**Philipp Schmitt**

# Strict quantization of certain classes of analytic functions

PhD thesis • Department of Mathematical Sciences • University of Copenhagen

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This PhD thesis has been submitted to the PhD School of the Faculty of Science, University of Copenhagen, on August 31, 2020.

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ISBN: 978-87-7125-033-6

*To my family,*
*and anyone else who supported me*
*while writing this thesis.*

*Abstract:*

This thesis studies strict quantizations in a Fréchet-algebraic setting.

In the Introduction, we review the quantization problem and different approaches to its solution: formal deformation quantization, strict deformation quantization in the sense of Rieffel, Berezin–Toeplitz quantization, and strict quantization in a Fréchet-algebraic setting.

In Paper I, which is joint with M. Schötz, we study strict Fréchet-algebraic quantizations of a family of manifolds $M_{\mathrm{red}}$ that can be obtained via phase space reduction from $\mathbb{C}^{1+n}$ with the Wick product in different signatures. In particular, we show how to reduce the formal Wick star product to $M_{\mathrm{red}}$, compute its defining bidifferential operators explicitly, and prove that it restricts to a strict product on a subalgebra of polynomial functions. We prove that this product extends to a continuous product on the Fréchet algebra of certain analytic functions that admit a holomorphic extension to a larger space. We obtain an isomorphism between the Fréchet-algebraic quantizations for different signatures, which is similar to a Wick rotation.

In Paper II, we obtain strict quantizations for semisimple coadjoint orbits $\mathcal{O}$ of semisimple connected Lie groups $G$. We give an explicit formula for the canonical element of the Shapovalov pairing, which was used by Alekseev–Lachowska to define a formal $G$-invariant star product on $\mathcal{O}$. We show that the formal star product converges on polynomials, and, using the explicit formula for the canonical element, we show that it extends to a strict $G$-invariant product on the Fréchet algebra of all functions that admit a holomorphic extension to the complexification of $\mathcal{O}$. In this setting, we also have an analogue of a Wick rotation.

In the Appendix, we show that all the reduced manifolds $M_{\mathrm{red}}$ are coadjoint orbits, and that the strict star products obtained for $M_{\mathrm{red}}$ via phase space reduction as in Paper I coincide with the strict star products obtained in Paper II.

*Résumé:*

I denne afhandling behandler vi streng kvantisering ud fra en Fréchet-algebraisk tilgang.

I indledningen indfører vi kvantiseringsproblemet og diskuterer forskellige løsningstilgange: formel deformationskvantisering, streng deformationskvantisering som indført af Rieffel, Berezin–Toeplitz kvantisering, og streng kvantisering ud fra en Fréchet-algebraisk tilgang.

I Paper I (sammen med M. Schötz) studerer vi strenge Fréchet-algebraiske kvantiseringer af en familie af mangfoldigheder $M_{\mathrm{red}}$, som konstrueres ved hjælp af faserumreduktion fra $\mathbb{C}^{1+n}$ med Wick-produktet i forskellige signaturer: vi demonstrerer hvordan man reducerer Wick-produktet til $M_{\mathrm{red}}$, beregner bidifferentialoperatorerne som definerer det og beviser at det restringerer til en algebra af polynomiale funktioner. Derudover viser vi at produktet kan udvides til fuldstændiggørelsen af denne algebra, som er en Fréchet algebra og består af funktioner med en udvidelse til holomorfe funktioner på et større rum. Vi indfører en isomorfi mellem de Fréchet-algebraiske kvantiseringer med forskellige signaturer, som ligner en Wick rotation.

I Paper II konstruerer vi strenge kvantiseringer af semisimple koadjungerede baner $\mathcal{O}$ af semisimple sammenhængende Lie grupper $G$. Vi giver en eksplicit formel for det kanoniske element af Shapovalov-parringen, som Alekseev–Lachowska brugte til at definerer formelle $G$-invariante stjerneprodukter. Vi beviser at det formelle stjerneprodukt konvergerer på polynomier og med formlen for det kanoniske element viser vi at produktet udvides til et strengt $G$-invariant produkt på Fréchet algebraen af alle funktioner som kan udvides til holomorphe funktioner på kompleksificeringen af $\mathcal{O}$. Der eksisterer også en Wick rotation i den her formalisme.

I appendikset beviser vi, at alle de reducerede mangfoldigheder $M_{\mathrm{red}}$ er koadjungerede baner og at det strenge produkt, som vi konstruerede på $M_{\mathrm{red}}$ ved hjælp af faserumreduktion i Paper I, er det samme, som det strenge produkt vi konstruerede i Paper II.

# Acknowledgements

First and most important, my warmest thanks to my adviser Ryszard Nest. You were always there when I had questions, no matter whether it was about complex analysis, algebraic topology, or mathematical logic. Your insight and intuitive way of explaining things have changed my own understanding of mathematics drastically. Thank you!

Thanks to Pierre Bieliavsky and Alexander Gorokhovsky for the warm hospitality that I received during my stays at the Université catholique de Louvain and the University of Colorado, Boulder. I learned a lot from you during this time and broadened my mathematical horizons.

Thanks to Chiara Esposito, Matthias Schötz, and Stefan Waldmann for their interest in my work, for solving problems together, and for all the other support that I received in the last three years after leaving Würzburg.

My officemates made my average working day much more enjoyable, and I really missed you during the lockdown. Thank you for all the mathematical and non-mathematical discussions.

Writing this thesis would not have been possible without the support of my family. Thanks for always being there, and for running to the German authorities whenever I needed some documents stamped.

Special thanks to the only two people whom I saw regularly in person during the lockdown, and not just on a computer screen: Calista and Clemens, you are amazing! Thanks for all the boardgaming and the fun time that we spent together, for proofreading some parts of this thesis, and for preventing me from going crazy during the lockdown.

Thanks also to everyone else who made this world a bit more normal again, in these unusual times!

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# Preface

The quantization problem is a good example to illustrate the close relationship between mathematics and theoretical physics. This problem asks to quantize a classical mechanical system, which can be formalized as the task of associating non-commutative (quantum) algebras to commutative Poisson algebras, in such a way that the commutator corresponds to the Poisson bracket. Formal deformation quantization is a mathematical theory which tries to study this problem in a simplified formal setting. One of the most important theorems in this context is Kontsevich's formality theorem, which ensures the existence of a formal deformation quantization on any Poisson manifold. On the one hand, these mathematical results can be applied to help to understand the original physical problem, but on the other hand they are also of independent mathematical interest. For example, formal deformation quantizations can be used to formulate an algebraic analogue of the Atiyah–Singer index theorem.

More generally, the mathematical fields of non-commutative geometry and quantum groups emerged from the need of having a good formalism to study non-commutative spaces and their symmetries, as required in order to better understand quantum mechanics. There are numerous applications to quite different mathematical problems, ranging from number theory to representation theory.

The aim of this thesis is to study the quantization problem from a mathematical perspective. We will introduce the problem in detail and discuss some of the numerous approaches to its solution. In the main part of this thesis we present new results on a Fréchet-algebraic approach: We obtain quantizations of algebras of certain analytic functions. Our hope is that these results will lead to a better understanding of some of the other approaches to quantization, too. For example, the Fréchet algebras interpolate between the quantum algebras obtained via Berezin–Toeplitz quantization. It also seems reasonable that they are related to $C^*$-algebraic quantizations, at least in some well-behaved examples.

This thesis consists of an introduction, two research articles, and an appendix. The research articles can be found at arxiv:1911.12118 and arxiv:1907.03185. They have both been edited to comply with the conventions used in the rest of this thesis, and a few footnotes referring to other parts of this thesis were added.

When referencing within a certain chapter, we only indicate the number of the corresponding theorem or equation, e.g. Theorem 3.18 or (2.1). When referencing to a different chapter, this number is prefixed by I or II for the research articles, and Intro or Appendix for the introduction or appendix, e.g. Theorem II.3.18 or (Intro.2.1).

Numbers in brackets like (2.1) always refer to equations.

Throughout this work we use $\mathbb{N}$ to denote the natural numbers $\{1, 2, 3, \dots\}$ and let $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. As usual $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote integers, rational numbers, real numbers, and complex numbers. Formal power series over a ring $R$ are denoted by $R[[\nu]]$ and Laurent series by $R[\nu^{-1}, \nu]]$.

We use standard differential geometric notation throughout this thesis. $\mathscr{C}^\infty(M)$ stands for (complex) smooth functions on a manifold, $\mathrm{T}M$ is the (real) tangent bundle of the manifold $M$, $\mathrm{T}^{\mathbb{C}}M$ its complexification, and $\mathrm{T}^*M$ the (real) cotangent bundle. If $V$ is a vector space, then we denote its tensor algebra by $\mathrm{T}^\bullet V$, and the subspaces of symmetric and antisymmetric tensors by $\mathrm{S}^\bullet V$ and $\Lambda^\bullet V$, respectively. This notation applies also to vector bundles $E$ over $M$, e.g. $\Lambda^k E$ denotes the vector bundle obtained as the $k$-fold antisymmetric tensor product of $E$. $\Gamma^\infty(E)$ denotes the smooth sections of $E$. In particular, $\Gamma^\infty(\Lambda^k \mathrm{T}^*M)$ denotes $k$-forms on $M$.

Lie groups are denotes by $G$ or $H$, or by $K$ if they are compact. We denote Lie algebras always by gothic letters, e.g. $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$. $\mathscr{U}\mathfrak{g}$ stands for the universal enveloping algebra.

# Introduction

This introduction is divided into three parts. The first part consists of a very short non-technical summary of the motivation of this thesis, aimed at a reader without any mathematical background. The second part is a mathematical introduction to the topic of quantization. We give a brief review of classical and quantum mechanics, and introduce the quantization problem. Then we discuss several approaches to solve this problem: Formal deformation quantization, strict quantization in a $C^*$-algebraic context, and a Fréchet-algebraic approach. This part is supposed to give the reader the necessary background knowledge to understand the broader context, in which this thesis is placed. The third part consists of a more detailed summary of previous results that are directly relevant to the author's work, a description of the main contributions of this thesis, and an outlook on possible future directions.

## 1 Quantization: A non-technical introduction

In this short section we attempt to explain what the central problem of this thesis, the *quantization problem*, is about, without assuming any background knowledge. We give a brief description of the development of quantum mechanics, and explain why a good theory of quantization is desirable.

### Classical physics

At the end of the 19th century, physicists believed to have a rather accurate description of the world. Most of the existent phenomena could be described and predicted by their theories, and there was good progress on the remaining questions. One important theory concerned the understanding of electromagnetism, i.e. light, which was described by Maxwell's equations. These equations describe light as a wave that propagates through space, similar to a water wave, although the precise description is more complicated: A light wave consists of oscillating magnetic and electric fields, that interact with each other and with matter, and only in vacuum will those fields be synchronous, so that they can essentially be treated as one. Maxwell's equations could explain and accurately predict many phenomena known at that time, like refraction or polarization.

### Of waves and particles

However, in the beginning of the 20th century experiments showed that, under certain circumstances, light behaved more as a collection of particles and not as a wave, contradicting the existing physical theories. For example, if light behaved as a wave and was shone on a metal, then one would expect that it knocks out electrons after a certain amount of time needed for the wave to deposit enough energy, and that this time is shorter if the light is more intense ("brighter"). However, when performing this experiment, one finds that there is no waiting time, but electrons are only emitted if the frequency of the light is high, i.e. if the wave oscillates quickly. The intensity has no influence on whether electrons are knocked out or not. Einstein explained this by postulating that light consists of particles, whose energy is high if the frequency is high. Electrons are hit by only one of those particles at a time, and get dislodged if the frequency is above a certain threshold. The intensity of the light is the number of particles in the beam, and therefore only affects the number of dislodged electrons, but not whether electrons can be dislodged or not.

Additionally, new experiments showed that particles, which were believed to be localized in space and essentially described as small balls moving around, behaved as waves in certain experiments. Shooting a beam of electrons through two slits produced an interference pattern on a screen, similar to the pattern one would expect for waves: One can think of each of the slits as emitting a wave and the intensity varies along the screen, depending on whether these waves reach a certain point in phase or not: Two wave peaks add up to high intensity, whereas a peak and a trough cancel each other. Such a cancelling, meaning that electrons do not hit certain parts of the screen at all, cannot be explained in a particle picture.

More surprisingly, when physicists acquired the technology to shoot only a single electron at a time, that electron appeared at a random position on the screen, following the intensity pattern described above. So the single electron seems to pass through both slits, waves emerging from the two slits interfere with each other, and the electron appears at a random position with probability proportional to (the square of) the amplitude of the superposition of these waves.

### Quantum mechanics is born

The picture that physicists had made of the world changed drastically with those observations. The classical description, where electrons behave like particles and light behaves as a wave described by Maxwell's equations, became known as *classical physics*, whereas the new theory describing both in a uniform way became known as *quantum physics*. In this new theory, all particles (and light, which is essentially a bunch of particles, called *photons*) are described by waves, called probability waves. Such a wave propagates through space and may interfere with itself, just as a water wave. When measuring where a particle is, it will appear at a random position with probability proportional to (the square of the absolute value of) its probability wave. It is impossible to predict where exactly the particle will appear.

Even worse, this new theory also predicts that certain properties of the particle cannot be measured simultaneously. If one tries to determine its position, then the velocity becomes more uncertain, and vice versa, when determining the velocity, the particle spreads out more and more. This is not a problem coming from a lack of good measurement devices, it is inherent to the theory: Measuring certain properties of a particle, will change this particle's state and always influence other properties.

Physicists were of course aware of the fact that such a theory sounds crazy, as we cannot observe any of those quantum effects in the macroscopic world that we observe on a daily basis. The thesis that you are currently reading is somewhere in space, probably on your desk, and if anyone told you that it is a wave, interfering with itself and appearing at random positions when you look at it, then you would probably be inclined to send that person to a psychiatric ward. Similarly, many physicists were unsatisfied with quantum theory, and hoped for different explanations. Einstein famously said *"God does not play dice with the universe."* to express that he was repelled by the probabilistic aspects of this theory, Niels Bohr said *"Hvis kvantemekanikken ikke gør dig svimmel, har du ikke forstået noget som helst."* (*"If quantum mechanics hasn't profoundly shocked you, you haven't understood it."*), and the similar quote *"If you think you understand quantum mechanics, you don't understand quantum mechanics."* is commonly attributed to Richard Feynman.

#### Quantization

But no matter how abstruse quantum mechanics might sound, until now no better theory has been found. On the contrary, quantum mechanics agrees with the experiments to a high degree and can make extremely accurate (probabilistic) predictions, but none of the proposed alternatives does. This, of course, poses a problem to physicists: How does one find a good mathematical description of a quantum system, if quantum mechanics is so far from our intuition? One idea is to start with an analogous classical system, which we can describe very well, and then "quantize" this theory by replacing certain objects in the formalism with other quantum objects. This process is usually called *quantization*.

There is no universal theory of quantization that would allow to get quantum systems out of any classical input. On the contrary, there are a few theorems asserting that if one demands too many similarities between a classical and a quantum system, then there does not exist a quantization procedure. So far, many different approaches to quantization have been proposed and studied extensively.

In this thesis, we discuss and relate some of these approaches. We show how one of them can be applied to so-called *coadjoint orbits*, a class of classical systems which possess a lot of symmetries.

#### Quantization in mathematics

The quantization problem, i.e. the problem of finding a good quantization procedure to quantize classical systems, is a good example of a problem that originated in physics and which has led to important developments in pure mathematics. Many notions, as for example formal deformation quantizations that we introduce in Subsection 2.4, can be used to understand other mathematical problems better.

More generally, the whole mathematical fields of non-commutative geometry and quantum groups were motivated by quantum mechanics and the fact that quantum observables do not commute. Many of the mathematical developments in those fields have then inspired new physical developments, showing the close interaction between mathematics and physics.

## 2 Mathematical introduction

In this section we give a mathematical introduction to quantization. This should provide the broad context, in which to see the results of this thesis. We start by briefly outlining classical mechanics and quantum mechanics in Subsection 2.1 and Subsection 2.2, focussing on their observable algebras. The observable algebras will be the starting point for many, but not all, quantization theories. We formulate the quantization problem, and present some no-go theorems in Subsection 2.3.

We then discuss possible approaches to solve the quantization problem. In Subsection 2.4 we introduce formal deformation quantization, which tries to neglect analytic aspects of the problem, and thereby makes it accessible to algebraic methods. We present the main ideas needed to obtain existence and classification results, both in the symplectic and in the more general Poisson case. In Subsection 2.5 we discuss strict deformation quantization in the sense of Rieffel, presenting a very prominent construction of such strict deformation quantizations and discussing why this construction, as many others, cannot be applied to the 2-sphere. In Subsection 2.6 we give a more general definition of strict quantizations, that also covers examples like the Berezin–Toeplitz quantization.

To illustrate the importance of formal deformation quantizations, also in other areas of mathematics, we describe the algebraic index theorem in Subsection 2.7. This theorem is the starting point for many results about obstructions to the existence of strict quantizations. Finally, we describe a Fréchet-algebraic approach to quantization in Subsection 2.8.

### 2.1 Classical mechanics

In this subsection, we recall briefly the Hamiltonian formalism in classical mechanics. Our description is, of course, far from complete. For more details, see [Arn89, Gol91, Tak08, Wal07].

There are two powerful formulations of classical mechanics, namely *Lagrangian* and *Hamiltonian mechanics*. Essentially, those formulations are equivalent to Newton's law of motion, but offer more freedom in the choice of coordinates to describe the problem. The Hamiltonian formalism will be more useful for us, as it makes the similarities to quantum mechanics more apparent. However, the Lagrangian formalism is more commonly used when describing relativistic theories, which is due to the fact that it does not distinguish time and energy, and therefore is manifestly Lorentz invariant. There are interesting physical systems, for example dissipative systems, that cannot be described in either framework.

In Hamiltonian mechanics, the time evolution of a classical mechanical system is determined by its current *state*, consisting of the positions and momenta of all particles. The set of all allowed positions and momenta is usually referred to as the *phase space* $M$ of the system, and assumed to be a smooth manifold. We would like to compute how any point in phase space evolves with time.

In many examples, the positions of the particles are described by some manifold $Q$, and the phase space is just the cotangent bundle $M = \mathrm{T}^*Q$. Choose coordinates $q^1, \ldots, q^d$ on $Q$. These coordinates induce coordinates $x^1, \ldots, x^d, p_1, \ldots, p_d$ on $\mathrm{T}^*Q$, given by $x^i(\alpha_q) = q^i(q)$ and $p_i(\alpha_q) = \alpha_q\big(\frac{\partial}{\partial q^i}\big|_q\big)$ where $\alpha_q \in \mathrm{T}^*_q Q$. Now given a smooth real function $H \in \mathscr{C}^\infty(M)$, called *Hamiltonian*, which corresponds to the energy of the system, we obtain the *equations of motion*

$$\frac{\mathrm{d}x^i(\gamma(t))}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}(\gamma(t)) \quad \text{and} \quad \frac{\mathrm{d}p_i(\gamma(t))}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}(\gamma(t)), \tag{2.1}$$

where $\gamma$ is a curve in $M$, describing the time evolution of the state $\gamma(0)$. Let us give a coordinate independent formulation of these equations.

**Lemma 2.1** *With the notation above, $\omega = \sum_{i=1}^d \mathrm{d}x^i \wedge \mathrm{d}p_i$ defines a non-degenerate closed 2-form on $M = \mathrm{T}^*Q$, independent of the chosen coordinates $q^1, \ldots, q^d$.*

Non-degeneracy means that the map ${}^\flat\colon \mathrm{T}M \to \mathrm{T}^*M$, $v \mapsto v^\flat := \omega(v, \cdot\,)$ is an isomorphism. Denote its inverse by ${}^\sharp$. We can then define a *Poisson bracket*

$$\{\,\cdot\,,\,\cdot\,\}\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad \{f, g\} = \omega((\mathrm{d}f)^\sharp, (\mathrm{d}g)^\sharp), \tag{2.2}$$

meaning that $\{\,\cdot\,,\,\cdot\,\}$ is a Lie bracket, and also a derivation whenever one of its arguments is fixed. For any $H \in \mathscr{C}^\infty(M)$, we call the vector field $X_H$ corresponding to the derivation $\{H, \cdot\,\}$ of $\mathscr{C}^\infty(M)$ the *Hamiltonian vector field* of $H$. Equivalently, we could define it as $X_H = (\mathrm{d}H)^\sharp$. The equations of motion (2.1) mean that the Hamiltonian vector field $X_H$ coincides with the derivative of a trajectory in phase space for all times $t$, or, said differently, the time evolution of the physical system is given by the flow of $X_H$. In terms of the Poisson bracket, the equations of motion (2.1) can then be rewritten as

$$\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} = \{f, H\}(\gamma(t)) \tag{2.3}$$

where $f$ is any of the coordinates $x^1, \ldots x^d, p_1, \ldots, p_d$. It is easy to see that this equation remains true if $f \in \mathscr{C}^\infty(M)$.

Let us make some abstractions of the relevant structures above.

**Definition 2.2 (Symplectic manifold)** *A* symplectic manifold $(M, \omega)$ *is a manifold $M$ endowed with a non-degenerate closed 2-form $\omega \in \Gamma^\infty(\Lambda^2 \mathrm{T}^* M)$.*

There is a procedure of symplectic reduction, that allows to eliminate conserved quantities from the equations of motion. Starting with a cotangent bundle and doing symplectic reduction, we can end up with a symplectic manifold that is not a cotangent bundle. So symplectic manifolds are not just a mathematical generalization, they occur naturally in physics. Note that the definition of a Poisson structure from a symplectic form $\omega$ and the definition of Hamiltonian vector fields still work in this context.

More generally, we can only require the existence of a Poisson bracket $\{\cdot, \cdot\}$. In this case the definition of a Hamiltonian vector field as the derivation $\{H, \cdot\}$ still makes sense.

**Definition 2.3 (Poisson manifold)** *A* Poisson manifold $(M, \pi)$ *is a manifold $M$ endowed with a bivector field $\pi \in \Gamma^\infty(\Lambda^2 \mathrm{T}M)$ such that the bracket $\{f, g\} := \pi(\mathrm{d}f, \mathrm{d}g)$ satisfies the Jacobi identity. In this case, $\{\cdot, \cdot\}$ is called the* Poisson bracket induced *by $\pi$.*

Note that $\{\cdot, \cdot\}$ is automatically a derivation of any of its arguments when the other one is fixed. It satisfies the Jacobi identity if and only if $[\![\pi, \pi]\!] = 0$ where $[\![\cdot, \cdot]\!]$ denotes the Schouten–Nijenhuis bracket of multivector fields.

To summarize, the time evolution of a classical mechanical system can be computed from two pieces of data: a Poisson structure on the phase space $M$ and a Hamiltonian function $H \in \mathscr{C}^\infty(M)$ that is the energy of the system. The time evolution is then given by the flow of the Hamiltonian vector field $X_H$.

Finally, we would like to change perspective and view the equation of motion (2.3) not as an equation describing how the state of the system changes, but rather as an equation describing how the observables $f \in \mathscr{C}^\infty(M)$ change. We think of the state of the system as being time independent. To determine the value of any observable $f_0 \in \mathscr{C}^\infty(M)$ after time $t$ we evolve the observable according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \{f_t, H\} \tag{2.4}$$

and evaluate the resulting function on the constant state of the system. If $f_0$ itself depends on the time, this equation stays valid if we add $\frac{\partial}{\partial t} f_0$ on the right hand side.

The immediate advantage of this point of view is that we can define the state of the system as a functional on the observables, i.e. as a map $\mathscr{C}^\infty(M) \to \mathbb{C}$. The pure state that was given by a point $x \in M$ before then corresponds to the evaluation functional at $x$, and there are many more mixed states. To do statistical mechanics, we could even replace a state with a map $\mathscr{C}^\infty(M) \to \mathcal{P}(\mathbb{C})$ to the space of probability measures on $\mathbb{C}$.

## 2.2 Quantum mechanics

The idea to consider observables as the fundamental objects and to define states as linear functionals on the observables becomes more relevant in quantum mechanics. In fact, it is often unclear what a good definition of quantum state, that does not make use of observables, would be. In contrast, the quantum observables turn out to be (associated to) a C*-algebra, and therefore have a lot of structure. We start with an explicit example and then discuss an abstract formulation of quantum mechanics. Many more details can be found in [RS72, Rud91, Sch90, Thi02, Wal07].

**Example 2.4 (Particle in $\mathbb{R}^d$)** Let us look at the example of a particle moving in $\mathbb{R}^d$. This quantum system is usually described by the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$ of square integrable functions on $\mathbb{R}^d$. A pure state $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ should be thought of as a wave function describing the probability of measuring the particle at a certain position in $\mathbb{R}^d$. Note however that $\mathbb{R}^d$ is only the configuration space and not the whole phase space of the corresponding classical system, indicating that it could be difficult to define quantum states when the classical system is not a cotangent bundle.

An observable $A$ is a possibly unbounded self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^d)$. We will not go into any of the functional analytic difficulties arising from the fact that observables may not be bounded, but rather treat them in a naive way as if they were bounded. The position and momentum observables are the unbounded self-adjoint operators $\hat{X}^i$ and $\hat{P}_i$, defined (on appropriate domains) by $\hat{X}^i(f)(x) = x^i f(x)$ and $\hat{P}_i(f) = -\mathrm{i}\hbar \frac{\partial}{\partial x^i} f$. Their commutator

$$[\hat{X}^i, \hat{P}_j] = \mathrm{i}\hbar \delta^i_j \tag{2.5}$$

resembles the classical Poisson bracket $\{x^i, p_j\} = \delta^i_j$. Here $\hbar$ is Planck's constant, relating the angular frequency of a photon to its energy, and $\delta^i_j$ is 1 if $i = j$ and 0 otherwise. The non-commutativity of the quantum mechanical observables leads to the *uncertainty principle*: non-commuting observables cannot be measured with arbitrary precision at the same time. Measuring one of the observables influences the system and changes the expected measurement outcomes of the other.

The spectrum of an observable $A$ is the set of possible values when measuring $A$, and since $A$ is self-adjoint, it is a subset of $\mathbb{R}$. The probability $\mathcal{P}(E)$ of measuring a value in a Borel set $E \subseteq \mathbb{R}$ is given by

$$\mathcal{P}(E) = \langle \psi, P_A(E)\psi \rangle \,, \tag{2.6}$$

where $P_A(E)$ is the spectral projection associated to $A$, and $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ is the state of the system. As in the classical case, we can describe a quantum mechanical system by either evolving the state or the observables with time. These two points of view are called the *Schrödinger* or *Heisenberg picture* of quantum mechanics. In the Heisenberg picture, the time evolution of an observable $A_0$ is defined by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} A_t = \frac{1}{\mathrm{i}\hbar}[A_t, \hat{H}] \,, \tag{2.7}$$

and we can again add $\frac{\partial}{\partial t} A_0$ to the right hand side, if $A_0$ itself depends explicitly on time. This is clearly analogous to the classical case, and the observable $\hat{H}$ can be viewed as the quantum observable associated to the classical Hamiltonian $H$.

Abstracting properties of this example, the observables of a quantum mechanical system are given by the self-adjoint elements in an algebra $\mathcal{A}$ of possibly unbounded operators. A precise formulation can be obtained for example by using $\mathrm{O}^*$-algebras, see [Sch90], but for our purposes it will be enough to mention that, by considering bounded functions of the self-adjoint elements, it is usually possible to pass to bounded operators and $\mathrm{C}^*$-algebras. The time evolution of an observable is given as in (2.7).

States can be defined as positive linear functionals on $\mathcal{A}$. Note however, that this is not sufficient for defining the superposition of two pure states. Taking a linear combination of the linear functionals, that describe two states, would only give a probabilistic mixture. Instead, one has to represent the algebra $\mathcal{A}$ on a Hilbert space, and then one can define the superposition of two vector states as the vector state corresponding to the sum of vectors. So a concrete representation of $\mathcal{A}$ on a Hilbert space is necessary, and an important part of the quantum theory.

This raises the question to what extent the representation of $\mathcal{A}$ on a Hilbert space matters, or, in other words, to what extent $\mathcal{A}$ alone determines the quantum system. As a rule of thumb, the chosen representation is irrelevant when the physical system has only finitely many degrees of freedom: Consider the $\mathrm{C}^*$-algebra obtained by applying the function $t \mapsto \mathrm{e}^{\mathrm{i}t}$ to operators satisfying the canonical commutation relations (2.5). Then the Stone–von Neumann theorem (see e.g. [AM85, Theorem 5.4.25]) says that any two irreducible representations of this $\mathrm{C}^*$-algebra on a Hilbert space are unitarily equivalent, if they satisfy a natural continuity assumption. However, this changes drastically when passing to systems with infinitely many degrees of freedom, where it is the origin of interesting phenomena like spontaneous symmetry breaking.

## 2.3 The quantization problem

The *quantization problem* asks to associate a quantum mechanical system to a classical mechanical system. That is, given a phase space $M$ and a Hamilton function $H \in \mathscr{C}^\infty(M)$, we have to construct an operator algebra $\mathcal{A}_\hbar$ (ideally a $\mathrm{C}^*$-algebra), represented on a Hilbert space $\mathcal{H}$, and some correspondence between classical observables $f \in \mathscr{C}^\infty(M)$ and quantum observables $A \in \mathcal{A}_\hbar$.

Since classical mechanics is, when considering macroscopic objects, a very good approximation of quantum mechanics, we expect to recover the classical system in a *classical limit*: Note that physically we cannot change the value of Planck's constant $\hbar$, so this limit needs to be understood in the sense that other characteristic quantities of the system, that have the same dimension as $\hbar$, become large when compared to $\hbar$. Mathematically, we will simply treat $\hbar$ as a parameter, and ask to construct a family of quantum systems for different values of $\hbar$, accumulating at 0. The classical limit then becomes the limit $\hbar \to 0$.

The exposition in the previous two subsections has shown that the classical and quantum observable algebras $\mathscr{C}^\infty(M)$ and $\mathcal{A}_\hbar$ are similar: Both are $^*$-algebras and the actual observables are given by the self-adjoint elements. The classical algebra is commutative, and endowed with the extra structure of a Poisson bracket, which, in particular, is a Lie bracket. The quantum algebra is non-commutative, and therefore the commutator defines a Lie bracket.

Possible measurement outcomes are defined by the spectrum, and therefore directly related to the associative structure of the algebras. Comparing the time evolution equations (2.3) and (2.7), we see that the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ corresponds to the rescaled commutator $\frac{1}{\mathrm{i}\hbar}[\,\cdot\,,\,\cdot\,]$. Therefore, it is reasonable to try to construct some quantization map $Q_\hbar$, mapping classical observables to quantum observables, that respects the $^*$-algebra structures and intertwines the rescaled commutator with the Poisson bracket. That is, we would like some correspondence

$$\lambda Q_\hbar(f) + Q_\hbar(g) \leftrightsquigarrow Q_\hbar(\lambda f + g)\,, \tag{2.8a}$$

$$Q_\hbar(f)^* \leftrightsquigarrow Q_\hbar(\overline{f})\,, \tag{2.8b}$$

$$Q_\hbar(f)Q_\hbar(g) \leftrightsquigarrow Q_\hbar(fg)\,, \tag{2.8c}$$

$$\tfrac{1}{\mathrm{i}\hbar}[Q_\hbar(f), Q_\hbar(g)] \leftrightsquigarrow Q_\hbar(\{f,g\})\,. \tag{2.8d}$$

However, not all of these requirements can be implemented exactly, already in the simple example of a particle moving in $\mathbb{R}^d$. The following result says that there is no bijection between classical and quantum observables that implements (2.8a) and (2.8d) exactly. As the other result below, it follows from the work of Groenewold and van Hove [Gro46, vH51], see also [AM85]. We follow the exposition in [Wal07, Section 5.2.1].

**Theorem 2.5 (Groenewold–van Hove)** *There does not exist a unital algebra $\mathcal{A}$, for which the associated Lie algebra $(\mathcal{A}, \frac{1}{\mathrm{i}\hbar}[\,\cdot\,,\,\cdot\,])$ is isomorphic to the Lie algebra $(\mathrm{Pol}(\mathbb{R}^d), \{\,\cdot\,,\,\cdot\,\})$.*

Since we cannot find a bijection between observables satisfying (2.8a) and (2.8d) as shown in the previous theorem, we can ask whether we can at least extend the correspondence of monomials $x^1, \dots, x^d, p_1, \dots, p_d$ with $\hat{X}^1, \dots \hat{X}^d, \hat{P}_1, \dots, \hat{P}_d$ to a representation of $(\mathrm{Pol}(\mathbb{R}^d), \{\,\cdot\,,\,\cdot\,\})$. This is also not possible:

**Theorem 2.6 (Groenewold–van Hove)** *No faithful irreducible representation of the Lie algebra $(\mathrm{span}\{1, x^1, \dots, x^d, p_1, \dots, p_d\}, \{\,\cdot\,,\,\cdot\,\})$ can be extended to a representation of the Lie algebra $(\mathrm{Pol}(\mathbb{R}^{2d}), \{\,\cdot\,,\,\cdot\,\})$.*

By irreducible we mean that the commutant of the image of the representation consists only of multiples of the identity. The representation obtained by mapping $x^i$ to $\hat{X}^i$ and $p_i$ to $\hat{P}_i$ is faithful and irreducible and therefore satisfies the assumptions of Theorem 2.6.

Many quantization procedures overcome these problems by only implementing (2.8a) and (2.8b) exactly, and requiring (2.8c) and (2.8d) only in the classical limit $\hbar \to 0$.

## 2.4 Formal deformation quantization

After presenting the physical motivation behind the quantization problem in the previous subsections, our exposition will be more mathematical from here on. In this subsection we introduce the reader to formal deformation quantization that appeared first in [BFF+78]. Suppose for the moment that the maps $Q_\hbar$ are bijective. Then $Q_\hbar^{-1}(Q_\hbar(f)Q_\hbar(g))$ defines a new non-commutative product on $\mathscr{C}^\infty(M)$. Formal deformation quantization considers the expansion of this product into a formal power series around $\hbar = 0$.

**Definition 2.7 (Formal deformation quantization, star product)** *Let $(M, \pi)$ be a Poisson manifold. A* formal deformation quantization *of $M$ is an associative*

*$\mathbb{C}[[\nu]]$-bilinear product on formal power series of smooth functions*

$$\star\colon \mathscr{C}^\infty(M)[[\nu]] \times \mathscr{C}^\infty(M)[[\nu]] \to \mathscr{C}^\infty(M)[[\nu]]\,, \tag{2.9}$$

*such that, when expanded in the form*

$$f \star g = \nu^r \sum_{r=0}^{\infty} C_r(f,g)$$

*with $f, g \in \mathscr{C}^\infty(M)$ and $C_r\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, we have that*

*i.)* $C_0(f,g) = fg$,

*ii.)* $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$,

*iii.)* $f \star 1 = 1 \star f = f$ *where* $1$ *is the function on* $M$ *that is constantly* $1$, *and*

*iv.)* $C_r$ *is a bidifferential operator for every* $r \geq 0$.

*The product $\star$ is called a* formal star product.

The last property ensures that $\star$ is local and can be restricted to any open subset $U \subseteq M$. We call a product a *non-local star product* if it satisfies all properties of the last definition, except that $C_r$ might not be bidifferential. One can show that for any product satisfying all properties except *ii.)* in the previous definition, $\frac{1}{\mathrm{i}}(C_1(f,g) - C_1(g,f))$ defines a Poisson bracket on $M$. In this sense we require that the Poisson bracket given on $M$ coincides with the Poisson bracket defined by the quantization.

**Convention 2.8** From now on, we reserve $\hbar$ to denote an actual complex number. As in the previous definition, formal parameters that play the role of $\hbar$ are denoted by $\nu$.

Note that the definition of formal star products is purely algebraic, and the main difficulty when trying to find such formal star products is to ensure associativity, which can be written as a quadratic equation in the bidifferential operators $C_r$. As a consequence, a lot of algebraic tools are available to study formal deformations, and we discuss this in greater detail below. We may see a formal deformation quanization as a formal, that is arbitrary order infinitesimal, approximation to a well-behaved solution of the full quantization problem at $\hbar = 0$, similarly as we may view the expansion of a smooth function $f$ in a Taylor series at a point $x$ as the formal approximation to $f$ at $x$. In many cases, this provides a way to extract important properties from a quantization, and can help to understand it better.

**Example 2.9 (Weyl–Moyal and Wick star products)** Denote the standard coordinates on $\mathbb{R}^{2d}$ by $x^1, \ldots, x^d, p^1, \ldots, p^d$ and assume that $\mathbb{R}^{2d}$ is endowed with the standard symplectic form $\omega = \sum_{i=1}^d \mathrm{d}x^i \wedge \mathrm{d}p^i$. Choose $\beta \in \mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$, such that when viewed as a constant section $\beta \in \Gamma^\infty((\mathrm{T}^{\mathbb{C}}\mathbb{R}^{2d})^{\otimes 2})$ its antisymmetrization $\beta^{\mathrm{asym}} := \frac{1}{2}(\beta - \tau(\beta))$ coincides with the Poisson tensor associated to $\omega$. Here, $\tau$ is the map flipping the two tensor factors. Then

$$\begin{aligned}\star\colon \mathscr{C}^\infty(\mathbb{R}^{2d})[[\nu]] \times \mathscr{C}^\infty(\mathbb{R}^{2d})[[\nu]] \to \mathscr{C}^\infty(\mathbb{R}^{2d})[[\nu]]\,,\\ (f,g) \mapsto f \star g := \mu \circ \exp(\mathrm{i}\nu\beta) f \otimes g \quad (2.10)\end{aligned}$$

defines a formal deformation quantization. Here, $\mu$ denotes the map multiplying the two tensor factors together. If

$$\beta = \frac{1}{2}\sum_{i=1}^{d}\left(\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p^i} - \frac{\partial}{\partial p^i} \otimes \frac{\partial}{\partial x^i}\right) \tag{2.11}$$

is the Poisson tensor associated to $\omega$, then $\star$ is called the *Weyl–Moyal product.* If

$$\beta = \sum_{i=1}^{d} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p^i} \quad \text{or} \quad \beta = 2\mathrm{i}\sum_{i=1}^{d} \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial \overline{z}^i} \tag{2.12}$$

where $\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \mathrm{i}\frac{\partial}{\partial p^i}\right)$ and $\frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \mathrm{i}\frac{\partial}{\partial p^i}\right)$, then $\star$ is called *standard ordered* or the *Wick product*, respectively.

The class of star products, that separate holomorphic and antiholomorphic derivatives, like the Wick product, will be important when studying positivity of linear functionals. Therefore they deserve their own name.

**Definition 2.10 (Star product of Wick type)** *Let $M$ be a Kähler manifold. A formal star product that derives the first argument only in holomorphic directions and the second argument only in antiholomorphic directions is said to be of* Wick type.

We can interpret a formal deformation quantization as deforming the classical Poisson algebra into a non-commutative algebra. The idea of studying such deformations is not only relevant for the quantization problem, but can be applied much more universally. For any mathematical structure, for example associative algebras, it is an interesting question to ask what the space of all such structures looks like. This space is usually called a *moduli space* and can be very hard to understand. As a first step, one can try to study formal neighbourhoods of points, which in the example of an associative algebra $\mathcal{A}$ would correspond to associative $\mathbb{C}[[\nu]]$-bilinear products on $\mathcal{A}[[\nu]]$. Usually, one is only interested in isomorphism classes of such algebras. In the case of deformation quantizations this corresponds to isomorphism classes under the following equivalence relation.

**Definition 2.11 (Equivalence of formal deformations)** *Let $(M,\pi)$ be a Poisson manifold. Two formal deformation quantizations $\star_1$ and $\star_2$ of $M$ are called* equivalent *if there exist differential operators $T_r$ on $M$ for $r \in \mathbb{N}$ such that $T = 1 + \sum_{r=1}^{\infty} \nu^r T_r$ intertwines $\star_1$ and $\star_2$, i.e.*

$$T(f \star_1 g) = T(f) \star_2 T(g) \tag{2.13}$$

*holds for all $f, g \in \mathscr{C}^\infty(M)$.*

Note that since $\star_1$ and $\star_2$ are unital, we must automatically have $T1 = 1$, and therefore $T_r 1 = 0$ for all $r \in \mathbb{N}$. Note also that any formal power series starting with 1 is automatically invertible, in particular there is a series of differential operators $S_r$ such that $S = 1 + \sum_{r=1}^{\infty} \nu^r S_r$ is an inverse to $T$. As we shall see later, all products in Example 2.9 are equivalent.

Gerstenhaber introduced cohomological methods to study the deformation problem for associative algebras [Ger63]. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. The graded vector space $C^\bullet(\mathcal{A}) := \bigoplus_{n\geq -1} C^n(\mathcal{A})$ where $C^n(\mathcal{A})$ denotes $\mathbb{C}$-multilinear maps from $\mathcal{A}^{\otimes(n+1)}$ to $\mathcal{A}$ carries a graded Lie algebra structure defined as follows. For $\phi \in C^n(\mathcal{A})$, $\psi \in C^m(\mathcal{A})$, and $i \in \{0, 1, \dots, n\}$ define $\phi \circ_i \psi \in C^{n+m}(\mathcal{A})$ by

$$(\phi \circ_i \psi)(a_0, \dots, a_{n+m}) = \phi(a_0, \dots, a_{i-1}, \psi(a_i, \dots, a_{i+m}), a_{i+m+1}, \dots, a_{n+m}),$$

and set

$$\phi \circ \psi = \sum_{i=0}^{n} (-1)^{in} \phi \circ_i \psi \quad \text{and} \quad [\phi, \psi] = \phi \circ \psi - (-1)^{nm} \psi \circ \phi. \tag{2.14}$$

A "multiplication" $\mu \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is associative if and only if $[\mu, \mu] = 0$, and an associative multiplication defines a differential $\delta = [\mu, \cdot\,]$ on $C^\bullet(\mathcal{A})$. So if $\mathcal{A}$ is an associative algebra, then $C^\bullet(\mathcal{A})$ becomes a differential graded Lie algebra (dgla).

We will now reformulate the quantization problem in terms of dgla's. See [Wal07, Esp15] for a more detailed exposition. If $\mathfrak{g}$ is a dgla, then any $D \in \mathfrak{g}^0$ defines a homogeneous derivation $\operatorname{ad}(D) = [D, \cdot\,]$ of $\mathfrak{g}$.

**Definition 2.12 (Maurer–Cartan elements, gauge action)** *Let $\mathfrak{g}$ be a dgla. An element $m \in \nu\mathfrak{g}^1[[\nu]]$ is said to be a* (formal) Maurer–Cartan element *if*

$$\delta m + \frac{1}{2}[m, m] = 0. \tag{2.15}$$

*The group $G := \{\exp(\nu \operatorname{ad}(D)) \mid D \in \mathfrak{g}^0[[\nu]]\}$ of automorphisms of $\mathfrak{g}^1[[\nu]]$ acts on formal Maurer–Cartan elements by the* gauge action

$$\mathrm{e}^{\nu \operatorname{ad}(D)} \triangleright m := \mathrm{e}^{\nu \operatorname{ad}(D)}(m) - \nu \sum_{n=0}^{\infty} \frac{(\nu \operatorname{ad}(D))^n}{(n+1)!}(\delta D). \tag{2.16}$$

**Remark 2.13** It is useful to think about a Maurer–Cartan element $m$ as an element defining a "flat connection" $\nabla_m \colon \mathfrak{g}^\bullet \to \mathfrak{g}^{\bullet+1}$, $\nabla_m = \delta + [m, \cdot\,]$. Indeed, (2.15) says precisely that $\nabla_m^2 = 0$. The gauge action defined in (2.16) can then be obtained from the formula $\nabla_{\mathrm{e}^{\nu \operatorname{ad}(D)} \triangleright m} = \mathrm{e}^{\nu \operatorname{ad}(D)} \circ \nabla_m \circ \mathrm{e}^{-\nu \operatorname{ad}(D)}$.

**Example 2.14** Many deformation problems are equivalent to the problem of finding Maurer–Cartan elements in a dgla:

*i.)* If $(\mathcal{A}, \mu)$ is an associative algebra, and $(\mathcal{A}[[\nu]], \mu + m)$ is a formal deformation of $\mathcal{A}$, then $0 = [\mu+m, \mu+m] = 2[\mu, m] + [m, m]$, so $m$ is a formal Maurer–Cartan element in $C^\bullet(\mathcal{A})$. Vice versa, by reversing this argument, formal Maurer–Cartan elements in $C^\bullet(\mathcal{A})$ give rise to a formal deformation of $\mathcal{A}$. Furthermore, one can check that the equivalence relation induced by the gauge action of $G$ on Maurer–Cartan elements corresponds exactly to the equivalence of formal deformations of $\mathcal{A}$.

*ii.)* To formulate the problem of finding formal deformation quantization in terms of Maurer–Cartan elements, we need to modify our definitions slightly, in order to take care of the fact that deformation quantizations are always unital and defined by bidifferential operators in all orders of $\nu$. The differential and bracket on $C^\bullet(\mathscr{C}^\infty(M))$ restrict to the subspace $C^\bullet_{\mathrm{diff},1}(\mathscr{C}^\infty(M))$ of multidifferential maps (i.e. maps that are differential in each argument) that vanish on constant functions. Maurer–Cartan elements $m = \nu m_1 + \mathcal{O}(\nu^2) \in \nu C^1_{\mathrm{diff},1}(\mathscr{C}^\infty(M))[[\nu]]$ correspond precisely to formal deformations of $M$ with Poisson bracket $\{f, g\} := \frac{1}{\mathrm{i}}(m_1(f, g) - m_1(g, f))$, and the equivalence relation induced by the gauge action of $G := \{\exp(\nu \operatorname{ad}(D)) \mid D \in C^0_{\mathrm{diff},1}(\mathscr{C}^\infty(M))[[\nu]]\}$ corresponds to the equivalence of formal deformations defined in Definition 2.11.

*iii.)* The multivector fields $\mathfrak{X}^\bullet(M) := \Gamma^\infty(\Lambda^{\bullet+1}\mathrm{T}M)$ with Schouten–Nijenhuis bracket $[\![\,\cdot\,,\,\cdot\,]\!]$ and differential $\delta = 0$ also form a dgla, and Maurer–Cartan elements correspond to *formal Poisson structures*, i.e. elements $\pi \in \nu\mathfrak{X}^\bullet(M)[[\nu]]$ satisfying $[\![\pi, \pi]\!] = 0$. Note that such elements could be viewed as deforming the zero Poisson structure. Two formal Poisson structures $\pi_1$ and $\pi_2$ are equivalent, if there is a vector field $X \in \nu\mathfrak{X}^0(M)[[\nu]]$ such that $\pi_1 = \mathrm{e}^{\mathscr{L}_X}\pi_2$, with $\mathscr{L}_X$ denoting the Lie derivative, which is the case if and only if the corresponding Maurer–Cartan elements are gauge equivalent.

The Hochschild–Kostant–Rosenberg theorem (see [HKR62] for the original algebraic statement) asserts that the map of complexes

$$
\begin{aligned}
\mathrm{HKR}\colon \mathfrak{X}^\bullet(M) &\to C^\bullet_{\mathrm{diff},0}(\mathscr{C}^\infty(M))\,, \\
v_1 \wedge \cdots \wedge v_n &\mapsto \left( (f_1, \ldots, f_n) \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}(\sigma)} v_{\sigma(1)} f_1 \ldots v_{\sigma(n)} f_n \right)
\end{aligned}
\tag{2.17}
$$

induces a Lie algebra isomorphism on cohomology. However HKR is not itself a Lie algebra morphism and does therefore not necessarily map Maurer–Cartan elements to Maurer–Cartan elements.

This defect can be repaired. There is a weaker notion of an $L_\infty$-morphism $\mathcal{U}\colon \mathfrak{g} \to \mathfrak{h}$ between dgla's, or more generally $L_\infty$-algebras, $\mathfrak{g}$ and $\mathfrak{h}$, which consists of a sequence of linear maps $\mathcal{U}^{(r)}\colon \Lambda^r\mathfrak{g} \to \mathfrak{h}$ of degree $1 - r$. The idea is that $\mathcal{U}^{(1)}$ is a Lie algebra isomorphism up to higher homotopies, given by the maps $\mathcal{U}^{(r)}$.

Kontsevich proved in his famous formality theorem [Kon03] that $\mathcal{U}^{(1)} := \mathrm{HKR}$ can indeed be made into an $L_\infty$-morphism $\mathfrak{X}^\bullet(M) \to C^\bullet_{\mathrm{diff},0}(\mathscr{C}^\infty(M))$. If $\pi = \nu\pi_1 + \mathcal{O}(\nu^2) \in \nu\mathfrak{X}^1(M)[[\nu]]$ is a formal Poisson structure, then $\sum_{r=1}^\infty \mathcal{U}^{(r)}(\pi, \ldots, \pi) \in \nu C^1_{\mathrm{diff},0}(\mathscr{C}^\infty(M))[[\nu]]$ is again a Maurer–Cartan element and therefore defines a formal deformation quantization of $\pi_1$. Since $\mathcal{U}^{(1)}$ is an isomorphism on cohomology, it follows that there is another $L_\infty$-morphism $C^\bullet_{\mathrm{diff},0}(\mathscr{C}^\infty(M)) \to \mathfrak{X}^\bullet(M)$ inducing the inverse map on cohomology. In addition, $L_\infty$-morphisms map gauge equivalent Maurer–Cartan elements to gauge equivalent Maurer–Cartan elements, and therefore we obtain:

**Theorem 2.15 (Kontsevich)** *Every Poisson manifold $(M, \pi)$ admits a deformation quantization. Moreover, equivalence classes of deformation quantizations are in bijection with equivalence classes of formal deformations of the Poisson tensor $\pi$.*

We remark that given a Poisson manifold $(M, \pi)$, one obtains a canonical equivalence class of star products from the formal Poisson structure $\nu\pi$. However, there is no canonical choice of a star product in this equivalence class.

The main difficulty when quantizing Poisson manifolds or when proving Kontsevich's formality theorem, comes from the fact that there is no good local standard form of the Poisson tensor. Indeed, the construction of the maps $\mathcal{U}^{(r)}$ for $\mathbb{R}^d$ with an arbitrary Poisson structure is the hardest part of the proof, and was achieved by Kontsevich by encoding differential operators in graphs, and assigning certain weights to them. These weights can be interpreted using topological field theories [CF00]. Globalizing from $\mathbb{R}^d$ to arbitrary Poisson manifolds is then relatively easy [CFT02]. See also [Dol05] for a different approach to formality using operads.

For symplectic manifolds, the situation is much easier. Any symplectic manifold is locally symplectomorphic to $\mathbb{R}^{2d}$ with the standard symplectic form $\sum_{i=1}^d \mathrm{d}x^i \wedge \mathrm{d}p^i$. Existence [DL83, Fed94, OMY91] and classification [BCG97, NT95a, NT95b, WX91] results for star products on symplectic manifolds were obtained by many mathematicians and by rather different methods. Most notably, one should mention the *Fedosov construction* [Fed94], which, in a conceptually clear way, glues the local Weyl quantizations from Example 2.9 together to obtain a formal deformation quantization of $M$. In contrast to the case of Poisson manifolds, the formal star products on $M$ can be described more explicitly, and there is also an easier description of the equivalence classes of such star products.

**Theorem 2.16** *Let $M$ be a symplectic manifold. Then $M$ admits a deformation quantization, and equivalence classes of deformations are parametrized by the characteristic class*

$$\theta = \frac{1}{\mathrm{i}\nu}\omega + \theta_0 + \nu\theta_1 + \cdots \in \frac{1}{\mathrm{i}\nu}\omega + \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\nu]] \,. \tag{2.18}$$

The characteristic class can be obtained naturally from objects appearing in the Fedosov construction: It is the curvature of a lift of a certain connection. Since this curvature always has lowest order $\frac{1}{\mathrm{i}\nu}\omega$, one usually defines $\theta$ to take values in the affine space over this value. This is also convenient for formulating the algebraic index theorem in Subsection 2.7.

Note that Theorem 2.16 implies in particular that all formal star products considered in Example 2.9 are equivalent, since $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{R}^{2d}, \mathbb{C}) = 0$. The following theorem from [Kar96] classifies all star products of Wick type, and not just equivalence classes.

**Theorem 2.17 (Karabegov)** *Let $(M, \omega_0)$ be a Kähler manifold. Then formal star products of Wick type are classified by formal deformations of the Kähler form, that is by formal series*

$$\omega = \omega_0 + \nu\omega_1 + \nu^2\omega_2 + \cdots \in \omega_0 + \nu\Gamma^\infty(\Lambda^{(1,1)}\mathrm{T}^{\mathbb{C},*}M)[[\nu]] \tag{2.19}$$

*where $\omega_1, \omega_2, \ldots$ are 2-forms of type $(1,1)$, that are closed but not necessarily non-degenerate.*

The characteristic class from Theorem 2.16 can be obtained from Karabegov's class by rescaling with $(\mathrm{i}\nu)^{-1}$ and adding the Ricci curvature of a symplectic connection appearing in the Fedosov construction, see [Neu03].

## 2.5 Strict deformation quantization

Let us now discuss more complete solutions of the quantization problem, which construct actual C*-algebras varying with $\hbar$, and not just algebras over formal power series $\mathbb{C}[[\nu]]$. To make precise what it means for a C*-algebra to vary continuously with $\hbar$, recall the following definition from [Dix77].

**Definition 2.18 (Continuous field of C*-algebras)** *A* continuous field of C*-algebras *over a topological space $I$ is a family $(\mathcal{A}_\hbar)_{\hbar\in I}$ of C*-algebras together with a set $\Gamma \subseteq \prod_{\hbar\in I}\mathcal{A}_\hbar$ of* continuous sections*, such that*

*i.) $\Gamma$ is a ${}^*$-subalgebra of $\prod_{\hbar\in I}\mathcal{A}_\hbar$,*

*ii.) the set $\{x(\hbar) \mid x \in \Gamma\}$ is dense in $\mathcal{A}_\hbar$ for every $\hbar \in I$,*

*iii.) the function $\hbar \mapsto \|x(\hbar)\|_\hbar$ is continuous for every $x \in \Gamma$, and*

*iv.) if $x \in \prod_{\hbar\in I}\mathcal{A}_\hbar$ and if for every $\hbar \in I$ and every $\varepsilon > 0$ there exists an $x' \in \Gamma$ and a neighbourhood $I'$ of $\hbar$ in $I$ such that $\|x(\hbar') - x'(\hbar')\|_{\hbar'} < \varepsilon$ holds for all $\hbar' \in I'$, then $x \in \Gamma$.*

Usually, one only specifies a subset of $\Gamma$ when defining a continuous field of C*-algebras. The following lemma, which follows from [Dix77, Proposition 10.2.3] and [Dix77, Proposition 10.3.2], gives conditions for this to be enough.

**Lemma 2.19** *Given a family $(\mathcal{A}_\hbar)_{\hbar\in I}$ of C*-algebras and any subset $\Gamma' \subseteq \prod_{\hbar\in I}\mathcal{A}_\hbar$, such that the span of $\{x(\hbar) \mid x \in \Gamma'\}$ is dense in $\mathcal{A}_\hbar$ for all $\hbar \in I$, there is at most one set $\Gamma$ containing $\Gamma'$ that defines the structure of a continuous field on $(\mathcal{A}_\hbar)_{\hbar\in I}$. If $\Gamma'$ satisfies i.), ii.), and iii.) of the previous definition, then there exists a unique set $\Gamma$ containing $\Gamma'$ that defines the structure of a continuous field.*

The following definition of a strict deformation quantization is due to Rieffel [Rie89], and a special case of the more general Definition 2.28. We denote the continuous functions on a manifold $M$ vanishing at infinity by $\mathscr{C}_0(M)$ and the compactly supported smooth functions by $\mathscr{C}_c(M)$.

**Definition 2.20 (Strict deformation quantization)** *Let $(M,\pi)$ be a Poisson manifold. A* strict deformation quantization *on $M$ is specified by the following data:*

*i.) a dense ${}^*$-subalgebra $\mathscr{A}$ of $\mathscr{C}_0(M)$, which is closed under taking Poisson brackets,*

*ii.) an open interval $I \subseteq \mathbb{R}$ containing $0$,*

*iii.) for every $\hbar \in I$ a product $\star_\hbar$, an involution ${}^{*_\hbar}$ and a C*-norm $\|\cdot\|_\hbar$ (with respect to the product $\star_\hbar$ and involution ${}^{*_\hbar}$) on the underlying vector space of $\mathscr{A}$*

*such that*

*a.) for $\hbar = 0$ the product $\star_0$, involution ${}^{*_0}$, and norm $\|\cdot\|_0$ coincide with the commutative product, complex conjugation and maximum norm of $\mathscr{A}$,* *b.) the completions $\mathcal{A}_\hbar$ of $\mathscr{A}$ with respect to the norms $\|\cdot\|_\hbar$ form a continuous field of $C^*$-algebras over $I$, for which the constant sections $\hbar \mapsto a$ with $a \in \mathscr{A}$ are continuous,*

*c.) for $a, b \in \mathscr{A}$ we have*

$$\lim_{\hbar \to 0} \left\| \frac{1}{\hbar}(a \star_\hbar b - b \star_\hbar a) - \mathrm{i}\{a, b\} \right\|_\hbar = 0 \,. \tag{2.20}$$

*Such a strict deformation quantization is called* flabby *if $\mathscr{A}$ contains $\mathscr{C}_c^\infty(M)$.*

Note that it follows from Lemma 2.19 that the continuous field defined by the constant sections is necessarily unique (if it exists). However, since the constant sections do not necessarily form a $^*$-subalgebra, the continuity of $\hbar \mapsto \|a\|_\hbar$ for all $a \in \mathscr{A}$ is not enough to guarantee the existence of a continuous field structure (even though this is often claimed in the literature). It is sufficient to assume that $\hbar \mapsto \|a + b^{*_\hbar}\|_\hbar$ and $\hbar \mapsto \|a \star_\hbar b\|_\hbar$ are continuous for all $a, b \in \mathscr{A}$, see [Rie93, Proposition 9.1].

The standard examples of strict deformation quantizations are given by a strict version of the Weyl–Moyal product and by non-commutative tori, see Example 2.24 and Example 2.25. We will first describe Rieffel's construction of strict deformation quantizations from isometric actions of $\mathbb{R}^d$ on a C$^*$-algebra, and present the examples as special cases of this construction. Many known constructions of strict deformation quantizations rely in some way on oscillatory integrals and are therefore analytically demanding. Rieffel's construction starts with a Fréchet algebra.

**Definition 2.21 (Fréchet algebra)** *A* Fréchet space *$V$ is a Hausdorff complete topological vector space, whose topology is induced by a countable family of seminorms $\|\cdot\|_n$. A* Fréchet algebra *$\mathcal{A}$ is a Fréchet space, with a continuous multiplication $\cdot\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defining an associative algebra structure on $\mathcal{A}$.*

For an increasing family of seminorms $\|\cdot\|_n$ defining the topology on $\mathcal{A}$, continuity of $\cdot$ means that for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and $C \in \mathbb{R}^+$ such that $\|a \cdot b\|_n \leq C\|a\|_m\|b\|_m$ holds for all $a, b \in \mathcal{A}$. If there exists such a family for which we may choose $m = n$ then $\mathcal{A}$ is said to be *multiplicatively convex.*

Recall that an action $\alpha\colon \mathbb{R}^d \to \mathsf{Aut}(\mathcal{A})$ of the abelian group $\mathbb{R}^d$ on a Fréchet algebra $\mathcal{A}$ is called *isometric*, if there is a family of seminorms defining the topology of $\mathcal{A}$ such that, for all $g \in \mathbb{R}$, the map $\alpha(g)\colon \mathcal{A} \to \mathcal{A}$ is isometric with respect to every member of this family, and it is said to be *smooth*, if every $a \in \mathcal{A}$ is a smooth vector of the action, meaning that the map $\mathbb{R}^d \to \mathcal{A}$, $g \mapsto \alpha(g)(a)$ is smooth for every $a \in \mathcal{A}$.

**Theorem 2.22 (Rieffel)** *Let $\mathcal{A}$ be a Fréchet algebra endowed with a strongly continuous, isometric, and smooth action $\alpha\colon \mathbb{R}^d \to \mathsf{Aut}(\mathcal{A})$ of $\mathbb{R}^d$ by automorphisms, and let $J$ be any linear operator on $\mathbb{R}^d$. Then the oscillatory integral*

$$a \star_J b = \frac{1}{\pi^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \alpha_{Ju}(a)\alpha_v(b) \mathrm{e}^{2\mathrm{i}u\cdot v} \,\mathrm{d}u \,\mathrm{d}v \tag{2.21}$$

*is well-defined and*

*i.) the product $\star_J$ is associative and continuous,* *ii.)* *the action of* $\mathbb{R}^d$ *on* $\mathcal{A}$ *with the product* $\star_J$ *is by automorphisms,*

*iii.)* $a \star_0 b = ab$.

PROOF: See [Rie93, Proposition 2.2, Proposition 2.5, Corollary 2.8, and Theorem 2.14]. □

Let us make two important remarks: First, the product $\star_J$ is in general non-commutative, even if $\mathcal{A}$ is commutative. However, we do not need to assume that $\mathcal{A}$ is commutative: For a non-commutative Fréchet algebra $\mathcal{A}$ the previous theorem yields another possibly non-commutative product $\star_J$ on $\mathcal{A}$. Second, when given an action of $\mathbb{R}^d$ by automorphisms on a Fréchet algebra $\mathcal{A}$ that is isometric but not necessarily smooth, we can consider the smooth vectors $\mathcal{A}^\infty$. If we define seminorms on $\mathcal{A}^\infty$ that also take derivatives of the action into account as in [Rie93, Chapter 1], then $\mathcal{A}^\infty$ is again a Fréchet algebra, now with an isometric and smooth action of $\mathbb{R}^d$. So Theorem 2.22 applies to $\mathcal{A}^\infty$.

**Theorem 2.23 (Rieffel)** *Let* $\mathcal{A}$ *be a* $C^*$*-algebra endowed with an isometric action* $\alpha\colon \mathbb{R}^d \to \mathsf{Aut}(\mathcal{A})$ *of* $\mathbb{R}^d$ *by* $^*$*-automorphisms. Let* $J$ *be a skew-symmetric linear operator on* $\mathbb{R}^d$*. Then for any* $\hbar \in \mathbb{R}$

*i.)* *the previous theorem and the remark in the previous paragraph yield the existence of a product* $\star_{\hbar J}$ *on* $\mathcal{A}^\infty$*,*

*ii.)* *there is a representation of* $\mathcal{A}^\infty$ *with product* $\star_{\hbar J}$ *as bounded operators on a Schwartz space* $\mathcal{S}^{\mathcal{A}}$*,*

*iii.)* *this representation defines a* $C^*$*-norm* $\|\cdot\|_{\hbar J}$ *on* $\mathcal{A}^\infty$ *with respect to the product* $\star_{\hbar J}$*,*

*iv.)* *the topology defined by* $\|\cdot\|_{\hbar J}$ *is coarser than the Fréchet topology of* $\mathcal{A}^\infty$*, and*

*v.)* *endowing* $\mathcal{A}^\infty$ *with the product* $\star_{\hbar J}$*, the involution of* $\mathcal{A}$*, and the* $C^*$*-norm* $\|\cdot\|_{\hbar J}$ *we obtain a strict deformation quantization of* $\mathcal{A}$ *in the direction of the Poisson structure*

$$\{a, b\} = \sum_{j,k=1}^{d} J_{jk} \tilde{\alpha}_{e_j}(a) \tilde{\alpha}_{e_k}(b)\,, \tag{2.22}$$

*where* $a, b \in \mathcal{A}^\infty$*.*

In part *v.)* $\{e_1, \ldots, e_d\}$ is the standard basis of $\mathbb{R}^d$, and $\tilde{\alpha}_{e_i}(a) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \alpha_{te_i}(a)$ is the induced action of the Lie algebra $\mathbb{R}^d$ on $\mathcal{A}$.

PROOF: For parts *ii.)*, *iii.)*, and *iv.)* see [Rie93, Chapter 4], for part *v.)* see [Rie93, Theorem 9.3]. □

Again, the C$^*$-algebra $\mathcal{A}$ we start with might be non-commutative. In this case the Poisson structure defined in *v.)* is a Poisson structure on a non-commutative algebra as defined, for example, in [Xu94], and we need to generalize the definition of a strict deformation quantization to the non-commutative setting, which essentially means replacing (2.20) with $\big\|\frac{1}{\hbar} f \star_\hbar g - \frac{\mathrm{i}}{2}\{f, g\}\big\|_\hbar \to 0$, see [Rie93, Definition 9.2]. This becomes necessary since non-commutative Poisson structures are not guaranteed to be skew symmetric. Since we are only interested in deforming commutative algebras in the following, we shall not elaborate on this.

**Example 2.24 (Strict Weyl–Moyal quantization)** Let $\mathbb{R}^{2d}$ act on $\mathbb{R}^{2d}$ by translation, and consider the induced action on the C*-algebra $\mathscr{C}_0(\mathbb{R}^{2d})$ by pullbacks. Assume that, in standard coordinates $x^1, \ldots, x^d, p^1, \ldots, p^d$, we have $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with blocks of size $d \times d$. Then the product $\star_{\hbar J}$ obtained in Theorem 2.23 is a convergent version of the Weyl–Moyal product $\star_{\mathrm{WM}}$ from Example 2.9, in the sense that it has an asymptotic expansion in $\hbar$, which coincides with $\star_{\mathrm{WM}}$. For $f, g \in \mathscr{B}(\mathbb{R}^{2d})$, the space of smooth functions vanishing at $\infty$ for which all derivatives of arbitrary order are bounded, and $\hbar \neq 0$ we have

$$
\begin{aligned}
(f \star_{\hbar J} g)(x) &= \frac{1}{\pi^{2d}} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} f(x - \hbar J u) g(x - v) \mathrm{e}^{2\mathrm{i} u \cdot v} \, \mathrm{d}u \, \mathrm{d}v \\
&= \frac{1}{(\pi\hbar)^{2d}} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} f(u) g(v) \mathrm{e}^{\frac{2\mathrm{i}}{\hbar} J^{-1}(x-u)\cdot(x-v)} \, \mathrm{d}u \, \mathrm{d}v \\
&= \frac{1}{(\pi\hbar)^{2d}} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} f(u) g(v) \mathrm{e}^{\frac{2\mathrm{i}}{\hbar}(\omega(x,u)+\omega(u,v)+\omega(v,x))} \, \mathrm{d}u \, \mathrm{d}v \,. \qquad (2.23)
\end{aligned}
$$

Here $\omega(u, v) := J^{-1} u \cdot v$ coincides with the standard symplectic form $\omega = \sum_{i=1}^d \mathrm{d}x^i \wedge \mathrm{d}p^i$. Note that this formula reproduces the formulas obtained in [Fed96, Theorem 3.2.1] and [Wal07, Remark 5.3.25]. The completion of $\mathscr{B}(\mathbb{R}^{2n})$ with respect to the product $\star_{\hbar J}$ and norm $\|\cdot\|_{\hbar J}$ is just the C*-algebra of compact operators if $\hbar \neq 0$, and the C*-algebra $\mathscr{C}_0(\mathbb{R}^d)$ of continuous functions vanishing at $\infty$ if $\hbar = 0$, see [Rie93, Proposition 5.2].

Formula (2.23) is precisely the composition law of the pseudodifferential operators associated by the Weyl calculus to the symbols $f$ and $g$. More precisely, if we define the pseudo-differential operator associated to a sufficiently nice symbol $f$

$$
(\mathrm{Op}(f)u)(q) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \exp\left(\frac{\mathrm{i}}{\hbar} p(q - q')\right) f\left(\frac{q + q'}{2}, p\right) u(q') \, \mathrm{d}q' \, \mathrm{d}p \,, \qquad (2.24)
$$

where $u \in \mathcal{S}(\mathbb{R}^d)$ is a Schwartz function and $q \in \mathbb{R}^d$, then $\mathrm{Op}(f \star_{\hbar J} g) = \mathrm{Op}(f)\mathrm{Op}(g)$. Since it is not relevant in the following, we do not discuss what precisely "sufficiently nice symbol $f$" means, but only mention that (2.24) is obviously well-defined if $f$ is Schwartz, and that it can be extended to symbols in the sense of Hörmander [Hör05].

Note that one may replace $\frac{q+q'}{2}$ in the first argument of $f$ in (2.24) by $q$ to obtain a convergent version of the standard ordered product. There is a generalization of the pseudodifferential calculus (the association of a certain operator to a symbol as in (2.24)) to arbitrary cotangent bundles $\mathrm{T}^*Q$.

The phase in (2.23) can be interpreted as a multiple of the symplectic area of the triangle with vertices $x$, $u$, and $v$, or rather as a multiple of the symplectic area of a triangle with midpoints $x$, $u$, and $v$. This is the starting point of a construction of strict star products on solvable symmetric spaces [Bie02, Wei94].

**Example 2.25 (Non-commutative tori)** Consider the action of $\mathbb{R}^d$ on the torus $\mathbb{T}^d \cong \mathbb{R}^d/(2\pi\mathbb{Z})^d$ by translation, and the induced action on the C*-algebra $\mathscr{C}(\mathbb{T}^d)$

by pullbacks. It is more convenient to consider $\mathbb{T}^d \cong (\mathbb{S}^1)^d$ as a subset of $\mathbb{C}^d$ in the following. The smooth vectors of this action are just the smooth functions $\mathscr{C}^\infty(\mathbb{T}^d)$. Let $J$ be any skew-symmetric matrix. For every $n \in \mathbb{Z}^d$, define the element $u_n := (z \mapsto z^n)$ of $\mathscr{C}^\infty(\mathbb{T}^d)$, using the abbreviation $z^n := z_1^{n_1} \cdot \ldots \cdot z_d^{n_d}$. Note that $\alpha_v u_n = \mathrm{e}^{-\mathrm{i}v\cdot n} u_n$ for any $v \in \mathbb{R}^d$. Consequently

$$
\begin{aligned}
(u_n \star_{\hbar J} u_m) &= \frac{1}{\pi^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\alpha_{\hbar J v} u_n) \cdot (\alpha_w u_m) \cdot \mathrm{e}^{2\mathrm{i}v\cdot w} \,\mathrm{d}v \,\mathrm{d}w \\
&= \frac{1}{(2\pi)^d} u_n u_m \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\hbar J v\cdot n/2} \mathrm{e}^{-\mathrm{i}w\cdot m} \mathrm{e}^{\mathrm{i}v\cdot w} \,\mathrm{d}v \,\mathrm{d}w \\
&= \mathrm{e}^{-\mathrm{i}\hbar J m \cdot n/2} u_{n+m} \,. \qquad (2.25)
\end{aligned}
$$

We used the oscillatory distribution-valued integral $\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(v-m)\cdot w} \,\mathrm{d}w = \delta(v-m)$, but the result follows also in a more elementary way from [Rie93, Lemma 2.20].

Denote the C*-completion of $\mathscr{C}^\infty(\mathbb{T}^d)$ with respect to $\|\cdot\|_{\hbar J}$ by $\mathcal{A}_\hbar$. Note that $u_n^* = u_{-n}$, and therefore (2.25) shows that $u_n \in \mathcal{A}_\hbar$ is unitary for all $\hbar \in \mathbb{R}$. Furthermore, since $\operatorname{span}\{u_n \mid n \in \mathbb{Z}^d\}$ is dense in $\mathscr{C}^\infty(\mathbb{T}^d)$ with respect to its Fréchet topology, it is also dense in $\mathscr{C}^\infty(\mathbb{T}^d)$ with respect to the topology induced by $\|\cdot\|_{\hbar J}$ by Theorem 2.23 *iv.)*, and therefore dense in the C*-completion $\mathcal{A}_\hbar$. Thus (2.25) shows that $U_k := u_{e_k}$ with $1 \le k \le d$, where $e_k$ is the vector with entry 1 in its $k$-th component and all other components 0, generate $\mathcal{A}_\hbar$, and

$$
U_k \star_{\hbar J} U_\ell = \mathrm{e}^{-\mathrm{i}\hbar J e_\ell \cdot e_k/2} U_{k+\ell} = \mathrm{e}^{\mathrm{i}\hbar (J e_k\cdot e_\ell - J e_\ell \cdot e_k)/2} U_\ell \star_{\hbar J} U_k = \mathrm{e}^{\mathrm{i}\hbar J_{\ell k}} U_\ell \star_{\hbar J} U_k \,.
$$

The non-commutative torus $\mathcal{A}^{\mathbb{T}}_{\hbar J/(2\pi)}$ is defined as the universal C*-algebra generated by unitaries $V_k$ satisfying precisely these relations. Therefore there is a *-homomorphism $\pi\colon \mathcal{A}^{\mathbb{T}}_{\hbar J/(2\pi)} \to \mathcal{A}_\hbar$. This map is equivariant with respect to the action of $\mathbb{R}^d$ on $\mathcal{A}^{\mathbb{T}}_{\hbar J/(2\pi)}$, defined by $\alpha_v V_k = \mathrm{e}^{-\mathrm{i}v_k} V_k$ on the unitary generators. But since $\mathcal{A}^{\mathbb{T}}_{\hbar J/(2\pi)}$ does not have non-trivial ideals invariant under this action, $\pi$ must be an isomorphism.

However, there is a serious limitation to the Poisson manifolds that can be quantized using Rieffel's construction: The Poisson bracket must be of the form (2.22), i.e. it must be possible to define it using commuting derivations. This is not always the case. Note that up to scaling by a non-zero constant, there is a unique non-degenerate SO(3)-invariant Poisson structure on the 2-sphere.

**Proposition 2.26** *No non-degenerate* SO(3)*-invariant Poisson structure on the 2-sphere is induced by an action of* $\mathbb{R}^d$ *as in* (2.22)*, and therefore the 2-sphere with such a Poisson structure cannot be quantized as in Theorem 2.23.*

PROOF: Since any action of $\mathbb{R}^d$ on the 2-sphere has a fixed point, the Poisson structure defined in (2.22) must vanish at some point. But if it was also SO(3)-invariant, then it must vanish everywhere. □

There are several generalizations of Rieffel's approach. Bieliavsky–Gayral [BG15] show how to extend it to negatively curved Kählerian Lie groups, i.e. Lie groups that admit a left-invariant Kählerian structure of negative sectional curvature. Topologically, such groups are homeomorphic to $\mathbb{R}^d$, but they might be non-commutative. Examples are given by the groups $AN$, which are factors of the Iwasawa decomposition $\mathrm{SU}(1,n) = KAN$. In fact, any negatively curved Kählerian Lie group is an iterated semidirect product of such factors. See also [BGNT16, BGNT19] and the references therein for a different approach based on locally compact quantum groups and dual cocycles, and their equivalence.

**Obstructions to strict equivariant quantization of the 2-sphere**

The fact that Theorem 2.23 cannot be used to quantize the 2-sphere with a non-degenerate $\mathrm{SO}(3)$-equivariant Poisson structure does of course not mean that it could not be quantized by other means. However, the same problem occurs in other approaches, too. For example, as discussed in the Appendix, the 2-sphere is a homogeneous space $\mathrm{SU}(2)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ and one can therefore try to use the theory of quantum groups and quantum homogeneous spaces to obtain strict quantizations [DK94, She91]. However, since $\mathrm{SU}(2)$ is treated as a Poisson–Lie group in this approach, and the Poisson structure of a Poisson–Lie group is necessarily degenerate at the identity element, the induced Poisson structure on the sphere will vanish at least at one point and is therefore either trivial or not $\mathrm{SO}(3)$-invariant.

The following theorem shows that a $\mathrm{SO}(3)$-invariant strict deformation quantization does not exist at all. The theorem and its proof are motivated by Wassermann's result that every von Neumann algebra with an ergodic action of $\mathrm{SU}(2)$ must be of type I [Was88].

**Theorem 2.27 (Rieffel)** *Any product $\star$, involution $^*$ and C$^*$-norm $\|\cdot\|$ on $\mathscr{C}^\infty(\mathbb{S}^2)$, for which the usual action of $\mathrm{SO}(3)$ on $\mathscr{C}^\infty(\mathbb{S}^2)$ is by isometric $^*$-automorphisms, is commutative. In particular, the 2-sphere does not allow a $\mathrm{SO}(3)$-invariant flabby strict deformation quantization.*

We elaborate on the proof in [Rie89, Theorem 7.1], giving more details why (in the notation of the proof) $e$ is a unit in $\mathcal{A}$. This is required to apply the result from [EL77]. Note that we can replace $\mathrm{SO}(3)$ with $\mathrm{SU}(2)$ in the theorem, and the following proof still works.

PROOF: The proof depends crucially on the fact that we know how $\mathscr{C}^\infty(\mathbb{S}^2)$ decomposes into irreducible representations of $\mathrm{SO}(3)$: Irreducible representations $\mathcal{P}_k$ of $\mathrm{SO}(3)$ are labelled by non-negative integers $k \in \mathbb{N}_0$, are $(2k+1)$-dimensional, and every irreducible representation $\mathcal{P}_k$ appears in $\mathscr{C}^\infty(\mathbb{S}^2)$ with multiplicity exactly 1. Denote the C$^*$-completion of $\mathscr{C}^\infty(\mathbb{S}^2)$ by $\mathcal{A}$ and identify $\mathcal{P}_k$ with a subset of $\mathcal{A}$.

Take any $p_0 \in \mathcal{P}_0$. Since $\star$ is $\mathrm{SO}(3)$-equivariant, it follows that $g \triangleright (p_0 \star a) = (g \triangleright p_0) \star (g \triangleright a) = p_0 \star (g \triangleright a)$ holds for all $a \in \mathcal{A}$ and $g \in \mathrm{SO}(3)$. In other words $p_0 \star \cdot : \mathcal{A} \to \mathcal{A}$ is $\mathrm{SO}(3)$-equivariant and must therefore map $\mathcal{P}_k$ into $\mathcal{P}_k$. The same is true for the involution $^*$. In particular $(\mathcal{P}_0)^2 \subseteq \mathcal{P}_0$, and $(\mathcal{P}_0)^2 = 0$ would contradict that $\|\cdot\|$ is a C$^*$-norm. But since $\mathcal{P}_0$ is 1-dimensional, this implies $(\mathcal{P}_0)^2 = \mathcal{P}_0$ and there is a non-zero self-adjoint idempotent $e \in \mathcal{P}_0$.

From Schur's lemma it follows that for each $k \in \mathbb{N}_0$, the endomorphism $e \star \cdot$ of $\mathcal{P}_k$ is either the identity or zero. By pre- and postcomposing $e \star \cdot$ with the involution $^*$, it follows that $e\star\cdot\ =\ \cdot\star e$, so $e$ is central in $\mathcal{A}$. Consequently $e\star\mathcal{A}$ is a SO(3)-invariant 2-sided closed ideal in $\mathcal{A}$, and the quotient C$^*$-algebra $\mathcal{A}/e\star\mathcal{A}$ carries a representation of SO(3), in which the trivial representation has multiplicity 0. Denote the Haar measure on SO(3) by $\mu$, and let $a \in \mathcal{A}/e\star\mathcal{A}$. Then the element $\int_{\mathrm{SO}(3)}(g\triangleright a)^*\star(g\triangleright a)\mu \in \mathcal{A}/e\star\mathcal{A}$ is SO(3)-invariant, hence 0. But then $g\triangleright a = 0$ holds for all $g \in \mathrm{SO}(3)$ and $a \in \mathcal{A}/e\star\mathcal{A}$, so $\mathcal{A} = e\star\mathcal{A}$, and $e$ is a unit for $\mathcal{A}$.

Consider the commutator $[\,\cdot\,,\,\cdot\,]_\star$ with respect to $\star$. $[\mathcal{P}_1,\mathcal{P}_1]_\star$ is SO(3)-invariant, and since the commutator factors through the antisymmetric tensor product $\mathcal{P}_1 \wedge \mathcal{P}_1 \cong \mathcal{P}_1$ of the representation $\mathcal{P}_1$, this commutator is either $\mathcal{P}_1$ or 0. We want to show that it is 0. So, working towards a contradiction, assume that $[\mathcal{P}_1,\mathcal{P}_1]_\star = \mathcal{P}_1$. Choose a maximal torus of SO(3) and weight vectors $a_{-1}, a_0, a_1 \in \mathcal{P}_1$. Since $[\,\cdot\,,\,\cdot\,]_\star$ is SO(3)-equivariant, we can rescale the weight vectors so that we have $[a_0,a_1]_\star = 2a_1$, $[a_0,a_{-1}]_\star = -2a_{-1}$, and $[a_1,a_{-1}]_\star = a_0$. We would like to study the $^*$-subalgebra $\mathcal{B}$ of $\mathscr{C}^\infty(\mathbb{S}^2)$ generated by $\mathcal{P}_1$. To this end, set $\mathcal{B}_1 := \mathcal{P}_0 \oplus \mathcal{P}_1$ and define the spaces $\mathcal{B}_k := \mathcal{B}_{k-1} + \mathcal{B}_{k-1}\star\mathcal{P}_1$, spanned by products of up to $k$ many elements of $\mathcal{P}_1$, recursively. Note that $\mathcal{B}$ and $\mathcal{B}_k$ are SO(3)-invariant, and therefore direct sums of some $\mathcal{P}_\ell$'s. Since $\mathcal{P}_k \otimes \mathcal{P}_1 \cong \mathcal{P}_{k-1} \oplus \mathcal{P}_k \oplus \mathcal{P}_{k+1}$ if $k \in \mathbb{N}$, we have $\mathcal{B}_k \subseteq \mathcal{P}_0 \oplus \cdots \oplus \mathcal{P}_k$. If we had equality for every $k \in \mathbb{N}_0$, then $(a_1)^k \neq 0$ for all $k \in \mathbb{N}_0$. But then $[a_0,(a_1)^k]_\star = 2k(a_1)^k$, whence $[a_0,\,\cdot\,]_\star$ would be unbounded on $\mathcal{A}$. So there must be a smallest $k \in \mathbb{N}$ with $\mathcal{B}_k \subsetneq \mathcal{P}_0 \oplus \cdots \oplus \mathcal{P}_k$, and consequently $\mathcal{B}_{k-1} = \mathcal{B}_k = \cdots = \mathcal{B}$. Recall that the involution $^*$ preserves the $\mathcal{P}_\ell$'s, so $\mathcal{B}$ is a finite dimensional C$^*$-subalgebra of $\mathcal{A}$, thus a direct sum of matrix algebras. Central idempotents in $\mathcal{B}$ are projections to direct sums of matrix subalgebras, and therefore there are only finitely many central idempotents. Taking any central idempotent $f$, it is easy to check that $g \triangleright f$ is again a central idempotent for all $g \in \mathrm{SO}(3)$, depending continuously on $g$. Since the space of central idempotents is finite and hence discrete, this implies that every central idempotent $f$ is fixed by SO(3). But the trivial representation has multiplicity 1 in $\mathcal{B}$, so there is only one central idempotent and $\mathcal{B}$ is a full matrix algebra.

We saw before that $e$ is a unit in $\mathcal{A}$ and contained in the C$^*$-subalgebra $\mathcal{B}$. Therefore $\mathcal{A} \cong \mathcal{B} \otimes \mathcal{B}^{\mathrm{c}}$ by [EL77], where $\mathcal{B}^{\mathrm{c}}$ denotes the commutant of $\mathcal{B}$ in $\mathcal{A}$. Note that $\mathcal{B}^{\mathrm{c}}$ is SO(3)-invariant and the tensor product decomposition is preserved under the action of SO(3). So $\mathcal{B}^{\mathrm{c}}$ is a direct sum of $\mathcal{P}_\ell$'s, and if it contains any $\mathcal{P}_\ell$ with $\ell \geq 1$, then basic representation theory shows that $\mathcal{P}_\ell$ occurs both in the representations $\mathcal{P}_0 \otimes \mathcal{P}_\ell \cong \mathcal{P}_\ell$ and $\mathcal{P}_1 \otimes \mathcal{P}_\ell \cong \mathcal{P}_{\ell-1} \oplus \mathcal{P}_\ell \oplus \mathcal{P}_{\ell+1}$, and therefore occurs with multiplicity at least 2 in $\mathcal{B} \otimes \mathcal{B}^{\mathrm{c}}$. Consequently $\mathcal{B}^{\mathrm{c}}$ must be trivial, proving that $\mathcal{A} = \mathcal{B}$ and thereby contradicting that $\mathcal{B}$ is finite dimensional.

Consequently, we must have $[\mathcal{P}_1,\mathcal{P}_1] = 0$. Define the algebras $\mathcal{B}_k$ as before. If there is a smallest $k \in \mathbb{N}$ such that $\mathcal{B}_k \subsetneq \mathcal{P}_0 \oplus \cdots \oplus \mathcal{P}_k$, then $\mathcal{B}_{k-1} = \mathcal{B}_k = \cdots = \mathcal{B}$ is a commutative finite-dimensional C$^*$-algebra, and the argument above using central idempotents shows that $\mathcal{B}$ must be 1-dimensional, contradicting that $\mathcal{P}_0 \oplus \mathcal{P}_1 \subseteq \mathcal{B}$. So $\mathcal{B} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \ldots$ is commutative since it is generated by $\mathcal{P}_1$ and $[\mathcal{P}_1,\mathcal{P}_1] = 0$. Since $\mathcal{B}$ is dense in $\mathcal{A}$, this implies that $\mathcal{A}$ is commutative. □

In this thesis we would like to find a quantization procedure, that is also capable of quantizing the 2-sphere in a SO(3)-equivariant way. Therefore we will inevitable need to generalize our definition of strict quantizations.

## 2.6 Berezin–Toeplitz quantization

We now discuss a different quantization method, closely related to geometric quantization. This method works also for the 2-sphere with a SO(3)-invariant Poisson structure, and approximates $\mathscr{C}^\infty(M)$ by finite dimensional matrix algebras for values of $\hbar$ in the set $I = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. This requires a more general definition of strict quantizations than the one we gave in Definition 2.20. In particular, we need to allow non-injective quantization maps $\mathscr{A}_0 \to \mathcal{A}_\hbar$. See [Haw08] for a further discussion of definitions of strict quantizations present in the literature.

**Definition 2.28 (Strict quantization)** *Let $(M, \pi)$ be a Poisson manifold. A* strict quantization *on $M$ is specified by the following data:*

*i.) a subset $I \subseteq \mathbb{R}$ containing $0$ as a non-isolated point,*

*ii.) a collection $\mathcal{A}_\hbar$ of $C^*$-algebras with $\mathcal{A}_0 = \mathscr{C}_0(M)$,*

*iii.) a set $\Gamma \subseteq \prod_{\hbar \in I} \mathcal{A}_\hbar$ of sections,*

*iv.) a dense $^*$-subalgebra $\mathscr{A}_0$ of $\mathcal{A}_0$, and*

*v.) linear quantization maps $Q_\hbar \colon \mathscr{A}_0 \to \mathcal{A}_\hbar$,*

*such that*

*a.) $\{\mathcal{A}_\hbar\}_{\hbar \in I}$ together with $\Gamma$ defines a continuous field of $C^*$-algebras as in Definition 2.18,*

*b.) $\mathscr{A}_0$ is closed under taking Poisson brackets,*

*c.) for every $a \in \mathscr{A}_0$ the section $\hbar \mapsto Q_\hbar(a)$ is continuous, i.e. an element of $\Gamma$,*

*d.) for all $a, b \in \mathscr{A}_0$ we have*

$$\lim_{\hbar \to 0} \left\| \frac{1}{\mathrm{i}\hbar}[Q_\hbar(a), Q_\hbar(b)] - Q_\hbar(\{a, b\}) \right\|_\hbar = 0\,. \tag{2.26}$$

*A strict quantization is called*

*1.)* injective *if each $Q_\hbar$ is injective,*

*2.)* Hermitian *if $Q_\hbar(a^*) = Q_\hbar(a)^*$ for all $a \in \mathscr{A}_0$,*

*3.)* algebraically closed *if* $\mathrm{im}(Q_\hbar)$ *is an algebra for every $\hbar \in I$,*

*4.)* dense *if the $^*$-algebra generated by* $\mathrm{im}(Q_\hbar)$ *is dense in $\mathcal{A}_\hbar$ for every $\hbar \in I$,*

*5.)* unital *if $\mathscr{A}_0$ and all $\mathcal{A}_\hbar$ are unital, and the quantization maps $Q_\hbar$ are unital maps,*

*6.)* of order $n \in \mathbb{N}_0$ *if for all $a, b \in \mathscr{A}_0$ there exists a polynomial $a *^n b \in \mathscr{A}_0[\hbar]$ of degree $n$ such that*

$$\lim_{\hbar \to 0} \left\| \frac{1}{\hbar^n}\left(Q_\hbar(a)Q_\hbar(b) - Q_\hbar(a *^n b)\right) \right\| = 0\,. \tag{2.27}$$

In this terminology a strict deformation quantization as defined in Definition 2.20 is nothing else than an injective, algebraically closed, and dense strict quantization, for which $I$ is an open interval and $\mathrm{im}(Q_\hbar)$ is closed under $^*$.

**Berezin–Toeplitz quantization**

Let us now present the main idea of Berezin–Toeplitz quantization. As opposed to formal deformation quantization, it focusses more on the states, and directly constructs a Hilbert space $\mathcal{H}_\hbar$ together with a quantization map that associates a bounded operator to any smooth function on $M$. All the Hilbert spaces $\mathcal{H}_\hbar$ with $\hbar \neq 0$ are finite dimensional, but their dimension increases when $\hbar$ becomes small. In the limit $\hbar \to 0$ the bounded operators on $\mathcal{H}_\hbar$ approximate $\mathscr{C}^\infty(M)$.

Let $L \to M$ be a complex line bundle with a Hermitian metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on the fibers. Recall that a connection $\nabla$ on $L$ is *Hermitian* if it satisfies $X\langle s,t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$ for all vector fields $X \in \Gamma^\infty(\mathrm{T}M)$ and sections $s,t \in \Gamma^\infty(L)$, and that we may identify its curvature $R^\nabla$, which is a 2-form with values in $\mathsf{End}(L)$, with a complex 2-form on $M$.

**Definition 2.29** *A symplectic manifold $(M,\omega)$ is said to be* quantizable *if there exists a complex line bundle $L \to M$ with a Hermitian metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on the fibers and a Hermitian connection $\nabla$ on $L$, such that the curvature of $\nabla$ is $-\mathrm{i}\omega$.*

It can be shown that $M$ is quantizable if and only if $[\omega]/2\pi \in \mathrm{H}^2_{\mathrm{dR}}(M)$ lies in the image of the map $\mathrm{H}^2(M,\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(M)$. The Hermitian metric can be used to endow the space $\Gamma^\infty_c(L)$ of smooth compactly supported sections of $L$ with an inner product $\langle\!\langle s,t\rangle\!\rangle := \int_M \langle s,t\rangle \omega^d$. Let $\mathrm{L}^2\Gamma(L)$ denote the completion of $\Gamma^\infty_c(L)$ to a Hilbert space with respect to $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$.

Let $M$ be a quantizable compact Kähler manifold. For any quantizing line bundle $L \to M$ with connection $\nabla$, the antiholomorphic part of $\nabla$ can be used to make $L$ into a holomorphic line bundle. More precisely, there is a unique way to write $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ with $\nabla^{(1,0)} \colon \Gamma^\infty(L) \to \Gamma^\infty(\mathrm{T}^{*,(1,0)}M \otimes L)$ and $\nabla^{(0,1)} \colon \Gamma^\infty(L) \to \Gamma^\infty(\mathrm{T}^{*,(0,1)}M \otimes L)$. We can then define holomorphic sections as those sections $s \in \Gamma^\infty(L)$ that satisfy $\nabla^{(0,1)}s = 0$. The space $\Gamma_{\mathrm{hol}}(L)$ of holomorphic sections of $L$ is finite dimensional, and we may define the orthogonal projection $\Pi\colon \mathrm{L}^2\Gamma(L) \to \Gamma_{\mathrm{hol}}(L)$ and the *Toeplitz operators*

$$T_f \colon \Gamma_{\mathrm{hol}}(L) \to \Gamma_{\mathrm{hol}}(L)\,, \quad T_f = \Pi \circ M_f \tag{2.28}$$

for all $f \in \mathscr{C}^\infty(M)$. Here $M_f \colon \Gamma_{\mathrm{hol}}(L) \to \Gamma^\infty(L)$ denotes the multiplication by $f$.

Finally, consider tensor powers $L^{\otimes m}$ of the quantum line bundle, with the induced connection $\nabla^{(m)}$ and the induced inner product $\langle\,\cdot\,,\,\cdot\,\rangle^{(m)}$. Denote the Toeplitz operators on $L^{\otimes m}$, defined as in (2.28), by $T_f^{(m)}$.

Since $R^\nabla = -\mathrm{i}\omega$ it follows that $L$ is positive and therefore, by Kodaira's theorem, it is ample. This means that some tensor power of $L$ is very ample, in the sense that the global sections of this tensor power define an embedding of $M$ into projective space. It is a technical assumption in the following theorem that $L$ is already very ample, which we can achieve by replacing it with a tensor power and rescaling the symplectic form.

**Theorem 2.30 (Bordemann–Meinrenken–Schlichenmaier)** *Let $M$ be a compact Kähler manifold, with a very ample quantum line bundle $L$. Then* *i.) for every* $f \in \mathscr{C}^\infty(M)$ *there is a constant* $C$ *such that*

$$\|f\|_\infty - \frac{C}{m} \leq \|T_f^{(m)}\| \leq \|f\|_\infty \tag{2.29}$$

*holds for all* $m \in \mathbb{N}$*, where* $\|f\|_\infty$ *is the maximum norm of* $f$ *and* $\|T_f^{(m)}\|$ *is the operator norm of* $T_f^{(m)}$*,*

*ii.) for all* $f, g \in \mathscr{C}^\infty(M)$ *we have*

$$\lim_{m\to\infty} \|T_f^{(m)} T_g^{(m)} - T_{fg}^{(m)}\| = 0 \tag{2.30}$$

*and*

$$\lim_{m\to\infty} \|\mathrm{i}m[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}\| = 0\,. \tag{2.31}$$

*In particular, setting* $\hbar = \frac{1}{m}$*, the Berezin–Toeplitz quantization is a strict quantization on the set* $I = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$*. Furthermore, it is algebraically closed and of infinite order.*

PROOF: The proof of statements *i.)* and *ii.)* can be found in [BMS94]. Note that (2.29) implies that $\lim_{\hbar\to 0}\|T_f^{(1/\hbar)}\| = \|f\|_\infty$ and therefore, using the notation $Q_\hbar(f) = T_f^{(1/\hbar)}$ if $\hbar \neq 0$ and $Q_0(f) = f$, that $\hbar \mapsto \|Q_\hbar(f)\|_\hbar$ is continuous for every $f \in \mathscr{C}^\infty(M)$. It follows from (2.30) that $\hbar \mapsto \|Q_\hbar(f)Q_\hbar(g)\|_\hbar$ is continuous, and since $(T_g^{(1/\hbar)})^* = T_{\overline{g}}^{(1/\hbar)}$ we also have that $\hbar \mapsto \|Q_\hbar(f) + Q_\hbar(g)^*\|_\hbar = \|Q_\hbar(f + \overline{g})\|_\hbar$ is continuous. So by the discussion after Definition 2.20 it follows that the field of C$^*$-algebras is indeed continuous. The other properties of a strict quantization are easily verified.

It is proven in [BdMG81] that Toeplitz operators form a ring, so the quantization is algebraically closed, and in [Sch00] it is shown that the Berezin–Toeplitz quantization is of infinite order. □

To conclude this section, let us mention that Berezin–Toeplitz quantization is closely related to *geometric quantization*, where one considers the operators

$$\mathcal{Q}_f \colon \Gamma^\infty(L) \to \Gamma^\infty(L)\,, \quad \mathcal{Q}_f(s) = -\nabla_{X_f} s + \mathrm{i} f s \tag{2.32}$$

on sections of a quantum line bundle $L$ over a symplectic manifold $M$. However, the quantum state space $\mathrm{L}^2\Gamma(L)$ is too large for physical applications: If $M = \mathbb{R}^{2d}$ and $L$ is the trivial line bundle, then $\mathrm{L}^2\Gamma(L) = \mathrm{L}^2(\mathbb{R}^{2d})$, instead of $\mathrm{L}^2(\mathbb{R}^d)$. Therefore one needs to cut down its dimension by *polarizing*. If $M$ is a Kähler manifold we can use the complex polarization for this step.

The operators $Q_f \colon \Gamma_{\mathrm{hol}}(L) \to \Gamma_{\mathrm{hol}}(L)$, $Q_f = \Pi \circ \mathcal{Q}_f$ are related to Berezin–Toeplitz quantization by

$$Q_f = \mathrm{i} T_{f - \frac{1}{2}\Delta f}\,, \tag{2.33}$$

which is a result of Tuynman [Tuy87].

## 2.7 Obstructions to strict quantizations

The purpose of this subsection is two-fold. First, we want to show that formal deformation quantizations have important applications in other areas of mathematics, by demonstrating how they can be used to reformulate the Atiyah–Singer index theorem in a purely algebraic way, usually referred to as the *algebraic index theorem*. Second, we want to show how the algebraic index theorem can be used to obtain obstructions for the existence of strict quantizations.

The Atiyah–Singer index theorem [AS68] itself is certainly one of the most important mathematical results of the 20th century, revealing deep connections between analysis and algebraic topology. Let us only mention that it computes the index of an elliptic pseudodifferential operator $D$ between vector bundles over a compact manifold $M$ in terms of the symbol of $D$ and topological invariants of the manifold $M$, and refer to the literature [BGV04, Gil96] on the subject for further details.

The connection to the algebraic index theorem is given by the symbol calculus of pseudodifferential operators, see (2.24). Fixing a map which associates a pseudodifferential operator to a symbol defines a deformation quantization in a natural way. This can be used to derive the Atiyah–Singer index theorem from the algebraic index theorem [NT96].

In this subsection we use the abbreviation $\mathbb{A}$ for a formal deformation quantization $(\mathscr{C}^\infty(M)[[\nu]], \star)$, and denote the subalgebras $(\mathscr{C}^\infty_c(M)[[\nu]], \star)$ and $(\mathscr{C}^\infty_{\mathrm{const}}(M)[[\nu]], \star)$ of formal power series of functions with compact support and of formal power series of functions that are constant outside some compact set by $\mathbb{A}_c$ and $\mathbb{A}_{\mathrm{const}}$, respectively.

Any formal deformation quantization $\mathbb{A}$ has a trace [Fed96], that is a $\mathbb{C}[[\nu]]$-linear functional $\mathrm{tr}\colon \mathbb{A}_c \to \mathscr{C}^\infty[\nu^{-1}, \nu]]$ with values in Laurent series, such that $\mathrm{tr}(f \star g) = \mathrm{tr}(g \star f)$ holds for all $f \in \mathbb{A}_c$ and $g \in \mathbb{A}$. Such a trace is unique up to normalization, and the normalization can be fixed by local considerations. The algebraic index theorem computes the pairing of this trace with the compactly supported K-theory of $\mathbb{A}$. This theorem was obtained independently by Fedosov [Fed86, Fed95] and Nest–Tsygan [NT95a].

**Theorem 2.31 (Algebraic index theorem)** *Let $\mathbb{A}$ be a formal deformation quantization of a symplectic manifold $(M, \omega)$, and let $e, f \in M_{n\times n}(\mathbb{A})$ be idempotents in the matrix algebra over $\mathbb{A}$, such that their difference is of compact support. Then*

$$\mathrm{tr}(e - f) = \int_M (\mathrm{ch}(\sigma(e)) - \mathrm{ch}(\sigma(f)))\hat{A}(M)\mathrm{e}^{\mathrm{i}\theta}\,. \tag{2.34}$$

A precise understanding of the expression on the right hand side is not required in the following. It is only relevant that it depends on the symbols $\sigma(e), \sigma(f) \in M_{n\times n}(\mathscr{C}^\infty(M))$, obtained by setting $\hbar = 0$, topological data of the manifold, and the characteristic class of the deformation quantization. But for completeness let us mention that ch is the Chern character mapping idempotents in $M_{n\times n}(\mathscr{C}^\infty(M))$ to differential forms $\Omega^{\mathrm{even}}(M)$ of even degree, while $\hat{A}(M) \in \mathrm{H}^{4\bullet}_{\mathrm{dR}}(M)$ denotes the A roof genus of $M$, which is defined as a certain characteristic class of its (complexified) tangent bundle, and $\theta$ is the characteristic class from (2.18).

Let us now show how Theorem 2.31 can provide obstructions to the existence of certain quantizations. Note that given a strict quantization and an element $f \in$

$\mathscr{C}_c^\infty(M)$, we can ask whether the trace of $f$ viewed as an element of $\mathbb{A}_c$ can be computed from the elements $Q_\hbar(f)$. For the Weyl–Moyal product from Example 2.9 and Example 2.24 any $Q_\hbar(f)$ is of trace-class, and its trace agrees with the trace of $f$ in $\mathbb{A}_c$. This motivates the following definition, which is due to Fedosov [Fed96, Chapter 7], and abstracts the properties of the Weyl–Moyal product. In *ii.)* we require explicitly that the trace of an element $f \in \mathbb{A}_c$ is the formal expansion of the traces of the operators $\mathrm{Op}_{N,\hbar}(f)$. Write $\operatorname{tr} f|_{N-d}$ for the truncation of the formal Laurent series $\operatorname{tr} f = \sum_{k=-d}^{\infty} t_k \nu^k$ at order $N-d$ with $\nu$ substituted by $\hbar$, i.e. $\operatorname{tr} f|_{N-d} = \sum_{k=-d}^{N-d} t_k \hbar^k$.

**Definition 2.32 (Asymptotic operator representation)** *Let $(M,\omega)$ be a symplectic manifold of dimension $2d$ and let $I \subseteq (0,1]$ be a set with $0$ as limit point. An* asymptotic operator representation (AOR) *of a formal deformation quantization $\mathbb{A}$ of $M$ is a family of linear maps*

$$\mathrm{Op}_{N,\hbar}\colon \mathbb{A}_{\mathrm{const}}(M) \to B(\mathcal{H}) \tag{2.35}$$

*for all $N \in \mathbb{N}$ and $\hbar \in I$, where $\mathcal{H}$ is a fixed Hilbert space, satisfying that*

*i.) for all $N \in \mathbb{N}$ and $f, g \in \mathbb{A}_{\mathrm{const}}$ there are constants $C_1, C_2 \in \mathbb{R}$ such that*

$$\|\mathrm{Op}_{N,\hbar}(f) - \mathrm{Op}_{N+1,\hbar}(f)\| \leq C_1 \hbar^{N+1}\,, \tag{2.36}$$

$$\|\mathrm{Op}_{N,\hbar}(f \star g) - \mathrm{Op}_{N,\hbar}(f)\mathrm{Op}_{N,\hbar}(g)\| \leq C_2 \hbar^{N+1} \tag{2.37}$$

*hold for all $\hbar \in I$,*

*ii.) for all $N \in \mathbb{N}$ and $f \in \mathbb{A}_c$, there exist constants $C_3, C_4, C_5 \in \mathbb{R}$ such that the operator $\mathrm{Op}_{N,\hbar}(f)$ is trace class and*

$$\|\mathrm{Op}_{N,\hbar}(f)\|_1 \leq C_3 \hbar^{-d}\,, \tag{2.38}$$

$$\|\mathrm{Op}_{N,\hbar}(f) - \mathrm{Op}_{N+1,\hbar}(f)\|_1 \leq C_4 \hbar^{-d+N+1}\,, \tag{2.39}$$

$$\left|\operatorname{tr} \mathrm{Op}_{N,\hbar}(f) - \operatorname{tr} f|_{N-d}\right| \leq C_5 \hbar^{-d+N+1} \tag{2.40}$$

*hold for all $\hbar \in I$, and*

*iii.) for all $N > d$ and $f, g \in \mathbb{A}_{\mathrm{const}}$, there is a constant $C_6 \in \mathbb{R}$ such that the operator $\mathrm{Op}_{N,\hbar}(f \star g) - \mathrm{Op}_{N,\hbar}(f)\mathrm{Op}_{N,\hbar}(g)$ is trace class and*

$$\|\mathrm{Op}_{N,\hbar}(f \star g) - \mathrm{Op}_{N,\hbar}(f)\mathrm{Op}_{N,\hbar}(g)\|_1 \leq C_6 \hbar^{-d+N+1} \tag{2.41}$$

*holds for all $\hbar \in I$.*

The motivating example for the definition of an AOR is the Weyl–Moyal product. Note that for any $g \in \mathscr{C}^\infty(M)$ the pseudodifferential operator $\mathrm{Op}_g$ defined in Example 2.24 can be extended to $\mathrm{L}^2(\mathbb{R}^d)$, see [Fed96, Section 3.4]. If $f|_N = \sum_{k=0}^N f_k \hbar^k$ denotes the truncation of an element $f = \sum_{k=0}^\infty f_k \nu^k \in \mathbb{A}_{\mathrm{const}}$, then we may define $\mathrm{Op}_{N,\hbar}(f) = \mathrm{Op}_{f|_N}$, where the right hand side refers to the extension described above. As in this example, the parameter $N$ in the definition of an AOR can be thought of as a truncation degree in powers of $\hbar$, and an AOR as the collection of all truncations at orders $N$ of an infinite order strict quantization. Note however that, starting with an AOR and fixing $\hbar$ and $f$, there is no reason for the sequence $\mathrm{Op}_{N,\hbar}(f)$ to converge.

There are good criteria for the existence of AORs. Let $K_c(M)$ be the compact $K$-theory of $M$, i.e. equivalence classes of formal differences $[e-f]$, where $e, f \in M_{n\times n}(\mathscr{C}^\infty(M))$ are idempotents with compactly supported difference. Denote the right hand side of (2.34) for $\xi \in K_c(M)$ by $\mathrm{ind}_{\mathrm{top}}(\xi)$, i.e.

$$\mathrm{ind}_{\mathrm{top}}(\xi) = \int_M \mathrm{ch}(\xi)\hat{A}(M)\mathrm{e}^{\mathrm{i}\theta}\,, \tag{2.42}$$

and let $\mathrm{ind}_{\mathrm{top}}(\xi)|_N$ be the formal expansion of $\mathrm{ind}_{\mathrm{top}}(\xi)$, truncated at order $N$ and with $\nu$ replaced by $\hbar$. Note that the only dependence of $\mathrm{ind}_{\mathrm{top}}(\xi)$ on $\nu$ is through the characteristic class $\theta$.

**Theorem 2.33 (Fedosov)** *Let $(M,\omega)$ be a symplectic manifold and $\xi_1,\dots,\xi_m \in K_c(M)$ be generators of the compact $K$-theory of $M$. Assume that $M$ has an AOR on the set $I$. Then we must have*

$$\mathrm{ind}_{\mathrm{top}}(\xi_k)|_{N-d} \in \mathbb{Z} \qquad \mathrm{mod}\ \mathcal{O}(\hbar^{N-d+1}) \tag{2.43}$$

*for all $N > d$ on the set $I$. In particular, if $\theta = \frac{1}{\mathrm{i}\hbar}\omega$, then*

$$\mathrm{ind}_{\mathrm{top}}(\xi_k) \in \mathbb{Z} \qquad \mathrm{mod}\ \mathcal{O}(\hbar^\infty)\,. \tag{2.44}$$

By $f(\hbar) \in \mathbb{Z} \mod \mathcal{O}(\hbar^N)$ we mean that the difference of $f(\hbar)$ to the nearest integer can be bounded by $C\hbar^N$ for some constant $C$, independent of $\hbar$. If $\theta = \frac{1}{\mathrm{i}\hbar}\omega$, then this theorem implies in particular that for every $\varepsilon > 0$ there is some $R > 0$ such that the difference of $\mathrm{ind}_{\mathrm{top}}(\xi_k)$ to the nearest integer is at most $\varepsilon$ for all $\hbar \in I \cap [-R, R]$.

Theorem 2.33 is not very surprising. Since $\xi_k$ can be represented by a difference of idempotents, vanishing outside of a compact set, the algebraic index theorem shows that $\mathrm{ind}_{\mathrm{top}}(\xi_k)$ equals the trace of $\xi_k$. But the trace can be computed asymptotically from the operator trace in the AOR, and this operator trace is integral for any $\hbar \in I$. If $\theta = \frac{1}{\mathrm{i}\hbar}\omega$ then $\mathrm{ind}_{\mathrm{top}}(\xi_k)$ is a Laurent polynomial concentrated in degrees $-d$ to 0, and therefore $\mathrm{ind}_{\mathrm{top}}(\xi_k)|_{N-d}$ is independent of $N$ if $N > d$.

Any symplectic manifold admits a compatible almost complex structure, and therefore its tangent bundle admits the structure of a $d$-dimensional complex vector bundle $\mathrm{T}_{\mathbb{C}}M$. We have

$$\hat{A}(M) = \exp\left(\frac{1}{2}c_1(\mathrm{T}_{\mathbb{C}}M)\right)\mathrm{Td}(\mathrm{T}_{\mathbb{C}}M) \tag{2.45}$$

with $c_1$ and Td denoting the first Chern class and the Todd class, respectively. If $\theta = \frac{\omega}{\mathrm{i}\hbar}$, then integrality of $\mathrm{ind}_{\mathrm{top}}(\xi_k)$ for all $\xi_k \in K_c(M)$ implies that $\frac{\omega}{\mathrm{i}\hbar} + \frac{1}{2}c_1(\mathrm{T}_{\mathbb{C}}M)$ is an integral cohomology class. On the other hand this condition is sufficient to guarantee the existence of AORs:

**Theorem 2.34 (Fedosov)** *Let $(M,\omega)$ be a compact symplectic manifold with formal deformation quantization $\mathbb{A}$. Assume that the characteristic class of $\mathbb{A}$ is $\theta = \frac{\omega}{\mathrm{i}\hbar}$. If $\hbar \in (0,1]$ is such that*

$$\frac{\omega}{\hbar} + \frac{1}{2}c_1(\mathrm{T}_{\mathbb{C}}M) \in \mathrm{im}\left(\mathrm{H}^2(M,\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})\right), \tag{2.46}$$

*then* $\operatorname{ind}_{\mathrm{top}}(\xi) \in \mathbb{Z}$ *for all* $\xi \in K_c(M)$ *and there is an AOR on the parameter set* $I = \{\frac{\hbar}{2k+1} \mid k \in \mathbb{N}_0\}$.

Proof: See [Fed96, Section 7.3]. □

Unfortunately, not every strict quantization, even if it is of infinite order, leads to an AOR in the sense of Fedosov. For the non-commutative torus from Example 2.25 there is a unique trace on the quantum algebras $\mathcal{A}_\hbar$, but the possible values of this trace when applied to idempotents are dense in $[0,1]$ [Rie81]. These operators are therefore not trace-class, as required in the definition of an AOR, and we do not get an integrality condition.

Nevertheless, both Fedosov's construction of AORs and his proof of the integrality condition have inspired other more general existence and obstruction results of strict quantizations. Natsume, Nest, and Peter generalized his construction to obtain strict quantizations of a rather general class of symplectic manifolds [NNP03], that does, however, not contain the 2-sphere.

**Theorem 2.35 (Natsume–Nest–Peter)** *Let* $M$ *be a closed symplectic manifold such that* $\pi_1(M)$ *is exact and* $\pi_2(M) = 0$. *Then* $M$ *has a dense strict quantization on a parameter set of the form* $I = [0, \varepsilon)$ *with* $\varepsilon \in \mathbb{R}^+$.

The basic idea of the proof is to pass to the universal cover $\tilde{M}$ of $M$. The assumption that $\pi_2(M) = 0$ implies that $\mathrm{H}^1(\tilde{M}) = \mathrm{H}^2(\tilde{M}) = 0$. For this reason the integrality condition of Fedosov becomes trivial. In fact, his construction of AORs can be generalized to yield a dense strict quantization of $\tilde{M}$. By considering the reduced crossed product of this strict quantization with the fundamental group $\Gamma$, which is the construction of "non-commutative quotients", and using that $\pi_1(M)$ is exact, it is possible to obtain a dense strict quantization of the quotient $M = \tilde{M}/\Gamma$.

Similar ideas of quantizing the universal cover had already been used to obtain quantizations of compact Riemann surfaces of genus at least 1, which are quotients of the hyperbolic plane by a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, see [KL92, NN99].

Hawkins shows how to obtain more general obstructions to the existence of strict quantizations of the 2-sphere that do not necessarily determine AORs in [Haw08], using similar ideas than Fedosov.

**Theorem 2.36 (Hawkins)** *If* $\{\mathcal{A}_\hbar\}_{\hbar \in I}$ *is a unital Hermitian second order strict quantization of the 2-sphere* $\mathbb{S}^2$ *with a non-degenerate* $\mathrm{SO}(3)$*-invariant Poisson structure, such that* $x^i \in \mathscr{A}_0$ *and such that* $\Delta f \in \mathscr{A}_0$ *implies* $f \in \mathscr{A}_0$ *for all* $f \in \mathscr{C}(\mathbb{S}^2)$, *then there is no connected neighbourhood of* 0 *in* $I$. *In particular,* $I$ *is not connected.*

Here $x^i$ denotes the coordinate functions on $\mathbb{R}^3$, restricted to $\mathbb{S}^2 \subseteq \mathbb{R}^3$, and $\Delta$ is the Laplacian.

## 2.8 Fréchet-algebraic approach to quantization

In the previous constructions of strict quantizations we obtained strict quantizations directly, without making use of formal deformation quantizations. In this subsection, we will use that a formal deformation quantization is usually easy to obtain, either by the results of Theorem 2.15 and Theorem 2.16 or by more explicit constructions for certain special cases, and ask whether we can make such a formal deformation quantization convergent. This idea goes back to the work of Beiser, Römer, and Waldmann [BRW07, BW14].

Note that, starting with a formal power series, it is not hard to construct a smooth function that has this series as Taylor expansion. But such a construction is certainly not canonical, and so we do not expect that there is any canonical way to make formal deformation quantizations convergent.

One possible approach is to search for subalgebras $\mathscr{P}$ of $\mathscr{C}^\infty(M)$, on which the star product of any two elements is a polynomial in $\nu$, or in other words the power series in $\nu$ has only finitely many non-zero elements. On such a subalgebra we can replace the formal parameter $\nu$ with any complex number $\hbar$, and obtain a family of non-formal products $\star_\hbar \colon \mathscr{P} \times \mathscr{P} \to \mathscr{P}$. While the existence of a non-trivial algebra $\mathscr{P}$ with these properties is in general not guaranteed, extra structure on $M$ determines such algebras in many concrete situations: For the formal star products introduced in Example 2.9 we may choose $\mathscr{P}$ to be the algebra of polynomials on $\mathbb{R}^{2d}$, and for a cotangent bundle we may try to take polynomials in the momentum variables.

Once such a subalgebra $\mathscr{P}$ is found, one can search for a topology with respect to which the product on $\mathscr{P}$ is continuous. Completing $\mathscr{P}$ with respect to this topology yields a larger algebra of interesting functions for which the product is still well-defined. Usually, such a topology is only locally convex and cannot be defined by multiplicative seminorms, so that the completion often becomes a Fréchet algebra that is not multiplicatively convex (see Definition 2.21), in particular it is not a $\mathrm{C}^*$-algebra. This is not very surprising, since $\mathscr{P}$ usually contains elements satisfying (some variant of) the canonical commutation relations, which cannot be realized in a $\mathrm{C}^*$-algebra, or even any multiplicatively convex algebra.

On the one hand, it might be desirable from a mathematical perspective to obtain $\mathrm{C}^*$-algebras, and one is led to the question whether there is a good way to associate $\mathrm{C}^*$-algebras to the Fréchet algebras obtained with this construction. On the other hand, for doing physics a Fréchet-algebraic quantization might be considered to be more direct, as it may already contain the relevant observables.

Let us now describe some of the examples in which this approach was carried out successfully.

**Example 2.37 (Star product of exponential type)** Let $V$ be a (not necessarily finite dimensional) vector space and let $\Lambda \colon V \times V \to \mathbb{C}$ be a bilinear form on $V$. Define $P_\Lambda \colon \mathrm{S}^m(V) \otimes \mathrm{S}^n(V) \to \mathrm{S}^{m-1}(V) \otimes \mathrm{S}^{n-1}(V)$ by

$$\begin{aligned} P_\Lambda(v_1 \vee \dots \vee v_m \otimes w_1 \vee \dots \vee w_n) = \sum_{j=1}^{m} \sum_{k=1}^{n} \Lambda(v_j, w_k) \cdot \\ \cdot v_1 \vee \dots \vee v_{j-1} \vee v_{j+1} \vee \dots \vee v_m \otimes w_1 \vee \dots \vee w_{k-1} \vee w_{k+1} \vee \dots \vee w_n \end{aligned} \tag{2.47}$$

and define the *strict star product of exponential type* as

$$\star_{\hbar\Lambda} \colon \mathrm{S}^\bullet(V) \times \mathrm{S}^\bullet(V) \to \mathrm{S}^\bullet(V)\,, \quad (v, w) \mapsto v \star_{\hbar\Lambda} w := \mu_\vee \circ \mathrm{e}^{\frac{1}{2}\mathrm{i} P_{\hbar\Lambda}}(v \otimes w)\,. \tag{2.48}$$

By $\mu_\vee$ we mean the map that multiplies the two tensor factors together, using the symmetric tensor product. Our convention concerning symmetric tensors is that $v \vee w = \mathrm{Sym}^\bullet(v \otimes w)$ where $\mathrm{Sym}^\bullet$ is the sum of the symmetrization operators

$$\mathrm{Sym}^k \colon \mathrm{T}^k(V) \to \mathrm{T}^k(V)\,, \quad \mathrm{Sym}^k(v_1 \otimes \ldots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)} \tag{2.49}$$

and $\mathrm{S}^\bullet(V)$ is identified with the image of $\mathrm{Sym}^\bullet$ in $\mathrm{T}^\bullet(V)$.

Note that (2.48) already defines a "strict product", in the sense that $\hbar$ is a complex number and not a formal parameter. If $V = \mathbb{R}^{2d}$ is finite dimensional, then $\mathrm{S}^\bullet(V)$ is isomorphic to the algebra of polynomials $\mathrm{Pol}(V^*)$, via the linear extension of the map $v_1 \vee \cdots \vee v_k \mapsto (\alpha \mapsto \alpha(v_1) \ldots \alpha(v_k))$. Choosing standard coordinates $x_1, \ldots, x_d, p_1, \ldots, p_d$ on $V^*$, and letting $\Lambda = \sum_{i=1}^d \bigl(\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial x_i}\bigr)$, the product (2.48) is precisely the restriction of the Weyl–Moyal product from Example 2.9 to the polynomials, with $\nu$ replaced by $\hbar$. Since differential operators are already determined by their behaviour on polynomials, one can also reconstruct the Weyl–Moyal product from (2.48).

Let $V$ be a locally convex vector space. Recall that the topology on the projective tensor product $V^{\otimes_\pi k}$ is the locally convex topology defined by all the seminorms

$$p^k(v) = \inf \left\{ \sum_{i=1}^{\ell} p(v_1^{(i)}) \ldots p(v_k^{(i)}) \;\middle|\; \ell \in \mathbb{N}\,,\, v = \sum_{i=1}^{\ell} v_1^{(i)} \otimes \ldots \otimes v_k^{(i)} \right\}, \tag{2.50}$$

obtained by letting $p$ run through the continuous seminorms on $V$.

**Definition 2.38 ($\mathbf{T}_{\boldsymbol{R}}$-topology)** *Let $R \in \mathbb{R}$ and $V$ be a locally convex vector space. Then the* $\mathrm{T}_R$-topology *on $\mathrm{T}^\bullet(V)$ is the locally convex topology determined by the seminorms*

$$p_R(v) = \sum_{k=0}^{\infty} (k!)^R p^k(v_k)\,, \tag{2.51}$$

*where $v = \sum_{k=0}^{\infty} v_k$ with $v_k \in V^{\otimes k}$ and $p$ is running through the continuous seminorms on $V$. The $\mathrm{T}_R$-topology on $\mathrm{S}^\bullet(V)$ is the subspace topology on $\mathrm{S}^\bullet(V)$, induced by the $\mathrm{T}_R$-topology on $\mathrm{T}^\bullet(V)$.*

The following theorem was obtained in [Wal14].

**Theorem 2.39 (Waldmann)** *Let $V$ be a locally convex vector space and $\Lambda \colon V \times V \to \mathbb{C}$ be a continuous bilinear form on $V$. For any $R \geq \frac{1}{2}$ and $\hbar \in \mathbb{C}$, the star product of exponential type $\star_{\hbar\Lambda}$ is continuous with respect to the $\mathrm{T}_R$-topology on $\mathrm{S}^\bullet(V)$.*

There is a version of this theorem for the case that all seminorms are Hilbert seminorms, meaning that they are induced by a not necessarily positive definite inner product. In this case, one can consider their Hilbert tensor product instead of the projective tensor product, see [SW18].

Note that, as opposed to the $\mathrm{C}^*$-algebraic constructions in the previous subsections, this example works well in infinite dimensions, which is important for possible applications in quantum field theory.

**Example 2.40 (Gutt star product)** Let $\mathfrak{g}$ be a Lie algebra, and denote its universal enveloping algebra by $\mathscr{U}\mathfrak{g}$. From the Poincaré–Birkhoff–Witt theorem it follows that summing the maps

$$q_{n,\hbar}\colon \mathrm{S}^n(\mathfrak{g}) \to \mathscr{U}\mathfrak{g}\,, \quad x_1 \vee \cdots \vee x_n \to \frac{(\mathrm{i}\hbar)^n}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)} \tag{2.52}$$

we obtain an isomorphism $q_\hbar\colon \mathrm{S}^\bullet(\mathfrak{g}) \to \mathscr{U}\mathfrak{g}$, $q_\hbar = \sum_{n=0}^\infty q_{n,\hbar}$ of vector spaces, but, unless $\mathfrak{g}$ is commutative, not of algebras. For $x \in \mathrm{S}^\bullet(\mathfrak{g})$ and $y \in \mathrm{S}^\bullet(\mathfrak{g})$ we define the *Gutt star product*

$$x \star_\hbar y = q_\hbar^{-1}(q_\hbar(x) q_\hbar(y))\,. \tag{2.53}$$

As in the previous example, if $\mathfrak{g}$ is finite dimensional, then this product is the restriction of a formal star product on $\mathfrak{g}^*$ to the polynomials $\mathrm{Pol}(\mathfrak{g}^*) \cong \mathrm{S}(\mathfrak{g})$. The formal star product can be reconstructed from $\star_\hbar$ and deforms the linear Poisson structure on $\mathfrak{g}^*$ obtained from the Lie algebra structure of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a locally convex Lie algebra. Then a continuous seminorm $q$ on $\mathfrak{g}$ is said to be an *asymptotic estimate* for a continuous seminorm $p$ if

$$p(w_n(x_1, \ldots, x_n)) \leq q(x_1) \cdot \ldots \cdot q(x_n) \tag{2.54}$$

holds for all $n \in \mathbb{N}$, all elements $x_1, \ldots, x_n \in \mathfrak{g}$, and words $w_n(x_1, \ldots, x_n)$ obtained by applying Lie brackets in arbitrary order to these elements (e.g. $[x_1, [[x_2, x_3], x_4]]$ or $[[[x_1, x_2], x_3], x_4]$). The Lie algebra $\mathfrak{g}$ is said to be *asymptotic estimate* if every continuous seminorm has an asymptotic estimate. For an asymptotic estimate Lie algebra, we have the following continuity result, see [ESW17]:

**Theorem 2.41 (Esposito–Stapor–Waldmann)** *Let $\mathfrak{g}$ be an asymptotic estimate Lie algebra. Then for any $R \geq 1$ and $\hbar \in \mathbb{C}$, the Gutt star product on $\mathfrak{g}$ is continuous with respect to the $\mathrm{T}_R$-topology on $\mathrm{S}^\bullet(\mathfrak{g})$.*

In both these examples the completion of $\mathrm{S}^\bullet(V)$ or $\mathrm{S}^\bullet(\mathfrak{g})$ with respect to the $\mathrm{T}_R$-topology can be described by power series with coefficients of a certain decay. However, there is no good geometric interpretation of this algebra. This is different in the following example. We describe this example in detail in Paper I, so we will only sketch the most relevant aspects.

**Definition 2.42** *The hyperbolic disc $\mathbb{D}^n$ is the $n$-dimensional Kähler manifold that is biholomorphic to the complex submanifold $\{z \in \mathbb{C}^n \mid |z^1| + \cdots + |z^n| < 1\}$ of $\mathbb{C}^n$, and whose Riemannian metric is given by*

$$g = \frac{\sum_{k=1}^n \mathrm{d}w^k \vee \mathrm{d}\overline{w}^k}{1 - \sum_{k=1}^n w^k \overline{w}^k} + \frac{\sum_{k,\ell=1}^n w^\ell \overline{w}^k \,\mathrm{d}w^k \vee \mathrm{d}\overline{w}^\ell}{(1 - \sum_{k=1}^n w^k \overline{w}^k)^2}\,, \tag{2.55}$$

*where $w^k(z) = z^k$ are the standard coordinates of $\mathbb{C}^n$ restricted to $\mathbb{D}^n$.*

The hyperbolic disc $\mathbb{D}^n$ can be obtained from $\mathbb{C}^{1+n}$ with a metric of signature $(1, n)$ via phase space reduction. A similar procedure allows one to define an extended disc $\hat{\mathbb{D}}^n$ by reduction from $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. The embedding $\mathbb{C}^{1+n} \ni z \mapsto (z, \overline{z}) \in \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ descends to an embedding $\mathbb{D}^n \to \hat{\mathbb{D}}^n$, and any holomorphic function on $\hat{\mathbb{D}}^n$ is uniquely determined by its restriction to $\mathbb{D}^n$. Denote the algebra of functions on $\mathbb{D}^n$, which are restrictions of a holomorphic function on $\hat{\mathbb{D}}^n$ by $\mathscr{A}(\mathbb{D}^n)$. Polynomials on $\mathbb{C}^{1+n}$ can be used to define an algebra $\mathscr{P}(\mathbb{D}^n)$ of polynomials on $\mathbb{D}^n$, and $\mathscr{P}(\mathbb{D}^n)$ is contained in $\mathscr{A}(\mathbb{D}^n)$.

There is a way to perform an equivalence transformation on the Wick star product on $\mathbb{C}^{1+n}$, introduced in Example 2.9, such that it can be reduced to a product $\star$ on the hyperbolic disc $\mathbb{D}^n$.

**Theorem 2.43 (Kraus–Roth–Schötz–Waldmann)** *The product $\star$ on $\mathbb{D}^n$, described in the last paragraph, restricts to a strict product $\star_\hbar$ on the polynomials $\mathscr{P}(\mathbb{D}^n)$, for all $\hbar \in \mathbb{C} \setminus \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$. It is continuous with respect to the topology on $\mathscr{P}(\mathbb{D}^n)$, induced by the $\mathrm{T}_R$-topology on $\mathrm{S}^\bullet((\mathbb{C}^{1+n})^*)$ for $R \geq 0$. For $R = 0$, this topology is precisely the topology of locally uniform convergence of the holomorphic extensions to $\hat{\mathbb{D}}^n$, and the completion of $\mathscr{P}(\mathbb{D}^n)$ with respect to this topology is precisely $\mathscr{A}(\mathbb{D}^n)$.*

The poles in this construction arise since $\star_\hbar$ is a finite power series with rational functions as coefficients, when applied to two polynomials.

# 3 Objectives

The previous section explained the general theory relevant for understanding the quantization problem. In this section, we will focus on the contributions of this thesis. In Subsection 3.1 we describe briefly what is known in the literature about the quantization of coadjoint orbits, Subsection 3.2 discusses the contributions of the author, and in Subsection 3.3 we give an outlook on possible future directions.

## 3.1 Existing results on coadjoint orbits

In this subsection, we review the definition of coadjoint orbits and results on their quantization.

Let $G$ be a Lie group. Then $G$ acts under the adjoint action on its Lie algebra $\mathfrak{g}$, and by dualizing also on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. This action is called the *coadjoint action* and its orbit through an element $\lambda \in \mathfrak{g}^*$ is called a *coadjoint orbit* and denoted by $\mathcal{O}_\lambda$. Note that we can always identify $\mathcal{O}_\lambda$ with the homogeneous space $G/G_\lambda$, where $G_\lambda$ is the stabilizer of $\lambda$. We give a brief introduction to coadjoint orbits in Section II.2.1, which we will not repeat here.

In many respects, coadjoint orbits behave better and have more relevant geometric structure than orbits of the adjoint action: First, a coadjoint orbit $\mathcal{O}_\lambda$ of a Lie group $G$ always admits a canonical $G$-invariant symplectic form, providing the necessary information to do classical mechanics on $\mathcal{O}_\lambda$. In addition, the group action of $G$ describes symmetries of $\mathcal{O}_\lambda$, and yields extra structure that we can use for the construction of a quantization. Since coadjoint orbits are subsets of the vector space

$\mathfrak{g}^*$, we can define polynomials on $\mathcal{O}_\lambda$ by restricting polynomials on $\mathfrak{g}^*$, and those polynomials can be the starting point for carrying out the Fréchet-algebraic approach outlined in Subsection 2.8. In addition, if $G$ is semisimple, then a lot of Lie algebraic and representation theoretic tools become available to study $\mathcal{O}_\lambda$.

Second, the class of coadjoint orbits contains many geometrically interesting examples. Among the coadjoint orbits of semisimple Lie groups are complex projective spaces $\mathbb{CP}^n$, including as the special case $n = 1$ the 2-sphere $\mathbb{S}^2$, and the hyperbolic discs $\mathbb{D}^n$ defined in Definition 2.42. Since Riemann surfaces are quotients of $\mathbb{D}^1$ by a Fuchsian group $\Gamma$, understanding $\Gamma$-equivariant quantizations of $\mathbb{D}^1$ provides a way to study the quantization problem for Riemann surfaces. In the Appendix we describe briefly how $\mathbb{D}^n$ and $\mathbb{CP}^n$ arise as coadjoint orbits. Note that $\mathbb{D}^n$ and $\mathbb{CP}^n$ are rather different spaces, with $\mathbb{CP}^n$ being compact and admitting a metric of positive sectional curvature, while $\mathbb{D}^n$ is non-compact and admits a metric of negative sectional curvature. For more examples of coadjoint orbits, see e.g. [MR99, Chapter 14].

Note that for a semisimple Lie algebra $\mathfrak{g}$ the Killing form $B$ is non-degenerate and therefore provides an isomorphism ${}^\flat\colon \mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto B(X, \cdot\,)$. Denote its inverse by ${}^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$.

**Definition 3.1** *Let $G$ be a complex semisimple Lie group. A coadjoint orbit $\mathcal{O}_\lambda$ of $G$ is called* semisimple *if $\lambda^\sharp$ is semisimple, i.e. diagonalisable under the adjoint action, and* nilpotent *if $\lambda^\sharp$ is nilpotent under the adjoint action. If $G$ is a real semisimple Lie group, then $\mathcal{O}_\lambda$ is* semisimple *or* nilpotent *if $\lambda^\sharp$ is semisimple or nilpotent in the complexification of $\mathfrak{g}$.*

Note that if any element of $\mathcal{O}_\lambda$ is semisimple (or nilpotent), then every element is semisimple (or nilpotent). It can be shown that complex semisimple coadjoint orbits are Zariski closed, and therefore in particular closed submanifolds of $\mathfrak{g}^*$, determined as the vanishing set of a finite family of polynomials. For a fixed semisimple Lie group, there are only finitely many nilpotent orbits. The Zariski closure of a nilpotent orbit is a union of nilpotent orbits, and contains 0.

In this thesis, we will only be interested in the case of semisimple coadjoint orbits of semisimple Lie groups. Note however, that formal deformation quantizations were also obtained for nilpotent coadjoint orbits [ABC94, AB02], where it is not required that the operators $C_r$ in Definition 2.7 are differential, and for coadjoint orbits of non-semisimple Lie groups like $\mathrm{GL}_n(\mathbb{C})$ [DM02].

One attempt to quantize semisimple coadjoint orbits is to start with the Gutt star product on $\mathfrak{g}^*$, introduced in Example 2.40, and to ask whether this product can be restricted to $\mathcal{O}_\lambda$. For this to be possible, the differential operators $C_r$ in Definition 2.7 would need to take derivatives only in directions tangential to the orbits, but not in transversal directions. There is a result of Cahen–Gutt–Rawnsley [CGR96] that there is no star product on any neighbourhood of the origin in $\mathfrak{g}^*$ that is tangential to all coadjoint orbits, including the zero orbit. (The original motivation of the authors when studying this question was to determine whether one might be able to quantize Poisson structures by quantizing all symplectic leaves, and glueing these quantizations together.)

However, it is still possible that a star product restricts to many (but not all) coadjoint orbits. So given an orbit $\mathcal{O}_\lambda$, we may try to perform an equivalence transformation as defined in Definition 2.11 on the Gutt star product, such that the resulting star product is tangential to $\mathcal{O}_\lambda$ and can therefore be restricted. Such an approach was attempted in an algebraic setting in [FL01], where the polynomial relations defining a semisimple orbit are deformed, and carried out in a much clearer geometrical setting by Karabegov [Ast99, Kar96, Kar98].

In the simplest form, Karabegov's construction works as follows. Let $K$ be a semisimple compact connected Lie group with coadjoint orbit $\mathcal{O}_\lambda \subseteq \mathfrak{k}^*$. Such a coadjoint orbit with the canonical symplectic form has a unique compatible $K$-invariant complex structure that makes it into a Kähler manifold. The map

$$\ell_\hbar \colon \mathfrak{k} \to \operatorname{DiffOp}(\mathcal{O}_\lambda), \quad X \mapsto X_{\mathcal{O}_\lambda}^{(1,0)} - \frac{\mathrm{i}}{\hbar} X \tag{3.1}$$

defines a representation of $\mathfrak{k}$ for all $\hbar \in \mathbb{C} \setminus \{0\}$. Here $X_{\mathcal{O}_\lambda}\big|_\xi := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \operatorname{Ad}^*_{\exp(-tX)} \xi \in \Gamma^\infty(\mathrm{T}\mathcal{O}_\lambda)$ is the *fundamental vector field* of $X$ and the superscript $(1,0)$ denotes its projection to the holomorphic tangent space. The $X \in \mathfrak{k} \subseteq \mathrm{S}\mathfrak{k}$ in the last term is interpreted as a polynomial on $\mathcal{O}_\lambda \subseteq \mathfrak{k}^*$.

Denote the complexification of $\mathfrak{k}$ by $\mathfrak{g}$. From the representation $\ell_\hbar$ we obtain a map $\Phi_\hbar \colon \mathscr{U}\mathfrak{g} \to \operatorname{Pol}(\mathcal{O}_\lambda)$ by extending $X_1 \dots X_k \mapsto \ell_\hbar(X_1) \dots \ell_\hbar(X_k)1$ complex linearly. Here 1 denotes the function on $\mathcal{O}_\lambda$ that is constantly 1. The kernel of $\Phi_\hbar$ is a two-sided ideal, and therefore we can push forward the non-commutative product of $\mathscr{U}\mathfrak{g}/\ker\Phi_\hbar$ to a product $*_\hbar$ on $\operatorname{im}\Phi_\hbar$. For all but countably many values of $\hbar$, the image of $\Phi_\hbar$ consists of all polynomials $\operatorname{Pol}(\mathcal{O}_\lambda)$. The dependence of $*_\hbar$ on $\hbar$ is rational with no pole at 0, and one can obtain a formal star product from the Taylor series expansion around $\hbar = 0$.

To see that we may interpret this construction as deforming the Gutt star product, note that we may extend $\Phi_\hbar$ to a map $\mathscr{U}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ by interpreting $X$ on the RHS of (3.1) as an element of $\mathrm{S}\mathfrak{g}$, and then view $\Phi_\hbar \circ q_\hbar$ as the equivalence transformation. Here $q_\hbar$ is the map introduced in Example 2.40. Note also the formal similarity of (3.1) with (2.32), which suggests that there is some relation with Berezin–Toeplitz quantization. This is made precise in [Kar99, Section 11]:

**Theorem 3.2 (Karabegov)** *Let $K$ be a compact connected semisimple Lie group $K$ with Lie algebra $\mathfrak{k}$, and assume that $\lambda \in \mathfrak{k}^*$ is chosen such that its stablizer Lie algebra $\mathfrak{k}_\lambda$ is a Cartan subalgebra of $\mathfrak{k}$ and such that $\lambda$ is a dominant weight. Then there is a quantizing line bundle for $\mathcal{O}_\lambda$, and for every $\hbar \in \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ Karabegov's star product $*_\hbar$ coincides with the quantization of $\mathcal{O}_\lambda$ through Berezin's covariant symbols, and is therefore related to the Berezin–Toeplitz quantization via the Berezin transform.*

The assumption that $\mathfrak{k}_\lambda$ is a Cartan subalgebra can be replaced by a weaker requirement of $\lambda$ being invariant with respect to the Weyl group generated by a certain subset of the root system. But introducing the proper terminology to formulate this precisely would go too far. In any case, the main importance of Theorem 3.2 is that Karabov's quantization at the poles is intimately related to the Berezin–Toeplitz quantization, but unless one chooses more complicated representations in (3.1) does not reproduce the Berezin–Toeplitz quantization exactly. Since the precise quantization at the poles is irrelevant in the following, we refer the reader to [Kar99, Sch10] for a definition of Berezin's covariant symbols and the Berezin transform.

A seemingly different algebraic approach was developed by Alekseev–Lachowska [AL05]. They view a coadjoint orbit as a homogeneous space $\mathcal{O}_\lambda \cong G/G_\lambda$, and use that the space of $G$-invariant bidifferential operators on $G/G_\lambda$ is canonically isomorphic to $((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{g}_\lambda)^{\otimes 2})^{G_\lambda}$. By inverting a certain algebraic pairing between Verma modules, called the *Shapovalov pairing*, they obtain an element in $((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{g}_\lambda)^{\otimes 2})^{G_\lambda}[[\nu]]$, which defines an associative formal product on $\mathcal{O}_\lambda$. Here $\mathfrak{g}_\lambda$ denotes the Lie algebra of $G_\lambda$, the stabilizer of $\lambda$. We will present this construction in detail in Paper II.

The relation of the two approaches was studied in [ESW19]:

**Theorem 3.3 (Esposito–Schmitt–Waldmann)** *Let $G$ be a compact semisimple connected Lie group. The constructions of Alekseev–Lachowska and Karabegov lead to the same star products.*

To be more precise, there are some choices that need to be made in both constructions. It is shown in [ESW19] which choices need to be made to get the same star products.

In Paper II we focus on the algebraic construction of Alekseev–Lachowska since it is better suited for obtaining continuity estimates of the star product. However, it is good to keep the geometric interpretation in terms of Karabegov's construction in mind.

## 3.2 Contributions of the author

In this subsection, we give an account of the results obtained in this thesis. The first part contains the main motivation, whereas the second part contains a more detailed description and some intermediary results obtained along the way. Since the precise statements can be found in the introductions to the research articles, we try to be more qualitative in our description and refer to the articles whenever necessary.

### Motivation and main results

In a nutshell, the starting point of this thesis is Theorem 2.43. As mentioned in Subsection 2.8, the completion of the polynomials $\mathscr{P}(\mathbb{D}^n)$ on the hyperbolic disc with respect to the $\mathrm{T}_0$-topology can be described geometrically as the algebra of functions obtained by restricting holomorphic functions on an extended disc $\hat{\mathbb{D}}^n$. However, the construction of the extended hyperbolic disc $\hat{\mathbb{D}}^n$ in [KRSW19] is somewhat ad-hoc, and it was not clear whether such a construction could also be applied in other situations.

In Paper I we show that this construction is indeed not limited to hyperbolic discs, but works similarly for other manifolds $M_{\mathrm{red}}^{(s)}$ that can be obtained by a similar reduction procedure from $\mathbb{C}^{1+n}$, using a metric of signature $s \in \{1, \dots, n+1\}$. These manifolds include the complex projective spaces and hyperbolic discs. The construction of star products on $M_{\mathrm{red}}^{(s)}$ is analogous to the construction for the hyperbolic disc, see Main Theorem I.I and Main Theorem I.II. The novelty in our approach is that it allows us to compare the quantizations obtained for different signatures $s$, see Main Theorem I.IV. Indeed, every manifold $M_{\mathrm{red}}^{(s)}$ embeds "antidiagonally" into a complex manifold $\hat{M}_{\mathrm{red}}^{(s)}$, just as the hyperbolic disc $\mathbb{D}^n$ embeds into $\hat{\mathbb{D}}^n$, and the completion of the polynomials on $M_{\mathrm{red}}^{(s)}$ with respect to a topology induced from the $\mathrm{T}_0$-topology on

$\mathrm{S}^\bullet((\mathbb{C}^{1+n})^*)$ is given precisely by the restrictions of holomorphic functions on $\hat{M}^{(s)}_{\mathrm{red}}$. There is a holomorphic diffeomorphism between the extended spaces $\hat{M}^{(s)}_{\mathrm{red}}$, which is an analogue of the Wick rotation and essentially changes the signature of the metric. Extending from $M^{(s)}_{\mathrm{red}}$ to $\hat{M}^{(s)}_{\mathrm{red}}$, applying the Wick rotation, and restricting to a different $M^{(s')}_{\mathrm{red}}$ is compatible with the product structure and therefore gives an isomorphism of the quantum algebras for different manifolds $M^{(s)}_{\mathrm{red}}$, which is not compatible with the $^*$-structures.

This generalizes the construction for $\mathbb{D}^n$ to a larger class of examples, but it still remains somewhat unclear where the extended spaces actually come from. This problem is addressed in Paper II where we study semisimple coadjoint orbits of semisimple connected Lie groups. These coadjoint orbits admit a unique complexification. In fact, all the examples studied in Paper I are semisimple coadjoint orbits of semisimple connected Lie groups, and the extended spaces are precisely their complexifications. We show that the Alekseev–Lachowska star product on coadjoint orbits, which we introduced briefly in the last subsection, restricts to polynomials on the coadjoint orbit and can be extended by continuity to all holomorphic functions on the complexification, see Main Theorem II.II. This extension is quite non-trivial, and requires an explicit computation of the twist defining the star product and the application of some non-trivial theorems from complex analysis concerning the extension of holomorphic functions. We can restrict the quantization to real orbits, see Main Theorem II.III, and we also get an isomorphism generalizing the Wick rotation in this approach, see Main Theorem II.IV.

The main novelty in our construction is the *systematic use of complexifications.* Indeed, the construction of Alekseev–Lachowska yields both a product for polynomials on the real coadjoint orbit and for holomorphic polynomials on the complexification, and those products are intertwined by restriction. However, by working on the complexification many powerful tools from complex analysis become available. In particular, we have a good description of the decay of the coefficients in the Taylor series of any holomorphic function. These estimates combined with estimates for the coefficients of the twist allow us to prove the required continuity of the product on the complexification, with respect to the topology of locally uniform convergence. We need to invoke powerful complex analytic theorems again when proving that the completion of the holomorphic polynomials with respect to this topology really consists of all holomorphic functions on the complexification. The Wick rotation, which relates quantizations of different coadjoint orbits with the same complexification, is a natural consequence of this approach.

In the Appendix we show that the star products on the hyperbolic disc and the complex projective spaces, constructed in Paper I by phase space reduction and in Paper II by inverting the Shapovalov pairing, agree. In this sense many results obtained in Paper II are a generalization of the results obtained in Paper I. But note that the construction in Paper I is rather different and in some sense more geometric and explicit, and can therefore provide additional insights.

**More detailed description**

Let us start by giving a definition of the type of strict quantizations that we construct. Essentially, our constructions in Paper I and Paper II share most of the properties of the Fréchet-algebraic quantization of the hyperbolic disc, obtained in [KRSW19]. The following definition tries to capture these properties as closely as possible. Variations of this definition would certainly make sense, and might even be more natural. For example, there is no real need to require that the topology $\mathcal{T}$ does not depend on $\hbar$, but we decided to give the strongest sensible definition that is satisfied by our examples.

**Definition 3.4 (Fréchet-algebraic quantization)** *Let $(M, \pi)$ be a Poisson manifold. A* strict Fréchet-algebraic quantization *on $M$ is specified by the following data:*

*i.) a subalgebra $\mathscr{P}$ of $\mathscr{C}^\infty(M)$ which is closed under taking Poisson brackets,*

*ii.) a locally convex topology $\mathcal{T}$ on $\mathscr{P}$,*

*iii.) a countable subset $P \subseteq \mathbb{C} \setminus \{0\}$ accumulating only at zero,*

*iv.) for every $\hbar \in \mathbb{C} \setminus P$ an associative product $*_\hbar$ on the underlying vector space of $\mathscr{P}$*

*such that*

*a.) the product $*_0$ coincides with the commutative pointwise product of $\mathscr{C}^\infty(M)$,*

*b.) for every $\hbar \in \mathbb{C} \setminus P$, the product $*_\hbar$ is continuous with respect to the topology $\mathcal{T}$, and the unique extension of $*_\hbar$ to the completion $\mathscr{A}$ of $\mathscr{P}$ makes $(\mathscr{A}, *_\hbar)$ a Fréchet algebra,*

*c.) for $f, g \in \mathscr{P}$ we have*

$$\frac{1}{\hbar}(f *_\hbar g - g *_\hbar f) - \mathrm{i}\{f, g\} \xrightarrow{\hbar \to 0} 0 \tag{3.2}$$

*in the topology $\mathcal{T}$,*

*d.) for $f, g \in \mathscr{A}$ and every $x \in M$, the function $\hbar \mapsto (f *_\hbar g)(x)$ is holomorphic on $\mathbb{C} \setminus (P \cup \{0\})$.*

All our examples are constructed from a formal star product, which becomes a finite series with rational functions as coefficients when restricted to two elements of $\mathscr{P}$. We can therefore only specify $\hbar$ to values that do not coincide with the poles $P$ of these rational functions. The function $\hbar \mapsto (f \star_\hbar g)(x)$ with $f, g \in \mathscr{A}$ and $x \in M$ is usually not continuous at $\hbar = 0$ in this approach, as it can blow up around the poles $P$, which accumulate at 0, see [KRSW19, Example 4.2].

In all our examples, a Lie group $G$ acts on $M$ by Poisson maps, i.e. in a way that is compatible with the Poisson structure. The subspaces $\mathscr{P}$ and $\mathscr{A}$ are both $G$-invariant, and all products $*_\hbar$ with $\hbar \in \mathbb{C} \setminus P$ are $G$-equivariant. In many examples, there will be an involution $^*$ on $\mathscr{P}$ and $\mathscr{A}$, and for every $\hbar \in \mathbb{C} \setminus P$ we have $(f *_\hbar g)^* = g^* *_{\overline{\hbar}} f^*$ for all $f, g \in \mathscr{A}$.

Let us describe the intermediary results obtained in the research articles in more detail. First, we obtain *explicit formulas* for the star products in both articles. In Paper I, those formulas were already known for complex projective spaces [BBEW96a, BBEW96b] and the hyperbolic disc [BW14], but not for the manifolds $M_{\mathrm{red}}^{(s)}$ of signature $s \neq 1, n+1$. A formula describing the bidifferential operators of a Fedosov star product on the reduced manifold was obtained in [Löf10]. We computed the bidifferential operators directly via phase space reduction, and obtained the same formula, see Main Theorem I.I. This computation shows that the product from [Löf10] coincides indeed with the product obtained from phase space reduction, which was already suggested by the fact that their characteristic classes (see Theorem 2.16) agree.

For coadjoint orbits, the formula for the canonical element of the Shapovalov pairing of $\mathrm{SL}_2(\mathbb{C})$ appeared in [AL05], but to the author's best knowledge the formulas for $\mathrm{SL}_n(\mathbb{C})$ and an arbitrary semisimple Lie algebra, see Main Theorem II.I, are new. These formulas are of independent interest, but also allow one to give a fairly explicit description of the bidifferential operators defining the star products.

Especially in Paper II we tried to present some folklore results in an accessible way. Our appendix contains a detailed proof that $G$-invariant $k$-differential holomorphic operators on a complex homogeneous space $G/H$ are in bijection to certain invariant elements in a quotient of the universal enveloping algebra. We try to review results on the relation between polynomials on coadjoint orbits and holomorphic polynomials on the complexification, as well as on the relation between polynomials on the coadjoint orbit and invariant polynomials on the Lie group. The author is convinced that these results are not new, but is not aware of a good reference.

As we explained in Subsection 2.2, quantizing the observable algebra is not sufficient to define a quantum system. We do also need to represent this algebra on a Hilbert space. In most examples studied in Paper I and Paper II the complex conjugation is a star involution on the Fréchet algebras $(\mathscr{A}, *_\hbar)$. Given a positive linear functional, the GNS construction can then be used to obtain a representation. For the hyperbolic disc and more generally coadjoint orbits for which the root system satisfies some technical conditions, see Theorem II.5.28, we prove the existence of positive linear functionals, or more generally, we prove that all the point evaluations are positive linear functionals on the quantum algebras.

We prove in Proposition I.6.10 that the quantum algebra on the 2-sphere has no positive non-trivial linear functionals when $\hbar < -1$, because $-1$ can be written as a sum of squares. The same result holds for all $\hbar \in \mathbb{R}^+$ that are not poles and for all complex projective spaces (which is not proven in this thesis), implying that their Fréchet $^*$-algebras cannot be represented faithfully on a Hilbert space, and emphasizing that the $^*$-involution is an important piece of information of a strict quantization. Since there are non-trivial linear functionals for $\hbar < -1$ on the hyperbolic disc[1], this implies in particular that the Wick rotation cannot be an isomorphism of Fréchet $^*$-algebras, and more generally, that there cannot exist another $^*$-isomorphism between the Fréchet $^*$-algebras for $\mathbb{CP}^n$ and $\mathbb{D}^n$ either, see Proposition I.6.10.

[1] In Paper I we use conventions for which the Wick rotation is an isomorphism between the quantum algebras of different $M_{\mathrm{red}}^{(s)}$ for a fixed value of $\hbar$. As a consequence we consider the hyperbolic disc with a negative definite metric. Changing the sign of the metric, we would also need to change the sign of the symplectic form and of $\hbar$. In this sense $\hbar < -1$ in the conventions of Paper I corresponds to the case $\hbar > 1$ on the hyperbolic disc with a positive definite metric.

## 3.3 Future directions

In this thesis we focus on the behaviour away from the poles of the Alekseev–Lachowska star product and the star product obtained via phase space reduction. However, the explicit formulas that we obtain for those star products show that they stay well-defined at the poles for a restricted set of functions (depending on the pole), usually the polynomials up to a certain degree. With respect to the star product, these functions form an associative algebra. We have already observed this phenomenon for coadjoint orbits of compact Lie groups in Theorem 3.2. (Recall that Karabegov's star product coincides with the star product of Alekseev–Lachowska according to Theorem 3.3.)

It would certainly be interesting to gain a better understanding of how the Fréchet algebras "degenerate" into finite dimensional algebras at the poles. Note that the poles are intimately related to the representation theory of $G$. For example, the finite dimensional representations of a compact semisimple Lie group $K$ can be realized by the Borel–Weil–Bott theorem [Bot57] on the space of global holomorphic sections of a line bundle over $\mathcal{O}_\lambda$, and this line bundle can serve as a quantizing line bundle. In this sense, it would be interesting to see whether there is any good geometric interpretation of the Fréchet algebras interpolating between these finite dimensional representations.

There are two natural follow-up questions to the results obtained in Paper II. As mentioned in Subsection 2.2 it is crucial for physical applications to represent the constructed Fréchet algebras on a Hilbert space, which can be achieved through the GNS construction if there are enough positive linear functionals. As explained in the previous subsection, we know whether positive linear functionals exist for the quantum algebras of $\mathbb{CP}^n$ and $\mathbb{D}^n$, but for other coadjoint orbits this is a largely open question. For complex projective spaces, the question arises whether one might be able to modify the $^*$-involution on the quantum algebras so that there are positive linear functionals.

Closely related is the question whether one can associate $C^*$-algebraic deformation quantizations when one has found a representation on a Hilbert space. In certain examples, like the star product of exponential type from Theorem 2.39, it is possible to naively restrict to a $C^*$-algebra of bounded functions, as described in [Sch18]. It is unclear how this $C^*$-algebra relates to the other constructions of $C^*$-algebraic quantizations from Section 2, and whether more sophisticated constructions might yield better behaved $C^*$-algebras. For the hyperbolic disc, just looking at bounded elements does not seem to be enough. Rather one has to apply some sort of functional calculus to obtain a larger $C^*$-algebra. Since good $C^*$-algebraic quantizations of $\mathbb{D}^n$ are known, it would be interesting to investigate whether they can be obtained from our Fréchet-algebraic quantizations, with the aim of constructing $C^*$-algebras in similar ways for other coadjoint orbits.

It would certainly be interesting to explore in which other situations complexifications can provide new insights into constructions of strict quantizations. One class of examples could be semisimple Lie groups, since many of the tools we used in Paper II are available in this case, too. It could be helpful to have a more conceptual understanding of when a star products extends to all holomorphic functions, that does not rely on explicit computations as much as our methods in Paper II do.

In this context, it would be interesting to study Drinfel'd twists. Drinfel'd twists are elements of $(\mathscr{U}\mathfrak{g}\otimes\mathscr{U}\mathfrak{g})[[\nu]]$ satisfying a certain equation that is equivalent to saying that the $G$-invariant formal star product on $\mathscr{C}^\infty(G)[[\nu]]$, obtained by associating left-invariant differential operators on $G$ to elements of $\mathscr{U}\mathfrak{g}$, is associative. Such twists define associative products on $\mathscr{C}^\infty(M)[[\nu]]$ whenever $M$ is a manifold with an action of $\mathfrak{g}$. The construction of Drinfel'd twists from [Dri83] is in terms of the Baker–Campbell–Hausdorff series, and understanding the growth of the coefficients of that series was one of the main tools used to prove Theorem 2.41. Therefore it does not seem unreasonable that it is possible to obtain some topology in which twists obtained with this construction converge. If this was the case, it would be very interesting to investigate whether such twists can induce strict products on manifolds with an action of $\mathfrak{g}$.

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# Paper I:

# Wick rotations in deformation quantization

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**Abstract**

We study formal and non-formal deformation quantizations of a family of manifolds that can be obtained by phase space reduction from $\mathbb{C}^{1+n}$ with the Wick star product in arbitrary signature. Two special cases of such manifolds are the complex projective space $\mathbb{CP}^n$ and the complex hyperbolic disc $\mathbb{D}^n$. We generalize several older results to this setting: The construction of formal star products and their explicit description by bidifferential operators, the existence of a convergent subalgebra of "polynomial" functions, and its completion to an algebra of certain analytic functions that allow an easy characterization via their holomorphic extensions. Moreover, we find an isomorphism between the non-formal deformation quantizations for different signatures, linking e.g. the star products on $\mathbb{CP}^n$ and $\mathbb{D}^n$. More precisely, we describe an isomorphism between the (polynomial or analytic) function algebras that is compatible with Poisson brackets and the convergent star products. This isomorphism is essentially given by Wick rotation, i.e. holomorphic extension of analytic functions and restriction to a new domain. It is not compatible with the $^*$-involution of pointwise complex conjugation.

[1]Supported by the Danish National Research Foundation through the Centre of Symmetry and Deformation (DNRF92)

[2]Boursier de l'ULB and supported by the Fonds de la Recherche Scientifique (FNRS) and the Fonds Wetenschappelijk Onderzoek - Vlaaderen (FWO) under EOS Project n°30950721.

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# 1 Introduction

One way to study the quantization problem arising in physics, which asks how to associate a quantum mechanical system to a classical mechanical one, is *formal deformation quantization* as introduced in [2]. In this approach, the classical observable algebra is assumed to be the algebra $\mathscr{C}^\infty(M)$ of smooth functions on a Poisson manifold $M$ and one tries to find a so-called *formal star product* $\star$ that deforms the classical product. More precisely, $\star\colon \mathscr{C}^\infty(M)[[\nu]] \times \mathscr{C}^\infty(M)[[\nu]] \to \mathscr{C}^\infty(M)[[\nu]]$ is called a formal star product if it is $\mathbb{C}[[\nu]]$-bilinear, associative, has the constant 1-function as a unit, and if it can be expanded as $f \star g = \sum_{r=0}^\infty \nu^r C_r(f,g)$ with $\mathbb{C}[[\nu]]$-linear extensions of bidifferential operators $C_r\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ that satisfy that $C_0(f,g) = fg$ is the usual commutative pointwise product and that $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$ is (up to the factor i) the Poisson bracket of $f, g \in \mathscr{C}^\infty(M)$. Here $[[\nu]]$ denotes formal power series in the parameter $\nu$. We say that $\star$ deforms *in direction of* the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$. Such a star product is called *Hermitian* if pointwise complex conjugation is a $^*$-involution, i.e. if $\overline{f \star g} = \overline{g} \star \overline{f}$ holds for all

$f, g \in \mathscr{C}^\infty(M)$. In a sense, formal deformation quantization transfers the quantization problem to algebra and therefore allows to use powerful algebraic tools in its study. For example, existence and classification results follow from Kontsevich's formality theorem in the most general case of Poisson manifolds [16], but were already proven before in the special case of symplectic manifolds by various authors [5,10,14,20] and with the help of different techniques, e.g. the so-called Fedosov construction.

Formal deformation quantizations can also be studied in an equivariant setting. Assume $G$ is a Lie group acting on $M$. Then a star product is called *$G$-invariant* if all the bidifferential operators $C_r$ are $G$-invariant. For Hamiltonian $G$-actions there is a related notion of $G$-equivariance that considers the quantization of a momentum map as well. Existence and classification results are also available in this setting [4,11,21]. Some explicit examples of star products can easily be obtained on $\mathbb{C}^{1+n}$, namely the exponential star products like Weyl–Moyal or Wick star products. There are also explicit methods to obtain star products on more general spaces, like $\mathbb{CP}^n$ or $\mathbb{D}^n$. [3,7,8,17] use a construction via phase space reduction from one of the aforementioned products on $\mathbb{C}^{1+n}$. Alternatively, one can e.g. use Berezin dequantization [9], a Lie algebraic approach [1] or an explicit solution of the recursive equations coming from the Fedosov construction [18].

The drawback of considering formal power series is that one cannot easily replace the formal parameter $\nu$ by Planck's constant $\hbar$, as required in actual physical applications. Therefore *strict quantization* asks to find a field of well-behaved algebras, usually Fréchet $^*$-algebras or C$^*$-algebras, see [6,19,22], that depend nicely on a parameter $\hbar$ ranging over some subset of $\mathbb{C}$, and that reproduce the usual product and Poisson bracket in the zeroth and first order as above for $\hbar \to 0$. Usually, strict quantizations as in [6,22] are constructed by analytical methods, involving oscillatory integrals. If a strict quantization depends smoothly on the parameter $\hbar$, its asymptotic expansion around $\hbar = 0$ yields a formal deformation quantization. Conversely, one can ask to construct strict quantizations that have a given formal deformation quantization as their limit.

Some results in this direction were obtained by Waldmann and collaborators, who try to find some distinguished subalgebra $\mathscr{P}(M)$ of $\mathscr{C}^\infty(M)$, on which a star product converges trivially because the formal power series are finite. Such a choice usually comes from some extra structure, for example if $M = \mathrm{T}^*Q$ is a cotangent bundle then one can try to use functions that are polynomial in the momenta. One then tries to find some topology with respect to which a star product on $\mathscr{P}(M)$ is continuous, in order to complete $\mathscr{P}(M)$ to a more interesting algebra $\mathscr{A}(M)$, typically consisting of analytic functions. This approach has been worked out e.g. for star products of exponential type on possibly infinite-dimensional vector spaces [24, 26], for the Gutt star product on the dual of a Lie algebra [13], for the 2-sphere [12], for the hyperbolic disc $\mathbb{D}^n$ [3,17], and for semisimple coadjoint orbits of semisimple connected Lie groups [23]. In the case of the hyperbolic disc the completed algebra $\mathscr{A}$ has a nice geometric interpretation as functions that allow an extension to holomorphic functions on some fixed, larger space.

In this article we generalize the approach used in [17] for the hyperbolic disc to obtain formal and non-formal star products on a larger class of certain (pseudo-)Kähler manifolds. These manifolds depend on two parameters, dimension $n$ and signature $s$, and are obtained by using Marsden–Weinstein reduction for the canonical $\mathrm{U}(1)$-action on $\mathbb{C}^{1+n}$ endowed with a metric of signature $s$. Focussing on treating all these examples in a uniform way, we construct $\mathrm{U}(s, 1+n-s)$-invariant, Hermitian formal star products. Using ideas relating to Kähler reduction, we derive an explicit formula in Theorem 5.12:

**Main Theorem I** *For any of the reduced (pseudo-)Kähler manifolds $M_{\mathrm{red}}$ described above, the formula*

$$f \star_{\mathrm{red}} g = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\nu^r}{(1-\nu)(1-2\nu)\dots(1-(r-1)\nu)} \cdot \big\langle (D_{\mathrm{red}}^{\mathrm{sym}})^r f \otimes (D_{\mathrm{red}}^{\mathrm{sym}})^r g, H_{\mathrm{red}}^r \big\rangle \quad (1.1)$$

*defines a formal star product. Here $f, g \in \mathscr{C}^\infty(M_{\mathrm{red}})$, $D_{\mathrm{red}}^{\mathrm{sym}}$ is the symmetrized covariant derivative associated to the Levi-Civita connection of $M_{\mathrm{red}}$, and $H_{\mathrm{red}}$ is a certain bivector field on $M_{\mathrm{red}}$.*

This formula was already known in the special case of $\mathbb{CP}^n$ and $\mathbb{D}^n$, [18], where it was derived from the Fedosov construction. Our result therefore allows to compare this approach with phase space reduction without appealing to any abstract classification results, and generalizes it to a larger class of manifolds.

It will become clear from the construction that, at least outside of the poles appearing in (1.1), the star product $\star_{\mathrm{red}}$ converges trivially for a class of functions $\mathscr{P}(M_{\mathrm{red}})$ that is obtained by reducing polynomials on $\mathbb{C}^{1+n}$. All these functions can be (uniquely) extended to holomorphic functions on a larger complex manifold $\hat{M}_{\mathrm{red}}$ that can be obtained by an analogous reduction procedure from $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. We define the algebra $\mathscr{A}(M_{\mathrm{red}})$ of all functions that can be extended to holomorphic functions on $\hat{M}_{\mathrm{red}}$, thus obtaining an algebra of certain analytic functions. Using methods from complex analytic geometry, we prove that $\mathscr{P}(M_{\mathrm{red}})$ is dense in $\mathscr{A}(M_{\mathrm{red}})$ with respect to the topology of locally uniform convergence of the extensions to $\hat{M}_{\mathrm{red}}$. Then we obtain for all complex $\hbar$ outside of the poles of (1.1) our Theorem 5.26:

**Main Theorem II** *The strict product $\star_{\mathrm{red},\hbar}$ on $\mathscr{P}(M_{\mathrm{red}})$ obtained by replacing the formal parameter $\nu$ with $\hbar$ in (1.1), is continuous with respect to the topology of locally uniform convergence of the holomorphic extensions to $\hat{M}_{\mathrm{red}}$. It therefore extends uniquely to a continuous product on $\mathscr{A}(M_{\mathrm{red}})$.*

The geometries of the manifolds $M_{\mathrm{red}}$ can be quite different (e.g. sometimes compact, sometimes not). However, both the classical and quantum algebras of analytic functions cannot see this difference as we show in Theorem 6.4 and Theorem 6.7 using essentially a generalization of the Wick rotation:

**Main Theorem III** *The algebras $\mathscr{A}(M_{\mathrm{red}})$ (for the same dimension $n$ but different signatures $s$) with the pointwise product are all isomorphic as unital Fréchet algebras.*

**Main Theorem IV** *The algebras $\mathscr{A}(M_{\mathrm{red}})$ (for the same dimension $n$ but different signatures $s$) with the product $\star_{\mathrm{red},\hbar}$ and fixed $\hbar$ are all isomorphic as unital Fréchet algebras.*

Note that these last two results can also be proven in a more Lie algebraic context for coadjoint orbits [23]. However, the algebras $\mathscr{A}(M_{\mathrm{red}})$ are in general not $^*$-isomorphic (for real $\hbar$ and the $^*$-involution of pointwise complex conjugation), which demonstrates the importance of considering $^*$-algebras in strict deformation quantization. This can be shown by examining positive linear functionals on these $^*$-algebras, which encode information about their $^*$-representations on pre-Hilbert spaces.

The article is structured as follows: After introducing some notation in Section 2, we discuss the smooth and complex manifolds occurring at various stages of the construction in Section 3. The classical and quantum phase space reduction allow to construct Poisson brackets and formal star products on a reduced manifold $M_{\mathrm{red}}$ out of a constant Poisson bracket and the Wick star product on $\mathbb{C}^{1+n}$. This is achieved essentially by first restricting to the level set $Z$ of a momentum map $\mathscr{J} \in \mathscr{C}^\infty(\mathbb{C}^{1+n})$ and then dividing out the action of the group $\mathrm{U}(1)$ to obtain $M_{\mathrm{red}} \cong Z/\mathrm{U}(1)$. Depending on the choice of signature, $M_{\mathrm{red}}$ can e.g. be $\mathbb{CP}^n$ or $\mathbb{D}^n$. In order to be able to construct the spaces of analytic functions on which the non-formal star products can be defined, we introduce complex manifolds $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$, $\hat{Z}$, and $\hat{M}_{\mathrm{red}}$ into which $\mathbb{C}^{1+n}$, $Z$, and $M_{\mathrm{red}}$ can be embedded "anti-diagonally". The complex structure on $\mathbb{C}^{1+n}$ finally gives rise to a complex structure on $M_{\mathrm{red}}$, which in the special cases of $\mathbb{CP}^n$ and $\mathbb{D}^n$ coincides with the usual one. This also allows to obtain $M_{\mathrm{red}}$ by restricting first to an open subset $\mathbb{C}^{1+n}_+$ of $\mathbb{C}^{1+n}$ and then dividing out an action of the complexification $\mathbb{C}^* = \{z \in \mathbb{C} \,|\, z \neq 0\}$ of $\mathrm{U}(1)$, which simplifies some later considerations.

Section 4 deals with the algebras $\mathscr{C}^\infty(\dots)$, $\mathscr{A}(\dots)$ and $\mathscr{P}(\dots)$ of smooth, certain analytic, and polynomial functions on $\mathbb{C}^{1+n}$, $Z$ and $M_{\mathrm{red}}$. It is also discussed under which conditions and how additional structures given by bidifferential operators on $\mathbb{C}^{1+n}$ can be reduced to $M_{\mathrm{red}}$. This is then applied in Section 5 to the Poisson bracket and Wick star product on $\mathbb{C}^{1+n}$. We obtain the usual Fubini–Study structures as well as explicit formulae for the reduced star products both by means of bidifferential operators and by structure constants.

As the constructions for $\mathbb{CP}^n$, $\mathbb{D}^n$, and the other examples only differ by the choice of certain signs, it is not surprising that they yield closely related results: In Section 6 we construct isomorphisms between various function spaces on the reduced manifolds, which are compatible with both the Poisson brackets and the convergent star products, i.e. with the classical and quantum structures.

Finally, in Appendix A we discuss some details concerning the symmetrized covariant derivatives used for the explicit description of bidifferential operators in Section 5.

**Acknowledgements**

The authors would like to thank Simone Gutt, Ryszard Nest and Stefan Waldmann for some helpful discussions.

## 2 Notation and conventions

There are some conventions that will be used throughout the whole article: We fix two natural numbers $n \in \mathbb{N}$, $s \in \{1, \dots, 1+n\}$. These will be the complex dimension $n$ of the reduced manifold $M_{\mathrm{red}}$ and the choice of signature $s$. Nearly all objects will depend on this signature, but in order to keep the notation clean this dependence will usually not be made explicit. Only when it is necessary (especially when discussing the Wick rotation in Section 6) the choice of $s$ will be indicated by a superscript in brackets.

For a smooth manifold $M$, we denote by $\mathscr{C}^\infty(M)$ the unital $^*$-algebra of complex-valued smooth functions on $M$ with the pointwise operations. $\mathrm{T}M$ and $\mathrm{T}^*M$ are the real tangent and cotangent bundles of $M$, and $\mathrm{T}^{\mathbb{C}}M$ and $\mathrm{T}^{*,\mathbb{C}}M$ their complexifications. If $M$ is even a complex manifold with complex structure $I$, then $\mathrm{T}^{(1,0)}M$ and $\mathrm{T}^{(0,1)}M$ denote the linear subbundles of $+\mathrm{i}$ and $-\mathrm{i}$ eigenvectors of $I$, respectively, and $\mathrm{T}^{*,(1,0)}M$, $\mathrm{T}^{*,(0,1)}M$ their duals. The space of smooth sections of a complex vector bundle $E \to M$ over a smooth manifold $M$ is denoted by $\Gamma^\infty(E)$ and is a $\mathscr{C}^\infty(M)$-module. Tensor products between such spaces of sections are always tensor products over the ring $\mathscr{C}^\infty(M)$. If $M$ is endowed with an action of a group $G$, then $\mathscr{C}^\infty(M)^G \subseteq \mathscr{C}^\infty(M)$ denotes the $G$-invariant smooth functions on $M$. This notation is also applied to subspaces of $\mathscr{C}^\infty(M)$. A $k$-multilinear map $\Phi\colon \mathscr{C}^\infty(M) \times \dots \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ is called $G$-invariant if $\Phi(f_1 \triangleleft g, \dots, f_k \triangleleft g) \triangleleft g^{-1} = \Phi(f_1, \dots, f_k)$ holds for all $f_1, \dots, f_k \in \mathscr{C}^\infty(M)$ and all $g \in G$.

The tensor algebra over a vector space $V$ is denoted by $\mathrm{T}^\bullet V := \bigoplus_{k=0}^\infty \mathrm{T}^k V$ with $\mathrm{T}^k V$ the linear subspace of homogeneous tensors of degree $k \in \mathbb{N}_0$. The symmetric and antisymmetric tensor algebra are identified with the linear subspaces $\mathrm{S}^\bullet V$ and $\Lambda^\bullet V$ of $\mathrm{T}^\bullet V$ consisting of symmetric and antisymmetric tensors, respectively, with symmetric and antisymmetric tensor product $X \vee Y = \mathrm{Sym}^\bullet(X \otimes Y)$ for all $X, Y \in \mathrm{S}^\bullet V$ and $X \wedge Y = \mathrm{Asym}^\bullet(X \otimes Y)$ for all $X, Y \in \Lambda^\bullet V$. Here $\mathrm{Sym}^\bullet, \mathrm{Asym}^\bullet \colon \mathrm{T}^\bullet V \to \mathrm{T}^\bullet V$, the operators of symmetrization and antisymmetrization, are defined as the homogeneous projections onto $\mathrm{S}^\bullet V$ and $\Lambda^\bullet V$ fulfilling

$$\mathrm{Sym}^k\big(v_1 \otimes \dots \otimes v_k\big) = \frac{1}{k!} \sum_\sigma v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)} \tag{2.1}$$

and

$$\mathrm{Asym}^k\big(v_1 \otimes \dots \otimes v_k\big) = \frac{1}{k!} \sum_\sigma \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)} \tag{2.2}$$

for $k \in \mathbb{N}_0$ and $v_1, \dots, v_k \in V$, where the sum is over all permutations $\sigma$ of $\{1, \dots, k\}$. So especially $v \vee w = \frac{1}{2}(v \otimes w + w \otimes v)$ and $v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v)$ for all $v, w \in V$. Vector bundles and their sections are treated analogously.

By $\langle \cdot\,, \cdot \rangle \colon V^* \times V \to \mathbb{C}$ we denote the dual pairing between a complex vector space $V$ and its algebraic dual $V^*$, $\langle \omega, \alpha \rangle := \omega(\alpha)$ for all $\omega \in V^*$, $\alpha \in V$. This pairing is extended to higher tensor powers by demanding that

$$\big\langle \omega_1 \otimes \dots \otimes \omega_k, \alpha_1 \otimes \dots \otimes \alpha_k \big\rangle = \langle \omega_1, \alpha_1 \rangle \dots \langle \omega_k, \alpha_k \rangle \tag{2.3}$$

for all $k \in \mathbb{N}_0$ and $\omega_1, \dots, \omega_k \in V^*$, $\alpha_1, \dots, \alpha_k \in V$. Especially for symmetric tensor products this yields

$$\big\langle \omega_1 \vee \dots \vee \omega_k, \alpha_1 \vee \dots \vee \alpha_k \big\rangle = \frac{1}{k!} \sum_\sigma \langle \omega_1, \alpha_{\sigma(1)} \rangle \dots \langle \omega_k, \alpha_{\sigma(k)} \rangle \tag{2.4}$$

where again the sum is over all permutations $\sigma$ of $\{1, \dots, k\}$. If $\iota_\beta$ denotes the insertion derivation with a vector $\beta \in V$, i.e. the derivation of degree $-1$ of the symmetric tensor algebra over $V^*$ that fulfils $\iota_\beta \omega = \langle \omega, \beta \rangle$ for all $\omega \in V^*$, then by the above conventions,

$$\frac{1}{k} \big\langle \iota_\beta(\omega_1 \vee \dots \vee \omega_k), \alpha_1 \vee \dots \vee \alpha_{k-1} \big\rangle = \big\langle \omega_1 \vee \dots \vee \omega_k, \beta \vee \alpha_1 \vee \dots \vee \alpha_{k-1} \big\rangle \tag{2.5}$$

holds for all $k \in \mathbb{N}$, $\omega_1, \dots, \omega_k \in V^*$ and $\alpha_1, \dots, \alpha_{k-1} \in V$. Like before, vector bundles and their sections are treated analogously.

We will also make use of multiindices $P, Q \in \mathbb{N}_0^{1+n}$ and define as usual $P! \coloneqq \prod_{k=0}^n P_k!$ and

$$\binom{P}{Q} \coloneqq \frac{P!}{(P-Q)!Q!} \tag{2.6}$$

for $Q \leq P$ (the order is the elementwise one). Moreover, the elementwise minimum is

$$\min\{P, Q\} \coloneqq \big(\min\{P_0, Q_0\}, \dots, \min\{P_n, Q_n\}\big)\,. \tag{2.7}$$

# 3 Geometric background

In this section we will in detail explain the following commutative diagram, that describes the reduction procedures to obtain $M_{\mathrm{red}}$ and $\hat{M}_{\mathrm{red}}$:

$$\begin{tikzcd}
\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} & \hat{Z} \arrow[l, "\hat{\iota}"'] \arrow[r, "\hat{\mathrm{pr}}"] & \hat{M}_{\mathrm{red}} \\
\mathbb{C}^{1+n} \arrow[u, "\Delta"] & Z \arrow[l, "\iota"'] \arrow[r, "\mathrm{pr}"] \arrow[u, "\Delta_Z"] \arrow[d] & M_{\mathrm{red}} \arrow[u, "\Delta_{\mathrm{red}}"] \\
& \mathbb{C}^{1+n}_+ \arrow[ul] \arrow[ur, "\mathrm{Pr}"'] &
\end{tikzcd} \tag{3.1}$$

Note the similarity to the diagram considered in [17].

## Middle row

The middle row is a typical example of Marsden–Weinstein reduction, even though we will not yet discuss symplectic structures in this section. It consists of (at least) smooth manifolds endowed with an action of the real Lie group $\mathrm{G}_{\mathcal{J}}$, which is defined below, and of $\mathrm{G}_{\mathcal{J}}$-equivariant smooth maps.

On $\mathbb{C}^{1+n}$, let $z^0, \dots, z^n$ be the standard coordinates, i.e. $z^k(\rho) = \rho^k$ for all $k \in \{0, \dots, n\}$ and $\rho \in \mathbb{C}^{1+n}$. We define

$$\mathcal{J} \coloneqq \sum_{k=0}^{n} \sigma_k z^k \overline{z}^k = \sum_{k=0}^{s-1} z^k \overline{z}^k - \sum_{k=s}^{n} z^k \overline{z}^k \,, \tag{3.2}$$

where the coefficients $\sigma_k$ are $+1$ if $k \in \{0, \dots, s-1\}$ and $-1$ if $k \in \{s, \dots, n\}$. Note that we drop the dependence of $\mathcal{J}$ and $\sigma_k$ on $s$ from our notation as explained in the convention at the beginning of Section 2. The Lie group $\mathrm{GL}_{1+n}(\mathbb{C})$ of invertible complex $(1+n)\times(1+n)$ - matrices acts from the left on $\mathbb{C}^{1+n}$ as usual via $A \triangleright \rho \coloneqq A\rho$ for all $A \in \mathrm{GL}_{1+n}(\mathbb{C})$ and $\rho \in \mathbb{C}^{1+n}$. This left action $\,\cdot \triangleright \cdot\,$ on $\mathbb{C}^{1+n}$ induces a right action $\,\cdot \triangleleft \cdot\,$ on smooth functions and tensor fields by pullback. Especially for the coordinate functions, this yields $z^k \triangleleft A = \sum_{\ell=0}^{n} A^k{}_\ell \, z^\ell$.

The stabilizer of $\mathcal{J}$, i.e. the set of all $A \in \mathrm{GL}_{1+n}(\mathbb{C})$ fulfilling $\mathcal{J} \triangleleft A = \mathcal{J}$, is

$$\begin{aligned} \mathrm{G}_{\mathcal{J}} &\coloneqq \mathrm{U}(s, 1+n-s) \\ &= \Big\{ A \in \mathrm{GL}_{1+n}(\mathbb{C}) \,\Big|\, \textstyle\sum_{k=0}^{n} \sigma_k A^k{}_\ell \overline{A^k{}_m} = \delta_{\ell,m}\, \sigma_m \text{ for all } \ell, m \in \{0, \dots, n\} \Big\} \end{aligned} \tag{3.3}$$

with $\delta_{\ell,m}$ the usual Kronecker-$\delta$. Note that $\mathrm{G}_{\mathcal{J}}$ is a real Lie group and a subgroup of $\mathrm{GL}_{1+n}(\mathbb{C})$. Its Lie algebra is

$$\begin{aligned} \mathfrak{g}_{\mathcal{J}} &\coloneqq \mathfrak{u}(s, 1+n-s) \\ &= \Big\{ A \in \mathfrak{gl}_{1+n}(\mathbb{C}) \,\Big|\, \sigma_\ell \overline{A^\ell{}_m} + \sigma_m A^m{}_\ell = 0 \text{ for all } \ell, m \in \{0, \dots, n\} \Big\} \,, \end{aligned} \tag{3.4}$$

which is a real form of $\mathfrak{gl}_{1+n}(\mathbb{C}) = \mathbb{C}^{(1+n)\times(1+n)}$.

**Remark 3.1** Note that the Hamiltonian vector field of $-\mathcal{J}$ with respect to the symplectic form $\omega = \mathrm{i} \sum_{k=0}^{n} \sigma_k \,\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^k$ corresponding to the Poisson tensor considered in (5.6) is just the generator of the action of the U(1)-subgroup $\{\mathrm{e}^{\mathrm{i}\phi} \mathbb{1}_{1+n} \,|\, \phi \in \mathbb{R}\}$ of $\mathrm{G}_{\mathcal{J}}$ on $\mathbb{C}^{1+n}$, which is computed in (3.15). In other words, $-\mathcal{J}$ is a momentum map for the U(1)-action. Here $\mathbb{1}_{1+n}$ is the identity matrix. Our construction below can be understood as Marsden–Weinstein reduction with respect to this action.

We define $Z \coloneqq \mathcal{J}^{-1}(\{1\}) = \big\{ \rho \in \mathbb{C}^{1+n} \,\big|\, 1 + \sum_{k=s}^{n} |\rho^k|^2 = \sum_{k=0}^{s-1} |\rho^k|^2 \big\}$, the 1-level set of $\mathcal{J}$, and $\iota \colon Z \to \mathbb{C}^{1+n}$ as the canonical inclusion. Then the $\mathrm{G}_{\mathcal{J}}$-action on $\mathbb{C}^{1+n}$ restricts to $Z$ and $\iota$ is $\mathrm{G}_{\mathcal{J}}$-equivariant.

The next step is to divide out the action of the U(1)-subgroup $\{\mathrm{e}^{\mathrm{i}\phi} \mathbb{1}_{1+n} \,|\, \phi \in \mathbb{R}\}$ of $\mathrm{G}_{\mathcal{J}}$, which yields

$$M_{\mathrm{red}} \coloneqq Z \big/ \mathrm{U}(1) \,. \tag{3.5}$$

As the U(1)-subgroup of $\mathrm{G}_{\mathcal{J}}$ is central, the $\mathrm{G}_{\mathcal{J}}$-action remains well-defined on $M_{\mathrm{red}}$ and the canonical projection $\mathrm{pr} \colon Z \to M_{\mathrm{red}}$ is $\mathrm{G}_{\mathcal{J}}$-equivariant.

We note that, by mapping the U(1)-equivalence class $[\rho] \in M_{\mathrm{red}}$ of some $\rho \in Z$ to its $\mathbb{C}^*$-equivalence class $[\rho] \in \mathbb{CP}^n$, the manifold $M_{\mathrm{red}}$ can be identified with the well-defined open complex submanifold $\{[\rho] \in \mathbb{CP}^n \mid \mathcal{J}(\rho) > 0\}$ of $\mathbb{CP}^n$. Then $w^1, \dots, w^n \colon \{[\rho] \in M_{\mathrm{red}} \mid \rho^0 \neq 0\} \to \mathbb{C}$,

$$w^k\big([\rho]\big) := \frac{\rho^k}{\rho^0} \tag{3.6}$$

with $k \in \{1, \dots, n\}$ define the usual (complex) projective coordinates on $\{[\rho] \in M_{\mathrm{red}} \mid \rho^0 \neq 0\} \subseteq M_{\mathrm{red}}$ and it is easy to obtain an atlas by considering similar coordinates on $\{[\rho] \in M_{\mathrm{red}} \mid \rho^\ell \neq 0\}$ for $1 \leq \ell \leq n$. We will later see how the complex structure that $M_{\mathrm{red}}$ inherits from $\mathbb{CP}^n$ can also be obtained in a more natural way.

In the special case of the signature $s = 1 + n$, this construction yields $M^{(1+n)}_{\mathrm{red}} \cong \mathbb{CP}^n$ with the usual action of $\mathrm{U}(1+n)$ on it. For $s = 1$, one obtains the disc $M^{(1)}_{\mathrm{red}} \cong \mathbb{D}^n = \{\xi \in \mathbb{C}^n \mid \sum_{k=1}^n |\xi^k|^2 < 1\}$ with the action of $\mathrm{U}(1,n)$ by Möbius transformations. The holomorphic isomorphism from $M^{(1)}_{\mathrm{red}}$ to the disc is simply given by the coordinates $w^1, \dots, w^n$, which are global coordinates if $s = 1$.

Note that, in general, these projective coordinates $w^1, \dots, w^n$ describe a chart for $M_{\mathrm{red}}$ with dense domain of definition. Because of this, it is essentially sufficient to use only these coordinates for the explicit description of some tensors later on, but it is important to keep in mind that they describe $M_{\mathrm{red}}$ only up to a meagre subset.

## Top row

The top row consists of complex manifolds carrying a holomorphic action of a complex Lie group $\mathrm{G}_{\hat{\mathcal{J}}}$, and of $\mathrm{G}_{\hat{\mathcal{J}}}$-equivariant holomorphic maps. These complex manifolds will later be helpful for defining certain algebras of analytic functions on $\mathbb{C}^{1+n}$ and $M_{\mathrm{red}}$.

On $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$, the standard complex coordinate functions are denoted by $x^0, \dots, x^n, y^0, \dots, y^n$, and given by $x^k(\xi, \eta) := \xi^k$ as well as $y^k(\xi, \eta) := \eta^k$ for all $k \in \{0, \dots, n\}$ and $\xi, \eta \in \mathbb{C}^{1+n}$. Define the holomorphic polynomial

$$\hat{\mathcal{J}} := \sum_{k=0}^{n} \sigma_k x^k y^k = \sum_{k=0}^{s-1} x^k y^k - \sum_{k=s}^{n} x^k y^k \,. \tag{3.7}$$

Note that the polynomial $\mathcal{J}$ considered before is just the restriction of $\hat{\mathcal{J}}$ to the antidiagonal. More precisely, if $\Delta \colon \mathbb{C}^{1+n} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$,

$$\rho \mapsto \Delta(\rho) := (\rho, \overline{\rho}) \tag{3.8}$$

denotes the embedding along the antidiagonal, then $\mathcal{J} = \hat{\mathcal{J}} \circ \Delta = \Delta^*(\hat{\mathcal{J}})$. Similarly, $\Delta^*(x^k) = z^k$ and $\Delta^*(y^k) = \overline{z}^k$ for all $k \in \{0, \dots, n\}$.

The complex Lie group $\mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C})$ acts holomorphically from the left on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ as usual via $(A, B) \triangleright (\xi, \eta) := (A\xi, B\eta)$ for all $A, B \in \mathrm{GL}_{1+n}(\mathbb{C})$ and $\xi, \eta \in \mathbb{C}^{1+n}$, which induces a right action $\cdot \triangleleft \cdot$ by pullback on the spaces of holomorphic functions or holomorphic tensor fields. Especially for the coordinate functions, this yields $x^k \triangleleft (A, B) = \sum_{\ell=0}^n A^k{}_\ell \, x^\ell$ and $y^k \triangleleft (A, B) = \sum_{\ell=0}^n B^k{}_\ell \, y^\ell$.

The stabilizer $\mathrm{G}_{\hat{\mathcal{J}}}$ of $\hat{\mathcal{J}}$, i.e. the set of $(A,B) \in \mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C})$ fulfilling $\hat{\mathcal{J}} \triangleleft (A,B) = \hat{\mathcal{J}}$, is explicitly given by

$$\mathrm{G}_{\hat{\mathcal{J}}} = \Big\{ (A,B) \in \mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C}) \;\Big|\; \sum_{k=0}^{n} \sigma_k A^k{}_\ell B^k{}_m = \delta_{\ell,m}\, \sigma_m \text{ for all } \ell, m \in \{0,\dots,n\} \Big\}. \tag{3.9}$$

Note that for all $A \in \mathrm{GL}_{1+n}(\mathbb{C})$ there exists a unique $B \in \mathrm{GL}_{1+n}(\mathbb{C})$ such that $(A,B) \in \mathrm{G}_{\hat{\mathcal{J}}}$, namely $B^k{}_m = \sigma_k \sigma_m (A^{-1})^m{}_k$, so $\mathrm{G}_{\hat{\mathcal{J}}}$ is a complex Lie group and isomorphic to $\mathrm{GL}_{1+n}(\mathbb{C})$.

Similar to the definition of $Z$ we define $\hat{Z}$ as the 1-level set of $\hat{\mathcal{J}}$ in $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$, i.e.

$$\hat{Z} \coloneqq \hat{\mathcal{J}}^{-1}(\{1\}) = \Big\{ (\xi,\eta) \in \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \;\Big|\; 1 + \sum_{k=s}^{n} \xi^k \eta^k = \sum_{k=0}^{s-1} \xi^k \eta^k \Big\}. \tag{3.10}$$

Then $\hat{Z}$ is a complex submanifold of $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. The canonical inclusion of $\hat{Z}$ into $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ is denoted by $\hat{\imath}$. As $\hat{\mathcal{J}}$ is invariant under the action of $\mathrm{G}_{\hat{\mathcal{J}}}$, this action can be restricted to $\hat{Z}$ and $\hat{\imath}$ then is clearly $\mathrm{G}_{\hat{\mathcal{J}}}$-invariant. Moreover the inclusion $\Delta$ restricts to an inclusion $\Delta_Z \colon Z \to \hat{Z}$, which makes the upper left square in (3.1) commute.

The second step is to divide out the orbits of the Lie group $\mathbb{C}^* \coloneqq \mathbb{C} \setminus \{0\}$, more precisely of the subgroup $\{(\alpha \mathbb{1}_{1+n}, \alpha^{-1}\mathbb{1}_{1+n}) \mid \alpha \in \mathbb{C}^*\}$ of $\mathrm{G}_{\hat{\mathcal{J}}}$. So define

$$\hat{M}_{\mathrm{red}} \coloneqq \hat{Z} \big/ \mathbb{C}^*, \tag{3.11}$$

then $\hat{M}_{\mathrm{red}}$ can be identified with $\{([\xi],[\eta]) \in \mathbb{CP}^n \times \mathbb{CP}^n \mid \hat{\mathcal{J}}(\xi,\eta) \neq 0\}$, a well-defined open and dense complex submanifold of $\mathbb{CP}^n \times \mathbb{CP}^n$, via $\hat{M}_{\mathrm{red}} \ni [(\xi,\eta)] \mapsto ([\xi],[\eta]) \in \mathbb{CP}^n \times \mathbb{CP}^n$. As the $\mathbb{C}^*$-subgroup of $\mathrm{G}_{\hat{\mathcal{J}}}$ is central, the $\mathrm{G}_{\hat{\mathcal{J}}}$-action remains well-defined on $\hat{M}_{\mathrm{red}}$. The canonical projection from $\hat{Z}$ onto the quotient $\hat{M}_{\mathrm{red}}$ will be denoted by $\hat{\mathrm{pr}}$ and is again $\mathrm{G}_{\hat{\mathcal{J}}}$-equivariant by construction. Finally, one can check that $\Delta_{\mathrm{red}} \colon M_{\mathrm{red}} \to \hat{M}_{\mathrm{red}}$,

$$[\rho] \mapsto \Delta_{\mathrm{red}}\big([\rho]\big) \coloneqq \big[\Delta_Z(\rho)\big] = [(\rho,\overline{\rho})] \tag{3.12}$$

is well-defined and makes the upper right rectangle of (3.1) commute.

On $\hat{M}_{\mathrm{red}}$, we use the usual projective coordinates coming from $\mathbb{CP}^n \times \mathbb{CP}^n$, denoted by $u^1,\dots,u^n \colon \{[(\xi,\eta)] \in \hat{M}_{\mathrm{red}} \mid \xi^0 \neq 0\} \to \mathbb{C}$ and $v^1,\dots,v^n \colon \{[(\xi,\eta)] \in \hat{M}_{\mathrm{red}} \mid \eta^0 \neq 0\} \to \mathbb{C}$, and given by

$$u^k\big([(\xi,\eta)]\big) \coloneqq \frac{\xi^k}{\xi^0} \qquad \text{as well as} \qquad v^k\big([(\xi,\eta)]\big) \coloneqq \frac{\eta^k}{\eta^0} \tag{3.13}$$

for all $k \in \{1,\dots,n\}$. Note that it is again easy to obtain an atlas by considering similarly defined coordinates on $\{[(\xi,\eta)] \in \hat{M}_{\mathrm{red}} \mid \xi^i \neq 0\}$ and $\{[(\xi,\eta)] \in \hat{M}_{\mathrm{red}} \mid \eta^j \neq 0\}$ and that the relations $(\Delta_{\mathrm{red}})^*(u^k) = w^k$ and $(\Delta_{\mathrm{red}})^*(v^k) = \overline{w}^k$ hold for all $k \in \{1,\dots,n\}$. As before, one should also keep in mind that these coordinates form a chart with dense domain of definition.

## Bottom node

It turns out that the complex structure on $\mathbb{C}^{1+n}$ can be used to simplify the Marsden–Weinstein reduction in the middle row of (3.1). First, we define a complex structure on $M_{\mathrm{red}}$ that is compatible with the complex coordinates defined before. A more general treatment of this procedure can be found in [25]. Then we find a holomorphic projection map $\mathrm{Pr}\colon \mathbb{C}^{1+n}_+ \to M_{\mathrm{red}}$ from the open subset

$$\mathbb{C}^{1+n}_+ := \left\{ z \in \mathbb{C}^{1+n} \;\middle|\; \mathcal{J}(z) > 0 \right\} \tag{3.14}$$

of $\mathbb{C}^{1+n}$ to $M_{\mathrm{red}}$ making the bottom right triangle in (3.1) commute. Since restriction to an open subset is easy for almost any geometric structure, one can therefore avoid the restriction to a hypersurface that is needed in the Marsden–Weinstein reduction.

Denote the standard complex structure of $\mathbb{C}^{1+n}$ by $I$. For $A \in \mathfrak{gl}_{1+n}(\mathbb{C})$, let $X_A$ be the vector field on $\mathbb{C}^{1+n}$ obtained by differentiating the right action of $\mathrm{GL}_{1+n}(\mathbb{C})$ on $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ in the direction of $A$, i.e. $X_A(f) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f \triangleleft \exp(tA)$. In particular,

$$X_{\mathrm{i}} := X_{\mathrm{i}\mathbb{1}_{1+n}} = \sum_{k=0}^{n} \left( \mathrm{i} z^k \frac{\partial}{\partial z^k} - \mathrm{i}\overline{z}^k \frac{\partial}{\partial \overline{z}^k} \right) \tag{3.15}$$

is the generator of the (diagonal) $\mathrm{U}(1)$-action and

$$X_{\mathbb{1}} := X_{\mathbb{1}_{1+n}} = \sum_{k=0}^{n} \left( z^k \frac{\partial}{\partial z^k} + \overline{z}^k \frac{\partial}{\partial \overline{z}^k} \right) = -I X_{\mathrm{i}}\,. \tag{3.16}$$

Let $\langle\!\langle X_{\mathrm{i}} \rangle\!\rangle$ and $\langle\!\langle X_{\mathbb{1}} \rangle\!\rangle$ be the 1-dimensional vector subbundles of $\mathrm{T}\mathbb{C}^{1+n}_+$ spanned by $X_{\mathrm{i}}|_{\mathbb{C}^{1+n}_+}$ and $X_{\mathbb{1}}|_{\mathbb{C}^{1+n}_+}$, respectively. Moreover, for $\rho \in \mathbb{C}^{1+n}_+$ define

$$\Xi_\rho := \left\{ \alpha_\rho \in \mathrm{T}_\rho \mathbb{C}^{1+n}_+ \;\middle|\; \alpha_\rho(\mathcal{J}) = 0 \text{ and } \big(I\big|_\rho \alpha_\rho\big)(\mathcal{J}) = 0 \right\} \quad \text{and} \quad \Xi := \bigcup_{\rho \in \mathbb{C}^{1+n}_+} \Xi_\rho\,, \tag{3.17}$$

then one can check that $\Xi$ is a $2n$-dimensional vector subbundle of $\mathrm{T}\mathbb{C}^{1+n}_+$, and we get:

**Proposition 3.2** *The tangent bundle of $\mathbb{C}^{1+n}_+$ can be decomposed as the direct sum*

$$\mathrm{T}\mathbb{C}^{1+n}_+ = \langle\!\langle X_{\mathbb{1}} \rangle\!\rangle \oplus \langle\!\langle X_{\mathrm{i}} \rangle\!\rangle \oplus \Xi\,. \tag{3.18}$$

*Moreover, for all $\rho \in Z$, the map $\mathrm{T}_\rho \mathrm{pr} \circ (\mathrm{T}_\rho \iota)^{-1} \colon \Xi_\rho \to \mathrm{T}_{[\rho]} M_{\mathrm{red}}$ is a linear isomorphism.*

PROOF: The linear subspace $S_\rho := \left\{ \alpha_\rho \in \mathrm{T}_\rho \mathbb{C}^{1+n}_+ \;\middle|\; \alpha_\rho(\mathcal{J}) = 0 \right\}$ of $\mathrm{T}_\rho \mathbb{C}^{1+n}_+$ has codimension 1 for all $\rho \in \mathbb{C}^{1+n}_+$, and $\langle\!\langle X_{\mathbb{1}} \rangle\!\rangle|_\rho$ is a complement of $S_\rho$ in $\mathrm{T}_\rho \mathbb{C}^{1+n}_+$ because $X_{\mathbb{1}}(\mathcal{J}) = 2\mathcal{J}$. So $\mathrm{T}_\rho \mathbb{C}^{1+n}_+ = \langle\!\langle X_{\mathbb{1}} \rangle\!\rangle|_\rho \oplus S_\rho$. Moreover, $\mathrm{U}(1)$-invariance of $\mathcal{J}$ implies that $X_{\mathrm{i}}(\mathcal{J}) = 0$, so $\langle\!\langle X_{\mathrm{i}} \rangle\!\rangle|_\rho \subseteq S_\rho$, and $\Xi_\rho \subseteq S_\rho$ is clear. But as $(I X_{\mathrm{i}})(\mathcal{J}) = -X_{\mathbb{1}}(\mathcal{J}) = -2\mathcal{J}$, the sum of $\langle\!\langle X_{\mathrm{i}} \rangle\!\rangle|_\rho$ and $\Xi_\rho$ is direct, and therefore $S_\rho = \langle\!\langle X_{\mathrm{i}} \rangle\!\rangle|_\rho \oplus \Xi_\rho$ by counting dimensions. This proves the decomposition (3.18).

If $\rho \in Z$, then $S_\rho = \langle\!\langle X_\mathrm{i} \rangle\!\rangle|_\rho \oplus \Xi_\rho$ coincides with the image of $\mathrm{T}_\rho Z$ under $\mathrm{T}_\rho \iota$. Because of this, the map $\mathrm{T}_\rho \mathrm{pr} \circ (\mathrm{T}_\rho \iota)^{-1}$ is well-defined as a map from $\langle\!\langle X_\mathrm{i} \rangle\!\rangle|_\rho \oplus \Xi_\rho$ to $\mathrm{T}_{[\rho]} M_\mathrm{red}$ and is clearly surjective. The kernel of this map is $\langle\!\langle X_\mathrm{i} \rangle\!\rangle|_\rho$, so its restriction to $\Xi_\rho$ is an isomorphism. □

Note that $X_\mathrm{i}$, $X_\mathbb{1}$ and $\Xi$ do not depend on any choices but arise naturally from the $\mathrm{U}(1)$-action, the map $\mathscr{J}$, and the complex structure $I$ that $\mathbb{C}^{1+n}_+$ inherits from $\mathbb{C}^{1+n}$. By definition of $\Xi$, this complex structure restricts to $\Xi$. As it is also $\mathrm{U}(1)$-invariant, it gives rise to a well-defined (almost) complex structure $I_\mathrm{red}$ on $M_\mathrm{red}$:

**Definition 3.3** *Define the vector bundle endomorphism $I_\mathrm{red} \colon \mathrm{T}M_\mathrm{red} \to \mathrm{T}M_\mathrm{red}$, that maps any $\beta_{[\rho]} \in \mathrm{T}_{[\rho]} M_\mathrm{red}$ with $[\rho] \in M_\mathrm{red}$ to*

$$I_\mathrm{red}\big|_{[\rho]}(\beta_{[\rho]}) \coloneqq \big(\mathrm{T}_\rho \mathrm{pr} \circ (\mathrm{T}_\rho \iota)^{-1} \circ I|_\rho \circ (\mathrm{T}_\rho \mathrm{pr} \circ (\mathrm{T}_\rho \iota)^{-1})^{-1}\big)(\beta_{[\rho]}) .$$

It is clear that $I_\mathrm{red}$ squares to $-\mathsf{id}_{\mathrm{T}M_\mathrm{red}}$ and hence is an almost complex structure. In order to see that it is also integrable, we check that $I_\mathrm{red}$ coincides with the complex structure that $M_\mathrm{red}$ inherits from $\mathbb{CP}^n$. For a more general discussion, see [25]:

**Definition 3.4** *On $\mathbb{C}^{1+n}_+ \setminus \{\rho \in \mathbb{C}^{1+n}_+ \mid z^0(\rho) = 0\}$ we define the complex vector fields*

$$W_k \coloneqq z^0 \left( \frac{\partial}{\partial z^k} - \frac{\sigma_k \overline{z}^k}{\mathscr{J}} \sum_{\ell=0}^{n} z^\ell \frac{\partial}{\partial z^\ell} \right)\Bigg|_{\mathbb{C}^{1+n}_+ \setminus \{\rho \in \mathbb{C}^{1+n}_+ \,|\, z^0(\rho) = 0\}} \tag{3.19}$$

*for all $k \in \{1, \dots, n\}$.*

Note that, analogously to the projective coordinates $w^1, \dots, w^n$ on $M_\mathrm{red}$, the vector fields $W_1, \dots, W_n$ are only defined on a dense subset of $\mathbb{C}^{1+n}_+$. However, this will be completely sufficient for our purposes.

As $IW_k = \mathrm{i}W_k$ and $\langle \mathrm{d}\mathscr{J}, W_k \rangle = 0$ for all $k \in \{1, \dots, n\}$ on the domain of definition of $W_k$, these vector fields $W_k$, as well as their complex conjugates $\overline{W}_k$ with $k \in \{1, \dots, n\}$, are actually (local, densely defined) sections of $\Xi^\mathbb{C}$, the complex $2n$-dimensional vector subbundle of $\mathrm{T}^\mathbb{C}\mathbb{C}^{1+n}_+$ generated by $\Xi$. A short calculation shows that

$$W_k\big(z^\ell / z^0\big) = \delta^\ell_k \tag{3.20}$$

for all $k, \ell \in \{1, \dots, n\}$, so the sections $W_k$ are pointwise linearly independent and, by counting dimensions, they form a (local, densely defined) frame of $\Xi^\mathbb{C}$. Moreover, this immediately shows:

**Proposition 3.5** *If $\rho \in Z$, $z^0(\rho) \neq 0$, then*

$$\big(\mathrm{T}_\rho \mathrm{pr} \circ (\mathrm{T}_\rho \iota)^{-1}\big)(W_k|_\rho) = \frac{\partial}{\partial w^k}\bigg|_{[\rho]} \quad \text{and} \quad \big(\mathrm{T}_\rho \mathrm{pr} \circ (\mathrm{T}_\rho \iota)^{-1}\big)(\overline{W}_k|_\rho) = \frac{\partial}{\partial \overline{w}^k}\bigg|_{[\rho]} \tag{3.21}$$

*for all $k \in \{1, \dots, n\}$.*

As an immediate consequence we obtain:

**Corollary 3.6** *The reduced complex structure $I_{\mathrm{red}}$ satisfies $I_{\mathrm{red}}(\frac{\partial}{\partial w^k}) = \mathrm{i}\frac{\partial}{\partial w^k}$ and $I_{\mathrm{red}}(\frac{\partial}{\partial \overline{w}^k}) = -\mathrm{i}\frac{\partial}{\partial \overline{w}^k}$ for all $k \in \{1, \dots, n\}$, so $I_{\mathrm{red}}$ is indeed the standard complex structure of $M_{\mathrm{red}}$ interpreted as an open subset of $\mathbb{CP}^n$. In particular, $I_{\mathrm{red}}$ is integrable and really a complex structure.*

**Lemma 3.7** *If a holomorphic complex-valued map $\phi$ on a connected and open subset $S \subseteq \mathbb{C}^{1+n}$ with $S \cap Z \neq \emptyset$ vanishes on $S \cap Z$, then it already vanishes on all of $S$.*

PROOF: Indeed, as $\mathrm{T}_\rho\mathbb{C}^{1+n} = \langle\!\langle X_{\mathbb{1}} \rangle\!\rangle|_\rho \oplus (\mathrm{T}_\rho\iota)(\mathrm{T}_\rho Z)$ for all $\rho \in Z$, as $\alpha_\rho(\phi) = 0$ for all $\alpha_\rho \in (\mathrm{T}_\rho\iota)(\mathrm{T}_\rho Z)$ by assumption and as also $X_{\mathbb{1}}|_\rho(\phi) = X_{\mathrm{i}}|_\rho(-\mathrm{i}\phi) = 0$ because $\phi$ is holomorphic and $X_{\mathbb{1}}|_\rho \in (\mathrm{T}_\rho\iota)(\mathrm{T}_\rho Z)$, all first order partial derivatives of $\phi$ vanish on $S \cap Z$. This now extends to all arbitrarily high partial derivatives by using the same argument and thus the holomorphic $\phi$ vanishes on whole $S$. □

As a consequence, there is at most one holomorphic map $\mathrm{Pr}\colon \mathbb{C}^{1+n}_+ \to M_{\mathrm{red}}$ whose restriction to $Z$ coincides with pr. In the special case treated here it is not hard to guess this map:

**Proposition 3.8** *There exists a (unique) holomorphic map* $\mathrm{Pr}\colon \mathbb{C}^{1+n}_+ \to M_{\mathrm{red}}$ *whose restriction to $Z$ coincides with* pr. *It is explicitly given by*

$$\rho \mapsto \mathrm{Pr}(\rho) = [\rho/\sqrt{\mathcal{J}(\rho)}]\,. \tag{3.22}$$

*In coordinates, $w^k \circ \mathrm{Pr} = z^k/z^0$.*

PROOF: It is not hard to check the expression of (3.22) in coordinates, which also shows that Pr is holomorphic. Its restriction to $Z$ clearly coincides with pr. □

We also note that the domain $\mathbb{C}^{1+n}_+$ of Pr, which was chosen rather arbitrarily, is naturally determined from the $\mathrm{U}(1)$-action on $\mathbb{C}^{1+n}$ and the complex structure $I$: The action of the corresponding Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$ is given by its fundamental vector field $X_{\mathrm{i}}$, and the complex structure $I$ allows to extend this to an action of the complexified Lie algebra $\mathfrak{u}(1) \otimes \mathbb{C} \cong \mathbb{C}$ via the fundamental vector fields $X_{\mathrm{i}}$ and $X_{\mathbb{1}}$. This action even integrates to a unique holomorphic action of the corresponding complex Lie group $\mathbb{C}^*$ on $\mathbb{C}^{1+n}$, which is just given by multiplication with scalars. The orbit of $Z$ under the action of $\mathbb{C}^*$ is easily seen to be $\mathbb{C}^{1+n}_+$, and $\mathrm{Pr}\colon \mathbb{C}^{1+n}_+ \to M_{\mathrm{red}}$ is the quotient map that identifies $\mathbb{C}^{1+n}_+/\mathbb{C}^*$ with $M_{\mathrm{red}}$ as complex manifolds. From this point of view, the complex structure on $\mathbb{C}^{1+n}$ allows to replace the two steps of Marsden–Weinstein reduction (restriction to the level set $Z$ and taking $\mathrm{U}(1)$-equivalence classes) by restriction to the open complex submanifold $\mathbb{C}^{1+n}_+$ and taking equivalence classes with respect to the action of the complexification $\mathbb{C}^*$ of $\mathrm{U}(1)$.

For future use it will be helpful to be able to express the standard coordinate vectors $\frac{\partial}{\partial z^k}$ with $k \in \{0, \dots, n\}$ in terms of the holomorphic Euler vector field

$$E \coloneqq \frac{1}{2}(X_{\mathbb{1}} - \mathrm{i}X_{\mathrm{i}})\big|_{\mathbb{C}^{1+n}_+} = \sum_{k=0}^{n} z^k \frac{\partial}{\partial z^k}\bigg|_{\mathbb{C}^{1+n}_+} \tag{3.23}$$

and the $W_k$, $k \in \{1, \dots, n\}$. On their domain of definition, one gets (using that always $\sigma_0 = 1$)

$$\frac{\partial}{\partial z^0} = \frac{\overline{z}^0}{\mathcal{J}} E - \sum_{\ell=1}^{n} \frac{z^\ell}{(z^0)^2} W_\ell \qquad \text{and} \qquad \frac{\partial}{\partial z^k} = \frac{\sigma_k \overline{z}^k}{\mathcal{J}} E + \frac{1}{z^0} W_k \tag{3.24}$$

for all $k \in \{1, \dots, n\}$ and $(E, W_1, \dots, W_k)$ is a local, densely defined frame for $\mathrm{T}^{(1,0)}\mathbb{C}^{1+n}_+$. Together with its complex conjugates $(\overline{E}, \overline{W}_1, \dots, \overline{W}_n)$ we obtain a densely defined frame for the whole tangent space $\mathrm{T}^{\mathbb{C}}\mathbb{C}^{1+n}_+$. The dual frames are denoted by $(E^*, W_1^*, \dots, W_n^*)$ and $(\overline{E}^*, \overline{W}_1^*, \dots, \overline{W}_n^*)$, and (again only on the domain of definition of the vector fields $W_k$) we have

$$E^* = \frac{1}{\mathcal{J}} \sum_{k=0}^{n} \sigma_k \overline{z}^k \, \mathrm{d}z^k \,, \tag{3.25}$$

$$W_k^* = -\frac{z^k}{(z^0)^2} \, \mathrm{d}z^0 + \frac{1}{z^0} \, \mathrm{d}z^k = \mathrm{Pr}^*(\mathrm{d}w^k) \,, \tag{3.26}$$

$$\mathrm{d}z^0 = z^0 E^* - \frac{(z^0)^2}{\mathcal{J}} \sum_{k=1}^{n} \sigma_k \overline{z}^k W_k^* \,, \tag{3.27}$$

$$\mathrm{d}z^k = z^k E^* + z^0 \left( W_k^* - \frac{z^k}{\mathcal{J}} \sum_{\ell=1}^{n} v_\ell \overline{z}^\ell W_\ell^* \right) . \tag{3.28}$$

Note that $E$ and $\overline{E}$ are obtained from the $\mathrm{U}(1)$-action and complex structure of $\mathbb{C}^{1+n}_+$. Similarly, also $E^*$ and $\overline{E}^*$ can be obtained naturally as the $(1,0)$ and $(0,1)$-parts of $\mathrm{d}\mathcal{J}/\mathcal{J}$. Only the vector fields $W_1, \dots, W_n$ as well as their conjugates and duals depend on a choice of coordinates.

# 4 Algebraic point of view

The general reduction procedure from $\mathbb{C}^{1+n}$ to $M_{\mathrm{red}}$ by first restricting to the level set $Z$ and then dividing out the action of $\mathrm{U}(1)$ has a dual version that connects various function algebras on $\mathbb{C}^{1+n}$ and $M_{\mathrm{red}}$: First, one divides out the ideal of functions vanishing on $Z$ and then restricts to $\mathrm{U}(1)$-invariant equivalence classes. However, as every $\mathrm{U}(1)$-invariant equivalence class of functions also contains at least one $\mathrm{U}(1)$-invariant function, which can be obtained by averaging over the compact group $\mathrm{U}(1)$, a simplified procedure yields the same results: First, one restricts to $\mathrm{U}(1)$-invariant functions and then divides out the ideal of functions vanishing on $Z$. We will use this second approach throughout.

It is well-known that this way one can also construct algebraic structures on $M_{\mathrm{red}}$ out of such structures on $\mathbb{C}^{1+n}$, especially Poisson brackets and star products. In the following we will consider three types of function algebras: All smooth functions, polynomial functions, and certain analytic functions. While formal star products are defined on all smooth functions, their non-formal versions can only be defined on polynomial or some analytic functions. All these function algebras on $\mathbb{C}^{1+n}$ will also be endowed with the right-action of the stabilizer group $\mathrm{G}_{\mathcal{J}}$.

## 4.1 Smooth functions

The reduction procedure for smooth functions is well-known. In order to fix notation, it is helpful to shortly discuss some details again: Recall that $\mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ is the unital subalgebra of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ whose elements are the U(1)-invariant functions. It is easy to see that the following is well-defined:

**Definition 4.1** *Let $S$ be an open and* U(1)*-invariant subset of $\mathbb{C}^{1+n}$ such that $S \supseteq Z$. The* (classical) reduction map *is* $\cdot_{\mathrm{red}} \colon \mathscr{C}^\infty(S)^{\mathrm{U}(1)} \to \mathscr{C}^\infty(M_{\mathrm{red}})$, $f \mapsto f_{\mathrm{red}}$, *where*

$$f_{\mathrm{red}}([\rho]) := f(\rho) \tag{4.1}$$

*for all $\rho \in Z$.*

We will especially be interested in the two cases $S = \mathbb{C}^{1+n}$ and $S = \mathbb{C}^{1+n}_+$. Note that $f_{\mathrm{red}}$ is the unique smooth function on $M_{\mathrm{red}}$ that fulfils $\mathrm{pr}^*(f_{\mathrm{red}}) = \iota^*(f)$. From the algebraic point of view, smooth functions on $\mathbb{C}^{1+n}$ and $M_{\mathrm{red}}$ can be related as follows:

**Lemma 4.2** *For every $g \in \mathscr{C}^\infty(M_{\mathrm{red}})$ there exists an $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ such that $f_{\mathrm{red}} = g$, and $f$ can even be chosen in such a way that the following locality condition is fulfilled: Whenever $U$ is an open subset of $M_{\mathrm{red}}$ such that the restriction of $g$ to $U$ vanishes, then there exists an open subset $V$ of $\mathbb{C}^{1+n}$ such that $V \supseteq \mathrm{pr}^{-1}(U)$ and such that the restriction of $f$ to $V$ vanishes.*

Proof: This is well-known to be true in more generality, but in the present case it is also easy to construct such an $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ for every $g \in \mathscr{C}^\infty(M_{\mathrm{red}})$: Indeed, one can define $f(\rho) := 0$ for all $\rho \in \mathbb{C}^{1+n} \setminus \mathbb{C}^{1+n}_+$ and $f(\rho) := g(\mathrm{Pr}(\rho))\chi(\mathscr{J}(\rho))$ for all $\rho \in \mathbb{C}^{1+n}_+$, where $\chi\colon \left]0,\infty\right[ \to [0,1]$ is a smooth function with compact support that fulfils $\chi(1) = 1$. □

This lemma has the following consequence:

**Proposition 4.3** *For every* U(1)*-invariant open subset $S \subseteq \mathbb{C}^{1+n}$ containing $Z$, the reduction map $\cdot_{\mathrm{red}} \colon \mathscr{C}^\infty(S)^{\mathrm{U}(1)} \to \mathscr{C}^\infty(M_{\mathrm{red}})$ descends to an isomorphism between the unital $^*$-algebras $\mathscr{C}^\infty(S)^{\mathrm{U}(1)}/\{v \in \mathscr{C}^\infty(S)^{\mathrm{U}(1)} \mid \iota^*(v) = 0\}$ and $\mathscr{C}^\infty(M_{\mathrm{red}})$.*

We can now also construct algebraic structures on $\mathscr{C}^\infty(M_{\mathrm{red}})$ out of such structures on $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ or $\mathscr{C}^\infty(\mathbb{C}^{1+n}_+)$:

**Proposition 4.4** *Let $S$ be an open and* U(1)*-invariant subset of $\mathbb{C}^{1+n}$ such that $S \supseteq Z$, and let $C\colon \mathscr{C}^\infty(S) \times \mathscr{C}^\infty(S) \to \mathscr{C}^\infty(S)$ be a* U(1)*-invariant bilinear map, then the following is equivalent:*

- *There exists a bilinear map $C_{\mathrm{red}} \colon \mathscr{C}^\infty(M_{\mathrm{red}}) \times \mathscr{C}^\infty(M_{\mathrm{red}}) \to \mathscr{C}^\infty(M_{\mathrm{red}})$ such that*

  $$\big(C(f,g)\big)_{\mathrm{red}} = C_{\mathrm{red}}\big(f_{\mathrm{red}}, g_{\mathrm{red}}\big)$$

  *holds for all $f, g \in \mathscr{C}^\infty(S)^{\mathrm{U}(1)}$.*

- $C(f,v)|_Z = 0 = C(v,f)|_Z$ *holds for all* $f, v \in \mathscr{C}^\infty(S)^{\mathrm{U}(1)}$ *with* $\iota^*(v) = 0$.

*If one, hence both of these two conditions are fulfilled, then the bilinear map* $C_{\mathrm{red}}$ *from the first point is uniquely determined.*

PROOF: Using the existence of preimages under $\,\cdot_{\mathrm{red}}$ from Lemma 4.2, the equivalence of the two points and the uniqueness of $C_{\mathrm{red}}$ are standard results. □

**Definition 4.5** *Let* $S$ *be an open and* $\mathrm{U}(1)$*-invariant subset of* $\mathbb{C}^{1+n}$ *such that* $S \supseteq Z$*, and let* $C\colon \mathscr{C}^\infty(S) \times \mathscr{C}^\infty(S) \to \mathscr{C}^\infty(S)$ *be a* $\mathrm{U}(1)$*-invariant bilinear map, then* $C$ *is called* reducible *if one, hence both of the equivalent properties from the previous Proposition 4.4 are fulfilled. In this case, we also define the* reduced *map* $C_{\mathrm{red}}$ *like in the first point there.*

One example is of course the multiplication: Let $C$ be the pointwise multiplication of smooth functions on $\mathbb{C}^{1+n}$, then $C_{\mathrm{red}}$ is the pointwise multiplication of smooth functions on $M_{\mathrm{red}}$. For more interesting examples, however, the second point in Proposition 4.4 can still be hard to check. Luckily, there are some simplifications for bidifferential operators. Note that in the following it is no loss of generality to consider the special case of a $\mathrm{U}(1)$-invariant bidifferential operator $C\colon \mathscr{C}^\infty(\mathbb{C}^{1+n}_+) \times \mathscr{C}^\infty(\mathbb{C}^{1+n}_+) \to \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)$: A bidifferential operator on a different domain of definition can always be restricted and extended (in a not necessarily unique way) to a bidifferential operator on $\mathbb{C}^{1+n}_+$ which coincides with the original one in a neighbourhood of $Z$ and thus yields the same reduced map.

**Proposition 4.6** *Let* $C\colon \mathscr{C}^\infty(\mathbb{C}^{1+n}_+) \times \mathscr{C}^\infty(\mathbb{C}^{1+n}_+) \to \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)$ *be a* $\mathrm{U}(1)$*-invariant bidifferential operator. If* $C\big((\mathscr{J}-1)|_{\mathbb{C}^{1+n}_+} f, f'\big) = 0 = C\big(f, (\mathscr{J}-1)|_{\mathbb{C}^{1+n}_+} f'\big)$ *holds for all* $f, f' \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$*, then* $C$ *is reducible and*

$$\big(C(\mathrm{Pr}^*(g), \mathrm{Pr}^*(g'))\big)_{\mathrm{red}} = C_{\mathrm{red}}(g, g') \tag{4.2}$$

*holds for all* $g, g' \in \mathscr{C}^\infty(M_{\mathrm{red}})$.

PROOF: In order to show that $C$ is reducible, let $f, v \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$ with $\iota^*(v) = 0$ be given. For every $\epsilon \in (0,1)$ and using a bump function $\chi \in \mathscr{C}^\infty((0,\infty))$ with support in $[1-\epsilon, 1+\epsilon]$ fulfilling $\chi(r) = 1$ for all $r \in [1-\epsilon/2, 1+\epsilon/2]$, one can express $v$ as the sum $v = v\cdot(\chi\circ\mathscr{J}|_{\mathbb{C}^{1+n}_+}) + (\mathscr{J}-1)|_{\mathbb{C}^{1+n}_+}\tilde{v}$ of a function $v\cdot(\chi\circ\mathscr{J}|_{\mathbb{C}^{1+n}_+}) \in \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ with support in $\{\rho \in \mathbb{C}^{1+n}_+ \mid -\epsilon \leq \mathscr{J}(\rho) - 1 \leq \epsilon\}$ and the product of $(\mathscr{J}-1)|_{\mathbb{C}^{1+n}_+}$ with a function $\tilde{v} \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$. Then $C(f,v) = C\big(f, v\cdot(\chi\circ\mathscr{J}|_{\mathbb{C}^{1+n}_+})\big)$ and $C(v,f) = C\big(v\cdot(\chi\circ\mathscr{J}|_{\mathbb{C}^{1+n}_+}), f\big)$ have support in $\{\rho \in \mathbb{C}^{1+n}_+ \mid -\epsilon \leq \mathscr{J}(\rho) - 1 \leq \epsilon\}$. As $\epsilon \in (0,1)$ was arbitrary, even $C(v,f) = 0 = C(f,v)$ holds and $C$ is reducible. For Equation (4.2) we just note that $(\mathrm{Pr}^*(g))_{\mathrm{red}} = g$ for all $g \in \mathscr{C}^\infty(M_{\mathrm{red}})$. □

## 4.2 Polynomial functions

On polynomial functions it will be possible to construct non-formal star products in Section 5. Here we only discuss the basic definitions and the reduction procedure:

**Definition 4.7** *We write $\mathscr{P}(\mathbb{C}^{1+n})$ for the unital $^*$-subalgebra of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ that consists of all polynomial functions in $z^0, \dots, z^n, \overline{z}^0, \dots, \overline{z}^n$. We denote the image of $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ under $\cdot_{\mathrm{red}}$ by $\mathscr{P}(M_{\mathrm{red}})$ and call its elements* polynomials on $M_{\mathrm{red}}$.

One can check that $\mathscr{P}(M_{\mathrm{red}})$ is a unital $^*$-subalgebra of $\mathscr{C}^\infty(M_{\mathrm{red}})$ and so the reduction map restricts to a surjective unital $^*$-homomorphism from $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ to $\mathscr{P}(M_{\mathrm{red}})$. Its kernel are all U(1)-invariant polynomial functions on $\mathbb{C}^{1+n}$ which vanish on $Z$. We see that the unital $^*$-algebra $\mathscr{P}(M_{\mathrm{red}})$ is isomorphic to the quotient $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}/\{v \in \mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \,|\, \iota^*(v) = 0\}$ like in the smooth case. A basis of $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ yields a generating subset of $\mathscr{P}(M_{\mathrm{red}})$, a subset of which is a basis of $\mathscr{P}(M_{\mathrm{red}})$. We essentially follow [3, 17] and just check that the definitions and results there, which were made for the special case $s = 1$, actually work for all signatures:

**Definition 4.8** *For every pair of multiindices $P, Q \in \mathbb{N}_0^{1+n}$ we define the monomial on $\mathbb{C}^{1+n}$*

$$\mathrm{b}_{P,Q} := z^P \overline{z}^Q := (z^0)^{P_0} \dots (z^n)^{P_n} (\overline{z}^0)^{Q_0} \dots (\overline{z}^n)^{Q_n} \,. \tag{4.3}$$

The monomials $\mathrm{b}_{P,Q}$ with $P, Q \in \mathbb{N}_0^{1+n}$ are a basis of $\mathscr{P}(\mathbb{C}^{1+n})$, and those monomials with $|P| = |Q|$ are a basis of $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$. The resulting *reduced monomials* $\mathrm{b}_{P,Q;\mathrm{red}} \in \mathscr{P}(M_{\mathrm{red}})$ are, in the projective coordinates defined in (3.6) (and restricted to the dense domain of definition of these coordinates),

$$\mathrm{b}_{P,Q;\mathrm{red}} = \frac{w^{P'} \overline{w}^{Q'}}{(1 + \sum_{k=1}^n \sigma_k w^k \overline{w}^k)^{|P|}} := \frac{(w^1)^{P_1} \dots (w^n)^{P_n} (\overline{w}^1)^{Q_1} \dots (\overline{w}^n)^{Q_n}}{(1 + \sum_{k=1}^n \sigma_k w^k \overline{w}^k)^{|P|}} \tag{4.4}$$

for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and with $P' := (P_1, \dots, P_n) \in \mathbb{N}_0^n$, analogously for $Q$. To check this, note that the pullback with Pr of the right-hand side coincides with $\mathrm{b}_{P,Q}/\mathscr{J}^{|P|}$ on $\mathbb{C}_+^{1+n}$, hence with $\mathrm{b}_{P,Q}$ on $Z$. Even though the monomials $\mathrm{b}_{P,Q}$ on $\mathbb{C}^{1+n}$ are linearly independent, this does no longer hold for their counterparts $\mathrm{b}_{P,Q;\mathrm{red}}$ on $M_{\mathrm{red}}$. Because of this we introduce:

**Definition 4.9** *For all multiindices $P, Q \in \mathbb{N}_0^n$ we define the* fundamental monomial on $M_{\mathrm{red}}$

$$\mathrm{c}_{P,Q} := \begin{cases} \mathrm{b}_{(|Q|-|P|, P_1, \dots, P_n), (0, Q_1, \dots, Q_n);\mathrm{red}} & \textit{if } |P| \leq |Q| \,, \\ \mathrm{b}_{(0, P_1, \dots, P_n), (|P|-|Q|, Q_1, \dots, Q_n);\mathrm{red}} & \textit{if } |P| \geq |Q| \,. \end{cases} \tag{4.5}$$

Note that the fundamental monomials on $M_{\mathrm{red}}$—unlike the monomials on $\mathbb{C}^{1+n}$—are determined by $2n$ indices, not $2n+2$. Using projective coordinates on $M_{\mathrm{red}}$, they can be expressed as

$$\mathrm{c}_{P,Q} = \frac{w^P \overline{w}^Q}{(1 + \sum_{k=1}^n \sigma_k w^k \overline{w}^k)^{\max\{|P|,|Q|\}}} \tag{4.6}$$

for all $P, Q \in \mathbb{N}_0^n$. While the usual easy multiplication rules for monomials still hold for the $\mathrm{b}_{P,Q;\mathrm{red}}$, i.e. $\mathrm{b}_{P,Q;\mathrm{red}} \mathrm{b}_{R,S;\mathrm{red}} = \mathrm{b}_{P+R,Q+S;\mathrm{red}}$ for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$, this is no longer true for the fundamental monomials on $M_{\mathrm{red}}$. Their product can be obtained by rewriting them in terms of the reduced monomials, which can easily be multiplied, and by applying the following:

**Lemma 4.10** *For all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, the identity*

$$\mathrm{b}_{P,Q;\mathrm{red}} = \sum_{\substack{T\in\mathbb{N}_0^n \\ |T|\le\min\{P_0,Q_0\}}} (-1)^{|T|}\,\mathrm{sgn}(T)\binom{\min\{P_0,Q_0\}}{|T|}\frac{|T|!}{T!}\mathrm{c}_{P'+T,Q'+T} \tag{4.7}$$

*holds, where we use $P' := (P_1, \dots, P_n) \in \mathbb{N}_0^n$, $Q' := (Q_1, \dots, Q_n) \in \mathbb{N}_0^n$ and $\mathrm{sgn}(T) := \prod_{k=1}^n \sigma_k^{T_k}$.*

PROOF: For $k \in \{0, \dots, n\}$, let $E_k := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}_0^{1+n}$ be the tuple with 1 at position $k$. From $\mathrm{b}_{E_0,E_0;\mathrm{red}} = 1 - \sum_{k=1}^n \sigma_k \mathrm{b}_{E_k,E_k;\mathrm{red}}$ and the multinomial theorem it follows that

$$\big(\mathrm{b}_{E_0,E_0;\mathrm{red}}\big)^{\min\{P_0,Q_0\}} = \sum_{\substack{T\in\mathbb{N}_0^n \\ |T|\le\min\{P_0,Q_0\}}} (-1)^{|T|}\,\mathrm{sgn}(T)\binom{\min\{P_0,Q_0\}}{|T|}\frac{|T|!}{T!}\mathrm{c}_{T,T}\,.$$

Combining this with $\mathrm{b}_{P,Q;\mathrm{red}} = (\mathrm{b}_{E_0,E_0;\mathrm{red}})^{\min\{P_0,Q_0\}}\mathrm{c}_{P',Q'}$ yields the desired result.□

Analogous to [3,17], one can show that these fundamental monomials $\mathrm{c}_{P,Q}$ with $P, Q \in \mathbb{N}_0^n$ are a Hamel basis of $\mathscr{P}(M_{\mathrm{red}})$. We will come back to this problem later in Section 6.

## 4.3 Analytic functions

The polynomial algebras discussed in the previous Subsection 4.2 can be completed to algebras of certain analytic functions. More precisely, we are interested in the pullbacks with $\Delta\colon \mathbb{C}^{1+n} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ and $\Delta_{\mathrm{red}}\colon M_{\mathrm{red}} \to \hat{M}_{\mathrm{red}}$ of holomorphic functions:

**Definition 4.11** *By $\mathscr{O}(M)$ we denote the unital complex algebra of holomorphic functions on a complex manifold $M$. Moreover, we define the following subsets of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ and $\mathscr{C}^\infty(M_{\mathrm{red}})$, respectively:*

$$\mathscr{A}(\mathbb{C}^{1+n}) := \big\{\Delta^*(\hat{f}) \,\big|\, \hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})\big\} \tag{4.8}$$

*and*

$$\mathscr{A}(M_{\mathrm{red}}) := \big\{\Delta_{\mathrm{red}}^*(\hat{g}) \,\big|\, \hat{g} \in \mathscr{O}(\hat{M}_{\mathrm{red}})\big\}\,. \tag{4.9}$$

It is not hard to check that $\mathscr{A}(\mathbb{C}^{1+n})$ and $\mathscr{A}(M_{\mathrm{red}})$ are unital $^*$-subalgebras of $\mathscr{C}^\infty(\mathbb{C}^{1+n})$ and $\mathscr{C}^\infty(M_{\mathrm{red}})$, respectively. Especially for the $^*$-involution one finds: Given $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ or $\hat{g} \in \mathscr{O}(\hat{M}_{\mathrm{red}})$, then one can define $\hat{f}' \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ or $\hat{g}' \in \mathscr{O}(\hat{M}_{\mathrm{red}})$ as the functions

$$\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \ni (\xi, \eta) \mapsto \hat{f}'(\xi, \eta) := \overline{\hat{f}(\overline{\eta}, \overline{\xi})} \in \mathbb{C}$$

or

$$\hat{M}_{\mathrm{red}} \ni ([\xi, \eta]) \mapsto \hat{g}'([\xi, \eta]) := \overline{\hat{g}([\overline{\eta}, \overline{\xi}])} \in \mathbb{C}\,,$$

so that $\Delta^*(\hat{f}') = \overline{\Delta^*(\hat{f})}$ and $\Delta_{\mathrm{red}}^*(\hat{g}') = \overline{\Delta_{\mathrm{red}}^*(\hat{g})}$, respectively. As algebras, $\mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ and $\mathscr{A}(\mathbb{C}^{1+n})$ as well as $\mathscr{O}(\hat{M}_{\mathrm{red}})$ and $\mathscr{A}(M_{\mathrm{red}})$ are isomorphic:

**Proposition 4.12** *The pullbacks*

$$\Delta^* \colon \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \to \mathscr{A}(\mathbb{C}^{1+n}) \quad \text{and} \quad \Delta^*_{\mathrm{red}} \colon \mathscr{O}(\hat{M}_{\mathrm{red}}) \to \mathscr{A}(M_{\mathrm{red}})$$

*are isomorphisms of algebras.*

Proof: It is easy to check that $\Delta^*$ and $\Delta^*_{\mathrm{red}}$ are homomorphisms of algebras, and they are surjective by definition of $\mathscr{A}(\mathbb{C}^{1+n})$ and $\mathscr{A}(M_{\mathrm{red}})$, so only injectivity remains: Given $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ with $\Delta^*(\hat{f}) = 0$ or $\hat{g} \in \mathscr{O}(\hat{M}_{\mathrm{red}})$ with $\Delta^*_{\mathrm{red}}(\hat{g}) = 0$, then, in the coordinates introduced in Section 3,

$$\begin{aligned}
\left.\frac{\partial \hat{f}}{\partial x^k}\right|_{(\rho,\overline{\rho})} &= (\mathrm{T}_\rho \Delta)\left(\left.\frac{\partial}{\partial z^k}\right|_\rho\right)(\hat{f}) = \left.\frac{\partial}{\partial z^k}\right|_\rho \Delta^*(\hat{f}) = 0\,, \\
\left.\frac{\partial \hat{f}}{\partial y^k}\right|_{(\rho,\overline{\rho})} &= \left.\frac{\partial}{\partial \overline{z}^k}\right|_\rho \Delta^*(\hat{f}) = 0
\end{aligned}$$

or

$$\begin{aligned}
\left.\frac{\partial \hat{g}}{\partial u^\ell}\right|_{[(\rho,\overline{\rho})]} &= (\mathrm{T}_{[\rho]} \Delta_{\mathrm{red}})\left(\left.\frac{\partial}{\partial w^\ell}\right|_{[\rho]}\right)(\hat{g}) = \left.\frac{\partial}{\partial w^\ell}\right|_{[\rho]} \Delta^*_{\mathrm{red}}(\hat{g}) = 0\,, \\
\left.\frac{\partial \hat{g}}{\partial v^\ell}\right|_{[(\rho,\overline{\rho})]} &= \left.\frac{\partial}{\partial \overline{w}^\ell}\right|_{[\rho]} \Delta^*_{\mathrm{red}}(\hat{g}) = 0
\end{aligned}$$

hold for all $\rho \in Z$ with $z^0(\rho) \neq 0$ and all $k \in \{0, \dots, n\}$, $\ell \in \{1, \dots, n\}$, respectively. By iteration one finds that also all higher derivatives of $\hat{f}$ or $\hat{g}$ vanish, so that $\hat{f} = 0$ or $\hat{g} = 0$. □

It is well-known that the holomorphic functions $\mathscr{O}(M)$ on a complex manifold $M$ with the pointwise operations become a Fréchet algebra with the topology of locally uniform convergence (i.e. $\mathscr{O}(M)$ is complete and the multiplication continuous with respect to this metrizable locally convex topology). This locally convex topology can be described by all the seminorms $\|\cdot\|_K \colon \mathscr{O}(M) \to [0, \infty)$,

$$\hat{f} \mapsto \|\hat{f}\|_K \coloneqq \max_{z \in K} |\hat{f}(z)| \tag{4.10}$$

with $K$ a compact subset of $M$. From this we see immediately that $\mathscr{A}(\mathbb{C}^{1+n})$ and $\mathscr{A}(M_{\mathrm{red}})$ with the topology coming from $\mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ and $\mathscr{O}(\hat{M}_{\mathrm{red}})$, respectively, are Fréchet $^*$-algebras (Fréchet algebras endowed with a continuous $^*$-involution). It is a consequence of the Cauchy integral formula on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ that every $f \in \mathscr{A}(\mathbb{C}^{1+n})$ can be expressed in a unique way as an absolutely convergent series

$$f = \sum_{P,Q \in \mathbb{N}_0^{1+n}} f_{P,Q} \mathrm{b}_{P,Q} \tag{4.11}$$

with complex coefficients $f_{P,Q}$ fulfilling

$$\|f\|_r \coloneqq \sum_{P,Q \in \mathbb{N}_0^{1+n}} |f_{P,Q}| r^{|P|+|Q|} < \infty \tag{4.12}$$

for all $r \in [1, \infty)$, and that the topology of $\mathscr{A}(\mathbb{C}^{1+n})$ can equivalently be described by these seminorms $\|\cdot\|_r \colon \mathscr{A}(\mathbb{C}^{1+n}) \to [0, \infty)$. See e.g. [23, Proposition 3.5] for details. We will later in Proposition 6.11 obtain an analogous result also for $\mathscr{A}(M_{\mathrm{red}})$. Like for polynomials one also finds that the U(1)-invariant analytic functions $f$ are precisely those which fulfil $f_{P,Q} = 0$ for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| \neq |Q|$, e.g. by explicitly calculating the coefficients with the help of the Cauchy integral formula. Note that due to the completeness of $\mathscr{A}(\mathbb{C}^{1+n})$, averaging over the U(1)-action on $\mathscr{A}(\mathbb{C}^{1+n})$ is possible and yields for every $f \in \mathscr{A}(\mathbb{C}^{1+n})$ an $f_{\mathrm{av}} \in \mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$.

We observe that the reduction map $\cdot_{\mathrm{red}}$ can be defined analogously as before also for holomorphic functions:

**Lemma 4.13** *Assume that $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ is $\mathbb{C}^*$-invariant in the sense that $\hat{f} \triangleleft (\alpha \mathbb{1}_{1+n}, \alpha^{-1}\mathbb{1}_{1+n}) = \hat{f}$ holds for all $\alpha \in \mathbb{C}^*$. Then there exists a unique $\hat{f}_{\hat{\mathrm{red}}} \in \mathscr{O}(\hat{M}_{\mathrm{red}})$ for which*

$$\hat{\iota}^*(\hat{f}) = \hat{\mathrm{pr}}^*(\hat{f}_{\hat{\mathrm{red}}})$$

*holds.*

PROOF: As $\hat{\iota}^*(\hat{f})$ is $\mathbb{C}^*$-invariant, it descends to a well-defined function $\hat{f}_{\hat{\mathrm{red}}}$ on $\hat{M}_{\mathrm{red}} = \hat{Z}/\mathbb{C}^*$, which is automatically holomorphic. Uniqueness of $\hat{f}_{\hat{\mathrm{red}}}$ is clear. □

**Proposition 4.14** *The map $\cdot_{\mathrm{red}}$ restricts to a map $\mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \to \mathscr{A}(M_{\mathrm{red}})$. More precisely, given $f \in \mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ and $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ such that $\Delta^*(\hat{f}) = f$, then $\hat{f}$ is $\mathbb{C}^*$-invariant in the sense of the previous Lemma 4.13 and $f_{\mathrm{red}} = \Delta_{\mathrm{red}}^*(\hat{f}_{\hat{\mathrm{red}}}) \in \mathscr{A}(M_{\mathrm{red}})$.*

PROOF: Given such $f$ and $\hat{f}$, then

$$\Delta^*\big(\hat{f} \triangleleft (\mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_{1+n}, \mathrm{e}^{-\mathrm{i}\phi}\mathbb{1}_{1+n})\big) = \Delta^*(\hat{f}) \triangleleft \mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_{1+n} = f \triangleleft \mathrm{e}^{\mathrm{i}\phi}\mathbb{1}_{1+n} = f = \Delta^*(\hat{f})$$

holds for all $\phi \in \mathbb{R}$, so $\hat{f}$ is U(1)-invariant because $\Delta^*$ is an isomorphism between $\mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ and $\mathscr{A}(\mathbb{C}^{1+n})$. But since the action of the complex Lie group $\mathbb{C}^*$ on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ is holomorphic, $\hat{f}$ is even $\mathbb{C}^*$-invariant. Using the commutativity of the diagram in Section 3, one can now check that

$$\mathrm{pr}^*\big(\Delta_{\mathrm{red}}^*(\hat{f}_{\hat{\mathrm{red}}})\big) = \Delta_Z^*\big(\hat{\mathrm{pr}}^*(\hat{f}_{\hat{\mathrm{red}}})\big) = \Delta_Z^*\big(\hat{\iota}^*(\hat{f})\big) = \iota^*\big(\Delta^*(\hat{f})\big) = \iota^*(f)$$

holds, hence $f_{\mathrm{red}} = \Delta_{\mathrm{red}}^*(\hat{f}_{\hat{\mathrm{red}}}) \in \mathscr{A}(M_{\mathrm{red}})$. □

Using some deep results from complex analysis, the analytic functions on $M_{\mathrm{red}}$ and on $\mathbb{C}^{1+n}$ can be related in the same way as smooth or polynomial functions:

**Lemma 4.15** *For every $g \in \mathscr{A}(M_{\mathrm{red}})$ there exists an $f \in \mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ such that $f_{\mathrm{red}} = g$.*

PROOF: Given $g \in \mathscr{A}(M_{\mathrm{red}})$ and corresponding $\hat{g} \in \mathscr{O}(\hat{M}_{\mathrm{red}})$ such that $\Delta^*_{\mathrm{red}}(\hat{g}) = g$, then $\hat{\mathrm{pr}}^*(\hat{g})$ is a holomorphic function on $\hat{Z}$. Now note that $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ is a Stein manifold by [15, Sec. 5.1] and that $\hat{Z}$ is—in the language of [15, Def. 6.5.1]—an analytic submanifold thereof because it is the set of zeros of a holomorphic function on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. So [15, Thm. 7.4.8] applies and shows that there exists an extension $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ of $\hat{\mathrm{pr}}^*(\hat{g})$, i.e. $\hat{\iota}^*(\hat{f}) = \hat{\mathrm{pr}}^*(\hat{g})$. Therefore $f := \Delta^*(\hat{f})$ fulfils $\iota^*(f) = \mathrm{pr}^*(g)$ due to the commutativity of the diagram in Section 3. By averaging over the U(1)-action on $\mathscr{A}(\mathbb{C}^{1+n})$ we can even arrange that $f$ is U(1)-invariant. □

For an alternative proof one can also generalize the more constructive results obtained in [17, Sec. 3.2] for the case of signature $s = 1$, or use these results and the Wick rotation as discussed later in Section 6.

Clearly, $\{f \in \mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \,|\, \iota^*(f) = 0\}$ is the kernel of $\cdot_{\mathrm{red}}$ restricted to $\mathscr{A}(\mathbb{C}^{1+n})$ and therefore a closed $^*$-ideal of $\mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$. Similarly to the case of smooth or polynomial functions we get:

**Proposition 4.16** *The reduction map $\cdot_{\mathrm{red}}$ descends to a $^*$-isomorphism between the Fréchet $^*$-algebras $\mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}/\{f \in \mathscr{A}(\mathbb{C}^{1+n}) \,|\, \iota^*(f) = 0\}$ and $\mathscr{A}(M_{\mathrm{red}})$, that is in addition a homeomorphism.*

PROOF: Using Lemma 4.13 it is clear that that $\cdot_{\mathrm{red}}$ induces a $^*$-isomorphism. As $\|\hat{f}_{\hat{\mathrm{red}}}\|_K = \|\hat{f}\|_{B\cap\mathrm{pr}^*(K)}$ holds for every $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})^{\mathbb{C}^*}$ with $B \subseteq \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ a sufficiently large closed ball, the map $\cdot_{\hat{\mathrm{red}}} \colon \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})^{\mathbb{C}^*} \to \mathscr{O}(\hat{M}_{\mathrm{red}})$ from Lemma 4.13 is continuous with respect to the topologies of locally uniform convergence, thus $\cdot_{\mathrm{red}} \colon \mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \to \mathscr{A}(M_{\mathrm{red}})$ is continuous as well. It follows from the open mapping theorem that it is a homeomorphism. □

As the U(1)-invariant polynomials $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ are dense in $\mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, this immediately yields:

**Corollary 4.17** *The polynomials $\mathscr{P}(M_{\mathrm{red}})$ are dense in $\mathscr{A}(M_{\mathrm{red}})$.*

# 5 Poisson brackets and star products

In this section we introduce a Poisson bracket and star product on $\mathbb{C}^{1+n}$ and discuss their reduction to $M_{\mathrm{red}}$. First we consider formal star products, which make sense for formal power series of smooth functions. We present a method for reducing the (pseudo-)Wick product on $\mathbb{C}^{1+n}$ to $M_{\mathrm{red}}$ in Subsection 5.1 and derive more explicit formulas in Subsection 5.2. The other two sections deal with strict star products. In order to make the formal power series convergent, we restrict ourselves to polynomials in Subsection 5.3 and extend these results to analytic functions in Subsection 5.4.

## 5.1 The smooth case

We will now introduce the Wick star product on $\mathbb{C}^{1+n}$. The antisymmetrization of its first order gives rise to a Poisson structure on $\mathbb{C}^{1+n}$. Let $\nabla$ be the Euclidean covariant derivative of $\mathbb{C}^{1+n}$, $D$ its exterior covariant derivative and $D^{\mathrm{sym}}$ the corresponding symmetrized covariant derivative, see Appendix A for details. We define

$$H := \sum_{k=0}^{n} \sigma_k \frac{\partial}{\partial z^k} \otimes \frac{\partial}{\partial \overline{z}^k} \in \Gamma^\infty\big(\mathrm{T}^{(1,0)}\mathbb{C}^{1+n} \otimes \mathrm{T}^{(0,1)}\mathbb{C}^{1+n}\big) . \tag{5.1}$$

It is easy to see that $H$ is $\mathrm{U}(1)$-invariant, so that $H_{\mathrm{red}} \in \Gamma^\infty(\mathrm{T}^{(1,0)} M_{\mathrm{red}} \otimes \mathrm{T}^{(0,1)} M_{\mathrm{red}})$ can be defined as

$$H_{\mathrm{red}}\big|_{[\rho]} := (\mathrm{T}_\rho \operatorname{Pr})^{\otimes 2}(H|_\rho) \tag{5.2}$$

for all $[\rho] \in M_{\mathrm{red}}$ with representative $\rho \in Z$. An explicit formula for $H_{\mathrm{red}}$ in projective coordinates will be given later in Lemma 5.7. Using $H$ and symmetrized covariant derivatives, we can now define the well-known Wick star product:

**Definition 5.1** *The product*

$$\begin{aligned} \star \colon \mathscr{C}^\infty(\mathbb{C}^{1+n})[[\nu]] \times \mathscr{C}^\infty(\mathbb{C}^{1+n})[[\nu]] &\to \mathscr{C}^\infty(\mathbb{C}^{1+n})[[\nu]] , \\ (f,g) \mapsto f \star g &:= \sum_{r=0}^{\infty} \frac{\nu^r}{r!} \big\langle (D^{\mathrm{sym}})^r(f) \otimes (D^{\mathrm{sym}})^r(g), H^r \big\rangle \end{aligned} \tag{5.3}$$

*is the* (pseudo-)Wick star product[3] *on* $\mathbb{C}^{1+n}$. *Here* $H^r$ *denotes the* $r$*-th power of* $H$ *as an element of degree* $(1,1)$ *in the algebra* $\mathscr{S}^\bullet(\mathbb{C}^{1+n}) \otimes \mathscr{S}^\bullet(\mathbb{C}^{1+n})$ *with* $\mathscr{S}^\bullet(\mathbb{C}^{1+n}) := \bigoplus_{k=0}^\infty \Gamma^\infty(\mathrm{S}^k\, \mathrm{T}^{\mathbb{C}}\mathbb{C}^{1+n})$ *the algebra of complex symmetric multivector fields.*

Note that one can check that $\star$ is actually a $\mathrm{G}_{\mathcal{J}}$-invariant Hermitian formal star product constructed out of the bidifferential operators

$$C_r(f,g) = \frac{1}{r!} \big\langle (D^{\mathrm{sym}})^r(f) \otimes (D^{\mathrm{sym}})^r(g), H^r \big\rangle . \tag{5.4}$$

It deforms in direction of the Poisson bracket[4] with signature $s$

$$\frac{1}{\mathrm{i}}\big(C_1(f,g) - C_1(g,f)\big) = \frac{1}{\mathrm{i}} \sum_{k=0}^{n} \sigma_k \left( \frac{\partial f}{\partial z^k} \frac{\partial g}{\partial \overline{z}^k} - \frac{\partial g}{\partial z^k} \frac{\partial f}{\partial \overline{z}^k} \right) =: \{f,g\} \tag{5.5}$$

on $\mathbb{C}^{1+n}$ with Poisson tensor

$$\pi = -2\mathrm{i} \sum_{k=0}^{n} \sigma_k \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial \overline{z}^k} = 2 \operatorname{\mathsf{Im}}(H) , \tag{5.6}$$

where $\{f,g\} = \langle \mathrm{d}f \otimes \mathrm{d}g, \pi \rangle = \langle D^{\mathrm{sym}} f \otimes D^{\mathrm{sym}} g, \pi \rangle$. Note that (5.6) implies that $\pi$ is a real tensor.

---

[3] For signature $s = 1+n$ this product coincides (up to a rescaling of the Poisson bivector explained below) with the Wick product from Example Intro.2.9. Writing it with symmetrized covariant derivatives will be convenient for the reduction to $M_{\mathrm{red}}$.

[4] Note that from standard coordinates $z^0, \dots, z^{1+n}$ we get real coordinates $x^i = \operatorname{\mathsf{Re}}(z^i)$ and $p^i = \operatorname{\mathsf{Im}}(z^i)$. Then $\{x^i, p^j\} = \frac{1}{2}\delta^{i,j}$, so this is one half of the standard Poisson bracket. With this rescaling the set of poles of the strict star product becomes $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.

**Lemma 5.2** *The Poisson bracket* (5.5) *fulfils the condition for reducibility of Proposition 4.6.*

PROOF: First, $\{\,\cdot\,,\,\cdot\,\}$ is bidifferential, hence can be restricted to $\mathbb{C}_+^{1+n}$. As $H$ is U(1)-invariant, the Poisson bracket is U(1)-invariant, too. One also finds that $\{f, \mathcal{J}\} = X_{\mathrm{i}}(f)$ for all $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n})$, with $X_{\mathrm{i}}$ the generator of the U(1)-action as before. So if $f, g \in \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ are U(1)-invariant, then $\{f(\mathcal{J}-1), g\} = \{f, g\}(\mathcal{J}-1) - fX_{\mathrm{i}}(g) = \{f, g\}(\mathcal{J}-1)$ vanishes on $Z$, and similarly $\{f, g(\mathcal{J}-1)\}|_Z = 0$. □

Thus we can construct a reduced Poisson bracket on $M_{\mathrm{red}}$ by application of Definition 4.5 and get:

**Proposition 5.3** *For all $f, g \in \mathscr{C}^\infty(M_{\mathrm{red}})$ and $\rho \in Z$ the reduced Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}} : \mathscr{C}^\infty(M_{\mathrm{red}}) \times \mathscr{C}^\infty(M_{\mathrm{red}}) \to \mathscr{C}^\infty(M_{\mathrm{red}})$ is given by*

$$\{f, g\}_{\mathrm{red}}([\rho]) = \big\{\mathrm{Pr}^*(f), \mathrm{Pr}^*(g)\big\}(\rho) = \big\langle \mathrm{d}f \otimes \mathrm{d}g|_{[\rho]}, (\mathrm{T}_\rho \mathrm{Pr})^{\otimes 2}\pi|_\rho \big\rangle \tag{5.7}$$

*and the corresponding Poisson tensor $\pi_{\mathrm{red}}$ on $M_{\mathrm{red}}$ is simply*

$$\pi_{\mathrm{red}} = 2\,\mathsf{Im}(H_{\mathrm{red}})\,. \tag{5.8}$$

PROOF: Equation (5.7) is clear and (5.8) then follows from (5.2) and (5.6). □

This is just the Poisson-algebraic analog of the Marsden–Weinstein reduction scheme. However, the situation is a bit more difficult if one tries to reduce the bidifferential operators $C_r$ defining the Wick star product. One immediately sees that Proposition 4.4 cannot be applied directly: For example, $C_1(\mathcal{J}, \mathcal{J}-1) = \mathcal{J} \neq 0$. Following [7], this problem can be overcome by restricting to $\mathbb{C}_+^{1+n}$ and performing an equivalence transformation $S = \mathsf{id} + \sum_{k=1}^\infty \nu^k S_k$, with differential operators $S_k : \mathscr{C}^\infty(\mathbb{C}_+^{1+n}) \to \mathscr{C}^\infty(\mathbb{C}_+^{1+n})$ that vanish on constant functions, from $\star$ to a suitable new star product $\tilde{\star}$, i.e. $f \mathbin{\tilde{\star}} f' := S(S^{-1}(f) \star S^{-1}(f'))$, in such a way that $\tilde{\star}$ is reducible to a star product $\star_{\mathrm{red}}$ on $M_{\mathrm{red}}$ by application of Proposition 4.6. If this can be achieved, then $\mathrm{pr}^*(g \star_{\mathrm{red}} g') = (\mathrm{Pr}^*(g) \mathbin{\tilde{\star}} \mathrm{Pr}^*(g'))|_Z$ for all $g, g' \in \mathscr{C}^\infty(M_{\mathrm{red}})$. For this we require the following:

*i.)* $S$ should commute with $\overline{\,\cdot\,}$, since then $\tilde{\star}$ is again a Hermitian star product.

*ii.)* $S$ should be $\mathrm{G}_{\mathcal{J}}$-invariant, since then $\tilde{\star}$ is again $\mathrm{G}_{\mathcal{J}}$-invariant.

*iii.)* Moreover, $\tilde{\star}$ should fulfil $\mathcal{J} \mathbin{\tilde{\star}} f = \mathcal{J}f$ for all $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, hence also $f \mathbin{\tilde{\star}} \mathcal{J} = \overline{\mathcal{J} \mathbin{\tilde{\star}} \overline{f}} = \overline{\mathcal{J}\overline{f}} = f\mathcal{J}$ for all $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$. As a consequence, Proposition 4.6 can be applied to the bidifferential operator defining the $r$-th order of $\tilde{\star}$ for any $r$, so that $\star_{\mathrm{red}}$ as described above is indeed well-defined.

*iv.)* Finally, it would be helpful if $S$ (hence also $S^{-1}$) acts as the identity on $\mathbb{C}^*$-invariant functions, because this has the consequence that the formula for $\star_{\mathrm{red}}$ simplifies to

$$\mathrm{pr}^*(g \star_{\mathrm{red}} g') = \big(\mathrm{Pr}^*(g) \mathbin{\tilde{\star}} \mathrm{Pr}^*(g')\big)\big|_Z = \big(S\big(\mathrm{Pr}^*(g) \star \mathrm{Pr}^*(g')\big)\big)\big|_Z \tag{5.9}$$

for all $g, g' \in \mathscr{C}^\infty(M_{\mathrm{red}})$.

Let us define the rescaled vector field

$$\frac{\partial}{\partial \mathcal{J}} \coloneqq \frac{1}{2\mathcal{J}} X_{\mathbb{1}} \in \Gamma^\infty(\mathrm{T}\mathbb{C}^{1+n}_+) \tag{5.10}$$

on $\mathbb{C}^{1+n}_+$, which satisfies $\frac{\partial}{\partial \mathcal{J}}\mathcal{J} = 1$. Then Properties *i.)*, *ii.)* and *iv.)* are fulfilled if all the differential operators $S_k$ with $k \in \mathbb{N}$ are of the form $S_k = \sum_{\ell=1}^\infty (S_{k,\ell} \circ \mathcal{J})(\frac{\partial}{\partial \mathcal{J}})^\ell$ with smooth functions $S_{k,\ell} \colon (0,\infty) \to \mathbb{R}$, such that for every fixed $k \in \mathbb{N}$ there are only finitely many $\ell \in \mathbb{N}$ with $S_{k,\ell} \neq 0$.

We are interested in the inverse equivalence transformation $T = S^{-1}$, which then also contains only derivatives $\frac{\partial}{\partial \mathcal{J}}$ and coefficient functions dependent on $\mathcal{J}$, i.e. $T = \mathsf{id} + \sum_{k,\ell=1}^\infty \nu^k (T_{k,\ell} \circ \mathcal{J})(\frac{\partial}{\partial \mathcal{J}})^\ell$. Therefore recall the usual definition of the *falling* and *rising factorial* as

$$(\xi)_{\downarrow,r} \coloneqq \prod_{k=0}^{r-1}(\xi - k) \qquad \text{and} \qquad (\xi)_{\uparrow,r} \coloneqq \prod_{k=0}^{r-1}(\xi + k)\,, \tag{5.11}$$

respectively, for all elements $\xi$ of a ring with unit and all $r \in \mathbb{N}_0$. Here the empty product is of course $(\xi)_{\downarrow,0} = 1 = (\xi)_{\uparrow,0}$. For formal Laurent series in $\nu$ over the smooth functions $\mathscr{C}^\infty(M)$ of a manifold $M$ and a pointwise invertible $f \in \mathscr{C}^\infty(M)$ we see that

$$\frac{1}{(f/\nu)_{\downarrow,r}} = \frac{\nu^r}{\prod_{k=0}^{r-1}(f - k\nu)} \qquad \text{and} \qquad \frac{1}{(f/\nu)_{\uparrow,r}} = \frac{\nu^r}{\prod_{k=0}^{r-1}(f + k\nu)} \tag{5.12}$$

are actually formal power series because $f \pm k\nu \in \mathscr{C}^\infty(M)[[\nu]]$ are invertible.

**Proposition 5.4** *Let $T$ be a $\mathrm{U}(1)$-invariant equivalence transformation on $\mathbb{C}^{1+n}_+$ from a new star product $\tilde{\star}$ to $\star$, then the following is equivalent:*

- *$T(\mathcal{J}) = \mathcal{J}$ and $\mathcal{J} \,\tilde{\star}\, f = \mathcal{J} f$ for all $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$.*
- *$[T, \mathcal{J}](f) = \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T(f)$ for all $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$, where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator.*

*If $T$ fulfils one, hence both of these conditions, then*

$$T\big(\nu^r (\mathcal{J}/\nu)_{\downarrow,r}\big) = \mathcal{J}^r \qquad \textit{and} \qquad T\bigg(\frac{\mathcal{J}}{\nu^{r+1}(\mathcal{J}/\nu)_{\uparrow,r+1}}\bigg) = \mathcal{J}^{-r} \tag{5.13}$$

*for all $r \in \mathbb{N}_0$.*

PROOF: Assume $T(\mathcal{J}) = \mathcal{J}$ and $\mathcal{J} \,\tilde{\star}\, f = \mathcal{J} f$ for some $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$, then

$$\begin{aligned} T(\mathcal{J} f) = T(\mathcal{J} \,\tilde{\star}\, f) = T(\mathcal{J}) \star T(f) = \\ = \mathcal{J} \star T(f) = \big(\mathcal{J} + \nu \overline{E}\big) T(f) = \bigg(\mathcal{J} + \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}}\bigg) T(f) \end{aligned}$$

and so $[T, \mathcal{J}](f) = \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T(f)$. Conversely, if $[T, \mathcal{J}](f) = \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T(f)$ for all $f \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathrm{U}(1)}$, then especially for $f = 1$ one gets $T(\mathcal{J}) - \mathcal{J} T(1) = \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T(1)$, i.e.

$T(\mathcal{J}) - \mathcal{J} = 0$ because $T(1) = 1$ for the equivalence transformation $T$. Then one also checks that

$$\mathcal{J} \mathbin{\tilde{\star}} f = T^{-1}\left(\left(\mathcal{J} + \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}}\right) T(f)\right) = T^{-1}\big(\mathcal{J} T(f) + [T, \mathcal{J}](f)\big) = \mathcal{J} f \,.$$

Moreover, by induction one finds that indeed $T(\nu^r (\mathcal{J}/\nu)_{\downarrow,r}) = \mathcal{J}^r$ for all $r \in \mathbb{N}_0$: For $r = 0$ this is just $T(1) = 1$, and if it holds for one $r \in \mathbb{N}_0$, then

$$\begin{aligned}
T\big(\nu^{r+1} (\mathcal{J}/\nu)_{\downarrow,r+1}\big) &= \nu T\big(\nu^r (\mathcal{J}/\nu)_{\downarrow,r} (\mathcal{J}/\nu - r)\big) \\
&= [T, \mathcal{J}]\big(\nu^r (\mathcal{J}/\nu)_{\downarrow,r}\big) + \mathcal{J} T\big(\nu^r (\mathcal{J}/\nu)_{\downarrow,r}\big) - \nu r T\big(\nu^r (\mathcal{J}/\nu)_{\downarrow,r}\big) \\
&= \left(\nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} + \mathcal{J} - \nu r\right) T\big(\nu^r (\mathcal{J}/\nu)_{\downarrow,r}\big) \\
&= \left(\nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} + \mathcal{J} - \nu r\right) \mathcal{J}^r \\
&= \mathcal{J}^{r+1} \,.
\end{aligned}$$

In order to check the formula for $\mathcal{J}^{-r}$, we note first that

$$\begin{aligned}
(\mathcal{J} + r\nu) T\big(\mathcal{J} (\nu^{r+1} (\mathcal{J}/\nu)_{\uparrow,r+1})^{-1}\big) &= \\
&= [\mathcal{J} + r\nu, T]\big(\mathcal{J} (\nu^{r+1} (\mathcal{J}/\nu)_{\uparrow,r+1})^{-1}\big) + T\big(\mathcal{J} (\nu^r (\mathcal{J}/\nu)_{\uparrow,r})^{-1}\big) \\
&= -\nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T\big(\mathcal{J} (\nu^{r+1} (\mathcal{J}/\nu)_{\uparrow,r+1})^{-1}\big) + T\big(\mathcal{J} (\nu^r (\mathcal{J}/\nu)_{\uparrow,r})^{-1}\big) \,,
\end{aligned}$$

so

$$\left(\mathcal{J} + r\nu + \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}}\right) T\big(\mathcal{J} (\nu^{r+1} (\mathcal{J}/\nu)_{\uparrow,r+1})^{-1}\big) = T\big(\mathcal{J} (\nu^r (\mathcal{J}/\nu)_{\uparrow,r})^{-1}\big) \,.$$

Since $\mathcal{J}$ is an invertible function on $\mathbb{C}^{1+n}_+$ it follows that $\mathcal{J} + r\nu + \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}}$ is invertible on $\mathscr{C}^\infty(\mathbb{C}^{1+n}_+)[[\nu]]$. Since

$$\left(\mathcal{J} + r\nu + \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}}\right) \mathcal{J}^{-r} = \mathcal{J}^{-r+1} + r\nu \mathcal{J}^{-r} - r\nu \mathcal{J}^{-r} = \mathcal{J}^{-r+1}$$

for all $r \in \mathbb{N}$, we obtain $\big(\mathcal{J} + r\nu + \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}}\big)^{-1} (\mathcal{J}^{-r+1}) = \mathcal{J}^{-r}$. The statement now follows by induction because the base case $r = 0$ reduces to $T(1) = 1$ and is therefore fulfilled. □

**Proposition 5.5** *There exists a unique equivalence transformation $T$ on $\mathbb{C}^{1+n}_+$ of the form*

$$T = \mathsf{id} + \sum_{k=1}^{\infty} \sum_{\ell=1}^{2k} \nu^k (T_{k,\ell} \circ \mathcal{J}) \left(\frac{\partial}{\partial \mathcal{J}}\right)^\ell \tag{5.14}$$

*with $T_{k,\ell} \in \mathscr{C}^\infty((0,\infty))$ that has the properties from the previous Proposition 5.4. Its inverse $S = T^{-1}$ thus has all the properties i.) to iv.) discussed above and additionally fulfils $S(\mathcal{J}) = \mathcal{J}$.*

PROOF: By collecting terms in $\nu^k$ and $\big(\frac{\partial}{\partial \mathcal{J}}\big)^\ell$, the identity $[T, \mathcal{J}] = \nu \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T$ with $T$ like in (5.14) is equivalent to

$$T_{k+1,\ell+1} \circ \mathcal{J} = \frac{\mathcal{J}}{\ell+1}\big((T'_{k,\ell} + T_{k,\ell-1}) \circ \mathcal{J}\big)$$

for all $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$ with initial conditions $T_{0,0} = 1$ and $T_{0,\ell} = 0 = T_{k,0}$ for all $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, where $T'_{k,\ell} \in \mathscr{C}^\infty((0,\infty))$ is the derivative of $T_{k,\ell}$ and where $T_{k,-1} := 0$ for all $k \in \mathbb{N}_0$. □

So the equivalence transformation $S$ exists and is uniquely determined if we add to the four requirements *i.)* to *iv.)* above the fifth requirement that $S(\mathcal{J}) = \mathcal{J}$, which is just a convenience. We can now construct the reduced star product on $M_{\mathrm{red}}$:

**Definition 5.6** *The* transformed star product $\tilde{\star}$ *on* $\mathbb{C}^{1+n}_+$ *is the one obtained from* $\star$ *by application of the equivalence transformation* $S = T^{-1}$ *with* $T$ *like in Proposition 5.5. Explicitly,*

$$f \mathbin{\tilde{\star}} g = S\big(T(f) \star T(g)\big) \tag{5.15}$$

*for all* $f, g \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)[[\nu]]$. *Moreover, the* reduced star product $\star_{\mathrm{red}}$ *on* $M_{\mathrm{red}}$ *is defined as*

$$f \star_{\mathrm{red}} g := \sum_{r=0}^{\infty} \nu^r \tilde{C}_{r,\mathrm{red}}(f, g) \tag{5.16}$$

*for all* $f, g \in \mathscr{C}^\infty(M_{\mathrm{red}})$ *and extended to formal power series in* $\nu$*, where* $\tilde{C}_{r,\mathrm{red}}$ *on* $M_{\mathrm{red}}$ *are the reductions like in Definition 4.5 of the bidifferential operators* $\tilde{C}_r$ *on* $\mathbb{C}^{1+n}_+$ *that describe the transformed star product* $\tilde{\star}$ *on* $\mathbb{C}^{1+n}_+$.

Using the defining properties of the reduced bilinear maps $\tilde{C}_{r,\mathrm{red}}$ it is easy to check that $\star_{\mathrm{red}}$ is again associative and it is clear that the constant 1-function is the neutral element. It also follows from the construction that $\tilde{C}_{r,\mathrm{red}}$ are bidifferential operators on $M_{\mathrm{red}}$, but we will also show this by giving an explicit formula in the next subsection. As $T$ and thus also $S$ commute with the pointwise complex conjugation and the action of $\mathrm{G}_{\mathcal{J}}$, both $\tilde{\star}$ and $\star_{\mathrm{red}}$ are Hermitian and $\mathrm{G}_{\mathcal{J}}$-invariant. Note also that $\tilde{\star}$ still deforms in direction of the original Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ (or rather, its restriction to $\mathbb{C}^{1+n}_+$), so that it is easy to check that $\star_{\mathrm{red}}$ deforms in direction of the reduced Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ on $M_{\mathrm{red}}$.

## 5.2 Explicit formulae

We want to obtain an explicit expression for the reduced Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ and star product $\star_{\mathrm{red}}$ in terms of bidifferential operators on $M_{\mathrm{red}}$.

**Lemma 5.7** *The restriction to* $\mathbb{C}^{1+n}_+$ *of the tensor* $H$ *can be expressed as*

$$H\big|_{\mathbb{C}^{1+n}_+} = \frac{1}{\mathcal{J}} E \otimes \overline{E} + H_\Xi \tag{5.17}$$

*with some* $H_\Xi \in \Gamma^\infty(\Xi^{\mathbb{C}} \otimes \Xi^{\mathbb{C}})$*. Explicitly,*

$$H_\Xi = \frac{1}{z^0\overline{z}^0}\Bigg(\sum_{k,\ell=1}^{n} \frac{z^k\overline{z}^\ell}{z^0\overline{z}^0} W_k \otimes \overline{W}_\ell + \sum_{k=1}^{n} \sigma_k W_k \otimes \overline{W}_k\Bigg) \tag{5.18}$$

*on the domain of definition of the vector fields* $W_1, \dots, W_n$ *and consequently*

$$H_{\mathrm{red}} = \Bigg(1 + \sum_{k=1}^{n} \sigma_k w^k \overline{w}^k\Bigg)\Bigg(\sum_{k,\ell=1}^{n} w^k\overline{w}^\ell \frac{\partial}{\partial w^k} \otimes \frac{\partial}{\partial \overline{w}^\ell} + \sum_{k=1}^{n} \sigma_k \frac{\partial}{\partial w^k} \otimes \frac{\partial}{\partial \overline{w}^k}\Bigg) \tag{5.19}$$

*in projective coordinates on* $M_{\mathrm{red}}$*.*

PROOF: The first part is an easy computation using (3.24). The formula for $H_{\mathrm{red}}$ then follows since $(z^0\overline{z}^0)^{-1}|_Z = (\mathcal{J}/|z^0|^2)|_Z = \mathrm{Pr}^*(1 + \sum_{k=1}^n \sigma_k w^k \overline{w}^k)|_Z$, $(\mathrm{T}_\rho \mathrm{Pr})(W_k|_\rho) = (\mathrm{T}_\rho \mathrm{pr})(W_k|_\rho) = \frac{\partial}{\partial w^k}|_{[\rho]}$ by Proposition 3.5, and $(\mathrm{T}_\rho \mathrm{Pr})(E|_\rho) = 0$. □

As an immediate consequence we get from (5.8):

**Proposition 5.8** *The reduced Poisson tensor that determines* $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ *is*

$$\pi_{\mathrm{red}} = -2\mathrm{i}\Bigg(1 + \sum_{k=1}^{n} \sigma_k w^k \overline{w}^k\Bigg)\Bigg(\sum_{k,\ell=1}^{n} w^k\overline{w}^\ell \frac{\partial}{\partial w^k} \wedge \frac{\partial}{\partial \overline{w}^\ell} + \sum_{k=1}^{n} \sigma_k \frac{\partial}{\partial w^k} \wedge \frac{\partial}{\partial \overline{w}^k}\Bigg) \tag{5.20}$$

*in projective coordinates.*

For the signature $s = 1{+}n$, this is the usual Poisson tensor associated to the symplectic Fubini–Study form on $M_{\mathrm{red}}^{(1+n)} \cong \mathbb{CP}^n$. If $s = 1$, then one obtains (up to a sign) the Poisson tensor associated to the symplectic Fubini–Study form on the hyperbolic disc $M_{\mathrm{red}}^{(1)} \cong \mathbb{D}^n$.

Similarly to the Wick star product from Definition 5.1, the bidifferential operators defining the reduced star product should be expressed using symmetrized covariant derivatives. In order to define reduced symmetrized covariant derivatives we need the following:

**Definition 5.9** *We write* $\Theta_\Xi \colon \Gamma^\infty(\mathrm{T}^{\mathbb{C}}\mathbb{C}_+^{1+n}) \to \Gamma^\infty(\mathrm{T}^{\mathbb{C}}\mathbb{C}_+^{1+n})$ *for the projection on the subbundle* $\Xi^{\mathbb{C}}$ *of* $\mathrm{T}^{\mathbb{C}}\mathbb{C}_+^{1+n}$ *associated to the complexification of the decomposition* $\mathrm{T}\mathbb{C}_+^{1+n} = \langle\!\langle X_{\mathbb{1}} \rangle\!\rangle \oplus \langle\!\langle X_{\mathrm{i}} \rangle\!\rangle \oplus \Xi$ *from Proposition 3.2. Moreover, its dual will be denoted by* $\Theta_\Xi^* \colon \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}\mathbb{C}_+^{1+n}) \to \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}\mathbb{C}_+^{1+n})$*.*

Note that $\Theta_\Xi$ commutes with the complex structure $I$ of $\mathbb{C}_+^{1+n}$. Like in Proposition A.6 and Proposition A.8 we can construct a reduced exterior covariant derivative and a reduced symmetrized covariant derivative on $M_{\mathrm{red}}$ out of $D$ and $D^{\mathrm{sym}}$ on $\mathbb{C}_+^{1+n}$, because $D$ and $D^{\mathrm{sym}}$ are $\mathbb{C}^*$-invariant (even invariant under arbitrary linear automorphisms of $\mathbb{C}^{1+n}$):

**Definition 5.10** *By* $D_{\mathrm{red}} \colon (\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M_{\mathrm{red}}) \to (\mathscr{A} \otimes \mathscr{S})^{\bullet+1,\bullet}(M_{\mathrm{red}})$ *we denote the* reduced exterior covariant derivative *on* $M_{\mathrm{red}}$*, which is the one that fulfils*

$$\mathrm{Pr}^*\big(D_{\mathrm{red}}\Omega\big) = (\Theta_\Xi^*)^{\otimes(k+1+\ell)} D\, \mathrm{Pr}^*(\Omega) \tag{5.21}$$

*for all* $\Omega \in (\mathscr{A} \otimes \mathscr{S})^{k,\ell}(M_{\mathrm{red}})$, $k, \ell \in \mathbb{N}_0$, *and analogously, the* reduced symmetrized covariant derivative $D^{\mathrm{sym}}_{\mathrm{red}} \colon \mathscr{S}^\bullet(M_{\mathrm{red}}) \to \mathscr{S}^{\bullet+1}(M_{\mathrm{red}})$ *on* $M_{\mathrm{red}}$ *is determined by*

$$\Pr{}^* \left( D^{\mathrm{sym}}_{\mathrm{red}} \omega \right) = (\Theta^*_\Xi)^{\otimes(k+1)} D^{\mathrm{sym}} \Pr{}^*(\omega) \tag{5.22}$$

*for all* $\omega \in \mathscr{S}^k(M_{\mathrm{red}})$, $k \in \mathbb{N}_0$.

We will give a more explicit characterization of the corresponding covariant derivative on $M_{\mathrm{red}}$ later in Proposition 5.18. Note that $D$ is compatible with the complex structure on $\mathbb{C}^{1+n}$ in the sense of Definition A.9, and thus splits into $(1,0)$ and $(0,1)$-components $D = D_{\mathrm{hol}} + D_{\overline{\mathrm{hol}}}$ as in Definition A.10, analogously $D^{\mathrm{sym}} = D^{\mathrm{sym}}_{\mathrm{hol}} + D^{\mathrm{sym}}_{\overline{\mathrm{hol}}}$. This carries over to the reduced derivatives:

**Proposition 5.11** *The reduced exterior covariant derivative* $D_{\mathrm{red}}$ *is compatible with the complex structure and*

$$\Pr{}^*(D^{\mathrm{sym}}_{\mathrm{red,hol}} \omega) = (\Theta^*_\Xi)^{\otimes(k+1)} D^{\mathrm{sym}}_{\mathrm{hol}} \Pr{}^*(\omega) \tag{5.23}$$

*holds for all* $\omega \in \mathscr{S}^k(M_{\mathrm{red}})$.

PROOF: As a consequence of Proposition 3.2 the projection $\Theta^*_\Xi$ commutes with the complex structure $I$ on $\mathbb{C}^{1+n}_+$ and therefore $(\Theta^*_\Xi)^{\otimes(p+q)}$ commutes with the projection onto symmetric tensors of degree $(p,q)$. The projection onto such tensors also commutes with $\Pr^*$ since $\Pr$ is holomorphic. Therefore $D_{\mathrm{red}}$ is compatible with the complex structure and (5.23) follows immediately from (5.22). □

We can now formulate the main theorem of this section:

**Theorem 5.12** *The reduced Wick star product is*

$$f \star_{\mathrm{red}} g = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{(1/\nu)_{\downarrow,r}} \left\langle (D^{\mathrm{sym}}_{\mathrm{red}})^r f \otimes (D^{\mathrm{sym}}_{\mathrm{red}})^r g, H^r_{\mathrm{red}} \right\rangle \tag{5.24}$$

*for all* $f, g \in \mathscr{C}^\infty(M_{\mathrm{red}})$, *where* $H_{\mathrm{red}}|_{[\rho]} = (\mathrm{T}_\rho \Pr)^{\otimes 2} H|_\rho$ *was computed in Lemma 5.7. Moreover, if* $f \in \mathscr{P}(M_{\mathrm{red}})$ *or* $g \in \mathscr{P}(M_{\mathrm{red}})$, *then the series in* $r$ *in the product* $f \star_{\mathrm{red}} g$ *has only finitely many non-zero terms.*

Note that for complex projective spaces and hyperbolic discs this formula coincides (up to rescaling the formal parameter) with the formula derived in [18, Thm. 3.2.4] for a Fedosov star product with form $\Omega = 0$. For the proof of Theorem 5.12 we have to collect some intermediate results:

**Lemma 5.13** *On* $\mathbb{C}^*$*-invariant functions* $f, g \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathbb{C}^*}$ *the transformed Wick star product can be expressed as*

$$f \mathbin{\tilde{\star}} g = S(f \star g) = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\mathcal{J}/\nu}{(\mathcal{J}/\nu)_{\uparrow,r+1}} \left\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, H^r \mathcal{J}^r \right\rangle. \tag{5.25}$$

PROOF: The first equality in (5.25) follows from requirement *iv.)* for the equivalence transformation. For the second one we use that we can express $f \star g$ as

$$f \star g = \sum_{r=0}^{\infty} \frac{\nu^r}{\mathcal{J}^r r!} \big\langle (D^{\mathrm{sym}})^r f \otimes (D^{\mathrm{sym}})^r g, H^r \mathcal{J}^r \big\rangle \,.$$

Note that $\langle (D^{\mathrm{sym}})^r f \otimes (D^{\mathrm{sym}})^r g, H^r \mathcal{J}^r \rangle$ is $\mathbb{C}^*$-invariant, so that all its derivatives $\frac{\partial}{\partial \mathcal{J}}$ vanish. Now $\langle (D^{\mathrm{sym}})^r f \otimes (D^{\mathrm{sym}})^r g, H^r \mathcal{J}^r \rangle = \langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, H^r \mathcal{J}^r \rangle$ because of Proposition A.11 and since the first tensor factor of $H$ lies in $\mathrm{T}^{(1,0)}\mathbb{C}^{1+n}$ and the second one in $\mathrm{T}^{(0,1)}\mathbb{C}^{1+n}$. Then it only remains to apply the formula for $S(\mathcal{J}^{-r})$ from Proposition 5.4. □

If we restrict (5.25) to $Z$, we can substitute $\mathcal{J}$ by 1. In order to express $\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, H^r \rangle|_Z$ with $\mathbb{C}^*$-invariant functions $f$ and $g$ by differential operators on $M_{\mathrm{red}}$, we use formula (5.17) for $H$ and explicitly calculate the contribution of the vertical directions $E$ and $\overline{E}$:

**Lemma 5.14** *For $\mathbb{C}^*$-invariant $s \in \Gamma^\infty(\mathrm{S}^k\, \mathrm{T}^{*,(1,0)}\mathbb{C}^{1+n}_+)^{\mathbb{C}^*}$ and $k \in \mathbb{N}_0$, we have $[\iota_E, D^{\mathrm{sym}}_{\mathrm{hol}}](s) = -2ks$.*

PROOF: Let $k \in \mathbb{N}_0$ be given. For a multiindex $P \in \mathbb{N}_0^{1+n}$ we write $(\mathrm{d}z)^P := (\mathrm{d}z^0)^{P_0} \vee \cdots \vee (\mathrm{d}z^n)^{P_n}$, then a general element $s \in \Gamma^\infty(\mathrm{S}^k\, \mathrm{T}^{*,(1,0)}\mathbb{C}^{1+n}_+)^{\mathbb{C}^*}$ can be expanded as $s = \sum_{P \in \mathbb{N}_0^{1+n}, |P|=k} s_P (\mathrm{d}z)^P$ with coefficient functions $s_P \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)$. Note that $\mathbb{C}^*$-invariance of $s$ implies that its Lie derivative with $E$ vanishes, $\mathscr{L}_E s = 0$. As $\mathscr{L}_E \,\mathrm{d}z^\ell = \mathrm{d}z^\ell$ for all $\ell \in \{0, \dots, n\}$, hence $\mathscr{L}_E (\mathrm{d}z)^P = k(\mathrm{d}z)^P$, it follows that $\mathscr{L}_E(s_P) = -ks_P$ and thus

$$[\iota_E, D^{\mathrm{sym}}_{\mathrm{hol}}](s_P) = \iota_E\, \mathrm{d}(s_P) = \mathscr{L}_E(s_P) = -ks_P \,.$$

Moreover, $[\iota_E, D^{\mathrm{sym}}_{\mathrm{hol}}]\, \mathrm{d}z^\ell = -D^{\mathrm{sym}}_{\mathrm{hol}} \iota_E \,\mathrm{d}z^\ell = -\,\mathrm{d}z^\ell$ for all $\ell \in \{0, \dots, n\}$. As both $\iota_E$ and $D^{\mathrm{sym}}_{\mathrm{hol}}$ are derivations, their commutator $[\iota_E, D^{\mathrm{sym}}_{\mathrm{hol}}]$ is also a derivation and we obtain $[\iota_E, D^{\mathrm{sym}}_{\mathrm{hol}}]s = -2ks$. □

**Lemma 5.15** *For $f, g \in \mathscr{C}^\infty(\mathbb{C}^{1+n}_+)^{\mathbb{C}^*}$ and $r \in \mathbb{N}$ we have*

$$\iota_E (D^{\mathrm{sym}}_{\mathrm{hol}})^r f = -r(r-1)(D^{\mathrm{sym}}_{\mathrm{hol}})^{r-1} f \quad \text{and} \quad \iota_{\overline{E}} (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g = -r(r-1)(D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^{r-1} g \tag{5.26}$$

*as well as*

$$\begin{aligned} &\big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, H^r \big\rangle\big|_Z = \\ &\qquad = \sum_{k=1}^{r} \frac{r!(r-k)!}{k!} \binom{r-1}{k-1}^2 \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k g, (H_\Xi)^k \big\rangle\big|_Z \,, \end{aligned} \tag{5.27}$$

*where $H_\Xi$ is the component of $H$ in $\Xi^{\mathbb{C}} \otimes \Xi^{\mathbb{C}}$, as defined in (5.18).*

PROOF: For (5.26) it suffices to prove the first statement since the second one then follows by taking complex conjugates. Note that $(D^{\mathrm{sym}}_{\mathrm{hol}})^k f$ is $\mathbb{C}^*$-invariant, so the previous Lemma 5.14 yields

$$\begin{aligned}\iota_E (D^{\mathrm{sym}}_{\mathrm{hol}})^r f = \sum_{k=0}^{r-1} (D^{\mathrm{sym}}_{\mathrm{hol}})^{r-k-1} \big[\iota_E, D^{\mathrm{sym}}_{\mathrm{hol}}\big] (D^{\mathrm{sym}}_{\mathrm{hol}})^k f &= \\ = \sum_{k=0}^{r-1} (-2k)(D^{\mathrm{sym}}_{\mathrm{hol}})^{r-1} f &= -r(r-1)(D^{\mathrm{sym}}_{\mathrm{hol}})^{r-1} f\end{aligned}$$

for all $r \in \mathbb{N}_0$. With this and $H|_Z = E \otimes \overline{E}|_Z + H_\Xi|_Z$ from Lemma 5.7 we can now calculate

$$\begin{aligned}&\big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, H^r \big\rangle\big|_Z = \\ &= \sum_{k=0}^{r} \binom{r}{k} \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, (E \otimes \overline{E})^{r-k} (H_\Xi)^k \big\rangle\big|_Z \\ &\overset{(1)}{=} \sum_{k=0}^{r} \binom{r}{k} \left(\frac{k!}{r!}\right)^2 \big\langle (\iota_E)^{r-k} (D^{\mathrm{sym}}_{\mathrm{hol}})^r f \otimes (\iota_{\overline{E}})^{r-k} (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r g, (H_\Xi)^k \big\rangle\big|_Z \\ &\overset{(2)}{=} \sum_{k=1}^{r} \binom{r}{k} \left(\frac{(r-1)!}{(k-1)!}\right)^2 \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k g, (H_\Xi)^k \big\rangle\big|_Z \\ &= \sum_{k=1}^{r} \frac{r!(r-k)!}{k!} \binom{r-1}{k-1}^2 \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k f \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k g, (H_\Xi)^k \big\rangle\big|_Z \,.\end{aligned}$$

The factors appearing in step (1) are due to our conventions for the symmetric product, the dual pairing and the insertion derivation, see Equation (2.5). In (2) we used

$$(\iota_E)^{r-k} (D^{\mathrm{sym}}_{\mathrm{hol}})^r f = (-1)^{r-k} \frac{r!}{k!} \frac{(r-1)!}{(k-1)!} f$$

and its complex conjugate, which can be obtained by applying (5.26) several times. In the special case $k = 0$, (5.26) yields $(\iota_E)^r (D^{\mathrm{sym}}_{\mathrm{hol}})^r f = 0$. □

The combinatorial factors in (5.25) can be simplified using:

**Lemma 5.16** *For all $k \in \mathbb{N}$ the identity*

$$\sum_{s=0}^{\infty} \frac{1/\nu}{(1/\nu)_{\uparrow,k+s+1}} \binom{k+s-1}{k-1}^2 s! = \frac{1}{(1/\nu)_{\downarrow,k}} \tag{5.28}$$

*holds for a formal parameter $\nu$.*

PROOF: By multiplication with $\nu^{-k}$ we see that (5.28) is equivalent to the identity

$$\sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{k+s} (1+\nu\ell)} \binom{k+s-1}{k-1}^2 = \frac{1}{\prod_{\ell=1}^{k-1} (1-\nu\ell)} \tag{$*$}$$

of formal power series, which can be proven by induction over $k$: First note that

$$\sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{1+s}(1+\nu\ell)} = \sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{s}(1+\nu\ell)} \left(1 - \frac{\nu(s+1)}{1+\nu(s+1)}\right) =$$
$$= \sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{s}(1+\nu\ell)} - \sum_{s=0}^{\infty} \frac{\nu^{s+1}(s+1)!}{\prod_{\ell=1}^{s+1}(1+\nu\ell)},$$

which is a telescope sum that gives the result 1. So $(*)$ holds for $k=1$. Now assume that $(*)$ holds for some $k \in \mathbb{N}$, then we get for $k+1$:

$$\begin{aligned}
&\sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \binom{k+s}{k}^2 = \\
&= \frac{1}{1-\nu k} \sum_{s=0}^{\infty} \left( \frac{\nu^s s!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} - \frac{\nu^{s+1} s! k}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \right) \binom{k+s}{k}^2 \\
&= \frac{1}{1-\nu k} \sum_{s=0}^{\infty} \left( \frac{\nu^s s!}{\prod_{\ell=1}^{k+s}(1+\nu\ell)} \left(1 - \frac{\nu(k+s+1)}{1+\nu(k+s+1)}\right) \right. \\
&\qquad\qquad \left. - \frac{\nu^{s+1} s! k}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \right) \binom{k+s}{k}^2 \\
&= \frac{1}{1-\nu k} \sum_{s=0}^{\infty} \left( \frac{\nu^s s!}{\prod_{\ell=1}^{k+s}(1+\nu\ell)} - \frac{\nu^{s+1}(s+1)!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \frac{k+s+1}{s+1} \right. \\
&\qquad\qquad \left. - \frac{\nu^{s+1}(s+1)!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \frac{k}{s+1} \right) \binom{k+s}{k}^2 \\
&\overset{(**)}{=} \frac{1}{1-\nu k} \left( \frac{1}{\prod_{\ell=1}^{k}(1+\nu\ell)} \right. \\
&\qquad \left. + \sum_{s=0}^{\infty} \frac{\nu^{s+1}(s+1)!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \left( \frac{(k+s+1)^2}{(s+1)^2} - \frac{k+s+1}{s+1} - \frac{k}{s+1} \right) \binom{k+s}{k}^2 \right) \\
&= \frac{1}{1-\nu k} \left( \frac{1}{\prod_{\ell=1}^{k}(1+\nu\ell)} + \sum_{s=0}^{\infty} \frac{\nu^{s+1}(s+1)!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \binom{k+s}{k-1}^2 \right) \\
&= \frac{1}{1-\nu k} \left( \frac{1}{\prod_{\ell=1}^{k}(1+\nu\ell)} + \sum_{s=1}^{\infty} \frac{\nu^{s} s!}{\prod_{\ell=1}^{k+s}(1+\nu\ell)} \binom{k+s-1}{k-1}^2 \right) \\
&= \frac{1}{1-\nu k} \sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{k+s}(1+\nu\ell)} \binom{k+s-1}{k-1}^2 \\
&= \frac{1}{(1-\nu k)\prod_{\ell=1}^{k-1}(1-\nu\ell)} \\
&= \frac{1}{\prod_{\ell=1}^{k}(1-\nu\ell)}.
\end{aligned}$$

At $(**)$ we used that

$$
\begin{aligned}
&\sum_{s=0}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{k+s}(1+\nu\ell)} \binom{k+s}{k}^2 = \\
&\quad = \frac{1}{\prod_{\ell=1}^{k}(1+\nu\ell)} + \sum_{s=1}^{\infty} \frac{\nu^s s!}{\prod_{\ell=1}^{k+s}(1+\nu\ell)} \frac{(k+s)^2}{s^2} \binom{k+s-1}{k}^2 \\
&\quad = \frac{1}{\prod_{\ell=1}^{k}(1+\nu\ell)} + \sum_{s=0}^{\infty} \frac{\nu^{s+1} (s+1)!}{\prod_{\ell=1}^{k+s+1}(1+\nu\ell)} \frac{(k+s+1)^2}{(s+1)^2} \binom{k+s}{k}^2 . \qquad \square
\end{aligned}
$$

The last, crucial step is the following observation:

**Lemma 5.17** *We have*

$$
D_{\mathrm{hol}}^{\mathrm{sym}} E^* = -(E^*)^2 \qquad \textit{as well as} \qquad D_{\overline{\mathrm{hol}}}^{\mathrm{sym}} \overline{E}^* = -(\overline{E}^*)^2 \tag{5.29}
$$

*and consequently*

$$
(\Theta_{\Xi}^*)^{\otimes(k+1)} D_{\mathrm{hol}}^{\mathrm{sym}} (\Theta_{\Xi}^*)^{\otimes k} \omega = (\Theta_{\Xi}^*)^{\otimes(k+1)} D_{\mathrm{hol}}^{\mathrm{sym}} \omega \tag{5.30}
$$

*as well as*

$$
(\Theta_{\Xi}^*)^{\otimes(k+1)} D_{\overline{\mathrm{hol}}}^{\mathrm{sym}} (\Theta_{\Xi}^*)^{\otimes k} \overline{\omega} = (\Theta_{\Xi}^*)^{\otimes(k+1)} D_{\overline{\mathrm{hol}}}^{\mathrm{sym}} \overline{\omega} \tag{5.31}
$$

*for all $\omega \in \Gamma^\infty(\mathrm{S}^k\, \mathrm{T}^{*,(1,0)} \mathbb{C}_+^{1+n})$ with $k \in \mathbb{N}_0$.*

Proof: Again, it suffices to prove the first equalities since the second ones then follow by taking complex conjugates. Using (3.25), (3.26), (3.27), and (3.28), an easy computation shows

$$
\begin{aligned}
D_{\mathrm{hol}}^{\mathrm{sym}} E^* = D_{\mathrm{hol}}^{\mathrm{sym}} \bigg( \frac{1}{\mathcal{J}} \sum_{k=0}^{n} \sigma_k \overline{z}^k \, \mathrm{d}z^k \bigg) &= \big( D_{\mathrm{hol}}^{\mathrm{sym}} \mathcal{J}^{-1} \big) \sum_{k=0}^{n} \sigma_k \overline{z}^k \, \mathrm{d}z^k = \\
&= -\mathcal{J}^{-2} \bigg( \sum_{k=0}^{n} \sigma_k \overline{z}^k \, \mathrm{d}z^k \bigg)^2 = -(E^*)^2 .
\end{aligned}
$$

For (5.30) it is sufficient to consider the case $k = 1$, the general case then follows from the algebraic properties of $\Theta_{\Xi}^*$ and $D_{\mathrm{hol}}^{\mathrm{sym}}$ (i.e. being a projection and a derivation). If $k = 1$, then there is an $f \in \mathscr{C}^\infty(\mathbb{C}_+^{1+n})$ such that $\Theta_{\Xi}^*\omega - \omega = f E^*$, and thus

$$
D_{\mathrm{hol}}^{\mathrm{sym}} \Theta_{\Xi}^* \omega - D_{\mathrm{hol}}^{\mathrm{sym}} \omega = D_{\mathrm{hol}}^{\mathrm{sym}} (\Theta_{\Xi}^* \omega - \omega) = D_{\mathrm{hol}}^{\mathrm{sym}} f E^* = \mathrm{d}f \vee E^* - f (E^*)^2
$$

is in the kernel of $(\Theta_{\Xi}^*)^{\otimes 2}$. $\square$

Proof of Theorem 5.12: The reduced star product on $M_{\mathrm{red}}$ fulfils

$$
\mathrm{pr}^*(f \star_{\mathrm{red}} g) = \big( S\big( \mathrm{Pr}^*(f) \star \mathrm{Pr}^*(g) \big) \big)\big|_Z
$$

for all $f, g \in \mathscr{C}^\infty(M_{\mathrm{red}})$. Application of first Lemma 5.13 and then Lemma 5.15 now yields

$$\begin{aligned}
\mathrm{pr}^*(f \star_{\mathrm{red}} g) &= \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1/\nu}{(1/\nu)_{\uparrow,r+1}} \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^r \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^r \operatorname{Pr}^*(g), H^r \big\rangle\big|_Z \\
&= fg\big|_Z + \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1/\nu}{(1/\nu)_{\uparrow,r+1}} \frac{(r-k)!}{k!} \binom{r-1}{k-1}^2 \cdot \\
&\qquad \cdot \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle\big|_Z
\end{aligned}$$

for all $f, g \in \mathscr{C}^\infty(M_{\mathrm{red}})$. Collecting the $k$-th derivatives and using Lemma 5.16 we obtain

$$\begin{aligned}
\mathrm{pr}^*(f \star_{\mathrm{red}} g) &= fg\big|_Z + \sum_{k=1}^{\infty} \sum_{r=k}^{\infty} \frac{1/\nu}{(1/\nu)_{\uparrow,r+1}} \frac{(r-k)!}{k!} \binom{r-1}{k-1}^2 \cdot \\
&\qquad \cdot \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle\big|_Z \\
&= fg\big|_Z + \sum_{k=1}^{\infty} \sum_{s=0}^{\infty} \frac{1/\nu}{(1/\nu)_{\uparrow,k+s+1}} \frac{s!}{k!} \binom{k+s-1}{k-1}^2 \cdot \\
&\qquad \cdot \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle\big|_Z \\
&= fg\big|_Z + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(1/\nu)_{\downarrow,k}} \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle\big|_Z \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(1/\nu)_{\downarrow,k}} \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle\big|_Z .
\end{aligned}$$

As $(H_\Xi)|_\rho \in \Xi^{\mathbb{C}}_\rho \otimes \Xi^{\mathbb{C}}_\rho$ for all $\rho \in \mathbb{C}^{1+n}_+$, we may insert projections $\Theta^*_\Xi$ and get

$$\begin{aligned}
&\big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle = \\
&\qquad = \big\langle (\Theta^*_\Xi)^{\otimes k} (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (\Theta^*_\Xi)^{\otimes k} (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle .
\end{aligned}$$

Using Lemma 5.17 and Proposition 5.11 we obtain

$$\begin{aligned}
(\Theta^*_\Xi)^{\otimes k} (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) &= (\Theta^*_\Xi)^{\otimes k} D^{\mathrm{sym}}_{\mathrm{hol}} (\Theta^*_\Xi)^{\otimes (k-1)} D^{\mathrm{sym}}_{\mathrm{hol}} \dots \Theta^*_\Xi D^{\mathrm{sym}}_{\mathrm{hol}} \operatorname{Pr}^*(f) \\
&= \operatorname{Pr}^* \big( (D^{\mathrm{sym}}_{\mathrm{red,hol}})^k f \big)
\end{aligned}$$

and analogously for $g$, so that

$$\begin{aligned}
\mathrm{pr}^*(f \star_{\mathrm{red}} g) &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(1/\nu)_{\downarrow,k}} \big\langle (D^{\mathrm{sym}}_{\mathrm{hol}})^k \operatorname{Pr}^*(f) \otimes (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k \operatorname{Pr}^*(g), (H_\Xi)^k \big\rangle\big|_Z \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(1/\nu)_{\downarrow,k}} \big\langle \operatorname{Pr}^* \big( (D^{\mathrm{sym}}_{\mathrm{red,hol}})^k f \big) \otimes \operatorname{Pr}^* \big( (D^{\mathrm{sym}}_{\mathrm{red},\overline{\mathrm{hol}}})^k g \big), (H_\Xi)^k \big\rangle\big|_Z \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(1/\nu)_{\downarrow,k}} \mathrm{pr}^* \big( \big\langle (D^{\mathrm{sym}}_{\mathrm{red,hol}})^k f \otimes (D^{\mathrm{sym}}_{\mathrm{red},\overline{\mathrm{hol}}})^k g, H^k_{\mathrm{red}} \big\rangle \big)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(1/\nu)_{\downarrow,k}} \operatorname{pr}^*\big(\langle (D^{\mathrm{sym}}_{\mathrm{red}})^k f \otimes (D^{\mathrm{sym}}_{\mathrm{red}})^k g, H^k_{\mathrm{red}}\rangle\big)\,.$$

In the last step we used Proposition A.11 and that the first tensor factor of $H_{\mathrm{red}}$ lies in $\mathrm{T}^{(1,0)}M_{\mathrm{red}}$ whereas the second lies in $\mathrm{T}^{(0,1)}M_{\mathrm{red}}$.

It remains to check that only finitely many summands contribute to $f \star_{\mathrm{red}} g$ if $f \in \mathscr{P}(M_{\mathrm{red}})$ or $g \in \mathscr{P}(M_{\mathrm{red}})$: Indeed, if $f = \mathrm{b}_{P,Q;\mathrm{red}}$ with $P, Q \in \mathbb{N}_0^{1+n}$ and $|P| = |Q|$, then $\operatorname{Pr}^*(\mathrm{b}_{P,Q;\mathrm{red}}) = \mathcal{J}^{-|P|}\mathrm{b}_{P,Q}$, and as $D^{\mathrm{sym}}_{\mathrm{hol}}\mathcal{J} = \mathcal{J}E^*$ is in the kernel of $\Theta^*_\Xi$, we get

$$\begin{aligned}\operatorname{Pr}^*\big((D^{\mathrm{sym}}_{\mathrm{red,hol}})^k \mathrm{b}_{P,Q;\mathrm{red}}\big) &= (\Theta^*_\Xi)^{\otimes k} D^{\mathrm{sym}}_{\mathrm{hol}} (\Theta^*_\Xi)^{\otimes(k-1)} D^{\mathrm{sym}}_{\mathrm{hol}} \dots \Theta^*_\Xi D^{\mathrm{sym}}_{\mathrm{hol}} \mathcal{J}^{-|P|}\mathrm{b}_{P,Q} \\ &= \mathcal{J}^{-|P|}(\Theta^*_\Xi)^{\otimes k} D^{\mathrm{sym}}_{\mathrm{hol}} (\Theta^*_\Xi)^{\otimes(k-1)} D^{\mathrm{sym}}_{\mathrm{hol}} \dots \Theta^*_\Xi D^{\mathrm{sym}}_{\mathrm{hol}} \mathrm{b}_{P,Q} \\ &= \mathcal{J}^{-|P|}(\Theta^*_\Xi)^{\otimes k} (D^{\mathrm{sym}}_{\mathrm{hol}})^k \mathrm{b}_{P,Q}\,,\end{aligned}$$

which vanishes for $k > |P|$. The argument for $g$ is similar. □

Finally, we can also characterize the reduced covariant derivative as follows:

**Proposition 5.18** *The reduced exterior covariant derivative $D_{\mathrm{red}}$ on $M_{\mathrm{red}}$ is the one for the Levi-Civita connection associated to the (not necessarily definite) reduced metric $g_{\mathrm{red}} \in \mathscr{S}^2(M_{\mathrm{red}})$, which is defined by*

$$\operatorname{Pr}^*(g_{\mathrm{red}}) = (\Theta^*_\Xi)^{\otimes 2}\left(\sum_{k=0}^{n} \frac{\sigma_k\, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k}{\mathcal{J}}\bigg|_{\mathbb{C}^{1+n}_+}\right). \tag{5.32}$$

Proof: As $\sum_{k=0}^n \sigma_k\, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k/\mathcal{J}$ is $\mathbb{C}^*$-invariant, $g_{\mathrm{red}}$ is indeed well-defined. As $D$ is torsion-free, $D_{\mathrm{red}}$ is torsion free as well (see Proposition A.6). Now we calculate

$$\begin{aligned}\operatorname{Pr}^*\big(D_{\mathrm{red}}(g_{\mathrm{red}})\big) &= (\Theta^*_\Xi)^{\otimes 3} D \operatorname{Pr}^*(g_{\mathrm{red}}) \\ &= (\Theta^*_\Xi)^{\otimes 3} D (\Theta^*_\Xi)^{\otimes 2}\left(\sum_{k=0}^{n} \frac{\sigma_k\, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k}{\mathcal{J}}\bigg|_{\mathbb{C}^{1+n}_+}\right).\end{aligned}$$

Using (3.27) and (3.28) one can check that

$$\sum_{k=0}^{n} \sigma_k (\Theta^*_\Xi\, \mathrm{d}z^k) \vee (\Theta^*_\Xi\, \mathrm{d}\overline{z}^k) = \sum_{k=0}^{n} \sigma_k\, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k - \mathcal{J} E^* \vee \overline{E}^*\,.$$

It follows that

$$\operatorname{Pr}^*\big(D_{\mathrm{red}}(g_{\mathrm{red}})\big) = (\Theta^*_\Xi)^{\otimes 3} D\left(\sum_{k=0}^{n} \frac{\sigma_k\, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k}{\mathcal{J}}\bigg|_{\mathbb{C}^{1+n}_+}\right) - (\Theta^*_\Xi)^{\otimes 3} D\big(E^* \vee \overline{E}^*\big) = 0$$

because $\Theta^*_\Xi\, \mathrm{d}(\mathcal{J}^{-1}) = -\mathcal{J}^{-2}\Theta^*_\Xi\, \mathrm{d}\mathcal{J} = 0$ and because $(\Theta^*_\Xi)^{\otimes 3} D(E^* \vee \overline{E}^*) = 0$, so $D_{\mathrm{red}}(g_{\mathrm{red}}) = 0$. □

Note that $g_{\mathrm{red}}$ can be obtained from the standard (pseudo-)metric $g \coloneqq \sum_{k=0}^{n} \sigma_k \, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k$ on $\mathbb{C}^{1+n}$ in signature $s$ by first restricting $(\Theta_{\Xi}^*)^{\otimes 2} g$ to $Z$ and then projecting down on $M_{\mathrm{red}}$. In local coordinates on the domain of definition of the forms $W_k^*$ one finds that

$$(\Theta_{\Xi}^*)^{\otimes 2}\bigg(\sum_{k=0}^{n} \frac{\sigma_k \, \mathrm{d}z^k \vee \mathrm{d}\overline{z}^k}{\mathcal{J}}\bigg|_{\mathbb{C}_+^{1+n}}\bigg) =$$
$$= \frac{|z^0|^2}{\mathcal{J}} \sum_{k=1}^{n} \sigma_k W_k^* \vee \overline{W}_k^* - \frac{|z^0|^2}{\mathcal{J}^2} \sum_{k,\ell=1}^{n} \sigma_k \sigma_\ell z^\ell \overline{z}^k W_k^* \vee \overline{W}_\ell^* \tag{5.33}$$

and hence that

$$g_{\mathrm{red}} = \frac{\sum_{k=1}^{n} \sigma_k \, \mathrm{d}w^k \vee \mathrm{d}\overline{w}^k}{1 + \sum_{k=1}^{n} \sigma_k w^k \overline{w}^k} - \frac{\sum_{k,\ell=1}^{n} \sigma_k \sigma_\ell w^\ell \overline{w}^k \, \mathrm{d}w^k \vee \mathrm{d}\overline{w}^\ell}{(1 + \sum_{k=1}^{n} \sigma_k w^k \overline{w}^k)^2} \,. \tag{5.34}$$

In particular, in signature $s = 1+n$ one obtains that $g_{\mathrm{red}}$ is the usual Fubini–Study metric on $M_{\mathrm{red}}^{(1+n)} \cong \mathbb{CP}^n$, and for $s = 1$ it is the negative of the usual Fubini–Study metric on $M_{\mathrm{red}}^{(1)} \cong \mathbb{D}^n$.

## 5.3 The polynomial case

In this section we will replace the formal parameter $\nu$ by a complex number $\hbar$. In order to make sense of the convergence of the formal power series describing the star product, we restrict ourselves to polynomial functions.

From the definition of the Poisson bracket in (5.5) it is clear that it restricts to a well-defined map $\{\,\cdot\,,\,\cdot\,\}\colon \mathscr{P}(\mathbb{C}^{1+n}) \times \mathscr{P}(\mathbb{C}^{1+n}) \to \mathscr{P}(\mathbb{C}^{1+n})$ that is given by the same formula, and similarly for the Wick star product:

**Lemma 5.19** *In the basis $\mathrm{b}_{P,Q}$ defined in Definition 4.8, the Poisson bracket is*

$$\big\{\mathrm{b}_{P,Q}, \mathrm{b}_{R,S}\big\} = \frac{1}{\mathrm{i}} \sum_{k=0}^{n} \sigma_k (P_k S_k - Q_k R_k) \mathrm{b}_{P+R-E_k, Q+S-E_k} \tag{5.35}$$

*with $E_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}_0^{1+n}$ having the $1$ at position $k \in \{0, \dots, n\}$, and the product $\star$ from Definition 5.1 is*

$$\mathrm{b}_{P,Q} \star \mathrm{b}_{R,S} = \sum_{T=0}^{\min\{P,S\}} \operatorname{sgn}(T) \nu^{|T|} T! \binom{P}{T} \binom{S}{T} \mathrm{b}_{P+R-T, Q+S-T} \tag{5.36}$$

*with $\operatorname{sgn}(T) = \prod_{k=0}^{n} \sigma_k^{T_k} = \prod_{k=1}^{n} \sigma_k^{T_k}$ as in Lemma 4.10.*

So by setting $\nu$ to $\hbar \in \mathbb{C}$, this yields a well-defined map $\star_\hbar\colon \mathscr{P}(\mathbb{C}^{1+n}) \times \mathscr{P}(\mathbb{C}^{1+n}) \to \mathscr{P}(\mathbb{C}^{1+n})$. Next we consider the equivalence transformation $S$ from Proposition 5.5:

**Lemma 5.20** *For $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, the equivalence transformation $S$ is given by*

$$S(\mathrm{b}_{P,Q}) = \Big(\frac{\nu}{\mathcal{J}}\Big)^{|P|} \Big(\frac{\mathcal{J}}{\nu}\Big)_{\downarrow, |P|} \mathrm{b}_{P,Q} \,. \tag{5.37}$$

PROOF: As $\mathcal{J}^{-|P|}\mathrm{b}_{P,Q}$ is $\mathbb{C}^*$-invariant, we get using Proposition 5.4:

$$S(\mathrm{b}_{P,Q}) = S(\mathcal{J}^{|P|}\mathcal{J}^{-|P|}\mathrm{b}_{P,Q}) = S(\mathcal{J}^{|P|})\mathcal{J}^{-|P|}\mathrm{b}_{P,Q} = (\mathcal{J}/\nu)_{\downarrow,|P|}(\nu/\mathcal{J})^{|P|}\mathrm{b}_{P,Q}\,.$$

Note that this is indeed a well-defined formal power series in $\nu$ as the term $\mathcal{J}/\nu$ that occurs in $(\mathcal{J}/\nu)_{\downarrow,|P|}$ if $|P| \geq 1$ is cancelled. □

Replacing $\nu$ by $\hbar \in \mathbb{C}$ yields a rational expression in $\hbar$ and we have to be aware of some poles:

**Definition 5.21** *We define the open subset $\Omega$ of $\mathbb{C}$ as*

$$\Omega := \mathbb{C} \setminus \big(\{1/k \,|\, k \in \mathbb{N}\} \cup \{0\}\big)\,. \tag{5.38}$$

We have already seen in Theorem 5.12 that the reduced Wick star product $f \star_{\mathrm{red}} g$ of polynomials $f, g \in \mathscr{P}(M_{\mathrm{red}})$ is rational in $\hbar$ with poles in $\{1/k \,|\, k \in \mathbb{N}\}$. More precisely, we get:

**Proposition 5.22** *For $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$, the reduced star product from Definition 5.6 is given by*

$$\begin{aligned}
&\mathrm{b}_{P,Q;\mathrm{red}} \star_{\mathrm{red}} \mathrm{b}_{R,S;\mathrm{red}} = \\
&\quad = \sum_{T=0}^{\min\{P,S\}} \operatorname{sgn}(T) \frac{(1/\nu)_{\downarrow,|P+S-T|}}{(1/\nu)_{\downarrow,|P|}(1/\nu)_{\downarrow,|S|}} T! \binom{P}{T}\binom{S}{T} \mathrm{b}_{P+R-T,Q+S-T;\mathrm{red}}\,.
\end{aligned} \tag{5.39}$$

*Replacing $\nu$ by $\hbar \in \Omega$ gives a strict associative product $\star_{\mathrm{red},\hbar} \colon \mathscr{P}(M_{\mathrm{red}}) \times \mathscr{P}(M_{\mathrm{red}}) \to \mathscr{P}(M_{\mathrm{red}})$ and $\mathscr{P}(M_{\mathrm{red}})$ with this product and pointwise complex conjugation becomes a unital $^*$-algebra if $\hbar \in \Omega \cap \mathbb{R}$.*

PROOF: First we note that Lemma 5.19 and Lemma 5.20 show that

$$\begin{aligned}
&\left(\frac{\nu}{\mathcal{J}}\right)^{|P|}\left(\frac{\mathcal{J}}{\nu}\right)_{\downarrow,|P|}\mathrm{b}_{P,Q} \,\tilde{\star}\, \left(\frac{\nu}{\mathcal{J}}\right)^{|S|}\left(\frac{\mathcal{J}}{\nu}\right)_{\downarrow,|S|}\mathrm{b}_{R,S} = \\
&\quad = \sum_{T=0}^{\min\{P,S\}} \operatorname{sgn}(T)\nu^{|T|}T!\binom{P}{T}\binom{S}{T}\left(\frac{\nu}{\mathcal{J}}\right)^{|P+S-T|}\left(\frac{\mathcal{J}}{\nu}\right)_{\downarrow,|P+S-T|}\mathrm{b}_{P+R-T,Q+S-T}
\end{aligned}$$

holds for the transformed star product $\tilde{\star}$ and all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$. As

$$\mathrm{pr}^*\big(\nu^{|P|}(1/\nu)_{\downarrow,|P|}\mathrm{b}_{P,Q;\mathrm{red}}\big) = \iota^*\Big(\big(\nu/\mathcal{J}\big)^{|P|}\big(\mathcal{J}/\nu\big)_{\downarrow,|P|}\mathrm{b}_{P,Q}\Big)$$

we find that

$$\begin{aligned}
&\big(\nu^{|P|}(1/\nu)_{\downarrow,|P|}\mathrm{b}_{P,Q;\mathrm{red}}\big) \star_{\mathrm{red}} \big(\nu^{|S|}(1/\nu)_{\downarrow,|S|}\mathrm{b}_{R,S;\mathrm{red}}\big) = \\
&\quad = \sum_{T=0}^{\min\{P,S\}} \operatorname{sgn}(T)\nu^{|P+S|}T!\binom{P}{T}\binom{S}{T}(1/\nu)_{\downarrow,|P+S-T|}\mathrm{b}_{P+R-T,Q+S-T;\mathrm{red}}\,,
\end{aligned}$$

which yields (5.39) by $\mathbb{C}[[\nu]]$-linearity of $\star_{\mathrm{red}}$ and because $\nu^{|P|}(1/\nu)_{\downarrow,|P|} = (1-\nu)\dots(1-(|P|-1)\nu)$ is an invertible formal power series. Note that the right-hand side of (5.39) is indeed a well-defined formal power series in $\nu$ because the factor $1/\nu$ that occurs in $(1/\nu)_{\downarrow,|P+S-T|}$ for $|P+S-T| \geq 1$ is cancelled.

We can now substitute $\nu$ by $\hbar$: If $\hbar \in \Omega$, the falling factorials in the nominator are non-zero, thus (5.39) defines a well-defined product on the whole algebra $\mathscr{P}(M_{\mathrm{red}})$. Associativity and compatibility with pointwise complex conjugation follow from the properties of the Hermitian formal star product $\star_{\mathrm{red}}$, and the unit is the constant 1-function. □

Equation (5.39) immediately yields:

**Corollary 5.23** *For two fixed polynomials $f, g \in \mathscr{P}(M_{\mathrm{red}})$, the map $\hbar \mapsto f \star_{\mathrm{red},\hbar} g$ is rational and $\lim_{\hbar\to 0} f \star_{\mathrm{red},\hbar} g = fg$ holds pointwise.*

**Proposition 5.24** *For two polynomials $f, g \in \mathscr{P}(M_{\mathrm{red}})$, we have*

$$\lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}\big(f \star_{\mathrm{red},\hbar} g - g \star_{\mathrm{red},\hbar} f\big) = \{f, g\}_{\mathrm{red}} \tag{5.40}$$

*pointwise and with the reduced Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ on $M_{\mathrm{red}}$.*

PROOF: All terms with $|T| \geq 2$ in Equation (5.39) are at least of order $\hbar^2$ and the $T = 0$ term cancels out when taking the commutator. The first order in $\hbar$ of the terms with $|T| = 1$ produces

$$\lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}\big(\mathrm{b}_{P,Q;\mathrm{red}} \star_{\mathrm{red},\hbar} \mathrm{b}_{R,S;\mathrm{red}} - \mathrm{b}_{R,S;\mathrm{red}} \star_{\mathrm{red},\hbar} \mathrm{b}_{P,Q;\mathrm{red}}\big) = \\ = \frac{1}{\mathrm{i}} \sum_{k=0}^{n} \sigma_k (P_k S_k - Q_k R_k) \mathrm{b}_{P+R-E_k, Q+S-E_k;\mathrm{red}}\,,$$

where $E_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}_0^{1+n}$ has the 1 at position $k \in \{0, \dots, n\}$. This coincides by Definition 4.5 and Lemma 5.19 with $\{\mathrm{b}_{P,Q;\mathrm{red}}, \mathrm{b}_{R,S;\mathrm{red}}\}_{\mathrm{red}}$. □

## 5.4 The analytic case

The aim of this section is to obtain a strict star product on the algebra $\mathscr{A}(M_{\mathrm{red}})$. We achieve this by proving the continuity of the star product $\star_{\mathrm{red},\hbar}$ on $\mathscr{P}(M_{\mathrm{red}})$ with respect to the locally convex topology that $\mathscr{P}(M_{\mathrm{red}})$ inherits from $\mathscr{A}(M_{\mathrm{red}})$, i.e. the topology of locally uniform convergence of the holomorphic extensions to $\hat{M}_{\mathrm{red}}$. This then implies that $\star_{\mathrm{red},\hbar}$ extends uniquely to a continuous star product on $\mathscr{A}(M_{\mathrm{red}})$.

Recall from Proposition 4.16 that the topology on $\mathscr{A}(M_{\mathrm{red}})$ is just the quotient topology of the topology on $\mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ defined by locally uniform convergence of the holomorphic extensions to $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. For $\hbar \in \Omega$ define a product $*_\hbar$ on $\mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$ by bilinearly extending

$$\mathrm{b}_{P,Q} *_\hbar \mathrm{b}_{R,S} := \sum_{T=0}^{\min\{P,S\}} \mathrm{sgn}(T) \frac{(1/\hbar)_{\downarrow,|P+S-T|}}{(1/\hbar)_{\downarrow,|P|}(1/\hbar)_{\downarrow,|S|}} T! \binom{P}{T}\binom{S}{T} \mathrm{b}_{P+R-T,Q+S-T}$$

for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$. Note that this product might not be associative. However, from Proposition 5.22 it follows immediately that dividing out the vanishing ideal of $\mathscr{J} - 1$ is possible and reproduces the product $\star_{\mathrm{red},\hbar}$. Consequently, continuity of $*_\hbar$ with respect to the seminorms $\|\cdot\|_r$ with $r \in [1,\infty)$ defined in Equation (4.12) implies continuity of $\star_{\mathrm{red},\hbar}$.

**Proposition 5.25** *The product $*_\hbar$ is continuous with respect to the locally convex topology defined by the seminorms $\|\cdot\|_r$ with $r \in [1,\infty)$ as in Equation (4.12). More precisely, for every $r \in [1,\infty)$ and every compact subset $K$ of $\Omega$ there exists $r' \in [1,\infty)$ such that*

$$\|f *_\hbar g\|_r \le \|f\|_{r'} \|g\|_{r'} \tag{5.41}$$

*holds for all $\hbar \in K$ and all $f, g \in \mathscr{P}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$.*

PROOF: It is well-known that for any compact set $K' \subseteq \mathbb{C} \setminus \mathbb{N}_0$ there are constants $c, C > 0$ such that

$$c^n n! \le |(z)_{\downarrow,n}| \le C^n n!$$

holds for all $z \in K$ and all $n \in \mathbb{N}_0$. For a compact set $K \subseteq \Omega$ also $K' := \{z \in \mathbb{C} \setminus \{0\} \,|\, z^{-1} \in K\}$ is compact and a subset of $\mathbb{C} \setminus \mathbb{N}_0$. Therefore it follows for any $r \in [1,\infty)$ and $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$, and assuming without loss of generality that $C \ge 1$, that

$$\begin{aligned}
&\|\mathrm{b}_{P,Q} *_\hbar \mathrm{b}_{R,S}\|_r \\
&= \left\| \sum_{T=0}^{\min\{P,S\}} \mathrm{sgn}(T) \frac{(1/\hbar)_{\downarrow,|P+S-T|}}{(1/\hbar)_{\downarrow,|P|}(1/\hbar)_{\downarrow,|S|}} T! \binom{P}{T}\binom{S}{T} \mathrm{b}_{P+R-T,Q+S-T} \right\|_r \\
&\le \sum_{T=0}^{\min\{P,S\}} \left| \frac{(1/\hbar)_{\downarrow,|P+S-T|}}{(1/\hbar)_{\downarrow,|P|}(1/\hbar)_{\downarrow,|S|}} \right| T! \binom{P}{T}\binom{S}{T} r^{|P+Q+R+S-2T|} \\
&\le \sum_{T=0}^{\min\{P,S\}} \frac{C^{|P+S-T|}}{c^{|P+S|}} \frac{|P+S-T|!T!}{|P|!|S|!} 2^{|P+S|} r^{|P+Q+R+S-2T|} \\
&\le \sum_{T=0}^{\min\{P,S\}} (c^{-1}C)^{|P+S|} 4^{|P+S|} r^{|P+Q+R+S|} \\
&\le (8c^{-1}Cr)^{|P+Q+R+S|}.
\end{aligned}$$

Given U(1)-invariant polynomials $f = \sum_{P,Q} f_{P,Q} \mathrm{b}_{P,Q}$ and $g = \sum_{R,S} g_{R,S} \mathrm{b}_{R,S}$ on $\mathbb{C}^{1+n}$ with complex coefficients $f_{P,Q}$ and $g_{R,S}$, then

$$\begin{aligned}
\|f *_\hbar g\|_r &\le \sum_{\substack{P,Q \in \mathbb{N}_0^{1+n} \\ |P|=|Q|}} |f_{P,Q}| (8c^{-1}Cr)^{|P+Q|} \sum_{\substack{R,S \in \mathbb{N}_0^{1+n} \\ |R|=|S|}} |g_{R,S}| (8c^{-1}Cr)^{|R+S|} \\
&= \|f\|_{8c^{-1}Cr} \|g\|_{8c^{-1}Cr}.
\end{aligned}$$

□

We would like to remark that, similar as in [12], one can also use the description of the star product using bidifferential operators to prove its continuity.

**Theorem 5.26** *For every $\hbar \in \Omega$, the product $\star_{\mathrm{red},\hbar}$ on $\mathscr{P}(M_{\mathrm{red}})$ extends to a continuous associative product on $\mathscr{A}(M_{\mathrm{red}})$. Moreover $\mathscr{A}(M_{\mathrm{red}})$ becomes a unital Fréchet $^*$-algebra with this product and pointwise complex conjugation as $^*$-involution in the case that $\hbar \in \Omega \cap \mathbb{R}$. Finally, for any two fixed elements $f, g \in \mathscr{A}(M_{\mathrm{red}})$ and $[\rho] \in M_{\mathrm{red}}$, the map $\Omega \to \mathbb{C}$, $\hbar \mapsto (f \star_{\mathrm{red},\hbar} g)([\rho])$ is holomorphic.*

Proof: By the previous Proposition 5.25 and the discussion above, the associative product $\star_{\mathrm{red},\hbar}$ is continuous on $\mathscr{P}(M_{\mathrm{red}})$ with respect to the topology inherited from $\mathscr{A}(M_{\mathrm{red}})$, and thus extends to an associative and continuous product on $\mathscr{A}(M_{\mathrm{red}})$ because $\mathscr{P}(M_{\mathrm{red}})$ is dense in $\mathscr{A}(M_{\mathrm{red}})$ by Corollary 4.17. The constant 1-function remains the unit like on polynomials. Compatibility with the $^*$-involution is clear as well if $\hbar$ is additionally real.

Now recall that for polynomials $p, q \in \mathscr{P}(M_{\mathrm{red}})$, the map $\hbar \mapsto (p \star_{\mathrm{red},\hbar} q)([\rho])$ is rational by Corollary 5.23. Since the estimates in Proposition 5.25 are locally uniform in $\hbar$, it follows that $\hbar \mapsto (f \star_{\mathrm{red},\hbar} g)([\rho])$ is a locally uniform limit of rational functions and therefore holomorphic. $\square$

Note that $0 \notin \Omega$, so one would like to understand whether in the limit $\hbar \to 0$, the product $\star_{\mathrm{red},\hbar}$ yields the pointwise one, and whether its commutator yields the Poisson bracket also on $\mathscr{A}(M_{\mathrm{red}})$. Despite the results from Corollary 5.23 and Proposition 5.24 in the polynomial case, this is not so obvious because 0 is an accumulation point of the poles of $\star_{\mathrm{red},\hbar}$. We will come back to this question later in Proposition 6.12.

# 6 Wick rotation

The dependence on the choice of signature $s$ will now always be made explicit by a superscript "$^{(s)}$".

We have already seen that the construction of the formal and non-formal star products on $M_{\mathrm{red}}^{(s)}$ works completely independent of $s \in \{1, \dots, 1+n\}$. We will see now that the non-formal star product algebras are even all isomorphic as unital complex algebras. This will be proven by construction of a Wick transformation: A holomorphic isomorphism between the complex manifolds $\hat{M}_{\mathrm{red}}^{(s)}$ for different values of $s$ which gives rise to isomorphisms of the algebras $\mathscr{P}(M_{\mathrm{red}}^{(s)})$ and $\mathscr{A}(M_{\mathrm{red}}^{(s)})$ (with the pointwise product) and which are also compatible with the Poisson brackets and the non-formal star products, i.e. describe isomorphisms of Poisson algebras and associative algebras, respectively. However, we will also see that these isomorphisms are not compatible with the $^*$-involution which is given by pointwise complex conjugation, hence are not $^*$-isomorphisms. This demonstrates how important it is to consider $^*$-algebras and not just algebras in non-formal deformation quantization: After all, one would surely want to be able to distinguish the quantization of the complex projective space $\mathbb{CP}^n$ from the one of the hyperbolic disc $\mathbb{D}^n$.

## 6.1 Geometric Wick rotation

We start first with discussing the complex manifolds $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ and then proceed to $\hat{M}^{(s)}_{\mathrm{red}}$:

The Lie group $\mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C})$ acts on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ from the left via $\cdot \triangleright \cdot$, which induces a right action $\cdot \triangleleft \cdot$ on $\mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$. It is easy to check that $\Delta(A \triangleright \rho) = (A, \overline{A}) \triangleright \Delta(\rho)$ for all $\rho \in \mathbb{C}^{1+n}$ and all $A \in \mathrm{GL}_{1+n}(\mathbb{C})$, where $\overline{A}$ denotes the elementwise complex conjugate of $A$. For all $s \in \{1, \dots, 1+n\}$, let $W^{(s)} \in \mathrm{GL}_{1+n}(\mathbb{C})$ be $(W^{(s)})^k{}_\ell := 1$ if $k = \ell \in \{0, \dots, s-1\}$ and $(W^{(s)})^k{}_\ell := \mathrm{i}$ if $k = \ell \in \{s, \dots, n\}$, and otherwise $(W^{(s)})^k{}_\ell := 0$. Then the action of $(W^{(s)}, W^{(s)})$ on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ does not come from an action on $\mathbb{C}^{1+n}$, except in the trivial case that $s = 1+n$. However, the identity

$$\hat{\mathscr{J}}^{(s)} \triangleleft (W^{(s)}, W^{(s)}) = \hat{\mathscr{J}}^{(1+n)} \tag{6.1}$$

holds and thus the holomorphic automorphism of $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ that is given by the action of $(W^{(s)}, W^{(s)})$ restricts to a holomorphic isomorphism from $\hat{Z}^{(1+n)}$ to $\hat{Z}^{(s)}$. It is then immediate that this restriction even descends to a holomorphic isomorphism from $\hat{M}^{(1+n)}_{\mathrm{red}}$ to $\hat{M}^{(s)}_{\mathrm{red}}$, because $(W^{(s)}, W^{(s)})$ commutes with all elements of the $\mathbb{C}^*$-subgroup of $\mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C})$.

**Definition 6.1** *For every $s \in \{1, \dots, 1+n\}$ we define the map*

$$\alpha^{(s)} \colon \hat{M}^{(1+n)}_{\mathrm{red}} \to \hat{M}^{(s)}_{\mathrm{red}}, \quad [(\xi, \eta)] \mapsto \alpha^{(s)}\big([(\xi, \eta)]\big) := \big[(W^{(s)}, W^{(s)}) \triangleright (\xi, \eta)\big]. \tag{6.2}$$

The above discussion shows that $\alpha^{(s)}$ is well-defined and even more:

**Proposition 6.2** *The maps $\alpha^{(s)} \colon \hat{M}^{(1+n)}_{\mathrm{red}} \to \hat{M}^{(s)}_{\mathrm{red}}$ are holomorphic isomorphisms of complex manifolds for all $s \in \{1, \dots, 1+n\}$.*

Moreover, Equation (6.1) also shows that the inner automorphism of the Lie group $\mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C})$ that is given by conjugation with $(W^{(s)}, W^{(s)})$, i.e.

$$(A, B) \mapsto \big(W^{(s)} A (W^{(s)})^{-1}, W^{(s)} B (W^{(s)})^{-1}\big), \tag{6.3}$$

restricts to an isomorphism from $\mathrm{G}_{\hat{\mathscr{J}}^{(1+n)}}$ to $\mathrm{G}_{\hat{\mathscr{J}}^{(s)}}$. Note that we have already seen in Section 3 that $\mathrm{G}_{\hat{\mathscr{J}}^{(s)}}$ is isomorphic to $\mathrm{GL}_{1+n}(\mathbb{C})$ for all $s \in \{1, \dots, 1+n\}$.

As a final remark, we note that the isomorphisms of $\hat{M}^{(s)}_{\mathrm{red}}$ with different signature $s$ clearly do not descend to isomorphisms of $M^{(s)}_{\mathrm{red}}$. For example, $M^{(1+n)}_{\mathrm{red}} \cong \mathbb{CP}^n$ is compact while $M^{(1)}_{\mathrm{red}} \cong \mathbb{D}^n$ is not.

## 6.2 Algebraic Wick rotation

The isomorphism of the complex manifolds $\hat{M}^{(s)}_{\mathrm{red}}$ for different signatures from Proposition 6.2 immediately shows that the corresponding unital associative algebras $\mathscr{O}(\hat{M}^{(s)}_{\mathrm{red}})$ are isomorphic. By Proposition 4.12, the algebras $\mathscr{A}(\hat{M}^{(s)}_{\mathrm{red}})$ for different signatures are isomorphic as unital associative algebras as well (but not necessarily as $^*$-algebras).

**Definition 6.3** *For every $s \in \{1, \ldots, 1+n\}$ we define the maps*

$$\Phi^{(s)} \colon \mathscr{A}(\mathbb{C}^{1+n}) \to \mathscr{A}(\mathbb{C}^{1+n}), \qquad f \mapsto \Phi^{(s)}(f) := \Delta^*\big(\hat{f} \triangleleft (W^{(s)}, W^{(s)})\big) \tag{6.4}$$

*as well as*

$$\Phi^{(s)}_{\mathrm{red}} \colon \mathscr{A}(M^{(s)}_{\mathrm{red}}) \to \mathscr{A}(M^{(1+n)}_{\mathrm{red}}), \qquad g \mapsto \Phi^{(s)}_{\mathrm{red}}(g) := \big(\Delta^{(1+n)}_{\mathrm{red}}\big)^*(\hat{g} \circ \alpha^{(s)}), \tag{6.5}$$

*where $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$ and $\hat{g} \in \mathscr{O}(\hat{M}^{(s)}_{\mathrm{red}})$ satisfy $\Delta^*(\hat{f}) = f$ and $(\Delta^{(s)}_{\mathrm{red}})^*(\hat{g}) = g$. We will refer to $\Phi^{(s)}$ and $\Phi^{(s)}_{\mathrm{red}}$ as the* Wick rotation *and the* reduced Wick rotation*, respectively.*

Proposition 4.12, Proposition 6.2 and the observation that $(W^{(s)}, W^{(s)})$ commutes with the whole $\mathbb{C}^*$-subgroup of $\mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C})$ immediately show:

**Theorem 6.4** *The Wick rotation $\Phi^{(s)}$ is a well-defined homeomorphic automorphism of the unital associative Fréchet algebra $\mathscr{A}(\mathbb{C}^{1+n})$ that restricts to an automorphism of $\mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$. Moreover, the reduced Wick rotation $\Phi^{(s)}_{\mathrm{red}} \colon \mathscr{A}(M^{(s)}_{\mathrm{red}}) \to \mathscr{A}(M^{(1+n)}_{\mathrm{red}})$ is a well-defined homeomorphic isomorphism of unital associative Fréchet algebras.*

The Wick rotations are also compatible with the reduction procedure:

**Proposition 6.5** *Given $f \in \mathscr{A}(\mathbb{C}^{1+n})^{\mathrm{U}(1)}$, then*

$$\big(\Phi^{(s)}(f)\big)_{\mathrm{red}} = \Phi^{(s)}_{\mathrm{red}}(f_{\mathrm{red}}).$$

PROOF: By Proposition 4.14 there exists an $\hat{f} \in \mathscr{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})^{\mathbb{C}^*}$ as in Lemma 4.13 such that $\Delta^*(\hat{f}) = f$ and $(\Delta^{(s)}_{\mathrm{red}})^*(\hat{f}_{\hat{\mathrm{red}}}) = f_{\mathrm{red}}$. Using the commutativity of the diagram from Section 3 and the properties of the action of $(W^{(s)}, W^{(s)})$ one finds:

$$\begin{aligned}
(\iota^{(1+n)})^*\big(\Phi^{(s)}(f)\big) &= (\iota^{(1+n)})^*\big(\Delta^*\big(\hat{f} \triangleleft (W^{(s)}, W^{(s)})\big)\big) \\
&= (\Delta^{(1+n)}_Z)^*\big((\hat{\iota}^{(1+n)})^*\big(\hat{f} \triangleleft (W^{(s)}, W^{(s)})\big)\big) \\
&= (\Delta^{(1+n)}_Z)^*\big((\hat{\iota}^{(s)})^*(\hat{f}) \triangleleft (W^{(s)}, W^{(s)})\big) \\
&= (\Delta^{(1+n)}_Z)^*\big((\hat{\mathrm{pr}}^{(s)})^*(\hat{f}_{\hat{\mathrm{red}}}) \triangleleft (W^{(s)}, W^{(s)})\big) \\
&= (\Delta^{(1+n)}_Z)^*\big((\hat{\mathrm{pr}}^{(1+n)})^*\big(\hat{f}_{\hat{\mathrm{red}}} \circ \alpha^{(s)}\big)\big) \\
&= (\mathrm{pr}^{(1+n)})^*\big((\Delta^{(1+n)}_{\mathrm{red}})^*\big(\hat{f}_{\hat{\mathrm{red}}} \circ \alpha^{(s)}\big)\big) \\
&= (\mathrm{pr}^{(1+n)})^*\big(\Phi^{(s)}_{\mathrm{red}}(f_{\mathrm{red}})\big). \qquad \square
\end{aligned}$$

In the following we will see that the Wick rotations are not only isomorphisms of unital associative algebras, but also compatible with Poisson brackets and star products:

**Lemma 6.6** *Given $s \in \{1, \ldots, 1+n\}$, then the identity*

$$\Phi^{(s)}(\mathrm{b}_{P,Q}) = \mathrm{i}^{\sum_{k=s}^{n}(P_k+Q_k)}\mathrm{b}_{P,Q} \tag{6.6}$$

*holds for all* $P, Q \in \mathbb{N}_0^{1+n}$,

$$\Phi^{(s)}_{\mathrm{red}}\big(\mathrm{b}^{(s)}_{P,Q;\mathrm{red}}\big) = \mathrm{i}^{\sum_{k=s}^{n}(P_k+Q_k)}\mathrm{b}^{(1+n)}_{P,Q;\mathrm{red}} \tag{6.7}$$

*holds for all* $P, Q \in \mathbb{N}_0^{1+n}$ *with* $|P| = |Q|$, *and*

$$\Phi^{(s)}_{\mathrm{red}}\big(\mathrm{c}^{(s)}_{P,Q}\big) = \mathrm{i}^{\sum_{k=s}^{n}(P_k+Q_k)}\mathrm{c}^{(1+n)}_{P,Q} \tag{6.8}$$

*holds for all* $P, Q \in \mathbb{N}_0^{n}$, *where* $\mathrm{c}_{P,Q}$ *are the fundamental monomials from Definition 4.9. Moreover,* $\Phi^{(s)}$ *restricts to an automorphism of the unital subalgebra* $\mathscr{P}(\mathbb{C}^{1+n})$ *of* $\mathscr{A}(\mathbb{C}^{1+n})$, *and* $\Phi^{(s)}_{\mathrm{red}}$ *restricts to an isomorphism from the unital subalgebra* $\mathscr{P}(M^{(s)}_{\mathrm{red}})$ *of* $\mathscr{A}(M^{(s)}_{\mathrm{red}})$ *to the unital subalgebra* $\mathscr{P}(M^{(1+n)}_{\mathrm{red}})$ *of* $\mathscr{A}(M^{(1+n)}_{\mathrm{red}})$.

Proof: Using $\mathrm{b}_{P,Q} = \Delta^*(x^P y^Q)$ with $x^0, \dots, x^n, y^0, \dots, y^n \colon \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \to \mathbb{C}$ the standard coordinates, it is easy to check that Equation (6.6) holds. Equation (6.7) then follows by applying the previous Proposition 6.5, which gives Equation (6.8) as a special case. The rest is clear. □

**Theorem 6.7** *The Wick rotations remain isomorphisms of unital associative algebras also for the deformed products. More precisely, given* $s \in \{1, \dots, 1+n\}$, *then the identities*

$$\Phi^{(s)}\big(f \star^{(s)}_{\hbar} g\big) = \Phi^{(s)}(f) \star^{(1+n)}_{\hbar} \Phi^{(s)}(g) \tag{6.9}$$

*and*

$$\Phi^{(s)}\big\{f, g\big\}^{(s)} = \big\{\Phi^{(s)}(f), \Phi^{(s)}(g)\big\}^{(1+n)} \tag{6.10}$$

*hold for all* $f, g \in \mathscr{A}(\mathbb{C}^{1+n})$ *and all* $\hbar \in \mathbb{C}$. *Similarly, the identities*

$$\Phi^{(s)}_{\mathrm{red}}\big(f \star^{(s)}_{\mathrm{red},\hbar} g\big) = \Phi^{(s)}_{\mathrm{red}}(f) \star^{(1+n)}_{\mathrm{red},\hbar} \Phi^{(s)}_{\mathrm{red}}(g) \tag{6.11}$$

*and*

$$\Phi^{(s)}_{\mathrm{red}}\big\{f, g\big\}^{(s)}_{\mathrm{red}} = \big\{\Phi^{(s)}_{\mathrm{red}}(f), \Phi^{(s)}_{\mathrm{red}}(g)\big\}^{(1+n)}_{\mathrm{red}} \tag{6.12}$$

*hold for all* $f, g \in \mathscr{A}(M^{(s)}_{\mathrm{red}})$ *and all* $\hbar \in \Omega$.

Proof: First note that as a consequence of the previous Lemma 6.6, the identity

$$\Phi^{(s)}\big(\operatorname{sgn}^{(s)}(T)\mathrm{b}_{P+R-T,Q+S-T}\big) = \mathrm{i}^{\sum_{k=s}^{n}(P_k+Q_k+R_k+S_k)}\mathrm{b}_{P+R-T,Q+S-T}$$

holds for all $P, Q, R, S, T \in \mathbb{N}_0^{1+n}$ with $T \leq \min\{P, S\}$, and similarly,

$$\Phi^{(s)}_{\mathrm{red}}\big(\operatorname{sgn}^{(s)}(T)\mathrm{b}^{(s)}_{P+R-T,Q+S-T;\mathrm{red}}\big) = \mathrm{i}^{\sum_{k=s}^{n}(P_k+Q_k+R_k+S_k)}\mathrm{b}^{(1+n)}_{P+R-T,Q+S-T;\mathrm{red}}$$

holds for all $P, Q, R, S, T \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, $|R| = |S|$ and $T \leq \min\{P, S\}$. Using this and the explicit formulas from Lemma 5.19 and Proposition 5.22 and noting that $\operatorname{sgn}^{(1+n)}(T) = 1$ for all $T \in \mathbb{N}_0^{1+n}$, it is easy to check the identities for the star products in the special case that $f$ and $g$ are monomials. The identities for the Poisson brackets are an immediate consequence thereof due to the representation of the Poisson brackets as a limit of the star product commutator as in Proposition 5.24. The general case then follows by bilinearity and continuity of the star product and the Poisson bracket. □

While it is completely clear that the Wick rotations do not commute with the $^*$-involution given by pointwise complex conjugation, it is somewhat harder to show that the algebras $\mathscr{A}(M_{\mathrm{red}}^{(s)})$ with product $\star_{\mathrm{red},\hbar}^{(s)}$ are in general not $^*$-isomorphic, not even via some other isomorphism. One possibility to prove this is to examine their *positive linear functionals*: A linear functional $\phi\colon \mathscr{A}(M_{\mathrm{red}}^{(s)}) \to \mathbb{C}$ is called *positive* for the product $\star_{\mathrm{red},\hbar}^{(s)}$ with $\hbar \in \Omega \cap \mathbb{R}$ if

$$\phi\big(\overline{f} \star_{\mathrm{red},\hbar}^{(s)} f\big) \geq 0 \tag{6.13}$$

holds for all $f \in \mathscr{A}(M_{\mathrm{red}}^{(s)})$. It is easy to see that the pullback of a positive linear functional with a $^*$-homomorphism between two $^*$-algebras yields again a positive linear functional. In the special case of $s = 1$, i.e. $M_{\mathrm{red}}^{(1)} \cong \mathbb{D}^n$, the existence of non-trivial positive linear functionals for negative $\hbar$ is known:

**Proposition 6.8** *The evaluation functionals $\delta_{[\rho]}^{(1)} : \mathscr{A}(\mathbb{D}^n) \to \mathbb{C}$,*

$$f \mapsto \delta_{[\rho]}^{(1)}(f) := f([\rho])$$

*with $[\rho] \in M_{\mathrm{red}}^{(1)} \cong \mathbb{D}^n$ are positive linear functionals on $\mathscr{A}(\mathbb{D}^n)$ with product $\star_{\mathrm{red},\hbar}^{(1)}$ for all $\hbar \in (-\infty, 0)$.*

PROOF: Positivity of evaluation functionals has been proven in [3, Sec. 5.4] on an algebra containing (at least) $\mathscr{P}(\mathbb{D}^n)$ with a product $*_\hbar$ fulfilling $f \star_{\mathrm{red},\hbar}^{(1)} g = f *_{-\hbar/2} g$ for all $f, g \in \mathscr{P}(\mathbb{D}^n)$. By continuity of the evaluation functionals, the pointwise complex conjugation and the product $\star_{\mathrm{red},\hbar}^{(1)}$, this extends to whole $\mathscr{A}(\mathbb{D}^n)$. □

However, there are some limitations to the existence of positive linear functionals in the special case of $s = 1 + n$, i.e. $M_{\mathrm{red}}^{(1+n)} \cong \mathbb{CP}^n$, at least if $n = 1$:

**Lemma 6.9** *Consider only the case $n = 1$ and $s = 1 + n = 2$. Then the identity*

$$\sum_{i,j=0}^{1} \mathrm{b}_{E_i,E_j;\mathrm{red}}^{(2)} \star_{\mathrm{red},\hbar}^{(2)} \mathrm{b}_{E_j,E_i;\mathrm{red}}^{(2)} = 1 + \hbar \tag{6.14}$$

*holds for all $\hbar \in \Omega$, where $E_0 = (1, 0) \in \mathbb{N}_0^{1+1}$ and $E_1 = (0, 1) \in \mathbb{N}_0^{1+1}$.*

PROOF: Proposition 5.22 yields

$$\begin{aligned}
&\mathrm{b}_{E_i,E_j;\mathrm{red}}^{(2)} \star_{\mathrm{red},\hbar}^{(2)} \mathrm{b}_{E_j,E_i;\mathrm{red}}^{(2)} \\
&\qquad= \frac{(1/\hbar)_{\downarrow,2}}{(1/\hbar)_{\downarrow,1}(1/\hbar)_{\downarrow,1}} \mathrm{b}_{E_i+E_j,E_i+E_j;\mathrm{red}}^{(2)} + \frac{(1/\hbar)_{\downarrow,1}}{(1/\hbar)_{\downarrow,1}(1/\hbar)_{\downarrow,1}} \mathrm{b}_{E_j,E_j;\mathrm{red}}^{(2)} \\
&\qquad= (1-\hbar)\mathrm{b}_{E_i+E_j,E_i+E_j;\mathrm{red}}^{(2)} + \hbar \mathrm{b}_{E_j,E_j;\mathrm{red}}^{(2)}
\end{aligned}$$

for all $i, j \in \{0, 1\}$. By summation over $i$ and $j$ we get

$$\sum_{i,j=0}^{1} \mathrm{b}_{E_i,E_j;\mathrm{red}}^{(2)} \star_{\mathrm{red},\hbar}^{(2)} \mathrm{b}_{E_j,E_i;\mathrm{red}}^{(2)} =$$

$$= (1-\hbar)\big(\mathrm{b}^{(2)}_{2E_0,2E_0;\mathrm{red}} + 2\mathrm{b}^{(2)}_{E_0+E_1,E_0+E_1;\mathrm{red}} + \mathrm{b}^{(2)}_{2E_1,2E_1;\mathrm{red}}\big) \\ + 2\hbar\big(\mathrm{b}^{(2)}_{E_0,E_0;\mathrm{red}} + \mathrm{b}^{(2)}_{E_1,E_1;\mathrm{red}}\big)\,.$$

Keeping in mind that the reduced monomials are not linearly independent, this can be simplified: We find that $\mathrm{b}^{(2)}_{E_0,E_0;\mathrm{red}} + \mathrm{b}^{(2)}_{E_1,E_1;\mathrm{red}} = \mathscr{I}^{(2)}_{\mathrm{red}}$ is the constant 1-function, and the same is true for their pointwise square $(\mathrm{b}^{(2)}_{E_0,E_0;\mathrm{red}} + \mathrm{b}^{(2)}_{E_1,E_1;\mathrm{red}})^2 = \mathrm{b}^{(2)}_{2E_0,2E_0;\mathrm{red}} + 2\mathrm{b}^{(2)}_{E_0+E_1,E_0+E_1;\mathrm{red}} + \mathrm{b}^{(2)}_{2E_1,2E_1;\mathrm{red}}$. □

**Proposition 6.10** *Consider only the case $n = 1$ and $\hbar \in (-\infty, -1)$. For signature $s = 2$, the only linear functional $\phi\colon \mathscr{A}(\mathbb{CP}^1) \to \mathbb{C}$, which is positive for the product $\star^{(2)}_{\mathrm{red},\hbar}$, is $\phi = 0$. But for signature $s = 1$, the evaluation functionals from Proposition 6.8 are non-trivial positive linear functionals for the product $\star^{(1)}_{\mathrm{red},\hbar}$ on $\mathscr{A}(\mathbb{D}^1)$.*

*Consequently, the ${}^*$-algebra $\mathscr{A}(\mathbb{D}^1)$ with product $\star^{(1)}_{\mathrm{red},\hbar}$ and pointwise complex conjugation as ${}^*$-involution is not ${}^*$-isomorphic to the ${}^*$-algebra $\mathscr{A}(\mathbb{CP}^1)$ with product $\star^{(2)}_{\mathrm{red},\hbar}$ and pointwise complex conjugation as ${}^*$-involution.*

Proof: Let $s = 2$ and let $\phi\colon \mathscr{A}(\mathbb{CP}^1) \to \mathbb{C}$ be a positive linear functional for the product $\star^{(2)}_{\mathrm{red},\hbar}$. Then the previous Lemma 6.9 shows that there exist functions $f_1, \dots, f_4 \in \mathscr{A}(\mathbb{CP}^1)$ such that

$$0 \le \sum_{k=1}^{4} \phi\big(\overline{f_k} \star^{(2)}_{\mathrm{red},\hbar} f_k\big) = \phi(1+\hbar) = (1+\hbar)\,\phi(1) = (1+\hbar)\,\phi\big(\overline{1} \star^{(2)}_{\mathrm{red},\hbar} 1\big) \le 0$$

holds because $\hbar < -1$, so $\phi(1) = 0$. But then the Cauchy Schwarz inequality applied to the (possibly degenerate) inner product $\mathscr{A}(\mathbb{CP}^1) \ni (f,g) \mapsto \phi(\overline{f} \star^{(2)}_{\mathrm{red},\hbar} g) \in \mathbb{C}$ shows that

$$\big|\phi(f)\big|^2 = \big|\phi\big(\overline{1} \star^{(2)}_{\mathrm{red},\hbar} f\big)\big|^2 \le \phi\big(\overline{1} \star^{(2)}_{\mathrm{red},\hbar} 1\big)\,\phi\big(\overline{f} \star^{(2)}_{\mathrm{red},\hbar} f\big) = \phi(1)\,\phi\big(\overline{f} \star^{(2)}_{\mathrm{red},\hbar} f\big) = 0$$

holds for all $f \in \mathscr{A}(\mathbb{CP}^1)$, and therefore $\phi = 0$. The rest is clear. □

## 6.3 Applications

In this section we use the reduced Wick rotation to transfer some of the results obtained in [17] for the special case of the hyperbolic disc, i.e. $s = 1$, to general signatures. Note that one could also check that all the proofs in [17] work for an arbitrary signature, but the Wick rotation provides a more elegant way to generalize these results. We will again drop the superscripts ${}^{(s)}$ most of the time, the following is valid for every choice of signature $s \in \{1, \dots, 1+n\}$.

**Proposition 6.11** *The fundamental monomials $\mathrm{c}_{P,Q}$ with $P, Q \in \mathbb{N}_0^n$ form an absolute Schauder basis of $\mathscr{A}(M_{\mathrm{red}})$. More precisely, every $f \in \mathscr{A}(M_{\mathrm{red}})$ can be expanded in a unique way as an absolutely convergent series*

$$f = \sum_{P,Q \in \mathbb{N}_0^n} f_{P,Q}\mathrm{c}_{P,Q} \tag{6.15}$$

*with complex coefficients* $f_{P,Q}$ *that fulfil the estimate*

$$\|f\|_{\mathrm{red},r} := \sum_{P,Q \in \mathbb{N}_0^n} |f_{P,Q}| r^{|P|+|Q|} < \infty \tag{6.16}$$

*for all* $r \in [1, \infty)$. *Moreover, the topology of* $\mathscr{A}(M_{\mathrm{red}})$ *(i.e. the topology of locally uniform convergence of the holomorphic extensions to* $\hat{M}_{\mathrm{red}}$*) can equivalently be described by these seminorms* $\|f\|_{\mathrm{red},r}$ *and the coefficients* $f_{P,Q}$ *can be calculated explicitly by means of the integral formula*

$$f_{P,Q} = \frac{1}{(-4\pi^2)^n} \oint_C \cdots \oint_C \hat{f} \frac{\left(1 + \sum_{k=1}^n \sigma_k u^k v^k\right)^{\max\{|P|,|Q|\}-1}}{u^{P+(1,\dots,1)} v^{Q+(1,\dots,1)}} \,\mathrm{d}^n u \wedge \mathrm{d}^n v \tag{6.17}$$

*for all* $P, Q \in \mathbb{N}_0^n$. *Here* $f \in \mathscr{A}(M_{\mathrm{red}})$ *and* $\hat{f} \in \mathscr{O}(\hat{M}_{\mathrm{red}})$ *satisfies* $\Delta^*_{\mathrm{red}}(\hat{f}) = f$. *The coordinates* $u$ *and* $v$ *were defined in Equation* (3.13) *and* $C \subseteq \mathbb{C}$ *is, in these projective coordinates, a circle around zero with radius in* $(0, 1/\sqrt{n})$.

PROOF: For $s = 1$ this is exactly the statement of [17, Thm. 3.16]. Because the Wick rotation is a homeomorphic isomorphism and using Lemma 6.6, the generalization to arbitrary signatures is immediately clear for everything except the integral formula. In order to prove that (6.17) holds, we have to check that it is compatible with the holomorphic isomorphisms $\alpha^{(s)}$:

We have to use superscripts $^{(s)}$ again to indicate the signature $s$. As $u^{(s),k} \circ \alpha^{(s)} = u^{(1+n),k}$ and $v^{(s),k} \circ \alpha^{(s)} = v^{(1+n),k}$ for all $k \in \{1, \dots, s-1\}$ as well as $u^{(s),k} \circ \alpha^{(s)} = \mathrm{i}u^{(1+n),k}$ and $v^{(s),k} \circ \alpha^{(s)} = \mathrm{i}v^{(1+n),k}$ for all $k \in \{s, \dots, n\}$ hold, we get

$$\left(\alpha^{(s)}\right)^* (\mathrm{d}^n u^{(s)} \wedge \mathrm{d}^n v^{(s)}) = (-1)^{n+1-s} \,\mathrm{d}^n u^{(1+n)} \wedge \mathrm{d}^n v^{(1+n)}$$

as well as

$$\left(\alpha^{(s)}\right)^* \left(1 + \sum_{k=1}^n \sigma_k^{(s)} u^{(s),k} v^{(s),k}\right) = 1 + \sum_{k=1}^n \sigma_k^{(1+n)} u^{(1+n),k} v^{(1+n),k}$$

and

$$\begin{aligned}\left(\alpha^{(s)}\right)^* \left((u^{(s)})^{P+(1,\dots,1)} (v^{(s)})^{Q+(1,\dots,1)}\right) &= \\ &= \mathrm{i}^{\sum_{k=s}^n (P_k+Q_k+2)} (u^{(1+n)})^{P+(1,\dots,1)} (v^{(1+n)})^{Q+(1,\dots,1)}\end{aligned}$$

for all $P, Q \in \mathbb{N}_0^n$. So given $\hat{f} \in \mathscr{O}(\hat{M}^{(s)}_{\mathrm{red}})$, then the right-hand side of (6.17) for $\hat{f}$ in signature $s$, multiplied with the factor $\mathrm{i}^{\sum_{k=s}^n (P_k+Q_k)}$, gives the same result as for $\hat{f} \circ \alpha^{(s)} \in \mathscr{O}(\hat{M}^{(s)}_{\mathrm{red}})$ in signature $1+n$. This matches precisely with Lemma 6.6, which shows that

$$\Phi^{(s)}_{\mathrm{red}}(f) = \sum_{P,Q \in \mathbb{N}_0^n} f_{P,Q} \Phi^{(s)}_{\mathrm{red}}(\mathrm{b}^{(s)}_{P,Q;\mathrm{red}}) = \sum_{P,Q \in \mathbb{N}_0^n} f_{P,Q} \,\mathrm{i}^{\sum_{k=s}^n (P_k+Q_k)} \mathrm{b}^{(1+n)}_{P,Q;\mathrm{red}}$$

for all $f \in \mathscr{A}(M^{(s)}_{\mathrm{red}})$ with expansion coefficients $f_{P,Q}$. This way, one first sees that (6.17) holds not only for signature $s = 1$ but also for $s = 1 + n$, and then that it even holds for all $s \in \{1, \dots, n+1\}$. □

We would now like to generalize Corollary 5.23 and Proposition 5.24 for analytic functions. Because of the poles in $\hbar$ we only discuss one-sided limits: For some function $f\colon \Omega \cap \mathbb{R} \to \mathbb{C}$ the limit when $\hbar$ approaches 0 from the left is denoted by $\lim_{\hbar\to 0^-} f(\hbar)$ (if it exists).

**Proposition 6.12** *The limits* $\lim_{\hbar\to 0^-} f \star_{\mathrm{red},\hbar} g$ *and* $\lim_{\hbar\to 0^-} \frac{1}{\mathrm{i}\hbar}\big(f \star_{\mathrm{red},\hbar} g - g \star_{\mathrm{red},\hbar} f\big)$ *exist for any two analytic functions* $f, g \in \mathscr{A}(M_{\mathrm{red}})$. *They are given by*

$$\lim_{\hbar\to 0^-} f \star_{\mathrm{red},\hbar} g = fg \tag{6.18}$$

*and*

$$\lim_{\hbar\to 0^-} \frac{1}{\mathrm{i}\hbar}\big(f \star_{\mathrm{red},\hbar} g - g \star_{\mathrm{red},\hbar} f\big) = \{f, g\}_{\mathrm{red}} \tag{6.19}$$

*with the reduced Poisson bracket* $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$ *on* $M_{\mathrm{red}}$.

PROOF: This was proven in [17, Thm. 4.5] in the special case of signature $s = 1$ for a product $*_\hbar$ with $-\hbar \in \Omega$ fulfilling $f \star^{(1)}_{\mathrm{red},\hbar} g = f *_{-\hbar/2} g$ for all $f, g \in \mathscr{A}(M^{(1)}_{\mathrm{red}})$ and the corresponding Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_* = 2\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red}}$. The statements for arbitrary signatures $s$ follow immediately from Theorem 6.4 and Theorem 6.7. □

Note also that [17, Example 4.2] shows that there exist two functions $f, g \in \mathscr{A}(M_{\mathrm{red}})$ for which $f \star_{\mathrm{red},\hbar} g$ has non-trivial first order poles at all $\hbar = 1/m$ with $m \in \mathbb{N}$. As a consequence, the result of the above Proposition 6.12 cannot be generalized to limits over arbitrary sequences $(\hbar_k)_{k\in\mathbb{N}}$ in $\Omega$ with limit 0.

# A Symmetrized covariant derivatives

On a smooth manifold $M$ we define the spaces of complex tensor fields

$$\mathscr{S}^\ell(M) \coloneqq \Gamma^\infty(\mathrm{S}^\ell\, \mathrm{T}^{*,\mathbb{C}} M) \quad \text{and} \quad (\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M) \coloneqq \Gamma^\infty(\Lambda^k\, \mathrm{T}^{*,\mathbb{C}} M \otimes \mathrm{S}^\ell\, \mathrm{T}^{*,\mathbb{C}} M)$$

for all $k, \ell \in \mathbb{Z}$, as well as the $\mathbb{Z}$-graded algebra $\mathscr{S}^\bullet(M) \coloneqq \bigoplus_{\ell\in\mathbb{Z}} \mathscr{S}^\ell(M)$ with the usual pointwise symmetric tensor product $\vee$ and the $\mathbb{Z}^2$-graded algebra $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M) \coloneqq \bigoplus_{k,\ell\in\mathbb{Z}} (\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M)$ with product $\circ$ given by the combination of the pointwise antisymmetric and symmetric tensor products. In order to define graded commutators, a $\mathbb{Z}_2$-grading on these two algebras is needed: In the case of $\mathscr{S}^\bullet(M)$, this is the trivial one, in which all elements of $\mathscr{S}^\bullet(M)$ have even degree, and on $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ we consider the antisymmetric degree only, i.e. all elements of $(\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M)$ with $k \in 2\mathbb{Z}, \ell \in \mathbb{Z}$ have even degree and all elements of $(\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M)$ with $k \in 1 + 2\mathbb{Z}, \ell \in \mathbb{Z}$ have odd degree. This way, both $\mathscr{S}^\bullet(M)$ and $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ are graded commutative. For later use we also define the *total degree* Deg on $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ by setting

$$\operatorname{Deg}\Omega = (k+\ell)\Omega \tag{A.1}$$

for all $\Omega \in (\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M)$ with $k, \ell \in \mathbb{Z}$. Clearly Deg is a graded derivation of degree $(0,0)$. Note that $\mathscr{S}^0(M) \cong \mathscr{C}^\infty(M) \cong (\mathscr{A}\otimes\mathscr{S})^{0,0}(M)$ and that $\mathscr{S}^\bullet(M)$ is generated as a complex algebra by $\mathscr{S}^0(M) \oplus \mathscr{S}^1(M)$, whereas $(\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M)$ is generated as a complex algebra by $(\mathscr{A} \otimes \mathscr{S})^{0,0}(M) \oplus (\mathscr{A} \otimes \mathscr{S})^{1,0}(M) \oplus (\mathscr{A} \otimes \mathscr{S})^{0,1}(M)$.

We will need two other operators, the *Koszul differentials*: There are unique $\mathbb{C}$-linear graded derivations $\delta, \delta^*$ of $(\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M)$ of degree $(+1,-1)$ and $(-1,+1)$ that fulfil

$$\delta(1 \otimes \omega) = \omega \otimes 1 \qquad \text{as well as} \qquad \delta^*(\rho \otimes 1) = 1 \otimes \rho\,, \tag{A.2}$$

respectively, for all $\rho, \omega \in \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M)$. In local coordinates, $\delta(\rho \otimes \omega) = \sum_i (\mathrm{d}x^i \wedge \rho) \otimes (\iota_{\partial/\partial x^i}\omega)$ and $\delta^*(\rho \otimes \omega) = \sum_i (\iota_{\partial/\partial x^i}\rho) \otimes (\mathrm{d}x^i \vee \omega)$ hold for all $\rho \in \Gamma^\infty(\Lambda^k \mathrm{T}^{*,\mathbb{C}}M)$ and $\omega \in \Gamma^\infty(\mathrm{S}^\ell \mathrm{T}^{*,\mathbb{C}}M)$. Of course, $\delta$ and $\delta^*$ are not only $\mathbb{C}$-linear but even $\mathscr{C}^\infty(M)$-linear.

**Lemma A.1** *For the graded commutators we have*

$$[\delta,\delta] = 2\delta^2 = 0\,, \qquad [\delta^*,\delta^*] = 2(\delta^*)^2 = 0\,, \qquad [\delta,\delta^*] = [\delta^*,\delta] = \mathrm{Deg}\,,$$
$$[\mathrm{Deg},\delta] = -[\delta,\mathrm{Deg}] = 0 \qquad \textit{and} \qquad [\mathrm{Deg},\delta^*] = -[\delta^*,\mathrm{Deg}] = 0\,.$$

PROOF: One checks easily that this holds on $(\mathscr{A} \otimes \mathscr{S})^{0,0}(M)$, $(\mathscr{A} \otimes \mathscr{S})^{1,0}(M)$ and $(\mathscr{A} \otimes \mathscr{S})^{0,1}(M)$. But graded derivations are already uniquely determined by how they act on these spaces. □

One can also check that $\delta$ and $\delta^*$ commute with pullbacks. That is, whenever $\Psi : M \to N$ is smooth, then $\delta \circ \Psi^* = \Psi^* \circ \delta$ and $\delta^* \circ \Psi^* = \Psi^* \circ \delta^*$ where $\Psi^* : (\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(N) \to (\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M)$ denotes the usual pullback.

Next we consider the insertion of vector fields into the antisymmetric and symmetric part: Given $X \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$, then there exist unique $\mathbb{C}$-linear graded derivations $\iota^a_X, \iota^s_X$ of $(\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M)$ of degree $(-1,0)$ and $(0,-1)$ that fulfil

$$\iota^a_X(\rho \otimes 1) = \langle \rho, X\rangle \qquad \text{as well as} \qquad \iota^s_X(1 \otimes \omega) = \langle \omega, X\rangle\,, \tag{A.3}$$

respectively, for all $\rho, \omega \in \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M)$. Clearly, $\iota^a_X$ and $\iota^s_X$ are even $\mathscr{C}^\infty(M)$-linear and:

**Lemma A.2** *For the graded commutators we have*

$$[\iota^a_X,\iota^a_Y] = [\iota^a_X,\iota^s_Y] = [\iota^s_X,\iota^a_Y] = [\iota^a_X,\delta^*] = [\iota^s_X,\delta] = 0\,,$$
$$[\iota^a_X,\delta] = \iota^s_X\,, \quad [\iota^s_X,\delta^*] = \iota^a_X\,, \quad [\mathrm{Deg},\iota^a_X] = -\iota^a_X\,, \quad [\mathrm{Deg},\iota^s_X] = -\iota^s_X$$

*for all* $X, Y \in \mathscr{C}^\infty(M)$.

PROOF: These identities are easy to check on $(\mathscr{A} \otimes \mathscr{S})^{0,0}(M)$, $(\mathscr{A} \otimes \mathscr{S})^{1,0}(M)$ and $(\mathscr{A} \otimes \mathscr{S})^{0,1}(M)$. □

We see that the $\mathbb{C}$-linear span of $\delta$, $\delta^*$, Deg and all $\iota^a_X$ and $\iota^s_X$ with $X \in \mathscr{C}^\infty(M)$ in the graded Lie algebra of $\mathbb{C}$-linear graded derivations of $(\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M)$ is a graded Lie subalgebra. Now we can define exterior covariant derivatives:

**Definition A.3** *A $\mathbb{C}$-linear graded derivation $D$ of $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ of degree $(+1,0)$ that fulfils $D(\rho\otimes 1) = \mathrm{d}\rho\otimes 1$ for all $\rho\in\Gamma^\infty(\Lambda^\bullet\,\mathrm{T}^{*,\mathbb{C}}M)$ is called an* exterior covariant derivative *on $M$.*

For every covariant derivative $\nabla$ on $M$ there exists a unique exterior covariant derivative $D^\nabla$ on $M$ that fulfils

$$\iota^a_X D^\nabla(1\otimes\omega) = 1\otimes\nabla_X\omega \tag{A.4}$$

for all $\rho\in\Gamma^\infty(\Lambda^\bullet\,\mathrm{T}^{*,\mathbb{C}}M)$, $\omega\in\mathscr{S}^\bullet(M)$ and $X\in\Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$. In local coordinates,

$$D^\nabla(\rho\otimes\omega) = \mathrm{d}\rho\otimes\omega + \sum_i(\mathrm{d}x^i\wedge\rho)\otimes\nabla_{\partial/\partial x^i}\omega \tag{A.5}$$

for all $\rho\in\Gamma^\infty(\Lambda^\bullet\,\mathrm{T}^{*,\mathbb{C}}M)$ and $\omega\in\mathscr{S}^\bullet(M)$. Conversely, every exterior covariant derivative $D$ on $M$ determines a unique covariant derivative $\nabla^D$ on $M$ that fulfils

$$\langle\omega,\nabla^D_X Y\rangle = X\big(\langle\omega,Y\rangle\big) - \langle\nabla^D_X\omega,Y\rangle = X\big(\langle\omega,Y\rangle\big) - \iota^s_Y\iota^a_X D(1\otimes\omega) \tag{A.6}$$

for all $X,Y\in\Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$ and all $\omega\in\Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M)$. One can check that $\nabla^{D^\nabla} = \nabla$ for every covariant derivative $\nabla$ on $M$ and that $D^{\nabla^D} = D$ for every exterior covariant derivative on $M$. So there is a 1-to-1 correspondence between covariant derivatives and exterior covariant derivatives.

We say that an exterior covariant derivative $D$ is torsion-free if the associated covariant derivative $\nabla^D$ is torsion-free.

**Proposition A.4** *An exterior covariant derivative $D$ on $M$ is torsion-free if and only if $[D,\delta]=0$.*

PROOF: Denote the torsion of $\nabla^D$ by $T$. We compute

$$\begin{aligned}
\iota^a_Y\iota^a_X[D,\delta](1\otimes\omega) &= \iota^a_Y\iota^a_X(\mathrm{d}\omega\otimes 1) + \iota^a_Y\iota^a_X\delta D(1\otimes\omega)\\
&= 2\langle\mathrm{d}\omega, X\wedge Y\rangle - \iota^a_Y\delta\iota^a_X D(1\otimes\omega) + \iota^a_Y\iota^s_X D(1\otimes\omega)\\
&= 2\langle\mathrm{d}\omega, X\wedge Y\rangle - \iota^a_Y\delta(1\otimes\nabla_X\omega) + \iota^s_X\iota^a_Y D(1\otimes\omega)\\
&= 2\langle\mathrm{d}\omega, X\wedge Y\rangle - \langle\nabla_X\omega, Y\rangle + \langle\nabla_Y\omega, X\rangle\\
&= 2\langle\mathrm{d}\omega, X\wedge Y\rangle - X\big(\langle\omega,Y\rangle\big) + Y\big(\langle\omega,X\rangle\big) + \langle\omega,\nabla_X Y\rangle - \langle\omega,\nabla_Y X\rangle\\
&= \langle\omega, -[X,Y] + \nabla_X Y - \nabla_Y X\rangle\\
&= \langle\omega, T_{X,Y}\rangle\,.
\end{aligned}$$

In particular, if $[D,\delta]=0$, then $\nabla^D$ is torsion-free. Conversely, if $\nabla^D$ is torsion-free, then $[D,\delta]$ vanishes on $(\mathscr{A}\otimes\mathscr{S})^{0,1}(M)$ by the above calculation. But $[D,\delta]$ is a $\mathbb{C}$-linear graded derivation of $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ of degree $(+2,-1)$, so $[D,\delta]=0$ in this case. □

If $g\in\mathscr{S}^2(M)$ is real and non-degenerate, then there exists a unique exterior covariant derivative $D$ on $M$ that fulfils $D(1\otimes g) = 0 = [D,\delta]$, namely the one corresponding to the Levi-Civita connection. This *exterior Levi-Civita connection* will be interesting for us:

**Lemma A.5** *Let $M$ be a smooth manifold, $g \in \Gamma^\infty(\mathrm{S}^2\,\mathrm{T}^*M)$ a real and non-degenerate symmetric tensor with Levi-Civita connection $\nabla$, and $\Phi\colon M \to M$ a diffeomorphism. If $\nabla_X \Phi^*(g) = 0$ for all $X \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$, then the exterior Levi-Civita connection $D$ associated to $g$ commutes with the pullback $\Phi^*$, i.e. $D\Phi^*(\Omega) = \Phi^*(D\Omega)$ for all $\Omega \in (\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M)$.*

PROOF: It suffices to show that $D'\colon (\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M) \to (\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$, $\Omega \mapsto D'\Omega := (\Phi^{-1})^*(D\Phi^*(\Omega))$ is an exterior covariant derivative and fulfils $D'(1\otimes g) = 0 = [D', \delta]$: It is easy to see that $D'$ is a $\mathbb{C}$-linear graded derivation of $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ of degree $(+1, 0)$ that fulfils $D'(\rho\otimes 1) = \mathrm{d}\rho \otimes 1$ for all $\rho \in \Gamma^\infty(\Lambda^\bullet\,\mathrm{T}^{*,\mathbb{C}}M)$, hence an exterior covariant derivative. It commutes with $\delta$ (in the graded sense) because $\delta$ commutes with $D$ and all pullbacks. Finally, $D'(1\otimes g)$ holds because $\nabla_X\Phi^*(g) = 0$ for all $X \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$. □

Note that the condition $\nabla\Phi^*(g) = 0$ is fulfilled e.g. if $\Phi^*(g) = g$, but also more generally if $\Phi^*(g) = \lambda g$ with $\lambda \in \mathbb{C}$.

**Proposition A.6** *Let $M$ be a smooth manifold endowed with a free and proper action $\cdot \triangleright \cdot$ of a Lie group $G$ and a $G$-invariant exterior covariant derivative $D$ on $M$ (i.e. $D$ commutes with the action of $G$ on $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ by pullbacks like in the previous Lemma A.5). Moreover, write* $\mathrm{Pr}\colon M \to M/G$ *for the canonical projection onto the quotient manifold $M/G$ and assume we have chosen a smooth $G$-invariant complement $\Xi^{\mathbb{C}} = \bigcup_{p\in M}\Xi^{\mathbb{C}}_p$ of* $\ker(\mathrm{T}\,\mathrm{Pr})$*, i.e. a linear subbundle of* $\mathrm{T}^{\mathbb{C}}M$ *such that* $\mathrm{T}^{\mathbb{C}}M = \Xi^{\mathbb{C}} \oplus \ker(\mathrm{T}\,\mathrm{Pr})$ *and such that $\Xi^{\mathbb{C}}_{g\triangleright p} = \big(\mathrm{T}_p(\,\cdot \triangleleft g)\big)(\Xi^{\mathbb{C}}_p)$ for all $p \in M$. Let $\Theta_\Xi\colon \Gamma^\infty(\mathrm{T}^{\mathbb{C}}M) \to \Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$ be the corresponding projection on this subbundle $\Xi^{\mathbb{C}}$ and $\Theta^*_\Xi\colon \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M) \to \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M)$ its dual projection. Then*

$$\mathrm{Pr}^*\left(D_{\mathrm{red}}\Omega\right) := (\Theta^*_\Xi)^{\otimes(k+1+\ell)} D\,\mathrm{Pr}^*(\Omega) \tag{A.7}$$

*for all $\Omega \in (\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M/G)$, $k, \ell \in \mathbb{N}_0$ defines an exterior covariant derivative on $M/G$. If $D$ is torsion-free, then $D_{\mathrm{red}}$ also remains torsion-free.*

PROOF: Since $D$ and $\Xi^{\mathbb{C}}$ are $G$-invariant, it follows that $(\Theta^*_\Xi)^{\otimes(k+1+\ell)} D\,\mathrm{Pr}^*(\Omega)$ is $G$-invariant, so (A.7) does describe a well-defined $\mathbb{C}$-linear endomorphism $D_{\mathrm{red}}$ of $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M/G)$ of degree $(1, 0)$ and one can also check that $D_{\mathrm{red}}$ is again a graded derivation. Using that $\mathrm{T}\,\mathrm{Pr} \circ \Theta_\Xi = \mathrm{T}\,\mathrm{Pr}$ one sees that $D_{\mathrm{red}}(\rho\otimes 1) = \mathrm{d}\rho\otimes 1$ holds for all $\rho \in \Gamma^\infty(\Lambda^\bullet\mathrm{T}^{*,\mathbb{C}}M)$. As $\delta$ commutes with pullbacks and $\Theta^*_\Xi$, one also finds that $D_{\mathrm{red}}$ (graded) commutes with $\delta$ if $D$ does. □

Next we consider the graded commutator of an exterior covariant derivative $D$ on a smooth manifold $M$ with $\delta^*$, which is a $\mathbb{C}$-linear graded derivation of $(\mathscr{A}\otimes\mathscr{S})^{\bullet,\bullet}(M)$ of degree $(0, +1)$ and satisfies $[D, \delta^*](f) = \delta^* Df = 1 \otimes \mathrm{d}f$ for all $f \in \mathscr{C}^\infty(M)$. So $[D, \delta^*]$ restricts to a $\mathbb{C}$-linear derivation $D^{\mathrm{sym}}$ of $\mathscr{S}^\bullet(M)$ of degree 1.

**Definition A.7** *A $\mathbb{C}$-linear derivation $\Delta$ of $\mathscr{S}^\bullet(M)$ of degree 1 that fulfils $\Delta f = \mathrm{d}f$ for all $f \in \mathscr{C}^\infty(M)$ is called a* symmetrized covariant derivative*, and for every exterior* *covariant derivative $D$ of $M$ we define its induced symmetrized covariant derivative $D^{\mathrm{sym}}\colon \mathscr{S}^\bullet(M) \to \mathscr{S}^\bullet(M)$ by*

$$1 \otimes D^{\mathrm{sym}}\omega := [D, \delta^*](1 \otimes \omega) \tag{A.8}$$

*for all $\omega \in \mathscr{S}^\bullet(M)$.*

Given an exterior covariant derivative $D$, we compute that its induced symmetrized covariant derivative $D^{\mathrm{sym}}$ fulfils

$$\begin{aligned} 1 \otimes \iota_Y \iota_X D^{\mathrm{sym}}\omega &= \iota^s_Y \iota^s_X [D, \delta^*](1 \otimes \omega) \\ &= \iota^s_Y \iota^s_X \delta^* D(1 \otimes \omega) \\ &= \iota^s_Y \iota^a_X D(1 \otimes \omega) + \iota^s_Y \delta^* \iota^s_X D(1 \otimes \omega) \\ &= 1 \otimes \iota_Y \nabla^D_X \omega + \iota^a_Y \iota^s_X D(1 \otimes \omega) \\ &= 1 \otimes \big(\iota_Y \nabla^D_X \omega + \iota_X \nabla^D_Y \omega\big) \\ &= 1 \otimes \big(\langle \nabla^D_X \omega, Y\rangle + \langle \nabla^D_Y \omega, X\rangle\big) \end{aligned}$$

for all $\omega \in \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M)$. So in local coordinates, $D^{\mathrm{sym}}\omega = \mathrm{d}x^i \vee \nabla^D_{\partial/\partial x^i}\omega$.

Conversely, every $\mathbb{C}$-linear derivation $\Delta$ of $\mathscr{S}^\bullet(M)$ that fulfils $\Delta f = \mathrm{d}f$ for all $f \in \mathscr{C}^\infty(M)$ defines a covariant derivative $\nabla^\Delta$ on $M$ by

$$\big\langle \nabla^\Delta_X \omega, Y \big\rangle := \langle \Delta\omega, X \vee Y\rangle + \frac{1}{2}\big(X(\langle \omega, Y\rangle) - Y(\langle \omega, X\rangle) - \langle \omega, [X,Y]\rangle\big) \tag{A.9}$$

for all $\omega \in \Gamma^\infty(\mathrm{T}^{*,\mathbb{C}}M)$ and all $X, Y \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$. This covariant derivative $\nabla^\Delta$ then is torsion-free because

$$\langle \nabla^\Delta_X \omega, Y\rangle - \langle \nabla^\Delta_Y \omega, X\rangle = X(\langle \omega, Y\rangle) - Y(\langle \omega, X\rangle) - \langle \omega, [X,Y]\rangle \tag{A.10}$$

and fulfils

$$\langle \nabla^\Delta_X \omega, Y\rangle + \langle \nabla^\Delta_Y \omega, X\rangle = 2\langle \Delta\omega, X \vee Y\rangle = \iota_Y \iota_X \Delta\omega\,. \tag{A.11}$$

Consequently there is a 1-to-1-correspondence between torsion-free covariant derivatives (or their exterior covariant derivatives) and symmetrized covariant derivatives. For the reduction of symmetrized covariant derivatives we get:

**Proposition A.8** *Let $M$, $G$, $D$, $\mathrm{Pr}$ and $\Xi$ be as in Proposition A.6. Then $D^{\mathrm{sym}}_{\mathrm{red}}$, the symmetrized covariant derivative on $M/G$ constructed out of the reduced exterior covariant derivative $D_{\mathrm{red}}$, fulfils*

$$\mathrm{Pr}^*\left(D^{\mathrm{sym}}_{\mathrm{red}}\omega\right) = (\Theta^*_\Xi)^{\otimes(k+1)} D^{\mathrm{sym}}\,\mathrm{Pr}^*(\omega) \tag{A.12}$$

*for all $\omega \in \mathscr{S}^k(M/G)$, $k \in \mathbb{N}_0$.*

Proof: As $\delta^*$ commutes with the pullback $\mathrm{Pr}^*$ and the projection $\Theta^*_\Xi$ this follows immediately from (A.7). □

Being an endomorphism of $\mathscr{S}^\bullet(M)$, a symmetrized covariant derivative can be iterated. Given $k \in \mathbb{N}_0$, $X_0 \in \Gamma^\infty(\mathrm{S}^0\,\mathrm{T}^{\mathbb{C}}M)$, ..., $X_k \in \Gamma^\infty(\mathrm{S}^k\,\mathrm{T}^{\mathbb{C}}M)$, then

$$\mathscr{C}^\infty(M) \ni f \mapsto \sum_{r=0}^{k} \big\langle (D^{\mathrm{sym}})^r f, X_r \big\rangle \in \mathscr{C}^\infty(M)$$

is a differential operator of degree $k$. Conversely, by induction over their symbols, one can show that all differential operators of degree $k$ on $\mathscr{C}^\infty(M)$ are of this form. So symmetrized covariant derivatives allow to describe differential operators rather explicitly but without requiring a choice of coordinates.

So far, all results in the appendix would also make sense in a real setting. The decomposition of a covariant derivative on a complex manifold into holomorphic and antiholomorphic parts, which we introduce now, requires to use the complexified tangent space. Let $M$ be a complex manifold, so that its complexified tangent and cotangent space split into $(1,0)$ and $(0,1)$ parts. Consequently

$$\mathscr{S}^k(M) = \bigoplus_{p+q=k} \mathscr{S}^{(p,q)}(M) \tag{A.13}$$

and

$$(\mathscr{A}\otimes\mathscr{S})^{k,\ell}(M) = \bigoplus_{\substack{p+q=k\\ r+s=\ell}} (\mathscr{A}\otimes\mathscr{S})^{(p,q),(r,s)}(M) \tag{A.14}$$

also split into subspaces

$$\mathscr{S}^{(p,q)}(M) := \Gamma^\infty\big(\mathrm{S}^p\,\mathrm{T}^{*,(1,0)}M \vee \mathrm{S}^q\,\mathrm{T}^{*,(0,1)}M\big)\,, \tag{A.15}$$

$$\begin{aligned}(\mathscr{A}\otimes\mathscr{S})&^{(p,q),(r,s)}(M)\\ &:= \Gamma^\infty\big(\Lambda^p\,\mathrm{T}^{*,(1,0)}M \wedge \Lambda^q\,\mathrm{T}^{*,(0,1)}M \otimes \mathrm{S}^r\,\mathrm{T}^{*,(1,0)}M \vee \mathrm{S}^s\,\mathrm{T}^{*,(0,1)}M\big)\,.\end{aligned} \tag{A.16}$$

Note that $\delta$ and $\delta^*$ are compatible with this splitting in the sense that

$$\begin{aligned}\delta\big((\mathscr{A}\otimes\mathscr{S})&^{(p,q),(r,s)}(M)\big) \subseteq\\ &\subseteq (\mathscr{A}\otimes\mathscr{S})^{(p+1,q),(r-1,s)}(M) \oplus (\mathscr{A}\otimes\mathscr{S})^{(p,q+1),(r,s-1)}(M)\,,\end{aligned} \tag{A.17}$$

$$\begin{aligned}\delta^*\big((\mathscr{A}\otimes\mathscr{S})&^{(p,q),(r,s)}(M)\big) \subseteq\\ &\subseteq (\mathscr{A}\otimes\mathscr{S})^{(p-1,q),(r+1,s)}(M) \oplus (\mathscr{A}\otimes\mathscr{S})^{(p,q-1),(r,s+1)}(M)\end{aligned} \tag{A.18}$$

hold for all $p,q,r,s \in \mathbb{N}_0$. For exterior covariant derivatives, there is a similar compatibility condition:

**Definition A.9** *Let $M$ be a complex manifold and $D$ an exterior covariant derivative on $M$. Then $D$ is said to be* compatible with the complex structure *if*

$$D\big((\mathscr{A}\otimes\mathscr{S})^{(p,q),(r,s)}(M)\big) \subseteq (\mathscr{A}\otimes\mathscr{S})^{(p+1,q),(r,s)}(M) \oplus (\mathscr{A}\otimes\mathscr{S})^{(p,q+1),(r,s)}(M) \tag{A.19}$$

*holds for all $p,q,r,s \in \mathbb{N}_0$.*

If $D$ is the exterior covariant derivative associated to a covariant derivative $\nabla$, then it follows from Equations (A.5) and (A.6) that $D$ is compatible with the complex structure if and only if $\nabla_X$ preserves the holomorphic and antiholomorphic parts of the tangent bundle for all $X \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}}\mathbb{C}^{1+n})$, i.e. $\nabla_X(\Gamma^\infty(\mathrm{T}^{(1,0)}\mathbb{C}^{1+n})) \subseteq \Gamma^\infty(\mathrm{T}^{(1,0)}\mathbb{C}^{1+n})$ and $\nabla_X(\Gamma^\infty(\mathrm{T}^{(0,1)}\mathbb{C}^{1+n})) \subseteq \Gamma^\infty(\mathrm{T}^{(0,1)}\mathbb{C}^{1+n})$. As an example, this is well-known to be the case for the Levi-Civita covariant derivative on a Kähler manifold.

Using (A.18) it is easy to check that condition (A.19) implies that the symmetrized covariant derivative fulfils $D^{\mathrm{sym}}(\mathscr{S}^{(p,q)}(M)) \subseteq \mathscr{S}^{(p+1,q)}(M) \oplus \mathscr{S}^{(p,q+1)}(M)$.

**Definition A.10** *Let $M$ be a complex manifold and $D$ an exterior covariant derivative compatible with the complex structure. Then we define*

$$D_{\mathrm{hol}}, D_{\overline{\mathrm{hol}}} \colon (\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M) \to (\mathscr{A} \otimes \mathscr{S})^{\bullet,\bullet}(M) \tag{A.20}$$

*and*

$$D^{\mathrm{sym}}_{\mathrm{hol}}, D^{\mathrm{sym}}_{\overline{\mathrm{hol}}} \colon \mathscr{S}^\bullet(M) \to \mathscr{S}^\bullet(M) \tag{A.21}$$

*as the $(1,0)$ and $(0,1)$-components of $D$ and $D^{\mathrm{sym}}$, respectively, i.e.*

$$D_{\mathrm{hol}} := \sum_{p,q,r,s\in\mathbb{N}_0} \Theta^{*,(p+1,q),(r,s)} D \Theta^{*,(p,q),(r,s)}\,, \tag{A.22}$$

$$D_{\overline{\mathrm{hol}}} := \sum_{p,q,r,s\in\mathbb{N}_0} \Theta^{*,(p,q+1),(r,s)} D \Theta^{*,(p,q),(r,s)} \tag{A.23}$$

*and*

$$D^{\mathrm{sym}}_{\mathrm{hol}} := \sum_{p,q\in\mathbb{N}_0} \Theta^{*,(p+1,q)} D^{\mathrm{sym}} \Theta^{*,(p,q)}\,, \tag{A.24}$$

$$D^{\mathrm{sym}}_{\overline{\mathrm{hol}}} := \sum_{p,q\in\mathbb{N}_0} \Theta^{*,(p,q+1)} D^{\mathrm{sym}} \Theta^{*,(p,q)} \tag{A.25}$$

*with projections $\Theta^{*,(p,q),(r,s)} \colon (\mathscr{A} \otimes \mathscr{S})^{(\bullet,\bullet),(\bullet,\bullet)}(M) \to (\mathscr{A} \otimes \mathscr{S})^{(p,q),(r,s)}(M)$ and $\Theta^{*,(p,q)} \colon \mathscr{S}^{(\bullet,\bullet)}(M) \to \mathscr{S}^{(p,q)}(M)$ on graded subspaces.*

Because of the required compatibility with the complex structure one gets $D = D_{\mathrm{hol}} + D_{\overline{\mathrm{hol}}}$ and $D^{\mathrm{sym}} = D^{\mathrm{sym}}_{\mathrm{hol}} + D^{\mathrm{sym}}_{\overline{\mathrm{hol}}}$. Furthermore,

$$\big[D_{\mathrm{hol}}, \delta^*\big](1 \otimes \omega) = 1 \otimes D^{\mathrm{sym}}_{\mathrm{hol}}\omega \tag{A.26}$$

holds for all $\omega \in \mathscr{S}^\bullet(M)$, and analogously for the antiholomorphic part. Consequently:

**Proposition A.11** *Let $M$ be a complex manifold, $D$ an exterior covariant derivative on $M$ that is compatible with the complex structure, and $D^{\mathrm{sym}}$ its symmetrized covariant derivative. Then*

$$\Theta^{*,(k,0)}(D^{\mathrm{sym}})^k f = (D^{\mathrm{sym}}_{\mathrm{hol}})^k f \qquad \text{and} \qquad \Theta^{*,(0,k)}(D^{\mathrm{sym}})^k f = (D^{\mathrm{sym}}_{\overline{\mathrm{hol}}})^k f \tag{A.27}$$

*hold for all $f \in \mathscr{C}^\infty(M)$ and all $k \in \mathbb{N}_0$, with $\Theta^{*,(k,0)}$ as in the previous Definition A.10.*

PROOF: This follows immediately from the decomposition $D^{\mathrm{sym}} = D^{\mathrm{sym}}_{\mathrm{hol}} + D^{\mathrm{sym}}_{\overline{\mathrm{hol}}}$. $\square$

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# Paper II:

# Strict quantization of coadjoint orbits

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**Abstract**

For any semisimple coadjoint orbit $\hat{\mathcal{O}}$ of a complex connected semisimple Lie group $\hat{G}$, we obtain a family of strict $\hat{G}$-invariant products $\hat{*}_\hbar$ on the space of holomorphic functions on $\hat{\mathcal{O}}$. For any semisimple coadjoint orbit $\mathcal{O}$ of a real connected semisimple Lie group $G$, we obtain strict $G$-invariant products $*_\hbar$ on a space $\mathcal{A}(\mathcal{O})$ of certain analytic functions on $\mathcal{O}$ by restriction. $\mathcal{A}(\mathcal{O})$ endowed with one of the products $*_\hbar$ is a $G$-Fréchet algebra, and the formal expansion of the products around $\hbar = 0$ determines a formal deformation quantization of $\mathcal{O}$, which is of Wick type if $G$ is compact. We study a generalization of a Wick rotation, which provides isomorphisms between the quantizations obtained for different real orbits with the same complexification. Our construction relies on an explicit computation of the canonical element of the Shapovalov pairing between generalized Verma modules, and complex analytic results on the extension of holomorphic functions.

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[1]Supported by the Danish National Research Foundation through the Centre of Symmetry and Deformation (DNRF92)

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# 1 Introduction

The quantization problem in physics asks how to associate a quantum system to a classical mechanical system, such that the classical system can be recovered from the quantum system in a classical limit. Since both systems can be studied by their observable algebras, a first step is to quantize the classical observable algebra. This algebra is usually the Poisson algebra $\mathscr{C}^\infty(M)$ of smooth functions on a Poisson manifold $M$. The observable algebra of a quantum mechanical system is some non-commutative $^*$-algebra $\mathcal{A}$, which in many cases is obtained from a C$^*$-algebra. In a second step, the states of the quantum mechanical system can be obtained as normalized positive linear functionals on $\mathcal{A}$. To define their superposition, one has to represent $\mathcal{A}$ on a (pre) Hilbert space, so that the superposition of two vector states can be defined as the vector state corresponding to the sum of the two vectors.

*Formal deformation quantization*, as introduced in [2], has proven to be a fruitful theory for studying some aspects of the quantization problem. One views Planck's constant $\hbar$ as a formal parameter $\nu$ and tries to find so-called *formal star products* $\star$ on $\mathcal{A} = \mathscr{C}^\infty(M)[[\nu]]$, which may be thought of as the infinite jet of a full solution to the quantization problem at $\hbar = 0$. These star products are associative $\mathbb{C}[[\nu]]$-bilinear products for which $1 \in \mathscr{C}^\infty(M)$ is a unit and which satisfy the correct classical limit. To be more precise, if $f, g \in \mathscr{C}^\infty(M)$ and $f \star g = \sum_{r=0}^\infty \nu^r C_r(f,g)$ with operators $C_r \colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, then one requires $C_0$ to be the pointwise multiplication, $C_0(f,g) = fg$, and that the quantization is in the direction of the Poisson bracket, $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$. Usually one also requires the $C_r$ to be bidifferential operators, so that $\star$ is local and can be restricted to open subsets of $M$. Using formal power series means on the one hand that we cannot substitute $\nu$ with the real value of Planck's constant as required for direct physical applications, but on the other hand transfers the quantization problem to algebra by neglecting analytic aspects, such as convergence of the power series. Consequently, many powerful tools become available for its study, and existence and classification results were obtained in [5, 14, 18, 36] for symplectic manifolds, whereas in the more general case of Poisson manifolds they follow from Kontsevich's formality theorem [28]. One can also study formal star products that are equivariant with respect to the action of a Lie group, where the classification follows for example from [15].

A complete solution of the quantization problem consists of a Hilbert space $H$ together with a quantization map that associates a quantum observable, usually a self-adjoint operator on $H$, to any classical observable. This motivates the definition of a *strict quantization* [30,34,35,37], which is some field of "nice" $^*$-algebras $\mathcal{A}_\hbar$ (over $\mathbb{C}$) depending "nicely" on a parameter $\hbar$ ranging over some subset of $\mathbb{C}$, with $\mathcal{A}_0$ being a completion of the classical observable algebra and the deformation being in the direction of the Poisson bracket. However, strict quantizations are much harder to understand than formal deformation quantizations. There are many examples of strict quantizations in different contexts, and therefore there are several ways to formalize the above definition, i.e. specifying the parameter set and what "nice" actually means[2]. No general existence results are known, and a classification seems completely hopeless due to the increased complexity.

There are two prominent constructions of strict quantizations. The first is due to Rieffel [37] who, using oscillatory integrals, deforms the product on a Fréchet algebra endowed with an isometric action of $\mathbb{R}^d$. If the original algebra is a $\mathrm{C}^*$-algebra, then Rieffel constructs a $\mathrm{C}^*$-algebraic quantization. A generalization to negatively curved Kählerian Lie groups can be found in [6]. The second construction, due to Natsume, Nest, and Peter [35], essentially glues convergent versions of the Weyl product on charts to obtain a $\mathrm{C}^*$-algebraic quantization. However, both methods work only for some symplectic manifolds and fail for example for the 2-sphere with its $\mathrm{SO}(3)$-invariant symplectic structure [38]. They also make crucial use of the finite dimensionality of the classical mechanical system, so it remains unclear how to apply them to quantum field theories, despite such field theories fitting into the framework of formal deformation quantization.

Another approach to strict quantization was proposed by Beiser and Waldmann in [3,4,40]. They start with formal deformation quantizations, which are well-understood, and try to find subalgebras on which the formal power series converge. Such subalgebras are usually defined using additional geometric structures, and can be completed with respect to a topology in which the product is continuous. This approach was carried out explicitly for star products of exponential type on possibly infinite-dimensional vector spaces [39], for the linear Poisson structure on the dual of a Lie algebra [17], and for the hyperbolic disc $\mathbb{D}^n$ using an invariant star product obtained

[2] We attempted to give a definition that captures the most relevant cases in Definition Intro.2.28.

via phase space reduction [29]. See also [41] for a survey. In this paper, we extend this approach to semisimple coadjoint orbits of connected semisimple Lie groups, which gives a much larger class of geometrically interesting examples.

Coadjoint orbits play an important role in different areas of mathematics. In the representation theory of unitary Lie groups they appear e.g. in the Kirillov orbit method [27], while in symplectic geometry they are related to momentum maps. Basic examples of coadjoint orbits are hyperbolic discs and complex projective spaces, including the 2-sphere. Any coadjoint orbit $\mathcal{O}$ of a Lie group $G$ has a canonical $G$-invariant symplectic form, and if $\mathcal{O}$ is semisimple and $G$ is compact, connected, and semisimple then there is a unique compatible $G$-invariant complex structure that makes $\mathcal{O}$ a Kähler manifold.

Constructions of star products on coadjoint orbits are due to many authors [1, 8–11, 19, 25, 26]. In this paper, we focus on semisimple coadjoint orbits of connected semisimple Lie groups, and the algebraic construction of Alekseev–Lachowska [1]. The canonical element $F_\lambda$ of the Shapovalov pairing between certain generalized Verma modules satisfies an associativity equation generalizing that of a Drinfel'd twist. This twist induces a formal product for holomorphic functions on a complex orbit and a formal star product for smooth functions on a real orbit, and those products are compatible by restriction. It is very convenient that we can pass from one setting to the other: We will mainly work in the complex setting, which is more convenient for obtaining continuity estimates, and restrict to the real setting only in the very end.

Our first result uses methods developed by Ostapenko [32] to obtain an explicit formula for the canonical element of the Shapovalov pairing for a semisimple Lie algebra $\mathfrak{g}$:

**Main Theorem I** *The Shapovalov pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim}_{\lambda} \colon \mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathbb{C}$ is non-degenerate if $\lambda \in \tilde{\Lambda}$, and in this case its canonical element $F_\lambda \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \mathbin{\hat{\otimes}} \mathscr{U}(\tilde{\mathfrak{n}}^-)$ is given by*

$$F_\lambda = \sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w)\,. \tag{1.1}$$

The notation is explained in detail in Section 3. For now, it suffices to mention that the Shapovalov pairing is a pairing between the universal enveloping algebras of two nilpotent Lie subalgebras $\tilde{\mathfrak{n}}^\pm$ of $\mathfrak{g}$, depending on a parameter $\lambda \in \mathfrak{g}^*$. The sum is over a set of words $\tilde{W}$ related to the root system of $\mathfrak{g}$, the $p^w_\lambda(\alpha_w)$ are non-zero coefficients which are defined by an explicit formula, $X_w$ and $Y_w$ are elements of $\mathscr{U}\mathfrak{g}$ and $\tilde{\pi}^\pm_\lambda$ maps these elements to $\mathscr{U}(\tilde{\mathfrak{n}}^\pm)$. The element $F_\hbar$, which induces the star product, is obtained by rescaling $\lambda$, and doing so the coefficients $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1}$ will depend rationally on $\hbar$, with a countable set of poles $P$ that accumulate only at 0. It seems as if explicit formulas for deformation quantizations received special attention by various authors, and (1.1) provides such a formula that works in great generality.

As mentioned above, the formal expansion of $F_\hbar$ induces formal products in a complex and a real setting. Furthermore, we also obtain a family of actual (non-formal) products for holomorphic polynomial functions in the complex setting and for polynomial functions in the real setting, parametrized by $\mathbb{C} \setminus P$, since only finitely many elements of the infinite sum defining $F_\hbar$ are non-zero on polynomials. All these products are $G$-invariant, and under some conditions on the Cartan subalgebra used in the construction they are also Hermitian, meaning that $\overline{f *_\hbar g} = \overline{g} *_{\overline{\hbar}} \overline{f}$. In the real setting and for a compact semisimple connected Lie group $G$, the formal star product is of Wick type [24] with respect to the Kähler complex structure on the coadjoint orbit, meaning that it derives the first argument only in holomorphic directions and the second argument only in antiholomorphic directions.

The next major step after constructing the star product is to use the explicit formulas to prove its continuity in the complex setting with respect to the topology of locally uniform convergence. This topology is locally convex and we can extend the product to a continuous product on the completion of the holomorphic polynomials. Using methods from analytic geometry we identify this completion with the space of holomorphic functions.

**Main Theorem II** *For any semisimple coadjoint orbit $\hat{\mathcal{O}}$ of a connected semisimple complex Lie group $G$, there is a family of products $\hat{*}_\hbar \colon \mathrm{Hol}(\hat{\mathcal{O}}) \times \mathrm{Hol}(\hat{\mathcal{O}}) \to \mathrm{Hol}(\hat{\mathcal{O}})$ for $\hbar \in \mathbb{C} \setminus P$, where every product $\hat{*}_\hbar$ is $G$-invariant and continuous with respect to the topology of locally uniform convergence. The dependence of $\hat{*}_\hbar$ on $\hbar$ is holomorphic.*

This result is certainly interesting in its own right. However, as mentioned above, we can also restrict it to real coadjoint orbits $\mathcal{O} \subseteq \hat{\mathcal{O}}$. Denote by $\mathcal{A}(\mathcal{O})$ the class of functions on $\mathcal{O}$ that extend to holomorphic functions on $\hat{\mathcal{O}}$ (if a function extends, its extension is unique), which contains the polynomials. We define the topology of extended locally uniform convergence on $\mathcal{A}(\mathcal{O})$ by saying that a sequence of functions in $\mathcal{A}(\mathcal{O})$ converges if the corresponding sequence of extensions converges locally uniformly, so that $\mathcal{A}(\mathcal{O})$ is homeomorphic to $\mathrm{Hol}(\hat{\mathcal{O}})$.

**Main Theorem III** *For any semisimple coadjoint orbit $\mathcal{O}$ of a connected semisimple real Lie group $G$, there is a family of product $*_\hbar \colon \mathcal{A}(\mathcal{O}) \times \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\mathcal{O})$ for $\hbar \in \mathbb{C} \setminus P$, where every product $*_\hbar$ is $G$-invariant and continuous with respect to the topology of extended locally uniform convergence. The dependence of $*_\hbar$ on $\hbar$ is holomorphic. The formal expansion of $*_\hbar$ around $0$ is a formal star product deforming the $G$-invariant symplectic form of $\mathcal{O}$.*

For the hyperbolic disc the quantum algebra $(\mathcal{A}(\mathbb{D}^n), *_\hbar)$ agrees with the algebra obtained in [29] while for the 2-sphere, $(\mathcal{A}(\mathbb{S}^2), *_\hbar)$ is the algebra considered in [16].

Since we constructed a quantization of the holomorphic functions on a complex coadjoint orbit and the restriction $\mathrm{Hol}(\hat{\mathcal{O}}) \to \mathcal{A}(\mathcal{O})$ is an isomorphism, the quantizations of different real orbits with the same complexification are related:

**Main Theorem IV** *If $\mathcal{O}$ and $\mathcal{O}'$ are coadjoint orbits of real semisimple connected Lie groups with the same complexification and through one common semisimple element, then the algebras $(\mathcal{A}(\mathcal{O}), *_\hbar)$ and $(\mathcal{A}(\mathcal{O}'), *'_\hbar)$ are isomorphic.*

This isomorphism generalizes the classical Wick rotation, which can be interpreted as an isomorphism between the polynomial algebras $\mathrm{Pol}(\mathbb{CP}^n)$ and $\mathrm{Pol}(\mathbb{D}^n)$. However, this isomorphism does not necessarily respect the star involutions with which the algebras $\mathcal{A}(\mathcal{O})$ are equipped. In other words, the algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O}')$ are isomorphic as algebras, but not necessarily as $^*$-algebras.

In order to apply our quantization to physics, we should represent the Fréchet algebras $(\mathcal{A}(\mathcal{O}), *_\hbar)$ on a Hilbert space. Given a positive linear functional we can use the GNS representation to do so. For a formal star product of Wick type all point evaluation functionals are formally positive. However, formal positivity means only that the first non-vanishing order is positive and therefore, as in this case, might not survive the passage to strict products (where the contribution of higher orders can dominate the contribution of the first order). For certain coadjoint orbits we will prove that point evaluations stay positive.

One aspect that we do not discuss in this work is the relation to geometric or Berezin–Toeplitz quantization [8–11]. These theories construct a quantization by studying holomorphic sections of a quantizing line bundle over the manifold $M$. This line bundle needs to satisfy some integrality condition, which for compact $M$ means that only countably many values of $\hbar$, accumulating at 0, are allowed. The algebra $\mathscr{C}^\infty(M)$ is, in the limit $\hbar \to 0$, approximated by finite dimensional matrix algebras. The construction of Alekseev–Lachowska coincides with another more geometric construction of star products on semisimple coadjoint orbits by Karabegov [16,26], if $\hbar$ is not a pole. However, Karabegov's construction still makes sense at the poles, where it coincides with (a variant of) the Berezin–Toeplitz quantization [26]. In this sense our infinite dimensional Fréchet algebras $(\mathcal{A}(\mathcal{O}), *_\hbar)$ interpolate between the finite dimensional Berezin–Toeplitz algebras. It could be very interesting to study this in greater detail.

### Contents

In Section 2 we recall some well-known facts about coadjoint orbits. This includes the realizability of coadjoint orbits as orbits of matrix Lie groups, and a characterization of invariant multidifferential operators on homogeneous spaces. In Section 3 we introduce the Shapovalov pairing of (generalized) Verma modules and derive an explicit formula for its canonical element. From this, we obtain a product for holomorphic polynomials on complex coadjoint orbits. In Section 4 we show that this product is continuous with respect to the topology of locally uniform convergence, so that we can extend it to the completion, which consists of all holomorphic functions on the orbit. Finally, we restrict our results to real coadjoint orbits in Section 5. We will determine additional properties of the star products obtained in this way (e.g. being of Wick type or of standard ordered type), study positive linear functionals, and investigate isomorphisms of the algebras obtained for different real forms of the same complex coadjoint orbit. In Appendix A we give some remaining proofs and more details on complex structures.

### Acknowledgements

The author would like to thank Matthias Schötz for many valuable discussions and helpful comments on an earlier version of this article. He is extremely grateful to his advisor Ryszard Nest for many helpful and inspiring discussions on the content of this paper and related topics. The author was supported by the Danish National Research Foundation through the Centre of Symmetry and Deformation (DNRF92).

**Notation**

In the whole paper $G$ is either a real or complex Lie group, $\mathfrak{g}$ denotes the Lie algebra of $G$, and $\mathscr{U}\mathfrak{g}$ denotes the universal enveloping algebra of $\mathfrak{g}$. In Section 3 and Section 4, $G$ is always complex. In Section 5, $G$ refers to a real Lie group and $\hat{G}$ to a complexification of $G$. $K$ denotes a compact real Lie group. Coadjoint orbits through $\lambda \in \mathfrak{g}^*$ are denoted by $\mathcal{O}_\lambda$.

We write $\mathscr{C}^\infty(M)$ for the smooth complex-valued functions on a manifold $M$. If $M$ is a real manifold, $\mathrm{T}M$ denotes its (real) tangent bundle (so sections of $\mathrm{T}M$ are derivations of the algebra of real-valued smooth functions on $M$). The complexification of $\mathrm{T}M$ is denoted by $\mathrm{T}^{\mathbb{C}}M$ (so sections of $\mathrm{T}^{\mathbb{C}}M$ are derivations of $\mathscr{C}^\infty(M)$). If $M$ is a complex manifold, then the holomorphic tangent bundle is denoted by $\mathrm{T}^{(1,0)}M$.

# 2 Preliminaries

In this section we summarize some results that are needed in the rest of this article: We review the definition of coadjoint orbits and their realizability as orbits of matrix Lie groups in Subsection 2.1. In Subsection 2.2 we introduce invariant multidifferential operators on homogeneous spaces.

## 2.1 Coadjoint orbits

Let $G$ be a real or complex Lie group with Lie algebra $\mathfrak{g}$. We denote the *adjoint action* of $G$ on $\mathfrak{g}$ by $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$. For any $g \in G$, $\mathrm{Ad}_g := \mathrm{Ad}(g)$ is the tangent map of the conjugation $G \ni x \mapsto gxg^{-1} \in G$ by $g$. Its differential $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{end}(\mathfrak{g})$ is given by the Lie bracket, $\mathrm{ad}_X(Y) = [X, Y]$. The *coadjoint action* $\mathrm{Ad}^*\colon G \to \mathrm{End}(\mathfrak{g}^*)$ of $G$ on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is defined by $\mathrm{Ad}^*_g \xi = \xi \circ \mathrm{Ad}_{g^{-1}}$ for $\xi \in \mathfrak{g}^*$.

The *coadjoint orbit* $\mathcal{O}_\lambda$ of $G$ through an element $\lambda \in \mathfrak{g}^*$ is defined as

$$\mathcal{O}_\lambda = \{\xi \in \mathfrak{g}^* \mid \xi = \mathrm{Ad}^*_g \lambda \text{ for some } g \in G\}\,. \tag{2.1}$$

It is well-known that $\mathcal{O}_\lambda \cong G/G_\xi$ where $\xi \in \mathcal{O}_\lambda$ is any point on the coadjoint orbit and $G_\xi = \{g \in G \mid \mathrm{Ad}^*_g \xi = \xi\}$ is the *stabilizer subgroup* of $\xi$. If $G$ is a real (complex) Lie group, there is a unique smooth (complex) manifold structure on $G/G_\xi$ that makes the projection $\pi\colon G \to G/G_\xi$ a smooth (holomorphic) submersion, and we use it to define the structure of a smooth (complex) manifold on $\mathcal{O}_\lambda$. It does not depend on the choice of $\xi \in \mathcal{O}_\lambda$.

Fix a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ and let $C^k_{ij}$ be the structure constants with respect to this basis, i.e. $[e_i, e_j] = \sum_{k=1}^n C^k_{ij} e_k$. Then $\{f, g\}(\xi) = \sum_{i,j,k=1}^n C^k_{ij} \xi(e_k) \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_j}$ defines a linear Poisson structure on $\mathfrak{g}^*$, where $f, g \in \mathscr{C}^\infty(\mathfrak{g}^*)$ and where view the $e_i$ as global linear coordinates on $\mathfrak{g}^*$. The following proposition is well-known, see e.g. [12, Example 1.1.3].

**Proposition 2.1** *If the Lie group $G$ is connected, then the coadjoint orbits of $G$ are precisely the symplectic leaves of this linear Poisson structure. In particular, all connected Lie groups with the same Lie algebra have the same coadjoint orbits.*

**Corollary 2.2** *If the Lie group $G$ is semisimple and connected, then $G$ and its image under* $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$ *have the same coadjoint orbits.*

PROOF: Since $\mathfrak{g}$ is semisimple, it has trivial center and therefore $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{end}(\mathfrak{g})$ is injective. Consequently, $G$ and its image in $\mathrm{End}(\mathfrak{g})$ have the same Lie algebra. Since both are connected, the result follows by applying the previous proposition. □

It is easy to show that $G$ and its image under Ad do not only have the same coadjoint orbits, but that $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$ also intertwines the actions of $G$ and its image on the coadjoint orbits. Since the image of $G$ under Ad is a matrix Lie group, we can therefore, when studying coadjoint orbits of connected semisimple Lie groups, assume without loss of generality that such a Lie group is a matrix Lie group. Using the argument provided in [20, Theorem 9] we can even assume that $G$ is a closed matrix Lie group.

For $X \in \mathfrak{g}$, denote the *fundamental vector field* of $X$ for the coadjoint action by $X_{\mathcal{O}_\lambda}\big|_\xi \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{Ad}^*_{\exp(-tX)}\,\xi$, where $\xi \in \mathcal{O}_\lambda$. Note that the map $\mathfrak{g}/\mathfrak{g}_\xi \to \mathrm{T}_\xi\mathcal{O}_\lambda$, $X \mapsto X_{\mathcal{O}_\lambda}\big|_\xi$ is an isomorphism, where $\mathfrak{g}_\xi$ denotes the Lie algebra of $G_\xi$. Consequently,

$$\omega_{\mathrm{KKS}}(X_{\mathcal{O}_\lambda}, Y_{\mathcal{O}_\lambda})\big|_\xi = \xi([X,Y]) \tag{2.2}$$

determines a well-defined 2-form on $\mathcal{O}_\lambda$, which is called the Kirillov-Kostant-Souriau form. One can show that $\omega_{\mathrm{KKS}}$ is symplectic and $G$-invariant. By symplectic we mean that $\omega_{\mathrm{KKS}}$ is closed and that $\omega_{\mathrm{KKS}}\big|_\xi\colon \mathrm{T}_\xi\mathcal{O}_\lambda \times \mathrm{T}_\xi\mathcal{O}_\lambda \to \mathbb{k}$ is $\mathbb{k}$-bilinear, antisymmetric, and non-degenerate for all $\xi \in \mathcal{O}_\lambda$, where $\mathbb{k}$ is either $\mathbb{R}$ or $\mathbb{C}$, depending on whether $G$ is real or complex.

For a semisimple Lie algebra $\mathfrak{g}$, the Killing form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}$ is non-degenerate, giving an isomorphism $^\flat\colon \mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto X^\flat \coloneqq B(X,\,\cdot\,)$. We denote its inverse by $^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$. In the complex case we say that $\lambda \in \mathfrak{g}^*$ is semisimple if $\mathrm{ad}_{\lambda^\sharp} \in \mathfrak{end}(\mathfrak{g})$ is diagonalisable and in the real case $\lambda \in \mathfrak{g}^*$ is semisimple if the complex linear extension of $\lambda$ to the complexification of $\mathfrak{g}$ is semisimple. A coadjoint orbit $\mathcal{O}_\lambda$ is semisimple if $\lambda$ is semisimple.

**Proposition 2.3** *Let $G$ be a complex connected semisimple Lie group and $\lambda \in \mathfrak{g}^*$ be semisimple. Then $G_\lambda$ is connected.*

PROOF: The Lie algebra spanned by $\lambda^\sharp$ integrates to a connected commutative Lie subgroup $T'$ of $G$, and since $\lambda^\sharp$ is semisimple, all elements of $T'$ are diagonalisable in the adjoint representation. There is a smallest closed complex Lie group $T$ containing $T'$, that can be obtained as follows: Take the closure of $T'$ (which is a real Lie group), take the Lie algebra of this closure (which is a real Lie subalgebra of $\mathfrak{g}$), take the complex Lie algebra spanned by it, integrate this Lie algebra to a connected Lie subgroup of $G$, and possibly repeat these steps. $T$ is still connected and commutative, and all its elements are diagonalisable in the adjoint representation, so $T$ is a complex torus in $G$. Its centralizer is exactly $G_\lambda$, and centralizers of tori are connected. □

Note that the statement is also true for a real compact connected semisimple Lie group $K$, but might fail if the compactness assumption is dropped.

We denote the smooth functions on $G$ that are invariant under the action of $G_\lambda$ from the right by $\mathscr{C}^\infty(G)^{G_\lambda}$. That is, $f \in \mathscr{C}^\infty(G)^{G_\lambda}$ if and only if $f \in \mathscr{C}^\infty(G)$ and $f(gg') = f(g)$ for all $g \in G$ and $g' \in G_\lambda$. There is an algebra isomorphism

$$\pi^* \colon \mathscr{C}^\infty(G/G_\lambda) \to \mathscr{C}^\infty(G)^{G_\lambda}\,, \quad f \mapsto \pi^* f := f \circ \pi \tag{2.3}$$

and for a complex Lie group, this isomorphism restricts to an isomorphism on holomorphic functions. We denote the inverse by $\pi_* \colon \mathscr{C}^\infty(G)^{G_\lambda} \to \mathscr{C}^\infty(G/G_\lambda)$.

**Remark 2.4** This article is written mainly from a differential geometric perspective. Note however, that any complex connected semisimple Lie group $G$ has a unique structure of an algebraic group, see Theorem 6.3 and the preceding corollary in Chapter 1 of [31]. Any holomorphic representation of $G$ is polynomial. Consequently, if $G$ is realized as a subgroup of $\mathrm{GL}_N(\mathbb{C})$ it is automatically closed. The coadjoint action $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a morphism of algebraic varieties, and coadjoint orbits of $G$ are smooth subvarieties of $\mathfrak{g}^*$. A coadjoint orbit of $G$ is closed in the Zariski topology if and only if it is semisimple, see [13, Theorem 5.4]. In particular, semisimple coadjoint orbits of complex connected semisimple Lie groups are affine algebraic varieties.

Note however, that this is not necessarily true for real connected semisimple Lie groups (not even if they are linear). It is still true that real connected semisimple linear Lie groups and their coadjoint orbits are connected components (with respect to the usual topology) of affine algebraic varieties.

## 2.2 Invariant holomorphic $k$-differential operators

In the whole subsection $G$ is a complex Lie group, $H$ is a closed complex Lie subgroup of $G$, and $k \geq 1$ is an integer. We present some results on holomorphic $G$-invariant $k$-differential operators on the homogeneous space $G/H$, in particular we construct a bijection between the set $((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\otimes k})^H$ and the set of such operators. The results seem to be well-known, but proofs are hard to find in the literature.

A $k$-differential operator $D$ (see Appendix A.1 for a short review of the definition) on a manifold $M$ endowed with an action of a Lie group $G$ is said to be invariant under $G$ if $\phi_g^*(D\vec{f}) = D((\phi_g^*)^{\times k}\vec{f})$ for all $\vec{f} \in \mathscr{C}^\infty(M)^k$ and all $g \in G$. Here $\phi_g \colon M \to M$ is the diffeomorphism of $M$ given by the action of a fixed element $g \in G$, and the upper star denotes the pullback. We write $k\text{-}\mathrm{DiffOp}^G_{\mathcal{H}}(M)$ for the space of holomorphic $G$-invariant $k$-differential operators on a complex manifold $M$. A $k$-differential operator on $G$ is said to be left-invariant if it is invariant with respect to the left action $\mathrm{L}\colon G \times G \to G$, $(g, g') \mapsto gg' =: \mathrm{L}_g(g')$.

Let $M$ be a complex manifold with complex structure $I \colon \mathrm{T}M \to \mathrm{T}M$. For a vector field $V \in \Gamma^\infty(\mathrm{T}M)$ its holomorphic part is $V^{(1,0)} = \frac{1}{2}(V - \mathrm{i}IV) \in \Gamma^\infty(\mathrm{T}^{(1,0)}M)$. Let $\mathfrak{g}$ be the Lie algebra of $G$. For any $X \in \mathfrak{g}$ define the *left-invariant vector field*

$$X^{\mathrm{left}}\big|_g := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g\exp(tX) \in \Gamma^\infty(\mathrm{T}G)\,. \tag{2.4}$$

Its holomorphic part $X^{\mathrm{left},(1,0)} = \frac{1}{2}(X^{\mathrm{left}} - \mathrm{i}(\mathrm{i}X)^{\mathrm{left}}) \in \Gamma^\infty(\mathrm{T}^{(1,0)}G)$ induces a holomorphic left-invariant 1-differential operator $f \mapsto X^{\mathrm{left},(1,0)}f$ on $G$. Since the map

$(\,\cdot\,)^{\mathrm{left},(1,0)}\colon \mathfrak{g} \to \Gamma^\infty(\mathrm{T}^{(1,0)}G)$ is a Lie algebra homomorphism, it induces an algebra homomorphism $(\,\cdot\,)^{\mathrm{left},(1,0)}\colon \mathscr{U}\mathfrak{g} \to \mathrm{DiffOp}^G_{\mathcal{H}}(G)$.

In the following we extend various maps to $k$-fold products and still denote them by the same symbol,

$$\mathrm{Ad}_g\colon (\mathscr{U}\mathfrak{g})^{\otimes k} \to (\mathscr{U}\mathfrak{g})^{\otimes k}\,, \qquad u_1 \otimes \ldots \otimes u_k \mapsto \mathrm{Ad}_g\, u_1 \otimes \ldots \otimes \mathrm{Ad}_g\, u_k\,, \tag{2.5a}$$

$$\pi^*\colon \mathscr{C}^\infty(G/H)^k \to (\mathscr{C}^\infty(G)^H)^k\,, \qquad (f_1,\ldots,f_k) \mapsto (\pi^* f_1,\ldots,\pi^* f_k)\,, \tag{2.5b}$$

$$\begin{aligned}(\,\cdot\,)^{\mathrm{left},(1,0)}\colon (\mathscr{U}\mathfrak{g})^{\otimes k} &\to k\text{-}\mathrm{DiffOp}^G_{\mathcal{H}}(G)\,,\\ u_1 \otimes \ldots \otimes u_k &\mapsto \big((f_1,\ldots,f_k) \mapsto u_1^{\mathrm{left},(1,0)} f_1 \cdot \ldots \cdot u_k^{\mathrm{left},(1,0)} f_k\big)\,.\end{aligned} \tag{2.5c}$$

**Proposition 2.5** *The map $(\,\cdot\,)^{\mathrm{left},(1,0)}\colon (\mathscr{U}\mathfrak{g})^{\otimes k} \to k\text{-}\mathrm{DiffOp}^G_{\mathcal{H}}(G)$ is an isomorphism.*

PROOF: See Appendix A.1. □

Next, we want to describe holomorphic $G$-invariant $k$-differential operators on the homogeneous space $G/H$. Let $H$ be a closed Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$, and let $\mathscr{U}\mathfrak{g}\cdot\mathfrak{h} \subseteq \mathscr{U}\mathfrak{g}$ be the left ideal generated by $\mathfrak{h}$. Note that $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\otimes k}$ is isomorphic to $(\mathscr{U}\mathfrak{g})^{\otimes k}/I$ where $I = I_1 + \cdots + I_k$ and $I_i = (\mathscr{U}\mathfrak{g})^{\otimes(i-1)} \otimes \mathscr{U}\mathfrak{g}\cdot\mathfrak{h} \otimes (\mathscr{U}\mathfrak{g})^{\otimes(k-i)}$ is a left ideal in $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Introduce the set

$$\begin{aligned}U_{\mathrm{inv}} &= \{\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k} \mid [\vec{u}] \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\otimes k} \text{ is } H\text{-invariant}\}\\ &= \{\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k} \mid \mathrm{Ad}_h\,\vec{u} - \vec{u} \in I \text{ for all } h \in H\}\,.\end{aligned} \tag{2.6}$$

Here the action of $H$ on $(\mathscr{U}\mathfrak{g})^{\otimes k}$ is the diagonal action defined in (2.5a).

**Lemma 2.6** *Let $\vec{u} \in U_{\mathrm{inv}}$, $\vec{v} \in I$, and $\vec{f} \in (\mathscr{C}^\infty(G)^H)^k$. Then we have*

$$\vec{v}^{\,\mathrm{left},(1,0)}\vec{f} = 0 \quad \text{and} \quad \vec{u}^{\,\mathrm{left},(1,0)}\vec{f} \in \mathscr{C}^\infty(G)^H\,. \tag{2.7}$$

PROOF: Let $Y \in \mathfrak{h}$ and $f \in \mathscr{C}^\infty(G)^H$. Then we compute

$$(Y^{\mathrm{left}} f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g\exp(tY)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g) = 0\,.$$

By using that $Y^{\mathrm{left},(1,0)} = \frac{1}{2}(Y^{\mathrm{left}} - \mathrm{i}(\mathrm{i}Y)^{\mathrm{left}})$ this implies that $Y^{\mathrm{left},(1,0)} f = 0$, and therefore also $\vec{v}^{\,\mathrm{left},(1,0)}\vec{f} = 0$ for all $\vec{v} \in I$ and $\vec{f} \in (\mathscr{C}^\infty(G)^H)^k$. If $X \in \mathfrak{g}$, then

$$(X^{\mathrm{left}} f)(gh) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(gh\exp(tX)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g\exp(t\,\mathrm{Ad}_h X)) = ((\mathrm{Ad}_h X)^{\mathrm{left}} f)(g)$$

for all $f \in \mathscr{C}^\infty(G)^H$, $g \in G$, and $h \in H$. Consequently, we obtain $(X^{\mathrm{left},(1,0)} f)(gh) = ((\mathrm{Ad}_h X)^{\mathrm{left},(1,0)} f)(g)$, and extending to the universal enveloping algebra and to tensor products yields $(\vec{u}^{\,\mathrm{left},(1,0)}\vec{f})(gh) = ((\mathrm{Ad}_h\,\vec{u})^{\mathrm{left},(1,0)}\vec{f})(g)$ for all $\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k}$ and $\vec{f} \in (\mathscr{C}^\infty(G)^H)^k$. If $\vec{u} \in U_{\mathrm{inv}}$, then together with the first part we obtain

$$\begin{aligned}(\vec{u}^{\,\mathrm{left},(1,0)}\vec{f})(gh) &= ((\mathrm{Ad}_h\,\vec{u})^{\mathrm{left},(1,0)}\vec{f})(g) =\\ &= (\vec{u}^{\,\mathrm{left},(1,0)}\vec{f})(g) + ((\mathrm{Ad}_h\,\vec{u} - \vec{u})^{\mathrm{left},(1,0)}\vec{f})(g) = (\vec{u}^{\,\mathrm{left},(1,0)}\vec{f})(g)\,.\end{aligned}$$

□

Because of this lemma we can define

$$\tilde{\Psi}\colon U_{\mathrm{inv}} \to \mathrm{Map}(\mathscr{C}^\infty(G/H)^k, \mathscr{C}^\infty(G/H)), \quad \tilde{\Psi}(\vec{u})\vec{f} = \pi_*(\vec{u}^{\mathrm{left},(1,0)}(\pi^*\vec{f})).$$

Since $\pi^*$ and $\pi_*$ are algebra homomorphisms, it follows that $\tilde{\Psi}(\vec{u})$ satisfies essentially the same commutation relations with the operator of multiplying a component by a smooth function than $\vec{u}^{\mathrm{left},(1,0)}$ and consequently $\tilde{\Psi}(\vec{u})$ is $k$-differential and of the same order than $\vec{u}^{\mathrm{left},(1,0)}$ (see the definition of $k$-differential operators given in Definition A.1). Moreover, $\tilde{\Psi}(\vec{u})$ is $G$-invariant, because $\pi^*$ and $\pi_*$ are $G$-equivariant and $\vec{u}^{\mathrm{left},(1,0)}$ is $G$-invariant. Since $\pi\colon G \to G/H$ is a holomorphic map, it follows that $\tilde{\Psi}(\vec{u})$ is holomorphic, and $\tilde{\Psi}$ really maps into $k\text{-}\mathrm{DiffOp}^G_{\mathcal{H}}(G/H)$. The map $\tilde{\Psi}$ descends to a map

$$\Psi\colon ((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\otimes k})^H \to k\text{-}\mathrm{DiffOp}^G_{\mathcal{H}}(G/H) \tag{2.8}$$

because $\tilde{\Psi}(I) = 0$ according to the previous lemma.

**Proposition 2.7** *The map $\Psi$ defined in* (2.8) *is an isomorphism.*

Proof: The proof is given in Appendix A.1. □

The last result of this subsection gives a description of the $k$-differential operator $\Psi([\vec{u}])$ on the coadjoint orbit without using extensions to $G$. Let $S$ be the antipode of $\mathscr{U}\mathfrak{g}$ and extend the Lie algebra homomorphism $\mathfrak{g} \ni X \mapsto X_{\mathcal{O}_\lambda} \in \Gamma^\infty(\mathrm{T}\mathcal{O}_\lambda)$ defined just before (2.2) to an algebra homomorphism $\mathscr{U}\mathfrak{g} \to \mathrm{DiffOp}(\mathcal{O}_\lambda)$.

**Proposition 2.8** *Let $\mathcal{O}_\lambda \cong G/G_\lambda$ be a coadjoint orbit. For $\vec{u} = u_1 \otimes \ldots \otimes u_k \in U_{\mathrm{inv}}$ and $\vec{f} = (f_1, \ldots, f_k) \in \mathscr{C}^\infty(\mathcal{O}_\lambda)^k$ we have*

$$\Psi([\vec{u}])\vec{f}(\mathrm{Ad}^*_g \lambda) = (S(\mathrm{Ad}_g u_1))^{(1,0)}_{\mathcal{O}_\lambda} f_1(\mathrm{Ad}^*_g \lambda) \cdot \ldots \cdot (S(\mathrm{Ad}_g u_k))^{(1,0)}_{\mathcal{O}_\lambda} f_k(\mathrm{Ad}^*_g \lambda). \tag{2.9}$$

Proof: Defining the Lie algebra homomorphism $(\,\cdot\,)^{\mathrm{right}}\colon \mathfrak{g} \to \Gamma^\infty(\mathrm{T}G)$, $X \mapsto X^{\mathrm{right}}$ with $X^{\mathrm{right}}\big|_g := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(-tX)g$ and extending to $\mathscr{U}\mathfrak{g}$ as before, one checks that

$$\begin{aligned}
u^{\mathrm{left}} f(g) &= X_1^{\mathrm{left}} \ldots X_j^{\mathrm{left}} f(g) \\
&= \frac{\mathrm{d}}{\mathrm{d}t_1}\Big|_{t_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}t_j}\Big|_{t_j=0} f(g\exp(t_1X_1)\ldots\exp(t_jX_j)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t_1}\Big|_{t_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}t_j}\Big|_{t_j=0} f(\exp(t_1\,\mathrm{Ad}_g X_1)\ldots\exp(t_j\,\mathrm{Ad}_g X_j)g) \\
&= (-\mathrm{Ad}_g X_j)^{\mathrm{right}} \ldots (-\mathrm{Ad}_g X_1)^{\mathrm{right}} f(g) \\
&= (S(\mathrm{Ad}_g u))^{\mathrm{right}} f(g)
\end{aligned}$$

for $u = X_1 \ldots X_j \in \mathscr{U}\mathfrak{g}$ and similarly $u^{\mathrm{left},(1,0)} f(g) = (S(\mathrm{Ad}_g u))^{\mathrm{right},(1,0)} f(g)$. Furthermore, we have

$$\begin{aligned}
X^{\mathrm{right}}(\pi^* f)(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^* f(\exp(-tX)g) = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\mathrm{Ad}^*_{\exp(-tX)}\,\mathrm{Ad}^*_g \lambda) = X_{\mathcal{O}_\lambda} f(\mathrm{Ad}^*_g \lambda) = \pi^*(X_{\mathcal{O}_\lambda} f)(g)
\end{aligned}$$

for all $X \in \mathfrak{g}$, implying that $X^{\text{right},(1,0)} \circ \pi^* = \pi^* \circ X^{(1,0)}_{\mathcal{O}_\lambda}$, and therefore that $u^{\text{right},(1,0)} \circ \pi^* = \pi^* \circ u^{(1,0)}_{\mathcal{O}_\lambda}$ for all $u \in \mathscr{U}\mathfrak{g}$. Finally,

$$\begin{aligned}
\Psi([\vec{u}])\vec{f}(\operatorname{Ad}^*_g \lambda) &= (\vec{u}^{\text{left},(1,0)} \pi^* \vec{f})(g) \\
&= u_1^{\text{left},(1,0)}(\pi^* f_1)(g) \cdot \ldots \cdot u_k^{\text{left},(1,0)}(\pi^* f_k)(g) \\
&= (S(\operatorname{Ad}_g u_1))^{\text{right},(1,0)}(\pi^* f_1)(g) \cdot \ldots \cdot (S(\operatorname{Ad}_g u_k))^{\text{right},(1,0)}(\pi^* f_k)(g) \\
&= (S(\operatorname{Ad}_g u_1))^{(1,0)}_{\mathcal{O}_\lambda} f_1(\operatorname{Ad}^*_g \lambda) \cdot \ldots \cdot (S(\operatorname{Ad}_g u_k))^{(1,0)}_{\mathcal{O}_\lambda} f_k(\operatorname{Ad}^*_g \lambda) \,. \qquad \square
\end{aligned}$$

# 3 Quantizing complex coadjoint orbits

In this section we construct a formal associative product for holomorphic functions on a semisimple coadjoint orbit of a complex connected semisimple Lie group, and a strict associative product for polynomials. These products are induced by a twist, which is constructed using the Shapovalov pairing between generalized Verma modules. For the convenience of the reader we first consider the special case of regular semisimple orbits in Subsection 3.1, where we introduce the Shapovalov pairing between Verma modules and compute its canonical element. In Subsection 3.2 we generalize these results to non-regular semisimple orbits. In Subsection 3.3 we describe the induced formal and strict products in detail. We consider an example in Subsection 3.4.

Later, in Section 5, we will use the results of this section to obtain star products on semisimple coadjoint orbits of real connected semisimple Lie groups. From the example considered in this section, we will then obtain strict quantizations of the hyperbolic disc and the complex projective space.

## 3.1 Verma modules and the Shapovalov pairing

In this subsection we introduce the Shapovalov pairing between Verma modules. In case this pairing is non-degenerate, we derive an explicit formula for its canonical element, following [32]. A similar formula in the more general setting of quantum groups was obtained recently in [33]. The results allow us to quantize regular orbits.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Recall that a root is a non-zero element $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}^\alpha := \{X \in \mathfrak{g} \mid \operatorname{ad}_H X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ contains a non-zero element. Denote the set of roots by $\Delta$ and choose an ordering (i.e. a subset $\Delta^+$ of positive roots such that, setting $\Delta^- := -\Delta^+$, we have $\Delta^+ \cup \Delta^- = \Delta$, $\Delta^+ \cap \Delta^- = \emptyset$, and such that if the sum of positive roots is a root, then it is positive). Denote the simple roots (i.e. elements of $\Delta^+$ that cannot be written as a sum of two elements of $\Delta^+$) by $\Sigma$. Let $\mathfrak{n}^+$ and $\mathfrak{n}^-$ be the nilpotent Lie subalgebras of $\mathfrak{g}$ spanned by the positive respectively negative root spaces and define $\mathfrak{b}^+ := \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- := \mathfrak{h} \oplus \mathfrak{n}^-$ (the direct sum is as vector spaces, the Lie algebra structure on $\mathfrak{b}^\pm \subseteq \mathfrak{g}$ is obtained by restriction from $\mathfrak{g}$).

Note that 0 is not a root. However, it is convenient to introduce the notation $\mathfrak{g}^0 := \mathfrak{h}$. Then $\mathfrak{g}$ is $(\Delta \cup \{0\})$-graded, in the sense that $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}^\alpha$ and $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$ for any $\alpha, \beta \in \Delta \cup \{0\}$. Consequently the tensor algebra $\mathrm{T}\mathfrak{g}$ is $\mathbb{Z}\Delta$-graded, where the so-called root lattice $\mathbb{Z}\Delta$ is the set of linear combinations of roots. The two-sided ideal generated by elements of the form $X\otimes Y-Y\otimes X-[X,Y]$ with $X,Y\in\mathfrak{g}$ is homogeneous and therefore the universal enveloping algebra $\mathscr{U}\mathfrak{g}=\mathrm{T}\mathfrak{g}/\langle X\otimes Y-Y\otimes X-[X,Y]\rangle$ is also $\mathbb{Z}\Delta$-graded. Denote the degree of a homogeneous element $w\in\mathscr{U}\mathfrak{g}$ by $d(w)\in\mathbb{Z}\Delta$.

Given a linear functional $\lambda\in\mathfrak{h}^*$, the formula $H\triangleright z=\lambda(H)z$ makes $\mathbb{C}$ a left $\mathfrak{h}$-module, and since $\mathfrak{h}$ is commutative also a right $\mathfrak{h}$-module. We can extend this to a left or right $\mathfrak{b}^\pm$-module by noting that $\mathfrak{b}^\pm=\mathfrak{h}\oplus\mathfrak{n}^\pm$ and letting $\mathfrak{n}^\pm$ act trivially. Denote the corresponding left $\mathscr{U}(\mathfrak{b}^\pm)$-module by $\mathbb{C}^\pm_\lambda$ and the right $\mathscr{U}(\mathfrak{b}^-)$-module by $\mathbb{C}^*_\lambda$. Define the *Verma modules*

$$M_\lambda:=\mathscr{U}\mathfrak{g}\otimes_{\mathscr{U}(\mathfrak{b}^+)}\mathbb{C}^+_\lambda\,,\quad M^-_\lambda:=\mathscr{U}\mathfrak{g}\otimes_{\mathscr{U}(\mathfrak{b}^-)}\mathbb{C}^-_{-\lambda}\quad\text{and}\quad M^*_\lambda:=\mathbb{C}^*_\lambda\otimes_{\mathscr{U}(\mathfrak{b}^-)}\mathscr{U}\mathfrak{g}\,. \tag{3.1}$$

Note that $M_\lambda$ and $M^-_\lambda$ are left $\mathscr{U}\mathfrak{g}$-modules, whereas $M^*_\lambda$ is a right $\mathscr{U}\mathfrak{g}$-module. $M_\lambda$ is the most general left $\mathscr{U}\mathfrak{g}$-module of highest weight $\lambda$, meaning that any other left $\mathscr{U}\mathfrak{g}$-module of highest weight $\lambda$ can be obtained as a quotient of $M_\lambda$. $M^-_\lambda$ is the most general left $\mathscr{U}\mathfrak{g}$-module of lowest weight $-\lambda$.

There are canonical isomorphisms $M^*_\lambda\otimes_{\mathscr{U}\mathfrak{g}}M_\lambda\cong\mathbb{C}^*_\lambda\otimes_{\mathscr{U}(\mathfrak{b}^-)}\mathscr{U}\mathfrak{g}\otimes_{\mathscr{U}(\mathfrak{b}^+)}\mathbb{C}_\lambda\cong\mathbb{C}^*_\lambda\otimes_{\mathscr{U}\mathfrak{h}}\mathbb{C}_\lambda\cong\mathbb{C}$ since the left and right $\mathfrak{h}$-module structures on $\mathbb{C}$ coincide.

**Definition 3.1** *The pairing $\langle\,\cdot\,,\,\cdot\,\rangle'_\lambda\colon M^*_\lambda\times M_\lambda\to\mathbb{C}$ defined by $(x,y)\mapsto x\otimes_{\mathscr{U}\mathfrak{g}}y$ is called the* Shapovalov pairing *between $M^*_\lambda$ and $M_\lambda$.*

In the following it will be convenient to have alternative descriptions of $M_\lambda$, $M^-_\lambda$ and $M^*_\lambda$. Let $\{X_1,\dots,X_k\}$ be a basis of $\mathfrak{n}^+$, $\{Y_1,\dots,Y_k\}$ be a basis of $\mathfrak{n}^-$, and $\{H_1,\dots,H_r\}$ be a basis of $\mathfrak{h}$. Since $\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$ (as vector spaces) the Poincaré–Birkhoff–Witt theorem implies that

$$\{Y^IH^JX^K\mid I,K\in\mathbb{N}_0^k,J\in\mathbb{N}_0^r\}\quad\text{and}\quad\{X^KH^JY^I\mid I,K\in\mathbb{N}_0^k,J\in\mathbb{N}_0^r\} \tag{3.2}$$

are bases for $\mathscr{U}\mathfrak{g}$. Here we use the multiindex notation $Y^I:=Y_1^{I_1}\dots Y_k^{I_k}$ (and similarly for $H$ and $X$). Define maps

$$\pi^-_\lambda\colon\mathscr{U}\mathfrak{g}\to\mathscr{U}(\mathfrak{n}^-)\,,\quad\pi^-_\lambda(Y^IH^JX^K):=\lambda(H_1)^{J_1}\dots\lambda(H_r)^{J_r}Y^I\delta_{K,0}\,, \tag{3.3a}$$
$$\pi^+_\lambda\colon\mathscr{U}\mathfrak{g}\to\mathscr{U}(\mathfrak{n}^+)\,,\quad\pi^+_\lambda(X^KH^JY^I):=(-\lambda(H_1))^{J_1}\dots(-\lambda(H_r))^{J_r}X^K\delta_{I,0}\,, \tag{3.3b}$$
$$\pi^*_\lambda\colon\mathscr{U}\mathfrak{g}\to\mathscr{U}(\mathfrak{n}^+)\,,\quad\pi^*_\lambda(Y^IH^JX^K):=\lambda(H_1)^{J_1}\dots\lambda(H_r)^{J_r}X^K\delta_{I,0}\,, \tag{3.3c}$$

where $\delta_{K,0}$ is 1 if $K=(0,\dots,0)$ and is 0 otherwise. Note that $\pi^\pm_\lambda$ and $\pi^*_\lambda$ are independent of the choice of bases. Fix non-zero vectors $1\in\mathbb{C}^\pm_\lambda$ and $1\in\mathbb{C}^*_\lambda$ (thinking of $\mathbb{C}$ as a vector space, this choice is not canonical).

**Lemma 3.2** *The maps $\cdot\otimes1\colon\mathscr{U}(\mathfrak{n}^-)\to M_\lambda$, $v\mapsto v\otimes1$ and $\cdot\otimes1\colon\mathscr{U}(\mathfrak{n}^+)\to M^-_\lambda$, $u\mapsto u\otimes1$ define isomorphisms of left $\mathscr{U}(\mathfrak{n}^-)$-modules and $\mathscr{U}(\mathfrak{n}^+)$-modules, respectively. The map $1\otimes\cdot\,\colon\mathscr{U}(\mathfrak{n}^+)\to M^*_\lambda$, $u\mapsto1\otimes u$ defines an isomorphism of right $\mathscr{U}(\mathfrak{n}^+)$-modules. The $\mathscr{U}\mathfrak{g}$-module structures on $\mathscr{U}(\mathfrak{n}^\pm)$ obtained by transferring the module structures on the Verma modules with these isomorphisms are given explicitly by*

$$\triangleright^-_\lambda\colon\mathscr{U}\mathfrak{g}\times\mathscr{U}(\mathfrak{n}^-)\to\mathscr{U}(\mathfrak{n}^-)\,,\qquad(w,v)\mapsto w\triangleright^-_\lambda v:=\pi^-_\lambda(wv)\,, \tag{3.4a}$$

$$\triangleright_\lambda^+ \colon \mathscr{U}\mathfrak{g} \times \mathscr{U}(\mathfrak{n}^+) \to \mathscr{U}(\mathfrak{n}^+)\,, \qquad (w,u) \mapsto w \triangleright_\lambda^+ u := \pi_\lambda^+(wu)\,, \tag{3.4b}$$
$$\triangleleft_\lambda^* \colon \mathscr{U}(\mathfrak{n}^+) \times \mathscr{U}\mathfrak{g} \to \mathscr{U}(\mathfrak{n}^+)\,, \qquad (u,w) \mapsto u \triangleleft_\lambda^* w := \pi_\lambda^*(uw)\,. \tag{3.4c}$$

*Furthermore, $S(w \triangleright_\lambda^+ u) = S(u) \triangleleft_\lambda^* S(w)$, where $S$ denotes the antipode of $\mathscr{U}\mathfrak{g}$. Or, in other words, $S\colon \mathscr{U}(\mathfrak{n}^+) \to \mathscr{U}(\mathfrak{n}^+)$ is an isomorphism from the left $\mathscr{U}\mathfrak{g}$-module $(\mathscr{U}(\mathfrak{n}^+), \triangleright_\lambda^+)$ to the right $\mathscr{U}\mathfrak{g}$-module $(\mathscr{U}(\mathfrak{n}^+), \triangleleft_\lambda^*)$ over the map $S\colon \mathscr{U}\mathfrak{g} \to \mathscr{U}\mathfrak{g}$.*

Proof: One checks easily that the maps $M_\lambda \to \mathscr{U}(\mathfrak{n}^-)$, $w \otimes z1 \mapsto z \cdot \pi_\lambda^-(w)$ and $M_\lambda^- \to \mathscr{U}(\mathfrak{n}^+)$, $w \otimes z1 \mapsto z \cdot \pi_\lambda^+(w)$ as well as $M_\lambda^* \to \mathscr{U}(\mathfrak{n}^+)$, $z1 \otimes w \mapsto z \cdot \pi_\lambda^*(w)$ are all well-defined and inverses of the maps in the statement of the lemma. Consequently we have $w \triangleright_\lambda^- v = (\,\cdot \otimes 1)^{-1}(wv \otimes 1) = \pi_\lambda^-(wv)$, and (3.4b) and (3.4c) follow similarly. Finally, $\pi_\lambda^* \circ S = S \circ \pi_\lambda^+$, so $S(w \triangleright_\lambda^+ u) = S \circ \pi_\lambda^+(wu) = \pi_\lambda^* \circ S(wu) = \pi_\lambda^*(S(u)S(w)) = S(u) \triangleleft_\lambda^* S(w)$. □

The pairing of the left $\mathscr{U}\mathfrak{g}$-modules $(\mathscr{U}(\mathfrak{n}^\pm), \triangleright_\lambda^\pm)$ obtained from the Shapovalov pairing by composing with the isomorphisms $(\mathscr{U}(\mathfrak{n}^-), \triangleright_\lambda^-) \xrightarrow{\,\cdot \otimes 1\,} M_\lambda$ and $(\mathscr{U}(\mathfrak{n}^+), \triangleright_\lambda^+) \xrightarrow{S} (\mathscr{U}(\mathfrak{n}^+), \triangleleft_\lambda^*) \xrightarrow{1 \otimes \,\cdot\,} M_\lambda^*$ of the previous lemma, is

$$\langle\,\cdot\,,\,\cdot\,\rangle_\lambda \colon \mathscr{U}(\mathfrak{n}^+) \times \mathscr{U}(\mathfrak{n}^-) \to \mathbb{C}\,, \quad (u,v) \mapsto \langle u,v\rangle_\lambda := \langle 1 \otimes S(u), v \otimes 1\rangle'_\lambda\,. \tag{3.5}$$

In order to compute $\langle u,v\rangle_\lambda$ for $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$ one needs to write $S(u)v \in \mathscr{U}\mathfrak{g}$ in the form $\sum_i v'_i h'_i u'_i$ with $u'_i \in \mathscr{U}(\mathfrak{n}^+)$, $v'_i \in \mathscr{U}(\mathfrak{n}^-)$ and $h'_i \in \mathscr{U}\mathfrak{h}$. The pairing is then given by summing $\lambda(h'_i)$ for those summands that have $v'_i = u'_i = 1$. This is made more precise in the next lemma. Define $\pi_\lambda := \pi_\lambda^- \circ \pi_\lambda^* = \pi_\lambda^* \circ \pi_\lambda^- \colon \mathscr{U}\mathfrak{g} \to \mathbb{C}$, where $\mathbb{C}$ is identified with $\mathbb{C}1 \subseteq \mathscr{U}(\mathfrak{n}^\pm)$ and we have implicitly used the inclusion $\mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}\mathfrak{g}$ when composing the maps.

**Lemma 3.3** *For $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$ the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ defined in (3.5) can be computed as*

$$\langle u,v\rangle_\lambda = \pi_\lambda(S(u)v)\,. \tag{3.6}$$

*It is $\mathscr{U}\mathfrak{g}$-invariant, in the sense that $\langle w \triangleright_\lambda^+ u, v\rangle_\lambda = \langle u, S(w) \triangleright_\lambda^- v\rangle_\lambda$ for $u \in \mathscr{U}(\mathfrak{n}^+)$, $v \in \mathscr{U}(\mathfrak{n}^-)$ and $w \in \mathscr{U}\mathfrak{g}$. The pairing respects the degree $d$ defined in the beginning of this section, meaning that $\langle u,v\rangle_\lambda = 0$ for homogeneous elements $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$ with $d(u) \neq -d(v)$. Furthermore, if $d(u) = -d(v)$, then*

$$\langle u,v\rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^-)} = S(u) \triangleright_\lambda^- v \quad \textit{and} \quad \langle u,v\rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^+)} = S(v) \triangleright_\lambda^+ u\,. \tag{3.7}$$

Proof: By definition $\langle u,v\rangle_\lambda = 1 \otimes_{\mathscr{U}(\mathfrak{b}^-)} S(u)v \otimes_{\mathscr{U}(\mathfrak{b}^+)} 1$. So to prove (3.6) it suffices to check that $1 \otimes_{\mathscr{U}(\mathfrak{b}^-)} w \otimes_{\mathscr{U}(\mathfrak{b}^+)} 1 = \pi_\lambda(w)$ for all $w \in \mathscr{U}\mathfrak{g}$, which one can easily verify on the basis $\{Y^I H^J X^K \mid I,K \in \mathbb{N}_0^k, J \in \mathbb{N}_0^r\}$. The $\mathscr{U}\mathfrak{g}$-invariance follows by noting that $\langle\,\cdot\,,\,\cdot\,\rangle'_\lambda$ is $\mathscr{U}\mathfrak{g}$-invariant, meaning $\langle xw, y\rangle'_\lambda = \langle x, wy\rangle'_\lambda$ for $x \in M_\lambda^*$ and $y \in M_\lambda$, and using the isomorphisms of the previous lemma. For homogenous $u \in \mathscr{U}(\mathfrak{n}^+)$ and $v \in \mathscr{U}(\mathfrak{n}^-)$ with $d(u) \neq -d(v)$ it follows that $S(u)v$ is also homogeneous of degree $d(u) + d(v) \neq 0$ and therefore $\pi_\lambda(S(u)v) = 0$. Finally, if $d(u) = -d(v)$, then $d(S(u)v) = 0$ and $\langle u,v\rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^-)} = \pi_\lambda(S(u)v) 1_{\mathscr{U}(\mathfrak{n}^-)} = \pi_\lambda^-(S(u)v) = S(u) \triangleright_\lambda^- v$, implying the first equality of (3.7). The second one follows from applying $S$ on both sides of $\langle u,v\rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^+)} = \pi_\lambda(S(u)v) 1_{\mathscr{U}(\mathfrak{n}^+)} = \pi_\lambda^*(S(u)v) = S(\pi_\lambda^+(S(v)u)) = S(S(v) \triangleright_\lambda^+ u)$. □

If the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ is non-degenerate, we can pick bases $\{u_i\}_{i\in\mathbb{N}}$ of $\mathscr{U}(\mathfrak{n}^+)$ and $\{v_j\}_{j\in\mathbb{N}}$ of $\mathscr{U}(\mathfrak{n}^-)$ consisting of homogeneous elements with respect to $d$ and satisfying $\langle u_i, v_j\rangle_\lambda = \delta_{ij}$. Then the element $F_\lambda := \sum_{i=1}^\infty u_i \otimes v_i \in \mathscr{U}(\mathfrak{n}^+)\,\hat{\otimes}\,\mathscr{U}(\mathfrak{n}^-)$ is called the *canonical element* of the pairing. It is independent of the choice of bases. By $\mathscr{U}(\mathfrak{n}^+)\,\hat{\otimes}\,\mathscr{U}(\mathfrak{n}^-)$ we mean the completion of the tensor product with respect to the $\mathbb{Z}\Delta$-grading $d$ defined in the beginning of this subsection, which is needed to make sense of the infinite sum. The following lemma is a standard statement when working with canonical elements.

**Lemma 3.4** *Assume that $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ is non-degenerate, and let $F_\lambda = \sum_{i=1}^\infty u_i \otimes v_i \in \mathscr{U}(\mathfrak{n}^+)\,\hat{\otimes}\,\mathscr{U}(\mathfrak{n}^-)$ be its canonical element. Then*

$$\sum_{i=1}^\infty u_i \langle u, v_i\rangle_\lambda = u \quad \textit{and} \quad \sum_{i=1}^\infty v_i \langle u_i, v\rangle_\lambda = v \tag{3.8}$$

*hold for all $u \in \mathscr{U}(\mathfrak{n}^+)$ and all $v \in \mathscr{U}(\mathfrak{n}^-)$, and $F_\lambda$ is uniquely determined by this property.*

Note that $\langle u, v_i\rangle$ and $\langle u_i, v\rangle$ are non-zero for only finitely many indices $i$, so that the sums in (3.8) are both finite. The pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ is non-degenerate precisely when the Verma modules are irreducible, but we will not need this below. In order to determine $F_\lambda$ explicitly, we need to introduce some more notation.

Denote the Killing form of $\mathfrak{g}$ by $B$. Since $\mathfrak{g}$ is semisimple, $B$ is non-degenerate on $\mathfrak{g}$. Extending linear functionals on $\mathfrak{h}$ by 0 on the root spaces $\mathfrak{g}^\alpha$, we may view $\mathfrak{h}^*$ as a subspace of $\mathfrak{g}^*$. Since $B$ restricts to zero on $\mathfrak{h} \times \mathfrak{g}^\alpha$ for any $\alpha \in \Delta$, it follows that $B$ is non-degenerate on $\mathfrak{h}$ and that the maps ${}^\flat\colon \mathfrak{g} \to \mathfrak{g}^*$ and ${}^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$ defined in Subsection 2.1 restrict to mutually inverse isomorphisms ${}^\flat\colon \mathfrak{h} \to \mathfrak{h}^*$ and ${}^\sharp\colon \mathfrak{h}^* \to \mathfrak{h}$. For $\alpha, \beta \in \mathfrak{h}^*$, let $(\alpha, \beta) := B(\alpha^\sharp, \beta^\sharp)$.

Denote the positive roots by $\alpha_1, \ldots, \alpha_k$. For every positive root $\alpha_i \in \Delta^+$ choose elements $X_i := X_{\alpha_i} \in \mathfrak{g}^{\alpha_i}$ and $Y_i := Y_{\alpha_i} = X_{-\alpha_i} \in \mathfrak{g}^{-\alpha_i}$ such that $B(X_i, Y_i) = 1$. Then we have $[X_i, Y_i] = \alpha_i^\sharp$ since for all $H \in \mathfrak{h}$,

$$B([X_i, Y_i], H) = B(X_i, [Y_i, H]) = \alpha_i(H) B(X_i, Y_i) = \alpha_i(H) = B(\alpha_i^\sharp, H)$$

and the Killing form is non-degenerate on $\mathfrak{h}$. Note that $[\alpha_i^\sharp, X_i] = \alpha_i(\alpha_i^\sharp) X_i = (\alpha_i, \alpha_i) X_i$ and similarly $[\alpha_i^\sharp, Y_i] = -(\alpha_i, \alpha_i) Y_i$, so $X_i' = 2(\alpha_i, \alpha_i)^{-1} X_i$, $Y_i' = Y_i$ and $H_i' = 2(\alpha_i, \alpha_i)^{-1} \alpha_i^\sharp$ satisfy the commutation relations $[X_i', Y_i'] = H_i'$, $[H_i', X_i'] = 2X_i'$ and $[H_i', Y_i'] = -2Y_i'$ of the usual generators of $\mathfrak{sl}_2(\mathbb{C})$, the special linear Lie algebra in 2 dimensions.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ be the *half-sum* of all positive roots. Denote non-negative integral linear combinations of positive roots by $\mathbb{N}_0 \Delta^+$. For $\lambda \in \mathfrak{h}^*$ fixed, and $\mu \in \mathfrak{h}^*$ define the number

$$p_\lambda(\mu) := \frac{1}{2}(\mu, \mu) - (\rho, \mu) - (\lambda, \mu)\,. \tag{3.9}$$

Recall that for a representation $\varrho\colon \mathfrak{g} \to V$ and $\mu \in \mathfrak{h}^*$ we define $V^\mu := \{v \in V \mid \varrho(H)v = \mu(H)v \text{ for all } H \in \mathfrak{h}\}$. If $V^\mu \neq \{0\}$, then we call $\mu$ a *weight* and any $v \in V^\mu$

is called a *weight vector* of weight $\mu$. $V$ is called a weight module if $V = \bigoplus_{\mu\in\mathfrak{h}^*} V^\mu$. A highest weight module is a weight module generated by a vector $v \in V$ satisfying $X_\alpha v = 0$ for all $\alpha \in \Delta^+$. It is said to be of highest weight $\mu$ if $v \in V^\mu$.

**Lemma 3.5 (Ostapenko, [32, Lemma 2])** *Let $V$ be a highest weight module of highest weight $\lambda$, assume $\mu \in \mathbb{N}_0\Delta^+$, and let $v \in V^{\lambda-\mu}$. Then*

$$-p_\lambda(\mu)v = \sum_{\alpha\in\Delta^+} Y_\alpha X_\alpha v\,. \tag{3.10}$$

PROOF: Choose an orthonormal basis $\{H_1,\dots,H_r\}$ of $\mathfrak{h}$ with respect to the Killing form. The Casimir element

$$c = \sum_{\alpha\in\Delta^+}(X_\alpha Y_\alpha + Y_\alpha X_\alpha) + \sum_{i=1}^r H_i H_i = \sum_{\alpha\in\Delta^+}(2Y_\alpha X_\alpha + \alpha^\sharp) + \sum_{i=1}^r H_i H_i$$

acts as a scalar on $V$ because $V$ is generated by a highest weight vector and $c$ is central in $\mathscr{U}\mathfrak{g}$. Evaluating it on a highest weight vector the $Y_\alpha X_\alpha$-part vanishes and we obtain that $c$ acts as multiplication by $\sum_{\alpha\in\Delta^+}(\alpha,\lambda) + \sum_{i=1}^r \lambda(H_i)\lambda(H_i) = (2\rho,\lambda) + (\lambda,\lambda)$. Therefore

$$(2\rho,\lambda)v + (\lambda,\lambda)v = 2\sum_{\alpha\in\Delta^+} Y_\alpha X_\alpha v + (2\rho,\lambda-\mu)v + (\lambda-\mu,\lambda-\mu)v$$

holds for any $v \in V^{\lambda-\mu}$, and rearranging this equation proves the lemma. □

Let $W$ be the set of words with letters from $\{1,\dots,k\}$. For any $w = (w_1,\dots,w_{|w|}) \in W$, we define $w^{\mathrm{opp}} := (w_{|w|},\dots,w_1)$, $w_{i\dots j} := (w_i,\dots,w_j)$, $X_w := X_{w_1}\dots X_{w_{|w|}} \in \mathscr{U}(\mathfrak{n}^+)$, $Y_w := Y_{w_1}\dots Y_{w_{|w|}} \in \mathscr{U}(\mathfrak{n}^-)$ and $\alpha_w := \alpha_{w_1} + \dots + \alpha_{w_{|w|}}$. We use $w_{i\dots j} := \emptyset$ if $j < i$, $X_\emptyset := 1$, $Y_\emptyset := 1$ and $\alpha_\emptyset := 0$. Furthermore let

$$p_\lambda^w(\mu) := \prod_{i=0}^{|w|-1} p_\lambda(\mu - \alpha_{w_{1\dots i}})\,. \tag{3.11}$$

We call a set $T$ of words a *tree* if $w = (w_1,\dots,w_{|w|}) \in T$ implies that $w_{1\dots i} \in T$ for all $i = 0,\dots,|w|-1$ and $(w_1,w_2,\dots,w_{|w|-1},x) \in T$ for all $x \in \{1,\dots,k\}$. See Figure 3.1 for a visualization of a tree. For a tree $T$ we denote by $\max T$ the set of elements $w \in T$ such that $w \neq w'_{1\dots i}$ for any $w' \in T$ and any $i \in \{0,\dots,|w'|-1\}$. Finally a tree is said to be *$\mu$-admissible* if $p_\lambda(\mu - \alpha_w) \neq 0$ for all $w \in T\setminus\max T$, or equivalently if $p_\lambda^w(\mu) \neq 0$ for all $w \in T$.

**Lemma 3.6 (Ostapenko, [32, Theorem 3])** *Let $V$ be a highest weight module of highest weight $\lambda$, assume $\mu \in \mathbb{N}_0\Delta^+$, and let $v \in V^{\lambda-\mu}$. For a $\mu$-admissible tree $T$ we have*

$$v = \sum_{w\in\max T} (-1)^{|w|} p_\lambda^w(\mu)^{-1} Y_w X_{w^{\mathrm{opp}}} v\,. \tag{3.12}$$

PROOF: Apply the previous lemma repeatedly. □

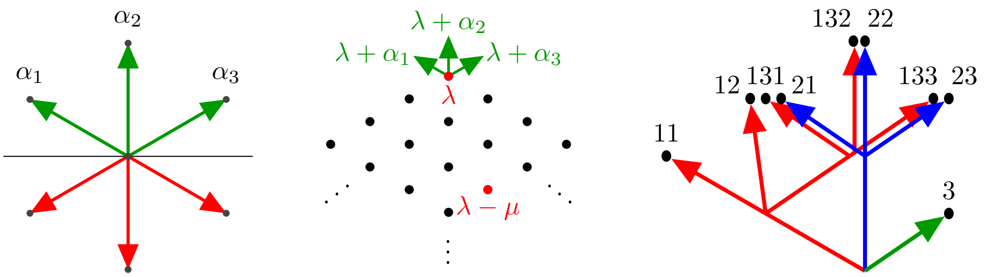


Figure 3.1: Left: The roots of $\mathfrak{sl}_3(\mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_3(\mathbb{C})$ is 2-dimensional and there are six 1-dimensional root spaces. The picture shows the real subspace of $\mathfrak{h}^*$ spanned by the roots. The positive roots are denoted by $\alpha_1$, $\alpha_2$ and $\alpha_3$ and drawn in green, negative roots are drawn in red. Middle: The weights in a highest weight module of highest weight $\lambda$. The picture shows again the real subspace of $\mathfrak{h}^*$ spanned by the roots. Weights are indicated by black dots, and $\mu = 3\alpha_1 + 2\alpha_3$. Note that since $\lambda$ is a highest weight, the spaces $V^{\lambda+\alpha_1}$, $V^{\lambda+\alpha_2}$ and $V^{\lambda+\alpha_3}$ must all be trivial. Right: Visualization of the tree $T = \{\emptyset, 1, 2, 3, 11, 12, 13, 21, 22, 23, 131, 132, 133\}$. The elements of $\max T = \{3, 11, 12, 21, 22, 23, 131, 132, 133\}$ are indicated by black dots. Words starting with a 1 are coloured red, words starting with a 2 blue, and words starting with a 3 green.

**Lemma 3.7** *Let $V$ be a lowest weight module of lowest weight $-\lambda$, assume $\mu \in \mathbb{N}_0\Delta^+$, and let $v \in V^{-\lambda+\mu}$. Then $\sum_{\alpha\in\Delta^+} X_\alpha Y_\alpha v = -p_\lambda(\mu)v$ and for a $\mu$-admissible tree $T$ we have*

$$v = \sum_{w\in\max T} (-1)^{|w|} p_\lambda^w(\mu)^{-1} X_w Y_{w^{\mathrm{opp}}} v\,. \tag{3.13}$$

PROOF: Similar to the proof of Lemma 3.5 and Lemma 3.6. □

Define the set

$$\Lambda := \{\lambda \in \mathfrak{h}^* \mid p_\lambda(\mu) \neq 0 \ \ \forall \mu \in \mathbb{N}_0\Delta^+ \setminus \{0\}\}\,. \tag{3.14}$$

**Proposition 3.8** *The Shapovalov pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda \colon \mathscr{U}(\mathfrak{n}^+) \times \mathscr{U}(\mathfrak{n}^-) \to \mathbb{C}$ is non-degenerate for $\lambda \in \Lambda$, and in this case its canonical element $F_\lambda \in \mathscr{U}(\mathfrak{n}^+) \hat{\otimes} \mathscr{U}(\mathfrak{n}^-)$ is given by*

$$F_\lambda = \sum_{w\in W} p_\lambda^w(\alpha_w)^{-1} X_w \otimes Y_w = \sum_{w\in W} \prod_{i=1}^{|w|} p_\lambda(\alpha_{w_{i\ldots|w|}})^{-1} X_w \otimes Y_w\,. \tag{3.15}$$

PROOF: We check that $F_\lambda$ satisfies the property given in Lemma 3.4. We decompose $v \in \mathscr{U}(\mathfrak{n}^-)$ as $v = \sum_{\mu\in\mathbb{N}_0\Delta^+} v_{-\mu}$ where $v_{-\mu}$ is homogeneous of degree $-\mu$ with respect to the $\mathbb{Z}\Delta$-grading. For $\mu \in \mathbb{N}_0\Delta^+$ let $W_\mu$ be the set of words $w \in W$ satisfying $\alpha_w = \mu$. Then

$$\sum_{w\in W} p_\lambda^w(\alpha_w)^{-1} Y_w \langle X_w, v\rangle_\lambda = \sum_{w\in W} p_\lambda^w(\alpha_w)^{-1} Y_w \triangleright_\lambda^- S(X_w) \triangleright_\lambda^- v_{-\alpha_w} =$$

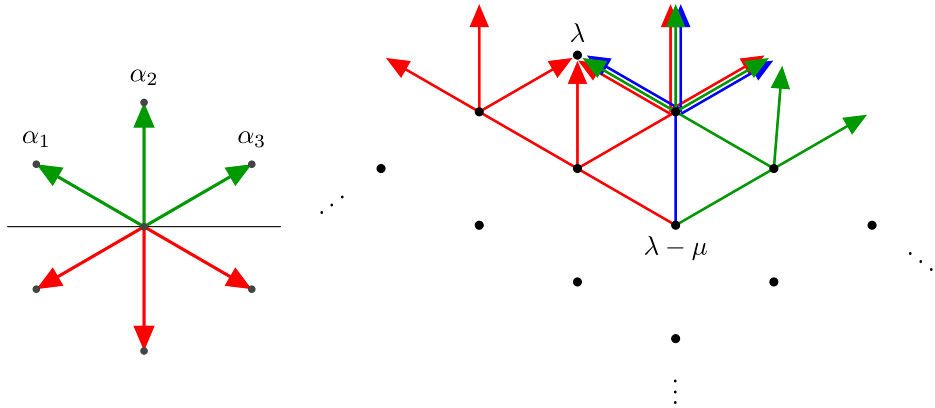


Figure 3.2: The tree $T$ used in the proof of Proposition 3.8 for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\mu = 2\alpha_1 + \alpha_3$. Elements of the tree starting with 1, 2 and 3 are coloured red, blue and green, respectively. Note that all weight spaces of maximal elements of this tree are trivial, except for $V^\lambda$. All non-maximal weight spaces are non-trivial.

$$= \sum_{\mu \in \mathbb{N}_0\Delta^+} \sum_{w \in W_\mu} (-1)^{|w|} p^w_\lambda(\alpha_w)^{-1} Y_w \triangleright^-_\lambda X_{w^{\mathrm{opp}}} \triangleright^-_\lambda v_{-\mu} = \sum_{\mu \in \mathbb{N}_0\Delta^+} v_{-\mu} = v\,.$$

The first equality holds because $Y_w\langle X_w, v\rangle_\lambda = Y_w \triangleright^-_\lambda (\langle X_w, v_{-\alpha_w}\rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^-)}) = Y_w \triangleright^-_\lambda S(X_w) \triangleright^-_\lambda v_{-\alpha_w}$ by Lemma 3.3. The second equality is true by basic manipulations. The third equality follows from Lemma 3.6 because we can rewrite the sum over all $w \in W_\mu$ as a sum over $\max T$ for a $\mu$-admissible tree $T$ as follows: Define

$$T := \{\emptyset\} \cup \{w \in W \mid \exists w' \in W_\mu \text{ and } 0 \le i \le |w'| - 1 \text{ such that } w_{1\dots|w|-1} = w'_{1\dots i}\}\,,$$

which is the smallest tree containing $W_\mu$. Since $\lambda \in \Lambda$ this tree is $\mu$-admissible. Clearly $W_\mu \subseteq \max T$. Furthermore, any element $w \in \max T$ satisfies either $\alpha_w = \mu$, so that $w \in W_\mu$, or there does not exist any $w' \in W_\mu$ and $i \in \{0, \dots, |w'|\}$ with $w = w'_{1\dots i}$, so that $\mu - \alpha_w \notin \mathbb{N}_0\Delta^+$ and therefore $X_{w^{\mathrm{opp}}} v_{-\mu} = 0$.

Similarly, for $u = \sum_{\mu \in \mathbb{N}_0\Delta^+} u_\mu \in \mathscr{U}(\mathfrak{n}^+)$ with $d(u_\mu) = \mu$ we compute that

$$\sum_{w \in W} p^w_\lambda(\mu)^{-1} X_w\langle u, Y_w\rangle_\lambda = \sum_{w \in W} p^w_\lambda(\mu)^{-1} X_w \triangleright^+_\lambda S(Y_w) \triangleright^+_\lambda u_{\alpha_w} =$$
$$= \sum_{\mu \in \mathbb{N}_0\Delta^+} \sum_{w \in W_\mu} (-1)^{|w|} p^w_\lambda(\mu)^{-1} X_w \triangleright^+_\lambda Y_{w^{\mathrm{opp}}} \triangleright^+_\lambda u_\mu = \sum_{\mu \in \mathbb{N}_0\Delta^+} u_\mu = u\,,$$

using $X_w\langle u, Y_w\rangle_\lambda = X_w \triangleright^+_\lambda (\langle u_{\alpha_w}, Y_w\rangle_\lambda 1_{\mathscr{U}(\mathfrak{n}^+)}) = X_w \triangleright^+_\lambda S(Y_w) \triangleright^+_\lambda u_{\alpha_w}$, and that the sum over $w \in W_\mu$ can be rewritten as a sum over maximal elements of a tree $T$ in a similar way than before. □

Using the inclusion $\mathscr{U}(\mathfrak{n}^+) \mathbin{\hat{\otimes}} \mathscr{U}(\mathfrak{n}^-) \to (\mathscr{U}\mathfrak{g})^{\hat{\otimes}2}$ and passing to the quotient, we can map the element $F_\lambda$ from (3.15) to $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\hat{\otimes}2}$. Note that $\mathscr{U}\mathfrak{g}\cdot\mathfrak{h}$ is a homogeneous ideal in $\mathscr{U}\mathfrak{g}$ with respect to the degree $d$, so the quotient $\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h}$ is still graded. The completed tensor product is defined with respect to this grading. The action of $\mathfrak{h}$ on $(\mathscr{U}\mathfrak{g})^{\otimes 2}$ given by $H \triangleright (w \otimes w') = \mathrm{ad}_H w \otimes w' + w \otimes \mathrm{ad}_H w'$ with $H \in \mathfrak{h}$ and $w, w' \in \mathscr{U}\mathfrak{g}$ stays well-defined on the quotient and preserves the degree, so extends uniquely to a continuous action on the completed tensor product. Denote the coproduct of the Hopf algebra $\mathscr{U}\mathfrak{g}$ by $\Delta$. It is defined by extending the assignment $\mathfrak{g} \ni X \mapsto X \otimes 1 + 1 \otimes X \in \mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g}$ to an algebra homomorphism $\Delta\colon \mathscr{U}\mathfrak{g} \to \mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g}$.

**Proposition 3.9 (Alekseev–Lachowska [1])** *Let $\lambda \in \Lambda$. Then the element $F_\lambda \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\hat{\otimes} 2}$ is $\mathfrak{h}$-invariant and satisfies*

$$((\mathsf{id} \otimes \Delta)F_\lambda)1 \otimes F_\lambda = ((\Delta \otimes \mathsf{id})F_\lambda)F_\lambda \otimes 1 \tag{3.16}$$

*in $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\hat{\otimes} 3}$.*

PROOF: See the proof of Theorem 3.23. □

Using the results of Subsection 2.2, elements of $((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{h})^{\otimes 2})^H$ determine bidifferential operators on a complex coadjoint orbit for which $\mathfrak{g}_\lambda = \mathfrak{h}$. Such orbits are of maximal dimension among all coadjoint orbits and called *regular*. Note that $H$ is automatically connected by Proposition 2.3, so $\mathfrak{h}$-invariance of $F_\lambda$ implies $H$-invariance, but $F_\lambda$ is only an element of the completed tensor product. So applying the construction from Subsection 2.2 naively gives a sum of bidifferential operators of increasing orders. To make sense of this sum, we can either introduce a formal parameter $\nu$ in the construction in such a way that we obtain a formal power series of bidifferential operators, or we can restrict ourselves to applying these operators to some class of polynomials, for which only finitely many of the bidifferential operators appearing in the sum give a non-zero contribution.

We will now proceed as follows: In Subsection 3.2, we generalize the construction of $F_\lambda$ to work for arbitrary stabilizers $\mathfrak{g}_\lambda$ (and not just $\mathfrak{h}$). In Subsection 3.3 we will give details on how to construct bidifferential operators out of $F_\lambda$, both in the formal and polynomial setting mentioned above.

## 3.2 Generalization to non-regular orbits

The aim of this subsection is to generalize the results of the last subsection to non-regular semisimple coadjoint orbits. To achieve this, we need to replace $\mathfrak{h}$ by a possibly larger stabilizer $\mathfrak{g}_\lambda$ and define a generalization of the Shapovalov pairing. When this pairing is non-degenerate, we derive an explicit formula for its canonical element, which satisfies (3.16).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra acting under the coadjoint action, i.e. the action dual to the adjoint action, on its dual $\mathfrak{g}^*$. We assume that $\lambda \in \mathfrak{g}^*$ is semisimple (as defined in Subsection 2.1) with stabilizer $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \mathrm{ad}^*_X \lambda = 0\}$. We fix a Cartan subalgebra $\mathfrak{h}$ containing $\lambda^\sharp$ (which is possible since $\lambda$ is semisimple) and denote the corresponding root system by $\Delta$. Since any $H \in \mathfrak{h}$ commutes with $\lambda^\sharp$, it follows that $\mathrm{ad}^*_H \lambda = \lambda([-H, \cdot\,]) = -B(\lambda^\sharp, [H, \cdot\,]) = -B([\lambda^\sharp, H], \cdot\,) = 0$, so $\mathfrak{h} \subseteq \mathfrak{g}_\lambda$. We let

$$\Delta' := \{\alpha \in \Delta \mid (\alpha, \lambda) = 0\} \quad \text{and} \quad \hat{\Delta} := \{\alpha \in \Delta \mid (\alpha, \lambda) \neq 0\} = \Delta \setminus \Delta'. \tag{3.17}$$

One checks easily that $\mathfrak{g}_\lambda = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta'} \mathfrak{g}^\alpha$. Given an ordering on $\Delta$ with $\Delta^\pm$ being the set of positive respectively negative roots, define $\hat{\Delta}^\pm = \Delta^\pm \cap \hat{\Delta}$ and $(\Delta')^\pm = \Delta^\pm \cap \Delta'$. Furthermore, let $\tilde{\mathfrak{n}}^\pm := \bigoplus_{\alpha\in\hat{\Delta}^\pm} \mathfrak{g}^\alpha$ and $\tilde{\mathfrak{b}}^\pm := \mathfrak{g}_\lambda \oplus \tilde{\mathfrak{n}}^\pm$.

**Definition 3.10** *An ordering of $\Delta$ is called* invariant *if for any $\alpha \in \hat{\Delta}^+$ and $\beta \in \Delta'$ such that $\alpha+\beta$ is again a root, this root $\alpha+\beta$ is in $\hat{\Delta}^+$.*

Note that since the sum of two roots in $\Delta'$ is again in $\Delta'$ (if it is a root), it is automatic that $\alpha+\beta \in \hat{\Delta}$. The important part of the previous definition is that $\alpha+\beta$ should again be positive. See Figure 3.3 for an example of invariant and non-invariant orderings.

**Lemma 3.11** *An ordering of $\Delta$ is invariant if and only if for any $\alpha,\beta \in \hat{\Delta}^+$ with $\alpha+\beta \in \Delta$ we have $\alpha+\beta \in \hat{\Delta}^+$.*

In the condition of the lemma it is automatic that $\alpha+\beta$ is positive and the important part is that it lies in $\hat{\Delta}$.

PROOF: Assume the condition of the lemma is false, i.e. $\alpha,\beta \in \hat{\Delta}^+$ and $\alpha+\beta \in \Delta\setminus\hat{\Delta}^+$. Since $\alpha+\beta$ is positive we must then have $\alpha+\beta \in \Delta'$. Consequently $\alpha+(-(\alpha+\beta)) = -\beta \notin \hat{\Delta}^+$, so the ordering is not invariant.

Conversely, if the ordering is not invariant, then we can find $\alpha \in \hat{\Delta}^+$ and $\beta \in \Delta'$ such that $\alpha+\beta \in \Delta \setminus \hat{\Delta}^+$. Then we must have $\alpha+\beta \in \hat{\Delta}^-$ and therefore $\alpha+(-(\alpha+\beta)) = -\beta \notin \hat{\Delta}^+$, so the condition of the lemma is not fulfilled. □

Intuitively the invariance of an ordering means that roots in $\Delta'$ are close to being simple, or more precisely that they are linear combinations of simple roots in $\Delta'$. Indeed, if $\alpha \in (\Delta')^+$, then $\alpha$ is a non-negative linear combination of simple roots. By the lemma at least one of those simple roots, say $\sigma$, must be in $\Delta'$, so $\alpha = \sigma$ or $\alpha - \sigma \in (\Delta')^+$ and we can apply induction.

**Corollary 3.12** *If the ordering of $\Delta$ is invariant, then $\tilde{\mathfrak{n}}^\pm$ and $\tilde{\mathfrak{b}}^\pm$ are both Lie subalgebras of $\mathfrak{g}$. Moreover, $[\mathfrak{g}_\lambda, \tilde{\mathfrak{n}}^\pm] \subseteq \tilde{\mathfrak{n}}^\pm$ and $[\mathfrak{g}_\lambda, \tilde{\mathfrak{b}}^\pm] \subseteq \tilde{\mathfrak{b}}^\pm$.*

PROOF: The condition in the previous lemma says precisely that $[\tilde{\mathfrak{n}}^\pm, \tilde{\mathfrak{n}}^\pm] \subseteq \tilde{\mathfrak{n}}^\pm$, i.e. that $\tilde{\mathfrak{n}}^\pm$ is a Lie subalgebra of $\mathfrak{g}$. The defining property of an invariant ordering means that $[\mathfrak{g}_\lambda, \tilde{\mathfrak{n}}^\pm] \subseteq \tilde{\mathfrak{n}}^\pm$. The statements for $\tilde{\mathfrak{b}}^\pm$ are then clear. □

**Definition 3.13** *We say an ordering is* standard *if there is a set $S \subseteq \mathbb{C}\setminus\{0\}$, closed under addition and satisfying $S\cap(-S) = \emptyset$, $S\cup(-S) = \mathbb{C}\setminus\{0\}$ such that $\alpha \in \hat{\Delta}$ is positive if and only if $(\alpha,\lambda) \in S$.*

Standard invariant orderings exist always since we can construct them as follows. First, take any ordering on the set $\Delta'$ (meaning a subset $(\Delta')^+$ such that if the sum of two elements of $(\Delta')^+$ is in $\Delta'$, then it is in $(\Delta')^+$ and such that for $(\Delta')^- := -(\Delta')^+$ we have $(\Delta')^+ \cup (\Delta')^- = \Delta'$ and $(\Delta')^+ \cap (\Delta')^- = \emptyset$). Then choose a set $S$ that is closed under addition and satisfies $S\cap(-S) = \emptyset$ and $S\cup(-S) = \mathbb{C}\setminus\{0\}$, e.g. $S = \{z \in \mathbb{C}\setminus\{0\} \mid \mathsf{Re}(z) > 0 \text{ or } z \in i\mathbb{R}^+\}$. Let $\alpha \in \Delta$ be positive if $\alpha \in (\Delta')^+$ or $(\alpha,\lambda) \in S$.

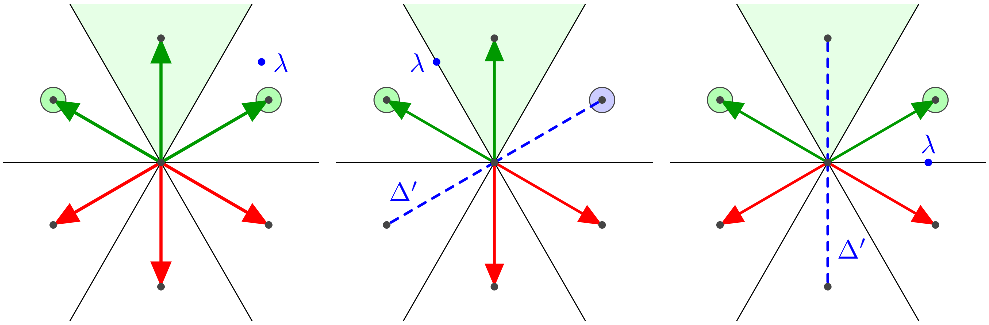


Figure 3.3: Invariant and non-invariant orderings. As in the left picture of Figure 3.1 the roots of $\mathfrak{sl}_3(\mathbb{C})$ are shown. Simple roots are encircled. Roots in $\Delta'$ are drawn with blue dashed lines. Roots in $\hat{\Delta}$ are drawn in green if they are positive, and in red if they are negative. The fundamental Weyl chamber has a light green background. A regular orbit of $\mathrm{SL}_3(\mathbb{C})$ is shown on the left, the other two pictures are of non-regular orbits. In the right picture the ordering on $\Delta$ is not invariant, since adding the negative root in $\Delta'$ (the lower blue dashed line) to one of the positive roots (a green arrow) gives a negative root (a red arrow). The ordering in the middle picture is invariant and standard, the ordering in the left picture is invariant, but not standard. It would be standard if $\lambda$ was in the fundamental Weyl chamber.

For real coadjoint orbits standard invariant orderings are the ones which induce star products of pseudo Wick type (under some further assumptions, see Proposition 5.21), and therefore the orderings we are mainly interested in. However, the construction below works also for other (possibly non-standard) invariant orderings.

Before generalizing the results of the last subsection, we would like to mention the following technical lemma for later use:

**Lemma 3.14** *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\lambda \in \mathfrak{g}^*$ be semisimple, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\lambda^\sharp$. Assume that we have chosen an invariant ordering defining sets $\Delta^+$, $\hat{\Delta}$, and $\Delta'$ as above. Then there is a constant $M \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ the sum of $m$ positive roots in $\hat{\Delta}^+$ and at least $Mm$ positive roots in $(\Delta')^+$ is not in $\mathbb{N}_0\hat{\Delta}^+$.*

PROOF: Label the simple roots by $\sigma_1, \ldots, \sigma_r$ such that the first $r'$ simple roots $\sigma_1, \ldots, \sigma_{r'}$ are in $\Delta'$ and the remaining simple roots are in $\hat{\Delta}$. Label all roots in $\hat{\Delta}^+$ by $\alpha_1, \ldots, \alpha_{\tilde{k}}$. Then there are unique non-negative integers $c_j^i \in \mathbb{N}_0$ such that $\alpha_j = \sum_{i=1}^r c_j^i \sigma_i$. Set $M' = \max_{j \in \{1,\ldots,\tilde{k}\}} \sum_{i=1}^{r'} c_j^i$, $M'' = \max_{j\in\{1,\ldots,\tilde{k}\}} \sum_{i=r'+1}^r c_j^i$ and $M = M'M'' + 1$.

Since $\alpha_j \in \hat{\Delta}^+$ we have $\sum_{i=r'+1}^r c_j^i \geq 1$, and $\sum_{i=1}^{r'} c_j^i \leq M' \leq M' \sum_{i=r'+1}^r c_j^i$ for any $j \in \{1, \ldots, \tilde{k}\}$. Note that any element $\beta \in \mathbb{N}_0\hat{\Delta}^+$ can be written uniquely as $\beta = \sum_{i=1}^r \beta^i \sigma_i$ with $\beta^i \in \mathbb{N}_0$, and the coefficients satisfy the same inequality $\sum_{i=1}^{r'} \beta^i \leq M' \sum_{i=r'+1}^r \beta^i$.

Recall that any root in $(\Delta')^+$ is a linear combination of simple roots in $(\Delta')^+$. So if $\sum_{i=1}^r d^i\sigma_i \in (\Delta')^+$, then $d^i = 0$ for all $i = r'+1, \ldots, r$. Therefore, if $\gamma$ is the sum of $m$ roots from $\hat{\Delta}^+$ and at least $Mm$ roots from $(\Delta')^+$, and $\gamma = \sum_{i=1}^r \gamma^i \sigma_i$, then $M' \sum_{i=r'+1}^r \gamma^i \leq M'M''m < Mm \leq \sum_{i=1}^{r'} \gamma^i$, so $\gamma$ cannot be in $\mathbb{N}_0 \hat{\Delta}^+$. □

Note that for a regular coadjoint orbit, we have $\Delta' = \emptyset$. Consequently $\hat{\Delta} = \Delta$, $\mathfrak{g}_\lambda = \mathfrak{h}$, $\tilde{\mathfrak{n}}^+ = \mathfrak{n}^+$ and $\tilde{\mathfrak{n}}^- = \mathfrak{n}^-$. In this case every ordering is invariant, and the generalized Shapovalov pairing, that we will introduce now, coincides with the Shapovalov pairing introduced in the last subsection. Since $\mathfrak{g}_\lambda = \mathfrak{h}$ when $\Delta' = \emptyset$, we usually denote an element of $\mathfrak{g}_\lambda$ by $H$.

Let $\lambda \in \mathfrak{g}_\lambda^*$ be the restriction of $\lambda \in \mathfrak{g}^*$ to $\mathfrak{g}_\lambda$. Then $\lambda([H', H]) = \mathrm{ad}_H^* \lambda(H') = 0$ for all $H, H' \in \mathfrak{g}_\lambda$, so $H \triangleright z = \lambda(H)z$ makes $\mathbb{C}$ a left or right $\mathfrak{g}_\lambda$-module. Extending trivially along $\tilde{\mathfrak{n}}^\pm$ gives a left or right $\tilde{\mathfrak{b}}^\pm$-module, and we denote the corresponding left $\mathscr{U}(\tilde{\mathfrak{b}}^\pm)$-module by $\tilde{\mathbb{C}}_\lambda^\pm$ and the right $\mathscr{U}(\tilde{\mathfrak{b}}^-)$-module by $\tilde{\mathbb{C}}_\lambda^*$. Define the *generalized Verma modules*

$$\tilde{M}_\lambda = \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^+)} \tilde{\mathbb{C}}_\lambda^+ , \quad \tilde{M}_\lambda^- = \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^-)} \tilde{\mathbb{C}}_{-\lambda}^* , \quad \text{and} \quad \tilde{M}_\lambda^* = \tilde{\mathbb{C}}_\lambda^* \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^-)} \mathscr{U}\mathfrak{g} . \tag{3.18}$$

$\tilde{M}_\lambda$ and $\tilde{M}_\lambda^-$ are left $\mathscr{U}\mathfrak{g}$-modules, $\tilde{M}_\lambda^*$ is a right $\mathscr{U}\mathfrak{g}$-module. Most of the results of the previous subsection have obvious analogues in this setting.

Let $\{X_1, \ldots, X_{\tilde{k}}\}$ be a basis of $\tilde{\mathfrak{n}}^+$, $\{Y_1, \ldots, Y_{\tilde{k}}\}$ be a basis of $\tilde{\mathfrak{n}}^-$, and $\{H_1, \ldots, H_{\tilde{r}}\}$ be a basis of $\mathfrak{g}_\lambda$. Since $\mathfrak{g} = \tilde{\mathfrak{n}}^+ \oplus \mathfrak{g}_\lambda \oplus \tilde{\mathfrak{n}}^-$ the Poincaré–Birkhoff–Witt theorem implies that

$$\{Y^I H^J X^K \mid I, K \in \mathbb{N}_0^{\tilde{k}}, J \in \mathbb{N}_0^{\tilde{r}}\} \quad \text{and} \quad \{X^K H^J Y^I \mid I, K \in \mathbb{N}_0^{\tilde{k}}, J \in \mathbb{N}_0^{\tilde{r}}\} \tag{3.19}$$

are bases for $\mathscr{U}\mathfrak{g}$. Define maps

$$\tilde{\pi}_\lambda^- \colon \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^-) , \qquad \tilde{\pi}_\lambda^-(Y^I H^J X^K) := \lambda(H_1)^{J_1} \ldots \lambda(H_{\tilde{r}})^{J_{\tilde{r}}} Y^I \delta_{K,0} , \tag{3.20a}$$

$$\tilde{\pi}_\lambda^+ \colon \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^+) , \qquad \tilde{\pi}_\lambda^+(X^K H^J Y^I) := (-\lambda(H_1))^{J_1} \ldots (-\lambda(H_{\tilde{r}}))^{J_{\tilde{r}}} X^K \delta_{I,0} , \tag{3.20b}$$

$$\tilde{\pi}_\lambda^* \colon \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^+) , \qquad \tilde{\pi}_\lambda^*(Y^I H^J X^K) := \lambda(H_1)^{J_1} \ldots \lambda(H_{\tilde{r}})^{J_{\tilde{r}}} X^K \delta_{I,0} . \tag{3.20c}$$

Note that they are compatible with the maps $\pi_\lambda^-$, $\pi_\lambda^+$, and $\pi_\lambda^*$ in the sense that $\tilde{\pi}_\lambda^- \circ \pi_\lambda^- = \tilde{\pi}_\lambda^-$, $\tilde{\pi}_\lambda^+ \circ \pi_\lambda^+ = \tilde{\pi}_\lambda^+$, and $\tilde{\pi}_\lambda^* \circ \pi_\lambda^* = \tilde{\pi}_\lambda^*$. On the left hand sides, we are implicitly using the inclusion $\mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}\mathfrak{g}$. Note that this inclusion is not a $\mathscr{U}\mathfrak{g}$-module map.

**Lemma 3.15** *The maps* $\cdot \otimes 1 \colon \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \tilde{M}_\lambda$, $v \mapsto v \otimes 1$ *and* $\cdot \otimes 1 \colon \mathscr{U}(\tilde{\mathfrak{n}}^+) \to \tilde{M}_\lambda^-$, $u \mapsto u \otimes 1$ *define isomorphisms of left* $\mathscr{U}(\tilde{\mathfrak{n}}^-)$*-modules and* $\mathscr{U}(\tilde{\mathfrak{n}}^+)$*-modules, respectively. The map* $1 \otimes \cdot \colon \mathscr{U}(\tilde{\mathfrak{n}}^+) \to M_\lambda^*$, $u \mapsto 1 \otimes u$ *is an isomorphism of right* $\mathscr{U}(\tilde{\mathfrak{n}}^+)$*-modules. The* $\mathscr{U}\mathfrak{g}$*-module structures on* $\mathscr{U}(\tilde{\mathfrak{n}}^\pm)$ *obtained by transferring the module structures on the generalized Verma modules with these isomorphisms are given explicitly by*

$$\tilde{\triangleright}_\lambda^- \colon \mathscr{U}\mathfrak{g} \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathscr{U}(\tilde{\mathfrak{n}}^-) , \qquad (w, v) \mapsto w \,\tilde{\triangleright}_\lambda^-\, v := \tilde{\pi}_\lambda^-(wv) , \tag{3.21a}$$

$$\tilde{\triangleright}_\lambda^+ \colon \mathscr{U}\mathfrak{g} \times \mathscr{U}(\tilde{\mathfrak{n}}^+) \to \mathscr{U}(\tilde{\mathfrak{n}}^+) , \qquad (w, u) \mapsto w \,\tilde{\triangleright}_\lambda^+\, u := \tilde{\pi}_\lambda^+(wu) , \tag{3.21b}$$

$$\tilde{\triangleleft}_\lambda^* \colon \mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}\mathfrak{g} \to \mathscr{U}(\tilde{\mathfrak{n}}^+) , \qquad (u, w) \mapsto u \,\tilde{\triangleleft}_\lambda^*\, w := \tilde{\pi}_\lambda^*(uw) . \tag{3.21c}$$

*Furthermore,* $S(w \,\tilde{\triangleright}_\lambda^+\, u) = S(u) \,\tilde{\triangleleft}_\lambda^*\, S(w)$*, where* $S$ *denotes the antipode of* $\mathscr{U}\mathfrak{g}$.

PROOF: Similar to the proof of Lemma 3.2. $\square$

Note that since $\mathscr{U}(\tilde{\mathfrak{n}}^{\pm})$ is a $\mathscr{U}\mathfrak{g}$-module, we must have

$$\tilde{\pi}^{\pm}_{\lambda}(w\tilde{\pi}^{\pm}_{\lambda}(w')) = w \,\tilde{\triangleright}^{\pm}_{\lambda}\, (w' \,\tilde{\triangleright}^{\pm}_{\lambda}\, 1) = (ww') \,\tilde{\triangleright}^{\pm}_{\lambda}\, 1 = \tilde{\pi}^{\pm}_{\lambda}(ww') \tag{3.22}$$

and

$$\tilde{\pi}^{*}_{\lambda}(\tilde{\pi}^{*}_{\lambda}(w)w') = \tilde{\pi}^{*}_{\lambda}(ww') \tag{3.23}$$

for all $w, w' \in \mathscr{U}\mathfrak{g}$. In particular, this implies that the map $\tilde{\pi}^{\pm}_{\lambda}\big|_{\mathscr{U}(\mathfrak{n}^{\pm})} \colon \mathscr{U}(\mathfrak{n}^{\pm}) \to \mathscr{U}(\tilde{\mathfrak{n}}^{\pm})$ is a $\mathscr{U}\mathfrak{g}$-module homomorphism (with respect to the module structures given by $\triangleright^{\pm}_{\lambda}$ and $\tilde{\triangleright}^{\pm}_{\lambda}$). Indeed, for the plus case we have

$$\tilde{\pi}^{+}_{\lambda}(w \triangleright^{+}_{\lambda} u) = \tilde{\pi}^{+}_{\lambda}\pi^{+}_{\lambda}(wu) = \tilde{\pi}^{+}_{\lambda}(wu) = \tilde{\pi}^{+}_{\lambda}(w\tilde{\pi}^{+}_{\lambda}u) = w \,\tilde{\triangleright}^{+}_{\lambda}\, \tilde{\pi}^{+}_{\lambda}u$$

for all $w \in \mathscr{U}\mathfrak{g}$ and $u \in \mathscr{U}(\mathfrak{n}^{+})$ and the minus case is similar. Define $\mathfrak{g}^{\pm}_{\lambda} := \bigoplus_{\alpha\in(\Delta')^{\pm}} \mathfrak{g}^{\alpha} = \mathfrak{g}_{\lambda} \cap \mathfrak{n}^{\pm}$. Note that $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}^{\pm}_{\lambda} = \{w \triangleright^{\pm}_{\lambda} X \mid w \in \mathscr{U}\mathfrak{g}, X \in \mathfrak{g}^{\pm}_{\lambda}\}$ is a $\mathscr{U}\mathfrak{g}$-submodule of $\mathscr{U}(\mathfrak{n}^{\pm})$. Since $\tilde{\pi}^{\pm}_{\lambda}$ is a map of $\mathscr{U}\mathfrak{g}$-modules and vanishes on $\mathfrak{g}^{\pm}_{\lambda}$, $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}^{\pm}_{\lambda}$ is in its kernel.

**Lemma 3.16** *The induced maps $\tilde{\pi}^{\pm}_{\lambda} \colon \mathscr{U}(\mathfrak{n}^{\pm})/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}^{\pm}_{\lambda} \to \mathscr{U}(\tilde{\mathfrak{n}}^{\pm})$ are isomorphisms of $\mathscr{U}\mathfrak{g}$-modules.*

PROOF: It is easy to check that the quotient map induced by the inclusion $\mathscr{U}(\tilde{\mathfrak{n}}^{\pm}) \to \mathscr{U}(\mathfrak{n}^{\pm})$ defines an inverse. $\square$

As before there are isomorphisms $\tilde{M}^{*}_{\lambda} \otimes_{\mathscr{U}\mathfrak{g}} \tilde{M}_{\lambda} \cong \tilde{\mathbb{C}}^{-}_{\lambda} \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^{-})} \mathscr{U}\mathfrak{g} \otimes_{\mathscr{U}(\tilde{\mathfrak{b}}^{+})} \tilde{\mathbb{C}}_{\lambda} \cong \tilde{\mathbb{C}}^{-}_{\lambda} \otimes_{\mathscr{U}(\mathfrak{g}_{\lambda})} \tilde{\mathbb{C}}_{\lambda} \cong \mathbb{C}$, which we use to define the Shapovalov pairings $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim\prime}_{\lambda} \colon \tilde{M}^{*}_{\lambda} \times \tilde{M}_{\lambda} \to \mathbb{C}$, $(x, y) \mapsto \langle x, y\rangle^{\sim\prime}_{\lambda} := x \otimes y$ and

$$\langle\,\cdot\,,\,\cdot\,\rangle^{\sim}_{\lambda} \colon \mathscr{U}(\tilde{\mathfrak{n}}^{+}) \times \mathscr{U}(\tilde{\mathfrak{n}}^{-}) \to \mathbb{C}\,, \quad \langle u, v\rangle^{\sim}_{\lambda} = \langle 1 \otimes S(u), v \otimes 1\rangle^{\sim\prime}_{\lambda} = 1 \otimes S(u)v \otimes 1\,. \tag{3.24}$$

In the same way as in Lemma 3.3 one proves that this pairing can be computed by

$$\langle u, v\rangle^{\sim}_{\lambda} = \pi_{\lambda}(S(u)v)\,. \tag{3.25}$$

Note that $\tilde{\pi}^{-}_{\lambda} \circ \tilde{\pi}^{*}_{\lambda} = \tilde{\pi}^{*}_{\lambda} \circ \tilde{\pi}^{-}_{\lambda} = \pi^{*}_{\lambda} \circ \pi^{-}_{\lambda} = \pi_{\lambda}$, so there is no need to introduce a $\tilde{\pi}_{\lambda}$.

**Lemma 3.17** *Let $u \in \mathscr{U}(\mathfrak{n}^{+})$ and $v \in \mathscr{U}(\mathfrak{n}^{-})$. Then we have $\langle \tilde{\pi}^{+}_{\lambda}u, \tilde{\pi}_{\lambda}v\rangle^{\sim}_{\lambda} = \langle u, v\rangle_{\lambda}$. In particular $\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}\big|_{\mathscr{U}(\mathfrak{n}^{+})\times\mathscr{U}\mathfrak{g}\cdot\mathfrak{g}^{-}_{\lambda}} = \langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}\big|_{\mathscr{U}\mathfrak{g}\cdot\mathfrak{g}^{+}_{\lambda}\times\mathscr{U}(\mathfrak{n}^{-})} = 0$.*

PROOF: Using (3.22) twice, we compute

$$\begin{aligned}\langle \tilde{\pi}^{+}_{\lambda}u, \tilde{\pi}^{-}_{\lambda}v\rangle^{\sim}_{\lambda} &= \pi_{\lambda}(S(\tilde{\pi}^{+}_{\lambda}u)\tilde{\pi}^{-}_{\lambda}v) = \tilde{\pi}^{*}_{\lambda} \circ \tilde{\pi}^{-}_{\lambda}(\tilde{\pi}^{*}_{\lambda}(Su)\tilde{\pi}^{-}_{\lambda}v) = \tilde{\pi}^{*}_{\lambda} \circ \tilde{\pi}^{-}_{\lambda}(\tilde{\pi}^{*}_{\lambda}(Su)v) = \\ &= \tilde{\pi}^{-}_{\lambda} \circ \tilde{\pi}^{*}_{\lambda}(\tilde{\pi}^{*}_{\lambda}(Su)v) = \tilde{\pi}^{-}_{\lambda} \circ \tilde{\pi}^{*}_{\lambda}(S(u)v) = \pi_{\lambda}(S(u)v) = \langle u, v\rangle_{\lambda}\,.\end{aligned}$$ $\square$

Define the set

$$\tilde{\Lambda} = \{\lambda \in \mathfrak{h}^{*} \mid p_{\lambda}(\mu) \neq 0 \;\; \forall \mu \in \mathbb{N}_{0}\hat{\Delta}^{+} \setminus \{0\}\}\,. \tag{3.26}$$

Furthermore, let $\tilde{W}$ be the set of words $w \in W$ such that $\alpha_{w_{i\ldots|w|}} \in \mathbb{N}_{0}\hat{\Delta}^{+}$ for all $i = 1, \ldots, |w|$. Since $\tilde{\pi}^{+}_{\lambda}(X_{w}) = 0$ and $\tilde{\pi}^{-}_{\lambda}(Y_{w}) = 0$ for $w \in W \setminus \tilde{W}$, the following theorem is not surprising.

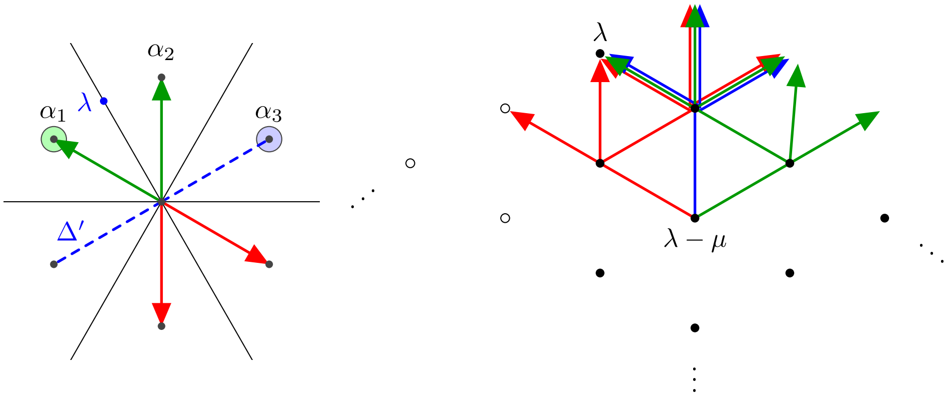


Figure 3.4: The tree $T$ used in the proof of Theorem 3.18 for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\mu = 2\alpha_1 + \alpha_3$. Compare this with Figure 3.2. Elements of the tree starting with 1, 2 and 3 are coloured red, blue and green, respectively. Only the weight spaces marked with filled dots are non-trivial (but might have a different dimension than in the case where $\Delta' = \emptyset$), and all weight spaces marked with circles only contain 0. In particular, the weight spaces at maximal elements of the tree are trivial, except for $V^\lambda$. All non-maximal weight spaces are non-trivial.

**Theorem 3.18** *Let $\lambda \in \tilde{\Lambda}$. Then the Shapovalov pairing $\langle\,\cdot\,,\,\cdot\,\rangle^\sim_\lambda : \mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathbb{C}$ is non-degenerate and its canonical element $F_\lambda \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \,\hat{\otimes}\, \mathscr{U}(\tilde{\mathfrak{n}}^-)$ is given by*

$$F_\lambda = \sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w) = \sum_{w \in \tilde{W}} \prod_{i=1}^{|w|} p_\lambda(\alpha_{w_{i\dots|w|}})^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w)\,. \tag{3.27}$$

Proof: It suffices to prove that $\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^-_\lambda(Y_w) \langle \tilde{\pi}^+_\lambda(X_w), \tilde{v}\rangle^\sim_\lambda = \tilde{v}$ for all $\tilde{v} \in \mathscr{U}(\tilde{\mathfrak{n}}^-)$ and that $\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \langle \tilde{u}, \tilde{\pi}^-_\lambda(Y_w)\rangle^\sim_\lambda = \tilde{u}$ for all $\tilde{u} \in \mathscr{U}(\tilde{\mathfrak{n}}^+)$ by using an analogue of Lemma 3.4. Let $v \in \mathscr{U}(\mathfrak{n}^-)$ be the image of $\tilde{v}$ under the inclusion $\mathscr{U}(\tilde{\mathfrak{n}}^-) \to \mathscr{U}(\mathfrak{n}^-)$, so that $\tilde{\pi}^-_\lambda(v) = \tilde{v}$. Assume that $v = \sum_{\mu \in \mathbb{N}_0 \hat{\Delta}^+} v_{-\mu}$ is the weight decomposition of $v$. Then

$$\begin{aligned}
&\sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^-_\lambda(Y_w) \langle \tilde{\pi}^+_\lambda(X_w), \tilde{v}\rangle^\sim_\lambda \\
&\qquad = \sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} \tilde{\pi}^-_\lambda(Y_w) \langle X_w, v\rangle_\lambda \\
&\qquad = \tilde{\pi}^-_\lambda\left( \sum_{w \in \tilde{W}} p^w_\lambda(\alpha_w)^{-1} Y_w \langle X_w, v_{-\alpha_w}\rangle_\lambda \right) \\
&\qquad = \tilde{\pi}^-_\lambda\left( \sum_{\mu \in \mathbb{N}_0 \hat{\Delta}^+} \sum_{w \in \tilde{W}_\mu} (-1)^{|w|} p^w_\lambda(\alpha_w)^{-1} Y_w \triangleright^-_\lambda X_{w^{\mathrm{opp}}} \triangleright^-_\lambda v_{-\mu} \right),
\end{aligned}$$

where $\tilde{W}_\mu = \{w \in \tilde{W} \mid \alpha_w = \mu\}$. We claim that there is an admissible tree $T$ and $v' \in \mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^-$ such that

$$\sum_{w \in \tilde{W}_\mu} (-1)^{|w|} p_\lambda^w(\alpha_w)^{-1} Y_w \triangleright_\lambda^- X_{w^{\mathrm{opp}}} \triangleright_\lambda^- v_{-\mu} =$$

$$= v' + \sum_{w \in \max T} (-1)^{|w|} p_\lambda^w(\alpha_w)^{-1} Y_w \triangleright_\lambda^- X_{w^{\mathrm{opp}}} \triangleright_\lambda^- v_{-\mu},$$

which would finish the proof by using Lemma 3.6. Indeed, let

$$T = \{\emptyset\} \cup \{w \in W \mid \exists w' \in \tilde{W}_\mu \text{ and } 0 \leq i \leq |w'| - 1 \text{ such that } w_{1\ldots|w|-1} = w'_{1\ldots i}\}$$

be the smallest tree containing $\tilde{W}_\mu$. Since $\lambda \in \tilde{\Lambda}$, this tree is admissible. Furthermore $\tilde{W}_\mu \subseteq \max T$ and any element $w \in \max T$ satisfies exactly one of the following two conditions. Either $\alpha_w = \mu$, so that $w \in \tilde{W}_\mu$ appears in the sum on the left hand side of the above equation. Or $\mu - \alpha_w \notin \mathbb{N}_0 \hat{\Delta}^+$, so that $X_{w^{\mathrm{opp}}} v_{-\mu}$ would have to be of weight $\alpha_w - \mu \notin -\mathbb{N}_0 \hat{\Delta}^+$ and does therefore either vanish or lie in $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda^-$. The statement for $\tilde{u}$ is proven similarly. □

Using the inclusions $\mathscr{U}(\tilde{\mathfrak{n}}^\pm) \to \mathscr{U}\mathfrak{g}$ and the projection $\mathscr{U}\mathfrak{g} \to \mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda$, we map $F_\lambda$ to $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\hat{\otimes} 2}$. Note that, as before, $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda$ is a homogeneous ideal in $\mathscr{U}\mathfrak{g}$, so the grading of $\mathscr{U}\mathfrak{g}$ stays well-defined on the quotient. The action of $\mathfrak{g}_\lambda$ on $(\mathscr{U}\mathfrak{g})^{\otimes 2}$ also passes to the quotient and extends to a continuous action on the completed tensor product.

**Theorem 3.19 (Alekseev–Lachowska [1])** *Let $\lambda \in \tilde{\Lambda}$. Then the element $F_\lambda \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\hat{\otimes} 2}$ is $\mathfrak{g}_\lambda$-invariant and satisfies*

$$((\mathsf{id} \otimes \Delta)F_\lambda) 1 \otimes F_\lambda = ((\Delta \otimes \mathsf{id})F_\lambda) F_\lambda \otimes 1 \tag{3.28}$$

*in $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\hat{\otimes} 3}$.*

PROOF: Note that the $\mathfrak{g}$-invariance of the Shapovalov pairing (proven similarly as in Lemma 3.3) implies that $F_\lambda \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \hat{\otimes} \mathscr{U}(\tilde{\mathfrak{n}}^-)$ is also $\mathfrak{g}$-invariant. Then $F_\lambda \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\hat{\otimes} 2}$ is $\mathfrak{g}_\lambda$-invariant since the map $\mathscr{U}(\tilde{\mathfrak{n}}^+) \times \mathscr{U}(\tilde{\mathfrak{n}}^-) \to (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\hat{\otimes} 2}$ is $\mathfrak{g}_\lambda$-equivariant. Equation (3.28) is proven in [1, Section 4]. □

It will be convenient in the following to write $F_\lambda$ as a sum of elements that are all invariant under $\mathfrak{g}_\lambda$.

**Lemma 3.20** *Let $\lambda \in \tilde{\Lambda}$. Then there is a partition of $\tilde{W}$ into finite subsets $\tilde{W}_\ell$, $\ell \in \mathbb{N}_0$ such that*

$$F_{\lambda,\ell} := \sum_{w \in \tilde{W}_\ell} p_\lambda^w(\alpha_w)^{-1} \tilde{\pi}_\lambda^+(X_w) \otimes \tilde{\pi}_\lambda^-(Y_w) \tag{3.29}$$

*is $\mathfrak{g}_\lambda$-invariant.*

PROOF: It will be convenient to introduce a different grading $d'$ on $\mathfrak{g}$, for which $\mathfrak{g}_\lambda$ is of degree 0. To this end, let $\mathfrak{h}$ and the root spaces of simple roots in $\Delta'$ be of degree 0, and let the root spaces of simple roots in $\hat{\Delta}$ be of degree 1. Since any root is a unique linear combination of simple roots this assignment extends to a grading on $\mathfrak{g}$. More explicitly, if $\sigma_1, \dots, \sigma_r \in \Delta$ are the simple roots, with $\sigma_1, \dots, \sigma_{r'} \in \Delta'$, then the root space of a root $\alpha = \sum_{i=1}^r c^i \sigma_i$ is of degree $d'(\alpha) = \sum_{i=r'+1}^r c^i$. Since $\mathfrak{g}_\lambda$ is spanned by $\mathfrak{h}$ and the root spaces of roots in $\Delta'$, and since the invariance of the ordering implies that any root in $\Delta'$ is a linear combination of simple roots in $\Delta'$, it follows that every element of $\mathfrak{g}_\lambda$ is homogeneous of degree 0. This grading is coarser than the grading given by $d$, in the sense that the graded components with respect to the new grading $d'$ are direct sums of the graded components with respect to $d$. The restrictions of the maps $\tilde{\pi}^\pm_\lambda$ to $\mathscr{U}(\mathfrak{n}^\pm)$ are homogeneous of degree 0 with respect to (the restriction of) the $\mathbb{Z}$-grading on $\mathscr{U}\mathfrak{g}$ induced by $d'$.

For $w \in W$ set $d'(w) := d'(\alpha_{w_1}) + \dots + d'(\alpha_{w_{|w|}})$, and define $\tilde{W}_\ell := \{w \in \tilde{W} \mid d'(w) = \ell\}$. It follows from Lemma 3.14 that $\tilde{W}_\ell$ is finite for every $\ell$. The elements $F_{\lambda,\ell}$ defined from $\tilde{W}_\ell$ as in (3.29) have a nice description in terms of the grading $d'$. Since all graded components of $\tilde{\mathfrak{n}}^+$ resp. $\tilde{\mathfrak{n}}^-$ are of degree $\geq 1$ resp. $\leq -1$, $d'$ induces a grading of $\mathscr{U}(\tilde{\mathfrak{n}}^+) \otimes \mathscr{U}(\tilde{\mathfrak{n}}^-)$ by $\mathbb{N}_0 \times (-\mathbb{N}_0)$. Using the homogeneity of $\tilde{\pi}^\pm_\lambda$, it follows directly from the definition of $\tilde{W}_\ell$ that $F_{\lambda,\ell}$ is precisely the component of $F_\lambda$ of degree $(\ell, -\ell)$ with respect to this grading. Since $\mathfrak{g}_\lambda$ is of degree 0, the action of $\mathfrak{g}_\lambda$ on $\mathscr{U}(\tilde{\mathfrak{n}}^+) \otimes \mathscr{U}(\tilde{\mathfrak{n}}^-)$ preserves the graded components, and the $\mathfrak{g}_\lambda$-invariance of $F_\lambda$ implies that all the graded components $F_{\lambda,\ell}$ must also be $\mathfrak{g}_\lambda$-invariant. □

## 3.3 The induced formal and strict products

In this subsection we construct associative products from the element $F_\lambda$ obtained at the end of the last subsection. We will rescale $\lambda$ in order to introduce a parameter playing the role of Planck's constant in the construction. Then we would like to use the results of Subsection 2.2 to obtain bidifferential operators from (the rescaled) $F_\lambda$. However, since $F_\lambda$ is only in the completed tensor product, applying these results naively would give a sum of bidifferential operators of increasing orders and we have to deal with its convergence. There are essentially two solutions to this problem: Firstly, we can take a formal expansion in the parameter $\hbar$, which will give us a well-defined power series of bidifferential operators of increasing order. Secondly, we can restrict ourselves to applying these operators only to some polynomial functions, for which only finitely many terms of the infinite sum give a non-zero contribution. We discuss both approaches in detail, starting with the formal one.

Let us first introduce the rescaling. Define the set

$$P_\lambda = \{0\} \cup \{\hbar \in \mathbb{C} \setminus \{0\} \mid \mathrm{i}\lambda/\hbar \notin \tilde{\Lambda}\}\,, \tag{3.30}$$

and for $\hbar \in \mathbb{C} \setminus P_\lambda$ set $F_\hbar := F_{\mathrm{i}\lambda/\hbar}$ and $F_{\hbar,\ell} := F_{\mathrm{i}\lambda/\hbar,\ell}$, where $F_{\mathrm{i}\lambda/\hbar}$ was computed in Theorem 3.19 and $F_{\mathrm{i}\lambda/\hbar,\ell}$ was defined in Lemma 3.20. Note that $\mathfrak{g}_{\mathrm{i}\lambda/\hbar} = \mathfrak{g}_\lambda$, so $F_\hbar \in ((\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g}\cdot\mathfrak{g}_\lambda)^{\hat{\otimes}2})^{\mathfrak{g}_\lambda}$ holds for all $\hbar \in \mathbb{C} \setminus P_\lambda$. Furthermore, the projections $\tilde{\pi}^\pm_{\mathrm{i}\lambda/\hbar}\big|_{\mathscr{U}(\mathfrak{n}^\pm)} \colon \mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}(\tilde{\mathfrak{n}}^\pm)$ are independent of $\hbar$, which one can easily see from their definition in (3.20).

**Proposition 3.21** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\lambda \in \mathfrak{h}^*$. Fix an invariant ordering on $\Delta$, and assume that $(\lambda, \mu) \neq 0$ for all $\mu \in \mathbb{N}_0 \hat{\Delta}^+$ satisfying $\frac{1}{2}(\mu, \mu) = (\rho, \mu)$. Then the set $P_\lambda$ is countable and accumulates only at zero.*

PROOF: From the definition of $P_\lambda$ we obtain

$$P_\lambda = \{0\} \cup \{\hbar \in \mathbb{C} \setminus \{0\} \mid p_{\mathrm{i}\lambda/\hbar}(\mu) = 0 \text{ for some } \mu \in \mathbb{N}_0 \hat{\Delta}^+ \setminus \{0\}\}.$$

Under our assumptions the function $\hbar \mapsto p_{\mathrm{i}\lambda/\hbar}(\mu) = \frac{1}{2}(\mu, \mu) - (\rho, \mu) - \frac{\mathrm{i}}{\hbar}(\lambda, \mu)$ has the only root $\mathrm{i}(\lambda, \mu)/(\frac{1}{2}(\mu, \mu) - (\rho, \mu))$ if $\frac{1}{2}(\mu, \mu) - (\rho, \mu) \neq 0$ and no root otherwise. Therefore $P_\lambda$ is countable since $\mathbb{N}_0 \hat{\Delta}^+ \setminus \{0\}$ is countable. Furthermore, $P_\lambda$ accumulates only at zero since

$$\left| \frac{\mathrm{i}(\lambda, \mu)}{\frac{1}{2}(\mu, \mu) - (\rho, \mu)} \right| \leq \frac{\|\lambda\| \|\mu\|}{\frac{1}{2}\|\mu\|^2 - \|\mu\| \|\rho\|} = \frac{\|\lambda\|}{\frac{1}{2}\|\mu\| - \|\rho\|}$$

if $\|\mu\| > 2\|\rho\|$. Note that there are only finitely many elements $\mu \in \mathbb{N}_0 \hat{\Delta}^+$ with $\|\mu\| \leq 2\|\rho\|$. $\square$

**Remark 3.22** If the ordering in the previous proposition is standard, then any element $\mu \in \mathbb{N}_0 \hat{\Delta}^+$ automatically satisfies $(\lambda, \mu) \neq 0$: For all $\alpha \in \hat{\Delta}^+$ we have $(\lambda, \alpha) \in S$ and since $S$ is closed under addition this implies $(\lambda, \mu) \in S$ for all $\mu \in \mathbb{N}_0 \hat{\Delta}^+$. Note that $0 \notin S$, so in particular $(\lambda, \mu) \neq 0$.

Note also that $\frac{1}{2}(\mu, \mu) = (\rho, \mu)$ implies $\|\mu\| \leq 2\|\rho\|$, so there can only be finitely many elements $\mu \in \mathbb{N}_0 \Delta$ satisfiying $\frac{1}{2}(\mu, \mu) = (\rho, \mu)$. Among those are all simple roots and the element $2\rho$. However, simple roots which are in $\mathbb{N}_0 \hat{\Delta}$ are by definition not orthogonal to $\lambda$. An example of an element that is not a simple root and not $2\rho$ in the case of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ with root system as in Figure 3.1 is $\mu = \alpha_1 + \alpha_2$.

We say that $F_\hbar$ depends rationally on $\hbar$ if all the $F_{\hbar,\ell}$ depend rationally on $\hbar$. This makes sense since $F_{\hbar,\ell}$ takes values in a finite dimensional subspace of $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 2}$ that is independent of $\hbar$.

**Theorem 3.23 (Alekseev–Lachowska [1])** *Let $\lambda \in \mathfrak{h}^*$ and assume that $P_\lambda$ is countable. Then $F_\hbar$ depends rationally on $\hbar$, with no pole at zero. In particular, the Taylor series expansion of $F_\hbar$ around $0$ makes sense, and it gives an element $F \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 2}[[\nu]]$, where the tensor product is the usual (not completed) tensor product. Furthermore, $F$ satisfies (3.28) in $(\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 3}[[\nu]]$ and is $\mathfrak{g}_\lambda$-invariant.*

PROOF: As mentioned before, $\mathfrak{g}_{\mathrm{i}\lambda/\hbar}$ and $\tilde{\pi}^{\pm}_{\mathrm{i}\lambda/\hbar}\big|_{\mathscr{U}(\mathfrak{n}^\pm)} : \mathscr{U}(\mathfrak{n}^\pm) \to \mathscr{U}(\tilde{\mathfrak{n}}^\pm)$ are independent of $\hbar$, so only the coefficients $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1}$ in the formula for $F_{\mathrm{i}\lambda/\hbar}$ obtained in Theorem 3.18 depend on $\hbar$. Since they are products of elements of the form

$$p_{\mathrm{i}\lambda/\hbar}(\mu)^{-1} = \left( \frac{1}{2}(\mu, \mu) - (\rho, \mu) - \left( \frac{\mathrm{i}\lambda}{\hbar}, \mu \right) \right)^{-1} = \frac{\hbar}{\left(\frac{1}{2}(\mu, \mu) - (\rho, \mu)\right)\hbar - (\mathrm{i}\lambda, \mu)}$$

with $\mu \in \mathbb{N}_0 \hat{\Delta}^+ \setminus \{0\}$, their dependence on $\hbar$ is rational without a pole at zero. (Observe that $\frac{1}{2}(\mu, \mu) - (\rho, \mu)$ and $(\mathrm{i}\lambda, \mu)$ cannot vanish simultaneously since $P_\lambda$ is assumed to be countable.) Consequently, we may take the Taylor expansion of $F_{\mathrm{i}\lambda/\hbar}$ around $\hbar = 0$. To see that this yields an element in the usual tensor product, note that the formal expansion of $p_{\mathrm{i}\lambda/\hbar}(\mu)^{-1}$ is a multiple of $\nu$ unless $(\lambda, \mu) = 0$. Now $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} = \prod_{i=1}^{|w|} p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{i\ldots|w|}})^{-1}$, and if the formal expansions of both $p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{i\ldots|w|}})^{-1}$ and $p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{i+1\ldots|w|}})^{-1}$ are not multiples of $\hbar$, then $(\lambda, \alpha_{w_i}) = 0$, i.e. $\alpha_{w_i} \in \Delta'$. However, Lemma 3.14 ensures that this cannot happen too often: If $M$ is the constant obtained in that lemma, then at least $\lceil |w|/(M+1) \rceil$ many elements in the formal expansion of $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1}$ are multiples of $\nu$, so this expansion is of order at least $\nu^{\lceil |w|/(M+1) \rceil}$. Consequently, only finitely many words contribute to a given order in $\nu$, so that we do not need to complete the tensor product. Since every $F_\hbar$ satisfies (3.28) and is $\mathfrak{g}_\lambda$-invariant, this is also true for the formal expansion $F$. □

Let us now apply this theorem to quantize complex coadjoint orbits. Let $G$ be a complex connected semisimple Lie group with coadjoint orbit $\mathcal{O}_\lambda$ through a semisimple element $\lambda \in \mathfrak{g}^*$. Pick a Cartan subalgebra $\mathfrak{h}$ containing $\lambda^\sharp$. Choose an invariant ordering for which $P_\lambda$ is countable (e.g. a standard invariant ordering).

By Proposition 2.3 we know that $G_\lambda$ is connected. Therefore the $\mathfrak{g}_\lambda$-invariance of the elements $F$ and $F_\hbar$ constructed previously implies their $G_\lambda$-invariance. Consequently we can apply the results of Subsection 2.2 in order to obtain holomorphic $G$-invariant bidifferential operators on $\mathcal{O}_\lambda \cong G/G_\lambda$. Define the formal product

$$\star\colon \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]] \times \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]] \to \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]], \quad (f, g) \mapsto f \star g := \Psi(F)(f, g), \tag{3.31}$$

and note that this product is well-defined since the previous theorem asserts that $F \in (\mathscr{U}\mathfrak{g}/\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda)^{\otimes 2}[[\nu]]$.

**Proposition 3.24** *The product $\star$ is associative and restricts to a product*

$$\star\colon \mathrm{Hol}(\mathcal{O}_\lambda)[[\nu]] \times \mathrm{Hol}(\mathcal{O}_\lambda)[[\nu]] \to \mathrm{Hol}(\mathcal{O}_\lambda)[[\nu]] \tag{3.32}$$

*on power series of holomorphic functions. Moreover, $\star$ is $G$-invariant, in the sense that $(g \triangleright f_1) \star (g \triangleright f_2) = g \triangleright (f_1 \star f_2)$ holds for all $g \in G$ and $f_1, f_2 \in \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]]$.*

Proof: It is a standard argument that the twist condition (3.28) translates into associativity of the induced product. That $\star$ restricts to power series of holomorphic functions and is $G$-invariant is immediate since the image of $\Psi$ consists of holomorphic $G$-invariant bidifferential operators. □

In order to define strict star products from $F_\hbar$ directly, i.e. without taking a formal power series expansion, we need to ensure that $\Psi(F_\hbar)$ is well-defined. To do that we introduce polynomials on the coadjoint orbit. It will turn out that only finitely many elements of the infinite sum defining $F_\hbar$ contribute non-trivially when $\Psi(F_\hbar)$ is applied to polynomials.

Recall from Subsection 2.1 that we may assume without loss of generality that $G$ is a closed complex Lie subgroup of $\mathrm{GL}_N(\mathbb{C})$. We fix a way to realize $G$ as such a matrix Lie group once and for all. In particular, the Lie algebra $\mathfrak{g}$ of $G$ is realized as a complex Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$.

**Definition 3.25 (Polynomials on $\mathcal{O}_\lambda$)** *Let $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ be a complex coadjoint orbit. Then*

$$\mathrm{Pol}(\mathcal{O}_\lambda) = \{p\colon \mathcal{O}_\lambda \to \mathbb{C} \mid p = P\big|_{\mathcal{O}_\lambda} \text{ for some holomorphic polynomial } P \text{ on } \mathfrak{g}^*\} \tag{3.33}$$

*is called the algebra of polynomials on $\mathcal{O}_\lambda$.*

Recall that the symmetric algebra $\mathrm{S}\mathfrak{g}$ of $\mathfrak{g}$ is isomorphic (as an algebra) to the algebra $\mathrm{Pol}(\mathfrak{g}^*)$ of polynomials on $\mathfrak{g}^*$. The isomorphism sends an element $X_1 \vee \cdots \vee X_j \in \mathrm{S}^j\mathfrak{g}$ to $\xi \mapsto \xi(X_1)\ldots\xi(X_j)$.

**Definition 3.26 (Polynomials on $G$)** *For a complex linear Lie group $G$, the algebra of polynomials* $\mathrm{Pol}(G)$ *is the unital complex subalgebra of $\mathscr{C}^\infty(G)$ generated by the functions $P_{ij}\colon G \to \mathbb{C}$, $g \mapsto g_{ij}$.*

Polynomials on a complex Lie group $G$ are holomorphic. In the case of semisimple connected Lie groups both the Lie group itself and the coadjoint orbit are affine algebraic varieties, see Remark 2.4, and our definition of polynomials coincides with the definition of regular functions on algebraic varieties. If $G$ is connected and semisimple, then the definition of polynomials on $G$ is independent of the way in which $G$ is realized as a linear group, which can be proven as outlined in Appendix A.2.

**Proposition 3.27** *Assume that the complex linear Lie group $G$ is semisimple and connected. Then $\pi^*\colon \mathrm{Hol}(\mathcal{O}_\lambda) \cong \mathrm{Hol}(G/G_\lambda) \to \mathrm{Hol}(G)^{G_\lambda}$ restricts to an isomorphism $\pi^*\colon \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(G)^{G_\lambda}$.*

PROOF: Since the Lie algebra $\mathfrak{g}$ is semisimple, we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e. every element of $\mathfrak{g}$ can be written as a sum of commutators. Consequently the trace of any element of $\mathfrak{g}$ is zero. Therefore any element in a sufficiently small neighbourhood of the identity of $G$ must have determinant 1, and consequently $G$ is a Lie subgroup of $\mathrm{SL}_N(\mathbb{C})$.

Let $E_{ij} \in \mathfrak{gl}_N(\mathbb{C})$ be the matrix that is 1 at position $(i,j)$ and 0 otherwise. Extend $\lambda$ to a linear functional $\tilde{\lambda} \in \mathfrak{gl}_N(\mathbb{C})^*$. For an element $X \in \mathfrak{g} = \mathrm{S}^1\mathfrak{g}$, which we identify with a polynomial on $\mathfrak{g}^*$, we compute

$$\begin{aligned}\pi^*\big(X\big|_{\mathcal{O}_\lambda}\big)(g) = X\big|_{\mathcal{O}_\lambda}(\pi(g)) = X\big|_{\mathcal{O}_\lambda}(\mathrm{Ad}_g^*\,\lambda) = X\big|_{\mathcal{O}_\lambda}(\lambda(g^{-1}\cdot g)) = \\ = \lambda(g^{-1}Xg) = \sum_{i,j} \tilde{\lambda}((g^{-1}Xg)_{ij}E_{ij}) = \sum_{i,j,k,\ell} (g^{-1})_{ik} g_{\ell j} X_{k\ell} \tilde{\lambda}(E_{ij})\,.\end{aligned}$$

Since $\det g = 1$ we can write $(g^{-1})_{ik}$ as a polynomial in the entries of $g$, so that $\pi^*\big(X|_{\mathcal{O}_\lambda}\big)$ itself is a polynomial in the entries of $g$. Since $\mathrm{Pol}(\mathcal{O}_\lambda)$ is generated by $X|_{\mathcal{O}_\lambda}$ and $\pi^*$ is an algebra homomorphism, it follows that $\pi^*p \in \mathrm{Pol}(G)$ for any $p \in \mathrm{Pol}(\mathcal{O}_\lambda)$. Injectivity of $\pi^*$ is immediate. Surjectivity is harder to prove. One can either use methods from algebraic geometry (making use of Remark 2.4, see for example [23, Chapter 12]) or work in a more differential geometric setting using $G$-finite functions as outlined in Appendix A.2. □

Recall the degree $d'$ introduced in the proof of Lemma 3.20.

**Lemma 3.28** *For any polynomial $p \in \mathrm{Pol}(\mathrm{GL}_N(\mathbb{C}))$, there is a constant $N_p \in \mathbb{N}$ such that for any $u \in \mathscr{U}(\tilde{\mathfrak{n}}^+) \subseteq \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$ of degree $d'$ greater $N_p$ and any $v \in \mathscr{U}(\tilde{\mathfrak{n}}^-) \subseteq \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$ of degree $d'$ smaller $-N_p$ we have $u^{\mathrm{left},(1,0)}p = v^{\mathrm{left},(1,0)}p = 0$.*

PROOF: Using the Leibniz rule we may assume that $p = P_{k\ell}$ in the notation of Definition 3.26. Let $E_{ij} \in \mathfrak{gl}_N(\mathbb{C})$ be the matrix that is 1 at position $(i,j)$ and 0 otherwise. It is easy to check that $E_{ij}^{\mathrm{left}} P_{k\ell} = \delta_{j\ell} P_{ki}$ and therefore $X^{\mathrm{left}} P_{k\ell} = \big(\sum_{i,j} X_{ij} E_{ij}\big)^{\mathrm{left}} P_{k\ell} = \sum_i X_{i\ell} P_{ki}$ for all $X \in \mathfrak{gl}_N(\mathbb{C})$. Since $P_{k\ell}$ is holomorphic, this implies that also $X^{\mathrm{left},(1,0)} P_{k\ell} = X^{\mathrm{left}} P_{k\ell} = \sum_i X_{i\ell} P_{ki}$. Consequently, if $u = u_1 \dots u_M \in \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$ with $u_1, \dots, u_M \in \mathfrak{gl}_N(\mathbb{C})$, then

$$
\begin{aligned}
u^{\mathrm{left},(1,0)} P_{k\ell} &= \sum_{i_M} (u_1 \dots u_{M-1})^{\mathrm{left},(1,0)} (u_M)_{i_M \ell} P_{k i_M} = \\
&= \sum_{i_{M-1}, i_M} (u_1 \dots u_{M-2})^{\mathrm{left},(1,0)} (u_{M-1})_{i_{M-1} i_M} (u_M)_{i_M \ell} P_{k i_{M-1}} = \dots = \\
&= \sum_{i_1, \dots, i_M} (u_1)_{i_1 i_2} \dots (u_{M-1})_{i_{M-1} i_M} (u_M)_{i_M \ell} P_{k i_1} = \sum_i (u_1 \dots u_M)_{i\ell} P_{ki} \,.
\end{aligned}
$$

Since $\mathrm{ad}_X$ is nilpotent for any $X \in \tilde{\mathfrak{n}}^+$ it follows that $0 = (\mathrm{ad}\, X)_s = \mathrm{ad}(X_s)$ for $X \in \tilde{\mathfrak{n}}^+$, where the index $s$ stands for the semisimple part of the Jordan decomposition. Since $\mathfrak{g}$ is semisimple this implies $X_s = 0$, so every $X \in \tilde{\mathfrak{n}}^+$ is realized by a nilpotent matrix. It follows from Engel's theorem that any matrix Lie algebra consisting of nilpotent matrices is nilpotent as an algebra, so there exists a constant $M \in \mathbb{N}$ such that products of $M$ or more elements of $\tilde{\mathfrak{n}}^+$ vanish. Therefore, if $u$ is a product of at least $M$ elements of $\tilde{\mathfrak{n}}^+$ the above calculation shows that $u^{\mathrm{left}} P_{k\ell} = 0$. If $M'$ is an upper bound for the degree $d'$ of elements of $\tilde{\mathfrak{n}}^+$ then we can set $N_{P_{k\ell}} := MM'$. It is easy to check that this constant also works for $\tilde{\mathfrak{n}}^-$. □

**Corollary 3.29** *For all polynomials $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$ and all $\hbar \in \mathbb{C} \setminus P_\lambda$, the sum $\sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q)$ is finite, and $\sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q) \in \mathrm{Pol}(\mathcal{O}_\lambda)$.*

PROOF: Proposition 3.27 implies that $\pi^* p$ and $\pi^* q$ are polynomials. By Lemma 3.20 the components $F_{\hbar,\ell}$ are of degree $(\ell, -\ell)$, and then the previous lemma implies that only finitely many summands of $\sum_{\ell=0}^\infty F_{\hbar,\ell}^{\mathrm{left},(1,0)}(\pi^* p, \pi^* q)$ are non-zero. Its proof shows that $\sum_{\ell=0}^\infty F_{\hbar,\ell}^{\mathrm{left},(1,0)}(\pi^* p, \pi^* q)$ is again a polynomial. The components $F_{\hbar,\ell}$ are $\mathfrak{g}_\lambda$-invariant, and therefore, since $G_\lambda$ is connected by Proposition 2.3, also $G_\lambda$-invariant. Consequently $\sum_{\ell=0}^\infty F_{\hbar,\ell}^{\mathrm{left},(1,0)}(\pi^* p, \pi^* q)$ is $G_\lambda$-invariant by Lemma 2.6. Then Proposition 3.27 yields that $\sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q) = \sum_{\ell=0}^\infty \pi_*(F_{\hbar,\ell}^{\mathrm{left},(1,0)}(\pi^* p, \pi^* q))$ is a polynomial. □

**Corollary 3.30** *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a complex connected semisimple Lie group $G$ with Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\lambda^\sharp$, and that we have chosen an invariant ordering. Then for any $\hbar \in \mathbb{C} \setminus P_\lambda$,*

$$
*_\hbar \colon \mathrm{Pol}(\mathcal{O}_\lambda) \times \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)\,, \quad (p,q) \mapsto p *_\hbar q := \sum_{\ell=0}^\infty \Psi(F_{\hbar,\ell})(p,q) \tag{3.34}
$$

*defines an associative and $G$-invariant product (where $G$-invariant means that $(g \triangleright p) *_\hbar (g \triangleright q) = g \triangleright (p *_\hbar q)$ holds for any $g \in G$ and $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$). For $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, $p *_\hbar q$ depends rationally on $\hbar$, and the formal expansion of $*_\hbar$ around $\hbar = 0$ coincides with the formal product $\star$.*

PROOF: As in the formal case, it is a standard argument to show that (3.28) implies the associativity of $*_\hbar$. Since the codomain of $\Psi$ consists of $G$-invariant bidifferential operators, it is clear that $*_\hbar$ is $G$-invariant. Since the dependence of $F_\hbar$ on $\hbar$ is rational without pole at 0, it follows that $*_\hbar$ also depends rationally on $\hbar$ without pole at 0, and since $\star$ was constructed from the formal expansion of $F_\hbar$, it coincides with the formal expansion of $*_\hbar$. $\square$

**Remark 3.31** When considering $\Psi(F_{\hbar,\ell})$, we may leave out the projections $\tilde{\pi}^\pm_\lambda$ in the formula for $F_{\hbar,\ell}$ from Lemma 3.20 to obtain the same result. Indeed, by Lemma 3.16 the difference of $F_{\hbar,\ell}$ and

$$F'_{\hbar,\ell} \coloneqq \sum_{w \in \tilde{W}_\ell} p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} X_w \otimes Y_w \in \mathscr{U}(\mathfrak{n}^+) \otimes \mathscr{U}(\mathfrak{n}^-) \tag{3.35}$$

is an element in the ideal $\mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda \otimes \mathscr{U}\mathfrak{g} + \mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g} \cdot \mathfrak{g}_\lambda$ and therefore contained in the kernel of $\Psi$ by Lemma 2.6.

Recall that we obtained a condition for $P_\lambda$ being countable in Proposition 3.21, and that this condition is satisfied in particular when the ordering is standard, see Remark 3.22.

**Proposition 3.32** *Assume that $P_\lambda$ is countable. Then the first order commutator of $\star$ coincides with the Poisson bracket induced by the KKS form $\omega_{\mathrm{KKS}}$ defined in* (2.2).

PROOF: Note that the formal expansion of

$$p_{\mathrm{i}\lambda/\hbar}(\mu)^{-1} = \left(\frac{1}{2}(\mu,\mu) - (\rho,\mu) - \frac{\mathrm{i}}{\hbar}(\lambda,\mu)\right)^{-1} = \mathrm{i}\hbar\left(\frac{\mathrm{i}\hbar}{2}(\mu,\mu) - \mathrm{i}\hbar(\rho,\mu) + (\lambda,\mu)\right)^{-1}$$

is of order $\nu$ if $(\lambda,\mu) \neq 0$. It follows from Theorem 3.18 that the element $F$ is the formal expansion of

$$\sum_{\substack{w \in \tilde{W} \\ |w| \leq 1}} p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w) + \sum_{\substack{w \in \tilde{W} \\ |w| \geq 2}} p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} \tilde{\pi}^+_\lambda(X_w) \otimes \tilde{\pi}^-_\lambda(Y_w)\,.$$

Using that the words $w \in \tilde{W}$ with $|w| \leq 1$ are precisely the empty word and the one-letter words $(\ell)$ with $\alpha_\ell \in \hat{\Delta}^+$, i.e. $(\lambda, \alpha_\ell) \neq 0$, it follows that the first sum expands to $1 + \mathrm{i}\nu \sum_{\alpha \in \hat{\Delta}^+} (\lambda,\alpha)^{-1} X_\alpha \otimes Y_\alpha + \mathcal{O}(\nu^2)$. Let us argue why the formal expansion of the second sum is of order $\nu^2$. By definition $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} = \prod_{i=1}^{|w|} p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{i\ldots|w|}})^{-1}$. Since, by definition of $\tilde{W}$, we have $\alpha_{w_{|w|}} \in \hat{\Delta}^+$, it is clear that the formal expansions of all summands with $(\lambda, \alpha_{w_{|w|-1}} + \alpha_{w_{|w|}}) \neq 0$ are of order $\nu^2$ (because both $p_{\mathrm{i}\lambda/\hbar}(\alpha_{(w_{|w|-1}, w_{|w|})})^{-1}$ and $p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{|w|}})^{-1}$ are of order $\nu$). So assume

$(\lambda, \alpha_{w_{|w|-1}} + \alpha_{w_{|w|}}) = 0$, in which case $\alpha_{w_{|w|-1}} \in \hat{\Delta}^+$ and, by invariance of the ordering, $\alpha_{w_{|w|-1}} + \alpha_{w_{|w|}}$ is not a root. Therefore $X_{w_{|w|-1}} X_{w_{|w|}} = X_{w_{|w|}} X_{w_{|w|-1}}$, and if $w' = (w_1, \ldots, w_{|w|-2}, w_{|w|}, w_{|w|-1})$ is the word obtained form $w$ by switching the last two letters then $X_w = X_{w'}$. Similarly $Y_w = Y_{w'}$. Furthermore, by definition of $\alpha_w$, we have $\alpha_{w_{i\ldots|w|}} = \alpha_{w'_{i\ldots|w'|}}$ for all $i < |w|$ and

$$p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} + p^{w'}_{\mathrm{i}\lambda/\hbar}(\alpha_{w'})^{-1} = \\ = \left(p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{|w|}})^{-1} + p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{|w|-1}})^{-1}\right) \prod_{i=1}^{|w|-1} p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{i\ldots|w|}})^{-1} .$$

But under our assumptions $(\alpha_{w_{|w|}}, \lambda)^{-1} + (\alpha_{w_{|w|-1}}, \lambda)^{-1} = 0$, and therefore the formal expansion of $p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{|w|}})^{-1} + p_{\mathrm{i}\lambda/\hbar}(\alpha_{w_{|w|-1}})^{-1}$ is $\mathrm{i}\nu(\alpha_{w_{|w|}}, \lambda)^{-1} + \mathrm{i}\nu(\alpha_{w_{|w|-1}}, \lambda)^{-1} + \mathcal{O}(\nu^2) = \mathcal{O}(\nu^2)$. Consequently, the summands which could potentially be of order $\nu$ in the sum over $w \in \tilde{W}$ with $|w| \geq 2$ cancel out, and this sum is therefore of order $\nu^2$ as claimed.

To conclude the proof, note that antisymmetrizing the first order gives indeed

$$F^{\text{antisym}}_{(1)} = \mathrm{i} \sum_{\alpha \in \hat{\Delta}^+} \lambda(\alpha^\sharp)^{-1}(X_\alpha \otimes Y_\alpha - Y_\alpha \otimes X_\alpha) = \mathrm{i} \sum_{\alpha \in \hat{\Delta}} \lambda([X_\alpha, Y_\alpha])^{-1} X_\alpha \otimes Y_\alpha = \mathrm{i}\pi_{\mathrm{KKS}} ,$$

where $\pi_{\mathrm{KKS}}$ denotes the Poisson tensor associated to the KKS symplectic form. □

We conclude this subsection by saying a bit more about the directions in which $\star$ and $*_\hbar$ differentiate.

**Lemma 3.33** *For any $\xi = \mathrm{Ad}^*_g \lambda \in \mathcal{O}_\lambda$, the subspaces*

$$L_{+,\xi} = \operatorname{span}\left\{ (\mathrm{Ad}_g X_\alpha)_{\mathcal{O}_\lambda}\big|_\xi \,,\ \alpha \in \hat{\Delta}^+ \right\} \subseteq \mathrm{T}_\xi \mathcal{O}_\lambda \,, \tag{3.36a}$$

$$L_{-,\xi} = \operatorname{span}\left\{ (\mathrm{Ad}_g X_\alpha)_{\mathcal{O}_\lambda}\big|_\xi \,,\ \alpha \in \hat{\Delta}^- \right\} \subseteq \mathrm{T}_\xi \mathcal{O}_\lambda \tag{3.36b}$$

*are independent of the choice of $g \in G$.*

PROOF: Any two choices $g, g' \in G$ differ by an element of $G_\lambda$, that is $g' = gx$ with $x \in G_\lambda$. So it suffices to prove that $\operatorname{span}\{\mathrm{Ad}_x X_\alpha \,,\ \alpha \in \hat{\Delta}^\pm\} = \operatorname{span}\{X_\alpha \,,\ \alpha \in \hat{\Delta}^\pm\}$. This follows from the invariance of the ordering and the connectedness of $G_\lambda$. □

Therefore the distributions $L_+$ and $L_-$ in $\mathrm{T}\mathcal{O}_\lambda$ spanned by $L_{+,\xi}$ and $L_{-,\xi}$, respectively, are well-defined.

**Corollary 3.34** *The star product $*_\hbar$ derives the first argument only in the directions of $L_+^{(1,0)}$ and the second argument only in the directions of $L_-^{(1,0)}$.*

PROOF: This follows from the explicit formula for $F_\hbar$ obtained in Theorem 3.18, from Remark 3.31, and from Proposition 2.8. □

## 3.4 Examples

In this subsection we derive formulas for $F_\hbar$ in the case $G = \mathrm{SL}_{1+n}(\mathbb{C})$ for the largest non-trivial stabilizer $G_\lambda$. When restricting to real coadjoint orbits in Subsection 5.4, this example allows us to obtain quantizations of complex projective spaces and hyperbolic discs.

**Example 3.35 ($\mathbf{SL_{1+n}(\mathbb{C})}$)** Let $G = \mathrm{SL}_{1+n}(\mathbb{C})$ be the Lie group of matrices with determinant 1. Its Lie algebra $\mathfrak{g} = \mathfrak{sl}_{1+n}(\mathbb{C})$ consists of matrices with trace 0. Number the rows and columns of a matrix $X \in \mathfrak{g}$ by $0, \dots, n$. Let $\lambda \colon \mathfrak{g} \to \mathbb{C}$, $X \mapsto -\mathrm{i}rX_{0,0}$ where $r \in \mathbb{C}$. Using that the Killing form $B$ satisfies $B(X,Y) = 2(n+1)\operatorname{tr}(XY)$, where tr is the usual (not normalized) matrix trace, it follows that $\lambda^\sharp$ is a multiple of the diagonal matrix $\operatorname{diag}(n, -1, \dots, -1)$, and therefore

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{sl}_{1+n}(\mathbb{C}) \mid X_{0,i} = X_{i,0} = 0 \text{ for } 1 \le i \le n\}\,, \tag{3.37a}$$
$$G_\lambda = \{g \in \mathrm{SL}_{1+n}(\mathbb{C}) \mid g_{0,i} = g_{i,0} = 0 \text{ for } 1 \le i \le n\}\,. \tag{3.37b}$$

We choose the Cartan subalgebra $\mathfrak{h}$ consisting of the diagonal matrices in $\mathfrak{g}$. The roots are then given by $\alpha_{i,j} = L_i - L_j$ for $0 \le i, j \le n$ with $i \ne j$, where $L_i \in \mathfrak{h}^*$, $L_i(X) = X_{i,i}$. If we let the roots $\alpha_{i,j}$ with $i < j$ be positive, then the simple roots are $\alpha_{0,1}, \alpha_{1,2}, \dots, \alpha_{n-1,n}$. As before, denote the matrix with entry 1 at position $(i,j)$ by $E_{i,j}$, and define $X_{i,j} \coloneqq E_{i,j} \in \mathfrak{g}^{\alpha_{i,j}}$ and $Y_{i,j} \coloneqq E_{j,i} \in \mathfrak{g}^{\alpha_{j,i}} = \mathfrak{g}^{-\alpha_{i,j}}$. Note that $B(X_{i,j}, Y_{i,j}) = 2(n+1)\operatorname{tr}(X_{i,j}Y_{i,j}) = 2(n+1)$, so we use a different normalization than in Subsection 3.1.

If $n = 1$, it is easy to simplify the formula for $F_\hbar$ obtained in Theorem 3.18: There is only one positive root $\alpha = \alpha_{0,1}$, and there is a unique word $w_\ell$ of a given length $\ell \in \mathbb{N}_0$. Note that $\lambda = -\mathrm{i}r\alpha/2$ and $\rho = \alpha/2$, so $p_{\mathrm{i}\lambda/\hbar}(m\alpha) = \frac{1}{2}m^2(\alpha,\alpha) - \frac{1}{2}m(\alpha,\alpha) - \frac{1}{2\hbar}mr(\alpha,\alpha) = \frac{1}{4}m(m - 1 - \frac{r}{\hbar})$. Therefore

$$p^{w_\ell}_{\mathrm{i}\lambda/\hbar}(\alpha_{w_\ell}) = \prod_{m=1}^{\ell} \frac{4}{m\big(m - 1 - \frac{r}{\hbar}\big)} = \frac{(-4)^\ell}{\ell!\,\frac{r}{\hbar}\big(\frac{r}{\hbar} - 1\big)\dots\big(\frac{r}{\hbar} - (\ell-1)\big)}\,.$$

We set $X \coloneqq X_{0,1}$ and $Y \coloneqq Y_{0,1}$. Since $B(X,Y) = 4$ we have to plug the normalized elements $X/2$ and $Y/2$ into (3.27), and obtain

$$F_\hbar = \sum_{\ell \in \mathbb{N}_0} \frac{(-1)^\ell}{\ell!\,\frac{r}{\hbar}\big(\frac{r}{\hbar} - 1\big)\dots\big(\frac{r}{\hbar} - (\ell-1)\big)} X^\ell \otimes Y^\ell\,. \tag{3.38}$$

This result was already obtained in [1, Example 4.16], but the following result for arbitrary $n$ is new. We prove it by computing the canonical element of the Shapovalov pairing directly, instead of simplifying (3.27).

**Proposition 3.36** *For $G = \mathrm{SL}_{1+n}(\mathbb{C})$, the same $\lambda$ and the same ordering as above, we have*

$$F_\hbar = \sum_{\ell \in \mathbb{N}_0} \frac{(-1)^\ell}{\ell!\,\frac{r}{\hbar}\big(\frac{r}{\hbar} - 1\big)\dots\big(\frac{r}{\hbar} - (\ell-1)\big)} \big(X_{0,1} \otimes Y_{0,1} + \dots + X_{0,n} \otimes Y_{0,n}\big)^\ell\,. \tag{3.39}$$

PROOF: The Lie algebras $\tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{n}}^-$ are commutative Lie algebras spanned by $X_{0,1},\dots,X_{0,n}$ and $Y_{0,1},\dots,Y_{0,n}$ respectively, and therefore $\{X^I \coloneqq X_{0,1}^{I_1}\dots X_{0,n}^{I_n} \mid I \in \mathbb{N}_0^n\}$ and $\{Y^J \coloneqq Y_{0,1}^{J_1}\dots Y_{0,n}^{J_n} \mid J \in \mathbb{N}_0^n\}$ are bases of $\mathscr{U}(\tilde{\mathfrak{n}}^+)$ and $\mathscr{U}(\tilde{\mathfrak{n}}^-)$. The Lie algebra $\mathfrak{n}^+$ is spanned by $X_{i,j}$ with $i<j$ and we can view $X^I$ also as an element of $\mathscr{U}(\mathfrak{n}^+)$. Then $\tilde{\pi}^+_\lambda(X^I) = X^I$ and similarly $\tilde{\pi}^-_\lambda(Y^J) = Y^J$. Consequently $\langle X^I, Y^J\rangle^\sim_{\mathrm{i}\lambda/\hbar} = \langle X^I, Y^J\rangle_{\mathrm{i}\lambda/\hbar}$. For degree reasons the bases $\{X^I\}$ and $\{Y^J\}$ are orthogonal, meaning that $\langle X^I, Y^J\rangle_{\mathrm{i}\lambda/\hbar} = 0$ for $I \neq J$. Indeed, $X^I$ and $Y^J$ are homogeneous with respect to the degree $d$ defined in the beginning of Subsection 3.1, $d(X^I) = I_1 d(X_{0,1}) + \dots + I_n d(X_{0,n}) = I_1\alpha_{0,1} + \dots + I_n\alpha_{0,n}$, and $d(Y^J) = -(J_1\alpha_{0,1} + \dots + J_n\alpha_{0,n})$. Since the $\alpha_{0,i}$ are linearly independent, Lemma 3.3 implies the claimed orthogonality.

Therefore it suffices to determine the normalization $\langle X^I, Y^I\rangle_{\mathrm{i}\lambda/\hbar}$. Define $H_i \coloneqq [X_{0,i}, Y_{0,i}] = E_{0,0} - E_{i,i}$. Given a multiindex $I \in \mathbb{N}_0^n$ we can form a sequence that starts with $I_1$ many 1's, then has $I_2$ many 2's, ..., then $I_n$ many $n$'s. Denote the $k$-th element of this sequence by $I_{\{k\}}$. Introduce the projection $(\,\cdot\,)_0$ to $\mathscr{U}\mathfrak{h}$ in the decomposition $\mathscr{U}\mathfrak{g} = \mathscr{U}\mathfrak{h} \oplus (\mathfrak{n}^- \cdot \mathscr{U}\mathfrak{g} + \mathscr{U}\mathfrak{g}\cdot\mathfrak{n}^+)$, so that $\pi_\lambda(u) = \lambda((u)_0)$. Then we claim that

$$(X^I Y^I)_0 = I! \prod_{\ell=0}^{|I|-1} (H_{I_{\{\ell\}}} - \ell)\,. \tag{3.40}$$

To see that this formula implies the proposition, note that

$$\langle X^I, Y^I\rangle_{\mathrm{i}\lambda/\hbar} = \pi_{\mathrm{i}\lambda/\hbar}(S(X^I)Y^I) = (-1)^{|I|}\left(\frac{\mathrm{i}}{\hbar}\lambda\right)((X^I Y^I)_0)$$

and that $\frac{\mathrm{i}}{\hbar}\lambda(H_i) = \frac{r}{\hbar}$ for all $i = 1,\dots,n$. So

$$F_\hbar = \sum_{I\in\mathbb{N}_0^n} \frac{1}{\langle X^I, Y^I\rangle_{\mathrm{i}\lambda/\hbar}} X^I \otimes Y^I = \sum_{I\in\mathbb{N}_0^n} \frac{(-1)^{|I|}}{I!\frac{r}{\hbar}\left(\frac{r}{\hbar}-1\right)\dots\left(\frac{r}{\hbar}-(|I|-1)\right)} X^I \otimes Y^I$$

and an application of the multinomial theorem gives (3.39).

It remains to prove (3.40). For $n = 1$ this is the statement of [16, Lemma 5.2]. Note that this also means that $Z \coloneqq X_{0,n}^{I_n} Y_{0,n}^{I_n} - I_n! H_n(H_n-1)\dots(H_n - I_n + 1) \in \mathscr{U}(\operatorname{span}\{X_{0,n}, Y_{0,n}, H_n\})$ satisfies $(Z)_0 = 0$. We proceed by induction and assume that (3.40) holds for $n-1$. Writing $I_- = (I_1,\dots,I_{n-1},0)$ and noting that $[H_n, X_{0,i}] = X_{0,i}$ for $1 \le i \le n-1$, we compute

$$\begin{aligned}
(X^I Y^I)_0 &= (X^{I_-} X_{0,n}^{I_n} Y_{0,n}^{I_n} Y^{I_-})_0 \\
&= (X^{I_-}(I_n! H_n(H_n-1)\dots(H_n-I_n+1) + Z)Y^{I_-})_0 \\
&= I_n!((H_n - |I_-|)(H_n - |I_-| - 1)\dots(H_n - |I_-| - I_n + 1)X^{I_-}Y^{I_-})_0 + (X^{I_-} Z Y^{I_-})_0 \\
&= I_n!(H_n - |I_-|)(H_n - |I_-| - 1)\dots(H_n - |I_-| - I_n + 1)(X^{I_-}Y^{I_-})_0 + (X^{I_-} Z Y^{I_-})_0.
\end{aligned}$$

Since $(Z)_0 = 0$ and $d(Z) = d(X_{0,n}^{I_n} Y_{0,n}^{I_n} - I_n! H_n(H_n-1)\dots(H_n-I_n+1)) = 0$ we can write $Z = Y_{0,n} Z' X_{0,n}$ for some $Z' \in \mathscr{U}(\operatorname{span}\{X_{0,n}, Y_{0,n}, H_n\})$. Since $Y_{0,n} \in \mathfrak{g}^{\alpha_{n,0}}$ any commutator of $Y_{0,n}$ with elements of $\mathfrak{g}^{\alpha_{0,1}},\dots,\mathfrak{g}^{\alpha_{0,n-1}}$ has degree $d$ equal to $L_n -$

$\sum_{i=0}^{n-1} c_i L_i$ for some $c_i \in \mathbb{Z}$, so must either be 0 or in a negative root space. Therefore $(X^{I_-} Z Y^{I_-})_0 = 0$, and the claim follows by applying the induction hypothesis to the first summand in the equation above. $\square$

**Corollary 3.37** *Let $G = \mathrm{SL}_{1+n}(\mathbb{C})$ and $\lambda$ be as above, but choose the opposite ordering, for which $\alpha_{i,j}$ with $i > j$ is positive. Then*

$$F_\hbar = \sum_{\ell \in \mathbb{N}_0} \frac{1}{\ell! \frac{r}{\hbar}\left(\frac{r}{\hbar}+1\right)\dots\left(\frac{r}{\hbar}+(\ell-1)\right)} (Y_{0,1} \otimes X_{0,1} + \cdots + Y_{0,n} \otimes X_{0,n})^\ell . \tag{3.41}$$

Proof: The only change in the computation above is that the roles of $X_{0,i}$ and $Y_{0,i}$ are swapped. Now $[Y_{0,i}, X_{0,i}] = E_{i,i} - E_{0,0}$, so $\frac{\mathrm{i}}{\hbar}\lambda([Y_{0,i}, X_{0,i}]) = -\frac{r}{\hbar}$, which means that $r$ changes sign. $\square$

# 4 Continuity

In this section, we extend the product $*_\hbar \colon \mathrm{Pol}(\mathcal{O}_\lambda) \times \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ obtained in Corollary 3.30 to a product $*_\hbar \colon \mathrm{Hol}(\mathcal{O}_\lambda) \times \mathrm{Hol}(\mathcal{O}_\lambda) \to \mathrm{Hol}(\mathcal{O}_\lambda)$ on all holomorphic functions on the coadjoint orbit, that is continuous with respect to the topology of locally uniform convergence. More precisely, we prove the following theorem.

**Theorem 4.1** *Let $\mathcal{O}_\lambda$ be a complex semisimple coadjoint orbit of a complex semisimple connected Lie group $G$. Then for any $\hbar \in \mathbb{C} \setminus P_\lambda$ the product $*_\hbar$ on $\mathrm{Pol}(\mathcal{O}_\lambda)$ is continuous with respect to the topology of locally uniform convergence and extends to a continuous and $G$-invariant product $*_\hbar \colon \mathrm{Hol}(\mathcal{O}_\lambda) \times \mathrm{Hol}(\mathcal{O}_\lambda) \to \mathrm{Hol}(\mathcal{O}_\lambda)$ on the space of all holomorphic functions on $\mathcal{O}_\lambda$.*

The proof of this theorem proceeds as follows: In Subsection 4.1 we prove the continuity of $*_\hbar$ with respect to a topology that we call the reduction-topology and in Subsection 4.3 we prove that the reduction-topology coincides with the topology of locally uniform convergence. Consequently $*_\hbar$ extends to the completion of the space of polynomials on $\mathcal{O}_\lambda$. Using the results of Subsection 4.2 we prove in Subsection 4.3 that this completion is the space $\mathrm{Hol}(\mathcal{O}_\lambda)$ of all holomorphic functions on $\mathcal{O}_\lambda$.

In the whole section we assume that the complex connected semisimple Lie group $G$ is concretely realized as a complex Lie subgroup of $\mathrm{GL}_N(\mathbb{C})$ for some $N \in \mathbb{N}$, as explained in Subsection 2.1. In particular, since $G$ is semisimple, it is a closed submanifold of $\mathbb{C}^{N \times N}$.

## 4.1 Continuity in the reduction-topology

In this subsection we prove the continuity of the star product $*_\hbar$ with respect to a topology that we call the reduction-topology, and which is defined below. Recall that a sequence of functions $f_i \colon X \to \mathbb{C}$ on a topological space $X$ is said to be locally uniformly convergent if for every $x \in X$ there is a neighbourhood $U \subseteq X$ such that $f_i$ converges uniformly to $f$ on $U$, i.e. $\lim_{i\to\infty} \sup_{y\in U} |f_i(y) - f(y)| = 0$. In this work, $X$ will always be a manifold. Then the topology of locally uniform convergence coincides with the topology of compact convergence (for every compact subset $K \subseteq X$, $f_i$ converges uniformly on $K$), and is therefore a locally convex topology, defined by the seminorms $\|f\|_K \coloneqq \sup_K |f|$.

Denote the ideal of polynomials in $\mathrm{Pol}(\mathbb{C}^{N\times N})$ whose restriction to $G$ vanishes by $\mathcal{I}(G)$.

**Definition 4.2 (Reduction-topology)** *The topology $\mathcal{T}_{\mathrm{lc}}$ of locally uniform convergence on the space $\mathrm{Pol}(\mathbb{C}^{N\times N})$ of polynomials on $\mathbb{C}^{N\times N}$ induces a quotient topology on the space $\mathrm{Pol}(G) \cong \mathrm{Pol}(\mathbb{C}^{N\times N})/\mathcal{I}(G)$ of polynomials on $G$, and we call the subspace topology on the space $\mathrm{Pol}(\mathcal{O}_\lambda) \cong \mathrm{Pol}(G)^{G_\lambda}$ of polynomials on the coadjoint orbit $\mathcal{O}_\lambda$ the reduction-topology.*

In Subsection 4.3 we will prove that the reduction-topology coincides with the topology of locally uniform convergence on $\mathcal{O}_\lambda$.

This topology is convenient for obtaining continuity estimates for $*_\hbar$, since we gave a description of $\Psi(F_\hbar)$ via bidifferential operators on $G$ in Subsection 2.2. Since we assume that the Lie group $G$ is concretely realized as a complex Lie subgroup of $\mathrm{GL}_N(\mathbb{C})$, its Lie algebra $\mathfrak{g}$ is realized as a Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$. Considering the element $F'_{\hbar,\ell}$ defined in (3.35) as an element of $\mathscr{U}(\mathfrak{gl}_N(\mathbb{C})) \otimes \mathscr{U}(\mathfrak{gl}_N(\mathbb{C}))$, we let

$$*'_\hbar \colon \mathrm{Pol}(\mathbb{C}^{N\times N}) \times \mathrm{Pol}(\mathbb{C}^{N\times N}) \to \mathrm{Pol}(\mathbb{C}^{N\times N}),$$
$$(p,q) \mapsto p *'_\hbar q \coloneqq \sum_{\ell=0}^{\infty} (F'_{\hbar,\ell})^{\mathrm{left},(1,0)}(p,q)\,, \quad (4.1)$$

which is well-defined because Lemma 3.28 implies that the sum over $\ell$ is finite, and that $(F'_{\hbar,\ell})^{\mathrm{left},(1,0)}(p,q)$ is again a polynomial. Note that $*'_\hbar$ is (in general) not associative since $\sum_{\ell=0}^{\infty} F'_{\hbar,\ell}$ satisfies (3.28) only after passing to the quotient. However, since $F'_{\hbar,\ell}$ lies in the subspace $\mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g}$ it induces a product on $\mathrm{Pol}(G) \cong \mathrm{Pol}(\mathbb{C}^{N\times N})/\mathcal{I}(G)$. As in Remark 3.31 it follows that the restriction of this product to $\mathrm{Pol}(G)^{G_\lambda} \cong \mathrm{Pol}(\mathcal{O}_\lambda)$ coincides with $*_\hbar$.

**Theorem 4.3** *For $\hbar \in \mathbb{C} \setminus P_\lambda$ the product $*'_\hbar$ on $\mathrm{Pol}(\mathbb{C}^{N\times N})$ is continuous with respect to the topology of locally uniform convergence $\mathcal{T}_{\mathrm{lc}}$.*

Before proving this theorem in the rest of this section, we would like to note the following consequence, which motivates the definition of the reduction-topology given above.

**Corollary 4.4** *For $\hbar \in \mathbb{C} \setminus P_\lambda$ the product $*_\hbar$ on $\mathrm{Pol}(\mathcal{O}_\lambda)$ is continuous with respect to the reduction-topology.*

PROOF: This follows immediately from the previous theorem and the construction of the reduction-topology. □

**Remark 4.5** It is interesting to point out that the proof of Theorem 4.3 will not use anything about the actual Lie algebra structure but semisimplicity and the form of the element $F_\hbar$. In fact, we only need that the coefficients of $F_\hbar$ behave like $p^w_\lambda(\alpha_w) \approx |w|^2$ for large $|w|$. The rest of the proof consists in counting terms and checking that there are not too many.

The strategy to prove Theorem 4.3 is as follows. We first introduce a different locally convex topology that is better suited for obtaining continuity estimates. Then we prove that this topology is equivalent to the topology of locally uniform convergence and we prove the continuity of $*'_\hbar$ with respect to this topology.

Set $m = N^2$. Let $B = \{b_1, \dots, b_m\}$ be the standard basis of $\mathbb{C}^m$ and denote the dual basis of $(\mathbb{C}^m)^*$ by $B^* = \{b_1^*, \dots, b_m^*\}$. Elements of $\mathrm{Pol}(\mathbb{C}^m) \cong \mathrm{S}((\mathbb{C}^m)^*)$ (where S denotes the symmetric tensor algebra) can be written uniquely in the form $\sum_{I\in\mathbb{N}_0^m} a_I b_I^*$. Here $I \in \mathbb{N}_0^m$ is a multiindex, $b_I^* = (b_1^*)^{\vee I_1} \vee \dots \vee (b_m^*)^{\vee I_m}$ and only finitely many of the coefficients $a_I \in \mathbb{C}$ are non-zero. For any $R \in \mathbb{R}^+$ define a norm

$$\left|\!\left|\!\left| \sum\nolimits_{I\in\mathbb{N}_0^m} a_I b_I^* \right|\!\right|\!\right|_R := \sum\nolimits_{I\in\mathbb{N}_0^m} |a_I| R^{|I|} \,. \tag{4.2}$$

Note that these norms coincide with the $T_0$-norms with respect to the basis $B^*$, studied for example in [40]. We denote the locally convex topology given by endowing $\mathrm{Pol}(\mathbb{C}^m) \cong \mathrm{S}((\mathbb{C}^m)^*)$ with the seminorms $|\!|\!|\cdot|\!|\!|_R$ by $\mathcal{T}_{|\!|\!|\cdot|\!|\!|}$. This topology can equivalently be defined by the countable set of norms $|\!|\!|\cdot|\!|\!|_R$ with $R \in \mathbb{N}$.

Note that $|\!|\!|\cdot|\!|\!|_R$ is submultiplicative with respect to the classical product:

$$\begin{aligned}
\left|\!\left|\!\left| \Big(\sum\nolimits_{I\in\mathbb{N}_0^m} a_I b_I^*\Big) \vee \Big(\sum\nolimits_{J\in\mathbb{N}_0^m} a'_J b_J^*\Big) \right|\!\right|\!\right|_R &= \left|\!\left|\!\left| \sum\nolimits_{I,J\in\mathbb{N}_0^m} a_I a'_J b_I^* \vee b_J^* \right|\!\right|\!\right|_R \le \\
\le \sum\nolimits_{I,J\in\mathbb{N}_0^m} |a_I||a'_J| R^{|I|+|J|} &= \Big(\sum\nolimits_{I\in\mathbb{N}_0^m} |a_I| R^{|I|}\Big)\Big(\sum\nolimits_{J\in\mathbb{N}_0^m} |a'_J| R^{|J|}\Big) = \\
&= \left|\!\left|\!\left| \sum\nolimits_{I\in\mathbb{N}_0^m} a_I b_I^* \right|\!\right|\!\right|_R \left|\!\left|\!\left| \sum\nolimits_{J\in\mathbb{N}_0^m} a'_J b_J^* \right|\!\right|\!\right|_R .
\end{aligned}$$

**Proposition 4.6** *The topologies $\mathcal{T}_{|\!|\!|\cdot|\!|\!|}$ and $\mathcal{T}_{\mathrm{lc}}$ coincide.*

PROOF: Assume $p = \sum_{I\in\mathbb{N}_0^m} a_I b_I^* \in \mathrm{Pol}(\mathbb{C}^m)$ is a polynomial. Given $K \subseteq \mathbb{C}^m$ compact, choose $R \in \mathbb{R}$ such that $|z| \le R$ holds for all $z \in K$. Then on the one hand we have

$$\|p\|_K = \max_{z\in K} |p(z)| \le \sum_{I\in\mathbb{N}_0^m} |a_I| R^{|I|} = |\!|\!|p|\!|\!|_R \,.$$

On the other hand, if $D_R = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid |z_i| \le R \text{ for all } i = 1, \dots, m\} \subseteq \mathbb{C}^m$ denotes a closed polydisc of radius $R$, then Cauchy's integral formula yields

$$\begin{aligned}
|a_I| = \frac{1}{I!} |\partial_I p(0)| = \frac{1}{(2\pi)^m} \left| \int_{|z_i|=R} \frac{p(z)}{z^{I+(1,\dots,1)}} \,\mathrm{d}z^I \right| &\le \\
\le \max_{z\in D_R} |p(z)| \frac{R^m}{R^{|I+(1,\dots,1)|}} &= \frac{1}{R^{|I|}} \max_{z\in D_R} |p(z)| \,.
\end{aligned}$$

Applying this estimate for a polydisc of radius $2mR$ yields

$$|\!|\!|p|\!|\!|_R = \sum_{I\in\mathbb{N}_0^m} |a_I| R^{|I|} \le \sum_{I\in\mathbb{N}_0^m} \frac{1}{(2mR)^{|I|}} R^{|I|} \max_{z\in D_{2mR}} |p(z)| \le$$

$$\leq \max_{z\in D_{2mR}} |p(z)| \sum_{I\in\mathbb{N}_0^m} (2m)^{-|I|} \leq 2 \max_{z\in D_{2mR}} |p(z)| = 2\|p\|_{D_{2mR}}\,.$$

Consequently we can estimate any norm of $\mathcal{T}_{|||\cdot|||}$ by a seminorm of $\mathcal{T}_{\mathrm{lc}}$ and vice versa, so the topologies $\mathcal{T}_{|||\cdot|||}$ and $\mathcal{T}_{\mathrm{lc}}$ coincide. □

Because of the previous proposition we can and will work with the norms $|||\cdot|||_R$ instead of the seminorms $\|\cdot\|_K$ in the following. To obtain continuity estimates, we need to estimate the coefficients $p_\lambda(\mu)$ defined in (3.9).

**Lemma 4.7 (Estimates for $p_\lambda$)** *For any fixed compact set $K \subseteq \mathfrak{h}^*$ there are constants $C > 0$ and $M$ such that $p_\lambda(\alpha_w)$ defined in* (3.9) *satisfies*

$$|p_\lambda(\alpha_w)| \geq C|w|^2 \tag{4.3}$$

*for all words $w \in W$ of length $|w| \geq M$ and all $\lambda \in K$.*

Proof: Assume that the positive roots $\alpha_1, \dots, \alpha_k \in \Delta^+$ are ordered in such a way that $\alpha_1, \dots, \alpha_r$ are the simple roots. Write $\alpha_w = \sum_{i=1}^r c_{w,i}\alpha_i$ as a linear combination of simple roots, where $c_{w,i} \in \mathbb{N}_0$ satisfy $|w| \leq \sum_{i=1}^r c_{w,i} \leq c|w|$ with $c$ depending only on the root system. Since $(\rho, \alpha_i) > 0$ for all $1 \leq i \leq r$ we can choose $c_\rho, C_\rho \in \mathbb{R}^+$ such that $c_\rho \leq (\rho, \alpha_i) \leq C_\rho$ holds for all $1 \leq i \leq r$. Similarly, there is $C' \in \mathbb{R}^+$ with $|(\lambda, \alpha_i)| \leq C'$ for all $\lambda \in K$ and $1 \leq i \leq r$. Then

$$(\alpha_w, \alpha_w) \geq \frac{1}{(\rho,\rho)}(\alpha_w, \rho)^2 = \frac{1}{(\rho,\rho)}\left(\sum_{i=1}^r (c_{w,i}\alpha_i, \rho)\right)^2 \geq$$
$$\geq \frac{c_\rho^2}{(\rho,\rho)}\left(\sum_{i=1}^r c_{w,i}\right)^2 \geq \frac{c_\rho^2}{(\rho,\rho)}|w|^2$$

and for all $\lambda \in K$ we obtain

$$|(\rho+\lambda, \alpha_w)| \leq \sum_{i=1}^r c_{w,i}(|(\rho,\alpha_i)| + |(\lambda,\alpha_i)|) \leq (C_\rho + C')\sum_{i=1}^r c_{w,i} \leq c(C_\rho + C')|w|\,.$$

Setting $C := \frac{1}{4(\rho,\rho)}c_\rho^2$, $C_1 := c(C_\rho + C')$, and $M := \frac{C_1}{C}$, and assuming $|w| \geq M$ we obtain

$$|p_\lambda(\alpha_w)| \geq \frac{1}{2}(\alpha_w,\alpha_w) - |(\rho+\lambda, \alpha_w)| \geq 2C|w|^2 - C_1|w| \geq 2C|w|^2 - C|w|^2 = C|w|^2\,. \ □$$

**Corollary 4.8 (Estimates for $p_\lambda^w$)** *Fix $\lambda \in \mathfrak{h}^*$. For any compact set $K \subseteq \mathbb{C} \setminus P_\lambda$ there is a constant $C_p > 0$ such that $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)$ defined in* (3.9) *satisfies*

$$|p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1}| \leq \frac{C_p^{|w|}}{(|w|!)^2} \tag{4.4}$$

*for all words $w \in \tilde{W}$ and all $\hbar \in K$.*

PROOF: Note that $K' = \{i\lambda/\hbar \mid \hbar \in K\}$ is a compact subset of $\tilde{\Lambda}$. Let $M$ and $C$ be the constants obtained by applying the previous lemma to $K'$, so $|p_{\lambda'}(\alpha_w)| \geq C|w|^2$ for all $w \in W$ with $|w| \geq M$ and all $\lambda' \in K'$. Since $i\lambda/\hbar \in \tilde{\Lambda}$, we have $\min_{w\in\tilde{W},|w|<M}|p_{i\lambda/\hbar}(\alpha_w)| > 0$ for all $\hbar \in K$. Since this quantity depends continuously on $\hbar$ the minimum for $\hbar \in K$ exists and must also be positive. Hence we may decrease the constant $C$ such that $|p_{i\lambda/\hbar}(\alpha_w)| \geq C|w|^2$ also holds for the finitely many words $w \in \tilde{W}$ with $|w| < M$. Consequently $|p_{i\lambda/\hbar}(\alpha_w)| \geq C|w|^2$ holds for all words $w \in \tilde{W}$. Setting $C_p \coloneqq 1/C$, the corollary follows by rearranging. $\square$

We have now collected all the results needed to prove Theorem 4.3.

PROOF OF THEOREM 4.3: First, we note that it suffices to prove the existence of a constant $M$ such that for any multiindices $I, J \in \mathbb{N}_0^m$ we have $|||b_I^* *'_\hbar b_J^*|||_R \leq (RM)^{|I|+|J|}$. Indeed, this statement implies the continuity of $*'_\hbar$ since for $p = \sum_{I\in\mathbb{N}_0^m} p_I b_I^*$ and $q = \sum_{I\in\mathbb{N}_0^m} q_I b_I^*$ we estimate

$$
\begin{aligned}
|||p *'_\hbar q|||_R &= \Big|\Big|\Big|\sum\nolimits_{I\in\mathbb{N}_0^m} p_I b_I^* *'_\hbar \sum\nolimits_{J\in\mathbb{N}_0^m} q_J b_J^*\Big|\Big|\Big|_R \\
&\leq \sum_{I\in\mathbb{N}_0^m} \sum_{J\in\mathbb{N}_0^m} |p_I||q_J| |||b_I^* *'_\hbar b_J^*|||_R \\
&\leq \sum_{I\in\mathbb{N}_0^m} \sum_{J\in\mathbb{N}_0^m} |p_I||q_J|(RM)^{|I|+|J|} \\
&= \Big|\Big|\Big|\sum\nolimits_{I\in\mathbb{N}_0^m} p_I b_I^*\Big|\Big|\Big|_{RM} \Big|\Big|\Big|\sum\nolimits_{J\in\mathbb{N}_0^m} q_J b_J^*\Big|\Big|\Big|_{RM} \\
&= |||p|||_{RM} |||q|||_{RM} \,.
\end{aligned}
$$

Using the notation $I_{\{j\}}$ introduced in the proof of Proposition 3.36 we estimate

$$
\begin{aligned}
|||b_I^* *'_\hbar b_J^*|||_R &= \Big|\Big|\Big|\sum\nolimits_{\ell=0}^{\infty} (F'_{\hbar,\ell})^{\mathrm{left},(1,0)}(b_I^*, b_J^*)\Big|\Big|\Big|_R \\
&\underset{(1)}{\leq} \Big|\Big|\Big|\sum\nolimits_{w\in\tilde{W}} p^w_{i\lambda/\hbar}(\alpha_w)^{-1}(X_w \otimes Y_w)^{\mathrm{left},(1,0)}(b_I^*, b_J^*)\Big|\Big|\Big|_R \\
&\leq \sum_{w\in\tilde{W}} |p^w_{i\lambda/\hbar}(\alpha_w)^{-1}| \cdot \\
&\qquad \cdot \sum_{w_{(1)},\dots,w_{(|I|)}} \sum_{w'_{(1)},\dots,w'_{(|J|)}} \Big|\Big|\Big|X^{\mathrm{left},(1,0)}_{w_{(1)}} b^*_{I_{\{1\}}}\Big|\Big|\Big|_R \cdots \Big|\Big|\Big|X^{\mathrm{left},(1,0)}_{w_{(|I|)}} b^*_{I_{\{|I|\}}}\Big|\Big|\Big|_R \cdot \\
&\qquad\qquad \cdot \Big|\Big|\Big|Y^{\mathrm{left},(1,0)}_{w'_{(1)}} b^*_{J_{\{1\}}}\Big|\Big|\Big|_R \cdots \Big|\Big|\Big|Y^{\mathrm{left},(1,0)}_{w'_{(|J|)}} b^*_{J_{\{|J|\}}}\Big|\Big|\Big|_R \\
&\overset{(2)}{\leq} \sum_{w\in\tilde{W}} \frac{C_p^{|w|}}{(|w|!)^2} |I|^{|w|}|J|^{|w|} C^{2|w|} R^{|I|+|J|} \\
&\overset{(3)}{\leq} R^{|I|+|J|} \sum_{\ell=0}^{\infty} (kC_pC^2)^\ell \frac{|I|^\ell |J|^\ell}{(\ell!)^2}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(4)}{\leq} R^{|I|+|J|} \sum_{\ell=0}^{\infty} \frac{(k^{1/2} C_p^{1/2} C|I|)^{\ell}}{\ell!} \sum_{\ell'=0}^{\infty} \frac{(k^{1/2} C_p^{1/2} C|J|)^{\ell'}}{\ell'!} \\
&\leq R^{|I|+|J|} \mathrm{e}^{k^{1/2} C_p^{1/2} C|I|} \mathrm{e}^{k^{1/2} C_p^{1/2} C|J|} \\
&= (R \mathrm{e}^{k^{1/2} C_p^{1/2} C})^{|I|+|J|} .
\end{aligned}
$$

The sum $\sum_{w_{(1)},\dots,w_{(|I|)}}$ introduced in (1) is over all partitions of $w \in \tilde{W}$ into words $w_{(1)},\dots,w_{(|I|)}$. To be more precise, consider a partition $P_1,\dots,P_{|I|}$ of $\{1,\dots,|w|\}$ into $|I|$ many subsets. If $P_i = \{p_{i,1},\dots,p_{i,j_i}\}$ with $p_{i,1} < \cdots < p_{i,j_i}$, then associate the word $w_{(i)} = w_{p_{i,1}} w_{p_{i,2}} \dots w_{p_{i,j_i}}$. Then we sum over all partitions. The other sum is defined similarly. We also used submultiplicativity of $|\!|\!| \cdot |\!|\!|_R$ in this step. To justify (2), we note that for any $Z \in \mathfrak{gl}_N(\mathbb{C})$, $Z^{\mathrm{left},(1,0)} b_i^*$ is of degree 1, so that $X_{w_{(\ell)}}^{\mathrm{left},(1,0)} b_{I_{\{\ell\}}}^*$ is of degree 1. Defining $C \coloneqq \max_{i\in\{1,\dots,m\},\alpha\in\Delta} |\!|\!| X_\alpha^{\mathrm{left},(1,0)} b_i^* |\!|\!|_1$ we obtain

$$
\left|\!\left|\!\left| X_{w_{(\ell)}}^{\mathrm{left},(1,0)} b_{I_{\{\ell\}}}^* \right|\!\right|\!\right|_R \leq C^{|w_{(\ell)}|} R .
$$

The sum over $w_{(1)},\dots,w_{(|I|)}$ has $|I|^{|w|}$ many terms, since for each letter of $w$ we can choose in which of the $|I|$ many sets we want to have it. Similarly for the other sum. In (3) we used that there are at most $k^{|w|}$ many words of a given length $|w|$ in $\tilde{W}$ and (4) holds, because we just added some positive extra terms. □

**Remark 4.9** For a fixed compact set $K \subseteq \mathbb{C} \setminus P_\lambda$ the proof above shows that there is a constant $M \in \mathbb{R}^+$ such that for any $\hbar \in K$ we have

$$
|\!|\!| p *'_\hbar q |\!|\!|_R \leq |\!|\!| p |\!|\!|_{RM} |\!|\!| q |\!|\!|_{RM} \tag{4.5}
$$

since Corollary 4.8 gives uniform estimates for all $\hbar \in K$.

## 4.2 Stein manifolds and extension of holomorphic functions

In this subsection, we discuss extension properties of holomorphic functions on closed complex submanifolds of Stein manifolds or, more generally, on analytic subsets of Stein manifolds. We will use the results in the next subsection to identify the reduction-topology with the topology of locally uniform convergence and to determine the completion of the space of polynomials with respect to this topology.

Since analytic subsets in a Stein manifold are a very natural setting to prove the extendability results, we formulate them in this generality (even though we only need the case of closed submanifolds most of the time). The content of this subsection has been known for long and can be found e.g. in the textbook [22].

Recall that for a complex manifold $M$, we denote the vector space of holomorphic functions on $M$ by $\mathrm{Hol}(M)$.

**Definition 4.10 (Holomorphic convex hull)** *For a compact subset $K$ of a complex manifold $M$ we define its* holomorphic convex hull *to be the set*

$$
\hat{K}_M = \{ z \in M \mid |f(z)| \leq \sup_K |f| \text{ for all } f \in \mathrm{Hol}(M) \} . \tag{4.6}
$$

**Definition 4.11 (Stein manifold)** *A complex manifold $M$ of dimension $n$ is said to be* Stein *if*

*i.) for any compact subset $K \subseteq M$ its holomorphic convex hull $\hat{K}_M$ is compact,*

*ii.) for every $z \in M$ there are functions $f_1, \ldots, f_n \in \mathrm{Hol}(M)$ that form a coordinate system around $z$.*

Stein manifolds should be thought of as domains of holomorphicity for holomorphic functions of several complex variables. Clearly $\mathbb{C}^n$ is Stein.

**Definition 4.12** *A subset $V \subseteq M$ of a complex manifold is called* analytic*, if for every point $z \in M$ there is a neighbourhood $U \subseteq M$ of $z$ such that there is a family of holomorphic functions $f_j \in \mathrm{Hol}(U)$ with $j \in J$, $J$ some index set, such that*

$$V \cap U = \{z \in U \mid f_j(z) = 0 \text{ for all } j \in J\}. \tag{4.7}$$

**Example 4.13** Any closed complex submanifold $M$ of $\mathbb{C}^n$ is an analytic subset. Indeed, around any $z \in M$ we can find a submanifold chart, that is a neighbourhood $U$ and coordinates $z = (z_1, \ldots, z_n)$ such that $M \cap U$ is given by the vanishing of the first $n - \dim M$ coordinates. Therefore we can take $f_j = z_j$ for $j = 1, \ldots, n - \dim M$ in Definition 4.12. Around any $z \notin M$ there is a neighbourhood $U$ such that $U \cap M = \emptyset$ and we may pick $f_1 = 1$ in Definition 4.12.

**Definition 4.14** *A function $f \colon V \to \mathbb{C}$ on an analytic subset $V \subseteq M$ of a complex manifold is called holomorphic, if for every point $z \in V$ there is a neighbourhood $U \subseteq M$ of $z$ and a holomorphic function $g \in \mathrm{Hol}(U)$ such that $g\big|_{U\cap V} = f\big|_{U \cap V}$.*

**Example 4.15** If $V$ is a closed complex submanifold of $\mathbb{C}^n$ as in Example 4.13, then this definition of a holomorphic function coincides with the usual definition. Indeed, in any submanifold chart $(U, z)$ as in Example 4.13, a holomorphic function on $U \cap V$ can be extended constantly along the first $n - \dim M$ variables to a holomorphic function on $U$. The reverse implication is clear.

**Proposition 4.16** *Let $V$ be an analytic subset of a Stein manifold $M$. Then* $\mathrm{Hol}(V)$ *endowed with the topology of locally uniform convergence is a Fréchet space.*

PROOF: It follows from the definition of analytic subsets that $V$ is closed. Therefore the restriction of any compact exhaustion of $M$ to $V$ gives a compact exhaustion $K_i$ of $V$. The seminorms $\|f\|_{K_i} = \sup_{K_i} |f|$ define a countable system of seminorms inducing the topology of locally uniform convergence. The completeness of $\mathrm{Hol}(V)$ with respect to this topology is a non-trivial result and proved in [22, Theorem 7.4.9].$\square$

The crucial property of an analytic subset $V$ of a Stein manifold is the following extendability property for any holomorphic function on $V$.

**Theorem 4.17 (Extendability of holomorphic functions)** *Let $V$ be an analytic subset of a Stein manifold $M$. Any holomorphic function $f \in \mathrm{Hol}(V)$ can be extended to a holomorphic function $f \in \mathrm{Hol}(M)$. In other words, the restriction map* $\mathrm{Hol}(M) \to \mathrm{Hol}(V)$ *is surjective.*

PROOF: See [22, Theorem 7.4.8]. □

For an analytic subset $V$ of a complex manifold $M$ we denote the subspace of $\mathrm{Hol}(M)$ consisting of functions that vanish on $V$ by $\mathcal{I}(V)$. Note that the restriction map $\mathrm{Hol}(M) \to \mathrm{Hol}(V)$ descends to a map on the quotient, $r\colon \mathrm{Hol}(M)/\mathcal{I}(V) \to \mathrm{Hol}(V)$. This map is clearly injective by definition of $\mathcal{I}(V)$, and if $M$ is Stein it is surjective by the previous theorem.

**Corollary 4.18** *Assume that $M$ is Stein and that $V \subseteq M$ is an analytic subset. If $\mathrm{Hol}(M)/\mathcal{I}(V)$ is endowed with the quotient topology of the topology of locally uniform convergence and $\mathrm{Hol}(V)$ is endowed with the topology of locally uniform convergence then the map $r\colon \mathrm{Hol}(M)/\mathcal{I}(V) \to \mathrm{Hol}(V)$ is a homeomorphism.*

PROOF: We know that $r$ is bijective, so it only remains to prove the continuity of $r$ and $r^{-1}$. Both $\mathrm{Hol}(M)$ and $\mathrm{Hol}(V)$ are Fréchet spaces (for $\mathrm{Hol}(M)$ this is well-known, for $\mathrm{Hol}(V)$ it is the statement of Proposition 4.16). Since $\mathcal{I}(V)$ is closed, $\mathrm{Hol}(M)/\mathcal{I}(V)$ is also a Fréchet space. Clearly the locally uniform convergence of a sequence $f_i \in \mathrm{Hol}(M)$ implies the locally uniform convergence of the sequence of restrictions $f_i\big|_V \in \mathrm{Hol}(V)$, so the map $r$ is continuous. The statement then follows from the open mapping theorem for Fréchet spaces. □

## 4.3 Characterizing the reduction-topology

In this subsection, we show that the reduction-topology on $\mathcal{O}_\lambda$ as defined in Subsection 4.1 is the topology of locally uniform convergence and that the completion of the space of polynomials $\mathrm{Pol}(\mathcal{O}_\lambda)$ on $\mathcal{O}_\lambda$ with respect to this topology is exactly the space of holomorphic functions $\mathrm{Hol}(\mathcal{O}_\lambda)$ on $\mathcal{O}_\lambda$.

**Proposition 4.19** *The reduction topology $\mathcal{T}_{\mathrm{red}}$ on $\mathcal{O}_\lambda$ coincides with the topology of locally uniform convergence.*

PROOF: By the assumption at the beginning of this section (see also Subsection 2.1), $G$ is a closed complex submanifold of $\mathbb{C}^{N\times N}$, hence an analytic subset by Example 4.13. Applying Corollary 4.18 yields that the quotient topology on $\mathrm{Hol}(G)$ induced by the topology of locally uniform convergence on $\mathbb{C}^{N\times N}$ is precisely the topology of locally uniform convergence on $G$.

By Definition 4.2 the reduction-topology is the restriction of this topology to the subspace of right $G_\lambda$-invariant holomorphic functions. Using that this subspace is closed, and that a sequence $f_i \in \mathrm{Hol}(\mathcal{O}_\lambda)$ converges locally uniformly if and only if the sequence $\pi^*(f_i) \in \mathrm{Hol}(G)^{G_\lambda}$ converges locally uniformly, one can easily check that the reduction-topology coincides with the topology of locally uniform convergence on $\mathrm{Hol}(\mathcal{O}_\lambda)$. □

Finally, we would like to determine the completion $\widehat{\mathrm{Pol}}(\mathcal{O}_\lambda)$ of $\mathrm{Pol}(\mathcal{O}_\lambda)$ with respect to the topology of locally uniform convergence.

**Proposition 4.20** *We have $\widehat{\mathrm{Pol}}(\mathcal{O}_\lambda) = \mathrm{Hol}(\mathcal{O}_\lambda)$.*

PROOF: The inclusion $\widehat{\mathrm{Pol}}(\mathcal{O}_\lambda) \subseteq \mathrm{Hol}(\mathcal{O}_\lambda)$ is trivial, since $\mathrm{Pol}(\mathcal{O}_\lambda) \subseteq \mathrm{Hol}(\mathcal{O}_\lambda)$ and the limit of a locally uniformly convergent sequence of holomorphic functions is again holomorphic.

The other inclusion is easy to see if one uses that semisimple coadjoint orbits are affine algebraic varieties, see Remark 2.4: In particular they are analytic subsets of the Stein manifold $\mathfrak{g}^*$ and therefore we can use Theorem 4.17 to extend any $f \in \mathrm{Hol}(\mathcal{O}_\lambda)$ to a holomorphic function $\tilde{f} \in \mathrm{Hol}(\mathfrak{g}^*)$, which can be approximated by polynomials. Restricting these approximating polynomials to $\mathcal{O}_\lambda$ gives a sequence of polynomials in $\mathrm{Pol}(\mathcal{O}_\lambda)$ converging locally uniformly to $f$.

Alternatively, we know that $G$ is a closed submanifold of the Stein manifold $\mathbb{C}^{N\times N}$, so the same argument yields that any $f \in \mathrm{Hol}(G)$ can be approximated by some $p_n \in \mathrm{Pol}(G)$. Assume that $f \in \mathrm{Hol}(G)^{G_\lambda}$. Let $K_\lambda$ be a maximal compact subgroup of $G_\lambda$. Averaging $p_n$ over $K_\lambda$ gives a sequence $p'_n \in \mathrm{Pol}(G)^{K_\lambda}$ that converges locally uniformly to $f$. Now $p'_n$ is even $G_\lambda$-invariant since the action of $G$ is holomorphic, so $\pi_* p'_n \in \mathrm{Pol}(\mathcal{O}_\lambda)$ converges to $\pi_* f \in \mathrm{Hol}(\mathcal{O}_\lambda)$. □

We are now able to prove the main theorem stated in the introduction to this section.

PROOF OF THEOREM 4.1: From Subsection 4.1 we know that the product $*_\hbar$ is continuous with respect to the reduction-topology. We showed in Proposition 4.19 that the reduction-topology coincides with the topology of locally uniform convergence on $\mathcal{O}_\lambda$. The previous proposition shows that the completion of $\mathrm{Pol}(\mathcal{O}_\lambda)$ in this topology is $\mathrm{Hol}(\mathcal{O}_\lambda)$. Finally $G$-invariance of the product on the completion is clear since the action of $G$ on $\mathrm{Pol}(\mathcal{O}_\lambda)$ is continuous with respect to the topology of locally uniform convergence. □

We close this section by the following proposition, which asserts that the dependence of $*_\hbar$ on $\hbar$ is holomorphic.

**Proposition 4.21 (Holomorphic dependence on $\hbar$)** *For two fixed holomorphic functions $p, q \in \mathrm{Hol}(\mathcal{O}_\lambda)$ and $x \in \mathcal{O}_\lambda$ the map $\mathbb{C} \setminus P_\lambda \to \mathbb{C}$, $\hbar \mapsto p *_\hbar q(x)$ is holomorphic.*

PROOF: By construction of $*_\hbar$ in Section 3, the map $\mathbb{C} \setminus P_\lambda \to \mathbb{C}$, $\hbar \mapsto p' *_\hbar q'(x)$ is rational for $p', q' \in \mathrm{Pol}(\mathcal{O}_\lambda)$. Assume that $p_n$, $q_n$ are sequences of polynomials on $\mathcal{O}_\lambda$ such that $p_n \to p$ and $q_n \to q$ locally uniformly. Since the estimates of Subsection 4.1 are locally uniform in $\hbar$, see Remark 4.9, it follows that $p_n *_\hbar q_n \to p *_\hbar q$ locally uniformly in $\hbar$. But clearly the evaluation at $x$ is continuous, so that $\hbar \mapsto p *_\hbar q(x)$ is a locally uniform limit of rational functions and therefore holomorphic. □

# 5 Quantizing real coadjoint orbits

We have seen in the previous sections how to construct (formal and strict) quantizations of complex coadjoint orbits. In this section, we will use these results to obtain (formal and strict) quantizations of real coadjoint orbits.

In Subsection 5.1 and Subsection 5.2 we collect some prelimary results on the complexification of a real coadjoint orbit $\mathcal{O}_\lambda$ and a real Lie group $G$. We define a certain class of analytic functions that we denote by $\mathcal{A}(\mathcal{O}_\lambda)$ and $\mathcal{A}(G)$. In Subsection 5.3 we construct a quantization of real orbits by restricting the quantization of a complexification. We discuss the examples of complex projective spaces and hyperbolic discs in Subsection 5.4. Finally, we show that point evaluation functionals are positive for certain coadjoint orbits in Subsection 5.5 and compare the quantum algebras obtained for coadjoint orbits of real Lie groups with the same complexification in Subsection 5.6. Most results in the later subsections follow almost directly from the results in the complex case.

From now on, all complex Lie groups and Lie algebras will be denoted with a hat and letters without decoration will be used to denote real objects. We will also use hats for maps between complex objects, e.g. we rename the map defined in (2.8) to $\hat{\Psi}$.

## 5.1 Complexification

In this subsection we define the complexification of a real coadjoint orbit $\mathcal{O}_\lambda$ and a real Lie group $G$, and show how they are related.

For a real Lie algebra $\mathfrak{g}$, denote the space of real-valued real-linear functionals on $\mathfrak{g}$ by $\mathfrak{g}^*$. As before, $\hat{\mathfrak{g}}^*$ denotes the space of complex-valued complex-linear functionals on a complex Lie algebra $\hat{\mathfrak{g}}$. In the following, we will always assume that $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ is the complexification of $\mathfrak{g}$. In this case, any element of $\mathfrak{g}^*$ has a unique extension to an element of $\hat{\mathfrak{g}}^*$. We will perform this extension implicitly whenever necessary, without mentioning it. For example, in the following proposition, the coadjoint orbit $\hat{\mathcal{O}}_\lambda$ is really the coadjoint orbit through the extension of $\lambda \in \mathfrak{g}^*$ to an element of $\hat{\mathfrak{g}}^*$.

**Proposition 5.1** *Let $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ be a coadjoint orbit of a real connected Lie group, and assume that $\hat{\mathfrak{g}}$ is the complexification of $\mathfrak{g}$. Then $\mathcal{O}_\lambda$ is a submanifold of a unique complex coadjoint orbit $\hat{\mathcal{O}}_\lambda \subseteq \hat{\mathfrak{g}}^*$ of a complex connected Lie group with Lie algebra $\hat{\mathfrak{g}}$. The tangent space $\mathrm{T}_\xi\hat{\mathcal{O}}_\lambda$ of this orbit $\hat{\mathcal{O}}_\lambda$ is the complexification of $\mathrm{T}_\xi\mathcal{O}_\lambda$ for every $\xi \in \mathfrak{g}^*$.*

PROOF: By Proposition 2.1 the coadjoint orbit $\mathcal{O}_\lambda$ is the symplectic leaf through $\lambda$ of the linear Poisson structure on $\mathfrak{g}^*$ defined just before Proposition 2.1. Similarly the coadjoint orbits in $\hat{\mathfrak{g}}^*$ are symplectic leaves of the linear Poisson structure on $\hat{\mathfrak{g}}^*$, and the symplectic leaf containing $\lambda \in \hat{\mathfrak{g}}^*$ contains the whole orbit $\mathcal{O}_\lambda$. This proves existence and uniqueness of $\hat{\mathcal{O}}_\lambda$.

As in Subsection 2.1, we can identify $\mathrm{T}_\xi\mathcal{O}_\lambda$ with $\mathfrak{g}/\mathfrak{g}_\xi$ (as real vector spaces) and $\mathrm{T}_\xi\hat{\mathcal{O}}_\lambda$ with $\hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\xi$ (as complex vector spaces) for all $\xi \in \mathcal{O}_\lambda$. Therefore $\mathrm{T}_\xi\hat{\mathcal{O}}_\lambda$ is indeed the complexification of $\mathrm{T}_\xi\mathcal{O}_\lambda$. $\square$

We refer to the complex coadjoint orbit $\hat{\mathcal{O}}_\lambda$ of the previous proposition as the *complexification* of $\mathcal{O}_\lambda$. We will show how to realize it explicitly as the coadjoint orbit of some Lie group $\hat{G}$.

**Definition 5.2** *Let $G$ be a real Lie group. A* complexification *of $G$ is a complex connected Lie group $\hat{G}$ together with an embedding $\iota\colon G \to \hat{G}$, such that the corresponding Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$ and such that the map $\mathrm{T}_e\iota\colon \mathfrak{g} \to \hat{\mathfrak{g}}$ corresponds to the injection $X \mapsto X \otimes 1$ under this isomorphism.*

Note that a complexification according to this definition may fail to exist or may not be unique, if it exists. See the paragraph after Proposition 5.8 for an example of a Lie group with non-unique complexification. For a connected semisimple Lie group $G$ a complexification exists if and only if the group can be realized as a linear group: Existence for linear Lie groups is shown below, and the reverse implication follows since semisimplicity of $G$ implies semisimplicity of the complexification and complex connected semisimple Lie groups are always matrix Lie groups, see Remark 2.4. There is a different notion of a universal complexification that does always exist, but that does not enjoy the property that $\hat{\mathfrak{g}} \cong \mathfrak{g} \otimes \mathbb{C}$. We will not use universal complexifications in this paper.

**Proposition 5.3** *If $G$ is a real connected closed linear Lie group, then it admits a complexification $\hat{G}$.*

PROOF: We may assume that both $G$ and its Lie algebra $\mathfrak{g}$ are realized by real matrices. Then the complexification $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ is a Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$. We can use the exponential map to construct an immersed complex Lie subgroup $\hat{G}$ of $\mathrm{GL}_N(\mathbb{C})$ containing $G$ as a subgroup and having $\hat{\mathfrak{g}}$ as Lie algebra, see e.g. [21, Chapter 5.9]. Since $G$ is a closed subgroup of $\mathrm{GL}_N(\mathbb{C})$, it is also a closed subgroup of $\hat{G}$. $\square$

Note that we did not claim that $\hat{G}$ is a closed subgroup of $\mathrm{GL}_N(\mathbb{C})$. For semisimple Lie groups this follows automatically from Remark 2.4.

**Lemma 5.4** *Let $G$ be a real connected Lie group with complexification $\hat{G}$ and let $\mathcal{O}_\lambda$ be a coadjoint orbit of $G$ with complexification $\hat{\mathcal{O}}_\lambda$. Then $\hat{\mathcal{O}}_\lambda$ is a coadjoint orbit of $\hat{G}$ and the embedding $\iota\colon G \to \hat{G}$ descends to an embedding $\mathcal{O}_\lambda \cong G/G_\lambda \to \hat{G}/\hat{G}_\lambda \cong \hat{\mathcal{O}}_\lambda$.*

PROOF: Since $\hat{G}$ is connected and has the Lie algebra $\hat{\mathfrak{g}}$, it follows from Proposition 2.1 that its coadjoint orbit through $\lambda$ is $\hat{\mathcal{O}}_\lambda$. We identify $G$ with a subgroup of $\hat{G}$. Since the coadjoint action of $\hat{G}$ on $\hat{\mathfrak{g}}$ is holomorphic, $\hat{G}_\lambda$ is a complexification of $G_\lambda = \hat{G}_\lambda \cap G$. So the map $\iota$ descends to a map $\mathcal{O}_\lambda \cong G/G_\lambda \to \hat{G}/\hat{G}_\lambda \cong \hat{\mathcal{O}}_\lambda$ that is still injective. To see that it is an embedding, note that the actions of $G_\lambda$ and $\hat{G}_\lambda$ on $\hat{G}$ are proper and free, so $\hat{G}$ is a principal $G_\lambda$ resp. $\hat{G}_\lambda$ bundle over $\hat{G}/G_\lambda$ resp. $\hat{G}/\hat{G}_\lambda$. This implies first that $G/G_\lambda \to \hat{G}/G_\lambda$ is still an embedding, and then that $G/G_\lambda \to \hat{G}/\hat{G}_\lambda$ also is. $\square$

## 5.2 Polynomials and analytic functions

In this subsection we introduce polynomials $\mathrm{Pol}(\mathcal{O}_\lambda)$ and a certain class of analytic functions $\mathcal{A}(\mathcal{O}_\lambda)$ on a real coadjoint orbit $\mathcal{O}_\lambda$. $\mathcal{A}(\mathcal{O}_\lambda)$ consists of restrictions of holomorphic functions on the complexification. In analogy to the complex case, $\mathcal{A}(\mathcal{O}_\lambda)$ is the completion of $\mathrm{Pol}(\mathcal{O}_\lambda)$ with respect to some locally convex topology.

All our polynomials are complex-valued. So for a real finite dimensional vector space $V$ we define $\mathrm{Pol}(V)$ to be the unital complex subalgebra of $\mathscr{C}^\infty(V)$ generated by the linear maps. (Remember that $\mathscr{C}^\infty(V)$ consists of smooth functions $V \to \mathbb{C}$.) So $\mathrm{Pol}(V) \cong \mathrm{S}(V^*_\mathbb{C})$ where $V^*_\mathbb{C}$ is the complexification of $V^* = \{\phi\colon V \to \mathbb{R},\ \phi \text{ linear}\}$.

**Definition 5.5 (Polynomials)** *Let $\mathcal{O}_\lambda$ be a coadjoint orbit of a real connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Then*

$$\mathrm{Pol}(\mathcal{O}_\lambda) = \{p\colon \mathcal{O}_\lambda \to \mathbb{C} \mid p = P\big|_{\mathcal{O}_\lambda} \text{ for some polynomial } P \text{ on } \mathfrak{g}^*\} \tag{5.1}$$

*is called the algebra of polynomials on $\mathcal{O}_\lambda$.*

Note that polynomials on a complex orbit $\hat{\mathcal{O}}_\lambda$ were assumed to be holomorphic and do therefore not coincide with polynomials on the underlying real orbit. We will always use holomorphic polynomials on complexifications, so this will hopefully not cause any confusion.

Denote the ideal of polynomials on $\mathfrak{g}^*$ resp. $\hat{\mathfrak{g}}^*$ vanishing on $\mathcal{O}_\lambda$ resp. $\hat{\mathcal{O}}_\lambda$ by $\mathcal{I}(\mathcal{O}_\lambda)$ resp. $\mathcal{I}(\hat{\mathcal{O}}_\lambda)$. It is clear that the maps $\mathrm{Pol}(\mathfrak{g}^*)/\mathcal{I}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ and $\mathrm{Pol}(\hat{\mathfrak{g}}^*)/\mathcal{I}(\hat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\hat{\mathcal{O}}_\lambda)$ are isomorphisms. We would now like to relate polynomials on $\mathcal{O}_\lambda$ and $\hat{\mathcal{O}}_\lambda$.

**Proposition 5.6** *Let $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ be a real coadjoint orbit with complexification $\hat{\mathcal{O}}_\lambda \subseteq \hat{\mathfrak{g}}^*$. Then the restriction map $(\,\cdot\,)\big|_{\mathcal{O}_\lambda}\colon \mathscr{C}^\infty(\hat{\mathcal{O}}_\lambda) \to \mathscr{C}^\infty(\mathcal{O}_\lambda)$ restricts to an isomorphism $(\,\cdot\,)\big|_{\mathcal{O}_\lambda}\colon \mathrm{Pol}(\hat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$.*

Proof: Since restriction to $V$ is a bijection between complex linear maps $V \otimes \mathbb{C} \to \mathbb{C}$ and real linear maps $V \to \mathbb{C}$ for any finite dimensional real vector space $V$, it follows that the restriction map $\mathrm{Pol}(\hat{\mathfrak{g}}^*) \to \mathrm{Pol}(\mathfrak{g}^*)$ is an isomorphism. If we can prove that the restriction map $\mathcal{I}(\hat{\mathcal{O}}_\lambda) \to \mathcal{I}(\mathcal{O}_\lambda)$ is also an isomorphism, then we are done since $\mathrm{Pol}(\hat{\mathcal{O}}_\lambda) \cong \mathrm{Pol}(\hat{\mathfrak{g}}^*)/\mathcal{I}(\hat{\mathcal{O}}_\lambda) \to \mathrm{Pol}(\mathfrak{g}^*)/\mathcal{I}(\mathcal{O}_\lambda) \cong \mathrm{Pol}(\mathcal{O}_\lambda)$ would be an isomorphism.

Since any map vanishing on $\hat{\mathcal{O}}_\lambda$ vanishes in particular on $\mathcal{O}_\lambda \subseteq \hat{\mathcal{O}}_\lambda$, the restriction map $\mathcal{I}(\hat{\mathcal{O}}_\lambda) \to \mathcal{I}(\mathcal{O}_\lambda)$ is well-defined and it is injective since it is the restriction of the injective map $\mathrm{Pol}(\hat{\mathfrak{g}}^*) \to \mathrm{Pol}(\mathfrak{g}^*)$. So we only need to prove surjectivity, meaning that if a polynomial $p$ on $\mathfrak{g}^*$ vanishes on $\mathcal{O}_\lambda$, then its unique extension to a polynomial $\hat{p}$ on $\hat{\mathfrak{g}}^*$ vanishes on $\hat{\mathcal{O}}_\lambda$. Since $\hat{\mathcal{O}}_\lambda$ is a complex submanifold of $\hat{\mathfrak{g}}^*$, the restriction of $\hat{p}$ to $\hat{\mathcal{O}}_\lambda$ is holomorphic. As such it is determined by its derivatives (of all orders) at $\lambda$. It is even determined by its derivatives in the direction of $\mathrm{T}_\lambda\mathcal{O}_\lambda$ since $\mathrm{T}_\lambda\hat{\mathcal{O}}_\lambda$ is the complexification of $\mathrm{T}_\lambda\mathcal{O}_\lambda$. But all these derivatives vanish since the restriction of $\hat{p}$ to $\mathcal{O}_\lambda$ vanishes. □

**Definition 5.7** *Let $G$ be a linear real Lie group. Its algebra of polynomials* $\mathrm{Pol}(G)$ *is the unital complex subalgebra of $\mathscr{C}^\infty(G)$ generated by the functions $P_{ij}\colon G \to \mathbb{C}$, $g \mapsto g_{ij}$.*

In contrast to the complex case, the algebra of polynomials $\mathrm{Pol}(G)$ may depend on the way in which $G$ is realized as a linear group, even in the semisimple case. We will give an instructive example after stating the following proposition, which can be proven in a similar way than Proposition 5.6.

**Proposition 5.8** *Let $G \subseteq \mathrm{GL}_N(\mathbb{R})$ be a linear connected Lie group with complexification $\hat{G} \subseteq \mathrm{GL}_N(\mathbb{C})$. Then the restriction map $(\,\cdot\,)\big|_G\colon \mathscr{C}^\infty(\hat{G}) \to \mathscr{C}^\infty(G)$ restricts to an isomorphism $(\,\cdot\,)\big|_G\colon \mathrm{Pol}(\hat{G}) \to \mathrm{Pol}(G)$.*

The reason why the algebra of polynomials $\mathrm{Pol}(G)$ may depend on the linear structure of $G$, is essentially that $G$ may not have a unique complexification. Consider the linear semisimple Lie group $\mathrm{SL}_3(\mathbb{R}) \subseteq \mathrm{GL}_3(\mathbb{R})$, which has $\mathrm{SL}_3(\mathbb{C})$ as a complexification. The images of $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{C})$ under Ad are again semisimple Lie groups. Furthermore, $\mathrm{Ad}(\mathrm{SL}_3(\mathbb{R})) \cong \mathrm{SL}_3(\mathbb{R})$ since $\mathrm{SL}(3,\mathbb{R})$ has trivial center, and $\mathrm{Ad}(\mathrm{SL}_3(\mathbb{C})) \cong \mathrm{SL}_3(\mathbb{C})/\{1, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}\}$ is a complexification of $\mathrm{Ad}(\mathrm{SL}_3(\mathbb{R}))$. By the previous proposition $\mathrm{Pol}(\mathrm{Ad}(\mathrm{SL}_3(\mathbb{R}))) \cong \mathrm{Pol}(\mathrm{SL}_3(\mathbb{C})/\{1, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}\}) \to \mathrm{Pol}(\mathrm{SL}_3(\mathbb{C})) \cong \mathrm{Pol}(\mathrm{SL}_3(\mathbb{R}))$ where the map in the middle is not surjective, since there are polynomials on $\mathrm{SL}_3(\mathbb{C})$ that are not constant on $\{1, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}\}$.

We denote the inverses of the isomorphisms in Proposition 5.6 and Proposition 5.8 by

$$\hat{\cdot}\colon \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\hat{\mathcal{O}}_\lambda) \quad \text{and} \quad \hat{\cdot}\colon \mathrm{Pol}(G) \to \mathrm{Pol}(\hat{G})\,. \tag{5.2}$$

**Lemma 5.9** *Let $G$ be a real connected linear Lie group with complexification $\hat{G}$, and let $\lambda \in \mathfrak{g}^*$ be such that $\hat{G}_\lambda$ is connected. If $f \in \mathrm{Pol}(\hat{G})$ satisfies $f\big|_G \in \mathrm{Pol}(G)^{G_\lambda}$ then $f \in \mathrm{Pol}(\hat{G})^{\hat{G}_\lambda}$.*

PROOF: Let $f$ be as in the statement of the lemma. Since $f\big|_G = (g \triangleright f)\big|_G$ holds for all $g \in G_\lambda$ it follows from the injectivity of $(\,\cdot\,)\big|_G$ that $f = g \triangleright f$, so $f \in \mathrm{Pol}(\hat{G})^{G_\lambda}$. Therefore $f$ is in particular invariant under $\mathfrak{g}_\lambda$, thus also under $\hat{\mathfrak{g}}_\lambda$ since the action is holomorphic. Since $\hat{G}_\lambda$ is connected we obtain that $f$ is $\hat{G}_\lambda$-invariant. □

**Corollary 5.10** *Let $G$ be a real connected semisimple linear Lie group with complexification $\hat{G}$, and assume that $\lambda \in \mathfrak{g}^*$ is semisimple. In this case the restriction map $(\,\cdot\,)\big|_G\colon \mathrm{Pol}(\hat{G})^{\hat{G}_\lambda} \to \mathrm{Pol}(G)^{G_\lambda}$ is an isomorphism.*

PROOF: $G_\lambda$ is connected by Proposition 2.3, so this is an immediate consequence of Proposition 5.8 and Lemma 5.9. □

**Corollary 5.11** *Let $G$ be a real connected semisimple linear Lie group with complexification $\hat{G}$, and assume that $\lambda \in \mathfrak{g}^*$ is semisimple. Then the map $\pi^*\colon \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(G)^{G_\lambda}$ is an isomorphism.*

PROOF: The composition $\mathrm{Pol}(\mathcal{O}_\lambda) \xrightarrow{\hat{\cdot}} \mathrm{Pol}(\hat{\mathcal{O}}_\lambda) \xrightarrow{\hat{\pi}^*} \mathrm{Pol}(\hat{G})^{\hat{G}_\lambda} \xrightarrow{(\,\cdot\,)|_G} \mathrm{Pol}(G)^{G_\lambda}$ equals $\pi^*$ and is an isomorphism because of Proposition 5.6, Proposition 3.27, and Corollary 5.10. □

**Corollary 5.12** *Let $G$ be a real connected semisimple linear Lie group with complexification $\hat{G}$, and assume that $\lambda \in \mathfrak{g}^*$ is semisimple. Then the following diagram commutes and all arrows are isomorphisms:*

$$\begin{tikzcd}
\mathrm{Pol}(\hat{G})^{\hat{G}_\lambda} \arrow[r, shift left, "\hat{\pi}_*"] \arrow[d, shift left, "(\cdot)|_G"] & \mathrm{Pol}(\hat{\mathcal{O}}_\lambda) \arrow[l, shift left, "\hat{\pi}^*"] \arrow[d, shift left, "(\cdot)|_{\mathcal{O}_\lambda}"] \\
\mathrm{Pol}(G)^{G_\lambda} \arrow[r, shift left, "\pi_*"] \arrow[u, shift left, "\hat{\cdot}"] & \mathrm{Pol}(\mathcal{O}_\lambda) \arrow[l, shift left, "\pi^*"] \arrow[u, shift left, "\hat{\cdot}"]
\end{tikzcd} \tag{5.3}$$

Next, we want to introduce a class of analytic functions, that becomes the closure of the polynomials with respect to a certain locally convex topology. To this end, assume that $\mathcal{O}_\lambda$ is a coadjoint orbit with complexification $\hat{\mathcal{O}}_\lambda$, and that $G$ is a real connected Lie group with complexification $\hat{G}$. Then define

$$\mathcal{A}(\mathcal{O}_\lambda) = \operatorname{im}\Big( (\,\cdot\,)\big|_{\mathcal{O}_\lambda} : \operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \to \mathscr{C}^\infty(\mathcal{O}_\lambda) \Big) \tag{5.4}$$

and

$$\mathcal{A}(G) = \operatorname{im}\Big( (\,\cdot\,)\big|_{G} : \operatorname{Hol}(\hat{G}) \to \mathscr{C}^\infty(G) \Big)\,. \tag{5.5}$$

Note that an element $f \in \mathcal{A}(\mathcal{O}_\lambda)$ determines a unique element $\hat{f} \in \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$: Existence follows by definition of $\mathcal{A}(\mathcal{O}_\lambda)$ and $\hat{f}$ is determined by all its derivatives at $\lambda$. Since the complexification of $\mathrm{T}_\lambda\mathcal{O}_\lambda$ is just $\mathrm{T}_\lambda\hat{\mathcal{O}}_\lambda$, see Lemma 5.4, it suffices to take derivatives in the direction of $\mathrm{T}_\lambda\mathcal{O}_\lambda$. But these derivatives are determined by $f$. A similar reasoning holds for $G$ and $\hat{G}$. We obtain a commuting square that is similar to the square for polynomials obtained in Corollary 5.12.

**Proposition 5.13** *The following diagram is commutative and all arrows are isomorphisms:*

$$\begin{tikzcd}
\operatorname{Hol}(\hat{G})^{\hat{G}_\lambda} \arrow[r, shift left, "\hat{\pi}_*"] \arrow[d, shift left, "(\cdot)|_G"] & \operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \arrow[l, shift left, "\hat{\pi}^*"] \arrow[d, shift left, "(\cdot)|_{\mathcal{O}_\lambda}"] \\
\mathcal{A}(G)^{G_\lambda} \arrow[r, shift left, "\pi_*"] \arrow[u, shift left, "\hat{\ }"] & \mathcal{A}(\mathcal{O}_\lambda) \arrow[l, shift left, "\pi^*"] \arrow[u, shift left, "\hat{\ }"]
\end{tikzcd} \tag{5.6}$$

Proof: We know from Subsection 2.1 that $\hat{\pi}^* : \operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \to \operatorname{Hol}(\hat{G})^{\hat{G}_\lambda}$ is an isomorphism. In the previous paragraph we explained that $\hat{\ } : \mathcal{A}(\mathcal{O}_\lambda) \to \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$ and $\hat{\ } : \mathcal{A}(G) \to \operatorname{Hol}(\hat{G})$ are isomorphisms and as in Lemma 5.9 it follows that the same is true for $\hat{\ } : \mathcal{A}(G)^{G_\lambda} \to \operatorname{Hol}(\hat{G})^{\hat{G}_\lambda}$. Composing these isomorphisms we obtain that $\pi^* : \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(G)^{G_\lambda}$ is an isomorphism. $\square$

Since $\operatorname{Pol}(\hat{\mathcal{O}}_\lambda) \subseteq \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$ it follows that $\operatorname{Pol}(\mathcal{O}_\lambda) \subseteq \mathcal{A}(\mathcal{O}_\lambda)$. We can define a topology $\mathcal{T}_{\widehat{\mathrm{lu}}}$ of *extended locally uniform convergence* on $\mathcal{A}(\mathcal{O}_\lambda)$ as follows: A sequence $f_n \in \mathcal{A}(\mathcal{O}_\lambda)$ converges to some $f \in \mathcal{A}(\mathcal{O}_\lambda)$ if and only if the sequence $\hat{f}_n \in \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$ converges locally uniformly to $\hat{f} \in \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$. Clearly the maps $\hat{\ } : \mathcal{A}(\mathcal{O}_\lambda) \to \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$ and $(\,\cdot\,)\big|_{\mathcal{O}_\lambda} : \operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$ are both homeomorphisms. From Proposition 4.20 it follows that the closure of $\operatorname{Pol}(\mathcal{O}_\lambda)$ with respect to the topology of extended locally uniform convergence is $\mathcal{A}(\mathcal{O}_\lambda)$.

## 5.3 Formal and strict star products on real coadjoint orbits

In a sense all constructions in Section 2, Section 3, and Section 4 are compatible with the restriction to real forms. In this subsection we want to make this statement precise. In particular, we will show that we can restrict formal and strict products from a complexification $\hat{\mathcal{O}}_\lambda$ of a semisimple coadjoint orbit $\mathcal{O}_\lambda$ of a real connected semisimple Lie group $G$ to formal and strict star products on $\mathcal{O}_\lambda$. These star products can—as before—be computed by applying fundamental vector fields or by passing to the Lie group by using the maps $\pi^*$ and $\pi_*$. We will determine when the star products on $\mathcal{O}_\lambda$ are of (pseudo) Wick type or of standard ordered type.

**Proposition 5.14** *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected real Lie group $G$. By Lemma 5.4 it has a complexification $\hat{\mathcal{O}}_\lambda$, and there are strict products $\hat{*}_\hbar \colon \operatorname{Pol}(\hat{\mathcal{O}}_\lambda) \times \operatorname{Pol}(\hat{\mathcal{O}}_\lambda) \to \operatorname{Pol}(\hat{\mathcal{O}}_\lambda)$ with extensions $\hat{*}_\hbar \colon \operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \times \operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \to \operatorname{Hol}(\hat{\mathcal{O}}_\lambda)$ constructed in Corollary 3.30 and Theorem 4.1, where $\hbar \in \mathbb{C} \setminus P_\lambda$. These products restrict to $G$-invariant strict products*

$$*_\hbar \colon \operatorname{Pol}(\mathcal{O}_\lambda) \times \operatorname{Pol}(\mathcal{O}_\lambda) \to \operatorname{Pol}(\mathcal{O}_\lambda) \quad \text{and} \quad *_\hbar \colon \mathcal{A}(\mathcal{O}_\lambda) \times \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda) \tag{5.7}$$

*for all $\hbar \in \mathbb{C} \setminus P_\lambda$. For fixed $p, q \in \operatorname{Pol}(\mathcal{O}_\lambda)$, the dependence of $p *_\hbar q$ on $\hbar$ is rational with no pole at zero, and for fixed $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$ and $x \in \mathcal{O}_\lambda$, the dependence of $f *_\hbar g(x)$ on $\hbar$ is holomorphic. Both products are continuous with respect to the topology of extended locally uniform convergence defined at the end of Subsection 5.2.*

PROOF: Since the restriction maps $\operatorname{Pol}(\hat{\mathcal{O}}_\lambda) \to \operatorname{Pol}(\mathcal{O}_\lambda)$ and $\operatorname{Hol}(\hat{\mathcal{O}}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$ are both homeomorphisms (with respect to the topology of locally uniform convergence on the domains and the topology of extended locally uniform convergence on the codomains), the statement follows trivially from the corresponding statements for $\hat{*}_\hbar$, obtained in Corollary 3.30, Theorem 4.1, and Proposition 4.21. □

We would like to compute these star products without passing to the complexification. The construction of bidifferential operators from Subsection 2.2 works completely similarly in the real setting. Recall that our differential operators act on complex-valued functions, and therefore any complex vector field $\Gamma^\infty(\mathrm{T}^{\mathbb{C}}M)$ defines a first order differential operator on $M$.

**Proposition 5.15** *Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$, and let $\hat{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. The map*

$$(\,\cdot\,)^{\mathrm{left}} \colon (\mathscr{U}\hat{\mathfrak{g}})^{\otimes k} \to k\text{-}\mathrm{DiffOp}^G(G)$$

*obtained by extending $\hat{\mathfrak{g}} \ni X \mapsto X^{\mathrm{left}} \in \Gamma^\infty(\mathrm{T}^{\mathbb{C}}G)$ to an algebra homomorphism $\mathscr{U}\hat{\mathfrak{g}} \to \mathrm{DiffOp}^G(G)$ and further to tensor products as in (2.5c) is an isomorphism. If $H$ is a closed Lie subgroup of $G$, then the map*

$$\Psi \colon ((\mathscr{U}\hat{\mathfrak{g}}/\mathscr{U}\hat{\mathfrak{g}} \cdot \hat{\mathfrak{h}})^{\otimes k})^H \to k\text{-}\mathrm{DiffOp}^G(G/H)\,, \quad \Psi([\vec{u}])(\vec{f}) = \pi_*(\vec{u}^{\mathrm{left}}(\pi^*\vec{f})) \tag{5.8}$$

*is also an isomorphism.*

PROOF: With the obvious modifications the proofs of Proposition 2.5 and Proposition 2.7 given in Appendix A.1 apply also to the real situation. □

To be consistent with the notation of this chapter, we denote the map defined in (2.8) by $\hat{\Psi}$.

**Lemma 5.16** *Let $G$ be a real Lie group with closed subgroup $H$ and assume that the complex Lie group $\hat{G}$ is a complexification of $G$ and contains a complex closed subgroup $\hat{H}$ that is a complexification of $H$. The maps $(\,\cdot\,)^{\mathrm{left}}$ and $\Psi$ are compatible with the maps $(\,\cdot\,)^{\mathrm{left},(1,0)}$ and $\hat{\Psi}$ in the sense that the diagrams*

$$\begin{tikzcd} \mathrm{Hol}(\hat{U})^k \arrow[r, "(\cdot)|_U^{\times k}"] \arrow[d, "\vec{u}^{\mathrm{left},(1,0)}"'] & \mathscr{C}^\infty(U)^k \arrow[d, "\vec{u}^{\mathrm{left}}"] \\ \mathrm{Hol}(\hat{U}) \arrow[r, "(\cdot)|_U"'] & \mathscr{C}^\infty(U) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Hol}(\hat{V})^k \arrow[r, "(\cdot)|_V^{\times k}"] \arrow[d, "\hat{\Psi}([\vec{v}])"'] & \mathscr{C}^\infty(V)^k \arrow[d, "\Psi([\vec{v}])"] \\ \mathrm{Hol}(\hat{V}) \arrow[r, "(\cdot)|_V"'] & \mathscr{C}^\infty(V) \end{tikzcd} \tag{5.9}$$

*commute for all open subsets $\hat{U} \subseteq \hat{G}$ and $\hat{V} \subseteq \hat{G}/\hat{H}$, with $U := \hat{U} \cap G$ and $V := \hat{V} \cap G/H$, and all elements $\vec{u} \in (\mathscr{U}\hat{\mathfrak{g}})^{\otimes k}$ and $\vec{v} \in ((\mathscr{U}\hat{\mathfrak{g}}/\mathscr{U}\hat{\mathfrak{g}} \cdot \hat{\mathfrak{h}})^{\otimes k})^{\hat{H}}$.*

PROOF: The commutativity of the second diagram follows easily from commutativity of the first, since the restrictions are compatible with $\pi^*$ and $\pi_*$. To prove commutativity of the first diagram, assume that $k = 1$ and $\vec{u} = X \in \hat{\mathfrak{g}} \subseteq \mathscr{U}\hat{\mathfrak{g}}$. The tangent map of a holomorphic function commutes with the multiplication by i. We compute

$$\begin{aligned} X^{\mathrm{left},(1,0)} f(g) &= \frac{1}{2}(\mathrm{T}_g f \circ \mathrm{T}_e \mathrm{L}_g(X) - \mathrm{i}\mathrm{T}_g f \circ \mathrm{T}_e \mathrm{L}_g(\mathrm{i}X)) = \\ &= \mathrm{T}_g f \circ \mathrm{T}_e \mathrm{L}_g(X) = X^{\mathrm{left}} f\big|_U(g) \end{aligned}$$

for $f \in \mathrm{Hol}(\hat{U})$ and $g \in U$. The general case follows from this computation by the way in which $(\,\cdot\,)^{\mathrm{left},(1,0)}$ and $(\,\cdot\,)^{\mathrm{left}}$ are extended to $(\mathscr{U}\hat{\mathfrak{g}})^{\otimes k}$. □

**Corollary 5.17** *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected real Lie group $G$. For $\hbar \in \mathbb{C} \setminus P_\lambda$ and $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, the product $*_\hbar \colon \mathrm{Pol}(\mathcal{O}_\lambda) \times \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(\mathcal{O}_\lambda)$ defined in Proposition 5.14 can be computed by*

$$p *_\hbar q = \sum_{\ell=0}^{\infty} \Psi(F_{\hbar,\ell})(p, q)\,. \tag{5.10}$$

PROOF: The previous lemma implies

$$p *_\hbar q = (\hat{p} \mathbin{\hat{*}_\hbar} \hat{q})\big|_{\mathcal{O}_\lambda} = \sum_{\ell=0}^{\infty} \hat{\Psi}(F_{\hbar,\ell})(\hat{p}, \hat{q})\big|_{\mathcal{O}_\lambda} = \sum_{\ell=0}^{\infty} \Psi(F_{\hbar,\ell})(p, q)\,.$$

Note that the sum over $\ell$ is finite by Corollary 3.29. □

**Theorem 5.18** *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected real Lie group $G$. The product $\star \colon \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]] \times \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]] \to \mathscr{C}^\infty(\mathcal{O}_\lambda)[[\nu]]$ defined by $f \star g = \Psi(F)(f, g)$ where $F$ was obtained in Theorem 3.23 is a $G$-invariant formal star product. In particular, it is associative and deforms the KKS symplectic form on $\mathcal{O}_\lambda$. Furthermore, $p \star q$ coincides with the formal power series expansion of $p *_\hbar q$ around $\hbar = 0$ for $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, and $f \star g = \hat{f} \mathbin{\hat{\star}} \hat{g}\big|_{\mathcal{O}_\lambda}$ for $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$.*

PROOF: It is immediate from the definition of $F$ and $\Psi$ that every order of $\star$ is given by a $G$-invariant bidifferential operator. Since $F$ is the formal power series expansion of $F_\hbar$ around $\hbar = 0$ and $p *_\hbar q$ is rational with no pole at 0 for $p, q \in \mathrm{Pol}(\mathcal{O}_\lambda)$, it follows that $p \star q$ coincides with the formal power series expansion of $p *_\hbar q$. The compatibility with $\hat{\star}$ is immediate from Lemma 5.16. Since bidifferential operators are uniquely determined by their behaviour on $\mathrm{Pol}(\mathcal{O}_\lambda) \subseteq \mathcal{A}(\mathcal{O}_\lambda)$, the compatibility with $\hat{\star}$ implies that $\star$ is associative and, using Proposition 3.32, that it deforms the KKS symplectic form. $\square$

Recall that we proved in Corollary 3.34 that the product $\hat{*}_\hbar$ separates variables with respect to the distributions $L_+$ and $L_-$, which we call $\hat{L}_+$ and $\hat{L}_-$ in this section. In the real case, those distributions may have further properties. They can be real, or the holomorphic and antiholomorphic tangent spaces with respect to a complex structure. Before giving further details let us make the following definitions.

**Definition 5.19 (Star products of standard ordered type)** *A star product $*_\hbar$ on a symplectic manifold $M$ is said to be of* standard ordered type *if there are two Lagrangian distributions $L_1, L_2 \subseteq \mathrm{T}M$ spanning the real tangent bundle $\mathrm{T}M$ of $M$ such that the first argument of the star product is derived only in directions of $L_1$ and the second argument only in directions of $L_2$.*

**Definition 5.20 (Star products of (pseudo) Wick type)** *A star product $*_\hbar$ on a complex manifold $M$ that is also symplectic is said to be of* pseudo Wick type *if the first argument is derived only in holomorphic directions and the second argument only in antiholomorphic directions. A star product of pseudo Wick type on a Kähler manifold is said to be of* Wick type.

For formal star products of Wick type and with respect to the usual $^*$-involution given by complex conjugation, point evaluations are positive linear functionals, which is not necessarily the case for formal star products of pseudo Wick type. Note that the situation is more complicated for strict star products, as we shall see in Subsection 5.5.

Let us briefly recall some results on the existence of invariant complex structures on coadjoint orbits. See Appendix A.3 for more details. Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a real connected semisimple Lie group $G$ with Lie algebra $\mathfrak{g}$, and assume that $G_\lambda$ is compact. Choose a real Cartan subalgebra $\mathfrak{h}$ containing $\lambda^\sharp$. Since $\mathfrak{h} \subseteq \mathfrak{g}_\lambda$, it follows that $\mathfrak{h}$ is compact (meaning that it integrates to a subgroup of $G$ with compact closure). Then there are $G$-invariant complex structures on $\mathcal{O}_\lambda$, and these structures are in bijection to invariant orderings of $\hat{\Delta}$ (we say an ordering on $\hat{\Delta}$ is invariant if it is the restriction of an invariant ordering of $\Delta$ as defined in Definition 3.10) as follows. Recall that $\mathrm{T}^{\mathbb{C}}_\lambda \mathcal{O}_\lambda \cong \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\lambda \cong \bigoplus_{\alpha \in \hat{\Delta}} \mathfrak{g}^\alpha$. So given an invariant ordering we can define a map $I_\lambda \colon \mathrm{T}^{\mathbb{C}}_\lambda \mathcal{O}_\lambda \to \mathrm{T}^{\mathbb{C}}_\lambda \mathcal{O}_\lambda$ by letting $I_\lambda X_\alpha = \mathrm{i} X_\alpha$ if $\alpha \in \hat{\Delta}^+$, and $I_\lambda X_\alpha = -\mathrm{i} X_\alpha$ if $\alpha \in \hat{\Delta}^-$. The map $I_\lambda$ extends $G$-invariantly to an endomorphism $I$ of the complexified tangent bundle $\mathrm{T}^{\mathbb{C}} \mathcal{O}_\lambda$ and restricts to an endomorphism of the real tangent bundle $\mathrm{T}\mathcal{O}_\lambda$, thus it defines a complex structure.

If $G$ is compact, there is a unique ordering that makes $\mathcal{O}_\lambda$ with the complex structure $I$ and the KKS symplectic form $\omega_{\mathrm{KKS}}$ a Kähler manifold. This ordering is characterized by $\alpha \in \hat{\Delta}$ being positive iff $(\alpha, \mathrm{i}\lambda) > 0$. In particular it is standard. See Appendix A.3 for more details.

**Proposition 5.21** *For a semisimple coadjoint orbit $\mathcal{O}_\lambda$ of a real connected semisimple linear Lie group $G$, the product $*_\hbar$ obtained in Proposition 5.14*

*i.) has poles $P_\lambda \subseteq \mathbb{R}$ if $\mathfrak{h}$ is compact,*

*ii.) is of pseudo Wick type if $G_\lambda$ is compact and the same ordering is used in the construction of the star product and the definition of the complex structure,*

*iii.) is of standard ordered type with poles $P_\lambda \subseteq \mathrm{i}\mathbb{R}$ if $\mathrm{i}\mathfrak{h} \subseteq \hat{\mathfrak{g}}$ is compact.*

*In particular, if $G$ is compact and, in the construction of $*_\hbar$, we choose the ordering that makes $\mathcal{O}_\lambda$ with the induced complex structure $I$ a Kähler manifold, then $*_\hbar$ is of Wick type.*

PROOF: Roots take purely imaginary values on a compact Lie subalgebra of $\mathfrak{h}$. Since $\lambda \in \mathfrak{g}^*$ is by definition real on $\mathfrak{h} \subseteq \hat{\mathfrak{h}}$, it follows that $(\lambda, \mu) \in \mathrm{i}\mathbb{R}$ if $\mathfrak{h}$ is compact and $(\lambda, \mu) \in \mathbb{R}$ if $\mathrm{i}\mathfrak{h}$ is compact. Since $\frac{1}{2}(\mu,\mu) - (\rho,\mu) \in \mathbb{R}$, this implies that the roots (with respect to $\hbar$) of $p_{\mathrm{i}\lambda/\hbar}(\mu) = \frac{1}{2}(\mu,\mu) - (\rho,\mu) - \frac{\mathrm{i}}{\hbar}(\lambda,\mu)$ are real if $\mathfrak{h}$ is compact and purely imaginary if $\mathrm{i}\mathfrak{h}$ is compact.

Recall the definition of the distributions $L_+$ and $L_-$, which we denote by $\hat{L}_+$ and $\hat{L}_-$ in this section, made just after Lemma 3.33. Restricting them to $\mathcal{O}_\lambda \subseteq \hat{\mathcal{O}}_\lambda$ gives two distributions $L_+, L_- \subseteq \mathrm{T}^{\mathbb{C}}\mathcal{O}_\lambda$ of the complexified tangent bundle. An analogue of Proposition 2.8 in the real case and the explicit formula for $F_\hbar$ from Theorem 3.18 together with Remark 3.31 show that $\star$ derives the first argument only in directions of $L_+$, and the second argument only in directions of $L_-$.

Assume that $\mathfrak{g}_\lambda$ is compact. The holomorphic tangent space $\mathrm{T}^{(1,0)}_\lambda\mathcal{O}_\lambda$ is, under the isomorphism $\mathrm{T}^{\mathbb{C}}_\lambda\mathcal{O}_\lambda \cong \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\lambda$, spanned by $X_\alpha - \mathrm{i}I_\lambda X_\alpha$ for $\alpha \in \hat{\Delta}$. If $I_\lambda$ is defined using the ordering chosen in the construction of $*_\hbar$ as described above, then $X_\alpha - \mathrm{i}I_\lambda X_\alpha = X_\alpha - \mathrm{i}\cdot\mathrm{i}X_\alpha = 2X_\alpha$ if $\alpha \in \hat{\Delta}^+$, and $X_\alpha - \mathrm{i}I_\lambda X_\alpha = X_\alpha - \mathrm{i}\cdot(-\mathrm{i})X_\alpha = 0$ if $\alpha \in \hat{\Delta}^-$, so $\mathrm{T}^{(1,0)}_\lambda\mathcal{O}_\lambda = \operatorname{span}\{(X_\alpha)_{\mathcal{O}_\lambda}\big|_\lambda, \alpha \in \hat{\Delta}^+\}$. This coincides exactly with $L_+\big|_\lambda$, and by $G$-invariance it follows that $L_+$ coincides with $\mathrm{T}^{(1,0)}\mathcal{O}_\lambda$. Similarly, $L_-$ coincides with $\mathrm{T}^{(0,1)}\mathcal{O}_\lambda$. Therefore $\star$ is of pseudo Wick type.

If $\mathrm{i}\mathfrak{h}$ is compact, then every $\mathrm{ad}_H$ for $H \in \mathfrak{h}$ is self-adjoint. Since they are all commuting we can find simultaneous eigenvectors in $\mathfrak{g}$ of all $\mathrm{ad}_H$ (without complexifying $\mathfrak{g}$). But then we can pick $X_\alpha$ and $Y_\alpha$ to lie in $\mathfrak{g}$ so that $L_1 = L_+ \cap \mathfrak{g}$ and $L_2 = L_- \cap \mathfrak{g}$ are Lagrangian distributions satisfying Definition 5.19. □

**Remark 5.22** Assume that $\mathfrak{g}_\lambda$ is compact as in part *ii.)* of the previous proposition. If one uses different invariant orderings in the construction of the star product and in the definition of a complex structure, then the distributions $L_+$ and $L_-$ may both contain holomorphic and antiholomorphic directions. Since we are mainly interested in star products of (pseudo) Wick type (these are the ones for which we would hope to find positive linear functionals on the star product algebra, see Subsection 5.5), we will usually assume that the two orderings agree.

## 5.4 Examples: complex projective spaces and hyperbolic discs

Recall that we have computed the canonical element of the Shapovalov pairing for $\mathrm{SL}_{1+n}(\mathbb{C})$ and a certain choice of $\lambda$ in Subsection 3.4. Let us now specialize this result to the real forms $\mathrm{SU}(1+n)$ and $\mathrm{SU}(1,n)$.

**Example 5.23 ($\mathbb{CP}^n$)** The coadjoint orbit of $\mathrm{SU}(1+n)$ through $\lambda\colon \mathfrak{su}_{1+n} \to \mathbb{R}$, $X \mapsto -\mathrm{i}rX_{0,0}$ with $r \in \mathbb{R}^+$ is the complex projective space $\mathbb{CP}^n$. $\mathrm{SL}_{1+n}(\mathbb{C})$ is a complexification of $\mathrm{SU}(1+n)$. Using the notation $\hat{\mathfrak{h}}$ for the Cartan subalgebra of $\mathfrak{sl}_{1+n}(\mathbb{C})$ introduced in Subsection 3.4, we obtain a compact Cartan subalgebra $\mathfrak{h} := \mathfrak{su}_{1+n} \cap \hat{\mathfrak{h}}$ of $\mathfrak{su}_{1+n}$. Proposition A.10 tells us that the Kähler complex structure is defined by the ordering of $\hat{\Delta}$ for which $\alpha \in \hat{\Delta}^+$ iff $(\mathrm{i}\lambda, \alpha) > 0$. This ordering is the restriction of the ordering on $\Delta$ for which all $\alpha_{i,j}$ with $i < j$ are positive. Therefore the element $F_\hbar$ from Proposition 3.36 induces a Wick type star product on $\mathbb{CP}^n$. This product has poles at $\{\frac{1}{n}r \mid n \in \mathbb{N}\}$.

**Example 5.24 ($\mathbb{D}^n$)** Denote the complex hyperbolic disc in $n$ dimensions by $\mathbb{D}^n$. Recall that $\mathrm{SU}(1,n)$ denotes the group of isometries of the indefinite scalar product $g(v,w) = -v_0w_0 + \sum_{i=1}^n v_iw_i$ on $\mathbb{R}^{1+n}$. The coadjoint orbit of $\mathrm{SU}(1,n)$ through $\lambda\colon \mathfrak{su}_{1,n} \to \mathbb{R}$, $X \mapsto -\mathrm{i}rX_{0,0}$ with $r \in \mathbb{R}^+$ is the hyperbolic disc $\mathbb{D}^n$. $\mathrm{SL}_{1+n}(\mathbb{C})$ is a complexification of $\mathrm{SU}(1,n)$. Again, $\mathfrak{h} := \mathfrak{su}_{1,n} \cap \hat{\mathfrak{h}}$ defines a compact Cartan subalgebra of $\mathfrak{su}_{1,n}$. Now all roots are non-compact, so that according to Corollary A.11 the Kähler complex structure is defined by the ordering on $\hat{\Delta}$ for which $\alpha \in \hat{\Delta}^+$ iff $(\mathrm{i}\lambda, \alpha) < 0$. This ordering is the restriction of the ordering on $\Delta$ for which all $\alpha_{i,j}$ with $i > j$ are positive. Therefore the element $F_\hbar$ from Corollary 3.37 induces a Wick type star product on $\mathbb{D}^n$. This product has poles at $\{-\frac{1}{n}r \mid n \in \mathbb{N}\}$.

**Remark 5.25** A star product of Wick type on the hyperbolic disc was also studied in [29], where it was obtained from a star product of Wick type on $\mathbb{C}^{1+n}$ using phase space reduction. This product coincides[3] with the star product obtained in Example 5.24. To see this, one checks that monomials of degree 1 generate the star product algebra, so that it suffices to compare the two formulas for a degree 1 monomial and an arbitrary monomial. But for a degree 1 monomial only very few summands are non-zero in both constructions and one can explicitly check that the expressions agree.

## 5.5 Positive linear functionals

In this subsection we prove that for certain coadjoint orbits and certain values of $\hbar$ the point evaluation functionals of the star product algebras constructed in Subsection 5.3 are positive. In order to have a meaningful notion of positivity we need a star involution on $(\mathcal{A}(\mathcal{O}_\lambda), *_\hbar)$. Of course, this star involution should be the restriction of the complex conjugation of $\mathscr{C}^\infty(\mathcal{O}_\lambda)$, but we need to prove that this restriction is well-defined.

Assume that $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ is the complexification of a Lie algebra $\mathfrak{g}$. The complex conjugation $\overline{\,\cdot\,}\colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, $X \otimes z \mapsto X \otimes \overline{z}$ is an antilinear involution on $\hat{\mathfrak{g}}$. Then $\overline{\,\cdot\,}\colon \hat{\mathfrak{g}}^* \to \hat{\mathfrak{g}}^*$, $\phi \mapsto \overline{\phi} := \overline{\,\cdot\,} \circ \phi \circ \overline{\,\cdot\,}$ defines an antilinear involution on $\hat{\mathfrak{g}}^*$. Note that on the right hand side, we first apply the involution of $\hat{\mathfrak{g}}$, then $\phi$, and then the complex conjugation of $\mathbb{C}$. Therefore the right hand side defines a complex linear functional $\overline{\phi} \in \hat{\mathfrak{g}}^*$. The map $\phi \mapsto \overline{\phi}$ is antilinear.

[3]More generally, the products obtained in Paper I coincide with the products obtained by considering different real forms of $\mathrm{SL}_{1+n}(\mathbb{C})$. We show this in the Appendix.

**Lemma 5.26** *Let $G \subseteq \mathrm{GL}_N(\mathbb{R})$ be a real linear Lie group with complexification $\hat{G} \subseteq \mathrm{GL}_N(\mathbb{C})$, assume $\lambda \in \mathfrak{g}^*$, and let $\hat{\mathcal{O}}_\lambda$ be the coadjoint orbit of $\hat{G}$ through $\lambda$. Then the map $\overline{\cdot}\colon \hat{\mathfrak{g}}^* \to \hat{\mathfrak{g}}^*$ restricts to an antilinear involution $\overline{\cdot}\colon \hat{\mathcal{O}}_\lambda \to \hat{\mathcal{O}}_\lambda$.*

PROOF: Note that since $\lambda \in \mathfrak{g}^*$ we have $\overline{\lambda} = \lambda$. Therefore we compute

$$\overline{\mathrm{Ad}_g^* \lambda} = \overline{\lambda \circ \mathrm{Ad}_{g^{-1}}} = \overline{\lambda} \circ \overline{\cdot} \circ \mathrm{Ad}_{g^{-1}} \circ \overline{\cdot} = \lambda \circ \mathrm{Ad}_{\overline{g}^{-1}} = \mathrm{Ad}_{\overline{g}}^* \lambda\,.$$

Here $\overline{g}$ denotes the entrywise complex conjugate of $g \in \hat{G}$. Since the exponential map $\hat{\mathfrak{g}} \to \hat{G}$ commutes with the complex conjugation, it follows that $\hat{G}$ is closed under entrywise complex conjugation, and therefore $\overline{g} \in \hat{G}$ and $\mathrm{Ad}_{\overline{g}}^* \lambda \in \hat{\mathcal{O}}_\lambda$. This proves that $\overline{\cdot}$ restricts to $\hat{\mathcal{O}}_\lambda$, and the restriction is clearly still an antilinear involution. □

Note that $\mathrm{T}_\xi \overline{\cdot} \circ I_\xi = (I_{\overline{\xi}})^{-1} \circ \mathrm{T}_\xi \overline{\cdot}$ holds for $\xi \in \hat{\mathfrak{g}}^*$, where $\mathrm{T}_\xi \overline{\cdot}\colon \mathrm{T}_\xi \hat{\mathfrak{g}}^* \to \mathrm{T}_{\overline{\xi}} \hat{\mathfrak{g}}^*$ is the tangent map to the complex conjugation of $\hat{\mathfrak{g}}^*$ and $I_\xi\colon \mathrm{T}_\xi \hat{\mathfrak{g}}^* \to \mathrm{T}_\xi \hat{\mathfrak{g}}^*$ is the complex structure at $\xi$. Since the complex structure $I$ and the complex conjugation $\overline{\cdot}$ of $\mathcal{O}_\lambda$ are both obtained by restriction from $\hat{\mathfrak{g}}^*$, they satisfy the same relation.

For any $f \in \mathrm{Hol}(\hat{\mathcal{O}}_\lambda)$ consider the function $f^* := \overline{\cdot} \circ f \circ \overline{\cdot}$, where the left $\overline{\cdot}$ is the complex conjugation of $\mathbb{C}$ and the right $\overline{\cdot}$ is the antilinear involution obtained in the previous lemma. Denote the complex structure of $\mathbb{C}$ by $J$, and identify the tangent space of $\mathbb{C}$ with $\mathbb{C}$. Then

$$\begin{aligned} \mathrm{T}_\xi f^* \circ I_\xi &= \overline{\cdot} \circ \mathrm{T}_{\overline{\xi}} f \circ \mathrm{T}_\xi \overline{\cdot} \circ I_\xi = \overline{\cdot} \circ \mathrm{T}_{\overline{\xi}} f \circ I_{\overline{\xi}}^{-1} \circ \mathrm{T}_\xi \overline{\cdot} = \\ &= \overline{\cdot} \circ J^{-1} \circ \mathrm{T}_{\overline{\xi}} f \circ \mathrm{T}_\xi \overline{\cdot} = J \circ \overline{\cdot} \circ \mathrm{T}_{\overline{\xi}} f \circ \mathrm{T}_\xi \overline{\cdot} = J \circ \mathrm{T}_\xi f^* \end{aligned}$$

shows that $f^*$ is holomorphic. Since $\overline{\cdot}$ restricts to the identity on $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$, it follows that $f^*|_{\mathcal{O}_\lambda} = \overline{f|_{\mathcal{O}_\lambda}}$. Consequently, the restriction of ${}^*\colon \mathrm{Hol}(\hat{\mathcal{O}}_\lambda) \to \mathrm{Hol}(\hat{\mathcal{O}}_\lambda)$ to $\mathcal{A}(\mathcal{O}_\lambda)$ is just the complex conjugation $\overline{\cdot}\colon \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$. In other words, the complex conjugation is well-defined on $\mathcal{A}(\mathcal{O}_\lambda)$.

**Proposition 5.27** *Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a connected semisimple real Lie group $G$. Assume that the Cartan subalgebra $\mathfrak{h}$ used in the construction of a star product $*_\hbar$ is compact. Then $\overline{f *_\hbar g} = \overline{g} *_{\overline{\hbar}} \overline{f}$ holds for all $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$.*

PROOF: As in the proof of Proposition 5.21 one argues that since $\mathfrak{h}$ is compact the coefficients $p^w_{\mathrm{i}\lambda}(\alpha_w)$ are real and more generally $\overline{p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)} = p^w_{\mathrm{i}\lambda/\overline{\hbar}}(\alpha_w)$. From (A.3) we obtain that $\overline{X_\alpha \otimes Y_\alpha} = Y_\alpha \otimes X_\alpha = \tau(X_\alpha \otimes Y_\alpha)$ for both a compact and a non-compact root $\alpha \in \hat{\Delta}^+$, and the same formula holds when $\alpha$ is replaced by a word $w \in \tilde{W}$. Here $\overline{\cdot}$ is the complex conjugation of $\hat{\mathfrak{g}}$ with respect to $\mathfrak{g}$, extended to $(\mathscr{U}\hat{\mathfrak{g}})^{\otimes 2}$, and $\tau\colon (\mathscr{U}\hat{\mathfrak{g}})^{\otimes 2} \to (\mathscr{U}\hat{\mathfrak{g}})^{\otimes 2}$ is the flip of the two tensor factors. Note that $\tau$ stays well-defined on $(\mathscr{U}\hat{\mathfrak{g}}/\mathscr{U}\hat{\mathfrak{g}} \cdot \hat{\mathfrak{g}}_\lambda)^{\otimes 2}$, and therefore the formula for $F_\hbar$ obtained in Theorem 3.18, Remark 3.31, and the computations above imply $\overline{F_{\hbar,\ell}} = \tau(F_{\overline{\hbar},\ell})$. Consequently

$$\overline{f *_\hbar g} = \sum_{\ell=0}^{\infty} \overline{\Psi(F_{\hbar,\ell})(f,g)} = \sum_{\ell=0}^{\infty} \Psi(\overline{F_{\hbar,\ell}})(\overline{f},\overline{g}) =$$

$$= \sum_{\ell=0}^{\infty} \Psi(\tau(F_{\overline{\hbar},\ell}))(\overline{f},\overline{g}) = \sum_{\ell=0}^{\infty} \Psi(F_{\overline{\hbar},\ell})(\overline{g},\overline{f}) = \overline{g} *_{\overline{\hbar}} \overline{f}$$

holds for all $f, g \in \operatorname{Pol}(\mathcal{O}_\lambda)$ and extends to $\mathcal{A}(\mathcal{O}_\lambda)$ by continuity. $\square$

A linear functional $\phi$ on a $^*$-algebra $\mathscr{A}$ is said to be positive if $\phi(a^*a) \geq 0$ for all $a \in \mathscr{A}$. In the following we formulate our results for the star algebra $\mathscr{A}_\hbar := (\mathcal{A}(\mathcal{O}_\lambda), *_\hbar, \overline{\cdot})$, but would like to point out that they also hold for $(\operatorname{Pol}(\mathcal{O}_\lambda), *_\hbar, \overline{\cdot})$.

**Theorem 5.28** *Assume that $\mathcal{O}_\lambda$ is a semisimple coadjoint orbit of a real connected semisimple Lie group $G$. Assume further that $\mathfrak{h}$ is a compact Cartan subalgebra, and that all roots (with respect to the complexification $\hat{\mathfrak{h}}$ of $\mathfrak{h}$) in $\hat{\Delta}$ are non-compact. Let $*_\hbar$ be the star product constructed with respect to the ordering for which $\alpha \in \hat{\Delta}$ is positive if and only if $(\alpha, \mathrm{i}\lambda) < 0$. Then there is a constant $M > 0$ such that for all $\xi \in \mathcal{O}_\lambda$ and $\hbar \in (0, M) \setminus P_\lambda$ the point evaluation at $\xi$ is a positive linear functional $\mathrm{ev}_\xi \colon \mathscr{A}_\hbar \to \mathbb{C}$.*

PROOF: Since $(\alpha, \mathrm{i}\lambda) < 0$ for all $\alpha \in \hat{\Delta}^+$, it follows that $-\mathrm{i}(\lambda, \mu) > 0$ holds for all $\mu \in \mathbb{N}_0\hat{\Delta}^+ \setminus \{0\}$. There are only finitely many $\mu \in \mathbb{N}_0\hat{\Delta}^+$ with $(\rho, \mu) - \frac{1}{2}(\mu, \mu) > 0$, thus we can choose $M > 0$ such that $-\frac{\mathrm{i}}{\hbar}(\lambda, \mu) > (\rho, \mu) - \frac{1}{2}(\mu, \mu)$ holds for all $\mu \in \mathbb{N}_0\hat{\Delta}^+ \setminus \{0\}$ and $\hbar \in (0, M) \setminus P_\lambda$. But this says precisely that $p_{\mathrm{i}\lambda/\hbar}(\mu) > 0$, and therefore $p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w) > 0$ for all $w \in \tilde{W}$. For a non-compact root we have $\overline{X_\alpha} = Y_\alpha$ according to (A.3b). Consequently, if $g \in G$ is such that $\xi = \mathrm{Ad}^*_g(\lambda)$, then

$$\begin{aligned}
\mathrm{ev}_\xi(f *_\hbar \overline{f}) &= \sum_{\ell=0}^{\infty} \Psi\Bigg( \sum_{w \in \tilde{W}_\ell} p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} \tilde{\pi}^+(X_w) \otimes \tilde{\pi}^-(Y_w) \Bigg)(f, \overline{f})(\xi) \\
&= \sum_{\ell=0}^{\infty} \sum_{w \in \tilde{W}_\ell} p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} X^{\mathrm{left}}_w(\pi^* f)(g) \cdot Y^{\mathrm{left}}_w(\pi^* \overline{f})(g) \\
&= \sum_{\ell=0}^{\infty} \sum_{w \in \tilde{W}_\ell} p^w_{\mathrm{i}\lambda/\hbar}(\alpha_w)^{-1} X^{\mathrm{left}}_w(\pi^* f)(g) \cdot \overline{X^{\mathrm{left}}_w(\pi^* f)(g)} \\
&\geq 0
\end{aligned}$$

holds for all $f \in \mathcal{A}(\mathcal{O}_\lambda)$. $\square$

**Example 5.29 ($\mathbb{D}^n$)** It is straightforward to check that the choices made to quantize the hyperbolic disc in Example 5.24 are such that $\mathfrak{h}$ is compact, such that every root in $\hat{\Delta}$ is non-compact, and such that $\alpha \in \hat{\Delta}$ is positive iff $(\alpha, \mathrm{i}\lambda) < 0$. Therefore the previous theorem implies the existence of a constant $M > 0$ such that all point evaluation functionals are positive if $\hbar \in (0, M)$.

We can prove a stronger result by using the formula for $F_\hbar$ derived in Corollary 3.37. If $\hbar \in (0, \infty)$ then all the coefficients appearing in this formula are positive, and so point evaluations are positive for all $\hbar \in (0, \infty)$.

Note that a similar proof does not work for $\mathbb{CP}^n$ since some of the coefficients in (3.39) are negative. Indeed, one can use the appearing negative coefficients to show that no point evaluation functional is positive on $\mathbb{CP}^n$ for $\hbar \in (0, \infty) \setminus P_\lambda$.

## 5.6 A generalized Wick rotation

In this subsection we want to state an immediate corollary of the construction in the previous sections. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two real semisimple Lie algebras with the same complexification $\hat{\mathfrak{g}}$. Assume $\lambda \in \mathfrak{g}_1^* \cap \mathfrak{g}_2^*$ where we view $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ as subspaces of $\hat{\mathfrak{g}}^*$. Denote the coadjoint orbits in $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ through $\lambda$ by $\mathcal{O}_\lambda^1$ and $\mathcal{O}_\lambda^2$, respectively. There is an isomorphism $\mathrm{Pol}(\mathcal{O}_\lambda^1) \to \mathrm{Pol}(\mathcal{O}_\lambda^2)$ given by composing the map $\mathrm{Pol}(\mathcal{O}_\lambda^1) \ni p \mapsto \hat{p} \in \mathrm{Pol}(\hat{\mathcal{O}}_\lambda)$ with the restriction to $\mathcal{O}_\lambda^2$. Here $\hat{\mathcal{O}}_\lambda$ is the complex extension of $\mathcal{O}_\lambda$. It turns out that this isomorphism is still an isomorphism of both the uncompleted and completed quantum algebras.

**Theorem 5.30** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two real semisimple Lie algebras with a common complexification $\hat{\mathfrak{g}}$ and assume that $\lambda \in \mathfrak{g}_1^* \cap \mathfrak{g}_2^*$ is semisimple. Then the algebras $(\mathrm{Pol}(\mathcal{O}_\lambda^1), *_\hbar^1)$ and $(\mathrm{Pol}(\mathcal{O}_\lambda^2), *_\hbar^2)$, and also the algebras $(\mathcal{A}(\mathcal{O}_\lambda^1), *_\hbar^1)$ and $(\mathcal{A}(\mathcal{O}_\lambda^2), *_\hbar^2)$, constructed with respect to the same Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_1 \cap \mathfrak{g}_2$ and the same ordering, are isomorphic.*

PROOF: Both algebras are isomorphic to $(\mathrm{Pol}(\hat{\mathcal{O}}_\lambda), \hat{*}_\hbar)$ or $(\mathrm{Hol}(\hat{\mathcal{O}}_\lambda), \hat{*}_\hbar)$. □

**Example 5.31 ($\mathbb{CP}^n$ and $\mathbb{D}^n$)** We know from Example 5.23 and Example 5.24 that $\mathbb{CP}^n$ and $\mathbb{D}^n$ are coadjoint orbits of the Lie groups $\mathrm{SU}(1+n)$ and $\mathrm{SU}(1,n)$ through the same element, and that $\mathrm{SL}_{1+n}(\mathbb{C})$ is a common complexification. So the previous proposition implies that the star product algebras on $\mathbb{CP}^n$ and $\mathbb{D}^n$ are isomorphic if we choose the same ordering in the construction of the star products.

The ordering that induces a Kähler complex structure on $\mathbb{CP}^n$, induces the complex structure on $\mathbb{D}^n$ that is the opposite of the Kähler complex structure. Therefore the associated star product on $\mathbb{D}^n$ is of pseudo Wick type with respect to this opposite complex structure, and therefore of anti-Wick type for the Kähler complex structure. (A star product is of anti-Wick type if the first argument is derived in antiholomorphic directions and the second argument is derived in holomorphic ones.) Consequently, the algebra $\mathcal{A}(\mathbb{CP}^n)$ with the Wick type star product is isomorphic to the algebra $\mathcal{A}(\mathbb{D}^n)$ with the anti-Wick type star product. Similarly, the algebra $\mathcal{A}(\mathbb{CP}^n)$ with the anti-Wick type star product is isomorphic to the algebra $\mathcal{A}(\mathbb{D}^n)$ with the Wick type star product.

One can also construct an isomorphism between the Wick type star product for $\hbar$ and the anti-Wick type star product for $-\hbar$, both on the hyperbolic disc and the complex projective space. Composing with these isomorphisms shows that the Wick type star product for $\hbar$ on $\mathbb{CP}^n$ is isomorphic to the Wick type star product for $-\hbar$ on $\mathbb{D}^n$.

Note that Theorem 5.30 only gives an algebra homomorphism between $\mathrm{Pol}(\mathcal{O}_\lambda^1)$ and $\mathrm{Pol}(\mathcal{O}_\lambda^2)$, or between $\mathcal{A}(\mathcal{O}_\lambda^1)$ and $\mathcal{A}(\mathcal{O}_\lambda^2)$. If we view these algebras as $^*$-algebras with the star involution considered in Subsection 5.5 then they are in general not $^*$-isomorphic! One can see this for example by proving that the point evaluation functionals on $\mathbb{CP}^n$ are not positive for $\hbar \in (0, \infty) \setminus P_\lambda$.

# A Proofs, $G$-finite functions, and complex structures

In Appendix A.1 we prove Proposition 2.5 and Proposition 2.7. In Appendix A.2 we prove Proposition 3.27 using the concept of $G$-finite functions. Finally we recall some facts about complex structures on coadjoint orbits in Appendix A.3.

## A.1 Proofs of Proposition 2.5 and Proposition 2.7

Let $M$ be a manifold. For $f \in \mathscr{C}^\infty(M)$ we define $M_f \colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, $f' \mapsto ff'$ and $M_f^i = \mathsf{id}^{\times(i-1)} \times M_f \times \mathsf{id}^{\times(k-i)} \colon \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M)^k$.

**Definition A.1** *Let $M$ be a manifold. For a multiindex $K = (K_1, \dots, K_k) \in \mathbb{Z}^k$ we define $k$-$\mathrm{DiffOp}_K(M) = \{0\}$ if some $K_i < 0$ and otherwise we define inductively*

$$k\text{-}\mathrm{DiffOp}_K(M) = \{D \colon \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M) \mid M_f \circ D - D \circ M_f^i \in k\text{-}\mathrm{DiffOp}_{K-E_i}(M) \text{ for all } f \in \mathscr{C}^\infty(M) \text{ and } 1 \leq i \leq k\}. \tag{A.1}$$

*Here $(K - E_i)_j = K_j - \delta_{ij}$ where $\delta_{ij}$ is $1$ if $i = j$ and $0$ otherwise. Elements of $k$-$\mathrm{DiffOp}_K(M)$ are called* $k$-differential operators of degree $K$. *A map $D \colon \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M)$ is said to be a* $k$-differential operator *if it is a $k$-differential operator of some degree $K$. The space of $k$-differential operators is denoted by $k$-$\mathrm{DiffOp}(M)$.*

It follows that a $k$-differential operator is local in every argument, so that it can be restricted to any open subset. In a chart $U \subseteq M$ with local coordinates $(x^1, \dots, x^n)$, a $k$-differential operator $D$ of degree $K$ can be written as

$$D(f_1, \dots, f_k) = \sum\nolimits_{I_1,\dots,I_k \in \mathbb{N}_0^n} c_{I_1,\dots,I_k} \partial_x^{I_1} f_1 \cdot \ldots \cdot \partial_x^{I_k} f_k \tag{A.2}$$

where $c_{I_1,\dots,I_k} \in \mathscr{C}^\infty(M)$ and $c_{I_1,\dots,I_k} = 0$ if $|I_i| > K_i$ for some $1 \leq i \leq k$. For a multiindex $J \in \mathbb{N}_0^n$ we used $\partial_x^J := \partial_{x^1}^{J_1} \dots \partial_{x^n}^{J_n}$ and $\partial_{x^i} := \frac{\partial}{\partial x^i}$. Conversely, an operator $D \colon \mathscr{C}^\infty(M)^k \to \mathscr{C}^\infty(M)$ that has this form in any chart is $k$-differential of order $K$. A $k$-differential operator $D$ on a complex manifold $M$ is holomorphic if, in local holomorphic coordinates $(z^1, \dots, z^n)$, we have

$$D(f_1, \dots, f_k) = \sum\nolimits_{I_1,\dots,I_k \in \mathbb{N}_0^n} c_{I_1,\dots,I_k} \partial_z^{I_1} f_1 \cdot \ldots \cdot \partial_z^{I_k} f_k$$

with all $c_{I_1,\dots,I_k}$ being holomorphic. Here $\partial_z^J = \partial_{z^1}^{J_1} \dots \partial_{z^n}^{J_n}$ and $\partial_{z^i} = \frac{\partial}{\partial z^i}$. Equivalently, $D$ is holomorphic if $D$ maps $\mathrm{Hol}(U)^k$ into $\mathrm{Hol}(U)$ and $D\big|_U \circ M_{\bar f}^i - M_{\bar f} \circ D\big|_U = 0$ for all open subsets $U \subseteq M$ and all antiholomorphic functions $f$ on $U$. We write $k$-$\mathrm{DiffOp}_{\mathcal{H}}(M)$ for the space of holomorphic $k$-differential operators.

We say a $k$-differential operator is of order $K \in \mathbb{Z}^k$ at a point $p \in M$ if, when written in a local chart $U$ around $p$ as in (A.2), we have $c_{I_1,\dots,I_k}(p) = 0$ whenever $|I_j| > K_j$ for some $1 \leq j \leq k$.

If $I_1, \dots, I_k, J, K \in \mathbb{N}_0^n$ are all multiindices, we write $J \leq K$ if $J_i \leq K_i$ for all $1 \leq i \leq n$. If $X_1, \dots, X_n \in \mathfrak{g}$, then we use $X^J$ as a shorthand for $X_1^{J_1} \dots X_n^{J_n} \in \mathscr{U}\mathfrak{g}$ and $X^{I_1 \otimes \dots \otimes I_k}$ as a shorthand for $X^{I_1} \otimes \dots \otimes X^{I_k} \in (\mathscr{U}\mathfrak{g})^{\otimes k}$.

Proof of Proposition 2.5: Choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$. It follows from the Poincaré–Birkhoff–Witt theorem that $\{X^{I_1 \otimes \dots \otimes I_k} \mid I_1, \dots, I_k \in \mathbb{N}_0^n\}$ is a basis of $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Moreover, $\{X_1^{\mathrm{left},(1,0)}\big|_e, \dots, X_n^{\mathrm{left},(1,0)}\big|_e\}$ is a basis of the tangent space $\mathrm{T}_e^{(1,0)} G$ and we can choose a complex chart $U$ around $e$ with local coordinates $(z^1, \dots, z^n)$ such that $\partial_{z^i}\big|_e = X_i^{\mathrm{left},(1,0)}\big|_e$.

Assume $\vec{u} = \sum_{I_1,\dots,I_k \in \mathbb{N}_0^n} c_{I_1,\dots,I_k} X^{I_1 \otimes \dots \otimes I_k} \neq 0$ with only finitely many $c_{I_1,\dots,I_k} \neq 0$. Choose $I_1, \dots, I_k$ in such a way that $c_{I_1,\dots,I_k} \neq 0$ and $c_{J_1,\dots,J_k} = 0$ whenever $I_i \leq J_i$ and $(I_1, \dots, I_k) \neq (J_1, \dots J_k)$. For $\vec{f} = (z^{I_1}, \dots, z^{I_k}) \in \mathscr{C}^\infty(U)^{\times k}$ we compute $\vec{u}^{\mathrm{left},(1,0)} \vec{f}(e) = I_1! \dots I_k! c_{I_1,\dots,I_k} \neq 0$. So $\vec{u}^{\mathrm{left},(1,0)} \neq 0$ and $(\,\cdot\,)^{\mathrm{left},(1,0)}$ is injective.

Note that $(X^{I_1})^{\mathrm{left},(1,0)} f_1 \cdot \ldots \cdot (X^{I_k})^{\mathrm{left},(1,0)} f_k = \partial_z^{I_1} f_1 \cdot \ldots \cdot \partial_z^{I_k} f_k + D'(f_1, \dots, f_k)$ where $D'$ is a holomorphic $k$-differential operator whose order at $e$ is strictly smaller than $(|I_1|, \dots, |I_k|)$. For any holomorphic $k$-differential operator $D$ we can therefore, by induction, find coefficients $c_{I_1,\dots,I_k} \in \mathbb{C}$, only finitely many of which are non-zero, such that

$$D(f_1, \dots, f_k)(e) = \sum_{I_1,\dots,I_k \in \mathbb{N}_0^n} c_{I_1,\dots,I_k} (X^{I_1})^{\mathrm{left},(1,0)} f_1(e) \cdot \ldots \cdot (X^{I_k})^{\mathrm{left},(1,0)} f_k(e)$$

holds for all $f_1, \dots, f_k \in \mathscr{C}^\infty(G)$. In other words, $D$ and the differential operator $\sum_{I_1,\dots,I_k \in \mathbb{N}_0^n} (c_{I_1,\dots,I_k} X^{I_1 \otimes \dots \otimes I_k})^{\mathrm{left},(1,0)}$ agree at $e$. So if $D$ is also left-invariant, then these operators agree everywhere on $G$, proving surjectivity. $\square$

The proof of Proposition 2.7 is similar. We need the following lemma to simplify the local calculations.

**Lemma A.2** *Let $G$ be a complex Lie group with Lie algebra $\mathfrak{g}$, and assume that $H$ is a closed complex Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$. Given a basis $B = \{X_1, \dots, X_n\}$ of $\mathfrak{g}$ such that $B' = \{X_{n-r+1}, \dots, X_n\}$ is a basis of $\mathfrak{h}$ we can choose a neighbourhood $U$ of $e$ in $G$ and complex coordinates $z = (z^1, \dots, z^n)$ on $U$ such that*

*i.) for any $g \in U$ its fiber $gH \cap U$ is given locally as $(\{z(g)\} + \{0\} \times \mathbb{C}^r) \cap z(U)$,*

*ii.) the left-invariant holomorphic vector fields agree with coordinate vector fields at $e \in G$, that is $X_i^{\mathrm{left},(1,0)}\big|_e = \partial_{z^i}\big|_e$.*

Proof: It is well known that $\pi\colon G \to G/H$ is a principal bundle. Therefore we can choose a local trivialization $\chi\colon \pi^{-1}(V) \to V \times H$ on a small neighbourhood $V$ of $eH$ in $G/H$. Choosing coordinates on $V$ (after possibly shrinking $V$ first) and on a neighbourhood $W$ of the identity in $H$, we obtain coordinates $z'$ on $U := \chi^{-1}(V \times W) \subseteq G$ satisfying property *i.)*. Since all $X_i^{\mathrm{left},(1,0)}$ are linearly independent we can write $X_i^{\mathrm{left},(1,0)}\big|_e = A_{ij} \partial_{(z')^j}\big|_e$ for some invertible matrix $A$ and since $X_i^{\mathrm{left},(1,0)}$ is tangential to $H \subseteq G$ for $i > n-r$, it follows that $A_{ij} = 0$ for $i > n-r$, $j \leq n-r$. Then the coordinates $z := (A^{-1})^{\mathrm{T}} z'$ satisfy both properties of the lemma. $\square$

Let $\pi\colon G \to G/H$. Given coordinates as in the previous lemma we may identify $\pi(U)$ locally with $\{(z^1(g), \dots, z^{n-r}(g), 0, \dots, 0) \mid g \in U\}$. Then $(z^1, \dots, z^{n-r})$ descend to coordinates on $\pi(U)$ and $\pi$ is, with respect to these coordinates, given by the projection to the first $n-r$ coordinates.

**Lemma A.3** *The map $\Psi$ from Proposition 2.7 is injective.*

PROOF: Let $r = \dim \mathfrak{h}$ and $n = \dim \mathfrak{g} \geq r$. We can choose a basis $B = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that $B' = \{X_{n-r+1}, \ldots, X_n\}$ is a basis of $\mathfrak{h}$. Recall from the proof of Proposition 2.5 that $\{X^{I_1 \otimes \ldots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n\}$ is a basis of $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Furthermore,

$$\{X^{I_1 \otimes \ldots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n, (I_i)_j > 0 \text{ for some } 1 \leq i \leq k \text{ and some } j > n - r\}$$

is a basis of the ideal $I$ defined just before Lemma 2.6 and

$$\begin{aligned}\{X^{I_1 \otimes \ldots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^n, (I_i)_j = 0 \text{ for all } 1 \leq i \leq k,\ j > n - r\} = \\ = \{X^{I_1 \otimes \ldots \otimes I_k} \mid I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}\}\end{aligned}$$

is a basis of a complement $C$ of $I$ in $(\mathscr{U}\mathfrak{g})^{\otimes k}$. Injectivity of $\Psi$ means that 0 is the only element of $C$ on which $\Psi$ vanishes.

So to prove that $\Psi$ is injective, it suffices to find, for any non-zero

$$\vec{u} = \sum_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} c_{I_1, \ldots, I_k} X^{I_1 \otimes \ldots \otimes I_k} \in C\,,$$

some open subset $U \subseteq G/H$ and some $k$-tuple of functions $\vec{f} \in \mathscr{C}^\infty(U)^k$ such that $\Psi([\vec{u}])(\vec{f}) \neq 0$. Fix $\vec{u} \in C \setminus \{0\}$ and assume that $I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}$ are chosen such that $c_{I_1, \ldots, I_k} \neq 0$ and such that for any multiindices $J_1, \ldots, J_k \in \mathbb{N}_0^{n-r}$ satisfying $I_i \leq J_i$ and $(I_1, \ldots, I_k) \neq (J_1, \ldots, J_k)$ we have $c_{J_1, \ldots, J_k} = 0$. Choose coordinates $z = (z^1, \ldots, z^n)$ around $e$ on $G$ as in the previous lemma, and note that, as described just after this lemma, $(z^1, \ldots, z^{n-r})$ descend to coordinates $(y^1, \ldots, y^{n-r})$ on $G/H$. Set $\vec{f} = (y^{I_1}, \ldots, y^{I_k})$, so that $\pi^* \vec{f} = (z^{I_1}, \ldots, z^{I_k})$. This implies that

$$\Psi([\vec{u}])(\vec{f})(eH) = \vec{u}^{\mathrm{left},(1,0)}(\pi^* \vec{f})(e) = I_1! \ldots I_k! c_{I_1, \ldots, I_k} \neq 0\,. \qquad \square$$

**Lemma A.4** *The map $\Psi$ from Proposition 2.7 is surjective.*

PROOF: We claim that for any holomorphic $k$-differential operator $D$ on $G/H$ we can find $\vec{u} \in (\mathscr{U}\mathfrak{g})^{\otimes k}$ such that

$$\vec{u}^{\mathrm{left},(1,0)}(\pi^* \vec{f})(e) = \pi^*(D\vec{f})(e)$$

holds for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$. We prove this claim by induction on the order $K \in \mathbb{Z}^k$ of $D$ at $eH$. If $K_i < 0$ for some $1 \leq i \leq k$, then $D = 0$ and we can use $\vec{u} = 0$. For the induction step, assume that the claim is already proven for every holomorphic $k$-differential operator of order strictly smaller than $K$ at $eH$. Choose coordinates $z = (z^1, \ldots, z^n)$ around $e$ on $G$ as in Lemma A.2 and denote the coordinates on $G/H$ induced by $(z^1, \ldots, z^{n-r})$ by $y \coloneqq (y^1, \ldots, y^{n-r})$. Locally we can write

$$D(f_1, \ldots, f_k) = \sum\nolimits_{I_1, \ldots, I_k \in \mathbb{N}_0^{n-r}} c_{I_1, \ldots, I_k} \cdot \partial_y^{I_1} f_1 \cdot \ldots \cdot \partial_y^{I_k} f_k$$

with $c_{I_1,\dots,I_k} \in \mathscr{C}^\infty(G/H)$ satisfying $c_{I_1,\dots,I_k}(eH) = 0$ whenever $|I_i| > K_i$ for some $1 \leq i \leq k$. Define a holomorphic $k$-differential operator $D_G$ on $G$ by

$$D_G(f'_1,\dots,f'_k) = \sum\nolimits_{I_1,\dots,I_k \in \mathbb{N}_0^{n-r}} (c_{I_1,\dots,I_k} \circ \pi) \cdot \partial_z^{I_1} f'_1 \cdot \ldots \cdot \partial_z^{I_k} f'_k \,.$$

Then $D_G(\pi^*\vec{f})(e) = \pi^*(D\vec{f})(e)$. Set $\vec{u}_1 \coloneqq \sum_{I_1,\dots,I_k \in \mathbb{N}_0^{n-r}} c_{I_1,\dots,I_k}(\pi(e)) X^{I_1} \otimes \ldots \otimes X^{I_k} \in (\mathscr{U}\mathfrak{g})^{\otimes k}$. Note that $D'_G \coloneqq D_G - \vec{u}_1^{\mathrm{left},(1,0)}$ has a strictly smaller order than $D_G$ at $e$ since $X_i^{\mathrm{left},(1,0)}\big|_e = \partial_{z^i}\big|_e$. There are functions $c'_{I_1,\dots,I_k} \in \mathscr{C}^\infty(G)$ such that we can express $D'_G$ in local coordinates as

$$D'_G(f'_1,\dots,f'_k) = \sum\nolimits_{I_1,\dots,I_k \in \mathbb{N}_0^{n}} c'_{I_1,\dots,I_k} \cdot \partial_z^{I_1} f'_1 \cdot \ldots \cdot \partial_z^{I_k} f'_k \,.$$

We obtain a $k$-differential operator $D'$ on $G/H$ of strictly smaller order than $D$ at $eH$ by letting

$$D'(f_1,\dots,f_k) = \sum\nolimits_{I_1,\dots,I_k \in \mathbb{N}_0^{n-r}} c'_{I_1,\dots,I_k}(\,\cdot\,,0) \partial_y^{I_1} f_1 \cdot \ldots \cdot \partial_y^{I_k} f_k \,.$$

It fulfils $D'_G(\pi^*\vec{f})(e) = \pi^*(D'\vec{f})(e)$. Using the induction hypothesis we find $\vec{u}' \in (\mathscr{U}\mathfrak{g})^{\otimes k}$ such that $\vec{u}'^{\mathrm{left},(1,0)}(\pi^*\vec{f})(e) = \pi^*(D'\vec{f})(e)$. Now

$$\begin{aligned}(\vec{u}_1 + \vec{u}')^{\mathrm{left},(1,0)}(\pi^*\vec{f})(e) &= (D_G - D'_G)(\pi^*\vec{f})(e) + \pi^*(D'\vec{f})(e) = \\ &= \pi^*(D\vec{f})(e) - \pi^*(D'\vec{f})(e) + \pi^*(D'\vec{f})(e) = \pi^*(D\vec{f})(e)\,,\end{aligned}$$

proving the claim.

Assume that $D$ is in addition left-invariant. Writing $\mathrm{L}_g \colon G/H \to G/H$ also for the action of $g \in G$ on $G/H$ we compute

$$\begin{aligned}\vec{u}^{\mathrm{left},(1,0)}(\pi^*\vec{f})(g) &= \mathrm{L}_g^*\vec{u}^{\mathrm{left},(1,0)}(\pi^*\vec{f})(e) = \vec{u}^{\mathrm{left},(1,0)}(\mathrm{L}_g^{\times k})^*(\pi^*\vec{f})(e) = \\ &= \vec{u}^{\mathrm{left},(1,0)}\pi^*((\mathrm{L}_g^{\times k})^*\vec{f})(e) = \pi^*(D(\mathrm{L}_g^{\times k})^*\vec{f})(e) = \\ &= \pi^*(\mathrm{L}_g^* D\vec{f})(e) = \mathrm{L}_g^*\pi^*(D\vec{f})(e) = \pi^*(D\vec{f})(g)\,.\end{aligned}$$

Thus $\vec{u}^{\mathrm{left},(1,0)}(\pi^*\vec{f}) = \pi^*(D\vec{f})$ holds for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$. Finally, we need to show that $\vec{u}$ has the correct invariance properties under the adjoint action of $H$. Define $\mathrm{R}_g \colon G \to G$, $\mathrm{R}_{g'}(g) \coloneqq gg'$. Since $\mathrm{R}_h^*\pi^*(D\vec{f}) = \pi^*(D\vec{f})$ for all $h \in H$ we obtain $\mathrm{R}_h^*\vec{u}^{\mathrm{left},(1,0)}\pi^*\vec{f} = \vec{u}^{\mathrm{left},(1,0)}\pi^*\vec{f}$ and therefore

$$(\mathrm{Ad}_h\,\vec{u})^{\mathrm{left},(1,0)}(\pi^*\vec{f})(g) = (\vec{u}^{\mathrm{left},(1,0)}\pi^*\vec{f})(gh) = \mathrm{R}_h^*\vec{u}^{\mathrm{left},(1,0)}\pi^*\vec{f}(g) = \vec{u}^{\mathrm{left},(1,0)}\pi^*\vec{f}(g)$$

for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$ and all $g \in G$, where the first equality follows as in the proof of Lemma 2.6. This means that $(\mathrm{Ad}_h\,\vec{u} - \vec{u})^{\mathrm{left},(1,0)}(\pi^*\vec{f}) = 0$ for all $\vec{f} \in \mathscr{C}^\infty(G/H)^k$, and therefore the proof of injectivity implies $\mathrm{Ad}_h\,\vec{u} - \vec{u} \in I$, or in other words $\vec{u} \in U_{\mathrm{inv}}$. □

## A.2 $G$-finite functions

In this subsection we introduce $G$-finite functions on a Lie group $G$ and use them to prove Proposition 3.27. The definition of $G$-finite functions uses only abstract properties of the Lie group $G$, and is therefore independent of whether $G$ is explicitly realized by matrices or not. For complex semisimple connected Lie groups a function is $G$-finite if and only if it is a polynomial, and therefore $G$-finite functions give a characterization of polynomials that is independent of the representation.

**Definition A.5 ($G$-finite functions)** *Let $M$ be a manifold with an action of a Lie group $G$. Then $f \in \mathscr{C}^\infty(M)$ is said to be $G$-finite if the vector space* $\operatorname{span}\{g \triangleright f \mid g \in G\}$ *is finite dimensional. We denote the space of $G$-finite functions on $M$ by* $\operatorname{Fin}^G(M)$ *or just by* $\operatorname{Fin}(M)$ *if $G$ is clear from the context.*

Here $g \triangleright f$ denotes the smooth function on $M$ defined by $(g \triangleright f)(m) = f(g^{-1} \triangleright m)$. Below, we use this definition only for $M = G$ and the action L or for $M = \hat{\mathcal{O}}_\lambda$ and the coadjoint action, and will therefore not mention these actions explicitly.

**Lemma A.6** *Let $G$ be a real or complex matrix Lie group and let $\mathcal{O}_\lambda$ be a coadjoint orbit of $G$. Then polynomials on $G$ are $G$-finite, and polynomials on $\mathcal{O}_\lambda$ are also $G$-finite.*

PROOF: Let $P_{ij} \colon G \to \mathbb{C}$, $X \mapsto X_{ij}$, and call such polynomials elementary in this proof. We compute $(g \triangleright P_{ij})(h) = P_{ij}(g^{-1}h) = \sum_k (g^{-1})_{ik} h_{kj} = \sum_k (g^{-1})_{ik} P_{kj}(h)$ for $g \in G$, so $g \triangleright P_{ij}$ is a linear combination of some elementary polynomials. If $p = P_{i_1 j_1} \dots P_{i_n j_n} \in \operatorname{Pol}(G)$ is a product of $n$ elementary polynomials, then $g \triangleright p$ is in the linear span of products of $n$ many elementary polynomials, which is a finite dimensional space. The statement for arbitrary polynomials follows by taking linear combinations.

The action of $G$ on $\operatorname{Pol}(\mathcal{O}_\lambda)$ is obtained by restricting the adjoint action of $G$ on $\mathrm{S}\mathfrak{g} \cong \operatorname{Pol}(\mathfrak{g}^*)$. The adjoint action preserves the degree of a symmetric tensor, so $\operatorname{span}\{\operatorname{Ad}_g X \mid g \in G\}$ is finite dimensional for any $X \in \mathrm{S}\mathfrak{g}$, and therefore $\operatorname{span}\{g \triangleright p \mid g \in G\}$ is finite dimensional for any $p \in \operatorname{Pol}(\mathcal{O}_\lambda)$. $\square$

**Proposition A.7** *Let $G$ be a complex semisimple connected Lie group with coadjoint orbit $\mathcal{O}_\lambda$. Then $G$-finite holomorphic functions on $\mathcal{O}_\lambda$ are polynomials.*

PROOF: $\operatorname{Hol}(\mathcal{O}_\lambda)$ is isomorphic to $\operatorname{Hol}(G)^{G_\lambda}$ as a $G$-module. The restriction to a maximal compact Lie subgroup $K \subseteq G$ is an injective $K$-module homomorphism to $\mathrm{L}^2(K)$, the square-integrable functions on $K$ with respect to the left-invariant Haar measure, so that we may view $\operatorname{Hol}(\mathcal{O}_\lambda)$ as a $K$-submodule of $\mathrm{L}^2(K)$. In particular, it is completely reducible as a $K$-module and therefore also as a $G$-module. Each irreducible module of highest weight $\nu$ appears only finitely many times in $\mathrm{L}^2(K)$ and thus also in $\operatorname{Hol}(\mathcal{O}_\lambda)$.

The scalar product of $\mathrm{L}^2(K)$ is $K$-invariant and therefore any irreducible modules of different highest weights are orthogonal. Restricting the scalar product to $\operatorname{Hol}(\mathcal{O}_\lambda)$ gives that $\operatorname{Hol}(\mathcal{O}_\lambda)^\nu$ is orthogonal to $\operatorname{Hol}(\mathcal{O}_\lambda)^{\nu'}$ if $\nu \neq \nu'$.

Assume $f \in \mathrm{Fin}(\mathcal{O}_\lambda)$ is holomorphic and not in $\mathrm{Pol}(\mathcal{O}_\lambda)$. We can without loss of generality assume that $f \in \mathrm{Fin}(\mathcal{O}_\lambda)^\nu$ for some weight $\nu$. (Indeed, we can write $f = \sum_\mu f^\mu$ with $f^\mu \in \mathrm{Fin}(\mathcal{O}_\lambda)^\mu$ and only finitely many $f^\mu$ are non-zero because $f$ is $G$-finite. One of these $f^\mu$ is not in $\mathrm{Pol}(\mathcal{O}_\lambda)$.) We can choose $f$ orthogonal to $\mathrm{Pol}(\mathcal{O}_\lambda)^\nu$ (which is finite dimensional) and therefore orthogonal to $\mathrm{Pol}(\mathcal{O}_\lambda)$. However, this space is dense in $\mathrm{Hol}(\mathcal{O}_\lambda)$ because polynomials on $K$ are dense in $\mathrm{L}^2(K)$. So $f = 0$, a contradiction. □

**Corollary A.8** *Let $G$ be a complex semisimple connected Lie group. Then the pull-back $\pi^*\colon \mathrm{Pol}(\mathcal{O}_\lambda) \to \mathrm{Pol}(G)^{G_\lambda}$ is an isomorphism.*

Proof: We have seen in the proof of Proposition 3.27 that $\pi^*$ is well-defined and injective, so it only remains to show that $\pi^*$ is surjective. Any element $f \in \mathrm{Pol}(G)^{G_\lambda}$ is $G$-finite by Lemma A.6. Then its image under the $G$-equivariant isomorphism $\pi_*\colon \mathrm{Hol}(G)^{G_\lambda} \to \mathrm{Hol}(\mathcal{O}_\lambda)$ is also $G$-finite because finite dimensionality of $\mathrm{span}\{g \triangleright f \mid g \in G\}$ implies finite dimensionality of $\mathrm{span}\{g \triangleright \pi_* f \mid g \in G\} = \mathrm{span}\{\pi_*(g \triangleright f) \mid g \in G\}$. The previous proposition implies that the $G$-finite element $\pi_* f \in \mathrm{Pol}(\mathcal{O}_\lambda)$ is a polynomial. It is mapped to $f$ by $\pi^*$. □

With similar methods as in this subsection one can prove that $G$-finite functions on a complex semisimple connected Lie group $G$ coincide with polynomials on $G$. Since the definition of $G$-finite functions does not depend on a representation of $G$ as a linear group, it follows that our definition of polynomials in Definition 3.26 is indeed independent of the representation. The same result is true for a compact semisimple connected Lie group $K$.

## A.3 Complex structures on real coadjoint orbits

We have seen in Subsection 2.1 that a coadjoint orbit of a real Lie group $G$ always admits a $G$-invariant symplectic structure, in particular its dimension is even. In this subsection, we will see that a semisimple coadjoint orbit $\mathcal{O}_\lambda$ of a connected semisimple real Lie group $G$ admits a $G$-invariant complex structure if $G_\lambda$ is compact, and that the set of such complex structures is in bijection to invariant orderings. If $G$ is compact, then there is a unique $G$-invariant complex structure that makes $\mathcal{O}_\lambda$ a Kähler manifold. If $G$ is not compact, then $\mathcal{O}_\lambda$ might or might not admit a Kähler structure. All results of this subsection are classical and well-known, see for example [7] for a summary.

Let $G$ be a real connected semisimple Lie group. Assume that $\lambda \in \mathfrak{g}^*$ is semisimple and that $G_\lambda$ is compact. Then any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ containing $\lambda^\sharp$ is contained in $\mathfrak{g}_\lambda$ and therefore compact. As usual, we denote the complexification of $\mathfrak{g}$ by $\hat{\mathfrak{g}}$ and let $\overline{\cdot}$ be the complex conjugation of $\hat{\mathfrak{g}}$ with respect to $\mathfrak{g}$.

Recall that a root $\alpha \in \mathfrak{h}^*$ is called *compact* if the Killing form $B$ is negative definite on $\mathfrak{g} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$, and *non-compact* if it is positive definite. (The root spaces $\mathfrak{g}^\alpha$ are subspaces of the complexification $\hat{\mathfrak{g}}$ of $\mathfrak{g}$.) We can always choose $X_\alpha \in \mathfrak{g}^\alpha$ such that $B(X_\alpha, X_{-\alpha}) = 1$ and if $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$, then $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ (see [7, Section 3]). In this case,

$$-X_{-\alpha} = \overline{X}_\alpha \quad \text{and} \quad \mathrm{i}(X_\alpha + X_{-\alpha}), X_\alpha - X_{-\alpha} \in \mathfrak{g} \quad \text{if } \alpha \text{ is compact,} \tag{A.3a}$$

$$X_{-\alpha} = \overline{X}_\alpha \quad \text{and} \quad \mathrm{i}(X_\alpha - X_{-\alpha}), X_\alpha + X_{-\alpha} \in \mathfrak{g} \quad \text{if } \alpha \text{ is non-compact.} \qquad \text{(A.3b)}$$

Recall that $\hat{\Delta}$ is the set of roots that are not orthogonal to $\lambda$.

**Theorem A.9** *Let $\mathcal{O}_\lambda$ be a coadjoint orbit of a real connected semisimple Lie group $G$. Assume that $G_\lambda$ is compact, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\lambda^\sharp$. Then $G$-invariant complex structures on $\mathcal{O}_\lambda$ are in bijection with invariant orderings of $\hat{\Delta}$ (i.e. choices of positive roots $\hat{\Delta}^+$ that arise as $\hat{\Delta}^+ = \hat{\Delta} \cap \Delta^+$ from an invariant ordering of $\Delta$ as defined in Definition 3.10).*

Proof (Sketch): Introduce $\mathfrak{m} = \bigoplus_{\alpha \in \hat{\Delta}} \mathfrak{g}^\alpha \cong \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_\lambda$. Since taking fundamental vector fields (see Subsection 2.1) gives an isomorphism $\mathfrak{g}/\mathfrak{g}_\lambda \to \mathrm{T}_\lambda\mathcal{O}_\lambda$, $\mathfrak{m}$ is isomorphic to the complexified tangent space $\mathrm{T}^{\mathbb{C}}_\lambda\mathcal{O}_\lambda$ and $\mathfrak{g} \cap \mathfrak{m}$ is isomorphic to $\mathrm{T}_\lambda\mathcal{O}_\lambda$.

Given an invariant ordering of $\hat{\Delta}$, see Definition 3.10, define $I\colon \mathfrak{m} \to \mathfrak{m}$ by extending $X_\alpha \mapsto \mathrm{i}X_\alpha$ if $\alpha \in \hat{\Delta}^+$, $X_\alpha \mapsto -\mathrm{i}X_\alpha$ if $\alpha \in \hat{\Delta}^-$ linearly. Clearly $I^2 = -\mathsf{id}$. For both a compact and a non-compact root $\alpha$, $I$ restricts to an endomorphism of $\mathfrak{g} \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$, from which it follows that $I$ restricts to a map $\mathfrak{g} \cap \mathfrak{m} \to \mathfrak{g} \cap \mathfrak{m}$, squaring to $-\mathsf{id}$. To prove that it extends to a $G$-invariant almost complex structure on $\mathcal{O}_\lambda$, it suffices to prove that $I$ is $G_\lambda$-invariant. By applying the analogue of Proposition 2.3 for compact connected semisimple Lie groups to a maximally compact subgroup of $G$ containing $G_\lambda$, it follows that $G_\lambda$ is connected, and it suffices to prove that $I$ is $\mathfrak{g}_\lambda$-invariant, in the sense that $I([A, B]) = [A, I(B)]$ holds for all $A \in \mathfrak{g}_\lambda$ and $B \in \mathfrak{m}$. This identity holds for $A \in \mathfrak{h}$ since $I$ preserves the root spaces. So we only need to check it for $A = X_\alpha$ and $B = X_\beta$ with $\alpha \in \Delta'$ and $\beta \in \hat{\Delta}$, which is equivalent to the invariance of the ordering. Finally, one uses that $\alpha + \beta$ is positive if $\alpha, \beta \in \hat{\Delta}$ are positive to compute that the Nijenhuis torsion of $I$ vanishes, so $I$ is indeed a complex structure.

Vice versa, a $G$-invariant complex structure $I$ on $\mathcal{O}_\lambda$ determines a $\mathfrak{g}_\lambda$-invariant map $I\colon \mathfrak{m} \to \mathfrak{m}$ with $I^2 = -\mathsf{id}$ by restricting to the tangent space at $\lambda$ and complexifying. In particular $I$ is $\mathfrak{h}$-invariant, and therefore preserves the root spaces, so $X_\alpha \mapsto \mathrm{i}c_\alpha X_\alpha$ with $c_\alpha = \pm 1$. Since $I$ preserves the real tangent space, we must have $c_\alpha = -c_{-\alpha}$. The Nijenhuis torsion of the complex structure vanishes, which implies that $\hat{\Delta}^+ = \{\alpha \in \hat{\Delta} \mid c_\alpha = 1\}$ defines an ordering. Finally invariance under the whole Lie algebra $\mathfrak{g}_\lambda$ gives that this ordering is invariant. □

**Proposition A.10** *If $\mathcal{O}_\lambda$ is a coadjoint orbit of a compact connected semisimple Lie group $K$, then $\mathcal{O}_\lambda$ has a unique $K$-invariant complex structure $I$ that makes $(\mathcal{O}_\lambda, I, \omega_{\mathrm{KKS}})$ a Kähler manifold, and this complex structure corresponds to an ordering for which $\alpha \in \hat{\Delta}$ is positive if and only if $(\alpha, \mathrm{i}\lambda) > 0$.*

Note that $\alpha$ attains purely imaginary values on $\mathfrak{k}$, whereas $\lambda$ attains real values. Therefore $(\alpha, \mathrm{i}\lambda) \in \mathbb{R}$. The ordering for which $\alpha \in \hat{\Delta}$ is positive if $(\alpha, \mathrm{i}\lambda) > 0$ is standard (see Subsection 3.2).

Proof: Since $K$ is compact, it follows that any root is compact. Given a $K$-invariant complex structure $I$, we associate the (not necessarily positive definite) metric $g(v, w) = \omega_{\mathrm{KKS}}(v, Iw)$ and $\mathcal{O}_\lambda$ is a Kähler manifold if $g$ is positive definite. Since $I$ and $\omega_{\mathrm{KKS}}$

are $K$-invariant, so is $g$ and we may check positive definiteness on $\mathrm{T}_\lambda \mathcal{O}_\lambda$. Identifying $\mathrm{T}^{\mathbb{C}}_\lambda \mathcal{O}_\lambda$ with $\mathfrak{m}$ as in the proof of the previous proposition and extending $g$ complex linearly, we compute that $g(X_\alpha, X_\beta) = \omega_{\mathrm{KKS}}(X_\alpha, IX_\beta) = c_\beta \lambda([X_\alpha, X_\beta])$ for all $\alpha, \beta \in \hat{\Delta}$. This expression is non-zero only if $\alpha = -\beta$, and in this case $g(X_\alpha, X_{-\alpha}) = -\mathrm{i} c_\alpha \lambda(\alpha^\sharp) = -\mathrm{i} c_\alpha \cdot (\alpha, \lambda)$. Then

$$g(\mathrm{i}(X_\alpha + X_{-\alpha}), \mathrm{i}(X_\alpha + X_{-\alpha})) = 2\mathrm{i} c_\alpha \cdot (\alpha, \lambda)$$

and

$$g(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = 2\mathrm{i} c_\alpha \cdot (\alpha, \lambda)\,.$$

So $g$ is positive definite if and only if $c_\alpha = 1$ for all $\alpha \in \hat{\Delta}$ with $(\alpha, \mathrm{i}\lambda) > 0$. □

Note that the situation is more complicated if $G$ is non-compact, but $G_\lambda$ is compact, since we may then have both compact and non-compact roots. The condition for $g$ being positive definite then becomes $c_\alpha = 1$ if either $\alpha$ is a compact root and $(\alpha, \mathrm{i}\lambda) > 0$ or if $\alpha$ is a non-compact root and $(\alpha, \mathrm{i}\lambda) < 0$. If these conditions define an invariant ordering, then $\mathcal{O}_\lambda$ has a $G$-invariant Kähler structure (which is automatically unique). One can give more explicit criteria for when the conditions above define an invariant ordering, see [7], but we only need the following easy case.

**Corollary A.11** *Let $\mathcal{O}_\lambda$ be a coadjoint orbit of a connected semisimple Lie group $G$. Assume that $G_\lambda$ is compact, and that $\mathfrak{h}$ is a Cartan subalgebra containing $\lambda^\sharp$. If all roots in $\hat{\Delta}$ are non-compact, then $(\mathcal{O}_\lambda, I, \omega_{\mathrm{KKS}})$ is a Kähler manifold, where $I$ is the complex structure corresponding to the ordering for which $\alpha \in \hat{\Delta}$ is positive if and only if $(\alpha, \mathrm{i}\lambda) < 0$.*

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# Appendix A

# Comparison between the two constructions

In this appendix we prove that the strict star products obtained in Paper I via phase space reduction and in Paper II by inverting the Shapovalov pairing agree, both for complex projective spaces and hyperbolic discs.

In the notation of Paper I, let us fix the signature $s \coloneqq 1 + n$. Note that we have seen in Section I.3 that the manifolds $M_{\mathrm{red}}^{(1+n)}$ and $\mathbb{CP}^n$ are diffeomorphic, and in Section I.5.2 that the metric and symplectic form on $M_{\mathrm{red}}^{(1+n)}$ coincide with the standard Fubini–Study metric and symplectic form on $\mathbb{CP}^n$ (at least up to a scalar). Recall that $\mathrm{G}_{\mathcal{J}} = \mathrm{U}(1+n)$ acts on $\mathbb{C}^{1+n}$, that this action restricts to a transitive action on $Z = \mathbb{S}^{2n+1}$, and descends to the quotient of $Z$ by the action of the central subgroup $\mathrm{U}(1) \subseteq \mathrm{U}(1+n)$, i.e. to $\mathbb{CP}^n$. The action of the subgroup $\mathrm{SU}(1+n)$ is still transitive.

Let $\lambda \colon \mathfrak{su}_{1+n} \to \mathbb{R}$, $X \mapsto -\mathrm{i} r X_{0,0}$ be the linear functional on $\mathfrak{su}_{1+n}$ considered in Example II.5.23.

**Lemma A.1** *The stabilizer of* $[1{:}0{:}\dots{:}0]$ *under the* $\mathrm{SU}(1+n)$*-action on* $\mathbb{CP}^n$ *is*

$$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) \coloneqq \{g \in \mathrm{SU}(1+n) \mid g_{0,i} = g_{i,0} = 0 \text{ for } 1 \leq i \leq n\}\,, \tag{A.1}$$

*and therefore coincides with the stabilizer of* $\lambda \in \mathfrak{su}_{1+n}^*$.

Proof: It is clear that any matrix in $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ stabilizes $[1{:}0{:}\dots{:}0]$. To see the reverse implication, let $e_0 = (1, 0, \dots, 0) \in \mathbb{C}^{1+n}$, and assume that $g \in \mathrm{SU}(1+n)$ stabilizes $[1{:}0{:}\dots{:}0]$. Then $ge_0 = \phi e_0$ for some $\phi \in \mathbb{C}$ with $|\phi| = 1$, and therefore $g_{i,0} = 0$ for $1 \leq i \leq n$. But since $g \in \mathrm{SU}(1+n)$ this implies that $g_{0,i} = 0$, i.e. $g \in \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$.

Since the stabilizer of $\lambda$ is just the intersection of the stabilizer $\{g \in \mathrm{SL}_{1+n}(\mathbb{C}) \mid g_{0,i} = g_{i,0} = 0 \text{ for } 1 \leq i \leq n\}$ from (II.3.37b) with $\mathrm{SU}(1+n)$, it coincides with $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$. □

It turns out that the naive map $\mathrm{SU}(1+n) \to \mathbb{CP}^n$, $g \mapsto g \triangleright [1{:}0{:}\dots{:}0]$ descends to an antiholomorphic map from the coadjoint orbit $\mathcal{O}_\lambda \cong \mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$, through the element $\lambda \in \mathfrak{su}^*_{1+n}$ defined above, to $\mathbb{CP}^n$. We therefore consider the smooth map

$$\tilde{\Phi} \colon \mathrm{SU}(1+n) \to \mathbb{CP}^n\,, \quad g \mapsto (g^{\mathrm{T}})^{-1} \triangleright [1{:}0{:}\ldots{:}0]\,. \tag{A.2}$$

Note that on $\mathrm{SU}(1+n)$ the map $g \mapsto (g^{\mathrm{T}})^{-1}$ is a group homomorphism, coincides with the entrywise complex conjugation, and maps the stabilizer $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ of $[1{:}0{:}\ldots{:}0] \in \mathbb{CP}^n$ to itself. Therefore $\tilde{\Phi}$ descends to a bijection

$$\Phi \colon \mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) \to \mathbb{CP}^n\,. \tag{A.3}$$

Since the projection $\mathrm{SU}(1+n) \to \mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ is a smooth submersion, $\Phi$ is also smooth. Since $\Phi$ is also an immersion (this follows e.g. by considering fundamental vector fields), the inverse of $\Phi$ is also smooth, implying that $\Phi$ is a diffeomorphism.

Consider the left action of $\mathrm{SU}(1+n)$ on itself. The map $\tilde{\Phi}$ is clearly equivariant with respect to the actions of $\mathrm{SU}(1+n)$ over the group homomorphism $g \mapsto (g^{\mathrm{T}})^{-1}$, meaning that $\tilde{\Phi}(gg') = (g^{\mathrm{T}})^{-1}\tilde{\Phi}(g')$ holds for all $g, g' \in \mathrm{SU}(1+n)$. The same equivariance property holds for $\Phi$.

**Lemma A.2** *The map $\Phi$ is holomorphic.*

PROOF: Recall that $E_{ij}$ denotes a matrix with entry 1 at position $(i,j)$ and all other entries 0. The elements $F_{0j} := E_{0j} - E_{j0} \in \mathfrak{su}_{1+n}$ and $G_{0j} := \mathrm{i}E_{0j} + \mathrm{i}E_{j0} \in \mathfrak{su}_{1+n}$ with $1 \le j \le n$ define a complement of the stabilizer $(\mathfrak{su}_{1+n})_\lambda$ in $\mathfrak{su}_{1+n}$. Identifying $\mathfrak{su}_{1+n}/(\mathfrak{su}_{1+n})_\lambda$ with the tangent space of $\mathcal{O}_\lambda \subseteq \mathfrak{su}^*_{1+n}$ at $\lambda$ via fundamental vector fields as usual, Example II.5.23 implies that the Kähler complex structure maps $E_{0j}$ to $\mathrm{i}E_{0j}$ and $E_{j0}$ to $-\mathrm{i}E_{j0}$, and therefore maps $F_{0j}$ to $G_{0j}$ and $G_{0j}$ to $-F_{0j}$ for all $1 \le j \le n$. The tangent map of $\mathrm{SU}(1+n) \ni g \mapsto (g^{\mathrm{T}})^{-1} \in \mathrm{SU}(1+n)$ is just $\mathfrak{su}_{1+n} \ni X \mapsto -X^{\mathrm{T}} \in \mathfrak{su}_{1+n}$, and therefore maps $F_{0j}$ to itself and $G_{0j}$ to $-G_{0j}$. It is straightforward to compute that, with respect to the coordinates $w^i = \frac{z^i}{z^0}$ on $\mathbb{CP}^n$ around $[1{:}0{:}\ldots{:}0]$, the tangent map of the standard action of $\mathrm{SU}(1+n)$ on $\mathbb{CP}^n$ at $\lambda$ maps $F_{0j}$ to $-\frac{\partial}{\partial w^j}$ and $G_{0j}$ to $\mathrm{i}\frac{\partial}{\partial w^j}$. This shows that the composition $\Phi$ maps $F_{0j}$ to $-\frac{\partial}{\partial w^j}$ and $G_{0j}$ to $-\mathrm{i}\frac{\partial}{\partial w^j}$, and therefore maps the holomorphic tangent space of $\mathcal{O}_\lambda$ at $\lambda$ to the holomorphic tangent space of $\mathbb{CP}^n$ at $[1{:}0{:}\ldots{:}0]$. By $\mathrm{SU}(1+n)$-invariance of the complex structures it follows that $\Phi$ is holomorphic. □

Recall the definitions of $\mathscr{P}(\mathbb{CP}^n)$ from Definition I.4.7, of $\mathrm{Pol}(\mathbb{CP}^n)$ from Definition II.5.5, and of $\mathrm{Pol}(\mathrm{SU}(1+n))$ from Definition II.5.7: The polynomials $\mathscr{P}(\mathbb{CP}^n)$ are defined via the reduction map, viewing $\mathbb{CP}^n$ as a quotient of $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{1+n}$ by the action of $\mathrm{U}(1)$, whereas the polynomials $\mathrm{Pol}(\mathbb{CP}^n)$ are defined by restricting polynomials on $\mathfrak{su}^*_{1+n}$ to $\mathbb{CP}^n$ viewed as a coadjoint orbit. $\mathrm{Pol}(\mathrm{SU}(n+1))$ was defined as the algebra spanned by matrix coefficients for any representation of $\mathrm{SU}(n+1)$ by real matrices (and, since $\mathrm{SU}(n+1)$ is compact, does not depend on the chosen representation). We obtain such a representation e.g. by replacing an entry $x + \mathrm{i}y \in \mathbb{C}$ in the standard representation with a 2-by-2 matrix with entries $x$ on the diagonal, and $\pm y$ in the upper right and lower left corners. Polynomials on $\mathrm{SU}(n+1)$ are therefore precisely polynomials in the real and imaginary parts of the matrix entries, or equivalently polynomials in the matrix entries of the standard representation and their complex conjugates. Using the defining relation $U^\dagger U = 1$ of $\mathrm{SU}(n+1)$, the complex conjugates of matrix entries are polynomials of matrix entries, so $\mathrm{Pol}(\mathrm{SU}(n+1))$ also coincides with polynomials in the matrix entries of the standard representation.

**Lemma A.3** *The polynomials $\mathscr{P}(\mathbb{CP}^n)$ are precisely the $\mathrm{SU}(1+n)$-finite functions on $\mathbb{CP}^n$.*

PROOF: Any polynomial $p_{\mathrm{red}} \in \mathscr{P}(\mathbb{CP}^n)$ lies in the image of the reduction map $(\,\cdot\,)_{\mathrm{red}}$ from Definition I.4.1, so is induced by a $\mathrm{U}(1)$-invariant polynomial $p$ on $\mathbb{C}^{1+n}$. Using the degree, it is clear that $\operatorname{span}\{g \triangleright p \mid g \in \mathrm{SU}(1+n)\}$ is finite dimensional, and since $(\,\cdot\,)_{\mathrm{red}}$ is $\mathrm{SU}(1+n)$-equivariant it follows that $p_{\mathrm{red}}$ is $\mathrm{SU}(1+n)$-finite.

To see the converse, note that every $\mathrm{SU}(1+n)$-finite function on $\mathbb{CP}^n$ extends to a $\mathrm{U}(1)$-invariant $\mathrm{SU}(1+n)$-finite function on $\mathbb{S}^{2n+1}$. Projecting to the zeroth column gives a $\mathrm{SU}(1+n)$-equivariant map $\mathrm{SU}(1+n) \to \mathbb{S}^{2n+1}$, so we may view $\mathrm{L}^2(\mathbb{S}^{2n+1})$ as a submodule of the left-regular representation of $\mathrm{SU}(1+n)$ on itself. Modifying the proof of Proposition A.7 gives that all $\mathrm{SU}(1+n)$-finite functions on $\mathbb{S}^{2n+1}$ are polynomials. $\square$

We saw in Section II.A.2 that $\mathrm{Pol}(\mathbb{CP}^n) \cong \mathrm{Pol}(\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n)))$ also consists precisely of the $\mathrm{SU}(1+n)$-finite functions, and since $\Phi$ is a $\mathrm{SU}(1+n)$-equivariant diffeomorphism (over an automorphism of $\mathrm{SU}(1+n)$), we obtain the following corollary.

**Corollary A.4** *The pullback $\Phi^*\colon \mathscr{C}^\infty(\mathbb{CP}^n) \to \mathscr{C}^\infty(\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n)))$ restricts to an isomorphism $\Phi^*\colon \mathscr{P}(\mathbb{CP}^n) \to \mathrm{Pol}(\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n)))$.*

We will usually identify $\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$ with the corresponding coadjoint orbit, which we also denote by $\mathbb{CP}^n$, so that the isomorphism becomes $\Phi^*\colon \mathscr{P}(\mathbb{CP}^n) \to \mathrm{Pol}(\mathbb{CP}^n)$.

We are now ready to compare the strict star products on $\mathbb{CP}^n$, obtained in Paper I and Paper II.

**Theorem A.5** *The map $\Phi^*\colon \mathscr{P}(\mathbb{CP}^n) \to \mathrm{Pol}(\mathbb{CP}^n)$ intertwines the two star products $\star_{\mathrm{red},\hbar}$ defined in Proposition I.5.22 and $*_\hbar$ for $r = 1$ defined in Example II.5.23, meaning that for all $p, q \in \mathscr{P}(\mathbb{CP}^n)$ we have*

$$\Phi^*(p \star_{\mathrm{red},\hbar} q) = \Phi^*(p) *_\hbar \Phi^*(q)\,. \tag{A.4}$$

PROOF: Both star products are $\mathrm{SU}(1+n)$-equivariant, so it suffices to prove (A.4) at the point $\lambda \in \mathfrak{su}^*_{1+n}$ defined above, which corresponds to $e\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n)) \in \mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$, and therefore to $[1{:}0{:}\dots{:}0] \in \mathbb{CP}^n$ under the isomorphism $\Phi$. Since $\Phi^*(p \star_{\mathrm{red},\hbar} q)(\lambda) = (p \star_{\mathrm{red},\hbar} q)([1{:}0{:}\dots{:}0])$ we have to prove that

$$(p \star_{\mathrm{red},\hbar} q)([1{:}0{:}\dots{:}0]) = (\Phi^*(p) *_\hbar \Phi^*(q))(\lambda)\,.$$

Recall the definition of the monomials $\mathrm{b}_{P,Q}$ for $P, Q \in \mathbb{N}_0^{1+n}$ from Definition I.5.6, and that the $\mathrm{b}_{P,Q;\mathrm{red}}$ with $|P| = |Q|$ span the space $\mathscr{P}(\mathbb{CP}^n)$. It is therefore enough to prove the above equation for $p = \mathrm{b}_{P,Q;\mathrm{red}}$ and $q = \mathrm{b}_{R,S;\mathrm{red}}$ with $P, Q, R, S \in \mathbb{N}_0^{1+n}$, $|P| = |Q|$, and $|R| = |S|$.

The explicit formula for $\star_{\mathrm{red},\hbar}$ obtained in Proposition I.5.22 shows that $\mathrm{b}_{P,Q;\mathrm{red}}$ with $|P| = |Q| = 1$ generate the whole star product algebra, meaning that elements of the form $\mathrm{b}_{P_1,Q_1;\mathrm{red}} \star \dots \star \mathrm{b}_{P_k,Q_k;\mathrm{red}}$ with $k \in \mathbb{N}$ and $|P_1| = \dots = |P_k| = |Q_1| = \dots = |Q_k| = 1$ span $\mathscr{P}(\mathbb{CP}^n)$. Since both $\star_{\mathrm{red},\hbar}$ and $*_\hbar$ are associative and unital, it therefore suffices to prove that

$$(\mathrm{b}_{P,Q;\mathrm{red}} \star_{\mathrm{red},\hbar} \mathrm{b}_{R,S;\mathrm{red}})([1{:}0{:}\dots{:}0]) = (\Phi^*(\mathrm{b}_{P,Q;\mathrm{red}}) *_\hbar \Phi^*(\mathrm{b}_{R,S;\mathrm{red}}))(\lambda) \tag{A.5}$$

for $P, Q, R, S \in \mathbb{N}_0^{n+1}$ with $|P| = |Q| = 1$ and $|R| = |S|$. ▽

We will compute both sides of that equation explicitly, but before we derive a lemma that simplifies this computation. In analogy to the notation used in Definition II.5.7 we define the polynomials $P_{ij} \colon \mathrm{SU}(1+n) \to \mathbb{C}$, $g \mapsto g_{ij}$. We explained above that the $P_{ij}$ generate $\mathrm{Pol}(\mathrm{SU}(1+n))$. As in Proposition II.3.36, if $1 \leq j \leq n$ we define $X_{0j} := E_{0j}$ and $Y_{0j} := E_{j0}$ in $\mathfrak{sl}_{1+n}(\mathbb{C})$, the complexification of $\mathfrak{su}_{1+n}$.

**Lemma A.6** *For all $j \in \{1, \dots, n\}$ and $k, \ell \in \{0, \dots, n\}$ we have*

$$X_{0j}^{\mathrm{left}} P_{k0} = 0\,, \qquad Y_{0j}^{\mathrm{left}} P_{k\ell} = \delta_{0\ell} P_{kj}\,, \tag{A.6a}$$

$$X_{0j}^{\mathrm{left}} \overline{P_{k\ell}} = -\delta_{0\ell} \overline{P_{kj}}\,, \qquad Y_{0j}^{\mathrm{left}} \overline{P_{k0}} = 0\,. \tag{A.6b}$$

PROOF: Note that $F_{0j} := E_{0j} - E_{j0}$ and $G_{0j} := \mathrm{i}E_{0j} + \mathrm{i}E_{j0}$ with $1 \leq j \leq n$ all lie in $\mathfrak{su}_{1+n}$, and that $X_{0j} = \frac{1}{2}(F_{0j} - \mathrm{i}G_{0j})$. Therefore

$$\begin{aligned}
X_{0j}^{\mathrm{left}} P_{k\ell}(g) &= \frac{1}{2}\left( \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_{k\ell}(g \exp(tF_{0j})) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_{k\ell}(g \exp(tG_{0j})) \right) \\
&= \frac{1}{2}(P_{k\ell}(gF_{0j}) - \mathrm{i}P_{k\ell}(gG_{0j})) \\
&= \frac{1}{2}(g_{k0}\delta_{j\ell} - g_{kj}\delta_{0\ell} - \mathrm{i}\cdot\mathrm{i}g_{k0}\delta_{j\ell} - \mathrm{i}\cdot\mathrm{i}g_{kj}\delta_{0\ell}) \\
&= g_{k0}\delta_{j\ell} \\
&= \delta_{j\ell} P_{k0}(g)\,,
\end{aligned}$$

and the first equation is the special case $\ell = 0$. The other equation including $X_{0j}^{\mathrm{left}}$ follows from a similar computation with $\overline{P_{k\ell}}$ instead of $P_{k\ell}$. The statements for $Y_{0j}^{\mathrm{left}}$ follow by taking conjugates: Since $\mathrm{SU}(n+1)$ is compact, all roots are compact, and therefore $\overline{X_{0j}} = -Y_{0j}$ by (II.A.3a) (with $\overline{\,\cdot\,}$ denoting the complex conjugation of $\mathfrak{sl}_{1+n}(\mathbb{C})$ with respect to the real form $\mathfrak{su}_{1+n}$, not the entrywise complex conjugation of a matrix). □

CONTINUATION OF THE PROOF OF THEOREM A.5: We attempt to compute both sides of (A.5) explicitly, starting with the left hand side. Since $|P| = |Q| = 1$, let $0 \leq p, q \leq n$ be such that $P = E_p$ and $Q = E_q$. Note that $\mathrm{b}_{E_i,0}(1, 0, \dots, 0) = \delta_{i,0}$ and $\mathrm{b}_{0,E_j}(1, 0, \dots, 0) = \delta_{j,0}$, and consequently $\mathrm{b}_{R,S;\mathrm{red}}([1{:}0{:}\dots{:}0]) = \delta_{R',0}\delta_{S',0}$ for any multiindices $R = (R_0, \dots, R_n), S = (S_0, \dots, S_n) \in \mathbb{N}_0^{1+n}$ with $|R| = |S|$ and with truncations $R' = (R_1, \dots, R_n)$, $S' = (S_1, \dots, S_n)$. Then

$$
\begin{aligned}
&(\mathrm{b}_{E_p,E_q;\mathrm{red}} \star_{\mathrm{red},\hbar} \mathrm{b}_{R,S;\mathrm{red}})([1{:}0{:}\ldots{:}0]) \\
&\quad\overset{(1)}{=} \sum_{T=0}^{\min\{E_p,S\}} \frac{\hbar(1/\hbar)_{\downarrow,|E_p+S-T|}}{(1/\hbar)_{\downarrow,|S|}} T! \binom{E_p}{T}\binom{S}{T} \delta_{E'_p+R'-T',0}\delta_{E'_q+S'-T',0} \\
&\quad\overset{(2)}{=} \sum_{t=0}^{\min\{S_p,1\}} \frac{\hbar(1/\hbar)_{\downarrow,|S|+1-t}}{(1/\hbar)_{\downarrow,|S|}} \binom{S_p}{t} \delta_{E'_p+R'-(tE_p)',0}\delta_{E'_q+S'-(tE_p)',0} \\
&\quad\overset{(3)}{=} \hbar(1/\hbar - |S|)\delta_{E'_p+R',0}\delta_{E'_q+S',0} + \hbar S_p \delta_{R',0}\delta_{E'_q+S'-E'_p,0} \\
&\quad\overset{(4)}{=} \delta_{E'_p+R',0}\delta_{E'_q+S',0} - \hbar|S|\delta_{E'_p+R',0}\delta_{E'_q+S',0} + \hbar S_p \delta_{R',0}\delta_{E'_q+S'-E'_p,0} \\
&\quad\overset{(5)}{=} \delta_{p,0}\delta_{q,0}\delta_{R',0}\delta_{S',0} - \hbar S_0 \delta_{p,0}\delta_{q,0}\delta_{R',0}\delta_{S',0} \\
&\qquad\qquad + \hbar S_0 \delta_{p,0}\delta_{q,0}\delta_{R',0}\delta_{S',0} + \sum_{i=1}^{n} \hbar S_i \delta_{p,i}\delta_{q,0}\delta_{R',0}\delta_{S',E'_i} \\
&\quad\overset{(6)}{=} \delta_{p,0}\delta_{q,0}\delta_{R',0}\delta_{S',0} + \hbar \sum_{i=1}^{n} \delta_{p,i}\delta_{q,0}\delta_{R',0}\delta_{S',E'_i}\,.
\end{aligned}
$$

Here we used the explicit formula for $\star_{\mathrm{red},\hbar}$ from Proposition I.5.22 in (1). In (2), the sum over $T$ is a sum over the multiindex 0 and, if $S_p \geq 1$, the multiindex $E_p$. In (3) we wrote out the sum and simplified the expression, and in (4) we multiplied out the first product. In (5) we used that all our multiindices have non-negative entries, so e.g. $\delta_{E'_p+R',0} = \delta_{E'_p,0}\delta_{R',0} = \delta_{p,0}\delta_{R',0}$. When doing a similar computation for the last summand, we distinguish the two cases $p = 0$ and $p \geq 1$, yielding the third and fourth summand. Finally (6) only involves cancelling.

To compute the right hand side of (A.5), note that

$$
(\pi^* \circ \Phi^*)(\mathrm{b}_{R,S;\mathrm{red}})(g) = \mathrm{b}_{R,S;\mathrm{red}}((g^{\mathrm{T}})^{-1} \triangleright [1{:}0{:}\ldots{:}0]) = \mathrm{b}_{R,S;\mathrm{red}}(\overline{g} \triangleright [1{:}0{:}\ldots{:}0])
$$

holds for all $R, S \in \mathbb{N}_0^{1+n}$ with $|R| = |S|$ and for all $g \in \mathrm{SU}(1+n)$. Therefore we have

$$
(\pi^* \circ \Phi^*)(\mathrm{b}_{R,S;\mathrm{red}}) = \overline{P_{\cdot\,0}}^R (P_{\cdot\,0})^S
$$

where $\overline{P_{\cdot\,0}}^R$ is a shorthand for $\overline{P_{00}}^{R_0}\overline{P_{10}}^{R_1}\ldots\overline{P_{n0}}^{R_n}$ and $(P_{\cdot\,0})^S$ is a shorthand for $(P_{00})^{S_0}(P_{10})^{S_1}\ldots(P_{n0})^{S_n}$. Clearly $\overline{P_{\cdot\,0}}^R(e) = \delta_{R',0}$ and $(P_{\cdot\,0})^S(e) = \delta_{S',0}$ with $e$ denoting the identity element of $\mathrm{SU}(1+n)$.

From (A.6a) and (A.6b) it is immediate that applying two or more $X_{0j}^{\mathrm{left}}$ to $(\pi^* \circ \Phi^*)(\mathrm{b}_{P,Q;\mathrm{red}})$ with $|P| = |Q| = 1$ gives 0, so that all summands in (II.3.39) with $\ell \geq 2$ vanish. Therefore

$$
\begin{aligned}
&(\Phi^*(\mathrm{b}_{E_p,E_q;\mathrm{red}}) *_\hbar \Phi^*(\mathrm{b}_{R,S;\mathrm{red}}))(\lambda) \\
&\quad\overset{(1)}{=} \sum_{\ell=0}^{1} \frac{(-1)^\ell}{\ell!\frac{1}{\hbar}\big(\frac{1}{\hbar}-1\big)\ldots\big(\frac{1}{\hbar}-(\ell-1)\big)} \cdot \\
&\qquad\qquad \cdot \big((X_{0,1}\otimes Y_{0,1} + \cdots + X_{0,n}\otimes Y_{0,n})^\ell\big)^{\mathrm{left}}\big(\overline{P_{p,0}}P_{q,0}, \overline{P_{\cdot\,,0}}^R(P_{\cdot\,,0})^S\big)(e)
\end{aligned}
$$

$$\begin{aligned}
&\overset{(2)}{=} (\overline{P_{p,0}} P_{q,0} \overline{P_{\cdot,0}}^{R} (P_{\cdot,0})^S)(e) \\
&\qquad\qquad - \hbar \sum_{i=1}^{n} (X_{0,i}^{\mathrm{left}} \overline{P_{p,0}})(e) P_{q,0}(e) \overline{P_{\cdot,0}}^{R}(e) (Y_{0,i}^{\mathrm{left}} (P_{\cdot,0})^S)(e) \\
&\overset{(3)}{=} \delta_{p,0}\delta_{q,0}\delta_{R',0}\delta_{S',0} + \hbar \sum_{i=1}^{n} \delta_{p,i}\delta_{q,0}\delta_{R',0}\delta_{S',E_i} .
\end{aligned}$$

In (1) we used that $f *_\hbar g = \pi_*(F_\hbar^{\mathrm{left}}(\pi^* f, \pi^* g))$, inserted (II.3.39) with $r = 1$ for $F_\hbar$, and truncated the sum at $\ell = 1$ as described above. In (2) we wrote out the sum over $\ell$, and used (A.6a) and (A.6b). Finally, we used these equations again in step (3): $X_{0,i}^{\mathrm{left}} \overline{P_{p,0}}(e) = -\overline{P_{p,i}}(e) = -\delta_{p,i}$ and similarly $Y_{0,i}^{\mathrm{left}}(P_{\cdot,0})^S(e) = \delta_{S',E_i}$.

These computations prove (A.5) and therefore Theorem A.5. □

**Corollary A.7** *The formal star product $\star_{\mathrm{red}}$ defined in Definition I.5.6 and the formal star product $\star$, obtained from the asymptotic expansion of the element $F_\hbar$ in Proposition II.3.36, are intertwined by $\Phi^*$.*

PROOF: Since both formal star products are the asymptotic expansions of the corresponding strict star products $\star_{\mathrm{red},\hbar}$ and $*_\hbar$ (recall that differential operators are uniquely determined by their behaviour on polynomials), this follows from the previous theorem. □

**Corollary A.8** *The map $\Phi\colon \mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n)) \to \mathbb{CP}^n$ defined in the beginning of this section is a symplectomorphism.*

PROOF: The symplectic forms on $\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$ and $\mathbb{CP}^n$ can be recovered from the star products $\star$ and $\star_{\mathrm{red}}$ as the antisymmetrized first order. Since $\Phi^*$ intertwines the star products, it must also intertwine the symplectic forms. □

Recall the setup from Section I.3, where we defined complex manifolds $\hat{M}_{\mathrm{red}}$ and an antidiagonal embedding $\Delta_{\mathrm{red}}\colon M_{\mathrm{red}} \to \hat{M}_{\mathrm{red}}$. Remember that these definitions depend on the signature $s$, which we omit from the notation for better readability. We did not define extended products $\hat{\star}_{\mathrm{red},\hbar}$ in Paper I, so we will do so now. Similarly to the definition of polynomials on $M_{\mathrm{red}}$ in Section I.4.2, we can define holomorphic polynomials $\mathscr{P}(\hat{M}_{\mathrm{red}})$ as the image of $\mathbb{C}^*$-invariant holomorphic polynomials on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ under the reduction map $\cdot_{\hat{\mathrm{red}}}$. It is easy to verify that $(\Delta_{\mathrm{red}})^*\colon \mathscr{P}(\hat{M}_{\mathrm{red}}) \to \mathscr{P}(M_{\mathrm{red}})$ is a bijection. Consequently there is a uniquely determined product $\hat{\star}_{\mathrm{red},\hbar}\colon \mathscr{P}(\hat{M}_{\mathrm{red}}) \times \mathscr{P}(\hat{M}_{\mathrm{red}}) \to \mathscr{P}(\hat{M}_{\mathrm{red}})$ that restricts to $\star_{\mathrm{red},\hbar}$ on $M_{\mathrm{red}}$.

Recall also the definition of the stabilizer $\mathrm{G}_{\hat{\mathcal{J}}}$ from Section I.3. Assembling the $\sigma_j$ into a diagonal matrix $\sigma := \mathrm{diag}(\sigma_0, \dots, \sigma_n)$, we have $\mathrm{G}_{\hat{\mathcal{J}}} = \{(g', g) \in \mathrm{GL}_{1+n}(\mathbb{C}) \times \mathrm{GL}_{1+n}(\mathbb{C}) \mid g' = \sigma(g^{-1})^{\mathrm{T}}\sigma\}$. $\mathrm{G}_{\hat{\mathcal{J}}}$ acts on $\hat{M}_{\mathrm{red}}$ by $(g', g) \triangleright [(x, y)]_{\mathbb{C}^*} = [(g'x, gy)]_{\mathbb{C}^*}$. Since the subgroup $\{(z\mathbb{1}_{1+n}, z^{-1}\mathbb{1}_{1+n}) \mid z \in \mathbb{C}^*\} \subseteq \mathrm{G}_{\hat{\mathcal{J}}}$ acts trivially, we will usually only consider the action of the subgroup

$$\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}}) = \{(g', g) \in \mathrm{SL}_{1+n}(\mathbb{C}) \times \mathrm{SL}_{1+n}(\mathbb{C}) \mid g' = \sigma(g^{-1})^{\mathrm{T}}\sigma\}$$

of $\mathrm{G}_{\hat{\mathcal{J}}}$ in the following. Note that $\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}})$ still acts transitively on $\hat{M}_{\mathrm{red}}$.

Let $e_0 = (1, 0, \dots, 0) \in \mathbb{C}^{1+n}$. The stabilizer of $\mu := [e_0, e_0]_{\mathbb{C}^*} \in \hat{M}_{\mathrm{red}}$ is given by $\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}})_\mu := \{(\sigma(g^{-1})^{\mathrm{T}}\sigma, g) \mid g \in \mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))\}$, where

$$\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) := \{g \in \mathrm{SL}_{1+n}(\mathbb{C}) \mid g_{0,i} = g_{i,0} = 0 \text{ for } 1 \leq i \leq n\} \subseteq \mathrm{SL}_{1+n}(\mathbb{C})$$

denotes matrices with determinant 1 that are block diagonal with blocks of size 1 and $n$. In the following we identify $\mathrm{SL}_{1+n}(\mathbb{C})$ with $\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}})$ via the map $g \mapsto (\sigma(g^{-1})^{\mathrm{T}}\sigma, g)$. Under this correspondence, the subgroup $\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))$, that also appeared in (II.3.37b), is identified with $\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}})_\mu$. Therefore, similarly to the definition of $\Phi$, we obtain a holomorphic diffeomorphism

$$\begin{aligned} \hat{\Phi} \colon \mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) &\to \hat{M}_{\mathrm{red}}\,, \\ g \cdot \mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) &\mapsto (\sigma(g^{\mathrm{T}})^{-1}\sigma, g) \triangleright \mu = [\sigma(g^{\mathrm{T}})^{-1}\sigma e_0, g e_0]_{\mathbb{C}^*}\,. \end{aligned} \tag{A.7}$$

Indeed, surjectivity is clear by the transitivity of the action of $\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}})$ on $\hat{M}_{\mathrm{red}}$, injectivity follows since we divided out the stabilizer, being a diffeomorphism follows as before, and being holomorphic is obvious since all involved maps like transposition, inverse, and the action on $\hat{M}_{\mathrm{red}}$ are holomorphic. $\hat{\Phi}$ is also $\mathrm{SL}_{1+n}(\mathbb{C})$-equivariant (with respect to the left action of $\mathrm{SL}_{1+n}(\mathbb{C})$ on $\mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))$ and the action of $\mathrm{SL}_{1+n}(\mathbb{C})$ on $\hat{M}_{\mathrm{red}}$, obtained through the identification of $\mathrm{SL}_{1+n}(\mathbb{C})$ with $\mathrm{S}(\mathrm{G}_{\hat{\mathcal{J}}})$).

To better distinguish the next corollary from the following statements, we write out the signatures.

**Corollary A.9** *The pullback with the map $\hat{\Phi}^{(n+1)}$ defined in* (A.7) *intertwines the product $\hat{\star}^{(1+n)}_{\mathrm{red},\hbar}$ on $\mathscr{P}(\hat{M}^{(1+n)}_{\mathrm{red}})$ with the product $\hat{*}_\hbar$ on* $\mathrm{Pol}(\mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})))$ *defined by $F_\hbar$ from (II.3.39), meaning that*

$$(\hat{\Phi}^{(1+n)})^*(p \,\hat{\star}^{(1+n)}_{\mathrm{red},\hbar}\, q) = (\hat{\Phi}^{(1+n)})^*(p) \,\hat{*}_\hbar\, (\hat{\Phi}^{(1+n)})^*(q) \tag{A.8}$$

*holds for all $p, q \in \mathscr{P}(\hat{M}^{(1+n)}_{\mathrm{red}})$.*

Proof: For better readability we drop the signature $s = 1+n$ in this proof. The inclusion $\iota \colon \mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) \to \mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))$ is intertwined with $\Delta_{\mathrm{red}}$ by $\Phi$ and $\hat{\Phi}$, in the sense that $\Delta_{\mathrm{red}} \circ \Phi = \hat{\Phi} \circ \iota$. Indeed,

$$\begin{aligned} (\hat{\Phi} \circ \iota)(g \cdot \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))) &= [(g^{\mathrm{T}})^{-1} e_0, g e_0]_{\mathbb{C}^*} = [\overline{g} e_0, g e_0]_{\mathbb{C}^*} = \\ &= \Delta_{\mathrm{red}}([\overline{g} e_0]_{\mathrm{U}(1)}) = (\Delta_{\mathrm{red}} \circ \Phi)(g \cdot \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))) \end{aligned}$$

holds for all $g \in \mathrm{SU}(1+n)$. Recall that elements of $\mathscr{P}(M_{\mathrm{red}})$ extend uniquely to holomorphic polynomials $\mathscr{P}(\hat{M}_{\mathrm{red}})$ whereas elements of $\mathrm{Pol}(\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)))$ extend uniquely to holomorphic polynomials $\mathrm{Pol}(\mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})))$. Since the pullback $\Phi^* \colon \mathscr{P}(M_{\mathrm{red}}) \to \mathrm{Pol}(\mathrm{SU}(1+n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)))$ is an isomorphism according to Corollary A.4, and $\hat{\Phi}$ is holomorphic, it follows that $\hat{\Phi}^*$ is a bijection between $\mathscr{P}(\hat{M}_{\mathrm{red}})$ and $\mathrm{Pol}(\mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})))$.

The products $\hat{\star}_{\mathrm{red},\hbar}$ and $\hat{*}_\hbar$ are uniquely characterized by the property that they restrict to $\star_{\mathrm{red},\hbar}$ and $*_\hbar$ under the embeddings $\Delta_{\mathrm{red}}$ and $\iota$. Since $\Phi$ intertwines $\star_{\mathrm{red},\hbar}$ and $*_\hbar$ by Theorem A.5, it follows that $\hat{\Phi}$ intertwines $\hat{\star}_{\mathrm{red},\hbar}$ and $\hat{*}_\hbar$. □

We will use the Wick rotation to transfer this result to other signatures. Recall that the geometric Wick rotation $\alpha^{(s)}\colon \hat{M}^{(1+n)}_{\mathrm{red}} \to \hat{M}^{(s)}_{\mathrm{red}}$ defined in Definition I.6.1 intertwines $\hat{\star}^{(1+n)}_{\mathrm{red},\hbar}$ and $\hat{\star}^{(s)}_{\mathrm{red},\hbar}$.

**Corollary A.10** *The pullback with the map $\hat{\Phi}^{(s)}$ defined in* (A.7) *intertwines the product $\hat{\star}^{(s)}_{\mathrm{red},\hbar}$ on $\mathscr{P}(\hat{M}^{(s)}_{\mathrm{red}})$ with the product $\hat{*}_\hbar$ on* $\mathrm{Pol}(\mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})))$ *defined by $F_\hbar$ from (II.3.39), meaning that*

$$(\hat{\Phi}^{(s)})^*(p \mathbin{\hat{\star}^{(s)}_{\mathrm{red},\hbar}} q) = (\hat{\Phi}^{(s)})^*(p) \mathbin{\hat{*}_\hbar} (\hat{\Phi}^{(s)})^*(q) \tag{A.9}$$

*holds for all $p, q \in \mathscr{P}(\hat{M}^{(s)}_{\mathrm{red}})$.*

PROOF: We will determine which map $\alpha^{(s)}$ induces on $\mathrm{SL}_{1+n}(\mathbb{C})$, from which the result will follow since this map leaves $\hat{*}_\hbar$ invariant. The details are as follows.

Recall the definition of $W^{(s)}$ from the beginning of Section I.6.1. Let $\mathrm{L}_W$ be the map $\mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \to \mathrm{SL}_{1+n}(\mathbb{C})/\mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))$ given by the left action of $\det(W^{(s)})^{-1}W^{(s)} \in \mathrm{SL}_{1+n}(\mathbb{C})$. Then $\hat{\Phi}^{(s)} \circ \mathrm{L}_W = \alpha^{(s)} \circ \hat{\Phi}^{(1+n)}$. Indeed, we compute

$$\begin{aligned}
&(\hat{\Phi}^{(s)} \circ \mathrm{L}_W)(g \cdot \mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})))\\
&\qquad = \hat{\Phi}^{(s)}(\det(W^{(s)})^{-1}W^{(s)}g \cdot \mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})))\\
&\qquad = [\sigma^{(s)}((\det(W^{(s)})^{-1}W^{(s)}g)^{\mathrm{T}})^{-1}\sigma^{(s)}e_0, \det(W^{(s)})^{-1}W^{(s)}ge_0]_{\mathbb{C}^*}\\
&\qquad = [W^{(s)}(g^{\mathrm{T}})^{-1}e_0, W^{(s)}ge_0]_{\mathbb{C}^*}\\
&\qquad = (\alpha^{(s)} \circ \hat{\Phi}^{(1+n)})(g)
\end{aligned}$$

for all $g \in \mathrm{SL}_{1+n}(\mathbb{C})$ since $\sigma^{(s)}e_0 = e_0$, $(W^{(s)})^{\mathrm{T}} = W^{(s)}$, $(W^{(s)})^{-1} = \sigma^{(s)}W^{(s)}$, and $[zx, z^{-1}y]_{\mathbb{C}^*} = [x,y]_{\mathbb{C}^*}$ holds for $z \in \mathbb{C}^*$ and $x, y \in \mathbb{C}^{1+n}$. Applying the previous corollary to the pullbacks $(\alpha^{(s)})^*p$ and $(\alpha^{(s)})^*q$, and using that $\alpha^{(s)}$ intertwines $\hat{\star}^{(1+n)}_{\mathrm{red},\hbar}$ and $\hat{\star}^{(s)}_{\mathrm{red},\hbar}$, we obtain

$$\begin{aligned}
&((\hat{\Phi}^{(1+n)})^* \circ (\alpha^{(s)})^*(p)) \mathbin{\hat{*}_\hbar} ((\hat{\Phi}^{(1+n)})^* \circ (\alpha^{(s)})^*(q))\\
&\qquad = (\hat{\Phi}^{(1+n)})^*((\alpha^{(s)})^*(p) \mathbin{\hat{\star}^{(1+n)}_{\mathrm{red},\hbar}} (\alpha^{(s)})^*(q))\\
&\qquad = (\hat{\Phi}^{(1+n)})^* \circ (\alpha^{(s)})^*(p \mathbin{\hat{\star}^{(s)}_{\mathrm{red},\hbar}} q)\\
&\qquad = (\mathrm{L}_W)^* \circ (\hat{\Phi}^{(s)})^*(p \mathbin{\hat{\star}^{(s)}_{\mathrm{red},\hbar}} q)\,.
\end{aligned}$$

A direct computation gives

$$\begin{aligned}
&((\hat{\Phi}^{(1+n)})^* \circ (\alpha^{(s)})^*(p)) \mathbin{\hat{*}_\hbar} ((\hat{\Phi}^{(1+n)})^* \circ (\alpha^{(s)})^*(q))\\
&\qquad = ((\mathrm{L}_W)^* \circ (\hat{\Phi}^{(s)})^*(p)) \mathbin{\hat{*}_\hbar} ((\mathrm{L}_W)^* \circ (\hat{\Phi}^{(s)})^*(q))
\end{aligned}$$

$$= (\mathrm{L}_W)^*((\hat{\Phi}^{(s)})^*(p) \mathbin{\hat{*}_\hbar} (\hat{\Phi}^{(s)})^*(q))\,,$$

where we used in the last step that $\hat{*}_\hbar$ is $\mathrm{SL}_{1+n}(\mathbb{C})$-invariant. Equating the expressions and applying the pullback with the left action of $\det(W^{(s)}) \cdot (W^{(s)})^{-1} \in \mathrm{SL}_{1+n}(\mathbb{C})$ proves the corollary. □

**Corollary A.11** *The pullback with the restriction of $\hat{\Phi}^{(s)}$ to a map*

$$\Phi^{(s)} \colon \mathrm{SU}(s, 1+n-s)/\mathrm{S}(\mathrm{U}(s) \times \mathrm{U}(1+n-s)) \to M^{(s)}_{\mathrm{red}}$$

*intertwines the star product $\star^{(s)}_{\mathrm{red},\hbar}$ defined in Proposition I.5.22 with the star product $*^{(s)}_\hbar$ defined by the element $F_\hbar$ from Proposition II.3.36 for $r = 1$. That is, for all $p, q \in \mathscr{P}(M^{(s)}_{\mathrm{red}})$ we have*

$$(\Phi^{(s)})^*(p \star^{(s)}_{\mathrm{red},\hbar} q) = (\Phi^{(s)})^*(p) *^{(s)}_\hbar (\Phi^{(s)})^*(q)\,. \tag{A.10}$$

Proof: This follows immediately from the previous corollary by restriction. □

Note that for $s = 1$ the star product $*^{(1)}_\hbar$ is not the Wick type star product from Example II.5.24 which is induced by the element $F_\hbar$ from Corollary II.3.37, but is the anti-Wick type star product induced by the element $F_\hbar$ from Proposition II.3.36. We have already seen this in Example II.5.31.