigarrow w$ be a path in $\mathcal{D}^X$. Then, p is homotopic to a sequence of directed edges $(e_0, \ldots, e_l)$. Given a directed edge e in $\mathcal{D}^X$ starting at $v = \{a,b\}$ and ending at $w = \{a,c\}$, we get an induced bijection $e : v \to w$ that preserves the common element of v and w. Recall that we fixed an identification $S \cong \langle n \rangle$ and hence an ordering of the generators. The map e is therefore either order preserving ($sgn(e) = 1$) or order reversing ($sgn(e) = -1$). The coefficient system $\mathscr{L}$ is the functor $\Pi_1(\mathcal{D}^X) \to$ **R-mod** sending all vertices to $R_{(1+q)}$ and a path $[p]$ represented by $(e_0, \ldots, e_l)$ to multiplication by $sgn([p]) = sgn(e_0) \ldots sgn(e_l)$.

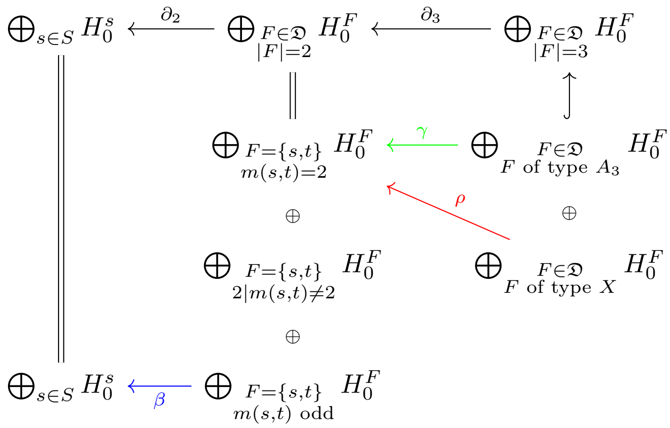
By Lemma 36 and refining the proof of Proposition 4.7 in [**Boy20**], we obtain the following theorem.

THEOREM 37. *Let $q \neq -1$ be a unit in a commutative unital ring R satisfying $T_{(1+q)}(R) = 0$. Let $\mathcal{H} = \mathcal{H}(W, S)$ be an associated Iwahori–Hecke algebra with $|S| < \infty$. Then, there exists an isomorphism*

$$H_2(\mathcal{H}, \mathbb{1}^{(q)}) \cong H_0(\mathcal{D}^X_{A_3}, R_{(1+q,2)}) \oplus H_0(\mathcal{D}^X_{\neg A_3}, \mathscr{L}) \oplus R_{(1+q)}\{E(\mathcal{D}_{even})\} \oplus H_1(\mathcal{D}_{odd}, R_{(1+q)})$$

where the graphs $\mathcal{D}^X_{A_3}, \mathcal{D}^X_{\neg A_3}, \mathcal{D}_{even}$ and $\mathcal{D}_{odd}$ are as in Definition 46, $\mathscr{L}$ is the local coefficient system described after this definition and $E(\mathcal{D}_{even})$ denotes the set of edges of $\mathcal{D}_{even}$.

Proof: We consider the Iwahori–Hecke analogue of the diagram in the proof of Proposition 4.7 of [**Boy20**], which describes the complex $C_{1\leq\star\leq 3}$. We use the notation $H_0^F := H_0(\mathcal{H}_F, \mathbb{1}^{(-1)})$ and, in accordance with the X in $\mathcal{D}^X$ in item *iii*) of Definition 46, we will call a Coxeter group W_F of type X, if it is of type $I_2(p) \times A_1$ with $p \geq 2$ odd.



The first two columns of the diagram are direct sum decompositions of the chain modules. The third column displays the summands of C_3 on which ∂_3 is nontrivial (this follows from Lemma 36). The arrows β, γ and ρ indicate on which summands the differentials ∂_2 and ∂_3 are nontrivial and show into which summand of the target they map (this follows from Lemma 34 for ∂_2 and from Lemma 36 for ∂_3). Using that $H_0^F \cong R_{(1+q)}$ (see Lemma 31, part *ii*)), it follows that

$$H_2(C_\star) = \ker(\beta) \oplus \bigoplus_{\substack{F=\{s,t\}\\ 2|m(s,t)\neq 2}} R_{(1+q)} \oplus \Big(\bigoplus_{\substack{F=\{s,t\}\\ m(s,t)=2}} R_{(1+q)}\Big)/\operatorname{im}(\gamma\oplus\rho)$$

The proof of Theorem 35 shows that $\ker(\beta) \cong H_1(\Gamma_{odd}, R_{(1+q)})$. In the definition of item *ii*) above, we saw that the edge set $E(\mathcal{D}_{even})$ of the graph $\mathcal{D}_{even}$ is exactly the indexing set of the second summand in the last equation. It remains to show that

$$\Big(\bigoplus_{\substack{F=\{s,t\}\\ m(s,t)=2}} R_{(1+q)}\Big)/\operatorname{im}(\gamma\oplus\rho) \cong H_0(\mathcal{D}^X_{A_3}, R_{(1+q,2)}) \oplus H_0(\mathcal{D}^X_{\neg A_3}, \mathscr{L}).$$

We start by arguing that the subcomplex

$$\rho : \bigoplus_{\substack{F\in\mathfrak{D}\\ F \text{ of type } X}} H_0^F \to \bigoplus_{\substack{F=\{s,t\}\\ m(s,t)=2}} H_0^F$$

identifies with the chain complex of graph $\mathcal{D}^X$ with coefficients in the functor $\mathscr{L} : \Pi_1(\mathcal{D}^X) \to$ **R-mod** defined above. Recall that a vertex $v = \{a, b\} \in \mathcal{D}^X$ consists of two distinct elements in S. We fixed an ordering on S, therefore v is a tuple $v = (\max(v), \min(v)) \in S \times S$. The lexicographic order on $S \times S$ restricts to an ordering on the vertex set of $\mathcal{D}^X$. Hence, $\mathcal{D}^X$ is an ordered simplicial graph. For an undirected edge $e : v \leftrightsquigarrow w$ in $\mathcal{D}^X$, let $e_c \in S$ denote the element which v and w have in common. Note that for a directed edge $e : v \rightsquigarrow w$, we have: $\mathscr{L}(e) = sgn(e) = 1$ if and only if $e_c = \max(e)$ or $e_c = \min(e)$. We obtain the following identification

$$\begin{array}{ccccc} C_\star(\mathcal{D}^X, \mathscr{L}) : & & \oplus_{m_{s,t}=2} R_{(1+q)} & \xleftarrow{\delta^{\mathscr{L}}} & \oplus_{\substack{F \in \mathfrak{D} \\ F \text{ of type } X}} R_{(1+q)} \\ \big\downarrow \cong & & \big\| & & \big\downarrow \phi \\ \text{The complex } \rho & & \oplus_{m_{s,t}=2} R_{(1+q)} & \xleftarrow{\rho} & \oplus_{\substack{F \in \mathfrak{D} \\ F \text{ of type } X}} R_{(1+q)} \end{array}$$

with ϕ is summand-wise defined by: $1_e \mapsto (-1) \cdot 1_e$, if $e_c = \min(e)$, and $1_e \mapsto sgn(e) \cdot 1_e$ otherwise. Indeed, let $e = \{v, w\}$ be an edge whose vertices are ordered as $v < w$. Then Lemma 36 implies that

$$\rho(e) = \begin{cases} w - v & \text{if } e_c = \max(e) \\ (-1)(w - v) & \text{if } e_c = \min(e) \\ w + v & \text{otherwise} \end{cases}$$

On the other hand, the differential $\delta^{\mathscr{L}}$ of $C_\star(\mathcal{D}^X, \mathscr{L})$ evaluates as follows:

$$\delta^{\mathscr{L}}(e) = \mathscr{L}(e) d_0(e) - d_1(e) = \mathscr{L}(e) w - v = \begin{cases} w - v & \text{if } e_c = \max(e) \text{ or } e_c = \min(e) \\ (-1)(w + v) & \text{otherwise} \end{cases}$$

This finishes the proof, if there are no $F \in \mathfrak{D}$ with $|F| = 3$ and F of type A_3.

For the general case: Let us assume that there is some $F = \{a, b, c\} \in \mathfrak{D}$ with $|F| = 3$ and F of type A_3, so that $m_{a,b} = m_{b,c} = 3$ and $m_{a,c} = 2$. Then $z = \{a, c\}$ is a vertex in $\mathcal{D}^X$ and we write Z for the connected component of z in $\mathcal{D}^X$. The A_3-case of Lemma 36, which has been discussed in detail at the beginning of this subsection, implies that

$$2 \cdot [z] = 0 \in \Big(\bigoplus_{F \in V(Z)} R_{(1+p)} \Big) / \operatorname{im}(\gamma \oplus \rho)$$

The connectedness of Z implies that $[z] = \pm[z']$ where $z' \in V(Z)$ is an arbitrary vertex, hence $2 \cdot [z'] = 0$ as well. It follows that

$$\Big(\bigoplus_{F \in V(Z)} R_{(1+p)} \Big) / \operatorname{im}(\gamma \oplus \rho) \cong R_{(1+q,2)}.$$

This completes the proof. □

Corollary 20. *For $q = 1$ and $R = \mathbb{Z}$, this is exactly Boyd's proof of* [**Boy20**]*, Theorem A. Note that $[2] = [0]$ in $R_{(1+1)}$ implies that $\mathscr{L}$ is trivial and that the first two summands in Theorem 37 give the term $H_0(\mathcal{D}^X, \mathbb{Z}_2)$ appearing in* [**Boy20**]*, Theorem A.*

7. Appendix: Additional Details

The top-homology of the Coxeter complex of a finite Coxeter group. We give a proof of Lemma 26.

Statement (Lemma 26). *Let (W, F) be a Coxeter system with W finite and $S \cong \langle n \rangle = \{0, \dots, n-1\}$ for $n \in \mathbb{N}$. Let $CP(W, \mathfrak{C}) \in$* **ss(G-set)** *be the Coxeter poset of (W, S) in the sense of Chapter 1, Example 1. Then,*

$$|CP(W, \mathfrak{C})| \cong S^{n-1}.$$

The class of the cycle $\Delta_{\langle n \rangle}$, defined below, is a generator of $H_{n-1}(CP(W, \mathfrak{C}); R)$.

$$\Delta_{\langle n \rangle} = \sum_{(w, y_\bullet) \in W \times N_{n-1}(\mathfrak{C})} (-1)^{-|w|} \operatorname{sgn}_{\langle n \rangle}(y_\bullet)(w \otimes_{y_0} 1)$$

where $N_{n-1}(\mathfrak{C})$ *is the set of flags* $y_\bullet = (\emptyset = y_0 \subsetneq \cdots \subsetneq y_{n-1})$ *in* $\mathfrak{C}$ *and* $\mathrm{sgn}_{\langle n\rangle}(y_\bullet)$ *is the signum of the permutation of* $\langle n\rangle$ *that sends* $i \in \langle n\rangle$ *to the unique element in* $y_{i+1} - y_i$*, where* $y_n := \langle n\rangle$*.*

Proof: The first part, $|CP(W, \mathfrak{C})| \cong S^{n-1}$, is well known (see [**AB08**], Definition 3.1. and Proposition 1.108, or [**BB05**], §3 Exercise 16.(h), for example). For the second part, note that the sum in the definition of the homology class is taken over all top-dimensional simplices. These triangulate a sphere and hence, two distinct top-dimensional simplices $(a, A_\bullet)$ and $(b, B_\bullet)$ are, if at all, glued together along exactly one common codimension-one faces $d_i(a, A_\bullet) = d_j(b, B_\bullet)$. We therefore only need to argue that we picked the correct signs for each sum term. For checking this, recall that $(a, A_\bullet)$ can be identified with the flag of cosets $\{a\} = aW_{A_0} \subsetneq \cdots \subsetneq aW_{A_{n-1}}$.

Case 1: Let $a \neq b$. We must have $i = j = 0$ and $aW_{A_\bullet} = bW_{B_\bullet}$ for $\bullet > 0$. As $A_0 = B_0 = \emptyset$, this implies that $A_\bullet = B_\bullet$, so $\mathrm{sgn}_{\langle n\rangle}(A_\bullet) = \mathrm{sgn}_{\langle n\rangle}(B_\bullet)$. Further, $\bullet = 1$ implies $b = as$ for some $s \in S$, hence $(-1)^{-|a|} \neq (-1)^{-|b|}$.

Case 2: Let $a = b$ and $A_\bullet \neq B_\bullet$. Then we must have $i = j$ for cardinalilty reasons. We have $A_t = B_t$ for $t \neq i$, $A_{i+1} - A_{i-1} = B_{i+1} - B_{i-1} = \{x, y\}$ and, without loss of generality, $A_i = A_{i-1} \cup \{x\}$ and $B_i = B_{i-1} \cup \{y\}$. It follows that the two signs are unequal because $\mathrm{sgn}_{\langle n\rangle}(A_\bullet) \neq \mathrm{sgn}_{\langle n\rangle}(B_\bullet)$. □

GAP3 computation for A_3, B_3 and H_3. The following code for GAP3 [**S⁺97**] can be used to verify Lemma 36 for A_3, B_3 and H_3.

```
#!/bin/gap
T := [ "A", "B", "H" ];
S := [[1,2], [1,3], [2,3]];
for t in T do
    W := CoxeterGroup( t, 3 );
    for s in S do
        H := ReflectionSubgroup( W, s);
        Print(H, "\n");
        minimallengthrepresentatives := List(ReducedRightCosetRepresentatives(W, H), x -> CoxeterWord(W,x));
        Print(minimallengthrepresentatives, "\n");
        lengthvector := List(minimallengthrepresentatives, x -> Length(x));
        Print(lengthvector, "\n");
        transfer := 0;
        for l in lengthvector do
            transfer := transfer + E(2)^l;
        od;
        Print(transfer, "\n");
    od;
od;
```

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CHAPTER 4

Deformations of the free E_∞-algebra on a point

The content of this chapter is based on joint work with Richard Hepworth and Jeremy Miller. All mistakes contained in this write-up have been made by the author of the thesis.

Summary: In this chapter, we explain how one can associate an E_k-algebra in simplicial R-modules to certain families of R-algebras that assemble into an enriched monoidal category by extending techniques we learned from Berger, Berger–Fresse, Balteanu–Fiedorowicz–Schwänzl–Vogt and Fiedorowicz to the enriched setting. We use this criterion to construct E_2-algebras from certain families of Iwahori–Hecke algebras. The resulting E_2-algebras can be thought of as deformations of the free E_∞-algebra on a point. We verify that in general the E_2-structure cannot be extended to an E_3-structure.

1. Introduction

The classical method for proving the existence of stability patterns in the homology of a family of groups or spaces originates from work of Quillen [**Qui**]. Hepworth [**Hep20**], Boyd–Hepworth [**BH20**] and Boyd–Hepworth–Patzt [**BHP20**] demonstrated that one can, in a similar fashion as for groups and spaces, study the homology of families of augmented algebras using Quillen's method. This chapter is closely related to the following question of Boyd–Hepworth [**BH20**].

QUESTION 1 ([**BH20**], page 9). *Can the homological stability machinery of Galatius, Kupers and Randal-Williams* [**GKRW19**] *be applied in the setting of algebras?*

The approach to homological stability developed by Galatius–Kupers–Randal-Williams usually requires an E_k-algebra as an input. If one is interested in studying the homology of a sequence of discrete groups $\{G_n\}_{n\in\mathbb{N}}$, one way to verify that such a structure exists on the disjoint union of classifying spaces $\bigsqcup_{n\in\mathbb{N}} B_\bullet G_n$ is to prove the following. The groupoid $\mathcal{G}$ with object set $\mathbb{N}$, in which all morphisms are automorphisms and $\mathcal{G}(n,n) = G_n$, admits a braided monoidal structure. It is demonstrated in work of Randal-Williams–Wahl [**RWW17**] that for many sequences of groups $\{G_n\}_{n\in\mathbb{N}}$ that satisfy homological stability $\mathcal{G}$ admits this structure. By work of Fiedorowicz [**Fie**], the nerve of such a braided monoidal groupoid $\mathcal{G}$,

$$N_\bullet \mathcal{G} = \bigsqcup_{n\in\mathbb{N}} B_\bullet G_n,$$

is an E_2-algebra in the category of simplicial sets. This also follows from work of Balteanu–Fiedorowicz–Schwänzl–Vogt [**BFSV03**] on classifying spaces of iterated monoidal categories and work of Berger [**Ber99**] using the Smith filtration [**Smi89**] of the Barratt–Eccles operad [**BE74a**].

The main technical result of this chapter, Theorem A stated below, shows that it is possible to apply these techniques to construct E_2-algebras $N_\bullet\mathcal{A}$ from a sequence of augmented algebras $\{(A_n,\epsilon_n)\}_{n\in\mathbb{N}}$ in a similar manner, if a set of extra conditions is satisfied. To explain this in greater detail, we will now fix the setting in which we will be working throughout this chapter.

DEFINITION 47. *Let R be a commutative unital ring. An augmented R-algebra (A,ϵ) is an R-algebra A together with a morphism of R-algebras $\epsilon : A \to R$. Using the augmentation ϵ, we can give R the structure of a left and right A-module, respectively. We will denote both representations by $\mathbb{1}$ and call them the trivial representations. With this notation, the homology groups of (A,ϵ) with trivial coefficients are defined as*

$$H_\star(A,\mathbb{1}) = \mathrm{Tor}^A_\star(\mathbb{1},\mathbb{1}).$$

The E_2-algebra $N_\bullet\mathcal{A}$, that Theorem A associates to certain sequences of R-algebras

$$\{(A_n,\epsilon_n)\}_{n\in\mathbb{N}},$$

will be an element in the category of simplicial R-modules, which has the following structure

$$N_\bullet\mathcal{A} = \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1},A_n,\mathbb{1}).$$

Here $B_\bullet(\mathbb{1},A_n,\mathbb{1})$ denotes the two-sided bar construction on A_n with respect to its trivial representations $\mathbb{1}$. Observe that, as in the group case, the simplicial object $N_\bullet\mathcal{A}$ is $\mathbb{N}$-graded. We refer to this grading as the charge. It is well-known that the homology groups of the bar-construction in charge n are exactly the homology groups of (A_n,ϵ_n) with trivial coefficients, if A_n is R-projective (see [**CE99**], Ch. II Proposition 5.3 and Ch. IX §6),

$$H_{\star,n}(N_\bullet\mathcal{A}) = H_\star(B_\bullet(\mathbb{1},A_n,\mathbb{1})) \cong \mathrm{Tor}^{A_n}_\star(\mathbb{1},\mathbb{1}) = H_\star(A_n,\mathbb{1}).$$

In particular, the E_2-algebra $N_\bullet\mathcal{A}$ will in this case remember the homology groups of the individual augmented algebras (A_n, ϵ_n) and in future work, it might be possible to use the E_2-structure to study them by applying, for example, the homological stability techniques developed by Galatius, Kupers and Randal-Williams [**GKRW19**].

In order to state Theorem A, we will now explain the construction of the Mod_R-enriched analogue of the groupoid $\mathcal{G}$ that appeared in the discussion for sequences of groups $\{G_m\}_{n\in\mathbb{N}}$.

The Mod_R-category $\mathcal{A}$. Let $(A_n, \epsilon_n)_{n\in\mathbb{N}}$ be a sequence of augmented R-algebras (see Definition 47). Then, we can construct a Mod_R-enriched category $\mathcal{A}$ (see Definition 56) with underlying set of objects $\mathbb{N}$ and morphism modules $\mathcal{A}(m,n) = 0$, if $m \neq n$, and $\mathcal{A}(n,n) = A_n$. The composition is given by the multiplication on the algebras A_n and the identity morphisms are defined by $id_n : R \to A_n : 1 \mapsto 1_{A_n}$.

It is known that the notion of braided monoidal and symmetric monoidal Set-categories carry over to Mod_R-categories. We refer the reader to [**JS93**], Section 1 and 2, for a definition of these terms in Set-categories and to ([**Kel05**], §1.4) for the definition of the tensor product of two Mod_R-categories as well as the notion of a Mod_R-bifunctor. Before proving Theorem A in section 3, we will give more details. In the appendix, we included a definition of Mod_R-natural transformations (see Definition 58).

We can now state our first main theorem.

THEOREM A. *Let $\{(A_n, \epsilon_n)\}_{n\in\mathbb{N}}$ be a sequence of augmented R-algebras (see Definition 47). Assume that this family assembles, in the sense above, into a strict braided monoidal category $(\mathcal{A}, \square, b_{-,-})$, which is enriched in R-modules and whose monoidal structure on the object-set $\mathbb{N}$ is given by addition. If the augmentations $\epsilon_n : A_n \to R$ satisfy the following two conditions*

i) $\{\epsilon_n\}$ is monoidally stable:
For all $h \in A_n$ and $m, o \in \mathbb{N} : \epsilon_{m+n+o}(id_m \square h \square id_o) = \epsilon_n(h)$.

ii) The braidings $b_{m,n} : m \square n \to n \square m \in A_{m+n}$ act by 1:
For all $m, n \in \mathbb{N} : \epsilon_{m+n}(b_{m,n}) = 1 \in R$.

then the simplicial R-module

$$N_\bullet\mathcal{A} := \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1}, A_n, \mathbb{1})$$

is an E_2-algebra in the category of simplicial R-modules i.e. $N_\bullet\mathcal{A} \in Alg_{E_2}(sMod_R)$. If the braiding is symmetric, the E_2-structure extends to an E_∞-structure.

We will prove this theorem using techniques that we learned from Berger [**Ber99**], Berger–Fresse [**BF04**], Balteanu–Fiedorowicz–Schwänzl–Vogt [**BFSV03**] and Fiedorowicz [**Fie**] in the unenriched setting. Our approach will be similar to Berger's [**Ber99**], which itself is inspired by May's work on symmetric monoidal categories and E_∞-algebras in [**May74**]. Our main contribution is to specify conditions in which their ideas can be applied to sequence of augmented R-algebras.

After proving Theorem A, we will show that the criterion applies to sequences of Iwahori–Hecke algebras $\{(\mathcal{H}(\Sigma_n, q), \epsilon_n)\}_{n\in\mathbb{N}}$ associated to the sequence of symmetric groups $\{\Sigma_n\}_{n\in\mathbb{N}}$, deformation parameter $q \in R$ a unit (see Chapter 3, Definition 37) and $\epsilon_n : \mathcal{H}(\Sigma_n, q) \to R$ the trivial augmentation (see Chapter 3, Definition 39). We studied the low-dimensional homology of these algebras in Chapter 3. Invoking Theorem A, we obtain a family of E_k-algebras $\{N_\bullet\mathcal{H}^q\}_{q\in R^\times}$ that is parameterized over the units in R. Recall that if $q = 1$, then the Iwahori–Hecke algebra $\mathcal{H}(\Sigma_n, q) = R[\Sigma_n]$ is the group ring of the symmetric group Σ_n. In this case, $q = 1$, it will immediately follow from the proof of Theorem A that

$$N_\bullet\mathcal{H}^1 = R[E_\infty(*)]$$

is the R-linearization of the free E_∞-algebra of a point $*$. The family $\{N_\bullet\mathcal{H}^q\}_{q\in R^\times}$ can hence be conceptualized as a deformation of the free E_∞-algebra on a point. For other parameters $q \neq 1$, Theorem A will only provide an E_2-structure. We show that for the parameter $q = -1 \neq 1 \in R$, this E_2-structure does not extend to an E_3-structure. These results are summarized in the following theorem, which we will prove in the second part of this chapter.

THEOREM B. *Let R be a commutative unital ring and let $q \in R$ be a deformation parameter. Consider the associated sequence of Iwahori–Hecke algebras $\{(\mathcal{H}^q(\Sigma_n), \epsilon_n)\}_{n\in\mathbb{N}}$, where Σ_n denotes the symmetric group permuting n letters and $\epsilon_n : \mathcal{H}^q(\Sigma_n) \to R$ denotes the trivial augmentation (see Chapter 3, Definition 37 and Definition 39).*

If $q \in R$ is a unit, then there exists an E_2-algebra

$$N_\bullet\mathcal{H}^q = \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1}, \mathcal{H}^q(\Sigma_n), \mathbb{1}) \in Alg_{E_2}(sMod_R)$$

in the category of simplicial R-modules whose homology groups in charge n are exactly $H_\star(\mathcal{H}^q(\Sigma_n), \mathbb{1})$, the homology groups of $\mathcal{H}^q(\Sigma_n)$ with trivial coefficients. Furthermore:

- *i) If $q = 1$, then the E_2-structure extends to an E_∞-structure and $N_\bullet\mathcal{H}^1 = R[E_\infty(*)]$ is the R-linearization of the free E_∞-algebra on a point $*$.*
- *ii) If $q = -1 \neq 1 \in R$, then the E_2-structure of $N_\bullet\mathcal{A}$ does not extend to an E_3-structure.*

To prove part *ii*) of Theorem B, we will show that the E_2-Browder bracket of the E_2-algebra $N_\bullet\mathcal{H}^q$ is in general non-trivial. This method was also employed by Fiedorowicz–Song in ([**FS97**], Theorem 2.5) to show that the E_2-structure on the disjoint union of mapping class groups does not extend to an E_3-structure.

Future work. The two results presented in this chapter are part of a joint project with Richard Hepworth and Jeremy Miller in which try to study stability patterns in the homology of Iwahori–Hecke algebras using the E_k-cellular perspective of Galatius–Kupers–Randal-Williams. Theorem A can likely be applied to various other families of R-algebras.

REMARK 10. *Galatius–Kupers–Randal-Williams [**GKRW19**], Section 17, contains a different method to construct E_k-algebras from families of groups that does not make use of the combinatorial E_k-operads that we will work with in this chapter.*

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2. Combinatorial E_k-operads

This section introduces the E_k-operads $\Gamma^{(k)}$ that we will use to prove Theorem A. After introducing the permutation operad Σ, we will define the Barratt–Eccles operad $\Gamma^{(\infty)}$ [**BE74a**]. This E_∞-operad admits a filtration by suboperads

$$\Gamma^{(0)} \hookrightarrow \Gamma^{(1)} \hookrightarrow \cdots \hookrightarrow \Gamma^{(k)} \hookrightarrow \cdots \hookrightarrow \Gamma^{(\infty)}$$

which is called the Smith filtration [**Smi89**]. Berger (see [**Ber96**] and [**Ber97**]) showed that for any k the operad $\Gamma^{(k)}$ is an E_k-operad.

2.1. The permutation operad. Let Σ_n be the symmetric group permuting n letters $\{1, \ldots, n\}$. Any permutation $\sigma \in \Sigma_n$ is specified by a sequence $\sigma = [\sigma(1), \ldots, \sigma(n)]$. The identity permutation $1_n \in \sigma_n$ corresponds to the sequence $1_n = [1, \ldots, n]$, for example. Given numbers $(n_1, \ldots, n_p) \in \mathbb{N}^p$ and some element $\sigma \in \Sigma_p$, we write

$$\sigma(n_1, \ldots, n_p) \in \Sigma_{n_1+\cdots+n_p}$$

for the associated block permutation, which shuffles the p blocks

$$\{1+\sum_{j=1}^{i-1} n_j, \dots, \sum_{j=1}^{i} n_j\} \hookrightarrow \{1, \dots, n_1 + \dots + n_p\} \text{ for } i \in \{1, \dots, p\}$$

according to σ.[1]

Definition 48 (e.g. see [**Ber97**], Example 1.3 and 1.15). *The collection of symmetric groups $\Sigma = \{\Sigma_n\}$ forms an operad in the category of sets* Set. *Its unit is $1 \in \Sigma_1$ and the structure maps are defined as*

$$\begin{aligned} \mu^{\Sigma}_{n_1,\dots,n_p} : \Sigma_p \times (\Sigma_{n_1} \times \dots \times \Sigma_{n_p}) &\to \Sigma_{n_1+\dots+n_p} \\ (\sigma; \sigma_1, \dots, \sigma_p) &\mapsto \sigma(n_1, \dots, n_p) \circ (\sigma_1 \oplus \dots \oplus \sigma_p) \end{aligned}$$

The operad Σ is called the permutation operad.

2.2. The Barratt–Eccles operad and the Smith filtration. We will now define the Barratt–Eccles operad $\Gamma^{(\infty)}$ and its filtration by the suboperads $\{\Gamma^{(k)}\}_{k\in\mathbb{N}}$, the Smith filtration. This section is largely based on work of Balteanu–Fiedorowicz–Schwänzl–Vogt [**BFSV03**], p. 346 et seqq., Berger [**Ber99**], Section 1, and Berger–Fresse [**BF04**], §1.1., 1.5. and 1.6..

Definition 49. *Let $T\Sigma_n$ be the translation category of the symmetric group Σ_n acting on itself from the left via the group structure. Its objects are the group elements of Σ_n and there is a unique isomorphism*

$$f : \sigma_0 \to f \cdot \sigma_0 = \sigma_1$$

for any two objects $\sigma_0, \sigma_1 \in \Sigma_n$.

The right translation action of the symmetric group Σ_n on itself induces a natural right action on the translation category $T\Sigma_n$:

$$\begin{aligned} T\Sigma_n \times \Sigma_n &\to T\Sigma_n \\ (\sigma, g) &\mapsto \sigma \cdot g \\ (\sigma_0 \xrightarrow{f} \sigma_1, g) &\mapsto \sigma_0 \cdot g \xrightarrow{f} \sigma_1 \cdot g \end{aligned}$$

It follows that $T\Sigma = \{T\Sigma_n\}_{n\in\mathbb{N}}$ forms a symmetric sequence in the category of small categories. The structure maps of the permutation operad (see Definition 48)

$$\begin{aligned} \mu^{\Sigma}_{n_1,\dots,n_p} : \Sigma_p \times (\Sigma_{n_1} \times \dots \times \Sigma_{n_p}) &\to \Sigma_{n_1+\dots+n_p} \\ (\sigma; \sigma_1, \dots, \sigma_p) &\mapsto \sigma(n_1, \dots, n_p) \circ (\sigma_1 \oplus \dots \oplus \sigma_p) \end{aligned}$$

induce structure maps on $\{T\Sigma_n\}_{n\in\mathbb{N}}$ that make $T\Sigma$ an operad in the category of small categories (e.g. see §1.2.0.2 and Example 1.2.7. of Wahl's PhD Thesis [**Wah01**]). Taking nerves is compatible with products, hence $\Gamma^{(\infty)} = \{N(T\Sigma_n)\}_{n\in\mathbb{N}}$ is an operad in the category of simplicial sets.

Definition 50. *The operad $\Gamma^{(\infty)} = \{N(T\Sigma_n)\}_{n\in\mathbb{N}}$ is called the Barratt–Eccles operad.*

This E_∞-operad has been introduced in Barratt–Eccles' work on infinite loop spaces [**BE74a, BE74b, BE74c**]. Smith defined a filtration of $\Gamma^{(\infty)}$ by suboperads $\{\Gamma^{(k)}\}_{k\in\mathbb{N}}$ [**Smi89**]. The following account is similar to [**BFSV03**], p. 346. A chain of permutations

$$\sigma_0 \to \dots \to \sigma_d$$

in the translation category $T\Sigma_n$ corresponds to a chain of orderings of the set $\{1, \dots, n\}$,

$$[\sigma_0(1), \dots, \sigma_0(n)] \to \dots \to [\sigma_d(1), \dots, \sigma_d(n)].$$

[1] These are essentially the conventions used in [**BFSV03**].

Given any two elements $a \neq b \in \{1, \dots, n\}$, we say that the relative order of a and b changes at σ_i if the order in which a and b appear in $[\sigma_i(1), \dots, \sigma_i(n)]$ is different from the order in which they appear in $[\sigma_{i+1}(1), \dots, \sigma_{i+1}(n)]$. We give an example to illustrate this definition.

EXAMPLE 5. *Consider the following sequence of orderings of the set* $\{1, 2, 3\}$*,*

$$[3, 1, 2] \to [1, 2, 3] \to [1, 3, 2].$$

The relative order of 1 *and* 2 *never changes. The relative order of* 1 *and* 3 *changes in the first step, but not in the second step. The relative order of* 2 *and* 3 *changes in the first and in the second step.*

Given a sequence of permutations $\sigma_0 \to \cdots \to \sigma_d$, we say that the relative order of a and b changes at most $k-1$ times, if

$$|\{i : \text{ relative order of } a, b \text{ changes at } \sigma_i\}| \leq k - 1.$$

EXAMPLE 6. *In Example 5, the relative order of* 2 *and* 3 *changes at most two or more times, but it does not change at most one time.*

DEFINITION 51. *Let* $\sigma_\bullet = (\sigma_0 \to \cdots \to \sigma_d)$ *be a* d*-simplex of* $\Gamma^{(\infty)}_d(n) = N_d T\Sigma_n$*, then*

$$\sigma_\bullet \in \Gamma^{(k)}(n)$$

if the relative order of any two elements

$$a \neq b \in \{1, \dots, n\}$$

changes at most $k-1$ *times in* $\sigma_\bullet$*.*

EXAMPLE 7. *The sequence in Example 5 is a* 2*-simplex in* $\Gamma^{(3)}(3)$ *but not in* $\Gamma^{(2)}(3)$*.*

Smith introduced this filtration $\{\Gamma^{(k)}\}_{k\in\mathbb{N}}$ of the Barratt–Eccles operad $\Gamma^{(\infty)}$ by suboperads in [**Smi89**]. Berger ([**Ber96**], [**Ber97**]) proved that the operad $\Gamma^{(k)}$ is an E_k-operad. This was indicated by work of Kashiwabara, who showed that the spaces $\Gamma^{(k)}(n)$ have the correct homotopy type [**Kas93**]. An alternative proof of Berger's result can be found in ([**BFSV03**], Theorem 3.16.).

In this chapter, we will work in the category of simplicial R-modules $sMod_R$. The R-linearization functor

$$R[-] : sSet \to sMod_R$$

preserves the monoidal product. Therefore, we obtain a family of E_k-operads, which we will also denote by $\Gamma^{(k)}$ in the category $sMod_R$ through R-linearization.

3. Constructing E_k-algebras from sequences of augmented R-algebras

The goal of this section is the proof of Theorem A, which specifies conditions that allow one to construct an E_k-algebra from a sequence of augmented R-algebras $\{(A_n, \epsilon_n)\}_{n\in\mathbb{N}}$ (see Definition 47). We start the proof, by recall the notation and conditions.

Proof of Theorem A: Let $\mathcal{A}$ be the Mod_R-category constructed from the sequence $\{(A_n, \epsilon_n)\}_{n\in\mathbb{N}}$ with object set $\mathbb{N}$ and with morphism modules given by

$$\mathcal{A}(m, n) = \begin{cases} A_n & , \text{ if } m = n \\ 0 & , \text{ if } m \neq n. \end{cases}$$

The first assumption in Theorem A is that this category is equipped with a strict braided monoidal structure $(\mathcal{A}, \square, b_{-,-})$. Here,

$$\square : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

is a Mod_R-enriched bi-functor (see [**JS93**], Section 1, and [**Kel05**], §1.4). We assumed that this bi-functor is given by addition

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

on the set of objects. In particular, the left and right identity object is given by 0. On morphism modules, this bifunctor gives rise to maps

$$- \square - : A_m \otimes A_n \to A_{m+n}.$$

The strictness assumption amounts to the equality $(-\square(-\square-)) = ((-\square-)\square-)$. The braiding (see [**JS93**], Definition 2.1) is a Mod_R-enriched natural isomorphism

$$b_{-,-} : -_1 \square -_2 \Longrightarrow -_2 \square -_1$$

which is specified by elements $b_{m,n} = b_{m,n}(1) \in A_{m+n}$ for any pair (m, n) of natural numbers (see Section 7, Definition 58). The hexagon relations (see [**JS93**], Definition 2.1) of the braiding then state that for any triple (m, n, o) of natural numbers, these elements satisfy

$$(id_n \square b_{m,o}) \cdot (b_{m,n} \square id_o) = b_{m,n+o}$$

$$(b_{m,o} \square id_n) \cdot (id_m \square b_{n,o}) = b_{m+n,o}.$$

We will use the monoidal product and the braiding to define an explicit E_k-algebra structure the simplicial R-module

$$N_\bullet \mathcal{A} := \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1}, A_n, \mathbb{1}).$$

Here, $B_\bullet(\mathbb{1}, A_n, \mathbb{1})$ is the two-sided bar construction on A_n, its module of k-simplices are given by

$$B_k(\mathbb{1}, A_n, \mathbb{1}) = \mathbb{1} \otimes A_n^{\otimes k} \otimes \mathbb{1}$$

and its face maps are given by the module structure and the product in A_n, the degeneracy maps are given by insertion of identities.

We will recall the two assumptions on the augmentations $\{\epsilon_n\}_{n\in\mathbb{N}}$ of the algebras $\{A_n\}_{n\in\mathbb{N}}$ in Theorem A later, when they are needed.

Let β_n denote the braid group on n strands and let $R[\beta_n]$ denote the group ring. The E_k-structure will be constructed by acting with the following β_n-representation on elements that are monoidal products in the algebra A_n (e.g. $(h_{n_1} \square h_{n_2}) \in A_n$, for $h_{n_1} \in A_{n_1}$, $h_{n_2} \in A_{n_2}$ and $n_1 + n_2 = n$).

For any $n \in \mathbb{N}$, the braided monoidal structure of $\mathcal{A}$ induces maps

$$\widehat{\cdot} : R[\beta_n] \to A_n$$

and similar, if the braiding is symmetric,

$$\widehat{\cdot} : R[\Sigma_n] \to A_n.$$

Indeed, recall that $\beta_2 \cong \mathbb{Z}$. Hence, the braiding $b_{1,1} \in A_2$ gives rise to a map $R[\beta_2] \to A_2$. For any $n \in \mathbb{N}$, we find elements $\{id_m \square b_{1,1} \square id_{n-m-2}\}_{\{0\leq m\leq n-2\}}$ contained in A_n that satisfy the braid relations and therefore a map $R[\beta_n] \to A_n$.

The action of the E_2-operad $\Gamma^{(2)}$ on $N_\bullet\mathcal{A}$ will be defined via the β_n-representations above. Recall that $\Gamma^{(2)}$ is defined in terms of elements of the symmetric group. In order to define the E_2-structure maps on $N_\bullet\mathcal{A}$, we will need to interpret elements of the symmetric group Σ_n as elements of the braid group β_n. We will explain this now.

For every $n \in \mathbb{N}$, there is a surjection $\pi_n : \beta_n \to \Sigma_n$ mapping the standard generator s_i of the symmetric group on n letters Σ_n to the i-th standard generator γ_i of the braid group on n strands β_n.

DEFINITION 52. *The surjection $\pi : \beta_n \to \Sigma_n$ admits a set-section $\nu : \Sigma_n \to \beta_n$ defined as follows:*

i) ν maps the i-th generator s_i of Σ_n to the i-th generator γ_i of β_n,

$$\nu(s_i) = \gamma_i.$$

ii) For each $\sigma \in \Sigma$, we fix reduced spelling $\sigma = s_{i_0} \dots s_{i_m}$ representing σ (see Chapter 3, Definition 33) and set

$$\nu(\sigma) = \gamma_{i_0} \dots \gamma_{i_m}.$$

Lemma 38. *The section ν does not depend on the choice of reduced word made in ii) of Definition 52. I.e. if $\sigma = s_{j_0} \dots s_{j_m} = s_{i_0} \dots s_{i_m}$ are two reduced words representing $\sigma \in \Sigma_n$, then*

$$\nu(\sigma) = \gamma_{j_0} \dots \gamma_{j_m} = \gamma_{i_0} \dots \gamma_{i_m} \in \beta_n.$$

Proof: This follows from a theorem of Matsumoto, [**Mat99**] 1.8 Theorem, which states that two reduced words in Σ_n represent the same element if and only if one can be transformed into the other using only the braid relations. □

Observe that the section $\nu : \Sigma_n \to \beta_n$ preserves the word length i.e. for all

$$\sigma \in \Sigma_n : |\sigma| = |\nu(\sigma)|.$$

We will use the section ν to view elements of the symmetric group as braids i.e. the section only plays a role if the braiding is not symmetric.

Fix $\vec{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and let $n := \sum_1^k n_i$. Let σ_0, σ_1 and $f \in \Sigma_k$ be such that $f \cdot \sigma_0 = \sigma_1$. We denote by $L_{\vec{n}}(f) \in \Sigma_n$ the block permutation

$$L_{\vec{n}}(f) := \sigma_1(n_1, \dots, n_k) \circ \sigma_0(n_1, \dots, n_k)^{-1} = (\sigma_1 \sigma_0^{-1})(n_{\sigma_0(1)}, \dots, n_{\sigma_0(k)}).$$

Before defining the structure maps on

$$N_\bullet \mathcal{A} = \bigoplus_{n \in \mathbb{N}} B_\bullet(\mathbb{1}, A_n, \mathbb{1}),$$

we will explain how we denote elements in various bar-constructions. We will often use two different notations for elements in $B_l(\mathbb{1}, A_n, \mathbb{1}) = \mathbb{1} \otimes A_n^{\otimes l} \otimes \mathbb{1}$ in order to increase the readability of formulas. Let $a, b \in \mathbb{1}$ and $h_1, \dots, h_l \in A_n$. Elements in the domain for a map are usually denoted by

$$a \otimes n \xrightarrow{h_1} \dots \xrightarrow{h_l} n \otimes b \in B_l(\mathbb{1}, A_n, \mathbb{1}).$$

This makes it visible that the element is in charge n. Elements in the target of a map are usually denoted by

$$a \otimes h_1 \otimes \dots \otimes h_l \otimes b \in B_l(\mathbb{1}, A_n, \mathbb{1})$$

Their charge can be calculated from the domain and is left implicit.

With this convention, we can now define the structure maps on $N_\bullet \mathcal{A}$. The following formula makes the intuition of "acting on elements that are monoidal products via the β_n-representations" that we mentioned before precise.

Definition 53. *Fix a simplicial degree $l \in \mathbb{N}$ and an arity degree $k \in \mathbb{N}$. We may define maps*

$$\alpha_l^k : \Gamma_l^{(2)}(k) \otimes (N_l \mathcal{A})^{\otimes k} \to N_l \mathcal{A}$$

by mapping a basis l-simplex on the left hand side

$$(a \otimes \sigma_0 \xrightarrow{f_1} \dots \xrightarrow{f_l} \sigma_l \otimes b) \otimes \bigotimes_{i=1}^k (a^i \otimes n_i \xrightarrow{h_1^i} \dots \xrightarrow{h_l^i} n_i \otimes b^i)$$

to the following l-simplex in charge $n = n_1 + \dots + n_k$ on the right hand side

$$a(\prod_{i=1}^k a^i) \otimes \bigotimes_{j=1}^l (\widehat{L_{\vec{n}}(f_j)} \cdot [h_j^{\sigma_{j-1}(1)} \square \dots \square h_j^{\sigma_{j-1}(k)}]) \otimes b(\prod_{i=1}^k b^i)$$

where we view $L_{\vec{n}}(f_j) \in \Sigma_n$ as an element in β_n using the section $\nu : \Sigma_n \to \beta_n$ in Definition 52. If the braiding is symmetric, the same formula defines maps:

$$\alpha_l^k : \Gamma_l^{(\infty)}(k) \otimes (N_l \mathcal{A})^{\otimes k} \to N_l \mathcal{A}$$

The two conditions on the augmentations $\{\epsilon_n\}_{n\in\mathbb{N}}$ of the algebras $\{A_n\}_{n\in\mathbb{N}}$ ensure the maps defined in Definition 53 assemble to simplicial maps. Once this is verified, the proof that they define an E_2-algebra or E_∞-structure, respectively, on $N_\bullet\mathcal{A}$ is completely analogous to the proof in the unenriched setting. We recall the two conditions as part of the following proposition.

PROPOSITION 13. *Assume that the augmentations $\epsilon_n : A_n \to R$ satisfy the following two conditions*

i) $\{\epsilon_n\}$ is monoidally stable:
For all $h \in A_n$ and $m, o \in \mathbb{N} : \epsilon_{m+n+o}(id_m \square h \square id_o) = \epsilon_n(h)$.

ii) The braidings $b_{m,n} : m \square n \to n \square m \in A_{m+n}$ act by 1:
For all $m, n \in \mathbb{N} : \epsilon_{m+n}(b_{m,n}) = 1 \in R$.

Then the maps $\{\alpha^k_l\}_{l\in\mathbb{N}}$ in Definition 53 assemble to a map of simplicial R-modules

$$\alpha^k_\bullet : \Gamma^{(2)}(k) \otimes (N_\bullet C)^{\otimes k} \to N_\bullet C$$

for any $k \geq 0$.

Proof: We need to check the simplicial identities. We will do this on basis simplices using the notation introduced in Definition 53. For $r \neq 0, l$, the equality $d_r \circ \alpha^k_l = \alpha^k_{l-1} \circ d_r$ holds if and only if the following equality holds in A_n

$$\begin{aligned}&(\widehat{L_{\vec{n}}(f_{r+1})} \cdot [h_{r+1}^{\sigma_r(1)} \square \ldots \square h_{r+1}^{\sigma_r(k)}])(\widehat{L_{\vec{n}}(f_r)} \cdot [h_r^{\sigma_{r-1}(1)} \square \ldots \square h_r^{\sigma_{r-1}(k)}])\\ &=\\ &\widehat{L_{\vec{n}}(f_{r+1}f_r)} \cdot [h_{r+1}^{\sigma_{r-1}(1)} h_r^{\sigma_{r-1}(1)} \square \ldots \square h_{r+1}^{\sigma_{r-1}(k)} h_r^{\sigma_{r-1}(k)}].\end{aligned}$$

Verifying this does not use the two conditions in Proposition 13. It follows from the naturality of the braiding that

$$\begin{aligned}&(\widehat{L_{\vec{n}}(f_{r+1})} \cdot [h_{r+1}^{\sigma_r(1)} \square \ldots \square h_{r+1}^{\sigma_r(k)}])(\widehat{L_{\vec{n}}(f_r)} \cdot [h_r^{\sigma_{r-1}(1)} \square \ldots \square h_r^{\sigma_{r-1}(k)}])\\ &=\\ &(\widehat{L_{\vec{n}}(f_{r+1})} \cdot \widehat{L_{\vec{n}}(f_r)})([h_{r+1}^{\sigma_{r-1}(1)} \square \ldots \square h_{r+1}^{\sigma_{r-1}(k)}] \cdot [h_r^{\sigma_{r-1}(1)} \square \ldots \square h_r^{\sigma_{r-1}(k)}]).\end{aligned}$$

Bifunctoriality of the monoidal structure $\square : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ implies that

$$\begin{aligned}&[h_{r+1}^{\sigma_{r-1}(1)} \square \ldots \square h_{r+1}^{\sigma_{r-1}(k)}] \cdot [h_r^{\sigma_{r-1}(1)} \square \ldots \square h_r^{\sigma_{r-1}(k)}]\\ &=\\ &[h_{r+1}^{\sigma_{r-1}(1)} h_r^{\sigma_{r-1}(1)} \square \ldots \square h_{r+1}^{\sigma_{r-1}(k)} h_r^{\sigma_{r-1}(k)}].\end{aligned}$$

We therefore only need to argue that

$$\widehat{L_{\vec{n}}(f_{r+1})} \cdot \widehat{L_{\vec{n}}(f_r)} = \widehat{L_{\vec{n}}(f_{r+1}f_r)}.$$

Note that the equation $L_{\vec{n}}(f_{r+1}) \circ L_{\vec{n}}(f_r) = L_{\vec{n}}(f_{r+1}f_r)$ holds in Σ_n

$$\begin{aligned}L(f_{r+1}) \circ L(f_r) =&(\sigma_{r+1}(n_1, \ldots, n_k) \circ \sigma_r(n_1, \ldots, n_k)^{-1}) \circ (\sigma_r(n_1, \ldots, n_k) \circ \sigma_{r-1}(n_1, \ldots, n_k)^{-1})\\ &= \sigma_{r+1}(n_1, \ldots, n_k) \circ \sigma_{r-1}(n_1, \ldots, n_k)^{-1}\\ &= L(f_{r+1}f_r).\end{aligned}$$

If the braiding is symmetric, we are done. If the braiding is not symmetric, we need to verify that the equality

$$\nu(L_{\vec{n}}(f_{r+1})) \circ \nu(L_{\vec{n}}(f_r)) = \nu(L_{\vec{n}}(f_{r+1}f_r))$$

holds for the lifts to the braid group β_n. Recall that $|f_i|$ denotes the word length of a permutation $f_i \in \Sigma_k$. The definition of $\Gamma^{(2)}$ (see Definition 51) implies that:

$$\text{If } \sigma_0 \xrightarrow{f_1} \ldots \xrightarrow{f_l} \sigma_l \in \Gamma^{(2)}, \text{ then } |f_1 \circ \cdots \circ f_l| = \sum_1^l |f_i|$$

This implication also holds for the associated block permutations:

$$|L_{\vec{n}}(f_1) \circ \cdots \circ L_{\vec{n}}(f_l)| = \sum_1^l |L_{\vec{n}}(f_i)|$$

It follows that if $L_{\vec{n}}(f_{r+1}) = s_{i_1} \dots s_{i_m}$ and $L_{\vec{n}}(f_r) = s_{i_1} \dots s_{i_z}$ are reduced words representing $L_{\vec{n}}(f_{r+1})$ and $L_{\vec{n}}(f_r)$ respectively, then

$$L_{\vec{n}}(f_{r+1} f_r) = L_{\vec{n}}(f_{r+1}) \circ L_{\vec{n}}(f_r) = s_{i_1} \dots s_{i_m} \cdot s_{i_1} \dots s_{i_z}$$

is a reduced word representing $L_{\vec{n}}(f_{r+1} f_r)$. Hence, Lemma 38 implies that

$$\nu(L_{\vec{n}}(f_{r+1})) \circ \nu(L_{\vec{n}}(f_r)) = \nu(L_{\vec{n}}(f_{r+1} f_r))$$

This finishes the proof for $r \neq 0, l$.

For $r = 0$ and analogous for $r = l$, the equality $d_0 \circ \alpha_l^k = \alpha_{l-1}^k \circ d_0$ holds if and only if the following equality holds in R

$$a \cdot (\prod_{i=1}^k a^i \epsilon_{n_i}(h_1^i)) = (a \prod_{i=1}^k a^i) \cdot \epsilon_n(\widehat{L_{\vec{n}}(f_1)} \cdot [h_1^{\sigma_0(1)} \square \dots \square h_1^{\sigma_0(k)}]).$$

The fact that the augmentations are monoidally stable (Condition (1) in Proposition 13) and that the augmentations are algebra map implies that:

$$\begin{aligned} &\epsilon_n(\widehat{L(f_1)} \cdot [h_1^{\sigma_0^{-1}(1)} \square \dots \square h_1^{\sigma_0^{-1}(k)}]) \\ &= \epsilon_n(\widehat{L(f_1)}) \cdot \prod_1^k \epsilon_{n_i}(h_1^i) \end{aligned}$$

Now the claim follows from the fact that braidings act by 1 (Condition (2) in Proposition 13). This completes the proof. ∎

We now turn to the proof of that the structure maps defined in Definition 53 give $N_\bullet \mathcal{A}$ an E_2-algebra structure.

Proposition 14. *The maps of simplicial R-modules*

$$\alpha_\bullet^k : \Gamma^{(2)}(k) \otimes (N_\bullet C)^{\otimes k} \to N_\bullet C$$

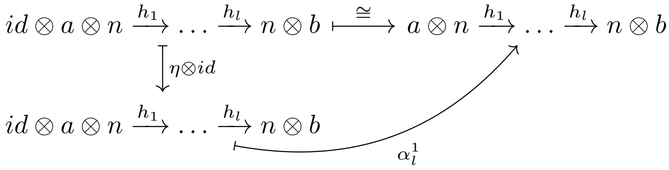
for $k \geq 0$ constructed in Definition 53 and Proposition 13 define an E_2–algebra structure on $N_\bullet \mathcal{A}$. Similarly, for the E_∞-case.

Proof of Proposition 14: We need check the associativity, equivariance and unit axiom as stated in (e.g. [**KM95**], Definition 2.1.). All axioms can be check level-wise for a fixed simplicial degree $l \in \mathbb{N}$. We will do this using the maps and the notation in Definition 53.

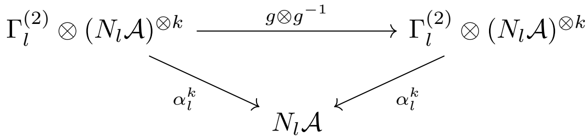
Unit axiom: The unit object in the symmetric monoidal category $sMod_R$ is the simplicial R-module given by $\kappa_l = R\{id\}$. The unit map $\eta : \kappa \to \Gamma^{(2)}(1)$ of the operad $\Gamma^{(2)}$ is the identity, as $\Gamma_l^{(2)}(1) = R\{id\}$. We need to verify that the following diagram commutes for all $l \in \mathbb{N}$.

$$\begin{array}{ccc} \kappa_l \otimes N_l \mathcal{A} & \xrightarrow{\cong} & N_l \mathcal{A} \\ \downarrow{\scriptstyle \eta \otimes id} & \nearrow{\scriptstyle \alpha_l^1} & \\ \Gamma^{(2)}(1) \otimes N_l \mathcal{A} & & \end{array}$$

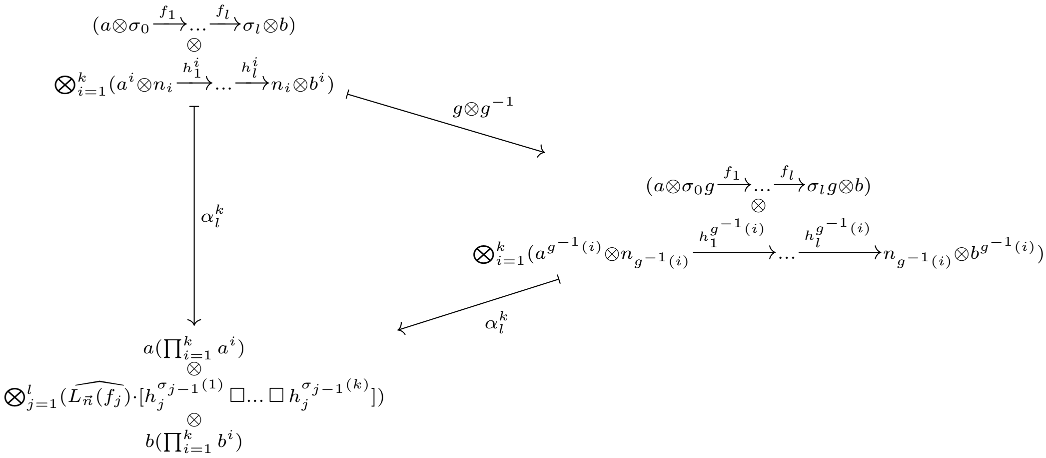
This holds because Definition 53 implies that:



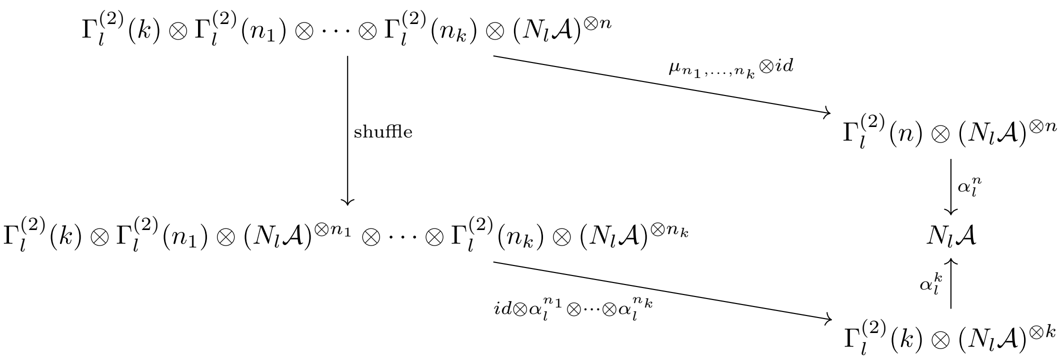
Equivariance axiom: We need to argue that for any $g \in \Sigma_k$ the following diagram commutes



where g acts on $\Gamma_l^{(2)}$ via the right action introduced after Definition 49 and g^{-1} acts by permuting the factors of $(N_l\mathcal{A})^{\otimes k}$. This holds because $(\sigma g) \cdot g^{-1} = \sigma$ and Definition 53 implies that



Associativity axiom: We need to argue that for any $n = \Sigma_1^k n_i$ the following diagram commutes:



We fix the following notation. Let

- $1 \otimes \sigma_0 \xrightarrow{f_1} \dots \xrightarrow{f_l} \sigma_l \otimes 1 \in \Gamma_l^{(2)}(k)$ be a R-basis element.
- $1 \otimes \tau_0^x \xrightarrow{g_1^x} \dots \xrightarrow{g_l^x} \tau_l^x \otimes 1 \in \Gamma_l^{(2)}(n_x)$ for $x \in \{1, \dots, k\}$ be R-basis elements.

- $1 \otimes m_{x,y} \xrightarrow{h_1^{x,y}} \dots \xrightarrow{h_l^{x,y}} m_{x,y} \otimes 1 \in N_l\mathcal{A}$ for all $x \in \{1,\dots,k\}$ and $y \in \{1,\dots,n_x\}$.

We will be omitting the outer $(1\otimes\cdots\otimes 1)$-part of these elements in the following calculation. Let $n = \sum_{x=1}^k n_x$, $m_x = \sum_{y=1}^{n_x} m_{x,y}$ for $x \in \{1,\dots,k\}$ and $m = \sum_{x=1}^k m_x$. Let furthermore $\vec{n} = (n_1,\dots,n_k), \vec{m}_x = (m_{x,1},\dots,m_{x,n_x})$ for $1 \le x \le k$ and $\vec{m} = (m_1,\dots,m_k)$. We will write $\vec{m}_{all} = \vec{m}_1 \circ \cdots \circ \vec{m}_k \in \mathbb{N}^m$ for the concatenation of the $\vec{m}_x$, $1 \le x \le k$. We will use the following notation for block permutations $(\sigma_w)_{\vec{n}} := \sigma_w(n_1,\dots,n_k)$. We start by evaluating the upper composition of the associativity diagram:

$$(\sigma_0 \xrightarrow{f_1} \dots \xrightarrow{f_l} \sigma_l) \otimes (\bigotimes_{x=1}^k (\tau_0^x \xrightarrow{g_1^x} \dots \xrightarrow{g_l^x} \tau_l^x)) \otimes (\bigotimes_{x=1}^k \bigotimes_{y=1}^{n_x} m_{x,y} \xrightarrow{h_1^{x,y}} \dots \xrightarrow{h_l^{x,y}} m_{x,y})$$

$$\Big\downarrow \mu_{n_1,\dots,n_k} \otimes id$$

$$(\bigotimes_{w=1}^l (\sigma_{w-1})_{\vec{n}} \circ (\tau_{w-1}^1 \oplus\cdots\oplus \tau_{w-1}^k) \xrightarrow{L_{\vec{n}}(f_w)\circ(g_w^{\sigma_{w-1}(1)}\oplus\cdots\oplus g_w^{\sigma_{w-1}(k)})} (\sigma_w)_{\vec{n}} \circ (\tau_w^1 \oplus\cdots\oplus\tau_w^k))$$
$$\otimes(\bigotimes_{x=1}^k \bigotimes_{y=1}^{n_x} m_{x,y} \xrightarrow{h_1^{x,y}} \dots \xrightarrow{h_l^{x,y}} m_{x,y})$$

$$\Big\downarrow \alpha_l^n$$

$$\bigotimes_{w=1}^l \overline{L_{\vec{m}_{all}}(L_{\vec{n}}(f_w) \circ (g_w^{\sigma_{w-1}(1)} \oplus \cdots \oplus g_w^{\sigma_{w-1}(k)}))}\, \square_{x=1}^k \square_{y=1}^{n_{\sigma_{w-1}(x)}} h_w^{\sigma_{w-1}(x),\tau_{w-1}^{\sigma_{w-1}(x)}(y)}$$

We continue by evaluating the lower composition of the of the associativity diagram:

$$(\sigma_0 \xrightarrow{f_1} \dots \xrightarrow{f_l} \sigma_l) \otimes (\bigotimes_{x=1}^k (\tau_0^x \xrightarrow{g_1^x} \dots \xrightarrow{g_l^x} \tau_l^x)) \otimes (\bigotimes_{x=1}^k \bigotimes_{y=1}^{n_x} m_{x,y} \xrightarrow{h_1^{x,y}} \dots \xrightarrow{h_l^{x,y}} m_{x,y})$$

$$\Big\downarrow \text{shuffle}$$

$$(\sigma_0 \xrightarrow{f_1} \dots \xrightarrow{f_l} \sigma_l) \otimes \bigotimes_{x=1}^k ((\tau_0^x \xrightarrow{g_1^x} \dots \xrightarrow{g_l^x} \tau_l^x) \otimes \bigotimes_{y=1}^{n_x} (m_{x,y} \xrightarrow{h_1^{x,y}} \dots \xrightarrow{h_l^{x,y}} m_{x,y}))$$

$$\Big\downarrow id\otimes\alpha_l^{n_1}\otimes\cdots\otimes\alpha_l^{n_k}$$

$$(\sigma_0 \xrightarrow{f_1} \dots \xrightarrow{f_l} \sigma_l) \otimes \bigotimes_{x=1}^k (\otimes_{w=1}^l \widehat{L_{\vec{m}_x}(g_w^x)}\, \square_{y=1}^{n_x} h_w^{x,\tau_{w-1}^x(y)})$$

$$\Big\downarrow \alpha_l^k$$

$$\bigotimes_{w=1}^l \widehat{L_{\vec{m}}(f_w)}(\square_{x=1}^k \overline{L_{\vec{m}_x}(g_w^{\sigma_{w-1}(x)})}\, \square_{y=1}^{n_{\sigma_{w-1}(x)}} h_w^{\sigma_{w-1}(x),\tau_{w-1}^{\sigma_{w-1}(x)}(y)})$$

To transform the term obtained from the lower composition into the term obtained from the upper composition, one first uses the bifunctoriality of the monoidal product to see that the following equation holds:

$$\begin{aligned}&\widehat{L_{\vec{m}}(f_w)}(\square_{x=1}^k \overline{L_{\vec{m}_x}(g_w^{\sigma_{w-1}(x)})}\, \square_{y=1}^{n_{\sigma_{w-1}(x)}} h_w^{\sigma_{w-1}(x),\tau_{w-1}^{\sigma_{w-1}(x)}(y)})\\ &= \widehat{L_{\vec{m}}(f_w)}(\square_{x=1}^k \overline{L_{\vec{m}_x}(g_w^{\sigma_{w-1}(x)})})(\square_{x=1}^k \square_{y=1}^{n_{\sigma_{w-1}(x)}} h_q^{\sigma_{w-1}(x),\tau_{w-1}^{\sigma_{w-1}(x)}(y)})\end{aligned}$$

We are left with checking that the following two braiding elements are the same:

$$\widehat{L_{\vec{m}}(f_w)}(\square_{x=1}^k \overline{L_{\vec{m}_x}(g_w^{\sigma_{w-1}(x)})}) = \overline{L_{\vec{m}_{all}}(L_{\vec{n}}(f_w) \circ (g_w^{\sigma_{w-1}(1)} \oplus \cdots \oplus g_w^{\sigma_{w-1}(k)}))}$$

It is easy to check that the two underlying permutations are the same i.e. that

$$L_{\vec{m}}(f_w)(\square_{x=1}^k L_{\vec{m}_x}(g_w^{\sigma_{w-1}(x)})) = L_{\vec{m}_{all}}(L_{\vec{n}}(f_w) \circ (g_w^{\sigma_{w-1}(1)} \oplus \cdots \oplus g_w^{\sigma_{w-1}(k)})) \in \Sigma_m.$$

If the braiding is symmetric, we are done. If the braiding is not symmetric, the equality of the lifts to β_m follows as in Proposition 13 using the definition of $\Gamma^{(2)}$ and Lemma 38. This completes the proof of Proposition 14. □

This also completes the proof of Theorem A. □

4. Deformations of the free E_∞-algebra on a point

4.1. Braided structures for quotient algebras. We will use the following criterion to turn certain strict monoidal categories into braided monoidal categories.

Lemma 39. *Let $\mathcal{A}$ be a strict monoidal category enriched in R-modules with object set $\mathbb{N}$ and monoidal structure on objects given by addition. Assume that there is a strict monoidal functor $Q : \beta \twoheadrightarrow \mathcal{A}$ from the braid groupoid that is the identity objects and such that the map on morphism modules $Q : R[\beta(m,n)] \to \mathcal{A}(m,n)$ is a surjection. In particular, $\mathcal{A}(m,n) = 0$, if $m \neq n$. Then, the push-forward of any braiding on β gives $\mathcal{A}$ the structure of a braided monoidal category.*

Proof: Let $\gamma_{-_1,-_2} : -_1 \square -_2 \Rightarrow -_2 \square -_1$ be a braiding in β i.e. is a Mod_R-natural isomorphism satisfying the two hexagon identities. By the strictness assumption, the hexagon identities amount to the following two equalities:

$$(id_n \square \gamma_{m,o}) \cdot (\gamma_{m,n} \square id_o) = \gamma_{m,n \square o}$$
$$(\gamma_{m,o} \square id_n) \cdot (id_m \square \gamma_{n,o}) = \gamma_{m \square n,o}$$

Applying the strict monodial functor Q to these equalities, we get that

$$(id_n \square Q(\gamma_{m,o})) \cdot (Q(\gamma_{m,n}) \square id_o) = Q(\gamma_{m,n \square o})$$
$$(Q(\gamma_{m,o}) \square id_n) \cdot (id_m \square Q(\gamma_{n,o})) = Q(\gamma_{m \square n,o}).$$

It follows that the set of morphisms $Q(\gamma_{m,n}) : m \square n \to n \square m \in \mathcal{A}_{m+n}$ satisfy the hexagon identities as well. Because Q is a functor and $\gamma_{m,n}$ is an isomorphism, any morphism $Q(\gamma_{m,n})$ is an isomorphism. We are left with verifying that $Q(\gamma_{-_1,-_2})$ defines a natural transformation. This follows from the surjectivity of the maps $Q_n : R[\beta_n] \to \mathcal{A}_n$ and the strict monoidality: Any diagram that needs to commute is the image of a commutative diagram in β. $\square$

4.2. E_k-algebras from Iwahori–Hecke algebras. Iwahori–Hecke algebras of a Coxeter group (W,S) have introduced in Chapter 3, Definition 37. We refer the reader to Chapter 3, Section 3, for the necessary background and definitions. Let R be a commutative ring and fix a deformation parameter $q \in R$. Let $\mathcal{H}_n^q = \mathcal{H}^q(\Sigma_n, \mathbb{1}^{(q)})$ be the Iwahori–Hecke algebra associated to the symmetric group (Σ_n, S) permuting n letters $\{1,\dots,n\}$. Here $\mathbb{1}^{(q)}$ is the trivial representation of $\mathcal{H}_n^q$ (see Chapter 3, Definition 39), which arises from the augmentation $\epsilon_n : \mathcal{H}_n^q \to R$ defined by $T_s \mapsto q$ for $s \in S$. Recall that for $q = 1$, the Iwahori–Hecke algebra $\mathcal{H}_n^q = R[\Sigma_n]$ is exactly the group algebra of Σ_n and $\mathbb{1}^q$ is the trivial Σ_n-module.

Definition 54. *Let $\mathcal{H}^q$ be the category enriched in R-modules with object set $\mathbb{N}$, automorphism modules defined by $\mathcal{H}^q(n,n) = \mathcal{H}_n^q$ and all other morphism modules $\mathcal{H}^q(m,n) = 0$ for $m \neq n$. The composition of morphisms is given by the multiplication of the algebras $\mathcal{H}_n^q$ for automorphism modules and the zero morphism otherwise. We call this category, the category of Iwahori–Hecke algebras.*

The following lemma defines a monoidal structure on $\mathcal{H}^q$.

Lemma 40. *The following data gives the category $\mathcal{H}^q$ a strict monoidal structure:*

- *On objects it is given the usual addition: $m \square n = m + n$.*
- *On morphisms it is given by*
$$\square : \mathcal{H}_m^q \otimes \mathcal{H}_n^q \to \mathcal{H}_{m+n}^q : a \otimes b \mapsto a \square b = a \cdot b,$$
where $\mathcal{H}_m^q$ is seen as a subalgebra of $\mathcal{H}_{m+n}^q$ via the inclusion
$$\{1,\dots,m\} \hookrightarrow \{1,\dots,n+m\}$$

and $\mathcal{H}^q_n$ is seen as a subalgebra of $\mathcal{H}^q_{m+n}$ via the injection

$$\{1,\ldots,n\} \hookrightarrow \{1,\ldots,n+m\} : r \mapsto r+m.$$

Proof: This is readily checked and completely analogous to the monoidal structure on Σ, the category of symmetric groups. □

The category of Iwahori–Hecke algebras admits a braiding. To see this, we will use the fact that Iwahori–Hecke algebras arise as quotients of the group rings of braid groups and invoke Lemma 39.

Lemma 41. *Let $Q_n : R[\beta_n] \to \mathcal{H}^q_n$ be the map that assigns the i-th standard generator of the braid group γ_i to the i-th standard generator T_{s_i} of the Iwahori–Hecke algebra. These maps are surjective for any $n \in \mathbb{N}$ and give rise to a strict monoidal functor $Q : \beta \to \mathcal{H}^q$ that is the identity on the object set.*

Proof: This follows from the definition of the Iwahori–Hecke algebra $\mathcal{H}^q_n$ (see Chapter 3, Definition 37 or [**GP00**], §4.4, for a definition that is “closer” to braid groups). □

Corollary 21. *Let $s \in \Sigma_2$ be the generator and $s(m,n) \in \Sigma_{m+n}$ denote the associated block permutation for any $m,n \in \mathbb{N}$. Then the maps $b_{m,n} : R \to \mathcal{H}^q_{m+n} : 1 \mapsto T_{s(m,n)}$ define a braiding on $\mathcal{H}^q$. This braiding is symmetric if and only if $q = 1$, because otherwise $T_s \neq T_s^{-1}$.*

Proof: This follows from the previous lemma, the fact that the braid groupoid β admits a braiding by the elements $\gamma(m,n) \in \beta_{m+n}$, where $\gamma \in \beta_2$ is a generator, and Lemma 39. □

We want to apply Theorem A to construct an E_k-algebra from $\mathcal{H}^q$. Note that the augmentations $\epsilon_n : \mathcal{H}^q_n \to R : T_{s_i} \mapsto q$ are monoidally stable. However, the second condition in Theorem A is not satisfied by the braiding obtained in Corollary 21 i.e. the braiding is not acting by one. If q is a unit, we can normalize to obtain a braiding which has this property.

Corollary 22. *Let q be a unit in R, then the maps $b_{m,n} : R \to \mathcal{H}^q_{m+n} : 1 \mapsto q^{-|s(m,n)|}T_{s(m,n)}$ define a braiding on $\mathcal{H}^q$. We call this braiding the normalized braiding.*

Proof: This follows from the previous lemma, the fact that the R-linearized braid groupoid β admits a braiding by the elements $q^{-|\gamma(m,n)|}\gamma(m,n) \in R[\beta_{m+n}]$ and Lemma 39. □

An application of Theorem A therefore yields the following result.

Theorem 42. *Let q be a unit in R and let $\mathcal{H}^q$ be the category of Iwahori–Hecke algebras with deformation parameter q equipped with the normalized braiding, then*

$$N_\bullet \mathcal{H}^q = \bigoplus_{n\in\mathbb{N}} B_\bullet(\mathbb{1}^{(q)}, \mathcal{H}^q_n, \mathbb{1}^{(q)})$$

is an E_2-algebra in simplicial R-modules. If $q = 1$, this E_2-structure extends to an E_∞-structure and $N_\bullet \mathcal{H}^q = \bigoplus_{n\in\mathbb{N}} R[B_\bullet(\Sigma_n)]$ is a free E_∞-algebra on a point.

Proof: This is a consequence of the results in this subsection and Theorem A. □

4.3. For $q = -1 \neq 1$, the E_2-structure cannot be extended to an E_3-structure. Throughout this subsection, we work in the setting of Theorem 42 and $N_\bullet\mathcal{H}^q$ denotes an E_k-algebra obtained by Theorem 42. We will show that the E_2-Browder bracket of the E_2-algebras $N_\bullet\mathcal{H}^q$ for the deformation parameter $q = (-1)$ is nontrivial whenever $1 \neq -1 \in R$.[2] This shows that the E_2-algebra structure of the simplicial R-modules $N_\bullet\mathcal{H}^q$ constructed in Theorem 42 does, in general, not extend to an E_3-structure. In future work we plan to settle the analogous question for other deformation parameters $q \neq 1$. The example $q = (-1)$ is particularly easy and does not require the introduction of extensive additional machinery.

[2]Note that $N_\bullet\mathcal{H}^q$ is an E_∞-algebra by Theorem 42, if $1 = -1$ in R.

In our exposition of the relevant homology operations and notation, we are following §16.1.1. of [**GKRW19**]. The Browder bracket was first studied in [**Bro60**]. For a detailed treatment of homology operations of E_k-algebras, we refer the reader to [**CLM76**].

Observe that before R-linearization the operations of arity two of the second term of the Smith filtration $\Gamma^{(2)}(2)$ (see Definition 51) is the simplicial set with 0-simplices

$$\Gamma_0^{(2)}(2) = \{id, s_1\}$$

and non-degenerate 1-simplices

$$\Gamma_1^{(2)}(2) = \{id \xrightarrow{s_1} s_1, id \xrightarrow{s_1} s_1\}.$$

All other simplices of $\Gamma^{(2)}(2)$ are degenerate. In particular, $|\Gamma^{(2)}(2)| \cong S^1$ and there are two nontrivial homology classes

$$c_0 = [id] \in H_0(\Gamma^{(2)}(2); R)$$

and

$$c_1 = [id \xrightarrow{s_1} s_1 + s_1 \xrightarrow{s_1} id] \in H_1(\Gamma^{(2)}(2); R).$$

Each of these classes gives rise to an arity two operation on the homology of $N_\bullet\mathcal{H}^q$. Using the equivariance axiom, the structure maps (see Definition 53 and Proposition 13) induce a map

$$\alpha^2 : \Gamma^{(2)}(2) \otimes_{\Sigma_2} (N_\bullet\mathcal{H}^q \otimes N_\bullet\mathcal{H}^q) \to N_\bullet\mathcal{H}^q$$

Passing to homology and precomposing with the external product, we obtain a map

$$\alpha^2_\star : H_\star(\Gamma^{(2)}(2); R) \otimes (H_{\star,\star}(N_\bullet\mathcal{H}^q) \otimes H_{\star,\star}(N_\bullet\mathcal{H}^q)) \to H_{\star,\star}(N_\bullet\mathcal{H}^q)$$

Definition 55. *We get two homology operations:*

i) The Pontryagin product is the map

$$- \cdot - = \alpha^2_\star(c_0 \otimes - \otimes -) : H_{n,d}(N_\bullet\mathcal{H}^q) \otimes H_{n',d'}(N_\bullet\mathcal{H}^q) \to H_{n+n',d+d'}(N_\bullet\mathcal{H}^q).$$

ii) The Browder bracket is (up to a sign[3]*) the map*

$$[-,-]_{E_2} = \alpha^2_\star(c_1 \otimes - \otimes -) : H_{n,d}(N_\bullet\mathcal{H}^q) \otimes H_{n',d'}(N_\bullet\mathcal{H}^q) \to H_{n+n',d+d'+1}(N_\bullet\mathcal{H}^q)$$

With these definitions and using homology calculations from Chapter 3, Lemma 31, the following theorem is readily verified.

Theorem 43. *Let $q = -1$ and assume that $2 \neq 0 \in R$, then the E_2-algebra algebra structure of $N_\bullet\mathcal{H}^q$ given by Theorem 42 does not extend to an E_3-structure.*

Proof: Let $\eta = [1 \otimes 1] \in H_{0,1}(N_\bullet\mathcal{H}^q) = H_0(\mathcal{H}_1^q; \mathbb{1}) \cong R$ be a generator. If follows from Chapter 3, Lemma 31 for $|F| = 1$, that

$$H_{1,2}(N_\bullet\mathcal{H}^{(-1)}) = H_1(\mathcal{H}_2^q; \mathbb{1}^{(-1)}) \cong R.$$

We will show that $[\eta, \eta]_{E_2} \in H_{1,2}(N_\bullet\mathcal{H}^q)$ is nonzero, then the claim follows.

$$\begin{aligned}[\eta,\eta] &= (\alpha^2)_\star([id \xrightarrow{s_1} s_1 + s_1 \xrightarrow{s_1} id] \otimes \eta \otimes \eta) \\ &= [q^{-1}T_{s_1}] + [q^{-1}T_{s_1}] \\ &= -2[T_{s_1}]\end{aligned}$$

But $[T_{s_1}] \in H_1(\mathcal{H}_2^q; \mathbb{1}) \cong R$ is a generator (see Chapter 3, Proof of Lemma 31) and by assumption $2 \neq 0$ in R. Hence, the claim follows. □

[3]The sign is not relevant for the purpose of this section.

5. Appendix: Enriched categories, functors and natural transformations

This appendix contains the basic notions of enriched category theory specialized to Mod_R-enriched categories.

DEFINITION 56 ([Rie14], Definition 3.3.1). *A small Mod_R-category $\mathcal{A}$ consists of*

i) a set of objects $\mathcal{A}$
ii) for each pair $x, y \in \mathcal{A}$, a morphism module $\mathcal{A}(x,y) \in Mod_R$
iii) for each $x \in \mathcal{A}$ a morphism $id_x : R \to \mathcal{A}(x,x)$ in Mod_R
iv) for each triple $x, y, z \in \mathcal{A}$, a morphism $(- \cdot -) : \mathcal{A}(y,z) \otimes \mathcal{A}(x,y) \to \mathcal{A}(x,z)$ in Mod_R

such that the following diagrams in Mod_R commute for all $x, y, z, w \in \mathcal{A}$:

$$\begin{array}{ccc}
\mathcal{A}(z,w) \otimes \mathcal{A}(y,z) \otimes \mathcal{A}(x,y) & \xrightarrow{1\otimes(-\cdot-)} & \mathcal{A}(z,w) \otimes \mathcal{A}(x,z) \\
\downarrow{\scriptstyle (-\cdot-)\otimes 1} & & \downarrow{\scriptstyle (-\cdot-)} \\
\mathcal{A}(y,w) \otimes \mathcal{A}(x,y) & \xrightarrow{(-\cdot-)} & \mathcal{A}(x,w)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{A}(x,y) \otimes R & \xrightarrow{1\otimes id_x} & \mathcal{A}(x,y) \otimes \mathcal{A}(x,x) \\
& \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle (-\cdot-)} \\
& & \mathcal{A}(x,y)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{A}(y,y) \otimes \mathcal{A}(x,y) & \xleftarrow{id_y\otimes 1} & R \otimes \mathcal{A}(x,y) \\
{\scriptstyle (-\cdot-)}\downarrow & \swarrow{\scriptstyle \cong} & \\
\mathcal{A}(x,y) & &
\end{array}$$

DEFINITION 57 ([Rie14], Definition 3.5.1). *A Mod_R-functor $F : \mathcal{A} \to \mathcal{A}'$ between Mod_R-categories is given by an object map $\mathcal{A} \ni x \mapsto Fx \in \mathcal{A}'$ together with morphisms in Mod_R*

$$F_{x,y} : \mathcal{A}(x,y) \to \mathcal{A}(Fx,Fy)$$

for each $x, y \in \mathcal{A}$ such that the following diagrams commute for all $x, y, z \in \mathcal{A}$:

$$\begin{array}{ccc}
\mathcal{A}(y,z) \otimes \mathcal{A}(x,y) & \xrightarrow{(-\cdot-)} & \mathcal{A}(x,z) \\
\downarrow{\scriptstyle F_{y,z}\otimes F_{x,y}} & & \downarrow{\scriptstyle F_{x,z}} \\
\mathcal{A}'(Fy,Fz) \otimes \mathcal{A}'(Fx,Fy) & \xrightarrow{(-\cdot-)} & \mathcal{A}'(Fx,Fz)
\end{array}
\qquad
\begin{array}{ccc}
R & \xrightarrow{id_x} & \mathcal{A}(x,x) \\
& \searrow{\scriptstyle id_{Fx}} & \downarrow{\scriptstyle F_{x,x}} \\
& & \mathcal{A}'(Fx,Fx)
\end{array}$$

DEFINITION 58 ([Rie14], Definition 3.5.8). *A Mod_R-natural transformation $\alpha : F \Rightarrow G$ between Mod_R-functors $F, G : \mathcal{A} \to \mathcal{A}'$ consists of morphisms $\alpha_x : R \to \mathcal{A}'(Fx, Gx)$ in Mod_R for each $x \in \mathcal{A}$ such that for all $x, y \in \mathcal{A}$ the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{A}(x,y) & \xrightarrow{F_{x,y}} & \mathcal{A}'(Fx,Fy) \\
{\scriptstyle G_{x,y}}\downarrow & & \downarrow{\scriptstyle (\alpha_y)_\star} \\
\mathcal{A}'(Gx,Gy) & \xrightarrow{(\alpha_x)^\star} & \mathcal{A}'(Fx,Gy)
\end{array}$$

Here,

$$(\alpha_y)_\star : \mathcal{A}'(Fx,Fy) \cong R \otimes \mathcal{A}'(Fx,Fy) \xrightarrow{\alpha_y\otimes 1} \mathcal{A}'(Fy,Gy) \otimes \mathcal{A}'(Fx,Fy) \xrightarrow{(-\cdot-)} \mathcal{A}'(Fx,Gy)$$

and $(\alpha_x)^\star$ is defined similarly.

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CHAPTER 5

An alternative proof of a theorem of Gunnells

The content of this chapter is based on joint work with Benjamin Brück and Peter Patzt. All mistakes contained in this write-up have been made by the author of the thesis.

Summary: This chapter implements an idea of Putman for a new proof of a theorem of Gunnells. The theorem states that the Steinberg module of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ is generated by integral apartment classes. We explain how one can combine connectivity results due to Putman with techniques developed by Church, Farb and Putman to prove this result and show that it implies that the rational top-cohomology of the symplectic group $H^{n^2}(\mathrm{Sp}_{2n}(\mathbb{Z});\mathbb{Q})$ vanishes. Here, n^2 is the virtual cohomological dimension of $\mathrm{Sp}_{2n}(\mathbb{Z})$.

1. Introduction

In 2014, Church, Farb and Putman [**CFP14**] formulated a set of vanishing conjectures for the high-dimensional unstable rational cohomology of special linear groups, mapping class groups of surfaces and automorphism groups of free groups.

CONJECTURE 1 ([**CFP14**]). *Let Γ_n be the special linear group $\mathrm{SL}_n(\mathbb{Z})$, the mapping class group MCG_n or the automorphism group of free groups $\mathrm{Aut}(\mathrm{F_n})$. Then*

$$H^{v_n-i}(\Gamma_n;\mathbb{Q})=0$$

whenever $n \gg i$ and where v_n is the virtual cohomological dimension of Γ_n.

Since then, Conjecture 1 has been investigated by many authors and in different settings (e.g. [**KMPW21, MPWY20**]). For special linear groups, the conjecture has been proven in codimension $i = 0$ by Lee–Szczarba [**LS76**], Theorem 4.1, and recently in codimension $i = 1$ by Church–Putman [**CP17**]. For mapping class groups, the conjecture has been disproved by Chan–Galatius–Payne [**CGP21**]. This chapter concerns the analogue of Conjecture 1 for the symplectic groups $\mathrm{Sp_{2n}}(\mathbb{Z})$.

It follows from work of Borel–Serre [**BS73**], §11.4., that the virtual cohomological dimension of $\mathrm{Sp_{2n}}(\mathbb{Z})$ is

$$\mathrm{vcd}(\mathrm{Sp_{2n}}(\mathbb{Z}))=n^2.$$

This implies that $H^k(\mathrm{Sp_{2n}}(\mathbb{Z});\mathbb{Q})=0$ for $k>n^2$.

QUESTION 2. *Is it true that*

$$H^{n^2-i}(\mathrm{Sp_{2n}}(\mathbb{Z});\mathbb{Q})=0$$

whenever $n \gg i$ and where n^2 is the virtual cohomological dimension of $\mathrm{Sp_{2n}}(\mathbb{Z})$?

In codimension $i = 0$, the main result of Gunnells' paper [**Gun00**], 4.11. Theorem, implies an affirmative answer to Question 2.

THEOREM 44 (cf. [**Gun00**], Corollary 4.12.). *Let $n \geq 1$, then $H^{n^2}(\mathrm{Sp_{2n}}(\mathbb{Z});\mathbb{Q})=0$.*

The goal of this chapter is to implement an idea of Putman [**Put21**] for a new proof of Gunnells' main result in the special case $\mathrm{Sp_{2n}}(\mathbb{Z})$ (see Theorem 45 and Remark 12) and explain how Theorem 44 can be deduced from it.

Gunnells' algorithmic strategy of proof in [**Gun00**] was inspired by work of Ash–Rudolph for special linear groups [**AR79**]. The argument presented in this chapter is in the same spirit as the approach that Church–Farb–Putman and Church–Putman used for proving versions of Conjecture 1 for special linear groups in [**CFP19**] and [**CP17**]. This strategy requires to show that a certain simplicial complex is highly connected. Finding a suitable complex and proving high connectivity is, in general, the hard part of the argument. For the codimension $i = 0$ case of the symplectic groups $\mathrm{Sp_{2n}}(\mathbb{Z})$ presented here, the main difficult connectivity calculations have already been carried out by Putman in [**Put09**].

Outline of this chapter. In the first section, we introduce the necessary background and state Gunnells' main theorem ([**Gun00**], 4.11. Theorem). The second section explains how Theorem 44 follows from Gunnells' theorem (cf. [**Gun00**], 4.12. Corollary). The third section introduces a new complex, which we call the restricted Tits building. Section 4 introduces the complex $\mathcal{I}_n^{\sigma,\delta}$, which Putman studied in [**Put09**], and the complex $\mathcal{IA}_n$, which we will use in the proof of Gunnells' theorem. We show that high-connectivity of $\mathcal{IA}_n$ follows from Putman's connectivity results for $\mathcal{I}_n^{\sigma,\delta}$. Furthermore, we explain how one can combine connectivity results obtained by Church–Putman [**CP17**] with the results deduced in Section 3 to give an alternative proof of the first step of Putman's connectivity calculation for

$\mathcal{I}_n^{\sigma,\delta}$. The last section contains the proof of Gunnells' theorem.

Future work. This chapter grew out of a project with Benjamin Brück and Peter Patzt in which we aim to address a question that Putman posed during the problem session of the conference 'Stability in Topology, Arithmetic, and Representation Theory 2020' [**MPP20**]. As part of this project, we try to answer Question 2 in codimension $i = 1$ using the approach of Church, Farb and Putman. The codimension $i = 0$ case presented in this chapter is the first step.

Acknowledgments. It is a pleasure to thank Andrew Putman for posing the question that led to this work, for sharing his idea for a new proof of Gunnells' theorem [**Put21**] and for his permission to implement it in this thesis chapter. I am grateful to my coauthors, Benjamin Brück and Peter Patzt, for many enlightening conversations and for allowing me to present parts of our joint work in this thesis.

2. Integral apartment classes and Gunnells' theorem

In this section, we introduce the symplectic group $\mathrm{Sp_{2n}}(\mathbb{Z})$ and the symplectic Steinberg module $\mathrm{St}^\omega_\mathrm{n}$, which is a certain $\mathrm{Sp_{2n}}(\mathbb{Z})$-module. To formulate Gunnells' theorem, Theorem 45, we explain the construction of the "integral apartment class map" following Section 3 of [**Gun00**]. This is a map of the form

$$\mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to \mathrm{St}^\omega_\mathrm{n}\,.$$

Gunnells' theorem states that this map is a surjection, if $n \geq 1$.

The symplectic Steinberg module. Consider the vector space $\mathbb{Q}^{2n}$ equipped with the standard symplectic form ω, i.e. the skew-symmetric, non-degenerate bilinear form which on the standard basis $\{e_0, \ldots, e_{n-1}, f_{n-1}, \ldots, f_0\}$ evaluates to

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \text{ for } i, j \in \{0, \ldots, n-1\}$$
$$\omega(e_i, f_j) = 0 \text{ for } i \neq j \in \{0, \ldots, n-1\}$$
$$\omega(e_i, f_i) = -\omega(f_i, e_i) = 1 \text{ for } i \in \{0, \ldots, n-1\}$$

The symplectic group $\mathrm{Sp_{2n}}(\mathbb{Q})$ is the group of linear automorphisms of $\mathbb{Q}^{2n}$ that preserve ω.

DEFINITION 59. *A subspace $V \subseteq \mathbb{Q}^{2n}$ is called isotropic, if $\omega|_V$ is zero.*

DEFINITION 60. *The symplectic Tits building $T^\omega_n(\mathbb{Q})$ is the poset of non-trivial isotropic subspaces of $\mathbb{Q}^{2n}$ ordered by inclusion. The order complex of this poset, which we also denote by $T^\omega_n(\mathbb{Q})$, is an ordered simplicial complex with k-simplices given by the following set of flags:*

$$\{0 \neq V_0 \subsetneq \cdots \subsetneq V_k : V_i \leq \mathbb{Q}^{2n} \textit{ isotropic}\}$$

The i-th face of a k-simplex is obtained by omitting the i-th isotropic subspace V_i of the flag.

This complex has dimension $n-1$ and admits a natural simplicial action by $\mathrm{Sp_{2n}}(\mathbb{Q})$. By a theorem of Solomon–Tits, the reduced homology of $T^\omega_n(\mathbb{Q})$ is concentrated in dimension $n-1$ (see [**Sol69**]). It follows that the top-dimensional homology group is free as a $\mathbb{Z}$-module.

DEFINITION 61. *The symplectic Steinberg module $\mathrm{St}^\omega_\mathrm{n}$ is the $\mathrm{Sp_{2n}}(\mathbb{Q})$-module that arises as the top-homology of the symplectic Tits building:*

$$\mathrm{St}^\omega_\mathrm{n} = \tilde{\mathrm{H}}_{n-1}(T^\omega_n(\mathbb{Q}); \mathbb{Z})$$

Integral apartment classes. The integral symplectic group $\mathrm{Sp_{2n}}(\mathbb{Z})$ is obtained by restriction from the group $\mathrm{Sp_{2n}}(\mathbb{Q})$. It is the group of linear automorphisms of $\mathbb{Z}^{2n}$ that preserve the symplectic form ω. Gunnells' theorem concerns a map $\mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to \mathrm{St}^\omega_\mathrm{n} : M \mapsto [M]$, which sends an integral symplectic matrix M to its integral apartment class $[M]$. We will explain the construction of this map now, following Section 3 of [**Gun00**].

DEFINITION 62. *Let* $[\![n]\!] := \{0, \dots, n-1, \overline{n-1}, \dots, \bar{0}\}$. *A nonempty subset* $I \subseteq [\![n]\!]$ *is called a standard subset if for all* $0 \leq a \leq n-1 : \{a, \bar{a}\} \not\subset I$. *We denote by* $\partial\beta_n$ *the following simplicial complex: Its vertex set is* $[\![n]\!]$ *and its* k*-simplices are the standard subsets* $I \subset [\![n]\!]$ *of size* $k+1$.

Observe that $\partial\beta_1 = \{0, \bar{0}\} \cong S^0$ and that the inclusion of the vertex sets $[\![n]\!] \subseteq [\![n+1]\!]$ induces an inclusion of simplicial complexes $\partial\beta_n \hookrightarrow \partial\beta_{n+1}$ for any $n \in \mathbb{N}$. It is readily verified that $\partial\beta_{n+1}$ is exactly the simplicial join $\partial\beta_n * \{n, \bar{n}\}$. It follows that $\partial\beta_n \cong *_0^{n-1} S^0$ is a sphere of dimension $n-1$ and that $\partial\beta_{n+1} = \partial\beta_n * \{n, \bar{n}\}$ is obtained from $\partial\beta_n$ by suspension. We fix a fundamental class $\xi = \xi_0 \in \tilde{\mathrm{H}}_0(\partial\beta_1; \mathbb{Z})$ once and for all. Using the suspension isomorphism, this class gives rise to fundamental classes $\xi = \xi_{n-1} \in \tilde{\mathrm{H}}_{n-1}(\partial\beta_n; \mathbb{Z})$ for all $n \in \mathbb{N}$.

Given an integral symplectic matrix $M \in \mathrm{Sp_{2n}}(\mathbb{Z}) \leq \mathrm{Sp_{2n}}(\mathbb{Q})$, its column vectors form a symplectic basis of $\mathbb{Q}^{2n}$. We may index the $2n$ column vectors from left to right by $[\![n]\!] = \{0, \dots, n-1, \overline{n-1}, \dots, \bar{0}\}$. By the definition of the symplectic form ω, this indexing $M = (M_a)_{a \in [\![n]\!]}$ has the property that:

If $I \in \partial\beta_n$ is a simplex, then $M_I = \mathrm{span}_{\mathbb{Q}}\{M_a : a \in I\} \leq \mathbb{Q}^{2n}$ is an isotropic summand.

This implies that for every $M \in \mathrm{Sp_{2n}}(\mathbb{Z})$, we can define a poset map

$$\partial M : P(\partial\beta_n) \to T_n^{\omega}(\mathbb{Q}) : I \mapsto M_I,$$

where $P(\partial\beta_n)$ denotes the poset of simplices of $\partial\beta_n$. Note that after passing to ordered simplicial complexes, this defines a simplicial embedding. Recall that the order complex of the poset $P(\partial\beta_n)$ is the barycentric subdivision of $\partial\beta_n$. It follows that the image of the map ∂M is a subcomplex that is homeomorphic to a $(n-1)$-sphere. Such a subcomplex is called an integral apartment of the Tits building T_n^{ω}. Taking homology, we obtain a map

$$\partial M_\star : \tilde{\mathrm{H}}_{n-1}(P(\partial\beta_n); \mathbb{Z}) \to \mathrm{St}_n^{\omega}.$$

Barycentric subdivision of simplicial complexes comes with a natural homology isomorphism on chain level $b : C_\star \to C_\star \circ P$, where $C_\star$ assigns an ordered simplicial complex its simplicial chain complex with trivial $\mathbb{Z}$-coefficients. Using the induced isomorphism

$$b : \tilde{\mathrm{H}}_{n-1}(\partial\beta_n; \mathbb{Z}) \to \tilde{\mathrm{H}}_{n-1}(P(\partial\beta_n); \mathbb{Z})$$

and we obtain a unique class $b(\xi) \in \tilde{\mathrm{H}}_{n-1}(P(\partial\beta_n); \mathbb{Z})$ for every $n \in \mathbb{N}$.

DEFINITION 63. *The integral apartment class* $[M] \in \mathrm{St}_n^{\omega}$ *of* $M \in \mathrm{Sp_{2n}}(\mathbb{Z})$ *is defined to be the value of* $\partial M_\star : \tilde{\mathrm{H}}_{n-1}(P(\partial\beta_n); \mathbb{Z}) \to \mathrm{St}_n^{\omega}$ *at* $b(\xi)$*:*

$$[M] := \partial M_\star(b(\xi))$$

This defines a map

$$[-] : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to \mathrm{St}_n^{\omega} : M \mapsto [M]$$

which we called the integral apartment class map.

REMARK 11. *The terminology that we use in this chapter is in analogy with the one that Church, Farb and Putman use in* [**CFP19**]*. In Gunnells' paper* [**Gun00**]*, the integral apartment classes introduced in Definition 63 are called symplectic unimodular symbols.*

With this construction, we can now state the main theorem that Gunnells proved in [**Gun00**].

THEOREM 45 (Gunnells' theorem; [**Gun00**] 4.11 Theorem). *The integral apartment class map*

$$[-] : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to \mathrm{St}_n^{\omega} : M \mapsto [M]$$

is a surjection. In other words, the set of integral apartment classes

$$\{[M]\}_{M \in \mathrm{Sp_{2n}}(\mathbb{Z})}$$

is a generating system for the abelian group $\mathrm{St}_\mathrm{n}^\omega$.

REMARK 12. *Theorem 45 is the special case* $\mathcal{O} = \mathbb{Z}$ *of Gunnells' result ([**Gun00**], 4.11 Theorem). In the paper [**Gun00**], Gunnells allows any euclidean ring of integers* $\mathcal{O}$ *of a number field* $K/\mathbb{Q}$.

REMARK 13. *The construction of apartment classes described above also works if one starts with an element in the rational symplectic group* $M \in \mathrm{Sp_{2n}}(\mathbb{Q})$. *This leads to the definition of a* rational *apartment class map*

$$[-] : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Q})] \to \mathrm{St}_\mathrm{n}^\omega\,.$$

*It follows from the proof of the Solomon–Tits theorem (see [**Sol69**] or [**Bro89**], IV.5 Theorem 2) that this map is a surjection. Gunnells' theorem states that the restriction of this map to the "much smaller" group ring* $\mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})]$ *is still a surjection.*

Gunnells' strategy of proof for Theorem 45 was inspired by work of Ash–Rudolph [**AR79**] and is based on the content of Remark 13. The general idea is to devise an algorithm that takes as input a *rational* apartment classes $[M] \in \mathrm{St}_\mathrm{n}^\omega$ for $M \in \mathrm{Sp_{2n}}(\mathbb{Q})$ and outputs a linear combination of *integral* apartment classes that is equal to $[M]$. In Section 6 of this chapter, we will give an alternative proof of Theorem 45 following ideas of Church, Farb and Putman.

3. Vanishing of the top-dimensional rational cohomology

The goal of this section is to explain how Gunnells' theorem (see Theorem 45) implies the vanishing of the top-dimensional rational cohomology of $\mathrm{Sp_{2n}}(\mathbb{Z})$ (see Theorem 44). The argument we present here is in analogy with the one given by Church–Farb–Putman in [**CFP19**].

Recall from the introduction that n^2 is the virtual cohomological dimension of $\mathrm{Sp_{2n}}(\mathbb{Z})$. By Theorem 11.4.1. of Borel–Serre [**BS73**] the group $\mathrm{Sp_{2n}}(\mathbb{Z})$ is a virtual duality group in the sense of Bieri–Eckmann [**BE73**], Definition 1.1. This means that there exists a rational dualizing $\mathbb{Q}\,\mathrm{Sp_{2n}}(\mathbb{Z})$-module D with the property that

$$H^{n^2-i}(\mathrm{Sp_{2n}}(\mathbb{Z}); M) \cong H_i(\mathrm{Sp_{2n}}(\mathbb{Z}); M \otimes_\mathbb{Q} D)$$

for any $\mathbb{Q}\,\mathrm{Sp_{2n}}(\mathbb{Z})$-module M and where $\mathrm{Sp_{2n}}(\mathbb{Z})$ acts diagonally on $M \otimes_\mathbb{Q} D$. Borel–Serre also identified the rational dualizing module. It is the rational top-homology of the symplectic Tits building introduced in Definition 61:

$$D = \mathbb{Q} \otimes_\mathbb{Z} \mathrm{St}_\mathrm{n}^\omega$$

For trivial rational coefficients, we therefore have that

$$H^{n^2-i}(\mathrm{Sp_{2n}}(\mathbb{Z}); \mathbb{Q}) \cong H_i(\mathrm{Sp_{2n}}(\mathbb{Z}); \mathbb{Q} \otimes_\mathbb{Z} \mathrm{St}_\mathrm{n}^\omega)$$

where $\mathrm{Sp_{2n}}(\mathbb{Z})$ acts diagonally on $\mathbb{Q} \otimes_\mathbb{Z} \mathrm{St}_\mathrm{n}^\omega$. This makes it possible to study the high-dimensional rational cohomology of symplectic groups by constructing suitable resolutions of the symplectic Steinberg module. Church–Farb–Putman used this perspective to study the high-dimensional cohomology of special linear groups. Here, we use the generating set provided by Theorem 45 to deduce Theorem 44 (cf. [**Gun00**], 4.12. Corollary).

Proof of Theorem 44. Let $n \geq 1$. The results of Borel–Serre discussed above imply that

$$H^{n^2}(\mathrm{Sp_{2n}}(\mathbb{Z}); \mathbb{Q}) \cong H_0(\mathrm{Sp_{2n}}(\mathbb{Z}); \mathbb{Q} \otimes_\mathbb{Z} \mathrm{St}_\mathrm{n}^\omega) \cong (\mathbb{Q} \otimes_\mathbb{Z} \mathrm{St}_\mathrm{n}^\omega)_{\mathrm{Sp_{2n}}(\mathbb{Z})}.$$

It therefore suffices to prove that the coinvariants of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{St}^{\omega}_{\mathrm{n}}$ vanish, i.e.

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{St}^{\omega}_{\mathrm{n}})_{\mathrm{Sp_{2n}}(\mathbb{Z})} = 0.$$

Gunnells' theorem (see Theorem 45) states that the set of integral apartment classes $\{[M]\}_{M \in \mathrm{Sp_{2n}}(\mathbb{Z})}$ is a generating set of $\mathrm{St}^{\omega}_{\mathrm{n}}$. Theorem 44 therefore follows, if for any integral apartment class $[M]$ and any element $q \in \mathbb{Q}$, it holds that

$$q \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} [M] = 0 \in \mathbb{Q} \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} \mathrm{St}^{\omega}_{\mathrm{n}} = (\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{St}^{\omega}_{\mathrm{n}})_{\mathrm{Sp_{2n}}(\mathbb{Z})}.$$

Let $M \in \mathrm{Sp_{2n}}(\mathbb{Z})$ be an integral symplectic matrix and index its column vectors $(M_a)_{a \in [\![n]\!]}$ as in Section 2 (see discussion after Definition 62). Consider the symplectic map

$$S : \mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$$

that "swaps" the column vectors M_0 and $M_{\bar{0}}$, i.e. the map given by:

$$\begin{cases} M_0 \mapsto -M_{\bar{0}} \\ M_{\bar{0}} \mapsto M_0 \\ M_a \mapsto M_a, \text{ if } a \neq 0 \\ M_{\bar{a}} \mapsto M_{\bar{a}}, \text{ if } a \neq 0 \end{cases}$$

Acting by S on the Steinberg module reverses the orientation of the apartment class $[M]$:

$$S \cdot [M] = -[M] \in \mathrm{St}^{\omega}_{\mathrm{n}}$$

It follows that

$$q \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} [M] = q \cdot S \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} [M] = q \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} S \cdot [M] = -(q \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} [M])$$

and therefore, that $q \otimes_{\mathbb{Z}\,\mathrm{Sp_{2n}}(\mathbb{Z})} [M] = 0$. □

4. The restricted Tits building of symplectic groups

Let $\{a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n, b_n\}$ be a symplectic basis of $\mathbb{Q}^{2n}$ with $\omega(a_i, b_j) = \delta_{i,j}$ and fix the subspace

$$W = \langle a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n \rangle_{\mathbb{Q}}$$

of $\mathbb{Q}^{2n}$. Let $T^{\omega}_n = T^{\omega}(\mathbb{Q}^{2n})$ be the symplectic Tits building (see Definition 60) and consider the subposet

$$\mathcal{T}^{\omega}(W) = \{V \in \mathcal{T}^{\omega}_n : V \subseteq W\}$$

of isotropic subspaces contained in W. We call $\mathcal{T}^{\omega}(W)$ the restricted Tits building of symplectic groups. The construction is motivated by the strategy that Putman employed to prove high connectivity of the complex $\mathcal{I}^{\sigma,\delta}_n$ in [**Put09**]. We will explain this in the next section. The goal of this section is to prove the following proposition.

Proposition 15. *$\mathcal{T}^{\omega}(W)$ is a contractible Cohen–Macaulay poset of dimension $n-1$.*

For any subspace $H \subseteq \mathbb{Q}^{2n}$, let

$$H^{\perp} = \{v \in \mathbb{Q}^{2n} : \omega(v, h) = 0 \text{ for all } h \in H\}$$

denote the symplectic complement of H in $\mathbb{Q}^{2n}$. The following two observations are the main ingredients of the proof of Proposition 15.

Observation 11. *If $V \in \mathcal{T}^{\omega}(W)$, then $\langle a_n \rangle + V \in \mathcal{T}^{\omega}(W)$.*

Proof: Observe that $W \subseteq \langle a_n \rangle^{\perp}$, hence $V \subseteq \langle a_n \rangle^{\perp}$. It follows that $\langle a_n \rangle + V \subseteq W$ is isotropic. □

The second observation is similar to Lemma 4.2 of [**SW20**][1].

[1]If $\langle a_n \rangle \subseteq Q$, then $\mathcal{T}^{\omega}(W)_{>Q}$ is the upper fiber in their building $\mathcal{T}(\mathbb{Q}^{2n}, \omega)$.

Observation 12. *If* $\langle a_n \rangle \subseteq Q \in \mathcal{T}^\omega(W)$*, then* $\mathcal{T}^\omega(W)_{>Q} \cong T^\omega(\langle a_n, b_n\rangle^\perp)_{>Q\cap\langle a_n,b_n\rangle^\perp}$*. For the case* $Q = \langle a_n \rangle$*, we set* $T^\omega(\langle a_n, b_n\rangle^\perp)_{>Q\cap\langle a_n,b_n\rangle^\perp} := T^\omega(\langle a_n, b_n\rangle^\perp)$*.*

Proof: Note that any $V \in \mathcal{T}^\omega(W)_{>Q}$ admits a direct sum decomposition

$$V = \langle a_n \rangle \oplus (V \cap \langle a_n, b_n\rangle^\perp).$$

The poset maps

$$\mathcal{T}^\omega(W)_{>Q} \to T^\omega(\langle a_n, b_n\rangle^\perp)_{>Q\cap\langle a_n,b_n\rangle^\perp} : V \mapsto V \cap \langle a_n, b_n\rangle^\perp$$

and

$$T^\omega(\langle a_n, b_n\rangle^\perp)_{>Q\cap\langle a_n,b_n\rangle^\perp} \to \mathcal{T}^\omega(W)_{>Q} : V \mapsto \langle a_n \rangle \oplus V$$

are therefore inverses of each other. □

Lemma 46. $\mathcal{T}^\omega(W)$ *is contractible.*

Proof: The poset map $f : \mathcal{T}^\omega(W) \to \mathcal{T}^\omega(W) : V \mapsto \langle a_n \rangle + V$ is well-defined by Observation 11 and has the property that $V \le f(V)$ for all $V \in \mathcal{T}^\omega(W)$. It follows from [**Qui78**], §1.5, that $\mathcal{T}^\omega(W)$ is homotopy equivalent to $\operatorname{im}(f)$ and that $\operatorname{im}(f)$ is contractible using the cone point $\langle a_n \rangle$. □

Lemma 47. $\mathcal{T}^\omega(W)$ *is a Cohen–Macaulay poset of dimension* $n-1$*.*

Proof: We need to see that for any $Q' \subseteq Q \in \mathcal{T}^\omega(W)$, $\mathcal{T}^\omega(W)_{<Q}$ is $(\dim Q - 2)$-spherical, the interval (Q', Q) is $(\dim Q - \dim Q' - 2)$-spherical and $\mathcal{T}^\omega(W)_{>Q}$ is $(n - \dim Q - 1)$-spherical.

Connectivity of the lower link and the interval: Note that $\mathcal{T}^\omega(W)_{<Q} \cong T(Q)$ is the poset of nontrivial proper subspaces of Q. This is the special linear Tits building on Q and a Cohen–Macaulay poset of dimension $(\dim Q - 2)$ (see [**Sol69**] and [**Bro89**], IV.5 Remark 2). Therefore $\mathcal{T}^\omega(W)_{<Q}$ is $(\dim Q - 2)$-spherical and (Q', Q) is $((\dim Q - 2) - (\dim Q' - 1) - 1) = (\dim Q - \dim Q' - 2)$-spherical.

Connectivity of the upper link: We consider two cases.

i) Assume that $\langle a_n \rangle \nsubseteq Q$. Then $\langle a_n \rangle + Q \in \mathcal{T}^\omega(W)_{>Q}$ is a cone point of the image of the monotone poset map $f : V \mapsto \langle a_n \rangle + V$ on $\mathcal{T}^\omega(W)_{>Q}$. It follows from [**Qui78**], §1.5, that $\mathcal{T}^\omega(W)_{>Q}$ is contractible and $(n - \dim Q - 1)$-spherical.

ii) Assume that $\langle a_n \rangle \subseteq Q$. Then Observation 12 yields the identification

$$\mathcal{T}^\omega(W)_{>Q} \cong T^\omega(\langle a_n, b_n\rangle^\perp)_{>Q\cap\langle a_n,b_n\rangle^\perp}.$$

But $T^\omega(\langle a_n, b_n\rangle^\perp)$ is Cohen–Macaulay of dimension $(n-2)$ (see [**Sol69**] and [**Bro89**], IV.5 Remark 2). Therefore, $\mathcal{T}^\omega(W)_{>Q}$ is $((n-2) - (\dim(Q \cap \langle a_n, b_n\rangle^\perp) - 1) - 1) = (n - \dim Q - 1)$-spherical.

□

5. From special linear groups to symplectic groups

This section introduces the simplicial complex $\mathcal{IA}_n$ that will play a key role in the proof of Gunnells' theorem (see Theorem 45) presented in the next section. The simplicial complex $\mathcal{IA}_n$ contains a subcomplex $\mathcal{I}_n^{\sigma,\delta} \hookrightarrow \mathcal{IA}_n$, which has been studied by Putman in [**Put09**]. The goal of this section is twofold. After defining simplicial complexes related to $\mathcal{I}_n^{\sigma,\delta}$ and $\mathcal{IA}_n$, we will outline the strategy that Putman used in [**Put09**] to prove that $\mathcal{I}_n^{\sigma,\delta}$ is highly connected. Our first goal is to give an alternative argument for the first step of Putman's argument by combining the results for the restricted Tits building $\mathcal{T}^\omega(W)$ obtained in the last section with connectivity calculations of Church–Putman [**CP17**]. Our second goal is to show that the complex $\mathcal{IA}_n$ can be constructed from $\mathcal{I}_n^{\sigma,\delta}$ by attaching simplices along highly connected links. As a consequence, Putman's connectivity result for $\mathcal{I}_n^{\sigma,\delta}$ implies that $\mathcal{IA}_n$ is highly connected as well. The high-connectivity of $\mathcal{IA}_n$ and the link structure of $\mathcal{IA}_n$ are exactly the properties that will make the induction argument in the proof of Gunnells' theorem in the next section work.

Let $\mathbb{Z}^{2n} \subset \mathbb{Q}^{2n}$ be equipped with the standard symplectic form ω and denote its standard symplectic basis by $\{e_0, \dots, e_{n-1}, f_{n-1}, \dots, f_0\}$.

Definition 64. *Let V_n be the set $V_n := \{L \leq \mathbb{Z}^{2n} : L \text{ is a rank 1 summand of } \mathbb{Z}^{2n}\}$. A subset*

$$\Delta = \{L_0, \dots, L_k\} \subset V_n$$

of $k+1$ lines $L_i = \langle \pm v_i \rangle_{\mathbb{Z}}$ is called

- *a standard k-simplex, if $\langle \pm v_i : 0 \leq i \leq k \rangle_{\mathbb{Z}}$ is an isotropic rank $k+1$ summand of $\mathbb{Z}^{2n}$.*
- *a delta k-simplex, if $v_0 = \pm v_1 \pm v_2$ and $\Delta - \{L_0\}$ is a standard $(k-1)$-simplex.*
- *a sigma k-simplex, if $\omega(v_k, v_{k-1}) = \pm 1, \omega(v_k, v_i) = 0$ for $0 \leq i \leq k-2$ and $\Delta - \{L_k\}$ is a standard $(k-1)$-simplex.*
- *a mixed k-simplex, if $\Delta - \{L_0\}$ is a sigma $(k-1)$-simplex, $\Delta - \{L_k\}$ is a delta $(k-1)$-simplex and $\omega(v_0, v_k) = 0$.*

Definition 65. *The simplicial complexes $\mathcal{I}_n, \mathcal{I}_n^{\delta}, \mathcal{I}_n^{\sigma,\delta}$ and $\mathcal{IA}_n$ have V_n as their vertex set and*

- *the k-simplices of $\mathcal{I}_n$ are all standard.*
- *the k-simplices of $\mathcal{I}_n^{\delta}$ are all either standard or delta.*
- *the k-simplices of $\mathcal{I}_n^{\sigma,\delta}$ are all either standard, delta or sigma.*
- *the k-simplices of $\mathcal{IA}_n$ are all either standard, delta, sigma or mixed.*

Definition 66. *Let $W = \langle e_0, f_0, \dots, e_{n-2}, f_{n-2}, e_{n-1} \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}^{2n}$ be as in Proposition 15. We write $\mathcal{I}^{\delta}(W)$ for the full subcomplex of $\mathcal{I}_n^{\delta}$ on the vertex set $V_n \cap W$.*

The complexes $\mathcal{I}_n, \mathcal{I}_n^{\delta}, \mathcal{I}_n^{\sigma,\delta}$ and $\mathcal{I}^{\delta}(W)$ have been defined and studied by Putman in [**Put09**]. The following theorem lists the three steps in Putman's proof that $\mathcal{I}_n^{\sigma,\delta}$ is spherical.

Theorem 48 (Putman, [**Put09**], 6.13 (3+4) and 6.11). *Let $n \geq 1$, then:*

i) $\mathcal{I}^{\delta}(W)$ is $(n-1)$-connected.
ii) $\mathcal{I}_n^{\delta} \hookrightarrow \mathcal{I}_n^{\sigma,\delta}$ is the zero map on π_k for $0 \leq k \leq n-1$.
iii) $\mathcal{I}_n^{\sigma,\delta}$ is n-dimensional and $(n-1)$-connected.

Putman made us aware that [**Put09**] contains some fixable issues. We will now give an alternative proof of part $i)$ of Theorem 48 using the restricted Tits building introduced in section 3 (see Proposition 15) and connectivity calculations obtained by Church–Putman in their work on Conjecture 1 for special linear groups [**CP17**].

Remark 14. *The proof presented here automatizes the first step of Putman's strategy for proving high connectivity of $\mathcal{I}_n^{\sigma,\delta}$ assuming one knows the connectivity of the complexes $\mathcal{BA}(\mathbb{Z}^n)$ (see Definition 67) appearing in the proof of the codimension $i = 1$ conjecture for special linear groups* [**CP17**].

We start with following definitions.

Definition 67. *Let $V \in T_n^{\omega}$ be an isotropic subspace of $\mathbb{Q}^{2n}$. Then $V \cap \mathbb{Z}^{2n} \leq \mathbb{Z}^{2n}$ is an isotropic summand of $\mathbb{Z}^{2n}$ (e.g.* [**CP17**], *Lemma 2.4).*

i) Let $\mathcal{B}(V \cap \mathbb{Z}^{2n})$ be the simplicial complex with vertex set

$$\{L \leq V \cap \mathbb{Z}^{2n} : L \text{ is a rank 1 summand of } V \cap \mathbb{Z}^{2n}\}$$

and in which all k-simplices are standard simplices in the sense of Definition 64.

ii) Let $\mathcal{BA}(V\cap\mathbb{Z}^{2n})$ be the simplicial complex on the same vertex set as $\mathcal{B}(V\cap\mathbb{Z}^{2n})$ and in which all k-simplices are standard or delta simplices in the sense of Definition 64.

Let $\mathbb{Z}^n = \langle e_0,\dots,e_{n-1}\rangle_{\mathbb{Z}} = \langle e_0,\dots,e_{n-1}\rangle_{\mathbb{Q}} \cap \mathbb{Z}^{2n}$. Maazen proved that $\mathcal{B}(\mathbb{Z}^n)$ is $(n-2)$-connected [**Maa79**]. This result was refined by Church–Putman (compare with [**CP17**], Section 4.1), who showed that $\mathcal{B}(\mathbb{Z}^n)$ is Cohen–Macaulay of dimension $(n-1)$. The connectivity properties of $\mathcal{BA}(\mathbb{Z}^n)$ have been studied by Church–Putman [**CP17**] and played an important role in their proof of Conjecture 1 in codimension $i=1$ for the special linear groups $\mathrm{SL}_n(\mathbb{Z})$. The results contained in [**CP17**] are summarized in the following theorem.

THEOREM 49 ([**CP17**], Theorem 4.2 and Theorem C).

i) Let $n\geq 1$, then $\mathcal{B}(\mathbb{Z}^n)$ is Cohen–Macaulay of dimension $(n-1)$.
ii) Let $n\geq 2$, then $\mathcal{BA}(\mathbb{Z}^n)$ is Cohen–Macaulay of dimension n.

The following generalization of Quillen's Theorem 9.1 [**Qui78**] due to van der Kallen–Looijenga will allow us to relate the complexes introduced above[2].

THEOREM 50 ([**vdKL11**], Corollary 2.2). *Let $f: X\to Y$ be a poset map, $\theta\in\mathbb{Z}$, and $t: Y\to\mathbb{Z}$ be an increasing (if $y'<y$, then $t(y')<t(y)$) but bounded function. Suppose that for every $y\in Y$, the lower fiber $f_{\leq y}=\{x\in X: f(x)\leq y\}$ is $(t(y)-2)$-connected and that the upper interval $Y_{>y}$ is $(\theta-t(y)-1)$-connected. Then the map f is θ-connected.*

Proof of part $i)$ of Theorem 48. Let $P(\mathcal{I}^\delta(W))$ denote the simplex poset of $\mathcal{I}^\delta(W)$ and let $\mathcal{T}^\omega(W)$ be the restricted Tits building. Consider the spanning map

$$f: P(\mathcal{I}^\delta(W))\to\mathcal{T}^\omega(W): \Delta\mapsto \operatorname{span}_{\mathbb{Q}}\Delta$$

Let $\theta=n$ and define $t:\mathcal{T}^\omega(W)\to\mathbb{Z}: V\mapsto\dim(V)+1$. By Proposition 15, we know that $\mathcal{T}^\omega(W)_{>V}$ is $(n-\dim(V)-2)=(\theta-t(V)-1)$-connected. Furthermore, $f_{\leq V}=\mathcal{BA}(V\cap\mathbb{Z}^{2n})$ is $(\dim(V)-1)=(t(V)-2)$-connected by Theorem 49 for $\dim(V)\geq 2$. For $\dim(V)=1$, $f_{\leq V}=\{V\cap\mathbb{Z}^{2n}\}$ is a point and hence 0-connected. Therefore Theorem 50 implies that f is n-connected. By Proposition 15, the target is contractible, hence $\mathcal{I}^\delta(W)$ is $(n-1)$-connected. □

We end this section by explaining how Putman's connectivity result for $\mathcal{I}_n^{\sigma,\delta}$ (see Theorem 48) implies high-connectivity of $\mathcal{IA}_n$.

COROLLARY 23. *Let $n\geq 1$, then $\mathcal{IA}_n$ is $(n-1)$-connected.*

In the proof of this corollary, we will use the following lemma for which we present an argument using the results above.

Lemma 51 (cf. [**Put09**], Proposition 6.13 (2), and [**vdKL11**], Proposition 1.2.)**.** *The complex $\mathcal{I}_n$ is Cohen–Macaulay of dimension $n-1$.*

Proof: Let $P(\mathcal{I}_n)$ denote the simplex poset of $\mathcal{I}_n$. The lemma follows by considering the spanning map $f: P(\mathcal{I}_n)\to T_n^\omega$ and invoking Corollary 9.7 of Quillen, [**Qui78**]. The application of Quillen's result relies on the fact that T_n^ω is Cohen–Macaulay of dimension $(n-1)$ (see [**Sol69**] and [**Bro89**], IV.5 Remark 2) and the observation that $f_{\leq V}=\mathcal{B}(V\cap\mathbb{Z}^{2n})$, which is Cohen–Macaulay of dimension $(\dim(V)-1)$ by Theorem 49. □

Proof of Corollary 23. Let $\Delta=\{\langle\pm v_0\pm v_1\rangle,\langle v_0\rangle,\langle v_1\rangle,\langle v_2\rangle,\langle w_2\rangle\}$ be a mixed 4-simplex in $\mathcal{IA}_n$ containing the sigma edge $\sigma=\{\langle v_2\rangle,\langle w_2\rangle\}$. Observe that

$$\operatorname{lk}_{\mathcal{IA}_n}(\Delta)\cong\operatorname{lk}_{\mathcal{I}(\sigma^\perp)}(\{\langle v_0\rangle,\langle v_1\rangle\}).$$

[2] We learned about this result from [**GKRW19**], Theorem 4.1.

The previous lemma implies that $\mathrm{lk}_{\mathcal{IA}_n}(\Delta)$ is $(n-5)$-connected. Therefore, the Mayer–Vietoris sequence associated to the homotopy pushout

$$\begin{array}{ccccc}
\bigsqcup_\Delta \Sigma^4 \,\mathrm{lk}(\Delta) & \xrightarrow{\cong} & \bigsqcup_\Delta st_{\mathcal{IA}_n}(\Delta) \cap \mathcal{I}_n^{\sigma,\delta} & \longrightarrow & \mathcal{I}_n^{\sigma,\delta} \\
 & & \big\downarrow & & \big\downarrow \\
 & & \bigsqcup_\Delta st_{\mathcal{IA}_n}(\Delta) & \longrightarrow & \mathcal{IA}_n
\end{array}$$

implies that the inclusion $\mathcal{I}_n^{\sigma,\delta} \hookrightarrow \mathcal{IA}_n$ is n-connected. The result therefore follows from part $iii)$ of Putman's theorem (see Theorem 48). ☐

6. Proof of Gunnells' Theorem

In this final section, we present an alternative proof of Gunnells' theorem, Theorem 45. We start by outlining the general strategy and will divide the proof into two separate steps, formulated below as Proposition 16 and Proposition 17. We will then explain the proof of each proposition.

6.1. Strategy of Proof.

We want to prove that the integral apartment class map

$$[-] : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to \mathrm{St}^\omega_\mathrm{n}$$

is surjective (see Theorem 45). Our strategy will be to factor the integral apartment class map into a composition of four maps and then verify that each of these maps is a surjection. This strategy is analogous to the one employed by Church–Farb–Putman in [**CFP19**]. Gunnells' theorem then follows from the following two propositions, whose proof we will explain in the remainder of this chapter.

Proposition 16. *If $n \geq 1$, there exists a commutative diagram of the following shape*

$$\begin{array}{ccccc}
\mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] & & & \xrightarrow{[-]} & \\
\big\downarrow \alpha & & & & \\
H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) & \xrightarrow{\delta} & H_{n-1}(\mathcal{I}_n^\delta) & & \\
 & & \big\downarrow b & & \\
 & & H_{n-1}(P(\mathcal{I}_n^\delta)) & \xrightarrow{s_\star} & H_{n-1}(T_n^\omega(\mathbb{Q})) = \mathrm{St}^\omega_\mathrm{n}
\end{array}$$

where $[-] : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to \mathrm{St}^\omega_\mathrm{n}$ is the integral apartment class map, δ is the connecting morphism of the long exact sequence of the pair $(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ and b is the homology isomorphism coming from barycentric subdivision.

The morphisms α and $s_\star$ in the statement of Proposition 16 will be defined below.

Proposition 17. *If $n \geq 1$, then the maps occurring in Proposition 16 satisfy:*

(1.) $\alpha : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ *is a surjection.*
(2.) $\delta : H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \to H_{n-1}(\mathcal{I}_n^\delta)$ *is a surjection.*
(3.) $b : H_{n-1}(\mathcal{I}_n^\delta) \to H_{n-1}(P(\mathcal{I}_n^\delta))$ *is an isomorphism.*
(4.) $s_\star : H_{n-1}(P(\mathcal{I}_n^\delta)) \to \mathrm{St}^\omega_\mathrm{n}$ *is an isomorphism.*

Proof of Theorem 45 (Gunnells' theorem): The result follows from Proposition 16 and Proposition 17. ☐

6.2. Definition of relevant morphisms. In this subsection, we will define the morphisms α and $s_\star$ in the statement of Proposition 16. The definition of α will involve the following simplicial complex, which is closely related to the complex $\partial\beta_n$ occurring in the definition of the apartment class map (see Definition 62 and Definition 63).

DEFINITION 68. *We call a nonempty subset $I \subset [\![n]\!]$ a sigma subset, if $\{n-1, \overline{n-1}\} \subset I$ and for all $0 \le a \le n-2 : \{a, \bar{a}\} \not\subset I$. Let β_n be the simplicial complex with vertex set $[\![n]\!]$ and k-simplices subsets $I \subset [\![n]\!]$ of size $k+1$, which are either standard (see Definition 62) or sigma subsets.*

Note that $\beta_1 \cong D^1$. Furthermore, $\beta_n \cong (*_0^{n-2} S^0) * D^1$ is homeomorphic to a disc of dimension n whose boundary sphere is triangulated by the subcomplex $\partial\beta_n \subset \beta_n$, i.e.

$$(|\beta_n|, |\partial\beta_n|) \cong (D^n, S^{n-1})$$

Definition of $\alpha : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$.

Let $M = (M_a)_{a \in [\![n]\!]} \in \mathrm{Sp_{2n}}(\mathbb{Z})$. Given some k-simplex I of β_n, we find an associated simplex $M_I^\alpha = \{\langle \pm M_a \rangle : a \in I\}$ of $\mathcal{IA}_n$: If I is a standard subset, then M_I^α is a standard simplex. If I is sigma subset, then M_I^α is a sigma simplex. The resulting map

$$M^\alpha : \beta_n \to \mathcal{IA}_n$$

is a simplicial embedding and the boundary

$$\partial M^\alpha : \partial\beta_n \to \mathcal{IA}_n$$

of this simplicial disc satisfies:

$$\partial M^\alpha : \partial\beta_n \to \mathcal{I}_n^\delta \to \mathcal{IA}_n$$

We define $\alpha(M)$ to be the image of the fundamental class $\xi \in H_{n-1}(\partial\beta_n)$ under the composition:

$$H_{n-1}(\partial\beta_n) \xleftarrow{\cong} H_n(\beta_n, \partial\beta_n) \xrightarrow{(M^\alpha, \partial M^\alpha)_\star} H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$$

where the first isomorphism is the connecting morphism associated to the pair $(\beta_n, \partial\beta_n)$.

Definition of $s_\star : H_{n-1}(P(\mathcal{I}_n^\delta)) \to \mathrm{St}_n^\omega$.

This is the map induced in homology by the spanning map

$$s : P(\mathcal{I}_n^\delta) \to T_n^\omega(\mathbb{Q}) : \Delta \mapsto \mathrm{span}_{\mathbb{Q}} \Delta$$

where $P(\mathcal{I}_n^\delta)$ denotes the poset of simplices of $\mathcal{I}_n^\delta$.

6.3. Proof of Proposition 16. Let $M \in \mathrm{Sp_{2n}}(\mathbb{Z})$, we need to verify that:

$$[M] = (s_\star \circ b \circ \delta \circ \alpha)(M)$$

Consider the following diagram:

$$\begin{array}{ccccc}
H_n(\beta_n, \partial\beta_n) & \xrightarrow{\cong} & H_{n-1}(\partial\beta_n) & \xrightarrow{b} & H_{n-1}(P(\partial\beta_n)) \\
\downarrow{\scriptstyle (M^\alpha, \partial M^\alpha)_\star} & & \downarrow{\scriptstyle \partial M^\alpha_\star} & & \downarrow{\scriptstyle P(\partial M^\alpha)_\star} \\
H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) & \xrightarrow{\delta} & H_{n-1}(\mathcal{I}_n^\delta) & \xrightarrow{b} & H_{n-1}(P(\mathcal{I}_n^\delta))
\end{array}$$

The left square commutes because the connecting morphism of the long exact sequence of a pair is a natural transformation. The right square commutes, because $b : C_\star \to C_\star \circ P$ is a natural homology isomorphism. It follows that:

$$(b \circ \delta \circ \alpha)(M) = (P(\partial M^\alpha)_\star \circ b)(\xi) \in H_{n-1}(P(\mathcal{I}_n^\delta))$$

To complete the proof, we need to see that:

$$(s \circ P(\partial M^\alpha) \circ b)_\star(\xi) = [M]$$

where $[M] = (\partial M \circ b)_\star(\xi)$ is as in Definition 63. This is holds, because the composition $(s \circ P(\partial M^\alpha))$ defined in this section is equal to the map ∂M defined in the paragraph before Definition 63. □

6.4. Proof of Proposition 17. The arguments for the conclusions (2.), (3.) and (4.) of Proposition 17 are similar to the arguments used by Church–Farb–Putman in [**CFP19**]. However, while the $\mathrm{SL}_n(\mathbb{Z})$-analogue of the surjectivity of $\alpha_n : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ (conclusion (1.) of Proposition 17) is rather immediate for special linear groups, this step is more involved for symplectic groups. The "pictorial reason" for this is that apartments in the Tits building for special linear groups can be identified with $\partial\Delta^n$ and can therefore be "filled" by gluing in a single simplex of dimension n. Apartments of the symplectic Tits building, even though homeomorphic to ∂D^n, have a different simplicial structure and require multiple simplices to be filled. In the complex $\mathcal{IA}_n$, this is achieved by sigma simplices. Observe that sigma simplices already occur in dimension one. Therefore, and in contrast to the analogous situation for special linear groups (see Step 1, 2.3 Proof of Theorem B, [**CFP19**]), the relative chain complex $C_\star(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is nontrivial in degree $\star = n-1$.

Proof of conclusion (2.). Corollary 23 implies that $H_{n-1}(\mathcal{IA}_n) = 0$. The long exact sequence of the pair $(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ therefore implies that $\delta : H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \to H_{n-1}(\mathcal{I}_n^\delta)$ is surjective. □

Proof of conclusion (3.). The map $b : H_{n-1}(\mathcal{I}_n^\delta) \to H_{n-1}(P(\mathcal{I}_n^\delta))$ is an isomorphism by definition. It is induced by a natural homology isomorphism of chain complexes. □

Proof of conclusion (4.). To verify that the map $s_\star : H_{n-1}(P(\mathcal{I}_n^\delta)) \to H_{n-1}(T_n^\omega(\mathbb{Q})) = \mathrm{St}_\mathrm{n}^\omega$ is an isomorphism, we can apply Theorem 50 once more. Let $\theta = n$. The symplectic Tits building $T_n^\omega(\mathbb{Q})$ is Cohen–Macaulay of dimension $n-1$ (see [**Sol69**] and [**Bro89**], IV.5 Remark 2). Let $V \in T_n^\omega(\mathbb{Q})$ and set $t(V) = \dim(V) + 1$. The upper interval $(T_n^\omega(\mathbb{Q}))_{>V}$ is $(n - \dim(V) - 2) = (n - t(V) - 1)$-connected. The fiber $f_{\leq V} = \mathcal{BA}(V \cap \mathbb{Z}^{2n})$ is a point, if $\dim(V) = 1$, and Cohen–Macaulay of dimension $\dim(V)$, if $\dim(V) \geq 2$, by the result of Church–Putman [**CP17**] stated as Theorem 49. In particular, all fibers $f_{\leq V}$ are $(\dim(V) - 1) = (t(V) - 2)$-connected. It follows from Theorem 50 that $s : P(\mathcal{I}_n^\delta) \to T_n^\omega(\mathbb{Q})$ is n-connected. In particular, the map $s_\star : H_{n-1}(P(\mathcal{I}_n^\delta)) \to H_{n-1}(T_n^\omega(\mathbb{Q})) = \mathrm{St}_\mathrm{n}^\omega$ is an isomorphism. □

Proof of conclusion (1.). To see that the map

$$\alpha_n : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$$

is surjective, we will perform an induction on $n \geq 1$. In the following, E_n^ω will denote the set of all sigma edges in $\mathcal{IA}_n$. The induction beginning is the following lemma.

Lemma 52. *If $n = 1$, then $\alpha_n : \mathbb{Z}[\mathrm{Sp_{2n}}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is surjective.*

Proof: For $2n = 2$, it follows that $\mathcal{IA}_1$ is a one-dimensional connected simplicial complex, all edges are sigma edges and $\mathcal{I}_1^\delta = \mathcal{I}_1$ is exactly the 0-skeleton of $\mathcal{IA}_1$[3]. In particular,

$$H_1(\mathcal{IA}_1, \mathcal{I}_1^\delta) \cong \bigoplus_{\sigma \in E_1^\omega} \mathbb{Z}.$$

Given some $M \in \mathrm{Sp}_2(\mathbb{Z})$ with $M = (v, w)$, we see that $M^\alpha(\beta_1) \subset \mathcal{IA}_1$ is exactly the symplectic edge $\sigma = \{\pm v, \pm w\}$ and $M^\alpha(\partial\beta_1) \subset \mathcal{I}_1^\delta$ is exactly the boundary of this edge. Hence, under the identification above, α_1 maps the symplectic matrix M to a generator of the $\mathbb{Z}$-summand indexed by $\sigma = \{\pm v, \pm w\}$. Given any symplectic edge $\sigma = \{\pm v, \pm w\}$, we have that $\omega(v, w) = \pm 1$. Thus, for some choice of signs $(v, w) \in \mathrm{Sp}_2(\mathbb{Z})$. It follows that α_n is surjective for $n = 1$. □

Let $n > 1$. Assume that conclusion (1.) of Proposition 17 holds for $1 \leq k \leq n-1$. We will now explain how one can use induction to verify the surjectivity of $\alpha_n : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$. The first step is to show that the target $H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is a direct sum of "smaller" Steinberg modules.

Observe that $\mathcal{IA}_n$ is obtained from $\mathcal{I}_n^\delta$ via the following homotopy pushout diagram:

$$\begin{array}{ccccc} \coprod_{\sigma \in E_n^\omega} \Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma) & \xrightarrow{\cong} & \coprod_{\sigma \in E_n^\omega} \mathrm{st}_{\mathcal{IA}_n}(\sigma) \cap \mathcal{I}_n^\delta & \longrightarrow & \mathcal{I}_n^\delta \\ & & \big\downarrow & & \big\downarrow \\ & & \coprod_{\sigma \in E_n^\omega} \mathrm{st}_{\mathcal{IA}_n}(\sigma) & \longrightarrow & \mathcal{IA}_n \end{array}$$

It follows that the induced map on the vertical cofibers is a homotopy equivalence. In particular,

$$H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \cong \bigoplus_{\sigma \in E_n^\omega} H_n(\mathrm{st}_{\mathcal{IA}_n}(\sigma), \Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma)) \tag{1}$$

The contractibility of $\mathrm{st}_{\mathcal{IA}_n}(\sigma)$ implies that the connecting morphism of the pair

$$(\mathrm{st}_{\mathcal{IA}_n}(\sigma), \Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma))$$

is an isomorphism.

$$H_n(\mathrm{st}_{\mathcal{IA}_n}(\sigma), \Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma)) \xrightarrow{\delta_n} H_{n-1}(\Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma)) \tag{2}$$

The suspension isomorphism gives an identification.

$$H_{n-1}(\Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma)) \xrightarrow{\Sigma^{-1}} \tilde{H}_{n-2}(\mathrm{lk}_{\mathcal{IA}_n}(\sigma)) \tag{3}$$

Observe that $\mathrm{lk}_{\mathcal{IA}_n}(\sigma) = \mathcal{I}^\delta(\sigma^\perp)$. We therefore proved that:

$$H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) \cong \bigoplus_{\sigma \in E_n^\omega} H_{n-2}(\mathcal{I}^\delta(\sigma^\perp)) \cong \bigoplus_{\sigma \in E_n^\omega} \mathrm{St}^\omega(\sigma^\perp) \tag{4}$$

where the last isomorphism is obtained by invoking conclusion (3.) and (4.) of Proposition 17 and $\mathrm{St}^\omega(\sigma^\perp)$ denotes the Steinberg module of the symplectic subspace $\sigma^\perp \subset \mathbb{Q}^{2n}$.

This completes the first step. The second step of the proof, that $\alpha_n : \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})] \to H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ is surjective, is to decompose the domain $\mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})]$ in a compatible way and identify the resulting map on each summand. This is the content of the following lemma.

[3] In fact, $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{IA}_1$ isomorphic to the 1-dimensional complex of partial frames $\mathcal{B}(\mathbb{Z}^2)$. The complex $\mathcal{B}(\mathbb{Z}^2)$ is discussed in detail in the introduction of [**CP17**] (see paragraph "Improving connectivity: the complex of partial augmented frames").

Lemma 53. *Let $\sigma = \{\langle \pm v\rangle, \langle \pm w\rangle\} \in E_n^\omega$ and let $\omega(v,w) = 1$, $\tilde{\sigma} = (v,w)$ an ordered pair. Let $\mathbb{Z}[\mathrm{Sp}(\tilde{\sigma}^\perp)] \subset \mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})]$ be the $\mathbb{Z}$-summand spanned by symplectic matrices $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ satisfying $M_{n-1} = v$ and $M_{\overline{n-1}} = w$. The sequence of identifications above yields a map*

$$[-]_{\tilde{\sigma}} : \mathbb{Z}[\mathrm{Sp}(\tilde{\sigma}^\perp)] \to H_{n-1}(\mathcal{IA}(\sigma^\perp), \mathcal{I}^\delta(\sigma^\perp)) \to H_{n-2}(\mathcal{I}^\delta(\sigma^\perp)) \to \mathrm{St}^\omega(\sigma^\perp)$$

which is exactly the integral apartment class map of the group $\mathrm{Sp}(\sigma^\perp)$ of symplectic automorphisms of the summand $\sigma^\perp \subset \mathbb{Z}^{2n}$.

Before proving the lemma, we explain how this finishes the proof of the induction step. Lemma 53 implies that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{Sp}_{2n}] & = & \bigoplus_{\tilde{\sigma}=(M_{n-1}, M_{\overline{n-1}})} \mathbb{Z}[\mathrm{Sp}(\tilde{\sigma}^\perp)] \\ \downarrow \alpha_n & & \downarrow \oplus [-]_{\tilde{\sigma}} \\ H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta) & \xrightarrow{\cong} & \bigoplus_{\sigma \in E_n^\omega} \mathrm{St}^\omega(\sigma^\perp) \end{array}$$

The induction hypothesis, Lemma 53 and the conclusions (2-4.) of Proposition 17 imply that the integral apartment class maps occurring on the right side

$$[-]_{\tilde{\sigma}} : \mathbb{Z}[\mathrm{Sp}(\tilde{\sigma}^\perp)] \to \mathrm{St}^\omega(\sigma^\perp)$$

are surjective. For any $\sigma \in E_n^\omega$, there exists an ordered pair $\tilde{\sigma} = (M_{n-1}, M_{\overline{n-1}})$ such that $\sigma = \{\langle \pm M_{n-1}\rangle, \langle \pm M_{\overline{n-1}}\rangle\}$. It follows that the right vertical map in the diagram is surjective. Therefore, α_n is surjective as well. □

Proof of Lemma 53. It suffices to consider the case where $(v,w) = (e_{n-1}, f_{n-1})$ consists of the last symplectic pair of the standard symplectic basis. All other cases can be reduced to this case by applying a symplectic matrix that sends (v,w) to (e_{n-1}, f_{n-1}). Let $\sigma = \{\langle \pm e_{n-1}\rangle, \langle \pm f_{n-1}\rangle\}$, $\tilde{\sigma} = (e_{n-1}, f_{n-1})$ and $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ a symplectic matrix with $M_{n-1} = e_{n-1}$ and $M_{\overline{n-1}} = f_{n-1}$. The symplectic relations imply that the e_{n-1}- and f_{n-1}-coordinates of all other column vectors $M_a, M_{\bar{a}}$, $a \in \{0, \ldots, n-2\}$, of M are zero. In particular, M corresponds to a unique element $\widetilde{M}$ of the symplectic group $\mathrm{Sp}(\sigma^\perp)$ of the summand $\sigma^\perp \subset \mathbb{Z}^{2n}$ and vice versa. Recall that the class $\alpha(M) \in H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$ was defined using the map of pairs:

$$(M^\alpha, \partial M^\alpha) : (\beta_n, \partial\beta_n) \to (\mathcal{IA}_n, \mathcal{I}_n^\delta)$$

This map factors through the pair:

$$(\mathrm{st}_{\mathcal{IA}_n}(\sigma), \mathrm{st}_{\mathcal{IA}_n}(\sigma) \cap \mathcal{I}_n^\delta) \cong (\mathrm{st}_{\mathcal{IA}_n}(\sigma), \Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma))$$

The naturality of connecting morphisms yields a commutative diagram:

$$\begin{array}{ccccc} H_n(\beta_n, \partial\beta_n) & \xrightarrow{\delta} & H_{n-1}(\partial\beta_n) & \xrightarrow{\Sigma^{-1}} & H_{n-2}(\partial\beta_{n-1}) \\ \downarrow (M^\alpha, \partial M^\alpha)_\star & & \downarrow \partial M^\alpha_\star & & \downarrow \partial \widetilde{M}^\alpha_\star \\ H_n(\mathrm{st}_{\mathcal{IA}_n}(\sigma), \Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma)) & \xrightarrow{\delta} & H_{n-1}(\Sigma^1 \mathrm{lk}_{\mathcal{IA}_n}(\sigma)) & \xrightarrow{\Sigma^{-1}} & H_{n-2}(\mathrm{lk}_{\mathcal{IA}_n}(\sigma)) \end{array}$$

Hence, under the identifications in Equation (1), Equation (2) and Equation (3), the class

$$\alpha(M) \in H_n(\mathcal{IA}_n, \mathcal{I}_n^\delta)$$

is mapped to

$$\partial \widetilde{M}^\alpha_\star(\xi_{n-2}) \in H_{n-2}(\mathrm{lk}_{\mathcal{IA}_n}(\sigma)) = H_{n-2}(\mathcal{I}^\delta(\sigma^\perp)).$$

The following commuting square proves that the class $\partial \widetilde{M}^\alpha_\star(\xi_{n-2})$ is exactly $(\delta \circ \alpha_{n-1})(\widetilde{M})$:

$$\begin{array}{ccc} H_n(\beta_{n-1}, \partial\beta_{n-1}) & \xrightarrow{\delta} & H_{n-1}(\partial\beta_{n-1}) \\ \big\downarrow{\scriptstyle (\widetilde{M}^\alpha, \partial\widetilde{M}^\alpha)_\star} & & \big\downarrow{\scriptstyle \partial\widetilde{M}^\alpha_\star} \\ H_n(\mathcal{IA}(\sigma^\perp), \mathcal{I}^\delta(\sigma^\perp)) & \xrightarrow{\delta} & H_{n-1}(\mathcal{I}^\delta(\sigma^\perp)) \end{array}$$

Hence, the final identification used in Equation (4) and Proposition 16 yield that $\alpha_n(M)$ is mapped to

$$(s_\star \circ b \circ \delta \circ \alpha_{n-1})(\widetilde{M}) = [\widetilde{M}] \in \mathrm{St}^\omega(\sigma^\perp)$$

□

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