The motivic zeta functions of Hilbert schemes of points on surfaces

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Abstract

The main object of this thesis is the motivic zeta function for Calabi-Yau varieties defined over a non Archimedean valued field, focusing on Hilbert schemes of points on surfaces. We use the tools from logarithmic geometry for the construction of semistable models of the surfaces with trivial canonical sheaf. We exploit such construction in order to give a recipe for constructing weak Néron models of their Hilbert schemes of points and deduce from this a formula for computing their motivic zeta function. We use the formula developed in this way in order to prove that the Hilbert schemes of points have the monodromy property if their underlying surfaces have it.

Resumé

Hovedobjektet i denne afhandling er den motiviske zeta funktion for Calabi-Yau varieteter defineret over et legeme med en ultrametrisk absolut værdi, med fokus på Hilbert skemaer af punkter på flader. Vi bruger redskaberne fra logaritmisk geometri til at konstruere semistabile modeller af fladerne med trivielt kanonisk knippe. Vi udnytter sådanne konstruktioner til at give en opskrift på at konstruere svage Néron modeller af deres Hilbert skemaer af punkter og udleder heraf en formel til at udregne deres motiviske zeta funktion. Vi bruger formlen udledt på denne måde til at bevise, at Hilbert skemaer af punkter har monodromiegenskaben, hvis deres underliggende flader har den.

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Contents

Introduction

$p-$adic and motivic zeta functions

Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial and fix a prime p. A way to study the asymptotic behavior of the number N_d of solutions of f modulo p^{d+1}, for $d \gg 1$, consists on studying the *Poincaré series* $P_f(t) := \sum_{d \in \mathbb{N}} \frac{N_d}{p^{md}} t^d$. In particular, one has that P_f is rational if and only if N_d/p^{md} is a linear recurrence sequence and, in such case, the poles of P_f are the roots of the charachteristic polynomial of the sequence. The $p-$adic Igusa zeta function was born in order to address this number-theoretical problem: on one hand the $p-$adic zeta function has an intrinsic expression as a $p-$adic integral, on the other hand it is related to the Poincaré series by the equation (1.1.1); this equation implies that the Zeta function is rational in p^{-s} if and only if the Poincaré series is rational. Using the Hironaka resolution of singularities, Igusa proved the rationality of the zeta function, hence that of the Poincaré series.

After Kontsevich invented the theory of motivic integration, Denef and Loeser adapted the definition of Igusa zeta function to a totally new context, inventing the motivic zeta function, a powerful invariant attached to hypersurface singularities. This invariant is a formal series with coefficients in a Grothendieck ring of varieties which has been proved to be rational by Denef and Loeser in [10], in a sense that will become clear in §3.6. A further variant of the motivic integration theory was developed by Loeser and Sebag. In their theory, rather than studying the hypersurface singularities, one studies their tubular neighbourhoods, i.e. families of smooth varieties defined over a punctured disk, or over the spectrum of a field endowed with a non trivial ultrametric absolute value for an algebro-geometric version. The motivic integral, in this case, measures in what way the family can be filled over the puncture, i.e. what central fibres should weak Néron models over the disk should have. The motivic zeta function for Calabi-Yau varieties collects these motivic integrals for all the possible finite étale covers of the punctured disk and puts the results together in a formal series.

Monodromy conjecture

For all these settings, a question about the poles of these rational functions naturally arises. The most discussed open question concerning the motivic zeta function is the *Monodromy Conjecture* which claims the existence of a relationship between the poles of the zeta functions and the eigenvalues of the local monodromy operator associated either to the hypersurface singularity (for Denef and Loeser's setting) or to the deck tansformations of the universal cover of the punctured disk (for Loeser and Sebag's setting). In the situation we will deal with, i.e. Loeser and Sebag's setting, we consider a Calabi-Yau variety $X \to \operatorname{Spec} K$, for some field K with an ultrametric valuation, the étale fundamental group of $\operatorname{Spec} K$, i.e. the absolute Galois group of K, acts on X, inducing an action on its cohomology with coefficients in $\mathbb{Q}_l$. Under the assumption that the wild inertia acts trivially on X, we can identify the action of $\mathrm{Gal}(\overline{K}|K)$ with the action of its tame quotient, which admits a topological generator σ.

The corresponding cohomology operator $\sigma^*\colon H^*(X_{\overline{K}}, \mathbb{Q}_l) \to H^*(X_{\overline{K}}, \mathbb{Q}_l)$ is a quasi-unipotent linear operator; its eigenvalues are thus roots of the unity. The monodromy conjecture relates these eigenvalues to the poles of the motivic zeta function of X. Halle and Nicaise proved in [17] that the monodromy conjecture holds when X is an Abelian variety; in such case the motivic zeta function has always a single pole. Jaspers has proved in [23] that whenever X is a K3 surface admitting a Crauder-Morrison model, then the monodromy conjecture holds for X. In [36], Overkamp proved that all the Kummer surfaces satisfy the monodromy property. Lunardon proved in [26] that K3 surfaces admitting a model with ADE singularities satisfy the monodromy conjecture. All these works provide a large class of surfaces with the monodromy property, giving sense to our investigation on Hilbert schemes of points over surfaces.

Hilbert schemes

Hilbert schemes are the answer to one of the most natural moduli problem that mathematician study: parametrizing subschemes of a given variety. Hilbert schemes of points on surfaces with trivial canonical bundle are Calabi-Yau varieties as well; moreover if the surface is K3, then its Hilbert schemes of points are Irreducible Holomorphic Symplectic varieties. In this thesis we will construct weak Néron models of Hilbert schemes by accurately using the moduli problem they solve. Starting with a degeneration of surfaces we will use techniques from logarithmic geometry in order to construct other degenerations over the extensions of the base ring. Applying the Hilbert functor to these degenerations one gets proper, but singular, models for the Hilbert schemes of points on the surfaces. Starting with these models for Hilbert schemes, applying the theory of Weil restriction of scalars, it is possible to obtain weak Néron models of Hilbert schemes over an arbitrary base change of K and, moreover, it is possible to study accurately their central fibre.

The main result

This construction lead us to a formula for the motivic zeta function of the Hilbert schemes in terms of the zeta functions of the underlying surface and of its base changes over the finite extensions of K. From that formula it will be possible to deduce the main theorem of this manuscript:

Theorem. *Let X be a surface with trivial canonical bundle satisfying the monodromy conjecture. Then the conjecture holds also for* $\mathrm{Hilb}^n(X)$, $\forall n \in \mathbb{N}$.

This formula has sees application not only in the context of Hilbert schemes of points, but also in some moduli spaces of sheaves over the K3 surface, as we will see in the end of the thesis.

Outline of the thesis

In chapter 1 we give an historical introduction on $p-$adic zeta function and on its corresponding monodromy conjecture.

In chapter 2 we introduce the Grothendieck rings of varieties and their equivariant versions, using them to define the main actors of the manuscript.

In chapter 3 we recall the notion of motivic integration and the definition of the Motivic Zeta Function in two settings: the setting of hypersurface singularities and the setting of Calabi-Yau varieties. Then we explain the notion of rationality for power series with coefficients in $\mathscr{M}_k$, in $\mathscr{M}_k\left[(\mathbb{L}^r-1)^{-1}\colon 0<r\in\mathbb{N}\right]$ and in $\widehat{\mathscr{M}_k}$, giving also a definition of a pole of such functions. We conclude by stating the Monodromy Conjecture in two forms.

In chapter 4 we recall the notion of a weak Néron model and show some techniques that we will use in order to construct them.

In chapter 5 we introduce the notion of logarithmic scheme and the facts that we need later on in the thesis.

In chapter 6 we define toric schemes over a DVR and explore some similarities with the theory of toric varieties.

In chapter 7 we adapt the theory of potential semistable reduction of families of surfaces in our case. We construct a specific equivariant semistable model satisfying a good property with respect to the Galois action of the extension.

In chapter 8 we discuss some facts concerning the poles of rational functions with coefficients in our motivic ring. These properties will be useful when applied to the formula that we will produce for the motivic zeta function of Hilbert schemes.

In chapter 9 we state basic facts about the objects we are mostly interested in, Hilbert schemes.

In chapter 10 we give a construction of the weak Néron models of Hilbert schemes of points on surfaces and use those models to compute their motivic zeta function.

We finally discuss the monodromy property in chapter 11.

In chapter 12 we compute explicitly the poles of the zeta function of the Hilbert scheme of two points of a quartic K3 surface.

In chapter 13 we prove birationality between some moduli spaces of sheaves over a K3 surface and a Hilbert scheme of the appropriate dimension; we then use the birational invariance of the motivic zeta function to prove the monodromy conjecture for those moduli spaces.

Notation and conventions

Throughout the thesis, unless differently stated, R will be a DVR, K its fraction field and k its residue field, which we will assume to be algebraically closed. We fix an algebraic closure $\overline{K}$ of K, so that whenever we consider extensions of K and R we think of algebraic extensions of K in $\overline{K}$ and integral extensions of R in $\overline{K}$. When m is an integer coprime with the characteristic exponent of k, we denote by $K(m)$ the unique extension of K of degree m and by $R(m)$ the integral closure of R in $K(m)$. By variety over a field F we denote a reduced, separated scheme of finite type over F. The set of natural numbers $\mathbb{N}$ contains 0.

Part I

Zeta functions

Chapter 1

Zeta functions in algebraic geometry

A large class of algebro-geometric invariants are often denoted by the term ”zeta functions”

1.1 The $p-$adic Igusa zeta function.

1.1.1 The Igusa zeta function was originally defined as an invariant attached to hypersurface of some affine spece $\mathbb{A}^n_{\mathbb{Z}_p}$.

Definition 1.1.2. Let $f \in \mathbb{Z}_p[x_1, x_2, \ldots, x_n]$ be a nonconstant polynomial and consider the function of complex variable s, in the half-plane $\{s\colon \Re s > 0\}$:

$$Z_f(s) \coloneqq \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, \mathrm{d}\mu \ ,$$

where the $p-$adic value is normalized by $|p|_p = p^{-1}$ and $\mathrm{d}\mu$ denotes the Haar measure of the compact group $\mathbb{Z}_p^n$ so normalized that the total measure is 1. This define an holomorphic function in the domain.

Remark 1.1.3. It is possible to replace $\mathbb{Z}_p$ with its integral closure in any finite extension K of $\mathbb{Q}_p$ and define essentially the same theory with the appropriate Haar measure.

1.1.4 The Haar measure of the oversets, i.e. $\mu(\{x \in \mathbb{Z}_p^n\colon |f(x)|_p \geq p^{-d}\})$, for $d \in \mathbb{N}$, is related to the numbers $N_d \coloneqq |\{a \in (\mathbb{Z}/p^d\mathbb{Z})^n\colon f(a) \cong 0 \pmod{p^d}\}|$, with $N_0 = 1$, by the equation

$$\mu(\{x \in \mathbb{Z}_p^n\colon |f(x)|_p \geq p^{-d}\}) = 1 - \frac{N_{d+1}}{p^{(d+1)n}} \, .$$

Because of this property the Igusa zeta function of f becomes a useful invariant for computing the number of solutions of f modulo all the powers of p; this can be made more explicit via a functional equation involving the Poincaré power series $P_f(p^{-s})$, where:

$$P_f(t) \coloneqq \sum_{d \geq 0} \frac{N_d}{p^{nd}} t^d \, ,$$

indeed the two functions satisfy the equation:

$$P_f(t) = \frac{1 - tZ_f(s)}{1 - t} \, , \tag{1.1.1}$$

after the substitution $t = p^{-s}$.

1.1.5 Despite the two functions Z_f and P_f are, as we will see soon, useful invariants of singularities, they are not very interesting if f has no critical points, for the numbers N_d, $d \geq 1$, are all equal because of Hensel's Lemma. On the other hand, this means that P_f is a rational function of t as soon as f is a smooth polynomial; similarly Z_f will be a rational function of p^{-s} in the same case. Combining this fact with Hironaka's resolution of singularities, Igusa proved in [22] that $Z_f(s)$ is rational for an arbitrary f, taking advantage of the change of variable formula that holds for $p-$adic integrals. It follows that P_f is always a rational function, which was stated as a conjecture by Borevich and Shafarevich in [2]. Once established that the zeta function is rational, the most natural task for the Mathematician is determining its poles, possibly with multiplicity, and understanding the relations existing between them and the "nature" of $V(f)$. Indeed this question found a partial answer in the *p-adic monodromy conjecture* formulated by Igusa, which links the arithmetic nature of Z_f with the topological nature of the Milnor fibration of $V(f) \subseteq \mathbb{C}^n$.

1.2 Milnor fibration and monodromy action

1.2.1 In order to explain the monodromy conjecture, we need to briefly introduce the Milnor fibration and the monodromy action on it. Consier the polynomial f as a map $\mathbb{C}^n \to \mathbb{C}$ and fix a point $x \in V(f)$. Let $0 < \delta \ll \epsilon \ll 1$ be small positive real numbers and consider the disk $\Delta = D(0, \delta) \subseteq \mathbb{C}$ and the ball $B = B(x, \epsilon) \subseteq \mathbb{C}^n$; consider the punctured disk $\Delta^* := \Delta \backslash \{0\}$ and the tubular neighbourhood $B^* := B \cap f^{-1}(\Delta^a st)$. Restricting f results in a locally trivial fibration $f_x \colon B^* \to \Delta^*$, called the *Milnor fibration* of f at x. Consider the universal cover $\widetilde{\Delta^*} \to \Delta^*$; the base-change $F_x := B \times_{\Delta^*} \widetilde{\Delta^*}$ is called *the (universal) Milnor fibre* of f at x. The group $\pi_1(\Delta^*)$ of deck transformations of $\widetilde{\Delta^*} \to \Delta^*$ induces automorphism of F_x, which, in turns, induces a linear action of $\pi_1(\Delta^*)$ on the singular cohomology

$$\pi_1(\Delta^*) \curvearrowright \bigoplus_{i \geq 0} H^i_{\text{sing}}(F_x, \mathbb{Z}) \, .$$

The action on $H^*_{\text{sing}}(F_x, \mathbb{Z})$ of the canonical generator of $\pi_1(\Delta^*)$, i.e. the transformation corresponding to one single counterclockwise loop, will be simply called the *monodromy action* of f at x and the eigenvalues of this map are called *monodromy eigenvalues*. The $p-$adic monodromy conjecture links these with the poles of the Igusa zeta function.

Conjecture 1.2.2 ($p-$adic monodromy conjecture)**.** *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial. Assume that s is a pole of Z_f, where f is considered as an element of $\mathbb{Z}_p[x_1, \ldots, x_n]$, then* $\exp(2i\pi\Re(s))$ *is a monodromy eigenvalue of f, where f is considered as an element of* $\mathbb{C}[x_1, \ldots, x_n]$.

1.2.3 Notice that if f is smooth at x, then f_x is a trivial fibration and the monodromy action coincide with the identity, while if x is a critical point of f, the monodromy eigenvalues are more interesting. In any case all the monodromy eigenvalues are roots of 1, [8, Théorème de monodromie 2.1]; the computation of these eigenvalues is simplyfied by the means of the so called *monodromy zeta function*.

1.3 Monodromy zeta function and $p-$adic monodromy conjecture.

1.3.1 Despite the characteristic polynomial of a linear operator already carries complete information about the eigenvalues of such operator, it is not very suitable for practical purposes. For this reason we will introduce the following monodromy zeta function, which will provide significant computational advantages:

Definition 1.3.2. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a non-constant analytic map. The *monodromy zeta function* of f at x is

$$\zeta_{f,x} \coloneqq \prod_{i\geq 0} \det(Id - T\cdot M_{x,i}|H^i(F_x,\mathbb{C}))^{(-1)^{i+1}}.$$

1.3.3 All the zeroes and the poles of this function are monodromy eigenvalues at x, nevertheless some of the eigenvalues might be missing due to cancellation between eigenvalues that appear in different cohomology groups. On the other hand, Denef proved in [9] that if γ is a monodromy eigenvalue at x, then there is at least one point $y \in V(f)$, possibly different than x, such that γ is either a zero or a pole of $\zeta_{f,y}$. Thus the monodromy zeta function, globally speaking, covers the same informations as the charactristic polynomial, in spite of missing some local information.

1.3.4 The following theorem, due to A'Campo [1, Théorème 3], provides and effective and efficient way to compute the monodromy zeta functions in terms of resolutions of singularities:

Theorem 1.3.5 (A' Campo)**.** *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a non-constant analytic map, let $X_0 = V(f)$ and let $g\colon Y \to \mathbb{A}^n_{\mathbb{C}}$ be a embedded resolution of singularities of f, with $Y_0 = g^{-1}(X_0)$. Let $\{E_i\}_{i\in I}$ be the set of irreducible components of Y_0, where each E_i has multiplcity N_i, and set $E_i^\circ \coloneqq E_i \backslash \left(\bigcup_{j\in I\backslash\{i\}} E_j\right)$. Then the following equality gives an expression for the monodromy zeta function:*

$$\zeta_{f,x} = \prod_{i\in I}(1-T^{N_i})^{-\chi\left(E_i^\circ \cap g^{-1}(x)\right)},$$

where χ is the topological Euler characteristic.

Chapter 2

Grothendick ring of varieties

In this section we introduce the rings containing the coefficients of the formal series we will study later on.

2.1 A motivic ring

2.1.1 Fix a field k and consider the category of algebraic varieties Var_k. Let $K_0(\mathrm{Var}_k)$ be the group whose generators are isomorphism classes in Var_k and whose relations, called *scissor relations*, are generated by elements in the form

$$X - Y - (X \backslash Y)\,,$$

whenever X is an algebraic variety and $Y \subseteq X$ is a closed subvariety. We denote by $[X]$ the class of $X \in \mathrm{Var}_k$ in $K_0(\mathrm{Var}_k)$.

2.1.2 There is a unique ring structure on $K_0(\mathrm{Var}_k)$ such that for all $X, Y \in \mathrm{Var}_k$ one has $[X] \cdot [Y] = [\overline{X \times_k Y}]$, where by $\overline{X \times_k Y}$ we denote the reduced scheme associated to the product $X \times_k Y$. With this ring structure, $K_0(\mathrm{Var}_k)$ is called *the Grothendieck ring of varieties.* It is also characterized by the following universal property:

Universal property of $K_0(\mathrm{Var}_k)$. Let R be a ring and let $\Psi\colon \mathrm{Var}_k \to R$ a multiplicative and additive invariant, i.e. a function, constant on isomorphims classes, which associates to a variety X an element $\Psi(X) \in R$ such that $\Psi(\overline{X \times_k Y}) = \Psi(X)\Psi(Y)$ and if $X = Y \cup Z$, then $\Psi(X) + \Psi(Y \cap Z) = \Psi(Y) + \Psi(Z)$. Then there is a unique ring homomorphism $\varphi\colon K_0(\mathrm{Var}_k) \to R$ such that

$$\forall X \in \mathrm{Var}_k, \text{ one has that } \varphi([X]) = \Psi(X)\,.$$

2.1.3 We give a couple of examples of this property:

Example 2.1.4. Assume $k = \mathbb{C}$, let $\chi\colon \mathrm{Var}_k \to \mathbb{Z}$ be the topological Euler characteristic; then χ factors through a map $\overline{\chi}\colon K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$.

Example 2.1.5. If $k = \mathbb{C}$, we can associate to a variety $X \to \operatorname{Spec} k$ its Poincaré polynomial $p(X, v) = \displaystyle\sum_{n=0}^{2\dim X} b_n(X)v^n \in \mathbb{Z}[v]$, where $b_n(X)$ is the $n-$th Betti number of X. The Poincaré polynomial induces a ring homomorphism $K_0(\mathrm{Var}_k) \to \mathbb{Z}[v]$, called *the Poincaré specialization map.*

Example 2.1.6. If $k = \mathbb{F}_q$ is a finite field, then the point counting map $X \mapsto \#X(\mathbb{F}_q)$ induces a ring homomorphsm $K_0(\mathrm{Var}_k) \to \mathbb{Z}$.

2.2 Localised Grothendieck ring

2.2.1 A ring that is worth some consideration is obtained as a localization of $K_0(\mathrm{Var}_k)$.

Definition 2.2.2 (Localised Grothendieck ring of varieties)**.** Let us denote by $\mathbb{L}$ the element $[\mathbb{A}^1_k] \in K_0(\mathrm{Var}_k)$.

The localisation of $K_0(\mathrm{Var}_k)$ with respect to $\mathbb{L}$,

$$\mathscr{M}_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]\,,$$

is called *the localised Grothendieck ring of varieties.*

2.2.3 By combining the universal property of localization and of $K_0(\mathrm{Var}_k)$, one can define $\mathscr{M}_k$ as a universal ring for all the invariants $\mathrm{Var}_k \to R$ which send $\mathbb{A}^1_k$ in R^*.

2.3 The Grothendieck ring of algebraic stacks

2.3.1 Consider all the elements of the form $1 - \mathbb{L}^a \in \mathscr{M}_k$. A key role in this manuscript will be played by the ring obtained by inverting those elements.

Definition 2.3.2. The ring $\mathscr{M}[(1-\mathbb{L}^a)^{-1} \colon a \in \mathbb{N}]$ is called *the Grothendieck ring of algebraic stacks.*

2.3.3 The name of this ring is due to the fact that it can be obtained with the $K-$theoretic construction from the category of algebraic stacks. Anyway we will not need this construction.

2.4 Completed Grothendieck ring

2.4.1 Consider the filtration $\mathcal{F}^n\mathscr{M}_k := \langle \mathbb{L}^r[X] | r \in \mathbb{Z}\,,\ \dim[X] + r \leq -n\rangle_{\mathbb{Z}}$.

Definition 2.4.2 (Completed Grothendieck ring of varieties)**.** The completed Grothendieck ring of varieties $\widehat{\mathscr{M}_k}$ is the completion of $\mathscr{M}_k$ with respect to the filtration $\mathcal{F}^\bullet$.

2.5 Equivariant setting

2.5.1 All the four rings above have an equivariant version, i.e. can be constructed in the category Var^G_k of algebraic varieties endowed with the action of a finite group G.

Definition 2.5.2. A $G-$action on a variety X is said to be *good* if every point of X admits a $G-$invariant affine open neighbourhood. We say that X is a good $G-$variety if it is endowed with a good $G-$action.

2.5.3 Quasi projective $G-$varieties are always good. If not differently stated we assume that $G-$varieties are good.

2.5.4 As before, the *equivariant Grothendieck group of varieties* $K_0(\mathrm{Var}^G_k)$ is the group generated on the isomorphism classes of good $G-$varieties with relation of two kinds:

Scissor relations Let X be a $G-$variety and Y a $G-$invariant closed subscheme, then

$$X - Y - (X \backslash Y)\,,$$

is 0 in $K_0(\mathrm{Var}^G_k)$.

Trivializing relations Let $S \in \mathrm{Var}_k^G$ and let $V \to S$ be a $G-$equivariant affine bundle of rank d. Then

$$V - (S \times \mathbb{A}_k^d) \in \mathrm{Var}_k^G$$

is set to 0, where G acts trivially on the second factor of $S \times \mathbb{A}_k^d$.

There is a unique ring structure on $K_0(\mathrm{Var}_k^G)$ such that for every two $X, Y \in \mathrm{Var}_k^G$, we have $[X] \cdot [Y] := [X \times_k Y]$, where the action of G on $X \times_k Y$ is the diagonal action.

2.5.5 We use again the symbol $\mathbb{L} := [\mathbb{A}_k^1]$, where the group G acts trivially on the affine line; thus the trivializing relations tell us nothing more than:

$$[V] = \mathbb{L}^d[S]\,,$$

whenever $V \to S$ is an equivariant affine bundle of rank d.

2.5.6 Similarly to what we did in the previous sections, we define the localisation $\mathscr{M}_k^G := K_0(\mathrm{Var}_k^G)[\mathbb{L}^{-1}]$ and its completion $\widehat{\mathscr{M}_k^G}$ with respect to the filtration

$$\mathcal{F}^n\mathscr{M}_k := \langle \mathbb{L}^r[X] | r \in \mathbb{Z}\,,\ \dim[X] + r \leq -n \rangle_{\mathbb{Z}}\ .$$

Chapter 3

Motivic zeta functions

3.1 Jet spaces and arc spaces

3.1.1 Taking inspiration from the $p-$adic zeta function, it is possible to define an invariant, called *motivic zeta function*, in a different context, namely for varieties defined over a field or for varieties defined over a DVR of equicharacteristic 0. This is possible after the work of Kontsevich, who used the notion of arc spaces in order to construct a measure, defined on constructible subsets of a variety, that take values in a suitable Grothendieck ring of varieties.

3.1.2 Let k be a field and $X \to \operatorname{Spec} k$ a variety. Consider the functors J_n, with $n \in \mathbb{N}$, defined by

$$\begin{aligned} J_n(X)\colon \ \mathrm{Sch}_k^{\mathrm{opp}} &\to \mathrm{Sets} \\ T &\mapsto X(T \times_k k[t]/(t^{n+1}))\,. \end{aligned}$$

Proposition 3.1.3 ([29, Proposition 4.3])**.** *Each functor defined above is represented by separated schemes of finite type $\mathcal{L}_n(X)$, called $n-$th jet scheme of X. If X is affine, then so is $\mathcal{L}_n(X)$.*

3.1.4 For $m \geq n$, we have a truncation map

$$k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$$

that induces a natural transformation of functors $J_m \to J_n$, and in turns a morphism of schemes, also called *truncation map*:

$$\pi_n^m\colon \mathcal{L}_m(X) \to \mathcal{L}_n(X)\,,$$

the composition of these functors follows the rule of the composition of truncation maps, i.e. for $l \geq m \geq n$ one has that $\pi_n^m \circ \pi_m^l = \pi_n^l$. Since the maps $\pi_0^m\colon J_m(X) \to J_0(X) = X$ are affine, it follows that all the truncation maps are affine as well. It follows form this that the projective limit

$$\mathcal{L}(X) \coloneqq \varprojlim_n \mathcal{L}_n(X)$$

exists in the category of schemes and it is called the *arc scheme* of X.

Example 3.1.5 (Jet scheme of an affine space.)**.** Let A be a $k-$algebra. Then $J_n(\mathbb{A}_k^d)(A) = \mathbb{A}_k^d(A[t]/(t^{m+1}))$, which is a $d-$tuple of elements in $A[t]/(t^{m+1})$, thus $\mathcal{L}_m(A_k^d) = \mathbb{A}_k^{(m+1)d}$. The map π_0^m is just the projection on the coordinates corresponding to the constant coefficients of each element in the $d-$uple of $A[t]/(t^{m+1})$. In the same spirit, the truncation map $A[t]/(T^{n+1}) \to A[t]/(T^{m+1})$ induces a projection $\mathbb{A}_k^{d(n+1)} \to \mathbb{A}_k^{d(m+1)}$.

3.1.6 A morphism of varieties $h\colon Y \to X$ induces by pushforwars a natural transformation among the jest functors and consequently morphisms $\mathcal{L}(h)\colon \mathcal{L}(Y) \to \mathcal{L}(X)$ are defined. These pass to the limit, giving a morphism of arc spaces:

$$\mathcal{L}(h)\colon \mathcal{L}(Y) \to \mathcal{L}(X)\,.$$

Proposition 3.1.7. *If X is smooth of pure dimension d, then $\forall n \geq m\, \mathcal{L}_n(X) \to \mathcal{L}_m(X)$ is an affine bundle of rank $d \cdot (n-m)$.*

Proof. Up to replacing X with a Zariski open cover, we may assume that there is an étale map $f\colon X \to \mathbb{A}^d_k$. This induces a commutative diagram

$$\begin{tikzcd} \mathcal{L}_n(X) \arrow[r, "\mathcal{L}_n(f)"] \arrow[d, "\pi^n_m(X)"'] & \mathbb{A}_k^{(n+1)d} \arrow[d, "\pi^n_m(\mathbb{A}^d)"] \\ \mathcal{L}_m(X) \arrow[r, "\mathcal{L}_m(f)"] & \mathbb{A}_k^{(m+1)d} \end{tikzcd},$$

where the horizontal maps are étale, while the rightmost is a Zariski localy trivial fibration with fibre isomorphic to $\mathbb{A}^{d(n-m)}$. It follows from the cartesianity of the diagram, which shall be explained soon, that also the leftmost map is a $\mathbb{A}^{d(n-m)}-$bundle in the Zariski topology.

In order to prove that the diagram above is cartesian it is enough to do it for $m = 0$. In such case, let A be a $k-$algebra and consider a commutative diagram of the form:

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{L}_n(\mathbb{A}^d_k) \arrow[d, "\pi^n_0(\mathbb{A}^d)"] \\ X \arrow[r, "f"] & \mathbb{A}^d_k \end{tikzcd},$$

which corresponds, by definition of $\mathcal{L}_n(\mathbb{A}^d_k)$, to the following diagram of solid arrows:

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \operatorname{Spec} A[t]/(t^{n+1}) \arrow[d] \arrow[dl, dashed] \\ X \arrow[r, "f"] & \mathbb{A}^d_k \end{tikzcd},$$

which induces, by étaleness of $X \to \mathbb{A}^d_k$, a unique dashed arrow, which in turns is the morphism $\operatorname{Spec} A \to \mathcal{L}_n(X)$ expected from the universal property of fibered products, showing the desired cartesianity. □

3.2 Kontsevich motivic measure

3.2.1 We now introduce the core of Kontsevich's theory

Definition 3.2.2. A subset $C \subseteq \mathcal{L}(X)$ is called a cylinder if, for some $m \in \mathbb{N}$ there is a constructible subset $C_m \subseteq \mathcal{L}_m(X)$ such that $C = (\pi_m)^{-1}(C_m)$.

3.2.3 Clearly, if $C_m \subseteq \mathcal{L}_m(X)$ is constructible, then so is $(\pi_m^n)^{-1}(C_m) \subseteq \mathcal{L}_n(X)$, for π_m^n is a locally trivial fibration with fibre $\mathbb{A}_k^{d(n-m)}$. It follows that the set of cylinders is a boolean algebra and that we can define a *motivic measure* as follows:

Definition 3.2.4. Let $C = (\pi_m)^{-1}(C_m) \subseteq \mathcal{L}(X)$ be a cylinder, for some $m \in \mathbb{N}$ and $C_m \subseteq \mathcal{L}_m(X)$. We define the motivic measure of C

$$\mu(C) \coloneqq [C_m]\mathbb{L}^{-d(m+1)} \in \mathscr{M}_k\,.$$

Remark 3.2.5. The good definition follows, indeed, from the fact that, for $n \geq m$, $[C_n] = [C_m] \cdot \mathbb{L}^{d(n-m)}$, as shown in Poposition 3.1.7. On the other hand, if X is not regular, it is no longer true that $[C_n] = [C_m] \cdot \mathbb{L}^{d(n-m)}$; nevertheless the theory of motivic integration can be approached in a different way.

3.2.6 We use this measure in order to define the class of integrable functions, according to this measure, and the value of their *motivic integral*:

Definition 3.2.7. We say that a function

$$\alpha\colon \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$$

is integrable if it takes finitely many values and if $\forall i \in \mathbb{N}$, the fibre $\alpha^{-1}(i)$ is a cylinder.

In such case the motivic integral of α is

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} \coloneqq \sum_{i \in \mathbb{N}} \mu(\alpha^{-1}(i))\mathbb{L}^{-i} \in \mathscr{M}_k\,.$$

3.2.8 We define a parametrical version of the motivic integral, whose output is a formal series with coefficients in $\mathscr{M}_k$. This will be used for the definition of the motivic zeta function.

Definition 3.2.9. Consider two functions

$$\alpha, \beta\colon \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}\,.$$

The couple (α, β) is said to be integrable if $\forall i \in \mathbb{N}$, the fibres $\alpha^{-1}(i), \beta^{-1}(i)$ are cylinders and if β takes finitely many values on each cylinder $\alpha^{-1}(i)$.

In such case the motivic integral of $T^\alpha \mathbb{L}^{-\beta}$ is the formal series

$$\int_{\mathcal{L}(X)} T^\alpha \mathbb{L}^{-\beta} \coloneqq \sum_{i,j \in \mathbb{N}} \mu(\alpha^{-1}(i) \cap \beta^{-1}(j))T^i\mathbb{L}^{-j} \in \mathscr{M}_k[[T]]\,.$$

Remark 3.2.10. The finiteness of $\{\beta(\alpha^{-1}(i))\}$ guarantee that the coefficient of T^i is well defined in $\mathscr{M}_k$.

3.2.11 For historical reasons, the formal variable T is identified with the symbol $\mathbb{L}^{-s}$, where s is (again) a formal variable; the linear function $\alpha s + \beta$ is said to be integrable if the couple (α, β) is integrable in our notation. With this language, the function α has the same role of the absolute value in $p-$adic integration and the variable s is often considered a complex variable. Indeed, this analogy is more clear when α is interpreted as the evaluation with respect to a Cartier divisor $\mathscr{F}$ in X, i.e. $\forall$ arc $\psi\colon F[[t]] \to X$, we set $\operatorname{ord}_t \mathscr{F}(\psi) \coloneqq \min_{f \in \mathscr{F}} \operatorname{ord}_t(\psi^*(f))$. The following theorem, due to Denef and Loeser, is one of the most powerful tools in motivic integration, which provides a good link with singularity theory:

Theorem 3.2.12 (Change of varialbes formula)**.** *Assume that k is a perfect field. Let $h\colon Y \to X$ be a proper birational morphism and assume that $Y \to \operatorname{Spec} k$ is smooth. Let Jac_h be its Jacobian sheaf. If (α, β) is integrable on X and if Jac_h takes finitely many values on $\alpha^{-1}(i)$, for $i \in \mathbb{N}$, then also $(\alpha \circ \mathcal{L}(h), \beta \circ \mathcal{L}(h))$ is integrable on Y and the equality*

$$\int_{\mathcal{L}(X)} T^{-\alpha}\mathbb{L}^{-\beta} = \int_{\mathcal{L}(Y)} T^{-\alpha\circ\mathcal{L}(h)}\mathbb{L}^{-(\beta\circ\mathcal{L}(h)+\mathrm{ord}_t \mathrm{Jac}_h)}$$

holds in $\mathscr{M}_k[[T]]$.

Let $F \in \mathscr{M}_k[[T]]$, we say that F is rational if there is a finite set $S \subseteq \mathbb{N} \times \mathbb{N}_+$ such that $F \in \mathscr{M}_k\left[T, \dfrac{1}{1-\mathbb{L}^{-a}T^b} \colon (a,b) \in S\right]$.

3.3 Rational functions in $\mathscr{M}_k[[T]]$

3.3.1 The notions discussed in this paragraph are introduced, for instance, in [32].

Definition 3.3.2. Let $F \in \mathscr{M}_k[[T]]$, we say that F is rational if there is a finite set $S \subseteq \mathbb{N}\times\mathbb{N}_+$ such that $F \in \mathscr{M}_k\left[T, \dfrac{1}{1-\mathbb{L}^{-a}T^b} \colon (a,b) \in S\right]$.

In such case, we say that F has a pole of order at most $n \in \mathbb{N}$ in $q \in \mathbb{Q}$ if there exist a finite set $S' \in \mathbb{N} \times \mathbb{N}_+$ such that $\dfrac{a}{b} = q \Rightarrow (a,b) \notin S'$ and a positive integer N such that

$$(1-\mathbb{L}^{-qN}T^N)^n F \in \mathscr{M}_k\left[T, \frac{1}{1-\mathbb{L}^{-a}T^b} \colon (a,b) \in S'\right] .$$

We say that F has a pole of order $n \geq 1$ in $q \in \mathbb{Q}$ if F has a pole of order at most n, but not a pole of order at most $n-1$.

3.3.3 The definition can be simplified, provided that we work on a ring $\mathscr{R}$ endowed with a map $\mathscr{M}_k \to \mathscr{R}$ such that the images of all the elements of the form $\mathbb{L}^r - 1$, with $r \in \mathbb{N}\backslash\{0\}$, are invertible. The minimal such choice for $\mathscr{R}$ is, clearly, the localization in $\mathscr{M}_k$ with respect to that set of elements, i.e. the *Grothendieck ring of algebraic stacks*, for it can be obtained by repeating the construction we have seen in §2 starting with the category of algebraic stacks. Another natural choice for $\mathscr{R}$ is the completed Grothendieck ring of varietes: $\widehat{\mathscr{M}_k}$, where the inverse of $1-\mathbb{L}^r$ is $1+\mathbb{L}^r+\mathbb{L}^{2r}+\cdots$. The following lemma clarifies why in this case it is easier to define a pole:

Lemma 3.3.4. *Let $\mathscr{R}$ be a ring as above and $F \in \mathscr{R}[[T]]$ any rational function. Then $\exists N > 0$ a positive integer and a finite set $S \subseteq \mathbb{Q}$ such that:*

$$F(T) = g(T) + \sum_{q\in S} \frac{f_q(T)}{(1-\mathbb{L}^{-qN}T^N)^{a_q}} , \tag{3.3.1}$$

for some polynomials $g, f_q \in \mathscr{R}[T]$ and positive integers a_q.

Proof. Since F is rational, it admits an expression of the form

$$\frac{h(T)}{\prod_{i\in I}(1-\mathbb{L}^{m_i}T^{n_i})^{a_i}} ,$$

where $h \in \mathscr{R}[T]$, I is a finite set and $m_i, n_i \in \mathbb{N}$. Let $N := \mathrm{lcm}(n_i \colon i \in I)$, then

$$F(T) = \frac{h'(T)}{\prod_{i\in I}(1-\mathbb{L}^{\frac{m_i}{n_i}N}T^N)^{a_i}} ,$$

for some $h' \in \mathscr{R}[T]$. We begin by noticing that, given $\mu > \nu$ positive integers, one has that

$$\frac{1}{(1-\mathbb{L}^{\mu}T^N)(1-\mathbb{L}^{\nu}T^N)} = \frac{1}{1-\mathbb{L}^{\mu-\nu}}\left(\frac{1}{1-\mathbb{L}^{\nu}T^N} - \frac{\mathbb{L}^{\mu-\nu}}{1-\mathbb{L}^{\mu}T^N}\right),$$

where $N \in \mathbb{N}$.

By applying the previous step with an induction on $a+b$, one obtains the following identity

$$\frac{1}{(1-\mathbb{L}^{\mu}T^N)^a(1-\mathbb{L}^{\nu}T^N)^b} = \frac{u(T^N)}{(1-\mathbb{L}^{\mu}T^N)^a} + \frac{v(T^N)}{(1-\mathbb{L}^{\nu}T^N)^b},$$

for positive integers a, b and polynomials $u, v \in \mathbb{Z}\left[\mathbb{L}, \frac{1}{1-\mathbb{L}^{\mu-\nu}}, t\right]$ such that $\deg_t(u) < a$, $\deg_t(v) < b$. In particular any rational function in $\mathscr{R}[[T]]$ with two candidate poles is the sum of two functions with a single pole. We conclude the proof by induction on the number of poles of F. $\square$

Definition 3.3.5. If F is written as in (3.3.1) and $q \in S$, then we say that F has a pole of order at most a_q in q.

If, moreover, there is no integer $N' \in \mathbb{N}_+$ such that $f_q \in (1-\mathbb{L}^{-qN'}T^{N'})\mathscr{R}[T]$, then F has a pole of order exactly a_q in q.

Remark 3.3.6. Let $F \in \mathscr{M}_k[[T]]$ and let $\widetilde{F} \in \mathscr{R}[[T]]$ be the image of F under the completion map (or localisation map).

Because of Lemma 3.3.4, any pole q of $\widetilde{F}$ of order $a \in \mathbb{N}$ is also a pole of F of order greater or equal than a. Indeed, if by contradiction q were a pole of order at most $a-1$ of F, then Lemma 3.3.4 would provide an expression of $\tilde{F}$ where q appears as a pole of order lower than a.

3.4 Denef and Loeser's motivic zeta function

3.4.1 We are now ready to define the zeta function:

Definition 3.4.2. Let $X \to \operatorname{Spec} k$ be a smooth variety of pure dimension and let $f\colon X \to \mathbb{A}^1_k$ be a $k-$morphism. We define the *motivic zeta function* as:

$$Z_f(T) := \int_{\mathcal{L}(X)} T^{\operatorname{ord}_t f} \in \mathscr{M}_k[[T]]\,.$$

3.4.3 This function has been proven to be rational by Denef and Loeser, if $\operatorname{char} k = 0$. Another result of Denef and Loeser, that we are going to illustrate, provides a way to compute $Z_f(T)$ in terms of an embedded resolution of singularities for f. Let $h\colon Y \to X$ be such a resolution and let $\{E_i\}_{i\in I}$ be the set of irreducible component of the *snc* divisor $(f \circ h) \subseteq Y$; let N_i be the multiplicity of E_i and $\nu_i := \operatorname{ord}_{E_i}(\operatorname{Jac}_h) + 1$. For every subset $J \subseteq I$ we define $E_J := \bigcap_{j\in J} E_j$ and $E_J^{\circ} := E_j \backslash \left(\bigcup_{i\in I\backslash J} E_i\right)$.

Theorem 3.4.4 (Denef-Loeser)**.** *Keeping the notation introduced above, the following identity holds:*

$$Z_f(T) = \mathbb{L}^{-d}\sum_{J\subseteq I}[E_J^{\circ}]\prod_{j\in J}(\mathbb{L}-1)\frac{\mathbb{L}^{-\nu_j}T^{N_j}}{1-\mathbb{L}^{-\nu_j}T^{N_j}} \in \mathscr{M}_k[[T]]\,.$$

3.5 Motivic and $p-$adic zeta functions

3.5.1 We conclude the section by showing a few interaction that exist between the motivic and the $p-$adic worlds.

Motivic Poincaré series Similarly to what happen for the $p-$adic zeta function, one method for computing the value of the integral $\int_{\mathcal{L}(X)} T^{\mathrm{ord}_t(f)}$ consists in measuring the subsets of $\mathcal{L}(X)$ where f has a given order. For this purpos a motivic analog of the Poincaré series has been introduced; it is defined as follows:

$$Q_{mot}(T) := \sum_{m\geq 0} [\mathcal{L}_m(V_f)] T^{m+1} .$$

After observing that, for a $k-$algebra A, $\mathcal{L}_m(V_f)(A)$ consists of the morphisms $A[t]/(t^{m+1}) \to X$ that split through V_f, one deduces that $(\pi_m)^{-1}(\mathcal{L}_m(V_f)) \subseteq \mathcal{L}(X)$ is a cylinder containg all an only the arcs where f vanishes with order at least $m+1$; thus we obtain the identity:

$$\frac{Z_f(T) - Z_f(0)}{1 - \mathbb{L}^{-d}T} = Q_{mot}(\mathbb{L}^{-d}T) .$$

Specialization from the motivic to $p-$adic zeta functions Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $X \to \operatorname{Spec} K$ be a variety and let $\mathfrak{X} \to \mathcal{O}_K$ one of its models. For $P \subseteq \mathcal{O}_K$ maximal ideal, consider the number of rational points of its reduction modulo P, i.e. $\mathfrak{X}_P(k_P)$. The sequence of these numbers depends on the choice model, but it gives a well defined element of the adelic ring A_K, i.e. two such sequences differ for a finite number of elements. Since the number of rational points is additive with respect to the disjoint union and multiplicative with respect to the fibre product, it extends to a ring homomorphism $K_0(\mathrm{Var}_K) \to A_K$. This induces another morphism, between the ring of rational functions with coefficients in

$$\mathscr{N} : \mathscr{M}_K \left[\frac{\mathbb{L}^{-b}T^a}{1 - \mathbb{L}^{-b}T^a} : (a,b) \in \mathbb{Z}_{>0} \right] \to A'_K ,$$

where

$$A'_K = \left(\prod_{P \subseteq \mathcal{O}_K \ \max.} \mathbb{Q} \left[\frac{|k_P|^{-as-b}}{1 - |k_P|^{-as-b}} \right]_{(a,b \in \mathbb{Z}^2_{>0})} \right) / \left(\bigoplus_{P \subseteq \mathcal{O}_K \ \max.} \mathbb{Q} \left[\frac{|k_P|^{-as-b}}{1 - |k_P|^{-as-b}} \right]_{(a,b \in \mathbb{Z}^2_{>0})} \right) .$$

Let $f \in K[x_1, \dots, x_d]$ be a polynomial; it defines a morphism $f \colon \mathbb{A}^d_K \to \mathbb{A}^1_K$. For all but a finite number of maximal ideals $P \subseteq \mathcal{O}_K$, f defines also a morphism $\widehat{\mathcal{O}_K}^d \to \widehat{\mathcal{O}_K}$; the collection of the associated $P-$adic zeta functions $Z_{f,P}(s)$, as P vaires, define an element of A'_K.

Consider also the motivic zeta function associated to f, $Z_f(T)$. Then, a theorem of Denef and Loeser explains how this is related to the $P-$adic zeta functions above:

Theorem 3.5.2 ([29, Theorem 5.5])**.** *With the notation introduced above, the map $\mathscr{N} : \mathscr{M}_K \to A'_K$ sends $Z_f(T)$ in the sequence $(Z_{f,P}(s))_{P_{max}} \in A'_K$.*

3.6 Motivic integration over discretely valued fields

3.6.1 Let us fix a complete DVR R and let K denote its fraction field, while k denotes its residue field, which we assume to be algebraically closed. Denote by $\Delta := \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$.

3.6.2 Let $Y \to \Delta^*$ be a smooth Calabi Yau variety and let ω be a volume form on Y. Fix a weak Néron model, as in Definition 4.1.5, $\mathfrak{Y} \to \Delta$ of Y. For a connected component $C \in \pi_0(\mathfrak{Y}_0)$ let $\mathrm{ord}_C(\omega)$ be the order of ω, considered as a meromorphic function, on the generic point of C.

3.6.3 It follows from a result of Loeser and Sebag (see [27, Proposition 4.3.1]), that the following definition does not depend on the choice of the weak Néron model of Y:

Definition 3.6.4. (Motivic integral) With the same notation introduced in this paragraph, we call motivic integral of the volume form ω on Y the element of $\mathscr{M}_k$ given by the following sum:

$$\int_Y \omega d\mu = \sum_{C \in \pi_0 \mathfrak{Y}_0} [C] \mathbb{L}^{-\operatorname{ord}_C(\omega)}.$$

3.6.5 Now fix a Calabi Yau variety $X \to \Delta^*$, together with a volume form $\omega \in \omega_X(X)$.

For every positive integer m, define $X(m) \coloneqq X \times_{\Delta^*} \Delta^*(m)$. Since the map $\Delta^*(m) \to \Delta^*$ is an étale map, the basechange map $X(m) \to X$ is étale as well. The pull-back of ω through that map is thus a volume form on $X(m)$, which we denote by $\omega(m)$. Using this construction, a formal series with coefficients in $\mathscr{M}_k$ is defined:

Definition 3.6.6 (Motivic Zeta Function)**.** Keep the notation of this paragraph. The *Motivic Zeta Function* of X with respect to the volume form ω is the formal series

$$Z_{X,\omega}(T) \coloneqq \sum_{\substack{m \geq 1 \\ \operatorname{char} k \nmid m}} \left(\int_{X(m)} \omega(m) d\mu T^m \right).$$

Remark 3.6.7. Bultot and Nicaise [5, Definition 5.2.2] gave an alternative definition of the zeta function that involves also the motivic integrals over wild extensions of K. Their definition depends on the choice of the uniformizer of R if $\operatorname{char} k > 0$.

3.6.8 This formal series is know to be rational if X admits a log-smooth model over Δ. In such case [5, Theorem 5.3.1] provides an explicit formula for computing the zeta function.

3.6.9 The motivic integral and the Zeta function have an equivariant counterpart in $\mathscr{M}_k^G[[T]]$, provided that the volume form of the Calabi-Yau variety is chosen to be $G-$equivariant.

3.6.10 We conclude this section by explaining the main problem we are going to face. Let $\mathscr{R}$ be one of the three rings $\mathscr{M}_k, \mathscr{M}_k\left[(\mathbb{L}^r - 1)^{-1} \colon 0 < r \in \mathbb{N}\right]$ or $\widehat{\mathscr{M}_k}$. Assume that the wild inertia group of K acts trivially on X, so that the action of the absolute Galois group of K on X is identified with the action of its tame quotient.

Definition 3.6.11. Let σ a topological generator of $\operatorname{Gal}(\overline{K}|K)$. The induced operator $\sigma^* \colon H^*(X_{\overline{K}}, \mathbb{Q}_l) \to H^*(X_{\overline{K}}, \mathbb{Q}_l)$ is called *the monodromy operator* on the cohomology of X.

3.6.12 The monodromy operator is known to be quasi-unipotent, i.e. there are integers $a, b \in \mathbb{N}$ such that $((\sigma^*)^a - \mathrm{id})^b = 0$, so its eigenvalues, called *monodromy eigenvalues* are roots of the unity; the *monodromy conjecture* states that there is a relation between these eigenvalues and the poles of the zeta function of X. We give a statement of the monodromy conjecture that depends on how the ring of coefficients for the zeta function is interpreted:

Conjecture 3.6.13 (Monodromy conjecture in $\mathscr{R}$)**.** *Let $X \to \operatorname{Spec} K$ be a Calabi-Yau variety and let ω be a volume form on it. Let $q \in \mathbb{Q}$ be a pole of $Z_{X,\omega}(T) \in \mathscr{R}[[T]]$, then $e^{2\pi i q}$ is a monodromy eigenvalue of X.*

3.6.14 The monodromy conjecture in $\mathscr{M}_k$ implies the monodromy conjecture in $\mathscr{M}_k\left[(\mathbb{L}^r - 1)^{-1} \colon 0 < r \in \mathbb{N}\right]$ by the Remark 3.3.6. In turn, the monodromy conjecture in $\mathscr{M}_k\left[(\mathbb{L}^r - 1)^{-1} \colon 0 < r \in \mathbb{N}\right]$ implies the version in $\widehat{\mathscr{M}_k}$.

3.6.15 The monodromy conjecture has been proven in several classes of varieties: Halle and Nicaise proved it for Abelian varieties, [17], and for Hilbert schemes of points of $K3$ surfaces with potential good reduction, [18]. Jaspers proved it in [23] when X is a K3 surface admitting a Crauder-Morrison model and Overkamp proved it for Kummer K3 surfaces in [36]. Yet we do not know whether all the K3 surfaces satisfy the Monodromy conjecture. We are going to prove later that if the monodromy conjecture holds for a surface X, then it also holds for $\mathrm{Hilb}^n(X)$ for all $n \in \mathbb{N}$, if $\operatorname{char} k = 0$ or for all $n \leq p$ if $\operatorname{char} k = p > 0$.

Part II

Models of varieties

Chapter 4

Weak Néron models

4.1 Definition and basic constructions

4.1.1 In order to define the motivic zeta function we will need to introduce the notion of a weak Néron model and to develop some techniques involved in the construction of such models.

4.1.2 The results we are going to state in this section hold in the context of algebraic spaces, but we will not work in such generality, thus we state them only in the context of schemes.

Definition 4.1.3. Let R be a DVR and K its fraction field and denote by $\Delta := \operatorname{Spec} R$. Let $X \to \operatorname{Spec} K$ be a smooth morphism of schemes. A model for X over Δ is a flat morphism $\mathfrak{X} \to \Delta$ of schemes together with an isomorphism $\mathfrak{X}_K \tilde{\to} X$ in Sch_K.

Moreover we say that the model $\mathfrak{X}$ has a property $\mathbf{P}$ (e.g. smooth, proper) if the morphism $\mathfrak{X} \to \Delta$ has such property.

4.1.4 The notion we are mostly interested in is that of *Weak Néron Model*:

Definition 4.1.5. We say that a model $\mathfrak{X} \to \Delta$ of X has the weak extension property if for any finite étale morphism $Z \to \Delta$, there is a bijection $\operatorname{Hom}_\Delta(Z, \mathfrak{X}) \tilde{\to} \operatorname{Hom}_K(Z_K, X)$, $(f : Z \to \mathfrak{X}) \mapsto f|_{Z_K}$.

A smooth model $\mathfrak{X} \to \Delta$ of $X \to \operatorname{Spec} K$ that satisfies the weak extension property is said to be a *weak Néron model* of X.

Remark 4.1.6. Let $X \to \operatorname{Spec} K$ be a smooth scheme. A weak Néron model of X over Δ always exists: the following example shows a way to construct weak Néron models starting from proper models.

Example 4.1.7. Let $X \to \operatorname{Spec} K$ be a smooth and proper variety and let $\mathcal{X} \to \operatorname{Spec} R$ be a proper regular model of X, then the smooth locus of $\mathcal{X} \to \operatorname{Spec} R$ is a weak Néron model of X. Consider an étale map $Z \to \Delta$ and a map $Z_K \to X$; due tu the valuative criterion for properness, there is an extension $Z \to \mathfrak{X}$; we need to show that the image of such map is contained in $\mathfrak{X}_{\mathrm{sm}}$. Assume by contradiction that this map meets $\mathfrak{X}_{\mathrm{sing}}$, then the composition $Z \to \mathfrak{X} \to \Delta$ would be ramified, contraddicting the étaleness of $Z \to \Delta$.

4.2 Weil restriction of scalars

4.2.1 In this paragraph we study some generalities about the functor of the restriction of scalars. The main content of this paragraph is Proposition 4.2.6, which shall allow us to construct weak Néron models of a finite, tamely ramified base-change of a given scheme $X \to K$.

Definition 4.2.2. Let $S' \to S$ be a morphism of schemes and let $\mathcal{Y} \to S'$ be a scheme. The functor $\mathrm{Res}_{S'/S}(\mathcal{Y})\colon (\mathrm{Sch}_S)^{\mathrm{opp}} \to \mathrm{Sets}$ defined by $T \mapsto \mathcal{Y}(T \times_S S')$ is called the Weil restriction of scalars of $\mathcal{Y}$ along $S' \to S$.

When $\mathrm{Res}_{S'/S}(\mathcal{Y})$ is represented by a scheme, we say that the Weil restriction of $\mathcal{Y}$ along $S' \to S$ exists.

4.2.3 It follows from [3, Theorem 7.6] that if $S' \to S$ is a finite, flat and locally of finite presentation and $\mathcal{Y} \to S'$ is quasi-projective, then the Weil restriction of $\mathcal{Y}$ along $S' \to S$ exists. This will always be the case, throughout this manuscript.

Remark 4.2.4. Consider arbitrary morphisms of schemes $S' \to S$ and $\mathcal{X} \to S$; then the universal property of fibered products implies the following:

$$\mathrm{Res}_{S'/S}(\mathcal{X} \times_S S') = \underline{\mathrm{Hom}}_S(S', \mathcal{X})\,,$$

where $\underline{\mathrm{Hom}}_S(S', \mathcal{X})$ is the fpqc-sheaf $T \mapsto \mathrm{Hom}_T(S' \times_S T, \mathcal{X} \times_S T)$.

4.2.5 Let R be a complete DVR and let K be its fraction field and k be its residue field. We assume k is algebraically closed. Let $K \subseteq L$ be a finite, tame, Galois extension with $\mathcal{G} := \mathrm{Gal}(L|K)$ and denote by R_L the integral closure of R in L. Let $X \to \mathrm{Spec}\, R_L$ be a $\mathcal{G}-$equivariant morphism. For an arbitrary scheme $T \to \mathrm{Spec}\, R$, the action of $\mathcal{G}$ on $\mathrm{Spec}\, R_L$ induces an action on T_{R_L}, thus, as constructed in [11, Construction 2.4], a right action on the Weil restriction, $\mathrm{Res}_{R_L/R}(\mathcal{X})$: more precisely, given $g \in \mathcal{G}$, it induces an automorphisms $\rho_{T_{R_L}}(g)\colon T_{R_L} \to T_{R_L}$ and an automorphism $\rho_X(g)\colon X \to X$; the action of $\mathcal{G}$ sends the point corresponding to the morphism $\psi\colon T_{R_L} \to \mathcal{X}$ to the composition $g(\psi) := \rho_X(g) \circ \psi \circ \rho_{T_{R_L}}(g)^{-1}$. The following proposition, already proved in [19, Theorem 3.1], provides a recipe that we will use for constructing weak Néron models of varieties:

Proposition 4.2.6. *Let $\mathfrak{X}' \to R_L$ be a $\mathcal{G}-$equivariant weak Néron model for X_L, then $\mathfrak{X} := \left(\mathrm{Res}_{R_L/R}\, \mathfrak{X}'\right)^{\mathcal{G}}$ is a weak Néron model for X.*

Proof. It follows from [6, Proposition A.5.2] that the operations of taking the generic fibre and taking the restriction of scalars commute, therefore

$$(\mathrm{Res}_{R_L/R}\, \mathfrak{X}')_K = \mathrm{Res}_{L/K}\, X_L = \underline{\mathrm{Hom}}_K(\mathrm{Spec}\, L, X)\,,$$

where the last equality follows from Remark 4.2.4.

Let $T \to \mathrm{Spec}\, K$ be a scheme, then a morphism $T_L \to X \times_{\mathrm{Spec}\, K} T$ is $\mathcal{G}-$invariant if and only if it factors through $T_L \to T_L/\mathcal{G} = T$, this gives a bijection between $\mathrm{Hom}_T(T_L, X \times_{\mathrm{Spec}\, K} T)$ and the set of sections of $X \times_{\mathrm{Spec}\, K} T \to T$, which in turn is $\mathrm{Hom}_K(T, X) = X(T)$. Therefore we have that

$$\left((\mathrm{Res}_{R_L/R}\, \mathfrak{X}')^{\mathcal{G}}\right)_K \cong X\,.$$

Since $\mathfrak{X}' \to \mathrm{Spec}\, R_L$ is a weak Néron model for X_L, it is in particular a smooth morphism, thus, by [6, Proposition A.5.2] also $\mathrm{Res}_{R_L/R}(\mathfrak{X}') \to \mathrm{Spec}\, R$ is smooth. It follows by [11, Proposition 3.4] that the $\mathcal{G}-$fixed locus $\mathfrak{X}$ is smooth as well.

In order to conclude the proof, we only need to show that all the $K-$valued points of X extend to $R-$valued points of $\mathfrak{X}$, since we assumed R to be complete and k to be algebraically closed.

A morphism $\mathrm{Spec}\, K \to X$ induces a unique $\mathcal{G}-$equivariant morphism $\mathrm{Spec}\, L \to X_L$. Since $\mathfrak{X}'$ is a weak Néron model for X_L, such map extends to a unique map $\mathrm{Spec}\, R_L \to \mathfrak{X}'$, which is also $\mathcal{G}-$equivariant and correspond, by the definition of the restriction of scalars, to a $\mathcal{G}-$invariant map $\mathrm{Spec}\, R_L \to \mathrm{Res}_{R_L/R}\, \mathfrak{X}'$ which, by $\mathcal{G}-$invariance, factors through a map $\mathrm{Spec}\, R_L \to \mathfrak{X} \subseteq \mathrm{Res}_{R_L/R}\, \mathfrak{X}'$. $\square$

4.2.7 The following lemma says that this construction is well behaved with respect to a tower of extensions:

Lemma 4.2.8. *Let $K \subseteq F \subseteq L$ be a tower of finite tame extensions such that also $K \subseteq L$ is normal. Let $G := \mathrm{Gal}(L|K)$, $N := \mathrm{Gal}(L/F)$ and $G/N = H := \mathrm{Gal}(F|K)$. Denote by R_F and R_L the integral closures of R in F and L, respectively, and Δ_F, Δ_L their spectra. Let $\mathcal{F}\colon \mathrm{Sch}^{\mathrm{opp}}_{\Delta_L} \to \mathrm{Sets}$ be a functor endowed with an action of G compatible with its action on Δ_L.*

Then the following two functors are naturally isomorphic:

$$\left(\mathrm{Res}_{\Delta_L/\Delta}\,\mathcal{F}\right)^G \cong \left(\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)^N\right)^H .$$

Proof. The left hand side is equal to

$$\left(\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)\right)^G = \left(\left(\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)\right)^N\right)^H ,$$

where N acts as a subgroup of G and $H = G/N$ inherits the action of G on the $N-$invariant locus. Thus, we only need to show that there is an $H-$equivariant isomorphism of functors:

$$\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)^N \cong \left(\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)\right)^N .$$

Given a scheme morphism $T \to \Delta$, we have that

$$\begin{aligned}
\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)^N(T) &= \left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)^N(T \times_\Delta \Delta_F) \\
&= \left(\mathcal{F}(T \times_\Delta \Delta_L)\right)^N \\
&= \left(\mathrm{Res}_{\Delta_F/\Delta}\left(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F}\right)(T)\right)^N ,
\end{aligned}$$

and we are done. □

4.2.9 The construction above can be made more explicit: in the following paragraphs we will describe the central fibre and the canonical divisor of $\mathfrak{X}$ in terms of $\mathfrak{X}'$.

4.3 Weil restriction and the central fibre

4.3.1 This subsection and the next one summarize some results contained in an unpublished manuscript of Lars Halle and Johannes Nicaise. I am grateful to them for letting me use these results which are crucial for the computation of the motivic integral in §10.3. We keep the notation of the previous paragraph.

4.3.2 The inclusion $\mathfrak{X} \subseteq \mathrm{Res}_{R_L/R}\,\mathfrak{X}'$, as in Proposition 4.2.6, corresponds, according to the definition of the restriction of scalars, to a map of R_L-schemes

$$h\colon \mathfrak{X} \times_R R_L \to \mathfrak{X}' ,$$

which gives, over the special fibres a morphism of $k-$schemes:

$$h_k\colon \mathfrak{X}_k \to \mathfrak{X}'_k .$$

4.3.3 On the other hand, we can characterize $\mathfrak{X}_k$ in a different way, using the *Greenberg schemes*. The following definition will cover the cases we will use:

Definition 4.3.4. Let $d = [L : K]$ and let $\mathfrak{m}$ be the maximal ideal of R_L; for $i \in \{0, \dots, d-1\}$, let $R_{L,i} \coloneqq R_L/(\mathfrak{m}^{i+1})$. For a separated, smooth morphism $\mathcal{A} \to \operatorname{Spec} R_L$, consider the functor

$$\operatorname{Gr}_i(\mathcal{A}) \coloneqq \operatorname{Res}_{R_{L,i}/k}(\mathcal{A} \times_{R_L} R_{L,i}) \,,$$

which is representable by a separated, smooth scheme, as it follows from the proof of [3, Proposition 7.6]; this is also called the *level i Greenberg scheme of $\mathcal{A}$*.

Remark 4.3.5. Clearly, $\operatorname{Gr}_0(\mathcal{A}) = \mathcal{A}_k$, while $\operatorname{Gr}_{d-1}(\mathcal{A}) = \operatorname{Res}_{R_L/R}(\mathcal{A})_k$, since $d = [L : K]$ is also the ramification index of $\operatorname{Spec} R_L \to \operatorname{Spec} R$ at their closed points.

Remark 4.3.6. The rings $R_{L,i}$ inherit from R_L a $\mathcal{G}-$action, thus we get a $\mathcal{G}-$action on the Greenberg schemes as in §4.2.5.

4.3.7 In our case we have that $\mathfrak{X}_k = (\operatorname{Gr}_{d-1}(\mathfrak{X}'))^{\mathcal{G}}$. Indeed, if T is a $k-$scheme, we have that

$$\mathfrak{X}_k(T) = \left\{f \colon T \times_R R_L \to \mathfrak{X}'\right\}^{\mathcal{G}} = (\operatorname{Gr}_{d-1}(\mathfrak{X}')(T))^{\mathcal{G}} \,.$$

4.3.8 The natural truncation maps of Greenberg schemes define $\mathcal{G}-$equivariant affine bundles, in particular $\operatorname{Gr}_{d-1}(\mathfrak{X}') \to \operatorname{Gr}_0(\mathfrak{X}') = \mathfrak{X}'_k$ is a composition of affine bundles. By taking the $\mathcal{G}-$invariant loci of this map, we get a description, at least locally, of $\mathfrak{X}_k$ as an affine bundle over $(\mathfrak{X}'_k)^{\mathcal{G}}$, in the sense that for each connected component $C \subseteq \mathfrak{X}_k$, there is a connected component $C' \subseteq (\mathfrak{X}'_k)^{\mathcal{G}}$ such that C is an $\mathbb{A}^r_k-$bundle over C', where $r = \dim(\mathfrak{X}'_k) - \dim C'$. In particular the following relation holds in $K_0(\mathrm{Var}_k)$:

$$[C] = \mathbb{L}^{\dim(\mathfrak{X}_k) - \dim C'}[C'] \,.$$

4.4 Weil restriction and canonical divisor

4.4.1 Let us keep the notation introduced in the previous paragraph, but we also assume that X is a Calabi-Yau variety, i.e. it has trivial canonical bundle, and that a volume form $\omega \in \Omega^{\dim X}_{X/K}(X)$ is given. Let $\omega_L \in \Omega^{\dim X}_{X_L/L}(X_L)$ be the pull-back of ω under the base-change map. In this paragraph we will study the order of vanishing of ω on each component of $\mathfrak{X}_k$, which we will define as follows, adapting the definition given in [27, §4.1].

4.4.2 Let $p \in \mathfrak{X}_k$ be a closed point. Since R is a Henselian ring and since $\mathfrak{X} \to \operatorname{Spec} R$ is smooth, there is at least a section $\psi \colon \operatorname{Spec} R \to \mathfrak{X}$ such that $\psi(0) = p$. Consider the line bundle $L \coloneqq \psi^*\Omega^{\dim X}_{\mathfrak{X}/R}$ over $\operatorname{Spec} R$. There is $a \in \mathbb{Z}$ such that $\pi^a\omega$ extends to a global section $\omega' \in \Omega^{\dim X}_{\mathfrak{X}/R}(\mathfrak{X})$, where $\pi \in R$ is the uniformizer. So its pull-back $\psi^*(\omega')$ is a global section of L. Let $M \coloneqq L/\psi^*\omega'\mathcal{O}_{\operatorname{Spec} R}$ be the quotient of $\mathcal{O}_{\operatorname{Spec} R}-$modules.

Definition 4.4.3. The order of ω at p is defined as:

$$\operatorname{ord}_p(\omega) \coloneqq \inf\{b \in \mathbb{N} \colon \pi^b M = 0\} - a \,.$$

If $C \subseteq \mathfrak{X}_k$ is a connected component, then $\operatorname{ord}_p(\omega)$ does not depend on the coice of the closed point $p \in C$, so we define $\operatorname{ord}_C(\omega)$ as the order of ω at any of its closed point. If $\operatorname{ord}_C(\omega) > 0$ we say that C is a *zero* of ω, if $\operatorname{ord}_C(\omega) < 0$ we say that it is a *pole* of ω.

Remark 4.4.4. If ω extends to a global section of $\Omega^{\dim X}_{\mathfrak{X}/R}$, then the order of ω at p is

$$\operatorname{ord}_p(\omega) = \operatorname{length} L/\psi^*(\omega)\mathcal{O}_{\operatorname{Spec} R} \,.$$

4.4.5 Let Z be a smooth scheme defined over a field F and let $V \to Z$ be a vector bundle over Z. If $\operatorname{char} k = p > 0$ assume $\gcd(p, d) = 1$ and consider the cyclic group $G \cong \mu_d$ acting equivariantly on $V \to Z$ and let $z \in Z$ be a fixed point. There is a unique sequence of integers $(j_1, j_2, \ldots, j_{\operatorname{rk} V})$ such that $0 \le j_1 \le j_2 \le \cdots \le j_{\operatorname{rk} V} \le d-1$ such that V_z has a basis $v_1, \ldots, v_{\operatorname{rk} V}$ of eigenvectors such that $\zeta \star v_i = \zeta^{-j_i} \cdot v_i$ (where ζ is any generator of μ_d); the tuple $(j_i)_i$ is called the *tuple of exponents* of the $G-$action.

Definition 4.4.6. We define the *conductor* of the action of G in z as the sum:

$$c(V, z) := \sum_{i=1}^{\operatorname{rk} V} j_i \, .$$

If $C \subseteq Z^G$ is an irreducible subscheme, then for all $z, z' \in C$ one has that $c(V, z) = c(V, z')$, so we simply denote by $c(V, C)$ either of the conductors. Moreover we denote $c(Z, C)$ the conductor $c(T_Z, C)$.

Lemma 4.4.7. *Let C be a connected component of $\mathfrak{X}_k$ and let $C' = h(C)$, where h is the map defined in §4.3.2. Then:*

$$\operatorname{ord}_C(\omega) = \frac{\operatorname{ord}_{C'}(\omega_L) - c(\mathfrak{X}'_k, C')}{[L\colon K]} \, .$$

Proof. Let $\psi'\colon \operatorname{Spec} R_L \to \mathfrak{X}'$ be a section that lifts ψ. The map $\mathfrak{X} \times_R R_L \to \mathfrak{X}'$ induces a monomorphism

$$\alpha\colon (\psi')^* \Omega_{\mathfrak{X}'/R_L} \to \psi^* \Omega_{\mathfrak{X}/R} \otimes_R R_L$$

sending ω_L to $\omega \otimes 1$; in particular

$$\operatorname{length}\left(\alpha\left((\psi')^*\Omega_{\mathfrak{X}'/R_L}\right)/\langle \omega \otimes 1\rangle\right) = [L:K] \operatorname{length}\left((\psi')^*\Omega_{\mathfrak{X}'/R_L}/\langle \omega_L \rangle\right) \, ,$$

thus the statement shall follow from the fact that $\operatorname{length} \operatorname{coker} \alpha = c(\mathfrak{X}'_k, C')$.

On the other hand, under the identification

$$T_{\mathfrak{X}/R} = \underline{\operatorname{Hom}}_R(R[\varepsilon]/(\varepsilon^2), \mathfrak{X}) = \underline{\operatorname{Hom}}_{R_L}(R_L[\varepsilon]/(\varepsilon^2), \mathfrak{X}')^{\operatorname{Gal}(L|K)} = T_{\mathfrak{X}'/R_L}^{\operatorname{Gal}(L|K)} \, ,$$

the tangent map $T_h\colon T_{\mathfrak{X}/R \times_R R_L} \to T_{\mathfrak{X}'/R_L}$ induces a map

$$\beta\colon (\psi')^*(T_{\mathfrak{X}'/R_L})^{\operatorname{Gal}(L|K)} \otimes_R R_L \to (\psi')^* T_{\mathfrak{X}'/R_L} \, .$$

Fix a base of eigenvectors of $T_{\mathfrak{X}'_k}$; the upcoming Lemma 4.4.8, applied to the subspaces generated by each element of the base, implies that $\operatorname{coker}(\beta) = \oplus_{i=1}^d R_L/\mathfrak{m}_L^{j_i}$, hence

$$\operatorname{coker}(\alpha) = \bigwedge_{i=1}^{d} R_L/\mathfrak{m}_L^{j_i} = R_L/\mathfrak{m}_L^{c(\mathfrak{X}, C')} \, .$$

□

Lemma 4.4.8. *Let M a free R_L-module of rank 1. Assuming that $\operatorname{Gal}(L|K)$ acts $R-$linearly on M from the left. Let j be the exponent of the action induced on $M \otimes_{R_L} k$, as in Definition 4.4.6. Then the natural morphism $M^{\operatorname{Gal}(L|K)} \otimes_R R_L \to M$ has cokernel isomorphic to $R_L/\mathfrak{m}_L^j$.*

Proof. Let us fix an element $v \in M$ such that $0 \neq v \otimes 1 \in M \otimes_{R_L} k$; by our hypothesis we have that $(\zeta * v) \otimes 1 = \zeta^{-j} v \otimes 1$. Let π_L a uniformizer for R_L such that $\pi_L^d \in K$; if $0 \le b \le d-1$ the vectors $v_b := \pi_L^b v \otimes 1 \in M \otimes_R k$ form a base of the vector space $M \otimes_R k \cong M \otimes_{R_L} R_L/\mathfrak{m}_L^d$. Moreover that one is a base of $G-$eigenvectors, for $\zeta * v_b = \zeta^{b-j} v_b$.

By Henselianity we can lift the base $\{v_b\}$ to an $R-$base $\{w_b\colon 0 \leq b \leq d-1\}$ of M such that $\zeta * w_b = \zeta^{b-j} w_b$.

Now let $x = a_0 w_0 + \cdots + a_{d-1} w_{d-1} \in M$ be an arbitrary element. We have that $x \in M^G$ iff $x - \zeta * x = 0$, i.e. iff

$$\sum_{b=0}^{d-1} a_b (1 - \zeta^{b-j}) w_b = 0 \,,$$

therefore, the $R-$module M^G is generated by w_j.

It follows that $M^G \otimes R_L$ is sent onto $\langle w_j \rangle \subseteq M$, which leads to our coveted statemet. $\square$

Chapter 5

Fundaments of Logarithmic geometry

Logarithmic geometry was introduced with the purpose of dealing with the algebro-geometric version of differential manifolds with boundary, i.e. for studying the compactification of non-proper varieties and the degenerations of families.

The datum of a logarithmic scheme consist in a scheme X and a sheaf of monoids $\mathcal{M}_X$, called *log structure*, together with a morphism $\mathcal{M}_X \to \mathcal{O}_X$. This datum is meant to keep track of what is the "ineer part" of the algebraic variety and which one is the "boundary". In this manuscript, we will be dealing with manifolds defined over a discretely valued field K; the role of "variety with boundary" is played by models over the ring of integers $R \subseteq K$, so that the central fibre will be the boundary. We will see how a theory of resolution of singularities has been developed in the context of log schemes; this is broadly used for constructing semistable models, which is indeed what we will aim for.

We intend to follow Ogus' book [35] and Gabber's and Ramero's notes [16], which cover all the material that we shall use in the subsequent parts.

All the monoids we will encounter througout this thesis shall be abelian, thus we denote by Mon the category of abelian monoids with their morphisms.

5.1 Operations with monoids

Definition 5.1.1. Let P be a monoid and let $u_i \colon P \to Q_i$, for $i = 1, 2$, be morphisms of monoids. The *amalgamate sum* of u_1, u_2 (or more simply of Q_1 and Q_2 if no confusion may arise) is denoted by $Q_1 \oplus_P Q_2$ and is defined as the colimit of the diagram:

$$Q_1 \leftarrow P \to Q_2 \,.$$

5.1.2 If $P = \{1\}$, then the amalgamated sum coincide with the direct sum of Q_1, Q_2; in general it is isomorphic to the quotient of $Q_1 \oplus Q_2$ with respect to the congruence relation generated by $\{u_1(p) \sim u_2(p) \colon p \in P\}$; it coincides with the coproduct in the category of monoids "under" P. One has a remarkable case for $Q_2 = \{1\}$, since $Q_1 \oplus_P \{1\}$ is the *cokernel* of the morphism u_1 and can be denoted also as $\operatorname{coker} u_1 = Q_1/P$.

Example 5.1.3. Let $P = \mathbb{N}$, let $Q_1 = \mathbb{N}^2$ with $u_1(1) = (1, 1)$ and let $Q_2 = \mathbb{N}$ with $u_2(1) = 2$. Then $Q_1 \oplus_P Q_2 = \mathbb{N}^3/\sim$, where $(a, a, 0) \sim (0, 0, 2a)$, for $a \in \mathbb{N}$; in other words, the result is isomorphic to the submonoid of $\mathbb{Q}^2$ generated by $(1, 0), (0, 1)$ and $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

5.1.4 Let $M^\times$ be the set of invertible elements of M; it inherits a structure of monoid from M and is, in fact, a group. If $M^\times = \{1\}$, then M is said to be sharp. In any case it is always possible to construct a sharp monoid as the cokernel $M^\sharp := M/M^\times$. The monoid $M^\sharp$ constructed in this way is called the *sharpification* of M.

Definition 5.1.5. A morphism of monoids $\varphi\colon M \to N$ is said to be local if $M^\times = \varphi^{-1}(N^\times)$.

5.1.6 For a monoid M, we denote by $M^\vee := \mathrm{Hom}(M, \mathbb{N})$ its *dual monoid*, while $M^{\vee,\mathrm{loc}}$ will denote its submonoid of local morphisms.

5.1.7 Let M a monoid and $S \subseteq M$ a submonoid. The *localisation* of M with respect to S is the map of monoids $M \to S^{-1}M$ that satisfy the following universal property: ***If*** $\varphi\colon M \to N$ ***is a morphism of monoids, such that*** $\varphi^{-1}(N^\times) \subseteq M^\times S$***, then*** φ ***factors in a unique way as*** $M \to S^{-1}M \to N$***.*** If $f \in M$ and $S = \{f^n : n \in \mathbb{N}\}$, we use the notation $M_f := S^{-1}M$. If $S = M$, then $S^{-1}M$ is an abelian group, called the *groupification* of M and denoted by M^{grp}. It follows direclty from the universal properties that we have a canonical isomorphism:

$$(Q_1 \oplus_P Q_2)^{\mathrm{grp}} \cong Q_1^{\mathrm{grp}} \oplus_{P^{\mathrm{grp}}} Q_2^{\mathrm{grp}} \,,$$

for any pair of morphisms $u_i\colon P \to Q_i$.

5.2 Properties of monoids

5.2.1 We introduce here some terminology concerning monoids that will be crucial in the context of the logarithmic geometry.

Definition 5.2.2. A monoid M is said to be finitely generated if there is a finite number of elements $x_1, \dots, x_n$ such that for all $x \in M$ there are $a_1, \dots, a_n \in \mathbb{N}$ such that $x = \prod_{j=1}^n x_j^{a_j}$.

Definition 5.2.3. A monoid M is *integral* if the morphism $M \to M^{\mathrm{grp}}$ is injective. Morover, if M is finitely generated and integral, it is called *fine*.

Definition 5.2.4. Let M be an integral monoid. If $\forall x \in M^{\mathrm{grp}}$, the implication

$$\text{"}\ (\exists n \in \mathbb{N}\backslash\{0\} \colon x^n \in M) \Rightarrow x \in M\text{"}$$

holds, then M is said *saturated*. We use the abbreviation *fs* for the monoids that are at the same time fine and saturated (or, equivalently, finitely generated and saturated), which are the most relevant in our dissertation.

5.2.5 If M is a monoid, we can construct an integral monoid, M^{int}, out of it simply taking $\mathrm{im}(M \to M^{\mathrm{grp}})$. Moreover we can construct its saturation M^{sat} as

$$\{x \in M^{\mathrm{grp}} \colon \exists \alpha \in \mathbb{N}^*, x^\alpha \in M^{\mathrm{int}}\}\,.$$

Definition 5.2.6. A morphism of monoids $P \to M$ is said to be *integral* (resp. *saturated*), if, for every *integral* (resp. *saturated*) monoid N and morphism $P \to N$, the amalgamated sum $M \oplus_P N$ is *integral* (resp. *saturated*).

5.3 Ideals, faces and localisation

5.3.1 In this section we will introduce some fundamental notions that will allow us to define geometrical properties of monoids.

Definition 5.3.2. Let $(M, \cdot)$ be a monoid. A subset $I \subseteq M$ is an *ideal* if $\forall a \in M$, $aI \subseteq I$. A proper ideal is called *prime* if $\forall x, y \in M$ such that $xy \in I$, either $x \in I$ or $y \in I$. A submonoid of M such that $\forall x, y \in M$, $xy \in I \Rightarrow x, y \in I$, is called a *face*. If M is *fs*, we cal a non-empty subset $I \subseteq M^{\text{grp}}$ a *fractional ideal* if there exist a finite collection of elements $a_1, \dots, a_r \in I$ such that $I = \bigcup_{i=1}^{r} a_i M$.

5.3.3 No proper ideal of M contains an invertible element, thus M admits a maximum proper ideal $M^+ := M \backslash M^\times$, which is also prime. Notice that a morphism of monoids $f \colon M \to N$ is local if and only if $f^{-1}(N^+) = M^+$, coherently with the therminology of local morphisms of local rings.

5.3.4 Notice that a face is nothing else than the complement of a prime ideal and the set of units is the smallest face of M. In particular we can define the localisation of M with respect to any prime $\mathfrak{p} \subseteq M$ as follows:

$$M_{\mathfrak{p}} := (M \backslash \mathfrak{p})^{-1} M \,.$$

Definition 5.3.5. Let $\mathfrak{p} \subseteq M$ be a prime ideal.

- The height of $\mathfrak{p}$, $\text{ht}(\mathfrak{p})$, is the maximum length h of a chain of primes:

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_h \,.$$

- The dimension of M is the maximum lenght $\dim M = d$ of ascending chains:

$$\emptyset = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = M^+ \,.$$

5.3.6 Similarly to the case of the Krull dimension for rings, we have the following relation between height of a prime and dimension of a localised monoid:

$$\dim M_{\mathfrak{p}} = \text{ht}(\mathfrak{p}) \,.$$

Example 5.3.7. If $\text{ht}(\mathfrak{p}) = 1$, then $M_{\mathfrak{p}}^\sharp \cong \mathbb{N}$. This isomorphism induces (by composition with the localisation map and quotient by $M_{\mathfrak{p}}^\times$) a map $v_{\mathfrak{p}} \colon M \to \mathbb{N}$ called the *valuation* with respect to $\mathfrak{p}$ and it extends to a group homomorphism $v_{\mathfrak{p}} \colon M^{\text{grp}} \to \mathbb{Z}$.

5.4 Monoidal spaces

5.4.1 Before moving towards the core of logarithmic geometry, i.e. log schemes, we introduce the notion of monoidal spaces and study some of their properties. The logarithmic schemes will be a particular case of monoidal space, i.e. schemes whose underlying topological space has a monoidal structure interacting with the ringed structure.

5.4.2 A *monoidal space* $(T, \mathcal{M}_T)$ is a topological space T endowed with a sheaf of monoids $\mathcal{M}_T$. Morphisms of monoidal spaces $(T', \mathcal{M}_{T'}) \to (T, \mathcal{M}_T)$ consist of a continuous map $f \colon T' \to T$ and a map of sheaves $f^{-1}\mathcal{M}_T \to \mathcal{M}_{T'}$ whose induced map on the stalks is a local morphism of monoids.

5.4.3 A monoidal space is called *sharp* if the monoids associated to all the open sets are sharp. Every monoidal space $(T, \mathcal{M}_T)$ can be replaced by a sharp monoidal space, called its *sharpification* $(T, \mathcal{M}_T^\sharp)$, obtained as the sheafification of

$$U \mapsto \mathcal{M}_T(U)^\sharp .$$

Definition 5.4.4. A *chart* for $(T, \mathcal{M}_T)$ subordinate to the monoid P is a monoid homomorphism $P \to \Gamma(T, \mathcal{M}_T)$ that induces an isomorphism

$$\underline{P} \oplus_{\theta^{-1}(\mathcal{M}_T^\times)} \mathcal{M}_T^\times \xrightarrow{\sim} \mathcal{M}_T ,$$

where $\underline{P}$ is the $P-$valued constant sheaf of monoids and $\theta \colon \underline{P} \to \mathcal{M}_T$ is the associated map. A chart subordinate to P is said to be *coherent* (resp. *integral, fine, saturated*) if P is finitely generated (resp. integral, fine, saturated).

5.4.5 A basic example of a monoidal space is the spectrum of a monoid P, which as a topological space is the set $\operatorname{Spec} P$ of its prime ideals, endowed with the topology generated by $\{D(f) \colon f \in P\}$, where $D(f) = \{\mathfrak{p} \in \operatorname{Spec} P \colon f \notin \mathfrak{p}\}$; equivalently we could define the family of closed sets of this topology, i.e. $\{Z(I) \colon I \text{ is an ideal of } P\}$, where $Z(I) = \{\mathfrak{p} \supseteq I\}$; in analogy with the terminology of rings, this is called the *Zariski topology.* Let $\mathcal{M}_P$ be the sheaf of monoids such that $\mathcal{M}_P(D(f)) = P_f$. The space $(\operatorname{Spec} P, \mathcal{M}_P)$, or simply $\operatorname{Spec} P$, is called the *spectrum* of P. By construction we have that $\forall \mathfrak{p} \in \operatorname{Spec} P$, the stalk of the sheaf $M_{P,\mathfrak{p}}$ coincide with $P_\mathfrak{p}$. We can sharpify this construction and obtain $(\operatorname{Spec} P)^\sharp$, called the *sharp spectrum* of P.

5.4.6 We can glue together spectra of monoids and construct all the monoidal space that we shall be interested in:

Definition 5.4.7. A *fan* is a sharp monoidal space $(F, \mathcal{M}_F)$ such that every point admits a neighbourhood isomorphic to a spectrum of a monoid. We call spectra of monoids *affine fans.*

A fan is said to be

- *locally finite* (resp. *finite*) if it can be covered with (resp. a finite number of) spectra of finitely generated monoids;
- *saturated* if it can be covered with spectra of saturated monoids;
- *locally fs* (resp. *fs*) if it can be covered with (resp. a finite number of) spectra of fs monoids.

5.4.8 There is a notion of regularity and of resolution of fans that recalls the theory already developed for rings and schemes:

Definition 5.4.9. A fan F is said to be regular at $t \in F$ if the stalk $M_{F,t}$ is isomorphic to $\mathbb{N}^{\dim M_{F,t}}$. We denote by F_{reg} the set of regular points.

A fan is regular if it is regular at every point.

Definition 5.4.10. Let $(F, \mathcal{M}_F)$ be a fan. A subdivision is a morphism of fans $\varphi \colon (F', \mathcal{M}_{F'}) \to (F, \mathcal{M}_F)$ such that:

- For every $t \in F'$, the map induced on stalks $\mathcal{M}^{\text{grp}}_{F,\varphi(t)} \to \mathcal{M}^{\text{grp}}_{F,t}$ is surjective;
- The composition with φ induces a bijection $\operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F') \to \operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F)$.

Proposition 5.4.11 ([16, Theorem 3.6.31])**.** *Let F be a locally fs fan. There is a subdivision $\varphi \colon F' \to F$ such that F' is regular and $\varphi^{-1}(F_{\text{reg}}) \to F_{\text{reg}}$ is an isomorphism.*

5.4.12 A method for constructing subdivisions of fans is given by the blowing up of fractional ideals.

Definition 5.4.13. Let F be a fan. A subsheaf $\mathcal{I} \subseteq \mathcal{M}_F^{\mathrm{grp}}$ is a fractional ideal sheaf if $\mathcal{M}_F\mathcal{I} \subseteq \mathcal{I}$ and, for $U \subseteq F$ affine, the subset $\Gamma(U,\mathcal{I}) \subseteq \Gamma(U,\mathcal{M}_F^{\mathrm{grp}})$ is a fractional ideal and $\mathcal{I}|_U = \Gamma(U,\mathcal{I})\mathcal{M}_F|_U$.

5.4.14 Let P be a monoid and $I \subseteq P^{\mathrm{grp}}$ a fractional ideal, we define the blow-up of $\operatorname{Spec} P$ at I as the fan obtained in the following way: For $a \in I$ let $P_a := \bigcup_{n=0}^{\infty} a^{-n}I^n$. For $a, b \in I$ we have $P_a[(a/b)^{-1}] = P_b[(b/a)^{-1}]$, the blow up of $\operatorname{Spec} P$ at I is the union $\bigcup_{a\in I} \operatorname{Spec} P_a$, glued along $\{\operatorname{Spec} P_a[(a/b)^{-1}]\}_{a,b\in I}$.

5.4.15 Let F be a fan and $\mathcal{I}$ a fractional ideal sheaf. The blow-up of F at $\mathcal{I}$ is the fan $F_{\mathcal{I}}$ obtained by glueing the blow-ups of its affine subfans U at $\mathcal{I}|_U$. $F_{\mathcal{I}}$ is a subdivision of F.

5.5 Analogy with toric geometry

5.5.1 The role of fans in logarithmic geometry is comparable to the role of cone complexes in toric geometry. One should think of the spectrum of a monoid as a cone in the lactice of $1-$parametre subgroup. Regular fans correspond to cone complexes whose cones are regular, i.e. whose intersection with the lactice is isomorphic to a free monoid; subdivision of fans do the same job as subdivision of cones.

5.5.2 More precisely let $(F, \mathcal{M}_F)$ be a fan and let $U \subseteq F$ be an open affine subfan. We consider the monoid $P_U := \Gamma(U, M_F)$ and its associated real vector space $V_U := P^{\mathrm{grp}} \otimes_{\mathbb{Z}} \mathbb{R}$. We have the polyhedral cone $\sigma_U := \operatorname{Hom}(P_U, \mathbb{R}_{\geq 0}) \subseteq V_U^{\vee}$. If $U_1 \subseteq U_2$ are two affine open subfans of F, then we get a map $P_{U_1} \to P_{U_2}$ which induces a map $\sigma_{U_2} \to \sigma_{U_1}$; by gluing all the σ_Us with respect to these map, we get a polyhedral cone complex Δ_F.

5.5.3 For $U \subseteq F$ open affine subfan, we consider the lattice $N_U := \operatorname{Hom}(P_U^{\mathrm{grp}}, \mathbb{Z}) \subseteq V_U^{\vee}$. The family $\{N_U\}$ is called *integral structure* of Δ_F. There is a narrow link between regularity of fans and their integral structure, namely a fan F is regular if and only if F can be covered with affine open subfans $U_j \subseteq F$ such that each monoid $\sigma_{U_j} \cap N_{U_j}$ is generated by a base of N_{U_j}.

5.5.4 If $\varphi\colon F_1 \to F_2$ is a map of fan, $U_2 \subseteq F_2$ and $U_1 \subseteq \varphi^{-1}(U_2)$ are affine subfans, then the map $\Gamma(U_2, M_{F_2}) \to \Gamma(\varphi^{-1}(U_2), M_{F_1}) \to \Gamma(U_1, M_{F_1})$ induces a map $\sigma_{U_1} \to \sigma_{U_2}$. These maps glue together giving $\Delta_{F_1} \to \Delta_{F_2}$. If $F' \to F$ is a subdivision of fans, then $\Delta_{F'} \to \Delta_F$ is a subdivision of polyhedral complexes.

5.5.5 Given a function $f\colon \Delta_F \to \mathbb{R}$ satysfying the following:

1. $f(\lambda x) = \lambda f(x)$ for $\lambda \in \mathbb{R}_{\geq 0}, x \in \Delta_F$;
2. f is continuous and picewise linear;
3. For $U \subseteq F$ affine open, $f(\sigma_U \cap N_U) \subseteq \mathbb{Z}$;
4. f is convex on each σ_U, i.e. $\forall x, y \in \sigma_U$, $\forall \lambda, \mu \in \mathbb{R}_{\geq 0}$, one has $f(\lambda x + \mu y) \geq \lambda f(x) + \mu f(y)$.

it is possible to define a fractional ideal $\mathcal{I}_f$ such that for $U \subseteq F$ affine we have

$$\Gamma(U, \mathcal{I}_f) = \{a \in P_U^{\mathrm{grp}} \colon \forall x \in \sigma_U, x(a) \geq f(x)\}.$$

It is possible to choose f with the properties above such that the subdivision $F_{\mathcal{I}_f} \to F$ is regular, [38, Lemma 2.3].

5.6 Logarithmic schemes

5.6.1 We will now see how to combine the theory of ringed spaces and the theory of monoidal spaces; logarithmic geometry studies topologycal spaces endowed with two sheaves. In practice, this construction turns out to be useful for dealing with singular schemes: one usually endows a scheme with a suitable monoidal sheaf, then construct a fan (that can be topologically embedded in the scheme) containing the relevant information on the monoid, and, by focusing on that fan, one is able to manipulate the scheme and "improve" its singularities.

Definition 5.6.2. Let X be a scheme. A *pre-logarithmic structure* on X consists of a sheaf of monoids $\mathcal{M}_X$ on X and a homomorphism of sheaves of monoids $\alpha\colon \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$.

If α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$, then the datum is a *logarithmic structure.* We will call *logarithmic schemes* the pairs $X^\dagger = (X, \alpha\colon \mathcal{M} \to \mathcal{O}_X)$ of schemes and log structures over them. For the sake of simplicity, when confusion may not arise, we shall only write the monoid and the scheme in order to define a log scheme, namely $X^\dagger = (X, \mathcal{M})$.

5.6.3 The divisorial log structure will be our leading example:

Example 5.6.4 (Divisorial logarithmic structure)**.** Let X be a scheme and D a divisor. For all open subsets $V \subseteq X$, set

$$\mathcal{M}_X(V) \coloneqq \{f \in \mathcal{O}_X(V)\colon f|_{V\setminus D} \text{ is invertible}\}\,,$$

then $\mathcal{M}_X$ is a log structure, called the *divisorial log structure* associated to D.

If $D = 0$, then $\mathcal{M}_X$ coincide with $\mathcal{O}_X^\times$, which is the *trivial logarithmic structure.*

5.6.5 As we will see, every pre-log structure can be enhanced to a log structure. Since pre-log structures contain most of the relevant information, it is preferable to work with them rather than with log structures, which contain unessential information. Given a pre-log structure $\mathcal{M}_X$ on X we construct its *associated logarithmic structure* $\mathcal{M}_X^a$, defining it as the sheafification of the presheaf

$$U \mapsto \mathcal{M}_X(U) \oplus_{\alpha_U^{-1}(\mathcal{O}_X(U)^\times)} \mathcal{O}_X(U)^\times\,.$$

5.6.6 A *morphism of log schemes* $f^\dagger\colon X^\dagger \to Y^\dagger$ is the datum of a morphism of schemes between their underlying schemes $f\colon X \to Y$ and a morphim of sheaves of monoids $f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ coherent with the maps $\alpha_X\colon \mathcal{M}_X \to \mathcal{O}_X$ and $\alpha_Y\colon \mathcal{M}_Y \to \mathcal{O}_Y$.

5.6.7 In general $f^{-1}\mathcal{M}_Y$ provides a pre-log structure on X and the map, $f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ factors via the associated log structure $f^{-1}(\mathcal{M}_Y) \to (f^{-1}\mathcal{M}_Y)^a \to \mathcal{M}_X$. We say that f is *strict* if the second map is an isomorphism.

5.6.8 The datum of a chart 5.4.4 $P \to \mathcal{M}_X(X)$ is equivalent to the datum of a strict morphism $c\colon X \to \operatorname{Spec} P$. The datum of a morphism of monoids $u\colon Q \to P$ and charts $c_X\colon X \to \operatorname{Spec} P$ and $c_Y\colon Y \to \operatorname{Spec} Q$ is a *chart for the morphism* $f\colon X \to Y$ if the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "c_P"] \arrow[d, "f"'] & \operatorname{Spec}\mathbb{Z}[P] \arrow[d, "\operatorname{Spec}\mathbb{Z}[u]"] \\
Y \arrow[r, "c_Q"] & \operatorname{Spec}\mathbb{Z}[Q]
\end{tikzcd},$$

5.6.9 In our thesis we shall mainly deal with divisorial log structure coming from snc divisors.

Example 5.6.10. Let X be a scheme and D a strict normal crossing divisor on it, let $X^\dagger$ be the associated log structure (and let us use similar notation for subspaces of X). Fix a point $x \in X$ and choose an open neighbourhood $x \in U \subseteq X$ where the ideal defining D admits a generator $f = y_1 \cdots y_r$, where the y_i are irreducible and the sequence $(y_1, \ldots, y_r)$ can be extended to a regular one. Then $U^\dagger$ admits a chart $\mathbb{N}^r \to \mathcal{M}_U(U)$ such that $e_j \mapsto y_j$.

In the special case of the spectrum of a DVR $(R, \mathfrak{p})$, $X = \operatorname{Spec} R$, and $D = \mathfrak{p}$, there is a global chart $\mathbb{N} \to \mathcal{M}_X$ that sends $1 \in \mathbb{N}$ onto the uniformizer $\pi \in \mathfrak{p}$.

5.6.11 Now let us consider a morphism of schemes $\mathfrak{X} \to \operatorname{Spec} R$, where R is a DVR and $\mathfrak{X}$ is a *sncd* scheme with central fibre $\mathfrak{X}_0 = \bigcup_{j \in J} E_j$, where each E_j is an irreducible component of multiplicity N_j. In this situation, each point $x \in \bigcap_{i \in I} E_i$, for $I \subseteq J$, admits a neighbourhood $x \in U \to \operatorname{Spec} R$ where the morphism is given by $\pi \mapsto u \cdot \prod_{i \in I} y_i^{N_i}$, where $u \in \mathcal{O}_{\mathfrak{X}}(U)$ is invertible. It is not always possible to cancel u and find a morphism $\mathbb{N} \to \mathbb{N}^I$ giving a chart for the morphism of log schemes $\mathfrak{X}^\dagger \to \operatorname{Spec} R^\dagger$; one can replace $\mathbb{N}^I \to \mathcal{M}_U(U)$ with $\mathbb{N}^I \oplus \mathbb{Z} \to \mathcal{M}_U(U)$ sending $(0,1) \mapsto u$; at this point there is a chart $\mathbb{N} \to \mathbb{N}^I \oplus \mathbb{Z}$, $1 \mapsto ((N_i)_{i \in I}, 1)$. It is not always possible to remove the factor u in order to simplify the chart as $\mathbb{N} \to \mathbb{N}^J$; if not all the multiplicities N_i are divisible by $\operatorname{char} k$, then this can be done after replacing $\mathfrak{X}$ with a suitable finite étale cover of its.

5.6.12 Together with schemes, logarithmic structures can be pulled back along morphisms of log schemes. Some advantages of this construction will become more evident when we will restrict this operation to suitable subcategories, namely the one of fs log schemes, since this will provide an efficient shortcut towards the resolution of singularities. Let us consider two morphisms of log schemes $f_i \colon X_i \to Y$ for $i = 1, 2$ and the fibered product of their underlying schemes:

$$\begin{tikzcd} & X_1 \times_Y X_2 \arrow[dl, "p_1"'] \arrow[dd, "p"] \arrow[dr, "p_2"] & \\ X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd},$$

then we complete the construction of the *fibered product* by endowing the scheme with the monoidal sheaf $\mathcal{M}_{X_1} \oplus_{\mathcal{M}_Y} \mathcal{M}_{X_2}$. Moreover if the morphisms $X_i \to Y$ admits charts $P \to Q_i$, then the resulting scheme comes together with a chart $X_1 \times_Y X_2 \to \operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]$.

5.6.13 Even if $X_i \to Y$ are fine (resp. fs), their fibered product does not necessairily preserve that property. On the other hand if $u_i \colon P \to Q_i$ give fine (res. fs) charts of such morphisms, the construction

$$X_1 \times_Y^{\text{int}} X_2 \coloneqq (X_1 \times_Y X_2) \times_{\operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]} \operatorname{Spec} \mathbb{Z}[(Q_1 \oplus_Y Q_2)^{\text{int}}]$$

$$\left(\text{resp.} \quad X_1 \times_Y^{\text{fs}} X_2 \coloneqq (X_1 \times_Y X_2) \times_{\operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]} \operatorname{Spec} \mathbb{Z}[(Q_1 \oplus_Y Q_2)^{\text{sat}}]\right),$$

correspond to the fibered product in the category of fine (res. fs) logarithmic schemes and the projection on the second factor provides a chart for its log structure. Notice that, since these operations are fibered products in a suitable category, they have nice properties with respect to composition, e.g. if $X \to Z$ and $Y_1 \to Y_2 \to Z$ are morphisms of integral (resp. fs) logarithmic schemes, then

$$X \times_Z^{\mathrm{int}} Y_1 = (X \times_Z^{\mathrm{int}} Y_2) \times_{Y_2}^{\mathrm{int}} Y_1 ,$$

resp.

$$X \times_Z^{\mathrm{fs}} Y_1 = (X \times_Z^{\mathrm{fs}} Y_2) \times_{Y_2}^{\mathrm{fs}} Y_1 .$$

Example 5.6.14. Let R be a DVR and $R \subseteq R_d$ a totally ramified extension of degree d. Let $\mathfrak{X} \to \operatorname{Spec} R$ be a regular *sncd* scheme. Let $\mathfrak{X}^\dagger$ be the log scheme with the divisorial log structure induced by $\mathfrak{X}_0$, $\operatorname{Spec} R^\dagger$ the divisorial log structure induced by 0 and similarly $\operatorname{Spec} R_d^\dagger$. Then the underlying scheme of $\mathfrak{X}^\dagger \times^{\mathrm{fs}}_{\operatorname{Spec} R^\dagger} \operatorname{Spec} R_d^\dagger$ coincides with the normalization of $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R_d$, [5, Proposition 3.7.1].

5.7 Local properties of log maps

5.7.1 In this section we are going to talk about the notions of log smoothnes and étaleness.

Definition 5.7.2. Let $f\colon X \to Y$ be a map of logarithmic schemes. Fix a geometric point $x \to X$ and let $y = f(x)$ and assume that $u\colon Q \to P$ is a chart for f around $x \mapsto y$. We say that f is a logarithmically étale (resp. smooth) at x, if:

- $\ker u$ is a finite group whose order is invertible in $k(x)$;
- $\operatorname{coker} u$ (resp. its torsion) is a finite group whose order is invertible in $k(x)$;
- There is some neighbourhood $x \in U$ such that $f|_U$ factors via

  $$U \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P] \to Y ,$$

  where the first map is étale (resp. smooth).

Proposition 5.7.3 ([35, IV.3.1.2])**.** *Log smooth and log étale maps are stuble under base change and composition. Those notion are equivalent to their non logarithmic versions for strict morphisms.*

Definition 5.7.4. A morphism of integral log schemes $f\colon X \to Y$ is called integral if $\forall x \in X$, the morphism of monoids $\mathcal{M}_{Y,f(x)} \to \mathcal{M}_{X,x}$ is integral.

Proposition 5.7.5 (Illuise-Ogus)**.** *Let $f\colon X \to Y$ be a logarithmically smooth morphism and, for $x \in X$, denote by $\dim_x f$ the relative dimension of f at x. The sheaf $\Omega_{X^\dagger/Y^\dagger}$ is locally free of finite rank.*

Moreover, if X and Y are fine, then we have that $\forall\, x \in X$,

$$\operatorname{rk} \Omega_{X^\dagger/Y^\dagger} \leq \dim_x f ,$$

with the equality holding if f is integral.

Definition 5.7.6. Let $X^\dagger = (X, \mathcal{M}_X)$ be a locally Noetherian fs logarithmic scheme. We say that $X^\dagger$ is log regular at a point $x \in X$ if the following conditions:

- The ring $\mathcal{O}_{X,x}/\mathcal{M}^+_{X,x}\mathcal{O}_{X,x}$ is regular;
- $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}/\mathcal{M}^+_{X,x}\mathcal{O}_{X,x} + \dim \mathcal{M}_{X,x}$.

are satisfied. A log structure $X^\dagger$ is logarithmically regular if it is log regular at each of its points.

Example 5.7.7. Let $\Delta = \operatorname{Spec} R$, with R being a discrete valuation ring, and consider the divisorial log structure associated to its closed point $0 \in \Delta$. Then $\mathcal{M}^+_{\Delta,0}\mathcal{O}_{\Delta,0} = \pi R$, where π is the uniformizer of R, thus $\mathcal{O}_{\Delta,0}/\mathcal{M}^+_{\Delta,0}$ is the residue field of R. Moreover $\mathcal{M}_{\Delta,0} \cong \mathbb{N}$ has dimension 1, as Δ, hence it is a log regular scheme.

Proposition 5.7.8 ([24, Theorem 6.2])**.** *Let X be a locally Noetherian log scheme and fix a point $x \in X$ admitting a sharp fs local chart $P \to \mathcal{M}_X(X)$. Assume that $\mathcal{O}_{X,x}$ contains a field k. Then X is log regular at x if and only if $\mathcal{O}_{X,x}/\mathcal{M}^+_{X,x}\mathcal{O}_{X,x}$ is a regular local ring and the map $k[P] \to \mathcal{O}_{X,x}$, induced by the chart, is flat.*

Proposition 5.7.9. *Let $x \in X$ be a point on a log regular scheme, if $\mathfrak{p} \subseteq \mathcal{M}_{X,x}$ is a prime ideal, then $\mathfrak{p}\mathcal{O}_{X,x}$ is prime of the same height as $\mathfrak{p}$.*

Proposition 5.7.10. *Let $X \to Y$ be a log smooth morphism of locally fs log schemes. If Y is log regular then so is X.*

Example 5.7.11. Let $\mathfrak{X} \to \Delta$ a log smooth morphism of log schemes. Since $(\Delta, 0)$ is log regular, then also $(\mathfrak{X}, \mathfrak{X}_0)$ is.

5.7.12 We now associate to any logarithmically regular scheme X a topological subspace

$$F = F(X) \coloneqq \{x \in X \colon \mathcal{M}^+_{X,x}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}\}\,,$$

endowed with the monoidal sheaf obtained by pulling back $\mathcal{M}_X/\mathcal{O}^\times_X$ along its inclusion in X. Kato shows in [24, Proosition 10.1] that F is a fs fan, said to be the *fan associated to* X.

5.7.13 The construction of $F(X)$ comes together with a map of monoidal spaces $\pi \colon X \to F$ which associate to $x \in X$ the point $\pi(x) \in F$ corresponding to the prime ideal of $\mathcal{O}_{X,x}$ generated by $\mathcal{M}^+_{X,x}$. It is a continuous and open map of monoidal spaces that admits the inclusion $F \subseteq X$ as a section; this endows X with the structure of monoidal space over $F(X)$. Given a subdivision $F' \to F$, one can define a new logarithmic scheme $X \times_{F(X)} F'$ as the final object in the category of logarithmic schemes admitting a commutative diagram of the following form:

$$\begin{tikzcd}
(X', \mathcal{M}_{X'}/\mathcal{O}^\times_{X}) \arrow[r, "\pi'"] \arrow[d, "f"'] & F' \arrow[d] \\
(X, \mathcal{M}_X/\mathcal{O}^\times_X) \arrow[r, "\pi"] & F
\end{tikzcd}\;.$$

The map $X \times_F F' \to X$ is birational and log smooth, thus $X \times_F F'$ is log regular, because of Proposition 5.7.10, and $F(X \times_F F') = F'$.

5.7.14 Assuming that X is quasi-compact, so that $F(X)$ is finite. Because of 5.4.11, there is a regular subdivision $F' \to F(X)$. Then $X \times_{F(X)} F'$ is a regular scheme because of [33, Theorem 4.7 and Lemma 5.2].

Chapter 6

Toric schemes

In this chapter we introduce the notion of a toric scheme and provide its main properties. Let R be a DVR with uniformizer π, K its fraction field and k its residue field. Let $\Delta = \operatorname{Spec} R$. Let F be a free abelian group of finite rank, a torus is a group scheme of the form $T = \operatorname{Spec} \mathbb{Z}[t_1^{\pm 1} \dots, t_n^{\pm 1}]$ for some positive integer n. For a lattice N, i.e. a free f.g. abelian group, we denote by $N_{\mathbb{Q}} \coloneqq N \otimes_{\mathbb{Z}} \mathbb{Q}$.

6.1 Construction of toric schemes

Definition 6.1.1. A normal toric scheme is an integral normal scheme $\mathfrak{X} \to \operatorname{Spec} R$, separated and of finite type over R whose generic fibre $\mathfrak{X}_K$ contains a torus T_K and it is endowed with a group action $T_R \times \mathfrak{X} \to \mathfrak{X}$ extending the multiplication of T_K.

6.1.2 Let N be a free abelian group of rank 1, i.e. a lactice. Let $\tilde{N} \coloneqq N \oplus \mathbb{Z}$; consider the exact sequence $0 \to N \to \tilde{N} \xrightarrow{p} \mathbb{Z} \to 0$ and the dual sequence $0 \to \mathbb{Z} \to \tilde{M} \to M \to 0$; call $e \in \tilde{M}$ the element corresponding to $(0,1) \in M \oplus \mathbb{Z}$. Let $\tilde{N}_{\mathbb{Q}}^{+}$ be the preimage of $\mathbb{Q}_{\geq 0}$ under the map $\tilde{N}_{\mathbb{Q}} \xrightarrow{p} \mathbb{Q}$. Consider a cone $\sigma \subset \tilde{N}_{\mathbb{Q}}$ and its dual $\sigma^{\vee} \subseteq \tilde{M}_{\mathbb{Q}}$, which always contain e because $\sigma \subseteq \tilde{N}_{\mathbb{Q}}^{+}$. For a submonoid $S \subseteq \tilde{M}$ denote by $A[S]$ the algebra $R[\pi^r \chi^m \colon (m, r) \in S]$, where we are considering the isomorphism $M \times \mathbb{Z} \cong \tilde{M}$. Then $\mathfrak{X}_{\sigma} \coloneqq \operatorname{Spec} A[\sigma^{\vee} \cap \tilde{M}]$ is an affine toric scheme, as shown in [37, Proposition 2.1.4].

6.1.3 The inclusion of cones $\tau \subseteq \sigma$ induces inclusions $\sigma^{\vee} \subseteq \tau^{\vee}$, which, in turn induces open inclusions $\mathfrak{X}_{\tau} \subseteq \mathfrak{X}_{\sigma}$. Fix a fan, i.e. a cone complex, $\Gamma \subseteq \tilde{N}_{\mathbb{Q}}^{+}$, each face of Γ induces an affine toric scheme; consider two faces $\sigma, \tau \in \Gamma$; it is possible to glue $\mathfrak{X}_{\sigma}$ and $\mathfrak{X}_{\tau}$ along their common open subset $\mathfrak{X}_{\sigma \cap \tau}$; we call $\mathfrak{X}(\Gamma)$ the toric scheme constructed in this way.

6.2 Orbits and stratifications

6.2.1 Similarly to the theory of toric varieties, the action of the main torus induces a canonical stratification on toric schemes. Let $\sigma \subseteq \tilde{N}_{\mathbb{Q}}^{+}$ be a cone. Denote by O_{σ} the scheme

$$O_{\sigma} \coloneqq \operatorname{Spec} A[\sigma^{\perp} \cap M]\,,$$

if $\sigma \subseteq N_{\mathbb{Q}}$, or

$$O_{\sigma} \coloneqq \operatorname{Spec} k[\sigma^{\perp} \cap M]\,,$$

otherwise. In the first case O_{σ} is an orbit of T_K, thus a torus over K, while in the second case an orbit of T_k, thus a torus over k. The following proposition, [37, Proposition 2.1.13]

Proposition 6.2.2. *There is a bijection between orbits of T_K in $\mathfrak{X}(\Gamma)_K$ and faces of Γ in $N_{\mathbb{Q}}$ and a bijection between orbits of T_k in $\mathfrak{X}(\Gamma)_k$ and faces of Γ in $\tilde{N}_{\mathbb{Q}}$ not contained in $N_{\mathbb{Q}}$.*

Moreover $\tau \subseteq \sigma$ if and only if $O_\sigma \subseteq \overline{O}_\tau$, where the closure is taken in $\mathfrak{X}(\Gamma)$.

Chapter 7

Equivariant semistable reduction

Throughout this chapter let K be a field with an ultrametric absolute value $|\cdot|$, let R be its valuation ring and let k be its residue field, which we assume to be algebraically closed. Let $\Delta = \operatorname{Spec} R$. Let $R(m)$ the totally ramified extension of R of degree m, let $\Delta(m) = \operatorname{Spec} R(m)$.

7.1 The construction of a semistable model

7.1.1 Let $\Delta^\dagger$ be the log scheme supported on Δ with the divisorial log structure associated to $0 \in \Delta$. Similarly $\Delta^\dagger(m)$ shall denote the scheme $\Delta(m)$ introduced above endowed with the divisorial log structure for $0 \in \Delta(m)$. Let $S \to \Delta$ be a *sncd* model of a smooth surface S_K and let $S^\dagger$ be the divisorial logarithmic structure associated to the central fibre S_k. In particular $S^\dagger \to \Delta^\dagger$ is logarithmically smooth and $S^\dagger$ is logarithmically regular. Let us fix a positive integer N such that the multiplicity of any irreducible component of S_0 divides N and let m be an arbitrary positive integer; in case $\operatorname{char} k = p > 0$ let us assume that the multiplicities of the irreducible components of S_0 are coprime with p, choose N, m coprime with p. Let Γ be the fan associated to the logarithmic structure of $S^\dagger$.

7.1.2 The fs-basechange $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger \to \Delta^\dagger(mN)$ which is a normal space (see Example 5.6.14), yet not necessarily regular, is endowed with an equivariant $\mu_{mN}-$action induced by the trivial action on the first factor and the Galois action on the second one. This action induces an action on the fan of $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger$ (it is log regular because of Propositions 5.7.3 and 5.7.10), which we call $\Gamma(mN)$.

7.1.3 In the literature it is well known how to perform and embedded resolution of a log reguar scheme, for instance following the proof of [41, Theorem 4.8], it is possible to construct a $\mu_{mN}-$equivariant logarithmic resolution of singularities of $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger$, which gives an equivariant semistable model $S(mN) \to \Delta(mN)$ of $S_{K(mN)}$. Despite in [41] characteristic 0 is assumed, the semistable reduction works also in positive and mixed characteristic.

7.1.4 We conclude the section with the following lemma which grasp the most important, at least for our purpose, property of the action of μ_{mN} on $S(mN)$; in fact this is the only reason why we did perform the construction in this way:

Definition 7.1.5. For each point $p \in S(mN)_k$ let $\operatorname{Stab}_p \subseteq \mu_{mN}$ be the subgroup consisting of the elements that fix p.

Lemma 7.1.6. Stab_{-} *is locally constant on* $S(mN)_{k,\mathrm{sm}}$*. In particular, if a point* $p \in S(mN)_{k,\mathrm{sm}}$ *is fixed under the action of* μ_N*, then the whole connected component of* $S(mN)_{k,\mathrm{sm}}$ *containing* p *is fixed under such action.*

Proof. Each point of S admits an étale neighbourhood $U \twoheadrightarrow W \subseteq S$ (with W being a Zariski open subset of S) such that the map $U \to \Delta$ splits through a smooth map $U \to V \subseteq \operatorname{Spec} R[P]$, where P is a torsion-free monoid giving a local chart $P \to \mathcal{M}$ for the logarithmic structure on U. The embedding $V \hookrightarrow \operatorname{Spec} R[P]$ is given by the ideal $(\chi^v - \pi) \subseteq R[P]$, where $v \in P$ is the image of 1 under the map of monoids $\mathbb{N} \to P$ giving the chart for $U^\dagger \to \Delta^\dagger$.

Let $U(mN) \to W(mN) \subseteq S(mN)$ be the base change of $U \to W \subseteq S$ with respect to the map $S(mN) \to S$ arising from the construction above and let $V(mN) \to V \times_\Delta \Delta(mN)$ be the toric resolution arising from the same subdivision that we performed above; in particular there is a smooth map $\psi \colon U(mN) \to V(mN)$ which is equivariant with respect to the natural action of $\mu_{mN} = \operatorname{Gal}(K(mN)|K)$.

Let us omit the symbol † from the log-schemes, all the object we deal with are interpreted in the category of log-schemes unless differently stated. Consider the following equivariant map:

$$V(mN) \to V \times_\Delta^{\mathrm{fs}} \Delta(mN) \subset \operatorname{Spec} R\left[\left(P \oplus_{\mathbb{N}} \frac{1}{mN}\mathbb{N}\right)^{\mathrm{sat}}\right] = \operatorname{Spec} R[P] \times_\Delta^{\mathrm{fs}} \Delta(mN)\,,$$

where the amalgamated sum is taken with respect to the obvious inclusion $\mathbb{N} \subseteq \frac{1}{mN}\mathbb{N}$ and $\mathbb{N} \to P$, $1 \mapsto v \in P$ defined above. This map is an isomorphism outside the central fibre of the two schemes.

There is an integer d and a primitive element $v_1 \in P$ such that $v = d \cdot v_1$ (in our case, i.e. of a log structure arising from an *snc* divisor, d coincides with the *g.c.d.* of the multiplicities of the components containing p, in general it was defined as *root index* in [5]); by our assumption on N, we have that $d|mN$, thus the following identity holds:

$$V \times_\Delta^{\mathrm{fs}} \Delta(mN) = (V \times_\Delta^{\mathrm{fs}} \Delta(d)) \times_{\Delta(d)}^{\mathrm{fs}} \Delta(mN)\,.$$

We can, thus, consider the base-changes separately; the undelying scheme of $V \times_\Delta^{\mathrm{fs}} \Delta(d)$ is the normalization of the base-change in the category of schemes. We have that $V \times_\Delta \Delta(d) \cong \operatorname{Spec} R(d)\,[P]\,/(\chi^{dv_1} - \varpi^d)$, where $\varpi \in R(d)$ is a uniformizer such that $\varpi^d = \pi$. The chart of $\operatorname{Spec} R(d)[P]/(\chi^{dv_1} - \varpi^d)$ is given by the map of monoids

$$\begin{aligned} P \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N} &\to R(d)\,[P]\,/(\chi^{dv_1} - \varpi^d) \\ \left(0, \frac{1}{d}\right) &\mapsto \varpi\,, \end{aligned}$$

since ϖ is the $d-$th root of π. As shown in Lemma 7.1.9, we have that $\left(P \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N}\right)^{\mathrm{sat}} \cong P \oplus \mathbb{Z}/d\mathbb{Z}$. Then $V \times_\Delta^{\mathrm{fs}} \Delta(d) \cong \coprod_{i=0}^{d-1} \operatorname{Spec} R(d)\,[P]\,/(\chi^{v_1} - \zeta_d^i \varpi)$, and $\operatorname{Gal}(K(d)|K)$ acts on it via a cyclic permutation of the components.

We now assume that $d = 1$, i.e. that v is primitive. In this case the monoid $P(mN) := (P \oplus_{\mathbb{N}} \frac{1}{mN}\mathbb{N})^{\mathrm{sat}}$ is sharp [5, Proposition 2.2.2 (3)] and the inclusion $P^{\mathrm{grp}} \subseteq (P \oplus_{\mathbb{N}} \frac{1}{mN}\mathbb{N})^{\mathrm{grp}}$ induces an étale map of tori $T(mN) := \operatorname{Spec} R\left[(P \oplus_{\mathbb{N}} \frac{1}{mN}\mathbb{N})^{\mathrm{grp}}\right] \to T := \operatorname{Spec} R[P^{\mathrm{grp}}]$ of degree mN, which is the quotient with respect to the action of μ_{mN}; in particular the group μ_{mN} acts freely and transitively on the kernel of such map and there is an exact sequence of group schemes over $\operatorname{Spec} R$:

$$1 \to \mu_{mN} \to T(mN) \to T \to 1\,. \tag{7.1.1}$$

On the other hand, let us consider the $1-$codimensional subtorus $T' \subseteq T(mN)$ corresponding to the quotient $P(mN)^{\mathrm{grp}} \to P(mN)^{\mathrm{grp}}/\langle u \rangle$, where u is the image of the generator of $\frac{1}{mN}\mathbb{N}$ in $P(mN)$.

Claim 7.1.7. *The action of $T(mN)_R$ on $\operatorname{Spec} R[P(mN)]$ induces an action*

$$T'_R \times_\Delta (V \times^{\mathrm{fs}}_\Delta \Delta(mN)) \to V \times^{\mathrm{fs}}_\Delta \Delta(mN)$$

making $V \times^{\mathrm{fs}}_\Delta \Delta(mN) \to \Delta$ a toric scheme with respect to the torus T'.

Remark 7.1.8. Even though we are considering schemes over $R(mN)$, they have the structure of toric schemes over R.

The equivariant toric resolution $V(mN) \to V \times^{\mathrm{fs}}_\Delta \Delta(mN)$ is an isomorphims over an open set containing the dense toric orbit, hence $\mathrm{Gal}(K(mN)|K)$ acts also on $V(mN)$ as a subgroup of T'.

Let $O \subseteq V(mN)$ be a locally closed stratum in the canonical stratification of $V(mN)$, i.e. an orbit for the action of T', as described in [37, 2.1.13]; then a suitable quotient $T' \twoheadrightarrow \overline{T}$ acts freely on O and, thus, the image of μ_{mN} in $\overline{T}$ acts freely on O, hence the stabilizer of an arbitrary point $q \in O$ acts trivially on the whole orbit.

In the general case, we may apply the above argument to the map $\Delta(mN) \to \Delta(d)$, obtaining the following chain of maps:

$$V(mN) \to V \times^{\mathrm{fs}}_\Delta \Delta(mN) \to V \times^{\mathrm{fs}}_\Delta \Delta(d) \to V\,,$$

where the first map is a map of toric schemes (over $\Delta(d)$) whose main torus fits in the sequence

$$1 \to \mu_{\frac{mN}{d}} \to T(mN) \to T(d) \to 1$$

and the last map is just the collapse of d copies of V. Since the generator of μ_{mN} acts on $V(mN)$ by permuting the d connected components, then the stabilizer of each point is contained in the subgroup generated by its $d-$th power, i.e. $\mu_{\frac{mN}{d}}$, in particular the fact that the stabilizer of a point is locally constant on each orbit follows from what said for the $d=1$ case.

Let $q \in U(mN)_{k,\mathrm{sm}}$ be a closed point and let O be the stratum of $V(mN)$ containing $\psi(q)$; and let $Q := \psi^{-1}(O) \subseteq U(mN)$. Since $q \in U(mN)_{k,\mathrm{sm}}$, then O is a connected component of $V(mN)_{k,\mathrm{sm}}$, therefore Q is open in $U(mN)_{k,\mathrm{sm}}$. Since the map $U(mN) \to V(mN)$ is equivariant, we have that $\mathrm{Stab}_q \subseteq \mathrm{Stab}_{\psi(q)}$, thus Stab_q acts trivially on O. The étale map $Q \to O$ induces a Stab_q-equivariant étale map of the complete local rings $\widehat{\mathcal{O}_{O,\psi(q)}} \to \widehat{\mathcal{O}_{Q,q}}$, which is an isomorphism since k is algebraically closed. In particular Stab_q acts trivially on $\widehat{\mathcal{O}_{Q,q}}$ which is the formal completion of the local ring $\mathcal{O}_{Q,q}$, in particular Stab_q acts trivially in a neighbourhood of q. Since the fixed locus of Stab_q is also closed, it acts trivially on the whole connected component containing q. It follows that $\mathrm{Stab}_{\_}$ is locally constant on $U(mN)_{k,\mathrm{sm}}$.

In order to conclude that the stabilizer is locally constant on $W(mN)_{k,\mathrm{sm}}$ as well, we will prove that each point $q \in U(mN)_{k,\mathrm{sm}}$ has the same stabilizer as its image $p \in W(mN)$. Let $q_0 \in U$ be the image of q under $U(mN) \to U$ and similarly let $p_0 \in W$ be the image of p under $W(mN) \to W$. Up to replacing $U \to V$ with an open subset $U' \subseteq U \to V$ we can assume that q_0 is the only preimage of p_0 under $U \to W$; in particular the orbit of $q \in U(mN)$ is sent bijectively onto the orbit of $p \in W(mN)$ under the $\mu_{mN}-$equivariant map $U(mN) \to W(mN)$, so their stabilizers must coincide. □

Proof of Claim 7.1.7. Consider the composition of maps of affine schemes

$$T' \times_R (V \times^{\mathrm{fs}}_\Delta \Delta(mN)) \to T(mN) \times_R \operatorname{Spec} R[P(mN)] \to \operatorname{Spec} R[P(mN)]\,,$$

corresponding to the following composition of maps of rings

$$\begin{array}{rcl} R[P(mN)] \to & R[P^{\mathrm{grp}}(mN)] \otimes_R R[P(mN)] \to & R[P(mN)^{\mathrm{grp}}/\langle u\rangle] \otimes_R R[P(mN)]/(\chi^u - \pi) \\ \chi^x \mapsto & \chi^x \otimes \chi^x \mapsto & \chi^{\overline{x}} \otimes \overline{\chi^x}\,, \end{array}$$

where $\overline{x} \in P^{\mathrm{grp}}(mN)/\langle u\rangle$ denotes the projection of $x \in P(mN)$ and $\overline{\chi^x}$ denotes the the projection of $\chi^x \in R[P(mN)]$ into $R[P(mN)]/(\chi^u - \pi)$. Since $\overline{u} = 0$, one sees that $\chi^u - \pi$ is sent to $1 \otimes \overline{\chi^u} - 1 \otimes \pi = 0$, thus the map factors throug

$$\begin{aligned} R[P(mN)]/(\chi^u - \pi) &\to R[P(mN)^{\mathrm{grp}}/\langle u\rangle] \otimes_R R[P(mN)]/(\chi^u - \pi) \\ \overline{\chi^x} &\mapsto \chi^{\overline{x}} \otimes \overline{\chi^x}\,, \end{aligned}$$

giving an action $T'_R \times_\Delta V \times^{\mathrm{fs}}_\Delta \Delta(mN) \to V \times^{\mathrm{fs}}_\Delta \Delta(mN)$.

We conclude by showing that $V \times^{\mathrm{fs}}_\Delta \Delta(mN)$ admit a dense orbit with respect to the action of T'_K.

Let $y \in (V \times^{\mathrm{fs}}_\Delta \Delta(mN)) \cap T(mN)_K$ be a closed point. Since $\mathrm{codim}_{T(mN)_K} T'_K = 1$, then $T'_K \cdot y \subseteq T(mN)_K \cdot y = T(mN)_K$ is a closed subscheme of codimension at most 1. Density of $T'_K \cdot y$ follows from the fact that $(V \times^{\mathrm{fs}}_\Delta \Delta(mN)) \cap T(mN)_K$ is irreducible and has codimension 1 in $T(mN)_K$. □

Lemma 7.1.9. *The identity* $\left(P \oplus_{\mathbb{N}} \dfrac{1}{d}\mathbb{N}\right)^{\mathrm{sat}} \cong P \oplus \mathbb{Z}/d\mathbb{Z}$, *holds.*

Proof. We have that

$$\left(P \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N}\right)^{\mathrm{grp}} \cong \left(P^{\mathrm{grp}} \oplus \frac{1}{d}\mathbb{Z}\right)/(v,-1)$$

and

$$\begin{aligned} P^{\mathrm{grp}} \oplus \mathbb{Z}/d\mathbb{Z} &\cong \left(P^{\mathrm{grp}} \oplus \frac{1}{d}\mathbb{Z}\right)/(v,-1) \\ (0,1) &\mapsto \left(-v_1, \frac{1}{d}\right), \end{aligned}$$

oreover the monoid $P \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N} \subseteq \left(P^{\mathrm{grp}} \oplus \frac{1}{d}\mathbb{Z}\right)/(v,-1)$ corresponds to $\langle P, (v_1, 1)\rangle$. On one hand, we have that $P \oplus \mathbb{Z}/d\mathbb{Z} \subseteq \left(P \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N}\right)^{\mathrm{sat}}$. On the other hand, if $(x,a) \in P^{\mathrm{grp}} \oplus \mathbb{Z}/d\mathbb{Z}$ is such that $t(x,a) \in \langle P, (v_1, 1)\rangle$ for some $t \in \mathbb{N}$, then $td(x,a) = (tdx, 0)$, so $x \in \mathbb{P}$ because P is saturated. Thus the identity is proved. □

Part III

Hilbert schemes

Chapter 8

Formal series in the motivic rings

In this section, the symbol $\mathscr{R}$ shall denote one of the rings $\mathscr{M}_k, \mathscr{M}_k\left[(\mathbb{L}^r-1)^{-1}\colon 0<r\in\mathbb{N}\right]$ or $\widehat{\mathscr{M}_k}$, unless differently specified.

We will discuss some properties of power series with coefficients in $\mathscr{R}$, then we will describe some operations that shall be useful for computing the Motivic Zeta Function in our case.

8.1 Quotient by a group action

8.1.1 Let us fix a finite group G, let $N \trianglelefteq G$ and let $H = G/N$ be its quotient. Consider the equivariant versions of the ring $\mathscr{R}$, which we call $\mathscr{R}^G$ and $\mathscr{R}^H$, as in §2.5. For an arbitrary variety X endowed with a good action of G, we consider the quotient X/N which is again an algebraic space endowed with an action of the group H, namely the quotient action.

8.1.2 Even if X/N is not a scheme, there exists an open affine subscheme $\operatorname{Spec} A \subseteq X$ which is invariant under the action of G, so that $\operatorname{Spec} A^N \subseteq X/N$ is a scheme. We may, thus, repeat the argument for the closed $G-$invariant subscheme $X\setminus \operatorname{Spec} A$ and, by Noetherian induction, we stratify X as a union of $G-$schemes whose quotients with respect to the action of N are $H-$schemes; moreover a $G-$invariant stratification of each stratum induces an $H-$invariant stratification of its quotient. Thus there is a well defined map of groups $\pi_N\colon \mathscr{R}^G \to \mathscr{R}^H$ by $[X] \mapsto [X/N]$; this map does not preserve the products.

8.1.3 In the following definition we extend the map above to a map $\mathscr{R}^G[[T]] \to \mathscr{R}^H[[T]]$ and in the subsequent proposition we show that rationality of any power series is well behaved under this map.

Definition 8.1.4. If $F = \sum_n A_n T^n \in \mathscr{R}^G[[T]]$, we define the *series of the quotients* with respect to N associated to F as

$$(F/N)(T) = \sum_n (A_n/N) T^n \in \mathscr{R}^H[[T]]\,.$$

Proposition 8.1.5. *For $f, g \in \mathscr{R}^G[T]$, with g being of the form $\prod_{j\in J}(1-\mathbb{L}^{a_j}T^{b_j})$, let F be the rational function $F(T) \coloneqq \dfrac{f(T)}{g(T)}$. Then $F/N = \dfrac{f/N}{g}$.*

In particular all the poles of F/N belong to the set of poles of F.

Proof. It suffices to show the statement in the case when $f = \alpha \in \mathscr{R}^G$ is a constant. Let $\dfrac{1}{g(T)} = \sum_n A_n T^n$, where $A_n \in \mathscr{R}$ has a trivial $G-$action. Then

$$\left(\frac{\alpha}{g(T)}\right) \Big/ N = \sum_n (\alpha A_n)/NT^n = \sum_n \alpha/N A_n T^n = \frac{\alpha/N}{g(T)}\,.$$

□

8.2 Power structures

8.2.1 We need to define a map on the Grothendieck rings which extends the symmetric product of a variety, allowing us to talk about the symmetric product of a "difference of varieties"; in order to do so, we need to use a power structure on $K_0(\mathrm{Var}_k)$, thus we begin by recalling what a power structure is, as introduced in [15]:

Definition 8.2.2. Let A be a ring. A power structure on A is a map

$$(1 + tA[[t]]) \times A \to 1 + tA[[t]]$$
$$(F(t), X) \mapsto F(t)^X$$

satisfying the following conditions for all $F, G \in 1 + tA[[t]]$ and $X, Y \in A$:

- $F(t)^0 = 1$;
- $F(t)^1 = F(t)$;
- $(F(t)G(t))^X = F(t)^X \cdot G(t)^X$;
- $F(t)^{X+Y} = (F(t))^X (F(t))^Y$;
- $F(t)^{XY} = \left(F(t)^X\right)^Y$;
- $(1+t)^X \in 1 + Xt + t^2 A[[t]]$;
- $F(t)^X|_{t \to t^n} = F(t^n)^X$.

In fact, the last two properties are not part of the original definition, but other authors include them in their definition.

8.2.3 In the rest of the section, for $F = \sum_n F_n T^n \in K_0(\mathrm{Var}_k)[[T]]$, with $F_0 = 1$, and for $X \in K_0(\mathrm{Var}_k)$ we denote by $F(T)^X$ the power structure introduced by Gusein-Zade, Luengo and Melle-Hernandez in [15] that we will describe as follows:

Definition 8.2.4. Let $n \geq 0$ be an arbitrary natural number and $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}^{\mathbb{N}_{>0}}$. We say that α is a *partition* of n if $\sum_{i>0} i\alpha_i = n$. In this case we write $\alpha \dashv n$ or $|\alpha| = n$. We define the lenght of a partition α as $||\alpha|| := \sum_{i>0} \alpha_i$. We say that $\alpha \in \mathbb{N}^{\mathbb{N}_{>0}}$ is a partition if it is a partition of n for some $n \in \mathbb{N}$.

Definition 8.2.5. Let $A(T) = 1 + \sum_{i>0} A_i T^i \in 1 + tK_0(\mathrm{Var}_k)[[T]]$ and let M be a variety. We define $A(T)^{[M]}$ as

$$A(T)^{[M]} := 1 + \sum_{\alpha \text{ partition}} \pi_{G_\alpha} \left(\left[\prod_i M^{\alpha_i} \backslash \Delta \right] \prod_i A_i^{\alpha_i} \right) t^{|\alpha|}\,,$$

where $G_\alpha = \prod_i \Sigma_{\alpha_i}$ acts simultaneously on $\prod_i M^{\alpha_i}$ and $\prod_i A_i^{\alpha_i}$ by permuting the factors, $\Delta \subseteq \prod_i M^{\alpha_i}$ is the large diagonal, i.e. the subscheme of points at least two equal entries and $\pi_{G_\alpha} \colon \prod_i M^{\alpha_i} \to \mathrm{Sym}^{||\alpha||}(M)$ is the canonical projection.

8.3 Symmetric powers

8.3.1 Keep the notation introduced in the previous paragraph. Let us begin with the following definition:

Definition 8.3.2 (Symmetric power in GRV)**.** Let $\alpha \in K_0(\mathrm{Var}_k)$ and let $r \in \mathbb{N}$. The *$r-th$ symmetric power of* α is the element $\mathrm{Sym}^r(\alpha) \in K_0(\mathrm{Var}_k)$ defined as

$$\mathrm{Sym}^r(\alpha) \coloneqq [t^r](1-t)^{-\alpha}\,.$$

8.3.3 In particular, if $\alpha = [U]$, we have that $\mathrm{Sym}^r([U]) = [\mathrm{Sym}^r(U)]$; in this sense $\mathrm{Sym}^\bullet$ exends the notion of symmetric power of an algebraic variety. In general, for $\alpha = [U] - [V]$ one gets an explicit formula by analyzing the coefficients of

$$(1-t)^{[V]-[U]} = (1-t)^{[V]} \cdot (1 + t + t^2 + \cdots)^{[U]}\,.$$

Example 8.3.4. In order to show how this computation can be handled, we compute explicitly the coefficient of t^2, that is, the expression for $\mathrm{Sym}^2([U] - [V])$. We know, from [15, Theorem 1], that

$$(1 + t + t^2 + \cdots)^{[U]} = 1 + [U]t + [\mathrm{Sym}^2(U)]t^2 + o(t^2)\,,$$

thus we get

$$(1-t)^{[V]} = (1 + [V]t + [\mathrm{Sym}^2(V)]t^2 + o(t^2))^{-1} = 1 - [V]t + ([V^2] - [\mathrm{Sym}^2(V)])t^2 + o(t^2)\,.$$

It follows that

$$\mathrm{Sym}^2([U] - [V]) = [\mathrm{Sym}^2(U)] - [\mathrm{Sym}^2(V)] + [V]^2 - [U][V]\,.$$

8.3.5 It is possible to extend the map $\mathrm{Sym}^r \colon K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$ to a map $\mathrm{Sym}^r \colon \mathscr{M}_k \to \mathscr{M}_k$ by $\mathrm{Sym}^r \mathbb{L}^{-s}\alpha \coloneqq \mathbb{L}^{-rs}\,\mathrm{Sym}^r(\alpha)$. In order to ensure that this map is well defined, we only need to show that, for $\alpha \in K_0(\mathrm{Var}_k)$ and for $s \in \mathbb{N}$, we have $\mathbb{L}^{-rs}\,\mathrm{Sym}^r(\alpha) = \mathbb{L}^{-r(s+1)}\,\mathrm{Sym}^r(\mathbb{L}\alpha)$.

Indeed, recalling that $\mathrm{Sym}^r(\mathbb{A}^n) \cong \mathbb{A}^{nr}$, we get

$$(1-t)^{\mathbb{L}\alpha} = \Big((1-t)^{-\mathbb{L}}\Big)^{-\alpha} = (1 + \mathbb{L}t + \mathbb{L}^2t^2 + \cdots)^{-\alpha} = (1 - \mathbb{L}t)^{\alpha}\,,$$

which implies that $\forall \alpha \in K_0(\mathrm{Var}_k)$, $\mathrm{Sym}^r(\mathbb{L}\alpha) = \mathbb{L}^r\,\mathrm{Sym}^r(\alpha)$.

8.3.6 The map Sym^r can also be defined at the level of $\widehat{\mathscr{M}_k}$; indeed, for $\alpha, \beta \in K_0(\mathrm{Var}_k)$, we have that:

$$\forall p \in \mathbb{N}, \quad (1-t)^{\alpha + \mathbb{L}^p\beta} - (1 - t^{\alpha}) = (1-t)^{\alpha} \cdot \Big((1 - \mathbb{L}^p t)^{\beta} - 1\Big) \in \mathbb{L}^p K_0(\mathrm{Var}_k)[[t]]\,,$$

thus $\mathrm{Sym}^r(\alpha + \mathbb{L}^p\beta) \equiv \mathrm{Sym}^r(\alpha) \pmod{\mathbb{L}^p}$.

8.3.7 We will also define a version of Sym^r defined over $\mathscr{M}_k\left[(\mathbb{L}^n-1)^{-1}\colon 0<n\in\mathbb{N}\right]$. Since Sym^1 is already defined as the identity map we can proceed inductively on r. Let us assume that all the maps Sym^i are defined for $1\le i\le r-1$. Let us first check that for $\alpha\in\mathscr{M}_k$, the value of

$$\mathrm{Sym}^r\left(\frac{\alpha}{1}\right):=\frac{\mathrm{Sym}^r(\alpha)}{1}\in\mathscr{M}_k\left[(\mathbb{L}^n-1)^{-1}\colon 0<n\in\mathbb{N}\right]$$

is well defined, i.e. that if $\dfrac{\alpha}{1}=\dfrac{\beta}{1}$, then $\dfrac{\mathrm{Sym}^r(\alpha)}{1}=\dfrac{\mathrm{Sym}^r(\beta)}{1}$. If $\gamma\in\mathscr{M}_k$ is such that $(\mathbb{L}^n-1)\gamma=0$ for some $n\in\mathbb{N}$, then

$$\mathrm{Sym}^r(\mathbb{L}^n\gamma)=\mathrm{Sym}^r(\gamma+(\mathbb{L}^n-1)\gamma)=\sum_{j=0}^{r}\mathrm{Sym}^j(\gamma)\,\mathrm{Sym}^{r-j}(0)=\mathrm{Sym}^r(\gamma)\,,$$

thus $(\mathbb{L}^{nr}-1)\,\mathrm{Sym}^r(\gamma)=0$. By an inductive argument one proves that if $\dfrac{\gamma}{1}=0$, then also $\dfrac{\mathrm{Sym}^r(\gamma)}{1}=0$. It follows that, whenever $\dfrac{\alpha}{1}=\dfrac{\beta}{1}$, then

$$\frac{\mathrm{Sym}^r(\beta)}{1}=\frac{\mathrm{Sym}^r(\alpha+(\beta-\alpha))}{1}=\sum_{j=1}^{r}\frac{\mathrm{Sym}^j(\alpha)}{1}\,\frac{\mathrm{Sym}^{r-j}(\beta-\alpha)}{1}=\frac{\mathrm{Sym}^r(\alpha)}{1}\,.$$

Then we define, recursively:

$$\mathrm{Sym}^r\left(\frac{\alpha}{\mathbb{L}^n-1}\right):=(\mathbb{L}^{nr}-1)^{-1}\sum_{i=1}^{r}\mathrm{Sym}^i(\alpha)\,\mathrm{Sym}^{r-i}\left(\frac{\alpha}{\mathbb{L}^n-1}\right)\,.$$

Example 8.3.8. We show how to compute this map in a specific case. For $\alpha=\dfrac{[U]}{1-\mathbb{L}^n}$, we have that $\mathrm{Sym}^2(\alpha)=\dfrac{[\mathrm{Sym}^2(U)]}{1-\mathbb{L}^{2n}}+\dfrac{\mathbb{L}^n\cdot[U]^2}{(1-\mathbb{L}^n)(1-\mathbb{L}^{2n})}$.

8.3.9 For a power series $F(T)=\sum A_nT^n\in\mathscr{R}[[T]]$, let us consider the power series obtained by plugging each coefficient of F into the above mentioned maps Sym^r:

$$\mathrm{Sym}^r(F)(T):=\sum\mathrm{Sym}^r(A_n)T^n\,.$$

These maps have very interesting properties when the function F is rational, indeed in this case we are able to control the poles of Sym^r thanks to the upcoming results:

Lemma 8.3.10. *Let* $F=\dfrac{\alpha T^h}{(1-\mathbb{L}^{-qN}T^N)^e}\in\mathscr{R}[[T]]$, *where* $q\in\mathbb{Q}$ *is such that* $qN\in\mathbb{Z}$. *Then, for* $r>0$, *we have that* $\mathrm{Sym}^r(F)$ *has at most one pole of order (at most)* $r(e-1)+1$ *in* rq.

Proof. Let $F=\sum_{m\ge0}A_mT^{mN+h}$; then $A_m=\binom{m+e-1}{e-1}\alpha\mathbb{L}^{-mqN}$. It follows that

$$\mathrm{Sym}^r(A_m)=\sum_{\beta\dashv r}\binom{\binom{m+e-1}{e-1}}{\beta_1,\dots,\beta_r}\alpha^{\beta_1}\cdot(\mathrm{Sym}^2\alpha)^{\beta_2}\cdots(\mathrm{Sym}^r\alpha)^{\beta_r}\mathbb{L}^{-rmqN}\,;$$

once β is fixed, $\binom{\binom{m+e-1}{e-1}}{\beta_1,\dots,\beta_r}$ is either 0 $\forall m\in\mathbb{Z}$ (this happens only if $e=1$ and $\beta_1+\cdots+\beta_r>1$) or a polynomial in m of degree $(e-1)(\beta_1+\cdots+\beta_r)$.

It follows (from the fact that $\dfrac{1}{(1-x)^n} = \sum_m p_n(m)x^m$, where $p_n \in \mathbb{Q}[t]$ is a polynomial such that $\deg p_n = n-1$ and with coefficients in $\dfrac{1}{m!}\mathbb{Z}$) that

$$\sum_{m\geq 0} \binom{\binom{m+e-1}{e-1}}{\beta_1,\dots,\beta_r} \alpha^{\beta_1}\cdot(\operatorname{Sym}^2\alpha)^{\beta_2}\cdots(\operatorname{Sym}^r\alpha)^{\beta_r}\mathbb{L}^{-rmqN}T^{mN+h}$$

is a suitable combination with integer coefficients of

$$\left\{\frac{\alpha^{\beta_1}\cdot(\operatorname{Sym}^2\alpha)^{\beta_2}\cdots(\operatorname{Sym}^r\alpha)^{\beta_r}T^h}{(1-\mathbb{L}^{-rqN}T^N)^j}\right\}_{j=0}^{(e-1)(\beta_1+\cdots+\beta_r)+1}.$$

Among all the partitions of r, the one which gives the highest possible order of the pole is the one maximizing $\beta_1+\cdots+\beta_r$, namely $\beta=(r,0,0,\dots)$, which gives a pole of order at most $r(e-1)+1$. $\square$

8.3.11 For the rest of the section, we denote by $\mathscr{R}$ one of the two rings $\mathscr{M}_k\left[(\mathbb{L}^n-1)^{-1}\colon 0<n\in\mathbb{N}\right]$ or $\widehat{\mathscr{M}_k}$: the proofs we will present do not hold for functions with coefficients in $\mathscr{M}_k$.

Proposition 8.3.12. *For $i=1,\dots,s$ let $F_i=\sum_{m\geq 0}A_m^{[i]}T^m\in\mathscr{R}[[T]]$ be rational functions and let $\mathcal{Q}_i\subseteq\mathbb{Q}$ be the set of poles of F_i. Let $F=\sum_{m\geq 0}A_m^{[1]}\cdots A_m^{[s]}T^m\in\mathscr{R}[[T]]$. Then F is also rational and its set of poles, $\mathcal{Q}$, is contained in $\mathcal{Q}_1+\mathcal{Q}_2+\cdots+\mathcal{Q}_s$.*

Moreover, for each $q\in\mathcal{Q}$, we have that

$$\operatorname{ord}_q(F)\leq\max\left\{1-s+\sum_{i=1}^s\operatorname{ord}_{q_i}(F_i)\colon q_i\in\mathcal{Q}_i \text{ and } \sum q_i=q\right\}.$$

Remark 8.3.13. This statement holds, with the same proof, also if we consider functions $F_i\in\mathscr{M}_k[[T]]$, provided that each of them is sum of functions with a single pole.

Proof. Let us assume for a moment that, $\forall i$, $F_i=\frac{\alpha_iT^{h_i}}{(1-\mathbb{L}^{-q_iN}T^N)^{e_i}}$, for some $\alpha_i\in\mathscr{R}$, $0\leq h_i<N$ integers, $q_i\in\mathbb{Q}$, $0<e_i\in\mathbb{N}$. In such case $A_{mN+h_i}^{[i]}=\binom{m+e_i-1}{e_i-1}\alpha_i\mathbb{L}^{-mq_iN}$. Thus $F=0$ unless $h_i=h$ $\forall i$, while in this case we have that

$$\prod_{i=1}^s A_{mN+h}^{[i]}=\left(\prod_{i=1}^s\binom{m+e_i-1}{e_i-1}\alpha_i\right)\mathbb{L}^{-mqN},$$

where $q=q_1+\cdots+q_s$. The degree of $\prod_{i=1}^s\binom{m+e_i-1}{e_i-1}$, seen as a polynomial in m, is $\sum e_i-s$, thus we get the desired result in this case.

For the general case it is enough to consider $F_1=F_1'+F_1''$, where $F_1'=\sum B_mT^m$ and $F_1''=\sum C_mT^m$; then, setting $F'\coloneqq\sum B_mA_m^{[2]}\cdots A_m^{[s]}T^m$ and $F''\coloneqq\sum C_mA_m^{[2]}\cdots A_m^{[s]}T^m$, we have that $F=F'+F''$ and if our statement holds for both F' and F'' then it holds also for F, in particular writing all the F_i as in Equation (3.3.1), the proposition follows by an induction on the number of their summands. $\square$

Lemma 8.3.14. *Let $F\in\mathscr{R}[[T]]$ be a rational function whose set of poles is $\mathcal{Q}\subseteq\mathbb{Q}$, or let $F\in\mathscr{M}_k[[T]]$ be the sum of functions with at most one pole. For all $r\in\mathbb{N}$, let $\Sigma^r\mathcal{Q}$ be the set of rational numbers that are sum of r elements of $\mathcal{Q}$. Then $\operatorname{Sym}^r F$ is also rational and its set of poles is contained in $\Sigma^r\mathcal{Q}$.*

Moreover, for each $q \in \mathcal{Q}$, we have that

$$\operatorname{ord}_q(\operatorname{Sym}^r F) \leq \max\left\{1 - r + \sum_{i=1}^{s} \operatorname{ord}_{q_i}(F)\colon q_i \in \mathcal{Q} \textit{ and } \sum q_i = q\right\}.$$

Proof. If the Lemma holds for F and G, then by Proposition 8.3.12, it holds also for $F + G$, since $\operatorname{Sym}^r(F+G) = (\operatorname{Sym}^r F) + (\operatorname{Sym}^{r-1} F)G + \cdots + (\operatorname{Sym}^r G)$. Thus it is enough to write F as in Equation (3.3.1) and notice that for each addendum the statement coincides with Lemma 8.3.10. $\square$

Chapter 9

Hilbert schemes

In this chapter we collect a few of the basic facts about Hilbert schemes that can be useful to undertand the construction that shall appear in the incoming section.

9.1 The moduli problem

9.1.1 Hilbert schemes are the answer to one of the most fundamental moduli problems that nathematcians happen to face: classifying subschemes of a given variety.

Definition 9.1.2. Let $X \to S$ be a projective morphism of schemes.

The *Hilbert Functor* of $S-$subschemes of X is the functor:

$$\mathscr{H}(X/S) \colon \mathrm{Sch}_S \to \mathrm{Sets}$$

which associate to any $S-$scheme T the set

$$\mathscr{H}(X/S)(T) \coloneqq \{V \subseteq X \times_S T | V \to T \text{ is proper and flat}\}$$

and associate to each morphism $T' \to T$ of $S-$schemes the map of sets:

$$\begin{aligned}&\mathscr{H}(X/S)(T) \to \mathscr{H}(X/S)(T')\\ &V \subseteq X \times_S T \mapsto V \times_T T' \subseteq (X \times_S T) \times_T T' \cong X \times_S T' \,.\end{aligned}$$

9.1.3 This functor is actually represented by a scheme $\mathrm{Hilb}(X/S)$ which is called the *Hilbert Scheme* of X over S. The scheme that can be constructed in this way is arguably unmanageable, for it has, typically, i.e. when $X \to S$ is not finite, an infinite amount of connected components whose dimension is not even bounded. It is often considered convenient to stratify this scheme as a disjoint union of locally closed subschemes each of them parametrizing subschemes of X with similar properties, i.e. classifying the subschemes of X according to an invariant:

Definition 9.1.4. Fix a relatively ample line bundle $\mathcal{L}$ over $f \colon X \to S$. Let $\mathcal{F} \in \mathrm{Coh}(X)$ an $S-$flat sheaf; then $f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})$ is a locally free $\mathcal{O}_S-$module, $\forall m \in \mathbb{Z}$. There is a polynomial $h_{X/S,\mathcal{F},\mathcal{L}} \in \mathbb{Q}[t]$ such that for $m >> 1$, $\mathrm{rk}(f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})) = h_{X/S,\mathcal{F},\mathcal{L}}(m)$, which is called *Hilbert polynomial* of $\mathcal{F}$.

9.1.5 Let $T \to S$ be a morphism of schemes and denote by $g \colon X_T \coloneqq X \times_S T \to X$ the morphism induced by the base-change. The set of $T-$flat and proper (also over T) subschemes of X_T is in natural bijection with the set of quotients of $\mathcal{O}_{X \times_S T}$ which are $T-$flat and have proper (over T) support, i.e.

$$\mathscr{H}(X/S)(T) = \{V \subseteq X \times_S T | V \to T \text{ is proper and flat}\} \cong \{\mathcal{O}_{X \times_S T} \twoheadrightarrow \mathcal{F} | \mathcal{F} \text{ is flat and proper over } T\} \,.$$

Hence, given a polynomial $p \in \mathbb{Q}[t]$ and a relatively ample line bundle $\mathcal{L}$ over X, we can define a subfunctor of $\mathscr{H}(X/S)$ as:

$$\mathscr{H}^p_{\mathcal{L}}(X/S)(T) = \{\mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F} | T-\text{flat, with proper support and such that } h_{X_T/T,\mathcal{F},g^*\mathcal{L}} = p\}\,,$$

which is represented by an open and closed subscheme of $\mathrm{Hilb}(X/S)$ which we denote by $\mathrm{Hilb}^p_{\mathcal{L}}(X/S)$. We have, moreover, that

$$\mathrm{Hilb}(X/S) = \bigsqcup_{p\in\mathbb{Q}[t]} \mathrm{Hilb}^p_{\mathcal{L}}(X/S)\,.$$

9.1.6 For the purposes of this thesis, we will need only to study only the components corresponding to constant polynomials, which are called *Hilbert schemes of points*, since they parametrize $0-$dimensional subschemes of X. Since the Hilbert polynomial of a finite subscheme of X is independent on the choice of the relatively ample line bundle $\mathcal{L}$, we can omit it from the notation and denote by $\mathrm{Hilb}^n(X/S)$ the Hilbert scheme corresponding to the constant polynomial n, which is often called *Hilbert scheme of n points on X*, with a slight abuse of language.

9.1.7 We will need a Lemma which describes how the Hilbert schemes behave under base-change.

Lemma 9.1.8. *Let $T \to S$ and $X \to S$ be morphisms of schemes, $\mathcal{L}$ an $S-$ample line bundle over X and let $p \in \mathbb{Q}[t]$. Let $f\colon X_T \to X$ the basechange morphism with respect to $T \to S$.*

Then $\mathrm{Hilb}^p_{f^*\mathcal{L}}(X_T/T) \cong \mathrm{Hilb}^p_{\mathcal{L}}(X/S) \times_S T$ in Sch_T.

In particular, if F is a field and $\operatorname{Spec} F \to S$ is a point of F, the fibre over F of $\mathrm{Hilb}^p_{\mathcal{L}}(X/S)$ coincide with the Hilbert scheme of the fibre: $\mathrm{Hilb}^p_{\mathcal{L}|_{X_F}}(X_F/\operatorname{Spec} F)$.

Proof. Let us begin by showing that the two functors $\mathscr{H}(X_T/T)$ and $\mathscr{H}(X/S)|_{\mathrm{Sch}_T}$ coincide.

Let $U \to T$ be a morphism of schemes. Then

$$\begin{aligned}\mathscr{H}(X_T/T)(U) &= \{V \subseteq X_T \times_T U | V \to U \text{ is proper and flat}\}\\ &= \{V \subseteq (X\times_S T)\times_T U | V \to U \text{ is proper and flat}\}\\ &= \{V \subseteq X \times_S U | V \to U \text{ is proper and flat}\}\\ &= \mathscr{H}(X/S)(U)\,.\end{aligned}$$

It remains to show that the two stratifications induced by the Hilbert polynomials coincide, but this follows from the fact that, given the composition $U \xrightarrow{g} T \xrightarrow{f} S$, one has that

$$(f\circ g)^* = g^* \circ f^*\,.$$

□

9.2 Properties of the Hilbert scheme of points

9.2.1 Let $X \to S$ be a flat morphism of finite type of Noetherian schemes and let $n \in \mathbb{N}^+$ a positive integer. Consider the scheme $\mathrm{Sym}^n(X/S) \coloneqq X^n/\Sigma_n$, where the symmetric group acts on the product by permuting the factors. There is a natural morphism

$$\mathrm{Hilb}^n(X/S) \to \mathrm{Sym}^n(X/S)$$

called Hilbert-to-Chow morphism, sending a subscheme of X to its underlying $0-$cycle.

9.2.2 If $X \to S$ is a smooth map of relative dimension 2, then $\mathrm{Hilb}^n(X/S)$ is a smooth scheme of relative dimension $2n$, in particular the Hilbert-to-Chow morphism is a resolution of singularties of $\mathrm{Sym}^n(X/S)$; the generic points of both $\mathrm{Hilb}^n(X/S)$ and $\mathrm{Sym}^n(X/S)$ parametrize reduced subschemes of X. This is no longer true if $\dim_S(X) > 2$ and $n > 2$, unless the couple $(n, \dim_S(X)) = (3, 3)$.

9.2.3 The main reason for studying the Hilbert schemes of points on a K3 surface is the following:

Theorem 9.2.4. *If $X \to \operatorname{Spec} K$ is a K3 surface, then $\mathrm{Hilb}^n(X/K)$ is an irreducible holomorphic symplectic variety of dimension $2n$.*

9.2.5 Also the case of an abelian surface A is interesting, because in that case the Hilbert scheme is anyway a smooth Calabi-Yau variety; moreover all IHS varieties of Kummer type can be constructed as subschemes of $\mathrm{Hilb}^n(A/K)$.

Chapter 10

Hilbert schemes of points on a surface

10.1 Construction of a weak Néron model

10.1.1 Let $X \to \operatorname{Spec} K$ be a smooth surface with trivial canonical divisor and let $\omega \in \omega_{X/K}(X)$ be a volume form on it. Let $\mathfrak{X} \to \Delta$ be a regular model whose central fibre $\mathfrak{X}_k$ is a strict normal crossing divisor of $\mathfrak{X}$. Let us keep the notation of Chapter 7 concerning the field extension over K and the corresponding base-changes. If $\operatorname{char} k = p > 0$, we add the further assumption d that the central fibre $\mathfrak{X}_k$ has no components with multiplicity divisible by p.

10.1.2 The aim of this section is to provide a closed formula for the zeta function of $\operatorname{Hilb}^n(X)$ in terms of the zeta functions of $X(i)$ for $1 \leq i \leq n$.

10.1.3 Let a be the *lcm* of the multiplicities of the irreducible components of $\mathfrak{X}_k$. For $n \in \mathbb{N}$ (and $n < \operatorname{char} k$ if the latter is positive), let $\tilde{n} := a \operatorname{lcm}(1, 2, \ldots, n)$ and let $K(\tilde{n})$ be the unique totally ramified extension of K whose degree is $\tilde{n}$ (by our assumptions, if $\operatorname{char} k = p > 0$, then $\gcd(\tilde{n}, p) = 1$), so that $\operatorname{Gal}(K(\tilde{n})/K) = \mu_{\tilde{n}}$.

10.1.4 For all $0 < m \in \mathbb{N} <$ denote by $\mathfrak{X}(m\tilde{n})$ be the semistable model of $X(m\tilde{n})$ obtained from $\mathfrak{X}$ using the construction of §7.1. As usual we denote by $\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}$ the smooth locus of $\mathfrak{X}(m\tilde{n}) \to \Delta(m\tilde{n})$. Since $\operatorname{Hilb}^n(\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}/\Delta(m\tilde{n})) \to \Delta(m\tilde{n})$ is a smooth model of $\operatorname{Hilb}^n(X(m\tilde{n}))$, we have that

$$\mathfrak{X}^{[n]}(m) := \left(\operatorname{Res}_{\Delta(m\tilde{n})/\Delta(m)} \operatorname{Hilb}^n(\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}/\Delta(m\tilde{n}))\right)^{\mu_{\tilde{n}}} \to \Delta(m)$$

is a smooth model of $\operatorname{Hilb}^n(X(m))$.

Proposition 10.1.5. *Assume that either K is perfect or $\operatorname{char} K > n$, then $\mathfrak{X}^{[n]}(m) \to \Delta(m)$ is a weak Néron model of $\operatorname{Hilb}^n(X(m))$.*

Proof. We assume that $a = 1$, i.e. that X has semistable reduction on K. The proof in the general case will descend from Proposition 4.2.6 and Lemma 4.2.8. We just need to show that every point $\operatorname{Spec} K(m) \to \operatorname{Hilb}^n(X(m)) \subseteq \mathfrak{X}^{[n]}(m)$ extends to a morphism $\Delta(m) \to \mathfrak{X}^{[n]}(m)$.

Consider a point $\operatorname{Spec} K(m) \to \operatorname{Hilb}^n(X(m))$ and let $Z \subseteq X(m\tilde{n})$ the ($\mu_{\tilde{n}}-$invariant) subscheme representing such point. Either the closure of Z in $\mathfrak{X}(m\tilde{n})$ is contained in $\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}$ or at least one point $P \in \operatorname{supp} Z$ specializes to the singular locus $\mathfrak{X}(m\tilde{n})_{k,\mathrm{sing}}$. In the first case $\overline{Z}$ represents a morphism $\Delta(m) \to \mathfrak{X}^{[n]}(m)$ extending the given $\operatorname{Spec} K(m) \to \operatorname{Hilb}^n(X(m))$. If $P \in \operatorname{supp} Z$ specializes to $\mathfrak{X}(m\tilde{n})_{k,\mathrm{sing}}$, then its residue field $k(P)$ contains strictly $K(m\tilde{n})$, for $\mathfrak{X}(m\tilde{n})$ is a regular model of $X(m\tilde{n})$ (See example 4.1.7).

Let $Q \in X(m)$ be the image of P under the map $\operatorname{Spec} k(P) \hookrightarrow X(m\tilde{n}) \to X(m)$. The degree $[k(Q) : K(m)]$ cannot divide $\tilde{n}$, otherwise $k(P)$ would be $K(m\tilde{n})$, thus $[k(Q) \colon K(m)] \geq n+1$.

Since Z is $\mu_{\tilde{n}}-$invariant, then it contains the whole orbit of P which is the reduced scheme associated to the preimage of Q under the map $\pi \colon X(m\tilde{n}) \to X(m\tilde{n})/\mu_{\tilde{n}} \cong X(m)$. We have that $\pi^{-1}(Q) = \operatorname{Spec}(k(Q) \otimes_{K(m)} K(m\tilde{n}))$, which is a reduced $K(m\tilde{n})-$algebra of dimension $[k(Q) : K(m)] > n$.

This contradicts the fact that Z is a subscheme of length n, thus this case cannot occur and we are done. □

10.1.6 Our goal is to study the motivic integral of $\operatorname{Hilb}^n(X(m))$ using the models $\mathfrak{X}^{[n]}(m)$ constructed above. In order to simplify notation, we perform the following computations only for $m = 1$, working with $\mathfrak{X}^{[n]} = \mathfrak{X}^{[n]}(1)$; similar arguments apply when $m > 1$. Even though $\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{\mathrm{sm}}/\Delta(\tilde{n}))$ is not a weak Néron model of $X(\tilde{n})$, it is a smooth model and the results of §4.3 apply. It follows that the connected components of $\mathfrak{X}^{[n]}_k$ are in bijection with the connected components of $\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}}$, moreover if $C \subseteq \mathfrak{X}^{[n]}_k$ is sent to C' via this bijection, then $[C] = \mathbb{L}^{2n - \dim C'}[C']$.

10.1.7 For any $d \in \mathbb{N}$ dividing $\tilde{n}$ we denote by $Y(d) \subseteq \mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}$ the subscheme consisting of the points whose stabilizer is exacty $\mu_{\tilde{n}/d}$. Then $Y(d)$ is $\mu_{\tilde{n}}-$invariant, since $\mu_{\tilde{n}}$ is an abelian group and points in the same orbit have the same the stabilizer. Because of Lemma 7.1.6, $Y(d)$ is at the same time an open and closed subscheme of $\mathfrak{X}_{k,\mathrm{sm}}$ and we have that:

$$\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}^{\mu_{\tilde{n}/d}} = \bigsqcup_{d'|d} Y(d')\,.$$

Remark 10.1.8. Since $\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}^{\mu_{\tilde{n}/d}}$ is a scheme of pure dimension 2, it can be identified with the central fibre of $\operatorname{Res}_{\Delta(\tilde{n})/\Delta(d)}(\mathfrak{X}(\tilde{n})_{\mathrm{sm}})^{\mu_{\tilde{n}/d}}$, which is a weak Néron model of $X(d)$, via the map h_k described in §4.3.2. Let $C \subseteq \operatorname{Res}_{\Delta(\tilde{n})/\Delta(d)}(\mathfrak{X}(\tilde{n})_{\mathrm{sm}})^{\mu_{\tilde{n}/d}}$ be a connected component. It follows from Lemma 4.4.7 and from the fact that $\operatorname{Gal}(K(\tilde{n})|K(d))$ acts trivially on $T_{h_k(C)}$ that, ,

$$\operatorname{ord}_{h_k(C)}(\omega(\tilde{n})) = \frac{\tilde{n}}{d}\operatorname{ord}_C(\omega(d))\,.$$

10.1.9 The following statement gives a decomposition of the central fibre as a union of closed and open subschemes; it is an *ad hoc* partition that is more suitable, for our computation, than the "canonical" stratification of the Hilbert schemes whose strata are related to the combinatorics of the underlying $0-$cycle:

Proposition 10.1.10. $\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}}$ *admits the following decomposition as disconnected union of subschemes:*

$$\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}} \cong \bigsqcup_{\alpha \dashv n}\prod_{j=1}^{n}\operatorname{Hilb}^{\alpha_j}(Y(j)/\mu_j)\,.$$

Moreover, for a fixed partition $\alpha \dashv n$, the isomorphism above restricts to an isomorphim between $\prod_{j=1}^{n}\operatorname{Hilb}^{\alpha_j}(Y(j)/\mu_j)$ and a finite union of connected components of $\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}}$.

Proof. Fix a $k-$scheme S. For any closed subscheme $Z \subseteq \mathfrak{X}(\tilde{n})_S$ endowed with a finite map $Z \to S$, set $Z_j \coloneqq Z \cap Y(j)_S$. If Z is stable under the action of $\mu_{\tilde{n}}$, then every Z_j, which is the intersection of two stable schemes, is stable as well; moreover the induced action of $\mu_j = \mu_{\tilde{n}}/\mu_{\tilde{n}/j}$ on $Y(j)_S$ is free, thus the induced maps $\pi_j \colon Y(j)_S \to Y(j)_S/\mu_j$ and $Z_j \to \pi_j(Z_j)$ are étale of degree j.

In this way, from any invariant finite $S-$subscheme of $\mathfrak{X}(\tilde{n})_S$ we construct a finite subscheme in each $Y(j)_S/\mu_j$; on the other hand given a sequence of finite $S-$subschemes of $Y(j)_S/\mu_j$ of length α_j we get a unique $\mu_{\tilde{n}}-$stable $S-$subscheme of $\mathfrak{X}(\tilde{n})_S$ whose length is $\sum_j j\alpha_j$.

The last statement follows directly from the fact that the $Y(j)-$s are themselves open and closed subschemes of $\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}$. $\square$

10.1.11 Thus, recalling that $\mathrm{Hilb}^{\alpha_j}(Y(j)/\mu_j)$ is pure of dimension $2\alpha_j$, we conclude that $\mathfrak{X}_k^{[n]} = \bigsqcup_{\alpha \dashv n} \mathfrak{X}_{k,\alpha}^{[n]}$, where $\mathfrak{X}_{k,\alpha}^{[n]}$ is an affine bundle of rank $\sum_j 2(j-1)\alpha_j$ on

$$\mathrm{Hilb}^{\alpha_1}(Y(1)) \times \cdots \times \mathrm{Hilb}^{\alpha_n}(Y(n)/\mu_n)\,.$$

We thus have the following:

Corollary 10.1.12. *The following equation holds in the Grothendieck ring of varieties:*

$$\left[\mathfrak{X}_k^{[n]}\right] = \sum_{\alpha \dashv n} \prod_{j=1}^{n} \mathbb{L}^{2(j-1)\alpha_j} [\mathrm{Hilb}^{\alpha_j}(Y(j)/\mu_j)]\,.$$

10.2 The volume form on $\mathfrak{X}^{[n]}$

10.2.1 There is a volume form, $\omega^{[n]}$, on $\mathrm{Hilb}^n(X)$ that naturally arises from the given $\omega \in \omega_{X/K}$. In this paragraph we will recall its construction and compute its zeroes and poles on $\mathfrak{X}^{[n]}$.

10.2.2 Let $\mathrm{pr}_i \colon X^n \to X$, for $i \in \{1, \ldots, n\}$, denote the projections on the factors. Then $\mathrm{pr}_1^* \omega \wedge \cdots \wedge \mathrm{pr}_n^* \omega$ is a global section of $\omega_{X^n/K}$ which, being invariant under the permutation of coordinates, descends to a global section of $\omega_{\mathrm{Sym}^n X/K}$, which we denote by φ. Let finally $\omega^{[n]}$ be the pull-back of φ through the Hilbert-Chow morphism, thus $\omega^{[n]} \in H^0(\mathrm{Hilb}^n(X), \omega_{\mathrm{Hilb}^n(X)})$ is a volume form on $\mathrm{Hilb}^n(X)$.

10.2.3 Now we will compute the zeroes and poles of $\omega^{[n]}$ seen as a rational section of $\omega_{\mathfrak{X}^{[n]}/\Delta}$. Since it is a volume form on the generic fibre of $\mathfrak{X}^{[n]}$, its zeroes or poles are all irreducible components of the central fibre. Let us fix a connected component $C \subseteq \mathfrak{X}_k^{[n]}$ and let us denote by C' the connected component of $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}}$ such that $C \to C'$ is the affine bundle described in §4.3.8. In the following lemma we compute the conductor of the action of $\mu_{\tilde{n}}$ at points of C' in terms of the partition of n corresponding to the stratum of $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})/\Delta(\tilde{n}))^{\mu_{\tilde{n}}}$ containing C'.

Lemma 10.2.4. *Consider the decomposition of* $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}}$ *of Proposition 10.1.10 and fix a point* $[Z]$ *lying inside the stratum corresponding to* $\alpha \dashv n$*. Then:*

1. *The conductor of the action of* $\mu_{\tilde{n}}$ *at* $[Z]$ *is*

$$c\left(\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}), [Z]\right) = \tilde{n} \sum_{j=1}^{n} (j-1)\alpha_j\,.$$

2. *If we denote by* $[Z_j]$ *the point of* $\mathrm{Hilb}^{\alpha_j}(Y(j)/\mu_j)$ *corresponding to* Z_j/μ_j*, then one has that:*

$$\mathrm{ord}_{[Z]}(\omega^{[n]}(\tilde{n})) = \tilde{n} \sum_{j=1}^{n} \mathrm{ord}_{[Z_j/\mu_j]}(\omega^{[\alpha_j]}(j))\,,$$

where $Y(j)$ are considered as part of the central fibre of a weak Néron model for $X(j)$ as in Remark 10.1.8.

Proof. Since the values of $c\left(\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}), [Z]\right)$ and $\operatorname{ord}_{[Z]}(\omega^{[n]}(\tilde{n}))$ depend only on the connected component containing $[Z]$ and since the generic point of each connected component corresponds to a reduced scheme, we may compute them with the additional assumption that Z is a reduced subscheme of $\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}$. In particular Z is the disjoint union of α_1 orbits of length 1, α_2 orbits of length 2 and so on. There is an equivariant isomorphism

$$T_{[Z]}\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}) \cong \bigoplus_{p\in \operatorname{supp} Z} T_p\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}\,.$$

Let $\zeta = \zeta_{\tilde{n}}$ be a primitive root of unity and let σ be the unique generator of $\mu_{\tilde{n}}$ such that σ acts on $R(\tilde{n})$ by multiplying the uniformizing parametre by ζ. Consider an orbit of points $p_0, \dots, p_{j-1} \in Z$, let e_1, e_2 be two generators of $T_{p_0}\mathfrak{X}(\tilde{n})_k$, so that $\sigma^l(e_1), \sigma^l(e_2)$ will give a basis of $T_{p_l}\mathfrak{X}(\tilde{n})_k$ for each $l = 0, \dots, j-1$. Notice that $\sigma^j(e_h) = e_h$ for $h = 1, 2$ since $\mu_{\tilde{n}/j}$ acts trivially on the whole connected component cointaining p_0.

For $i = 0, \dots, j-1$, $h = 1, 2$ we have that

$$(e_h, \zeta^{i\tilde{n}/j}\sigma(e_h), \zeta^{2i\tilde{n}/j}\sigma^2(e_h), \dots, \zeta^{(j-1)i\tilde{n}/j}\sigma^{j-1}(e_h)) \in T_{p_0}\mathfrak{X}(\tilde{n})_k \oplus \cdots \oplus T_{p_{j-1}}\mathfrak{X}(\tilde{n})$$

is an eigenvector with eigenvalue $\zeta^{-i\tilde{n}/j}$. In total there are $2j$ of such eigenvectors, which constitute a basis for $T_{p_0}\mathfrak{X}(\tilde{n}) \oplus \cdots \oplus T_{p_{j-1}}\mathfrak{X}(\tilde{n})_k$.

The sum of the exponents of this base is

$$2\sum_{i=0}^{j-1} -\frac{i\tilde{n}}{j} = -(j-1)\tilde{n}$$

We construct eigenvectors of $T_{[Z]}\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_k)$ by putting a vector such as the above one at the coordinates corresponding to an orbit and 0 at the other coordinates. Running through all the possible orbits, we get a base of eigenvectors of $T_{[Z]}\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_k)$. Thus summing the exponents among all the eigenvectors will lead to the desired result for the conductor.

Concerning the order of the volume form, we have that

$$\begin{aligned}
\operatorname{ord}_{[Z]}(\omega^{[n]}(\tilde{n})) &= \sum_{p\in \operatorname{supp} Z} \operatorname{ord}_p(\omega(\tilde{n})) \\
&= \sum_{j=1}^{n} \frac{\tilde{n}}{j} \sum_{p\in \operatorname{supp} Z_j} \operatorname{ord}_p(\omega(j)) \\
&= \sum_{j=1}^{n} \tilde{n} \sum_{p\in Z_j/\mu_j} \operatorname{ord}_p(\omega(j)) \\
&= \tilde{n}\sum_{j=1}^{n} \operatorname{ord}_{Z_j/\mu_j}(\omega(j)^{[\alpha_j]})\,.
\end{aligned}$$

Where the first and the last equality follow from the fact that the stalk of the canonical bundle at a point with reduced support of a Hilbert scheme are the tensor products of the stalks of the canonical bundle of the surface at every point in the support. The second equality follows from Remark 10.1.8. The third equality follow from the fact that $\omega(j)$ has the same order on all the j points of an orbit of μ_j. $\square$

10.2.5 We are able, now, to compute the order of $\omega^{[n]}$ at any point $z \in \mathfrak{X}_k^{[n]}$:

Corollary 10.2.6. *Let* $\pi\colon \mathfrak{X}_k^{[n]} \to \mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})$ *and assume that*

$$\begin{aligned}\pi(z) &= (\pi^1(z), \pi^2(z), \dots, \pi^n(z)) \\ &\in \mathrm{Hilb}^{\alpha_1}(Y(1)) \times \mathrm{Hilb}^{\alpha_2}(Y(2)/\mu_2) \times \cdots \times \mathrm{Hilb}^n(Y(n)/\mu_n)\,.\end{aligned}$$

We have that $\mathrm{ord}_z(\omega^{[n]}) = \sum_{j=1}^{n} \Big((j-1)\alpha_j + \mathrm{ord}_{\pi^j(z)}(\omega(j)^{[\alpha_j]})\Big)$.

Proof. As a direct consequence of Lemma 4.4.7 and of the previous lemma we get:

$$\mathrm{ord}_z(\omega^{[n]}) = \frac{\mathrm{ord}_{\pi(z)}(\omega(\tilde{n})^{[n]}) - c\left(\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}), [Z]\right)}{\tilde{n}} = \sum_{j=1}^{n} \Big(-(j-1)\alpha_j + \mathrm{ord}_{\pi^j(z)}(\omega(j)^{[\alpha_j]})\Big)\,.$$

□

10.3 Motivic integral

10.3.1 We keep the convention on $\mathscr{R}$ being one of the three rings $\mathscr{M}_k$, $\mathscr{M}_k\left[(\mathbb{L}^r-1)^{-1}\colon 0<r\in\mathbb{N}\right]$ or $\widehat{\mathscr{M}_k}$. We are now ready to perform the main computation of the manuscript; by using the models we constructed above we are able to compute a generating function for the motivic integrals of all the Hilbert schemes of points of a surface with trivial canonical bundle. More precisely the formula we are going to prove is the content of the following proposition:

Theorem 10.3.2. *The following identity holds true in* $\mathscr{R}[[q]]$ *if* $\mathrm{char}\,k = 0$*, while it holds true in* $\mathscr{R}[q]/(q^p)$ *if* $\mathrm{char}\,k = p > 0$*:*

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} q^n\right) = \prod_{m\geq 1}\left(\left(1-\mathbb{L}^{m-1}q^m\right)^{-\left(\int_{X(m)}\omega(m)\right)/\mu_m}\right)\,.$$

Corollary 10.3.3. *Assume that either* $\mathrm{char}\,k = 0$ *or* $\mathrm{char}\,k > n$*, then the following equation holds:*

$$\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} = \sum_{\alpha\dashv n}\prod_{j=1}^{\infty}\left(\mathbb{L}^{(j-1)\alpha_j}\,\mathrm{Sym}^{\alpha_j}\left(\left(\int_{X(j)}\omega(j)\right)/\mu_j\right)\right)\,.$$

Proof. Since

$$\left(1-\mathbb{L}^{m-1}q^m\right)^{-\left(\int_{X(m)}\omega(m)\right)/\mu_m} = \sum_{l=0}^{\infty}\mathbb{L}^{(m-1)l}\,\mathrm{Sym}^{l}\left(\left(\int_{X(m)}\omega(m)\right)/\mu_m\right)q^{ml}\,,$$

moreover, given a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ such that $\alpha_j = 0$ for $j \gg 1$, one has that

$$\deg_q\left(\prod_{j=1}^{\infty}\left(\mathbb{L}^{(j-1)\alpha_j}\,\mathrm{Sym}^{\alpha_j}\left(\left(\int_{X(j)}\omega(j)\right)/\mu_j\right)q^{j\alpha_j}\right)\right) = \sum_{j=1}^{\infty} j\alpha_j\,.$$

We get the desired result after identifying the coefficients of q^n from Theorem 10.3.2. □

10.3.4 Before facing the theorem, let us introduce a piece of notation that will help facing the computation more smoothly: If $Z \to \operatorname{Spec} k$ is a scheme and $\Theta\colon Z \to \mathcal{M}_k$ is a locally constant function, we denote by

$$\int_Z \Theta(z)dz := \sum_{\substack{C \subseteq \mathbb{Z} \\ \text{connected component}}} [C]\Theta(p) \in \mathcal{M}_k \,,$$

where the p in the sum above is an arbitrary point of the connected component C. We state a lemma that will be useful for the proof of the theorem:

Lemma 10.3.5. *Let $Y \to \operatorname{Spec} k$ be a smooth surface endowed with a locally constant function $\nu\colon |Y| \to \mathscr{R}$. Suppose that functions $\nu^{[n]}\colon |\operatorname{Hilb}^n(Y)| \to \mathscr{R}$ and $\nu'^{[n]}\colon |\operatorname{Sym}^n(Y)| \to \mathscr{R}$ are defined in such a way that, for a given subscheme $Z \subseteq Y$ of length n we have*

$$\nu^{[n]}(Z) = \prod_{p \in \operatorname{supp}(Z)} \nu(p)^{\operatorname{length}(\mathscr{O}_{Z,p})}$$

and $\nu^{[n]} = \nu'^{[n]} \circ p_n$, where $p_n\colon \operatorname{Hilb}^n(Y) \to \operatorname{Sym}^n(Y)$ is the Hilbert-Chow morphism. Then, for an arbitrary natural number $\alpha \in \mathbb{N}$, the following identity holds:

$$\int_{\operatorname{Hilb}^\alpha(Y)} \nu^{[\alpha]}(z)dz = \sum_{\beta \dashv \alpha} \left(\prod_{l \geq 1} \left(\mathbb{L}^{(l-1)\beta_l} \int_{\operatorname{Sym}^{\beta_l}(Y)} \nu'^{[l\beta_l]}(z)dz \right) \right) .$$

Proof. We first suppose that Y is connected and, thus, $\nu \equiv \lambda \in \mathscr{R}$ is constant. Thus we simply have that

$$\int_{\operatorname{Hilb}^\alpha(Y)} \nu^{[\alpha]}(z)dz = \lambda^\alpha [\operatorname{Hilb}^\alpha(Y)] \,.$$

It follows from a well known result, for instance [39, §2.2.3], that

$$[\operatorname{Hilb}^\alpha(Y)] = \sum_{\beta \dashv \alpha} \left(\prod_{l \geq 1} \left(\mathbb{L}^{(l-1)\beta_l} [\operatorname{Sym}^{\beta_l}(Y)] \right) \right) ,$$

thus we deduce the desired statement, at least when Y is connected.

Now suppose that $C \subseteq Y$ is a connected component and that the statement holds for $Y \backslash C$.

Using the fact that $\mathrm{Hilb}^\alpha(Y)=\bigsqcup_{j=0}^{\alpha}\left(\mathrm{Hilb}^{\alpha-j}(Y\backslash C)\times \mathrm{Hilb}^j(C)\right)$, we deduce that

$$\begin{aligned}
\int_{\mathrm{Hilb}^\alpha(Y)}\nu^{[\alpha]}(z)dz =&\sum_{j=0}^{\alpha}\left(\int_{\mathrm{Hilb}^{\alpha-j}(Y\backslash C)}\nu^{[\alpha-j]}(z)dz\right)\cdot\left(\int_{\mathrm{Hilb}^j(C)}\nu^{[j]}dz\right)\\
=&\sum_{j=0}^{\alpha}\left(\sum_{\delta^j\dashv\alpha-j}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\delta_l^j}\int_{\mathrm{Sym}^{\delta_l^j}(Y\backslash C)}\nu^{\prime l[\delta_l^j]}(z)dz\right)\right)\cdot\right.\\
&\left.\cdot\sum_{\gamma^j\dashv j}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\gamma_l^j}\int_{\mathrm{Sym}^{\gamma_l^j}(C)}\nu^{\prime l[\gamma_l^j]}(z)dz\right)\right)\right)\\
=&\sum_{j=0}^{\alpha}\left(\sum_{\substack{\delta^j\dashv\alpha-j\\ \gamma^j\dashv j}}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)(\delta_l^j+\gamma_l^j)}\int_{\mathrm{Sym}^{\beta_l^j}(Y\backslash C)\times\mathrm{Sym}^{\gamma_l^j}(C)}\nu^{\prime l[\delta_l^j+\gamma_l^j]}(z)dz\right)\right)\right)\\
=&\sum_{\beta\dashv\alpha}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\beta_l}\sum_{i=0}^{\beta_l}\int_{\mathrm{Sym}^{\beta_l-i}(Y\backslash C)\times\mathrm{Sym}^{i}(C)}\nu^{\prime l[\beta_l]}(z)dz\right)\right)\\
=&\sum_{\beta\dashv\alpha}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\beta_l}\int_{\mathrm{Sym}^{\beta_l}(Y)}\nu^{\prime l[\beta_l]}(z)dz\right)\right),
\end{aligned}$$

which concludes the proof. □

Proof of Theorem 10.3.2. We begin our computation using some identities we proved in the previous section.

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)}\omega^{[n]}q^n\right)=\sum_{n\geq 1}\left(\int_{\mathfrak{X}_k^{[n]}}\mathbb{L}^{-\operatorname{ord}_z(\omega^{[n]})}dzq^n\right),$$

by decomposing $\mathfrak{X}_k$ as union of its strata $\{\mathfrak{X}_{k,\alpha}\}_{\alpha\dashv n}$, we get:

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)}\omega^{[n]}q^n\right)=\sum_{n\geq 1}\left(\sum_{\alpha\dashv n}\left(\int_{\mathfrak{X}_{k,\alpha}^{[n]}}\mathbb{L}^{-\operatorname{ord}_z(\omega^{[n]})}dz\right)q^n\right),$$

by Corollary 10.2.6 we obtain:

$$\begin{aligned}
\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)}\omega^{[n]}q^n\right)=&\sum_{n\geq 1}\left(\sum_{\alpha\dashv n}\left(\int_{\mathfrak{X}_{k,\alpha}^{[n]}}\prod_{j\geq 1}\left(\mathbb{L}^{(j-1)\alpha_j-\operatorname{ord}_{\pi^j(z)}(\omega^{[\alpha_j]}(j))}q^{j\alpha_j}\right)dz\right)\right)\\
=&\sum_{\alpha\in\mathbb{N}^{\oplus\mathbb{N}_{\geq 1}}}\left(\prod_{j\geq 1}\left(\mathbb{L}^{(j-1)\alpha_j}q^{j\alpha_j}\int_{\mathrm{Hilb}^{\alpha_j}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(j)^{[\alpha_j]})}dz\right)\right)\\
=&\prod_{j\geq 1}\left(\sum_{\alpha_j\geq 0}\left(\mathbb{L}^{(j-1)\alpha_j}q^{j\alpha_j}\int_{\mathrm{Hilb}^{\alpha_j}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(j)^{[\alpha_j]})}dz\right)\right).
\end{aligned}$$

We plug the lemma above in our chain of equalities using $Y=Y(j)/\mu_j$, $\nu:=\mathbb{L}^{-\operatorname{ord}_z(\omega(j))}$, recalling also that $l\operatorname{ord}_z(\omega(j))=\operatorname{ord}_z(\omega(jl))$ and that $Y(j)$ naturally embeds in the central fibre of some weak Néron model of $X(lj)$ (as in Remark 10.1.8 with $d={}^{\tilde{n}}/{}_{lj}$), we get:

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)}\omega^{[n]}q^n\right)=$$

$$=\prod_{j\geq 1}\left(\sum_{\alpha_j\geq 0}\left(\mathbb{L}^{(j-1)\alpha_j}\sum_{\beta^j\dashv\alpha_j}\left(q^{j\alpha_j}\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\beta^j_l}\int_{\mathrm{Sym}^{\beta^j_l}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(lj)^{[\beta^j_l]})}dz\right)\right)\right)\right)$$

$$=\prod_{j\geq 1}\left(\sum_{\beta^j\in\mathbb{N}^{\oplus\mathbb{N}_{\geq 1}}}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(jl-1)\beta^j_l}q^{jl\beta^j_l}\int_{\mathrm{Sym}^{\beta^j_l}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(lj)^{[\beta^j_l]})}dz\right)\right)\right)$$

$$=\prod_{j,l\geq 1}\left(\sum_{\beta^j_l\geq 0}\left(\mathbb{L}^{(jl-1)\beta^j_l}q^{jl\beta^j_l}\int_{\mathrm{Sym}^{\beta^j_l}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(lj)^{[\beta^j_l]})}dz\right)\right).$$

Recalling that, in the sum above, there is only a finite number of nonvanishing coefficients of q^n, for every positive integer n, we are allowed to group such summands in a different order; since the map

$$\begin{aligned}\mathbb{N}_+\times\mathbb{N}_+&\to\mathbb{N}_+\times\mathbb{N}_+\\(j,l)&\mapsto(j\cdot l,j)\end{aligned}$$

is injective and its image is $\{(m,j)\colon j|m\}$, after the substitution $\lambda^m_j:=\beta^j_l$, we get the equivalent expression:

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)}\omega^{[n]}q^n\right)=$$

$$=\prod_{m\geq 1}\left(\prod_{j|m}\left(\sum_{\lambda^m_j\geq 0}\left(\mathbb{L}^{(m-1)\lambda^m_j}q^{m\lambda^m_j}\int_{\mathrm{Sym}^{\lambda^m_j}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(m)^{[\lambda^m_j]})}dz\right)\right)\right)$$

$$=\prod_{m\geq 1}\left(\sum_{\lambda^m\in\mathbb{N}^{\mathrm{Div}(m)}}\left(\left(\mathbb{L}^{(m-1)}q^m\right)^{\sum_{j|m}\lambda^m_j}\prod_{j|m}\left(\int_{\mathrm{Sym}^{\lambda^m_j}(Y(j)/\mu_j)}\mathbb{L}^{-\operatorname{ord}_z(\omega(m)^{[\lambda^m_j]})}dz\right)\right)\right)$$

$$=\prod_{m\geq 1}\left(\sum_{r_m\geq 0}\left(\left(\mathbb{L}^{(m-1)}q^m\right)^{r_m}\int_{\mathrm{Sym}^{r_m}\left(\left(\sqcup_{j|m}Y(j)\right)/\mu_m\right)}\mathbb{L}^{-\operatorname{ord}_z(\omega(m)^{[r_m]})}dz\right)\right)$$

$$=\prod_{m\geq 1}\left(\sum_{r_m\geq 0}\left(\left(\mathbb{L}^{(m-1)}q^m\right)^{r_m}\mathrm{Sym}^{r_m}\left(\int_{\left(\sqcup_{j|m}Y(j)\right)/\mu_m}\mathbb{L}^{-\operatorname{ord}_z(\omega(m))}dz\right)\right)\right)$$

$$=\prod_{m\geq 1}\left(\sum_{r_m\geq 0}\left(\left(\mathbb{L}^{(m-1)}q^m\right)^{r_m}\mathrm{Sym}^{r_m}\left(\left(\textstyle\int_{X(m)}\omega(m)\right)/\mu_m\right)\right)\right)$$

$$=\prod_{m\geq 1}\left(\left(1-\mathbb{L}^{m-1}q^m\right)^{-\left(\int_{X(m)}\omega(m)\right)/\mu_m}\right). \tag{10.3.1}$$

□

10.3.6 Applying this identity to every coefficient of the zeta function $Z_{X,\omega}(T)$, we obtain a formula for the motivic zeta function for the Hilbert schemes of points of X:

Theorem 10.3.7. *Assume that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > n$*, then the following equation holds:*

$$Z_{\mathrm{Hilb}^n(X),\omega^{[n]}} = \sum_{\alpha \dashv n} \prod_{j=1}^{\infty} \left(\mathbb{L}^{(j-1)\alpha_j} \operatorname{Sym}^{\alpha_j} \left(Z_{X(j),\omega(j)}/\mu_j \right) \right) . \tag{10.3.2}$$

Proof. It follows after an application of Corollary 10.3.3 to each coefficient of the zeta function. $\square$

Chapter 11

Proof of the conjecture

11.1 Poles of the Zeta function

11.1.1 Througout this section, we denote by $\overline{\mathscr{R}}$ one of the two rings $\mathscr{M}_k\left[(\mathbb{L}^r-1)^{-1}\colon 0<r\in\mathbb{N}\right]$ or $\widehat{\mathscr{M}_k}$, while $\mathscr{R}$ will denote either $\overline{\mathscr{R}}$ or $\mathscr{M}_k$.

The aim of this section is to study the poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$ in terms of those of $Z_{X,\omega}$ and deduce the following:

Theorem 11.1.2 (Monodromy conjecture for Hilbert schemes)**.** *Let X be a surface with trivial canonical bundle satisfying the monodromy conjecture in $\overline{\mathscr{R}}$.*
If $\operatorname{char} k=0$, then the same holds for $\mathrm{Hilb}^n(X)$, $\forall n\in\mathbb{N}$. If $\operatorname{char} k=p>0$ and X admits a model as in §10.1, then the monodromy conjecture in $\overline{\mathscr{R}}$ holds for $\mathrm{Hilb}^n(X)$, $\forall n<\operatorname{char} k$.

11.1.3 Despite not being the aim of our discussion, we report here the following statement, which can be obtained as a byproduct of the argments we have developed so far:

Proposition 11.1.4. *Let Y,Z be two Calabi-Yau varieties endowed with volume forms ω_1,ω_2 satisfying the monodromy conjecture in $\overline{\mathscr{R}}$. Let ω be the volume form on $Y\times Z$ defined as $\omega:=\mathrm{pr}_Y^*\,\omega_1\wedge\mathrm{pr}_Z^*\,\omega_2$. Then also $Y\times Z$, endowed with the volume form ω, satisfies the monodromy conjecture in $\overline{\mathscr{R}}$.*

11.1.5 For an arbitrary positive integer $l>0$ we have that

$$l\cdot Z_{X(l),\omega(l)}(T^l)=\sum_{i=0}^{l-1}Z_{X,\omega}(\zeta_l^iT)\,,$$

where we consider the functions as power series with coefficient in an algebraic extensions of $\mathscr{R}$ containing the $l-$th roots of unity (though after the due cancellations, the above equation involves only elements of $\mathscr{R}$). Thus, by writing $Z_{X,\omega}(T)$ in the form Equation (3.3.1), with N divisible by l, we see that the set of poles of $Z_{X(l),\omega(l)}(T)$ is contained in $l\cdot\mathcal{P}$.

11.1.6 Using this remark and the results from §8.3 we get an upper bound on the set of poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$:

Corollary 11.1.7. *Let X be a surface with trivial canonical bundle and ω a volume form on it. Assume that $Z_{X,\omega}(T)\in\mathscr{R}[[T]]$ can be written as a sum of functions with only one pole. Let $\mathcal{P}$ be the set of poles of $Z_{X,\omega}$. Then all the poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$ are contained in $\Sigma^n\mathcal{P}$.*

Proof. Let us write $Z_{X(j),\omega(j)}/\mu_j(T) = \sum_{i\geq 0} A_i^{(j)} T^i$. For each $\alpha \dashv n$, let

$$F_\alpha(T) := \sum_{i>0} \left(\mathrm{Sym}^{\alpha_1} A_i^{(1)}\right) \cdots \left(\mathrm{Sym}^{\alpha_n} A_i^{(n)}\right) T^i .$$

According to Equation (10.3.1), we have that $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T) = \sum_{\alpha \dashv n} \mathbb{L}^{n-|\alpha|} F_\alpha(T)$, thus we only need to prove that F_α has only poles inside $\Sigma^n \mathcal{P}$. Lemma 8.3.14 implies that $(\mathrm{Sym}^{\alpha_j} Z_{X(j),\omega(j)}/\mu_j)(T)$ has poles in $\Sigma^{\alpha_j}(j\mathcal{P}) \subseteq \Sigma^{j\alpha_j}\mathcal{P}$; our statement follows from Proposition 8.3.12 and from the identity

$$\Sigma^{\alpha_1}\mathcal{P} + \Sigma^{2\alpha_2}\mathcal{P} + \cdots + \Sigma^{n\alpha_n}\mathcal{P} = \Sigma^n \mathcal{P} .$$

□

11.1.8 We are now ready to prove Theorem 11.1.2:

Proof of Theorem 11.1.2. Let q be a pole of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$ and let $\sigma \in \mathrm{Gal}(\overline{K}|K)$ be a topological generator of the tame Galois subgroup. Consider poles $q_1, \ldots, q_n$ of $Z_{X,\omega}(T)$ such that $q = q_1 + \cdots + q_n$. Since the monodromy conjecture holds for X, there are elements $v_1, v_2, \ldots, v_n \in H^*(X_{\overline{K}}, \mathbb{Q}_l)$ such that $\sigma(v_j) = e^{2\pi i q_j} v_j$. Let us consider the Galois-equivariant isomorphism from [14, Theorem 2]:

$$H^*(\mathrm{Hilb}^n(X), \mathbb{Q}_l) \cong \bigoplus_{\alpha \dashv n} H^*(\mathrm{Sym}^{|\alpha|}(X), \mathbb{Q}_l)(n - |\alpha|) ;$$

focusing on the summand $H^*(\mathrm{Sym}^n(X), \mathbb{Q}_l) \cong H^*(X^n, \mathbb{Q}_l)^{\Sigma_n}$, where the action of Σ_n on $H^*(X^n, \mathbb{Q}_l) \cong H^*(X, \mathbb{Q}_l)^{\otimes n}$ is induced by the usual action $\Sigma_n \curvearrowright X^n$ given by permutation of the factors. Thus the element

$$\sum_{\rho \in \Sigma_n} v_{\rho(1)} \otimes \cdots \otimes v_{\rho(n)}$$

is a non-zero eigenvector of $H^*(\mathrm{Hilb}^n(X), \mathbb{Q}_l)$ for the eigenvalue $\prod_{j=1}^n e^{2\pi i q_j}$. □

11.1.9 And similarly:

Proof of Proposition 11.1.4. We have that $\displaystyle\int_{Y(n)\times Z(n)} \omega(n) = \left(\int_{Y(n)} \omega_1(n)\right)\left(\int_{Z(n)} \omega_2(n)\right)$.

Hence Proposition 8.3.12 implies that $\forall q$ pole of $Z_{Y\times Z,\omega}(T)$ there are a pole q_1 of $Z_{Y,\omega_1}(T)$ and a pole q_2 of $Z_{Z,\omega_2}(T)$ such that $q = q_1 + q_2$.

Since Y and Z satisfy the monodromy conjecture, there are nonzero eigenvectors $v \in H^*(Y, \mathbb{Q}_l)$ with eigenvalue $\exp(2\pi i q_1)$ and $w \in H^*(Z, \mathbb{Q}_l)$ with eigenvalue $\exp(2\pi i q_2)$, so that the element $v \otimes w \in H^*(Y, \mathbb{Q}_l) \otimes H^*(Z, \mathbb{Q}_l) \cong H^*(Y \times Z, \mathbb{Q}_l)$ is an eigenvector with eigenvalue $\exp(2\pi i q)$. □

11.1.10 We are not able to say much about the monodromy conjecture in $\mathscr{M}_k$ for all the Hilbert schemes of points on a surface, since it is not always possible to write $Z_{X,\omega}(T) \in \mathscr{M}_k[[T]]$ as a sum of functions with a single pole. However, there are a few remarkable classes of surfaces whose zeta function has a unique pole. In these cases, such a condition is automatically satisfied, so also $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}$ has a unique pole which is n times the pole of $Z_{X,\omega}$ and $\mathrm{Hilb}^n(X)$ will then satisfy the monodromy property.

Example 11.1.11. We list a few classes of surfaces satisfying the property above:

- Assume that X is an abelian surface; according to [17], $Z_{X,\omega}$ has a unique pole which coincides with Chai's basechange conductor of X;

- If $X \to \operatorname{Spec} K$ is a $K3$ surface admitting an equivariant Kulikov model after the base change with respect to a finite extension F/K, then Halle and Nicaise proved in [18] that $Z_{X,\omega}$ has a unique pole;
- Assume that X is a Kummer surface constructed from an abelian surface A; then Overkamp proved in [36] that $Z_{X,\omega}$ has a unique pole.

Moreover all the surfaces in this list satisfy the monodromy property.

Corollary 11.1.12. *Let X be a surface in the list above, then the monodromy conjecture holds for $\operatorname{Hilb}^n(X)$, provided that either $\operatorname{char} k = 0$ or $\operatorname{char} k > n$, with the usual assumptions on the models of X.*

Similarly, the monodromy conjecture holds for a product $X_1 \times \cdots \times X_n$, where all the X_i are surfaces in the list above.

Proof. The first statement follows from 11.1.2, while the latter follows from 11.1.4. $\square$

Part IV

Examples and future perspectives

Chapter 12

Hilbert scheme of two points on a surface

12.1 Computation of the poles

12.1.1 It is still an open question whether all the sums of n poles of $Z_{X,\omega}(T)$ are actually poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$, or if cancellation might occur. It is reasonable to expect that given a very general $K3$ surface X and a positive integer n, if $\mathcal{P}$ is the set of poles of $Z_{X,\omega}$, then the set of poles of $Z_{Hilb^n(X),\omega^{[n]}}$ coincides with $\Sigma^n\mathcal{P}$. We expect this in virtue of the fact that the expression for $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}$, obtained by following the algorithm of Corollary 11.1.7 and §8.3, contains terms having a pole in each element of $\Sigma^n\mathcal{P}$ and the cancellation among them "should happen only exceptionally".

12.1.2 Let $K := k((t))$, $R := k[[t]]$, where $\operatorname{char} k = 0$. Let $X \subseteq \mathbb{P}^3_K$ (with homogeneous coordinates $[w : x : y : z]$) be the surface defined by the quartic polynomial:

$$w^2x^2 + w^2y^2 + w^2z^2 + x^4 + y^4 + z^4 + tw^4\,,$$

and let, finally, ω be an arbitrary volume form over it; this example was already studied in [18]. It is possible to prove that $Z_{X,\omega}$ has two poles and satisfies the monodromy conjecture in $\mathscr{M}_k$. A direct computation (relying on a construction we will sketch later) shows that the poles of $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}}$ are actually the three expected poles. Let $\mathfrak{Y} \subseteq \mathbb{P}^3_R$ the model of X obtained by the above equation considered as a polynomial with coefficients in R. The model constructed in this way is a regular model whose central fibre is an irreducible surface with only a singular point O of type A_1. After blowing up $O \in \mathfrak{Y}$ one obtains a regular model with strict normal crossing divisor, whose central fibre consists of two components: a regular $K3$ surface D (the strict transform of $\mathfrak{Y}_k$) and a copy of $\mathbb{P}^2_k$ with multiplicity 2, which we denote by E, their intersection is a rational curve C which sits in E as a conic. After semistable reduction, one gets a model $\mathfrak{X}(2) \to R(2)$ of $X(2)$ whose central fibre consists of a smooth $K3$ surface (mapped isomorphically onto D) intersecting $\tilde{E} = \mathbb{P}^1 \times \mathbb{P}^1$ along its diagonal, which is mapped isomorphically onto C.

12.1.3 In [28], Nagai shows that if $\mathfrak{S}$ is a semistable model of S whose central fibre is a chain of surfaces, then $\mathrm{Hilb}^2(\mathfrak{S}, \Delta)$ can be desingularizd by blowing up some components of $\mathfrak{S}_0$, obtaining a semistable model of $\mathrm{Hilb}^2(S)$ over Δ. If there is a group action on $\mathfrak{S}$ that stabilizes the components of $\mathfrak{S}_0$, then it is possible to desingularize $\mathrm{Hilb}^2(\mathfrak{S}, \Delta)$ in an equivariant way with respect to such action.

12.1.4 Using the construction of Nagai one gets a semistable model for $\mathrm{Hilb}^2(X(2m))$ over $R(2m)$ which is Galois equivariant with respect to the action of μ_2 and after Weil-restricting its smooth locus it is possible to compute the motivic integral of $\mathrm{Hilb}^2(X(m))$ for all $m \in \mathbb{N}$ and thus $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}} \in \mathscr{M}_k[[T]]$. After specializing the zeta function using the Poincaré polynomial one sees that all the three possible poles are indeed poles for $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}}$.

12.1.5 The motivic zeta function of $\mathrm{Hilb}^2(X)$ is

$$\begin{aligned} Z_{\mathrm{Hilb}^2(X),\omega^{[2]}}(T) =& (\mathbb{L}-1)^2[C]^2\frac{\mathbb{L}^{-3}T^5}{(1-T)(1-\mathbb{L}^{-2}T^2)(1-\mathbb{L}^{-1}T^2)} + [D^\circ][\tilde{E}^\circ]\frac{\mathbb{L}^{-1}T^2}{1-\mathbb{L}^{-1}T^2} \\ &+ (\mathbb{L}-1)[C][D^\circ]\frac{\mathbb{L}^{-1}T^3}{(1-T)(1-\mathbb{L}^{-1}T^2)} + (\mathbb{L}-1)[C][\tilde{E}^\circ]\frac{\mathbb{L}^{-3}T^4}{(1-\mathbb{L}^{-2}T^2)(1-\mathbb{L}^{-1}T^2)} \\ &+ [\mathrm{Hilb}^2(C\times\mathbb{G}_m)]\frac{\mathbb{L}^{-2}T^3}{(1-T)(1-\mathbb{L}^{-2}T^2)} + [\mathrm{Hilb}^2(D^\circ)]\frac{T}{1-T} \\ &+ [\mathrm{Hilb}^2(\tilde{E}^\circ)]\frac{\mathbb{L}^{-2}T^2}{1-\mathbb{L}^{-2}T^2} + \mathbb{L}(\mathbb{L}-1)[C]\frac{\mathbb{L}^{-1}T^2}{(1-T)(1-\mathbb{L}^{-1}T^2)} \\ &+ \mathbb{L}[E^\circ]\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-2}T^2}\,. \end{aligned}$$

This expression shows that $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}}(T)$ has at most simple poles in $0, \frac{1}{2}, 1$. We can check that all of them are actually poles via the Poincaré specialization, obtained by replacing the classes of varieties by their Poincaré polynomial. We show how we computed the Poincaré polynomial of $\mathrm{Hilb}^2(C\times\mathbb{G}_m)$:

Example 12.1.6. One has that $C\times\mathbb{G}_m$ is obtained by $\mathbb{P}^1\times\mathbb{P}^1$ by removing two copies of $\mathbb{P}^1$, hence

$$\begin{aligned} [\mathrm{Hilb}^2(C\times\mathbb{G}_m)] &= [\mathrm{Hilb}^2(\mathbb{P}^1\times\mathbb{P}^1)] - 2[\mathbb{P}^1]\cdot[\mathbb{P}^1\times\mathbb{G}_m] - [\mathbb{P}^1]^2 - 2([\mathrm{Sym}^2(\mathbb{P}^1)] + \mathbb{L}[\mathbb{P}^1]) \\ &= [\mathrm{Hilb}^2(\mathbb{P}^1\times\mathbb{P}^1)] - 2\mathbb{L}^3 - 7\mathbb{L}^2 - 4\mathbb{L} - 1 \end{aligned}$$

It follows from [13, Theorem 0.1] that the Poincaré polynomial of $\mathrm{Hilb}^2(\mathbb{P}^1\times\mathbb{P}^1)$ is

$$p(\mathrm{Hilb}^2(\mathbb{P}^1\times\mathbb{P}^1),v) = \frac{1}{2}p(\mathbb{P}^1\times\mathbb{P}^1,v^2) + v^2p(\mathbb{P}^1\times\mathbb{P}^1,v) + \frac{1}{2}p(\mathbb{P}^1\times\mathbb{P}^1,v)^2\,,$$

thus

$$p(\mathrm{Hilb}^2(C\times\mathbb{G}_m),v) = v^8 + v^6 - v^4 - v^2\,.$$

12.1.7 Similarly one obtains that

$$p(\mathrm{Hilb}^2(D^\circ),v) = p(\mathrm{Hilb}^2(D),v) - (v^2+1)p(D,v) - v^4\,,$$

hence, by replacing $p(D,v) = v^4 + 22v^2 + 1$, one obtains

$$p(\mathrm{Hilb}^2(D^\circ),v) = v^8 + 22v^6 + 252v^4 + 1\,.$$

Finally

$$p(\mathrm{Hilb}^2(\tilde{E}^\circ),v) = v^8 + 2v^6 + 2v^4\,.$$

12.1.8 Thus, the Poincaré specialization of the zeta function is:

$$\begin{aligned}
P_{\mathrm{Hilb}^2(X),\omega^{[2]}}(v,T) =& \frac{(v^8-2v^4+1)v^{-6}T^5}{(1-T)(1-v^{-4}T^2)(1-v^{-2}T^2)} + \frac{(v^6+22v^4+21v^2)T^2}{1-v^{-2}T^2} \\
&+ \frac{(v^6+21v^4-v^2-21)T^3}{(1-T)(1-v^{-2}T^2)} + \frac{(v^6+v^4-v^2-1)v^{-4}T^4}{(1-v^{-4}T^2)(1-v^{-2}T^2)} \\
&+ \frac{(v^6+v^4-v^2-1)v^{-2}T^3}{(1-T)(1-v^{-4}T^2)} + \frac{(v^8+22v^6+252v^4+1)T}{1-T} \\
&+ \frac{(v^4+2v^2+2)T^2}{1-v^{-4}T^2} + \frac{(v^4-1)T^2}{(1-T)(1-v^{-2}T^2)} \\
&+ \frac{v^4T}{1-v^{-4}T^2}\,.
\end{aligned}$$

0 is a pole In order to check that 0 is a pole, let us rewrite the above expression as:

$$\begin{aligned}
(1-T)P_{\mathrm{Hilb}^2(X),\omega^{[2]}}(v,T) =& \frac{(v^8-2v^4+1)v^{-6}T^5}{(1-v^{-4}T^2)(1-v^{-2}T^2)} + \frac{(v^6+21v^4-v^2-21)(1-v^{-4}T^2)T^3}{(1-v^{-2}T^2)(1-v^{-4}T^2)} \\
&+ \frac{(v^6+v^4-v^2-1)v^{-2}(1-v^{-2}T^2)T^3}{(1-v^{-2}T^2)(1-v^{-4}T^2)} \\
&+ \frac{(v^8+22v^6+252v^4+1)(1-v^{-2}T^2)(1-v^{-4}T^2)T}{(1-v^{-2}T^2)(1-v^{-4}T^2)} \\
&+ \frac{(v^4-1)(1-v^{-4}T^2)T^2}{(1-v^{-2}T^2)(1-v^{-4}T^2)} + (1-T)f(T)\,,
\end{aligned}$$

where $f(T)$ is a function that has not a pole in $T=1$. We then replace $T\mapsto 1$ in the RHS and check that it does not vanish.

$\frac{1}{2}$ is a pole Similarly, we write

$$\begin{aligned}
(1-v^{-2}T)P_{\mathrm{Hilb}^2(X),\omega^{[2]}}(v,T) =& \frac{(v^8-2v^4+1)v^{-6}T^5}{(1-T)(1-v^{-4}T^2)} + \frac{(v^6+22v^4+21v^2)(1-T)(1-v^{-4}T^2)T^2}{(1-T)(1-v^{-4}T^2)} \\
&+ \frac{(v^6+21v^4-v^2-21)(1-v^{-4}T^2)T^3}{(1-T)(1-v^{-4}T^2)} \\
&+ \frac{(v^6+v^4-v^2-1)v^{-4}(1-T)T^4}{(1-T)(1-v^{-4}T^2)} \\
&+ \frac{(v^4-1)(1-v^{-4}T^2)T^2}{(1-T)(1-v^{-4}T^2)} + (1-v^{-2}T^2)g(T)\,,
\end{aligned}$$

where $g(T)$ is a function without poles in $T=v$. Then replacing $T\mapsto v$ in the RHS we check that it does not vanish, thus Z has a pole in $\frac{1}{2}$.

1 is a pole Similarly, we write

$$\begin{aligned}
(1-v^{-4}T)P_{\mathrm{Hilb}^2(X),\omega^{[2]}}(v,T) =& \frac{(v^8-2v^4+1)v^{-6}T^5}{(1-T)(1-v^{-2}T^2)} + \frac{(v^6+v^4-v^2-1)v^{-4}(1-T)T^4}{(1-T)(1-v^{-2}T^2)} \\
&+ \frac{(v^6+v^4-v^2-1)v^{-2}(1-v^{-2}T^2)T^3}{(1-T)(1-v^{-2}T^2)} \\
&+ \frac{(v^4+2v^2+2)(1-T)(1-v^{-2}T^2)T^2}{(1-T)(1-v^{-2}T^2)} \\
&+ \frac{v^4(1-T)(1-v^{-2}T^2)T}{(1-T)(1-v^{-2}T^2)} + (1-v^{-4}T^2)h(T)\,,
\end{aligned}$$

where $h(T)$ is a function without poles in $T = v^2$. Then we replace $T \mapsto v^{-2}$ in the RHS and we check that it does not vanish, hence Z has actually a pole in 1.

Chapter 13

Moduli spaces of stable sheaves

We list a couple of examples where we show that our result sees applications in a broader context. Namely we prove the monodromy conjecture for a new class of varieties. We begin proving that some moduli spaces of sheaves on a $K3$ surface are birationally equivalent to some Hilbert schemes of points on the same $K3$ surface, then we compute their motivic zeta function using its birational invariance and finally we use the birational invariance of the monodromy eigenvalues to conclude the proof.

13.1 Moduli spaces of sheaves of rank 2

13.1.1 Let X be a $K3$ surface and H an ample divisor on it. For $nin\mathbb{N}$ let us consider the moduli space of $H-$slope semistable sheaves $M_H(2, \mathcal{O}_X(H), k(n))$, with $k(n) = (n^2+n+1/2)c_1^2(H)+3$. This consists of rank 2 semistable sheaves whose determinant is H and whose second Chern class is $k(n)$; setting $l(n) = (2n^2 + 2n + 1/2)c_1^2(H) + 3$, we are going to prove the following theorem:

Theorem 13.1.2. *The moduli space $M(2, H, k(n))$ is birational to the Hilbert scheme of points* $\mathrm{Hilb}^{l(n)}(X)$ *for $n \gg 1$.*

Proof. For n large enough, $M_H(2, \mathcal{O}_X(H), k(n))_{\overline{K}}$ is irreducible and its generic point is a stable locally free sheaf by [20, Theorems 9.3.4 and 9.3.2]; this implies that $M_H(2, H, k(n))$ itself is irreducible; moreover the stability and locally freeness of its generic sheaf can be checked after base-change, so $M_H(2, \mathcal{O}_X(H), k(n))$ admits a dense open subset N containing stable locally free sheaves. For dimensional reasons, in order to prove birationality, it is enough to provide a generically injective rational map $\mathrm{Hilb}^{l(n)}(X) \dashrightarrow N$. By the Hirzebruch-Rieman-Roch formula, for $K3$ surfaces:

$$\chi(D) = 2 + \frac{D^2}{2}\,,$$

for a divisor D on X. If n is large enough, $\chi(\mathcal{O}_{X_{\overline{K}}}((2n+1)H)) = h^0(X, \mathcal{O}_{X_{\overline{K}}}((2n+1)H)s)$, thus

$$h^0(X_{\overline{K}}, \mathcal{O}_{X_{\overline{K}}}((2n+1)H)) = \frac{(2n+1)^2}{2}H^2 + 2 = l(n) - 1\,.$$

Hence the global sections $\mathcal{O}_{X_{\overline{K}}}((2n+1)H)$ do not vanish on the generic point of $\mathrm{Hilb}^{l(n)}(X_{\overline{K}})$, so $H^0(X_{\overline{K}}, \mathcal{I}_Z((2n+1)H) = 0$. By [25, Corollary 5.27], it follows that $H^0(X_{\overline{K}}, \mathcal{O}_{X_{\overline{K}}}((2n+1)H)) = H^0(X, \mathcal{O}_X((2n+1)H)) \otimes_K \overline{K}$ and $H^0(X_{\overline{K}}, \mathcal{I}_Z((2n+1)H)) = H^0(X, \mathcal{I}_Z((2n+1)H)) \otimes_K \overline{K}$; hence from the exact sequence

$$0 \to H^0(X, \mathcal{O}_X((2n+1)H) \to H^0(X, \mathcal{O}_Z) \to H^1(X, \mathcal{I}_Z((2n+1)H)) \to 0\,,$$

it follows that $h^1(X, \mathcal{I}_Z((2n+1)H)) = 1$ for generic $[Z] \in \mathrm{Hilb}^{l(n)}(X)$. This means that there is a unique non trivial extension

$$0 \to \mathcal{O}_X \to F_Z \to \mathcal{I}_Z((2n+1)H) \to 0\,,$$

which is locally free because of [20, Thehorem 5.1.1]. We will prove that F_Z is $H-$slope stable for generic Z. If not, there is a line bundle $L \subseteq F_Z$ with $c_1(L) \cdot H \geq \frac{2n+1}{2} H^2$. Since L is not contained in $\mathcal{O}_X$, there is a curve $C \in |L^\vee((2n+1)H)|$ containing Z. In order to prove that this cannot happen for generic Z, it is enough to prove that $\dim |L^\vee((2n+1)H)| \leq l(n) - 1$. For a curve $C \in |L^\vee((2n+1)H)|$, we have that $h^0(\mathcal{O}_C(C)) = C^2/2 + 1 = \frac{2n+1}{2}((2n+1)H^2 - 2c_1(L) \cdot H) \leq \frac{c_1^2(L)}{2}$; which implies that

$$h^0(\mathcal{O}_X(C)) \leq \frac{c_1^2(L)}{2} + 2 \leq \frac{(c_1(L) \cdot H)^2}{2H^2} + 2 \leq \frac{(2n+1)^2 H^2}{2} + 2 = l(n) - 1\,.$$

The map $Z \mapsto F_Z$ is injective, where defined, because $h^0(X, F_Z) = 1$, so F_Z cannot fit in more than one exact sequence starting with $\mathcal{O}_X \to F_Z$. □

13.1.3 This result, together with [26, Theorem 5.1.12], allow us to compute, via the formula (10.3.2), the motivic zeta function of $M_H(2, H, k(n))$ for $n \gg 1$. Because of [12, Proposition 5.1] the monodromy eigenvalues of $M_H(2, H, k(n))$ coincide with those of $\mathrm{Hilb}^{l(n)}(X)$, thus $M_H(2, H, k(n))$ has the monodromy property if and only if $\mathrm{Hilb}^{l(n)}(X)$ has it. This allows us to conclude the following:

Corollary 13.1.4. *Let X be a $K3$ surfaces that satisfies the monodromy conjecture with coefficients in $\mathscr{R}$ and let H be an ample divisor, then the conjecture holds also for the moduli spaces $M_H(2, H, k(n))$, for $n \gg 1$.*

Chapter 14

What's next?

In this chapter we explain possible developements and generalisations that can be achieved by similar techniques than the one we used so far. We expect similar techniques can be used for computing the Zeta function of other symplectic varieties.

14.1 Generalized Kummer varieties

14.1.1 One first way to extend the results of this thesis would be by looking at the Generalized Kummer varieties. Given an Abelian variety $A \to \operatorname{Spec} K$, let $\mathcal{A} \to \Delta$ a proper model whose smooth locus is a Néron model of A. Then we can construct weak Néron models of $\mathrm{Hilb}^n(A)$ via the construction $\mathcal{A}^{[n]}$, the closure of $\mathrm{Kum}^{n-1}(A)$ inside $\mathcal{A}^{[n]}$ gives a wean Néron model of $\mathrm{Kum}^{n-1}(A)$. It is possible to use this construction in order to compute the motivic integrals and the motivic zeta functions of the generalized Kummer varieties.

Question 14.1.2. *Does the Monodromy conjecture hold for generalized Kummer varieties?*

14.2 Moduli spaces of sheaves on a K3 surface

14.2.1 Inaba studied in [21] studied moduli spaces of sheaves on *snc* varieties. It could be possible to use his construction in order to give a recipe for constructing weak Néron models of moduli spaces of sheaves on K3 surfaces and thus computing their motivic zeta functions. Also in this case the natural question to address is whether they satisfy the monodromy property.

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