On Equivariant Euler Characteristics and Spaces of Trees

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PhD Thesis

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Abstract

This thesis contains two parts. The first part studies general properties of equivariant Euler characteristics and several concrete calculations. I first review the basic notions of equivariant Euler characteristics introduced by Atiyah and Segal[AS89] and some useful properties and explanations of it. Then I calculate the equivariant Euler characteristics of Grothendieck constructions of G-functors and apply it to study the differences between the equivariant Euler characteristic of the centralizer $C_{\mathcal{S}_G^{p+*}}(\lambda)$ and the subposet $\mathcal{S}^{p+*}_{C_G(\lambda)}$. Moreover, I generalize the Tamanoi's result regarding the equivariant Euler characteristicof product of manifolds to poset cases. Lastly I determine the equivariant Euler characteristics of all subgroup complexes of symmetric groups in many cases.

The second part is a joint work with Greg Arone. We study the equivariant homotopy equivalence between a kind of space of trees and double suspension of the complex of not 2-connected graphs. This project was motivated by an easy observation that the homology of these two spaces as Σ_n-modules are same up to a sign representation. The way we prove it is by constructing a third space via a special homotopy colimits as a bridge. We show that these two spaces are both Σ_n-equivariant homotopy equivalent to the third space.

Resumé

Denne afhandling har to dele. I den første del handler om generelle egenskaber for ækvivariant Euler karakteristik. Jeg gennemgår først de grundlæggende begreber for ækvivariant Euler karakteristik som introduceret af Atiyah og Segal[AS89] og nævner og forklarer nogle nyttige egenskaber. Dernæst beregner jeg ækvivariant Euler karakteristik af Grothendieck konstruktioner af G-funktorer og anvender det til at studere forskellen mellem ækvivariant Euler karakteristik af centralisatoren $C_{\mathcal{S}_G^{p+*}}(\lambda)$ og underposet $\mathcal{S}_{C_G(\lambda)}^{p+*}$. Desuden generaliserer jeg Tamanois resultat om ækvivariant Euler karakteristik af produkter af mangfoldigheder til poset tilfældet. Til sidst bestemmer jeg den ækvivariant Euler karakteristik af undergruppe komplekset for de symmetriske grupper i mange tilfælde.

Den anden del er et samarbejde med Greg Arone. Vi studerer ækvivariant homotopi ækvivalens mellem en form for rum af træer og den dobbelte suspension af komplekset af ikke-2-sammenhængende grafer. Dette projekt er motiveret af den nemme observation at homologigrupperne for disse to rum er isomorfe Σ_n-moduler op til fortegns repræsentationen. Vi konstruerer et tredje rum, via en speciel homotopi colimes, der fungerer som bro. Og vi viser at disse to rum er Σ_n-ækvivariant homotopi ækvivalente til det tredje rum.

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Contents

Chapter 1

General Introduction

This thesis contains two projects. The first project studies the equivariant Euler characteristics and second project studies the homotopy equivalence between a kind of spaces of trees and complexes of not 2-connected graphs.

The equivariant Euler characteristics were introduced by Atiyah and Segal[AS89],

Definition 1.0.1. Given a finite group G and a finite space M with G-action, the r-th integral equivariant Euler characteristics is defined:

$$\chi_r(M,G) = \frac{1}{|G|}\sum_{X\in \mathrm{Hom}(\mathbb{Z}^r,G)} \chi(M^X)$$

In [AS89, Theorem 1], they expressed the second equivariant Euler characteristics could be represented as Euler characteristics in terms of the equivraint K-theory.

Theorem 1.0.2. *Given a compact G-manifold M, its second order integral equivariant Euler characteristics could be expressed using equivariant K theory:*

$$\chi_2(M,G) = dimK^0_G(M)\otimes\mathbb{C} - dimK^1_G(M)\otimes\mathbb{C}$$

They also conjectured the possibility of using equivariant elliptic cohomology to interpret the third order equivariant Euler characteristics, which was answered positively by Devoto[Dev96, Theorem 1.12]

Theorem 1.0.3. *If M is a compact G-manifold then*

$$\chi_3(M,G) = rank_{\mathbb{F}^*}[Ell^{even}_G\otimes\mathbb{F}^*] - rank_{\mathbb{F}^*}[Ell^{odd}_G\otimes\mathbb{F}^*]$$

Later this notion was generalized by Tamanoi[Tam01] by replace $\mathbb{Z}^{r-1}$ by any group K:

Definition 1.0.4. Given a finite group G and any group K, the generalized equivariant Euler characteristics of a finite G-space M is

$$\chi_K(M,G) = \frac{1}{|G|}\sum_{X\in \mathrm{Hom}(\mathbb{Z}\times K,G)} \chi(M^X)$$

when $K=\mathbb{Z}^r_p$, the associated p-primary equivariant Euler characteristics enjoy good properties such as its close relation with Morava K-theory[Tam01, Theorem B]

Theorem 1.0.5. *The $(r+1)$-th p-primary equivariant Euler characteristics of a compact G-manifold M is equal to equivariant Morava K-theory of M at height r, in other words:*

$$\chi^p_{r+1}(M;G) = \chi_{K_G(r)}(M)$$

Tamanoi calculated the equivariant Euler characteristic of the product of G-manifold M in terms of the equivariant Euler characteristic of M[Tam01, Theorem A][Tam03, Theorem C].

Theorem 1.0.6. *For any $r \geq 1$ and for any G-manifold M we have*

$$\sum_{n\geq 0} \chi_r(M^n; G\wr\Sigma_n)q^n = \big[\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\big]^{-\chi_r(M,G)}$$

where $j_d(\mathbb{Z}^{r-1})$ is the number of index d subgroups in $\mathbb{Z}^{r-1}$.

In this thesis we prove a similar result for bounded and half bounded G-posets:

Theorem 1.0.7. *1. If P is a bounded finite G-poset*

$$\sum_{n\geq 0} \widetilde{\chi}_r(\overline{P^n}; G\wr\Sigma_n)q^n = \big[\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\big]^{-\widetilde{\chi}_r(\overline{P},G)}$$

2. If P is a half bounded finite G-poset

$$\sum_{n\geq 0} \widetilde{\chi}_r(\overline{P^n}; G\wr\Sigma_n)q^n = -\big[\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\big]^{\widetilde{\chi}_r(\overline{P},G)}$$

The key ingredient for proving this theorem is the following technical combinatiorial result:

Lemma 1.0.8. *Let $\chi : G \longrightarrow \mathbb{Z}$ be a class function i.e. if c_1, c_2 are in a same conjugacy class of G then $\chi(c_1) = \chi(c_2)$. Then,*

$$\sum_{\sum\limits_{r,[c]} rm_r(c)=n} \sharp\{m_r(c)\} \prod_{[c]\in[G]} \chi(c)^{\sum\limits_r m_r(c)} = \big(\sum_{c\in G}\chi(c)\big)\big(\sum_{c\in G}\chi(c)+|G|\big)\cdots\big(\sum_{c\in G}\chi(c)+n|G|-|G|\big)$$

Where $|G|$ is the order of G,$\sharp\{m_r(c)\}_{[c]\in[G],1\leq r\leq n}$ means the number of group elements in $G\wr\Sigma_n$ with the same conjugacy class represented by the sequence $\{m_r(c)\}_{[c]\in[G],1\leq r\leq n}$ and the summation in the left hand side is actually taken over the conjugacy classes of $G\wr\Sigma_n$, i.e. the solution of the equation $\sum\limits_{r,[c]} rm_r(c) = n$ corresponds to the set of conjugacy classes of $G\wr\Sigma_n$.

The concrete calculations of equivariant Euler characteristics are intensely studied recently by Jesper Møller[Mø17b][Mø17a][Mø18][Mø19]. For example he determined the equivariant Euler characteristic of partition complex Π_n with respect to the symmetric group Σ_n action on it[Mø17a, Theorem 1.3].

Theorem 1.0.9. *The r-th reduced equivariant Euler characteristicof the Σ_n-poset Π_n is*

$$\widetilde{\chi}_r(\Pi_n,\Sigma_n) = \frac{c_r(n)}{n}$$

where the multiplicative function c_r is given by Dirichlet inverse

$$c_r = (\iota_2 * \pi_1 * \cdots * \pi_{r-1})^{-1}$$

of the iterated Dirichlet convolution of the function ι_2 and functions π_k. Here $\pi_k(n) = n^k$ for any $n \geq 1$ and ι_2 is the multiplicative function given by $\iota_2(n) = n$ if n is a power of 2 or $\iota(n) = 0$ for other n.

In this thesis, I calculate the equivariant Euler characteristics of all subgroup complex $\mathcal{S}^*_{\Sigma_n}$ for a series n:

Theorem 1.0.10. *The (reduced) equivariant Euler characteristics of all symmetric groups($n \geq 3$) are:*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = \frac{2^r}{n!}\mu(1, \Sigma_n)$$

Therefore in particular:

1. *If $n = p$ is an odd prime we have*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_p}, \Sigma_p) = 2^{r-1}$$

2. *If $n = 2p$ where p is an odd prime then*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = \begin{cases} -2^r & \text{if } n-1 \text{ is prime and } p \equiv 3 \pmod 4 \\ 2^{r-1} & \text{if } n = 22 \\ -2^{r-1} & \text{otherwise} \end{cases}$$

3. *If $n = 2^a$, for a a natural number then*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = -2^{r-1}$$

Besides that in this thesis I also calculate the equivariant Euler characteristicof the Grothendieck construction of a G-functor $\mathcal{S} : \mathcal{D} \to$ **POSET**:

Theorem 1.0.11.

$$\chi_r(\int_{\mathcal{D}} \mathcal{S}, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \sum_{d \in C_{\mathcal{D}}(X)} k^d(C_{\mathcal{D}}(X))\chi(C_{\mathcal{S}(d)}(X))$$

and apply it to study the difference of two posets in terms of its equivariant Euler characteristics.

Theorem 1.0.12. *The difference between the equivariant Euler characteristics of the $C_G(\lambda)$-posets $C_{\mathcal{S}^{p+*}_G}(\lambda)$ and $\mathcal{S}^{p+*}_{C_G(\lambda)}$ is*

$$\widetilde{\chi}_r(C_{\mathcal{S}^{p+*}_G}(\lambda), C_G(\lambda)) - \widetilde{\chi}_r(\mathcal{S}^{p+*}_{C_G(\lambda)}, C_G(\lambda)) = \widetilde{\chi}_r(\mathcal{D}^{p+*}_G(\lambda), C_G(\lambda)) - \widetilde{\chi}_r(\int_{\mathcal{D}^{p+*}_G(\lambda)} \mathcal{S}^{p+*}_{C_G(\lambda,-)}, C_G(\lambda))$$

A possible further research topic is trying to determine the equivariant Euler characteristics of p-subgroup complexes of symmetric groups. Since in general for any element λ two subposets $C_{\mathcal{S}^{p+*}_{\Sigma_n}}(\lambda)$ and $\mathcal{S}^{p+*}_{C_{\Sigma_n}(\lambda)}$ are not identical. The above theorem might be helpful to determine the equivariant Euler characteristics of p-subgroup complexes of symmetric groups.

The second project studies the reason of the similarity of the homology of the space of trees T_{n-1} and the complex of not 2-connected graphs on n vertices Δ^2_n in topological level.

Proposition 1.0.13. *[RW96, Theorem 3.1][BBL+99, Thereom 4.1]*

1. *The character of complex representation of Σ_n on the homology $H_{n-4}(T_{n-1}, \mathbb{C})$ is*

$$\epsilon \cdot (lie_{n-1} \uparrow^{\Sigma_n}_{\Sigma_{n-1}} - lie_n)$$

2. The character of complex representation of Σ_n on the homology $H_{2n-5}(\Delta_n^2, \mathbb{C})$ is

$$lie_{n-1} \uparrow_{\Sigma_{n-1}}^{\Sigma_n} - lie_n$$

So the character of these two homology are same up to a character of sign representation of Σ_n. We hope there is a homotopy equivalence between these two spaces to explain this phenomenon. However, because of dimension reason there is no homotopy equivalence of them directly. So we modify these two spaces T_{n-1} and $|\Delta_n^2|$ to get a new space of trees Q_{n-1} and double suspension $\Sigma S|\Delta_n^2|$ of $|\Delta_n^2|$. And the main result of this project is the following theorem:

Theorem 1.0.14. *The space of trees on n vertices: Q_{n-1} is Σ_n-equivariant homotopy equivalent to the double suspension of the complex of not 2-connected graphs on n vertices:$\Sigma S|\Delta_n^2|$. Where Σ means the reduced suspension and S means the unreduced suspension.*

However, since it is not easy to construct a map between them and show it is an equivariant homotopy equivalence, we construct a "black box" as a bridge to connect these two spaces. More concretely, let Y be be the total cofiber of the functor $F : \widehat{\mathcal{C}_n} \to \mathbf{Top}_*$ i.e. $Y = \text{hocofib}(\text{hocolim}_{\mathcal{C}_n} F \to P_n)$ where $\widehat{\mathcal{C}_n}$ is the category of connected acyclic hypergraphs on n-vertices, then we can show that this space Y is Σ_n-equivariant homotopy equivalent to both Q_{n-1} and $\Sigma S|\Delta_n^2|$.

Theorem 1.0.15. *There is an induced map $f_1 : Y \to Q_{n-1}$ which is a Σ_n-equivariant homotopy equivalence.*

Theorem 1.0.16. *There is an induced map $f_2 : Y \to \Sigma S|\Delta_n^2|$ which is a Σ_n-equivariant homotopy equivalence.*

Chapter 2

On Equivariant Euler Characteristics

This chapter is a study on some general properties and concrete calculations of the equivariant Euler characteristics introduced in [AS89] by Atiyah and Segal. It is a very interesting numerical topological invariant which has a close relation to orbifold theory[DHVW86][DHVW85], representation theory[Thé93], generalized cohomology theories[HKR00][Tam01] and so on. In this chapter we first introduce the definition of the equivariant Euler characteristics in section 2.1 and explain its different interpretations from combinatorics, geometry and topology in section 2.2. In section 2.3 we study a relation of a special kind of equivariant Euler characteristics with the representation theory of finite groups. In section 2.4 we calculate the equivariant Euler characteristics of the Grothendieck construction and apply it to study the equivariant Euler characteristics of p-subgroup complexes. In section 2.5 we generalize Tamanoi's result [Tam01, Theorem A] regarding the equivariant Euler characteristics of Cartesian products of manifolds to Cartesian products of posets. Finally in section 2.6 we calculate the equivariant Euler characteristics of all subgroup complexes of symmetric group Σ_n.

2.1 Introduction

2.1.1 Euler Characteristics of Spaces

To make this chapter self-contained, in this section we introduce what the Euler characteristic of a space X is. We first need to choose a good model for the spaces we are working on. There are several options of models of spaces with different advantages like: simplicial complexes, Δ-sets and simplicial sets. In this chapter we choose the Δ-sets as our model for spaces. Hence we don't distinguish spaces and Δ-sets.

Definition 2.1.1. Let Δ be the category with objects $[n]_+ := \{0, 1, \dots, n\}$ and morphisms strictly increasing maps. Then a Δ-set X is a presheaf of Δ on the category of sets. In other words, X is a functor from the category Δ^{op} to the category **SET**. We can collect all Δ-sets as objects to form a category **Δ-SET** with morphisms the natural transformations of functors.

Definition 2.1.2. Let G be a finite group. We denote by **BG** the category having as object just a single point and morphisms the whole group G. The composition of morphisms is just the multiplication of group elements. A G-Δ-set is a functor from **BG** to **Δ-SET**.

The advantage of choosing Δ-sets as our models for spaces is that they are invariant under taking quotients with a G-Δ-set X. In the case of simplicial complexes taking quotients may break the simplicial structures, so we need to pass to the subdivision or even the second subdivision to make sure it will equip the orbit spaces with simplicial structures. Therefore, the model Δ-sets successfully help us to avoid these subtleties. Furthermore we actually do not need degeneracy maps, hence the model Δ-sets are good enough for our purpose.

We say a Δ-set X is finite if there exists a natural number N such that X_n is an empty set for all $n > N$, and each X_n is a finite set. For simplicity, in this chapter we just consider the Euler characteristics of finite Δ-sets.

Definition 2.1.3. Given a finite Δ-set X, its Euler characteristic is defined as alternating sums of cardinality that of the set X_n $n \in \mathbb{N}$.

$$\chi(X) := \sum_{d=0}^{\infty} (-1)^d |X_d|$$

As in the case of simplicial sets we can also define the geometric realization of Δ-sets[GJ99, Chapter 1, Section 1.2].

Definition 2.1.4. Given a Δ-set X. Its geometric realization is defined as follows:

$$|X| = \left(\coprod_n X_n \times \Delta^n \right) / \sim$$

where the equivalence relation is generated by $(d_i x, t) \sim (x, \sigma_i t)$. $\sigma_i : \Delta^{n-1} \to \Delta^n$ is a face map sending $(t_1, t_2, \dots, t_{n-1})$ to $(t_1, t_2, \dots, t_{i-1}, 0, t_i, t_{i+1}, \dots, t_{n-1})$ where 0 is inserted in the i-th position and $d^i : X_n \to X_{n-1}$ is induced by the natural inclusion $d^i : [n-1]_+ \to [n]_+$ defined by:

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$$

Remark. A similar argument like showing that geometric realizations of simplicial sets are CW-complexes[GJ99, Chapter I, Section 1.2] could also be used to show that the geometric realizations of Δ-sets are CW-complexes.

A classical result of Euler characteristic of a space X is that it could be calculated via the alternating sum of Betti numbers i.e. the rank of (co)homology groups of X[Hat02, Theorem 2.44].

Theorem 2.1.5. *Given a finite Δ-set X we can use the alternating sum of the dimension of cohomology groups of X to compute its Euler characteristics. In other words,*

$$\chi(X) = \sum_{d=0}^{\infty} \operatorname{Rank} H^d(|X|, \mathbb{Z})$$

A direct corollary of this result is that the Euler characteristic is a homotopy invariance. Moreover the Euler characteristic enjoys the following two properties[Spa81, Page 481]

Proposition 2.1.6. *1.* $\chi(X \times Y) = \chi(X)\chi(Y)$

2. $\chi(X \coprod Y) = \chi(X) + \chi(Y)$

Remark. From this property we see that the Euler characteristic is a special additive and multiplicative function in the category of topological spaces with respect to addition as the disjoint union and multiplication as the Cartesian product. The reduced Euler Euler characteristic serves the same role for the category of pointed topological spaces with addition as the wedge sum of pointed spaces and multiplication as the smash products of pointed spaces.

Roughly speaking, the Euler characteristic is a homotopy invariant of spaces taking disjoint unions of spaces to addition and Cartesian products of spaces to multiplication. It enjoys similar properties of cardinality of finite sets. So we can also view the Euler characteristic as a way for counting in the homotopy theory world. The following two results could be viewed as an analogy between Euler characteristics and cardinality[Rot95, Theorem 3.22][Sha78, Page 127].

Theorem 2.1.7 (Burnside's counting lemma)**.** *Given a finite G-set X the number of G-orbits of X is*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

Theorem 2.1.8 (Lefschetz fixed point formula)**.** *Given a finite G-Δ-set X,*

$$\chi(X/G) = \frac{1}{|G|}\sum_{g\in G}\chi(X^g)$$

Remark. This theorem is still true when we replace the ordinary Euler characteristics by the reduced one.

2.1.2 Mobius Inversions and Euler Characteristics

We have introduced what the Euler characteristic is for a Δ-set. However, in this thesis, many mathematical objects are naturally described as posets.

Definition 2.1.9. Given a set P, we say a binary relation $\leq$ is a partial order on P if it satisfies the following three conditions:

1. Reflexivity: $a \leq a$ for all $a \in P$
2. Anti-symmetry: If $a \leq b$ and $b \leq a$ then $a = b$ for $a, b \in P$
3. Transitivity: If $a \leq b$ and $b \leq c$ then we have $a \leq c$

We call a set with a partial order a poset.

Definition 2.1.10. A poset map f between two posets P and Q is a function on sets which preserves the order i.e. for any $x \leq y$ in P we have $f(x) \leq f(y)$ in Q. We can form a new category **POSET** of objects of posets and morphism of poset maps between them.

It is convenient for us to define the Euler characteristics of posets directly. Given a poset S we can naturally view it as a Δ-set $\Delta(S)$ in which $\Delta(S)_d$ is the set of chains with length $d+1$. And the face maps $d_i\colon \Delta(S)_d \longrightarrow \Delta(S)_{d-1}$ are defined as follows:

$$d_i(a_1 \leq a_2 \leq \cdots \leq a_n) = a_1 \leq a_2 \leq \cdots \leq a_{i-1} \leq a_{i+1} \leq \cdots \leq a_n$$

So the Euler characteristic of a poset S is defined to be the Euler characteristic of its Δ-set $\Delta(S)$, i.e.

$$\chi(S) := \chi(\Delta(S))$$

And in this subsection we introduce a useful alternative description of Euler characteristics of posets via the language of Möbius inversions[Sta12, Section 3.8].

Definition 2.1.11. Given a finite poset P, $I(P)$ is the set of all functions $f : P \times P \to \mathbb{C}$ satisfying $f(x, y) = 0$ if $x \nleq y$. There is a natural binary operation on this set $I(P)$ defined as $f \cdot g(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$.

Remark. $I(P)$ is called incidence algebra equipped with this binary operation and the δ is the unit of this algebra:

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{others} \end{cases}$$

We define a natural function $\zeta : P \times P \to \mathbb{C}$ as :

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{others} \end{cases}$$

This function is called the zeta function of P or incidence matrix of P.

Definition 2.1.12. The Möbius inversion μ of P is the inverse of the zeta function ζ of P in $I(P)$.

Remark. Möbius inversion μ is equivalent as the inverse matrix of the incidence matrix of P.

Proposition 2.1.13. *Given a finite poest P and its two functions $f, g : P \to \mathbb{C}$ the following two equations are equivalent:*

1. $g(x) = \sum\limits_{y \leq x} f(y)$
2. $f(x) = \sum\limits_{y \leq x} g(y)\mu(y, x)$

Dually we have:

1. $g(x) = \sum\limits_{y \geq x} f(y)$
2. $f(x) = \sum\limits_{y \geq x} g(y)\mu(x, y)$

Proof. see [Sta12, Proposition 3.7.1][Sta12, Proposition 3.7.2]. □

The Möbius function is well-behaved under the product of posets

Proposition 2.1.14. *Given two finite posets P and Q we can naturally form its product poset $P \times Q$ by setting its elements to be exactly the elements of the Cartesian product of sets P and Q and $(x, y) \leq (x', y')$ if and only if $x \leq x'$ in P and $y \leq y'$ in Q. Suppose mu_P and μ_Q are the Möbius functions of these two posets respectively. Then by any pairs $(x, y) \leq (x', y')$ in $P \times Q$, the Möbius function of $P \times Q$ on it is given by*

$$\mu_{P \times Q}((x, y), (x', y')) = \mu_p(x, x')\mu_Q(y, y')$$

Proof. See [Sta12, Proposition 3.8.2] □

The theory of Möbius inversion formula for a partially order set was first introduced independently by Weisner[Wei35] and Hall[Hal34]. Later Rota[Rot64] showed it is extremely useful in combinatorics and other areas of mathematics and that it unifies many different mathematical phenomena. As a first example, let's see how the inversion formula recovers the classical principle in combinatorics: the Principle of Inclusion-Exclusion.

Example 2.1.15. [Qia08, Example 4.10] Given a positive integer n, let $[n]$ be the set with natural numbers from 1 to n. There is a natural poset of elements of subsets of $[n]$ and orders of inclusion of subsets which is called the Boolean poset of $[n]$ and denoted by B_n. Let's try to compute its Möbius function μ. First we observe that the poset B_n is isomorphic to the product of n-th copies of the poset $B := \{0 \leq 1\}$ given by the isomorphism $\varphi : B_n \to \{0 \leq 1\}^n$ where $\varphi(T) := (\chi_T(1), \dots, \chi_T(n))$ for T a subset of $[n]$. χ_T is the characteristic function of T over $[n]$, i.e. $\chi_T(i) = 1$ if $i \in T$ otherwise $\chi_T(i) = 0$.

The Möbius function μ_B of the single poset B is clearly $\mu_B(x, y) = (-1)^{y-x}$ for $0 \leq x \leq y \leq 1$. So given any two subsets S, T by applying Proposition 2.1.14 we have

$$\begin{aligned} \mu(S,T) &= \prod_{i=1}^{n} \mu_B(\chi_S(i), \chi_T(i)) \\ &= (-1)^{|T|-|S|} \end{aligned}$$

According to Proposition 2.1.13, we have

$$f(T) = \sum_{T \subset S} g(S) \iff g(T) = \sum_{T \subset S} f(S)\mu(T,S) = \sum_{T \subset S} (-1)^{|S|-|T|} f(S)$$

As an application of this formula, we can derive the principle of inclusion-exclusion. Given a finite set A and a sequence of its subsets $A_1, \dots, A_n$ and for any subset $T \subset [n]$ we define two functions $f(T) := |\cap_{i \in T} A_i|$ and $g(T) = |(\cap_{i \in T} A_i) \cap (\cap_{j \notin T} \bar{A}_j)|$, where $\bar{A}_j$ means the complement of A_j in A. Then we observe that

$$|\cap_{i \in T} A_i| = \sum_{T \subset S} |(\cap_{i \in S} A_i) \cap (\cap_{j \notin S} \overline{A_j})|$$

in other words: $f(T) = \sum_{T \subset S} g(S)$. Then by the Möbius inversion formula we have

$$\begin{aligned} |(\cap_{i \in T} A_i) \cap (\cap_{j \notin T} \bar{A}_j)| &= g(T) \\ &= \sum_{T \subset S} (-1)^{|S|-|T|} f(S) \\ &= \sum_{T \subset S} (-1)^{|S|-|T|} |\cap_{i \in S} A_i| \end{aligned}$$

In particular, if T is the empty set we have

$$|\cap_{i \in [n]} \bar{A}_i| = |A| - \sum_{i}^{n} |A_i| + \cdots + (-1)^n |\cap_{i \in [n]} A_i|$$

This is just the classical form of the principle of inclusion-exclusion.

Definition 2.1.16. Given a poset P, and its two elements $x \leq y$. Then we say the interval $[x, y]$ in P is the subposet of P with all objects z such that $x \leq z \leq y$.

The Möbius function of a finite poset has a very close relation with its Euler characteristic:

Proposition 2.1.17. *Given a finite poset P, its Euler characteristiccould be calculated via the Möbius function μ_p of P:*

$$\chi(P) = \sum_{x,y \in P} \mu_P(x, y)$$

Remark. We can also apply this formula as the definition of the ordinary Euler characteristics for finite posets or even finite categories. However, the disadvantage of this definition is that the Möbius function of a finite category may not exist. However, we will see in the next subsection that the existence of the Möbius function is not essential for describing a invariant with the same good properties as Euler characteristics.

Proof. By Definition 2.1.3 of Euler characteristics of a finite poset:

$$\chi(P) = \sum_k (-1)^k |P_k|$$

To prove the statement, we first give a description of the number of k-simplicies in terms of the Möbius function of P. In the incidence algebra of this finite poset P we observe that for any two elements $x \leq y$ in P the ζ function of P on them is $\zeta(x,y) = 1$. Moreover, by definition $\zeta^2(x,y) = \sum\limits_{x \leq z \leq y} \zeta(x,y)\zeta(y,z)$ is the cardinality of the interval $[x,y]$ in P. By induction on any non-negative integer k we have

$$\zeta^k(x,y) = \sum_{x = x_0 \leq x_1 \leq \cdots \leq x_k = y} 1$$

which is the number of k-chains from x to y. However, this is not the number of k-simplicies from x to y. Let I be the unit of the incidence algebra defined by $I(x,y) = 1$ if $x = y$ and $I(x,y) = 0$ if $x \neq y$. Therefore

$$(\zeta - I)^2(x,y) = \sum_{z \leq z \leq y} (\zeta - I)(x,z)(\zeta - I)(z,y) = \sum_{x < z < y} 1$$

which is the number of 2-simplicies from x to y. Then by induction we know that for any non-negative integer k the $(\zeta - I)^k(x,y)$ is the number of k-simplicies from x to y. Therefore,

$$|P_k| = \sum_{x,y \in P} (\zeta - I)^k(x,y)$$

Hence we have

$$\begin{aligned}
\chi(P) &= \sum_k (-1)^k \sum_{x,y \in P} (\zeta - I)^k(x,y) \\
&= \sum_{x,y \in P} \sum_k (-1)^k (\zeta - I)^k(x,y) \\
&= \sum_{x,y} \mu_P(x,y)
\end{aligned}$$

The last equation holds since $\mu_p(x,y) = \zeta^{-1}(x,y) = (I + \zeta - I)^{-1}(x,y) = \sum\limits_k (-1)^k (\zeta - I)^k(x,y)$. $\square$

Proposition 2.1.18. *[Sta12, Proposition 3.8.6] Möbius functions are local, in other words, given a poset P its Möbius function $\mu_P(x,y)$ for $x \leq y \in P$ is equal to the Möbius function $\mu_{[x,y]}(x,y)$ on the interval $[x,y]$ of P.*

Proof. Without losing any generality, we can assume that $x < y$. Then

$$\begin{aligned}
\mu_P(x,y) &= \sum_{k=0}^{\infty}(-1)^k(\zeta - I)^k(x,y) \\
&= (-1) + \sum_{k=2}^{\infty}(-1)^k(\zeta - I)^k(x,y) \\
&= (-1) + \sum_{k=2}^{\infty}|\text{k-simplicies in } (x,y)| \\
&= \widetilde{\chi}(x,y)
\end{aligned}$$

Where by same argument we know $\mu_{[x,y]}(x,y) = \widetilde{\chi}(x,y) = \mu_P(x,y)$ □

Remark. A direct consequence of this proposition is that for any two posets P and Q if there are two intervals $[x,y]$ and $[a,b]$ of P, Q respectively which are isomorphic then $\mu_P(x,y) = \mu_Q(a,b)$.

At the end of this subsection, we apply the theory of Möbius functions of posets to recover the classical Möbius inversion formula in number theory.

Example 2.1.19. [Qia08, Example 4.9] We first equip with a partial order the set of natural numbers $\mathbb{N}$ by division. In other words, we say two natural numbers d, n with $d \leq n$ if d divides n. So what's the Möbius function μ_D of $\mathbb{N}$ equipped with this partial order? Given any natural number $n = p_i^{a_1} \cdots p_k^{a_k}$ then we observe that the interval $[1, n]$ is isomorphic to the product of posets $\{0 \leq \cdots \leq a_1\} \times \cdots \times \{0 \leq \cdots \leq a_k\}$ given by the map $\varphi(d) = (b_1, \ldots, b_k)$ for any $d = p_1^{b_1} \cdots p_k^{b_k}$ where $b_i \leq a_i$ for $1 \leq i \leq k$. Therefore

$$\mu_D(1,n) = \prod_{i=1}^{k} \mu_i(0, a_i)$$

Where μ_i is the Möbius function on the poset $\{0 \leq \cdots \leq a_i\}$, and by an easy calculation we obtain $\mu_i(0, a_i) = (-1)^{a_i}$. Therefore we have

$$\mu_D(1,n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } a_1 = \cdots = a_k = 1 \\ 0, & \text{otherwise} \end{cases}$$

So $\mu_D(1,n)$ is just the classical Möbius function $\mu(n)$. Moreover, in the general case when d divides n then the interval $[d, n]$ is isomorphic to the interval $[1, \frac{n}{d}]$, therefore $\mu_D(d,n) = \mu_D(1, \frac{n}{d}) = \mu(\frac{n}{d})$. In this situation the Möbius inversion formula in Proposition 2.1.13 becomes:

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu_D(d,n) f(d) = \sum_{d|n} \mu(\frac{n}{d}) f(d)$$

This is the classical Möbius inversion formula in number theory.

2.1.3 Euler Characteristics of Categories

Tom Leinster[Lei08] generalizes the notion of Euler characteristics of spaces or posets to finite categories such that we can compute for example the Euler characteristic of the classifying space of a finite group G which is not well defined in the ordinary definition since it always have the homotopy type of infinite many cells. Recall that in the last subsection we defined what is a Möbius inversion for a poset by computing the inverse matrix of the matrix formed by the zeta-function of this poset. However, if this matrix is singular, then the inverse matrix doesn't exist. But we observe that we in fact just need a weaker notion than Möbius function to construct an invariant with similar properties like the Euler characteristics. The most content of this subsection is based on Tom Leinster's paper[Lei08].

Definition 2.1.20. Let $\mathcal{C}$ be a finite category. A function $k^\bullet : ob(\mathcal{C}) \to \mathbb{Q}$ such that

$$\sum_b \zeta_{\mathcal{C}}(a,b)k^b = 1$$

for any element $a \in ob(\mathcal{C})$ is called a weighting on $\mathcal{C}$. Where $\zeta_{\mathcal{C}}$ is the zeta function for this finite category $\mathcal{C}$ defined by $\zeta_{\mathcal{C}}(a,b) := |\mathcal{C}(a,b)|$. Dually, a co-weighting for $\mathcal{C}$ is a function $k_\bullet : ob(\mathcal{C}) \to \mathbb{Q}$ such that

$$\sum_a k_a \zeta_{\mathcal{C}}(a,b) = 1$$

for any element $b \in ob(\mathcal{C})$.

Remark.

1. The weighting or co-weighting of a finite category may not exist, and if they exist they might not be unique.
2. We call incidence matrix of a finite category $\mathcal{C}$ the matrix with entries $\mathcal{C}(a,b)$. If this matrix is non-singular then it's clear that the weighting and co-weighting exist and are unique since they can be computed via the inverse matrix of the matrix $(\zeta_{\mathcal{C}}(a,b))_{a,b\in ob(\mathcal{C})}$.
3. As an example, let $\mathcal{C}$ be a finite category with only two elements and two non-identity morphisms like this:

$$A \rightleftarrows B$$

Then the incidence matrix of this category is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It is clear that this matrix is a singular matrix but the weighting and co-weighting both exist and are not unique.

Lemma 2.1.21. *For a finite category $\mathcal{C}$ if its weighting $k^\bullet$ and co-weighting $k_\bullet$ both exist then*

$$\sum_{b\in ob(\mathcal{C})} k^b = \sum_{a\in ob(\mathcal{C})} k_a$$

Proof. Since $\mathcal{C}$ has a co-weighting, by definition we know that for any element b in $\mathcal{C}$, $\sum_a k_a \mathcal{C}(a,b) = 1$. Therefore,

$$\sum_b k^b = \sum_b \sum_a k_a \mathcal{C}(a,b)k^b = \sum_a k_a \sum_b \mathcal{C}(a,b)k^b = \sum_a k_a$$

□

Definition 2.1.22. We say a finite category $\mathcal{C}$ has an Euler characteristic if it has both weighting $k^\bullet$ and co-weighting $k_\bullet$ and in this case its Euler characteristics is calculated via

$$\chi(\mathcal{C}) = \sum_{b\in ob(\mathcal{C})} k^b = \sum_{a\in ob(\mathcal{C})} k_a$$

Remark. According to the remark of Definition 2.1.20 we know this definition coincides with the definition by Möbius functions of finite categories when they exist.

Proposition 2.1.23. 1. *If there is an adjunction between two finite categories $\mathcal{C}$, $\mathcal{D}$ and the Euler characteristics of both exist, then $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*

2. *If a finite category $\mathcal{C}$ has a initial object or a terminal object then its Euler characteristic exists and $\chi(\mathcal{C}) = 1$.*

3. *If two finite categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories then $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*

Proof. See [Lei08, Example 2.3 d],[Lei08, Propsition 2.2] □

2.1.4 Ordinary Equivariant Euler Characteristics

In 1980's Dixon, Harvey, Vafa and Witten [DHVW86] [DHVW85]pointed out that a "correct" version of the Euler characteristic for an orbifold M with a finite group G-action should look like

$$\chi^{orb}(M, G) = \frac{1}{|G|} \sum_{(g_1,g_2)'} \chi(M^{<g_1,g_2>})$$

for string theoretical reasons. Where $(g_1, g_2)'$ means a commuting pair of group elements in G. Atiyah and Segal [AS89] generalize this idea to give a higher order version of this orbifold Euler characteristics which they call the equivariant Euler characteristics.

Definition 2.1.24. Given a finite group G and a finite space M with G-action, the r-th integral equivariant Euler characteristics is defined:

$$\chi_r(M, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \chi(M^X)$$

and the reduced r-th integral equivariant Euler characteristics is defined similarly:

$$\widetilde{\chi}_r(M, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \widetilde{\chi}(M^X)$$

here M^X means the fixed points of M under the action of images of X. This could be also denoted as $C_M(X)$: the centralizer of M under X.

Remark.
- The second order equivariant Euler characteristics recovers the previous orbifold Euler characteristics.
- We can replace the notion of finite space by finite posets, compact manifold and so on.
- Actually equivariant Euler characteristic could be defined over finite G-categories with some restrictions on categories for example EI-categories i.e. each endomorphism in this category is isomorphism since in this case the Euler characteristics of each centralizer subcategory has Euler characteristics in terms of Tom Leinster as we introduced in Section 2.1.3 [GMl15, Section 3]. However, in this chapter we always dealt with equivariant Euler characteristics of Δ-sets unless otherwise specific.

The recursion lemma below is a very useful technique to simplify many concrete calculations of equivariant Euler characteristics.

Lemma 2.1.25. *[Tam01, Proposition 2-5]*

1. *The first integral equivariant Euler characteristics $\chi_1(M;G)$ is equal to $\chi(M/G)$.*
2. *For $r > 1$*

$$\chi_r(M;G) = \sum_{[g]\in[G]} \chi_{r-1}(M^g; C_G(g)) = \sum_{XG\in \mathrm{Hom}(\mathbb{Z}^{r-1},G)/G} \chi(M^X/C_G(X))$$

Where $[G]$ is the set of conjugate elements in G.

Proof. 1. By definition we have

$$\begin{aligned}\chi_1(M,G) &= \frac{1}{|G|}\sum_{g\in G}\chi(M^g)\\ &= \chi(M/G)\end{aligned}$$

The second equation holds by the Lefschetz fixed point formula i.e. Theorem 2.1.8

2. Any group homomorphism $X : \mathbb{Z}^r \to G$ associates to a unique pair of group homomorphisms $\mathbb{Z} \to G$ and $\mathbb{Z}^{r-1} \to G$, and we can see that the fixed points M^X could be viewed as the fixed points of M^{X_2} under X_1 or fixed points of M^{X_1} under X_2. So the r-th equivariant Euler characteristic is:

$$\begin{aligned}\chi_r(M,G) &= \frac{1}{|G|}\sum_{X\in \mathrm{Hom}(\mathbb{Z}^r,G)} \chi(M^X)\\ &= \frac{1}{|G|}\sum_{X_1\in \mathrm{Hom}(\mathbb{Z},G)}\ \sum_{X_2\in \mathrm{Hom}(\mathbb{Z}^{r-1},C_G(X_1))} \chi((M^{X_2})^{X_1})\\ &= \frac{1}{|G|}\sum_{X_1\in \mathrm{Hom}(\mathbb{Z},G)} |C_G(X_1)|\chi_{r-1}(M^{X_1},C_G(X_1))\\ &= \sum_{X_1\in \mathrm{Hom}(\mathbb{Z},G)} \frac{1}{|G:C_G(X_1)|}\chi_{r-1}(M^{X_1},C_G(X_1))\\ &= \sum_{X_1G\in \mathrm{Hom}(\mathbb{Z},G)/G} \chi_{r-1}(M^{X_1},C_G(X_1)) = \sum_{[g]\in[G]}\chi_{r-1}(M^g,C_G(g))\end{aligned}$$

Dually we have:

$$\begin{aligned}\chi_r(M,G) &= \frac{1}{|G|}\sum_{X\in \mathrm{Hom}(\mathbb{Z}^r,G)} \chi(M^X)\\ &= \frac{1}{|G|}\sum_{X_2\in \mathrm{Hom}(\mathbb{Z}^{r-1},G)}\ \sum_{X_1\in \mathrm{Hom}(\mathbb{Z},C_G(X_2))} \chi((M^{X_1})^{X_2})\\ &= \sum_{X_2G\in \mathrm{Hom}(\mathbb{Z}^{r-1},G)/G} \chi_1(M^{X_2},C_G(X_2)) = \sum_{X_2G\in \mathrm{Hom}(\mathbb{Z}^{r-1},G)/G} \chi(M^{X_2}/C_G(X_2))\end{aligned}$$

□

Remark. • The same recursion formula still holds if we replace ordinary Euler characteristics by the reduced one.

- $\chi_r(M,G)$ depends only on the equivariant homotopy type of G-space M.
- $\chi_r(*,G) = |\mathrm{Hom}(\mathbb{Z}^r,G)|/|G|$ where $*$ is just a single point with trivial G-action.
- $\widetilde{\chi}_r(M,G) = \chi_r(M,G) - \chi_r(*,G)$

We need a technical group-theoretic lemma[HKR00, Lemma 4.13] which will be used in the proof of the next proposition of the equivariant Euler characteristics and the section about the cohomology interpretation of equivariant Euler characteristics.

Lemma 2.1.26. *Let G, H be two groups with the order of G finite, then*

$$|\mathrm{Hom}(\mathbb{Z}\times H,G))|/|G| = |\mathrm{Hom}(H,G)/G|$$

Proof. There is a natural map π from $\mathrm{Hom}(\mathbb{Z}\times H,G)$ to $\mathrm{Hom}(H,G)$ by restriction. For any element $f\in\mathrm{Hom}(H,G)$ we claim that its fiber could be identified with the centralizer $C_G(f)$ which means the group elements in G fixing every element in image of f. In this case we have

$$\begin{aligned}|\mathrm{Hom}(\mathbb{Z}\times H,G)| &= \sum_{f\in\mathrm{Hom}(H,G)}|C_G(f)|\\ &= |G|\sum_{f\in\mathrm{Hom}(H,G)}\frac{|C_G(f)|}{|G|}\\ &= |G||\mathrm{Hom}(\mathbb{Z}\times H,G)/G|\end{aligned}$$

The third equality holds since for any discrete G-set X we have a canonical decomposition of X as a disjoint union of orbits, so when G-set X is $\mathrm{Hom}(H,G)$ we have

$$\mathrm{Hom}(H,G)\simeq\bigsqcup_{f\in\mathrm{Hom}(H,G)/G}G/C_G(f)$$

Hence

$$\begin{aligned}\sum_{f\in\mathrm{Hom}(H,G)}\frac{|C_G(f)|}{|G|} &= \sum_{f\in\mathrm{Hom}(H,G)/G}\frac{|G|}{|C_G(f)|}\frac{|C_G(f)|}{|G|}\\ &= |\mathrm{Hom}(\mathbb{Z}\times H,G)/G|\end{aligned}\tag{2.1.27}$$

Finally we need to prove the claim. We construct a map $\varphi:\pi^{-1}(f)\to C_G(f)$ be sending any element g to $g(1,e)$. Since $g\in\pi^{-1}(f)$, for any $h\in H$ we have $f(h)=g(0,h)$. Moreover by properties of group homomorphisms we know $g(1,e)g(0,h)=g(1,h)=g(0,h)g(1,e)$, hence the map φ is well-defined. As for the injectivity, if $g_1\neq g_2$ then of course $\varphi(g_1)=g_1(1,e)\neq g_2(1,e)=\varphi(g_2)$. And as for the surjectivity, if $a\in C_g(f)$, we simply assign $g(1,e)=a$, and since $g(0,-)=f(-)$, these two conditions will determine a group homomorphism g. □

Corollary 2.1.28. *When $H=\mathbb{Z}^{r-1}$ we have $|\mathrm{Hom}(\mathbb{Z}^r,G)|/|G| = |\mathrm{Hom}(\mathbb{Z}^{r-1},G)/G|$. In particular if $H=\mathbb{Z}$ we have $|\mathrm{Hom}(\mathbb{Z}^2,G)|/|G| = |\mathrm{Hom}(\mathbb{Z},G)/G| = k(G)$. Where $k(G)$ is the number of conjugacy classes of the finite group G.*

Proposition 2.1.29. *Let M be a finite G-space with G a finite group then for any integer $r\geq 2$*

$$\chi_r(M;G) = \frac{1}{|G|}\sum_{\sigma\in M}(-1)^{d(\sigma)}|\mathrm{Hom}(\mathbb{Z}^r,C_G(\sigma))| = \sum_{\sigma G\in M/G}(-1)^{d(\sigma)}|\mathrm{Hom}(\mathbb{Z}^{r-1},C_G(\sigma))/C_G(\sigma)|$$

Proof. We observe that for any homomorphism $X \in \mathrm{Hom}(\mathbb{Z}^r, G)$, a simplex $\sigma \in M^X$ if and only if the homomorphism $X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(\sigma))$. Then we have

$$\begin{aligned}
\chi_r(M,G) &= \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \chi(M^X) \\
&= \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \sum_{\sigma \in M^X} (-1)^{d(\sigma)} \\
&= \frac{1}{|G|} \sum_{\sigma \in M} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(\sigma))} (-1)^{d(\sigma)} = \frac{1}{|G|} \sum_{\sigma \in M} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^r, C_G(\sigma))|
\end{aligned}$$

As for the second equality of this statement

$$\begin{aligned}
\frac{1}{|G|} \sum_{\sigma \in M} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^r, C_G(\sigma))| &= \frac{1}{|G|} \sum_{\sigma \in M/G} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^r, C_G(\sigma))||G : C_G(\sigma)| \\
&= \sum_{\sigma \in M/G} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^r, C_G(\sigma))| / |C_G(\sigma)| \\
&= \sum_{\sigma G \in M/G} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma)/C_G(\sigma)|
\end{aligned}$$

where the last equality holds because of Lemma 2.1.26 □

Remark. When $r = 2$ we have:

$$\chi_2(M; G) = \sum_{\sigma G \in M/G} (-1)^{d(\sigma)} k(C_G(\sigma))$$

2.1.5 Generalized Equivariant Euler Characteristics

In the last subsection we gave the definition of integral equivariant Euler characteristics of a finite G-space set and investigated several basic properties. In this subsection we generalize this notion by replacing the group $\mathbb{Z}^{r-1}$ associated to the r-th level integral equivariant Euler characteristic by a more general group K. This generalization is due to Tamanoi[Tam01].

Definition 2.1.30. Given a finite group G and any group K, the generalized equivariant Euler characteristics of a finite G-space M is

$$\chi_K(M,G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z} \times K, G)} \chi(M^X)$$

The reduced version is defined similarly by replacing the ordinary Euler characteristics by reduced Euler characteristics. In particular the r-th integral equivariant Euler characteristics is taking K to be $\mathbb{Z}^{r-1}$.

Remark. There are two important families of generalized Euler characteristics other than the integral ones. The first one is called the r-th p**-primary** equivariant Euler Euler characteristic by letting $K := \mathbb{Z}_p^{r-1}$, in the next section we will interpret this specific case by the Morava K-Thories. Another is called **genus-**g equivariant Euler Euler characteristic by taking $K := \Gamma_g$ where Γ_g is the fundamental group of genus-g orientable surface Σ_g.

Proposition 2.1.31. *[Tam01, Proposition 2-1]*

1. *Like ordinary Euler characteristics, the generalized Euler*characteristics *are multiplicative in the following sense:*

$$\chi_K(M_1 \times M_2; G_1 \times G_2) = \chi_K(M_1; G_1) \cdot \chi_K(M_2, G_2)$$

2. *For any two group K, L we have*

$$\chi_{K\times L}(M, G) = \sum_{\phi G \in \mathrm{Hom}(K,G)/G} \chi_L(M^{\phi}, C_G(\phi))$$

Proof. 1. This statement holds simply because

$$\begin{aligned}\chi_K(M_1 \times M_2, G_1, G_2) &= \frac{1}{|G_1||G_2|} \sum_{X_1 \in \mathrm{Hom}(\mathbb{Z}\times K, G_1)} \sum_{X_2 \in \mathrm{Hom}(\mathbb{Z}\times K, G_2)} \chi((M_1 \times M_2)^{(X_1, X_2)}) \\ &= \frac{1}{|G_1|} \sum_{X_1 \in \mathrm{Hom}(\mathbb{Z}\times K, G_1)} \chi(M_1^{X_1}) \frac{1}{|G_2|} \sum_{X_2 \in \mathrm{Hom}(\mathbb{Z}\times K, G_2)} \chi(M_2^{X_2}) \\ &= \chi_K(M_1, G_1) \cdot \chi_K(M_2, G_2)\end{aligned}$$

2. We just need to modify the argument in the proof in Lemma 2.1.25 by replacing $\mathbb{Z}$ by K and $\mathbb{Z}^{r-1}$ by L.

□

2.2 Interpretations of Equivariant Euler Characteristics

2.2.1 Combinatorial Interpretation

According to Lemma 2.1.25 We know that for any finite G-poset P, $\chi_1(P; G)$ is equal to $\chi(\Delta(P)/G)$, in other words the first equivariant Euler characteristic of a poset could be expressed as an ordinary Euler characteristic of a space. In this subsection we generalize this observation that is, for any integer $r \geq 1$ and a G-poset P we will construct a Δ-set $\Delta_r(P, G)$ such that

$$\chi_{r-1}(P; G) = \chi(\Delta_r(P, G))/|G| \qquad \chi_r(P; G) = \chi(\Delta_r(P, G)/G)$$

This explanation was pointed out to me by my advisor Jesper Møller.

We can view the Δ-set $\Delta(P)$ as a poset with same objects and one object is less than another object if this object could be obtained by applying several face maps on another objects. Then we consider a pre-sheaf of sets on $\Delta(P)$ i.e a functor $C : \Delta(P)^{op} \to \mathbf{SET}$. For every face map $d_i : \Delta(P)_n \to \Delta(P)_{n-1}$ we have a unique morphism from $d_i(\sigma)$ to σ for any $\sigma \in \Delta(P)_n$, we still denote d_i the corresponding morphism from $C(\sigma)$ to $C(d_i(\sigma))$. We associate this functor a new Δ-set $\Delta(P, C)$ where

$$\Delta(P, C)_n := \{(\sigma, X) | \sigma \in \Delta(P)_n, X \in C(\sigma)\}$$

for any integer $n \geq 0$. As for the face maps $d_i : \Delta(P, C)_n \to \Delta(P, C)_{n-1}$ we assign every element (σ, X) via d_i to $(d_i(\sigma), d_i(X))$. By functoriality these maps satisfy the simplicial identity, therefore $\Delta(P, C)$ is a Δ-set. Moreover its ordinary Euler characteristic is

$$\chi(\Delta(P, C)) = \sum_{\sigma \in \Delta(P)} (-1)^{d(\sigma)} |C(\sigma)|$$

If P is a G-poset, we take the functor C to be $\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(-)) : \Delta(P)^{op} \to \mathbf{SET}$ by sending a simplex σ to a set $\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))$. And if $\sigma \subset \tau$ then $C_G(\tau) \subset C_G(\sigma)$ so we have $\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\tau)) \subset \mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))$

Definition 2.2.1. For any integer $r \geq 1$, $\Delta_r(P, C) := \Delta(P, \mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(-)))$, which is a G Δ-set by G-action by $g(\sigma, X) = (g\sigma, gXg^{-1})$.

Lemma 2.2.2. *The fiber over σG of the canonical map*

$$\pi : \Delta_r(P, G)/G \longrightarrow \Delta(P)/G$$

is $\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))/C_G(\sigma)$.

Proof. We construct a map $\varphi : \mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))/C_G(\sigma) \longrightarrow \pi^{-1}(\sigma G)$ by sending a coset $XC_G(\sigma)$ to a coset $(\sigma, X)G$. We claim it is a bijective map of sets. To show surjectivity, choose any representative (τ, Y) of a coset in the fiber $\pi^{-1}(\sigma G)$, i.e. $\pi(\tau, Y) = \sigma G$. By definition of π we know there is a group element g such that $g\sigma = \tau$. Hence $Y \in \mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(g\sigma)) = \mathrm{Hom}(\mathbb{Z}^{r-1}, gC_G(\sigma)g^{-1})$. It is clear that there is a unique map $X \in \mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))$ such that $gXg^{-1} = Y$. So in this case the coset $XC_G(\sigma)$ will be sent to the coset $(\tau, Y)G$. As for injectivity, assume we have two cosets $X_1C_G(\sigma)$ and $X_2C_G(\sigma)$ which have the same image $(\sigma, X)G$, by the construction of φ we see there is a group element g such that $g(\sigma, X_1) = (\sigma, X_2)$, then we have $g \in C_G(\sigma)$ and $gX_1g^{-1} = X_2$ which automatically mean that the two cosets $X_1C_G(\sigma)$ and $X_2C_G(\sigma)$ are the same. □

Corollary 2.2.3. *For any integer $r \geq 1$ and a finite G-poset P we have $\chi_{r-1}(P; G) = \chi(\Delta_r(P, G))/|G|$ and $\chi_r(P; G) = \chi(\Delta_r(P, G)/G)$.*

Proof. By Proposition 2.1.29 we know

$$\begin{aligned} \chi(\Delta_r(P, G))/|G| &= \frac{1}{|G|} \sum_{\sigma \in \Delta(P)} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))| \\ &= X_{r-1}(P; G) \end{aligned}$$

$$\begin{aligned} \chi(\Delta_r(P, G)/G) &= \sum_{\sigma G \in \Delta(P)/G} (-1)^{d(\sigma)} |\mathrm{Hom}(\mathbb{Z}^{r-1}, C_G(\sigma))/C_G(\sigma)| \\ &= \chi_r(P; G) \end{aligned}$$

□

2.2.2 Geometric Interpretation

In the last subsection we expressed the r-th equivariant Euler characteristic of a G-poset in terms of the usual Euler characteristic of a new poset $\Delta_r(P, G)$. In this subsection, we replace the poset P by a G-manifold M, then there is a geometric flavor interpretation of equivariant Euler characteristic due to Tamanoi[Tam01][Tam03] in terms of so called twisted mapping spaces. Here we briefly introduce Tamanoi's work on this interpretation. Let M be a G manifold, it is well-known that if the action is properly discontinuous and free then the quotient space M/G admits a smooth structure i.e. M/G is again a manifold. However, this is not true in the general case, and the study of singularities of M/G is a important task in string topology. In this subsection we assume the action is just properly discontinuous, in other words the quotient space M/G is an orbifold[Ras, Proposition 1]. Let's consider the free loop space $L(M/G) = \mathrm{Map}(S^1, M/G)$ of M/G. For each loop $\overline{\gamma} : S^1 \to M/G$ there is at least a g-periodic listing of this loop. More precisely, there is a loop

$\gamma : \mathbb{R} \to M$ which will pass to $\overline{\gamma}$ if we view $S^1 = \mathbb{R}/\mathbb{Z}$ such that $\gamma(t+1) = g^{-1}\gamma(t)$ for a fixed group element g and every $t \in \mathbb{R}$. So let's consider a space that consists of all the g-periodic loops which is called g-twisted free loop space:

$$L_g M := \{\gamma : \mathbb{R} \to M | \gamma(t+1) = g^{-1}\gamma(t), t \in \mathbb{R}\} \tag{2.2.4}$$

Then there is a canonical surjective map

$$p : \coprod_{g \in G} L_g(M) \twoheadrightarrow L(M/G)$$

Moreover, the space $\coprod_{g\in G} L_g(M)$ carries a natural left G-action on itself defined by $(g \cdot \gamma)(t) \colon = g\gamma(t)$. And it is clear that the map p will factor through the map $\bar{p} : \left(\coprod_{g\in G} L_g M\right)/G \twoheadrightarrow L(M/G)$. Since for each group element h the action induces a homeomorphism $L_g M \xrightarrow{\simeq} L_{hgh^{-1}} M$, we have an identification of this quotient $\left(\coprod_{g\in G} L_g M\right)/G \simeq \coprod_{[g]\in[G]} (L_g M/C_G(g))$ where $[G]$ denotes the set of all conjugacy classes of G.

Proposition 2.2.5.

$$\chi\big(\coprod_{[g]\in[G]} (L_g M/C_G(g))\big) = \chi_2(M, G)$$

Proof. The space $\coprod\limits_{[g]\in[G]} (L_g M/C_G(g))$ might be an infinite dimensional space. So we cannot calculate the usual Euler characteristic of it by definition. However, we observe that it carries a natural circle $\mathbb{T}$-action. We know there is a one-one correspondence between the set of group elements of G and the homomorphism $\phi : \mathbb{Z} \to G$. So we can rewrite the space as $\coprod_{[\phi]\in \mathrm{Hom}(\mathbb{Z},G)/G} (L_\phi M/C_G(\phi))$. Where the space $L_\phi M$ is defined as follows:

$$L_\phi M := \{\gamma \colon \mathbb{R} \to M | \gamma(t+m) = \phi(m)^{-1}\gamma(t), \quad m \in \mathbb{Z}, t \in \mathbb{R}\}$$

This space is also called ϕ-twisted free loop space. In order to define the $\mathbb{T}$-action we first need to identify the circle $\mathbb{T}$ as $\mathbb{R}/\bigcap_{\phi\in\mathrm{Hom}(\mathbb{Z},G)} \ker\phi$. Then for any $z \in \mathbb{T}$ and an element γ we define

$$(z \cdot \gamma)(t) = \gamma(t + z) \tag{2.2.6}$$

This is well-defined since if $z \in \bigcap_{\phi\in\mathrm{Hom}(\mathbb{Z},G)} \ker\phi$ then $\gamma(t+z) = \phi(z)^{-1}\gamma(t) = \gamma(t)$ in other words in this case $z \cdot \gamma = \gamma$. It is well known that the Euler characteristic of a topological space with a $\mathbb{T}$-action is equal to the Euler characteristic of its fixed points on $\mathbb{T}$ if it exists. Hence we have:

$$\begin{aligned}\chi\Big(\coprod_{[\phi]\in\mathrm{Hom}(\mathbb{Z},G)/G} (L_\phi M/C_G(\phi))\Big) &= \chi\Big(\big(\coprod_{[\phi]\in\mathrm{Hom}(\mathbb{Z},G)/G} (L_\phi M/C_G(\phi))\big)^{\mathbb{T}}\Big) \\ &= \sum_{[\phi]\in\mathrm{Hom}(\mathbb{Z},G)/G} \chi(M^\phi/C_G(\phi)) = \chi_2(M, G)\end{aligned}$$

The second equality holds since $(L_\phi M/C_G(\phi))^{\mathbb{T}} = M^\phi/C_G(\phi)$ and the last equality holds because of the Lemma 2.1.25. □

If the action of $C_G(g)$ on $L_g M$ is free for each $g \in G$ then $\bar{p}$ is a resolution of singularity of the free loop space $L(M/G)$. So we can view the map $\bar{p}$ as a mild resolution of $L(M/G)$. And we can iterate this procedure to get a resolution for $L_g M/C_G(g)$. Inspired by the argument in Proposition 2.2.5, in general, we can consider the following ϕ-twisted free loop space where $\phi\colon \mathbb{Z}^r \to G$ is a group homomorphism.

$$L_\phi M\colon = \{\gamma\colon \mathbb{R}^r \to M | \gamma(t+m) = \phi^{-1}(m)\gamma(t), \quad t \in \mathbb{R}^r, m \in \mathbb{Z}^r\}$$

Like in the 1-dimensional case we have the following canonical map:

$$p\colon \coprod_{\phi\colon \operatorname{Hom}(\mathbb{Z}^r,G)} L_\phi M \to L^r(M/G)$$

where $L^r(M/G)$ is the iterated free loop space of M/G. There is also a natural G-action on $\coprod_{\phi\colon \operatorname{Hom}(\mathbb{Z}^r,G)} L_\phi M$ and the map p factor through the map

$$\bar{p}\colon \big(\coprod_{\phi\in\operatorname{Hom}(\mathbb{Z}^r,G)} L_\phi M\big)/G \to L^r(M/G)$$

Unlike the 1-dimensional case this map $\bar{p}$ is neither injective nor surjective in general. But we can still view it as a mild resolution of the iterated free loop space $L^r(M/G)$. We denote $\mathbb{L}^r(M,G) = \big(\coprod_{\phi\in\operatorname{Hom}(\mathbb{Z}^r,G)} L_\phi M\big) = \coprod_{[\phi]\in\operatorname{Hom}(\mathbb{Z}^r,G)/G}(L_\phi M/C_G(\phi))$.

Proposition 2.2.7.

$$\chi(\mathbb{L}^r(M,G)) = \chi_{r+1}(M,G)$$

Proof. The argument here is completely parallel to the argument in the proof of Proposition 2.2.5. We identify $\mathbb{R}^r/\cap_{\phi\in\operatorname{Hom}(\mathbb{Z}^r,G)}\ker\phi = \mathbb{T}^r$: the r-dimensional torus. Then $\mathbb{T}^r$ acts naturally on the space $\coprod_{[\phi]\in\operatorname{Hom}(\mathbb{Z}^r,G)/G}(L_\phi M/C_G(\phi))$. Hence:

$$\begin{aligned}\chi(\coprod_{[\phi]\in\operatorname{Hom}(\mathbb{Z}^r,G)/G}(L_\phi M/C_G(\phi))) &= \chi((\coprod_{[\phi]\in\operatorname{Hom}(\mathbb{Z}^r,G)/G}(L_\phi M/C_G(\phi)))^{\mathbb{T}^r}) \\ &= \sum_{[\phi]\in\operatorname{Hom}(\mathbb{Z}^r,G)/G}\chi(M^\phi/C_G(\phi)) = \chi_r(M,G)\end{aligned}$$

□

Remark. Here we only give the explanation for the integral equivariant Euler characteristic of G-manifolds. A mild modification could be used to give a similar geometric explanation for generalized equivariant Euler characteristic of G-manifold. Readers could refer to [Tam01, Section 2][Tam03, Section 2] for details.

2.2.3 Cohomology Interpretation

In this subsection we use generalized cohomology theories to give an explanation of equivariant Euler characteristics in some special cases. More concretely, we know the ordinary Euler characteristic of a finite space is defined in terms of the alternating sums of rank of ordinary singular cohomology groups of this space in different dimensions. Hence, given a generalized cohomology theory we can replace the singular cohomology in the previous definition by this generalized cohomology theory. Then we say this is an Euler characteristic of this cohomology theory type. Our goal is to try and give an expression of equivariant Euler characteristics of a finite G-space M in terms of the usual Euler characteristics of a new cohomology theory type of its homotopy orbits M_{hG} or as the usual characteristics of a new equivariant cohomology theory type of this space M. In this subsection we always assume that for any cohomology theory E^* and any poset M, $E^*(M)$ actually means $E^*(|M|)$.

M.Atiyah and G.Segal[AS89, Theorem 1] give an interpretation of second order integral equivariant Euler characteristics by equivariant K-theory.

Theorem 2.2.8. *Given a finite G space M, its second order integral equivariant Euler characteristic could be expressed using equivariant K theory:*

$$\chi_2(M,G) = dim K^0_G(M)\otimes\mathbb{C} - dim K^1_G(M)\otimes\mathbb{C}$$

To prove this theorem we first need a technical lemma[AS89, Theorem 2][Kuh89, Theorem 6.4] in equivariant K theory:

Lemma 2.2.9. *Let M be a finite G-space, there is a natural isomorphism:*

$$\theta : K^*_G(M)\otimes\mathbb{C} \longrightarrow \bigoplus_{[g]\in[G]} \left(K^*(M^g)\otimes\mathbb{C}\right)^{C_G(g)}$$

Proof of Theorem 2.2.8. According to the the Chern character isomorphism:

$$\bigoplus_{[g]\in[G]} \left(K^*(M^g)\otimes\mathbb{C}\right)^{C_G(g)} \simeq \bigoplus_{[g]\in[G]} H^*(M^g;\mathbb{C})^{C_G(g)} \simeq \bigoplus_{[g]\in[G]} H^*(M^g/C_G(g);\mathbb{C})$$

The last equality holds because of [Bre72, Theorem III.2.4] So the Euler characteristic with respect to the equivariant K-theory is

$$\begin{aligned}
\dim K^0_G(M)\otimes\mathbb{C} - \dim K^1_G(M)\otimes\mathbb{C} &= \sum_{[g]\in[G]} \chi(M^g/C_G(g)) = \sum_{[g]\in[G]} \frac{1}{|C_G(g)|} \sum_{h\in C_G(g)} \chi(M^{(h,g)}) \\
&= \sum_{g\in G} \frac{1}{|G:C_G(g)|}\frac{1}{|C_G(g)|} \sum_{h\in C_G(g)} \chi(M^{h,g)}) \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{h\in C_G(g)} \chi(M^{(h,g)}) \\
&= \frac{1}{|G|}\sum_{X\in \mathrm{Hom}(\mathbb{Z}^2,G)} \chi(M^X) = \chi_2(M,G)
\end{aligned}$$

□

Remark. It is natural to ask if there exists a cohomology theory to explain higher order integral equivariant Euler characteristic. From the interpretation of the first and second order integral equivariant cohomology theory i.e using singular cohomology to explain the 1-st integral equivariant Euler characteristic and using K-theory to explain the 2-nd integral equivariant Euler characteristic , it seems like these invariants relate to the Chromatic homotopy theory. More concretely, its seems like an $(r+1)$-th integral equivariant Euler characteristic relates with the r-th level (integral) Morava K Theory. According to this, can we explain the 3-rd integral equivariant Euler characteristic in terms of the chromatic level 2 Morava K theory i.e the elliptic cohomology theory? This question has been answered positively by Deveto[Dev96, Theorem 1.12]. In his paper he showed a key result[Dev96, Theorem 6.3] which is very similar to Lemma 2.2.9 in terms of the decomposition of the equivariant K-theory.

$$\mathrm{Ell}_G(M)\otimes_{\mathrm{Ell}_*}\mathbb{F}^*_G \xrightarrow{\simeq} \bigoplus_{X\in\mathrm{Hom}(\mathbb{Z}^2,G)} [\mathrm{Ell}^*(M^X)\otimes_{\mathrm{Ell}^*}\mathbb{F}^*_G]^{C_G(X)} \tag{2.2.10}$$

Where Ell_G is a model for equivariant elliptic cohomology, Ell^* is the coefficient ring for elliptic cohomology and $\mathbb{F}_G^*$ is a special graded field[Dev96, Proposition 6.1]. Then a similar argument like in the proof of Theorem 2.2.8 shows the 3-rd equivariant Euler characteristic could be expressed in terms of the equivariant elliptic cohomology.

As for higher orders, it is still an open problem. For example, can we express the 4-th equivariant Euler characteristic in terms of equivariant K-3 cohomology[Szy10]?

Unlike the cohomology explanation of integral equivariant Euler characteristics, the p-primary equivariant Euler characteristics has a complete cohomology interpretation by HKR[HKR00, Proposition 4.11] and Tamanoi[Tam01, Theorem B] using the Morava K-Theory at the prime p. Recall that given a prime p we have for any non-negative integer n an generalized cohomology theory $K(n)$ with coefficient ring $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = -2(p^n - 1)$ a graded filed. Let G be a finite group the equivariant Morava K-theory of G-topological space X is defined via Borel construction, in other words:$K(n)_G^*(X) := K(n)^*(X_{hG})$, where $X_{hG} := X \times_G EG$. Since the coefficient ring $K(n)^*$ itself is a graded vector filed we can view $K_G^*(X)$ as a graded-vector space over $K(n)^*$ and count its dimension if it is finite. Therefore we can talk about the equivariant Morava K-theory Euler characteristic $\chi_{K_G(n)}(X)$ of a topological space with a finite group action.

Theorem 2.2.11. *The $(r+1)$-th p-primary equivariant Euler characteristics of a finite G-space M is equal to equivariant Morava K-theory of M at height r, in other words:*

$$\chi_{r+1}^p(M; G) = \chi_{K_G(r)}(M)$$

In order to prove the Theorem 2.2.11 let's first state a useful perspective to calculate the equivariant Euler characteristic.

Proposition 2.2.12. *[Mø17a, Section 5] Given any abelian group K, the K-generalized equivariant Euler characteristic could be expressed as:*

$$\chi_K(M, G) = \frac{1}{|G|} \sum_{A \in \mathcal{S}^{\mathrm{abe}}} \chi(M^A) \varphi_{\mathbb{Z} \times K}(A) \tag{2.2.13}$$

where $\mathcal{S}_G^{\mathrm{abe}}$ is the set of all abelian subgroups of G and $\varphi_{\mathbb{Z}\times K}(A)$ means the number of epimorphism from the group $\mathbb{Z} \times K$ to A.

Proof. By definition:

$$\chi_K(M, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z} \times K, G)} \chi(M^X)$$

Since the group K is an abelian group then for each homomorphism X its image is of course an abelian subgroup of G, so we can sum over all group homomorphisms X by summing over all abelian subgroups A of G multiplying the numbers of epimorphisms from $\mathbb{Z} \times K$ to A with the Euler characteristic of the fixed points M^A. □

We construct a new function $\mu_M(A)$ on all abelian subgroups A of G associated with a finite G space M by induction as follows:

$$\mu_M(A) := \sum_{B \in \mathcal{S}_G^{\mathrm{abe}}} \mu_G(A, B) \chi(M^B) \tag{2.2.14}$$

According to Proposition 2.1.13 it is equivalent to

$$\chi(M^A) = \sum_{B \in \mathcal{S}_G^{\text{abe}}} \xi_G(A, B)\mu_M(B) \tag{2.2.15}$$

And we can easily see that for any abelian group K we have:

$$|\text{Hom}(\mathbb{Z} \times K, B)| = \sum_{A \in \mathcal{S}_G^{\text{abe}}} \varphi_{\mathbb{Z}\times K}(A)\xi_G(A, B) \tag{2.2.16}$$

$$\varphi_{\mathbb{Z}\times K}(B) = \sum_{A \in \mathcal{S}_G^{\text{abe}}} |\text{Hom}(\mathbb{Z} \times K, A)|\mu_G(A, B) \tag{2.2.17}$$

Where the second equation is the Möbius inverse of the first one.

Proposition 2.2.18. *Given any abelian group K, the generalized equivariant Euler characteristic could be expressed using Möbius functions:*

$$\chi_K(M; G) = \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{\text{abe}}} |\text{Hom}(\mathbb{Z} \times K, A)|\mu_M(A)$$

In particular we have

1. $\chi_r(M; G) = \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{\text{abe}}} |A|^r \mu_M(A)$

2. $\chi_r^p(M; G) = \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{\text{abe}}} |A||A|_p^{r-1} \mu_M(A)$

Proof. According to Proposition 2.2.12, we know:

$$\chi_K(M, G) = \frac{1}{|G|} \sum_{B \in \mathcal{S}_G^{\text{abe}}} \chi(M^B)\varphi_{\mathbb{Z}\times K}(B) \tag{2.2.19}$$

By plugging in the equation 2.2.17 to the above equation we have

$$\begin{aligned}
\chi_K(M, G) &= \frac{1}{|G|} \sum_{B \in \mathcal{S}_G^{\text{abe}}} \chi(M^B) \sum_{A \in \mathcal{S}_G^{\text{abe}}} |\text{Hom}(\mathbb{Z} \times K, A)|\mu_G(A, B) \\
&= \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{\text{abe}}} |\text{Hom}(\mathbb{Z} \times K, A)| \sum_{B \le G} \chi(M^B)\mu_G(A, B) \\
&= \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{\text{abe}}} |\text{Hom}(\mathbb{Z} \times K, A)|\mu_M(A)
\end{aligned}$$

The last equality holds because of the equation 2.2.14. Whenever we take $K = \mathbb{Z}^{r-1}$ we have $|\text{Hom}(\mathbb{Z}^r, A)| = |A|^r$ and whenever we take $K = \mathbb{Z}_p^r$ then $|\text{Hom}(\mathbb{Z} \times \mathbb{Z}_p^r, A)| = |A||A|_p^r$. □

In [HKR00, Proposition 4.11], Hopkins-Kuhn-Ravenel calculated the Morava K-theory Euler characteristic in terms of the function $\mu_M(A)$:

Lemma 2.2.20. *Let $\chi^G_{r,p}$ denote the Euler characteristics in terms of the Morava K-theory at prime p and level r then:*

$$\chi^G_{r,p}(M) = \frac{1}{|G|}\sum_{A\in\mathcal{S}^{\mathrm{abe}}_G}|A|\chi^G_{r,p}(G/A)\mu_M(A)$$

Moreover $\chi^G_{r,p}(G/A) = |A|^r_p$

Hence Theorem 2.2.11 is a direct corollary of this lemma.

Remark. In general a p-primary equivariant Euler characteristic is closely related to Morava K-Theory in a very precise sense. As for a future potential project we can ask if we can find other natural cohomology theories to explain some other generalized equivariant Euler characteristics.

2.3 Relations with Representation Theory

p-subgroup complexes $\mathcal{S}^{p+*}_G$ were first introduced by Brown[Bro75] and systematically studied by Quillen[Qui78]. Given a finite group G we can associate a poset $\mathcal{S}^{p+*}_G$ of its non-trivial p-subgroups with the partial order defined by inclusion of subgroups for every prime p that divides the order of group G. Then the p-subgroup complex is the associated Δ-set of this poset. We adopt the notation $\mathcal{S}^{p+*}_G$ for both the poset and its associated complex if there is no confusion. Moreover, this complex carries a natural G-action given by conjugation action of G on its subgroups. Intuitively a p-subgroup complex is a geometry in the sense that the stabilizer of a simplex is an analogue of parabolic subgroups of G. If G is a finite group of Lie type, Quillen[Qui78, Theorem 3.1]showed that these p-subgroup complexes coincide with their Tits buildings of a given prime p. It turns out p-subgroup complexes could reflect many properties of modular representations of G. In this section we shall apply the notion of equivariant Euler characteristics of $\mathcal{S}^{p+*}_G$ to rephrase some conjectures in representation theory. The main content of this section is based on the work of J.Thévenaz in [Thé93] and of Jesper Møller in[Mø15, Section 5].

In the beginning, let's first mention some basic properties of p-subgroup complexes. In Brown's work[Bro75, Corollary 2], he studied the reduced Euler characteristics of of this complex:

Theorem 2.3.1. $|G|_p|\ \widetilde{\chi}(\mathcal{S}^{p+*}_G)$ *where $|G|_p$ means the p-part of the order of G.*

Quillen[Qui78, Proposition 2.4] showed that if G has a non-trivial normal p-subgroup then the p-subgroup complex $\mathcal{S}^{p+*}_G$ is G-equivariant contractible and it implies that for any integer r, $\widetilde{\chi}_r(\mathcal{S}^{p+*}_G, G) = 0$. Quillen conjectured that if $\mathcal{S}^{p+*}_G$ is G-equivariant contractible then G has a nontrivial p-subgroup. We can apply the notion of equivariant Euler characteristics to give a weaker conjecture of Quillen's:

Conjecture 2.3.2. If for any positive integer r we have $\widetilde{\chi}_r(\mathcal{S}^{p+*}_G; G) = 0$, then $O_p(G)$ is non trivial, where $O_p(G)$ is the largest normal p-subgroups of G[Gor68, Section 6.3] , in other words, G has a nontrivial normal p-subgroup.

When $r = 1$, Webb[Web87, Theorem 4.1] proved that $\widetilde{\chi}_1(\mathcal{S}^{p+*}_G; G)$ is 0 whenever $\mathcal{S}^{p+*}_G$ is non empty. Moreover, Symonds[Sym98] proved a much stronger result that the quotient space $\Delta(\mathcal{S}^{p+*}_G)/G$ is contractible when it is not empty.

We are going to study the second integral Equivariant Euler characteristic of $\mathcal{S}^{p+*}_G$ and apply the equivariant K Theory interpretation to relate it with Alperin's conjecture[Alp87] and Knörr-Robinson's conjecture[KR89][Thé93, Theorem 3.1].

We first review some notations in representation theory of finite groups. Given a finite group G and a prime number p, we denote by $k(G)$ the number of conjugacy classes of G and also the number of irreducible complex representations of G, in other words, the number of irreducible $\mathbb{C}G$-modules. We denote by $z_p(G)$ the number of irreducible complex representations of the dimension divisible by $|G|_p$. An element $g \in G$ is called a p-element if its order is a power of p, it is called a p'-element if its order is prime to p. A p'-conjugacy class C in G is a conjugacy class of G in which every element inside C is a p'-element and we denote $k_{p'}(G)$ the number of p'-conjugacy classes of G[K$\ddot{9}$1, (11.5)]. Let $\overline{\mathbb{F}}_p$ be the algebraic closure of finite field $\mathbb{F}_p$. If we consider $\overline{\mathbb{F}}_pG$-modules instead of $\mathbb{C}G$-modules, then we have another description of numbers $k_{p'}(G)$ and $z_p(G)$. In this situation $k_{p'}(G)$ could mean the number of irreducible $\overline{\mathbb{F}}_pG$-modules[Alp86, Chapter 1,Theorem 2] and z_p could mean the number of irreducible and projective $\overline{\mathbb{F}}_pG$-modules[Alp86, Corollary II.7][, Proposition 18.28]. Hence we have the following inequalities for these three numbers:

$$k(G) \geq k_{p'}(G) \geq z_p(G)$$

Conjecture 2.3.3. Given a finite group G and a prime number p :

$$\sum_{PG \in \mathcal{S}_G^{p+*}/G} z_p(N_G(P)/P) = k_{p'}(G) - z_p(G) \tag{2.3.4}$$

where $\mathcal{S}_G^{p+*}/G$ means the set of conjugacy classes of p-subgroups. It is believed that there is no morphism connecting them to explain the equality of these two numbers. This conjecture is called the Alperin's weight conjecture and denoted by $\mathrm{AWC}_p(G)$.

The Alperin weight conjecture has been verified for lots of cases. For example, when G is a p'-group i.e. $p \nmid |G|$. In this case $k_{p'}(G) = k(G)$ and $z_p(G) = k(G)$, so the right hand side of the equation 2.3.4 is just 0. On the left we have the empty sum which is of course 0. So $\mathrm{AWC}_p(G)$ is true in this case. If G is a trivial group, $z_p(G) = 1$. So in this case the right hand side of the equation 2.3.4 is 0 same as the left hand side, since the summation is over an empty set. When G is a non-trivial finite group, then $z_p(G) = 0$ according to [Alp86, Page 14]. So when G is a non-trivial p-group we have $k_{p'}(G) - z_p(G) = 1$. As for the left hand side of the equation 2.3.4, since G satisfies the normalizer condition[Rob96, 5.1.3,5.2.4], the groups are $N_G(P)/P$ are non-trivial p-groups for all proper subgroup P of G. Thus the terms are all 0 except when P is just the group G itself and in this case the term is 1.

Even the truth of $\mathrm{AWC}_p(G)$ has not been verified, but people has reduced the verification to check several conditions for all finite simple groups[NT11][Sch16][Spä13].

Knörr and Robinson[KR89][Thé93, Page 195] give an equivalent conjecture with Alperin's weight conjecture:

Conjecture 2.3.5. Given a finite group G and a prime number p we have the following equality:

$$\sum_{\sigma G \in \Delta(\mathcal{S}_G^{p+*})/G} (-1)^{d(\sigma)} k(C_G(\sigma)) = k(G) - z_p(G)$$

However the equivalence between these two conjectures is not case by case. That is, if one case is true for one conjecture it doesn't follow immediately that this case is also true for another conjecture and vice versa. The precise statement[Thé93, Theorem 3.1] of the equivalence between Alperin's weight conjecture and Knörr-Robinson's conjecture is as follows:

Theorem 2.3.6. *Let G be a finite group and $\mathcal{S}_G^p$ be the poset of all p-subgroups of G, then the following two statements are equivalent:*

1. *The Alperin's weight conjecture holds for G and $N_G(P)/P$ for every $P \in \mathcal{S}_G^p$.*

2. *The Knörr-Robinson's conjecture holds for G and $N_G(P)/P$ for every $P \in \mathcal{S}_G^p$*

According to Proposition 2.1.29 we can use the second integral equivariant Euler characteristic to reformulate the Knörr-Robinson's conjecture:

Conjecture 2.3.7. The Knörr-Robinson's conjecture is equivalent to

$$\chi_2(\mathcal{S}_G^{p+*}; G) = k(G) - z_p(G)$$

Proof. Take the finite space M of the remark in Proposition 2.1.29 to be the poset $\mathcal{S}_G^{p+*}$. In other words:

$$\chi_2(\mathcal{S}_G^{p+*}; G) = \sum_{\sigma G \in \Delta(\mathcal{S}_G^{p+*})/G} (-1)^{d(\sigma)} k(C_G(\sigma))$$

□

Remark. According to [Qui78, Proposition 2.1][TW91, Theorem 1], the p-subgroup complex $\mathcal{S}_G^{p+*}$ is G-equivariant homotopy equivalent to the complex of elementary abelian p-subgroups $\mathcal{S}_G^{p+eab+*}$ or radical p-subgroups $\mathcal{S}_G^{p+rad+*}$[Bou84], in other words we can replace $\mathcal{S}_G^{p+*}$ in the equivariant Euler characteristic by $\mathcal{S}_G^{p+rad+*}$.

Similarly the Knörr-Robinson's conjecture has been proved to be true for lots of case. For example when G is a p'-group i.e. $p \nmid |G|$, there are no non-trivial p-subgroups of G. So $\chi_2(\mathcal{S}_G^{p+*}, G) = \chi_2(\emptyset, G) = 0$. And in this case $z_p(G) = k(G)$ i.e the right hand side of the equation 2.3.4 is 0. Hence the Knörr-Robinson's conjecture holds when G is a p'-group. Thévenaz gave another example[Thé93, Example 1.4]: When $O_p(G) \neq 1$, $\mathcal{S}_G^{p+*}$ is G-equivariant contractible. So $\chi_2(\mathcal{S}_G^{p+*}, G) = \chi_2(*, G) = k(G)$. According to Ito's theorem[CR06, Corollary 53.18] then $O_p(G) \neq 1$ then $z_p(G) = 0$. We should mention here that this case has not been showed to be true for Alperin weight conjecture. So this is one example that the equivalence between the Alperin weight conjecture and the Knörr-Robinson's conjecture is not case by case.

2.4 Grothendieck Construction and its Equivariant Euler Characteristic

2.4.1 Equivariant Euler Characteristics of Discrete G-posets

In this subsection we determine the equivariant Euler characteristics of discrete G-posets. More precisely, let G be a finite group and K be a subgroup of it. Then $K\backslash G$ is a right transitive G-set and there is no non-trivial order relation between cosets. So it could be considered as a discrete G-poset. Then by definition

$$\chi_r(K\backslash G, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, K)} |C_{K\backslash G}(X)|$$

where $C_{K\backslash G}(X)$ consists of the K-cosets fixed by the image of the homomorphism X.

Theorem 2.4.1. *For any $r \geq 1$, $\chi_r(K\backslash G, G) = |\mathrm{Hom}(\mathbb{Z}^r, K)|/|K| = \chi_r(1, K)$ is the number of conjugacy classes of commuting $(r-1)$-tuples in K.*

Proof. The rth equivariant Euler characteristic is

$$\chi_r(K\backslash G, G) = \frac{1}{|G|}\sum_{A\in\mathcal{S}_G^{\mathrm{abe}}}\varphi_r(A)|C_{K\backslash G}(A)| = \sum_{A^G\in\mathcal{S}_G^{\mathrm{abe}}/G}\varphi_r(A)\frac{|C_{K\backslash G}(A)|}{|N_G(A)|}$$

where $\mathcal{S}_G^{\mathrm{abe}}$ is the poset of all abelian subgroups A of G, $\varphi_r(A)$ means the number of epimorphism from $\mathbb{Z}^r$ to A and $\mathcal{S}_G^{\mathrm{abe}}/G$ is the conjugacy classes of abelian subgroups A of G and by definition

$$C_{K\backslash G}(A) = K\backslash\{g\in G \mid A\leq K^g\}$$

is the fixed point set for the action of A on $K\backslash G$. Write $\mathcal{S}_G(A^G, K) = \{A^g \mid g\in G, A^g\leq K\}$ for the set of G-conjugates of A contained in K. The surjection

$$\{g\in G \mid A\leq K^g\}\twoheadrightarrow \mathcal{S}_G(A^G,K)\colon g\to A^{g^{-1}}$$

identifies two elements g_1 and g_2 of the domain if and only if $g_1g_2^{-1}\in N_G(A)$. So we have the identity

$$|K||C_{K\backslash G}(A)| = |N_G(A)||\mathcal{S}_G(A^G,K)|$$

and the disjoint union

$$\bigcup_{A^G\in\mathcal{S}_G^{\mathrm{abe}}/G}\mathcal{S}_G(A^G,K) = \mathcal{S}_K^{\mathrm{abe}}$$

we find that

$$\begin{aligned}|K|\chi_r(K\backslash G,G) &= \sum_{A^G\in\mathcal{S}_G^{\mathrm{abe}}/G}\varphi_r(A)\frac{|K||C_{K\backslash G}(A)|}{|N_G(A)|} = \sum_{A^G\in\mathcal{S}_G^{\mathrm{abe}}/G}\varphi_r(A)|\mathcal{S}_G(A^G,K)|\\ &= \sum_{A\in\mathcal{S}_K^{\mathrm{abe}}}\varphi_r(A) = |\operatorname{Hom}(\mathbb{Z}^r,K)|\end{aligned}$$

□

Corollary 2.4.2. $\widetilde{\chi}_r(K\backslash G,G) = \chi_r(1,K) - \chi_r(1,G)$

2.4.2 Equivariant Weightings and Grothendieck Construction

In Section 1.3 we introduced weightings and co-weightings for posets and how to use them to define the Euler characteristic of a finite poset. In this subsection we generalize the (co)weightings of a finite poset to the notion of equivariant weightings for G-posets and calculate the equivariant Euler characteristics of the Grothendieck constructions.

Definition 2.4.3. The r-th equivariant weighting on the G-poset P is the function $k_r^\bullet\colon P\to\mathbb{Q}$ given by

$$k_r^p = \sum_{X\in\operatorname{Hom}(\mathbb{Z}^r, C_G(p))} k^p(C_P(X)),\qquad p\in P$$

where $k^\bullet(C_P(X))\colon C_P(X)\to\mathbb{Q}$ is the weighting for the subposet $C_P(X)$.

Like the Euler characteristic case, the equivariant weightings could be used to express the equivariant Euler characteristic.

Proposition 2.4.4. $\frac{1}{|G|}\sum_{p\in P} k_r^p = \chi_r(P,G)$

Proof. The computation

$$\sum_{p\in P} k_r^p = \sum_{p\in P}\sum_{X\in \operatorname{Hom}(\mathbb{Z}^r, C_G(p))} k^p(C_P(X))$$
$$= \sum_{X\in \operatorname{Hom}(\mathbb{Z}^r, G)}\sum_{p\in C_P(X)} k^p(C_P(X)) = \sum_{X\in \operatorname{Hom}(\mathbb{Z}^r, G)} \chi(C_P(X)) = |G|\chi_r(G,P)$$

proves the lemma. □

We can apply this proposition to rewrite the proof of Theorem 2.4.1

Proof of Theorem 2.4.1. According to Proposition 2.4.4 we know

$$\begin{aligned}\chi_r(K\backslash G, G) &= \frac{1}{|G|}\sum_{Kg\in K\backslash G}\sum_{X\in \operatorname{Hom}(\mathbb{Z}^r, K^g)} 1\\ &= \frac{1}{|G|}\frac{|G|}{|K|}|\operatorname{Hom}(\mathbb{Z}^r, K)|\\ &= |\operatorname{Hom}(\mathbb{Z}^r, K)|/|K| = \chi_r(1, K)\end{aligned}$$

The first equality holds since the weighting $k^{Kg}(C_{K\backslash G}(X)) = 1$ for all $Kg \in C_{K\backslash G}(X)$.

□

Definition 2.4.5 (Grothendieck Construction)**.** Given a finite category $\mathcal{D}$ and a functor $\mathcal{S} : \mathcal{D} \to$ **POSET**, we associate a new poset $\int_{\mathcal{D}} \mathcal{S}$ called the Grothendieck construction of $\mathcal{S}$ with

1. Objects: (d, x) where $x \in \mathcal{S}(d)$
2. Morphisms: a morphism $\varphi : (d_1, x_1) \to (d_2, x_2)$ consists of a morphism $f : d_1 \to d_2$ in $\mathcal{D}$ and a morphism $g : \mathcal{S}(\varphi)(x_1) \to x_2$.

The following result is about the weighting of Grothendieck construction [Lei08, Lemma 1.14]

Lemma 2.4.6. *Let $\mathcal{D}$ be a finite category with weighting $k^\bullet$. Suppose we have a functor $\mathcal{S} : \mathcal{D} \to$ **POSET** such that its Grothendieck construction $\int_{\mathcal{D}} \mathcal{S}$ is also finite. If the weighting for every image $\mathcal{S}(d)$ exists and it is all written by $k^\bullet$ then its Grothendieck construction carries a weighting defined by $k^{(d,x)} = k^d k^x$ where $d \in \mathcal{D}, x \in \mathcal{S}(d)$.*

Proof. See [Lei08, Lemma 1.14] □

Corollary 2.4.7. *Under the same condition as in Lemma 2.4.6 and if Euler characteristic of the Grothendieck construction exists, then:*

$$\chi(\int_{\mathcal{D}} \mathcal{S}) = \sum_{d\in\mathcal{D}} k^d(\mathcal{D})\chi(\mathcal{S}(d))$$

where $k^\bullet(\mathcal{D})$ is the weighting for the category $\mathcal{D}$.

Proof. According to Lemma 2.4.6 we know

$$\chi(\int_{\mathcal{D}} \mathcal{S}) = \sum_{d\in\mathcal{D}, x\in\mathcal{S}(d)} k^{(d,x)} = \sum_{d\in\mathcal{D}} k^d(\mathcal{D}) \sum_{x\in\mathcal{S}(d)} k^x(\mathcal{S}(d)) = \sum_{d\in\mathcal{D}} k^d(\mathcal{D})\chi(\mathcal{S}(d))$$

□

Definition 2.4.8. [JSo01, Definition 2.2] Let $\mathcal{C}$ be a small category with a G-action and $\mathcal{D}$ an arbitrary category. We say a functor $F : \mathcal{C} \to \mathcal{D}$ a G-functor if there is a natural transformation $\Phi_g : F \to F \circ g$ for each $g \in G$ satisfying $\Phi_{g_1 g_2} = \Phi_{g_2} \circ \Phi_{g_1}$ for $g_1, g_2 \in G$.

Now given a G-functor $\mathcal{S} : \mathcal{D} \to \mathbf{POSET}$ where $\mathcal{D}$ is a finite G-poset and each image of $\mathcal{S}$ is also a finite poset. Its associated Grothendieck construction is a G-poset with the G-action given by $g(d, x) = (gd, \Phi_g(d)(x))$.

So we can talk about the fixed points of the Grothendieck construction. We observe that for any homomorphism $X : \mathbb{Z}^r \to G$ the centralizer:

$$C_{\int_{\mathcal{D}} \mathcal{S}}(X) = \int_{C_{\mathcal{D}(X)}} C_{\mathcal{S}}(X)$$

where $C_{\mathcal{S}}(X) : C_{\mathcal{D}}(X) \to \mathbf{POSET}$ is the fixed functor. Therefore we have

$$\chi(C_{\int_{\mathcal{D}} \mathcal{S}}(X)) = \chi(\int_{C_{\mathcal{D}(X)}} C_{\mathcal{S}}(X)) = \sum_{d \in C_{\mathcal{D}}(X)} k^d(C_{\mathcal{D}}(X))\chi(C_{\mathcal{S}(d)}(X))$$

Where $C_{\mathcal{S}(d)}(X) = \{a \in \mathcal{S}(d) | \Phi_g(d)(a) = a, \forall g \in \mathrm{Im}(X)\}$. In case of a discrete poset $\mathcal{D}$, $k^d(C_{\mathcal{D}}(X)) = 1$ for all $d \in C_{\mathcal{D}}(X)$.

Theorem 2.4.9. *Under the same conditions as above we have:*

$$\chi_r(\int_{\mathcal{D}} \mathcal{S}, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \sum_{d \in C_{\mathcal{D}}(X)} k^d(C_{\mathcal{D}}(X))\chi(C_{\mathcal{S}(d)}(X))$$

Proof. By assumptions we know $\int_{\mathcal{D}} \mathcal{S}$ is a finite G-poset, so the equivariant Euler characteristic on it exists.

$$\begin{aligned}
\chi_r(\int_{\mathcal{D}} \mathcal{S}, G) &= \frac{1}{|G|} \sum_{(d,e)} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(d,e))} k^{(d,e)}(\int_{C_{\mathcal{D}(X)}} C_{\mathcal{S}}(X)) \\
&= \frac{1}{|G|} \sum_{(d,e)} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(d) \cap C_G(e))} k^d(C_{\mathcal{D}}(X)) k^e(C_{\mathcal{S}(d)}(X)) \\
&= \frac{1}{|G|} \sum_{d \in \mathcal{D}} \sum_{e \in \mathcal{S}(d)} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(d) \cap C_G(e))} k^d(C_{\mathcal{D}}(X)) k^e(C_{\mathcal{S}(d)}(X)) \\
&= \frac{1}{|G|} \sum_{d \in \mathcal{D}} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(d))} \sum_{e \in C_{\mathcal{S}(d)}(X)} k^d(C_{\mathcal{D}}(X)) k^e(C_{\mathcal{S}(d)}(X)) \\
&= \frac{1}{|G|} \sum_{d \in \mathcal{D}} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(d))} k^d(C_{\mathcal{D}}(X)) \sum_{e \in C_{\mathcal{S}(d)}(X)} k^e(C_{\mathcal{S}(d)}(X)) \\
&= \frac{1}{|G|} \sum_{d \in \mathcal{D}} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, C_G(d))} k^d(C_{\mathcal{D}}(X)) \chi(C_{\mathcal{S}(d)}(X)) \\
&= \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, G)} \sum_{d \in C_{\mathcal{D}}(X)} k^d(C_{\mathcal{D}}(X)) \chi(C_{\mathcal{S}(d)}(X))
\end{aligned}$$

The first equality holds since Proposition 2.4.4 and the second equality holds since Lemma 2.4.6. □

2.4.3 p-subgroup posets and Grothendieck Constructions

In this subsection we present a connection between p-subgroup posets and a specific Grothendieck construction in terms of their equivariant Euler characteristics. We study the equivariant Euler characteristics of p-subgroup complexes of symmetric groups. Let $H \trianglelefteq K \leq G$ and $\mathcal{S}_H$ the functor on the G-set $K\backslash G$ that takes Kg to $\mathcal{S}_{H^g}^{p+*}$. We denote $\mathcal{S}_G^{\mathrm{abe}}$ as the poset of all abelian subgroups of G.

Lemma 2.4.10. *For $r \geq 1$, $\chi_r(\int_{K\backslash G} \mathcal{S}_H, G) = \chi_r(\mathcal{S}_H^{p+*}, K)$.*

Proof. The r-th equivariant Euler characteristic of the G-poset $\int_{K\backslash G} \mathcal{S}_H$ is

$$\chi_r(\int_{K\backslash G} \mathcal{S}_H, G) = \sum_{A \in \mathcal{S}_G^{\mathrm{abe}}/G} \frac{\varphi_r(A)}{|N_G(A)|} \sum_{Kg \in C_{K\backslash G}(A)} \chi(C_{\mathcal{S}_{H^g}^{p+*}}(A))$$
$$= \frac{1}{|K|} \sum_{A \in \mathcal{S}_G^{\mathrm{abe}}/G} \frac{\varphi_r(A)}{|N_G(A)|} \sum_{\{g \in G \mid A \leq K^g\}} \chi(C_{\mathcal{S}_{H^g}^{p+*}}(A)) = \frac{1}{|K|} \sum_{A \in \mathcal{S}_G^{\mathrm{abe}}/G} \varphi_r(A) \sum_{B \in \mathcal{S}_G(A^G, K)} \chi(C_{\mathcal{S}_H^{p+*}}(B))$$
$$= \frac{1}{|K|} \sum_{A \in \mathcal{S}_K^{\mathrm{abe}}} \varphi_r(A) \chi(C_{\mathcal{S}_H^{p+*}}(A)) = \chi_r(\mathcal{S}_H^{p+*}, K)$$

□

Another proof of Lemma 2.4.10. The equivariant Euler characteristic of the Grothendieck construction $\int_{K\backslash G} \mathcal{S}_H^{p+*}$ is

$$\chi_r(\int_{K\backslash G} \mathcal{S}_H^{p+*}, G) = \frac{1}{|G|} \sum_{Kg \in K\backslash G} \sum_{X \in \mathrm{Hom}(\mathbb{Z}^r, K^g)} \chi(C_{\mathcal{S}_{H^g}^{p+*}}(X)) = \frac{1}{|G|} \sum_{Kg \in K\backslash G} |K^g| \chi_r(\mathcal{S}_{H^g}^{p+*}, K^g)$$
$$= \frac{1}{|G|} \frac{|G|}{|K|} |K| \chi_r(\mathcal{S}_H^{p+*}, K) = \chi_r(\mathcal{S}_H^{p+*}, K)$$

since the weighting $k^{Kg}(C_{K\backslash G}(X)) = 1$ for all $Kg \in C_{K\backslash G}(X)$. □

Let G be a finite group, p a prime, and λ an p-regular element of G(The order of λ is not divisible by p). In the rest of this section, we study the difference of equivariant Euler characteristics of two posets $C_{\mathcal{S}_G^{p+*}}(\lambda)$ and $\mathcal{S}_{C_G(\lambda)}^{p+*}$ in terms of a Grothendieck construction. It might be helpful for calculations on p-subgroup complexes of symmetric groups.

We want to compare the equivariant Euler characteristics of $C_{\mathcal{S}_G^{p+*}}(\lambda)$ and $\mathcal{S}_{C_G(\lambda)}^{p+*}$. We first define the 'opposite' of $\mathcal{S}_{C_G(\lambda)}^{p+*} = \{P \in \mathcal{S}_G^{p+*} \mid P^\lambda = P = C_P(\lambda)\}$.

Definition 2.4.11.

$$\mathcal{D}_G^{p+*}(\lambda) = \{P \in C_{\mathcal{S}_G^{p+*}}(\lambda) \mid [P, \lambda] = P\} = \{P \in \mathcal{S}_G^{p+*} \mid P^\lambda = P = [P, \lambda]\}$$

It is clear that $\mathcal{D}_G^{p+*}(\lambda)$ is a $C_G(\lambda)$-poset: Let $P \in \mathcal{D}_G^{p+*}(\lambda)$ and $g \in C_G(\lambda)$. Then $P^g \in \mathcal{D}_G^{p+*}(\lambda)$ because $(P^g)^\lambda = P^{g\lambda} = P^{\lambda g} = (P^\lambda)^g = P^g$ and $[P^g, \lambda] = [P^g, \lambda^g] = [P, \lambda]^g = P^g$.

We also need the Grothendieck construction $\int_{\mathcal{D}_G^{p+*}(\lambda)} \mathcal{S}_{C_G(\lambda,-)}^{p+*}$ for the $C_G(\lambda)$- functor $\mathcal{S}_{C_G(\lambda,-)}^{p+*}$ on the $C_G(\lambda)$-poset $\mathcal{D}_G^{p+*}(\lambda)$ taking $P \in \mathcal{D}_G^{p+*}(\lambda)$ to the poset $\mathcal{S}_{C_G(\lambda,P)}^{p+*}$ of non-trivial p-subgroups of $C_G(\lambda,P) = C_G(\lambda)\cap C_G(P)$. For $P\in\mathcal{D}_G^{p+*}(\lambda)$ and $g\in C_G(\lambda)$, this functor takes P^g to $\mathcal{S}^{p+*}_{C_G(\lambda,P^g)} = \mathcal{S}^{p+*}_{C_G(\lambda^g,P^g)} = \mathcal{S}^{p+*}_{C_G(\lambda,P)^g}$. The Grothendieck construction is a $C_G(\lambda)$-poset whose objects are pairs (P,E), $P\in\mathcal{D}_G^{p+*}(\lambda)$, $E\in\mathcal{S}^{p+*}_{C_G(\lambda,P)}$, and $(P,E)^g=(P^g,E^g)$ for all $g\in C_G(\lambda)$.

The following theorem follows essentially because any λ-normalized p-subgroup P of G splits uniquely as $P = C_P(\lambda)\times[P,\lambda]$ [Gor68, Chp 5, Theorem 2.3].

Theorem 2.4.12. *The difference between the equivariant Euler characteristics of the $C_G(\lambda)$-posets $C_{\mathcal{S}_G^{p+*}}(\lambda)$ and $\mathcal{S}^{p+*}_{C_G(\lambda)}$ is*

$$\widetilde{\chi}_r(C_{\mathcal{S}_G^{p+*}}(\lambda),C_G(\lambda)) - \widetilde{\chi}_r(\mathcal{S}^{p+*}_{C_G(\lambda)},C_G(\lambda)) = \widetilde{\chi}_r(\mathcal{D}_G^{p+*}(\lambda),C_G(\lambda)) - \widetilde{\chi}_r(\int_{\mathcal{D}_G^{p+*}(\lambda)}\mathcal{S}^{p+*}_{C_G(\lambda,-)},C_G(\lambda))$$

A similar result holds for the difference between the equivariant Euler characteristics of the $C_G(\lambda)$-posets $C_{\mathcal{S}_G^{\mathrm{elab}+p+*}}(\lambda)$ and $\mathcal{S}^{\mathrm{elab}+p+*}_{C_G(\lambda)}$ of elementary abelian p-subgroups.

Proof. Since the $C_G(\lambda)$-poset $C_{\mathcal{S}_G^{p+*}}(\lambda) = \mathcal{S}_1\cup\mathcal{S}_2$ is the union of the two upward closed $C_G(\lambda)$-ideals[Mø15, Section 3.3]

$$\mathcal{S}_1 = \{P\in C_{\mathcal{S}_G^{p+*}}(\lambda)\mid [P,\lambda]\neq 1\}\qquad \mathcal{S}_2 = \{P\in C_{\mathcal{S}_G^{p+*}}(\lambda)\mid C_P(\lambda)\neq 1\}$$

we have $\widetilde{\chi}_r(C_{\mathcal{S}_G^{p+*}}(\lambda),C_G(\lambda)) - \widetilde{\chi}_r(\mathcal{S}_2,C_G(\lambda)) = \widetilde{\chi}_r(\mathcal{S}_1,C_G(\lambda)) - \widetilde{\chi}_r(\mathcal{S}_1\cap\mathcal{S}_2,C_G(\lambda))$ by the Mayer-Vietoris relations. Moreover there are equivariant deformation retractions

$$\mathcal{D}_G^{p+*}(\lambda)\ \underset{[P,\lambda]\leftarrow P}{\rightleftarrows}\ \mathcal{S}_1\qquad\qquad \mathcal{S}^{p+*}_{C_G(\lambda)}\ \underset{C_P(\lambda)\leftarrow P}{\rightleftarrows}\ \mathcal{S}_2$$

where we use that $[P,\lambda,\lambda]=[P,\lambda]$ [Gor68, Chp 5, Theorem 3.6] for the identification of $\mathcal{S}_1$. Moreover, there is a $C_G(\lambda)$-poset morphism

$$\mathcal{S}_1\cap\mathcal{S}_2\xrightarrow{P\to[P,\lambda]}\mathcal{D}_G^{p+*}(\lambda)$$

for which the fiber over any $D\in\mathcal{D}_G^{p+*}(\lambda)$ are the groups $V\times D$ where V is any nontrivial p-subgroup of $C_G(\lambda)\cap C_G(D)$. Thus the $C_G(\lambda)$-poset $\mathcal{S}_1\cap\mathcal{S}_2$ is the Grothendieck construction of the functor $\mathcal{S}^{p+*}_{C_G(\lambda,-)}$ on $\mathcal{D}_G^{p+*}(\lambda)$. □

2.5 Macdonald Type Equations

2.5.1 Symmetric products

Symmetric products of a space M: $SP(M)$ plays an important role in algebraic topology. Let's first review the definition of symmetric product of a space.

Definition 2.5.1. Given a topological space X we define the n-th symmetric product of this space X as

$$SP_n(X) := X\times\cdots\times X/\Sigma_n$$

where Σ_n is the n-th symmetric groups acting on $X^{\times n}$ by switching of coordinates. If X is pointed we say the infinite symmetric product of X is

$$SP(X) := \mathrm{colim}_n\, SP_n(X)$$

where the colimits are taken under the map $SP_n(X)$ to $SP_{n+1}(X)$ by sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, e)$ where e is a chosen base point of X.

Remark. It turns out that the infinite symmetric product construction of pointed topological spaces is a functor from the category of pointed topological spaces to the category of topological commutative monoids and it is left adjoint to the forgetful functor.

A famous application of infinite symmetric product of spaces is **Dold-Thom** theorem which expresses the singular homology by the homotopy groups[Hat02, Theorem 4K.6].

Theorem 2.5.2. *If X is a based connected CW complex the functor n-th homotopy groups of its infinite symmetric product $SP(X)$ coincides with the functor of the n-th singular homology groups of X.*

In this section we will review the calculations of the ordinary Euler characteristics of the symmetric product of manifolds and we will generalize it to the equivariant Euler characteristics for bounded and half bounded posets. Our main arguments here are based on Tamanoi's treatments for the case of manifolds. In the end, as an application we use this general result to calculate the equivariant Euler characteristics of Coxeter complexes of type A and B.

Theorem 2.5.3 (**Macdonald**[Mac62])**.** *The generating function of ordinary Euler characteristics of symmetric products of a finite CW complex M is*

$$\sum_{n\geq 0} \chi(SP_n(M))u^n = \frac{1}{(1-u)^{\chi(M)}}$$

We call a concise formula involving a generating series of the Euler characteristics looks like this a Macdonald's type equation.

Proof. See [GZ17, Theorem 1]. □

Now consider a manifold M with a G-action on it where G is a finite group, Tamanoi[Tam01, Theorem A] give a Macdonald's type equation for the equivariant Euler characteristics of M^n with $G \wr \Sigma_n$-action.

Theorem 2.5.4. *For any $r \geq 1$ and for any G-manifold M we have*

$$\sum_{n\geq 0} \chi_r(M^n; G \wr \Sigma_n)q^n = \Big[\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\Big]^{-\chi_r(M,G)}$$

where $j_d(\mathbb{Z}^{r-1})$ is the number of index d subgroups in $\mathbb{Z}^{r-1}$. The explicit formula is

$$j_d(\mathbb{Z}^{r-1}) = \sum_{a_1 a_2 \ldots a_{r-1} = d} a_2 a_3^2 \cdots a_{r-1}^{r-2}$$

In this section we give a similar Macdonald's type equation for bounded and half-bounded posets with group actions. We first recall some well-known results about the conjugacy class and centralizer of an element of a special wreath product. Then we will study the fixed points of products of posets under the action by the wreath product. Next we will cook it up to prove our Macdonald's type equations for several cases mentioned before. And finally we will apply these results to calculate the equivariant Euler characteristics of Coxeter complexes of type A and B.

2.5.2 Centralizers of wreath products and fixed points

In this subsection we study the centralizer of wreath products and the associated actions on fixed points. Most of this stuff is treated in[Tam01]. We first describe the conjugacy classes and centralizers in symmetric groups then we generalize them to the wreath products of symmetric groups with a finite group G: $G \wr \Sigma_n$.

Definition 2.5.5. Given a permutation $\sigma \in \Sigma_n$ we define m_r to be the number of cycles with length r of σ. And we call the sequence $\{m_r\}_{1\leq r\leq n}$ the type of σ. We denote $P(n)$ as the set of all types associated to every element of Σ_n.

Proposition 2.5.6. *[Rot95, Theorem 3.5] Conjugacy classes in Σ_n are completely classified by $P(n)$. More precisely, two elements in Σ_n are conjugate to each other if and only if they have same type.*

Consider the d-cycle $a(d) = (1, 2, \dots, d) \in \Sigma_d$, the centralizer of $a(d)$ in Σ_d is a cyclic generated by $a(d)$, that is

$$C_{\Sigma_d}(a(d)) = \langle a(d)\rangle \simeq C_d$$

Moreover if an element $\sigma \in \Sigma_{de}$ consists of e d-cycles, i.e. its conjugacy class could be represented by the sequence $\{m_r\}_{1\leq r\leq m}$ where $m_d = e$ and others are all 0. Then the centralizer of σ is

$$C_{\Sigma_{de}}(\sigma) \simeq C_d \wr \Sigma_e$$

And In general let $\sigma \in \Sigma_n$ with type $\{m_r\}_{1\leq r\leq n}$, then the centralizer of σ is

$$C_{\Sigma_n}(\sigma) \simeq \prod_{1\leq r\leq n} C_r \wr \Sigma_{m_r}$$

Definition 2.5.7. Given an element (g, σ) in the wreath product $G \wr \Sigma_n$, where $g = (g_1, \dots, g_n)$. Let $\sigma = \sigma_1 \dots \sigma_k$ be the decomposition of σ by disjoint cycles $\sigma_i, 1 \leq i \leq k$. For each cycle $\sigma_i = (a_1, \dots, a_{m_i})$ the associated group element $g_{a_{m_i}} \cdots g_{a_2} g_{a_1}$ is called the cycle product of this cycle. Let $m_r(c)$ be the number of r-cycles in σ with cycle products in the conjugacy class $[c]$ in G. Then we say the sequence $\{m_r(c)\}_{r,[c]}$,where r runs from 1 to n and $[c]$ runs all conjugacy classes of G(we sometimes omit the subscript for simplicity when there is no confusion), the type of element σ. Let $P(n, G)$ be the set of all type of elements in $G \wr \Sigma_n$.

Proposition 2.5.8. *[Mac15, Chapter I, Appendix B] Conjugacy classes in $G \wr \Sigma_n$ are completely classified by the set $P(n, G)$. In other words, any two elements σ, τ in $G \wr \Sigma_n$ are conjugate to each other if and only if they have same type.*

Remark. It's equivalent to say that the conjugacy classes of $G \wr \Sigma_n$ is classified by sequences $\{m_r(c)\}_{[c]\in[G],1\leq r\leq n}$ such that $\sum_{[c]\in[G],1\leq r\leq n} r m_r(c) = n$ where $[G]$ means the set of conjugacy classes of the finite group G.

Example 2.5.9. Let's see several concrete examples of centralizers in wreath products:

1. Let $\sigma = (g, 1, 1, \cdots, 1; (12\cdots d)) \in G \wr \Sigma_n$. We have $\sigma^d = (g, g, \cdots, g; id) \in C_G(g) \leq G \leq G \leq G \wr \Sigma_d$, then the centralizer of σ is

$$C_{G\wr\Sigma_n}(\sigma) = \langle C_G(g), \sigma\rangle$$

   of order $|C_{G\wr\Sigma_n}(\sigma)| = d|C_G(g)|$[Mac15, Appexdix B, (3.1)]

2. Let $\sigma = (\overbrace{\underbrace{g, 1, \cdots 1}_{d}, \cdots, \underbrace{g, 1, \cdots, 1}_{d}}^{e}; \overbrace{(1 \cdots d) \cdots (d(e-1)+1 \cdots de)}^{e}) \in G \wr \Sigma_{de}$, then the centralizer of σ is

$$C_{G \wr \Sigma_{de}}(\sigma) = \langle C_G(g), x \rangle \wr \Sigma_e$$

of order

$$|C_{G \wr \Sigma_{de}}(\sigma)| = (d|C_G(g)|)^e |\Sigma_e| = |C_G(g)|^e d^e e!$$

where $x = (g, 1, \cdots, 1; (1 \cdots d))$

In general, the centralizer in wreath product is[Tam01, Theorem 3.5]:

Proposition 2.5.10. *Let $(g, \sigma) \in G \wr \Sigma_n$ with type $\{m_r(c)\}$ then we have*

$$C_{G \wr \Sigma_n}((g, \sigma)) \simeq \prod_{[c] \in [G]} \prod_{r \geq 1} \{(C_G(c) \cdot \langle a_{r,c} \rangle) \wr \Sigma_{m_r}(c)\}$$

where $(a_{r,c})^r = c \in C_G(c)$ and $[a_{r,c}, C_G(c)] = 1$ i.e. $a_{r,c}$ commutes with each group element in $C_G(c)$. In particular the order of this centralizer is

$$|C_{G \wr \Sigma_n}((g, \sigma))| = \prod_{[c] \in [G]} \prod_{r \geq 1} (r|C_G(c)|)^{m_r(c)} m_r(c)!$$

Proof. See [Tam01, Theorem 3.5] □

Definition 2.5.11. For a finite poset P we say this poset is **bounded** if there are both maximal and minimal elements $\hat{0}, \hat{1}$ in P and we denote $\overline{P} = P - \{\hat{0}, \hat{1}\}$. Also we say it is half bounded if there exists either a maximal or minimal element but not all of them simultaneously. Then in this case $\overline{P}$ is the subposet of P obtained by P minus the minimal or maximal element.

Proposition 2.5.12. *Let P be a G-poset either bounded or half bounded, $(g, \sigma) \in G \wr \Sigma_n$ with type $\{m_r(c)\}$, then*

$$(\overline{P^n})^{\langle (g,\sigma) \rangle} \simeq \overline{\prod_{[c] \in [G]} (P^{\langle c \rangle})^{\Sigma_r m_r(c)}}$$

where the product is taken over the conjugacy classes of G.

Proof. It is equivalent to prove

$$(P^n)^{(g,\sigma)} \simeq \prod_{[c] \in [G]} (P^{\langle c \rangle})^{\Sigma_r m_r(c)}$$

which has been showed a similar formula in [Tam01, Proposition 3.2] for manifolds. That argument works exactly same here for posets. □

Proposition 2.5.13. *1. If $P_1, P_2, \ldots, P_n$ are all bounded posets then*

$$|\overline{P_1 \times \cdots \times P_n}| \simeq S^{n-2} * |\overline{P_1}| * \cdots * |\overline{P_n}|$$

2. If $P_1, P_2, \ldots, P_n$ are all half bounded posets and we assume they all have a maximal element then

$$|\overline{P_1 \times \cdots \times P_n}| \simeq |\overline{P_1}| * \cdots * |\overline{P_n}|$$

Proof. 1. See [Aro15, Proposition 2.8]

2. See [Qui78, Proposition 1.9]

□

Corollary 2.5.14. *1. If $P_1, \ldots, P_n$ are all bounded posets we have*

$$\widetilde{\chi}(\overline{P_1 \times \cdots \times P_n}) = \prod_{i=1}^{n} \widetilde{\chi}(\overline{P_i})$$

2. If $P_1, \ldots, P_n$ are all half bounded posets and we assume they all have a maximal element then

$$\widetilde{\chi}(\overline{P_1 \times \cdots \times P_n}) = (-1)^{n-1} \prod_{i=1}^{n} \widetilde{\chi}(\overline{P_i})$$

Proof. Apply the reduced Euler characteristic functor to both sides. □

Remark. So for bounded posets the reduced over characteristics functor is distributive over the Cartesian products and for half bounded posets the minus Euler characteristic functor is distributive over the Cartesian products.

2.5.3 Macdonald's type equations

In this subsection we will determine the generating functions for equivariant Euler characteristic of products of bounded and half bounded G-posets . We first need a technical combinatorial result which could be viewed as a generalization of the explanation of Stirling number of first kind of rising factorial.

Lemma 2.5.15. *Let $\chi : G \longrightarrow \mathbb{Z}$ be a class function i.e. if c_1, c_2 are in a same conjugacy class of G then $\chi(c_1) = \chi(c_2)$. Then,*

$$\sum_{\sum\limits_{r,[c]} rm_r(c)=n} \sharp\{m_r(c)\} \prod_{[c]\in[G]} \chi(c)^{\sum\limits_r m_r(c)} = \Big(\sum_{c\in G} \chi(c)\Big)\Big(\sum_{c\in G} \chi(c) + |G|\Big) \cdots \Big(\sum_{c\in G} \chi(c) + n|G| - |G|\Big) \tag{2.5.16}$$

Where $|G|$ is the order of G,$\sharp\{m_r(c)\}_{[c]\in[G],1\le r\le n}$ means the number of group elements in $G \wr \Sigma_n$ with the same conjugacy class represented by the sequence $\{m_r(c)\}_{[c]\in[G],1\le r\le n}$ and the summation in the left hand side is actually taken over the conjugacy classes of $G \wr \Sigma_n$, i.e. the solution of the equation $\sum\limits_{r,[c]} rm_r(c) = n$ corresponds to the set of conjugacy classes of $G \wr \Sigma_n$.

Proof. We prove it by induction on n. When $n = 1$, then left hand side and right hand side are both equal to $\sum\limits_{c\in G} \chi(c)$. In other word the identity holds for $n = 1$. Now we assume this identity works for $n - 1$, we are going to show that this identity also works for n. By inductive hypothesis it suffices to show the left hand side of 2.5.16 equals to:

$$\left(\sum_{\sum\limits_{r,[c]} rm_r(c)=n-1} \sharp\{m_r(c)\}_{[c]\in[G],1\le r\le n-1} \prod_{[c]\in[G]} \chi(c)^{\sum\limits_r m_r(c)}\right)\left(\sum_{c\in G} \chi(c) + n|G| - |G|\right) \tag{2.5.17}$$

To show the equality we just need to show the equality of coefficients of each monomials of both sides. Now we denote $|G| = m$ and we denote the m-conjugacy classes of of G by $1, \ldots, m$ simply. Then given a sequence of non-negative integers $k_1, \ldots, k_m$ the coefficient of monomial $\chi(1)^{k_1} \cdots \chi(m)^{k_m}$(here if c_1, c_2 is in the same conjugacy classes then we view $\chi(c_1)$ and $\chi(c_2)$ as a same variable.) in the left hand side of 2.5.16 is

$$\sum_{\substack{\sum_{r,[c]} rm_r(c)=n \\ \sum_r m_r(c)=k_c \text{ for all } c}} \sharp\{m_r(c)\}_{[c]\in[G],1\le r\le n} \tag{2.5.18}$$

the coefficient of monomial $\chi(1)^{k_1} \cdots \chi(m)^{k_m}$ in 2.5.17 is:

$$(n-1)m \left(\sum_{\substack{\sum_{r,[c]} rm_r(c)=n-1 \\ \sum_r m_r(c)=k_c}} \sharp\{m_r(c)\}_{[c]\in[G],1\le r\le n-1} \right) + \sum_{1\le i\le m} \sum_{\substack{\sum_{r,[c]} rm_r(c)=n \\ \sum_r m_r(c)=k_c \text{ for } c\neq i \\ \sum_r m_r(i)=k_i-1}} \sharp\{m_r(c)\}_{[c]\in[G],1\le r\le n-1} \tag{2.5.19}$$

Where the formula 2.5.18 just counts the number of group elements in $G \wr \Sigma_n$ with number of cycles with cycle product in $[c]$ equals k_c. On the other hand, the set of group elements satisfy these properties could be separated into two cases. The first case the number n forms a cycle itself, then its cycle product has exactly m possibility, the number of group elements in this case corresponds exactly the second term in the formula 2.5.19; the second case is that the number n doesn't form a cycle itself then we first form cycles from the remaining $n-1$ elements then insert the number n somehow to one of these cycles and the way of inserting is exactly $(n-1)m$. However since in the first term of formula 2.5.19 the summation is over the cases of numbers of cycles with cycle product in $[c]$ is k_c then if we insert a number n carrying a specific group element g it will change the conjugate type of the cycle it inserts. But here I claim that the first term of formula 2.5.19 still counts the number of group elements in $G \wr \Sigma_n$ in which the number n doesn't form an individual cycle and the number of cycles with cycle products lies in $[c]$ is exactly k_c. It suffices to construct a bijection from the set A of the group elements in $G \wr \Sigma_n$ where the number n doesn't form an individual cycle and if we ignore the number n the number of cycles with cycle products in $[c]$ is exactly k_c to the set B of the group elements in $G \wr \Sigma_n$ where the number n doesn't form an individual cycle and the number of cycles with cycle products in $[c]$ is exactly k_c. Suppose $(c_1, c_2, \ldots, c_l)$ is an individual cycle with cycle product in $[c]$. Let $g_1, g_2, \ldots, g_l$ be the group elements associated to $c_1, c_2, \ldots, c_l$, so we can assume that $g_l g_{l-1} \cdots g_1 = c$ without losing any generality. So in this case if we insert the number n with a group element g in this cycle say $(c_1, c_2, \ldots, c_l, n)$. However after inserting the cycle product of this new cycle is gc, then we can change the group element g_l associated to the number c_l by $g^{-1}g_l$. In this case the cycle product is $gg^{-1}g_l g_{l-1} \cdots g_1 = g_l g_{l-1} \cdots g_1 = c$. In other words we construct a map f from the set A to B, on the other hand if the cycle $(c_1, c_2, \ldots, c_l, n)$ contains the number n and we assume the associated group elements are $g_1, g_2, \ldots, g_l, g$ such that the cycle product $gg_l \cdots g_1 = c$ .Then we can change the group element g_l associated to the number c_l by gg_l then the cycle $(c_1, c_2, \ldots, c_l, n)$ with new associated group elements such that if we ignore the number n then the cycle product of the remaining cycle $(c_1, c_2, \ldots, c_l)$ is $gg_l \ldots g_1 = c$. Hence we construct a map g from the set B to A. And we observe that the composition of these two maps $f \circ g$ and $g \circ f$ are both equal to the identity map on B and A respectively. Therefore e first term of formula 2.5.19 counts the number of set B. So both two formulas count same number: the number of group elements in $G \wr \Sigma_n$ with number of cycles with cycle product in $[c]$ equals k_c. In other words, the coefficient of monomial $\chi(1)^{k_1} \cdots \chi(m)^{k_m}$ agrees on both sides. □

Corollary 2.5.20. *1. Let P be a finite bounded G-poset then*

$$\sum_{n \geq 0} \widetilde{\chi}_1(\overline{P^n}, G \wr \Sigma_n) q^n = (1-q)^{-\widetilde{\chi}_1(\overline{P}, G)} \tag{2.5.21}$$

We assume here the $n = 0$ term is 1.

2. Let P be a finite half-bounded G-poset then

$$\sum_{n \geq 0} \widetilde{\chi}_1(\overline{P^n}, G \wr \Sigma_n) q^n = -(1-q)^{\widetilde{\chi}_1(\overline{P}, G)} \tag{2.5.22}$$

We assume here the $n = 0$ term is -1.

Proof. 1. It suffices to prove the coefficient of q^n on both side agree. The coefficient in the left hand side is

$$\widetilde{\chi}_1(\overline{P^n}, G \wr \Sigma_n) = \frac{1}{|G \wr \Sigma_n|} \sum_{(g,\sigma) \in G \wr \Sigma_n} \widetilde{\chi}((\overline{P^n})^{(g,\sigma)})$$

If the conjugacy type of element (g, σ) is represented by a sequence $\{m_r(c)\}_{[c] \in [G], 1 \leq r \leq n}$ then in this situation according to Proposition 2.5.12 and Corollary 2.5.14 we know

$$\widetilde{\chi}((\overline{P^n}))^{\langle (g,\sigma) \rangle} \simeq \prod_{[c] \in [G]} \widetilde{\chi}((\overline{P^{\langle c \rangle}}))^{\sum_r m_r(c)}$$

And the number of elements in $G \wr \Sigma_n$ with same conjugacy type $\{m_r(c)\}_{[c] \in [G], 1 \leq r \leq n}$ is $\sharp\{m_r(c)\}_{[c] \in [G], 1 \leq r \leq n}$. Therefore the coefficient of q^n in left hand side is

$$\frac{1}{|G \wr \Sigma_n|} \sum_{\sum\limits_{r,[c]} r m_r(c) = n} \widetilde{\chi}(\overline{P^{\langle c \rangle}})^{\sum_r m_r(c)} \sharp\{m_r(c)\}_{[c] \in [G], 1 \leq r \leq n} \tag{2.5.23}$$

On the other hand the coefficient of q^n in right hand side is simply expressed by the binomial coefficient

$$\begin{aligned}
(-1)^n \binom{-\widetilde{\chi}_1(\overline{P}, G)}{n} &= (-1)^n \binom{-\frac{1}{|G|} \sum\limits_{g \in G} \widetilde{\chi}(\overline{P^g})}{n} \\
&= (-1)^n \frac{(-\frac{1}{|G|} \sum\limits_{g \in G} \widetilde{\chi}(\overline{P^g})) \dots (-\frac{1}{|G|} \sum\limits_{g \in G} \widetilde{\chi}(\overline{P^g}) - n + 1)}{n!} \\
&= \frac{(\sum\limits_{g \in G} \widetilde{\chi}(\overline{P^g})) \dots (\sum\limits_{g \in G} \widetilde{\chi}(\overline{P^g}) + n|G| - |G|)}{|G|^n n!}
\end{aligned}$$

Then according to Lemma 2.5.15 we know the coefficient of q^n on both sides coincide.

2. The proof of the second statement is very similar like the argument for first statement. The only difference is that the reduced Euler characteristic of product of half-bounded posets. More concretely, when P is a half-bounded poset according to Corollary 2.5.14

$$(-1)(-1)^{\sum_r m_r(c_1)}\widetilde{\chi}(\overline{P^{\langle c_1\rangle})^{\sum_r m_r(c_1)}}\dots(-1)^{\sum_r m_r(c_k)}\widetilde{\chi}(\overline{P^{\langle c_k\rangle})^{\sum_r m_r(c_k)}}$$

where k denotes the number of conjugacy classes of G. So If we replace the reduced Euler characteristic by its minus and multiply -1 of the left hand side of 2.5.23 then we get the coefficient of q^n of left hand side of the equality of the second statement. So we just need to modify the right hand side by replace the reduced Euler characteristic by its minus one and multiply -1 then what we get is exactly the coefficient of q^n of right hand side of the equality of second statement.

□

Lemma 2.5.24. *[Tam01, Lemma 4.1] Let $G\cdot\langle a\rangle$ be a group generated by a finite group G and an element a such that a commutes with every element in G and $\langle a\rangle\cap G=\langle a^m\rangle$ for some integer $m\geq 1$. Suppose the element a acts trivially on a G-poset P either bounded or half-bounded. Then we have*

$$\widetilde{\chi}_r(\overline{P},G\cdot\langle a\rangle)=m^{r-1}\widetilde{\chi}_r(\overline{P},G) \tag{2.5.25}$$

Proof. We first observe that two elements ga^i and ha^j in $G\cdot\langle a\rangle$ commutes if and only if g,h commutes in G. Therefore we have

$$\widetilde{\chi}_r(\overline{P},G\cdot\langle a\rangle)=\frac{1}{m|G|}\sum_{(g_1,\dots,g_r),0\leq i_l<m}\widetilde{\chi}(\overline{P}^{\langle g_1a^{i_1},\dots,g_ra^{i_r}\rangle})$$

where the summation runs over all tuple $(g_1,\dots,g_r)$ of mutual commuting elements in G and l from 1 to r. Since a acts trivially on the poset P, the fixed points $\overline{P}^{\langle g_1a^{i_1},\dots,g_ra^{i_r}\rangle}$ is just $\overline{P}^{\langle g_1,\cdots,g_r\rangle}$, therefore

$$\widetilde{\chi}_r(\overline{P},G\cdot\langle a\rangle)=\frac{m^r}{m|G|}\sum_{(g_1,\dots,g_r)}\widetilde{\chi}(\overline{P}^{\langle g_1,\dots,g_r\rangle})=m^{r-1}\widetilde{\chi}_r(\overline{P},G)$$

□

Theorem 2.5.26. *If P is a bounded finite G-poset*

$$\sum_{n\geq 0}\widetilde{\chi}_r(\overline{P^n};G\wr\Sigma_n)q^n=\Big[\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\Big]^{-\widetilde{\chi}_r(\overline{P},G)}$$

Proof. We prove it by induction on r. When $r=1$, this formula holds because of Corollary 2.5.20. Then we assume this formula holds for $r\leq k-1$. Now let's try to prove this equality also holds when $r=k$. According to Lemma 2.1.25, we know

$$\sum_{n\geq 0}\widetilde{\chi}_k(\overline{P^n},G\wr\Sigma_n)q^n=\sum_{n\geq 0}\sum_{[(g,\sigma)]\in[G\wr\Sigma_n]}\widetilde{\chi}_{k-1}(\overline{P^n}^{(g,\sigma)},C_{G\wr\Sigma_n}((g,\sigma))q^n$$

According to Proposition 2.5.12 and Proposition 2.5.10 we have two isomorphisms:

$$C_{G\wr\Sigma_n}((g,\sigma))\simeq\prod_{[c]\in[G]}\prod_{r\geq 1}\{(C_G(c)\cdot\langle a_{r,c}\rangle)\wr\Sigma_{m_r}(c)\}$$

$$(\overline{P^n})^{\langle(g,\sigma)\rangle}\simeq\overline{\prod_{[c]\in[G]}(P^{\langle c\rangle})^{\sum_r m_r(c)}}$$

And the group action of the centralizer on fixed points is compatible with the isomorphisms. If we use the sequence $\{m_r(c)\}_{[c]\in[G],1\leq r\leq n}$ to represent the conjugacy type $[(g,\sigma)]$ then we have

$$\begin{aligned}\sum_{n\geq 0}\widetilde{\chi}_k(\overline{P^n}, G\wr\Sigma_n)q^n &= \sum_{n\geq 0} q^n \sum_{\sum\limits_{r,[c]} rm_r(c)=n}\prod_{[c],r}\widetilde{\chi}_{k-1}(\overline{(P^{\langle c\rangle})^{m_r(c)}}, C_G(c)\cdot\langle a_{r,c}\rangle\wr\Sigma_{m_r(c)}) \\ &= \prod_{[c],r}\sum_{m_r(c)\geq 0}(q^r)^{m_r(c)}\widetilde{\chi}_{k-1}(\overline{(P^{\langle c\rangle})^{m_r(c)}}, C_G(c)\cdot\langle a_{r,c}\rangle\wr\Sigma_{m_r(c)})\end{aligned} \tag{2.5.27}$$

By induction hypothesis we know for each conjugacy class $[c]$ in G we have

$$\sum_{m_r(c)\geq 0}(q^r)^{m_r(c)}\widetilde{\chi}_{k-1}(\overline{(P^{\langle c\rangle})^{m_r(c)}}, C_G(c)\cdot\langle a_{r,c}\rangle\wr\Sigma_{m_r(c)}) = \left(\prod_{d\geq 1}(1-(q^r)^d)^{j_d(\mathbb{Z}^{k-2})}\right)^{-\widetilde{\chi}_{k-1}(\overline{P^{\langle c\rangle}}, C_G(c)\cdot\langle a_{r,c}\rangle)}$$

Apply this equality to 2.5.27 we get

$$\begin{aligned}\sum_{n\geq 0}\widetilde{\chi}_k(\overline{P^n}, G\wr\Sigma_n)q^n &= \prod_{r,[c]}\left(\prod_{d\geq 1}(1-(q^r)^d)^{j_d(\mathbb{Z}^{k-2})}\right)^{-\widetilde{\chi}_{k-1}(\overline{P^{\langle c\rangle}}, C_G(c)\cdot\langle a_{r,c}\rangle)} \\ &= \prod_{[c]}\left(\prod_{d\geq 1, r, a_1a_2\cdots a_{k-2}=d}(1-(q^{rd}))^{a_2a_3^2\cdots a_{k-2}^{k-3}}\right)^{-\widetilde{\chi}_{k-1}(\overline{P^{\langle c\rangle}}, C_G(c)\cdot\langle a_{r,c}\rangle)} \\ &= \prod_{[c]}\left(\prod_{a_1a_2\cdots a_{k-2}r=dr}(1-q^{dr})^{a_2a_3^2\cdots a_{k-2}^{k-3}r^{k-2}}\right)^{-\widetilde{\chi}_{k-1}(\overline{P^{\langle c\rangle}}, C_G(c))} \\ &= \left(\prod_{a_1a_2\cdots a_{k-2}r=dr}(1-q^{dr})^{a_2a_3^2\cdots a_{k-2}^{k-3}r^{k-2}}\right)^{-\sum\limits_{[c]}\widetilde{\chi}_{k-1}(\overline{P^{\langle c\rangle}}, C_G(c))} \\ &= \left(\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{k-1})}\right)^{-\widetilde{\chi}_k(\overline{P}, G)}\end{aligned}$$

□

Theorem 2.5.28. *If P is a half bounded finite G-poset*

$$\sum_{n\geq 0}\widetilde{\chi}_r(\overline{P^n}; G\wr\Sigma_n)q^n = -\big[\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\big]^{\widetilde{\chi}_r(\overline{P}, G)}$$

Proof. The proof of this theorem is very similar to the proof in Theorem 2.5.26. We still write some key steps here. When $r=1$, this equality holds because of Corollary 2.5.20. Then we assume this equality holds for $r \leq k-1$, let's try to show this equality still holds when $r = k$.

$$\begin{aligned}\sum_{n\geq 0} \widetilde{\chi}_k(\overline{P^n}, G \wr \Sigma_n) q^n &= \prod_{[c],r} \sum_{m_r(c)\geq 0} (q^r)^{m_r(c)} \widetilde{\chi}_{k-1}(\overline{(P^{\langle c\rangle})^{m_r(c)}}, C_G(c)\cdot\langle a_{r,c}\rangle \wr \Sigma_{m_r(c)}) \\ &= -\prod_{r,[c]} \left(\prod_{d\geq 1}(1-(q^r)^d)^{j_d(\mathbb{Z}^{k-2})}\right)^{\widetilde{\chi}_{k-1}(\overline{P^{\langle c\rangle}}, C_G(c)\cdot\langle a_{r,c}\rangle)} \\ &= -\left(\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{k-1})}\right)^{\widetilde{\chi}_k(\overline{P}, G)}\end{aligned}$$

□

2.5.4 Coxeter Complexes and its equivariant Euler characteristics

In this subsection we determine the equivariant Euler characteristic of Coxeter complexes of type A and B. In the beginning let's give here a crash introduction to Coxeter complexes. For more details people could refer [Wac06, Lecture 1][Pet15, Chapter 11][AB08, Chapter 3].

Definition 2.5.29. Given V a finite dimensional Euclidean space over $\mathbb{R}$. Let $\mathcal{H}$ be a finite collection of hyper-planes $\{H_i\}_{i\in I}$ with $|I| < \infty$. We call it a hyper-plane arrangement of V. Then for any hyperplane H_i there is a linear functional $f_i \in V^*$ such that the kernal of this linear functional is exactly H_i. Then we call H_i itself having sign 0, the region $f_i > 0$ having sign + and the region $f_i < 0$ having sign − with respect to f_i respectively. Therefore these linear functionals divide V into many regions with a sequence of signs and we call it cells. Let $\Sigma(\mathcal{H})$ to be the set of all cells except the minimal cell with the sign sequence $(0, 0, \ldots, 0)$. We can equip face relation on $\Sigma\mathcal{H}$ i.e $A \leq B \in \Sigma(\mathcal{H})$ if and only if $A \subset B$. So $\Sigma(\mathcal{H})$ is a poset. Then the Δ-set of this poset is called the associated complex of this hyper-plane arrangement.

Definition 2.5.30. Given a finite hyper-plane arrangement $\mathcal{H}$ on V we associate a group generated by reflections with respect to each hyper-plane in $\mathcal{H}$. This is called the Weyl-group of this arrangement. Moreover, this Weyl-group has a natural action on the poset $\Sigma(\mathcal{H})$.

Example 2.5.31. Let V be $\mathbb{R}^n$, the type A Coxeter arrangement $\mathcal{H}_{A_n}$ is the collection of hyper-planes:

$$H_{i,j} = \{x \in \mathbb{R}^n : x_i = x_j\}$$

for $1 \leq i < j \leq n$. This is the hyper-plane arrangement associated to the Coxeter group of type A[Wac06, Example 1.3.3]: the symmetric group Σ_n, which is also the Weyl-group of this arrangement. And the Δ-set Σ_{A_n} of associated poset $\Sigma(\mathcal{H}_{A_n})$ is called the Coxeter complex of type A with the action by Σ_n.

Example 2.5.32. Let V be $\mathbb{R}^n$, the type B Coxeter arrangement $\mathcal{H}_{B_n}$ is the collection of hyper-planes:

$$H^+_{i,j} = \{x \in \mathbb{R}^n : x_i = x_j\}, \quad H^-_{i,j} = \{x \in \mathbb{R}^n : x_i = -x_j\}, \quad H_i = \{x \in \mathbb{R}^n : x_i = 0\}$$

for $1 \leq i < j \leq n$. This is the hyper-plane arrangement associated to the Coxeter group of type B[Wac06, Example 1.3.4] : the wreath product $C_2 \wr \Sigma_n$, which is also the Weyl group of this arrangement. And the Δ-set Σ_{B_n} of associated poset $\Sigma(\mathcal{H}_{B_n})$ is called the Coxeter complex of type B with the action by $C_2 \wr \Sigma_n$.

Theorem 2.5.33. *1. The generating function associated reduced equivariant Euler characteristics of Coxeter complex of type A is:*

$$\sum_{n\geq 0}\widetilde{\chi}_r(\Sigma_{A_n},\Sigma_n)q^n = \prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})} \tag{2.5.34}$$

This result also appears in [Mø17a, Section 1]

2. The generating function associated to reduced equivariant Euler characteristics of Coxeter compex of type B is

$$\sum_{n\geq 0}\widetilde{\chi}_r(\Sigma_{B_n},C_2\wr\Sigma_n)q^n = -\big(\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\big)^{1-2^{r-1}}. \tag{2.5.35}$$

In other words, for any $r\geq 1, n\geq 1$ then $\widetilde{\chi}_r(\Sigma_{B_n},C_2\wr\Sigma_n)=0$. When we consider the restriction action of Σ_n on Σ_{B_n}, the generating function in this case is

$$\sum_{n\geq 0}\widetilde{\chi}_r(\Sigma_{B_n},\Sigma_n)q^n = -\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})} \tag{2.5.36}$$

Proof. According to [Wac06, Example 1.3.3] Σ_{A_n} is the subdivision of the boundary of the standard simplex Δ^{n-1}. Hence it is the order complex of Boolean lattice without two extreme elements $\{\hat{0},\hat{1}\}$: B_n^*. We already knew in Example 2.1.15 that $B_n\simeq I^n$ where I is the poset $\{\hat{0}<\hat{1}\}$. Hence $\widetilde{\chi}_r(\Sigma_{A_n},\Sigma_n)=\widetilde{\chi}_r(\overline{I^n},\Sigma_n)$. Apply Theorem 2.5.26 and $\widetilde{\chi}(\overline{I})=-1$ we know

$$\sum_{n\geq 0}\widetilde{\chi}_r(\Sigma_{A_n},\Sigma_n)q^n = \prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}$$

As for the second statement, according to [Wac06, Example 1.3.4] Σ_{B_n} is the subdivision of the boundary of n-cube i.e the order complex of face poset of n-cube which is denoted by $\mathcal{C}_n$. We observe that the poset C_n is isomorphic to $\overline{\Lambda^n}$ where Λ is a half bounded poset consisting 3 elements $\{a<\hat{1}>b\}$ with C_2-action on it by switching a and b[Sta12, Exercise 71]. Then by definition we get $\widetilde{\chi}_r(\overline{\Lambda},C_2)=1-2^{r-1}$. Then according to Theorem 2.5.28

$$\sum_{n\geq 0}\widetilde{\chi}_r(\Sigma_{B_n},C_2\wr\Sigma_n)q^n = \sum_{n\geq 0}\widetilde{\chi}_r(\overline{\Lambda^n},C_2\wr\Sigma_n)q^n = -\big(\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}\big)^{1-2^{r-1}}$$

If we only consider action by Σ_n on Σ_{B_n} i.e we replace C_2 by the trivial group e then definition $\widetilde{\chi}_r(\overline{\Lambda},e)=\widetilde{\chi}(\overline{\Lambda})=1$. Hence:

$$\sum_{n\geq 0}\widetilde{\chi}_r(\Sigma_{B_n},\Sigma_n)q^n = -\prod_{d\geq 1}(1-q^d)^{j_d(\mathbb{Z}^{r-1})}$$

□

2.6 Equivariant Euler Characteristics of All Subgroup Complexes of Symmetric Groups

In this section we study the ordinary equivariant Euler characteristics of the poset of all proper non-trivial subgroups of symmetric groups and several variants of it. Actually this poset is not just a poset, it enjoys a richer structure called the lattice structure.

Definition 2.6.1. If L is a poset and S is a subset of L, then we say an element $u \in L$ is an **upper bound** of S if $s \leq u$ for any $s \in S$. Moreover if there is an upper bound u of S such that all other upper bounds x of S with $x \geq u$, then we call u the **join** of S. Dually we can define **lower bound** and **meet** of any subset S. Then a poset L is called a lattice if any two elements a, b in L have a join and a meet. The join operation is denoted by $\bigvee$ and the meet operation is denoted by $\bigwedge$. A lattice is called bounded if it has both maximal element $\hat{1}$ and minimal element $\hat{0}$.

Definition 2.6.2. Given a finite group G, all subgroups of G form a lattice denoted by $\mathcal{S}_G$. Any number of subgroups H_i enjoys a meet $\bigwedge H_i$ defined by their intersection $\bigcap H_i$, and a join $\bigvee H_i$ defined by the subgroup generated by the union of all of them together. Moreover $\mathcal{S}_G^* := S_G - \{e, G\}$ denotes the subposet of all non trivial proper subgroups of G and its associated Δ-set will be called the **all subgroup complex** of G.

Remark. The lattice $\mathcal{S}_G$ also carries a natural G-action by conjugation.

Like p-subgroup complexes, the all subgroup lattice reflects many important properties of G itself by its topological or combinatorial properties. And in this section we focus on the calculation of the (reduced)equivariant Euler characteristics of $\mathcal{S}_{\Sigma_n}^*$: the all subgroup lattice of symmetric group Σ_n for some series of n. In order to do this calculation we first recall a useful tool for calculations of equivariant Euler characteristics:

Proposition 2.6.3. *Let G be a finite group and P is a finite G-poset then*

$$\widetilde{\chi}_r(P, G) = \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{\text{abe}}} \widetilde{\chi}(C_P(A)) \varphi_r(A)$$

Where $\mathcal{S}_G^{\text{abe}}$ means the set of all abelian subgroups of G and $\varphi_r(A)$ is the number of all epimorphism from $\mathbb{Z}^r$ to the abelian group A.

From this proposition we know that the key step is trying to compute the ordinary Euler characteristics of some centralized sub-posets or furthermore trying to analyze the homotopy type of these centralized sub-posets. Here comes a key observation of the centralized sub-posets of $S_{\Sigma_n}^*$ under the action of abelian subgroup A of G.

Lemma 2.6.4. *If A is an abelian subgroup of symmetric group Σ_n with order greater than 2, then the centralized subposet $C_{S_{\Sigma_n}^*}(A)$ is $C_{\Sigma_n}(A)$-equivariant contractible.*

In the rest of this section A is always an abelian subgroup of Σ_n with order greater than 2. We use three steps to prove this theorem. As for the first step, we consider the abelian group C_m which is the cyclic group generated by a m-cycle where $2 < m \leq n$.

Let's first recall a very useful lemma in equivariant poset homtopy theory [TW91, Proposition 1.1]

Lemma 2.6.5. *Let P be a G-poset and $f, g : P \to P$ be two self G-maps of P such that $f(x) \geq g(x)$ for all $x \in P$ then f, g are G-homotopic.*

Next let's introduce a convenient conception.

Definition 2.6.6. An element $C \in C_{S_{\Sigma_n}^*}(A)$ which is a subgroup of A is a **contractor** for this abelian subgroup A if $CK \neq \Sigma_n$ for all $K \in C_{S_{\Sigma_n}^*}(A)$

Remark. In this case CK is equal to the subgroup generated by the union of K and C since C is a subgroup of A which normalizes K by definition.

Proposition 2.6.7. *Given a centralizer subposet $C_{S^*_{\Sigma_n}}(A)$, if we have a contractor then this centralizer subposet is $C_{\Sigma_n}(A)$-equivariant contractible.*

Proof. Let C be the contractor, since C is a subgroup of A then for any element K in this centralizer subposet we have CP a group and by assumption we know $CK \neq \Sigma_n$. Hence CK lies in $C_{S^*_{\Sigma_n}}(A)$. Since the contractor is normalized by $C_{\Sigma_n}(A)$, we have $C_{\Sigma_n}(A)$-equivariant double cones as follows:

$$K \leq CK \geq C$$

According to Lemma 2.6.5 we know $C_{S^*_{\Sigma_n}}(A)$ is $C_{\Sigma_n}(A)$-equivariant contractible. □

Lemma 2.6.8. *Let C_m denote the cyclic group generated by an m-cycle in Σ_n where m is an odd number. Then $C_{S^*_{\Sigma_n}}(C_m)$ is $C_{\Sigma_n}(C_m)$-equivariant contractible.*

Proof. Our basic strategy is to find a contractor. We claim that the cyclic group C_m itself is a contractor. To show it we just need to show that for any K in the centralizer subposet we have $C_mK \neq \Sigma_n$.

Assume it is not true which means there is a group K in this centralizer subposet satisfies: $C_mK = \Sigma_n$. Without losing any generality we can just assume that the m-cycle is $\lambda = (1, 2, .., m)$.

If $C_mK = \Sigma_n$, there is a element $k \in K$ and an integer i such that

$$k = (1, 2)\lambda^i$$

As λ normalizes K, it follows that $\lambda^i(1, 2)$ and $\lambda^i(1, 2)(1, 2)\lambda^i = \lambda^{2i}$ belong to K. Now since m is odd we can write $m = 2z - 1$, in this case we have $m \mid (2z - 1)i$ which means $2iz \equiv i \mod m$. Then $\lambda^i = (\lambda^{2i})^z \in K$, in other words the transposition (12) is also in K. Then even $\lambda = (m - 1, m) \cdots (2, 3)(1, 2)$ is in K because K contains $(1, 2)$ and its C_m-conjugates $(2, 3), \ldots, (m - 1, m)$. But then $C_mK = K$ contradicting $C_mK = \Sigma_n$. Therefore C_m is a contractor, then apply the Proposition 2.6.7 we finish the proof. □

Our previous argument can be generalized to a little bit general situation where the abelian group A is a cyclic group C with odd order m.

Corollary 2.6.9. *Given a cyclic subgroup C with odd order m, then $C_{S^*_{\Sigma_n}}(C)$ is $C_{\Sigma_n}(C)$-equivariant contractible.*

Proof. This proof is almost same as what we did previously. Let σ be a generator of this cyclic group C. We can express $\sigma = \sigma_1 \cdots \sigma_j$ where σ_i for i from 1 to j are disjoint cycles. Since the order of σ is odd, then the order if any σ_i is an cycle with odd order. Then we want to show that the cyclic group C is itself a contractor, which means we need to show that for any K in the centralizer subposet we have $CK \neq \Sigma_n$. Assume it is not true, that is $CK = \Sigma_n$, without losing any generality we can assume $\sigma_1 = (1, 2, \ldots, m_1)$, then there is an element $k \in K$ and an integer i such that

$$k = (1, 2) \cdot \sigma^i$$

Since $\sigma \cdot K \cdot \sigma^{-1} = K$, we have $\sigma^{2i} \in K$, then same number theoretic argument shows that $\sigma^i \in K$. Therefore $(1, 2) \in K$. Then apply conjugation action of σ we have $\sigma_1 \in K$. Repeat out argument we can show that $\sigma_i \in K$ for any i from 1 to j. So finally we have $\sigma \in K$, which means $CK = K$. But it is a contradiction. So we proved that C is a contractor. Therefore $C_{S^*_{\Sigma_n}}(C)$ is $C_{\Sigma_n}(C)$-equivariant contractible by same argument. □

Proposition 2.6.10. *Let C_2 be a cyclic of order 2 generated by even permutation σ, then $C_{S^*_{\Sigma_n}}(C_2)$ is $C_{\Sigma_n}(C_2)$-equivariant contractible.*

Proof. We just need to show that $C = C_2$ itself is the contractor we want. Since C is generated by a even permutation, we know $CA_n = A_n \neq \Sigma_n$. So we just need to consider the group K in the centralizer subposet which is not the alternating group A_n. Since $|K| < \frac{n!}{2}$, then we have $|K \cap C| = \frac{|C||K|}{|CK|} < \frac{2\cdot\frac{n!}{2}}{n!} = 1$. This is a contradiction because $|K \cap C| \geq 1$. □

The second step we want to study the abelian group C which is a cyclic group with order than 2.

Lemma 2.6.11. *Given a cyclic group C with order greater than 2, then $C_{S^*_{\Sigma_n}}(C)$ is $C_{\Sigma_n}(C)$-equivariant contractible.*

Proof. Let σ be a generator of this cyclic group C. Let m denote the order of this cyclic group C. We have proven the case when m is an odd number in Corollary 2.6.9. So next let's consider the case the order of this cyclic group is even. We need to find a new contractor for this situation, if the order m is not the power of 2, then there is a subgroup $C_{m'} \subset C$ where m' is an odd number. Then apply the argument in Corollary 2.6.9 we can show that this cyclic group $C_{m'}$ is the contractor we want. Lastly, if the order m is the power of 2, then there is a cyclic subgroup $C_2 \subset C$ with a even permutation as the generator since this generator is an even power of σ. Since C is an abelian subgroup we know $C_2 \in C_{S^*_{\Sigma_n}}(C)$ and C_2 is $N_{\Sigma_n}(C)$-normalized. Then apply the argument in Proposition 2.6.10, we can show that C_2 is the contractor we want for this case. Based on all previous discussion we proved that $C_{S^*_{\Sigma_n}}(C)$ is $C_{\Sigma_n}(C)$-equivariant contractible. □

The last step let's consider the general case the abelian subgroup A with order greater than 2.

Theorem 2.6.12. *Let A be a abelian subgroup of symmetric group Σ_n with order greater than 2. Then the centralizer subposet $C_{S^*_{\Sigma_n}}(A)$ is $C_{\Sigma_n}(A)$-equivariant contractible.*

Proof. Since A is a finitely-generated abelian group we can express A as a direct product of cyclic groups as follows:

$$A \cong C_{m_1} \times \cdots \times C_{m_j}$$

We need to use previous results to find a suitable contractor for this $C_{\Sigma_n}(A)$-subposet $C_{S^*_{\Sigma_n}}(A)$.

If there is an odd number m_i, we claim we can just choose C_{m_i} as the contractor. Firstly, $C_{m_i} \in C_{S^*_{\Sigma_n}}(A)$ since A is an abelian group. And C_{m_i} is $N_{\Sigma_n}(A)$-normalized by commutativity. Then we follow the argument in Corollary 2.6.9 we see for any group K in this centralizer subposet we have $C_{m_i}K \neq \Sigma_n$. Therefore the centralizer subposet $C_{S^*_{\Sigma_n}}(A)$ is $C_{\Sigma_n}(A)$-equivariant contractible in this case.

If there is an even number m_i greater or equal to 4. Then follow the argument in Proposition 2.6.10 and the argument previously we can choose the subgroup $C_2 \subset C_{m_i}$ as the contractor we want for this case. So we proved the statement in this case.

Finally, if all numbers are 2, that is

$$A \cong C_2 \times \cdots \times C_2$$

Let σ_1 and σ_2 be the generators of first two cyclic subgroups of order 2. Since these two generators commutes each other we can consider the cyclic subgroup C_2 generated by $\sigma_1 \cdot \sigma_2$. Same argument shows that $C_2 \in C_{S^*_{\Sigma_n}}(A)$ and it is $N_{\Sigma_n}(A)$-normalized. Moreover Since $\sigma_1 \cdot \sigma_2$ is a product of two permutations which means it must be a even permutation, then apply the Proposition 2.6.10 we know this cyclic subgroup C_2 is the contractor we want for this case. So we finished all the proof. □

After proving the basic technical Lemma 2.6.4 let's introduce some notions in poset topology [BW83]

Definition 2.6.13. Let $\widehat{L}$ be a finite bounded lattice and L is $\widehat{L}-\{\hat{0},\hat{1}\}$. Given any element $a \in L$ a **complement** of a in $\widehat{L}$ consists of elements $c \in \widehat{L}$ with

$$a \wedge c = \hat{0}; a \vee c = \hat{1}$$

And we denote the complement of a by $a^{\perp}$.

We recall that a poset P is called an anti-chain if for any $c, d \in P$ satisfy $c \leq d$, then we must have $c = d$.

Theorem 2.6.14. *[BW83, Theorem 4.2] Let $\widehat{L}$ be a finite bounded lattice and L be its proper part. Given an element $a \in L$, if its complement $a^{\perp}$ is an anti-chain then we have*

$$|L| \simeq \bigvee_{c \in a^{\perp}} \Sigma(|L_{<c}| * |L_{>c}|)$$

And in particular if $a^{\perp} = \emptyset$ then $|L|$ is contractible.

A direct consequence we have is the **Crapo's complementation formula**[Cra66, Theorem 3] by applying the Möbius function to this decomposition.

Corollary 2.6.15. *If $\widehat{L}$ is a finite bounded lattice and L is its proper part, and if an element $a \in L$ with $a^{\perp}$ is an anti-chain then we have*

$$\mu(L) = \sum_{c \in a^{\perp}} \mu(L_{<c})\mu(L_{>c})$$

And in particular if $a^{\perp} = \emptyset$ then $\mu(L) = 0$, where $\mu(L) = \mu_{\widehat{L}}(\hat{0},\hat{1})$ is the Möbius function of the poset $\widehat{L}$.

Remark. Actually we don't need the assumption that $a^{\perp}$ is an anti-chain in Corollary 2.6.15 the general Crapo's complementation formula is

$$\mu(L) = \sum_{c,c' \in a^{\perp}} \mu(L_{<c})z(c,c')\mu(L_{>c'})$$

where z is the zeta function of poset L.

Proposition 2.6.16. *[KT85, Lemma 4.6] Suppose G is a finite group and H is a subgroup of G, if N is normalized by H and is also in the complement of H in S_G, then we have two isomorphisms of posets as follows:*

$$(C_{\mathcal{S}^*_G}(H))_{>N} \longrightarrow C_{\mathcal{S}^*_H}(H)$$

and

$$(C_{\mathcal{S}^*_G}(H))_{<N} \longrightarrow (\mathcal{S}^*_G)_{>H}$$

Proof. Here we just prove the first case and the second case is similar. We construct a poset map from $(C_{\mathcal{S}^*_G}(H))_{>N}$ to $C_{\mathcal{S}^*_H}(H)$ by sending a subgroup X to $X \cap H$. It is clear this map is a well-defined poset map. Then we try to construct an inverse poset map from $C_{\mathcal{S}^*_H}(H)$ to $(C_{\mathcal{S}^*_G}(H))_{>N}$ by sending a subgroup $Y \in C_{\mathcal{S}^*_H}(H)$ to $YN \in (C_{\mathcal{S}^*_G}(H))_{>N}$. We need to show that the two poset maps we constructed are inverse to each other. On the one hand we just need to prove that given any subgroup $X \in (C_{\mathcal{S}_G^*}(H))_{>N}$ we have $(X \cap H)N = X$. We observe that $X \geq N$ and $X \geq X \cap H$ by definition, and moreover since $X \cap H$ normalizes N we know $(X \cap H)N$ is a subgroup of G. Hence $(X \cap H)N$ is also a subgroup of X. Furthermore we know

$$\begin{aligned}|(X \cap H)N| &= \frac{|X \cap H||N|}{|X \cap H \cap N|} = |X \cap H||N| \\ &= \frac{|X||H|}{|XH|}|N| = \frac{|X||H|}{|NH|}|N| \\ &= |X|\end{aligned}$$

Where the second equality holds since N is in the complement of H i.e. $N \cap H = e$ and $< N, H >= NH = G$; and the forth equality holds since N is already a subgroup of X, then $NH = G$ forces $XH = NH = G$.

Hence $(X \cap H)N = X$. Similar cardinality argument shows that $YN \cap H$ is equal to Y for any $Y \in C_{\mathcal{S}_H^*}(H)$. □

This proposition can help us to calculate the (reduced)Euler characteristics or the Möbius function of centralizers[Thé87, Proposition 4.3]:

Corollary 2.6.17. *Let G be a finite group and H be a non-trivial proper subgroup of G*

$$\mu(C_{\mathcal{S}_G^*}(H)) = \sum_{N \in H^\perp} \mu(C_{\mathcal{S}_N^*}(H))\mu(C_{\mathcal{S}_H^*}(H)) = Card(H^\perp)\mu(H, G)\mu(C_{\mathcal{S}_H^*}(H))$$

Where $\mu(H, G)$ is the Möbius function of the poset $\mathcal{S}_G$, $H^\perp$ is the complement of H inside the sublattice $C_{\mathcal{S}_G}(H)$ and Card$(H^\perp)$ means the cardinality of the set $H^\perp$.

Proof. First we notice that for any subgroup $N \in C_{\mathcal{S}_G^*}(H)$ inside $H^\perp$ we have $HN = G$ and $N \cap H = e$ by definition, which implies $|N||H| = |G|$. In other words $|N| = |G|/|H|$. Therefore $H^\perp$ is an anti-chain because of cardinality. Then according to Corollary 2.6.15 we know

$$\mu(C_{\mathcal{S}_G^*}(H)) = \sum_{N \in H^\perp} \mu((C_{\mathcal{S}_G^*}(H))_{<N})\mu((C_{\mathcal{S}_G^*}(H))_{>N})$$

then we replace $(C_{\mathcal{S}_G^*}(H))_{>N}, (C_{\mathcal{S}_G^*}(H))_{<N}$ by $C_{\mathcal{S}_H^*}(H)$ and $(\mathcal{S}_G^*)_{>H}$ respectively according to Proposition 2.6.16 we have

$$\begin{aligned}\mu(C_{\mathcal{S}_G^*}(H)) &= \sum_{N \in H^\perp} \mu((\mathcal{S}_G^*)_{>H})\mu(C_{\mathcal{S}_H^*}(H)) \\ &= Card(H^\perp)\mu(H, G)\mu(C_{\mathcal{S}_H^*}(H))\end{aligned}$$

□

As an easy application we can try to analyze several specific cases of sub-poset of $\mathcal{S}_{\Sigma_n}^*$:

Proposition 2.6.18. *If $\lambda \in \Sigma_n$ is an even involution then $(\mathcal{S}_{\Sigma_n}^*)_{>\lambda}$ is contractible. Here λ also indicates the cyclic subgroup with order 2 generated by λ. If λ is an odd permutation then*

$$\widetilde{\chi}(C_{\mathcal{S}_{\Sigma_n}^*}(\lambda)) = -\mu(\lambda, \Sigma_n)$$

Proof. Let $L = (\mathcal{S}^*_{\Sigma_n})_{>\lambda}$, Since λ is a an even involution we know the alternating group A_n is in L. Now we apply the Theorem 2.6.14 by choosing $a = A_n$. Since in $\mathcal{S}^*_{\Sigma_n}$ the complement of A_n are odd involutions, the complement of A_n in L is empty. So by Theorem 2.6.14 we know L is contractible.

When λ is an odd involution applying Corollary 2.6.17 we have

$$\widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = \mu(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = Card(\lambda^{\perp})\mu(\lambda, \Sigma_n)\mu(C_{\mathcal{S}^*_{\lambda}}(\lambda))$$

Since $\lambda^{\perp}$ is the complement of λ inside the centralizer $C_{\mathcal{S}_{\Sigma_n}}(\lambda)$, then for any subgroup $H \in \lambda^{\perp}$ we know $\langle\lambda\rangle \cdot H = \langle\lambda, H\rangle = \Sigma_n$ which implies $|H| = \frac{n!}{2}$. So the subgroup H must be the alternating subgroup A_n, in other words $Card(\lambda^{\perp}) = 1$. And we know $\mu(C_{\mathcal{S}^*_{\lambda}}(\lambda)) = -1$. So we have $\widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = -\mu(\lambda, \Sigma_n)$. □

The following result is an easy consequence of Corollary 2.6.17 which we have already proven in Corollary 2.6.10 in a little bit different way.

Corollary 2.6.19. *If $\lambda \in \Sigma_n$ is an even involution, then $\widetilde{\chi}(C_{S^*_{\Sigma_n}}(\lambda)) = 0$*

Proof. By Corollary 2.6.17 We know

$$\widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = \mu(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = Card(\lambda^{\perp})\mu(\lambda, \Sigma_n)\mu(C_{\varphi^*_{\lambda}}(\lambda))$$

Since λ is an even permutation we have $(\mathcal{S}^*_{\Sigma_n})_{>\lambda}$ is contractible, it follows that

$$\mu(\lambda, \Sigma_n) = \widetilde{\chi}((\mathcal{S}^*_{\Sigma_n})_{>\lambda}) = 0$$

Therefore $\widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = 0$. □

Now we need to cite a well-known result before the final calculations. This well-known result is about the Möbius function of subgroup lattices of symmetric groups by Shareshian in [Sha97, Theorem 1.6, Theorem 1.8,Theorem 1.10].

Theorem 2.6.20. *Let μ be the Möbius function of the poset $\mathcal{S}_{\Sigma_n}$.*

1. *If $n = p$ is a prime then*

$$\mu(1, \Sigma_p) = (-1)^{p-1}\frac{p!}{2}$$

2. *If $n = 2p$ where p is an odd prime then*

$$\mu(1, \Sigma_n) = \begin{cases} -n! & \text{if } n-1 \text{ is prime and } p \equiv 3 \pmod 4 \\ \frac{n!}{2} & \text{if } n = 22 \\ -\frac{n!}{2} & \text{otherwise} \end{cases}$$

3. *If $n = 2^a$, for a a natural number then*

$$\mu(1, \Sigma_n) = -\frac{n!}{2}$$

Proposition 2.6.21.

$$\sum_{\lambda \in S_{odd}} \mu(\lambda, \Sigma_n) = -\mu(1, \Sigma_n)$$

Where S_{odd} denotes the set of all odd involutions in Σ_n.

Proof. We apply the general version of Crapo's complement formula by choosing $x = A_n$ the alternating subgroup of Σ_n. Since $A_n^{\perp} = S_{odd}$ we get

$$\mu(1, \Sigma_n) = \sum_{c,c' \in S_{odd}} \mu(1, c) z(c, c') \mu(c', \sigma_n)$$

Since c is an involution we have $(1, c) = \emptyset$ i.e $\mu(1, c) = -1$, and since S_{odd} is an anti-chain we have

$$\mu(1, \Sigma_n) = -\sum_{c \in S_{odd}} \mu(c, \Sigma_n)$$

□

Theorem 2.6.22. *The (reduced) equivariant Euler characteristics of all symmetric groups($n \geq 3$) are:*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = \frac{2^r}{n!} \mu(1, \Sigma_n)$$

Therefore in particular:

1. *If $n = p$ is an odd prime we have*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_p}, \Sigma_p) = 2^{r-1}$$

2. *If $n = 2p$ where p is an odd prime then*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = \begin{cases} -2^r & \text{if } n-1 \text{ is prime and } p \equiv 3 (\mod 4) \\ 2^{r-1} & \text{if } n = 22 \\ -2^{r-1} & \text{otherwise} \end{cases}$$

3. *If $n = 2^a$,for a a natural number then*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = -2^{r-1}$$

Proof. For any integer $n \geq 3$, by Proposition 2.6.3 we know

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = \frac{1}{|\Sigma_n|} \sum_{A \in S^{\text{abe}}_{\Sigma_n}} \widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(A)) \varphi_r(A)$$

Then apply Lemma 2.6.4 and Corollary 2.6.19 we get

$$\frac{1}{|\Sigma_n|} \sum_{A \in \mathcal{S}^{\text{abe}}_{\Sigma_n}} \widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(A)) \varphi_r(A) = \frac{1}{|\Sigma_n|} \widetilde{\chi}(\mathcal{S}^*_{\Sigma_n}) \varphi_r(id) + \frac{1}{|\Sigma_n|} \sum_{\lambda \in S_{odd}} \widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) \varphi_r(C_2)$$

Hence by Proposition 2.6.18 and Proposition 2.6.21

$$\sum_{\lambda \in S_{odd}} \widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_n}}(\lambda)) = -\sum_{\lambda \in S_{odd}} \mu(\lambda, \Sigma_n) = \mu(1, \Sigma_n)$$

Therefore

$$\frac{1}{|\Sigma_n|} \widetilde{\chi}(\mathcal{S}^*_{\Sigma_p}) \varphi_r(id) + \frac{1}{|\Sigma_n|} \sum_{\lambda \in S_{odd}} \widetilde{\chi}(C_{\mathcal{S}^*_{\Sigma_p}}(\lambda)) \varphi_r(C_2) = \frac{1}{|\Sigma_n|} (1 + \varphi_r(C_2)) \mu(1, \Sigma_n)$$

Finally by definition $\varphi_r(id) = 1$ and $\varphi_r(C_2) = 2^r - 1$, we have

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = \frac{2^r}{n!} \mu(1, \Sigma_n)$$

□

Remark. In [Sha97, Proposition 1.3], Shareshian determined a general formula but not so explicit for computing $\mu(1, \Sigma_n)$ for any n:

$$\mu(1, \Sigma_n) = (-1)^{n-1}\frac{n!}{2} - \sum_{H \in \mathbf{C}_n} \mu(1, H)$$

where $\mathbf{C}_n$ consists of proper subgroups of Σ_n which is transitive and contains an odd involution. So in general we can express our (reduced) equivariant Euler characteristics of subgroup lattice of symmetric groups as:

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n) = (-1)^{n-1}2^{r-1} - \frac{2^r}{n!}\sum_{H \in \mathbf{C}_n} \mu(1, H)$$

This project was suggested by a magma computation given by my supervisor

$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n}, \Sigma_n)$	n=2	n=3	n=4	n=5	n=6	n=7	n=8
r=1	-1	1	-1	1	-2	1	-1
r=2	-2	2	-2	2	-4	2	-2
r=3	-4	4	-4	4	-8	4	-4
r=4	-8	8	-8	8	-16	8	-8
r=5	-16	16	-16	16	-32	16	-16

(2.6.23)

$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n} - \langle(1,2)\rangle^{\Sigma_n})$	n=2	n=3	n=4	n=5	n=6	n=7	n=8
r=1	-1	0	0	0	-1	0	0
r=2	-2	0	0	0	-2	0	0
r=3	-4	0	0	0	-4	0	0
r=4	-8	0	0	0	-8	0	0
r=5	-16	0	0	0	-16	0	0

(2.6.24)

$\widetilde{\chi}_r(\mathcal{S}^*_{A_n}; A_n)$	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9
r=1	-1	-1	1	-1	2	-1	1	$\frac{1}{\lvert A_9\rvert}\widetilde{\chi}(S^*_{A_9})$
r=2	-1	-3	3	-1	2	-1	1	
r=3	-1	-9	9	-1	2	-1	1	
r=4	-1	-27	27	-1	2	-1	1	
r=5	-1	-81	81	-1	2	-1	1	
r=6	-1	-243	243	-1	2	-1	1	
r=7	-1	-729	729	-1	2	-1	1	
r=8	-1	-2187	2187	-1	2	-1	1	
r=9	-1	-6561	6561	-1	2	-1	1	
r=10	-1	-19683	19683	-1	2	-1	1	

(2.6.25)

These tables give us several potential further research projects such like how to explain the numerical result of equivariant Euler characteristics of all subgroups complex of alternating groups A_n? Or how to explain the numerical result of the sublattice $\mathcal{S}^*_{\Sigma_n} - \langle(12)\rangle^{\Sigma_n}$? Here we first give a partial answer to the second question and then we give an explanation of the alternating group cases in the end of this section.

Theorem 2.6.26. *Where n is a power of 2 or odd prime p except $p \equiv 3 \mod 4$, then $\mathcal{S}^*_{\Sigma_n} - \langle(12)\rangle^{\Sigma_n}$ is Σ_n-equivariant contractible. Therefore*

$$\widetilde{\chi}_r(\mathcal{S}^*_{\Sigma_n} - \langle(12)\rangle^{\Sigma_n}; \Sigma_n) = 0$$

For any integer r.

In order to prove this result we need a little bit more basic properties on equivariant homotopy theory on posets. The first one we need is a baby toy of equivariant version of decomposition theorem[Wel95, Corollary 2.4].

Lemma 2.6.27. *Let $\widehat{P}$ be a bounded G-lattice with P as its proper non-trivial part. If $a \in P$ is a G-invariant element then the subposet $P - a^{\perp}$ is G-equivariant contractible.*

In particular let's apply this lemma to the all subgroup lattice $\mathcal{S}^*_{\Sigma_n}$. It's clear the alternating group $A_n \in \mathcal{S}^*_{\Sigma_n}$ is a Σ_n invariant element. Therefore we have

Corollary 2.6.28. *For $n \geq 3$, $S^*_{\Sigma_n} - A_n^{\perp}$ is Σ_n-equivariant contractible. Where $A_n^{\perp}$ is just S_{odd} the anti-chain of all odd involutions in Σ_n.*

Now let's consider a natural inclusion:

$$i : \mathcal{S}^*_{\Sigma_n} - A_n^{\perp} \hookrightarrow \mathcal{S}^*_{\Sigma_n} - \langle (12) \rangle^{\Sigma_n}$$

Our idea is applying the equivariant version of Quillen's fibre lemma[TW91, Theorem 1] to prove the inclusion i is a Σ_n-equivariant homotopy equivalence.

Lemma 2.6.29. *Let G be a group, and X, Y two G-posets with a G-poset map $\varphi : X \to Y$, if either*

- *for all $y \in Y$, we have $\varphi^{-1}(Y_{\leq y})$ is $C_G(y)$-contractible or*
- *for all $y \in Y$, we have $\varphi^{-1}(Y_{\geq y})$ is $C_G(y)$-contractible*

Then φ is a G-equivariant homotopy equivalence.

Now we apply this lemma to our inclusion of subgroup posets. It suffices to show that when $t \in \mathcal{S}^*_{\Sigma_n} - \langle (12) \rangle^{\Sigma_n}$ which is an odd involution not a transposition then the pre-image of $(\mathcal{S}^*_{\Sigma_n} - \langle (12) \rangle^{\Sigma_n})_{\geq \langle t \rangle} = (\langle t \rangle, \Sigma_n)$ is $C_{\Sigma_n}(\langle t \rangle)$-contractible when $n = 2^a$ or an odd prime p except $p \equiv 3 \mod 4$.

For any $t \in A_n^{\perp}$ we set[Sha97, Page 146]

$$NT^t = \{H \in (\langle t \rangle, \Sigma_n) | H \text{ is not transitive}\}$$

$$NP^t = \{H \in (\langle t \rangle, \Sigma_n) | H \text{ is not primitive}\}$$

Where a subgroup H in symmetric group is called transitive if the induced action of H on the set $[n] := \{1, 2, \ldots, n\}$ is transitive. Moreover a subgroup H in symmetric group is called primitive if it is first transitive and H will not preserve any non-trivial partition of $[n]$. Where a non-trivial partition of $[n]$ means that that isn't a partition into singleton sets or partition into one set $[n]$.

We want to use these two subposets to study the equivariant homotopy type of $(\langle t \rangle, \Sigma_n)$. First we study the equivariant homotopy type of NT^t.

Definition 2.6.30. [Mø17a, Definition 2.1]

1. A **partition** π on $[n]$ is an equivalence relation $\sim_\pi$ on $[n]$. For a block in π it is an equivalence class of $\sim_\pi$. Let $P(n)$ denote the set of all partitions on $[n]$.

2. For two partitions π_1, π_2 on $[n]$ we write $\pi_1 \leq \pi_2$ if each block in π_1 is contained in a block in π_2. $P(n)$ is a poset equipped with this order and its associated simplicial complex is called partition complex which is also denoted as $P(n)$.

Proposition 2.6.31. *Assume $n \geq 3$, if t is an odd involution which is not a transposition then NT^t is $C_{\Sigma_n}(\langle t \rangle)$-contractible.*

Proof. First we introduce a useful map **orb** from $\mathcal{S}^*_{\Sigma_n}$ to $P(n)$. For any subgroup H of Σ_n, we define the image $\mathbf{orb}(H)$ to be the partition whose parts are the orbits of the action of H on $[n]$. We list some simple properties of this function[Sha03, Lemma 3.2]:

- The function **orb** is order preserving, in other words, it is a poset map.
- For any partition $\pi \in P(n)$, we have $\mathbf{orb}(\Sigma_\pi) = \pi$. Here $\Sigma_\pi = \Sigma_{\pi_1} \times \cdots \times \Sigma_{\pi_k}$ where $\pi = [\pi_1|...|\pi_k]$.
- For $H \in \mathcal{S}^*_{\Sigma_n}$, we have $\mathbf{orb}(H) \leq \pi$ if and only if H is a subgroup of Σ_π.

Furthermore, we observe that this map is in fact Σ_n-equivariant. For any $H \in \mathcal{S}^*_{\Sigma_n}$ and choose any element $g \in \Sigma_n$. By the definition of group actions on $P(n)$, two numbers $x, y \in [n]$ are in same part of the partition $g \cdot \mathbf{orb}(H)$if and only if $g^{-1}x$ and $g^{-1}y$ are in same part of the partition $\mathbf{orb}(H)$, in other words there is an element $h \in H$ such that $g^{-1}x = h \cdot g^{-1}y$. Meanwhile, two numbers $x, y \in [n]$ are in same component i the partition $\mathbf{orb}(H^g$ if and only if there is element $h \in H$ such that $x = ghg^{-1}y$ which is just $g^{-1}x = h'g^{-1}y$. Hence we showed

$$\mathbf{orb}(H^g) = g \cdot \mathbf{orb}(H)$$

In other words, the poset map **orb** is a Σ_n-equivariant poset map.

Now consider the restriction of **orb** to the subposet NT^t. We denote this restriction map as $\mathbf{orb}_t$.

Since every element in NT^t is not a transitive subgroup of Σ_n by definition which means the image of these elements could not reach the maximum element in $P(n)$.

Moreover, when t it an odd involution but not a transposition then the $\langle t \rangle$ is a proper subgroup of $\Sigma_{\mathbf{orb}(\langle t \rangle)}$. And the image of these two subgroups under **orb** are same. Hence when t is an odd involution but not a transposition,

$$\mathrm{Im}(\mathbf{orb}_t) = [\mathbf{orb}(\langle t \rangle), \hat{1})$$

where $\hat{1}$ denotes the maximal partition of $[n]$.

We denote $G = C_{\Sigma_n}(\langle t \rangle)$. By equivariance of the map **orb**, we know the image of $\mathbf{orb}_t$ is also a G-subposet. Hence we have an equivariant map $\mathbf{orb}_t$:

$$NT^t \longrightarrow \mathrm{Im}(\mathbf{orb}_t) = [\mathbf{orb}(\langle t \rangle), \hat{1})$$

And because $\mathbf{orb}(\langle t \rangle)$ is a minimum element and G-invariant, it implies $\mathrm{Im}(\mathbf{orb}_t)$ is G-equivariant contractible. And for any non-trivial partition π in $\mathrm{Im}(\mathbf{orb}_t)$, we have

$$\mathbf{orb}_t^{-1}(\leq \pi) = (\langle t \rangle, \Sigma_\pi]$$

Where Σ_π is the maximum element of this poset and it is $C_G(\pi)$-invariant. Hence it is $C_G(\pi)$-contractible. And by the equivariant Quillen's fibre lemma we know

$$NT^t \cong_G \mathrm{Im}(\mathbf{orb}_t)$$

where $\cong_G$ means G-homotopy equivalence. In other words NT^t is $C_{\Sigma_n}(\langle t \rangle)$-contractible. □

Corollary 2.6.32. *When n is an odd prime expect $p \equiv 3 \mod 4$ then $\mathcal{S}^*_{\Sigma_n} - \langle(12)\rangle^{\Sigma_n}$ is Σ_n-contractible.*

Proof. We just need to prove $(\langle t\rangle, \Sigma_p)$ is $C_{\Sigma_n}(t)$-contractible if t is an odd involution but not a transposition under the assumption. According to [Sha97, Corollary 3.2] we know $(\langle t\rangle, \Sigma_p) = NT^t$ unless $p \equiv 3 \mod 4$, and according to Proposition 2.6.31 we know NT^t is $C_{\Sigma_n}(t)$-contractible . Therefore $(\langle t\rangle, \Sigma_p)$ is also $C_{\Sigma_n}(t)$-contractible unless $p \equiv 3 \mod 4$. □

In addition we give here an equivariant version of a key lemma in [Sha03, Lemma 2.2]. Given a finite group G, let $\widehat{L}$ be a bounded finite G-lattice where the group action is assumed to be (co)-atoms preserving, mapping $\{\hat{0}, \hat{1}\}$ to $\{\hat{0}, \hat{1}\}$ and if $gx_1 = gx_2$ then we have $x_1 = x_2$ for any $g \in G; x_1, x_2 \in \widehat{L}$. Moreover, let $\widehat{P}$ be any G-subposet of $\widehat{L}$ which contains $\widehat{L^*} \cup \{\hat{0}\}$ where $\widehat{L^*}$ consists of all elements in $\widehat{L}$ which can be obtained by taking the meet of a set of coatoms of L(We allow this set to be an empty set which means $\hat{1} \in \widehat{L^*}$). Coatoms in a bounded lattice are those elements x with $x < \hat{1}$ but there is no third object y with $x < y < \hat{1}$.Atoms are defined dually. And for any finite bounded lattice $\widehat{A}$ we use A denotes the subposet of $\widehat{A}$ removed the two extreme elements $\{\hat{0}, \hat{1}\}$

Lemma 2.6.33. *Based on the assumptions given above, we have*

- *If $\hat{0} \in \widehat{L^*}$ then*
$$P \simeq_G L^*$$
- *If $\hat{0} \notin \widehat{L^*}$, then P is G-contractible.*

Proof. First we set $M := L^*$ if $\hat{0} \in \widehat{L^*}$ and $M := L^* \setminus \{\hat{1}\}$ if $\hat{0} \notin \widehat{L^*}$. Now consider the G-equivariant inclusion
$$i : M \to P$$
For any element $x \in P$, let $x^* := \bigwedge_{a\in S} a$ where S is the set of all coatoms lie above x. For any coatom a greater or equal to x, $ga \geq gx = x$ where $g \in C_G(x)$. Since the group action is assumed to be coatoms preserving:
$$g(\bigwedge_{a\in S} a) = \bigwedge_{a\in S} ga = x^*$$
In other words the element x^* is $C_G(x)$-invariant. Moreover, given any element $x \in P$ we have $i^{-1}(P_{\geq x}) = M_{\geq x^*}$. Since x^* is a minimal element in $M_{\geq x^*}$ and x^* is $C_G(x)$-invariant, $M_{\geq x^*}$ is $C_G(x)$-contractible. According to the equivariant Quillen's fibre lemma i.e. Lemma 2.6.29 we know this inclusion is in fact G-equivariant homotopy equivalence. So the first claim is true. Moreover, if $\hat{0} \notin \widehat{L^*}$ which means the meet of all coatoms which is G-invariant is not $\hat{0}$, so M is G-contractible to this non-zero minimal element. □

Lemma 2.6.34. *When n is a power of 2 and $t \in \Sigma_n$ which is a involution but not a transposition then NT^t is $G = C_{\Sigma_n}(t)$-equivariant homotopy equivalent to NP^t.*

Proof. Consider a third object $\widehat{NP^t} := NP^t \cup \{t, \Sigma_n\}$ and let $\widehat{NP^t}^*$ consists of the elements could be obtained by taking meet of a collection of coatoms in $\widehat{NP^t}$. Then we define $\widehat{M_t} := \widehat{NP^t}^* \cup NT^t$ and set $M_t = \widehat{M_t} - \{\langle t\rangle, \Sigma_n\}$ if $\langle t\rangle \in \widehat{NP^t}^*$ or $M_t = \widehat{M_t} - \{\Sigma_n\}$ if $\langle t\rangle \notin \widehat{NP^t}^*$. When $\langle t\rangle \notin \widehat{NP^t}^*$ then according to Lemma 2.6.33 NP^t is G-contractible, so it is G-equivariant homotopy equivalent to NT^t since it is also G-contractible. On the other hand, if $\langle t\rangle \in \widehat{NP^t}^*$ we set $(NP^t)^* = \widehat{NP^t}^* - \{\langle t\rangle, \Sigma_n\}$. Then also according to Lemma 2.6.33 we know

$$NP^t \simeq_G (NP^t)^* \simeq_G M_t$$

Now consider the inclusion $i : NT^t \hookrightarrow M_t$, it suffices to show that for any subgroup $H \in M_t \backslash NT^t$ the preimage $i^{-1}((M_t)_{\leq H}) = (\langle t\rangle, H) \cap NT^t$ is $C_G(H)$-contractible. We study this object under the orbit map

$$\mathbf{orb}_H : i^{-1}((M_t)_{\leq H}) \longrightarrow \Pi(H)$$

where $\Pi(H)$ be the image of $i^{-1}((M_t)_{\leq H})$ under the map **orb** :. Then for each partition $\pi \in \Pi(H)$ we observe that $H \cap \Sigma_\pi$ is the minimal element in $\mathbf{orb}_H^{-1}(\Pi(H)_{\leq\pi}))$ and it is $C_G(H,\pi)$-invariant. Hence according to Lemma 2.6.29 we know $\mathbf{orb}_H$ is a $C_G(H)$-equivariant homotopy equivalence. Moreover, according to [Sha03, Proof of Lemma 3.8] we know there always exists a subgroup $K \in (\langle t\rangle, \Sigma_n)$ such that $\mathbf{orb}(K) = \mathbf{orb}(\langle t\rangle)$, in other words $\mathbf{orb}(\langle t\rangle)$ is the minimal element in $\Pi(H)$ and is $C_G(H)$-invariant. Hence $\Pi(H)$ is $C_G(H)$-contractible. Therefore apply the Lemma 2.6.29 once more we know NT^t is G-equivariant homotopy equivalent to M_t, so NT^t is G-equivariant homotopy equivalent to NP^t. □

Theorem 2.6.35. *When n is a power of 2 and $t \in \Sigma_n$ which is a involution but not a transposition then $(\langle t\rangle, \Sigma_n)$ is G-contractible. Therefore $\mathcal{S}^*_{\Sigma_n} - \langle(12)\rangle^{\Sigma_n}$ is Σ_n-contractible.*

Proof. We already showed that NT^t is equivariant homotopy equivalent to NP^t under these assumptions. Here we claim that the inclusion $NP^t \hookrightarrow (\langle t\rangle, \Sigma_n)$ is also a $G = C_{\Sigma_n}(t)$-equivariant homotopy equivalence. To prove this claim it suffices to show that when H is a primitive proper subgroup of Σ_{2^a} which contains an odd involution then $(\langle t\rangle, H)$ is $C_G(H)$-contractible. In [Sha97, Lemma 6.10] Shareshian shows in this case $p = 2^a - 1$ is a prime and $H \simeq \mathrm{PGL}_2(p)$. Then we define $\widehat{L}$ to be the bounded lattice of t-invariant subgroups of $\mathrm{PSL}_2(p)$, then according to the proof in [Sha03, Lemma 3,9] its non-trivial proper part L is isomorphic to the interval $(\langle t\rangle, \mathrm{PGL}_2(p))$ with the isomorphism given by sending M to $\langle t\rangle M$(this map is clearly $C_G(H)$-equivariant). Let $C := C_{\mathrm{PSL}_2(p)}(t)$. We first observe C is a $C_G(H)$-invariant element, then again according to the proof in [Sha03, Lemma 3.9] we know the complement of C in $\widehat{L^*}$ is empty, therefore L^* is $C_G(H)$-contractible by 2.6.27. Finally apply Lemma 2.6.33 we know L is $C_G(H)$-contractible, so $(\langle t\rangle, \mathrm{PGL}_2(p))$ is also $C_G(H)$-contractible. □

In the end let's state and prove an easy result to partially explain the numerical result of equivariant Euler characteristics of all subgroup complex of alternating groups $\widetilde{\chi}_r(\mathcal{S}^*_{A_n}, A_n)$.

Theorem 2.6.36. *For $n \geq 5$, $\widetilde{\chi}_r(\mathcal{S}^*_{A_n}, A_n) = \frac{\widetilde{\chi}(\mathcal{S}^*_{A_n})}{|A_n|}$.*

Proof. We first claim when $n \geq 5$ then for any $\lambda \in A_n$ which is not equal to identity the centralizer $C_{\mathcal{S}^*_{A_n}}(\lambda)$ is $C_{A_n}(\lambda)$-contractible. If $\lambda \neq id$ then for any proper non-trivial subgroup $H \leq A_n$ we observe that $\langle\lambda\rangle H$ is not equal to A_n. If it is, i.e $\langle\lambda\rangle H = A_n$. However H is a normal subgroup of $\langle\lambda\rangle H$, which means H is also a normal subgroup of A_n. It is well-known that when $n \geq 5$ A_n is a simple group. That is a contradiction. Hence $\langle\lambda\rangle H \neq A_n$, in other words $\langle\lambda\rangle$ is a contractor for $C_{\mathcal{S}^*_{A_n}}(\lambda)$. Then according to $C_{\mathcal{S}^*_{A_n}}(\lambda)$ is $C_{A_n}(\lambda)$-contractible. According to Lemma 2.1.25 we know

$$\widetilde{\chi}_r(\mathcal{S}^*_{A_n}, A_n) = \widetilde{\chi}_{r-1}(\mathcal{S}^*_{A_n}, A_n) = \cdots = \widetilde{\chi}_1(\mathcal{S}^*_{A_n}, A_n) = \frac{\widetilde{\chi}(\mathcal{S}^*_{A_n})}{|A_n|}$$

□

Corollary 2.6.37. *For $n \geq 5$ we always have $|A_n| \mid \widetilde{\chi}(\mathcal{S}^*_{A_n})$.*

Proof. According to Lemma 2.1.25 we know all ordinary equivariant Euler characteristics are valued in integers. Then for $n \geq 5$ we know $\frac{\widetilde{\chi}(\mathcal{S}^*_{A_n})}{|A_n|}$ is an integer, in other words $|A_n| \mid \widetilde{\chi}(\mathcal{S}^*_{A_n})$. □

Chapter 3

Spaces of Trees and Complexes of Not 2 Connected Graphs

3.1 Introduction and Main Result

This chapter is a joint-work with Gregory Arone in Stockholm University. In this chapter we study the relation between the spaces of trees and complexes of not 2-connected graphs. The spaces of trees first appear in the work in [BHV01] and Sarah Whitehouse's thesis[Whi94] for studying Γ-homology and E_∞-obstruction theory. She and Alan Robinson determined the homology of the space of trees as a Σ_n-module[RW96, Theorem 3.1]

Theorem 3.1.1. *Let T_{n-1} be the space of fully grown $(n-1)$-trees. Then the character of complex representation of Σ_n on the homology $H_{n-4}(T_{n-1}, \mathbb{C})$ is*

$$\epsilon \cdot (lie_{n-1} \uparrow_{\Sigma_{n-1}}^{\Sigma_n} - lie_n) \tag{3.1.2}$$

where ϵ is the character of the sign representation of Σ_n and lie_n is the character of the complex Lie representation Lie_n[Aro15, Section 3].

The general notion of complex of (not) i-connected graphs was first introduced by Vassilie in [Vas14][Vas99] during his research on knot invariants. In [BBL+99, Theorem 4.1] the authors determined the homology of complex of not 2-connected graphs : Δ_n^2 as a $\mathbb{C}[\Sigma_n]$-module.

Theorem 3.1.3. *The character of complex representation of Σ_n on the homology $H_{2n-5}(\Delta_n^2, \mathbb{C})$ is*

$$lie_{n-1} \uparrow_{\Sigma_{n-1}}^{\Sigma_n} - lie_n \tag{3.1.4}$$

So we observe that these two modules $H_{n-4}(T_{n-1}), H_{2n-5}(\Delta_n^2)$ coincide up to a sign representation of the symmetric group Σ_n and a degree shift. The following result is the main theorem of this chapter which asserts that this is not a coincidence, in other words, these two spaces T_{n-1} and Δ_n^2 are equivariant homotopy equivalent in some sense.

Theorem 3.1.5. *The space of trees on n vertices : Q_{n-1} is Σ_n-equivariant homotopy equivalent to the double suspension of the complex of not 2-connected graphs on n vertices :$\Sigma S|\Delta_n^2|$. Where Σ means the reduced suspension and S means the non-reduced suspension.*

To prove these two spaces are equivariant homotopy equivalent we introduce the third space Y which is called the total cofiber.

Definition 3.1.6. Let $\widehat{\mathcal{C}_n}$ be the category of connected acyclic hypergraphs on n vertices and $F\colon \widehat{\mathcal{C}_n} \to \mathbf{Top}_*$ be a functor which sends any hyper-graph $H = (E_1, \dots, E_k)$ to the pointed space $P_{E_1} \wedge \cdots \wedge P_{E_k}$. Then we define a new topological space called the total cofiber associated to this functor F.

$$Y := \operatorname{hocofib}(\operatorname*{hocolim}_{\mathcal{C}_n} F \to P_n)$$

where $\operatorname*{hocolim}_{\mathcal{C}_n} F$ is a pointed homotopy colimit which is taken over the category $\mathcal{C}_n := \widehat{\mathcal{C}_n} - \{\hat{1}\}$, the element $\hat{1}$ is the final object in $\widehat{\mathcal{C}_n}$ which consists only of one hyper-edge on whole vertices. We will introduce the category $\widehat{\mathcal{C}_n}$ in section 3.4 and the functor F in section 3.5.

Actually we can construct a zigzag map as follows:

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ Q_{n-1} & & \Sigma S|\Delta_n^2| \end{array} \tag{3.1.7}$$

The following two theorems state that two morphisms in the zigzag map are both equivariant homotopy equivalences. Therefore by two out of three properties of homotopy equivalences the two spaces Q_{n-1} and $\Sigma S|\Delta_n^2|$ are Σ_n-equivariant homotopy equivalent. This is exactly the content of our main result Theorem 3.1.5.

Theorem 3.1.8. *There is an induced map $f_1 : Y \to Q_{n-1}$ which is a Σ_n-equivariant homotopy equivalence.*

Since the image $P_{E_1} \wedge \cdots \wedge P_{E_k}$ of functor F on each hyper-graph $H = (E_1, \dots, E_k)$ could be expressed as $\Sigma\partial|P(E_1) \times P(E_2) \times \cdots \times P(E_k)|$ and the reduced suspension functor commutes with homotopy colimits, we know the total cofiber admits a de-suspension X.

Theorem 3.1.9. *There is an induced map $f_2 : X \to S|\Delta_n^2|$ which is a Σ_n-equivariant homotopy equivalence.*

This chapter is organized as follows

- In section 3.2 we introduce the complex of not 2-connected graphs Δ_n^2 and its homology.
- In section 3.3 we introduce the classical definition of spaces of trees T_{n-1} and how do we modify it for our purpose.
- In section 3.4 we introduce the notion of the category of connected acyclic hyper-graphs: $\widehat{\mathcal{C}_n}$, construct the functor $F\colon\ :\widehat{\mathcal{C}_n} \to \mathbf{Top}_*$ and the associated total cofiber Y of it.
- In section 3.5 we prove the Theorem 3.1.8.
- In section 3.6 we prove the Theorem 3.1.9.

3.2 Complexes of Not 2-Connected Graphs

Complexes of not i-connected graphs and their homology arose in the study of Vassiliev on knot invariants[Vas14][Vas99]. This section we give a concise introduction to this object and state some basic properties, for more details people can refer to [BBL+99].

Definition 3.2.1. A graph G contains a non-empty vertex set $V(G)$ and a non-empty edge set $E(G)$ which is a subset of the set of unordered pairs (x, y) where $x, y \in V(G)$ and we require $x \neq y$. The adjacent edges of a vertex $x \in V(G)$ are those edges with the form $(x, y) \in E(G)$ where $y \in V(G)$.

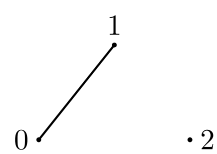
Definition 3.2.2. A graph G is called not connected if there exist two vertices v, v' of G such that there is no path of edges from v to v'. Moreover, a graph is called not i-connected for a positive integer i which is less than $|V(G)|$, if there exist j vertices $v_1 \dots v_j \in V(G)$, $j < i$, such that the graph G' obtained from G by deleting these vertices and their adjacent edges is not connected.

In the following discussion we use $[n-1]_+ = \{0, 1, \dots, n-1\}$ to denote the standard set of n vertices.

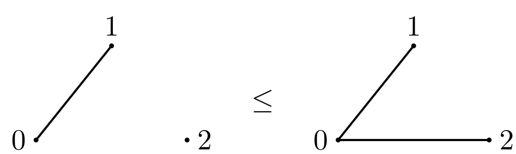
Definition 3.2.3. We say a graph G is complete if for any two vertices v_i, v_j in the vertex set V of G there is an edge connecting these two vertices. Moreover a completion of a graph G is the complete graph on the vertex set of G.

Definition 3.2.4. Δ_n^i denotes the poset of not i-connected graphs on $[n-1]_+$ with order relation when $G \leq G'$ if $E(G) \subset E(G')$.

Example 3.2.5. 1. Here is an easy example of not 1-connected graphs with 3 vertices $\{0, 1, 2\}$ and edge set $\{\{1, 2\}\}$:



2. Here are two examples of not 2-connected graphs with order relations where the edge set of second graph is $\{\{0, 1\}, \{0, 2\}\}$:



In the beginning, let's investigate some properties of Δ_n^1. We observe that Δ_n^1 is homotopy equivalent to the partition complex Π_n(see Definition 3.3.6) via the homotopy equivalence φ by sending each connected component in a graph to a block with elements of vertices of this component(Quillen's fiber lemma).

Theorem 3.2.6. *[BBL+99, Proposition 2.1] The homotopy type of Δ_n^1 is a wedge of $(n-1)!$ spheres with same dimension of $n-3$.*

Moreover, the symmetric group Σ_n could be viewed as the permutation group on $[n-1]_+$, it induces a natural Σ_n-action on the complexes of not i-connected graphs Δ_n^i. Their homology are not just abelian groups but also $\mathbb{C}[\Sigma_n]$-modules.

Definition 3.2.7. [Aro15, Section 3]

1. Given a finite set $V = \{x_0, x_1, \ldots, x_{n-1}\}$, let $\mathcal{L}[x_0, x_1, \cdots, x_{n-1}]$ be the free Lie algebra over $\mathbb{C}$ generated by V. Elements in $\mathcal{L}[x_0, x_1, \cdots, x_{n-1}]$ could be expressed by a linear combination of parenthesized monomials.

2. For each monomial in $\mathcal{L}[x_0, x_1, \cdots, x_{n-1}]$ its degree is a sequence of numbers $(d_0, \ldots, d_{n-1})$ where d_i means the number of copies of x_i appears in this monomial. Let Lie_n be the submodule of $\mathcal{L}[x_0, x_1, \cdots, x_{n-1}]$ generated by monomials of degree $(1, \cdots, 1)$. There is a natural Σ_n-action on Lie_n induced by Σ_n action on the finite set V. We denote lie_n as the character of Lie_n as a complex representation of Σ_n.

Theorem 3.2.8. *[Rob04, Theorem 4.1] There is an isomorphism for two Σ_n-modules:*

$$\widetilde{H}_*(\Delta_n^1; \mathbb{C}) \simeq \epsilon \otimes Lie_n^*$$

Where ϵ is the complex sign representation of Σ_n and Lie_n^* is the linear dual of Lie_n. Equivalently we have an isomorphism of Σ_n-modules on cohomology level

$$\widetilde{H}^*(\Delta_n^1; \mathbb{C}) \simeq \epsilon \otimes Lie_n$$

Now we move on to the main object this chapter studies: Δ_n^2, the complexes of not 2 connected graphs. Its non-equivariant homotopy type and its homology as a complex representation of Σ_n was studied by E.Babson and other co-authors in[BBL+99]. We list their main results here. The first result is about the non-equivariant homotopy type of Δ_n^2[BBL+99, Theorem 3.1].

Theorem 3.2.9. *When $n \geq 3$, the homotopy type of Δ_n^2 is a wedge of $(n-2)!$ of spheres of same dimension of $2n-5$.*

Similarly, Δ_n^2 also enjoys a natural Σ_n-action. In their paper[BBL+99, Theorem 4.1] they determined the character ω_n^2 of complex representation of Σ_n on $\widetilde{H}_*(\Delta_n^2; \mathbb{C})$. We let Σ_{n-1} be the stabilizer of $0 \in [n-1]_+$ under the Σ_n-action.

Theorem 3.2.10. *The character ω_n^2 is given by*

$$\omega_n^2 = lie_{n-1} \uparrow_{\Sigma_{n-1}}^{\Sigma_n} - lie_n$$

Where lie_n is the character of linear Σ_n-representation Lie_n, and $lie_{n-1} \uparrow_{\Sigma_{n-1}}^{\Sigma_n}$ means the induced character of lie_{n-1}.

3.3 Spaces of Trees

In this section we first recall the notion of trees used by A.Robinson and S.Whitehouse[Whi94]. Later we show how we modify it into a new space of trees for our purpose.

Definition 3.3.1. A (reduced) **tree** is a compact contractible 1-dimensional polyhedron X with its unique coarsest triangulation in which no vertex lies on exactly 2 edges. A vertex that incidents more than 2 edges is called an **internal** vertex and a vertex which only meets one edge is called **external** vertex. An edge is called **internal** if both its vertices are internal vertices and an edge in which one vertex of it is external is called **leaf**.

Remark. With the coarsest triangulation on a tree X, we can view it as a graph defined in the Definition 3.2.1 with vertex set including all vertices in the triangulation and edge set including all edges in the triangulation.

Definition 3.3.2. A metric tree is a tree equipped with length (possibly 0) for each edge. Moreover we say an n-tree with $n+1$ external vertices is a metric tree when:

1. The length l_e of every internal edge e is greater than 0 and less or equal to 1, and the length of each leaf is required to be exactly 1.

2. The external vertices are labeled by the set $[n]_+$.

Two n-trees are called isomorphic if there is an isometry between them which preserves the labels. A n-tree is called **fully grown** if there is an internal edge with length 1.

Remark. We say a tree is a rooted tree if we pick a distinguished external edge of this tree and we call it as a root.

Definition 3.3.3. [KMM04, Definition 2.9] We say a space $X \subset \mathbb{R}^k$ is a cubical complex if it is a union of elementary cubes such that the intersection of any two elementary cubes is either empty or a common face of these two elementary cubes where elementary cubes are those spaces homeomorphic to d-dimension cubes for $d \leq k$.

Let $\widetilde{T_n}$ be the set of isomorphic classes of n-trees. From now on when we say a tree we always mean the isomorphic classes of this tree.

Proposition 3.3.4. *[RW96, Page 246] $\widetilde{T_n}$ is a cubical complex where two trees belong to the same open cube if there is a homeomorphism between them which preserves labels and edges of length 1. The coordinates of each point inside the cube are determined by the length of the internal edges.*

Remark. We observe that any n-tree could be shrunk along its internal edges to a tree without internal edges. Hence $\widetilde{T_n}$ is a cone topologically with an apex by the n-tree with no internal edges.

Definition 3.3.5. Let T_n be the base of $\widetilde{T_n}$ which is the subspace consisting of fully grown n-trees. Moreover T_n and $\widetilde{T_n}$ both carry a natural Σ_{n+1}-action induced by actions on $[n]_+$.

We have introduced what is the partition poset $P(n)$ in Definition 2.6.30 of chapter 2. For self-contained of this chapter let's recall what's a partition and what is a partition poset or complex on a given finite set E here again.

Definition 3.3.6.
1. A **partition** π on a given finite set E is an equivalence relation $\sim_\pi$ on E. For a block in π it is an equivalence class of $\sim_\pi$. Let $P(E)$ denote the set of all partitions on E.

2. For two partitions π_1, π_2 on E we write $\pi_1 \leq \pi_2$ if each block in π_1 is contained in a block in π_2. $P(E)$ is a poset equipped with this order and its associated simplicial complex is called partition complex which is also denoted as $P(E)$. Specially if the cardinality of the set E is n then $P(E)$ could be denoted simply as $P(n)$.

3. Let $\Pi_n = P(n) - \{\hat{0}, \hat{1}\}$ where $\hat{0}, \hat{1}$ are two extreme points in $P(n)$. We will call both of them partition complexes if there is no confusion.

Theorem 3.3.7. *The geometric realization of the partition poset Π_n is Σ_n-equivariant homeomorphic to the space T_n of fully grown n-trees.*

Proof. See [Rob04, Proposition 2.7]. □

Corollary 3.3.8. *The quotient space $\widetilde{T_n}/T_n$ is homotopy equivalent to a wedge sum of $(n-1)!$ copies of S^{n-2}. And via the homeomorphism in Theorem 3.3.7 between $|\Pi_n|$ and T_n we can transform the natural Σ_{n+1}-action of T_n to $|\Pi_n|$. In other words, we have a model for partition complexes with richer symmetries.*

Proof. Since $\widetilde{T_n}$ is contractible, $\widetilde{T_n}/T_n$ is homotopy equivalent to the suspension of T_n: ΣT_n. Then according to the above theorem and the well-known result of the homotopy type of the partition complex[Sha03, Proposition 3.1] i.e. $|\Pi_n| \simeq \bigvee_{(n-1)!} S^{n-3}$ we know $\widetilde{T_n}/T_n \simeq \bigvee_{(n-1)!} S^{n-2}$ □

Theorem 3.3.9. *[RW96, Theorem 3.1] The homology $H_{n-2}(\widetilde{T_n}/T_n, \mathbb{C})$ as a Σ_{n+1}-module has character:*

$$\epsilon \cdot (lie_n \uparrow_{\Sigma_n}^{\Sigma_{n+1}} - lie_{n+1})$$

where ϵ means the character of the sign representation of Σ_{n+1}.

From now let's modify the space of trees such that leaves can have variable length.

Definition 3.3.10. Let $\widetilde{W_n}$ be defined in the same way as $\widetilde{T_n}$ except that all edges including leaves can have length between 0 and 1. In other words elements in $\widetilde{W_n}$ can have length between 0 and 1 for each edge. Moreover let W_n be the subspace of $\widetilde{W_n}$ consisting of trees that are either fully grown(i.e. at least one edge including leaves has length 1) or have at least one leaf with length 0. Let Q_n be the quotient space $\widetilde{W_n}/W_n$ with base point W_n. Like the space T_n, the space Q_n also carries a natural Σ_{n+1}-action.

Remark. From now on, elements of $\widetilde{W_n}$ are also called n-trees. In the rest of this chapter when we say a tree we always refer to a tree in $\widetilde{W_n}$ or its quotient space Q_n. Furthermore we call Q_n the space of trees on labels $[n]_+$. In general $Q_E := \widetilde{W_E}/W_E$ where $\widetilde{W_E}, W_E$ are the spaces of the trees like $\widetilde{W_n}, W_n$ respectively but with label set E. In particular $Q_n = Q_{[n]_+}$.

Proposition 3.3.11. *The space Q_n is Σ_{n+1}-equivariant homeomorphic to the space $\widetilde{T_n}/T_n \wedge S^{n+1}$. The Σ_{n+1} action on S^{n+1} is just the permutation of coordinates if we view S^{n+1} as $n+1$ copies of the smash product: $S^1 \wedge \cdots \wedge S^1$.*

Proof. We construct a Σ_{n+1}-equivariant map here $\tilde{h} : \widetilde{W_n} \to \widetilde{T_n} \times I^{n+1}$ by sending a given metric tree $T \in \widetilde{W_n}$ to a new metric tree $\tilde{T} \in \widetilde{T_n}$ which is just setting all length of leaves in T to be 1 and a $(n+1)$-tuple of numbers which represents of the length of all $n+1$ leaves in T. It is clearly a homeomorphism between two spaces. And we observe that by the definition of the subspace $W_n \subset \widetilde{W_n}$ the image of this subspace under the map $\tilde{h}$ is exactly the subspace $\widetilde{T_n} \times \partial I^{n+1} \coprod_{T_n \times \partial I^{n+1}} T_n \times I^{n+1}$, moreover the pre-image of this subspace is also exactly W_n. So the homeomorphism $\tilde{h}$ will pass to a homeomorphism:

$$h : \widetilde{W_n}/W_n \to \widetilde{T_n} \times I^{n+1} / \widetilde{T_n} \times \partial I^{n+1} \coprod_{T_n \times \partial I^{n+1}} T_n \times I^{n+1} \simeq \widetilde{T_n}/T_n \wedge S^{n+1}$$

on the other hand the space Q_n is Σ_{n+1}-equivariant homeomorphic to the space $\widetilde{T_n}/T_n \wedge S^{n+1}$. □

Remark. By Künneth theorem[Hat02, Theorem 3B.6] for homology we know:

$$H_{2n-1}(Q_n) \simeq H_{n-2}(\widetilde{T_n}/T_n) \otimes_{\mathbb{C}} H_{n+1}(S^{n+1})$$

This is an isomorphism of Σ_{n+1}-modules. On the right hand side the homology $H_{n+1}(S^{n+1})$ as a Σ_{n+1}-module is just the sign representation of Σ_{n+1}. In other words the character of $H_{2n-1}(Q_n)$ as a Σ_{n+1}-module is $lie_n \uparrow_{\Sigma_n}^{\Sigma_{n+1}} - lie_{n+1}$ which coincides with the character of $H_{2n-1}(\Sigma S|\Delta^2_{n+1}|)$.

Corollary 3.3.12. *It follows from the Proposition 3.3.11 that Q_n is homotopy equivalent to a wedge sum of $(n-1)!$ copies of S^{2n-1}.*

Definition 3.3.13. Given an n-tree T. For some point on this tree, the distance from it to the set of leaves is the minimal distance from this point to a leaf. Then the radius $r(T)$ of this n-tree T is defined to be the maximal distance from a point to the set of leaves.

Proposition 3.3.14. *There is a Σ_{n+1}-equivariant homeomorphism from the space $\widetilde{W_n}$ to the space of trees with radius at most 1. Furthermore this homeomorphism take the subspace W_n to the space of trees that either have radius exactly one or has at least one leaf of length zero.*

Proof. Given a metric tree T in $\widetilde{W_n}$ we first denote by $\max T$ the maximal length among all edges of T. Then construct a new metric tree $\widetilde{T}$ by adjusting the length of each edge by multiplying $\max T$ and dividing $r(T)$. Then it is clear that the radius $r(\widetilde{T})$ of $\widetilde{T}$ is just $\max T$ which is at most 1 and this map is of course a bijective map, then since both sides are compact Hausdorff spaces we know the map we constructed is an equivariant homeomorphism. □

Definition 3.3.15. According to Proposition 3.3.14 we can re-define Q_n as the space of all n-trees of radius at most one quotient out by the subspace of trees either having radius exactly one or have at least one leaf with length zero.

Definition 3.3.16. An n-tree is **centered** if it has a point whose distance from every leaf is exactly the radius of the tree. Note that if such point exists than it is unique and we call it as the center of this centered tree. And we also call this point as the center associated with external vertices. Let P_{n+1} be the subspace of Q_n consisting of all centered trees and the base point of P_{n+1} inherited from the base point of Q_n. In general given a finite set E, P_E is the space of all centered trees with radius at most one with leaves labeled by E. In particular $P_{n+1} = P_{[n]_+}$.

Remark. We should note here that the center of a centered tree is a point determined by this tree but usually not one of vertices of this tree.

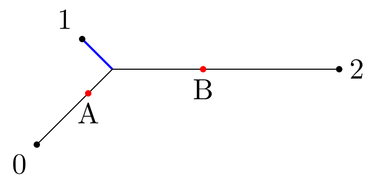
The following lemma is a key result in this chapter which will be used in section 3.5.

Lemma 3.3.17. *Any tree $T \in \widetilde{W_n}$ is a union of several centered trees inside T.*

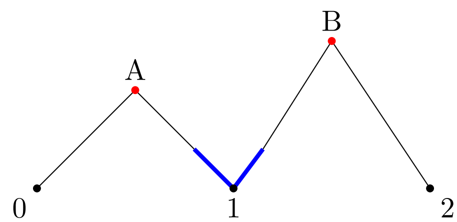
Proof. Given any tree $T \in \widetilde{W_n}$, we have defined the distance of a given point in T to the set of leaves of T(see Definition 3.3.13). Therefor we have a distance function $d : T \to I$ where I denotes the standard interval $[0,1]$. Then we take all locally maximal points of this distance function $x_1, \ldots, x_k$. We observe that all locally maximums lie in different edges, i.e. not two different locally maximums lie in a same edge in tree T. For each locally maximal point we can take a centered tree inside T by all leaves with same minimal distance to the locally maximum and the inner edges which is contained in the path connecting the locally maximum and the leaves. So the tree T is of course the union of those centered trees with centers $x_1, \ldots, x_k$. □

Definition 3.3.18. We say a decomposition of the tree $T \in \widetilde{W_n}$ is a set of locally maximal points, the centered trees associated with those points as centers and the labels sets of these centered trees. Moreover we say the intersection parts of this decomposition is the intersections of those centered trees.

Example 3.3.19. The following tree has 2 locally maximal points which are decorated as red points:



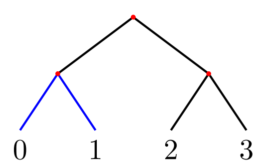
We observe there are 2 centered trees inside this tree with A, B as centers respectively and moreover the blue part is the intersection part of the decomposition of this tree. And the decomposition data of the above drawing is:



The two labels sets in above drawing are $E_1 = \{0, 1\}$ and $E_2 = \{1, 2\}$.

Definition 3.3.20. We call all vertices and the center of a centered tree as **branching** points of this tree. A **branching tree** associated to a branching point consists of the all subpaths connecting this branching point and external vertices which would not across the center(We view the centered tree itself also a branching tree).

Example 3.3.21. The following is a centered tree with red points as branching points:



And the blue part is one of branching tree inside the centered tree.

Proposition 3.3.22. *The space of centered trees $P_n \subset Q_{n-1}$ is equivariant homeomorphic to the reduced suspension of boundary of the partition complex: $\Sigma\partial|P(n)|$, here boundary of $|P(n)|$ is the geometric realization of the simplicial complex consisting of chains in $P(n)$ which cannot not contain the minimal and maximal elements simultaneously.*

Proof. We mainly follow the argument in [Rob04, Proposition 2.7] by constructing an explicit homeomorphism φ between P_n and $|P(n)|/\partial|P(n)|$. Since $|P(n)|$ is Σ_n-equivariant contractible we know $|P(n)|/\partial|P(n)| \simeq_{\Sigma_n} \Sigma\partial|P(n)|$, here $\sim_{\Sigma_n}$ means Σ_n-equivariant homotopy equivalence. By definition an element x in P_n is a centered tree with n-leaves labeled by $[n-1]_+$. And with the help of this center we can view this centered tree as a rooted tree with a root attached to this center with length $1 - r(x)$, where $0 \leq r(x) \leq 1$ is the radius of this tree x. From the starting point of this root there is a unique path connecting this root and a leaf i for every $i \in [n-1]_+$ and since x itself is a metric tree we can parametrize this path by a unit speed, in other words we have a continuous function $\gamma_i : [0, 1] \to x$(to be more precise here: x should be a tree in the isomorphic class and itself be viewed as a metric space) and we set this function is constant at the leaf i after it reaches the leaf i. The coordinate in interval $[0, 1]$ is called time.

According to the definition of geometric realization of a poset:

$$|P(n)| = \coprod_{d\geq 0} P(n)_d \times \Delta^d / \sim$$

where $\sim$ is generated by face and degenaracy relations. $P(n)_d$ is the set of chains of length $d+1$ in $P(n)$ and here we use standard simplex as the model for Δ^d i.e.

$$\Delta^d := \{(t_0, t_1, \dots, t_d) \in \mathbb{R}^{d+1} | \sum_{i=0}^{d} t_i = 1\}$$

The coordinates of point in this standard simplex Δ^d are called barycentric coordinates.

Given a centered metric tree x. The tree determines finer and finer partitions when time increases. More precisely for each time $t \in [0,1]$ we define the partition $\Pi_x(t)$ to be the partition on $[n-1]_+$ defined by the following equivalence relations:

$$i \sim j \Longleftrightarrow \gamma_i(t) = \gamma_j(t)$$

This gives the chain of partitions of the image $\varphi(x)$. And the barycentric coordinates of $\varphi(x)$ with respect to any partition π in the chain of partitions of $\varphi(x)$ is the length of the interval $\{t \in [0,1] | \Pi_x(t) = \pi\}$. When the metric tree x has radius 1 or has a leaf with length 0(all these points are collapsed to the base point in P_n) then the image $\varphi(x)$ has barycentric coordinates 0 with respect to minimal or maximal partitions, which means $\varphi(x) \in \partial|P(n)|$. In other words φ is a base point preserving map. And this map is clearly an equivariant map. The reason φ is a homeomorphism is simply because any metric tree x is completely determined by its associated family of partitions in $P(n)$.

□

Since this section contains many definition and notations, we summarize those notations here :

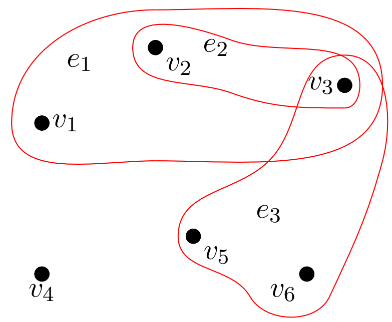
- $\widetilde{T_n}$: Whitehouse's space of trees labeled by $[n]_+ = \{0, 1, \dots, n\}$; T_n: subspace of $\widetilde{T_n}$ which are fully grown trees.
- $\widetilde{W_n}$: space of generalized trees labeled by $[n]_+$; W_n: subspace of $\widetilde{W_n}$ containing trees which are either fully grown or having one leaf with length 0.
- Q_n: the quotient space $\widetilde{W_n}/W_n$; Q_E: the quotient space $\widetilde{W_E}/W_E$.
- P_n: the subspace of Q_{n-1} of all centered trees; P_E: the subspace of Q_E all centered trees labeled by E.
- $P(E)$: the poset of all partition on a given finite set E.

3.4 Acyclic Hypergraphs and a Total Cofiber Construction

To construct the total cofiber Y, we need first introduce the notion of hypergraph and the connectness and acyclicity of it. Roughly speaking hyper graphs are generalizations of graph which edges can join not just two points.

Definition 3.4.1. A hypergraph is a pair $H = (V, E)$ where V is a non empty set of vertices and E is a non-empty subset of $P(V) - \{\emptyset\}$, where $P(V)$ is the power set of V. Elements in E are called hyperedges. We say a hypergraph $H = (V, E)$ is trivial if its edge set E consists of just one element which is just the vertex set V, we call other hypergraphs as proper hypergraphs.

Example 3.4.2. Here is an example of a hypergraph with vertex set $\{v_1, v_2, \ldots, v_6\}$ and edge set $\{e_1, e_2, e_3\}$:



Definition 3.4.3. A hypergraph is called connected if for any two elements $v, v' \in V$ there is a collection of hyperedges $E_1, \ldots, E_m$ with $E_i \cap E_{i+1} \neq \emptyset$ such that E_1 contains v and E_m contains v'.

There are several not equivalent notions of acyclicity of hypergraphs like α-acyclic[TY84], β-acyclic[Fag83] and so on[BFMY83][CJLT12]. In this chapter we use the notion of acyclicity given by Claude Berg[Ber85, Page 391]. Let's first recall the definition of acyclic graphs[Ber85, Page 12].

Definition 3.4.4. In a graph $G = (V, E)$ a loop is a sequence $\mu = (v_1, E_1, v_2, E_2, \ldots, v_k, E_k, v_1)$ such that:

1. All edges E_i and vertices v_j are distinct,
2. $v_i, v_{i+1} \in E_i$ for each $i = 1, 2, \ldots, k$ (here we assume that $v_{k+1} = v_1$).

Then we call a graph without any loops as an acyclic graph.

Remark. It's clear that any tree is a connected and acyclic graph, since otherwise the tree is not a contractible space. On the other hand if we view a connected acyclic a metric space then it is a tree since it is contractible.

Definition 3.4.5. Given a hypergraph $H = (V, E)$, a path of length k in this hypergraph is defined to be a sequence $(v_1, E_1, v_2, E_2, \ldots, v_k, E_k, v_{k+1})$ such that

1. $v_1, v_2, \ldots, v_k$ are all distinct vertices,
2. $E_1, E_2, \ldots, E_k$ are all distinct edges,
3. $v_i, v_{i+1} \in E_i$ for all $1 \leq i \leq k$.

When $k \geq 1$ and $v_1 = v_{k+1}$ then we say this path a loop in H.

Definition 3.4.6. We say a hypergraph $H = (V, E)$ is acyclic if there is no loop in H.

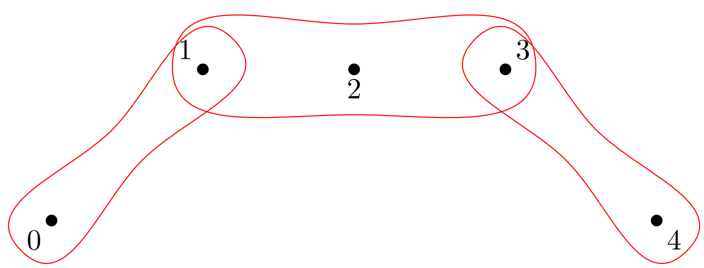
Remark. It is easy to observe that for any acyclic hypergraph the intersection of any two hyperedges are at most one element.

Definition 3.4.7. Given two hypergraphs $H_1 = (V, E), H_2 = (V, E')$. We say $H_1 \leq H_2$ if for any hyperedge $E_1 \in E$ there is a hyperedge $E_2 \in E'$ such that $E_1 \subset E_2$. Based on this order relation the set of all connected acyclic hypergraphs over the set $[n-1]_+$ forms a poset or a category and we denote this poset as $\widehat{\mathcal{C}_n}$. As for each element in $\widehat{\mathcal{C}_n}$ we simply use blocks for each hyperedge to denote it. For example 01/12/234 is an acyclic hyper graph with hyperedges $E_1 = \{0, 1\}, E_2 = \{1, 2\}, E_3 = \{2, 3, 4\}$.

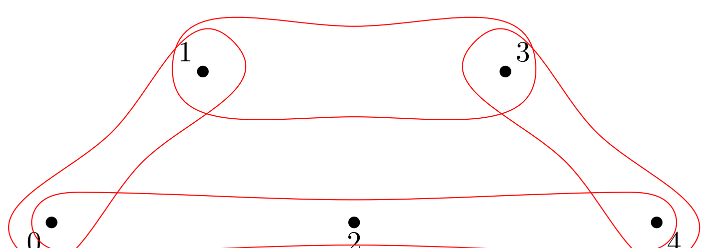
Proposition 3.4.8. *Given two connected acyclic hypergraphs $H_1 = (V, E), H_2 = (V, E')$ with $H_1 \leq H_2$, then for any hyperedge $E_2 \in E'$, then there is a collection of hyperedges $F_1, F_2, \ldots, F_k$ such that $\cup_{i=1}^{k} F_i = E_2$ and the new hypergraph H_{E_2} consisting of all vertices of E_2 as the vertex set and $\{F_1, F_2, \ldots, F_k\}$ as edge set is a connected acyclic hypergraphs.*

Proof. Take any two elements a_1, a_2 in E_2. Since H_1, H_2 share the same vertex set, a_1, a_2 are both vertices in H_1. According to our assumption that H_1 is a connected acyclic hypergraph we know there is a unique path $(a_1, F_1, b_1, F_2, \ldots, b_{m-1}, F_m, a_2)$ in H_1. Then we claim all hyperedges F_i in this sequence satisfying $F_i \subset E_2$. If this claim is true, then we collect all hyperedges $\{F_1, F_2, \ldots, F_k\}$ associated to the sequence of any two elements in E_2. It is clear that the new hypergraph H_{E_2} formed by these hyperedges $\{F_1, F_2, \ldots, F_k\}$ is connected and acyclic, moreover $\cup_{i=1}^{k} F_i = E_2$. Let's prove the truth of this claim. Since $H_1 \leq H_2$, for any hyperedge F_i there is a hyperedge F_i' such that $F_i \subset F_i'$. And there is no other hyperedge F_i'' with this property, since otherwise $F_i \subset F_i' \cap F_i''$ which consists of more than one element. But this is a contradiction since H_2 is an acyclic hypergraph. Therefore we have a sequence in H_2: $(a_1, F_1', b_1, F_2', \ldots, b_{m-1}, F_m', a_2)$. If all hyperedges in this sequence are distinct, then we can form a sequence $(a_2, E_2, a_1, F_1', b_1, F_2', \ldots, b_{m-1}, F_m', a_2)$. Since H_2 is acyclic then there is at least one hyperedge F_i such that $F_i = E_2$. We can find the least i such that $F_i = E_2$, then in this case we have a new sequence $(a_1, F_1', \ldots, b_{i-1}, E_2, a_1)$ which is a loop in H_2, so it is a contradiction. Hence there is at least two hyperedges $F_i' = F_j'$. If two consecutive edges F_s', F_{s+1}' are actually same in this sequence then we can just shorten this sequence by replacing F_s', b_s, F_{s+1}' by just F_{s+1}'. Without losing any generality we can assume that for all edges F_t' with $i < t < j$ then $F_t' \neq F_i'$. If F_i' and F_j' are not consecutive edges then after doing the shortening procedure we observe that the two hyperedges F_i', F_j' become consecutive since otherwise there is a loop $(b_i, F_{i+1}', \ldots, b_{j-1}, F_i', b_i)$. This is a contradiction. So we can continue to shorten the sequence to obtain a new sequence. And according to the previous argument we imply that all hyperedges $F_1', \ldots, F_m'$ are same and it is equal to E_2 since otherwise $E_2 \cap F_1'$ consists more than one element. In other words $F_i \subset E_2$ for any i. □

Example 3.4.9. 1. This is an example of connected acyclic hypergraph on vertices $\{0, 1, \ldots, 4\}$



2. This is an example of connected but not acyclic hypergraph



We are going to introduce a homotopy colimits model for our later use. The homotopy colimits model we choose is the so called Bousfield-Kan model[BK72]. And in the rest of this chapter all homotopy colimits we take is based on this model.

Now let **Top** be the category of compactly generated spaces and $\mathbf{Top}_*$ be the category of pointed compactly generated spaces.

Definition 3.4.10. A simplicial based space is a functor $X_\bullet : \Delta^{op} \to \mathbf{Top}_*$. Like simplicial space we can define the geometric realization of it:

$$|X_\bullet| := \left(\coprod_n X_n \times \Delta^n \right) / \sim \tag{3.4.11}$$

where the equivalence relation $\sim$ is generated by the following relations:

$$(d_i(x), (t_1, \ldots, t_{n-1})) \sim (x, d^i(t_1, \ldots, t_{n-1}))$$
$$(s_j(x), (t_1, \ldots, t_n)) \sim (x, s^j(t_1, \ldots, t_n))$$

where $d^i : \Delta^{n-1} \to \Delta^n; s^j : \Delta^n \to \Delta^{n-1}$ are face and degeneracy maps. Because the existence of base point on each level of based simplicial space the subspace

$$\coprod_n * \times \Delta^n / \sim \tag{3.4.12}$$

will be collapsed to one point, so we can rewrite the geometric realization as

$$|X_\bullet| = \left(\bigvee_n X_n \wedge \Delta^n_+ \right) / \sim \tag{3.4.13}$$

Definition 3.4.14. Consider a functor $G \colon \mathcal{D} \to \mathbf{Top}_*$ where $\mathcal{D}$ is a small category. We can equip a based simplicial space $X^G_\bullet$ of G defined on n-th level as

$$X^G_n := \bigvee_{d_0 \to d_1 \to \cdots \to d_n} G(c_0)$$

As for any morphism $\phi : [m]_+ \to [n]_+$ in Δ the induced map $\phi^* : X^G_n \to X^G_m$ is defined on the component indexed by a chain $d_0 \to d_1 \to \cdots \to d_n$ by $G(d_0 \to d_1 \to \cdots \to d_{\phi(0)}) : G(d_0) \to G(d_{\phi(0)})$ where the target is in the component indexed by the chain $d_{\phi(0)} \to \cdots \to d_{\phi(m)}$. Then we define the homotopy colimits of G as the geometric realization of $X^G_\bullet$, i.e. $\operatorname{hocolim}_{\mathcal{D}} G := |X^G_\bullet|$.

We have introduced what is the total cofiber of the functor F in Definition 3.1.6, let's recall it here with another form.

Definition 3.4.15. If $\widehat{\mathcal{D}}$ is a poset/category with a final object $\hat{1}$. And $G : \widehat{\mathcal{D}} \to \mathbf{Top}_*$ is a functor, then we say a total cofiber construction associated to the functor G is as follows

$$\operatorname{Tot}(G) := \operatorname*{hocolim}_{\widehat{\mathcal{D}}} G / \operatorname*{hocolim}_{\widehat{\mathcal{D}} \backslash \hat{1}} G \tag{3.4.16}$$

where $\widehat{\mathcal{D}} \backslash \hat{1}$ is the full subcategory which consists of all objects without the final object $\hat{1}$. It is clear $\operatorname{Tot}(G)$ is homotopy equivalent to $\operatorname{hocofib}(\operatorname*{hocolim}_{\widehat{\mathcal{D}} \backslash \hat{1}} G \to G(\hat{1}))$.

3.5 Proof of Theorem 3.1.8

In this section we construct the main functor $F : \widehat{\mathcal{C}_n} \to \mathbf{Top}_*$ of this chapter and prove the Theorem 3.1.8.

Proposition 3.5.1. *Let $H = (E_1, E_2, \dots, E_k)$ be a connected and acyclic hypergraph. For any point $x \in P_{E_1} \wedge \cdots \wedge P_{E_k}$ i.e. a sequence of centered trees with leaf sets labeled by $E_1, \dots, E_k$, if we take the union of these trees and identify the external vertices with same labels of those centered trees, then the graph we got is actually a tree.*

Proof. Since H is connected then it is clear the graph we got is a connected graph. Suppose the graph is not a tree, in other words there is a loop in this graph say

$$(a_1, A_1, c_1, B_1, a_2, \dots, a_m, A_m, c_m, B_m, a_1) \tag{3.5.2}$$

where a_i are distinct vertices in H, c_i are the centers of those shortest paths connecting a_i and a_{i+1} in a centered tree. A_i are those paths connecting c_i and the vertex a_i and B_i are those paths connecting c_i and a_{i+1}(here we ask $a_{m+1} = a_1$). Then we can get a sequence $(a_1, E_1, a_2, \dots, a_k, E_m, a_1)$ in the hypergraph H where E_i is the hyperedge which labels the centered tree with c_i belongs to. We first observe that it is not possible that all hyperedges E_i are equal, since otherwise the loop 3.5.2 is actually a loop in the centered tree labeled by E_i, but this is a contradiction. If all hyperedges are distinct then the sequence $(a_1, E_1, a_2, \dots, a_k, E_m, a_1)$ is just a loop in H which is a contradiction. So there is at least two edges $E_i = E_j$ and without losing any generality we can assume $j > i + 1$ i.e. these are not consecutive hyperedges since if they are just consecutive hyperedges then we can shorten the sequence $(a_1, E_1, a_2, \dots, a_k, E_m, a_1)$ by replacing E_i, a_{i+1}, E_{i+1} by just E_{i+1}. Similarly we can also assume that for any $i < t < j$, all hyperedges E_t satisfying $E_t \neq E_i$ and they are all distinct. Then in this case we have a loop $(a_i, E_{i+1}, a_{i+1}, \dots, a_j, E_i, a_i)$ in H but this is also a contradiction. Therefore there is no loop on the graph we got, in other words this graph is actually a tree. ☐

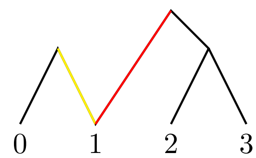
In the rest of this section, given a point x in $P_{E_1} \wedge \cdots \wedge P_{E_k}$ i.e. a k-tuple of centered trees with leaf sets $E_1, \dots, E_k$, we could also view this sequence of centered trees as a single tree by identifying external vertices of leaves with same labels.

Construction 3.5.3. Now let's construct the functor F we want. Let $\widehat{\mathcal{C}_n}$ be the poset of connected acyclic hypergraphs on n-vertices labeled by $[n-1]_+$ and $\mathcal{C}_n$ be the poset of proper connected acyclic hypergraphs on n-vertices with same labels. We construct a functor $F : \widehat{\mathcal{C}_n} \to \mathbf{Top}_*$ by sending a hypergraph $c \in \widehat{\mathcal{C}_n}$ where c has hyperedges $E_1, E_2, \dots, E_k$ to a space $P_{E_1} \wedge \cdots \wedge P_{E_k}$. According to the Proposition 3.4.8, for a morphism $c_0 < c_1$ in $\widehat{\mathcal{C}_n}$ any hyperedge E of c_1 is a union of several hyperedges in c_0 and those hyperedges also form an connected acyclic hypergraph with vertices of E. Since the image of the functor on a point in $F(c_0)$ is a sequence of centered tree labeled by hyperedges in c_1, it is enough to describe the map on a specific hyperedge of c_1. In other words, to describe what this functor does on morphisms it is basically enough to describe a map

$$P_{E_1} \wedge \cdots \wedge P_{E_k} \longrightarrow P_n \tag{3.5.4}$$

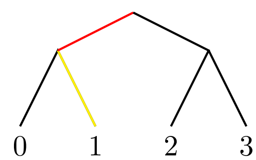
The map is defined as follows: A point in $P_{E_1} \wedge \cdots \wedge P_{E_k}$ is represented by a k-tuple of centered trees with leaf sets labeled by $E_1, \dots, E_k$. Then according to the Proposition 3.5.1 we can view it as a single tree with leaf set labeled by $[n-1]_+$. Moreover, whenever E_i, E_j share a vertex with same label, glue the two path starting at the shared vertex and leading to centers of i-th and j-th tree by a length preserving map(i.e. identify the shorter path with the sub-path of the longer one with the same length). The obtained tree is again a centered tree i.e. an element of P_n. We still need to check this is a well-defined map, if one of centered tree T_i with leaf set labeled by E_i has radius 1 or has a leaf with length 0. Then after gluing the new centered tree in P_n still has radius 1 or has a leaf with length 0, in other words the new tree is the base point in P_n.

Example 3.5.5. Let's consider a connected acyclic hypergraph $c_0 = (E_1, E_2)$ with $E_1 = \{0, 1\}$ and $E_2 = \{1, 2, 3\}$ and $c_1 = (E)$ with $E = \{0123\}$. The following tree is a point $x \in F(c_0) = P_{E_1} \wedge P_{E_2}$.



(3.5.6)

Then the image of this point x in morphism $F(c_0) = P_{E_1} \wedge P_{E_2} \to F(c_1) = P_4$ is:



i.e. we identify the yellow path to the sub-path of the red path with same length.

Construction 3.5.7. We are going to construct two natural maps $\tilde{f}$ from $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$ to $Q_{n-1} = \widetilde{W_{n-1}}/W_{n-1}$ and f from the total cofiber: $Y = \operatorname{Tot} F = \operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F / \operatorname{hocolim}_{\mathcal{C}_n} F$ to Q_{n-1}. Actually the map $\tilde{f}$ factor through the map f.

The construction is based on the Bousfield-Kan model of homotopy colimits of $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$ 3.4.14 i.e.

$$\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F = |X_\bullet^F| = \Big(\bigvee_{n\geq 0} \bigvee_{c_0 < c_1 < \cdots < c_n} F(c_0) \wedge \Delta_+^n\Big)/\sim \tag{3.5.8}$$

Here we view the simplex Δ^k as subspace in $\mathbb{R}^k$ defined as:

$$\Delta^k := \{(t_1, t_2, \ldots, t_k) | 1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 0\} \tag{3.5.9}$$

For $0 \leq i \leq k$ and $1 \leq j \leq k-1$ the face maps and degeneracy maps are defined as follows:

$$\begin{aligned} d^i :& \Delta^{k-1} \longrightarrow \Delta^k \\ (t_1, t_2, \ldots, t_{k-1}) &\mapsto (t_1, \ldots, t_i, t_i, \cdots, t_{k-1}) \\ s^j :& \Delta^k \longrightarrow \Delta^{k-1} \\ (t_1, t_2, \ldots, t_k) &\mapsto (t_1, \cdots, \widehat{t_{j+1}}, \ldots, t_k) \end{aligned} \tag{3.5.10}$$

where d^0, d^k are actually sending $(t_1, \ldots, t_{k-1})$ to $(1, t_1, \ldots, t_{k-1})$ and $(t_1, \ldots, t_{k-1}, 0)$ respectively, and $\widehat{t_j}$ means this coordinate disappears. Then we can express the equivalence relation in the formula 3.5.8 as:

$$\begin{aligned} (d_i(x), (t_1, \ldots, t_{k-1})) &\sim (x, d^i(t_1, \ldots, t_{k-1})) = (x, (t_1, \ldots, t_i, t_i, \cdots, t_{k-1})) \\ (s_j(x), (t_1, \ldots, t_k)) &\sim (x, s^j(t_1, \ldots, t_k)) = (x, (t_1, \cdots, \widehat{t_{j+1}}, \ldots, t_k)) \end{aligned} \tag{3.5.11}$$

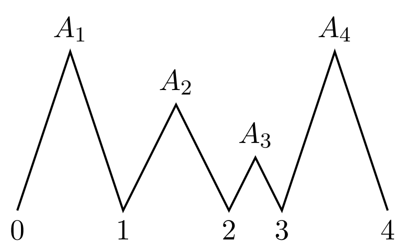
where $d_i : X_k^F \to X_{k-1}^F, s_j : X_{k-1}^F \to X_k^F$ are face and degeneracy maps respectively.

Given a chain $c_0 < c_1 < \cdots < c_k$ in $\widehat{\mathcal{C}_n}$ and an element $x \in F(c_0)$. We call $\{x\} \wedge \Delta_+^k$ a simplex in $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$ indexed by the chain $c_0 < c_1 < \cdots < c_k$. We define the map $\tilde{f} : \operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F \to Q_{n-1}$ on each such simplex in $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$. Consider any element $a = (x, (t_1, t_2, \ldots, t_k)) \in \{x\} \wedge \Delta_+^k$ then the image $\tilde{f}(a)$ is a tree in Q_{n-1}. We are going to describe how do we obtain the tree $\tilde{f}(a)$ from the tree $x \in F(c_0)$. We separate into several steps to get the tree $\tilde{f}(a)$, and we notice here that in the following step we work in the space $\widetilde{W_{n-1}}$ and finally we a tree $\tilde{f}(a) \in Q_{n-1}$, in other words after each step the tree we get is actually in $\widetilde{W_{n-1}}$ and when we finish all steps we just pass the tree we get to the quotient space Q_{n-1}.

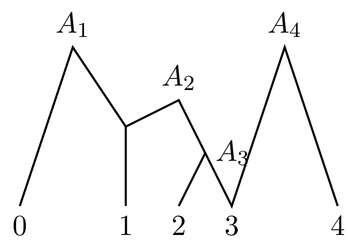
In the first step, along the morphism $c_0 < c_1$ in $\widehat{\mathcal{C}_n}$, according to the Proposition 3.4.8 each hyperedge E of c_1 is a union of several hyperedges $F_1, F_2, \ldots, F_j$ in c_0 and these hyperedges also form a connected acyclic hypergraph with vertices in E. For each external vertex whose label is located in hyperedge E which is actually lying in at least two of the sets $F_1, F_2, \ldots, F_j$ i.e. there are more than one path from this external vertex to the associated centers(For such vertex since it belongs to more that one centered trees, the associated centers are not single), we glue those paths starting the shared vertex by a length preserving map along the length t_1. We repeat this process for each vertex in $F_1, F_2, \ldots, F_j$ and each hyperedge E in c_1. In short we call this first step as gluing process with respect to the morphism $c_0 < c_1$.

To describe the second step we first need to make sense of what does it mean to be the centers of an external vertex with respect to each hyperedge of c_1. In the beginning case since what we have is a sequence of centered trees (which we view it as a single tree) and the centers of each external vertex are just the centers of each centered tree with label sets containing the label of the external vertex. After the first step gluing, some hyperedges $F_1, F_2, \ldots, F_j$ in c_0 are merged to a single hyperedge E in c_1. However, the new tree T_E with label set E obtained by gluing those centered trees with labels sets $F_1, F_2, \ldots, F_j$ might not be a centered tree. If this tree with label set E is already a centered tree then we say the center of this centered tree is the new center of all external vertices in E. If this tree is not a centered tree according to Lemma 3.3.17 we can use the distance function to find the locally maximal points $x_1, x_2, \ldots, x_m$ inside the tree T_E and we can decompose T_E as several centered trees inside it with centers as $x_1, x_2, \ldots, x_m$. If the external vertex with label in E just belongs to only one centered tree of the decomposition of T_E then the center of this centered tree is the new center of this external vertex. On the other hand if the external vertex with label in E belongs to more than one centered tree inside T_E then we can pick any center of those centered trees as its new center.

For example consider a chain $c_0 = 01/12/23/34 < c_1 = 0123/34 < c_2 = 01234$ and the tree $x \in F(c_0)$ as follows:



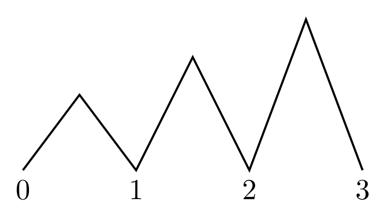
where A_1, A_2, A_3, A_4 are centers of the four centered trees. The vertex with label 0 associates with center A_1 with respect to the hyperedge 01, the vertex with label 1 associates with center A_1 with respect to the hyperedge 01 and the center A_2 with respect to the hyperedge 12, the vertex with label 2 associates with center A_2 with respect to the hyperedge 12 and the center A_3 with respect to the hyperedge 23, the vertex with label 3 associates with center A_3 with respect to the hyperedge 23 and the center A_4 with respect to the hyperedge 34, and vertex with label 4 associates with center A_4 with respect to the hyperedge 34. We take the radii of this sequence of centered trees from left to right to be $0.7, 0.5, 0.2, 0.7$. Then after gluing the length $t_1 = 0.4$ with respect to the morphism $c_0 < c_1$ we get a tree as follows:



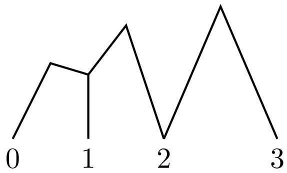
Then in this tree with respect to each hyperedge in c_1 the center associated with the vertex labeled by 0 is A_1(with respect to the hyper edge 01), the centers associated with the vertex labeled by 1 is A_1(with respect to the hyper edge 01) and A_2(with respect to the hyper edge 123), the center associated with the vertex labeled by 2 is A_2(with respect to the hyper edge 123), the centers associated with the vertex labeled by 3 are A_2(with respect to the hyper edge 123) and A_4(with respect to the hyper edge 34) and the center associated with the vertex labeled by 4 is A_4(with respect to the hyper edge 34).

Then the second step is similar, along the morphism $c_1 < c_2$ each hyperedge E' of c_2 is the union of several hyperedges $F_1', F_2', \dots, F_i'$ of c_1. Then for each external vertex with label located in hyperedge E which is actually lying in at least two of sets in $F_1', F_2', \dots, F_i'$ i.e. there are more than one path from this external vertex to the associated centers, we glue those paths starting the shared vertex by a length preserving map along the length t_2. And we repeat this process for each vertex in $F_1', F_2', \dots, F_j'$ and each hyperedge E' in c_2. Since $t_2 \leq t_1$ we know the choice of centers for some external vertices in the paragraph before the previous example is independent. And after the second step gluing we need to assign new centers for each external vertex in the hyperedges of c_2. We just iterate this process until the last morphism $c_{k-1} < c_k$ then we get the tree $\tilde{f}(a)$ we want. And in short we describe the total gluing process from x to $\tilde{f}(a)$ as gluing the length t_1 with respect to $c_0 < c_1$, gluing the length t_2 with respect to $c_1 < c_2$ until gluing the length t_k with respect to $c_{k-1} < c_k$.

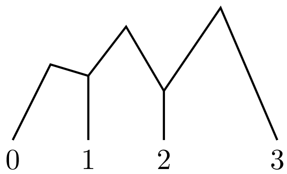
For example given a chain $c_0 = 01/12/23 < c_1 = 012/23 < c_2 = 0123$ and a tree $x \in F(c_0)$ like follows:



where the radius of centered trees with labels sets $E_1 = \{0, 1\}, E_2 = \{1, 2\}, E_3 = \{2, 3\}$ are $0.5, 0.7, 0.9$ respectively. Then the image of $\tilde{f}$ of the point $a = (x, (0.3, 0))$ would be the following tree(this tree is actually collapsed to the base point of Q_{n-1} by definition since the leaf with label 2 having length 0):



And the image of the point $a = (x, (0.3, 0.2))$ is the following tree:



We need to check this definition is compatible for faces and degeneracies relations. For simplicity here we just check the compatibility for face map d_0 and degeneracy map s_0, the compatibility for other maps are similar. As for the face map d_0, we first observe that for any tree $x \in F(c_0)$, $d_0(x)$ would be the tree x^1 obtained by the edge joining the length 1 from c_0 to c_1. The equivalence relation asks the two elements $(d_0(x), (t_1, t_2, \ldots, t_{k-1}))$ and $(x, (1, t_1, t_2, \ldots, t_{k-1}))$ being identified in $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$. And according to the definition of $\tilde{f}$ we know the image of these two points are same under the map $\tilde{f}$. As for the degeneracy map s_0, for any tree $x \in F(c_0)$, $s_0(x) = x$. And the index in summand will become $c_0 < c_0 < c_1 < \cdots < c_k$. The equivalence relation asks the two elements $(s_0(x), (t_0, t_1, t_2, \ldots, t_k))$ and $(x, (t_1, t_2 \ldots, t_k))$ being identified. The images of these two points under $\tilde{f}$ are same because after first gluing process staring the point $(s_0(x), (t_0, t_1, t_2, \ldots, t_k))$ along the morphism $c_0 \to c_0$ are just the original tree $x \in F(c_0)$. Moreover for any points $a = (x, (t_1, t_2, \cdots, t_k))$ where $x \in F(c_0)$ is represented by a sequence of trees in which at least one tree having radius 0 or 1, then all those points are just the base point in the homotopy colimits $\operatorname{hocolim}_{\widehat{\mathcal{C}}} F$ according to the Definition 3.4.10. Then the tree $\tilde{f}(a)$ we got is also the base point of Q_{n-1} since $\tilde{f}(a)$ is radius 1 or having a leaf with length 0. Therefore this map is a well-defined map. This map is also clearly a continuous map since the gluing procedure is continuous.

According to this definition we can see that all points in subspace $\operatorname{hocolim}_{\mathcal{C}_n} F$ are sent to the base point of Q_{n-1} since the image of each point having at least one leaf with 0 length. Hence we have an induced map $f : Y = \operatorname{Tot}(F) \to Q_{n-1} = \widetilde{W_{n-1}}/W_{n-1}$.

Remark. It's clear this construction could be extended to a little bit more general case in the sense that we replace the standard set $[n-1]_+$ by a set E. In this case, let $\widehat{\mathcal{C}_E}$ be the category of connected and acyclic hypergraphs on the vertex set E then following the exact construction we have the natural map $\tilde{f} : \operatorname{hocolim}_{\widehat{\mathcal{C}_E}} F \to Q_E$.

After the introduction of these preliminary notations, we are going to prove the main result Theorem 3.1.8.

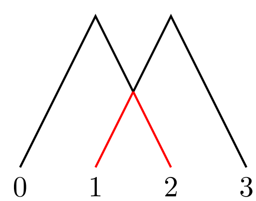
Theorem 3.5.12. *The total cofiber of the functor $F : \widehat{\mathcal{C}_n} \to \boldsymbol{Top}_*$ is Σ_n-equivariant homotopy equivalent to Q_{n-1}.*

i.e. we want to show that the induced map f is a Σ_n-equivariant homotopy equivalence. And we will use Y to denote the total cofiber for simplicity.

Lemma 3.5.13. *The map $\tilde{f}$ is a surjective map, in other words the induced map f is a surjective map.*

Proof. It suffices to show that for any $T \in Q_{n-1}$ which is not a base point then its preimage under the map $\tilde{f}$ is non-empty. So we can assume we have a tree $T \in \widetilde{W_{n-1}}$. According to the Definition 3.3.18, let $\{E_1, E_2, \ldots, E_k\}$ be the decomposition data of T, where E_i is the labels set of a centered tree associated with a locally maximal point for each i. Then for each individual centered tree with labels set E_i in the decomposition, we can view it as an element in the space P_{E_i}, in other words the decomposition data of the tree T could be viewed as an element in the space $P_{E_1} \wedge P_{E_2} \wedge \cdots \wedge P_{E_k}$ i.e. a sequence of centered trees with labels sets $E_1, \ldots, E_k$.

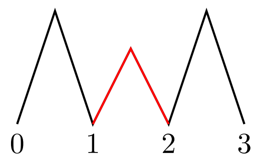
However, the hypergraph with edge set as $\{E_1, \ldots, E_k\}$ may not be a connected cyclic hypergraph because there might be an intersection part having more than one label. For example, the decomposition data of the following tree is two centered trees with labels set $E_1 = \{0, 1, 2\}$ and $\{1, 2, 3\}$.



However, the intersection part of the decomposition is the red part of the above drawing which contains 2 labels $1, 2$. In other words the hyper graph with hyper-edges E_1, E_2 are not acyclic because there are two paths connecting vertices 0 and 3.

Given the decomposition data, let $E_1, \ldots, E_k$ be the labels sets, if there are two sets E_i, E_j with intersection more than one element, in other words the two centered trees with labels sets E_i, E_j having intersection part with more than one label. Let T_i, T_j be the centered trees with labels sets E_i, E_j respectively. Then we take the intersection set $E_{ij} = E_i \cap E_j$ and consider the centered trees T_{ij} inside T_i with label set E_{ij} and centered trees T_{ji} inside T_j with label set E_{ij}. We claim two trees T_{ij} and T_{ji} are identical inside the original tree T. Because for any two external vertices with labels inside E_{ij} say u, v then two paths connecting these two vertices with the center of tree T_i would be merged from a branching point c_{ij} in T_i and it is clear this branching point is also a vertex in T_{ij}. Similarly there is also a branching point v_{ji} inside the tree T_j and T_{ji}. These two points c_{ij}, c_{ji} and the associated subpaths connecting two vertices u, v and c_{ij}, c_{ji} respectively must be identical since otherwise there would be a loop inside the tree T which is a clearly a contradiction. Hence the vertex c_{ij} and the subpaths connecting c_{ij} and vertices u, v are all inside the intersection part of two centered trees T_i and T_j. Moreover since all paths inside tree T_{ij} or T_{ji} comes from this way we imply that T_{ij} and T_{ji} are identical inside T and of course inside the intersection part of two trees T_i, T_j. Moreover we take $E_{i \backslash j} = E_i \backslash E_j + v$, $E_{j \backslash i} = E_j \backslash E_i + v'$ for any two elements $v, v' \in E_i \cap E_j$(we notice here that the choice of v, v' is arbitrary), these two new labels sets correspond to two new centered trees $T_{i\backslash j}, T_{j \backslash i}$ inside the original two centered trees T_i, T_j with labels set $E_{i\backslash j}, E_{j\backslash i}$ respectively. Then we replace E_i, E_j with T_i, T_j in decomposition data by $E_{ij}, E_{i\backslash j}, E_{j \backslash i}$ with centered trees $T_{ij}, T_{i\backslash j}, T_{j\backslash i}$ respectively.

We can iterate this procedure for the new decomposition data until we get a collection of sets $F_1, F_2, \ldots, F_m$ such that $F_i \cap F_j$ either empty or has just one element. And each label set associates a centered tree inside T with it as labels. We call a collection of centered trees with labels set obtained like this a complete decomposition data. For example, a complete decomposition data of the above drawing with red intersection part is as follows:



where the labels sets are $E_1 = \{0,1\}; E_2 = \{1,2\}; E_3 = \{2,3\}$.

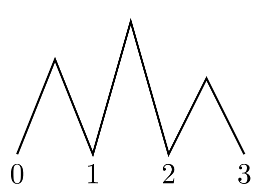
We claim that the hypergraph formed by the edge set $\{F_1,\dots,F_m\}$ of this further decomposition data is actually connected and acyclic. The connectness is clear. As for the acyclicity, since now the intersection of hyperedges is either empty or just one element then if this hypergraph is not acyclic only if there is a sequence of hyper edges forming a loop $v_1, F_1, v_2, F_2, \dots, v_q, F_q, v_1$ such that $q > 2$. Since if $q = 2$, i.e. we have a loop v_1, F_1, v_2, F_2, v_1, then by definition we know $F_1 \cap F_2$ contains more than one element. This is a contradiction. So we have two paths from v_1 to v_q: $v_1, F_1, v_2, F_2, \dots, v_{q-1}, F_{q-1}, v_q$ and v_1, F_q, v_q and we denote these two paths as L_1, L_2 respectively. From the path L_1 we can get a path $v_1, A_1, c_1, B_1, v_2, A_2, c_2, B_2, v_3, \dots, c_{q-1}, B_{q-1}, v_q$ where A_i, B_i are edges inside tree T and c_i are inner vertices of T for all i. Similarly from path L_2 we can get a path v_1, A_q, c_q, B_q, v_q. We observe that all vertices c_i are distinct. Since if $c_i = c_j$ for some i, j then the vertices $v_{i-1}, v_i, v_{j-1}, v_j$ have same distance to the vertex c_i(here we assume $v_{q+1} = v_1$). However, then these four vertices $v_{i-1}, v_i, v_{j-1}, v_j$ would be in a same labels set. This is a contradiction. Similarly reasons show all edges A_i, B_i are distinct. So these two paths can give a loop on v_1 which is a contradiction since T is a tree.

Now the sequence of the centered trees on labels set $F_1, \dots, F_m$ could be viewed as an element $x \in P_{F_1} \wedge P_{F_2} \wedge \cdots \wedge P_{F_m}$. And let $l_1 \geq l_2 \geq \cdots \geq l_s$ be the length of intersection parts with respect to the further decomposition data $\{F_1, \dots, F_m\}$. Then T is the image of the point $a = (x, (l_1, l_2, \dots, l_s))$ under the map $\tilde{f}$. Therefore the map $\tilde{f}$ is a surjective map. □

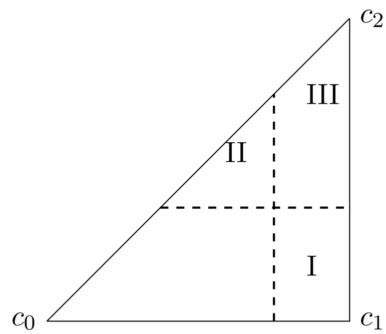
The way we prove f is an equivariant homotopy equivalence is by identifying the "kernal" of the map f i.e. identifying the preimage of each point in Q_{n-1} to get a quotient space Y' which is equivariant homeomorphic by definition. Then we prove the quotient map $\pi\colon Y \to Y'$ is actually a Σ_n-equivariant homotopy equivalence.

Let's first discuss the points making the map f not injective intuitively and then we will construct several functions to describe it precisely. Given any morphism say $c_0 < c_1$ in $\widehat{\mathcal{C}_n}$ and a tree $x \in F(c_0)$, then the map f describes how the tree x glues along the edges joining same labels. However, since x is actually union of several centered trees and each tree has radius less or equal 1, so after a time $t \leq 1$ the images under the map f are stable. Moreover, because of this in a given chain $c_0 < c_1 < \cdots < c_k$ and $x \in F(c_0)$ there is a stable region in the simplex $\{x\} \wedge \Delta^k_+$ because of the stable phenomenon with respect to the morphism $c_0 \to c_1$. Similarly in the simplex there are also other stable regions because of the stable phenomenon with respect to morphisms $c_i < c_{i+1}$ for each i.

Example 3.5.14. let's consider a 2-simplex in $\operatorname{hocolim}_{\widehat{\mathcal{C}_4}} F$ where $c_0 = 01/12/23, c_1 = 012/23, c_2 = 0123$ and the starting tree $x \in F(c_0)$ looks like follows:

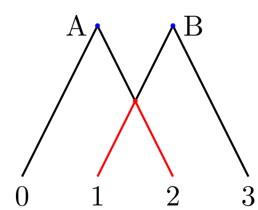


Then the region I+III is the stable region associated to the stable range with respect to morphism $c_0 < c_1$, and the region II+III is the stable region with respect to the morphism $c_1 < c_2$, and the region III is the stable region with respect to both morphisms $c_0 < c_1, c_1 < c_2$.



Of course these stable regions will make the induced map not an injective map, However this is not the only reason why map f is not an injective map, inside each simplex of Y there are special points we call it ambiguous points whose images are trees in Q_{n-1} which are not centered trees and the associated decomposition having an intersection part which has more or equal than 2 labels.

Example 3.5.15. The following tree is an example which is not centered tree but its decomposition having an intersection part which has more or equal than 2 labels.



Since after taking decomposition we can see that this given tree is the union of two sub centered trees: one is a centered tree centering at the point A with labels $\{0, 1, 2\}$ and another one is a centered tree centering at the point B with labels $\{1, 2, 3\}$. Therefore the intersection of two centered trees is the red part in the above figure which consists of 2 labels.

In summary there are two type of points making f not injective: degenerate points and ambiguous points.

Construction 3.5.16. Let $I = [0, 1]$, we list here several easy functions we will be used later.

1. For each morphism $c_0 < c_1$ in $\widehat{\mathcal{C}_n}$ we associate a function

$$T_{c_0,c_1} : F(c_0) \to I \tag{3.5.17}$$

   which sends each tree $x \in F(c_0)$ to the time when the gluing process with respect to $c_0 < c_1$ begins stable. More precisely if $x \in F(c_0)$ is the base point then we just ask $T_{c_0,c_1}(x) = 0$. On the other hand, if $x \in F(c_0)$ is not the base point then according to the Construction 3.5.7 we locally have a restriction map $\tilde{f} : \{x\} \wedge \Delta^1_+ \to \widetilde{W_{n-1}}$ on the simplex $\{x\} \wedge \Delta^1_+$ indexed by the chain $c_0 < c_1$, then $T_{c_0,c_1}(x)$ is the time when the images of the map $\tilde{f}$ on this 1-simplex become constant.

2. For each pair of morphisms with a common target $a_0 < c_1$ and $b_0 < c_1$ take $X_{a_0,b_0;c_1}$ as the pull-back of the following diagram:

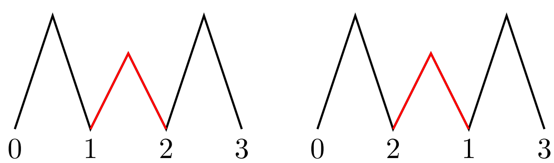
$$F(a_0) \to F(c_1) \leftarrow F(b_0)$$

in other words a point in $X_{a_0,b_0;c_1}$ could be represented by a pair (x, y) where $x \in F(a_0), y \in F(b_0)$ such that x and y will be sent to a same point in $F(c_1)$ under the morphism $F(a_0) \to F(c_1)$ and $F(b_0)$ to $F(c_1)$ respectively. Moreover we associate a function

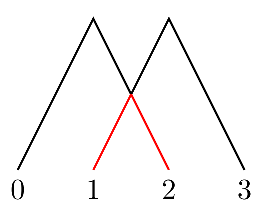
$$\mu_{a_0,b_0;c_1} : X \to I \tag{3.5.18}$$

by sending a pair (x, y) to the time when these two trees x, y become to the same tree during the gluing process with respect to the morphisms $a_0 < c_1$ and $b_0 < c_1$ respectively. More precisely, if either x or y are base point then the common image $z \in F(c_1)$ is also the base point, and in this case we ask $\mu_{a_0,b_0;c_1}(x, y) = 0$. On the other hand if both x, y are not base points then we locally have restriction maps $\tilde{f} : \{x\} \wedge \Delta^1_+ \to \widetilde{W_{n-1}}$ and $\tilde{f} : \{y\} \wedge \Delta^1_+ \to \widetilde{W_{n-1}}$ on two simplices $\{x\} \wedge \Delta^1_+$ and $\{y\} \wedge \Delta^1_+$ indexed by the chains $a_0 < c_1$ and $b_0 < c_1$ respectively. Then in this case $\mu_{a_0,b_0;c_1}(x, y)$ is the smallest time $t \in [0, 1]$ such $\tilde{f}(x, t) = \tilde{f}(y, t)$. Where (x, t) is in the 1-simplex $\{x\} \wedge \Delta^1_+$ indexed by the chain $a_0 < c_1$ and (y, t) is in the simplex $\{y\} \wedge \Delta^1_+$ indexed by the chain $b_0 < c_1$.

   For example, $a_0 = 01/12/23, b_0 = 02/12/13$ and $c_1 = 0123$, suppose we have the following two trees $x \in F(a_0), y \in F(b_0)$ where the red parts are centered trees with radius 0.5.



   Then after time 0.5, these two trees x, y will become to the same following tree:



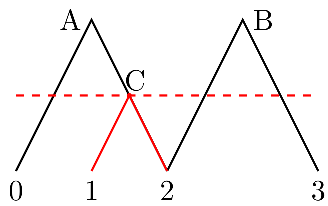
   in other words in this case $\mu_{a_0,b_0;c_1}(x, y) = 0.5$.

3. For each element $c \in \widehat{\mathcal{C}_n}$ we equip a function which is called the rank of a tree:

$$r_c \colon F(c) \to I \tag{3.5.19}$$

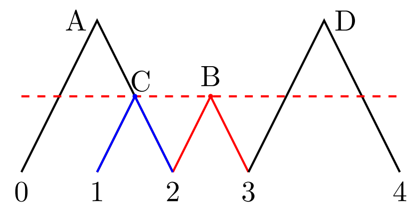
by sending a tree $t \in F(c)$ to the minimal length of branching points in the sequence of centered trees inside t to its associated leaves(see Definition 3.3.20). We require that if $* \in F(c)$ is the base point then $r_c(*) = 0$.

   For example, the rank of this following tree is 0.5:



where the length from the point C to the leaves 1 or 2 are both 0.5.

As another example the rank of this following tree is 0.3:



where the red parts is a centered tree with radius 0.3 and the blue part is a branching tree of the centered tree with branching point C, and the radius of this branching tree is also 0.3.

These functions satisfy the following easy observations:

Observation 3.5.20. 1. For each morphism $c_0 < c_1$ and a tree $x \in F(c_0)$ we always have:

$$r_{c_0}(x) \leq T_{c_0,c_1}(x)$$

This is simply because $T_{c_0,c_1}(x)$ is greater or equal to the radius of one centered tree(here centered trees include the branching trees inside the sequence of centered trees in x) in x which is by definition greater or equal to $r_{c_0}(x)$.

2. For any pair of morphisms with a common target: $a_0 < c_1, b_0 < c_1$ where $a_0 \neq b_0$ and a point (x, y) in $X_{a_0,b_0;c_1} \subset F(a_0) \times F(b_0)$ we have:

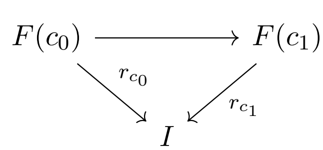
$$\max\{r_{a_0}(x), r_{b_0}(y)\} \leq \mu_{a_0,b_0;c_1}(x, y) \leq \min\{T_{a_0,c_1}(x), T_{b_0,c_1}(y)\}$$

Without losing any generality we can assume x, y are not base points. If the gluing time $\mu_{a_0,b_0;c_1}(x, y)$ is strictly less than $r_{a_0}(x)$ or $r_{b_0}(y)$ then there is a unique decomposition for the new tree after gluing which is a contradiction because in this case the pre-image should be not unique. As for the second inequality, this is simply because after time $T_{a_0,c_1}(x)$ or $T_{b_0,c_1}(y)$ of gluing, the tree x or y will become to the same centered tree z, hence $\mu_{a_0,b_0;c_1}(x, y) \leq \min\{T_{a_0,c_1}(x), T_{b_o,c_1}(y)\}$.

3. If $x \in F(c)\backslash *$ where $*$ is the basepoint in $F(c)$ then

$$r_c(x) > 0$$

4. For each morphism $c_0 < c_1$, we have the following commutative diagram:



in other words, the rank of a tree is stable with respect to the morphism $F(c_0) \to F(c_1)$. This is because if the rank r is the radius of a branching tree then it is clear this tree will be not changed under the morphism $c_0 < c_1$.

5. For a chain $c_0 < c_1 < c_2$ in $\widehat{\mathcal{C}_n}$ and a tree $x \in F(c_0)$ we have:

$$T_{c_0,c_2}(x) \geq T_{c_0,c_1}(x)$$

This is just because the gluing process with respect to the morphism $c_0 < c_2$ contains the gluing process with respect to the morphism $c_0 < c_1$.

Construction 3.5.21. We construct an equivalence relation on $\bigvee_{n\geq 0} \bigvee_{c_0<c_1<\cdots<c_n} F(c_0) \wedge \Delta_+^k$ which is compatible with the faces and degeneracies relations inside geometric realizations. In other words, we construct an equivalence relation on $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$. The equivalence relation is generated by the following two type relations.

Firstly, given a chain $c_0 < c_1 < \cdots < c_k$ and a point $x \in F(c_0)$. For some j let x^{j-1} is the image of x under of the morphism $F(c_0)$ to $F(c_{j-1})$ and we suppose both t_j, t_j' are greater or equal to $T_{c_{j-1},c_j}(x^{j-1})$ then we say two points $(x, (t_1, \ldots, t_j, \ldots, t_k))$ and $(x, (t_1, \ldots, t_j', \ldots, t_k))$ are equivalent. We call this relation as the type A relation. Intuitively, this just asks that two point in degenerate region which is in the line parallel to the stable "direction" should be viewed as equivalent points. More precisely according to the definition of the map f we observe that if two points a, b are type A equivalent then $\tilde{f}(a) = \tilde{f}(b)$.

Secondly, suppose we have two chains

$$a_0 < a_1 < \cdots < a_{k-1} < c_k$$

$$b_0 < b_1 < \cdots < b_{k-1} < c_k$$

and a point $(x, y) \in X_{a_0,b_0;c_k}$. If the k-tuple $(t_1, t_2, \ldots, t_k)$ satisfies the condition:

$$\begin{cases} t_1 \geq \mu_{a_0,b_0;c_k}(x, y) \\ t_2 \geq \mu_{a_1,b_1;c_k}(x^1, y^1) \\ \quad \vdots \\ t_k \geq \mu_{a_{k-1},b_{k-1};c_k}(x^{k-1}, y^{k-1}) \end{cases} \tag{3.5.22}$$

Similarly here x^j, y^j denotes the image of x, y under the morphism $F(a_0) \to F(a_j)$ and the morphism $F(b_0) \to F(b_j)$ respectively. Then the point $(x, (t_1, t_2, \ldots, t_k)) \in F(a_0) \wedge \Delta_+^k$ is asked to be equivalent to the point $(y, (t_1, t_2, \ldots, t_k)) \in F(b_0) \wedge \Delta_+^k$. We call this relation as type B relation. Intuitively this relation just ask two ambiguous points with the same image should be equivalent. More precisely, according to definition of $\mu_{a_0,b_0;c_k}(x, y)$ if we glue the tree x, y by time t_1 with respect to the morphisms $a_0 < c_k$ and $b_0 < c_k$ respectively then the two trees becoming same. However, instead of doing that we first ask two trees x, y gluing by time t_1 with respect to the morphisms $a_0 < a_1$ and $b_0 < b_1$ respectively. We observe that after doing that there is no "essential" branching points being merged if we further glue with respect to the morphisms $a_0 < a_1$ and $b_0 < b_1$ to get trees x^1, y^1 respectively. Here essential branching points mean those branching points which make the tree x, y finally becoming the same tree during the gluing process with respect to morphisms $a_0 < c_k$ and $b_0 < c_k$ respectively. Since otherwise t_1 will be strictly less than $\mu_{a_0,b_0;c_k}(x, y)$ by definition. Now let $x' = f(x, (t_1, t_2, \ldots, t_{k-1})), y' = f(y, (t_1, t_2, \ldots, t_{k-1}))$ i.e. x', y' are the trees gluing from x, y by time t_1 with respect to the morphisms $a_0 \to a_1, b_0 \to b_1$ respectively, by time

t_2 with respect the morphisms $a_1 < a_2, b_1 < b_2$ respectively and so on finally by time t_{k-1} with respect to the morphisms $a_{k-2} < a_{k-1}, b_{k-2} < b_{k-1}$ respectively. Then by same reason there is no essential branching points being merged when we glue the trees x', y' further to get x^{k-1}, y^{k-1}. And according to the definition of $\mu_{a_{k-1},b_{k-1};c_k}(x^{k-1}, y^{k-1})$ we know if we glue the trees x^{k-1}, y^{k-1} by time t_k with respect to the morphism $a_{k-1} \to c_k$ and $b_{k-1} \to c_k$ respectively we get a same tree. Then if we glue trees x', y' with respect to the morphisms $a_{k-1} < c_k$ and $b_{k-1} < c_k$ we will also get a same tree, this is just because if the trees we got are not same then the two trees we have by further gluing x', y' to the trees in $F(c_k)$ are not same either since there is no essential branching points being merged in this gluing process. However this is contradiction since we already showed that those two trees in $F(c_k)$ we got are same. In other words $\tilde{f}(a) = \tilde{f}(b)$.

However we need to check that this is a well-defined equivalence relation i.e. we need to show that this relation is compatible with faces and degeneracies relations. For simplicity Let's check it is compatible with face relations for the type A relations. Given a chain $c_0 < c_1 < \cdots < c_k$ and a tree $x \in F(c_0)$. According to definition the face map d_0 is just the morphism $F(c_0) \to F(c_1)$ i.e. $d_0(x) = x^1$ and for $i > 0$ the face map d_i is the identity map on $F(c_0)$. So when $i = 0$, suppose we have two points $a = (d_0(x), (t_1, t_2, \ldots, t_{k-1})), b = (d_0(x), (t_1', t_2, \ldots, t_{k-1}))$ with $t_1, t_1' \geq T_{c_1,c_2}(d_0(x))$ i.e. $a \sim_A b$. Then according to relations 3.5.11 in the definition of the geometric realization we need to check that two new points $a' = (x, (1, t_1, t_2, \ldots, t_{k-1})), b' = (x, (1, t_1', t_2, \ldots, t_{k-1}))$ are still equivalent in terms of type A relations. According to the definition in this case we need to check that $t_1, t_1' \geq T_{c_1,c_2}(x^1)$, however we already knew that $d_0(x) = x^1$ then it is automatically true. So the type A relation is compatible with d_0. For $i = 1$, then suppose we have two points $a = (d_1(x), (t_1, t_2, \ldots, t_{k-1})), b = (d_1(x), (t_1', t_2, \ldots, t_{k-1}))$ with $t_1, t_1' \geq T_{c_0,c_2}(d_1(x))$ i.e. $a \sim_A b$. Similarly according to the relations 3.5.11 we need to check that two new points $a' = (x, (t_1, t_1, t_2, \ldots, t_{k-1})), b' = (x, (t_1', t_1', t_2, \ldots, t_{k-1}))$ are still equivalent in terms of type A relations. According to the 5-th statement of the Observation 3.5.20 We know $T_{c_0,c_2}(d_1(x)) = T_{c_0,c_2}(x) \geq T_{c_0,c_1}(x)$, therefore $t_1, t_1' \geq T_{c_0,c_1}(x)$ in other words $a' \sim_A b'$. For all i-th the type A relations are still compatible with d_i or s_i for similar reasons. As for the type B relations for simplicity let's check it here that it is compatible with d_0 and the compatibility for other faces or degeneracies are similar. Let $a_0 < a_1 < \cdots < a_{k-1} < c_k, b_0 < b_1 < \cdots < b_{k-1} < c_k$ be two chains with the same target c_k and $(x, y) \in X_{a_0,b_0;c_k}$, and suppose we have two points $a = (d_0(x), (t_1, t_2, \ldots, t_{k-1})), b = (d_0(y), (t_1, t_2, \ldots, t_{k-1}))$ satisfy the inequalities 3.5.22. We notice here in this case $d_0(x) = x^1$ and generally $d_0(x)^j = x^{j+1}$. So $t_j \geq \mu_{a_j,b_j;c_k}(x^j, y^j)$ for any $1 \leq j \leq k-1$. Then we need to check that two new points $a' = (x, (1, t_1, t_2, \ldots, t_{k-1}), b' = (y, (1, t_1, t_2, \ldots, t_{k-1}))$ are equivalent in terms of type B relations. It suffices to show that the coordinates satisfy the inequalities 3.5.22 i.e. $t_i \geq \mu_{a_i,b_i;c_k}(x^i, y^i)$ for $1 \leq i \leq k-1$ and $1 \geq \mu_{a_0,b_0;c_k}(x, y)$. But these inequalities are all automatically true i.e. $a' \sim_B b'$. Hence the type B relations are compatible with d_0.

Remark. The equivalence relation we just constructed could be induced to an equivalence relation on the total cofiber $Y = \mathrm{Tot}(F)$, which we still denote it as $\sim$.

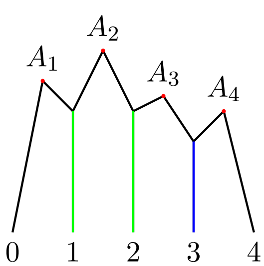
Definition 3.5.23. Let $Y' := Y/\sim$, where the equivalence relation here is the equivalence relation we constructed in Construction 3.5.21 and its remark. And the quotient map $Y \to Y'$ is denoted by π.

According to the Construction 3.5.21 we know if two points a, b are equivalent in Y then $f(a) = f(b)$, it implies we have a natural induced map $f' : Y' \to Q_{n-1}$ which is also a surjective map.

Theorem 3.5.24. *The induced map $f' : Y' \to Q_{n-1}$ is a Σ_n-equivariant homeomorphism.*

Proof. Since both spaces Y' and Q_{n-1} are compact and Hausdorff we just need to show that the map f' is a injective map. If $T \in Q_{n-1}$ is the base point, then according to the argument in the proof of the Lemma 3.5.13 we know all possible images are located in the subspace $\operatorname{hocolim}_{\mathcal{C}_n} F$ of $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$, in other words the preimage of T in Y' is just the base point. Then it suffices to prove that if two points a, b in total cofiber Y with $f(a) = f(b) = T$, where T is not the base point of Q_{n-1}, then a and b are equivalent in Y i.e. the preimage of T in this case is also a single point in Y'.

According to the Construction 3.5.21 we just need to prove the converse direction i.e. if $f(a) = f(b)$ then a is equivalent to b. For each point a in the homotopy colimits we can always find an equivalent point a' such that for each $1 \leq j \leq k$ its j-th coordinates t_j of it is strictly less than $T_{c_{j-1},c_j}(x^{j-1})$ since otherwise if one coordinate say t_j is greater or equal to $T_{c_{j-1},c_j}(x^{j-1})$ then it is equivalent to the point with same other coordinates except changing t_j to 1, in other words we can pass this point to a point in boundary. Moreover we can see that if several coordinates of a point are same then this point actually lives in the boundary of the simplex. So we can require that the coordinates of the point a' satisfies $t_1 > t_2 > \cdots > t_k$. So for those two points a and b we can first find equivalent points a' and b' respectively. As for its image T, if it is already a centered tree then the point a' must be of the form $a' = (T, 0)$. This is because if a' has the form that $a' = (x, (t_1, t_2, \ldots, t_k))$ and $f(a) = T$ is already a centered tree, in other word the gluing process is already stable, this is a contradiction because now $t_1 \geq T_{c_0,c_1}(x)$. So if T is not a centered tree, then according to the proof in Lemma 3.5.13 we can find more than one locally maximal points say $A_1, A_2, \ldots, A_m$. For each locally maximal point A_i there is a centered tree T_i in T with A_i as the center. Let $l_1 > l_2 > \cdots > l_{k'}$ be the arrangement of the number of length of the intersection parts i.e. l_1 is the length of longest intersection parts which of course may not be unique. l_i is defined similarly. For example, in the following tree



A_1, A_2, A_3, A_4 are four locally maximal points and the two green parts and the blue parts are all intersection parts. If the length of the green part is $0,7$ and the length of the blue part is 0.5 then in this case $l_1 = 0.7, l_2 = 0.5$.

Let $a' = (x, (t_1, t_2, \ldots, t_k))$, then according to the definition of the map f we know t_1 means the length of the intersection parts from c_0 to c_1 and since we already ask that $t_1 > t_2 > \cdots > t_k$ then t_1 must be equal to l_1. Then we just repeat this argument we get $k = k'$ and for each $1 \leq i \leq k$, $t_i = l_i$. This argument also words for $b = (y, (q_1.q_2, \cdots, q_m))$ for same reason, so we can rewrite $b' = (y, (l_1, l_2, \cdots, l_k))$. Finally, since $f(a') = f(b') = T$, then after time l_1 of gluing of trees x, y with respect to the morphisms $a_0 < c_k$ and $b_0 < c_k$ respectively the two trees $x \in F(a_0)$ $y \in F(b_0)$ will become the same tree T, in other words $l_1 \geq \mu_{a_0,b_0;c_k}(x, y)$. Similar argument shows that the coordinates of a', b' satisfy the inequalities 3.5.22 i.e. $a' \sim_B b'$. Therefore a is equivalent to b.

□

So finally we just need to prove the quotient map $\pi : Y \to Y'$ is a Σ_n-equivariant homotopy equivalence. The way we use to prove π is an equivariant homotopy equivalence is trying to construct a homotopy inverse of π. In order to do that we first try to construct two self maps of Y and Y' respectively.

Lemma 3.5.25. *If we can construct a continuous self map s_1 between the total cofiber Y satisfying two following conditions:*

1. *For any two points a, b in the total cofiber Y, if $\pi(a) = \pi(b)$ then $s_1(a) = s_1(b)$,*
2. *The self map s_1 is Σ_n-equivariant homotopy to the identity map id_Y between Y, i.e there is a base point preserving map $H : Y \times I \to Y$ from s_1 to the identity map between Y. Moreover we ask that for any time $s \in I$ the associated map $H_s := H(-, s) : Y \to Y$ satisfies the condition that for any two points a, b in the total cofiber Y, if $\pi(a) = \pi(b)$ then $\pi(H_s(a)) = \pi(H_s(b))$.*

Then the self map s_1 will pass to a self map s_2 which is also Σ_n-equivariant homotopy to the identity map $id_{Y'}$ between Y' and there is a continuous map $g : Y' \to Y$ making the two triangles commutes, in other words g is a homotopy inverse of π.

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & Y' \\ \Big\downarrow{\scriptstyle s_1} & \swarrow{\scriptstyle g} & \Big\downarrow{\scriptstyle s_2} \\ Y & \xrightarrow{\pi} & Y' \end{array}$$

Proof. Let's define the self map $s_2 \colon Y' \to Y'$ as $s_2(t) := \pi(s_1(x_t))$ where $x_t \in \pi^{-1}(t)$ is an element in the pre-image of t. According to condition (1) this is a well-defined function. But we still need to check it is a continuous map. Let $V \subset Y'$ be an open subset. First we observe that

$$s_2^{-1}(V) = \pi(s_1^{-1}(\pi^{-1}(V)))$$

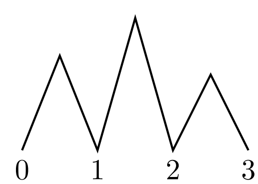
Since π, s_1 are two continuous map $s_1^{-1}(\pi^{-1}(V))$ is an open subset of Y. Moreover, since π is a quotient map, $s_2^{-1}(V) = \pi(s_1^{-1}(\pi^{-1}(V)))$ is also an open subset of Y'. Therefore s_2 is a continuous map.

Then since the self map s_1 is Σ_n-equivariant homotopy to the identity map, by definition we have a based homotopy $H : Y \times I \to Y$ between the identity map and the self map s_1. We define an equivalence relation on $Y \times I$ by $(a, s) \sim (b, s)$ if and only $\pi(a) = \pi(b)$, then it is clear with respect to this equivalence relation $Y \times I / \sim \simeq Y' \times I$. Then according to the condition (2) and the universal property of quotient maps we can pass the homotopy H to a homotopy $\widetilde{H} : Y \times I / \sim \to Y'$. In other words we find a based homtopy $\widetilde{H} : Y' \times I \to Y'$. So the self map s_2 is Σ_n equivariant homotopy equivalent to the identity map between Y'. Finally we define the map $g : Y' \to Y$ by $g(t) := s_1(x_t)$ where x_t is an element in $\pi^{-1}(t)$, same argument for continuity and well-defineness of s_2 works here. And the commutativity of two triangles is clear by construction. Then $g \circ \pi = s_1 \sim_{\Sigma_n} id_Y$ and $\pi \circ g = s_2 \sim_{\Sigma_n} id_{Y'}$ i.e. g is a homotopy inverse of π or π is a Σ_n-equivariant homotopy equivalence from Y to Y'. ☐

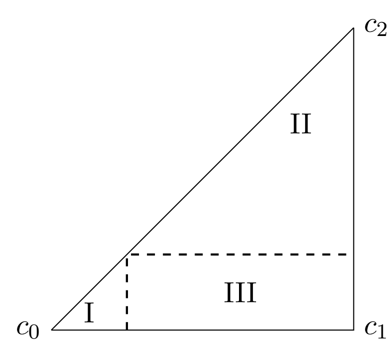
Construction 3.5.26. We try to construct a self map s on $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$ which will pass to a self map s_1 on the total cofiber $Y = \mathrm{Tot}(F)$ we want. In order to define a self map s we first need a technical triangulation on each simplex in $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$. Given a chain $c_0 < c_1 < \cdots < c_k$ and a point $x \in F(c_0)$, let $r = r_{c_0}(x)$. Then for the simplex $\{x\} \wedge \Delta_+^k$ indexed by the chain we can first subdivide this simplex to those following regions defined by inequalities:

$$\begin{cases} 1 \geq r \geq t_1 \geq \cdots \geq t_k \geq 0 \\ \quad \vdots \\ 1 \geq t_1 \geq \cdots t_i \geq r \geq t_{i-1} \cdots \geq t_k \geq 0 \\ \quad \vdots \\ 1 \geq t_1 \geq \cdots t_k \geq r \geq 0 \end{cases} \tag{3.5.27}$$

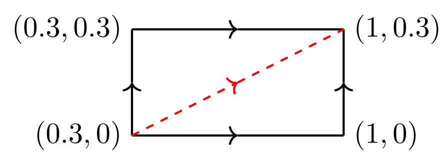
However, it is clear that not every region is a simplex, many regions here are general polyhedra. So we still need to further subdivide these polyhedra into simplices to get a triangulation. For each region defined by an inequality we observe that its vertices are completely determined by the inequality and we can actually give an ordering on those vertices by comparing its coordinates. More precisely if each coordinate of one point A is less or equal to the corresponding coordinate of another point B then we define $A \leq B$. So this will give an ordering on the set of vertices on each polyhedron. Then we just need to connect any two comparable vertices on each polyhedron to get a triangulation on the simplex $\{x\} \wedge \Delta_+^k$. As an example let's consider the chain $c_0 = 01/12/23 < c_1 = 012/23 < c_2 = 0123$ and the following tree $x \in F(c_0)$ with the rank $r = 0.3$



Then the inequalities subdivide the 2-simplex $\{x\} \wedge \Delta_+^2$ into 3-regions where regions I,II are already simplices but the region III is a rectangle.



However, the coordinates and ordering of its vertices of the region III looks like:



Then we just need to connect the points $(0.3, 0)$ and $(1, 0.3)$ with a line to get a triangulation.

According to the 4-th statement of the Observation 3.5.20 we know the number r is stable under taking boundary of a simplex, so this triangulation for each simplex is compatible with the boundary of simplices. Given a triangulation on each simplex then a self map on $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$ is completely determined by the assignment on each vertex in the triangulation. Concretely, in a simplex $\{x\} \wedge \Delta_+^k$ we know the coordinates of each vertex of the triangulation looks like $(x, (a_1, a_2, \ldots, a_k))$ where $1 \geq a_1 \geq a_2 \cdots \geq a_k \geq 0$ and $a_i = 1, r$, or 0 . If all coordinates of a vertex are not r, then this vertex is fixed under the self map, if there is at least one coordinate which is r then we assign this point to the point with the coordinates by replacing r by 1 for each r in the coordinates of the vertex. We need to check this self map is a well-defined map. Since each vertex may have different coordinates with respect to which simplex the vertex lives, we need to check that this self map is compatible with different coordinates of a vertex. For a vertex a lying in the simplex Δ say $\{x\} \wedge \Delta_+^k$ with the coordinates equal to $(x, (a_1, a_2, \ldots, a_k))$, if either $a_i = 0, 1$ or several coordinates are same then this point actually locates in a simplex of boundary of this simplex Δ. However, according to the definition of self map s those coordinates with values $0, 1$ remain same and those coordinates with values r are sending to 1, hence no matter which coordinates of the point a we choose the images under the self map s are just different coordinates of the same point i.e. the self map s is a well-defined map. As for the continuity issue of the self map s it might happen when $x \in F(c_0)$ is a base point. In this case actually the rank of x is 0 and the simplex $\{x\} \wedge \Delta_+^k$ indexed by any chain $c_0 < c_1 < \cdots c_k$ starting from c_0 has already been collapsed in $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$. So in this case the self map is just sending the base point to base point. Therefore the self map s is a base point preserving continuous map.

The necessary and sufficient condition of a point lies in $\operatorname{hocolim}_{\mathcal{C}_n} F$ is that if we view this point in a maximal simplex then the last coordinate must be 0, here a maximal simplex means a simplex in $\operatorname{hocolim}_{\widehat{\mathcal{C}_n}} F$ with maximal dimension. Then according to the construction of the self map s the last coordinate will not change. Hence this self map will pass to a self map s_1 on the total cofiber Y.

Lemma 3.5.28. *The self map s_1 on Y in Construction 3.5.26 satisfies the conditions in the Lemma 3.5.25*

Proof. First we claim that the self map s_1 is Σ_n-equivariant homotopy to the identity map between Y. We can construct the homotopy explicitly here:

$$H : Y \times I \to Y$$

For a point $a = (x, (t_1, t_2, \ldots, t_k))$ in a simplex say $\Delta = \{x\} \wedge \Delta_+^k$ with rank r, then for time $t \in I$ the homotopy $H(-, t)$ sends a to a point in same simplex but changes the coordinates of a which are r to coordinates $(1 - r)t + r$ and other coordinates remain same. According to the 4-th statement of the Observation 3.5.20 if the point is in the boundary Δ' of this simplex, then since the boundary still have same rank the image of a under the homotopy $H(-, t)$ when we view a is in Δ is as same as the image if a under the homotopy $H(-, t)$ if we view a is in the boundary Δ'. In other words the homotopy we defined is compatible with the boundary of simplices in homotopy colimits i.e. we have a well-defined homotopy H on Y. As for the continuity issue of H, as same as the self map s_1 it might happen when $x \in F(c_0)$ is a base point. And similarly in this case the simplex $\{x\} \wedge \Delta_+^k$ has already been collapse in Y to the base point. And the homotopy H here just sends the base point to base point, in other words H is a base point preserving map.

Moreover according to our construction of homotopy H when $t = 0$, $H(-, 0)$ is just the identity map between Y and when $t = 1$ then $H(-, 1)$ is equal to the self map s_1. Since the homotopy is clear Σ_n-equivariant, $s_1 \simeq_{\Sigma_n} id_Y$.

Then we need to check the remaining conditions. It suffices to check it for type A and type B relations in Construction 3.5.21. As for the type A relation, suppose $a \sim_A b$. Without losing any generality let's assume $a = (x, (t_1, t_2, \ldots, t_k))$ and $b = (x, (t_1', t_2, \ldots, t_k))$ both lie in the simplex $\{x\} \wedge \Delta_+^k$ associated with a chain $c_0 < c_1 < \cdots < c_k$ where $x \in F(c_0)$ and $t_1, t_1' \geq T_{c_0,c_1}(x)$. According to the first statement of the Observation 3.5.20 we know these two points are not in the region defined by the inequality:

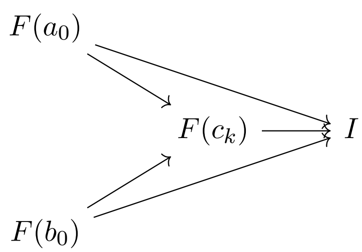
$$1 \geq r \geq t_1 \geq \cdots \geq t_k \geq 0$$

We can assume these two points locate in the region:

$$1 \geq t_1 \geq r \geq t_2 \geq \cdots \geq t_k \geq 0$$

According to the construction of the self map s_1, it is actually defined over each simplex in the triangulation. Under the self map s_1 the first coordinates t_1, t_1' of a, b respectively will be changed by 1. As for the other coordinates we know in each direction the self map actually enlarge r to 1, and since the coordinates of a and b are same except the first coordinate we know other coordinates of the images $s_1(a), s_1(b)$ will be also equal i.e $s_1(a) = s_1(b)$. As for the homotopy H associated to s_1 similar reason shows that for any times $s \in I$, the first coordinate of $H(a, s)$ and $H(b, s)$ are still greater or equal to $T_{c_0,c_1}(x)$ and other coordinates of $H(a, s)$ and $H(b, s)$ are same. So by definition we know $H(a, s) \sim_A H(b, s)$ for any time $s \in I$, in other words $\pi(H(a, s)) = \pi(H(b, s))$.

As for the type B relation, suppose $a \sim_B b$ where $a = (x, (t_1, t_2, \ldots, t_k))$, $b = (y, (t_1, t_2, \ldots, t_k))$ lie in the simplex $\{x\} \wedge \Delta_+^k$ associated with a chain $a_0 < a_1 < \cdots < a_{k-1} < c_k$ and the simplex $\{y\} \wedge \Delta_+^k$ associated with a chain $b_0 < b_1 < \cdots b_{k-1} < c_k$ respectively and $x \in F(a_0), y \in F(b_0)$, (x, y) is a point in the pullback $X_{a_0,b_0;c_k}$. Moreover the coordinates of a, b satisfy the inequality 3.5.22. As for the self map s_1, since (x, y) is a point in the pullback $X_{a_0,b_0;c_k}$ i.e. x, y will corresponds to the same point z under the morphism $F(a_0) \to F(c_k)$ and $F(b_0) \to F(c_k)$ respectively. According to the 4-th statement of the Observation 3.5.20 we have a commutative diagram as follows:



(3.5.29)

The commutativity of this diagram implies $r_{a_0}(x) = r_{b_0}(y) = r$. Then according to the second statement in the Observation 3.5.20 we know both points a, b locates in the region defined the inequality:

$$1 \geq t_1 \geq \cdots \geq t_k \geq r \geq 0 \qquad (3.5.30)$$

of the simplices $\{x\} \wedge \Delta_+^k$ and $\{y\} \wedge \Delta_+^k$ respectively. It is clear this region is already a simplex, and the coordinates of all vertices are either r or 1 by the inequality 3.5.30. So under the self map all these vertices will be sent to the point $(1, 1, \ldots, 1)$, in other words under the self map s_1, both points a and b will be sent to the pint $c = (z, (1, 1, \ldots, 1))$ i.e. $s_1(a) = s_1(b)$. As for the homotopy H, since the coordinates of a, b are exactly same and the homotopy on those two simplices are also completely parallel since $r_{a_0}(x) = r_{b_0}(y) = r$. Then according to the definition of the type B relation we know $H(a, s) \sim_B H(b, s)$ i.e. $\pi(H(a, s)) = \pi(H(b, s))$ for any $s \in I$. □

Corollary 3.5.31. *The induced map f is a Σ_n-equivariant homotopy equivalence.*

Proof. This is a direct consequence of the Lemma 3.5.25 and the Lemma 3.5.28. □

3.6 Proof of Theorem 3.1.9

We recall the Definition 3.4.15 that the total cofiber of the functor $F\colon \widehat{\mathcal{C}_n} \to \mathbf{Top}_*$ could be expressed as:

$$\mathrm{Tot}(F) = \mathrm{hocofib}(\operatorname*{hocolim}_{\mathcal{C}_n} F \to F(\hat{1})) \tag{3.6.1}$$

where this functor F sends a hypergraph $H = (E_1, E_2, \ldots, E_k)$ in to a based space $P_{E_1} \wedge P_{E_2} \wedge \cdots \wedge P_{E_k}$. According to the Proposition 3.3.22 we know that $P_n \simeq |P(n)|/\partial|P(n)| \simeq \Sigma\partial|P(n)|$ where $|P(n)|$ is the geometric realization of the poset of all partitions on $[n-1]_+ = \{0, 1, \ldots, n-1\}$. Furthermore we have:

$$\begin{aligned} P_{E_1} \wedge P_{E_2} \wedge \cdots \wedge P_{E_k} &\simeq |P(E_1) \times P(E_2) \times \cdots \times P(E_k)|/\partial|P(E_1) \times P(E_2) \times \cdots \times P(E_k)| \\ &\simeq \Sigma\partial|P(E_1) \times P(E_2) \times \cdots \times P(E_k)| \end{aligned} \tag{3.6.2}$$

Here $\simeq$ means homotopy equivalence. Since the reduced suspension functor Σ is a left Quillen functor between the model categories of pointed topological spaces ans itself[Hov99, Section 6.1], it commutes with homotopy colimits. Hence we can desuspend the total cofiber $\mathrm{Tot}(F)$ into a space X with the form:

$$X = \mathrm{hocofib}(\operatorname*{hocolim}_{\mathcal{C}_n} \Sigma^{-1}F \to \Sigma^{-1}F(\hat{1})) \tag{3.6.3}$$

Here $\Sigma^{-1}F$ denotes the new functor which sends a hypergraph $H = (E_1, E_2, \ldots, E_k)$ in to a unbased space $\partial|P(E_1) \times P(E_2) \times \cdots \times P(E_k)|$ and as the morphism it suffices to illustrate the case $H_1 = (E_1, E_2)$ and $H_2 = \hat{1}$ the trivial hypergraph. If we suppose two hyperedges E_1, E_2 intersect at the vertex $m \in [n-1]_+$ then the map $\partial|P(E_1) \times P(E_2)| \to \partial|P(n)|$ is induced by the poset map $P(E_1) \times P(E_2) \to P(n)$ by sending two partitions π_1, π_2 in $P(E_1)$ and $P(E_2)$ respectively into a new partition π such that if blocks in π_1, π_2 does not contain m then they will remain in the new partition π but if two blocks in π_1, π_2 containing m then they will be merged to a new block in π.

From now on let's fix some notations for this section. We still call the space X as total cofiber associated with the functor $\Sigma^{-1}F$ and we will still use F to denote this new functor $\Sigma^{-1}F$ since in the following we are all working in this desuspended space. Moreover, since we also need to dealt with both pointed and unpointed homotopy colimits we replace the notation $\operatorname*{hocolim}_{\mathcal{C}_n} F$ by $\mathrm{hocolim}_\bullet F$, and when we forget the base points of images of the functor F we can take the homotopy colimits of F over the unpointed topological spaces which is denoted as $\mathrm{hocolim}\, F$ in this section.

The main result of this section which already appeared in section 3.1 as the Theorem 3.1.9 is the following:

Theorem 3.6.4. *The total cofiber $X = \mathrm{hocofib}(\mathrm{hocolim}_\bullet F \to \partial|P(n)|)$ is Σ_n-equivariant homotopy equivalent to $S|\Delta_n^2|$, where S denotes the unreduced suspension functor.*

Definition 3.6.5. Given any graph G on n-vertices, we can decompose it into a union of several subgraphs. For a fixed decomposition D of G, we say the **hyper-cover** of this decomposition is a hypergraph where each hyperedge consists of points of vertices of one subgraph in this decomposition. Moreover if the hypergraph behind it is connected and acyclic then we call this hyper-cover a connected acyclic hyper-cover of G.

Lemma 3.6.6. *If G is a connected but not 2-connected graph, then we can always find a connected acyclic hyper-cover on G and there is a minimal element among all connected acyclic hyper-covers of G. And we call it the minimal hyper-cover of G.*

Proof. We first show the existence of a connected acyclic hyper-cover. Given a graph G we can get a set S containing all vertices $v_1, \ldots, v_k$ such that if you delete any vertex and its adjacent edges of those vertices then the remaining graph is not connected. And since G is a not 2-connected graph we know S is not an empty set. Moreover, Given this set S we can first take out v_1 of this graph then we have several not connected components, then we can group each component with v_1, so we get several hyperedges which intersect at v_1. Then in each hyper-edge we can repeat this process again to form new hyper-edges. If we get several new hyper-edges inside old hyper-edges then we just simply forget these old hyper-edges. So we repeat this procedure again and again until we reach all elements in S. We call the final hypergraph we get H_G. Since G is a connected graph, this hypergraph is connected. Moreover this hypergraph is acyclic since otherwise the graph G is a 2-connected graph. Therefore this hypergraph we constructed is a connected acyclic hyper-cover of G.

Now we are going to show that H_G is in fact minimal among the connected acyclic hyper-covers of G. Suppose we have another connected acyclic hyper-cover H of G, by requirements on this hyper-cover we know if two hyper-edges having intersection with one vertex then this vertex must be contained in S since if we delete this vertex the remaining graph is not connected. It suffices to show that every hyper-edge in H_G is contained in one hyper-edge of H. Suppose E is a hyper-edge of H_G we observe that the subgraph of G on this hyper-edge is a 2-connected graph since otherwise we can do the previous process again to separate this hyper-edge. Since it is a 2-connected graph then it must be contained in one hyper-edge of H since otherwise it would break the acyclicity of the hypergraph H. So H_G is a minimal element among the set of connected acyclic hyper-covers of G.

□

Definition 3.6.7. Let Z be a poset consists of elements (H, x) where $H \in \widehat{\mathcal{C}_n}$ is a connected acyclic hypergraph on n elements and x is a sequence of graphs on vertices of each hyperedge of H such that(it might be possible that a graph on one hyperedge of H is an empty graph i.e. there are no edges between any vertices in this hyperedge):

1. When H is the trivial hypergraph i.e it consists only one hyper edge which contains all n-elements, then the graph on this hyper-edge is asked to be non-connected and non-empty.

2. When H is a non-trivial hypergraph, then the union of graphs should be a non-empty graph.

And the order relation is $(H_1, x_1) \leq (H_2, x_2)$ if $H_1 \leq H_2$ as hypergraphs and $x_1 \leq x_2$ as union of graphs with the vertex set equaling to the vertex set of H i.e. the edge set of x_1 is a subset of edge set of x_2.

Let's first try to prove that the poset Z is actually Σ_n-equivariant homotopy equivalent to Δ_n^2. Then we try to show that Z is a desuspension model for the total cofiber X i.e. X is Σ_n-equivariant homotopy equivalent to suspension of the geometric realization of the poset Z.

Lemma 3.6.8. *Suppose we have a poset C, that decomposes as a disjoint union $C = A \coprod B$, with the property that if a is an element of A and b is an element of B, then either $a < b$ or a and b are incomparable. Suppose that $\hat{1}$ is a maximal element that can be adjoined to C. I.e., $\hat{1}$ is not an element of C, but it is the maximal element of $C \coprod \{\hat{1}\}$. Let CC be the poset of pairs (x, y) where x is in C satisfies:*

1. *$x \leq y$,*

2. *y is in $B \coprod \{\hat{1}\}$,*

3. if x is in B then $y < \hat{1}$. Equivalently, if $y = \hat{1}$ then x is in A.

The order in CC is defined by saying that $(x, y) \leq (x_1, y_1)$ if $x \leq x_1$ and $y \leq y_1$

Then the geometric realization of CC is naturally homotopy equivalent to the geometric realization of C.

Before proving the Lemma 3.6.8 let's recall what is the associated twisted poset of a given poset:

Definition 3.6.9. Let $tw(C)$ be the poset whose objects are pairs (x, y) of elements of C. The order in $tw(C)$ is defined as follows $(x, y) \leq (x_1, y_1)$ if $x_1 \leq x \leq y \leq y_1$. There is a natural transformations $tw(C) \Rightarrow C$ that sends $(x, y) \mapsto y$ and a natural transformation $tw(C) \Rightarrow C^{op}$, that sends $(x, y) \mapsto x$.

Remark. It is known that[Qui10, Page 94] both of these natural transformations induce equivalences of geometric realization based on Quillen's fiber lemma. It is also worth to point out that the geometric realization of $tw(C)$ is in fact homeomorphic to the geometric realization of C (but this homeomorphism is not induced by a functor) although we will not use this fact in this chapter.

Proof of Lemma 3.6.8. Now suppose $C = A \coprod B$ satisfying the conditions listed in Lemma 3.6.8. Then $tw(C)$ could be decomposed as a union of two posets. One consists of pairs (x, y) such that x is in A and the other consists of pairs such that y is in B. Let's denote these subposets P and Q respectively. Notice that $tw(C) = P \cup Q$ because the complement of $P \cup Q$ would consists of pairs (x, y) such that x is in B, y is in A and $x < y$, which is impossible. Moreover, if (x, y) is in $P \backslash Q$, and (x_1, y_1) is in $Q \backslash P$, then (x, y) and (x_1, y_1) are incomparable. Without losing any generality suppose $(x, y) \leq (x_1, y_1)$ then by definition it follows $x_1 \leq x \leq y \leq y_1$. However we already knew that $x_1 \in B$ which is a contradiction since $x \in A$ and two points $x \in A, x_1 \in B$ is non-comparable or $x \leq x_1$. So (x, y) and (x_1, y_1) are not comparable. It follows that the geometric realization $|tw(C)|$ is both the pushout and the homotopy pushout of the following diagram

$$|P| \leftarrow |P \cap Q| \rightarrow |Q| \tag{3.6.10}$$

Recall that CC consists of pairs (x, y), where x is in C, y is in $B \coprod \{\hat{1}\}$ and if $y = \hat{1}$ then x is in A. We note that CC decomposes as a union of two subposets. One consists of pairs (x, y) such that x is in A and the other consists of pairs (x, y) such that y is in B. Let us denote these subposets S and T respectively. Also, we observe that $CC = S \cup T$, and furthermore there are no relations between elements of $S \backslash T$ and $T \backslash S$. It follows that the geometric realization of CC is equivalent to the pushout and the homotopy pushout of the following diagram:

$$|S| \leftarrow |S \cap T| \rightarrow |T| \tag{3.6.11}$$

There is a natural poset map $\varphi_1 : P \rightarrow A^{\mathrm{op}}$ by sending a pair (x, y) to x. Similarly we have another natural poset map $\varphi_2 \colon Q \rightarrow B$ by sending a pair (x, y) to y. We claim that two poset maps φ_1 and φ_2 are homotopy equivalences. It suffices to show the truth for the poset map φ_1. Given any $c \in A$ we have an adjunction:

$$L \colon \varphi_1^{-1}(\geq c) \rightleftarrows \varphi_1^{-1}(c) : i \tag{3.6.12}$$

where $i : \varphi_1^{-1}(c) \rightarrow \varphi_1^{-1}(\geq c)$ is the inclusion functor and $L : \varphi_1^{-1}(\geq c) \rightarrow \varphi_1^{-1}(c)$ sends a pair (a, b) to a pair (a, c) here b is asked to be less or equal to c. Hence two categories $\varphi_1^{-1}(c)$ and $\varphi_1^{-1}(\geq c)$ are homotopy equivalent. Moreover we observe that the poset $\varphi_1^{-1}(c)$ has an initial object (c, c), it implies $\varphi_1^{-1}(c)$ is contractible. So $\varphi_1^{-1}(\geq c)$ is also contractible. Then according to the Quillen's fiber lemma we know φ_1 is a homotopy equivalence.

We recall the Thomason's homotopy colimits theorem here [Tho79, Theorem 1.2]. Given a functor $G : \mathcal{D} \to \mathbf{Cat}$ there is a natural homotopy equivalence:

$$\eta : \operatorname*{hocolim}_{\mathcal{D}} N(G) \to N(\int_{\mathcal{D}} G)$$

where $N(G)$ is the functor which sends each $d \in \mathcal{D}$ to the nerve $N(G(d))$.

Here we take $\mathcal{D} = B$ and $G = A^{\mathrm{op}}_{\leq} : B \to \mathbf{Cat}$ sending an element $b \in B$ to the poset $A^{\mathrm{op}}_{\leq b}$. And we observe there is a natural isomorphism between posets $P \cap Q$ and $\int_{b \in B} A^{\mathrm{op}}_{\leq b}$. In other words we have a a natural homotopy equivalence:

$$\eta : \operatorname*{hocolim}_{b \in B} |A^{\mathrm{op}}_{\leq b}| \to |P \cap Q|$$

and a commutative diagram as follows:

$$\begin{array}{ccccc} |P| & \longleftarrow & |P \cap Q| & \longrightarrow & |Q| \\ {\scriptstyle \varphi_1}\big\downarrow & & \big\uparrow{\scriptstyle \eta} & & {\scriptstyle \varphi_2}\big\downarrow \\ |A^{\mathrm{op}}| & \longleftarrow & \operatorname*{hocolim}_{b \in B} |A^{\mathrm{op}}_{\leq b}| & \longrightarrow & |B| \end{array} \tag{3.6.13}$$

where three vertical maps are all homotopy equivalences. Since η is a homotopy equivalence we can choose a homotopy inverse $\tilde{\eta}$ of it and we have a homotopy commutative diagram as follows:

$$\begin{array}{ccccc} |P| & \longleftarrow & |P \cap Q| & \longrightarrow & |Q| \\ {\scriptstyle \varphi_1}\big\downarrow & & {\scriptstyle \tilde{\eta}}\big\downarrow & & {\scriptstyle \varphi_2}\big\downarrow \\ |A^{\mathrm{op}}| & \longleftarrow & \operatorname*{hocolim}_{b \in B} |A^{\mathrm{op}}_{\leq b}| & \longrightarrow & |B| \end{array} \tag{3.6.14}$$

Since both two maps $|P \cap Q| \to |P|$ and $|P \cap Q| \to Q|$ are cofibrations we can replace maps φ_1, φ_2 by homotopic maps $\tilde{\varphi_1} : |P| \to |A^{\mathrm{op}}|$ and $\tilde{\varphi_2} : |Q| \to |B|$ respectively making the above homotopy commutative diagram commuting strictly:

$$\begin{array}{ccccc} |P| & \longleftarrow & |P \cap Q| & \longrightarrow & |Q| \\ {\scriptstyle \tilde{\varphi_1}}\big\downarrow & & {\scriptstyle \tilde{\eta}}\big\downarrow & & {\scriptstyle \tilde{\varphi_2}}\big\downarrow \\ |A^{\mathrm{op}}| & \longleftarrow & \operatorname*{hocolim}_{b \in B} |A^{\mathrm{op}}_{\leq b}| & \longrightarrow & |B| \end{array} \tag{3.6.15}$$

where the three vertical maps are still homotopy equivalences. Let W_1 denotes the homotopy push out of the second horizontal diagram. Then there is a homotopy equivalence between $|tw(C)|$ and W_1 since all three vertical maps are homotopy equivalences. We can do the exactly same thing to obtain a commutative diagram as follows where three vertical maps are homotopy equivalences:

$$\begin{array}{ccccc} |S| & \longleftarrow & |S \cap T| & \longrightarrow & |T| \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ |A| & \longleftarrow & \operatorname*{hocolim}_{b \in B} |A_{\leq b}| & \longrightarrow & |B| \end{array} \tag{3.6.16}$$

Let W_2 be the homotopy push out of the second horizontal diagram. Since all three vertical maps are homotopy equivalences we imply there is a homotopy equivalence from $|CC|$ to W_2. Clearly the second horizontal diagram is equivalent to the second horizontal diagram of the diagram 3.6.15. In other words two homotopy push outs W_1, W_2 are homotopy equivalent. Therefore $|tw(C)|$ is homotopy equivalent to $|CC|$. Moreover since $|tw(C)|$ is homotopy equivalent to $|C|$, $|CC|$ is also homotopy equivalent to $|C|$. □

Corollary 3.6.17. *Under the same assumptions as in the Lemma 3.6.8. Moreover let C be a G-poset and the two subposets A and B are also G-posets. In addition if the G-action on $\hat{1}$ is trivial then the geometric realization of the poset CC is G-equivariant homotopy equivalent to the geometric realization of C.*

Proof. We have already showed in the proof of the Lemma 3.6.8 that there is a homotopy equivalence $f : |CC| \to |C|$ which is just the composition of two homotopy equivalences $f_1 : |CC| \to |tw(C)|$ and $f_2 : |tw(C)| \to |C|$. We first claim tree maps $\varphi_1, \varphi_2, \eta$ in the diagram 3.6.13 are all G-equivariant homotopy equivalences. As for poset map $\varphi_1 : P \to A^{\text{op}}$, for any element $c \in A$, we observe the adjunction 3.6.12 is $C_G(c)$-adjunction[Mø15, Definition 2.2]. So $\varphi^{-1}(\geq c)$ is $C_G(c)$-equivariant homotopy equivalent to $\varphi_1^{-1}(c)$[Mø15, Proposition 2.3]. Moreover, the centralizer subposet $\varphi_1^{-1}(c)$ contains an initial object (c, c) which is fixed under $C_G(c)$, so $\varphi_1^{-1}(c)$ is $C_G(c)$-equivariant contractible, i.e. $\varphi_1^{-1}(\geq c)$ is $C_G(c)$-equivariant contractible. Then according to the equivariant Quillen's fiber lemma i.e. the Lemma 2.6.29 φ_1 is a G-equivariant homotopy equivalence. Similarly φ_2 is also a G-equivariant homotopy equivalence. As for the map η, for any subgroup $H \leq G$, the induced map $\eta^H : (\text{hocolim}_{b \in B} |A^{\text{op}}_{\leq b}|)^H \to |P \cap Q|^H$ could be identified as the map $\underset{b \in B^H}{\text{hocolim}} |(A^H_{b \leq B^H})^{\text{op}}| \to |P^H \cap Q^H|$ which is clearly a homotopy equivalence. So η is also a G-equivariant homotopy equivalence.

Since the inclusions $|P \cap Q| \to |P|, |P \cap Q| \to |Q|$ are both G-cofibration[tD87, Page 96] we can choose a G-homotopy inverse $\tilde{\eta}$ and two maps $\tilde{\varphi_1}, \tilde{\varphi_2}$ which are G-homotopy to maps φ_1, φ_2 respectively making the following diagram commute:

$$\begin{array}{ccccc} |P| & \longleftarrow & |P \cap Q| & \longrightarrow & |Q| \\ \tilde{\varphi_1}\big\downarrow & & \tilde{\eta}\big\downarrow & & \tilde{\varphi_2}\big\downarrow \\ |A^{\text{op}}| & \longleftarrow & \underset{b \in B}{\text{hocolim}} |A^{\text{op}}_{\leq b}| & \longrightarrow & |B| \end{array} \tag{3.6.18}$$

where all three vertical maps are all G-equivariant homotopy equivalences. And it is well known that[Mal, Proposition 1.2] when H is a finite group then the following map is a homeomorphism

$$\underset{A}{\text{hocolim}}\, X^H_\alpha \to (\underset{A}{\text{hocolim}}\, X_\alpha)^H \tag{3.6.19}$$

i.e. taking fixed points commutes with taking homotopy push-out. So given any subgroup $H \leq G$, we imply that the induced map $|tw(C)|^H \to W_1^H$ is a still a homotopy equivalence. Similaly the induced map $|CC|^H \to W_2^H$ is also homotopy equivalence. In other words, the induced map $f_1^H : |CC| \to |tw(C)|$ is homotopy equivalence. Then according to Bredon's Theorem[Bre67, Corollary II.5.5], f_1 is a G-equivariant homotopy equivalence. Similarly according to equivariant Quillen's fiber Lemma i.e. Lemma 2.6.29 we imply the map f_2: $|tw(C)| \to |C|$ is also a G-equivariant homotopy equivalence. Therefore the composition map f is also a G-equivariant homotopy equivalence. □

Corollary 3.6.20. *The poset Z is Σ_n-equivariant homotopy equivalent to the poset Δ_n^2 of not 2-connected graphs.*

Proof. Let $C = \Delta_n^2$, A be the subposet of non-connected graphs and B the subposet of connected graphs. And by definition the poset CC consists of pairs (G_1, G_2), where G_2 is a connected but not 2-connected graph, and G_1 a subgraph of G_2, or if G_1 is not connected, then G_2 can also be a new maximal element $\hat{1}$ that we can think of as the complete graphs on the n-vertices. If H is a proper connected acyclic hypergraph then we denote $\widetilde{H}$ to be the graph which is the union of complete subgraphs on each hyper-edge of H, it is clear that $\widetilde{H}$ is a connected but not 2-connected graph. Then we construct two poset maps $f : Z \to CC$ and $g : CC \to Z$, where f sends an element $(H, x) \in Z$ to $(x, \widetilde{H})$ if H is a non trivial hypergraph or sends an element $(1_H, x)$ to $(x, \hat{1})$ where 1_H means the trivial hypergraph; g sends $(G_1, G_2) \in CC$ to (H_{G_2}, G_1) if G_2 is a a connected graph, where H_{G_2} is the minimal hyper-cover of G_2 or it sends $(G, \hat{1})$ to $(1_H, G)$. Then we observe that when G_2 is a connected graph then $f(g((G_1, G_2))) = f(H_{G_2}, G_1) = (G_1, \widetilde{H_{G_2}}) \geq (G_1, G_2)$ and $f(g(G, \hat{1})) = f(1_H, G) = (G, \hat{1})$. Hence $f \circ g \geq id$ and it is clear this natural transformation from $f \circ g$ to id is Σ_n-equivariant. Similarly when H is a non-trivial hypergraph then $g(f(H, x)) = g(x, \widetilde{H}) = (H, x)$. And $g(f(1_H, x))) = g(x, \hat{1}) = (1_H, x)$. Hence $g \circ f = id$. Therefore g is a Σ_n equivariant homotopy inverse of f, in other words two posets CC and Z are Σ_n-equivariant homotopy equivalent. Then according to the Corollary 3.6.17 we know CC is also Σ_n-homotopy equivalent to C. Therefore Z is Σ_n-equivariant homotopy equivalent to $C = \Delta_n^2$. □

Now let's construct a desuspenison model for the total cofiber X in other words we want to find a space P such that $SP \simeq_{\Sigma_n} X$, where S means unreduced suspension. We first need a general desuspenion result regarding to a total cofiber:

Lemma 3.6.21. *Let $G : \widehat{\mathcal{D}} \to$ **Top** be a functor from a category $\widehat{\mathcal{D}}$ to the category of unbased topological spaces, where the category $\widehat{\mathcal{D}}$ has a final object $\hat{1}$, we let $\mathcal{D}$ be the full subcategory of $\widehat{\mathcal{D}}$ without the final object. Let $SG : \widehat{\mathcal{D}} \to$ **Top**$_*$ be the unreduced suspension of this functor which sends an element d to the unreduced suspension of the topological space $G(d)$ with the south pole as the base point. Then*

$$\operatorname{hocolim}_\bullet SG \simeq \operatorname{hocofib}(\operatorname{hocolim} G \xrightarrow{\pi} \operatorname{hocolim} *) \tag{3.6.22}$$

Both pointed and unpointed homotopy colimits are taken over category $\mathcal{D}$ and π is induced by the canonical map $G(d) \to *$ for each $d \in \mathcal{D}$. The equivalence $\simeq$ here is homotopy equivalence.

Proof. By definition of pointed homotopy colimits(see Definition 3.4.10) we know

$$\operatorname{hocolim}_\bullet SG \simeq \operatorname{hocolim} SG/|\mathcal{D}|$$

where $|\mathcal{D}| = \operatorname{hocolim} *$ is a subspace of $\operatorname{hocolim} SG$.

Then let's construct a continuous map:

$$\varphi : C(\operatorname{hocolim} G) \cup_\pi |\mathcal{D}| \to \operatorname{hocolim} SG/|\mathcal{D}| \tag{3.6.23}$$

where $C(\operatorname{hocolim} G)$ means the cone over $\operatorname{hocolim} G$. Suppose $(x, s) \in G(i_0) \times \Delta^n$ for a specific chain $i_0 \to \cdots \to i_n$ in $\mathcal{D}$ where $x \in G(i_0), s \in \Delta^n$. Let t means the coordinates in the cone in which $t = 1$ indicates the apex of the cone and $t = 0$ indicates the bottom of the cone. Then the coordinate $(t, (x, s))$ expresses a point in $C(\operatorname{hocolim} G) \cup_\pi |\mathcal{D}|$ where when $t = 1$ all coordinates are identified with the apex point $*$ and when $t = 0$ all points are identified with points in $|\mathcal{D}|$. Then φ sends an element $(t, (x, s))$ to the element $((1 - t, x), s)$ which is also associated to the chain $i_0 \to \cdots \to i_n$, here the first coordinates indicate the position in the suspension, in other words when the first coordinate equals 0 it is in the south pole and when the first coordinate equals 1 it is in the north pole. Finally we need to show that φ is a homeomorphism. It suffices to show that when $t = 0, 1$. When $t = 1$, the left hand side is just the apex of the cone. In the other hand, the first coordinate of its image is 0 which by definition is in the south pole. However south pole points are identified with the classifying space $|\mathcal{D}|$ and has been collapsed to a single point in $\operatorname{hocolim} SG/|\mathcal{D}|$, so the case when $t = 1$ is fine. When $t = 0$, in the left hand side all points are identified with points in $|\mathcal{D}|$ and its images in the right hand side are all north pole points which could be identified with the classifying space $|\mathcal{D}|$ just like the case of south pole i.e. the case $t = 0$ is also fine. Hence this map φ is actually a homeomorphism. In other words:

$$\operatorname{hocolim}_\bullet SG \simeq \operatorname{hocofib}(\operatorname{hocolim} G \to \operatorname{hocolim} *)$$

□

Lemma 3.6.24. *Let G be a functor from a category $\mathcal{D}$ to the category of topological spaces but take all values on CW-complexes. And let U also be a CW-complex. We form a homotopy pushout diagram in the following:*

$$\begin{array}{ccc} \operatorname{hocolim} G & \longrightarrow & U \\ \downarrow & & \downarrow \\ \operatorname{hocolim} * & \longrightarrow & P \end{array} \tag{3.6.25}$$

Then SP is homotopy equivalent to $\operatorname{hocofib}(\operatorname{hocolim}_\bullet SG \to SU)$

Proof. We consider a commutative diagram:

$$\begin{array}{ccc} \operatorname{hocolim} G & \longrightarrow & U \\ \downarrow & & \downarrow \\ \operatorname{hocolim} * & \longrightarrow & * \end{array} \tag{3.6.26}$$

The we take the homotopy cofiber of the canonical diagram map (3.6.25) → (3.6.26) which is

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & SP \end{array} \tag{3.6.27}$$

And according to Lemma 3.6.21 the total cofiber of the diagram 3.6.26 is $\operatorname{hocofib}(\operatorname{hocolim}_\bullet SG \to SU)$:

$$\begin{array}{ccccc} \operatorname{hocolim} G & \longrightarrow & U & & \\ \downarrow & & \downarrow & & \\ \operatorname{hocolim} * & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \\ \operatorname{hocolim}_\bullet SG & \longrightarrow & SU & \longrightarrow & \operatorname{hocofib}(\operatorname{hocolim}_\bullet SG \to SU) \end{array} \tag{3.6.28}$$

Similarly the total cofiber of diagram 3.6.27 is SP. And since the diagram 3.6.25 is a homotopy pushout diagram we know its total cofiber is just a point. Therefore, we have a cofibration sequence connecting the total cofiber of diagrams 3.6.25,3.6.26,3.6.27 as follows:

$$* \to \operatorname{hocofib}(\operatorname{hocolim}_\bullet SG \to SU) \to SP$$

In other words we have a weak equivalence $\mathrm{hocofib}(\mathrm{hocolim}_\bullet SG \to SU) \to SP$. Moreover, by assumption we know all three spaces $U, \mathrm{hocolim}\, *, \mathrm{hocolim}\, G$ are all CW-complexes, then the homotopy push-out P and the homotopy cofiber $\mathrm{hocofib}(\mathrm{hocolim}_\bullet SG \to SU)$ both have a homotopy type of a CW-complex. Then according to the Whitehead's theorem we know the weak equivalence is actually a homotopy equivalence i.e $SP \simeq \mathrm{hocofib}(\mathrm{hocolim}_\bullet SG \to SU)$. □

Since we can always write the space $|\partial(P(n))|$ as $S|\Pi_n|$ where S means unreduced suspension and $\Pi_n = \overline{P(n)}^{\min,\max}$(here this notation means we delete the minimal and the maximal element of this poset) is the poset of partitions on $[n-1]_+$ without the minimal and maximal elements. More general we have $|\partial(P(E_1) \times \cdots \times P(E_k))| = S|\overline{P(E_1) \times \cdots \times P(E_k)}^{\min,\max}|$. Let $\overline{F}^{\min,\max}$ be the new functor from $\mathcal{C}_n$ to the category of posets which sends a hypergraph $H = (E_1, E_2, \ldots, E_k)$ to the poset $\overline{P(E_1) \times \cdots \times P(E_k)}^{\min,\max}$, and as the morphisms it is defined exactly same as what we defined for the functor F.

Corollary 3.6.29. *Let G be the functor from $\mathcal{C}_n$ to **Top**$_*$ by taking geometric realization of the functor $\overline{F}^{\min,\max}$. Then the homotopy push-out P of the following homotopy pushout diagram is a desuspension model for the total cofiber X,i.e. $SP \simeq_{\Sigma_n} X$.*

$$\begin{array}{ccc} \mathrm{hocolim}\, G & \longrightarrow & |\Pi_n| \\ \downarrow & & \downarrow \\ \mathrm{hocolim}\, * & \longrightarrow & P \end{array} \tag{3.6.30}$$

Proof. According to Lemma 3.6.21 and Lemma 3.6.24 we know there is a homotopy equivalence $f : X \to SP$. As for the equivariance, let $H \leq \Sigma_n$ be any subgroup of Σ_n, according to the Bredon's result[Bre67, Corollary II], it suffices to show that the induced map $f^H : X^H \to SP^H$ is still a homotopy equivalence. Since taking fixed points commutes with taking homotopy colimits[Mal, Proposition]. Then the homotopy equivalence i.e. the formula 3.6.22 when the functor $G = \overline{F}^{\min,\max}$ is a Σ_n-equivariant homotopy equivalence. Since in this case $\mathrm{hocofib}(\mathrm{hocolim}\, G \xrightarrow{\pi} \mathrm{hocolim}\, *)^H \simeq \mathrm{hocofib}((\mathrm{hocolim}\, G)^H \xrightarrow{\pi} (\mathrm{hocolim}\, *)^H)$, and the homeomorphism 3.6.23 φ is clearly a Σ_n-equivariant map. Then we can apply the fixed point functor $(-)^H$ to all diagram in the proof of the Lemma 3.6.24 we know when $G = \overline{F}^{\min,\max}$, the induced map f^H is sill a homotopy equivalence. In other words SP is Σ_n-equivariant homotopy equivalent to X. □

Theorem 3.6.31. *The geometric realization of the poset Z is Σ_n-equivariant homotopy equivalent to the desuspension P of the total cofiber X.*

Proof. Now let

1. Z_1 be the subposet of Z consisting of elements with non-trivial hypergraph(i.e. it consists of more than one hyperedge) and graphs on each hyperedge such that the union of these graphs is not connected,

2. Z_2 be the subposet of Z consisting of elements with just non-trivial hypergraph,

3. Z_3 be the subposet of Z consisting of elements with the graphs on each hyperedge such that the union of these graphs is not-connected.

We observe that $Z_1 = Z_2 \cap Z_3$, Moreover for each element $x \in Z_2 \backslash Z_3$ i.e. this element x is in Z_2 not in Z_3 and an element $y \in Z_3 \backslash Z_2$, then x and y are non-comparable. Since the hypergraph behind

x is non-trivial and union graph of x is connected, however the hypergraph behind y is the trivial hyper graph and the union graph of y is not-connected. Hence the geometric realization of Z is the push-out or homotopy push-out of the following diagram:

$$|Z_2| \leftarrow |Z_1| \rightarrow |Z_3| \tag{3.6.32}$$

Apply Thomason's result[Tho79, Theorem 1.2] again we can replace the homotopy colimits in the diagram 3.6.30 by associated Grothendieck constructions. Then we construct three poset maps as follows:

1. $f_1 : Z_1 \to \int_{\mathcal{C}_n} G$ is defined by sending an element (H, x) in Z_1 to the element (H, π) where $\pi = (\pi_1, \pi_2, \ldots, \pi_k), \pi_i \in P(E_i)$ if $H = (E_1, E_2, \ldots, E_k)$. For subgraph of x in each hyperedge E_i the vertices of its connected component forms blocks of π_i.

2. $f_2 : Z_2 \to \int_{\mathcal{C}_n} *$ is defined by sending an element (H, x) to the hyper graph $H \in \int_{\mathcal{C}_n} *$

3. $f_3 : Z_3 \to \Pi_n$ is defined by sending an element (H, x) to the partition π with each block consisting of vertices in each connected component of x.

Moreover, we claim all poset maps f_1, f_2, f_3 are equivariant homotopy equivalences. Let's try to show it is true for f_1. Consider any element $(H, \pi) \in \int_{\mathcal{C}_n} G$, then the pre-image $f_1^{-1}(\leq (H, \pi))$ has a terminal object which is the element (H, x) with the same hyper graph H and the sub-graph of x in each hyper edge E_i is "locally complete" which means it is complete on each connected component and the connected components correspond to blocks in π_i . Then according to equivariant Quillen's fiber lemma i.e. Lemma 2.6.29 we know the vertical maps f_1 is a Σ_n-equivariant homotopy equivalence. Similarly f_2, f_3 are all Σ_n-equivariant homotopy equivalences. And these three maps f_1, f_2, f_3 will be served as a morphism between the following two diagrams:

$$\begin{array}{ccccc} |Z_2| & \longleftarrow & |Z_1| & \longrightarrow & |Z_3| \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ |\int_{\mathcal{C}_n} *| & \longleftarrow & |\int_{\mathcal{C}_n} G| & \longrightarrow & |\Pi_n| \end{array} \tag{3.6.33}$$

So we have a morphism which is level-wise equivariant homotopy equivalence. Then it will induce a Σ_n-equivariant homotopy equivalence from $|Z|$ to SP. Hence Z is a desuspension model of the total cofiber X. □

Corollary 3.6.34. *The total cofiber X is Σ_n-equivariant homotopy equivalent to $S|\Delta_n^2|$ i.e. the total cofiber Y is Σ_n-equivariant homotopy equivalent to $\Sigma S|\Delta_n^2|$*

Proof. We have proved the geometric realization of the poset Z is Σ_n-equivariant homotopy equivalent to the total cofiber X and $S|\Delta_n^2|$, in other words the total cofiber X is Σ_n-equivariant homotopy equivalent to $S|\Delta_n^2|$. Then we apply the reduced suspension functor we know the total cofiber Y is Σ_n-equivariant homotopy equivalent to $\Sigma S|\Delta_n^2|$. On the other hand we also showed that Y is Σ_n-equivariant homotopy equivalent to Q_{n-1}. Therefore Q_{n-1} is Σ_n-equivariant homotopy equivalent to $\Sigma S|\Delta_n^2|$. □

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