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Topological dynamics, groupoids and C*-algebras

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PhD thesis

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Abstract

The thesis addresses the interplay between topological dynamics, groupoids and C*-algebras. Ever since the inception of Cuntz–Krieger algebras (and later graph C*-algebras), symbolic dynamical systems have been exploited to exhibit new and interesting examples of operator algebras. Via a groupoid reconstruction theory of Kumjian and Renault (and later refined by many authors), we can now trace finer structures of the C*-algebras back to properties of the dynamical systems, and structure-preserving *-isomorphisms between C*-algebras back to conjugacies, orbit equivalences or flow equivalences of the dynamical systems. The first part of the thesis contains a review of the literature on this question specifically for shift spaces, while the second part contains the original contributions of the thesis.

Papers A and B (joint with Toke Meier Carlsen) concern orbit equivalences and flow equivalences between shift spaces, while paper C characterizes diagonal-preserving and gauge-intertwining *-isomorphisms of graph C*-algebras in terms of moves on the graphs. The paper D (joint with Eduardo Scarparo) studies the topological full group of groupoids and gives conditions for these groups to be C*-simple.

Resumé

Denne afhandling adresserer sammenspillet mellem topologisk dynamik, gruppoider og C*-algebraer. Siden indførelsen af Cuntz–Krieger algebraer (og senere graf-C*-algebraer), er symbolske dynamiske systemer blevet anvendt til at frembringe nye og interessante eksempler på operatoralgebraer. Via en gruppoid-rekonstruktionsteori af Kumjian og Renault (som senere er blevet raffineret af mange forfattere), kan vi nu spore visse finere strukturer ved C*-algebraerne tilbage til egenskaber ved de dynamiske systemer, og strukturbevarende *-isomorfier tilbage til konjugeringer, bane-ækvivalenser eller flow-ækvivalenser mellem de dynamiske systemer. Afhandlingens første del indeholder en gennemgang af litteraturen for dette spørgsmål med skiftrum som specifik ledetråd, mens anden del indeholder afhandlingens originale bidrag.

Artikel A og B (i samarbejde med Toke Meier Carlsen) omhandler baneækvivalenser og strømningsækvivalenser mellem skiftrum, mens artikel C karakteriserer diagonalbevarende og cirkelvirkningssammenflettende *-isomorfi af graf-C*-algebraer i termer af *moves* på graferne. Artiklen D (i samarbejde med Eduardo Scarparo) studerer den topologisk fulde gruppe af en gruppoid og giver betingelser for, hvornår disse grupper er C*-simple.

Contents

Preface

Prologar cuentos no leídos aún es tarea casi imposible,
ya que exige el análisis de tramas que no conviene anticipar.
J.L. Borges

The ability of mathematics to accurately model reality is an overwhelming and mysterious quality only paralleled by its propensity to isolate itself in engaged study of ... itself.

Dynamical systems is a branch of mathematics which in some sense aims to model time evolution of a system, be it physical or informational. For discrete systems, we chop up time into small quanta and see what happens as we let the clock run either forward or backward. We can think of such a system over the span of time as a string of symbols, each symbol representing the state of the system. We imagine these strings to be never-ending, in fact *infinitely* long, and this hypothesis of infinity actually serves to simplify matters — another curiosity of mathematics.

On the other hand, the theory of operator algebras, the inception of Hilbert space and operators on it originated most notably with John von Neumann and the attempt to find a rigorous mathematical foundation for the newly discovered quantum mechanics. Almost singlehandedly, he revolutionized mathematics with this gem of a discovery. The potential of operator algebras and C*-algebras was not overlooked and many brilliant minds have contributed to curiously investigating and refining this area of research. Even if operator algebras started with the study of physics, not much is left of this legacy when taught today, except for terminology perhaps. A prime characteristic however is its ability to connect with other branches in order to transport mutually beneficial ideas back and forth.

For a locally compact space X and a homeomorphism $\varphi\colon X \longrightarrow X$, the pair (X, φ) defines a topological dynamical system. By Gelfand duality, the complex-valued continuous functions vanishing at infinity $C_0(X)$ is a prominent example of a commutative C*-algebra and there is an induced *-isomorphism $\varphi^*\colon C_0(X) \longrightarrow C_0(X)$; by a slight stretch of the imagination we think of $C_0(X)$ as a system and φ^* as time evolution. The crossed product construction exemplifies the power of operator algebras by taking the pair $(C_0(X), \varphi^*)$ and constructing a *noncommutative* C*-algebra $C_0(X) \rtimes_{\varphi^*} \mathbb{Z}$ which captures the dynamics in a single object at the expense of commutativity. This fundamental idea is also at the core of this thesis!

A finite square and nonnegative integer matrix A determines a finite graph whose path space naturally carries the structure of a discrete dynamical system, a *topological Markov shift*. Vertices represent states and edges determine how one state can be followed by the next. In symbolic dynamics, the path space is an example of a shift space of *finite type*. For strongly connected graphs which are not just a single cycle, Cuntz and Krieger constructed a rich class of simple C*-algebras, today known as Cuntz–Krieger algebras. Their construction differs dramatically from that of the crossed product and this is reflected in the operator algebraic properties of these C*-algebras. Recent years have seen numerous developments and generalizations of this idea. The construction laid the ground for a firm connection between finite type symbolic dynamics and C*-algebras, a foundation on which this thesis is also built.

The most basic of questions is to determine *when a pair of dynamical systems are the same*. From the perspective of operator algebras, much work has then gone into invariance properties: If two systems are the same in a suitable sense, then the C*-algebras should be the same in a correspondingly suitable sense. This somehow works as a sanity check for the utility of a given construction. Cuntz and Krieger knew that conjugate two-sided shifts of finite type produces stably isomorphic C*-algebras even in a way which preserves a canonical abelian subalgebra called the diagonal. In fact, this is also true for the much coarser (but not less important) relation of flow equivalence. However, many years should pass before Matsumoto and Matui — with a brilliant use of groupoid techniques — proved that the stable isomorphism class of the Cuntz–Krieger algebra together with its canonical diagonal subalgebra completely remembers the flow class of the underlying dynamical system.

This groundbreaking discovery of Matsumoto and Matui has in recent years spawned a newfound rapture in the study of the finer structures of C*-algebras associated to dynamical systems and structure preserving *-isomorphism between them. More generally, topological groupoids and their C*-algebras have become mainstream and serve as a versatile picture of an increasingly (and surprisingly) large body of examples of C*-algebras. The present thesis with its contributions is to be viewed under this lens.

Shifts of finite type are but a small (but important) class of symbolic dynamical systems. The succes of associating C*-algebras to such systems and as a consequence *understanding* the dynamics involved naturally begs the question of whether a similar theory (with a similar payoff) can be developed for general symbolic dynamical systems. Authors such as Kengo Matsumoto, Toke Meier Carlsen and Klaus Thomsen have worked diligently to extend the results of finite type systems to general shift spaces. Unfortunately, there have been mistakes in the literature and various (nonisomorphic!) constructions. Hopefully, this document serves to clarify this story a bit.

Of course, the outline above is not strictly speaking true. Many authors have worked tirelessly over the coarse of several decades to produce results akin to the theorems described above. Similarly, there are many new perspectives flourishing which are beyond the scope of this thesis to include or mention. I apologize for that.

The thesis consists of two parts. Part 1 contains two chapters the first of which is a brief review of the story and literature of associating C*-algebras to general shift spaces. The second chapter describes various on-going projects and intriguing questions yet to be answered. Part 2 consists of four papers which comprise the original contributions of the thesis. Papers A and B are written joint with Toke Meier Carlsen, while paper D is written joint with Eduardo Scarparo. There is a single list of reference in the end of the document and this contains all references from both parts.

Acknowledgements

After reasonably many years of university studies, I do not claim to fully comprehend the nature of mathematics. At all. I have learned however to appreciate its versatility, beauty (to my own surprise) and wisdom. My advisor Søren Eilers has taught me to stay motivated and inspired, even when times were hard and the details just did not work out. This has significantly improved the quality of my time as a Ph.D. student, and for this I am deeply grateful, thank you. Thank you to my coauthors Eduardo Scarparo and Toke Meier Carlsen for sharing your ideas with me and for very enjoyable collaborations from which I have benefitted immensely. Also thank you James Gabe for your guidance over the years and the many discussions we have had.

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Gracias totales!

Kevin Aguyar Brix
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Copenhagen

Part 1

Introduction

CHAPTER 1

C*-algebras of shift spaces

Allt är kontruktion
The Narrator, Reconstruction

Cuntz–Krieger algebras arising from irreducible and nonpermutation $\{0, 1\}$-matrices provide a rich and interesting class of simple C*-algebras. This was later generalized to infinite matrices and general directed graphs (see, e.g., [**47, 65**]). and the study of graph C*-algebras occupies a prominent corner of C*-algebra theory because we can *see* the algebras in the graphs. On the other hand, the irreducible and nonpermutation $\{0, 1\}$-matrices are intimately connected to irreducible shifts of finite type. The question is: *how can we associate a* C*-*algebras to larger classes of shift spaces and, if so, how do they relate to the underlying dynamical systems?* In this chapter we sketch the story of associating a C*-algebra to a general shift spaces.

In Section 1.1, we give a detailed introduction to one-sided and two-sided shift spaces, and in Section 1.2 we describe some of the constructions of C*-algebras associated to symbolic dynamical systems. The question of how much the C*-algebra *remembers* of the underlying dynamics and the possibility of reconstructing the dynamical system (up to some coarse equivalence) using groupoids is described in Section 1.3.
Let $\mathbb{Z}$, $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$ be the integers, the nonnegative integers and the positive integers, respectively.

1.1. Symbolic dynamics

Let $\mathfrak{A}$ be a nonempty finite set equipped with the discrete topology. We call it the *alphabet* and its elements *letters* or *symbols*. If $\mathfrak{A}$ contains N elements, then the *full two-sided N-shift* is the space $\mathfrak{A}^{\mathbb{Z}}$ (with the product topology) together with the shift operation $\sigma\colon \mathfrak{A}^{\mathbb{Z}} \longrightarrow \mathfrak{A}^{\mathbb{Z}}$ given by $\sigma(\mathrm{x})_i = x_{i+1}$ for $i \in \mathbb{Z}$ and $\mathrm{x} = (x_i)_{i\in\mathbb{Z}} \in \mathfrak{A}^{\mathbb{Z}}$. This is a homeomorphism and $\mathfrak{A}^{\mathbb{Z}}$ is compact and Hausdorff. Replacing $\mathbb{Z}$ by $\mathbb{N}$, we obtain the *full one-sided N-shift* $\mathfrak{A}^{\mathbb{N}}$ whose shift operation $\sigma\colon \mathfrak{A}^{\mathbb{N}} \longrightarrow \mathfrak{A}^{\mathbb{N}}$ given by $\sigma(x)_i = x_{i+1}$ for $i \in \mathbb{N}$ and $x = (x_i)_{i\in\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ is a surjective local homeomorphism. Topologically, the full shifts are (homeomorphic to) Cantor sets. In classical symbolic dynamics, much effort has been devoted to the study of two-sided subshifts (reversible dynamics). From the point of view of operator algebras and groupoids, however, we shall see that it is more natural to consider one-sided shift spaces (irreversible dynamics). We introduce basic notation and results needed for this thesis without proofs. The interested reader is refered to [**68, 62**] for excellent introductions to the theory of symbolic dynamics.

1.1.1. Shift spaces. A *one-sided shift space* over $\mathfrak{A}$ is a closed subspace $\mathsf{X} \subseteq \mathfrak{A}^{\mathbb{N}}$ which is shift-invariant in the sense that $\sigma(\mathsf{X}) \subseteq \mathsf{X}$ (we do not assume equality) together with the restricted shift operation $\sigma_{\mathsf{X}} = \sigma|_{\mathsf{X}} \colon \mathsf{X} \longrightarrow \mathsf{X}$. Analogously, a *two-sided subshift* is a subspace $\Lambda \subseteq \mathfrak{A}^{\mathbb{Z}}$ which is closed and shift-invariant in the sense that $\sigma(\Lambda) = \Lambda$ with the restricted shift operation $\sigma_\Lambda = \sigma|_\Lambda \colon \Lambda \longrightarrow \Lambda$ which is a homeomorphism.

Let X be a one-sided shift space over $\mathfrak{A}$. If $x = (x_i)_{i\in\mathbb{N}} \in \mathsf{X}$ we write $x_{[i,j]} = x_i \cdots x_j$, for integers $0 \leqslant i \leqslant j$. Similarly, $x_{[i,j)} = x_{[i,j-1]}$ and $x_{(i,j]} = x_{[i+1,j]}$ whenever $i < j$. We extend this notation so that $x = x_{[0,\infty)}$. A *word* in X is a finite string $\alpha = \alpha_1 \cdots \alpha_n$ such that $\alpha_k \in \mathfrak{A}$ for all $k = 1, \ldots, n$ and $\alpha = x_{[i,j]}$ for some integers $0 \leqslant i \leqslant j$, and $|\alpha| = n$ is the *length* of α. Let $\mathsf{L}_n(\mathsf{X})$ be the set of all words in X of length $n \in \mathbb{N}$. The empty word ϵ is the unique element of $\mathsf{L}_0(\mathsf{X})$. The *language* of X is the collection of all finite words $\mathsf{L}(\mathsf{X}) = \bigcup_{n\geqslant 0} \mathsf{L}_n(\mathsf{X})$. Given two words $\mu, \nu \in \mathsf{L}(\mathsf{X})$ the concatenation $\mu\nu$ is again a word in X if and only if there exist $x \in \mathsf{X}$ and $i \in \mathbb{N}$ such that $x_{[i,i+|\mu|+|\nu|)} = \mu\nu$. The language determines its shift space completely in the sense that two languages are equal if and only if the shifts are the same. Furthermore, given a word $\mu \in \mathsf{L}(\mathsf{X})$ the *cylinder set* of μ is the compact and open subset

$$Z(\mu) = \{x \in \mathsf{X} \mid x_{[0,|\mu|)} = \mu\},$$

and the collection of cylinder sets constitutes a basis for the topology of X. A two-sided subshift Λ determines a one-sided shift X with surjective σ_{X} by

$$\mathsf{X} = \{\mathrm{x}_{[i,\infty)} \mid \mathrm{x} \in \Lambda, i \in \mathbb{Z}\},$$

and we write Λ_{X} to emphasize that X is the one-sided part of Λ. Conversely, a one-sided shift X for which σ_{X} is surjective determines a two-sided subshift Λ_{X} as the projective limit $\Lambda_{\mathsf{X}} = \varprojlim(\mathsf{X}, \sigma_{\mathsf{X}})$.

The language of a shift space contains the *allowed* words in X. Conversely, any collection of words $\mathfrak{F} \subseteq \mathsf{L}(\mathfrak{A}^{\mathbb{N}})$ determines a shift space $\mathsf{X}_{\mathfrak{F}}$ for which the elements of $\mathfrak{F}$ do not occur. We think of the elements of $\mathfrak{F}$ as the *forbidden* words. Different sets of forbidden words can determine the same shift space and any shift space has a (not necessarily unique) collection of forbidden words.

Definition 1.1.1. A shift space is of *finite type* if the collection of forbidden words can be chosen to be finite.

The shifts of finite type are arguably the most important class of shift spaces and the example below indicates this. We say that X is *irreducible* if for every ordered pair of words $\mu, \nu \in \mathsf{L}(\mathsf{X})$ there exists $\alpha \in \mathsf{L}(\mathsf{X})$ such that $\mu\alpha\nu \in \mathsf{L}(\mathsf{X})$.

Example 1.1.2. Let $E = (E^0, E^1, r, s)$ be a finite directed graph. That is, E^0 and E^1 are finite sets of vertices and edges, respectively, and $r, s \colon E^1 \longrightarrow E^0$ are the range and source maps, respectively. If E contains no sinks (that is, $s^{-1}(v) \neq \emptyset$ for every vertex $v \in E^0$), then the path space

$$E^\infty = \{x = (x_i)_i \in (E^1)^{\mathbb{N}} \mid r(x_i) = s(x_{i+1}), i \in \mathbb{N}\}$$

with the shift operation $\sigma_E \colon E^\infty \longrightarrow E^\infty$ given by $\sigma_E(x)_i = x_{i+1}$ for $x \in E^\infty$ and $i \in \mathbb{N}$ is a shift of finite type over the alphabet E^1 called the *edge shift* of E. The strength of

this picture is that many properties of the dynamical system can be seen directly on the graph. E.g., the shift operation is surjective if and only if E contains no sources (that is, $r^{-1}(v) \neq \emptyset$ for all $v \in E^0$), and the edge shift is irreducible if and only if the graph is strongly connected. The edge shift operation is a local homeomorphism; in fact, σ_{X} is a local homeomorphism exactly when X is of finite type, cf. [**96**].
Every shift space of finite type is (conjugate to) the edge shift of a finite directed graph, so these examples actually exhaust the class of finite type shifts.

The notion of a conjugacy determines when two systems are *the same*.

Definition 1.1.3. Let X and Y be a pair of one-sided shift spaces. A *conjugacy* is a homeomorphism $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ which intertwines the shift operations $h \circ \sigma_{\mathsf{X}} = \sigma_{\mathsf{Y}} \circ h$. A similar definition applies to two-sided subshifts.

It is not hard to see that the class of finite type shifts is invariant under conjugacy. Arguably, the biggest problem in the field of symbolic dynamics, however, is to determine when two shift spaces (of finite type) are conjugate. The seminal paper of Williams [**116**] made impressive progress on this question. For one-sided shifts of finite type there is a surprisingly simple answer: An amalgamation process on the adjacency matrices of the graphs which represent the shifts of finite type completely determines the conjugacy class, see also [**8, 62**]. For two-sided subshifts this problem seems much harder. Williams introduced the notion of *shift equivalence* (which is more combinatorial in flavor) between nonnegative integral matrices and thought this was equivalent to conjugacy of the corresponding subshifts. Although this was not the case (see the counterexamples of Kim and Roush [**60, 61**]), this relation is interesting in itself. Shift equivalence is weaker than conjugacy but it does not seem to be amenable to the approaches and techniques which have flourished in recent years and which play a pivotal rôle in this thesis (see Section 2.2 for a brief discussion of this relation and certain problems associated with it). In Paper C, we briefly relate the complexity of the conjugacy problem to a similar problem of one-sided finite type shifts.

There is a more general notion of morphism between shift spaces. A *sliding block code* $\varphi\colon \mathsf{X} \longrightarrow \mathsf{Y}$ is a continuous map satisfying $\varphi \circ \sigma_{\mathsf{X}} = \sigma_{\mathsf{Y}} \circ \varphi$. Since shift spaces are compact and Hausdorff, a conjugacy is a bijective sliding block code. Surjective sliding block codes are called *factor maps* while injective sliding block codes are called *embeddings*. The image of a shift space under a sliding block code is again a shift spaces, but the image of a finite type shift need not be of finite type. This is an intrinsic problem with the shifts of finite type. Weiss [**114**] studied the broader class of shift spaces which contains the finite type systems and is closed under taking images of sliding block codes and called them *sofic shifts*. Whereas finite type shifts are represented by finite directed graphs, sofic shifts are instead represented by finite directed and *labeled* graphs.

Example 1.1.4. Let $E = (E^0, E^1, r, s)$ be a finite directed graph and let $\mathfrak{A}$ be a finite set of labels with a surjective labeling map $\mathcal{L}\colon E^1 \longrightarrow \mathfrak{A}$. Then $\mathcal{L}$ extends to a map on the infinite path space $\mathcal{L}_\infty\colon E^\infty \longrightarrow \mathfrak{A}^{\mathbb{N}}$ by $\mathcal{L}_\infty(x) = \mathcal{L}(x_0)\mathcal{L}(x_1)\cdots$ for $x \in \mathsf{X}$. The *labeled path space*

$$\mathsf{X}_{E,\mathcal{L}} = \{\lambda \in \mathfrak{A}^{\mathbb{N}} \mid \exists x \in E^\infty : \lambda = \mathcal{L}_\infty(x)\} = \mathcal{L}_\infty(E^\infty)\}$$

with the obvious shift operation defines a one-sided sofic shift over the alphabet $\mathfrak{A}$. We say that the sofic shift $\mathsf{X}_{E,\mathcal{L}}$ is *represented* by the graph E. The same sofic shift can be represented by many different graphs and any sofic shift can be represented by a labeled graph [**51**].

There are several relevant constructions of graphs associated to sofic shifts, see, e.g., [**55**]. We shall now describe the *Krieger cover* which is of central importance here. Consider the *predecessor set*

$$P_l(x) = \{\mu \in \mathsf{L}_l(\mathsf{X}) \mid \mu x \in \mathsf{X}\},$$

for $l \in \mathbb{N}$ and $x \in \mathsf{X}$, and put $P_\infty(x) = \bigcup_{l\in\mathbb{N}} P_l(x)$. Two elements $x, x' \in \mathsf{X}$ are l*-past equivalent*, written $x \sim_l x'$, if $P_j(x) = P_j(x')$ for $j = 0, \dots, l$, cf. [**71**]. This formulation is slightly different from Matsumoto's because we want to include the case where σ_{X} is not surjective. This is an equivalence relation, and we let $[x]_l$ be the l-past equivalence class of x. Set $\Omega_l = \mathsf{X}/\sim_l$. Since $x \sim_l x'$ implies $x \sim_{l-1} x'$, the spaces $(\Omega_l)_{l\geqslant 0}$ define a projective system which has been studied by Matsumoto [**69**] among others. Weiss [**114**] has shown that X is sofic if and only if the collection of predecessor sets is finite (X is P-finitary); equivalently, the system $(\Omega_l)_{l\geqslant 0}$ is eventually constant.

Definition 1.1.5 ([**63**]). Suppose X is a sofic shift and choose $l \in \mathbb{N}$ such that $\Omega_l = \Omega_{l+i}$ for all $i \in \mathbb{N}$. Let m_{X} be the cardinality of Ω_l and let $E_1, \dots, E_{m_{\mathsf{X}}}$ be the l-past equivalence classes of X. The *left Krieger cover graph* of X is given as follows:
Let $\mathfrak{A} = \{1, \dots, m_{\mathsf{X}}\}$ and construct a graph K with vertex set $K^0 = \{E_1, \dots, E_{m_{\mathsf{X}}}\}$. There is an edge from E_k to E_i with label $j \in \mathfrak{A}$ if and only if there exists $x \in \mathsf{X}$ such that $E_k = P(jx)$ and $E_i = P(x)$.

Formally, a cover of a sofic shift X is a shift of finite type equipped with a factor map onto X. For our purposes, the left Krieger cover graph is the most important cover construction. Alternative constructions such as the Fischer cover [**51**] of X are irreducible whenever X is irreducible and enjoy a certain minimality property. Albeit this is not the case for the Krieger cover this is actually not a disadvantage; instead the Krieger cover detects certain properties that the Fischer cover does not see.

For general one-sided shifts, we propose a more general definition: A *cover* is a compact space equipped with a self-map which is a local homeomorphism (a Deaconu–Renault system [**110**]) and a factor map onto the shift space. In this sense, the space $\widetilde{\mathsf{X}}$ associated to a one-sided shift space X presented in Paper B is a cover which generalizes the left Krieger cover. We refer the reader to [**55**] and references therein for a discussion of different covers of sofic subshifts. To the author's knowledge the only theory of covers associated to general shift spaces are Carlsen's covers [**19**] (elaborated in Paper B) and Matsumoto's λ-graph systems [**74**] briefly described below.

Finally, we mention two conditions will be relevant in the next section.

Definition 1.1.6 ([**71**]). A one-sided shift space X satisfies *Matsumoto's condition (I)* if for any $l \in \mathbb{N}$ and $x \in \mathsf{X}$ there exists $y \in \mathsf{X}$ such that $y \neq x$ and $y \sim_l x$.

Matsumoto's condition (I) is a generalization of Cuntz and Krieger's condition (I) [**35**] introduced for shifts of finite type to ensure the nonexistence of isolated points. A point

x in a general one-sided shift X is *isolated in past equivalence* if $[x]_l$ is a singleton for some $l \in \mathbb{N}$. Matsumoto's condition (I) says that no points are isolated in past equivalence. In Paper B, we show that this is equivalent to the condition that the cover $\widetilde{\mathsf{X}}$ contains no isolated points. For certain purposes we shall instead consider the weaker condition that no *periodic points* be isolated in past equivalence.

Definition 1.1.7 ([**23**]). A two-sided subshift Λ satisfies *condition* $(*)$ if for each $l \in \mathbb{N}$ and each sequence of words $(\mu_i)_{i\in\mathbb{N}}$ satisfying $P_l(\mu_i) = P_l(\mu_j)$ for all $i, j \in \mathbb{N}$ there exists $x \in \mathsf{X}_\Lambda$ such that

$$P_l(x) = P_l(\mu_i),$$

for $i \in \mathbb{N}$.

1.2. C*-algebra constructions

In this section, we address the problem of associating C*-algebra to general shift spaces. As we shall see, this problem is intimately tied together with the association of covers to general shift spaces. To improve the readability, we shall denote the C*-algebras differently from the original papers in which they appear (since a few of them are defined using the same symbols). The original Fock space construction of Matsumoto [**69**] is here denoted $\mathcal{M}_{\Lambda^*}$, (originally $\mathcal{O}_\Lambda$) and the alternative construction of Carlsen and Matsumoto [**23**] is here denoted $\mathcal{M}_\Lambda$ (originally $\mathcal{O}_\Lambda$). Both of these constructions apply to two-sided subshifts Λ. However, we are mostly interested in constructions which work for one-sided shifts X. This was first introduced by Carlsen [**20**] and denoted $\mathcal{O}_{\mathsf{X}}$. The interested reader is refered to [**23, 27, 38**] for similar expositions.

1.2.1. Matsumoto's original construction. We describe Matsumoto's Fock space construction of a C*-algebra $\mathcal{M}_{\Lambda^*}$ of a two-sided subshift Λ as it appeared in [**69**].
Let Λ be a two-sided subshift over $\mathfrak{A}$. For any $n \in \mathbb{N}_+$, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Define

$$\begin{aligned} F_\Lambda^0 =& \mathbb{C}e_0, \\ F_\Lambda^k =& \operatorname{span}\{e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \mid \mu = \mu_1 \dots \mu_k \in \mathsf{L}_k(\Lambda)\}, \\ F_\Lambda =& \bigoplus_{k=1}^{\infty} F_\Lambda^k \end{aligned}$$

where e_0 is the *vacuum vector* and each F_Λ^k is understood to be a Hilbert space so that $\bigoplus$ denotes a sum of Hilbert spaces. For each $i \in \mathfrak{A}(\Lambda)$, the *creation operator of* e_i $T_i \in \mathbb{B}(F_\Lambda)$ is given as

$$T_i e_0 = e_i, \qquad T_i e_\mu = \begin{cases} e_i \otimes e_\mu & i\mu \in \mathsf{L}(\Lambda), \\ 0 & \text{otherwise.} \end{cases}$$

for $e_\mu \in \mathsf{L}(\Lambda)$. Note that T_i is a partial isometry. Let P_0 denote the projection onto the subspace spanned by the vacuum vector e_0. Then $\sum_{i\in\mathfrak{A}(\Lambda)} T_i T_i^* + P_0 = 1$ in $\mathbb{B}(F_\Lambda)$. The operators of the form $T_\mu P_0 T_\nu^*$ are rank one partial isometries taking e_ν to e_μ and the collection of these operators span the compact operators $\mathbb{K}(F_\Lambda)$. Let $\mathscr{T}_\Lambda$ be the C*-algebra generated by the operators $\{T_\mu\}_{\mu\in\mathsf{L}(\Lambda)}$ in $\mathbb{B}(F_\Lambda)$.

Definition 1.2.1 ([**69**]). The C*-algebra $\mathcal{M}_{\Lambda^*}$ associated to the two-sided subshift Λ is defined as the quotient $\mathscr{T}_\Lambda/\mathbb{K}(F_\Lambda)$. If S_i denotes the image of T_i in $\mathcal{M}_{\Lambda^*}$ for each $i \in \mathfrak{A}(\Lambda)$, then $\sum_i S_i S_i^* = 1$ in $\mathcal{M}_{\Lambda^*}$.

Let $\mathcal{H}_\Lambda$ be the Hilbert space with orthonormal basis $\{e_x \mid x \in \mathsf{X}_\Lambda\}$. For each $i \in \mathfrak{A}$, define operators $T_i \in \mathbb{B}(\mathcal{H}_\Lambda)$ by

$$T_i e_x = \begin{cases} e_{ix} & ix \in \mathsf{X}_\Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then each T_i is a partial isometry satisfying $\sum_{i\in\mathfrak{A}(\Lambda)} T_i T_i^* = 1$. In [**75**, Lemma 4.1], Matsumoto considers the representation $\mathcal{M}_{\Lambda^*} \longrightarrow \mathbb{B}(\mathcal{H}_\Lambda)$ given by sending $S_i \longmapsto T_i$ for $i \in \mathfrak{A}$ and claims that this is nondegenerate and *faithful*. This is however not the case unless Λ satisfies condition (I) and $(*)$ and this problem is related to the flawed statement that the C*-algebra $\mathscr{A}_l$ in $\mathcal{M}_{\Lambda^*}$ generated by the projections $S_\mu^* S_\mu$, for $\mu \in \mathsf{L}_l(\Lambda)$, is *-isomorphic to $C(\Omega_l)$. This observation did, however, inspire the next concrete definition of Carlsen and Matsumoto.

Definition 1.2.2 ([**23**]). The C*-algebra $\mathcal{M}_\Lambda$ is defined as the C*-algebra generated by the operators T_i in $\mathbb{B}(\mathcal{H}_\Lambda)$.

The C*-algebra $\mathcal{M}_\Lambda$ satisfies all the results of Matsumoto but in general it does not admit a gauge action. This is not satisfying.

1.2.2. Carlsen's construction. We recall a Cuntz–Pimsner construction associated to any one-sided shift space X over the alphabet $\mathfrak{A}$ [**20**]. Consider the commutative C*-algebra

$$\mathcal{D}_\mathsf{X} = \mathrm{C}^*\{1_{C(\mu,\nu)} \mid \mu, \nu \in \mathsf{L}(\mathsf{X})\}$$

inside the C*-algebra of bounded functions on X. The spectrum of $\mathcal{D}_\mathsf{X}$ is (homeomorphic to) the cover $\widetilde{\mathsf{X}}$ mentioned above. Let $\mathcal{D}_i \subseteq \mathcal{D}_\mathsf{X}$ be the ideal generated by the function $1_{\sigma_\mathsf{X}(Z(i))}$ and consider the right Hilbert $\mathcal{D}_\mathsf{X}$-module

$$\mathsf{H}_\mathsf{X} = \bigoplus_{i\in\mathfrak{A}} \mathcal{D}_i$$

with the inner product

$$\langle (f_i)_i, (g_i)_i \rangle = \sum_{i\in\mathfrak{A}} f_i^* g_i,$$

for $(f_i)_i, (g_i)_i \in \mathsf{H}_\mathsf{X}$. Since each $\mathcal{D}_i$ is an ideal there is a right action of $\mathcal{D}_\mathsf{X}$ given by

$$(f_i)_i \cdot f = f_i f_i$$

for $(f_i)_i \in \mathsf{H}_\mathsf{X}$ and $f \in \mathcal{D}_\mathsf{X}$. Furthermore, for each $i \in \mathfrak{A}$ there is a *-homomorphism on the bounded functions of X given by

$$\lambda_i(f)(x) = \begin{cases} f(ax) & \text{if } ax \in \mathsf{X}, \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in \mathsf{X}$. Let $\mathcal{L}(\mathsf{H}_\mathsf{X})$ denote the C*-algebra of adjointable operators on H_X. Then this defines a left action $\phi\colon \mathcal{D}_\mathsf{X} \longrightarrow \mathcal{L}(\mathsf{H}_\mathsf{X})$ by

$$\phi(f)((f_i)_i) = (\lambda_i(f) f_i)_i$$

for $f \in \mathcal{D}_{\mathsf{X}}$ and $(f_i)_i \in \mathsf{H}_{\mathsf{X}}$. Hence H_{X} is a C*-correspondence.

DEFINITION 1.2.3 ([**20**]). Let X be a one-sided shift space. Then $\mathcal{O}_{\mathsf{X}}$ is defined as the C*-algebra of the C*-correspondence H_{X}.

The C*-algebra $\mathcal{O}_{\mathsf{X}}$ enjoys the following universal property.

THEOREM 1.2.4 ([**20**]). *Let X be a one-sided shift space over $\mathfrak{A}$. Then $\mathcal{O}_{\mathsf{X}}$ is the universal unital C*-algebra generated by partial isometries $\{S_i\}_{i\in\mathfrak{A}}$ such that $S_\mu S_\nu = S_{\mu\nu}$ and such that the map*

$$1_{C(\mu,\nu)} \longmapsto S_\nu S_\mu^* S_\mu S_\nu^*$$

*extends to a *-homomorphism from $\mathcal{D}_{\mathsf{X}}$ into the C*-algebra generated by $\{S_i\}_i$, where $S_\mu = S_{\mu_1} \cdots S_{\mu_{|\mu|}}$ and $S_\nu = S_{\nu_1} \cdots S_{\nu_{|\nu|}}$.*

The next result relates the constructions we have seen hitherto.

THEOREM 1.2.5 ([**27**]). *Let Λ_{X} be a two-sided subshift and let X be the associated one-sided shift. Then there are surjective *-homomorphisms*

$$\mathcal{M}_{\Lambda^*} \longrightarrow \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{M}_\Lambda$$

which map the canonical generators to the canonical generators. The first map is injective if Λ satisfies condition $(*)$ while the second is injective if Λ satisfies Matsumoto's condition (I).

REMARK 1.2.6. Carlsen and Matsumoto remark that infinite and irreducible shifts of finite type satisfy both condition (I) and $(*)$. This is also the case for the class of β-shifts and for the context free shift. The shift consisting of a single point does not satisfy Matsumoto's condition (I) and the map is not injective. In [**23**, Section 4] there is an example of an irreducible sofic shift Λ which does no satisfy condition $(*)$ and such that the C*-algebras $\mathcal{M}_{\Lambda^*}$ and $\mathcal{M}_\Lambda$ are not stably isomorphic, the latter being simple while the first is not.

For our purposes, $\mathcal{O}_{\mathsf{X}}$ is the *right* C*-algebra to work with. There are several equivalent constructions available in the literature. Thomsen described it as a groupoid C*-algebra of a semi-étale groupoid (with unit space homeomorphic to X) [**113, 30**], Carlsen and Silvestrov viewed it as one of Exel's crossed product by an endomorphism and derived K-theory formulae [**27, 28**], while Dokuchaev and Exel used partial actions and characterized simplicity of $\mathcal{O}_{\mathsf{X}}$ [**38**] in terms of properties of X. In this sense, the C*-algebras of general shifts are well-studied and well-understood. However, there is a gap in the literature when it comes to understanding how structure preserving *-isomorphisms of the C*-algebras relate to dynamical properties of the spaces. In [**19**, Chapter 2], Carlsen constructed a cover $\widetilde{\mathsf{X}}$ from a one-sided shift X and built a Renault–Deaconu groupoid $\mathcal{G}_{\mathsf{X}}$ (with unit space homeomorphic to $\widetilde{\mathsf{X}}$) whose C*-algebra is $\mathcal{O}_{\mathsf{X}}$. This latter approach is the basis of Paper B. Starling later identified $\widetilde{\mathsf{X}}$ as the tight spectrum of a certain inverse semigroup $\mathcal{S}_{\mathsf{X}}$ associated to X and then identified $\mathcal{O}_{\mathsf{X}}$ with $C^*{}_{\text{tight}}(\mathcal{S}_{\mathsf{X}})$, Exel's tight C*-algebra of $\mathcal{S}_{\mathsf{X}}$, [**111, 45**]. Recently, Exel and Steinberg have further investigated semigroups of shift spaces and shown that there is a universal groupoid which can be suitably restricted to model either Matsumoto's C*-algebra or $\mathcal{O}_{\mathsf{X}}$ [**48**].

Despite the choice of particular construction, the space $\widetilde{\mathsf{X}}$ seems unavoidable — it is the *spectrum of the ill fated commutative algebra* $\mathcal{D}_{\mathsf{X}}$ *... which has evaded all attempts at analysis ...* and whose *descriptions are often somewhat terse and obscure.*[1]

1.2.3. Matsumoto's λ-graph systems. We briefly describe Matsumoto's λ-graph systems following [**74**]. Fix a two-sided subshift Λ. For each $l \in \mathbb{N}$, let F_i^l, $i = 1, \dots, m(l)$ be the l-past equivalence classes and define the matrix $I_{l,l+1}^{\Lambda}$ by

$$I_{l,l+1}^{\Lambda}(i,j) = \begin{cases} 1 & F_j^{l+1} \subseteq F_i^l, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, m(l)$ and $j = 1, \dots, m(l+1)$. Define an $m(l) \times m(l+1)$ matrix $\mathcal{M}_{l,l+1}^{\Lambda}$ by (the formal sums)

$$\mathcal{M}_{l,l+1}^{\Lambda}(i,j) = a_1 + \cdots + a_n,$$

for $i = 1, \dots, m(l)$ and $j = 1, \dots, m(l+1)$. Here, $\{a_1, \dots, a_n\} \subseteq \mathfrak{A}$ is the set of symbols for which $a_k x \in F_i^l$ for some $x \in F_j^{l+1}$. Matsumoto remarks that the matrices satisfy the commutation relation

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}. \tag{1.1}$$

The pair $(\mathcal{M}^{\Lambda}, I^{\Lambda}) := (\mathcal{M}_{l,l+1}^{\Lambda}, I_{l,l+1}^{\Lambda})_l$ is the *canonical symbolic matrix system* associated to Λ.

The symbolic matrix system defines a labeled Bratteli diagram $(E^{\Lambda}, V^{\Lambda}, \lambda^{\Lambda})$ as follows: For each $l \in \mathbb{N}_+$ put $V_l = \{F_i^l \mid i = 1, \dots, m(l)\}$ and $V^{\Lambda} = \bigcup_{l \in \mathbb{N}_+} V_l$. There is an edge $\lambda^{\Lambda}(a)$ from F_i^l to F_j^{l+1} with label $a \in \mathfrak{A}$ if and only if $ax \in F_i^l$ for some $x \in F_j^{l+1}$. Let $E_{l,l+1}$ be the collection of all such edges and put $E^{\Lambda} = \bigcup_{l \in \mathbb{N}_+} E_{l,l+1}$. Finally, given $j = 1 \dots, m(l+1)$, let $\iota_{l,l+1}^{\Lambda}(j) = i$ where $i = 1, \dots, m(l)$ is the unique integer with $I_{l,l+1}(i,j) = 1$. This defines the map $\iota^{\Lambda} = (\iota_{l,l+1}^{\Lambda})_l \colon V \setminus V_1 \longrightarrow V$ which we may call the shift operation. Then $\mathfrak{L}^{\Lambda} = (E^{\Lambda}, V^{\Lambda}, \lambda^{\Lambda}, \iota^{\Lambda})$ is the *λ-graph system* associated with $(\mathcal{M}^{\Lambda}, I^{\Lambda})$. The reader will notice the resemblance with the Krieger cover construction.

An abstract symbolic matrix system over $\mathfrak{A}$ is a pair $(\mathcal{M}, I)$ where $I = (I_{l,l+1})$ and each $I_{l,l+1}$ is a $\{0,1\}$-matrix with no zero rows and such that every column has a unique 1, and $\mathcal{M} = (\mathcal{M}_{l,l+1})$ where each $\mathcal{M}_{l,l+1}$ is a matrix whose entries are formal sums of elements of $\mathfrak{A}$ satisfying the commutation relation (1.1) above. A symbolic matrix system defines a labeled Bratteli diagram as above.

Paths (either labeled or not) in the Bratteli diagram starting in V_1 together with the map ι define dynamical systems. If the labeled Bratteli diagram is canonical to Λ (as above), then Λ can be recovered from this data. However, the same subshift Λ may have various λ-graph systems associated to it. Under mild conditions, Matsumoto has constructed groupoids and C*-algebras from the dynamics on λ-graphs systems and he has also used them to generalize both shift equivalence and strong shift equivalence.

[1] Text taken from [**38**].

1.3. Reconstruction

We have seen various constructions of C*-algebras arising from symbolic dynamical systems. One motivation for this endevour is simply to generate new examples of C*-algebras with interesting properties. Another motivation is to recover the underlying dynamics, possibly up to some coarse equivalence. We can naïvely ask: How much does the C*-algebra *remember* of the underlying dynamics? This is the rigidity question we are interested in and we shall pursue it via Kumjians philosophy and *belief that the structure of a* C**-algebra is illuminated by an understanding of the manner in which abelian subalgebras embed in it.*[2]

With the structure theory of C*-algebras in mind and inspired by Cartan subalgebras in von Neumann algebras (see, e.g., [**50**]), C*-*diagonals* were introduced by Kumjian in [**64**]. Let B be a commutative subalgebra of a C*-algebra A. The *normalizers* of B in A is the collection

$$N(A,B) = \{a \in A \mid aBa^* \cup a^*Ba \subseteq B\}$$

which contains B and is closed under multiplication and taking adjoints. The subalgebra B is *regular* in A if $N(A,B)$ generates A as a C*-algebra. A normalizer a is *free* if $a^2 = 0$; let $N_{\mathrm{f}}(A,B)$ be the collection of free normalizers of B in A.

Definition 1.3.1 ([**64**]). Let A be a unital C*-algebra. A commutative subalgebra B is a C*-*diagonal* if

(i) there is a conditional expectation $P\colon A \longrightarrow B$;
(ii) $\operatorname{span} N_{\mathrm{f}}(A,B)$ is dense in $\ker P$.

A commutative subalgebra B of a non-unital C*-algebra A is *diagonal* if the unitization of B is a diagonal in the unitization of A.

A C*-diagonal $B \subseteq A$ is maximal abelian and has the unique extension property (that is, every pure state on B extends to a pure state on A). The definition was modelled over the complex $n\times n$-matrices with the subalgebra of diagonal matrices. However, this notion is too restrictive to cover many interesting examples, e.g., C*-algebras of strongly connected directed graphs which are not just a single cycle. This leads us to Renault's notion of Cartan subalgebras.

Definition 1.3.2 ([**102, 101**]). Let A be a separable C*-algebra. A commutative subalgebra B is a *Cartan subalgebra* if

(i) B contains an approximate unit of A;
(ii) B is maximal abelian in A;
(iii) B is regular;
(iv) there is a conditional expectation $P\colon A \longrightarrow B$.

A C*-diagonal is exactly a Cartan subalgebra which has the unique extension property. The *raison d'être* for this definition is Renault's reconstruction result: For the reduced C*-algebra of an essentially principal, étale and Hausdorff groupoid $\mathcal{G}$, the commutative subalgebra $C_0(\mathcal{G}^{(0)})$ is a Cartan subalgebra — and every Cartan subalgebra arises in

[2]Taken from [**64**].

this way![3] Similarly, Kumjians C*-diagonals correspond to (twisted) principal étale groupoids.

Remark 1.3.3. For AF-algebras, Strătilă and Voiculescu introduced a notion of *diagonal* in [**112**]. Drinen [**39**] studied this construction and showed that these diagonals are C*-diagonals in the sense of Kumjian. Renault remarks that these diagonals are privileged Cartan subalgebras and that they are all conjugate by an automorphism of the ambient AF-algebra. There are, however, examples of Cartan subalgebras in AF-algebras which are not C*-diagonals, cf. [**102, 103**].

Renault's theory of reconstructing a groupoid from a pair of C*-algebras is exactly what Matsumoto and Matui use to show the following celebrated theorem which is in some respect the foundation for this thesis.

Theorem 1.3.4 ([**84, 22**]). *Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns. The following are equivalent:*

(i) *the one-sided shifts of finite type X_A and X_B are continuously orbit equivalent;*
(ii) *the groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$ are isomorphic;*
(iii) *the* C*-*pairs* $(\mathcal{O}_A, \mathcal{D}_A)$ *and* $(\mathcal{O}_B, \mathcal{D}_B)$ *are* **-isomorphic,*

and the following are equivalent:

(iv) *the two-sided subshifts Λ_A and Λ_B are flow equivalent;*
(v) *the groupoids $\mathcal{G}_A \times \mathcal{R}$ and $\mathcal{G}_B \times \mathcal{R}$ are isomorphic;*
(vi) *the* C*-*pairs* $(\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes c_0)$ *and* $(\mathcal{O}_B \otimes \mathbb{K}, \mathcal{D}_B \otimes c_0)$ *are* **-isomorphic.*

The versatility of reconstructing groupoids from C*-algebras with a distinguished diagonal subalgebra was hard to ignore. *The étale groupoid is the Rosetta stone*[4] which allows us to translate effectively between the dynamical realm and the C*-algebraic realm. A reconstruction theory refined specifically for graph C*-algebra and Leavitt path algebras was developed by Brownlowe–Carlsen–Whittaker [**15**] and Brown–Clark–an Huef [**14**], respectively. A reconstruction theory for Steinberg algebras was then formulated by Ara–Bosa–Hazrat–Sims [**2**]. Recently, an ambitious construction of Carlsen–Ruiz–Sims–Tomforde [**26**] reconstructs groupoids from a C*-algebra with a so-called *weakly Cartan* subalgebra and relates it back to orbit equivalences of the underlying Deaconu–Renault dynamical systems. This latter theory is exactly what we use in Paper B.

Furthermore, Carlsen and Rout [**24**] showed that for a graph E and its graph C*-algebra $C^*(E)$ the possible gauge actions come from cocycles on the graph groupoid, and a *-isomorphism which is diagonal preserving and intertwines the gauge actions comes from a groupoid isomorphism which intertwines these cocycles. This is then used to characterize one-sided eventual conjugacy of graphs and two-sided conjugacy of finite type subshifts. A general version of this is also present in [**26**].

We have now been invited into an investigation of the finer structures of possibly more general and abstract C*-algebras. Mediated by groupoids, we can *see* many properties

[3]This is only true up to a twist on the groupoid but we shall not bother with the details here, see [**102**].

[4]Taken from [**31**].

or relations attributed to the C*-algebras already reflected in the groupoid. Examples include a Brown–Green-Rieffel theorem for groupoids by Carlsen–Ruiz–Sims [**25**], a Pimsner–Voiculescu sequence by Ortega [**94**] and a Künneth formula by Matui [**89**]. Specifically for graph C*-algebras, a program for characterizing structure-preserving *-isomorphisms via moves on the graphs was recently initiated by Eilers and Ruiz [**43**]. Paper C is related to this last program.

Remark 1.3.5. There is an almost analogous story of encoding topological dynamical systems into either partial actions or inverse semigroups and building C*-algebras from them. In certain cases, the trinity of groupoids, partial actions and inverse semigroups is well-understood [**1**]. We shall not discuss this further here.

CHAPTER 2

Future prospects

2.1. Bowen–Franks invariants for C*-algebras with diagonals

This project, joint with James Gabe, is based on a profound realization of Matsumoto and Matui.

THEOREM 2.1.1 ([**84**]). *Let A and B be finite square and irreducible matrices which are not permutations. Then there is a *-isomorphism $\Phi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ if and only if there exists an isomorphism $\alpha\colon BF(A) \longrightarrow BF(B)$ satisfying $\alpha(u_A) = u_B$, and* $\det(I - A) = \det(I - B)$.

Here, u_A is the class of the unit in the Bowen–Franks group $\mathrm{BF}(A) = \mathbb{Z}^{|A|}/(I - A)\mathbb{Z}^{|A|}$. This group is naturally identified with the K_0-group $K_0(\mathcal{O}_A)$ in a way that preserves the class of the unit [**34**]. We can therefore compute it from the C*-algebra $\mathcal{O}_A$. The determinant condition however seems out of place: The theorem tells us that the C*-pair $(\mathcal{O}_A, \mathcal{D}_A)$ determines the value of the determinant $\det(I - A)$ but there is no indication of how to *compute* it from the C*-algebras. In this project, we aim to provide a strategy to compute the determinant from the C*-data alone.

The prototypical example of the problem is given by the irreducible matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_- = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

see, e.g., the seminal paper on classification theory of simple Cuntz–Krieger algebras [**106**] by Rørdam. The cognoscenti will note that A_- is the *Cuntz-splice* of A. We have $\det(I - A) = -1$ and $\det(I - A_-) = +1$ so the two-sided subshifts Λ_A and Λ_{A_-} determined by A and A_-, respectively, are *not* flow equivalent. Rørdam shows, however, that there is a *-isomorphism $\mathcal{O}_A \longrightarrow \mathcal{O}_{A_-}$ between the corresponding Cuntz–Krieger algebras, and that they are *-isomorphic to the Cuntz-algebra $\mathcal{O}_2$. We emphasize that only the existence of such a *-isomorphism is proved, there is no hope of actually *writing down the map*: Colloquially speaking, this *-isomorphism is not dynamical. This has been known all along, but in light of the Matsumoto–Matui result we can be more precise and say that no such *-isomorphism $\mathcal{O}_A \longrightarrow \mathcal{O}_{A_-}$ can be diagonal-preserving (and hence be induced by a continuous orbit equivalence between the graphs). Furthermore, no such *-isomorphism can intertwine the canonical gauge actions, cf. Section 2.2 below.

Let us briefly discuss two perspectives.

In [**80**], Matsumoto has developed a classification theory for simple Cuntz–Krieger algebras with a distinguished diagonal subalgebra.

THEOREM 2.1.2 ([**80**]). *Let A and B be finite square and irreducible matrices which are not permutations and let $\alpha\colon K_0(\mathcal{O}_A) \longrightarrow K_0(\mathcal{O}_B)$ be an isomorphism satisfying $\alpha([1_A]) = [1_B]$. Then* $\det(I - A) = \det(I - B)$ *if and only if there exists a *-isomorphism $\Phi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $K_0(\Phi) = \alpha$.*

The particular use of the condition $\det(I - A) = \det(I - B)$ is in relation to flow equivalence and work of Huang [**54**] on automorphisms of the Bowen–Franks group. The pure C*-algebraist would not consider the above result a (K-theoretic) classification theorem, but a proper understanding of the determinant condition will open up a classification theory of simple (and purely infinite) C*-algebras with distinguished subalgebra. The reconstruction theory of groupoid C*-algebras (see Section 1.3) nicely ties these ideas to a classification theory for groupoids.

A second perspective is connected to the class of sofic shift spaces. If X is a sofic shift then there is an associated C*-algebra $\mathcal{O}_{\mathsf{X}}$ [**18**]. This C*-algebra is canonically isomorphic to the one we consider in Paper B.

THEOREM 2.1.3 ([**18**]). *Let X be a sofic shift space and let G be its left Krieger cover graph. Then there is a *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathrm{C}^*(G)$ satisfying $\Psi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}(G)$.*

This is unfortunate: When we consider the class of sofic shifts which is much richer and harder to understand than the shifts of finite type, we obtain *only* graph algebras — even when we include the canonical diagonal! But there is hope. For a shift of finite type X, the spectrum of the canonical diagonal $\mathcal{D}_{\mathsf{X}}$ in $\mathcal{O}_{\mathsf{X}}$ is canonically homeomorphic to X. For strictly sofic shifts, however, this is not the case. The spectrum of $\mathcal{D}_{\mathsf{X}}$ is (homeomorphic to) the path space of the left Krieger cover graph, but the commutative C*-algebra $C(\mathsf{X})$ sits canonically inside of $\mathcal{D}_{\mathsf{X}}$ in $\mathcal{O}_{\mathsf{X}}$. Perhaps we can define a *determinant* of a C*-pair $(\mathcal{O}_{\mathsf{X}}, C(\mathsf{X}))$ for a strictly sofic X? This is still speculative.

2.2. Shift equivalence

Shift equivalence is a relation between (nonnegative) integral matrices first introduced by Williams in [**116**]. He *proved* that shift equivalence of nonnegative matrices was equivalent to two-sided conjugacy of the corresponding two-sided subshifts. Although conjugacy implies shift equivalence there was a mistake in the proof of the other direction. The *shift equivalence problem* — whether shift equivalence implies two-sided conjugacy — was thereafter one of the biggest open questions of symbolic dynamics. This was until Kim and Roush provided shift equivalent matrices which are not conjugate [**60, 61**] and thereby disproved the conjecture. The shift equivalence problem thus went from being a theorem to a conjecture to a false statement.

DEFINITION 2.2.1 ([**68**]). Let A and B be finite square matrices over $\mathbb{N}$ and let $\ell \in \mathbb{N}_+$. A *shift equivalence of lag ℓ* from A to B is a pair (R, S) of rectangular matrices over $\mathbb{N}$ satisfying

(i) $AR = RB, \qquad SA = BS,$

(ii) $A^\ell = RS, \qquad B^\ell = SR.$

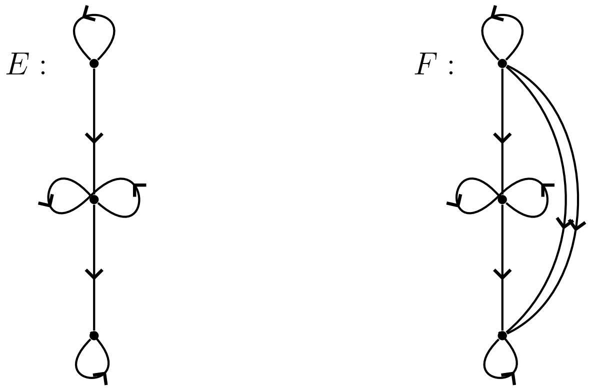
We write $(R,S) : A \sim B$ (lag ℓ). A *shift equivalence over $\mathbb{Z}$ with lag ℓ* from A and B is defined analogously by replacing $\mathbb{N}$ by $\mathbb{Z}$. In this case, we write $A \sim_{\mathbb{Z}} B$ (lag ℓ).

It is well-known that A and B are shift equivalent if and only if the two-sided subshifts Λ_A and Λ_B are eventually conjugate, cf. [**68**, Theorem 7.5.15]. For irreducible matrices (a square matrix A is *irreducible* if for every entry (i,j) in A there is $N \in \mathbb{N}_+$ such that $A^N(i,j) > 0$), shift equivalence is also equivalent to the condition that Λ_A and Λ_B are both factors of each other (this is known as weak conjugacy, cf. [**68**, Theorem 12.2.5]). For primitive matrices (a square matrix A is *primitive* if there is $N \in \mathbb{N}_+$ such that every entry of A^N is strictly positive), shift equivalence over $\mathbb{Z}$ implies shift equivalence. There is much to be said about this relation but we shall emphasize the following question.

Question 2.2.2. *Does shift equivalence imply flow equivalence?*

For irreducible nonnegative matrices A and B, it is well-known that the Bowen–Franks groups $\mathrm{BF}(A) = \mathbb{Z}^{|A|}/(I-A)\mathbb{Z}^{|A|}$ and $\mathrm{BF}(B) = \mathbb{Z}^{|B|}/(I-B)\mathbb{Z}^{|B|}$ are isomorphic and $\det(I-A) = \det(I-B)$, when A and B are shift equivalent (even over $\mathbb{Z}$), cf. [**68**, Section 7.4]. Since the Bowen–Franks group together with the determinant condition comprise a complete invariant for flow equivalence of irreducible subshifts of finite type by Parry–Sullivan, Bowen–Franks and Franks [**97, 5, 52**], it follows that shift equivalence (over $\mathbb{Z}$) implies flow equivalence. However, not much is known for general reducible subshifts of finite type.

Consider the graphs



determined by the adjacency matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

respectively. These graphs play a central rôle in [**42**] and are the motivation for the *Pulelehua move* in [**41**]. We know from [**42**, Lemma 5.1 and Example 5.10] that the two-sided edge subshifts of E and F are not flow equivalent. They do however give rise to stably isomorphic Cuntz–Krieger algebras, cf. [**42**, Example 6.9].

Proposition 2.2.3. *Shift equivalence over $\mathbb{Z}$ does not imply flow equivalence.*

PROOF. Let A and B be as in (2.1). We verify the claim by exhibiting a shift equivalence over $\mathbb{Z}$ of lag 2 between A and B. Consider the integer matrices

$$R = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

and note that

$$AR = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 4 & 5 \\ 0 & 0 & -1 \end{pmatrix} = RB, \qquad SA = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -1 \end{pmatrix} = BS.$$

Furthermore,

$$RS = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} = A^2, \qquad SR = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} = B^2.$$

Hence the pair (R, S) defines a shift equivalence over $\mathbb{Z}$ between A and B. □

A different example was recently found by Nyland. Since $C^*(E)$ and $C^*(F)$ are stably isomorphic we naturally ask the following question.

QUESTION 2.2.4. *Does shift equivalence over $\mathbb{Z}$ imply stable *-isomorphism of Cuntz–Krieger algebras?*

Unfortunately, we have not had the time to make any progress with this question.

We shall see below that A *and* B *defined above are not shift equivalent.* A computation shows that

$$A^\ell = \begin{pmatrix} 1 & 2^\ell - 1 & 2^\ell - 1 - \ell \\ 0 & 2^\ell & 2^\ell - 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B^\ell = \begin{pmatrix} 1 & 2^\ell - 1 & 2^\ell + 1 - \ell \\ 0 & 2^\ell & 2^\ell - 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

for $\ell \in \mathbb{N}_+$. Further computations show that if R and S are 3×3 nonnegative integer matrices satisfying $AR = RB$ and $SA = BS$, then they are of the form

$$R = \begin{pmatrix} 0 & a & b \\ 0 & a & a \\ 0 & 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & c & d \\ 0 & c & c \\ 0 & 0 & 0 \end{pmatrix},$$

for some $a, b, c, d \in \mathbb{N}$. But then the top left entry of RS is 0. Comparing this with (2.2) shows that no shift equivalence (R, S) between A and B can exist.

A possible strategy to attack the problem of shift equivalence and flow equivalence is via C^*-algebras and groupoids. The following result is a consequence of work of Bratteli and Kishimoto and relates shift equivalence to certain structure-preserving *-isomorphisms of their associated Cuntz–Krieger algebras.

THEOREM 2.2.5 ([**11**]). *Let A and B be* primitive *nonnegative integer matrices. Then A and B are shift equivalent if and only if there is a *-isomorphism $\Phi \colon \mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ satisfying $\Phi \circ (\gamma^A \otimes \mathrm{id}) = (\gamma^B \otimes \mathrm{id}) \circ \Phi$.*

Two subshifts Λ_A and Λ_B are flow equivalent if and only if there is a *-isomorphism $\Phi' \colon \mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ which satisfies $\Phi'(\mathcal{D}_A \otimes c_0) = \mathcal{D}_B \otimes c_0$, and this condition is again equivalent to the existence of a groupoid isomorphism $\mathcal{G}_A \times \mathcal{R} \longrightarrow \mathcal{G}_B \times \mathcal{R}$, cf. [**84, 22**]. Combining these results we arrive at the following statement.

THEOREM 2.2.6. *Let A and B be* primitive *nonnegative integer matrices. If there exists a* **-isomorphism* $\Phi \colon \mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ *satisfying*

$$\Phi \circ (\gamma^A \otimes \mathrm{id}) = (\gamma^B \otimes \mathrm{id}) \circ \Phi,$$

then there exists a (possibly different) **-isomorphism* $\Phi' \colon \mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ *satisfying*

$$\Phi'(\mathcal{D}_A \otimes c_0) = \mathcal{D}_B \otimes c_0.$$

REMARK 2.2.7. We emphasize that this holds for *primitive* matrices. On-going work of Eilers and Szabó might result in a generalization to irreducible matrices.

By [**24**], two-sided conjugacy is characterized by *-isomorphism of the stabilized Cuntz-Krieger algebras which maps the diagonal to the diagonal *and* intertwines the canonical gauge actions. Since we know that shift equivalence does not imply two-sided conjugacy, it is, in general, necessary to *change* the isomorphism in the above theorem. It is, however, not clear *how* to change the isomorphism in concrete examples.

The theorem is most interesting from a groupoid perspective. At least for primitive matrices, shift equivalence implies the existence of a groupoid isomorphism even though shift equivalence itself has evaded all attempts of groupoid descriptions. A thorough investigation of this question using this perspective will surely shed valuable light on both relations!

2.3. Open problems

Finally, we briefly discuss some open problems related to the theme of this thesis.

2.3.1. The decidability problem. We cannot discuss open problems related to symbolic dynamics without mentioning the *decidability problem* for two-sided conjugacy. Williams [**116**] introduced the notion of shift equivalence and showed that this was a decidable relation in an attempt to answer the above question in the affirmative. The existence (or nonexistence) of an algorithm which can decide whether two given subshifts of finite type are conjugate is still open. Althought this thesis contains no contributions to answer this question, Paper C shows that one-sided eventual conjugacy is surprisingly closely connected to two-sided conjugacy. Perhaps a further study of this relation is a viable path to approach the decidability problem. Or perhaps the decidability question of one-sided eventual conjugacy is just as complex as that of two-sided conjugacy?

2.3.2. β-shifts. For each real $\beta > 1$ there is an associated shift space X_β called the β-shift [**95, 105**]. The class of β-shifts is *transversal* in the sense that it intersects the finite type shifts (in particular, all the full shifts), the (strictly) sofic shifts and the nonsofic shifts without exhausting any of them. In [**58**], Katayama, Matsumoto and Watatani studied the C*-algebra $\mathcal{O}_\beta$ associated to a β-shift X_β. (the various constructions of C*-algebras from Section 1.2 coincide for β-shifts, cf. [**23**, Corollary 3.5]). They showed that $\mathcal{O}_\beta$ is simple and purely infinite for all $\beta > 1$, and using classification theory they show that there is a *-isomorphism $\mathcal{O}_\beta \longrightarrow \mathcal{O}_\infty$ whenever X_β is *not* sofic (this includes uncountably many choices of $\beta > 1$). However, this *-isomorphism cannot intertwine the canonical gauge actions since the fixed-point algebra $\mathcal{O}_\beta$ is simple while this is not the case for $\mathcal{O}_\infty$. Therefore, we cannot simply identify the two. Matsumoto and Matui later [**85**] realize $\mathcal{O}_\beta$ as a groupoid C*-algebra of an essentially principal, minimal and purely infinite étale groupoid[1] It is not clear, however, if this groupoid is isomorphic to the canonical groupoid associated to $\mathcal{O}_\infty$.

The question is: *How can we distinguish the* C**-algebras associated to the nonsofic* β*-shifts?* A first place to look is the canonical gauge action. The C*-algebra $\mathcal{O}_\beta$ admits a unique KMS-state only at the temperature $\log \beta$ and it is unique, cf. [**58**, Theorem C]. Another approach is to consider the pair $(\mathcal{O}_\beta, C(\mathsf{X}_\beta))$; when X_β is not sofic then the diagonal subalgebra $\mathcal{D}_\beta$ is not isomorphic to $C(\mathsf{X}_\beta)$.

2.3.3. Homology of groupoids. A homology theory for étale groupoids was introduced by Matui in [**87**] based on [**33**]. This was motivated by an attempt to understand and compute K-theory of C*-algebras. He showed that for simple Cuntz-Krieger algebras, the homology of the underlying groupoids coincide with the K-theory of the C*-algebras. This was later generalized to two conjectures about essentially principal, minimal and étale groupoids with unit spaces homeomorphic to the Cantor set [**89**]: The *HK-conjecture* relates the homology of the groupoid with the K-theory of the groupoid C*-algebra, while the *AH-conjecture* relates the homology groups to each other via the (abelianization of) the topological full group. The interested reader is refered to [**89**] for exact statements.

This has spawned much interest, since the conjectures have been verified for a surprisingly large body of examples, [**88, 49, 31**]. Although the HK-conjecture is known to be false as stated (see Scarparo's counterexample [**108**]), the AH-conjecture and the question of the precise relationship between homology and K-theory are left open. Interesting notions such as almost finiteness still need more exploration.

[1]Matsumoto and Matui study the associated topological full groups and identify them with the Higman–Thompson groups whenever X_β is sofic. In any case, it follows from work in Paper D that the topological full groups are C*-simple for all $\beta > 1$.

Part 2

Contributions

Summary

The original contributions of this thesis consist of four articles.

A. The paper *Cuntz–Krieger algebras and one-sided conjugacy of shift of finite type and their groupoids* is written joint with Toke Meier Carlsen and has been published in Journal of the Australian Mathematical Society [**12**]. We show that a pair of shifts of finite type X_A and X_B determined by finite square $\{0,1\}$ matrices A and B with no zero columns or rows are one-sided conjugate if and only if there is a *-isomorphism $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ which intertwines certain completely positive maps τ_A and τ_B on $\mathcal{O}_A$ and $\mathcal{O}_B$, respectively.

B. The paper C*-*algebras groupoids and covers of shift spaces* is written joint with Toke Meier Carlsen. We associate to every one-sided shift space X a cover $\widetilde{\mathsf{X}}$ and use this to construct a groupoid $\mathcal{G}_{\mathsf{X}}$ and a C*-algebra $\mathcal{O}_{\mathsf{X}}$. This setup allows us to classify various orbit equivalences between a pair of one-sided shift spaces X and Y in terms of certain diagonal-preserving *-isomorphisms of $\mathcal{O}_{\mathsf{X}}$ and $\mathcal{O}_{\mathsf{Y}}$ (as well as isomorphism of their groupoids). We also classify conjugacy and flow equivalence of two-sided shift spaces Λ_{X} and Λ_{Y} in terms of diagonal preserving *-isomorphism $\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ and certain circle actions (and isomorphism of the stabilized groupoids and certain cohomological data). The paper contains a theorem which generalizes the main result of [**12**].

C. The paper *Eventual conjugacy and the balanced in-split* is related to the recent preprint [**43**] of Eilers and Ruiz which addresses structure preserving *-isomorphisms of graph C*-algebras in relation to moves on the graphs. The paper contains a proof showing that one-sided eventual conjugacy between finite directed graphs with no sinks is generated by out-splits and balanced in-splits thus answering a question of Eilers and Ruiz. The balanced in-split is a refinement of the classical in-split move on graphs.

D. The paper C*-*simplicity and representations of topological full groups of groupoids* is written joint with Eduardo Scarparo and has been published in Journal of Functional Analysis [**13**]. We investigate the topological full group of ample groupoids via representations in the C*-algebras of the underlying groupoids. Using techniques from Le Boudec and Matte Bon, we provide conditions for when the topological full group is C*-simple. This paper is not directly related to any of the above articles.

The manuscripts appearing here only differ from the published versions in layout. This document contains a single list of references containing references from all papers.

ARTICLE A

Cuntz–Krieger algebras and one-sided conjugacy of shift of finite type and their groupoids

Kevin Aguyar Brix and Toke Meier Carlsen

Abstract

A one-sided shift of finite type (X_A, σ_A) determines on the one hand a Cuntz–Krieger algebra $\mathcal{O}_A$ with a distinguished abelian subalgebra $\mathcal{D}_A$ and a certain completely positive map τ_A on $\mathcal{O}_A$. On the other hand, (X_A, σ_A) determines a groupoid $\mathcal{G}_A$ together with a certain homomorphism ϵ_A on $\mathcal{G}_A$. We show that each of these two sets of data completely characterizes the one-sided conjugacy class of X_A. This strengthens a result of J. Cuntz and W. Krieger. We also exhibit an example of two irreducible shifts of finite type which are eventually conjugate but not conjugate. This provides a negative answer to a question of K. Matsumoto of whether eventual conjugacy implies conjugacy.

A.1. Introduction

In [**35**], J. Cuntz and W. Krieger initiated what has turned out to be a very fascinating study of the symbiotic relationship between operator algebras and symbolic dynamics. Given a finite square $\{0,1\}$-matrix A with no zero rows or zero columns, they construct a C^*-algebra $\mathcal{O}_A$ which is now called the *Cuntz–Krieger algebra* of A with a distinguished abelian C^*-subalgebra $\mathcal{D}_A$ called the *diagonal* subalgebra. Under a certain condition (I) (this is later generalized to condition (L) of graphs), $\mathcal{O}_A$ is a universal C^*-algebra and there is an action of the circle group $\mathbb{T} \curvearrowright \mathcal{O}_A$ called the *gauge action*.

The matrix A also determines both a one-sided and a two-sided shift space of finite type (see, e.g., [**62**], [**68**]) denoted (X_A, σ_A) and $(\bar{\mathsf{X}}_A, \bar{\sigma}_A)$, respectively. In fact, any one-sided (resp., two-sided) shift of finite type is conjugate to (X_A, σ_A) (resp., $(\bar{\mathsf{X}}_A, \bar{\sigma}_A)$) for some finite square $\{0,1\}$-matrix A with no zero rows or zero columns. The spectrum of the above mentioned abelian C^*-subalgebra $\mathcal{D}_A$ is homeomorphic to X_A in a natural way.

In [**35**, Proposition 2.17], J. Cuntz and W. Krieger proved that the data consisting of the C^*-algebra $\mathcal{O}_A$, the diagonal $\mathcal{D}_A$, the gauge action and the restriction to the diagonal of a certain completely positive map $\phi_A \colon \mathcal{O}_A \longrightarrow \mathcal{O}_A$ is an invariant of the one-sided conjugacy class of X_A provided A satisfies condition (I).

In [**34**], J. Cuntz also showed the following two results under the hypothesis that the defining matrices as well as their transposes satisfy condition (I): A flow equivalence between the two-sided shift spaces $(\bar{\mathsf{X}}_A, \bar{\sigma}_A)$ and $(\bar{\mathsf{X}}_B, \bar{\sigma}_B)$ implies the existence of a $*$-isomorphism $\mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ which sends $\mathcal{D}_A \otimes \mathcal{C}$ onto $\mathcal{D}_B \otimes \mathcal{C}$, where $\mathcal{C}$ is the canonical maximal abelian subalgebra of the C^*-algebra $\mathbb{K}$ of compact operators on an infinite-dimensional separable Hilbert space; a two-sided conjugacy between $(\bar{\mathsf{X}}_A, \bar{\sigma}_A)$ and $(\bar{\mathsf{X}}_B, \bar{\sigma}_B)$ implies the existence of a diagonal-preserving $*$-isomorphism between the stabilized C^*-algebras as above which also intertwines the gauge actions. Both of these results were also present in [**35**] under the additional assumptions that the defining matrices are irreducible and aperiodic.

From a one-sided shift space (X_A, σ_A) one can construct an amenable locally compact Hausdorff étale groupoid $\mathcal{G}_A$ whose C^*-algebra $C^*(\mathcal{G}_A)$ is isomorphic to $\mathcal{O}_A$ in a way that maps $\mathcal{D}_A$ onto $C(\mathcal{G}_A^{(0)})$. Using this groupoid, H. Matui and K. Matsumoto in [**84**] improved the work of J. Cuntz and W. Krieger when they showed that flow equivalence between shift spaces determined by irreducible and non-permutation $\{0,1\}$-matrices A and B is equivalent to the existence of a diagonal-preserving $*$-isomorphism between the stabilized Cuntz–Krieger algebras. This result was later proved in [**22**] to hold for any pair of finite square $\{0,1\}$-matrices A and B with no zero rows and no zero colums. In [**24**], the second-named author and J. Rout used a similar approach to prove that $(\bar{\mathsf{X}}_A, \bar{\sigma}_A)$ and $(\bar{\mathsf{X}}_B, \bar{\sigma}_B)$ are conjugate (only assuming that A and B have no zero rows or zero columns) if and only if there exists a diagonal preserving $*$-isomorphism between the stabilized Cuntz–Krieger algebras which intertwines the gauge actions. Thus, one can recover the two-sided shift space $(\bar{\mathsf{X}}_A, \bar{\sigma}_A)$ both up to flow equivalence and up to conjugacy from its Cuntz–Krieger algebra $\mathcal{O}_A$.

In [**78**], K. Matsumoto introduced the notion of continuous orbit equivalence of one-sided shift spaces of finite type and showed that for irreducible and non-permutation $\{0,1\}$-matrices A and B, the one-sided shift spaces (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if there is a diagonal preserving $*$-isomorphism (i.e., a $*$-isomorphism that maps $\mathcal{D}_A$ onto $\mathcal{D}_B$) between the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$. Building on the reconstruction theory of J. Renault in [**102**], H. Matui and K. Matsumoto observed in [**84**] that this is equivalent to isomorphism of the groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$. These results were in [**22**] shown to hold for any pair of finite square $\{0,1\}$-matrices A and B with no zero rows and no zero colums.

Given irreducible and non-permutation $\{0,1\}$-matrices A and B, K. Matsumoto proved that the stronger notion of eventual conjugacy of one-sided shifts is completely characterized by diagonal-preserving $*$-isomorphism of the Cuntz–Krieger algebras which intertwines the gauge actions, see [**81**]. This is generalized to any pair of finite square $\{0,1\}$-matrices A and B with no zero rows and no zero colums in [**24**]. K. Matsumoto then asks the question whether eventual conjugacy is equivalent to conjugacy. In the wake of the above mentioned results, the question can be rephrased as to whether diagonal preserving $*$-isomorphism of Cuntz–Krieger algebras intertwining the gauge actions actually characterizes conjugacy (see [**81**, Remark 3.6] or [**82**, p. 1139]).

In this paper, we address this question and the characterization of one-sided conjugacy of finite type shift spaces in relation to [**35**, Proposition 2.17]. Given any finite square $\{0,1\}$-matrix A with no zero rows or zero columns, we introduce a continuous groupoid homomorphism $\epsilon_A\colon \mathcal{G}_A \longrightarrow \mathcal{G}_A$ which induces a completely positive map $\tau_A\colon \mathcal{O}_A \longrightarrow \mathcal{O}_A$. This is different but related to the completely positive map $\phi_A\colon \mathcal{O}_A \longrightarrow \mathcal{O}_A$ considered in [**35**]. We show that for a pair of finite square $\{0,1\}$-matrices A and B with no zero rows and no zero colums, the one-sided shift spaces (X_A, σ_A) and (X_B, σ_B) are conjugate if and only if there is a diagonal preserving $*$-isomorphism $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ that intertwines τ_A and τ_B, if and only if there is a groupoid isomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ that intertwines ϵ_A and ϵ_B. We also show that these conditions are equivalent to the existence of a diagonal preserving $*$-isomorphism between $\mathcal{O}_A$ and $\mathcal{O}_B$ that intertwines both the gauge actions and $\phi_A|_{\mathcal{D}_A}$ and $\phi_B|_{\mathcal{D}_B}$, and thus show that [**35**, Proposition 2.17] holds also for matrices that do not satisfy condition (I). Finally, we exhibit an example of two irreducible shifts of finite type which are eventually conjugate but not conjugate. Another example of two irreducible shifts of finite type which are eventually conjugate but not conjugate was given in [**7**]. This shows that conjugacy is strictly stronger than eventual conjugacy and this answers K. Matsumoto's question in the negative.

A.2. Notation and preliminaries

In this section we briefly recall the definitions of the one-sided shift space X_A, the groupoid $\mathcal{G}_A$, and the Cuntz–Krieger algebra $\mathcal{O}_A$ together with the subalgebra $\mathcal{D}_A$ and the gauge action $\gamma\colon \mathbb{T} \curvearrowright \mathcal{O}_A$. We let $\mathbb{Z}$, $\mathbb{N} = \{0,1,2,\ldots\}$ and $\mathbb{C}$ denote the integers, the non-negative integers and the complex numbers, respectively. Let $\mathbb{T} \subseteq \mathbb{C}$ be the unit circle group.

A.2.1. One-sided shifts of finite type. Let N be a positive integer and let $A \in M_N(\{0,1\})$ be a matrix with no zero rows and no zero columns. The set

$$\mathsf{X}_A := \{x = (x_n)_{n\in\mathbb{N}} \in \{1,\ldots,N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1,\ n \in \mathbb{N}\}$$

is a second countable compact Hausdorff space in the subspace topology of $\{1,\ldots,N\}^{\mathbb{N}}$ (equipped with the product topology). Together with the shift operation $\sigma_A\colon \mathsf{X}_A \longrightarrow \mathsf{X}_A$ given by $(\sigma_A(x))_n = x_{n+1}$ for $x \in \mathsf{X}_A$ and $n \in \mathbb{N}$, the pair (X_A, σ_A) is *the one-sided shift of finite type* determined by A. In the literature (e.g., [**34**], [**35**], [**78**], [**84**]), (X_A, σ_A) is often refered to as the *one-sided topological Markov chain* determined by A. The reader is refered to [**68**] for an excellent introduction to the general theory of shift spaces.

A finite string $\alpha = \alpha_0\alpha_1\cdots\alpha_{n-1}$ with $\alpha_i \in \{1,\ldots,N\}$ is an *admissible word* (or just a *word*) of length $|\alpha| = n$ if $A(\alpha_{i-1}, \alpha_i) = 1$ for $i = 1,\ldots,n-1$. Equivalently, there exists $x = x_0x_1x_2\cdots \in \mathsf{X}_A$ and $j \in \mathbb{N}$ such that $\alpha = x_{[j,j+n)}$. If $\alpha = x_{[i,j)}$ and $\beta = y_{[i',j')}$ are words, then $\alpha\beta y_{[j',\infty)} \in \mathsf{X}_A$ if and only if $A(x_{j-1}, y_{i'}) = 1$ in which case the concatenation $\alpha\beta$ is again a word. The topology of X_A has a basis consisting of compact open sets of the form

$$Z_\alpha = \{x \in \mathsf{X}_A \mid x_{[0,|\alpha|)} = \alpha\} \subseteq \mathsf{X}_A$$

where α is a word. Let (X_A, σ_A) and (X_B, σ_B) be one-sided shifts of finite type and let $h\colon \mathsf{X}_A \longrightarrow \mathsf{X}_B$ be a homeomorphism. Then h is a *one-sided conjugacy* (or just a *conjugacy*) if $h \circ \sigma_A = \sigma_B \circ h$.

A.2.2. Groupoids. Let A be a finite square $\{0,1\}$-matrix with no zero rows and no zero columns. The *Deaconu-Renault groupoid* [**36**] associated to the one-sided shift of finite type (X_A, σ_A) is

$$\mathcal{G}_A := \{(x,n,y) \in \mathsf{X}_A \times \mathbb{Z} \times \mathsf{X}_A \mid \exists k,l \in \mathbb{N},\ n = k-l\colon \sigma_A^k(x) = \sigma_A^l(y)\}$$

with unit space $\mathcal{G}_A^{(0)} = \{(x,0,x) \in \mathcal{G}_A \mid x \in \mathsf{X}_A\}$. The range map is $r(x,n,y) = (x,0,x)$ and the source map is $s(x,n,y) = (y,0,y)$. The product $(x,n,y)(x',n',y')$ is well-defined if and only if $y = x'$ in which case it equals $(x, n+n', y')$ while inversion is given by $(x,n,y)^{-1} = (y,-n,x)$. We can specify a topology on $\mathcal{G}_A$ via a basis consisting of sets of the form

$$Z(U,k,l,V) := \{(x,k-l,y) \in \mathcal{G}_A \mid x \in U, y \in V\},$$

where $k,l \in \mathbb{N}$ and $U,V \subseteq \mathsf{X}_A$ are open such that $\sigma_A^k|_U$ and $\sigma_A^l|_V$ are injective and $\sigma_A^k(U) = \sigma_A^l(V)$. If α,β are words with the same final letter, we write

$$Z(\alpha,\beta) := Z(Z_\alpha, |\alpha|, |\beta|, Z_\beta).$$

With this topology, $\mathcal{G}_A$ is a second countable, étale (i.e., $s,r\colon \mathcal{G}_A \longrightarrow \mathcal{G}_A$ are local homeomorphisms onto $\mathcal{G}_A^{(0)}$) and locally compact Hausdorff groupoid. Throughout the paper, we identify the unit space $\mathcal{G}_A^{(0)}$ of $\mathcal{G}_A$ with X_A via the map $(x,0,x) \mapsto x$. By, e.g., [**110**, Lemma 3.5], $\mathcal{G}_A$ is amenable, so the reduced and the full groupoid C^*-algebras coincide. We shall refer to the groupoid homomorphism $c_A\colon \mathcal{G}_A \longrightarrow \mathbb{Z}$ given by $c_A(x,n,y) = n$ as the *canonical continuous cocycle*. The pre-image $c_A^{-1}(0) = \{(x,0,y) \in \mathcal{G}_A \mid x,y \in \mathsf{X}_A\}$ is a principal subgroupoid of $\mathcal{G}_A$.

A.2.3. Cuntz–Krieger algebras. Let A be an $N \times N$ matrix with entries in $\{0,1\}$ and no zero rows and no zero columns. The *Cuntz–Krieger algebra* [**35**] $\mathcal{O}_A$ is the universal unital C^*-algebra generated by partial isometries $s_1 \ldots, s_N$ subject to the conditions

$$s_i^* s_j = 0 \ (i \neq j), \qquad {s_i}^* s_i = \sum_{j=1}^N A(i,j) s_j {s_j}^*$$

for every $i = 1, \ldots, N$. A word $\alpha = \alpha_1 \cdots \alpha_{|\alpha|}$ defines a partial isometry $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_{|\alpha|}}$ in $\mathcal{O}_A$. The *diagonal* subalgebra $\mathcal{D}_A$ is the abelian C^*-algebra generated by the range projections of the partial isometries s_α inside $\mathcal{O}_A$. The algebras $\mathcal{D}_A$ and $C(\mathsf{X}_A)$ are isomorphic via the correspondence $s_\alpha {s_\alpha}^* \longleftrightarrow \chi_{Z_\alpha}$, where χ_{Z_α} is the indicator function on Z_α. If A is irreducible and not a permutation matrix, then $\mathcal{O}_A$ is simple and $\mathcal{D}_A$ is maximal abelian in $\mathcal{O}_A$; in fact, it is a Cartan subalgebra in the sense of [**102**]. The *gauge action* $\gamma^A\colon \mathbb{T} \curvearrowright \mathcal{O}_A$ is determined by $\gamma_z^A(s_i) = z s_i$, for every $z \in \mathbb{T}$ and $i = 1, \ldots, N$. The corresponding fixed point algebra is denoted $\mathcal{F}_A$.

The Cuntz–Krieger algebra is a groupoid C^*-algebra in the sense that there is a $*$-isomorphism $\mathcal{O}_A \longrightarrow C^*(\mathcal{G}_A)$ which sends the canonical generators s_i to the indicator functions $\chi_i = \chi_{Z(i,r(i))}$, for $i = 1, \ldots, N$ (see, e.g., [**65**]). This map takes $\mathcal{D}_A$ to $C(\mathcal{G}_A^{(0)})$ (the latter is identified with $C(\mathsf{X}_A)$) and $\mathcal{F}_A$ to $C^*(c_A^{-1}(0))$. The canonical cocycle $c_A\colon \mathcal{G}_A \longrightarrow \mathbb{Z}$ defines an action γ^{c_A} on $C_c(\mathcal{G}_A)$ as

$$\gamma_z^{c_A}(f)(x,n,y) = z^{c_A(x,n,y)} f(x,n,y) = z^n f(x,n,y),$$

for $f \in C_c(\mathcal{G}_A)$ and $(x,n,y) \in \mathcal{G}_A$. In particular, $\gamma_z^{c_A}(s_i) = zs_i$ for $i = 1, \dots, N$ so γ^{c_A} is the gauge action γ^A restricted to $C_c(\mathcal{G}_A)$.

Throughout the paper we shall suppress this $*$-isomorphism and simply identify the algebras. The existence of a linear injection $j\colon C^*(\mathcal{G}_A) \longrightarrow C_0(\mathcal{G}_A)$ allows us to think of elements in $\mathcal{O}_A$ as functions on $\mathcal{G}_A$ vanishing at infinity, cf. [**101**] or [**109**]. We shall do this whenever it be convenient. The inclusion $\iota\colon \mathcal{G}_A^{(0)} \longrightarrow \mathcal{G}_A$ induces a conditional expectation $d_A\colon \mathcal{O}_A \longrightarrow \mathcal{D}_A$, see [**35**, Remark 2.18] and [**101**, II, Proposition 4.8]. In light of the above, we can think of $d_A(f)$ as the restriction of $f \in \mathcal{O}_A \subseteq C_0(\mathcal{G}_A)$ to X_A where we identify X_A with $\mathcal{G}_A^{(0)}$.

A.3. The results

Let A be an $N \times N$ matrix with entries in $\{0,1\}$ and no zero rows and no zero columns, and let $s_1 \dots, s_N$ be the canonical generators of $\mathcal{O}_A$. In [**35**], J. Cuntz and W. Krieger consider a completely positive map $\phi_A\colon \mathcal{O}_A \longrightarrow \mathcal{O}_A$ given by

$$\phi_A(y) = \sum_{i=1}^N s_i y {s_i}^*,$$

for $y \in \mathcal{O}_A$. The map ϕ_A restricts to a $*$-homomorphism $\mathcal{D}_A \longrightarrow \mathcal{D}_A$. Under the identification of $\mathcal{D}_A$ and $C(\mathsf{X}_A)$ we have the relation $\phi_A(f)(x) = f(\sigma_A(x))$ for $f \in \mathcal{D}_A$ and $x \in \mathsf{X}_A$, cf. [**35**, Proposition 2.5].
Put $s := \sum_{i=1}^N s_i$. In this paper, we shall also consider the completely positive map $\tau_A\colon \mathcal{O}_A \longrightarrow \mathcal{O}_A$ defined by

$$\tau_A(y) := sys^* = \sum_{i,j=1}^N s_j y {s_i}^*, \tag{A.1}$$

for $y \in \mathcal{O}_A$. Note that $\mathcal{F}_A$ is generated by $\bigcup_{k=0}^\infty \tau_A^k(\mathcal{D}_A)$ and $\tau_A(\mathcal{F}_A) \subseteq \mathcal{F}_A$. On the level of groupoids, we consider the homomorphism $\epsilon_A\colon \mathcal{G}_A \longrightarrow \mathcal{G}_A$ given by

$$\epsilon_A(x,n,y) := (\sigma_A(x), n, \sigma_A(y)),$$

for $(x,n,y) \in \mathcal{G}_A$. Suppose $(x_i,n_i,y_i) \longrightarrow (x,n,y)$ in $\mathcal{G}_A$ as $i \longrightarrow \infty$ and suppose $Z(\mu,\nu)$ is any basic open set containing $(\sigma_A(x), n, \sigma_A(y))$. Then $n_i = n$ and $\sigma_A(x_i) \in Z_\mu$ and $\sigma_A(\nu) \in Z_\nu$ eventually by continuity of σ_A. Hence $(\sigma_A(x_i), n_i, \sigma_A(y_i)) \in Z(\mu,\nu)$ eventually, so $(\sigma_A(x_i), n_i, \sigma_A(y_i)) \longrightarrow (\sigma_A(x), n, \sigma_A(y))$ as $i \longrightarrow \infty$ and ε_A is continuous. This induces a map $\epsilon_A^*\colon C_c(\mathcal{G}_A) \longrightarrow C_c(\mathcal{G}_A)$ given by $\epsilon_A^*(f) = f \circ \epsilon_A$, for $f \in C_c(\mathcal{G}_A)$.

Lemma A.3.1. *The map $\tau_A\colon \mathcal{O}_A \longrightarrow \mathcal{O}_A$ extends the map ϵ_A^* defined above and we have $d_A \circ \tau_A|_{\mathcal{D}_A} = \phi_A|_{\mathcal{D}_A}$.*

Proof. The generators s_i in $\mathcal{O}_A$ correspond to the indicator functions χ_i in $C_c(\mathcal{G}_A)$. Inside the convolution algebra we thus have

$$\begin{aligned}\sum_{i,j=1}^N \chi_j \star (f \star {\chi_i}^*)(x,n,y) &= \sum_{j=1}^N \sum_{(z,m,y)\in(\mathcal{G}_A)} \chi_j(x, n-m, z) f(z, m+1, \sigma_A(y)) \\ &= f(\sigma_A(x), n, \sigma_A(y)),\end{aligned}$$

for $f \in C_c(\mathcal{G}_A)$ and $(x,n,y) \in \mathcal{G}_A$. Here, $\star$ denotes the convolution product in $C_c(\mathcal{G}_A)$. The maps ϵ_A^* and τ_A therefore agree on $C_c(\mathcal{G}_A)$. A computation similar to the above shows that $d_A(\tau_A(f)) = \phi_A(f)$, for $f \in \mathcal{D}_A$. □

For the proof of Theorem A.3.3, we need the following lemma which might be of interest on its own.

LEMMA A.3.2. *Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns, and let $\Psi\colon \mathcal{F}_A \longrightarrow \mathcal{F}_B$ be a $*$-isomorphism such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$. Then $\Psi(d_A(f)) = d_B(\Psi(f))$ for all $f \in \mathcal{F}_A$. If, in addition, $\Psi \circ \tau_A|_{\mathcal{F}_A} = \tau_B \circ \Psi$, then $\Psi \circ \phi_A|_{\mathcal{D}_A} = \phi_B \circ \Psi|_{\mathcal{D}_A}$.*

PROOF. The groupoids $c_A^{-1}(0)$ and $c_B^{-1}(0)$ are principal. By [**102**, Proposition 4.13] (see also [**26**, Theorem 3.3]) and [**87**, Proposition 5.7] and its proof, there is a groupoid isomorphism $\kappa\colon c_B^{-1}(0) \longrightarrow c_A^{-1}(0)$ and a groupoid homomorphism $\xi\colon c_A^{-1}(0) \longrightarrow \mathbb{T}$ such that

$$\Psi(f)(\eta) = \xi(\kappa(\eta)) f(\kappa(\eta)),$$

for $f \in \mathcal{F}_A$ and $\eta \in c_B^{-1}(0)$. In particular, $\Psi(f)(x) = f(\kappa(x))$ for $x \in \mathsf{X}_B$ when we identify $x \in \mathsf{X}_B$ with $(x,0,x) \in \mathcal{G}_B^{(0)}$. For $x \in \mathsf{X}_B$, we then see that

$$\Psi(f)(x) = f(\kappa(x)) = f|_{\mathsf{X}_A}(\kappa(x)) = \Psi(f|_{\mathsf{X}_A})(x),$$

that is, $\Psi \circ d_A = d_B \circ \Psi$. If, in addition, $\Psi \circ \tau_A|_{\mathcal{F}_A} = \tau_B \circ \Psi$, then

$$\Psi(\phi_A(f)) = \Psi(d_A(\tau_A(f))) = d_B(\tau_B(\Psi(f))) = \phi_B(\Psi(f)),$$

for $f \in \mathcal{D}_A$. □

We now arrive at our main theorem. If A and B are finite square $\{0,1\}$-matrices with no zero rows and no zero columns, then any isomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ restricts to a homeomorphism from $\mathcal{G}_A^{(0)}$ to $\mathcal{G}_B^{(0)}$ and thus induces a homeomorphism from X_A to X_B via the identification of X_A with $\mathcal{G}_A^{(0)}$ and the identification of X_B with $\mathcal{G}_B^{(0)}$. We denote the latter homeomorphism by $\Phi^{(0)}$.

THEOREM A.3.3. *Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns, and let $h\colon \mathsf{X}_A \longrightarrow \mathsf{X}_B$ be a homeomorphism. The following are equivalent.*

(i) *The homeomorphism $h\colon \mathsf{X}_A \longrightarrow \mathsf{X}_B$ is a conjugacy.*
(ii) *There is a groupoid isomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ satisfying $c_B \circ \Phi = c_A$, $\Phi^{(0)} = h$, and*
$$\Phi \circ \epsilon_A = \epsilon_B \circ \Phi. \tag{A.2}$$
(iii) *There is a groupoid isomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ satisfying $\Phi^{(0)} = h$ and* (A.2).
(iv) *There is a $*$-isomorphism $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Psi \circ \gamma_z^A = \gamma_z^B \circ \Psi$ for all $z \in \mathbb{T}$, $\Psi \circ d_A = d_B \circ \Psi$, $\Psi(f) = f \circ h^{-1}$ for all $f \in \mathcal{D}_A$, $\Psi \circ \phi_A|_{\mathcal{D}_A} = \phi_B \circ \Psi|_{\mathcal{D}_A}$ and*
$$\Psi \circ \tau_A = \tau_B \circ \Psi. \tag{A.3}$$
(v) *There is a $*$-isomorphism $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Psi(\mathcal{D}_A) = \mathcal{D}_B$, $\Psi(f) = f \circ h^{-1}$ for all $f \in \mathcal{D}_A$, and* (A.3).
(vi) *There is a $*$-isomorphism $\Theta\colon \mathcal{D}_A \to \mathcal{D}_B$ satisfying $\Theta(f) = f \circ h^{-1}$ for all $f \in \mathcal{D}_A$, and $\Theta \circ \phi_A|_{\mathcal{D}_A} = \phi_B \circ \Theta$.*

REMARK A.3.4. As we shall see in the proof, if $h\colon \mathsf{X}_A \longrightarrow \mathsf{X}_B$ is a conjugacy, then the groupoid isomorphisms $\Phi\colon \mathcal{G}_A \to \mathcal{G}_B$ in (ii) and (iii) can be chosen such that $\Phi((x,n,y)) = (h(x),n,h(y))$ for $(x,n,y) \in \mathcal{G}_A$. Also, if $\Phi\colon \mathcal{G}_A \to \mathcal{G}_B$ is a groupoid isomorphism as in (iii) (or (ii)), then the $*$-isomorphisms $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ in (iv) and (v) can be chosen to satisfy $\Psi(y)(\eta) = y(\Phi^{-1}(\eta))$ for $y \in \mathcal{O}_A$ and $\eta \in \mathcal{G}_B$.

PROOF. (i) $\implies$ (ii): If $h\colon \mathsf{X}_A \longrightarrow X_B$ is a conjugacy we can define a groupoid homomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ by $\Phi((x,n,y)) = (h(x),n,h(y))$ for each $(x,n,y) \in \mathcal{G}_A$. It is clear that Φ is a bijective algebraic homomorphism. In order to see that Φ is continuous, suppose $(x_i,n_i,y_i) \longrightarrow (x,n,y)$ in $\mathcal{G}_A$ as $i \longrightarrow \infty$ and pick $Z(\mu,\nu) \subseteq \mathcal{G}_B$ containing $(h(x),n,h(y))$. Note that n_i eventually equals n. As h is continuous, we have $h(x_i) \in Z(\mu)$ and $h(y_i) \in Z(\nu)$ eventually, hence $h(x_i,n,y_i) \in Z(\mu,\nu)$ eventually, so $h(x_i,n,y_i) \longrightarrow h(x,n,y)$ as $i \longrightarrow \infty$. The argument for Φ^{-1} is symmetric, so Φ is a groupoid isomorphism which satisfies $c_B \circ \Phi = c_A$ and $\Phi^{(0)} = h$. As h is a conjugacy, Φ also satisfies (A.2).

The implications (ii) $\implies$ (iii) and (iv) $\implies$ (v) are obvious.

(iii) $\implies$ (v) and (ii) $\implies$ (iv): A groupoid isomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ with $\Phi^{(0)} = h$ induces a $*$-isomorphism $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ with $\Psi \circ d_A = d_B \circ \Psi$. In particular, $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Psi(f) = f \circ h^{-1}$, for $f \in \mathcal{D}_A$. Since Φ satisfies (A.2), we also have $\Psi \circ \tau_A = \tau_B \circ \Psi$. This is (v). If, in addition, Φ satisfies $c_B \circ \Phi = c_A$, then $\Psi(s_i) = 1_{\Phi(Z(i,r(i)))} \in c_B^{-1}(\{0\})$ so

$$\Psi(\gamma_z^A(s_i)) = z\Psi(1_{\Phi(Z(i,r(i)))}) = \gamma_z^B(\Psi(s_i)),$$

for $i = 1,\dots,N$. It follows that $\Psi \circ \gamma_z^A = \gamma_z^B \circ \Psi$, for every $z \in \mathbb{T}$. In particular, this implies that $\Psi(\mathcal{F}_A) = \mathcal{F}_B$. By Lemma A.3.2, it follows that $\Psi \circ \phi_A|_{\mathcal{D}_A} = \phi_B \circ \Psi|_{\mathcal{D}_A}$. This is (iv).

(v) $\implies$ (vi): If Ψ preserves the diagonal and satisfies (A.3), then

$$\Psi\Big(\bigcup_{k=0}^{\infty} \tau_A^k(\mathcal{D}_A)\Big) = \bigcup_{k=0}^{\infty} \tau_B^k(\mathcal{D}_B).$$

As $\mathcal{F}_A$ is generated by $\bigcup_{k=0}^{\infty} \tau_A^k(\mathcal{D}_A)$ as a C*-algebra, it follows that $\Psi(\mathcal{F}_A) = \mathcal{F}_B$. Lemma A.3.2 then entails that $\Psi \circ \phi_A|_{\mathcal{D}_A} = \phi_B \circ \Psi|_{\mathcal{D}_A}$.

(vi) $\implies$ (i): The relation $\Theta \circ \phi_A|_{\mathcal{D}_A} = \phi_B \circ \Theta$ and the fact that $\phi_A(f)(x) = f(\sigma_A(x))$ for $f \in \mathcal{D}_A$ and $x \in \mathsf{X}_A$ ensures that h is a conjugacy by Gelfand duality. □

COROLLARY A.3.5. *Let A and B be finite square $\{0,1\}$-matrices with no zero rows and no zero columns. The following are equivalent.*

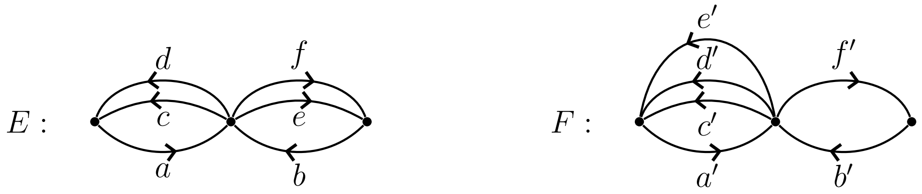
(i) *The one-sided shifts (X_A,σ_A) and (X_B,σ_B) are one-sided conjugate.*
(ii) *There is a groupoid isomorphism $\Phi\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ satisfying $\Phi \circ \epsilon_A = \epsilon_B \circ \Phi$.*
(iii) *There is a $*$-isomorphism $\Psi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Psi \circ \tau_A = \tau_B \circ \Psi$.*

One-sided shifts (X_A,σ_A) and (X_B,σ_B) are *eventually conjugate* if there exist a homeomorphism $h\colon \mathsf{X}_A \longrightarrow \mathsf{X}_B$ and $L \in \mathbb{N}$ such that

$$\sigma_B^L(h(\sigma_A(x)) = \sigma_B^{L+1}(h(x)), \qquad \sigma_A^L(h^{-1}(\sigma_B(y)) = \sigma_A^{L+1}(h^{-1}(y)),$$

for $x \in \mathsf{X}_A$ and $y \in \mathsf{X}_B$. The above theorem should be compared to [**81**, Corollary 3.5] and [**24**, Corollary 4.2] which characterize one-sided eventual conjugacy.

EXAMPLE A.3.6. Consider the following two graphs.



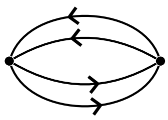
Let X_E be the one-sided edge shift of E and let X_F be the one-sided edge shift of F. Define a homeomorphism $h\colon \mathsf{X}_E \longrightarrow \mathsf{X}_F$ by sending $(x_n)_{n\geqslant 0} \in \mathsf{X}_E$ to $(y_n)_{n\geqslant 0} \in \mathsf{X}_F$ where

$$y_n = \begin{cases} a', & \text{if } n > 0 \text{ and } x_{n-1} = e, \\ (x_n)' & \text{otherwise.} \end{cases}$$

E.g., $h(ebcaf\ldots) = e'a'c'a'f'\ldots$ while $h(\sigma_E(ebcaf\ldots)) = h(bcaf\ldots) = b'c'a'f'\ldots$ and $h^{-1}(a'e'a'f'\ldots) = aebf\ldots$. Observe that

$$\sigma_F^2(h(x)) = \sigma_F(h(\sigma_E(x))), \qquad \sigma_E^2(h^{-1}(y)) = \sigma_E(h^{-1}(\sigma_F(y))),$$

for $x \in \mathsf{X}_E$ and $y \in \mathsf{X}_F$. Hence X_E and X_F are eventually conjugate. On the other hand, the total amalgamation of E is the graph



while the total amalgamation of F is F itself. It thus follows that X_E and X_F are *not* conjugate, cf. e.g., [**68**, Section 13.8] or [**62**, Theorem 2.1.10].

ARTICLE B

C*-algebras, groupoids and covers of shift spaces

Kevin Aguyar Brix and Toke Meier Carlsen

Abstract

To every one-sided shift space X we associate a cover $\widetilde{\mathsf{X}}$, a groupoid $\mathcal{G}_{\mathsf{X}}$ and a C*-algebra $\mathcal{O}_{\mathsf{X}}$. On the one hand, we characterize one-sided conjugacy, eventual conjugacy and (stabilizer preserving) continuous orbit equivalence between X and Y in terms of isomorphism of $\mathcal{G}_{\mathsf{X}}$ and $\mathcal{G}_{\mathsf{Y}}$, and diagonal preserving *-isomorphism of $\mathcal{O}_{\mathsf{X}}$ and $\mathcal{O}_{\mathsf{Y}}$. On the other hand, we characterize two-sided conjugacy and flow equivalence of the associated two-sided shift spaces Λ_{X} and Λ_{Y} in terms of isomorphism of the stabilized groupoids $\mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ and $\mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$, and diagonal preserving *-isomorphism of the stabilized C*-algebras $\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K}$ and $\mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$. Our strategy is to lift relations on the shift spaces to similar relations on the covers.

Introduction

In **[35]**, Cuntz and Krieger used finite type symbolic dynamical systems to construct a family of simple C*-algebras today known as *Cuntz–Krieger algebras*. Such a dynamical system is up to conjugacy determined by a finite square $\{0,1\}$-matrix A, and the C*-algebra $\mathcal{O}_A$ comes equipped with a distinguished commutative subalgebra $\mathcal{D}_A$ called the *diagonal* and a circle action $\gamma\colon \mathbb{T} \curvearrowright \mathcal{O}_A$ called the *gauge action*. This construction has allowed for new and fruitful discoveries in both symbolic dynamics and in operator algebras via translations of interesting problems and results.

One of the most important relations among two-sided subshifts besides conjugacy is flow equivalence. Cuntz and Krieger showed that if the subshifts Λ_A and Λ_B, determined by irreducible matrices which are not permutations A and B, are flow equivalent, then there is *-isomorphism between the stabilized Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ which maps $\mathcal{D}_A \otimes c_0$ onto $\mathcal{D}_B \otimes c_0$. Here, $\mathbb{K}$ is the C*-algebra of compact operators on separable Hilbert space and c_0 is the maximal abelian subalgebra of diagonal operators. The stabilized Cuntz–Krieger algebras together with their diagonal subalgebra therefore constitute an invariant of flow equivalence. However,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

are examples of irreducible and nonpermutation matrices which are not flow equivalent but whose Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A_-}$ are *-isomorphic, cf. [**106**]. This raised the question: Is it possible to characterize flow equivalence in terms of the associated C*-algebras?

In the striking paper [**84**] (see also [**86**]), Matsumoto and Matui employ topological groupoids to answer this question: Using Renault's groupoid reconstruction theory [**102**] (based on work of Kumjian [**64**]) they prove that Λ_A and Λ_B (determined by irreducible and nonpermutation $\{0,1\}$-matrices A and B) are flow equivalent if and only if there is a *-isomorphism $\Phi\colon \mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ satisfying $\Phi(\mathcal{D}_A \otimes c_0) = \mathcal{D}_B \otimes c_0$. In the particular case above, it follows that no *-isomorphism $\mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_{A_-} \otimes \mathbb{K}$ can be diagonal-preserving.

In [**78**] (see also [**79**]), Matsumoto introduces the notion of continuous orbit equivalence. He proves that one-sided shift spaces X_A and X_B (determined by irreducible and nonpermutation $\{0,1\}$-matrices A and B) are continuously orbit equivalent if and only if there is a *-isomorphism $\mathcal{O}_A \longrightarrow \mathcal{O}_B$ which carries $\mathcal{D}_A$ onto $\mathcal{D}_B$. For this reason, Matsumoto remarks that continuous orbit equivalence is a one-sided analog of flow equivalence. These results on flow equivalence and continuous orbit equivalence are generalized to include all finite type shifts in [**22**].

In the more general setting of directed graphs, the second-named author and Rout used groupoids to show that X_A and X_B (for $\{0,1\}$-matrices A and B with no zero rows and no zero columns) are one-sided eventually conjugate (see [**81, 82**]) if and only if there is *-isomorphism $\Phi\colon \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \gamma^A = \gamma^B \circ \Phi$. Furthermore, they show that Λ_A and Λ_B are conjugate if and only if there is a *-isomorphism $\Phi\colon \mathcal{O}_A \otimes \mathbb{K} \longrightarrow \mathcal{O}_B \otimes \mathbb{K}$ satisfying $\Phi(\mathcal{D}_A \otimes c_0) = \mathcal{D}_B \otimes c_0$ and $\Phi \circ (\gamma^A \otimes \mathrm{id}) = (\gamma^B \otimes \mathrm{id}) \circ \Phi$. From this we understand that one-sided eventual conjugacy is a one-sided analog of two-sided conjugacy. In a similar spirit, one-sided conjugacy for shifts of finite type was characterized using groupoids in terms of the Cuntz–Krieger algebra with its diagonal and a certain completely positive map by the authors [**12**]. Orbit equivalence of general directed graphs were studied in [**15, 3, 29**].

The aim of this paper is to study general shift spaces and provide similar characterizations in terms of groupoids and C*-algebras. When X is a shift space which is not of finite type then the shift operation σ_{X} is not a local homeomorphism [**96**] so $(\mathsf{X}, \sigma_{\mathsf{X}})$ is not a Deaconu–Renault system (in the sense of [**110**]). The Deaconu–Renault groupoid naturally associated to X therefore fails to be étale. Therefore, a naïve strategy to generalize Cuntz and Krieger's results does not work here. The bulk of the work is therefore to circumvent this problem.

Matsumoto is the first to associate a C*-algebra to a general two-sided subshift and study its properties, see [**69, 70, 71, 72, 73**]. Unfortunately, there was a mistake in one of the foundational results. Carlsen and Matsumoto [**23**] then provided a new construction which is in general not *-isomorphic to Matsumoto's algebra. This new construction lacks a universal property and therefore has the downside of not always admitting a gauge action. Carlsen finally introduced a C*-algebra $\mathcal{O}_{\mathsf{X}}$ associated to a general *one-sided* shift space X using a Cuntz–Pimsner construction [**20**] which satisfies Matsumoto's results and admits a gauge action. We refer the reader to [**23, 27, 38**] for a more detailed description of the story of associating a C*-algebra to general subshifts.

The C*-algebra $\mathcal{O}_{\mathsf{X}}$ has appeared in various guises throughout the literature. In [**113**] (see also [**30**]), Thomsen realized it as a groupoid C*-algebra of a semi-étale groupoid, Carlsen and Silvestrov describe it as one of Exel's crossed products [**27, 28**], while Dokuchaev and Exel use partial actions [**38**]. Matsumoto then took a slightly different approach and considered certain labeled Bratteli diagrams called λ-graph systems and associated to each λ-graph system $\mathfrak{L}$ a C*-algebra $\mathcal{O}_{\mathfrak{L}}$ [**74, 76, 77**]. Any two-sided subshift Λ has a canonical λ-graph system $\mathfrak{L}^{\Lambda}$ and the spectrum of the diagonal subalgebra of $\mathcal{O}_{\mathfrak{L}}$ is (homeomorphic to) the λ-graph $\mathfrak{L}^{\Lambda}$. Matsumoto then studied orbit equivalence, eventual conjugacy and two-sided conjugacy of these λ-graphs and how they are reflected in the C*-algebras [**79, 83**].

Our approach is based on [**19**]: To any one-sided shift space $(\mathsf{X}, \sigma_{\mathsf{X}})$, we construct a cover $\widetilde{\mathsf{X}}$ equipped with a local homeomorphism $\sigma_{\widetilde{\mathsf{X}}} \colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{X}}$ and a surjection $\pi_{\mathsf{X}} \colon \widetilde{\mathsf{X}} \longrightarrow \mathsf{X}$ which intertwines the shifts. The pair $(\widetilde{\mathsf{X}}, \sigma_{\widetilde{\mathsf{X}}})$ is a Deaconu–Renault system. From the cover, we construct the Deaconu–Renault groupoid $\mathcal{G}_{\mathsf{X}}$ which is étale and consider the associated groupoid C*-algebra $\mathcal{O}_{\mathsf{X}}$. Starling [**111**] constructed the space $\widetilde{\mathsf{X}}$ as the tight spectrum of a certain inverse semigroup $\mathcal{S}_{\mathsf{X}}$ associated to X and showed that $\mathcal{O}_{\mathsf{X}}$ is *-isomorphic to the tight C*-algebra of $\mathcal{S}_{\mathsf{X}}$. The construction of $\widetilde{\mathsf{X}}$ generalizes the left Krieger cover of a sofic shift. From [**18**], we therefore know that for sofic shifts $\mathcal{O}_{\mathsf{X}}$ is *-isomorphic to a graph C*-algebra.

The paper is structured in the following way: In Section B.2 we define the cover $\widetilde{\mathsf{X}}$ and the associated groupoid $\mathcal{G}_{\mathsf{X}}$. We characterize when $\mathcal{G}_{\mathsf{X}}$ is principal or essentially principal, respectively, in terms of conditions on X. In Section B.3, we show that any *-isomorphism $\mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ which maps $C(\mathsf{X})$ onto $C(\mathsf{Y})$ is in fact diagonal-preserving under mild conditions on X and Y. Sections B.4, B.5 and B.7 give complete characterizations of one-sided conjugacy (Theorem B.4.4), one-sided eventual conjugacy (Theorem B.5.3) and two-sided conjugacy (Theorem B.7.5), respectively, in terms of certain isomorphisms of groupoids and certain diagonal-preserving *-isomorphisms of C*-algebras. As opposed to Matsumoto, our results are not limited to the case where the groupoid is essentially principal, and we characterize the relations on the shift spaces and not only the covers (or the λ-graphs).

In Section B.6 we study continuous orbit equivalence. We characterize stabilizer preserving continuous orbit equivalence in terms of isomorphisms of groupoids which respect certain cocycles, and *-isomorphisms of C*-algebras which respect certain gauge actions (Theorem B.6.4). Section B.8 concerns flow equivalence. We can characterize flow equivalence in terms of isomorphism of stabilized groupoids which respects certain cohomological data, and *-isomorphism of stabilized C*-algebras which respect certain gauge actions suitably stabilized (Theorem B.8.9). When the groupoids involved are essentially principal, some of the conditions simplify. In particular, we find that continuous orbit equivalence between sofic shifts whose groupoids are essentially principal implies flow equivalence, cf. Corollary B.8.12.

In most sections we prove our results by lifting a relation on the shift spaces to a similar relation on the covers. We can then encode this relation into structure-preserving *-isomorphisms of the C*-algebras using groupoids as an intermediate step. The recent and more general reconstruction theory of Ruiz, Sims, Tomforde and the second-named author [**26**] allows us to reconstruct the groupoid from the C*-algebras.

B.1. Preliminaries

We let $\mathbb{Z}$ denote the integers and let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ denote the nonnegative and positive integers, respectively.

B.1.1. Symbolic dynamics. Let $\mathfrak{A}$ be a finite set of symbols (the *alphabet*) considered as a discrete space and let $|\mathfrak{A}|$ denote its cardinality. Then

$$\mathfrak{A}^{\mathbb{N}} = \{x = x_0 x_1 x_2 \cdots \mid x_i \in \mathfrak{A}, i \in \mathbb{N}\}$$

is a second countable, compact Hausdorff space when equipped with the subspace topology of the product topology on $\mathfrak{A}^{\mathbb{N}}$. The *shift-operation* $\sigma\colon \mathfrak{A}^{\mathbb{N}} \longrightarrow \mathfrak{A}^{\mathbb{N}}$ is the continuous surjection given by $\sigma(x)_n = x_{n+1}$, for $x \in \mathfrak{A}^{\mathbb{N}}$. A *one-sided shift space* is a pair $(\mathsf{X}, \sigma_{\mathsf{X}})$ in which $\mathsf{X} \subseteq \mathfrak{A}^{\mathbb{N}}$ is closed and *shift-invariant* in the sense that $\sigma(\mathsf{X}) \subseteq \mathsf{X}$ (we do not assume equality) and where $\sigma_{\mathsf{X}} := \sigma|_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathsf{X}$.

Let X be a one-sided shift space over the alphabet $\mathfrak{A}$. If $x = x_0 x_1 x_2 \cdots \in \mathsf{X}$, we write $x_{[i,j)} = x_i x_{i+1} \cdots x_{j-1}$ for $0 \leqslant i < j$ and $x_{[i,\infty)} = x_i x_{i+1} \cdots$ for $0 \leqslant i$. A finite *word* $\mu = \mu_1 \cdots \mu_k$ with $\mu_i \in \mathfrak{A}$, for each $i = 0, \ldots, k$, is *admissible* in X if $x_{[i,j)} = \mu$ for some $x \in \mathsf{X}$. Let $|\mu| = k$ denote the *length* of μ. The *empty word* ε is the unique word of length zero which satisfies $\varepsilon\mu = \mu = \mu\varepsilon$ for any word μ. The collection of admissible words in X of length l is denoted $\mathsf{L}_l(\mathsf{X})$ and the *language* of X is then the monoid consisting of the union $\mathsf{L}(\mathsf{X}) = \bigcup_{l \geqslant 0} \mathsf{L}_l(\mathsf{X})$; the product being concatenation of words.

The *cylinder set* of a word $\mu \in \mathsf{L}(\mathsf{X})$ is the compact and open set

$$Z_{\mathsf{X}}(\mu) = \{\mu x \in \mathsf{X} \mid x \in \mathsf{X}\},$$

and the collection of sets of the form $Z_{\mathsf{X}}(\mu)$ constitute a basis for the topology of X. A point $x \in \mathsf{X}$ is *isolated* if there is a $k \in \mathbb{N}$ such that $\{x\} = Z_{\mathsf{X}}(x_{[0,k)})$.

A point $x \in \mathsf{X}$ is *periodic* if there exists $p \in \mathbb{N}_+$ such that $\sigma_{\mathsf{X}}^p(x) = x$ and *eventually periodic* if there is an $n \in \mathbb{N}$ such that $\sigma_{\mathsf{X}}^n(x)$ is periodic. The *least period* of an eventually periodic point $x \in \mathsf{X}$ is

$$\operatorname{lp}(x) = \min\{p \in \mathbb{N}_+ \mid \exists n, m \in \mathbb{N} : p = n - m, \sigma_{\mathsf{X}}^n(x) = \sigma_{\mathsf{X}}^m(x)\}.$$

A point is *aperiodic* if it is not eventually periodic. The *stabilizer* of $x \in \mathsf{X}$ is the group $\operatorname{Stab}(x) = \{p \in \mathbb{Z} \mid \exists k, l \in \mathbb{N} : p = k - l, \sigma_{\mathsf{X}}^k(x) = \sigma_{\mathsf{X}}^l(x)\}$.

Following [**71**], we define for every $x \in \mathsf{X}$ and $l \in \mathbb{N}$ the *predecessor set* as

$$P_l(x) = \{\mu \in \mathsf{L}_l(\mathsf{X}) \mid \mu x \in \mathsf{X}\}.$$

Two points $x, y \in \mathsf{X}$ are *l-past equivalent* if $P_l(x) = P_l(y)$, in which case we write $x \sim_l y$. Let $[x]_l$ be the l-past equivalence class of x. A point $x \in \mathsf{X}$ is *isolated in past equivalence* if there is an $l \in \mathbb{N}$ such that $[x]_l$ is a singleton. A shift space X satisfies *Matsumoto's condition (I)* [**71**] if no points are isolated in past equivalence; this is a generalization of Cuntz and Krieger's condition (I). We shall also consider the slightly weaker condition that there are no *periodic* points which are isolated in past equivalence.

A *two-sided shift space* is a subset $\Lambda \subseteq \mathfrak{A}^{\mathbb{Z}}$ which is closed and shift invariant with respect to the shift operation $\sigma \colon \mathfrak{A}^{\mathbb{Z}} \longrightarrow \mathfrak{A}^{\mathbb{Z}}$ given by $\sigma(\mathrm{x})_n = \mathrm{x}_{n+1}$, for $\mathrm{x} = \ldots x_{-1}x_0x_1\ldots \in \Lambda$ and $n \in \mathbb{Z}$. Let $\sigma_\Lambda = \sigma|_\Lambda \colon \Lambda \longrightarrow \Lambda$. A pair of two-sided shift spaces (Λ_1, σ_1) and (Λ_2, σ_2) are *two-sided conjugate* if there is a homeomorphism $h \colon \Lambda_1 \longrightarrow \Lambda_2$ satisfying $h \circ \sigma_1 = \sigma_2 \circ h$. We shall consider conjugacy of two-sided shift spaces in Section B.7.

Given a two-sided shift space $(\Lambda, \sigma_\Lambda)$ there is a corresponding one-sided shift space defined by

$$\mathsf{X}_\Lambda = \{\mathrm{x}_{[0,\infty)} \in \mathfrak{A}^{\mathbb{N}} \mid \mathrm{x} \in \Lambda\}$$

together with the obvious shift operation. Conversely, if $(\mathsf{X}, \sigma_\mathsf{X})$ is a one-sided shift space and σ_X is surjective, then the pair consisting of the projective limit

$$\Lambda_\mathsf{X} = \varprojlim(\mathsf{X}, \sigma_\mathsf{X})$$

together with the induced shift homeomorphism $\sigma_\mathsf{X} \colon \Lambda_\mathsf{X} \longrightarrow \Lambda_\mathsf{X}$ given by $\sigma_\mathsf{X}(\mathrm{x})_n = \mathrm{x}_{n+1}$ for $\mathrm{x} \in \Lambda$ is the corresponding *two-sided shift space* (this is called the *natural extension* of X in [**37**, Section 9]). The two operations are mutually inverse to each other. See [**68, 62**] for excellent introductions to the general theory of symbolic dynamics.

B.1.2. C***-algebras of shift spaces.** To each shift space X, there is a universal unital C*-algebra $\mathcal{O}_\mathsf{X}$ which was first constructed as a Cuntz-Pimsner algebra [**20**]. In Section B.2, we follow [**19**] and construct a second countable, amenable, locally compact, Hausdorff and étale groupoid $\mathcal{G}_\mathsf{X}$ whose C*-algebra is canonically isomorphic to $\mathcal{O}_\mathsf{X}$. For an introduction to (étale) groupoid C*-algebras see [**101, 98**] or the introductory notes [**109**].

We briefly recall the universal description of $\mathcal{O}_\mathsf{X}$ [**20**]. Given words $\mu, \nu \in \mathsf{L}(\mathsf{X})$, consider the set

$$C_\mathsf{X}(\mu, \nu) := \{\nu x \in \mathsf{X} \mid \mu x \in \mathsf{X}\}$$

which is closed (but not necessarily open) in X. We shall refer to the commutative C*-algebra

$$\mathcal{D}_\mathsf{X} := \mathrm{C}^*\{1_{C_\mathsf{X}(\mu,\nu)} \mid \mu, \nu \in \mathsf{L}(\mathsf{X})\}$$

inside the C*-algebra of bounded functions on X as the *diagonal*. The C*-algebra $\mathcal{O}_\mathsf{X}$ is the universal unital C*-algebra generated by partial isometries $(s_\mu)_{\mu \in \mathsf{L}(\mathsf{X})}$ satisfying

$$s_\mu s_\nu = \begin{cases} s_{\mu\nu} & \mu\nu \in \mathsf{L}(\mathsf{X}), \\ 0 & \text{otherwise}, \end{cases}$$

and such that the map

$$1_{C(\mu,\nu)} \longmapsto s_\nu s_\mu^* s_\mu s_\nu^*,$$

for $\mu, \nu \in \mathsf{L}(\mathsf{X})$, extends to *-homomorphism $\mathcal{D}_\mathsf{X} \longrightarrow \mathrm{C}^*(s_\mu \mid \mu \in \mathsf{L}(\mathsf{X}))$. This map is injective and the projections $\{s_\nu s_\mu^* s_\mu s_\nu^*\}_{\mu,\nu}$ generate a commutative C*-subalgebra which is *-isomorphic to $\mathcal{D}_\mathsf{X}$ via the above map. We shall henceforth identify $\mathcal{D}_\mathsf{X}$ with this C*-subalgebra of $\mathcal{O}_\mathsf{X}$.

The universal property ensures that there is a *canonical gauge action* $\gamma^\mathsf{X} \colon \mathbb{T} \curvearrowright \mathcal{O}_\mathsf{X}$ of the circle group $\mathbb{T}$ given by

$$\gamma_z^\mathsf{X}(s_\mu) = z^{|\mu|} s_\mu,$$

for every $z \in \mathbb{T}$ and $\mu \in \mathsf{L}(\mathsf{X})$. The fixed point algebra under the gauge action is an AF-algebra which is denoted $\mathcal{F}_\mathsf{X}$. Note that $\mathcal{D}_\mathsf{X} \subseteq \mathcal{F}_\mathsf{X}$.

B.2. The approach

Let X be a one-sided shift space. In this section, we associate a cover $\widetilde{\mathsf{X}}$ to X and build a groupoid $\mathcal{G}_\mathsf{X}$ from the cover and its dynamical properties. This construction is due to the second-named author in [**19**, Chapter 2]. The C*-algebra $\mathcal{O}_\mathsf{X}$ is then constructed as a groupoid C*-algebra.

B.2.1. The cover $\widetilde{\mathsf{X}}$. Consider the set $\mathcal{I} = \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid k \leqslant l\}$ equipped with the partial order $\preceq$ given by

$$(k_1, l_1) \preceq (k_2, l_2) \iff k_1 \leqslant k_2 \text{ and } l_1 - k_1 \leqslant l_2 - k_2.$$

For every $(k,l) \in \mathcal{I}$ we define an equivalence relation on X by

$$x \overset{k,l}{\sim} x' \iff x_{[0,k)} = x'_{[0,k)} \text{ and } \bigcup_{l' \leqslant l} P_{l'}(\sigma_\mathsf{X}^k(x)) = \bigcup_{l' \leqslant l} P_{l'}(\sigma_\mathsf{X}^k(x')).$$

The (k,l)-equivalence class of $x \in \mathsf{X}$ is denoted ${}_k[x]_l$ and each ${}_k\mathsf{X}_l = \{{}_k[x]_l \mid x \in \mathsf{X}\}$ is a finite set. If $(k_1, l_1) \preceq (k_2, l_2)$, then

$$x \overset{k_2,l_2}{\sim} x' \implies x \overset{k_1,l_1}{\sim} x',$$

for every $x, x' \in \mathsf{X}$. Hence there is a well-defined map ${}_{(k_1,l_1)}Q_{(k_2,l_2)} \colon {}_{k_2}\mathsf{X}_{l_2} \longrightarrow {}_{k_1}\mathsf{X}_{l_1}$ given by

$${}_{(k_1,l_1)}Q_{(k_2,l_2)}({}_{k_2}[x]_{l_2}) = {}_{k_1}[x]_{l_1},$$

for every ${}_{k_2}[x]_{l_2} \in {}_{k_2}\mathsf{X}_{l_2}$. When the context is clear, we shall omit the subscripts of the map. The spaces ${}_k\mathsf{X}_l$ together with the maps Q thus define a projective system.

Definition B.2.1. Let X be a one-sided shift space. The *cover* of X is the second countable compact Hausdorff space $\widetilde{\mathsf{X}}$ defined as the projective limit $\varprojlim_{(k,l)\in\mathcal{I}} ({}_k\mathsf{X}_l, Q)$. We identify this with

$$\widetilde{\mathsf{X}} = \left\{ ({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}} \in \prod_{(k,l)\in\mathcal{I}} {}_k\mathsf{X}_l \mid (k_1,l_1) \preceq (k_2,l_2) \colon {}_{k_1}[{}_{k_1}x_{l_1}]_{l_1} = {}_{k_1}[{}_{k_2}x_{l_2}]_{l_1} \right\}$$

equipped with the subspace topology of the product topology of $\prod_{(k,l)\in\mathcal{I}} {}_k\mathsf{X}_l$.

The topology of $\widetilde{\mathsf{X}}$ is generated by compact open sets of the form

$$U(x,k,l) = \{\widetilde{x} \in \widetilde{\mathsf{X}} \mid {}_kx_l \overset{k,l}{\sim} x\},$$

for $x \in \mathsf{X}$ and $(k,l) \in \mathcal{I}$. In order to see that sets of the above form constitute a basis, let $\widetilde{x} \in U(y,k_1,l_1) \cap U(z,k_2,l_2)$. Set $k := \max\{k_1,k_2\}$ and $l := l_1 + l_2$. The pair (k,l) thus majorizes both (k_1,l_1) and (k_2,l_2), and

$$\widetilde{x} \in U({}_kx_l,k,l) \subseteq U(y,k_1,l_1) \cap U(z,k_2,l_2),$$

Given a word $\mu \in \mathsf{L}(\mathsf{X})$, we also consider the compact open sets

$$U_\mu := \bigcup_{x \in C(\mu)} U(x,|\mu|,|\mu|).$$

We shall now determine a *shift operation* on $\widetilde{\mathsf{X}}$ endowing it with the structure of a dynamical system. For any $(k,l) \in \mathcal{I}$ with $k \geqslant 1$, observe that

$$x \overset{k,l}{\sim} y \implies \sigma_{\mathsf{X}}(x) \overset{k-1,l}{\sim} \sigma_{\mathsf{X}}(y).$$

Therefore, there is a well-defined map ${}_k\sigma_l \colon {}_k\mathsf{X}_l \longrightarrow {}_{k-1}\mathsf{X}_l$ given by

$${}_k\sigma_l({}_k[x]_l) = {}_{k-1}[\sigma_{\mathsf{X}}(x)]_l,$$

for every ${}_k[x]_l \in {}_k\mathsf{X}_l$, $k \geqslant 1$. When the context is clear, we shall omit the subscripts. Furthermore, this shift operation intertwines the maps Q in the sense that the diagram

$$\begin{array}{ccc} {}_{k_2}\mathsf{X}_{l_2} & \xrightarrow{\sigma} & {}_{k_2-1}\mathsf{X}_{l_2} \\ Q\downarrow & & \downarrow Q \\ {}_{k_1}\mathsf{X}_{l_1} & \xrightarrow[\sigma]{} & {}_{k_1-1}\mathsf{X}_{l_1} \end{array}$$

commutes for every $(k_1,l_1),(k_2,l_2) \in \mathcal{I}$ with $(k_1,l_1) \preceq (k_2,l_2)$ and $k_1 \geqslant 1$. It follows that there is an induced shift operation $\sigma_{\widetilde{\mathsf{X}}} \colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{X}}$ given by

$${}_k\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})_l = {}_{k+1}\sigma_l({}_{k+1}[{}_{k+1}x_l]_l) = {}_k[\sigma_{\mathsf{X}}({}_{k+1}x_l)]_l,$$

for every $\widetilde{x} = ({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}} \in \widetilde{\mathsf{X}}$. The pair $(\widetilde{\mathsf{X}},\sigma_{\widetilde{\mathsf{X}}})$ is then a dynamical system.

There is a canonical continuous and surjective map $\pi_{\mathsf{X}} \colon \widetilde{\mathsf{X}} \longrightarrow \mathsf{X}$ given in the following way: If $\widetilde{x} \in \widetilde{\mathsf{X}}$, then $x = \pi_{\mathsf{X}}(\widetilde{x}) \in \mathsf{X}$ is the unique element with the property that $x_{[0,k)} = ({}_kx_l)_{[0,k)}$, for every $(k,l) \in \mathcal{I}$. This map intertwines the shift operations in the sense that

$$\sigma_{\mathsf{X}} \circ \pi_{\mathsf{X}} = \pi_{\mathsf{X}} \circ \sigma_{\widetilde{\mathsf{X}}}.$$

We shall refer to π_{X} as the *canonical factor map* associated to X. It is injective (and thus a homeomorphism) if and only if X is a shift of finite type.

On the other hand, there is a *function* $\iota_{\mathsf{X}} \colon \mathsf{X} \longrightarrow \widetilde{\mathsf{X}}$ given by sending $x \in \mathsf{X}$ to $\widetilde{x} \in \widetilde{\mathsf{X}}$ for which ${}_kx_l = x$, for every $(k,l) \in \mathcal{I}$. This satisfies the relation $\pi_{\mathsf{X}} \circ \iota_{\mathsf{X}} = \mathrm{id}_{\mathsf{X}}$. If $x \in \mathsf{X}$ is isolated, then $\pi_{\mathsf{X}}^{-1}(x) = \{\iota_{\mathsf{X}}(x)\}$. However, ι_{X} is in general *not* continuous.

EXAMPLE B.2.2. The even shift X_{even} is the strictly sofic one-sided shift space over the alphabet $\{0,1\}$ determined by the forbidden words $\mathscr{F} = \{10^{2n+1}1 \mid n \in \mathbb{N}\}$. The space X_{even} contains no isolated points, but 0^∞ is the unique element for which $P_2(0^\infty) = \{00, 10, 01\}$, so 0^∞ is isolated in past equivalence. Hence $\iota_{\text{even}}(0^\infty) \in \widetilde{\mathsf{X}}_{\text{even}}$ is isolated and ι_{even} is not continuous.

LEMMA B.2.3. *The shift operation* $\sigma_{\widetilde{\mathsf{X}}} \colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{X}}$ *is a local homeomorphism.*

PROOF. We show that $\sigma_{\widetilde{\mathsf{X}}}$ is open and locally injective. For the first part, let $z \in \mathsf{X}$ and $(k,l) \in \mathcal{I}$ with $k \geqslant 1$ and suppose $a = z_0 \in \mathfrak{A}$. We claim that

$$\sigma_{\widetilde{\mathsf{X}}}(U(z,k,l)) = U(\sigma_{\mathsf{X}}(z), k-1, l).$$

The left-to-right inclusion is straightforward. For the converse let $\widetilde{x} \in U(\sigma_{\mathsf{X}}(z), k-1, l)$ and note that $(0,1) \preceq (k-1, l)$. Since ${}_{k-1}x_l \overset{k-1,l}{\sim} \sigma_{\mathsf{X}}(z)$ it thus follows that ${}_{k-1}x_l \overset{0,1}{\sim} \sigma_{\mathsf{X}}(z)$. As $a \in P_1(\sigma_{\mathsf{X}}(z))$, we see that $a_{k-1}x_l \in \mathsf{X}$. A similar argument shows that $a_r x_s \in \mathsf{X}$, for every $(r,s) \in \mathcal{I}$. Put ${}_r y_s = a_r x_{s+1}$, for every $(r,s) \in \mathcal{I}$. Now, if $(k_1,l_1) \preceq (k_2,l_2)$ in $\mathcal{I}$, then $a_{k_2}x_{l_2+1} \overset{k_1,l_1}{\sim} a_{k_1}x_{l_1+1}$ and so

$${}_{(k_1,l_1)}Q_{(k_2,l_2)}({}_{k_2}[{}_{k_2}y_{l_2}]_{l_2}) = {}_{k_1}[a_{k_2}x_{l_2+1}]_{l_1} = {}_{k_1}[a_{k_1}x_{l_1+1}]_{l_1} = {}_{k_1}[{}_{k_1}y_{l_1}]_{l_1}.$$

Hence $\widetilde{y} = ({}_r[{}_r y_s]_s)_{(r,s)\in\mathcal{I}} \in \widetilde{\mathsf{X}}$. Observe now that

$${}_k y_l = a_k x_{l+1} \overset{k,l}{\sim} z,$$

showing that $\widetilde{y} \in U(z,k,l)$. Finally, we see that $\widetilde{x} = \sigma_{\widetilde{\mathsf{X}}}(\widetilde{y}) \in \sigma_{\widetilde{\mathsf{X}}}(U(z,k,l))$ so $\sigma_{\widetilde{\mathsf{X}}}$ is open.

In order to see that $\sigma_{\widetilde{\mathsf{X}}}$ is locally injective let $z \in \mathsf{X}$ with $a = z_0 \in \mathfrak{A}$. We claim that $\sigma_{\widetilde{\mathsf{X}}}$ is injective on $U(x,1,1)$. Indeed, suppose $\widetilde{x}, \widetilde{y} \in U(x,1,1)$ and $\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}(\widetilde{y})$. In particular, $({}_k x_l)_0 = z_0 = ({}_k y_l)_0$ for every $(k,l) \in \mathcal{I}$. Hence

$${}_k x_l = a\sigma_{\mathsf{X}}({}_k x_l) \overset{k,l}{\sim} a\sigma_{\mathsf{X}}({}_k y_l) = {}_k y_l$$

for every $(k,l) \in \mathcal{I}$ from which it follows that $\widetilde{x} = \widetilde{y}$. We conclude that $\sigma_{\widetilde{\mathsf{X}}}$ is a local homeomorphism. □

REMARK B.2.4. The cover $(\widetilde{\mathsf{X}}, \sigma_{\widetilde{\mathsf{X}}})$ is a *Deaconu–Renault system* in the sense of [**26**, Section 8], and the construction is a generalization of the left Krieger cover of a sofic shift space. In particular, the cover of a sofic shift is (conjugate to) a shift of finite type.

The next lemma shows how the topologies of X and $\widetilde{\mathsf{X}}$ interact.

LEMMA B.2.5. *Let* X *be a one-sided shift space and let* $k \colon \mathsf{X} \longrightarrow \mathbb{N}$ *be a map. Then the map* $k_{\widetilde{\mathsf{X}}} \colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ *satisfying* $k_{\widetilde{\mathsf{X}}} = k \circ \pi_{\mathsf{X}}$ *is continuous if and only if* k *is continuous.*

PROOF. Define $k_{\widetilde{\mathsf{X}}} \colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ by $k_{\widetilde{\mathsf{X}}} = k \circ \pi_{\mathsf{X}}$. If k is continuous, then $k_{\widetilde{\mathsf{X}}}$ is continuous.

Suppose k is not continuous. Then there is an element $x \in \mathsf{X}$ and a convergent sequence $(x_n)_n$ with limit x such that $k(x_n) \neq k(x)$ for all $n \in \mathbb{N}$. In particular, the set

$$C_k = \{x_n \mid n \in \mathbb{N}\} \cap Z(x_{[0,k)})$$

is nonempty. As π_{X} is surjective, $\widetilde{C}_k = \pi_{\mathsf{X}}^{-1}(C_k)$ is nonempty. Choose $\widetilde{t}_k \in \widetilde{C}_k$ for each $k \in \mathbb{N}$. Then $\pi_{\mathsf{X}}(\widetilde{t}_k) = x_{n_k}$ for some $n_k \in \mathbb{N}$ so $\widetilde{k}(\widetilde{t}_k) \neq k(x)$ for all $k \in \mathbb{N}$. Furthermore, the sequence $(\widetilde{t}_k)_k$ has a convergent subsequence $(\widetilde{t}_{k_j})_j$ with some limit $\widetilde{x}$ which satisfies

$$\pi_{\mathsf{X}}(\widetilde{x}) = \pi_{\mathsf{X}}(\lim_{j\to\infty} \widetilde{t}_{k_j}) = \lim_{j\to\infty} \pi_{\mathsf{X}}(\widetilde{t}_{k_j}) = \lim_{j\to\infty} x_{n_{k_j}} = x,$$

so $\widetilde{x} \in \pi_{\mathsf{X}}^{-1}(x)$. Then $\widetilde{t}_{k_j} \longrightarrow \widetilde{x}$ in $\widetilde{\mathsf{X}}$ and $k_{\widetilde{\mathsf{X}}}(\widetilde{x}) = k(x) \neq k_{\widetilde{\mathsf{X}}}(\widetilde{t}_{k_j})$ for every $j \in \mathbb{N}$, so $k_{\widetilde{\mathsf{X}}}$ is not continuous. □

The cover $\widetilde{\mathsf{X}}$ may contain isolated points even if X does not, cf. Example B.2.2. In [**22**, Lemma 4.3(1)], it is shown that every isolated point in a shift of finite type is eventually periodic. This is also the case for the class of sofic shift space but it need not be true in general.

Lemma B.2.6. *Let* X *be a one-sided* sofic *shift. If* $x \in \mathsf{X}$ *is isolated, then* x *is eventually periodic.*

Proof. Let $x \in \mathsf{X}$ be isolated. Then $\widetilde{x} \in \pi_{\mathsf{X}}^{-1}(x)$ is isolated. The cover $\widetilde{\mathsf{X}}$ is (conjugate to) a shift of finite type, so $\widetilde{x}$ is eventually periodic, cf. [**22**, Lemma 4.3(1)]. Hence $x = \pi_{\mathsf{X}}(\widetilde{x})$ is eventually periodic. □

Example B.2.7. Consider the shift space X_ω over the alphabet $\{0,1\}$ generated by the sequence

$$\omega = 01010010001000\cdots.$$

Since ω is not periodic, X_ω is infinite. The shift operation σ_ω is not surjective and X_ω is not minimal. We can identify X_ω with the orbit of ω together with all its accumulation points, i.e.,

$$\mathsf{X}_\omega = \{\sigma_\omega^i(\omega) : i \in \mathbb{N}\} \cup \{0^n 10^\infty : n \in \mathbb{N}\} \cup \{0^\infty\}$$

in which $\{\sigma^i(\omega) : i \in \mathbb{N}\}$ are exactly the isolated points of X_ω. In particular, $\omega \in \mathsf{X}_\omega$ is isolated and aperiodic. It follows from Lemma B.2.6 that X_ω is not sofic. Observe also that $0^\infty \in \mathsf{X}_\omega$ is periodic point isolated in past equivalence. In fact, every point in $\{0^n 10^\infty : n \in \mathbb{N}\}$ is isolated in past equivalence, so $\pi_{\mathsf{X}_\omega}^{-1}(x)$ contains an isolated point for every $x \in \mathsf{X}_\omega$.

B.2.2. The groupoid $\mathcal{G}_{\mathsf{X}}$. The pair $(\widetilde{\mathsf{X}}, \sigma_{\widetilde{\mathsf{X}}})$ is a Deaconu–Renault system in the sense of [**26**, Section 8]. The associated Deaconu–Renault groupoid [**36**] is

$$\mathcal{G}_{\mathsf{X}} = \{(\widetilde{x}, p, \widetilde{y}) \in \widetilde{\mathsf{X}} \times \mathbb{Z} \times \widetilde{\mathsf{X}} \mid \exists i,j \in \mathbb{N} \colon p = i - j,\ \widetilde{x}, \widetilde{y} \in \widetilde{\mathsf{X}},\ \sigma_{\widetilde{\mathsf{X}}}^i(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^j(\widetilde{x})\}.$$

The product of $(\widetilde{x}, p, \widetilde{y})$ and $(\widetilde{y'}, q, \widetilde{z})$ is defined if and only if $\widetilde{y} = \widetilde{y'}$ in which case

$$(\widetilde{x}, p, \widetilde{y})(\widetilde{y'}, q, \widetilde{z}) = (\widetilde{x}, p + q, \widetilde{z}),$$

while inversion is given by $(\widetilde{x}, p, \widetilde{y})^{-1} = (\widetilde{y}, -p, \widetilde{x})$. The range and source maps are given as

$$r(\widetilde{x}, p, \widetilde{y}) = (\widetilde{x}, 0, \widetilde{x}), \qquad s(\widetilde{x}, p, \widetilde{y}) = (\widetilde{y}, 0, \widetilde{y}),$$

respectively, for $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}}$. The topology of $\mathcal{G}_{\mathsf{X}}$ is generated by sets of the form

$$Z(U, i, j, V) = \{(\widetilde{x}, i - j, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}} \mid (\widetilde{x}, \widetilde{y}) \in U \times V\}$$

where $U, V \subseteq \widetilde{\mathsf{X}}$ are open subsets such that $\sigma_{\widetilde{\mathsf{X}}}^i|_U$ and $\sigma_{\widetilde{\mathsf{X}}}^j|_V$ are injective and $\sigma_{\widetilde{\mathsf{X}}}^i(U) = \sigma_{\widetilde{\mathsf{X}}}^j(V)$. We naturally identify the unit space $\mathcal{G}_{\mathsf{X}}^{(0)} = \{(\widetilde{x}, 0, \widetilde{x}) \in \mathcal{G}_{\mathsf{X}} \mid x \in \widetilde{\mathsf{X}}\}$ with the space $\widetilde{\mathsf{X}}$ via the map $(\widetilde{x}, 0, \widetilde{x}) \longmapsto \widetilde{x}$. Equipped with this topology, $\mathcal{G}_{\mathsf{X}}$ is topological groupoid which is second countable, locally compact Hausdorff and étale (in the sense that $r, s\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{X}}$ are local homeomorphism onto $\mathcal{G}_{\mathsf{X}}^{(0)}$). By, e.g., [**110**, Lemma 3.5], $\mathcal{G}_{\mathsf{X}}$ is also amenable.

The *isotropy* of a point $\widetilde{x} \in \widetilde{\mathsf{X}}$ is the set

$$\mathrm{Iso}(\widetilde{x}) = \{(\widetilde{x}, p, \widetilde{x}) \in \mathcal{G}_{\mathsf{X}}\}$$

which carries a natural group structure. In our case, the group $\mathrm{Iso}(\widetilde{x})$ is always (isomorphic to) 0 or $\mathbb{Z}$. The *stabilizer* is $\mathrm{Stab}(\widetilde{x}) = \{p \in \mathbb{Z} \mid (\widetilde{x}, p, \widetilde{x}) \in \mathrm{Iso}(\widetilde{x})\}$. The *isotropy subgroupoid* of $\mathcal{G}_{\mathsf{X}}$ is the group bundle

$$\mathrm{Iso}(\mathcal{G}_{\mathsf{X}}) = \bigcup_{\widetilde{x} \in \widetilde{\mathsf{X}}} \mathrm{Iso}(\widetilde{x}).$$

Following [**102**], the groupoid $\mathcal{G}_{\mathsf{X}}$ is *principal* if every point in $\mathcal{G}_{\mathsf{X}}^{(0)}$ has trivial isotropy, and *essentially principal* if the points with trivial isotropy are dense in $\mathcal{G}_{\mathsf{X}}^{(0)}$. Here, the latter is equivalent to $\mathcal{G}_{\mathsf{X}}$ being *effective*, i.e., that $\mathrm{Iso}(\mathcal{G}_{\mathsf{X}})^\circ = \mathcal{G}_{\mathsf{X}}^{(0)}$. Below we characterize when the groupoid $\mathcal{G}_{\mathsf{X}}$ is principal and essentially principal in terms of X. We first need a lemma.

LEMMA B.2.8. *Let X be a one-sided shift space and let $\widetilde{x}, \widetilde{y} \in \widetilde{\mathsf{X}}$.*

(i) *If $\pi_{\mathsf{X}}(\widetilde{x}) = \pi_{\mathsf{X}}(\widetilde{y})$ and $\sigma_{\widetilde{\mathsf{X}}}^k(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^k(\widetilde{y})$ for some $k \in \mathbb{N}$, then $\widetilde{x} = \widetilde{y}$.*

(ii) *If $\pi_{\mathsf{X}}(\widetilde{x}) = \pi_{\mathsf{X}}(\widetilde{y})$ is aperiodic and $\sigma_{\widetilde{\mathsf{X}}}^l(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^k(\widetilde{y})$ for some $k, l \in \mathbb{N}$, then $\widetilde{x} = \widetilde{y}$.*

PROOF. (i): Fix $k \in \mathbb{N}$ such that $\widetilde{\sigma}_{\mathsf{X}}^k(\widetilde{x}) = \widetilde{\sigma}_{\mathsf{X}}^k(\widetilde{y})$ and let $0 \leqslant r \leqslant s$ be integers with $r + k \leqslant s$. An (r, s)-representative of $\widetilde{\sigma}_{\mathsf{X}}^k(\widetilde{x})$ and $\sigma_{\widetilde{\mathsf{X}}}^k(\widetilde{y})$ is given by $\sigma_{\mathsf{X}}^k({}_{r+k}x_s)$ and $\sigma_{\mathsf{X}}^k({}_{r+k}y_s)$, respectively. So

$$\sigma_{\mathsf{X}}^k({}_{r+k}x_s) \overset{r,s}{\sim} \sigma_{\mathsf{X}}^k({}_{r+k}y_s).$$

Since $\pi_{\mathsf{X}}(\widetilde{x}) = \pi_{\mathsf{X}}(\widetilde{y})$ we also have ${}_{r+k}x_s \overset{r+k,s}{\sim} {}_{r+k}y_s$. It follows that $\widetilde{x} = \widetilde{y}$.

(ii): Let $x = \pi_{\mathsf{X}}(\widetilde{x}) = \pi_{\mathsf{X}}(\widetilde{y})$ be aperiodic. If $\sigma_{\widetilde{\mathsf{X}}}^l(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^k(\widetilde{y})$ for some $k, l \in \mathbb{N}$, then $\sigma_{\mathsf{X}}^l(x) = \sigma_{\mathsf{X}}^k(x)$, so $k = l$. Part (i) implies that $\widetilde{x} = \widetilde{y}$. ☐

Assertion (ii) may fail without the hypothesis of aperiodicity; this happens, e.g., for the even shift. Thus the preimage under π_{X} of an aperiodic element contains only aperiodic elements. The preimage under π_{X} of an eventually periodic point contains an eventually periodic point but we do not know if it consists only of eventually periodic points.

PROPOSITION B.2.9. *Let X be a one-sided shift space. The following conditions are equivalent:*

(i) *X contains no eventually periodic points;*

(ii) *$\widetilde{\mathsf{X}}$ contains no eventually periodic points;*

(iii) *$\mathcal{G}_{\mathsf{X}}$ is principal.*

Proof. (i) $\iff$ (ii): It follows from Lemma B.2.8 that if $x \in \mathsf{X}$ is aperiodic, then any $\widetilde{x} \in \pi_{\mathsf{X}}^{-1}(x) \in \widetilde{\mathsf{X}}$ is aperiodic. So if X consists only of aperiodic points, then $\widetilde{\mathsf{X}}$ contains only aperiodic points. Conversely, if $x \in \mathsf{X}$ is eventually periodic, then $\iota_{\mathsf{X}}(x) \in \widetilde{\mathsf{X}}$ is aperiodic.

The equivalence (ii) $\iff$ (iii) is obvious. □

Proposition B.2.10. *Let* X *be a one-sided shift space. The conditions*

(i) X *satisfies Matsumoto's condition (I);*
(ii) $\widetilde{\mathsf{X}}$ *contains no isolated points;*

are equivalent and strictly stronger that the following equivalent conditions

(iii) X *contains no periodic points isolated in past equivalence;*
(iv) $\widetilde{\mathsf{X}}$ *has a dense set of aperiodic points;*
(v) $\mathcal{G}_{\mathsf{X}}$ *is essentially principal;*

which are strictly stronger than

(vi) X *contains a dense set of aperiodic points.*

Proof. (i) $\iff$ (ii): Suppose $x \in \mathsf{X}$ is isolated in past equivalence so that $[x]_l = \{x\}$, for some $l \in \mathbb{N}$. Then $\{\iota(x)\} = U(x,0,l)$ so $\iota(x)$ is isolated in $\widetilde{\mathsf{X}}$. Conversely, if $\widetilde{x} \in \widetilde{\mathsf{X}}$ is isolated, say $\{\widetilde{x}\} = U(x,r,s)$ for some integers $0 \leqslant r \leqslant s$, then $\{\sigma_{\widetilde{\mathsf{X}}}^r(\widetilde{x})\} = U(\sigma_{\mathsf{X}}^r(x),0,s)$, so $\pi_{\mathsf{X}}(\sigma_{\widetilde{\mathsf{X}}}^r(\widetilde{x})) \in \mathsf{X}$ is isolated in s-past equivalence.

The implication (ii) $\implies$ (iii) is clear.

(iii) $\implies$ (iv): Let $\mathcal{EP}(\widetilde{\mathsf{X}})$ be the collection of eventually periodic points in $\widetilde{\mathsf{X}}$ and set

$$\mathcal{EP}_n^p = \{\widetilde{x} \in \mathcal{EP}(\widetilde{\mathsf{X}}) \mid \sigma_{\widetilde{\mathsf{X}}}^{n+p}(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^{n}(\widetilde{x})\},$$

for $n \in \mathbb{N}$ and $p \in \mathbb{N}_+$. Then $\mathcal{EP}(\widetilde{\mathsf{X}}) = \bigcup_{n,p} \mathcal{EP}_n^p$. If there is an open set $U \subseteq \widetilde{\mathsf{X}}$ consisting of eventually periodic points, then it follows from the Baire Category Theorem that Per_n^p has nonempty interior for some $n \in \mathbb{N}$ and $p \in \mathbb{N}_+$. In particular, there are an $x \in \mathsf{X}$ and integers $0 \leqslant r \leqslant s$ with $r \leqslant n$ such that $U(x,r,s) \subseteq \mathcal{EP}_n^p$. Since $\iota_{\mathsf{X}}(x) \in U(x,r,s)$ it follows that $\sigma_{\mathsf{X}}^n(x)$ is p-periodic. We claim that $\sigma_{\mathsf{X}}^n(x)$ is isolated in past equivalence.

Write $x = \mu\alpha^\infty$ for some words $\mu, \alpha \in \mathsf{L}(\mathsf{X})$ with $|\mu| = n$ and $|\alpha| = p$ and suppose

$$y \sim_{p+n+r-s} \sigma_{\mathsf{X}}^n(x) = \sigma_{\mathsf{X}}^{n+p}(x).$$

Then $\mu\alpha y \in \mathsf{X}$ and $\iota_{\mathsf{X}}(\mu\alpha y) \in U(x,r,s)$, so $\alpha y = y$. Hence $y = \sigma_{\mathsf{X}}^p(x)$ as wanted.

(iv) $\implies$ (iii): Suppose $x \in \mathsf{X}$ is a periodic point and there is an $l \in \mathbb{N}$ such that $[x]_l = \{x\}$. Then $U(x,0,l) = \{\iota_{\mathsf{X}}(x)\}$ is an open set consisting of points with nontrivial isotropy.

The equivalence (iv) $\iff$ (v) is obvious.

(iv) $\implies$ (vi): Suppose X contains an open set U consisting of eventually periodic points. Then

$$U = \bigcup_{\alpha,\beta\in\mathsf{L}(\mathsf{X})} \{\alpha\beta^\infty\} \cap U,$$

and by the Baire Category Theorem there are $\alpha, \beta \in \mathsf{L}(\mathsf{X})$ such that $\{\alpha\beta^\infty\}$ is an isolated eventually periodic point in U. Then $\iota_\mathsf{X}(\alpha\beta^\infty)$ is an isolated eventually periodic point in $\widetilde{\mathsf{X}}$.

To see that (iii) does not imply (i) observe that if X is the shift space generated by an aperiodic and primitive substitution, then X contains no eventually periodic points and $\mathcal{G}_\mathsf{X}$ is principal. However, X necessarily contains a point which is isolated in past equivalence, so it does not satisfy Matsumoto's condition (I).

Finally, the even shift is an example of a shift with a dense set of aperiodic points but it contains a periodic point which is isolated in past equivalence. $\square$

Any groupoid homomorphism is assumed to be continuous and a groupoid isomorphism is assumed to be a homeomorphism. A *continuous cocycle* on $\mathcal{G}_\mathsf{X}$ is a groupoid homomorphism $\mathcal{G}_\mathsf{X} \longrightarrow \mathbb{Z}$. Let $B^1(\mathcal{G}_\mathsf{X})$ be the collection of continuous cocycles on $\mathcal{G}_\mathsf{X}$. There is a map $\kappa_\mathsf{X} \colon C(\mathsf{X}, \mathbb{Z}) \longrightarrow B^1(\mathcal{G}_\mathsf{X})$ given by

$$\kappa_\mathsf{X}(f)(\widetilde{x}, p, \widetilde{y}) = \sum_{i=0}^{l} f(\pi_\mathsf{X}(\sigma^i_{\widetilde{\mathsf{X}}}(\widetilde{x}))) - \sum_{j=0}^{k} f(\pi_\mathsf{X}(\sigma^j_{\widetilde{\mathsf{X}}}(\widetilde{x}))),$$

for $f \in C(\mathsf{X}, \mathbb{Z})$, $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_\mathsf{X}$ and where $k, l \in \mathbb{N}$ satisfy $p = l - k$ and $\sigma^l_{\widetilde{\mathsf{X}}}(\widetilde{x}) = \sigma^k_{\widetilde{\mathsf{X}}}(\widetilde{y})$. Observe that $\kappa_\mathsf{X}(f)$ is the unique cocycle satisfying

$$\kappa_\mathsf{X}(f)(\widetilde{x}, 1, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})) = f(\pi_\mathsf{X}(\widetilde{x})),$$

for $\widetilde{x} \in \widetilde{\mathsf{X}}$. The *canonical continuous cocycle* $c_\mathsf{X} \colon \mathcal{G}_\mathsf{X} \longrightarrow \mathbb{Z}$ is defined by

$$c_\mathsf{X}(\widetilde{x}, p, \widetilde{y}) = p,$$

for $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_\mathsf{X}$. Note that $c_\mathsf{X} = \kappa_\mathsf{X}(1)$ and $c_\mathsf{X}^{-1}(0) = \{(\widetilde{x}, 0, \widetilde{y}) \in \mathcal{G}_\mathsf{X}\} \subseteq \mathcal{G}_\mathsf{X}$ is a clopen subgroupoid which is always principal.

B.2.3. The C*-algebra $\mathrm{C}^*(\mathcal{G}_\mathsf{X}) = \mathcal{O}_\mathsf{X}$. The groupoid $\mathcal{G}_\mathsf{X}$ is second countable, locally compact Hausdorff and étale. Let $C_c(\mathcal{G}_\mathsf{X})$ be the *-algebra consisting of compactly supported and complex-valued maps. As $\mathcal{G}_\mathsf{X}$ is also amenable, the full $\mathrm{C}^*(\mathcal{G}_\mathsf{X})$ and the reduced $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$ groupoid C*-algebras are canonically *-isomorphic, cf. [**101, 109**].

There is a canonical *-isomorphism $\mathcal{O}_\mathsf{X} \longrightarrow \mathrm{C}^*(\mathcal{G}_\mathsf{X})$ sending $s_a \longmapsto 1_{U_a}$ for each $a \in \mathfrak{A}$, cf. [**19**, Chapter 2]. Therefore, we identify the two C*-algebras and similarly we identify $\mathcal{D}_\mathsf{X}$ with $C(\widetilde{\mathsf{X}})$ and $\mathcal{F}_\mathsf{X}$ with $\mathrm{C}^*(c_\mathsf{X}^{-1}(0))$. The inclusion $\widetilde{\mathsf{X}} \longrightarrow \mathcal{G}_\mathsf{X}$ induces a conditional expectation $p_\mathsf{X} \colon \mathcal{O}_\mathsf{X} \longrightarrow \mathcal{D}_\mathsf{X}$ given by restriction.

Any continuous cocycle $c \in B^1(\mathcal{G}_\mathsf{X})$ induces a strongly continuous action $\beta^c \colon \mathbb{T} \curvearrowright \mathcal{O}_\mathsf{X}$ satisfying

$$\beta^c_z(f) = z^n f$$

for $z \in \mathbb{T}$ and $n \in \mathbb{N}$ and $f \in C_c(\mathcal{G}_\mathsf{X})$ with $\mathrm{supp}(f) \subseteq c^{-1}(\{n\})$. The canonical gauge action $\gamma^\mathsf{X} = \beta^{\kappa_\mathsf{X}(1)}$ is of the form

$$\gamma_z^\mathsf{X}(g)(\widetilde{x}, p, \widetilde{y}) = z^p g(\widetilde{x}, p, \widetilde{y}),$$

for every $z \in \mathbb{T}$, $g \in C_c(\mathcal{G}_\mathsf{X})$ and $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_\mathsf{X}$.

Finally, we let $\mathbb{K}$ denote the C*-algebra of compact operators on separable Hilbert space and let c_0 denote the canonical maximal abelian C*-subalgebra of diagonal operators in $\mathbb{K}$.

B.3. Preserving the diagonal

Let X and Y be one-sided shift spaces. A *-isomorphism $\Phi\colon \mathcal{O}_\mathsf{X} \longrightarrow \mathcal{O}_\mathsf{Y}$ is *diagonal-preserving* if $\Phi(\mathcal{D}_\mathsf{X}) = \mathcal{D}_\mathsf{Y}$. In this section we prove that a *-isomorphism $\Phi\colon \mathcal{O}_\mathsf{X} \longrightarrow \mathcal{O}_\mathsf{Y}$ satisfying $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ is diagonal-preserving, provided X and Y contain dense sets of aperiodic points. First we need some preliminary results.

LEMMA B.3.1. *Let X be a one-sided shift space. Then* $\mathrm{C}^*(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ) = \mathcal{D}_\mathsf{X}' \subseteq C(\mathsf{X})'$. *If X contains a dense set of aperiodic points, then $\mathcal{D}_\mathsf{X}' = C(\mathsf{X})'$.*

PROOF. Let $\xi \in C_c(\mathcal{G}_\mathsf{X})$. The condition that $\xi \star g = g \star \xi$ for all $g \in \mathcal{D}_\mathsf{X}$ means that ξ is supported on elements $\gamma \in \mathcal{G}_\mathsf{X}$ with $s(\gamma) = r(\gamma)$. It follows that $\mathrm{C}^*(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ) = \mathcal{D}_\mathsf{X}'$. The inclusion $\mathcal{D}_\mathsf{X}' \subseteq C(\mathsf{X})'$ follows from the inclusion $C(\mathsf{X}) \subseteq \mathcal{D}_\mathsf{X}$. □

Consider the equivalence relation $\sim$ on the space $\widetilde{\mathsf{X}} \times \mathbb{T}$ given by $(\widetilde{x}, \zeta) \sim (\widetilde{y}, \theta)$ if and only if $\widetilde{x} = \widetilde{y}$ and $\zeta^p = \theta^p$ for all $p \in \mathrm{Stab}(\widetilde{x})$. Then the quotient $\widetilde{\mathsf{X}} \times \mathbb{T}/\sim$ is compact and Hausdorff and as we shall see (homeomorphic to) the spectrum of $\mathrm{C}^*(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ)$.

LEMMA B.3.2. *Let $\sim$ be the equivalence relation on $\widetilde{\mathsf{X}} \times \mathbb{T}$ defined above. There is a *-isomorphism $\Xi\colon \mathrm{C}^*(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ) \longrightarrow C(\widetilde{\mathsf{X}} \times \mathbb{T}/\sim)$ given by*

$$\Xi(f)([\widetilde{x}, \zeta]) = \sum_{p \in \mathrm{Stab}(\widetilde{x})} f(\widetilde{x}, p, \widetilde{x})\zeta^n, \tag{B.1}$$

for $f \in C_c(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ)$ and $[\widetilde{x}, \zeta] \in \widetilde{\mathsf{X}} \times \mathbb{T}/\sim$.

PROOF. The map $\Xi\colon C_c(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ) \longrightarrow C(\widetilde{\mathsf{X}} \times \mathbb{T}/\sim)$ given in (B.1) is well-defined by the definition of $\sim$ and linear. If $f, g \in C_c(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ)$ and $[\widetilde{x}, z] \in \widetilde{\mathsf{X}} \times \mathbb{T}/\sim$, then

$$\begin{aligned}
\Xi(f)([\widetilde{x}, \zeta])\Xi(g)([\widetilde{x}, \zeta]) &= \sum_{k,l \in \mathrm{Stab}(\widetilde{x})} f(\widetilde{x}, k, \widetilde{x})g(\widetilde{x}, l, \widetilde{x})\zeta^{k+l} \\
&= \sum_{n,m \in \mathrm{Stab}(\widetilde{x})} f(\widetilde{x}, n-m, \widetilde{x})g(\widetilde{x}, m, \widetilde{x})\zeta^n \\
&= \Xi(f \star g)([\widetilde{x}, \zeta]),
\end{aligned}$$

so Ξ is multiplicative.

In order to see that Ξ is injective, let $f, g \in C_c(\mathrm{Iso}(\mathcal{G}_\mathsf{X})^\circ)$ such that both $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are bisections and suppose that $\Xi(f) = \Xi(g)$. Suppose $f(\widetilde{x}, p, \widetilde{x}) \neq 0$. The

$p \in \mathrm{Iso}(\widetilde{x})$ is necessarily unique because $\mathrm{supp}(f)$ is a bisection. The assumption implies the existence of a unique $q \in \mathrm{Iso}(\widetilde{x})$ such that $g(\widetilde{x}, q, \widetilde{x})$, and then

$$0 \neq f(\widetilde{x}, p, \widetilde{x}) = \Xi(f)([\widetilde{x}, 1]) = \Xi(g)([\widetilde{x}, 1]) = g(\widetilde{x}, q, \widetilde{x}).$$

Similarly,

$$0 \neq f(\widetilde{x}, p, \widetilde{x})\zeta^p = \Xi(f)([\widetilde{x}, \zeta]) = \Xi(g)([\widetilde{x}, \zeta]) = g(\widetilde{x}, q, \widetilde{x})\zeta^q.$$

for all $1 \neq \zeta \in \mathbb{T}$. It follows that $p = q$. Since $C_c(\mathrm{Iso}(\mathcal{G}_{\mathsf{X}})^\circ)$ is spanned by functions whose support is a bisection, we conclude that ξ is injective.

We show that Ξ separates points. First, if $[\widetilde{x}, \zeta] \neq [\widetilde{x}, \theta]$, then there is $p \in \mathrm{Iso}(\widetilde{x})$ such that $\zeta^p \neq \theta^p$. Choose a compact open bisection $U \subseteq \mathcal{G}_{\mathsf{X}}$ satisfying $U \cap \mathrm{Iso}(\widetilde{x}) = \{(\widetilde{x}, p, \widetilde{x})\}$ and observe that $\Xi(1_U)([\widetilde{x}, \zeta]) = \zeta^p$ and $\xi(1_U)([\widetilde{x}, \theta]) = \theta^p$. Second, if $\widetilde{x} \neq \widetilde{y}$ in $\widetilde{\mathsf{X}}$ then we choose a compact open bisection U satisfying $(\widetilde{x}, 0, \widetilde{x}) \in U$ and $\mathrm{Iso}(\widetilde{y}) \cap U = \emptyset$. Then $\Xi(1_U)([\widetilde{x}, \zeta]) = 1$ while $\Xi(1_U)([\widetilde{y}, \theta]) = 0$. By the Stone–Weierstrass theorem, the image of Ξ is dense in $C(\widetilde{\mathsf{X}} \times \mathbb{T}/\sim)$ and Ξ thus extends to a *-isomorphism as wanted. ☐

THEOREM B.3.3. *Let X and Y be one-sided shift spaces with dense sets of aperiodic points and let $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ be a *-isomorphism satisfying $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$. Then $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$.*

PROOF. If $\Psi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ is a *-isomorphism satisfying $\Psi(C(\mathsf{X})) = C(\mathsf{Y})$, then $\Psi(C(\mathsf{X})') = C(\mathsf{Y})'$. By Lemmas B.3.1 and B.3.2, there is a homeomorphism

$$h\colon \widetilde{\mathsf{X}} \times \mathbb{T}/\sim \longrightarrow \widetilde{\mathsf{Y}} \times \mathbb{T}/\sim$$

such that $\Psi(f) = f \circ h^{-1}$ for $f \in C(\widetilde{\mathsf{X}} \times \mathbb{T}/\sim)$.

Define the map $q_{\mathsf{X}}\colon \widetilde{\mathsf{X}} \times \mathbb{T}/\sim \longrightarrow \widetilde{\mathsf{X}}$ by $q_{\mathsf{X}}([\widetilde{x}, z]) = \widetilde{x}$. This is well-defined, continuous and surjective. Furthermore, q_{X} induces the inclusion $\mathcal{D}_{\mathsf{X}} \subseteq C(\mathsf{X})'$. Let $\widetilde{x} \in \widetilde{\mathsf{X}}$ and put $\widetilde{y}_{\widetilde{x}} = q_{\mathsf{Y}}(h([\widetilde{x}, 1])) \in \widetilde{\mathsf{Y}}$. The connected component of any $[\widetilde{x}, z]$ is the set $\{[\widetilde{x}, w] \mid w \in \mathbb{T}\}$, so since any homeomorphism will preserve connected components, we have $h(q_{\mathsf{X}}^{-1}(\widetilde{x})) = q_{\mathsf{Y}}^{-1}(h([\widetilde{x}, 1]))$. We may now define a map $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \mathsf{Y}$ by

$$\widetilde{h}(\widetilde{x}) = \widetilde{y}_{\widetilde{x}} = q_{\mathsf{Y}}(h([\widetilde{x}, 1]))$$

for $\widetilde{x} \in \widetilde{\mathsf{X}}$, which is well-defined, continuous and surjective. The above considerations show that h is also injective. As both $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{Y}}$ are compact and Hausdorff, $\widetilde{h}$ is a homeomorphism. The relation $\widetilde{h} \circ q_{\mathsf{X}} = q_{\mathsf{Y}} \circ h$ ensures that that $\Psi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$ as wanted. ☐

COROLLARY B.3.4. *Let X and Y be one-sided shift spaces and let $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ be a *-isomorphism satisfying $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ and $\Phi \circ \gamma^{\mathsf{X}} = \gamma^{\mathsf{Y}} \circ \Phi$. Then $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$.*

PROOF. This follows from the observation that $\mathcal{D}_{\mathsf{X}} = C(\mathsf{X})' \cap \mathcal{F}_{\mathsf{X}}$ and $\mathcal{D}_{\mathsf{Y}} = C(\mathsf{Y})' \cap \mathcal{F}_{\mathsf{Y}}$. ☐

REMARK B.3.5. Let X be any strictly sofic one-sided shift and let $\mathsf{Y} = \widetilde{\mathsf{X}}$ be its cover. Then Y is (conjugate to) a shift of finite type so $\mathcal{D}_{\mathsf{Y}} = C(\mathsf{Y})$ but $\mathcal{D}_{\mathsf{X}} = C(\mathsf{Y}) \not\supseteq C(\mathsf{X})$. The identity map is a *-isomorphism $\mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ with sends $\mathcal{D}_{\mathsf{X}}$ onto $\mathcal{D}_{\mathsf{Y}} = C(\mathsf{Y})$, but there is no *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ which satisfies $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$.

Below, we give a stabilized version of Theorem B.3.3. Consider the product $\widetilde{\mathsf{X}} \times \mathbb{N} \times \mathbb{T}$ equipped with the equivalence relation $\approx$ defined by $(\widetilde{x}, m_1, z) \approx (\widetilde{y}, m_2, w)$ if and only if $\widetilde{x} = \widetilde{y}$ and $m_1 = m_2$ and $z^n = w^n$ for all $n \in \mathrm{Iso}(\widetilde{x})$. The spaces $\widetilde{\mathsf{X}} \times \mathbb{N} \times \mathbb{T}/\approx$ and $(\widetilde{\mathsf{X}} \times \mathbb{T}/\sim) \times \mathbb{N}$ are now homeomorphic. An argument similar to the above then yields the following result.

Corollary B.3.6. *Let* X *and* Y *be one-sided shift spaces with dense sets of aperiodic points and let* $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ *be a* **-isomorphism satisfying* $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$. *Then* $\Phi(\mathcal{D}_{\mathsf{X}} \otimes c_0) = \mathcal{D}_{\mathsf{Y}} \otimes c_0$.

B.4. One-sided conjugacy

A pair of one-sided shift space X and Y are one-sided conjugate if there exists a homeomorphism $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ h$. A similar definition applies to the covers. If X and Y are shifts of finite type, then they are conjugate if and only if the groupoids $\mathcal{G}_{\mathsf{X}}$ and $\mathcal{G}_{\mathsf{Y}}$ are isomorphic in a way which preserves a certain endomorphism, if and only if the C*-algebras $\mathcal{O}_{\mathsf{X}}$ and $\mathcal{O}_{\mathsf{Y}}$ are *-isomorphic in a way which preserves a certain completely positive map [**12**]. In this section we characterize one-sided conjugacy of general one-sided shift spaces. We start by lifting a conjugacy on the shift spaces to a conjugacy on the covers. The cover construction is therefore *canonical*.

Lemma B.4.1 (Lifting lemma). *Let* X *and* Y *be one-sided shift spaces and let* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *be a homeomorphism. The following are equivalent:*

(i) *the map* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *is a conjugacy;*
(ii) *there is a conjugacy* $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ *satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$.

Proof. (i) $\implies$ (ii): Let $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a conjugacy and choose an integer $C \in \mathbb{N}$ such that

$$x_{[0,C+r)} = x'_{[0,C+r)} \implies h(x)_{[0,r)} = h(x')_{[0,r)}$$

for $r \in \mathbb{N}$ and $x, x' \in \mathsf{X}$. Given integers $0 \leqslant r \leqslant s$, we show that

$$\alpha x \overset{C+r,C+s}{\sim} \alpha x' \implies h(\alpha x) = h(\alpha x')$$

for $\alpha x, \alpha x' \in \mathsf{X}$ with $|\alpha| = r$. Start by writing $h(\alpha x) = \mu y$ and $h(\alpha x') = \mu y'$ for some $y \in \mathsf{Y}$ and $\mu \in \mathsf{L}(\mathsf{Y})$ with $|\mu| = s$ and observe that $h(x) = y$ and $h(x') = y'$ since h is a conjugacy. Assume now that $\nu y \in \mathsf{Y}$ for some $\nu \in \mathsf{L}(\mathsf{Y})$ with $|\nu| \leqslant s$. We need to show that $\nu y' \in \mathsf{Y}$.

Observe that $h^{-1}(\nu y) = \beta_\nu x$ for some $\beta_\nu \in \mathsf{L}(\mathsf{X})$ with $|\beta_\nu| = |\nu| \leqslant s$ and, by hypothesis, $\beta_\nu x' \in \mathsf{X}$. It is now easily verified that $h(\beta_\nu x') = \nu y'$ so that $\nu y' \in \mathsf{Y}$ as wanted.

This defines an induced map $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ determined by

$$\widetilde{h}\colon {}_{C+r}[x]_{C+s} \longmapsto {}_{r}[h(x)]_{s}$$

for integers $0 \leqslant r \leqslant s$. It is readily verified that $\widetilde{h}$ is a conjugacy satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$ using that h is a conjugacy.

(ii) $\implies$ (i): Given $x \in \mathsf{X}$ and any $\widetilde{x} \in \pi_{\mathsf{X}}^{-1}(x) \subseteq \widetilde{\mathsf{X}}$, we observe that

$$h(\sigma_{\mathsf{X}}(x)) = \pi_{\mathsf{Y}}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))) = \pi_{\mathsf{Y}}(\sigma_{\widetilde{\mathsf{Y}}}(\widetilde{h}(\widetilde{x}))) = \sigma_{\mathsf{Y}} h(x).$$

This shows that h is a conjugacy. □

Let X be a one-sided shift and let $\mathcal{G}_{\mathsf{X}}$ be the groupoid defined in Section B.2. The map $\epsilon_{\mathsf{X}} \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{X}}$ given by

$$\epsilon_{\mathsf{X}}(\widetilde{x}, p, \widetilde{y}) = (\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}), p, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{y})),$$

for $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}}$, is a continuous groupoid homomorphism. There is an induced homomorphism $\epsilon_{\mathsf{X}}^* \colon C_c(\mathcal{G}_{\mathsf{X}}) \longrightarrow C_c(\mathcal{G}_{\mathsf{X}})$ given by $\epsilon_{\mathsf{X}}^*(f) = f \circ \epsilon_{\mathsf{X}}$, for $f \in C_c(\mathcal{G}_{\mathsf{X}})$. We also consider two completely positive maps on $\mathcal{O}_{\mathsf{X}}$ as follows: Let $\{s_a\}_{a \in \mathfrak{A}}$ be the canonical generators of $\mathcal{O}_{\mathsf{X}}$ and consider $\phi_{\mathsf{X}} \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{X}}$ given by

$$\phi_{\mathsf{X}}(y) = \sum_{a \in \mathfrak{A}} s_a y s_a^*,$$

for $y \in \mathcal{O}_{\mathsf{X}}$, and map $\tau_{\mathsf{X}} \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{X}}$ given by

$$\tau_{\mathsf{X}}(y) = \sum_{a,b \in \mathfrak{A}} s_b y s_a^*,$$

for $y \in \mathcal{O}_{\mathsf{X}}$. The next lemma describes the relationship between these maps.

Lemma B.4.2 ([**12**]). *We have $\tau_{\mathsf{X}}(f) = f \circ \epsilon_{\mathsf{X}}$ for $f \in C_c(\mathcal{G}_{\mathsf{X}})$. Hence τ_{X} extends ϵ_{X}^* to $\mathcal{O}_{\mathsf{X}}$. Furthermore, $p_{\mathsf{X}} \circ \tau_{\mathsf{X}}|_{\mathcal{D}_{\mathsf{X}}} = \phi_{\mathsf{X}}|_{\mathcal{D}_{\mathsf{X}}}$.*

For the next lemma, recall that $\mathcal{F}_{\mathsf{X}} = \mathrm{C}^*(c_{\mathsf{X}}^{-1}(0))$ is the AF core inside $\mathcal{O}_{\mathsf{X}}$.

Lemma B.4.3 ([**12**]). *Let X and Y be one-sided shift spaces. If $\Phi \colon \mathcal{F}_{\mathsf{X}} \longrightarrow \mathcal{F}_{\mathsf{Y}}$ is a *-isomorphism satisfying $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$, then $\Phi(p_{\mathsf{X}}(f)) = p_{\mathsf{Y}}(\Psi(f))$ for $f \in \mathcal{F}_{\mathsf{X}}$. If, in addition, $\Phi \circ \tau_{\mathsf{X}} = \tau_{\mathsf{Y}} \circ \Phi$, then $\Phi \circ \phi_{\mathsf{X}}|_{\mathcal{D}_{\mathsf{X}}} = \phi_{\mathsf{Y}} \circ \Phi|_{\mathcal{D}_{\mathsf{X}}}$.*

We now characterize one-sided conjugacy of general one-sided shift spaces.

Theorem B.4.4. *Let X and Y be one-sided shift spaces and let $h \colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a homeomorphism. The following are equivalent:*

(i) *the map $h \colon \mathsf{X} \longrightarrow \mathsf{Y}$ is a one-sided conjugacy;*
(ii) *there is a conjugacy $\widetilde{h} \colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$.*
(iii) *there is a groupoid isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$, $c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Psi$ and*

$$\Psi \circ \epsilon_{\mathsf{X}} = \epsilon_{\mathsf{Y}} \circ \Psi; \tag{B.2}$$

(iv) *there is a groupoid isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ and* (B.2)*;*
(v) *there is a *-isomorphism $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ satisfying $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ with $\Phi(g) = g \circ h^{-1}$ for $g \in C(\mathsf{X})$, $\Phi \circ p_{\mathsf{X}} = p_{\mathsf{Y}} \circ \Phi$, $\Phi \circ \gamma_z^{\mathsf{X}} = \gamma_z^{\mathsf{Y}} \circ \Phi$ for $z \in \mathbb{T}$, $\Phi \circ \phi_{\mathsf{X}}|_{\mathcal{D}_{\mathsf{X}}} = \phi_{\mathsf{Y}} \circ \Phi|_{\mathcal{D}_{\mathsf{X}}}$, and*

$$\Phi \circ \tau_{\mathsf{X}} = \tau_{\mathsf{Y}} \circ \Phi. \tag{B.3}$$

(vi) *there is a *-isomorphism $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ satisfying $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ with $\Psi(g) = g \circ h^{-1}$ for $g \in C(\mathsf{X})$, and* (B.3)*.*

(vii) *there is a* **-isomorphism* $\Omega\colon \mathcal{D}_\mathsf{X} \longrightarrow \mathcal{D}_\mathsf{Y}$ *satisfying* $\Omega(C(\mathsf{X})) = C(\mathsf{Y})$, $\Omega(g) = g \circ h^{-1}$ *for* $g \in C(\mathsf{X})$ *and* $\Omega \circ \phi_\mathsf{X}|_{\mathcal{D}_\mathsf{X}} = \phi_\mathsf{Y} \circ \Omega$.

Proof. The equivalence (i) $\iff$ (ii) is Lemma B.4.1.

(ii) $\implies$ (iii): Let $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ be a conjugacy satisfying $h \circ \pi_\mathsf{X} = \pi_\mathsf{Y} \circ \widetilde{h}$. The map $\Phi\colon \mathcal{G}_\mathsf{X} \longrightarrow \mathcal{G}_\mathsf{Y}$ given by

$$\Phi(\widetilde{x}, p, \widetilde{y}) = (\widetilde{h}(\widetilde{x}), p, \widetilde{h}(\widetilde{y})),$$

for $(\widetilde{x}, p, \widetilde{y}) \in \widetilde{\mathsf{X}}$, is a groupoid isomorphism. Under the identification of $\Phi^{(0)}$ and $\widetilde{h}$, we then have $\pi_\mathsf{Y} \circ \Psi^{(0)} = h \circ \pi_\mathsf{X}$, $c_\mathsf{X} = c_\mathsf{Y} \circ \Psi$ and $\Psi \circ \epsilon_\mathsf{X} = \epsilon_\mathsf{Y} \circ \Psi$.

The implications (iii) $\implies$ (iv) and (v) $\implies$ (vi) are clear.

(iv) $\implies$ (vi) and (iii) $\implies$ (v): Let $\Psi\colon \mathcal{G}_\mathsf{X} \longrightarrow \mathcal{G}_\mathsf{Y}$ be a groupoid isomorphism as in (iv). This induces a *-isomorphism $\Phi\colon \mathcal{O}_\mathsf{X} \longrightarrow \mathcal{O}_\mathsf{Y}$ satisfying $\Phi \circ p_\mathsf{X} = p_\mathsf{Y} \circ \Phi$ and $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ with $\Phi(g) = g \circ h^{-1}$ for $g \in C(\mathsf{X})$. The relation (B.2) ensures that $\Phi \circ \tau_\mathsf{X} = \tau_\mathsf{Y} \circ \Phi$. This is (vi). If, in addition, $c_\mathsf{X} = c_\mathsf{Y} \circ \Phi$, then $\Phi \circ \gamma_z^\mathsf{X} = \gamma_z^\mathsf{Y} \circ \Phi$ for $z \in \mathbb{T}$. In particular, $\Psi(\mathcal{F}_\mathsf{X}) = \mathcal{F}_\mathsf{Y}$ and Lemma B.4.3 implies that $\Phi \circ \phi_\mathsf{X}|_{\mathcal{D}_\mathsf{X}} = \phi_\mathsf{Y} \circ \Psi|_{\mathcal{D}_\mathsf{X}}$. This is (v).

(vi) $\implies$ (vii): As Φ satisfies (B.3) and $\mathcal{F}_\mathsf{X}$ is generated as a C*-algebra by $\bigcup_{k=0}^\infty \tau_\mathsf{X}^k(\mathcal{D}_\mathsf{X})$, we also have $\Phi(\mathcal{F}_\mathsf{X}) = \mathcal{F}_\mathsf{Y}$. By Corollary B.3.4, $\Phi(\mathcal{D}_\mathsf{X}) = \mathcal{D}_\mathsf{Y}$. It therefore follows from Lemma B.4.3 that $\Phi \circ \phi_\mathsf{X}|_{\mathcal{D}_\mathsf{X}} = \phi_\mathsf{Y} \circ \Phi|_{\mathcal{D}_\mathsf{X}}$.

(vii) $\implies$ (ii): Let $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ be the homeomorphism induced by Ω via Gelfand duality. The relation $\Omega \circ \phi_\mathsf{X}|_{C(\mathsf{X})} = \phi_\mathsf{Y} \circ \Omega$ and the fact that $\phi_\mathsf{X}(f)(\widetilde{x}) = f(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))$ for $f \in \mathcal{D}_\mathsf{X}$ and $\widetilde{x} \in \widetilde{\mathsf{X}}$ ensures that $\widetilde{h}$ is a conjugacy. The condition $\Omega(C(\mathsf{X})) = C(\mathsf{Y})$ entails that $h \circ \pi_\mathsf{X} = \pi_\mathsf{Y} \circ \widetilde{h}$.

The final remark follows from Theorem B.3.3. □

If X_A and X_B are one-sided shifts of finite type determined by finite square $\{0,1\}$-matrices A and B with no zero rows and no zero columns, respectively, then we recover [**12**, Theorem 3.3].

Corollary B.4.5. *Let* X *and* Y *be one-sided shift spaces. The following are equivalent:*

(i) *the systems* X *and* Y *are one-sided conjugate;*
(ii) *there are a groupoid isomorphism* $\Psi\colon \mathcal{G}_\mathsf{X} \longrightarrow \mathcal{G}_\mathsf{Y}$ *and a homeomorphism* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *satisfying* $h \circ \pi_\mathsf{X} = \pi_\mathsf{Y} \circ \Psi^{(0)}$ *and* $\Psi \circ \epsilon_\mathsf{X} = \epsilon_\mathsf{Y} \circ \Psi$*;*
(iii) *there is a* **-isomorphism* $\Phi\colon \mathcal{O}_\mathsf{X} \longrightarrow \mathcal{O}_\mathsf{Y}$ *satisfying* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ *and* $\Phi \circ \tau_\mathsf{X} = \tau_\mathsf{Y} \circ \Phi$.

B.5. One-sided eventual conjugacy

Matsumoto has studied one-sided eventual conjugacy of shifts of finite type [**81**]. A pair of shifts of finite type X and Y are eventually conjugate if and only if the groupoids $\mathcal{G}_\mathsf{X}$ and $\mathcal{G}_\mathsf{Y}$ are isomorphic in a way which preserves the canonical cocycle, if and only if the C*-algebras $\mathcal{O}_\mathsf{X}$ and $\mathcal{O}_\mathsf{Y}$ are *-isomorphic in a way which preserves the canonical gauge actions. We characterize eventual conjugacy for general shift spaces in terms of groupoids and C*-algebras. We start by lifting an eventual conjugacy on the shift spaces to an eventual conjugacy on the covers.

Definition B.5.1. Two one-sided shift spaces X and Y are *eventually conjugate* if there exist a homeomorphism $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ and an integer $\ell \in \mathbb{N}$ such that

$$\sigma_{\mathsf{Y}}^{\ell}(h(\sigma_{\mathsf{X}}(x))) = \sigma_{\mathsf{Y}}^{\ell+1}h(x), \tag{B.4}$$

$$\sigma_{\mathsf{X}}^{\ell}(h^{-1}(\sigma_{\mathsf{Y}}(y))) = \sigma_{\mathsf{X}}^{\ell+1}h^{-1}(y), \tag{B.5}$$

for $x \in \mathsf{X}$ and $y \in \mathsf{Y}$. An eventual conjugacy h is a conjugacy if and only if we can choose $\ell = 0$.

A similar definition applies to the covers.

Lemma B.5.2 (Lifting lemma). *Let X and Y be one-sided shift spaces and let $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a homeomorphism. The following are equivalent:*

(i) *the map $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ is an eventual conjugacy;*
(ii) *there is an eventual conjugacy $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$.*

Proof. (i) $\implies$ (ii): Let $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be an eventual conjugacy and choose $\ell \in \mathbb{N}$ according to (B.4) and (B.5). Then there is a continuity constant $C \in \mathbb{N}$ with the property that

$$x_{[0,C+r)} = x'_{[0,C+r)} \implies h(x)_{[0,\ell+r)} = h(x')_{[0,\ell+r)},$$

for $x, x' \in \mathsf{X}$ and $r \in \mathbb{N}$. Fix integers $0 \leqslant r \leqslant s$ and put $K = C + 2\ell + s$. We will show that

$$\alpha x \overset{K+r,K+s}{\sim} \alpha x' \implies h(\alpha x) \overset{r,s}{\sim} h(\alpha x'),$$

where $|\alpha| = \ell + r$. Since $K \geqslant C$, we can write $h(\alpha x) = \mu y$ and $h(\alpha x') = \mu y'$ for some $y, y' \in \mathsf{Y}$ and $\mu \in \mathsf{L}(\mathsf{Y})$ with $|\mu| = r$. In particular, $y_{[0,2\ell)} = y'_{[0,2\ell)}$. Assume now that $\nu y \in \mathsf{Y}$ for some $\nu \in \mathsf{L}(\mathsf{Y})$ with $|\nu| \leqslant s$. We need to show that $\nu y' \in \mathsf{Y}$.

First observe that $h^{-1}(\nu y) = \beta_\nu x$ for some word $\beta_\nu \in \mathsf{L}(\mathsf{X})$ with $|\beta_\nu| = \ell + |\nu|$. This follows from the computation

$$x = \sigma_{\mathsf{X}}^{\ell+r}(\alpha x) = \sigma_{\mathsf{X}}^{\ell+r}(h^{-1}(\mu y)) = \sigma_{\mathsf{X}}^{\ell}(h^{-1}(y)) = \sigma_{\mathsf{X}}^{\ell+|\nu|}(h^{-1}(\nu y)).$$

By hypothesis, $\beta_\nu x' \in \mathsf{X}$ and we claim that $h(\beta_\nu x') = \nu y'$.
In order to verify the claim first observe that

$$\sigma_{\mathsf{Y}}^{2\ell+|\nu|}(h(\beta_\nu x')) = \sigma_{\mathsf{Y}}^{\ell}(h(x')) = \sigma_{\mathsf{Y}}^{2\ell+r}(h(\alpha x')) = \sigma_{\mathsf{Y}}^{2\ell}(y'),$$

and since $(\beta_\nu x)_{[0,C+2\ell+|\nu|)} = (\beta_\nu x')_{[0,C+2\ell+|\nu|)}$ we have

$$h(\beta_\nu x)_{[0,2\ell+|\nu|)} = h(\beta_\nu x')_{[0,2\ell+|\nu|)} = (\nu y)_{[0,2\ell+|\nu|)} = (\nu y')_{[0,2\ell+|\nu|)}.$$

Hence $h(\beta_\nu x') = \nu y'$. This shows that there is a well-defined map $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ given by

$$\widetilde{h}\colon {}_{K+r}[x]_{K+s} \longmapsto {}_{r}[h(x)]_{s},$$

for all integers $0 \leqslant r \leqslant s$. It is straightforward to check that $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$ and that $\widetilde{h}$ is an eventual conjugacy using the fact that h is an eventual conjugacy.

(ii) $\implies$ (i): Given $x \in \mathsf{X}$ and any $\widetilde{x} \in \pi_{\mathsf{X}}^{-1}(x) \subseteq \widetilde{\mathsf{X}}$, we have

$$\sigma_{\mathsf{Y}}^{\ell}(h(\sigma_{\mathsf{X}}(x))) = \pi_{\mathsf{Y}}(\sigma_{\widetilde{\mathsf{Y}}}^{\ell}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))) = \pi_{\mathsf{Y}}(\sigma_{\widetilde{\mathsf{Y}}}^{\ell+1}(\widetilde{h}(\widetilde{x}))) = \sigma_{\mathsf{Y}}^{\ell+1}(h(x)),$$

showing that h is an L-conjugacy. □

We can now characterize one-sided eventual conjugacy of general one-sided shifts spaces, cf. [**83**, Theorem 1.4]. The proof uses ideas of [**24**].

THEOREM B.5.3. *Let* X *and* Y *be one-sided shift spaces and let* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *be a homeomorphism. The following are equivalent:*

(i) *the map* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *is a one-sided eventual conjugacy;*
(ii) *there is an eventual conjugacy* $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ *such that* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$*;*
(iii) *there is a groupoid isomorphism* $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ *satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ *and*
$$c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Psi;$$
(iv) *there is a* **-isomorphism* $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ *satisfying* $\Phi \circ p_{\mathsf{X}} = p_{\mathsf{Y}} \circ \Phi$*,* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ *with* $\Phi(g) = g \circ h^{-1}$ *for* $g \in C(\mathsf{X})$ *and*
$$\Phi \circ \gamma_z^{\mathsf{X}} = \gamma_z^{\mathsf{Y}} \circ \Phi, \tag{B.6}$$
for $z \in \mathbb{T}$*;*
(v) *there is a* **-isomorphism* $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ *satisfying* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ *with* $\Phi(g) = g \circ h^{-1}$ *for* $g \in C(\mathsf{X})$ *and* (B.6)*.*

PROOF. The equivalence (i) $\iff$ (ii) is Lemma B.5.2.

(ii) $\implies$ (iii): Let $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ be an eventual conjugacy satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$. The map $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ given by

$$\Psi(\widetilde{x}, p, \widetilde{y}) = (\widetilde{h}(\widetilde{x}), p, \widetilde{h}(\widetilde{y}))$$

for $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}}$ is a groupoid isomorphism. Under the identification $\Psi^{(0)} = \widetilde{h}$, we have $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ and $c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Psi$.

(iii) $\implies$ (ii): Let $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ be a groupoid isomorphism satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ and $c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Psi$. Identify $\widetilde{\mathsf{X}} = \mathcal{G}_{\mathsf{X}}^{(0)}$, $\widetilde{\mathsf{Y}} = \mathcal{G}_{\mathsf{Y}}^{(0)}$ and $\widetilde{h} = \Psi^{(0)}$. Then Ψ is of the form

$$\Psi(\widetilde{x}, p, \widetilde{y}) = (\widetilde{h}(\widetilde{x}), p, \widetilde{h}(\widetilde{y})),$$

for $(\widetilde{x}, p, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}}$, and $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$. Let $\mathfrak{A}$ be the alphabet of X and consider the compact open bisection

$$A_a = (\sigma_{\widetilde{\mathsf{X}}}(U_a), 0, 1, U_a)$$

for $a \in \mathfrak{A}$. Here, $U_a = \bigcup_{x \in Z_{\mathsf{X}}(a)} U(x, 1, 1)$ in $\widetilde{\mathsf{X}}$. Then

$$\Psi(A_a) = \{(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})), -1, \widetilde{h}(\widetilde{x})) \mid \widetilde{x} \in U_a\}$$

is compact and open and contained in $c_{Y}^{-1}(\{-1\})$. Therefore

$$\Psi(A_a) = \bigcup_{j=1}^{n} (V_j, k_j, k_j + 1, W_j),$$

for some $n \in \mathbb{N}$ and some compact open and mutually disjoint subsets $V_1, \dots, V_n$, and compact open and mutually disjoint subsets $W_1, \dots, W_n$ of $\widetilde{\mathsf{X}}$ and integers $k_1, \dots, k_n \in \mathbb{N}$. In particular, $\widetilde{h}^{-1}(U_a)$ is the disjoint union $\widetilde{h}^{-1}(U_a) = \bigcup_{j=1}^n \widetilde{h}^{-1}(W_j)$ and

$$\sigma_{\widetilde{\mathsf{Y}}}^{k_j+1}(\widetilde{h}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}^{k_j}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$$

for $\widetilde{x} \in \widetilde{h}^{-1}(W_j) \subseteq U_a$. We can now define a continuous map $k_a \colon U_a \longrightarrow \mathbb{N}$ by $k_a(\widetilde{x}) = k_j$ for $\widetilde{x} \in \widetilde{h}^{-1}(W_j) \subseteq U_a$. Since $\widetilde{\mathsf{X}}$ is the disjoint union of U_a, $a \in \mathfrak{A}$, there is a continuous map $k \colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ given by $k(\widetilde{x}) = k_a(\widetilde{x})$ for $\widetilde{x} \in U_a \subseteq \widetilde{\mathsf{X}}$, and

$$\sigma_{\widetilde{\mathsf{Y}}}^{k(\widetilde{x})+1}(\widetilde{h}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}^{k(\widetilde{x})}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))),$$

for $\widetilde{x} \in \widetilde{\mathsf{X}}$. Similarly, there is a continuous map $k' \colon \widetilde{\mathsf{Y}} \longrightarrow \mathbb{N}$ which satisfies

$$\sigma_{\widetilde{\mathsf{X}}}^{k'(\widetilde{y})+1}(\widetilde{h}^{-1}(\widetilde{y})) = \sigma_{\widetilde{\mathsf{X}}}^{k'(\widetilde{y})}(\widetilde{h}^{-1}(\sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y}))),$$

for $\widetilde{y} \in \widetilde{\mathsf{Y}}$. Let $\ell = \max\{k(\widetilde{\mathsf{X}}), k'(\widetilde{\mathsf{Y}}))$. Then $\widetilde{h}$ is an eventual conjugacy satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$.

(iii) $\implies$ (iv): A groupoid isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ with $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ induces a *-isomorphism $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ satisfying $\Phi \circ p_{\mathsf{X}} = p_{\mathsf{Y}} \circ \Phi$ and $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ with $\Phi(g) = g \circ h^{-1}$ for $g \in C(\mathsf{X})$. The relation $c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Psi$ ensures that $\Phi \circ \gamma_z^{\mathsf{X}} = \gamma_z^{\mathsf{Y}} \circ \Phi$ for $z \in \mathbb{T}$.

The implication (iv) $\implies$ (v) is obvious.

(v) $\implies$ (iii): By Corollary B.3.4, $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$. The reconstruction theorem [**26**, Theorem 6.2] ensures the existence of a groupoid isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ satisfying $\Phi(f) = f \circ \widetilde{h}^{-1}$ for $f \in \mathcal{D}_{\mathsf{X}}$, where $\Psi^{(0)} = \widetilde{h} \colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ is the induced homeomorphism on the unit spaces, and $c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Phi$. Since $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ with $\Phi(g) = g \circ h^{-1}$ for $g \in C(\mathsf{X})$, the groupoid isomorphism Ψ satisfies $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$. ☐

When X_A and X_B are one-sided shifts of finite type determined by finite square $\{0,1\}$-matrices A and B with no zero rows and no zero columns, respectively, we recover [**24**, Corollary 4.2] (see also [**81**, Theorem 1.2]).

Corollary B.5.4. *Let X and Y be one-sided shift spaces. The following are equivalent:*

(i) *the systems X and Y are one-sided eventually conjugate;*
(ii) *there exist a groupoid isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ and a homeomorphism $h \colon \mathsf{X} \longrightarrow \mathsf{Y}$ satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ and $c_{\mathsf{X}} = c_{\mathsf{Y}} \circ \Psi$;*
(iii) *there is *-isomorphism $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ satisfying $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ and $\Phi \circ \gamma_z^{\mathsf{X}} = \gamma_z^{\mathsf{Y}} \circ \Phi$ for $z \in \mathbb{T}$.*

B.6. Continuous orbit equivalence

The notion of continous orbit equivalence among one-sided shift spaces was introduced by Matsumoto in [**78, 79**]. It is proven in [**22**, Corollary 6.1] (see also [**84**, Theorem 2.3]) that if X and Y are shifts of finite type, then X and Y are continuously orbit equivalent if and only if $\mathcal{G}_\mathsf{X}$ and $\mathcal{G}_\mathsf{Y}$ are isomorphic, and if and only if there is a diagonal-preserving *-isomorphism between $\mathcal{O}_\mathsf{X}$ and $\mathcal{O}_\mathsf{Y}$. In this section, we shall for general shift spaces X and Y look at the relationship between continuous orbit equivalence of X and Y, isomorphism of $\mathcal{G}_\mathsf{X}$ and $\mathcal{G}_\mathsf{Y}$, and diagonal-preserving *-isomorphism between $\mathcal{O}_\mathsf{X}$ and $\mathcal{O}_\mathsf{Y}$.

Definition B.6.1. Two one-sided shift spaces X and Y are *continuously orbit equivalent* if there exist a homeomorphism $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ and continuous maps $k_\mathsf{X}, l_\mathsf{X}\colon \mathsf{X} \longrightarrow \mathbb{N}$ and $k_\mathsf{Y}, l_\mathsf{Y}\colon \mathsf{Y} \longrightarrow \mathbb{N}$ satisfying

$$\sigma_\mathsf{Y}^{l_\mathsf{X}(x)}(h(x)) = \sigma_\mathsf{Y}^{k_\mathsf{X}(x)}(h(\sigma_\mathsf{X}(x))), \tag{B.7}$$

$$\sigma_\mathsf{X}^{l_\mathsf{Y}(y)}(h^{-1}(y)) = \sigma_\mathsf{X}^{k_\mathsf{Y}(y)}(h^{-1}(\sigma_\mathsf{Y}(y))), \tag{B.8}$$

for $x \in \mathsf{X}$ and $y \in \mathsf{Y}$. The underlying homeomorphism h is a *continuous orbit equivalence* and $(k_\mathsf{X}, l_\mathsf{X})$ and $(k_\mathsf{Y}, l_\mathsf{Y})$ are *cocycle pairs* for h.

Similar definitions apply to the covers of one-sided shift spaces. Our first aim is to show that a continuous orbit equivalence between X and Y can be lifted to a continuous orbit equivalence between $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{Y}}$.

Observe that if $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ is an orbit equivalence with cocycles $k_\mathsf{X}, l_\mathsf{X}\colon \mathsf{X} \longrightarrow \mathbb{N}$ and we define

$$k_\mathsf{X}^{(n)}(x) = \sum_{i=0}^{n-1} k_\mathsf{X}(\sigma_\mathsf{X}^i(x)), \qquad l_\mathsf{X}^{(n)}(x) = \sum_{i=0}^{n-1} l_\mathsf{X}(\sigma_\mathsf{X}^i(x)),$$

then $\sigma_\mathsf{Y}^{l_\mathsf{X}^{(n)}}(h(x)) = \sigma_\mathsf{Y}^{k_\mathsf{X}^{(n)}}(h(\sigma_\mathsf{X}^n(x)))$, for $x \in \mathsf{X}$.

We need some additional terminology. Let X and Y be one-sided shift spaces and let $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a continuous orbit equivalence with continuous cocycles $k_\mathsf{X}, l_\mathsf{X}\colon \mathsf{X} \longrightarrow \mathbb{N}$ and $k_\mathsf{Y}, l_\mathsf{Y}\colon \mathsf{Y} \longrightarrow \mathbb{N}$. We say that $(h, l_\mathsf{X}, k_\mathsf{X}, l_\mathsf{Y}, k_\mathsf{Y})$ is *least period preserving* if h maps eventually periodic points to eventually periodic points,

$$\mathrm{lp}(h(x)) = l_\mathsf{X}^{(p)}(x) - k_\mathsf{X}^{(p)}(x),$$

for any periodic point $x \in \mathsf{X}$ with $\mathrm{lp}(x) = p$, h^{-1} maps eventually periodic points to eventually periodic points, and

$$\mathrm{lp}(h^{-1}(y)) = l_\mathsf{Y}^{(q)}(y) - k_\mathsf{Y}^{(q)}(y),$$

for any periodic point $y \in \mathsf{Y}$ with $\mathrm{lp}(y) = q$. We say that $(h, l_\mathsf{X}, k_\mathsf{X}, l_\mathsf{Y}, k_\mathsf{Y})$ is *stabilizer preserving* if h maps eventually periodic points to eventually periodic points,

$$\mathrm{lp}(h(x)) = |l_\mathsf{X}^{(p)}(x) - k_\mathsf{X}^{(p)}(x)|,$$

for any periodic point $x \in \mathsf{X}$ with $\mathrm{lp}(x) = p$, h^{-1} maps eventually periodic points to eventually periodic points, and

$$\mathrm{lp}(h^{-1}(y)) = |l_\mathsf{Y}^{(q)}(y) - k_\mathsf{Y}^{(q)}(y)|,$$

for any periodic point $y \in \mathsf{Y}$ with $\mathrm{lp}(y) = q$. cf. [**22**, p. 1093] and [**26**, Definition 8.1]. There are analogous definitions for a continuous orbit equivalence between covers.

Remark B.6.2. Not every cocycle pair of a continuous orbit equivalence (even between finite type shifts) is least period preserving, cf. Remark B.6.5. However, we do not know if there is a continuous orbit equivalence which does not admit a least period/stabilizer preserving cocycle pair. In Example B.6.13, we exhibit an example of a continuous orbit equivalence between shifts of finite type which does not admit a cocycle pair which is least period preserving on all *eventually periodic* points.

Lemma B.6.3 (Lifting lemma). *Let* X *and* Y *be one-sided shift spaces and let* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *be a stabilizer preserving continuous orbit equivalence with continuous cocycles continuous cocycles* $l_\mathsf{X}, k_\mathsf{X}\colon \mathsf{X} \longrightarrow \mathbb{N}$ *and* $l_\mathsf{Y}, k_\mathsf{Y}\colon \mathsf{Y} \longrightarrow \mathbb{N}$. *Then there is a stabilizer preserving continuous orbit equivalence* $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ *with continuous cocycles* $l_{\widetilde{\mathsf{X}}} = l_\mathsf{X} \circ \pi_\mathsf{X}, k_{\widetilde{\mathsf{X}}} = k_\mathsf{X} \circ \pi_\mathsf{X}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ *and* $l_{\widetilde{\mathsf{Y}}} = l_\mathsf{Y} \circ \pi_\mathsf{Y}, k_{\widetilde{\mathsf{Y}}} = k_\mathsf{Y} \circ \pi_\mathsf{Y}\colon \widetilde{\mathsf{Y}} \longrightarrow \mathbb{N}$.

Proof. We first verify two claims which will allow us to define the map $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$. Then we show that $\widetilde{h}$ with the prescribed cocycles is stabilizer preserving.

Let $x \in \mathsf{X}$ and $\mu, \nu \in \mathsf{L}(\mathsf{X})$ and suppose $h(x) \in C_\mathsf{Y}(\mu, \nu)$.

Claim 1. There are integers $0 \leqslant k \leqslant l$ such that $x' \overset{k,l}{\sim} x \implies h(x') \in C_\mathsf{Y}(\mu, \nu)$.

Let $y := \sigma_\mathsf{Y}^{|\nu|}(h(x))$. Then $\nu y, \mu y \in \mathsf{Y}$ and $h(x) = \nu y$. From the cocycle relations (B.7) and (B.8) we have

$$\sigma_\mathsf{X}^{l_\mathsf{Y}^{(|\nu|)}(\nu y)}(h^{-1}\nu y)) = \sigma_\mathsf{X}^{k_\mathsf{Y}^{(|\nu|)}(\nu y)}(h^{-1}(y)),$$

and

$$\sigma_\mathsf{X}^{l_\mathsf{Y}^{(|\mu|)}(\mu y)}(h^{-1}\mu y)) = \sigma_\mathsf{X}^{k_\mathsf{Y}^{(|\mu|)}(\mu y)}(h^{-1}(y)).$$

Hence if

$$\alpha' = h^{-1}(\mu y)_{[0, l_\mathsf{Y}^{(|\mu|)}(\mu y) + k_\mathsf{Y}^{(|\nu|)}(\nu y))}, \quad \beta' = (h^{-1}(\nu y))_{[0, l_\mathsf{Y}^{(|\nu|)}(\nu y) + k_\mathsf{Y}^{(|\mu|)}(\mu y))},$$

then $h^{-1}(\mu y) = \alpha' z'$ and $h^{-1}(\nu y) = \beta' z'$, for some $z' \in \mathsf{X}$. If z' is eventually periodic, pick $q \in \mathbb{N}$ such that $z := \sigma_\mathsf{X}^q(z')$ is periodic and $\gamma := z'_{[0,q)}$; if z' is aperiodic, let $q = 0$ and let γ be the empty word. Set $\alpha := \alpha'\gamma$ and $\beta := \beta'\gamma$ and observe that $h(\alpha z) = \mu y$ and $h(\beta z) = \nu y$, and $x = \beta z$.

By the cocycle relations we have

$$\sigma_\mathsf{Y}^{l_\mathsf{X}^{(|\alpha|)}(\alpha z)}(\mu y) = \sigma_\mathsf{Y}^{l_\mathsf{X}^{(|\alpha|)}(\alpha z)}(h(\alpha z)) = \sigma_\mathsf{Y}^{k_\mathsf{X}^{(|\alpha|)}(\alpha z)}(h(z)),$$

and

$$\sigma_\mathsf{Y}^{l_\mathsf{X}^{(|\beta|)}(\beta z)}(\nu y) = \sigma_\mathsf{Y}^{l_\mathsf{X}^{(|\beta|)}(\beta z)}(h(\beta z)) = \sigma_\mathsf{Y}^{k_\mathsf{X}^{(|\beta|)}(\beta z)}(h(z)),$$

from which we deduce that

$$\sigma_\mathsf{Y}^{k_\mathsf{X}^{(|\alpha|)}(\alpha z) + |\mu| + l_\mathsf{X}^{(|\beta|)}(\beta z)}(h(z)) = \sigma_\mathsf{Y}^{l_\mathsf{X}^{(|\alpha|)}(\alpha z) + l_\mathsf{X}^{(|\beta|)}(\beta z)}(y) = \sigma_\mathsf{Y}^{k_\mathsf{X}^{(|\beta|)}(\beta z) + |\nu| + l_\mathsf{X}^{(|\alpha|)}(\alpha z)}(h(z)).$$

It now follows that In particular,

$$k_\mathsf{X}^{(|\alpha|)}(\alpha z) + |\mu| + l_\mathsf{X}^{(|\beta|)}(\beta z) - \big(k_\mathsf{X}^{(|\beta|)}(\beta z) + |\nu| + l_\mathsf{X}^{(|\alpha|)}(\alpha z)\big)$$

is a multiple of $\mathrm{lp}(h(z))$ — if $h(z)$ is aperiodic, we set $\mathrm{lp}(h(z)) = 0$. Without loss of generality, we may assume there is a nonnegative integer m such that

$$k_{\mathsf{X}}^{(|\alpha|)}(\alpha z)+|\mu|+l_{\mathsf{X}}^{(|\beta|)}(\beta z)-\big(k_{\mathsf{X}}^{(|\beta|)}(\beta z)+|\nu|+l_{\mathsf{X}}^{(|\alpha|)}(\alpha z)\big) = m\,\mathrm{lp}(h(z)) = m\big(l_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)-k_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)\big).$$

The final equality follows from the hypothesis that h is stabilizer preserving. Set

$$N := l_{\mathsf{X}}^{(|\beta|)}(\beta z) + l_{\mathsf{X}}^{(|\alpha|)}(\alpha z) + m l_{\mathsf{X}}^{(\mathrm{lp}(z))}(z).$$

Pick $r \in \mathbb{N}$ such that $k_{\mathsf{X}}^{(|\beta|)}$ and $l_{\mathsf{X}}^{(|\beta|)}$ are constant on $Z_{\mathsf{X}}(x_{[0,r)})$ and

$$h(Z_{\mathsf{X}}(x_{[0,r)})) \subseteq Z_{\mathsf{Y}}((\nu y)_{[0,|\nu|+N})).$$

Pick also $s \in \mathbb{N}$ such that $k_{\mathsf{X}}^{(|\alpha|)}$ and $l_{\mathsf{X}}^{(|\alpha|)}$ are constant on $Z_{\mathsf{X}}((\alpha z)_{[0,s)})$ and

$$h(Z_{\mathsf{X}}((\alpha z)_{[0,s)})) \subseteq Z_{\mathsf{Y}}((\mu y)_{[0,|\mu|+N)}),$$

and such that $l_{\mathsf{X}}^{(\mathrm{lp}(z))}$ and $k_{\mathsf{X}}^{(\mathrm{lp}(z))}$ are constant on $Z_{\mathsf{X}}(z_{[0,s)})$. Set $k := r + s + |\beta|$ and $l := r + s + |\beta| + |\alpha|$.

Let $x' \in \mathsf{X}$ and suppose $x' \overset{k,l}{\sim} x$. Then $x' \in Z_{\mathsf{X}}(x_{[0,r)})$, so $h(x') \in Z_{\mathsf{Y}}((\nu y)_{[0,|\nu|+N)})$ and $k_{\mathsf{X}}^{(|\alpha|)}(x') = k_{\mathsf{X}}^{(|\alpha|)}(x)$ and $l_{\mathsf{X}}^{(|\alpha|)}(x') = l_{\mathsf{X}}^{(|\alpha|)}(x)$. Put $x'' := \alpha z_{[0,m\,\mathrm{lp}(z))}\sigma_{\mathsf{X}}^{(|\beta|)}(x')$. Then $\sigma_{\mathsf{X}}^{|\alpha|+m\,\mathrm{lp}(z)}(x'') = \sigma_{\mathsf{X}}^{(|\beta|)}(x')$ and $x'' \in Z_{\mathsf{X}}((\alpha z)_{[0,s)})$ so $h(x'') \in Z_{\mathsf{Y}}((\mu y)_{[0,|\mu|+N)})$.

$$\begin{aligned}\sigma_{\mathsf{Y}}^{|\nu|+N}(h(x')) &= \sigma_{\mathsf{Y}}^{|\nu|+l_{\mathsf{X}}^{(|\beta|)}(x')+l_{\mathsf{X}}^{(|\alpha|)}(\alpha z)+ml_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(x'))\\ &= \sigma_{\mathsf{Y}}^{|\nu|+k_{\mathsf{X}}^{(|\beta|)}(x')+l_{\mathsf{X}}^{(|\alpha|)}(\alpha z)+ml_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(\sigma_{\mathsf{X}}^{|\beta|}(x')))\\ &= \sigma_{\mathsf{Y}}^{|\nu|+k_{\mathsf{X}}^{(|\beta|)}(\beta z)+l_{\mathsf{X}}^{(|\alpha|)}(\alpha z)+ml_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(\sigma_{\mathsf{X}}^{|\beta|}(x'))),\end{aligned}$$

and

$$\begin{aligned}\sigma_{\mathsf{Y}}^{|\mu|+N}(h(x'')) &= \sigma_{\mathsf{Y}}^{|\mu|+l_{\mathsf{X}}^{(|\beta|)}(\beta z)+l_{\mathsf{X}}^{(|\alpha|)}(x'')+ml_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(x''))\\ &= \sigma_{\mathsf{Y}}^{|\mu|+l_{\mathsf{X}}^{(|\beta|)}(\beta z)+k_{\mathsf{X}}^{(|\alpha|)}(x'')+mk_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(\sigma_{\mathsf{X}}^{|\alpha|+m\,\mathrm{lp}(z)}(x'')))\\ &= \sigma_{\mathsf{Y}}^{|\mu|+l_{\mathsf{X}}^{(|\beta|)}(\beta z)+k_{\mathsf{X}}^{(|\alpha|)}(\alpha z)+mk_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(\sigma_{\mathsf{X}}^{|\beta|}(x')))\\ &= \sigma_{\mathsf{Y}}^{|\nu|+k_{\mathsf{X}}^{(|\beta|)}(\beta z)+l_{\mathsf{X}}^{(|\alpha|)}(\alpha z)+ml_{\mathsf{X}}^{(\mathrm{lp}(z))}(z)}(h(\sigma_{\mathsf{X}}^{|\beta|}(x'))).\end{aligned}$$

Thus $\sigma_{\mathsf{Y}}^{|\nu|+N}(h(x')) = \sigma_{\mathsf{Y}}^{|\mu|+N}(h(x''))$ so

$$h(x'') \in C_{\mathsf{Y}}((\mu y)_{[0,|\mu|+N)}, (\nu y)_{[0,|\nu|+N)}) \subseteq C_{\mathsf{Y}}(\mu,\nu)$$

and this proves Claim 1.

Claim 2. For each $(k,l) \in \mathcal{I}$ there is $(m(k,l), n(k,l)) \in \mathcal{I}$ such that

$$x \overset{m(k,l),n(k,l)}{\sim} x' \implies h(x) \overset{k,l}{\sim} h(x').$$

Let $(k,l) \in \mathcal{I}$ and take $\mu,\nu \in \mathsf{L}(\mathsf{X})$ with $|\nu| = k$ and $|\mu| \leqslant l$ and $x \in C_{\mathsf{X}}(\mu,\nu)$. By Claim 1, we may choose $\big(r(\mu,\nu,x), s(\mu,\nu,x)\big) \in \mathcal{I}$ such that

$$h\big({}_{r(\mu,\nu,x)}[x]_{s(\mu,\nu,x)}\big) \subseteq C_{\mathsf{Y}}(\mu,\nu).$$

The topology on X generated by the sets $\{{}_r[x]_s \mid x \in \mathsf{X}, (r,s) \in \mathcal{I}\}$ is compact, so there is a finite set $F \subseteq \mathsf{L}(\mathsf{X}) \times \mathsf{L}(\mathsf{X}) \times \mathsf{X}$ such that

$$\bigcup_{(\mu,\nu,x)\in F} {}_{r(\mu,\nu,x)}[x]_{s(\mu,\nu,x)} = \mathsf{X}.$$

Set $m(k,l) := \max\{r(\mu,\nu,x) \mid (\mu,\nu,x) \in F\}$ and $n(k,l) := \max\{s(\mu,\nu,x) \mid (\mu,\nu,x) \in F\}$. Then the implication of Claim 2 holds.

We are now ready to prove the lemma. Let $(k,l) \in \mathcal{I}$ and set

$$\widetilde{m}(k,l) := \max\{m(k',l') \mid (k',l') \preceq (k,l)\}, \quad \widetilde{n}(k,l) := \max\{n(k',l') \mid (k',l') \preceq (k,l)\}.$$

Then there is a well-defined and continuous map $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ given by

$${}_k(\widetilde{h}(\widetilde{x}))_l = h({}_{\widetilde{m}(k,l)}[x]_{\widetilde{n}(k,l)}),$$

for $(k,l) \in \mathcal{I}$ and $\widetilde{x} = ({}_r[x]_s)_{(r,s)\in\mathcal{I}}$. A similar argument shows that there for $(k,l) \in \mathcal{I}$ is $(m'(k,l), n'(k,l)) \in \mathcal{I}$ such that

$$y \overset{m'(k,l),n'(k,l)}{\sim} y' \implies h^{-1}(y) \overset{k,l}{\sim} h^{-1}(y'),$$

and that there is a continuous map $\widetilde{h}'\colon \widetilde{\mathsf{Y}} \longrightarrow \widetilde{\mathsf{X}}$ given by

$${}_k(h'(\widetilde{y}))_l = h^{-1}({}_{\widetilde{m}'(k,l)}[y]_{\widetilde{n}'(k,l)}),$$

for $(k,l) \in \mathcal{I}$ and $\widetilde{y} = ({}_r[y]_s)_{(r,s)\in\mathcal{I}} \in \widetilde{\mathsf{Y}}$, where $\widetilde{m}'(k,l) = \max\{m'(k',l') \mid (k',l') \preceq (k,l)\}$ and $\widetilde{n}'(k,l) = \max\{n'(k',l') \mid (k',l') \preceq (k,l)\}$. Since h' is the inverse of $\widetilde{h}$, the latter map is a homeomorphism.

It is straightforward to check that $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$. Define $k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{X}}}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ and $k_{\widetilde{\mathsf{Y}}}, l_{\widetilde{\mathsf{Y}}}\colon \widetilde{\mathsf{Y}} \longrightarrow \mathbb{N}$ by $k_{\widetilde{\mathsf{X}}} = k_{\mathsf{X}} \circ \pi_{\mathsf{X}}$, $l_{\widetilde{\mathsf{X}}} = l_{\mathsf{X}} \circ \pi_{\mathsf{X}}$ and $k_{\widetilde{\mathsf{Y}}} = k_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$, $l_{\widetilde{\mathsf{Y}}} = l_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$. They are continuous. It is straightforward to check that $\sigma_{\widetilde{\mathsf{Y}}}^{l_{\widetilde{\mathsf{X}}}(\widetilde{x})}(\widetilde{h}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}^{k_{\widetilde{\mathsf{X}}}(\widetilde{x})}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$ for $\widetilde{x} \in \widetilde{\mathsf{X}}$, and that $\sigma_{\widetilde{\mathsf{X}}}^{l_{\widetilde{\mathsf{Y}}}(\widetilde{y})}(\widetilde{h}^{-1}(\widetilde{y})) = \sigma_{\widetilde{\mathsf{X}}}^{k_{\widetilde{\mathsf{Y}}}(\widetilde{y})}(\widetilde{h}^{-1}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$ for $\widetilde{y} \in \widetilde{\mathsf{Y}}$. Thus, $(\widetilde{h}, l_{\widetilde{\mathsf{X}}}, k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{Y}}}, k_{\widetilde{\mathsf{Y}}})$ is a continuous orbit equivalence.

We will now show that $(\widetilde{h}, l_{\widetilde{\mathsf{X}}}, k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{Y}}}, k_{\widetilde{\mathsf{Y}}})$ is stabilizer preserving. Pick a periodic element $\widetilde{x} \in \widetilde{\mathsf{X}}$ and let $x = \pi_{\mathsf{X}}(\widetilde{x}) \in \mathsf{X}$. Then x is periodic and if $\mathrm{lp}(x) = p$, then $\mathrm{lp}(\widetilde{x}) = np$ for some $n \in \mathbb{N}_+$. Since $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ is stabilizer preserving, $h(x) \in \mathsf{Y}$ is eventually periodic and $|l_{\mathsf{X}}^{(p)}(x) - k_{\mathsf{X}}^{(p)}(x)| = \mathrm{lp}(h(x))$. Furthermore,

$$|l_{\widetilde{\mathsf{X}}}^{(\mathrm{lp}(\widetilde{x}))}(\widetilde{x}) - \widetilde{k}_{\mathsf{X}}^{(\mathrm{lp}(\widetilde{x}))}(\widetilde{x})| = n\,\mathrm{lp}(h(x))$$

is a period for $\widetilde{h}(\widetilde{x})$. In particular, $\widetilde{h}(\widetilde{x})$ is eventually periodic and as above $\mathrm{lp}(\widetilde{h}(\widetilde{x})) = m\,\mathrm{lp}(h(x))$ for some $m \in \mathbb{N}_+$. The above computation shows that m divides n. A similar argument using h^{-1} instead of h shows that n divides m and thus that $n = m$. This shows that $\mathrm{lp}(\widetilde{h}(\widetilde{x})) = |l_{\widetilde{\mathsf{X}}}^{(\mathrm{lp}(\widetilde{x}))}(\widetilde{x}) - k_{\widetilde{\mathsf{X}}}^{(\mathrm{lp}(\widetilde{x}))}(\widetilde{x})|$. Since $\widetilde{h}$ maps periodic points to eventually periodic points, it also maps eventually periodic points to eventually periodic points. $\square$

We shall next find conditions on $\mathcal{G}_{\mathsf{X}}$ and $\mathcal{G}_{\mathsf{Y}}$ and for $\mathcal{O}_{\mathsf{X}}$ and $\mathcal{O}_{\mathsf{Y}}$ that are equivalent to the existence of a stabilizer preserving continuous orbit equivalence between X and Y.

THEOREM B.6.4. *Let* X *and* Y *be one-sided shift spaces, let* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *be a homeomorphism and let* $d_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{Z}$ *and* $d_{\mathsf{Y}}\colon \mathsf{Y} \longrightarrow \mathbb{Z}$ *be continuous maps. The following conditions are equivalent:*

(i) *there are continuous maps* $k_{\mathsf{X}}, l_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{N}$ *and* $k_{\mathsf{Y}}, l_{\mathsf{Y}}\colon \mathsf{Y} \longrightarrow \mathbb{N}$ *with* $d_{\mathsf{X}} = l_{\mathsf{X}} - k_{\mathsf{X}}$ *and* $d_{\mathsf{Y}} = l_{\mathsf{Y}} - k_{\mathsf{Y}}$ *such that* $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ *is a stabilizer preserving continuous orbit equivalence;*

(ii) *there are continuous maps* $k_{\mathsf{X}}, l_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{N}$ *and* $k_{\mathsf{Y}}, l_{\mathsf{Y}}\colon \mathsf{Y} \longrightarrow \mathbb{N}$ *with* $d_{\mathsf{X}} = l_{\mathsf{X}} - k_{\mathsf{X}}$ *and* $d_{\mathsf{Y}} = l_{\mathsf{Y}} - k_{\mathsf{Y}}$ *and continuous maps* $k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{X}}}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ *and* $k_{\widetilde{\mathsf{Y}}}, l_{\widetilde{\mathsf{Y}}}\colon \widetilde{\mathsf{Y}} \longrightarrow \mathbb{N}$ *with* $l_{\widetilde{\mathsf{X}}} = l_{\mathsf{X}} \circ \pi_{\mathsf{X}}$, $k_{\widetilde{\mathsf{X}}} = k_{\mathsf{X}} \circ \pi_{\mathsf{X}}$, $l_{\widetilde{\mathsf{Y}}} = l_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$, $k_{\widetilde{\mathsf{Y}}} = k_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$, *and a homeomorphism* $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ *such that* $(\widetilde{h}, l_{\widetilde{\mathsf{X}}}, k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{Y}}}, k_{\widetilde{\mathsf{Y}}})$ *is a stabilizer preserving continuous orbit equivalence satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$*;*

(iii) *there are*
- *a groupoid isomorphism* $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ *such that* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ *and* $\kappa_{\mathsf{X}}(d_{\mathsf{X}}) = \kappa_{\mathsf{Y}}(1) \circ \Psi$*;*
- *a groupoid isomorphism* $\Psi'\colon \mathcal{G}_{\mathsf{Y}} \to \mathcal{G}_{\mathsf{X}}$ *such that* $h^{-1} \circ \pi_{\mathsf{Y}} = \pi_{\mathsf{X}} \circ (\Psi')^{(0)}$ *and* $\kappa_{\mathsf{Y}}(d_{\mathsf{Y}}) = \kappa_{\mathsf{X}}(1) \circ \Psi'$*;*

(iv) *there are*
- *a* **-isomorphism* $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ *such that* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$, $\Phi(f) = f \circ h^{-1}$ *for* $f \in C(\mathsf{X})$ *and* $\Phi \circ \gamma_z^{\mathsf{X}} = \beta_z^{\kappa_{\mathsf{Y}}(d_{\mathsf{Y}})} \circ \Phi$ *for each* $z \in \mathbb{T}$*;*
- *a* **-isomorphism* $\Phi'\colon \mathcal{O}_{\mathsf{Y}} \longrightarrow \mathcal{O}_{\mathsf{X}}$ *such that* $\Phi'(C(\mathsf{Y})) = C(\mathsf{X})$, $\Phi'(f) = f \circ h$ *for* $f \in C(\mathsf{Y})$ *and* $\Phi' \circ \gamma_z^{\mathsf{Y}} = \beta_z^{\kappa_{\mathsf{X}}(d_{\mathsf{X}})} \circ \Phi'$ *for each* $z \in \mathbb{T}$.

PROOF. (i) $\implies$ (ii): This is Lemma B.6.3.

(ii) $\implies$ (iii): It follows from [**26**, Proposition 8.3] that there is a groupoid isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ satisfying

$$\Psi((\widetilde{x}, 1, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))) = (\widetilde{h}(\widetilde{x}), l_{\widetilde{\mathsf{X}}}(\widetilde{x}) - k_{\widetilde{\mathsf{Y}}}(\widetilde{x}), \widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$$

for $\widetilde{x} \in \widetilde{\mathsf{X}}$, and a groupoid isomorphism $\Phi'\colon \mathcal{G}_{\mathsf{Y}} \to \mathcal{G}_{\mathsf{X}}$ satisfying

$$\Phi'((\widetilde{y}, 1, \sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y}))) = (\widetilde{h}^{-1}(\widetilde{y}), l_{\widetilde{\mathsf{Y}}}(y) - k_{\widetilde{\mathsf{Y}}}(\widetilde{y}), \widetilde{h}^{-1}(\sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y})))$$

for $\widetilde{y} \in \widetilde{\mathsf{Y}}$. We then have that $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Phi^{(0)}$, $\kappa_{\mathsf{X}}(d_{\mathsf{X}}) = \kappa_{\mathsf{Y}}(1) \circ \Phi$, $h^{-1} \circ \pi_{\mathsf{Y}} = \pi_{\mathsf{X}} \circ (\Phi')^{(0)}$ and $\kappa_{\mathsf{Y}}(d_{\mathsf{Y}}) = \kappa_{\mathsf{X}}(1) \circ \Phi'$.

(iii) $\implies$ (i): Let $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ be a groupoid isomorphism satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ and $\kappa_{\mathsf{X}}(d_{\mathsf{X}}) = \kappa_{\mathsf{Y}}(1) \circ \Psi$. Put $\widetilde{h} := \Psi^{(0)}$. Then $\Psi(\widetilde{x}, 1, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})) = (\widetilde{h}(\widetilde{x}), d_{\mathsf{X}}(\pi_{\mathsf{X}}(\widetilde{x})), \widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$ and it follows from [**26**, Lemma 8.4] that the map $l_{\widetilde{\mathsf{X}}}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ given by

$$l_{\widetilde{\mathsf{X}}}(\widetilde{x}) = \min\{n \in \mathbb{N} \mid n \geqslant d_{\mathsf{X}}(\pi_{\mathsf{X}}(\widetilde{x})), \sigma_{\widetilde{\mathsf{Y}}}^n(\widetilde{h}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))\}$$

is continuous. We claim that

$$l_{\widetilde{\mathsf{X}}}(\widetilde{x}) = \min\{n \in \mathbb{N} \mid n \geqslant d_{\mathsf{X}}(\pi_{\mathsf{X}}(\widetilde{x})), \sigma_{\widetilde{\mathsf{Y}}}^n(\widetilde{h}(\pi_{\mathsf{X}}(\widetilde{x}))) = \sigma_{\widetilde{\mathsf{Y}}}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\pi_{\mathsf{X}}(\widetilde{x}))))\}. \tag{B.9}$$

By applying π_{Y}, it is easy to see that the left hand side is less than the right hand side. For the converse inequality, fix $\widetilde{x} \in \widetilde{\mathsf{X}}$ and suppose the right hand side of (B.9) equals n. Set $\widetilde{y} := \sigma_{\widetilde{\mathsf{Y}}}^n(\widetilde{h}(\widetilde{x}))$ and $\widetilde{y}' := \sigma_{\widetilde{\mathsf{Y}}}^{n - d_{\mathsf{X}}(\pi_{\mathsf{X}}(\widetilde{x}))}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$. Then $\pi_{\mathsf{Y}}(\widetilde{y}) = \pi_{\mathsf{Y}}(\widetilde{y}')$

by hypothesis, and since $(\widetilde{h}(\widetilde{x}), d_{\mathsf{X}}(\pi_{\mathsf{X}}(\widetilde{x})), \widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))) \in \mathcal{G}_{\mathsf{Y}}$ there is an $m \in \mathbb{N}$ such that $\sigma_{\widetilde{\mathsf{Y}}}^m(\widetilde{y}) = \sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y}')$. It now follows from Lemma B.2.8(i) that $\widetilde{y} = \widetilde{y}'$. This means that there is a map $l_{\mathsf{X}} \colon \mathsf{X} \longrightarrow \mathbb{N}$ such that $l_{\widetilde{\mathsf{X}}} = l_{\mathsf{X}} \circ \pi_{\mathsf{X}}$. This map is continuous by Lemma B.2.5. Set $k_{\mathsf{X}} := d_{\mathsf{X}} - l_{\mathsf{X}}$. Then k_{X} is a continuous map satisfying $d_{\mathsf{X}} = l_{\mathsf{X}} - k_{\mathsf{X}}$ and $\sigma_{\mathsf{Y}}^{l(x)}(h(x)) = \sigma_{\mathsf{Y}}^{k(x)}(h(\sigma_{\mathsf{X}}(x)))$ for $x \in \mathsf{X}$. A similar argument shows that there are continuous maps $l_{\mathsf{Y}}, k_{\mathsf{Y}} \colon \mathsf{Y} \longrightarrow \mathbb{N}$ satisfying $d_{\mathsf{Y}} = l_{\mathsf{Y}} - k_{\mathsf{Y}}$ and $\sigma_{\mathsf{X}}^{l_{\mathsf{Y}}(y)}(h^{-1}(y)) = \sigma_{\mathsf{X}}^{k_{\mathsf{Y}}(y)}(h^{-1}(\sigma_{\mathsf{Y}}(y)))$ for $y \in \mathsf{Y}$. Then $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ is a continuous orbit equivalence.

Finally, we show that $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ is stabilizer preserving. Observe first that an argument similar to the one used in the proof of [**26**, Lemma 8.6] shows that $(\widetilde{h}, l_{\widetilde{\mathsf{X}}}, k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{Y}}}, k_{\widetilde{\mathsf{Y}}})$ is stabilizer preserving. Fix an eventually periodic element $x \in \mathsf{X}$. Then $\widetilde{x} = \iota_{\mathsf{X}}(x) \in \widetilde{\mathsf{X}}$ is eventually periodic, so $\widetilde{h}(\widetilde{x})$ is eventually periodic. Hence $h(x) = \pi_{\mathsf{Y}}(\widetilde{h}(\widetilde{x})) \in \mathsf{Y}$ is eventually periodic. Now suppose x is periodic with $\mathrm{lp}(x) = p$. Then $\iota_{\mathsf{X}}(x) \in \widetilde{\mathsf{X}}$ is periodic with $\mathrm{lp}(\iota_{\mathsf{X}}(x)) = p$. Since $\widetilde{h}(\iota_{\mathsf{X}}(x)) = \iota_{\mathsf{Y}}(h(x))$ we also have $\mathrm{lp}(h(x)) = \mathrm{lp}(\widetilde{h(\widetilde{x})})$, and using that $\widetilde{h}$ is stabilizer preserving in the middle equality below we see that

$$|l_{\mathsf{X}}^{(p)}(x) - k_{\mathsf{X}}^{(p)}(x)| = |l_{\widetilde{\mathsf{X}}}^{(p)}(\widetilde{x}) - k_{\widetilde{\mathsf{X}}}^{(p)}(\widetilde{x})| = \mathrm{lp}(\widetilde{h}(\widetilde{x})) = \mathrm{lp}(h(x))$$

which shows that $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ is stabilizer preserving.

(iii) $\implies$ (iv): It follows from [**26**, Theorem 6.2] that there is *-isomorphism $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ such that $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$, $\Phi(f) = f \circ \widetilde{h}^{-1}$ for $f \in \mathcal{D}_{\mathsf{X}}$, and $\Phi \circ \gamma_z^{\mathsf{X}} = \beta_z^{\kappa_{\mathsf{Y}}(d_{\mathsf{Y}})} \circ \Phi$ for each $z \in \mathbb{T}$. Since $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Phi^{(0)}$, it follows that $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ and $\Psi(f) = f \circ h^{-1}$ for $f \in C(\mathsf{X})$. Similarly, there is a *-isomorphism $\Phi' \colon \mathcal{O}_{\mathsf{Y}} \longrightarrow \mathcal{O}_{\mathsf{X}}$ such that $\Phi'(C(\mathsf{Y})) = C(\mathsf{X})$, $\Phi'(f) = f \circ h$ for $f \in C(\mathsf{Y})$, and $\Phi' \circ \gamma_z^{\mathsf{Y}} = \beta_z^{\kappa_{\mathsf{X}}(d_{\mathsf{X}})} \circ \Phi'$ for each $z \in \mathbb{T}$.

(iv) $\implies$ (iii): An application of [**26**, Theorem 6.2] shows that there is a groupoid isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ such that $\kappa_{\mathsf{X}}(d_{\mathsf{X}}) = \kappa_{\mathsf{Y}}(1) \circ \Psi$. Since $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ and $\Psi(f) = f \circ h^{-1}$ for $f \in C(\mathsf{X})$, it follows that $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$. Similarly, there is a groupoid isomorphism $\Psi' \colon \mathcal{G}_{\mathsf{Y}} \longrightarrow \mathcal{G}_{\mathsf{X}}$ such that $h^{-1} \circ \pi_{\mathsf{Y}} = \pi_{\mathsf{X}} \circ (\Psi')^{(0)}$ and $\kappa_{\mathsf{Y}}(d_{\mathsf{Y}}) = \kappa_{\mathsf{X}}(1) \circ \Psi'$. □

Remark B.6.5. Notice that if $\mathsf{X} = \mathsf{Y}$ is the shift space with only one point, then $(\mathrm{id}, 1, 0, 0, 1)$ is a stabilizer preserving continuous orbit equivalence from X to Y (although it is not least period preserving), but there is no isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ such that $\kappa_{\mathsf{X}}(d_{\mathsf{X}}) = \kappa_{\mathsf{Y}}(1) \circ \Psi$ and $\kappa_{\mathsf{Y}}(d_{\mathsf{Y}}) = \kappa_{\mathsf{X}}(1) \circ \Psi^{-1}$. We do not know if there are shift spaces X and Y such that there is a stabilizer preserving continuous orbit equivalence from X to Y, but no isomorphism $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ such that $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$ for some homeomorphism $h \colon \mathsf{X} \to \mathsf{Y}$, and $\kappa_{\mathsf{Y}}(1) \circ \Psi = \kappa_{\mathsf{X}}(d_{\mathsf{X}})$ for some $d_{\mathsf{X}} \in C(\mathsf{X}, \mathbb{Z})$, and $\kappa_{\mathsf{X}}(1) \circ \Psi^{-1} = \kappa_{\mathsf{Y}}(d_{\mathsf{Y}})$ for some $d_{\mathsf{Y}} \in C(\mathsf{Y}, \mathbb{Z})$.

In [**79**, p. 61] (see also [**83**, p. 2]), Matsumoto introduces the notion of a continuous orbit equivalence between factor maps of two one-sided shift spaces X and Y satisfying condition (I) (implying that the groupoids $\mathcal{G}_{\mathsf{X}}$ and $\mathcal{G}_{\mathsf{Y}}$ are essentially principal). His factor maps can be more general than our π_{X} and π_{Y}. In this case, he proves a result ([**79**, Theorem 1.2] and [**83**, Theorem 1.3]) which is similar to the theorem below. Our results applies to all one-sided shifts.

THEOREM B.6.6. *Let* X *and* Y *be one-sided shift spaces and let* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *be a homeomorphism. The following conditions are equivalent:*

(i) *there is a stabilizer preserving continuous orbit equivalence* $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ *satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$*;*
(ii) *there is a groupoid isomorphism* $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ *satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$*;*
(iii) *there is a* **-isomorphism* $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ *satisfying* $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$ *and* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ *with* $\Phi(f) = f \circ h^{-1}$ *for* $f \in C(\mathsf{X})$*.*

If X *and* Y *contain dense sets of aperiodic points, then the condition* $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$ *in* (iii) *is superfluous. Moreover, if* h *is a stabilizer preserving continuous orbit equivalence, then the equivalent conditions above hold.*

PROOF. (i) $\implies$ (ii): Let $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ be a continuous orbit equivalence and let $k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{X}}}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ be continuous cocycles for $\widetilde{h}$. There is a groupoid homomorphism $\Phi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ given by

$$\Phi(\widetilde{x}, m-n, \widetilde{y}) = \big(\widetilde{h}(\widetilde{x}), l_{\widetilde{\mathsf{X}}}^{(m)}(\widetilde{x}) - k_{\widetilde{\mathsf{X}}}^{(m)}(\widetilde{x}) - l_{\widetilde{\mathsf{X}}}^{(n)}(\widetilde{y}) + k_{\widetilde{\mathsf{X}}}^{(n)}(\widetilde{y}), \widetilde{h}(\widetilde{y})\big)\big),$$

for $(\widetilde{x}, m-n, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}}$, The assumption that $\widetilde{h}$ be stabilizer preserving ensures that Φ is bijective, cf. [**26**, Lemma 8.8 and Proposition 8.3].

(ii) $\implies$ (i): A groupoid isomorphism $\Phi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ restricts to a homeomorphism $\widetilde{h} = \Phi^{(0)}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$. If c_{X} is the canonical cocycle for $\mathcal{G}_{\mathsf{X}}$ and c_{Y} is the canonical cocycle for $\mathcal{G}_{\mathsf{Y}}$, then the maps

$$\begin{aligned}
l_{\widetilde{\mathsf{X}}}(\widetilde{x}) &= \min\{l \in \mathbb{N} \mid \sigma_{\widetilde{\mathsf{Y}}}^{l}(\widetilde{h}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}^{l - c_{\mathsf{Y}}\Phi(\widetilde{x}, 1, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))\},\\
k_{\widetilde{\mathsf{X}}}(\widetilde{x}) &= l_{\widetilde{\mathsf{X}}}(\widetilde{x}) - c_{\mathsf{Y}}\Phi(\widetilde{x}, 1, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})),\\
l_{\widetilde{\mathsf{Y}}}(\widetilde{y}) &= \min\{l \in \mathbb{N} \mid \sigma_{\widetilde{\mathsf{X}}}^{l}(\widetilde{h}^{-1}(\widetilde{y})) = \sigma_{\widetilde{\mathsf{X}}}^{l - c_{\mathsf{X}}\Phi^{-1}(\widetilde{y}, 1, \sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y}))}(\widetilde{h}^{-1}(\sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y})))\},\\
k_{\widetilde{\mathsf{Y}}}(\widetilde{y}) &= l_{\widetilde{\mathsf{Y}}}(\widetilde{y}) - c_{\mathsf{X}}\Phi^{-1}(\widetilde{y}, 1, \sigma_{\widetilde{\mathsf{Y}}}(\widetilde{y}))
\end{aligned}$$

constitute continuous cocycles for $\widetilde{h}$ such that $(\widetilde{h}, l_{\widetilde{\mathsf{X}}}, k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{Y}}}, k_{\widetilde{\mathsf{Y}}})$ is a stabilizer preserving continuous orbit equivalence, cf. [**26**, Lemma 8.5 and Lemma 8.6]. The condition $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Phi^{(0)}$ implies that $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$.

The equivalence (ii) $\iff$ (iii) is [**26**, Theorem 8.2]. If $\Phi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ is a groupoid isomorphism, then the condition $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Phi^{(0)}$ translates to the condition that the *-isomorphism $\Psi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ satisfies $\Psi(C(\mathsf{X})) = C(\mathsf{Y})$.

The final remarks follow from Theorem B.3.3 and Lemma B.6.3, respectively. ☐

REMARK B.6.7. We do not know if there exist shift spaces X and Y such that the conditions in Theorem B.6.6 are satisfied, but there is no stabilizer preserving continuous orbit equivalence between X and Y.

Next, we show that we can relax the conditions in Theorem B.6.4 for certain classes of shift spaces. First we need a preliminary result concerning eventually periodic points.

In [**86**, Proposition 3.5], Matsumoto and Matui show that any continuous orbit equivalence between shift spaces containing a dense set of aperiodic points maps eventually periodic points to eventually periodic points. The result is only stated for shifts of finite type associated with irreducible and nonpermutation $\{0,1\}$-matrices, but the proof holds in this generality, cf. [**22**, Remark 3.1]. Below, we give a *point-wise* version of this result applicable to all shift spaces. We do not know if any continuous orbit equivalence between shift spaces preserves eventually periodic points, but we show that this problem hinges on whether there exists a continuous orbit equivalence which maps an aperiodic isolated point to an eventually periodic isolated point.

PROPOSITION B.6.8. *Let X and Y be one-sided shift spaces and let $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a continuous orbit equivalence. Then h maps nonisolated eventually periodic points to nonisolated eventually periodic points.*

PROOF. It suffices to verify the claim for nonisolated periodic points. Suppose $x = \alpha^\infty \in \mathsf{X}$ for some word $\alpha \in \mathsf{L}(\mathsf{X})$ with $|\alpha| = p \in \mathbb{N}_+$, and let $(x_n)_n$ be a sequence in X converging to x. We may assume that $x_n \in Z_{\mathsf{X}}(\alpha)$ for all n.

Suppose now that $k := k_{\mathsf{X}}^{(p)}(x) = l_{\mathsf{X}}^{(p)}(x)$. The cocycles $k_{\mathsf{X}}, l_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{N}$ for h are continuous, so there exists $N \in \mathbb{N}$ such that $k_{\mathsf{X}}^{(p)}(x_n) = l_{\mathsf{X}}^{(p)}(x_n) = k$ for $n \geqslant N$. In particular,

$$\sigma_{\mathsf{Y}}^k(h(x_n)) = \sigma_{\mathsf{Y}}^k(h(\sigma_{\mathsf{X}}^p(x_n))), \tag{B.10}$$

for $n \geqslant N$. The sequences $h(x_n)_n$ and $(h(\sigma_{\mathsf{X}}^p(x_n)))_n$ both converge to $h(x)$ in Y, so there exists an integer $M \geqslant N$ such that

$$h(x_n)_{[0,k)} = h(x)_{[0,k)} = h(\sigma_{\mathsf{X}}^p(x_n))_{[0,k)},$$

whenever $n \geqslant M$. This together with (B.10) means that $h(x_n) = h(\sigma_{\mathsf{X}}^p(x_n))$ and hence $x_n = \sigma_{\mathsf{X}}^p(x_n)$, for $n \geqslant M$. Since $x_n \in Z_{\mathsf{X}}(\alpha)$, the sequence $(x_n)_n$ is eventually equal to x, so we conclude that x is an isolated point. If x is not isolated, then $l_{\mathsf{X}}^{(p)}(x) \neq k_{\mathsf{X}}^{(p)}(x)$, and the observation

$$\sigma_{\mathsf{Y}}^{l_{\mathsf{X}}^{(p)}(x)}(h(x)) = \sigma_{\mathsf{Y}}^{k_{\mathsf{X}}^{(p)}(x)}(h(\sigma_{\mathsf{X}}^{(p)}(x))) = \sigma_{\mathsf{Y}}^{k_{\mathsf{X}}^{(p)}(x)}(h(x))$$

shows that $h(x)$ is eventually periodic. □

A similar result holds for covers.

Since any homeomorphism respects isolated points, we obtain the corollary below. Sofic shifts contain no aperiodic isolated points, so this result resolves the problem for this class of shift spaces.

COROLLARY B.6.9. *Let X and Y be one-sided shift spaces either containing no aperiodic isolated points or no isolated eventually periodic points, then any continuous orbit equivalence $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ maps eventually periodic points to eventually periodic points. In particular, this applies to sofic shift spaces.*

If X and Y contain no periodic points isolated in past equivalence, then X and Y as well as the covers $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{Y}}$ contain dense sets of aperiodic points. Hence the condition that a continuous orbit equivalence be stabilizer preserving is superfluous.

Theorem B.6.10. *Let* X *and* Y *be one-sided shift spaces with no periodic points isolated in past equivalence and let* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *be a homeomorphism. The following are equivalent:*

(i) *the map* $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *is a continuous orbit equivalence;*
(ii) *there is a continuous orbit equivalence* $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ *satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$;
(iii) *there is a groupoid isomorphism* $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ *satisfying* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$;
(iv) *there is a* **-isomorphism* $\Phi\colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ *satisfying* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$ *and* $\Phi(f) = f \circ h^{-1}$ *for* $f \in C(\mathsf{X})$.

Proof. (i) $\Longrightarrow$ (ii): Suppose $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ is a continuous orbit equivalence with continuous cocycles $k_{\mathsf{X}}, l_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{N}$. Since X and Y contain no periodic points isolated in past equivalence, it follows from Proposition B.2.10 that X and Y contain dense sets of aperiodic points. The proof of Theorem B.6.4 (i) $\Longrightarrow$ (ii) shows that there is a continuous orbit equivalence $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ with continuous cocycles $k_{\widetilde{\mathsf{X}}} = k_{\mathsf{X}} \circ \pi_{\mathsf{X}}$ and $l_{\widetilde{\mathsf{X}}} = l_{\mathsf{X}} \circ \pi_{\mathsf{X}}$ which satisfies $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{h}$.

(ii) $\Longrightarrow$ (iii): Let $\widetilde{h}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ be a continuous orbit equivalence and let $k_{\widetilde{\mathsf{X}}}, l_{\widetilde{\mathsf{X}}}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ be continuous cocycles for $\widetilde{h}$. The map $\Phi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ given by

$$\Phi(\widetilde{x}, m-n, \widetilde{y}) = \big(\widetilde{h}(\widetilde{x}), l_{\widetilde{\mathsf{X}}}^{(m)}(\widetilde{x}) - k_{\widetilde{\mathsf{X}}}^{(m)}(\widetilde{x}) - l_{\widetilde{\mathsf{X}}}^{(n)}(\widetilde{y}) + k_{\widetilde{\mathsf{X}}}^{(n)}(\widetilde{y}), \widetilde{h}(\widetilde{y})\big),$$

for $(\widetilde{x}, m-n, \widetilde{y}) \in \mathcal{G}_{\mathsf{X}}$, is a groupoid isomorphism satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$.

(iii) $\Longrightarrow$ (i): Let $\Psi\colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ be a groupoid isomorphism satisfying $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$. Then $\widetilde{h} := \Psi^{(0)}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ is a continuous orbit equivalence with continuous cocycles $l_{\widetilde{\mathsf{X}}}, k_{\widetilde{\mathsf{X}}}\colon \widetilde{\mathsf{X}} \longrightarrow \mathbb{N}$ and $l_{\widetilde{\mathsf{Y}}}, k_{\widetilde{\mathsf{Y}}}\colon \widetilde{\mathsf{Y}} \longrightarrow \mathbb{N}$ given as in the proof of Theorem B.6.6 (ii) $\Longrightarrow$ (i). We will show that there are continuous maps $l_{\mathsf{X}}, k_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{N}$ and $l_{\mathsf{Y}}, k_{\mathsf{Y}}\colon \mathsf{Y} \longrightarrow \mathbb{N}$ which are continuous cocycles for h such that $l_{\widetilde{\mathsf{X}}} = l_{\mathsf{X}} \circ \pi_{\mathsf{X}}$, $k_{\widetilde{\mathsf{X}}} = k_{\mathsf{X}} \circ \pi_{\mathsf{X}}$, $l_{\widetilde{\mathsf{Y}}} = l_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$ and $k_{\widetilde{\mathsf{Y}}} = k_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$. By Proposition B.2.10, X and Y have dense sets of aperiodic points, so it suffices to show that $l_{\widetilde{\mathsf{X}}}$ and $k_{\widetilde{\mathsf{X}}}$ are constant on $\pi_{\mathsf{X}}^{-1}(x)$ for an aperiodic $x \in \mathsf{X}$.

Let $x \in \mathsf{X}$ be aperiodic and take $\widetilde{x}, \widetilde{x}' \in \pi_{\mathsf{X}}^{-1}(x)$. Set $c_{\widetilde{\mathsf{X}}} = l_{\widetilde{\mathsf{X}}} - k_{\widetilde{\mathsf{X}}}$ and $k := \max\{k_{\widetilde{\mathsf{X}}}(\widetilde{x}), k_{\widetilde{\mathsf{X}}}(\widetilde{x}')\}$. If x is isolated, then $\pi_{\mathsf{X}}^{-1}(x)$ is a singleton, so we may assume that x is not isolated. Since $\sigma_{\widetilde{\mathsf{Y}}}^{c_{\widetilde{\mathsf{X}}}(\widetilde{x})+k}(\widetilde{h}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}^{k}(\widetilde{h}(\sigma_{\mathsf{X}}(\widetilde{x})))$ and $\sigma_{\widetilde{\mathsf{Y}}}^{c_{\widetilde{\mathsf{X}}}(\widetilde{x}')+k}(\widetilde{h}(\widetilde{x}')) = \sigma_{\widetilde{\mathsf{Y}}}^{k}(\widetilde{h}(\sigma_{\mathsf{X}}(\widetilde{x}')))$, it follows that

$$\sigma_{\mathsf{Y}}^{c_{\widetilde{\mathsf{X}}}(\widetilde{x})+k}(h(x)) = \sigma_{\widetilde{\mathsf{Y}}}^{c_{\widetilde{\mathsf{X}}}(\widetilde{x}')+k}(h(x)).$$

By Proposition B.6.8, we know that $h(x)$ is aperiodic, so $c_{\widetilde{\mathsf{X}}}(\widetilde{x}) = c_{\widetilde{\mathsf{X}}}(\widetilde{x}')$.

Set

$$\widetilde{v} := \sigma_{\widetilde{\mathsf{Y}}}^{l_{\widetilde{\mathsf{X}}}(\widetilde{x}')}(\widetilde{h}(\widetilde{x})), \quad \widetilde{w} := \sigma_{\widetilde{\mathsf{Y}}}^{l_{\widetilde{\mathsf{X}}}(\widetilde{x}')-c_{\widetilde{\mathsf{X}}}(\widetilde{x})}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x}))).$$

Since $c_{\widetilde{\mathsf{X}}}(\widetilde{x}) = c_{\widetilde{\mathsf{X}}}(\widetilde{x}')$ we have $\widetilde{w} = \sigma_{\widetilde{\mathsf{Y}}}^{k_{\widetilde{\mathsf{X}}}(\widetilde{x}')}(\widetilde{h}(\sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})))$, and $\pi_{\mathsf{Y}}(\widetilde{v}) = \pi_{\mathsf{Y}}(\widetilde{w})$. The point x is aperiodic, so Lemma B.2.8 implies that $\widetilde{v} = \widetilde{w}$. By minimality in the definition of $l_{\widetilde{\mathsf{X}}}$, it follows that $l_{\widetilde{\mathsf{X}}}(\widetilde{x}) \leqslant l_{\widetilde{\mathsf{X}}}(\widetilde{x}')$. A symmetric argument shows that $l_{\widetilde{\mathsf{X}}}(\widetilde{x}') \leqslant l_{\widetilde{\mathsf{X}}}(\widetilde{x})$. Hence $l_{\widetilde{\mathsf{X}}}$ and $k_{\widetilde{\mathsf{X}}}$ are constant on $\pi_{\mathsf{X}}^{-1}(x)$. There are therefore cocycles $l_{\mathsf{X}}, k_{\mathsf{X}} \colon \mathsf{X} \longrightarrow \mathbb{N}$ for h which satisfy $l_{\widetilde{\mathsf{X}}} = l_{\mathsf{X}} \circ \pi_{\mathsf{X}}$ and $k_{\widetilde{\mathsf{X}}} = k_{\mathsf{X}} \circ \pi_{\mathsf{X}}$, and they are continuous by Lemma B.2.5. A similar argument shows that there are continuous cocycles $l_{\mathsf{Y}}, k_{\mathsf{Y}} \colon \mathsf{Y} \longrightarrow \mathbb{N}$ satisfying $l_{\widetilde{\mathsf{Y}}} = l_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$ and $k_{\widetilde{\mathsf{Y}}} = k_{\mathsf{Y}} \circ \pi_{\mathsf{Y}}$. Hence h is a continuous orbit equivalence.

(iii) $\iff$ (iv): This is [**26**, Theorem 8.2]. Note that if $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ is a *-isomorphism as in (iv), then $\Phi(\mathcal{D}_{\mathsf{X}}) = \mathcal{D}_{\mathsf{Y}}$ by Theorem B.3.3. ☐

Corollary B.6.11. *Let* X *and* Y *be one-sided shift spaces with no periodic points which are isolated in past equivalence. The following are equivalent:*

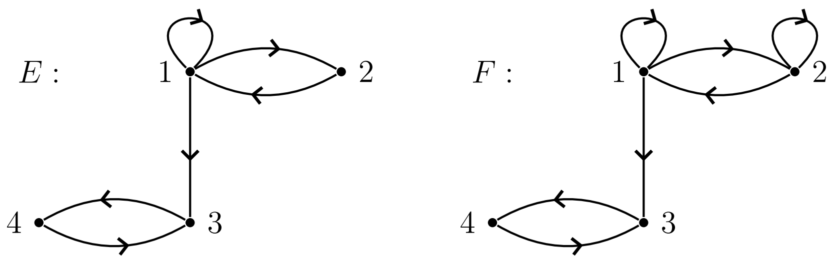
(i) *The systems* X *and* Y *are continuously orbit equivalent;*
(ii) *There is a groupoid isomorphism* $\Psi \colon \mathcal{G}_{\mathsf{X}} \longrightarrow \mathcal{G}_{\mathsf{Y}}$ *and a homeomorphism* $h \colon \mathsf{X} \longrightarrow \mathsf{Y}$ *such that* $h \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \Psi^{(0)}$*;*
(iii) *There is a* **-isomorphism* $\Phi \colon \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{Y}}$ *satisfying* $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$.

Remark B.6.12. In [**18**], the second-named author showed that for sofic shifts $\mathcal{O}_{\mathsf{X}}$ is *-isomorphic to a graph C*-algebra even in a diagonal preserving way. Corollary B.6.11 shows that if we consider the sofic shifts with no periodic points isolated in past equivalence, then $\mathcal{O}_{\mathsf{X}}$ remembers the continuously orbit equivalence class of X if we include the subalgebra $C(\mathsf{X})$ instead of the diagonal $\mathcal{D}_{\mathsf{X}}$.

B.6.1. Examples.

We consider a few examples.

Example B.6.13. Let X_E and X_F be the vertex shifts of the reducible graphs



Define a map $h \colon \mathsf{X}_E \longrightarrow \mathsf{X}_F$ by exchanging the word (21) with the word 2 except in the case $h(21(34)^\infty) = 21(34)^\infty$. Furthermore, $1(34)^\infty$ is fixed by h and $h((34)^\infty) = (43)^\infty$ and $h((43)^\infty) = (34)^\infty$. This is a homeomorphism. Consider the cocycles $k_E, l_E \colon \mathsf{X}_E \longrightarrow$

$\mathbb{N}$ and $k_F, l_F \colon \mathsf{X}_F \longrightarrow \mathbb{N}$ given by

$$\begin{cases} k_E|_{\mathsf{X}_E\setminus Z(2)} & = 0 \\ k_E|_{Z(2)} & = 1 \\ l_E(1(34)^\infty) & = 2 \\ l_E|_{Z(1)\setminus\{1(34)^\infty\}} & = 1 \\ l_E(21(34)^\infty) & = 2 \\ l_E|_{Z(2)\setminus\{21(34)^\infty\}} & = 1 \\ l_E((34)^\infty) & = 1 \\ l_E((43)^\infty) & = 1 \end{cases}, \qquad \begin{cases} k_F & = 0 \\ l_F(1(34)^\infty) & = 2 \\ l_F|_{Z(1)\setminus\{1(34)^\infty\}} & = 1 \\ l_F(21(34)^\infty) & = 1 \\ l_F|_{Z(2)\setminus\{21(34)^\infty\}} & = 2 \\ l_F((34)^\infty) & = 1 \\ l_F((43)^\infty) & = 1. \end{cases}$$

They are continuous and h is a continuous orbit equivalence with the specified cocycles. Hence X_A and X_B are continuously orbit equivalent.

We will show that no choice of continuous cocycles of h can be least period preserving on eventually periodic points. Let $k_E, l_E \colon \mathsf{X}_E \longrightarrow \mathbb{N}$ be any choice of continuous cocycles for h. Let $x = 1(34)^\infty \in \mathsf{X}_E$ and $z = \sigma_E(x)$. The computation

$$\sigma_F^{l_E(x)}(1(34)^\infty) = \sigma_F^{l_E(x)}(h(x)) = \sigma_F^{k_E(x)}(h(\sigma_E(x))) = \sigma_F^{k_E(x)}(43)^\infty$$

shows that $k_E(x)$ and $l_E(x)$ have the same parity. On the other hand,

$$\sigma_F^{l_A(z)}((43)^\infty) = \sigma_F^{l_A(z)}(h(z)) = \sigma_F^{k_A(z)}(h(\sigma_E(z))) = \sigma_F^{k_A(z)}(34)^\infty$$

shows that $k_E(z)$ and $l_E(z)$ have different parity. Then $l_E^{(2)}(x) - k_E^{(2)}(x)$ is odd while $\mathrm{lp}(x) = 2$.

Below we revisit an example of Matsumoto [**78**] of infinite and irreducible shifts of finite type that are continuously orbit equivalent. We show that they are *not* eventually conjugate.

EXAMPLE B.6.14. Let X be the full shift on the alphabet $\{1, 2\}$ and let Y be the golden mean shift determined by the single forbidden word $\{22\}$. Then X and Y are infinite and irreducible shifts of finite type which are continuously orbit equivalent, cf. [**78**, p. 213].

Suppose $h \colon \mathsf{X} \longrightarrow \mathsf{Y}$ is an eventual conjugacy and that $\ell \in \mathbb{N}$ is an integer in accordance with (B.4) and (B.5). Then both $\sigma_{\mathsf{Y}}^\ell(h(1^\infty))$ and $\sigma_{\mathsf{Y}}^\ell(h(2^\infty))$ are constant sequences in Y, so they are both equal to $1^\infty \in \mathsf{Y}$. However, then

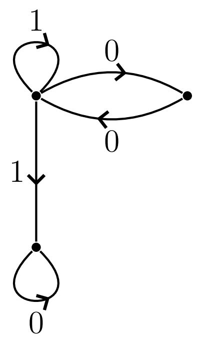
$$1^\infty = \sigma_{\mathsf{X}}^\ell(h^{-1}(\sigma_{\mathsf{Y}}^\ell(h(1^\infty)))) = \sigma_{\mathsf{X}}^\ell(h^{-1}(\sigma_{\mathsf{Y}}^\ell(h(2^\infty)))) = 2^\infty,$$

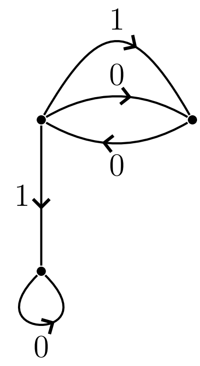
which cannot be the case. Therefore, X and Y are not eventually conjugate.

EXAMPLE B.6.15. Let $\mathsf{X} = \mathsf{X}_{\text{even}}$ and $\mathsf{Y} = \mathsf{Y}_{\text{odd}}$ be the even and the odd shift defined by the following sets of forbidden words

$$\mathscr{F}_{\text{even}} = \{ab^{2n+1}a : n \in \mathbb{N}\}, \qquad \mathscr{F}_{\text{odd}} = \{ab^{2n}a : n \in \mathbb{N}\},$$

respectively. The shift spaces are represented in the labeled graphs E and F below.

Even: 

Odd: 

Define a map $h\colon \mathsf{X} \longrightarrow \mathsf{Y}$ by exchanging the word 1 by the word (10). This is a homeomorphism. Furthermore, the cocycles $k_{\mathsf{X}}, l_{\mathsf{X}}\colon \mathsf{X} \longrightarrow \mathbb{N}$ and $k_{\mathsf{Y}}, l_{\mathsf{Y}}\colon \mathsf{Y} \longrightarrow \mathbb{N}$ given by

$$\begin{cases} k_{\mathsf{X}}|_{Z(0)} = 0, & l_{\mathsf{X}}|_{Z(0)} = 1, \\ k_{\mathsf{X}}|_{Z(1)} = 0, & l_{\mathsf{X}}|_{Z(1)} = 2 \end{cases}, \qquad \begin{cases} k_{\mathsf{Y}}|_{Z(0)} = 0, & l_{\mathsf{Y}}|_{Z(0)} = 1, \\ k_{\mathsf{Y}}|_{Z(1)} = 1, & l_{\mathsf{Y}}|_{Z(1)} = 1 \end{cases}$$

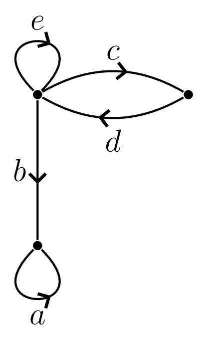
are continuous. Hence h is a continuous orbit equivalence and X and Y are continuously orbit equivalent. An argument similar to that of Example B.6.14 shows that X and Y are not one-sided eventually conjugate.

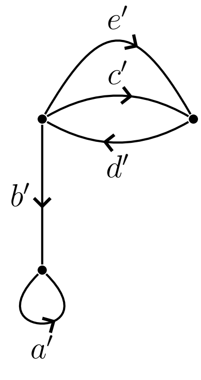
Observe that $h(0^\infty) = 0^\infty$ and $h(1^\infty) = (10)^\infty$ and

$$l_{\mathsf{X}}(0^\infty) - k_{\mathsf{X}}(0^\infty) = 1 = \mathrm{lp}(0^\infty),$$
$$l_{\mathsf{X}}(1^\infty) - k_{\mathsf{X}}(1^\infty) = 2 = \mathrm{lp}((10)^\infty),$$

so $(k_{\mathsf{X}}, l_{\mathsf{X}})$ is least period preserving. A similar computation shows that $(k_{\mathsf{Y}}, l_{\mathsf{Y}})$ is also least period preserving.

EXAMPLE B.6.16. Let X_E and X_F be the edge shifts determined by the reducible graphs

E : 

F : 

Define a map $h\colon \mathsf{X}_E \longrightarrow \mathsf{X}_F$ by exchanging any occurance of e by $e'd'$. This is a homeomorphism. The maps $k_E, l_E\colon \mathsf{X}_E \longrightarrow \mathbb{N}$ and $k_F, l_F\colon \mathsf{X}_F \longrightarrow \mathbb{N}$ given by

$$\begin{cases} k_E & = 0, \\ l_F|_{Z(a)\cup Z(c)\cup Z(d)} & = 1, \\ l_F|_{Z(b)\cup Z(e)} & = 2, \end{cases}, \begin{cases} k_F|_{Z(a')\cup Z(c')\cup Z(d')} & = 0, \\ k_F|_{Z(b')\cup Z(e')} & = 1, \\ l_F & = 1, \end{cases}$$

are continuous cocycles for h. Hence h is a continuous orbit equivalence and X_E and X_F are continuously orbit equivalent. A computation shows that (h, k_E, l_E, k_F, l_F) is least period preserving on periodic points but not on eventually periodic points.

In light of Example B.6.15 we can identify X_E and X_F with the covers $\widetilde{\mathsf{X}}_{\text{even}}$ and $\widetilde{\mathsf{Y}}_{\text{odd}}$, respectively, and the cocycles above are induced from the cocycles on the even and odd shifts. The maps $k_1, l_1, k_2, l_2 \colon \mathsf{X}_E \longrightarrow \mathbb{N}$ given by

$$\begin{cases} k_1 & = 0 \\ l_1|_{Z(a)} & = 0 \\ l_1|_{Z(b)\cup Z(c)\cup Z(d)} & = 1 \\ l_1|_{Z(e)} & = 2 \end{cases}, \quad \begin{cases} k_2 & = 0 \\ l_2|_{\mathsf{X}_E \setminus Z(e)} & = 1 \\ l_2|_{Z(e)} & = 2 \end{cases}$$

are continuous cocycles for h. Then (h, k_1, l_1, k_F, l_F) is not least period preserving and not constant on the preimages of π_{even} while (h, k_2, l_2, k_F, l_F) is least period preserving on all eventually periodic points but not constant on the preimages under π_{even}.

B.7. Two-sided conjugacy

In [**24**, Theorem 5.1], the second-named author and Rout show that two-sided subshifts of finite type Λ_{X} and Λ_{Y} are conjugate if and only if the groupoids $\mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ and $\mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ are isomorphic in a way which respects the canonical cocycle and if and only if $\mathcal{O}_{\mathsf{X}} \times \mathbb{K}$ and $\mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ are *-isomorphic in a way which intertwines the gauge actions suitably stabilized. In this section, we characterize when a pair of general two-sided shift spaces are conjugate in terms of isomorphism of the groupoids $\mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ and $\mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ and *-isomorphism of $\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K}$ and $\mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$.

Recall that if X is a one-sided shift space and σ_{X} is surjective, then the corresponding two-sided shift space Λ_{X} is constructed as the projective limit

$$\Lambda_{\mathsf{X}} = \varprojlim(\mathsf{X}, \sigma_{\mathsf{X}}).$$

We shall write elements of X as $x, y, z \cdots$ and elements of Λ_{X} as $\mathrm{x}, \mathrm{y}, \mathrm{z} \cdots$.

Lemma B.7.1. *Let X be a one-sided shift space and let $\widetilde{\mathsf{X}}$ be the associated cover. Then σ_{X} is surjective if and only if $\sigma_{\widetilde{\mathsf{X}}}$ is surjective.*

Proof. If $\sigma_{\widetilde{\mathsf{X}}}$ is surjective, then the relation $\sigma_{\mathsf{X}} \circ \pi_{\mathsf{X}} = \pi_{\mathsf{X}} \circ \sigma_{\widetilde{\mathsf{X}}}$ ensures that σ_{X} is surjective. On the other hand, suppose σ_{X} is surjective and let $\widetilde{x} \in \widetilde{\mathsf{X}}$. Take $x \in \mathsf{X}$ and integers $0 \leqslant r < s$ such that $\widetilde{x} \in U(x, r, s)$. Since σ_{X} is surjective, there exists $a \in \mathfrak{A}$ such that $ax \in \mathsf{X}$. We have

$$\widetilde{x} \in U(x, r, s) = U(\sigma_{\mathsf{X}}(ax), r, s) = \sigma_{\widetilde{\mathsf{X}}}(U(ax, r+1, s)).$$

In particular, we may pick $\widetilde{y} \in U(ax, r+1, s)$ such that $\sigma_{\widetilde{\mathsf{X}}}(\widetilde{y}) = \widetilde{x}$. □

Following [**25**], we let $\mathcal{R}$ be the full countable equivalence relation on $\mathbb{N} \times \mathbb{N}$. The product of $(n, m), (n', m') \in \mathcal{R}$ is defined exactly when $m = n'$ in which case

$$(n, m)(n', m') = (n, m').$$

Inversion is given as $(n,m)^{-1} = (m,n)$, and the source and range maps are

$$s(n,m) = m, \qquad r(n,m) = n,$$

respectively, for $(n,m) \in \mathcal{R}$.

Given a one-sided shift space X, we consider the product groupoid $\mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ whose unit space we shall identify with $\widetilde{\mathsf{X}} \times \mathbb{N}$ via the correspondence $((\widetilde{x},0,\widetilde{x}),(0,0)) \longmapsto (\widetilde{x},0)$. The canonical cocycle is the continuous map $\bar{c}_{\mathsf{X}} \colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathbb{Z}$ given by $\bar{c}_{\mathsf{X}}\big((\widetilde{x},k,\widetilde{y}),(n,m)\big) = k$, for $\big((\widetilde{x},k,\widetilde{y}),(n,m)\big) \in \mathcal{G}_{\mathsf{X}} \times \mathcal{R}$.

We start by describing two-sided conjugacy in terms of sliding block codes on the corresponding one-sided shift spaces. Recall that a sliding block code $\varphi \colon \mathsf{X} \longrightarrow \mathsf{Y}$ between one-sided shift spaces X and Y is a continuous map satisfying $\varphi \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \varphi$.

Definition B.7.2. Let X and Y be one-sided shift spaces and let $\varphi \colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a sliding block code. We say that φ is *almost injective* (with lag ℓ_1) if there exists $\ell_1 \in \mathbb{N}$ such that

$$\varphi(x) = \varphi(x') \implies \sigma_{\mathsf{X}}^{\ell_1}(x) = \sigma_{\mathsf{X}}^{\ell_1}(x'),$$

for every $x, x' \in \mathsf{X}$. We say that φ is *almost surjective* (with lag ℓ_2) if there exists $\ell_2 \in \mathbb{N}$ such that for each $y \in \mathsf{Y}$ there exists $x \in \mathsf{X}$ such that $\sigma_{\mathsf{Y}}^{\ell_2}(\varphi(x)) = \sigma_{\mathsf{Y}}^{\ell_2}(y)$.

Almost injective and almost surjective sliding block codes between covers is defined analogously.

Lemma B.7.3. *Let Λ_{X} and Λ_{Y} be two-sided subshifts. If Λ_{X} and Λ_{Y} are two-sided conjugate, then there is a surjective sliding block code $\varphi \colon \mathsf{X} \longrightarrow \mathsf{Y}$ which is almost injective. Conversely, if there exists a sliding block code $\varphi \colon \mathsf{X} \longrightarrow \mathsf{Y}$ which is almost injective and almost surjective, then Λ_{X} and Λ_{Y} are conjugate.*

Proof. If Λ_{X} and Λ_{Y} are two-sided conjugate, we may assume that there exist a two-sided conjugacy $H \colon \Lambda_{\mathsf{X}} \longrightarrow \Lambda_{\mathsf{Y}}$ and $\ell \in \mathbb{N}$ such that

$$\mathrm{x}_{[0,\infty)} = \mathrm{x}'_{[0,\infty)} \implies H(\mathrm{x})_{[0,\infty)} = H(\mathrm{x}')_{[0,\infty)},$$
$$\mathrm{y}_{[0,\infty)} = \mathrm{y}'_{[0,\infty)} \implies H^{-1}(\mathrm{y})_{[\ell,\infty)} = H^{-1}(\mathrm{y}')_{[\ell,\infty)}$$

for $\mathrm{x}, \mathrm{x}' \in \Lambda_{\mathsf{X}}$, $\mathrm{y}, \mathrm{y}' \in \Lambda_{\mathsf{Y}}$. Therefore, there is a well-defined map $\varphi \colon \mathsf{X} \longrightarrow \mathsf{Y}$ given by

$$\varphi(x) = H(\mathrm{x})_{[0,\infty)}, \tag{B.11}$$

for every $x \in \mathsf{X}$ and $\mathrm{x} \in \Lambda_{\mathsf{X}}$ with $x = \mathrm{x}_{[0,\infty)}$. The map φ is a surjective sliding block code. Furthermore, if $\varphi(x) = \varphi(x')$ then $\sigma_{\mathsf{X}}^{\ell}(x) = \sigma_{\mathsf{X}}^{\ell}(x')$ for $x, x' \in \mathsf{X}$.

Conversley, suppose $\varphi \colon \mathsf{X} \longrightarrow \mathsf{Y}$ is an almost injective and almost surjective sliding block code with lag ℓ. Define a map $h \colon \Lambda_{\mathsf{X}} \longrightarrow \Lambda_{\mathsf{Y}}$ by

$$h(\ldots, x_2, x_1, x_0) = (\ldots, \varphi(x_2), \varphi(x_1), \varphi(x_0)),$$

for $(\ldots, x_2, x_1, x_0) \in \Lambda_{\mathsf{X}}$. Note that $\sigma_{\mathsf{Y}}(\varphi(x_{i+1})) = \varphi(x_i)$ for $i \in \mathbb{N}$. Therefore h is a well-defined sliding block code. We will show that h is injective and surjective.

Suppose first that $(\dots,x_2,x_1,x_0),(\dots,x_2',x_1',x_0') \in \Lambda_{\mathsf{X}}$ and $\varphi(x_i) = \varphi(x_i')$ for every $i \in \mathbb{N}$. Then

$$x_i = \sigma_{\mathsf{X}}^{\ell}(x_{i+\ell}) = \sigma_{\mathsf{X}}^{\ell}(x_{i+\ell}') = x_i'$$

for $i \in \mathbb{N}$ so h is injective.

Now let $(\dots,y_2,y_1,y_0) \in \Lambda_{\mathsf{Y}}$ and choose $x,x' \in \mathsf{X}$ such that

$$\sigma_{\mathsf{Y}}^{\ell}(\varphi(x)) = \sigma_{\mathsf{X}}^{\ell}(y_{2\ell}) = y_\ell, \qquad \sigma_{\mathsf{Y}}^{\ell}(\varphi(x')) = \sigma_{\mathsf{X}}^{\ell}(y_{2\ell+1}) = y_{\ell+1}.$$

Note that

$$\varphi(\sigma_{\mathsf{X}}^{\ell+1}(x')) = y_\ell = \varphi(\sigma_{\mathsf{X}}^{\ell}(x))$$

so $\sigma_{\mathsf{X}}^{2\ell+1}(x') = \sigma_{\mathsf{X}}^{2\ell}(x)$ since φ is almost surjective with lag ℓ. Put $z_0 = \sigma_{\mathsf{X}}^{2\ell}(x)$ and $z_1 = \sigma_{\mathsf{X}}^{2\ell}(x')$ and observe that $\sigma_{\mathsf{X}}(z_1) = z_0$. Continuing this process inductively defines an sequence $(\dots,z_2,z_1,z_0) \in \Lambda_{\mathsf{X}}$ which is sent to $(\dots,y_2,y_1,y_0)$ via h. Hence h is surjective and thus a two-sided conjugacy. □

Next we lift surjective sliding block codes on one-sided shift spaces to surjective sliding block codes on the covers.

Lemma B.7.4. *Let X and Y be one-sided shift spaces and let $\varphi\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be a surjective sliding block code. Then there exists a surjective sliding block code $\widetilde{\varphi}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ satisfying $\varphi \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{\varphi}$.*

If, in addition, σ_{X} is surjective and φ is almost injective with lag ℓ, then $\widetilde{\varphi}$ is almost injective with lag ℓ.

Proof. Since φ is a sliding block code there exists an integer $K \in \mathbb{N}$ such that

$$x_{[0,r+K)} = x'_{[0,r+K)} \implies \varphi(x)_{[0,r)} = \varphi(x')_{[0,r)} \tag{B.12}$$

for $r \in \mathbb{N}$ and $x,x \in \mathsf{X}$. We want to show that

$$x \overset{r+K,s+K}{\sim} x' \implies \varphi(x) \overset{r,s}{\sim} \varphi(x'), \tag{B.13}$$

for $x,x' \in \mathsf{X}$ and integers $0 \leqslant r \leqslant s$.

Suppose $\nu\sigma_{\mathsf{Y}}^{r}(\varphi(x)) = \nu\varphi(\sigma_{\mathsf{X}}^{k}(x))$ where $\nu \in \mathsf{L}(\mathsf{Y})$ with $|\nu| \leqslant s$. We need to show that $\nu\sigma_{\mathsf{Y}}^{k}(\varphi(x')) \in \mathsf{Y}$. As φ is surjective and commutes with the shift, there exists a word $\mu \in \mathsf{L}(\mathsf{X})$ with $|\mu| = |\nu|$ such that $\mu\sigma_{\mathsf{X}}^{k}(x) \in \mathsf{X}$ and $\nu\varphi(\sigma_{\mathsf{X}}^{k}(x)) = \varphi(\mu\sigma_{\mathsf{X}}^{k}(x))$. By hypothesis, $\mu\sigma_{\mathsf{X}}^{k}(x') \in \mathsf{X}$ and we claim that $\varphi(\mu\sigma_{\mathsf{X}}^{k}(x')) = \nu\sigma_{\mathsf{Y}}^{k}(\varphi(x'))$. Indeed, $\mu\sigma_{\mathsf{X}}^{k}(x)_{[0,|\mu|+K)} = \mu\sigma_{\mathsf{X}}^{k}(x')_{[0,|\mu|+K)}$, so

$$\varphi(\mu\sigma_{\mathsf{X}}^{k}(x'))_{[0,|\nu|)} = \varphi(\mu\sigma_{\mathsf{X}}^{k}(x))_{[0,|\nu|)} = |\nu|$$

by the choice of K. This proves the claim.

Define $\widetilde{\varphi}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ by

$${}_r\widetilde{\varphi}(\widetilde{x})_s = \varphi({}_{r+K}x_{s+K}),$$

for $\widetilde{x} \in \widetilde{\mathsf{X}}$ and integers $0 \leqslant r \leqslant s$. It is straightforward to check that the induced map $\widetilde{\varphi}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ is a surjective sliding block code satisfying $\varphi \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{\varphi}$.

Suppose now that σ_{X} is surjective and that there is $\ell \in \mathbb{N}$ such that $\varphi(x) = \varphi(x')$ implies $\sigma_{\mathsf{X}}^{\ell}(x) = \sigma_{\mathsf{X}}^{\ell}(x')$ for all $x, x' \in \mathsf{X}$. Equivalently, there exists a surjective sliding block code $\rho\colon \mathsf{Y} \longrightarrow \mathsf{X}$ satisfying $\sigma_{\mathsf{X}}^{\ell} = \rho \circ \varphi$. An argument similar to the one above shows that there is an induced surjective sliding block code $\widetilde{\rho}\colon \widetilde{\mathsf{Y}} \longrightarrow \widetilde{\mathsf{X}}$ with $\rho \circ \pi_{\mathsf{Y}} = \pi_{\mathsf{X}} \circ \widetilde{\rho}$. It is straightforward to verify that $\sigma_{\widetilde{\mathsf{X}}}^{\ell} = \widetilde{\rho} \circ \widetilde{\varphi}$. Hence $\widetilde{\varphi}$ is almost injective. □

We now arrive at the main theorem of this section which characterizes two-sided conjugacy of general shift spaces. The proof uses ideas of [**24**]. Let $\pi_{\mathsf{X}\times\mathbb{N}}\colon \widetilde{\mathsf{X}} \times \mathbb{N} \longrightarrow \mathsf{X} \times \mathbb{N}$ be the map $\pi_{\mathsf{X}\times\mathbb{N}}(\widetilde{x}, n) = (\pi_{\mathsf{X}}(\widetilde{x}), n)$, for $(\widetilde{x}, n) \in \widetilde{\mathsf{X}} \times \mathbb{N}$.

THEOREM B.7.5. *Let Λ_{X} and Λ_{Y} be two-sided shift spaces. The following are equivalent:*

(i) *there is a sliding block code $\varphi\colon \mathsf{X} \longrightarrow \mathsf{Y}$ which is almost injective and almost surjective;*
(ii) *there is a two-sided conjugacy $h\colon \Lambda_{\mathsf{X}} \longrightarrow \Lambda_{\mathsf{Y}}$;*
(iii) *there are a groupoid isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ and a homeomorphism $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ satisfying $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Phi^{(0)}$ and*
$$\bar{c}_{\mathsf{X}} = \bar{c}_{\mathsf{Y}} \circ \Psi;$$
(iv) *there is a *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ satisfying $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$ and*
$$\Phi \circ (\gamma^{\mathsf{X}} \otimes \mathrm{id}) = (\gamma^{\mathsf{Y}} \otimes \mathrm{id}) \circ \Phi. \tag{B.14}$$

PROOF. The equivalence (i) $\Longleftrightarrow$ (ii) is Lemma B.7.3.

(ii) $\Longrightarrow$ (iii): Let $H\colon \Lambda_{\mathsf{X}} \longrightarrow \Lambda_{\mathsf{Y}}$ be a conjugacy as in the proof of Lemma B.7.3 and let $\varphi\colon \mathsf{X} \longrightarrow \mathsf{Y}$ be the surjective and almost injective sliding block code of (B.11). By Lemma B.7.4 there exists a surjective and almost injective sliding block code $\widetilde{\varphi}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ satisfying $\varphi \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{\varphi}$. Since φ is continuous, there exists $L \in \mathbb{N}$ such that
$$x_{[0,L)} = x'_{[0,L)} \implies \varphi(x)_{[0,\ell)} = \varphi(x')_{[0,\ell)},$$
for $x, x' \in \mathsf{X}$. Define an equivalence relation $\sim$ on words of length L in the following way: Two words $\mu, \nu \in \mathsf{L}_L(\mathsf{X})$ are $\sim$-equivalent, if there are $x \in Z(\mu)$ and $x' \in Z(\nu)$ such that $\varphi(x) = \varphi(x')$. Then $\varphi(x) = \varphi(x')$ if and only if $\sigma_{\mathsf{X}}^{\ell}(x) = \sigma_{\mathsf{X}}^{\ell}(x')$ and $x_{[0,L)} \sim x'_{[0,L)}$. For every $\sim$-equivalence class $[\mu] = \{\nu \in \mathsf{X} \mid \mu \sim \nu\}$, fix a partition
$$\mathbb{N} = \coprod_{\nu \in [\mu]} \mathbb{N}_\nu$$
and bijections $f_\nu\colon \mathbb{N}_\nu \longrightarrow \mathbb{N}$. Define $\omega\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathbb{N}$ by
$$\omega(x, n) = f^{-1}_{x_{[0,L)}}(n),$$
for $(x, n) \in \mathsf{X} \times \mathbb{N}$. Then $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ given by
$$\psi(x, n) = (\varphi(x), \omega(x, n)),$$
for $(x, n) \in \mathsf{X} \times \mathbb{N}$, is a homeomorphism. Furthermore, the map $\Phi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ given by
$$\Phi\big((\widetilde{x}, k), n, (\widetilde{y}, l)\big) = \big((\widetilde{\varphi}(\widetilde{x}), \omega(\pi_{\mathsf{X}}(\widetilde{x}), k)), n, (\widetilde{\varphi}(\widetilde{y}), \omega(\pi_{\mathsf{X}}(\widetilde{y}), l))\big)$$

for $\big((\widetilde{x},k),n,(\widetilde{y},l)\big) \in \mathcal{G}_{\mathsf{X}} \times \mathcal{R}$, is a groupoid isomorphism satisfying $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Phi^{(0)}$.

(iii) $\Longrightarrow$ (i): Suppose $\Phi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ is a groupoid isomorphism satisfying the hypotheses of (iii). Define a map $\widetilde{\kappa}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ by $\Phi^{(0)}(\widetilde{x},0) = (\widetilde{\kappa}(\widetilde{x}),n)$ for $\widetilde{x} \in \widetilde{\mathsf{X}}$ and some $n \in \mathbb{N}$. Then $\widetilde{\kappa}$ is well-defined and continuous since $\Phi^{(0)}$ is continuous. By an argument similar to one in the proof of [**24**, Theorem 5.1], there exists $L \in \mathbb{N}$ such that $\widetilde{\varphi} := \sigma_{\widetilde{\mathsf{Y}}}^{L} \circ \widetilde{\kappa}$ is a sliding block code which is almost injective and almost surjective, say with lag ℓ. Define also $\kappa\colon \mathsf{X} \longrightarrow \mathsf{Y}$ by $\psi(x,0) = (\varphi(x),m)$ for $x \in \mathsf{X}$ and some $m \in \mathbb{N}$. Then κ is continuous and $\kappa \circ \pi_{\mathsf{X}} = \pi_{\mathsf{Y}} \circ \widetilde{\kappa}$. It follows that $\varphi := \sigma_{\mathsf{Y}}^{L} \circ \kappa\colon \mathsf{X} \longrightarrow \mathsf{Y}$ is a sliding block code.

Let $y \in \mathsf{Y}$ and choose $\widetilde{y} \in \pi_{\mathsf{Y}}^{-1}(y)$. Pick $\widetilde{x} \in \widetilde{\mathsf{X}}$ such that $\sigma_{\widetilde{\mathsf{Y}}}^{\ell}(\widetilde{\varphi}(\widetilde{x})) = \sigma_{\widetilde{\mathsf{Y}}}^{\ell}(\widetilde{y})$. If $x = \pi_{\mathsf{X}}(\widetilde{x})$, then $\sigma_{\mathsf{Y}}^{\ell}(\varphi(x)) = \sigma_{\mathsf{Y}}^{\ell}(y)$ and φ is almost surjective.

In order to see that φ is almost injective, choose distinct $x, x' \in \mathsf{X}$ such that $y := \varphi(x) = \varphi(x')$. Choose distinct $n, n' \in \mathbb{N}$ such that $\psi(x,0) = (y,n)$ and $\psi(x',0) = (y,n')$ and pick $\widetilde{y} \in \pi_{\mathsf{Y}}^{-1}(y)$. Since $\Phi^{(0)}$ is a homeomorphism, there are unique and distinct $\widetilde{x}, \widetilde{x'} \in \mathsf{X}$ such that $\Phi^{(0)}(\widetilde{x},0) = (\widetilde{y},n)$ and $\Phi^{(0)}(\widetilde{x}',0) = (\widetilde{y},n')$. It follows that $\sigma_{\widetilde{\mathsf{X}}}^{\ell}(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^{\ell}(\widetilde{x}')$ since $\widetilde{\varphi}$ is almost injective. Since $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Phi^{(0)}$, we have $\pi_{\mathsf{X}}(\widetilde{x}) = x$ and $\pi_{\mathsf{X}}(\widetilde{x}') = x'$. Hence $\sigma_{\mathsf{X}}^{\ell}(x) = \sigma_{\mathsf{X}}^{\ell}(x')$ and φ is almost injective.

(iii) $\Longrightarrow$ (iv): A groupoid isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ induces a *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ satisfying $\Phi(\mathcal{D}_{\mathsf{X}} \otimes c_0) = \mathcal{D}_{\mathsf{Y}} \otimes c_0$. Since $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$, we also have $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$. The relation $\bar{c}_{\mathsf{X}} = \bar{c}_{\mathsf{Y}} \circ \Phi$ ensures that (B.14) is satisfied.

(iv) $\Longrightarrow$ (ii): By Corollary B.3.4, $\Phi(\mathcal{D}_{\mathsf{X}} \otimes c_0) = \mathcal{D}_{\mathsf{Y}} \otimes c_0$. From [**26**, Theorem 8.10] there is a groupoid isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ satisfying $\Phi(f) = f \circ \Psi^{-1} \in \mathcal{D}_{\mathsf{Y}} \otimes c_0$ for $f \in \mathcal{D}_{\mathsf{X}} \otimes c_0$, and $\bar{c}_{\mathsf{X}} = \bar{c}_{\mathsf{Y}} \circ \Psi$. Since $\Phi(g) = g \circ \widetilde{\psi}^{-1} \in C(\mathsf{Y}) \otimes c_0$ for $g \in C(\mathsf{X}) \otimes c_0$, there is a homeomorphism $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ such that $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$. ☐

If Λ_A and Λ_B are the two-sided subshifts associated to finite square $\{0,1\}$-matrices with no zero rows and no zero columns, then we recover [**24**, Corollary 5.2]. See also [**83**, Theorem 1.5].

Corollary B.7.6. *Let Λ_{X} and Λ_{Y} be two-sided shift spaces. The following are equivalent:*

(i) *the two-sided subshifts Λ_{X} and Λ_{Y} are two-sided conjugate;*
(ii) *there is a groupoid isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ and a homeomorphism $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ satisfying $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Phi^{(0)}$ and $\bar{c}_{\mathsf{X}} = \bar{c}_{\mathsf{Y}} \circ \Psi$;*
(iii) *there is a *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ satisfying $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$ and $\Phi \circ (\gamma^{\mathsf{X}} \otimes \mathrm{id}) = (\gamma^{\mathsf{Y}} \otimes \mathrm{id}) \circ \Phi$.*

B.8. Flow equivalence

It is proven in [**22**, Corollary 6.3] (see also [**84**, Theorem 2.3]) that if Λ_{X} and Λ_{Y} are two-sided subshifts of finite type, then Λ_{X} and Λ_{Y} are flow equivalent if and only if

$\mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ and $\mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ are isomorphic, and if and only if there is a diagonal preserving *-isomorphism $\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$. In this section, we shall for general shift spaces X and Y look at the relationship between flow equivalence of Λ_{X} and Λ_{Y}, isomorphism of $\mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ and $\mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$, and diagonal preserving *-isomorphism between $\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K}$ and $\mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$.

The *ordered cohomology* [**84**] of X is the group

$$H^{\mathsf{X}} = C(\mathsf{X}, \mathbb{Z})/\{f - f \circ \sigma_{\mathsf{X}} : f \in C(\mathsf{X}, \mathbb{Z})\},$$

with the positive cone

$$H^{\mathsf{X}}_{+} = \{[f] \in H^{\mathsf{X}} \mid f \geqslant 0\}.$$

The ordered cohomology of the cover $\widetilde{\mathsf{X}}$ is defined analogously. An isomorphism of cohomology groups is *positive* if it maps the positive cone onto the positive cone, and two maps $f, f' \in C(\mathsf{X}, \mathbb{Z})$ are *cohomologous* if $[f] = [f']$ in H^{X}.

Remark B.8.1. Recall that $B^1(\mathcal{G}_{\mathsf{X}})$ is the collection of groupoid homomorphisms from $\mathcal{G}_{\mathsf{X}}$ to $\mathbb{Z}$. The first cohomology group of $\mathcal{G}_{\mathsf{X}}$ is the group

$$H^1(\mathcal{G}_{\mathsf{X}}) = B^1(\mathcal{G}_{\mathsf{X}})/\{\partial(f) \mid f \in C(\mathcal{G}_{\mathsf{X}}^{(0)}, \mathbb{Z})\},$$

where $\partial \colon C(\mathcal{G}_{\mathsf{X}}^{(0)}, \mathbb{Z}) \longrightarrow B^1(\mathcal{G}_{\mathsf{X}})$ is $\partial(f)(\gamma) = f(r(\gamma)) - f(s(\gamma))$, for $f \in C(\mathcal{G}_{\mathsf{X}}^{(0)}, \mathbb{Z})$ and $\gamma \in \mathcal{G}_{\mathsf{X}}$, cf. [**84**]. There is a canonical isomorphism $\Theta \colon H^1(\mathcal{G}_{\mathsf{X}}) \longrightarrow H^{\widetilde{\mathsf{X}}}$ given by $\Theta([f]) = [g]$, where

$$g(\widetilde{x}) = f(\widetilde{x}, 1, \sigma_{\widetilde{\mathsf{X}}}(\widetilde{x})),$$

for $\widetilde{x} \in \widetilde{\mathsf{X}}$, cf. [**22**, Proposition 4,7].

The factor map $\pi_{\mathsf{X}} \colon \widetilde{\mathsf{X}} \longrightarrow \mathsf{X}$ induces a well-defined injective map $\pi_{\mathsf{X}}^* \colon H^{\mathsf{X}} \longrightarrow H^{\widetilde{\mathsf{X}}}$ given by $\pi_{\mathsf{X}}^*([f]) = [f \circ \pi_{\mathsf{X}}]$, for $f \in C(\mathsf{X}, \mathbb{Z})$. Note $\pi_{\mathsf{X}}^*(H^{\mathsf{X}}_{+}) \subseteq H^{\widetilde{\mathsf{X}}}_{+}$ and $\pi_{\mathsf{X}}^*([1_{\mathsf{X}}]) = 1_{\widetilde{\mathsf{X}}}$. The ordered cohomology $(H^{\Lambda_{\mathsf{X}}}, H^{\Lambda_{\mathsf{X}}}_{+})$ of a two-sided subshift Λ_{X} is defined analogously, and there is a canonical isomorphism $(H^{\mathsf{X}}, H^{\mathsf{X}}_{+}) \cong (H^{\Lambda_{\mathsf{X}}}, H^{\Lambda_{\mathsf{X}}}_{+})$. This was shown in [**84**, Lemma 3.1] for infinite irreducible shifts of finite type but as noted in [**22**, Section 2.5] the proof holds for general shifts.

If X is a one-sided shift space, then the *stabilization of* X is the space $\mathsf{X} \times \mathbb{N}$ with the shift operation $S_{\mathsf{X}} \colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{X} \times \mathbb{N}$ given by

$$S_{\mathsf{X}}(x, n) = \begin{cases} (x, n-1) & \text{if } n > 0, \\ (\sigma_{\mathsf{X}}(x), 0) & \text{if } n = 0, \end{cases}$$

for $(x, n) \in \mathsf{X} \times \mathbb{N}$. We define $S_{\widetilde{\mathsf{X}}} \colon \widetilde{\mathsf{X}} \times \mathbb{N} \longrightarrow \widetilde{\mathsf{X}} \times \mathbb{N}$ in a similar way.

The ordered cohomology for the stabilized system is the group

$$H^{\mathsf{X} \times \mathbb{N}} = C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})/\{f - f \circ S_{\mathsf{X}} : f \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})\},$$

with the positive cone

$$H^{\mathsf{X} \times \mathbb{N}}_{+} = \{[f] \in H^{\mathsf{X} \times \mathbb{N}} \mid f \geqslant 0\}.$$

The ordered cohomology is stable in the following sense.

LEMMA B.8.2. *Let* X *be a one-sided shift space and let* $\iota_0\colon \mathsf{X} \longrightarrow \mathsf{X}\times\mathbb{N}$ *be the inclusion given by* $\iota_0(x) = (x,0)$*. There is a surjective homomorphism* $\iota_0^*\colon C(\mathsf{X}\times\mathbb{N},\mathbb{Z}) \longrightarrow C(\mathsf{X},\mathbb{Z})$ *defined by* $\iota_0^*(\xi)(x) = \xi(x,0)$*, and an isomorphism* $H(\iota_0)\colon H^{\mathsf{X}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}}$ *such that* $H(\iota_0)([\xi]) = [\iota_0^*(\xi)]$*. Moreover,* $H(\iota_0)(H_+^{\mathsf{X}\times\mathbb{N}}) = H_+^{\mathsf{X}}$ *and* $H(\iota_0)([1_{\mathsf{X}\times\mathbb{N}}]) = [1_{\mathsf{X}}]$*.*

PROOF. It is straightforward to check that $\iota_0^*\colon C(\mathsf{X}\times\mathbb{N},\mathbb{Z}) \longrightarrow C(\mathsf{X},\mathbb{Z})$ is a surjective homomorphism. Since ι_0 is a sliding block code, there is a well-defined surjective map $H(\iota_0)\colon H^{\mathsf{X}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}}$ given by $H(\iota_0)([\xi]) = [\iota_0^*(\xi)]$. Any class $[\xi] \in H^{\mathsf{X}\times\mathbb{N}}$ can be represented by a map $\xi \in C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$ which is supported on $\mathsf{X}\times\{0\}$. If $\xi(x,0) = b(x) - b(\sigma_{\mathsf{X}}(x))$, for some $b \in C(\mathsf{X},\mathbb{Z})$ and $x \in \mathsf{X}$, we may take $\eta \in C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$ supported on $\mathsf{X}\times\{0\}$ such that $\eta(x,0) = b(x)$. Then $\xi = \eta - \eta\circ S_{\mathsf{X}}$ on $\mathsf{X}\times\{0\}$, so $H(\iota_0)$ is injective.

It is clear the $\iota_0^*(\xi) \geqslant 0$ if $\xi \geqslant 0$. Conversely, let $g \in C(\mathsf{X},\mathbb{Z})$ and assume that $g \geqslant 0$. Take $\eta \in C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$ supported on $\mathsf{X}\times\{0\}$ such that $\xi(x,0) = g(x)$ for $x \in \mathsf{X}$, and note that $\iota_0^*(\xi) = g$ and $\xi \geqslant 0$. Hence $H(\iota_0)(H_+^{\mathsf{X}\times\mathbb{N}}) = H_+^{\mathsf{X}}$. Finally, $\iota_0^*(1_{\mathsf{X}\times\mathbb{N}}) = 1_{\mathsf{X}}$. □

We will write an element of $\mathcal{G}_{\mathsf{X}}\times\mathcal{R}$ as $\big((\widetilde{x},k),n,(\widetilde{y},l)\big)$ instead of $\big((\widetilde{x},n,\widetilde{y}),(k,l)\big)$, where $(\widetilde{x},k),(\widetilde{y},l) \in \widetilde{\mathsf{X}}\times\mathbb{N}$ and $\sigma_{\widetilde{\mathsf{X}}}^j(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^i(\widetilde{y})$ for some $i,j \in \mathbb{N}$ with $n = j-i$. We then have that

$$\mathcal{G}_{\mathsf{X}}\times\mathcal{R} = \{\big((\widetilde{x},k),n,(\widetilde{y},l)\big) \mid \exists i,j\in\mathbb{N} : n = j-i,\ S_{\widetilde{\mathsf{X}}}^{k+j}(\widetilde{x},k) = S_{\widetilde{\mathsf{X}}}^{l+i}(\widetilde{y},l),\ \sigma_{\widetilde{\mathsf{X}}}^j(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^i(\widetilde{y})\}.$$

Let $\pi_{\mathsf{X}\times\mathbb{N}}\colon \widetilde{\mathsf{X}}\times\mathbb{N} \longrightarrow \mathsf{X}\times\mathbb{N}$ be the map defined by $\pi_{\mathsf{X}\times\mathbb{N}}(\widetilde{x},n) = (\pi_{\mathsf{X}}(\widetilde{x}),n)$, for $(\widetilde{x},n) \in \widetilde{\mathsf{X}}\times\mathbb{N}$. There is an injective homomorphism $\kappa_{\mathsf{X}\times\mathbb{N}}\colon C(\mathsf{X}\times\mathbb{N},\mathbb{Z}) \longrightarrow B^1(\mathcal{G}_{\mathsf{X}}\times\mathcal{R})$ defined by

$$\kappa_{\mathsf{X}\times\mathbb{N}}(f)\big((\widetilde{x},k),n,(\widetilde{y},l)\big) = \sum_{r=0}^{j+k} f\big(\pi_{\mathsf{X}\times\mathbb{N}}(S_{\widetilde{\mathsf{X}}}^r(\widetilde{x},k))\big) - \sum_{r=0}^{i+l} f\big(\pi_{\mathsf{X}\times\mathbb{N}}(S_{\widetilde{\mathsf{X}}}^r(\widetilde{y},l))\big)$$

where $i,j \in \mathbb{N}$ are such that $\sigma_{\widetilde{\mathsf{X}}}^j(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^i(\widetilde{y})$ and $n = j-i$. In particular, $\kappa_{\mathsf{X}\times\mathbb{N}}(f)\colon \mathcal{G}_{\mathsf{X}}\times\mathcal{R} \longrightarrow \mathbb{Z}$ it the unique cocycle satisfying

$$\kappa_{\mathsf{X}\times\mathbb{N}}(f)\big((\widetilde{x},k),1,S_{\widetilde{\mathsf{X}}}(\widetilde{x},k)\big) = f\big(\pi_{\mathsf{X}\times\mathbb{N}}(\widetilde{x},k)\big),$$

for $(\widetilde{x},k) \in \widetilde{\mathsf{X}}\times\mathbb{N}$.

If Λ_{X} and Λ_{Y} are conjugate subshifts, then they have isomorphic ordered cohomology. We give a one-sided decription below.

LEMMA B.8.3. *Let* Λ_{X} *and* Λ_{Y} *be two-sided subshifts that are conjugate. Then there exist*

(i)
- *a surjective and almost injective sliding block code* $\varphi\colon \mathsf{X} \longrightarrow \mathsf{Y}$ *and an injective homomorphism* $\varphi^*\colon C(\mathsf{Y},\mathbb{Z}) \longrightarrow C(\mathsf{X},\mathbb{Z})$ *given by* $\varphi^*(g) = g\circ\varphi$*;*
- *a positive isomorphism* $H(\varphi)\colon H^{\mathsf{Y}} \longrightarrow H^{\mathsf{X}}$ *satisfying* $H(\varphi)([1_{\mathsf{Y}}]) = [1_{\mathsf{X}}]$ *and* $H(\varphi)[g] = [g\circ\varphi]$ *for* $g \in C(\mathsf{Y},\mathbb{Z})$*;*

(ii) *a groupoid isomorphism* $\Psi\colon \mathcal{G}_{\mathsf{X}}\times\mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}}\times\mathcal{R}$ *and a homeomorphism* $\psi\colon \mathsf{X}\times\mathbb{N} \longrightarrow \mathsf{Y}\times\mathbb{N}$ *satisfying* $\psi\circ\pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}}\circ\Psi^{(0)}$*;*

(iii)
- *a homomorphism* $\psi^*\colon C(\mathsf{Y}\times\mathbb{N},\mathbb{Z}) \longrightarrow C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$ *such that* $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta)) = \kappa_{\mathsf{Y}\times\mathbb{N}}(\eta)\circ\Psi$ *for* $\eta \in C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$*,*

- *a homomorphism* $\psi^{\#}\colon C(\mathsf{X}\times\mathbb{N},\mathbb{Z}) \longrightarrow C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$ *such that* $\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^{\#}(\zeta)) = \kappa_{\mathsf{X}\times\mathbb{N}}(\zeta)\circ\Psi^{-1}$ *for* $\zeta \in C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$*; and*
- *a positive isomorphism* $H(\psi)\colon H^{\mathsf{Y}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ *such that* $H(\psi)([\eta]) = [\psi^*(\eta)]$ *for* $\eta \in C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$, $H(\psi)^{-1}([\zeta]) = [\psi^{\#}(\zeta)]$ *for* $\zeta \in C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$ *and* $H(\varphi)\circ H(\iota_0) = H(\iota_0)\circ H(\psi)$.

Proof. (i): Since Λ_{X} and Λ_{Y} are conjugate there is a surjective and almost injective sliding block code $\varphi\colon \mathsf{X} \longrightarrow \mathsf{Y}$, cf. Lemma B.7.3. The map $\varphi^*\colon C(\mathsf{Y},\mathbb{Z}) \longrightarrow C(\mathsf{X},\mathbb{Z})$ given by $\varphi^*(g) = g\circ\varphi$ for $g \in C(\mathsf{Y},\mathbb{Z})$ is an injective homomorphism.

Since φ is a sliding block code the map $H(\varphi)\colon H^{\mathsf{Y}} \longrightarrow H^{\mathsf{X}}$ given by $H(\varphi)[g] = [g\circ\varphi]$ is well-defined and injective. In order to see that $H(\varphi)$ is surjective, recall that φ is almost injective and pick $\ell \in \mathbb{N}$ accordingly. Take $f \in C(\mathsf{X},\mathbb{Z})$. Define a map $g\colon \mathsf{Y} \longrightarrow \mathbb{N}$ by $g(y) = f\circ\sigma_{\mathsf{X}}^{\ell}(\varphi^{-1}(y))$, for $y \in \mathsf{Y}$. Since φ is almost injective with lag ℓ this is well-defined and g is continuous, and $H(\varphi)[g] = [f\circ\sigma_{\mathsf{X}}^{\ell}] = [f]$. Hence $H(\varphi)$ is surjective. It is straightforward to verify that $H(\varphi)(H_+^{\mathsf{Y}}) = H_+^{\mathsf{X}}$ and $H(\varphi)([1_{\mathsf{Y}}]) = [1_{\mathsf{X}}]$.

(ii): By (the proof of) Theorem B.7.5, there is a surjective sliding block code $\widetilde{\varphi}\colon \widetilde{\mathsf{X}} \longrightarrow \widetilde{\mathsf{Y}}$ such that $\varphi\circ\pi_{\mathsf{X}} = \pi_{\mathsf{Y}}\circ\widetilde{\varphi}$ and a map $\omega\colon \mathsf{X}\times\mathbb{N} \longrightarrow \mathbb{N}$ such that the map $\Psi\colon \mathcal{G}_{\mathsf{X}}\times\mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}}\times\mathcal{R}$ defined by

$$\Psi\big((\widetilde{x},k),n,(\widetilde{y},l)\big) = \big((\widetilde{\varphi}(\widetilde{x}),\omega(\pi_{\mathsf{X}}(\widetilde{x}),k)),n,(\widetilde{\varphi}(\widetilde{y}),\omega(\pi_{\mathsf{X}}(\widetilde{y}),l))\big) \tag{B.15}$$

is a groupoid isomorphism, and the map $\psi\colon \mathsf{X}\times\mathbb{N} \longrightarrow \mathsf{Y}\times\mathbb{N}$ defined by

$$\psi(x,n) = (\varphi(x),\omega(x,n)),$$

is a homeomorphism satisfying $\psi\circ\pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}}\circ\Psi^{(0)}$.

(iii): Choose $\ell \in \mathbb{N}$ such that $\varphi(x) = \varphi(x') \implies \sigma_{\mathsf{X}}^{\ell}(x) = \sigma_{\mathsf{X}}^{\ell}(x')$ and let $\omega\colon \mathsf{X}\times\mathbb{N} \longrightarrow \mathbb{N}$ be the map from (B.15). Define $\omega'\colon \mathsf{Y}\times\mathbb{N} \longrightarrow \mathbb{N}$ by letting $\omega'(y,n) = m$ where $\psi^{-1}(y,n) = (x,m)$ for some $x \in \mathsf{X}$. Since ψ is a homeomorphism, ω' is continuous. Let $\psi^*\colon C(\mathsf{Y}\times\mathbb{N},\mathbb{Z}) \longrightarrow C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$ be the map defined by

$$\psi^*(\eta)(x,n) = \sum_{r=0}^{\omega(x,n)} \eta(S_{\mathsf{Y}}^{r}(\psi(x,n))) - \sum_{r=0}^{\omega(x,n-1)} \eta(S_{\mathsf{Y}}^{r}(\psi(x,n-1))),$$

for $\eta \in C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$ and $(x,n) \in \mathsf{X}\times\mathbb{N}$ with $n \geq 1$, and

$$\psi^*(\eta)(x,0) = \sum_{r=0}^{\omega(x,0)+1} \eta(S_{\mathsf{Y}}^{r}(\psi(x,0))) - \sum_{r=0}^{\omega(\sigma_{\mathsf{X}}(x),0)} \eta(S_{\mathsf{Y}}^{r}(\psi(\sigma_{\mathsf{X}}(x),0))),$$

for $\eta \in C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$ and $x \in \mathsf{X}$. Let $\psi^{\#}\colon C(\mathsf{X}\times\mathbb{N},\mathbb{Z}) \longrightarrow C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$ be the map defined by

$$\psi^{\#}(\zeta)(y,n) = \sum_{r=0}^{\omega'(y,n)+\ell} \zeta(S_{\mathsf{X}}^{r}(\psi^{-1}(y,n))) - \sum_{r=0}^{\omega'(y,n-1)+\ell} \zeta(S_{\mathsf{X}}^{r}(\psi^{-1}(y,n-1))),$$

for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $(y,n) \in \mathsf{Y} \times \mathbb{N}$ with $n \geq 1$, and

$$\psi^{\#}(\zeta)(y,0) = \sum_{r=0}^{\omega'(y,0)+\ell+1} \zeta(S_{\mathsf{X}}^r(\psi^{-1}(y,0))) - \sum_{r=0}^{\omega'(\sigma_{\mathsf{Y}}(y),0)+\ell} \zeta(S_{\mathsf{X}}^r(\psi^{-1}(\sigma_{\mathsf{Y}}(y),0))),$$

for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $y \in \mathsf{Y}$. It is straightforward to check that ψ^* and $\psi^{\#}$ are homomorphisms, and that $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta)) = \kappa_{\mathsf{Y}\times\mathbb{N}}(\eta) \circ \Psi$ for $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$, and $\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^{\#}(\zeta)) = \kappa_{\mathsf{X}\times\mathbb{N}}(\zeta) \circ \Psi^{-1}$ for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$.

Let $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ and observe that

$$\psi^*(\eta - \eta \circ S_{\mathsf{Y}})(x,n) = \eta(\omega(x,n)) - \eta(\omega(x,n-1)) = (\eta - \eta \circ S_{\mathsf{X}})(x,n)$$

for $(x,n) \in \mathsf{X} \times \mathbb{N}$ with $n \geq 1$, and

$$\psi^*(\eta - \eta \circ S_{\mathsf{Y}})(x,0) = \eta(\omega(x,0)) - \eta(\omega(\sigma_{\mathsf{X}}(x),0)) = (\eta - \eta \circ S_{\mathsf{X}})(x,0)$$

for $x \in \mathsf{X}$. Hence ψ^* induces a map $H(\psi)\colon H^{\mathsf{Y}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ given by $H(\psi)([\eta]) = [\psi^*(\eta)]$ for $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$.

Suppose $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ is supported on $\mathsf{Y} \times \{0\}$. Then

$$\iota_0^*(\psi^*(\eta))(x) = \eta(\varphi(x),0) = \varphi^*(\iota_0^*(\eta))(x)$$

for $x \in \mathsf{X}$. Since any element in $H^{\mathsf{Y}\times\mathbb{N}}$ can be represented by a map $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ which is supported on $\mathsf{Y} \times \{0\}$, it follows that $H(\varphi) \circ H(\iota_0) = H(\iota_0) \circ H(\psi)$. Therefore, $H(\psi)$ is a positive isomorphism.

Suppose $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ is is supported on $\mathsf{Y} \times \{0\}$. Then

$$\phi^*(\iota_0^*(\psi^{\#}(\zeta)))(x) = \psi^{\#}(\zeta)(\varphi(x),0) = \zeta(x,0) = \iota_0^*(\zeta)(x)$$

for $x \in \mathsf{X}$. It follows that $(H(\varphi) \circ H)(\iota_0)([\psi^{\#}(\zeta)]) = H(\iota_0)([\zeta])$. Since $H(\varphi) \circ H(\iota_0) = H(\iota_0) \circ H(\psi)$ and $H(\iota_0)$ is an isomorphism, we conclude that $H(\psi)^{-1}([\zeta]) = [\psi^{\#}(\zeta)]$. Any element in $H^{\mathsf{Y}\times\mathbb{N}}$ can be represented by a map $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ which is supported on $\mathsf{Y} \times \{0\}$, so it follows that $H(\psi)^{-1}([\zeta]) = [\psi^{\#}(\zeta)]$ for every $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$. □

Let $f\colon \mathsf{X} \longrightarrow \mathbb{N}_+$ be a continuous map. Following **[87]**, we consider the space

$$\mathsf{X}_f = \{(x,i) \in \mathsf{X} \times \mathbb{N} \mid i < f(x)\}$$

with the shift operation $\sigma_f\colon \mathsf{X}_f \longrightarrow \mathsf{X}_f$ given by

$$\sigma_f(x,i) = \begin{cases} (x, i-1) & i > 0, \\ \big(\sigma_{\mathsf{X}}(x), f(\sigma_{\mathsf{X}}(x)) - 1\big) & i = 0, \end{cases}$$

for $(x,i) \in \mathsf{X}_f$. We equip X_f with the subspace topology of $\mathsf{X} \times \mathbb{N}$ with the product topology where $\mathbb{N}$ is endowed with the discrete topology. Then X_f is compact and Hausdorff, and σ_f is surjective if and only if σ_{X} is surjective. If $\mathfrak{A}$ is the alphabet of X, then the pair (X_f, σ_f) is conjugate to a shift space $\mathsf{X}^f = j(\mathsf{X})$ over $\mathfrak{A} \times \{0,1,\ldots,\max\{f(x) \mid x \in \mathsf{X}\} - 1\}$ where $j\colon \mathsf{X}_f \longrightarrow (\mathfrak{A} \times \{0,1,\ldots,\max\{f(x) \mid x \in \mathsf{X}\} - 1\})^{\mathbb{N}}$ is the injective sliding block code given by

$$j(x,i) = (x_0,i)(x_0,i-1)\cdots(x_0,0)(x_1,f(\sigma_{\mathsf{X}}(x))-1)\cdots(x_1,0)\cdots$$

for $x = x_0x_1 \cdots \in \mathsf{X}$ and $i = 0, 1, \ldots, f(x) - 1$. By a slight abuse of notation, we shall identify X_f and X^f and consider the two-sided subshift Λ_{X_f} as well as the cover $\widetilde{\mathsf{X}_f}$. Note that Λ_X and Λ_{X_f} are flow equivalent, cf. [**22**, Section 5]. A similar construction applies to two-sided subshifts.

We shall make use of the following characterization of flow equivalence. This is probably known to experts but we have not been able to find a proper reference.

LEMMA B.8.4. *A pair of two-sided subshifts Λ_X and Λ_Y are flow equivalent, if and only if there are continuous maps $f \in C(\mathsf{X}, \mathbb{N}_+)$ and $g \in C(\mathsf{Y}, \mathbb{N}_+)$ such that Λ_{X_f} and Λ_{Y_g} are conjugate.*

PROOF. Suppose first that there are continuous maps $f \in C(\mathsf{X}, \mathbb{N}_+)$ and $g \in C(\mathsf{Y}, \mathbb{N}_+)$ such that Λ_{X_f} and Λ_{Y_g} are conjugate. It is well-known that Λ_X is flow equivalent to Λ_{X_f}, and that Λ_Y is flow equivalent to Λ_{Y_g}, cf. [**22**, Section 5], so it follows that Λ_{X_f} and Λ_{Y_g} are flow equivalent.

If Λ_X and Λ_Y are flow equivalent, then there is a compact metric space Z with a flow $\gamma\colon Z \times \mathbb{R} \longrightarrow Z$ and cross sections $\mathcal{X}$ and $\mathcal{Y}$ which are conjugate to Λ_X and Λ_Y, respectively, cf., e.g., [**97, 6**]. Let $h_\mathsf{X}\colon \Lambda_\mathsf{X} \longrightarrow \mathcal{X}$ and $h_\mathsf{Y}\colon \Lambda_\mathsf{Y} \longrightarrow \mathcal{Y}$ be such conjugacies.

Set $A = \mathcal{X} \cup \mathcal{Y}$. Consider the return time function $\tau_\mathcal{X}\colon Z \longrightarrow \mathbb{R}$ given by

$$\tau_\mathcal{X}(z) = \min\{t > 0 \mid \gamma(z, t) \in \mathcal{X}\},$$

for $z \in Z$, and define the map $\bar{f}\colon \Lambda_\mathsf{X} \longrightarrow \mathbb{N}$ by

$$\bar{f}(\mathrm{x}) = |\{t \in (0, \tau_\mathcal{X}(h_\mathsf{X}(\mathrm{x}))) \mid \gamma(h_\mathsf{X}(\mathrm{x}), t) \in \mathcal{Y}\}|$$

for $\mathrm{x} \in \Lambda_\mathsf{X}$. Then $\bar{f}$ is continuous and $f \geqslant 1$. Moreover, $(\Lambda_\mathsf{X})_{\bar{f}}$ is conjugate to A by construction. By continuity, there is an integer $n \in \mathbb{N}$ such that $\mathrm{x}_{[-n,n]} = \mathrm{x}_{[-n,n]}$ implies $\bar{f}(\mathrm{x}) = \bar{f}(\mathrm{x})$. It follows that there is a well-defined continuous map $f\colon \mathsf{X} \longrightarrow \mathbb{N}$ satisfying

$$f(\mathrm{x}_{[0,\infty)}) = \bar{f}(\sigma^n_{\Lambda_\mathsf{X}}(\mathrm{x}))$$

for $\mathrm{x} \in \Lambda_\mathsf{X}$. Then $(\Lambda_\mathsf{X})_{\bar{f}}$ is conjugate to $(\Lambda_\mathsf{X})_{\bar{f}\circ\sigma^n_{\Lambda_\mathsf{X}}}$, and $(\Lambda_\mathsf{X})_{\bar{f}\circ\sigma^n_{\Lambda_\mathsf{X}}}$ is conjugate to Λ_{X_f}. In particular, Λ_{X_f} is conjugate to A.

A similar argument shows that there is a continuous map $g\colon C(\mathsf{Y}, \mathbb{N}_+)$ such that Λ_{Y_g} is conjugate to A. It follows that Λ_{X_f} and Λ_{Y_g} are conjugate. □

LEMMA B.8.5. *Let X be a one-sided shift space and let $f\colon \mathsf{X} \longrightarrow \mathbb{N}_+$ be continuous. Then*

(i) *there are an injective sliding block code $\iota_f\colon \mathsf{X} \longrightarrow \mathsf{X}_f$ and a surjective homomorphism $\iota_f^*\colon C(\mathsf{X}_f, \mathbb{Z}) \longrightarrow C(\mathsf{X}, \mathbb{Z})$ given by*

$$\iota_f^*(\xi)(x) = \sum_{r=0}^{f(\sigma_\mathsf{X}(x))-1} \xi(\sigma_f^r(\iota_f(x))), \tag{B.16}$$

and a positive isomorphism $H(\iota_f)\colon H^{\mathsf{X}_f} \longrightarrow H^\mathsf{X}$ given by $H(\iota_f)([\xi]) = [\iota_f^*(\xi)]$;

(ii) *there are*

- *a groupoid isomorphism* $\Psi_f\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{X}_f} \times \mathcal{R}$ *and a homeomorphism* $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{X}_f \times \mathbb{N}$ *satisfying* $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi_f^{(0)}$*;*
- *a homomorphism* $\psi^*\colon C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X}\times\mathbb{N}, \mathbb{Z})$ *satisfying* $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\xi)) = \kappa_{\mathsf{X}_f\times\mathbb{N}}(\xi) \circ \Psi_f$ *for* $\xi \in C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$*;*
- *a homomorphism* $\psi^{\#}\colon C(\mathsf{X}\times\mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X}_f\times\mathbb{N}, \mathbb{Z})$ *satisfying* $\kappa_{\mathsf{X}_f\times\mathbb{N}}(\psi^{\#}(\zeta)) = \kappa_{\mathsf{X}\times\mathbb{N}}(\zeta) \circ \Psi_f^{-1}$ *for* $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$*;*
- *a positive isomorphism* $H(\psi)\colon H^{\mathsf{X}_f\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ *such that* $H(\psi)([\xi]) = [\psi^*(\xi)]$*,* $H(\psi)^{-1}([\zeta]) = [\psi^{\#}(\zeta)]$*, and* $H(\iota_f) \circ H(\iota_0) = H(\iota_0) \circ H(\psi)$*.*

PROOF. (i): The inclusion $\iota_f\colon \mathsf{X} \longrightarrow \mathsf{X}_f$ given by $\iota_f(x) = (x, 0)$ is an injective sliding block code, and $\iota_f^*\colon C(\mathsf{X}_f, \mathbb{Z}) \longrightarrow C(\mathsf{X}, \mathbb{Z})$ given by (B.16) is a surjective homomorphism. Since

$$\iota_f^*(\xi - \xi \circ \sigma_{\mathsf{X}})(x) = \xi(x, 0) - \xi(\sigma_{\mathsf{X}}(x), 0) = \iota_0^*(\xi)(x) - \iota_f^*(\xi)(\sigma_{\mathsf{X}}(x)),$$

for $\xi \in C(\mathsf{X}_f, \mathbb{Z})$ and $x \in \mathsf{X}$, the map ι_f^* induces a well-defined surjective map $H(\iota_f)\colon H^{\mathsf{X}_f} \longrightarrow H^{\mathsf{X}}$ given by $H(\iota_f)([\xi]) = [\iota_f^*(\xi)]$ for $\xi \in C(\mathsf{X}_f, \mathbb{Z})$.

To see that $H(\iota_f)$ is injective, notice that any element of H^{X_f} can be represented by a map $\xi \in C(\mathsf{X}_f, \mathbb{Z})$ which is supported on $\mathsf{X} \times \{0\} \subseteq \mathsf{X}_f$. Suppose $\xi \in C(\mathsf{X}_f, \mathbb{Z})$ is supported on $\mathsf{X} \times \{0\} \subseteq \mathsf{X}_f$ and $\iota_f^*(\xi)(x) = b(x) - b(\sigma_{\mathsf{X}}(x))$ for some $b \in C(\mathsf{X}, \mathbb{Z})$. Let $\eta \in C(\mathsf{X}_f, \mathbb{Z})$ be given by $\eta(x, n) = 0$ for $n > 0$ and $\eta(x, 0) = b(x)$. Then $\xi(x, 0) = \eta(x, 0) - \eta \circ \sigma_f^{f(\sigma_{\mathsf{X}}(x))}(x, 0)$, so ξ is cohomologous to zero.

Note that $\iota_f^*(\xi) \geqslant 0$ when $\xi \geqslant 0$. Conversely, let $g \in C(\mathsf{X}, \mathbb{Z})$ and take $\xi \in C(\mathsf{X}_f, \mathbb{Z})$ such that $\xi(x, i) = 0$ for all $i > 0$ and $\xi(x, 0) = g(x)$. Then $\iota_f^*(\xi) = g$ and $\xi \geqslant 0$ if $g \geqslant 0$. Hence $H(\iota_f^*)$ is a positive isomorphism.

(ii): Define $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{X}_f \times \mathbb{N}$ by

$$\psi(x, j) = \big((x, i), k\big)$$

where $i, k \in \mathbb{N}$ with $i < f(x)$ and $j = kf(x) + i$. Then ψ is a homeomorphism.

Define $\Psi_f\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{X}_f} \times \mathcal{R}$ by

$$\Psi_f\left((\widetilde{x}, j), p, (\widetilde{x}', j')\right) = \Big(\big((\widetilde{x}, i), k\big), l - l', \big((\widetilde{x}', i'), k'\big)\Big)$$

for $\big((\widetilde{x}, j), p, (\widetilde{x}', j')\big) \in \mathcal{G}_{\mathsf{X}} \times \mathcal{R}$ and $s, s' \in \mathbb{N}$ such that $\sigma_{\widetilde{\mathsf{X}}}^{s}(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^{s'}(\widetilde{x}')$ and $p = s - s'$. Here, $i, i', k, k' \in \mathbb{N}$ with $i < f(\pi_{\mathsf{X}}(\widetilde{x}))$ and $i' < f(\pi_{\mathsf{X}}(\widetilde{x}'))$, and $j = kf(\pi_{\mathsf{X}}(\widetilde{x})) + i$ and $j' = k' f(\pi_{\mathsf{X}}(\widetilde{x}')) + i'$, and

$$l = i + \sum_{r=1}^{s} f(\sigma_{\mathsf{X}}^{r}(\pi_{\mathsf{X}}(\widetilde{x}))), \quad l' = i' + \sum_{r=1}^{s'} f(\sigma_{\mathsf{X}}^{r}(\pi_{\mathsf{X}}(\widetilde{x}'))).$$

Then Ψ_f is a groupoid isomorphism such that $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi_f^{(0)}$.

(iii): Let $\psi^*\colon C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ be defined by

$$\psi^*(\xi)(x,j) = \sum_{r=0}^{k+1} \xi(S_f^r(\psi(x,j))) - \sum_{r=0}^{k} \xi(S_f^r(\psi(x,j-1)))$$

for $\xi \in C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$ and $(x,j) \in \mathsf{X} \times \mathbb{N}$ with $j \geq 1$, where k is the integer part of $j/f(x)$, and

$$\psi^*(\xi)(x,0) = \sum_{r=0}^{f(\sigma_{\mathsf{X}}(x))-1} \xi(S_f^r(\psi(x,0)))$$

for $\xi \in C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$ and $x \in \mathsf{X}$. Then ψ^* is a homomorphism such that $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\xi)) = \kappa_{\mathsf{X}_f\times\mathbb{N}}(\xi) \circ \Psi_f$ for $\xi \in C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$.

Define $\psi^\#\colon C(\mathsf{X} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$ by

$$\psi^\#(\zeta)((x,i),k) = \sum_{j=(k-1)f(x)+i+1}^{kf(x)+i} \zeta(x,j)$$

for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $\big((x,i),k\big) \in \mathsf{X}_f \times \mathbb{N}$ with $k \geq 1$,

$$\psi^\#(\zeta)\big((x,i),0\big) = \zeta(x,i)$$

for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $(x,i) \in \mathsf{X}_f$ with $i \geq 1$, and

$$\psi^\#(\zeta)\big((x,0),0\big) = \zeta(x,0) - \sum_{j=1}^{f(\sigma_{\mathsf{X}}(x))-1} \zeta(\sigma_{\mathsf{X}}(x),j)$$

for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $x \in \mathsf{X}$. Then $\psi^\#$ is a homomorphism such that $\kappa_{\mathsf{X}_f\times\mathbb{N}}(\psi^\#(\zeta)) = \kappa_{\mathsf{X}\times\mathbb{N}}(\zeta) \circ \Psi_f^{-1}$ for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$

Since

$$\psi^*(\xi - \xi \circ S_f)(x,j) = \xi((x,0),0) - \xi((\sigma_{\mathsf{X}}(x),0),0) = \xi((x,0),0) - \xi \circ S_{\mathsf{X}}^{f(\sigma_{\mathsf{X}}(x))}((x,0),0),$$

for $\xi \in C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$ and $(x,j) \in \mathsf{X} \times \mathbb{N}$, ψ^* induces a well-defined map $H(\psi)\colon H^{\mathsf{X}_f\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ given by $H(\psi)([\xi]) = [\psi^*(\xi)]$ for $\eta \in C(\mathsf{X}_f \times \mathbb{N}, \mathbb{Z})$. Since $\iota_f^* \circ \iota_0^* = \iota_0^* \circ \psi^*$, it follows that $H(\iota_f) \circ H(\iota_0) = H(\iota_0) \circ H(\psi)$. Since $H(\iota_0)$ and $H(\iota_f)$ are positive isomorphisms, $H(\psi)$ is also a positive isomorphism.

Suppose $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ is supported on $\mathsf{X} \times \{0\}$. Then

$$\iota_f^*(\iota_0^*(\psi^\#(\zeta)))(x) = \zeta(x,0) = \iota_0^*(\zeta)(x),$$

for every $x \in \mathsf{X}$. Since every element of $H_+^{\mathsf{X}\times\mathbb{N}}$ can be represented by a map $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ which is supported on $\mathsf{X} \times \{0\}$, this shows that $H(\iota_f) \circ H(\iota_0)([\psi^\#(\zeta)]) = H(\iota_0)([\zeta])$ for every $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$. Since $H(\iota_f) \circ H(\iota_0) = H(\iota_0) \circ H(\psi)$ and $H(\iota_0)$ is an isomorphism, it follows that $H(\psi)^{-1}([\zeta]) = [\psi^\#(\zeta)]$ for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$. □

Let us say that a stabilizer preserving continuous orbit equivalence $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ from X to Y is *positive* if $[l_{\mathsf{X}} - k_{\mathsf{X}}] \in H_+^{\mathsf{X}}$ and $[l_{\mathsf{Y}} - k_{\mathsf{Y}}] \in H_+^{\mathsf{Y}}$.

LEMMA B.8.6. *Let* X *and* Y *be one-sided shift spaces and let* $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ *be a positive stabilizer preserving continuous orbit equivalence from* X *to* Y*. Then* $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ *is least period preserving.*

PROOF. Since $[l_{\mathsf{X}} - k_{\mathsf{X}}] \in H_+^{\mathsf{X}}$ and $[l_{\mathsf{Y}} - k_{\mathsf{Y}}] \in H_+^{\mathsf{Y}}$, there are $b_{\mathsf{X}} \in C(\mathsf{X}, \mathbb{Z})$ and $n_{\mathsf{X}} \in C(\mathsf{X}, \mathbb{N})$ such that $l_{\mathsf{X}} - k_{\mathsf{X}} = n_{\mathsf{X}} + b_{\mathsf{X}} - b_{\mathsf{X}} \circ \sigma_{\mathsf{X}}$, and $b_{\mathsf{Y}} \in C(\mathsf{Y}, \mathbb{Z})$ and $n_{\mathsf{Y}} \in C(\mathsf{Y}, \mathbb{N})$ such that $l_{\mathsf{Y}} - k_{\mathsf{Y}} = n_{\mathsf{Y}} + b_{\mathsf{Y}} - b_{\mathsf{Y}} \circ \sigma_{\mathsf{Y}}$. If $x \in \mathsf{X}$ is periodic with $\mathrm{lp}(x) = p$, then

$$\begin{aligned} l_{\mathsf{X}}^{(p)}(x) - k_{\mathsf{X}}^{(p)}(x) &= \sum_{i=0}^{p-1} \big(l_{\mathsf{X}}(\sigma_{\mathsf{X}}^i(x)) - k_{\mathsf{X}}(\sigma_{\mathsf{X}}^i(x))\big) \\ &= \sum_{i=0}^{p-1} \big(n_{\mathsf{X}}(\sigma_{\mathsf{X}}^i(x) + b_{\mathsf{X}}(\sigma_{\mathsf{X}}^i(x) - b_{\mathsf{X}}(\sigma_{\mathsf{X}}^{i+1}(x))\big) \\ &= \sum_{i=0}^{p-1} n_{\mathsf{X}}(\sigma_{\mathsf{X}}^i(x) \geqslant 0. \end{aligned}$$

Since $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ is stabilizer preserving, we thus have that

$$l_{\mathsf{X}}^{(p)}(x) - k_{\mathsf{X}}(p)(x) = |l_{\mathsf{X}}^{(p)}(x) - k_{\mathsf{X}}(p)(x)| = \mathrm{lp}(h(x)).$$

A similar argument shows that $l_{\mathsf{Y}}^{(\mathrm{lp}(y))}(y) - k_{\mathsf{Y}}(\mathrm{lp}(y))(y) = \mathrm{lp}(h^{-1}(y))$ for any periodic $y \in \mathsf{Y}$. □

COROLLARY B.8.7. *Let* Λ_{X} *and* Λ_{Y} *be two-sided subshifts and suppose there is a positive stabilizer preserving continuous orbit equivalence from* X *to* Y*. Then* Λ_{X} *and* Λ_{Y} *are flow equivalent.*

PROOF. Let $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ be a positive stabilizer preserving continuous orbit equivalence from X to Y. It follows from Lemma B.8.6 that $(h, l_{\mathsf{X}}, k_{\mathsf{X}}, l_{\mathsf{Y}}, k_{\mathsf{Y}})$ is least period preserving, and thus from [**22**, Proposition 3.2] that Λ_{X} and Λ_{Y} are flow equivalent. □

The proof of [**86**, Theorem 5.11] shows that any continuous orbit equivalence between shifts of finite type with no isolated points is least period preserving and positive. However, if $\mathsf{X} = \mathsf{Y}$ is the shift space with only one point, then $(\mathrm{id}, 1, 0, 0, 1)$ is a stabilizer preserving continuous orbit equivalence from X to Y which is not positive. It follows from [**22**, Proposition 4.5 and Proposition 4.7] that if X and Y are shifts of finite type that are continuously orbit equivalent, then there is a least period preserving continuous orbit equivalence between X and Y. We do not know if there are shifts spaces X and Y that are continuously orbit equivalent, but for which there is no positive stabilizer preserving continuous orbit equivalence between X and Y.

REMARK B.8.8. Suppose $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid, Γ is an abelian group, and that $c\colon \mathcal{G} \longrightarrow \Gamma$ is a cocycle. In [**26**], a groupoid $\mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^c)$ consisting of equivalence classes of pairs (n, ϕ), where n is normaliser of $C_0(\mathcal{G}^{(0)})$ in $C_r^*(\mathcal{G})$ that is homogeneous with respect to c, and ϕ is a character of $C_0(\mathcal{G}^{(0)})$, is constructed, and it is shown in [**26**, Proposition 6.5] that there is an isomorphism $\theta_{(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^c)}\colon \mathcal{G} \longrightarrow \mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^c)$.

This is used in [**26**, Theorem 6.2] to prove that if $\mathcal{G}'$ is another second-countable locally compact Hausdorff étale groupoid, and $d\colon \mathcal{G}' \to \Gamma$ is a cocycle such that there is a *-isomorphism $\Phi\colon C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}')$ such that $\Phi(C_0(\mathcal{G}^{(0)})) = C_0((\mathcal{G}')^{(0)})$ and $\beta^d_\gamma \circ \Phi = \Phi \circ \beta^c_\gamma$ for all $\gamma \in \hat{\Gamma}$, then there is an isomorphism $\Psi\colon \mathcal{G} \longrightarrow \mathcal{G}'$ such that $d \circ \Psi = c$.

If we let c_0 denote the unique cocycle from $\mathcal{G}$ to the abelian group $\{0\}$, then any normaliser of $C_0(\mathcal{G}^{(0)}$ in $C_r^*(\mathcal{G})$ is homogeneous with respect to c_0. In particular, a normaliser n that is homogeneous with respect to c, is also homogeneous with respect to c_0, and there is a homomorphism $\Phi_\pi\colon \mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^c) \longrightarrow \mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^{c_0})$ that sends $[n, \phi]$ to $[n, \phi]$. Since $\theta_{(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^{c_0})} = \Phi_\pi \circ \theta_{(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}), \beta^c)}$, it follows that Φ_π is an isomorphism.

We thus have that the isomorphism $\Psi'\colon \mathcal{G} \longrightarrow \mathcal{G}'$ constructed in [**26**, Theorem 3.3] is equal to the isomorphism $\Psi\colon \mathcal{G} \longrightarrow \mathcal{G}'$ constructed in [**26**, Theorem 6.2] such that $d \circ \Psi = c$.

We are now ready to characterize flow equivalence of general two-sided subshifts. The equivalence (i) $\iff$ (iv) in Theorem B.8.9 below is a generalization of [**22**, Theorem 5.3 (5) $\iff$ (6)] which is formulated for shifts of finite type.

THEOREM B.8.9. *Let Λ_{X} and Λ_{Y} be two-sided subshifts. The following are equivalent:*

(i) *the two-sided subshifts Λ_{X} and Λ_{Y} are flow equivalent;*

(ii) *there are*
- *a groupoid isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ and a homeomorphism $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ such that $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$;*
- *a homomorphism $\psi^*\colon C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ such that*
$$\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta)) = \kappa_{\mathsf{Y}\times\mathbb{N}}(\eta) \circ \Psi, \tag{B.17}$$
for $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$;
- *a homomorphism $\psi^{\#}\colon C(\mathsf{X} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ such that*
$$\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^{\#}(\zeta)) = \kappa_{\mathsf{X}\times\mathbb{N}}(\zeta) \circ \Psi^{-1}, \tag{B.18}$$
for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$; and
- *a positive isomorphism $H(\psi)\colon H^{\mathsf{Y}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ such that $H(\psi)([\eta]) = [\psi^*(\eta)]$ $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$, and $H(\psi)^{-1}([\zeta]) = [\psi^{\#}(\zeta)]$ for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$.*

(iii) *there are*
- *a *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ such that $\Phi(\mathcal{D}_{\mathsf{X}} \otimes c_0) = \mathcal{D}_{\mathsf{Y}} \otimes c_0$ and $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$;*
- *a homomorphism $\psi^*\colon C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ such that*
$$\Phi \circ \beta_z^{\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta))} = \beta_z^{\kappa_{\mathsf{Y}\times\mathbb{N}}(\eta)} \circ \Phi,$$
for $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ and $z \in \mathbb{T}$;
- *a homomorphism $\psi^{\#}\colon C(\mathsf{X} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ such that*
$$\Phi \circ \beta_z^{\kappa_{\mathsf{X}\times\mathbb{N}}(\zeta)} = \beta_z^{\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^{\#}(\zeta))} \circ \Phi,$$
for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $z \in \mathbb{T}$;

- *a positive isomorphism* $H(\psi)\colon H^{\mathsf{Y}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ *such that* $H(\psi)([\eta]) = [\psi^*(\eta)]$ $\eta \in C(\mathsf{Y}\times\mathbb{N},\mathbb{Z})$, *and* $H(\psi)^{-1}([\zeta]) = [\psi^{\#}(\zeta)]$ *for* $\zeta \in C(\mathsf{X}\times\mathbb{N},\mathbb{Z})$*;*

(iv) *there are* $f \in C(\mathsf{X},\mathbb{N}_+)$ *and* $g \in C(\mathsf{Y},\mathbb{N}_+)$ *such that there is a positive stabiliser preserving continuous orbit equivalence between* X_f *and* Y_g*.*

If X *and* Y *contain dense sets of aperiodic points, then the condition* $\Phi(\mathcal{D}_\mathsf{X}\otimes c_0) = \mathcal{D}_\mathsf{Y}\otimes c_0$ *in* (iii) *is superfluous.*

Proof. (i) $\implies$ (ii): Suppose Λ_X and Λ_Y are flow equivalent. Then there are $f \in C(\mathsf{X},\mathbb{N}_+)$ and $g \in C(\mathsf{Y},\mathbb{N}_+)$ such that Λ_{X_f} and Λ_{Y_g} are conjugate. It therefore follows from Lemmas B.8.3 and B.8.5 that (ii) holds.

(ii) $\implies$ (iv): We shall identify $\mathcal{G}_\mathsf{X}\times\mathcal{R}^{(0)} = \widetilde{\mathsf{X}}\times\mathbb{N}$. Since $\widetilde{\mathsf{X}}$ is compact and Ψ is continuous, there is an integer $n\in\mathbb{N}$ such that $\Psi^{(0)}(\widetilde{\mathsf{X}}\times\{0\}) \subseteq \widetilde{\mathsf{Y}}\times\{0,\dots,n-1\}$.

Define $g \in C(\mathsf{Y},\mathbb{N}_+)$ to be constantly equal to n. Then $\phi_{\mathsf{Y}_g}\colon \mathsf{Y}\times\{0,\dots,n-1\} \longrightarrow \mathsf{Y}_g$ given by

$$\phi_{\mathsf{Y}_g}(y,k) = (y,k) \tag{B.19}$$

for $(y,k)\in\mathsf{Y}_g$, is a homeomorphism and $\Phi_{\mathsf{Y}_g}\colon \mathcal{G}_\mathsf{Y}\times\mathcal{R}|_{\widetilde{\mathsf{Y}}\times\{0,\dots,n-1\}} \longrightarrow \mathcal{G}_{\mathsf{Y}_g}$ defined by

$$\Phi_{\mathsf{Y}_g}\big((\widetilde{y},k),m,(\widetilde{y}',l)\big) = \big((\widetilde{y},k),k+mn-l,(\widetilde{y}',l)\big),$$

for $\big((\widetilde{y},k),m,(\widetilde{y}',l)\big) \in \mathcal{G}_\mathsf{Y}\times\mathcal{R}|_{\widetilde{\mathsf{Y}}\times\{0,\dots,n-1\}}$, is an isomorphism such that $\phi_{\mathsf{Y}_g}\circ\pi_{\mathsf{Y}\times\mathbb{N}} = \pi_{\mathsf{Y}_g}\circ\Phi_{\mathsf{Y}_g}^{(0)}$.

Define $\widetilde{f}\colon\widetilde{\mathsf{X}}\longrightarrow\mathbb{N}_+$ by

$$\widetilde{f}(\widetilde{x}) = |\{k\in\mathbb{N} : \Psi^{(0)}(\widetilde{x},k)\in\widetilde{\mathsf{Y}}\times\{0,\dots,n-1\}\}|$$

for $\widetilde{x}\in\widetilde{\mathsf{X}}$. Then $\widetilde{f}$ is continuous and $\widetilde{f}\geqslant 1$. Note that if $\pi_\mathsf{X}(\widetilde{x}) = \pi_\mathsf{X}(\widetilde{x}')$ then the condition $\psi\circ\pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}}\circ\Psi^{(0)}$ ensures that

$$\{k\in\mathbb{N}\mid\Psi^{(0)}(\widetilde{x},k)\in\widetilde{\mathsf{Y}}\times\{0,\dots,n-1\}\} = \{k'\in\mathbb{N}\mid\Psi^{(0)}(\widetilde{x}',k')\in\widetilde{\mathsf{Y}}\times\{0,\dots,n-1\}\},$$

so $\widetilde{f}(\widetilde{x}) = \widetilde{f}(\widetilde{x}')$. By Lemma B.2.5, there is a continuous map $f\colon\mathsf{X}\longrightarrow\mathbb{N}_+$ satisfying $\widetilde{f} = f\circ\pi_\mathsf{X}$.

For each $x\in\mathsf{X}$, there are exactly $f(x)$ integers $k(x,0),\dots,k(x,f(x)-1)\in\mathbb{N}$ such that $\psi(x,k(x,i))\in\mathsf{Y}\times\{0,\dots,n-1\}$. Arrange the integers in increasing order and define $\phi_{\mathsf{X}_f}\colon\mathsf{X}_f\longrightarrow\psi^{-1}(\mathsf{Y}\times\{0,\dots,n-1\})$ by

$$\phi_{\mathsf{X}_f}(x,i) = (x,k(x,i)), \tag{B.20}$$

for $(x,i)\in\mathsf{X}_f$. Define $\Phi_{\mathsf{X}_f}\colon\mathcal{G}_{\mathsf{X}_f}\longrightarrow\mathcal{G}_\mathsf{X}\times\mathcal{R}|_{\pi_\mathsf{X}^{-1}(\psi^{-1}(\mathsf{Y}\times\{0,\dots,n-1\}))}$ by

$$\Phi_{\mathsf{X}_f}\big((\widetilde{x},i),m,(\widetilde{x}',i')\big) = \big((\widetilde{x},k(\pi_\mathsf{X}(\widetilde{x}),i)),k-k',(\widetilde{x},k(\pi_\mathsf{X}(\widetilde{x}'),i'))\big)$$

where $k,k',\in\mathbb{N}$ are such that $\sigma_{\widetilde{\mathsf{X}}}^{k}(\widetilde{x}) = \sigma_{\widetilde{\mathsf{X}}}^{k'}(\widetilde{x}')$ and

$$m = i+\sum_{r=1}^{k} f(\sigma_\mathsf{X}^r(\pi_X(\widetilde{x}))) - i' - \sum_{r=1}^{k'} f(\sigma_\mathsf{X}^r(\pi_X(\widetilde{x}'))).$$

Then Φ_{X_f} is an isomorphism such that $\phi_{\mathsf{X}_f} \circ \pi_{\mathsf{X}_f} = \pi_{\mathsf{X}\times\mathbb{N}} \circ \Phi^{(0)}_{\mathsf{X}_f}$.

We have that $\Phi := \Phi_{\mathsf{Y}_g} \circ \Psi \circ \Phi_{\mathsf{X}_f} \colon \mathcal{G}_{\mathsf{X}_f} \longrightarrow \mathcal{G}_{\mathsf{Y}_g}$ is an isomorphism and $h := \phi_{\mathsf{Y}_g} \circ \psi \circ \phi_{\mathsf{X}_f} \colon \mathsf{X}_f \longrightarrow \mathsf{Y}_g$ is a homeomorphism such that $h \circ \pi_{\mathsf{X}_f} = \pi_{\mathsf{Y}_g} \circ \Phi^{(0)}$.

Let $\xi \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ be defined by

$$\xi(y,i) = \begin{cases} 1 & \text{if } i > 0, \\ n & \text{if } i = 0, \end{cases}$$

for $(y,i) \in \mathsf{Y} \times \mathbb{N}$. Then $\kappa_{\mathsf{Y}_g}(1) \circ \Phi_{\mathsf{Y}_g} = \kappa_{\mathsf{Y}\times\mathbb{N}}(\xi)$. Set $\eta := \psi^*(\xi) \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and define $d_{\mathsf{X}_f} \in C(\mathsf{X}_f, \mathbb{Z})$ by

$$d_{\mathsf{X}_f}(x,i) = \begin{cases} \sum_{j=k(x,i-1)+1}^{k(x,i)} \eta(x,j) & \text{if } i > 0, \\ \eta(x,0) - \sum_{j=1}^{k(\sigma_{\mathsf{X}}(x), f(\sigma_{\mathsf{X}}(x))-1)} \eta(\sigma_{\mathsf{X}}(x), j) & \text{if } i = 0, \end{cases}$$

for $(x,i) \in \mathsf{X}_f$. Then $\kappa_{\mathsf{Y}\times\mathbb{N}}(\xi) \circ \Psi = \kappa_{\mathsf{X}\times\mathbb{N}}(\eta)$ and $\kappa_{\mathsf{X}\times\mathbb{N}}(\eta) \circ \Phi_{\mathsf{X}_f} = \kappa_{\mathsf{X}_f}(d_{\mathsf{X}_f})$. We thus have $\kappa_{\mathsf{Y}_g}(1) \circ \Phi = \kappa_{\mathsf{X}_f}(d_{\mathsf{X}_f})$.

Similarly, $\kappa_{\mathsf{X}_f}(1) \circ \Phi^{-1}_{\mathsf{X}_f} = \kappa_{\mathsf{X}\times\mathbb{N}}(\rho)$ where $\rho \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ is defined by

$$\rho(x,j) = \begin{cases} f(\sigma(x)) & \text{if } j = 0, \\ 1 & \text{if } j = k(x,i) \text{ for some } i \in \{1, \dots, f(x)-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\chi = \psi^{\#}(\rho) \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$, and let $d_{\mathsf{Y}_g} \in C(\mathsf{Y}_g, \mathbb{Z})$ be defined by

$$d_{\mathsf{Y}_g}(y,i) = \begin{cases} \chi(y,i) & \text{if } i > 0, \\ \chi(y,0) - \sum_{j=1}^{n-1} \chi(\sigma_{\mathsf{Y}}(y), j) & \text{if } i = 0, \end{cases}$$

Then $\kappa_{\mathsf{X}\times\mathbb{N}}(\rho) \circ \Psi^{-1} = \kappa_{\mathsf{Y}\times\mathbb{N}}(\chi)$ and $\kappa_{\mathsf{Y}\times\mathbb{N}}(\chi) \circ \Phi_{\mathsf{Y}_g} = \kappa_{\mathsf{Y}_g}(d_{\mathsf{Y}_g})$. Hence, $\kappa_{\mathsf{X}_f}(1) \circ \Phi^{-1} = \kappa_{\mathsf{Y}_g}(d_{\mathsf{Y}_g})$.

It now follows from Theorem B.6.4 that there are continuous maps $k_{\mathsf{X}_f}, l_{\mathsf{X}_f} \colon \mathsf{X}_f \longrightarrow \mathbb{N}$ and $k_{\mathsf{Y}_g}, l_{\mathsf{Y}_g} \colon \mathsf{Y}_g \longrightarrow \mathbb{N}$ such that $(h, k_{\mathsf{X}_f}, l_{\mathsf{X}_f}, k_{\mathsf{Y}_g}, l_{\mathsf{Y}_g})$ is a stabilizer preserving continuous orbit equivalence from X_f to Y_g and $l_{\mathsf{X}_f} - k_{\mathsf{X}_f} = d_{\mathsf{X}_f}$ and $l_{\mathsf{Y}_g} - k_{\mathsf{Y}_g} = d_{\mathsf{Y}_g}$.

Note that $[\xi] \in H_+^{\mathsf{Y}\times\mathbb{N}}$ and $[\eta] = [\psi^*(\xi)] = H(\psi)([\xi]) \in H_+^{\mathsf{X}\times\mathbb{N}}$. Since $H(\iota_0) \colon H^{\mathsf{X}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}}$ is a positive isomorphism, it follows that there are continuous maps $\alpha, \beta \colon \mathsf{X}\times\mathbb{N} \longrightarrow \mathbb{N}$ such that α is supported on $\mathsf{X} \times \{0\}$ and $\eta = \alpha + \beta - \beta \circ S_{\mathsf{X}}$. Then $d_{\mathsf{X}_f}(x,i) = \beta(x, k(x,i)) - \beta(x, k(x,i-1))$ for $i > 0$ and

$$\begin{aligned} d_{\mathsf{X}_f}(x,0) &= \alpha(x,0) + \beta(x,0) - \beta(\sigma_{\mathsf{X}}(x), k(x, f(\sigma_{\mathsf{X}}(x))) - 1) \\ &= \alpha(x,0) + \beta(x,0) - \beta \circ \sigma_f(x,0), \end{aligned}$$

for $x \in \mathsf{X}$. Thus, $[l_{\mathsf{X}_f} - k_{\mathsf{X}_f}] = [d_{\mathsf{X}_f}] \in H_+^{\mathsf{X}_f}$.

Similarly, $[\rho] \in H_+^{\mathsf{X}\times\mathbb{N}}$ and $[\chi] = [\psi^\#(\rho)] = H(\psi)^{-1}([\rho]) \in H_+^{\mathsf{Y}\times\mathbb{N}}$, so there are continuous maps $\alpha', \beta' \colon \mathsf{Y}\times\mathbb{N} \longrightarrow \mathbb{N}$ such that γ is supported on $\mathsf{Y}\times\{0\}$ and $\chi = \alpha' + \beta' - \beta' \circ S_{\mathsf{Y}}$, and then $d_{\mathsf{Y}_g}(y,i) = \theta(y,i) - \theta(y,i-1)$ for $i > 0$, and $\tau(y,0) = \alpha(y,0) + \beta'(y,0) - \beta'(\sigma_{\mathsf{Y}}(y), n-1)$, for $y \in \mathsf{Y}$. This shows that $[l_{\mathsf{Y}_g} - k_{\mathsf{Y}_g}] = [\tau] \in H_+^{\mathsf{Y}_g}$.

We conclude that $(h, k_{\mathsf{X}_f}, l_{\mathsf{X}_f}, k_{\mathsf{Y}_g}, l_{\mathsf{Y}_g})$ is a positive stabilizer preserving continuous orbit equivalence.

(iv) $\implies$ (i): We have that Λ_{X_f} and Λ_{Y_g} are flow equivalent according to Corollary B.8.7. Since Λ_{X} and Λ_{X_f} are flow equivalent, and Λ_{Y} and Λ_{Y_g} are flow equivalent, it follows that Λ_{X} and Λ_{Y} are flow equivalent.

(ii) $\implies$ (iii): The isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ induces a *-isomorphism $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} = C^*_r(\mathcal{G}_{\mathsf{X}} \times \mathcal{R}) \longrightarrow C^*_r(\mathcal{G}_{\mathsf{Y}} \times \mathcal{R}) = \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ satisfying $\Phi(f) = f \circ \Psi^{-1}$, for $f \in C_c(\mathcal{G}_{\mathsf{X}} \times \mathcal{R})$. In particular, $\Phi(\mathcal{D}_{\mathsf{X}} \otimes c_0) = \mathcal{D}_{\mathsf{Y}} \otimes c_0$. The hypothesis, $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$ ensures that $\Phi(f) = f \circ \psi^{-1}$, for $f \in C(\mathsf{X}) \otimes c_0 \subseteq C(\widetilde{\mathsf{X}}) \otimes c_0 = \mathcal{D}_{\mathsf{X}} \otimes c_0$, and that $\Phi^{-1}(g) = g \circ \psi$ for $g \in C(\mathsf{Y}) \otimes c_0 \subseteq C(\widetilde{\mathsf{Y}}) \otimes c_0 = \mathcal{D}_{\mathsf{Y}} \otimes c_0$. Therefore, $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$.

Let $\eta \in C(\mathsf{Y}\times\mathbb{N}, \mathbb{Z})$ and suppose $f \in C_c(\mathcal{G}_{\mathsf{X}} \times \mathcal{R})$ has support in $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta))^{-1}(\{1\})$. By (B.17), $\Phi(f) = f \circ \Psi^{-1}$ has support in $\Psi(\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta))^{-1}(\{1\})) = \kappa_{\mathsf{Y}\times\mathbb{N}}(\eta)^{-1}(\{1\})$. It follows that

$$\Phi \circ \beta_z^{\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta))} = \beta_z^{\kappa_{\mathsf{Y}\times\mathbb{N}}(\eta)} \circ \Phi,$$

for $z \in \mathbb{T}$. A similar argument using (B.18) shows that $\Phi \circ \beta_z^{\kappa_{\mathsf{X}\times\mathbb{N}}(\zeta)} = \beta_z^{\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^\#(\zeta))} \circ \Phi$, for $\zeta \in C(\mathsf{X}\times\mathbb{N}, \mathbb{Z})$ and $z \in \mathbb{T}$.

(iii) $\implies$ (ii): It follows from [**26**, Theorem 3.3] that there is an isomorphism $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$.

Let $\eta \in C(\mathsf{Y}\times\mathbb{N}, \mathbb{Z})$. It then follows from [**26**, Theorem 6.2] that there is an isomorphism $\Psi_\eta\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ satisfying $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta)) = \kappa_{\mathsf{Y}\times\mathbb{N}}(\eta) \circ \Psi_\eta$, and according to Remark B.8.8, we have $\Psi = \Psi_\eta$. Therefore, $\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta)) = \kappa_{\mathsf{Y}\times\mathbb{N}}(\eta) \circ \Psi$, for every $\eta \in C(\mathsf{Y}\times\mathbb{N}, \mathbb{Z})$. A similar argument shows that $\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^\#(\zeta)) = \kappa_{\mathsf{X}\times\mathbb{N}}(\zeta) \circ \Psi^{-1}$, for every $\zeta \in C(\mathsf{X}\times\mathbb{N}, \mathbb{Z})$. Finally, the restriction $\Phi|_{C(\mathsf{X})\otimes c_0}\colon C(\mathsf{X}) \otimes c_0 \longrightarrow C(\mathsf{Y}) \otimes c_0$ induces a homeomorphism $\psi\colon \mathsf{X}\times\mathbb{N} \longrightarrow \mathsf{Y}\times\mathbb{N}$ such that $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$.

The final remark follows from Corollary B.3.6. □

If we restrict to the class of shift spaces which produce essentially principal groupoids, we can relax some of the conditions of Theorem B.8.9.

Theorem B.8.10. *Let Λ_{X} and Λ_{Y} be two-sided shift spaces such that X and Y contain no periodic points isolated in past equivalence. The following are equivalent:*

(i) *the systems Λ_{X} and Λ_{Y} are flow equivalent;*

(ii) *there is an isomorphism of groupoids* $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ *and a homeomorphism* $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ *satisfying* $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$ *and a positive isomorphism* $\theta\colon H^{\mathsf{X}\times\mathbb{N}} \longrightarrow H^{\mathsf{Y}\times\mathbb{N}}$ *satisfying* $\theta \circ \kappa_{\mathsf{X}\times\mathbb{N}} = \kappa_{\mathsf{Y}\times\mathbb{N}} \circ H^1(\Psi)$.

PROOF. (i) $\Longrightarrow$ (ii): This follows from the proof of Theorem B.8.9 (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i): Let $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ be a groupoid isomorphism and $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ be a homeomorphism satisfying $h \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$. As in the proof of Theorem B.8.9 (ii) $\Longrightarrow$ (iv) we choose $n \in \mathbb{N}_+$ and $f \in C(\mathsf{X}, \mathbb{N}_+)$. Let $g\colon \mathsf{Y} \longrightarrow \mathbb{N}$ be constantly equal to n. Then there is a groupoid isomorphism $\Psi'\colon \mathcal{G}_{\mathsf{X}_f} \longrightarrow \mathcal{G}_{\mathsf{Y}_g}$ and a homeomorphism $h = \phi_{\mathsf{Y}_g} \circ \psi \circ \phi_{\mathsf{X}_f}$ such that $h \circ \pi_{\mathsf{X}_f} = \pi_{\mathsf{Y}_g} \circ (\Psi')^{(0)}$.

It is not hard to see that the maps $\phi_{\mathsf{X}_f}\colon \mathsf{X}_f \longrightarrow \psi^{-1}(\mathsf{Y} \times \{0, \ldots, n-1\})$ and $\phi_{\mathsf{Y}_g}\colon \mathsf{Y}_g \longrightarrow \mathsf{Y} \times \{0, \ldots, n-1\}$ defined in (B.20) and (B.19), respectively, are positive continuous orbit equivalences. Since X and Y contain dense sets of aperiodic points, it follows from Theorem B.6.10 that $\psi\colon \psi^{-1}(\mathsf{Y} \times \{0, \ldots, n-1\}) \longrightarrow \mathsf{Y} \times \{0, \ldots, n-1\}$ is a continuous orbit equivalence. By the hypothesis in (ii), ψ is also positive. Hence h is a positive continuous orbit equivalence. It this follows from Corollary B.8.7 that Λ_{X_f} and Λ_{Y_g} are flow equivalent. Since Λ_{X} and Λ_{X_f} are flow equivalent, and Λ_{Y} and Λ_{Y_g} are flow equivalent, we conclude that Λ_{X} and Λ_{Y} are flow equivalent. ☐

Finally, we restrict to the class of sofic shifts whose groupoids are essentially principal.

THEOREM B.8.11. *Let* Λ_{X} *and* Λ_{Y} *be two-sided sofic shift spaces such that* X *and* Y *contain no periodic points isolated in past equivalence. The following are equivalent:*

(i) *the two-sided subshifts* Λ_{X} *and* Λ_{Y} *are flow equivalent;*
(ii) *there is an isomorphism* $\Psi\colon \mathcal{G}_{\mathsf{X}} \times \mathcal{R} \longrightarrow \mathcal{G}_{\mathsf{Y}} \times \mathcal{R}$ *and a homeomorphism* $\psi\colon \mathsf{X} \times \mathbb{N} \longrightarrow \mathsf{Y} \times \mathbb{N}$ *satisfying* $\psi \circ \pi_{\mathsf{X}\times\mathbb{N}} = \pi_{\mathsf{Y}\times\mathbb{N}} \circ \Psi^{(0)}$*;*
(iii) *there is a* **-isomorphism* $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$ *satisfying* $\Phi(C(\mathsf{X}) \otimes c_0) = C(\mathsf{Y}) \otimes c_0$.

PROOF. (i) $\Longrightarrow$ (ii): This follows from Theorem B.8.9.

(ii) $\Longrightarrow$ (i): As in the proof of (ii) $\Longrightarrow$ (iv) in Theorem B.8.9, there are $f \in C(\mathsf{X}, \mathbb{N}_+)$, $g \in C(\mathsf{Y}, \mathbb{N}_+)$, a groupoid isomorphism $\Psi'\colon \mathcal{G}_{\mathsf{X}_f} \longrightarrow \mathcal{G}_{\mathsf{Y}_g}$ and a homeomorphism $h\colon \mathsf{X}_f \longrightarrow \mathsf{Y}_g$ such that $h \circ \pi_{\mathsf{X}_f} = \pi_{\mathsf{Y}_g} \circ (\Psi')^{(0)}$. It follows from Theorem B.8.10 and its proof that h is a continuous orbit equivalence and that $(\Psi')^{(0)}\colon \widetilde{\mathsf{X}}_f \longrightarrow \widetilde{\mathsf{Y}}_g$ is a continuous orbit equivalence. Since X_f and Y_g are sofic shift spaces, the covers $\widetilde{\mathsf{X}}_f$ and $\widetilde{\mathsf{Y}}_g$ are (conjugate to) shifts of finite type. By hypothesis, X and Y have no periodic points isolated in past equivalence, so $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{Y}}$, and thus also $\widetilde{\mathsf{X}}_f$ and $\widetilde{\mathsf{Y}}_g$, have no isolated points. It therefore follows the proof of [**86**, Theorem 5.11] that the continuous orbit equivalence $(\Psi')^{(0)}$ is positive and least period preserving. It follows that h is also positive and least period preserving. It therefore follows from Corollary B.8.7 that X_f and Y_g are flow equivalent. Since X and X_f are flow equivalent, and Y and Y_g are flow equivalent, we conclude that X and Y are flow equivalent.

(ii) $\iff$ (iii): This is [**26**, Corollary 11.4]. Note that if $\Phi\colon \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \longrightarrow \mathcal{O}_{\mathsf{Y}} \times \mathbb{K}$ is a *-isomorphism as in (iii), then $\Phi(\mathcal{D}_{\mathsf{X}} \otimes c_0) = \mathcal{D}_{\mathsf{Y}} \otimes c_0$ by Corollary B.3.6. □

COROLLARY B.8.12. *Let* X *and* Y *be one-sided sofic shifts with no periodic points isolated in past equivalence. If* X *and* Y *are continuously orbit equivalent, then* Λ_{X} *and* Λ_{Y} *are flow equivalent.*

ARTICLE C

Eventual conjugacy and the balanced in-split

Kevin Aguyar Brix

Abstract

We show that one-sided eventual conjugacy between finite directed graphs with no sinks is generated by out-splits and balanced in-splits of the vertices in the graph while one-sided conjugacy is generated by out-splits. The balanced in-split is a variation of the classical in-split move introduced by Williams in order to study conjugacies of shifts of finite type. We also relate one-sided eventual conjugacies and two-sided conjugacies using block maps on the finite paths of the graphs.

C.1. Introduction

In order to study conjugacies of finite type shift spaces, Williams [**116**] introduced and studied certain state splittings of the symbolic dynamical systems. Any conjugacy can be decomposed into a finite sequence of such state splittings, and he successful translated conjugacy of the shift spaces into strong shift equivalence of the matrices which define the shift spaces. The coarser relation of *flow equivalence* was shown by Parry and Sullivan to be generated by conjugacies and a symbol expansions on the defining matrices [**97**]. The class of finite type shifts can be modeled by finite essential graphs, and the state splittings become splittings of vertices and their out-going edges (*out-splits*) or incoming edges (*in-splits*) [**68**], while the symbol expansion translates to a stretching (or delaying) of certain vertices.

Splitting and stretching vertices in a graph is an extremely useful tool in the study of graph C*-algebras and their K-theory. Bates and Pask [**4**] generalized these graphical constructions — which we now call *moves* — to arbitrary directed graphs and showed that whereas out-splits (Move `(O)`) produce *-isomorphic graph C*-algebras, the in-split (Move `(I)`) and the delay move only preserve the stable isomorphism class of the graph C*-algebras. In fact, it was known since the inception of the graph C*-algebras of finite essential graphs (irreducible and not just a cycle) — the *Cuntz–Krieger algebras* — that the stabilized graph C*-algebra together with its canonical diagonal subalgebra is invariant under flow equivalence [**35**].

In an ingenious use of groupoid techniques, Matsumoto and Matui [**84**] much later finally prove the converse statement: The stable isomorphism class of simple Cuntz–Krieger algebras *together with its diagonal* (suitably stabilized), entirely captures the flow class of the underlying shift of finite type. This was later generalized to all finite essential graphs [**22**]. A general program for understanding how specific moves on graphs are reflected in structure-preserving *-isomorphisms of the graph C*-algebras was recently initiated by Eilers and Ruiz [**43**].

The present paper addresses one-sided eventual conjugacy as studied by Matsumoto [**81**] and one-sided conjugacy of finite graphs with no sinks. The motivation is two-fold.

Firstly, work of recent years [**35, 81, 24**] have taught us that two-sided conjugacy of two-sided shifts of finite type and one-sided eventual conjugacy of one-sided shifts of finite type are *structurally* related in the following sense: Two finite essential graphs E and F are *two-sided conjugate* if and only if there is a *-isomorphism $\Psi\colon C^*(E)\otimes\mathbb{K}\longrightarrow C^*(F)\otimes\mathbb{K}$ which respects the diagonal $\Psi(\mathcal{D}(E)\otimes c_0)=\mathcal{D}(F)\otimes c_0$ and intertwines the gauge actions $\Psi\circ(\gamma^E\otimes\mathrm{id})=(\gamma^F\otimes\mathrm{id})\circ\Psi$. Here, $\mathbb{K}$ is the C*-algebra of compact operators on separable Hilbert space and c_0 is the C*-subalgebra of diagonal operators. On the other hand, E and F are *one-sided eventually conjugate* (even if we include sources) if and only if there is a *-isomorphism $\Psi\colon C^*(E)\longrightarrow C^*(F)$ which respects the diagonal $\Psi(\mathcal{D}(E))=\mathcal{D}(F)$ and intertwines the gauge actions $\Psi\circ\gamma^E=\gamma^F\circ\Psi$.

From a dynamical perspective, two-sided conjugacies are sliding block codes with a finite "window" (with memory and anticipation) which arise from block maps on finite paths of the graphs. Similarly, one-sided conjugacies are sliding block codes with anticipation but no memory. It is possible to make sense of one-sided sliding block codes with both anticipation and memory and, in fact, such sliding block codes induce all eventual conjugacies.

Theorem (Corollary C.3.9). Let E and F be finite graphs with no sinks. There is a one-to-one correspondence between eventual conjugacies $h\colon E^\infty\longrightarrow F^\infty$ and (ℓ,c)-block maps $\psi\colon E^{1+\ell+c}\longrightarrow F^{1+\ell}$ satisfying a bijectivity condition.

The connection between two-sided conjugacy and one-sided eventual conjugacy it seems is present before passing to infinite sequences.

The second motivation is a connection to the recent program of understanding the relationship between moves on graphs and certain structure-preserving *-isomorphisms of the corresponding graph C*-algebras [**43**]. We study a variation of Williams' classical in-split called the *balanced in-split* (Move (`I+`)) and show that it induces eventually conjugate graphs. As opposed to the classical in-split, this new move might introduce sources in the graphs, and if we consider the class of finite graphs with no sinks (but potentially with sources), we arrive at the following result.

Theorem (Theorem C.4.9). One-sided eventual conjugacy of finite graphs with no sinks is generated by out-splits and balanced in-splits.

The presence of sources is an essential part of the proof even if the graphs we start with do not have sources. It also follows from the proof that one-sided conjugacy is generated by out-splits alone, cf. [**116, 8, 62**]. Since work of Carlsen and Rout [**24**] shows that eventual conjugacy of the graphs is equivalent to diagonal-preserving *-isomorphism of the graph algebras which intertwines the gauge actions — this is a `111`-isomorphism in the terminology of [**43**] — the result characterizes this particular structure-preserving

*-isomorphism in terms of Move (0) and Move (I+).

Even if the present work is in part motivated by operator algebraic questions, our approach is purely dynamical. It seems reasonable that a thorough investigation of one-sided eventual conjugacy would shed light on questions concerning two-sided conjugacy as well.

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C.2. Preliminaries

We first recall the relevant concepts of finite graphs and the dynamics on their path spaces and fix notation. Let $\mathbb{Z}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$ denote the integers, the nonnegative integers and the positive integers, respectively.

C.2.1. Graphs. A *directed graph* (or just a graph) is a quadrupel $E = (E^0, E^1, r_E, s_E)$ where E^0 is the set of *vertices*, E^1 is the set of *edges* and $s_E, r_E \colon E^1 \longrightarrow E^0$ are the *source* and *range* maps, respectively. We shall omit the subscripts to simplify notation when the graph is understood. A vertex $v \in E^0$ is a *source* if $r^{-1}(v) = \emptyset$ and a *sink* if $s^{-1}(v) = \emptyset$. *In this paper we consider only finite graphs with no sinks.*

A *finite path* is a finite string $\mu = \mu_1 \cdots \mu_n$ of edges where $n \in \mathbb{N}_+$ and $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \ldots n-1$. The *length* of μ is $|\mu| = n$. By convention, the length of a vertex is zero. Let E^n be all finite paths of length n and let $E^* = \bigcup_{n \in \mathbb{N}} E^n$ be the collection of all finite paths. The source and range maps naturally extend to E^* by $s(\mu) = s(\mu_1)$ and $r(\mu) = r(\mu_{|\mu|})$. The *path space* of E is the set

$$E^\infty = \{x \in (E^1)^{\mathbb{N}} \mid r(x_i) = s(x_{i+1}), i \in \mathbb{N}\}$$

of all infinite paths of edges on the graph. The path space E^∞ is compact and Hausdorff in the subspace topology of the product topology on $(E^1)^{\mathbb{N}}$ where E^1 is discrete. The source map extends to E^∞ by putting $s(x) = s(x_0)$ for $x \in E^\infty$. Given $x \in E^\infty$ we write $x_{[i,j]} = x_i x_{i+1} \cdots x_j$ for $0 \leqslant i \leqslant j$, and put $x_{[i,j)} = x_{[i,j-1]}$ and $x_{(i,j]} = x_{[i+1,j]}$ when $i < j$. Note that any finite path $\mu \in E^*$ is of the form $x_{[i,i+|\mu|)}$ for some $x \in E^\infty$ and $i \in \mathbb{N}$. The *cylinder set* of a finite path μ is the compact open set

$$Z(\mu) = \{x \in E^\infty \mid x_{[0,n)} = \mu\},$$

and the collection of cylinder sets constitutes a basis for the topology of E^∞.

Define a *shift operation* $\sigma_E \colon E^\infty \longrightarrow E^\infty$ by $\sigma_E(x)_i = x_{i+1}$ for $x \in E^\infty$ and $i \in \mathbb{N}$. This is a local homeomorphism, and it is surjective if and only if E contains no sources. The dynamical system (E^∞, σ_E) is the *edge shift* of E. Two edge shifts (E^∞, σ_E) and (F^∞, σ_F) are *conjugate* if there exists a homeomorphism $h \colon E^\infty \longrightarrow F^\infty$ such that $h \circ \sigma_E = \sigma_F \circ h$. In this case we say the graphs E and F are conjugate and that h is a conjugacy. Edge shifts are examples of shifts of finite type and, in fact, every shift of finite type is (conjugate to) an edge shift of a finite graph [**68**].

The *N'th higher block graph* $E^{[N]}$ of a graph E has edges E^N and vertices E^{N-1}. The source and range of $\mu = \mu_1 \cdots \mu_N \in E^N$ is $s_N(\mu) = \mu_{[1,N)}$ and $r_N(\mu) = \mu_{(1,N]}$. Note that $E^{[1]} = E$, and that for any N there is a canonical conjugacy $\varphi\colon E^\infty \longrightarrow (E^{[N]})^\infty$ given by $\phi(x) = x_{[0,N)}x_{[1,N+1)}x_{[2,N+2)}\cdots$ for $x \in E^\infty$.

C.3. Eventual conjugacy and block maps

Let us fix two finite graphs with no sinks E and F and let E^∞ and F^∞ be their infinite path spaces, respectively.

C.3.1. Eventual conjugacy. A *sliding block code* $\varphi\colon E^\infty \longrightarrow F^\infty$ is a continuous map which intertwines the shift operations in the sense that $\varphi \circ \sigma_E = \sigma_F \circ \varphi$. A *conjugacy* is a bijective sliding block code. In order to describe eventual conjugacies on graphs we first modify the notion of sliding block codes slightly.

Definition C.3.1. Let $\ell \in \mathbb{N}$. An *ℓ-sliding block code* is a continuous map $\varphi\colon E^\infty \longrightarrow F^\infty$ such that $\sigma_E^\ell \circ \varphi$ is a sliding block code, that is,

$$\sigma_F^{\ell+1}(\varphi(x)) = \sigma_F^\ell(\varphi(\sigma_E(x))),$$

for every $x \in E^\infty$. If $\ell = 0$, then φ is a sliding block code in the usual sense. An *ℓ-conjugacy* is a bijective ℓ-sliding block code.

An ℓ-conjugacy is simply one half of an eventual conjugacy.

Definition C.3.2 ([**81**]). Two finite graphs with no sinks E and F are *eventually conjugate* if there exist a homeomorphism $h\colon E^\infty \longrightarrow F^\infty$ and $\ell, \ell' \in \mathbb{N}$ such that h and h^{-1} are ℓ- and ℓ'-conjugacies, respectively. That is,

$$\sigma_F^{\ell+1}(h(x)) = \sigma_F^\ell(h(\sigma_E(x))),$$
$$\sigma_E^{\ell'+1}(h^{-1}(y)) = \sigma_E^{\ell'}(h^{-1}(\sigma_F(y))),$$

for $x \in E^\infty$ and $y \in F^\infty$. We say that h is an *eventual conjugacy*

If $h\colon E^\infty \longrightarrow F^\infty$ is an ℓ-conjugacy, then h is an $(\ell+i)$-conjugacy for all $i \in \mathbb{N}$, and if $h'\colon F^\infty \longrightarrow G^\infty$ is an ℓ'-conjugacy, then $h' \circ h$ is an $(\ell+\ell')$-conjugacy. Furthermore, if h is an eventual conjugacy, then there is an $\ell \in \mathbb{N}$ such that h and h^{-1} are ℓ-conjugacies. Let $\ell \in \mathbb{N}$ and suppose $h\colon E^\infty \longrightarrow F^\infty$ is an ℓ-sliding block code. Then there is a $c \in \mathbb{N}$ such that

$$h(Z_E(x_{[0,k+c]})) \subseteq Z_F(h(x)_{[0,k]}) \tag{C.1}$$

for $k \geqslant \ell$ and $x \in E^\infty$. We shall refer to c as a *continuity constant for h (relative to ℓ).* Note that if c is a continuity constant for h (relative to ℓ), then $c+i$ is a continuity constant for h for any $i \in \mathbb{N}$. Moreover, if h and h^{-1} are ℓ- and ℓ'-conjugacies, then we can choose a continuity constant relative to $\max\{\ell, \ell'\}$.

Definition C.3.3. An *(ℓ, c)-sliding block code* is an ℓ-sliding block code $\varphi\colon E^\infty \longrightarrow F^\infty$ with continuity constant c relative to ℓ. An *(ℓ, c)-conjugacy* is a bijective (ℓ, c)-sliding block code.

The following lemma shows that we can always reduce the continuity complexity of an ℓ-conjugacy, at the expense of increasing the continuity complexity of the inverse.

LEMMA C.3.4. *Let E and F be finite graphs with no sinks and let $h\colon E^\infty \longrightarrow F^\infty$ be an (ℓ, c)-conjugacy. There exists a graph $\bar{E}$ with a conjugacy $\varphi\colon E \longrightarrow \bar{E}$ and an $(\ell, 0)$-conjugacy $\bar{h}\colon \bar{E}^\infty \longrightarrow F^\infty$ satisfying $\bar{h} \circ \varphi = h$.*

PROOF. Consider the higher block graph $\bar{E} = E^{[c]}$ and let $\varphi\colon E^\infty \longrightarrow \bar{E}^\infty$ be the canonical conjugacy. Define $\bar{h}\colon \bar{E}^\infty \longrightarrow F^\infty$ as $\bar{h} = h \circ \varphi^{-1}$, that is,

$$\bar{h}(x_{[0,c]}x_{[1,c+1]}\cdots) = h(x),$$

for $x \in E^\infty$. Then $\bar{h}$ is an ℓ-conjugacy. Since $\bar{h} = h \circ \varphi^{-1}$ and c is a continuity constant for h, we have

$$\bar{h}(Z_{\bar{E}}(\bar{x}_{[0,\ell]})) = h(Z_E(x_{[0,\ell+c]})) \subseteq Z_F(h(x)_{[0,\ell]})$$

for $\bar{x} \in \bar{E}^\infty$ with $\varphi^{-1}(\bar{x}) = x \in E^\infty$. Hence $\bar{h}$ is an $(\ell, 0)$-conjugacy. □

C.3.2. Block maps.

DEFINITION C.3.5. Let E and F be finite graphs with no sinks and let $\ell, c \in \mathbb{N}$. An (ℓ, c)-*block map* is a map $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ which is *compatible with E and F* in the sense that

$$\psi(x_{[0,\ell+c]})_\ell \psi(x_{[1,1+\ell+c]})_\ell \in F^2,$$

for $x \in E^\infty$.

Let $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ be an (ℓ, c)-block map. The compatibility condition ensures that there is an extension $\psi\colon E^{1+\ell+c+i} \longrightarrow F^{1+\ell+i}$ given by

$$\psi(x_{[0,\ell+c+i]}) = \psi(x_{[0,\ell+c]})\psi(x_{[1,\ell+c+1]})_\ell \cdots \psi(x_{[i,\ell+c+i]})_\ell$$

whenever $x \in E^\infty$ and $i \in \mathbb{N}$. We shall use the same symbol ψ for the block map and its extension, this should cause no confusion. Iterating this process *ad infinitum*, ψ extends to a map on the infinite path spaces $h = h_\psi\colon E^\infty \longrightarrow F^\infty$ by

$$h(x) = \psi(x_{[0,\ell+c]})\psi(x_{[1,\ell+c+1]})_\ell \cdots \psi(x_{[i,\ell+c+i]})_\ell \cdots,$$

for $x \in E^\infty$. In order to see that h is continuous, suppose $x^{(n)} \longrightarrow x$ in E^∞ as $n \longrightarrow \infty$. Given $i \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $x^{(n)} \in Z_E(x_{[0,\ell+c+i]})$ whenever $n \geqslant N$. Then

$$h(x^{(n)})_{[0,\ell+i]} = \psi(x^{(n)}{}_{[0,\ell+c+i]}) = \psi(x_{[0,\ell+c+i]}) = h(x)_{[0,\ell+i]},$$

so $h(x^{(n)}) \longrightarrow h(x)$ in F^∞ as $n \longrightarrow \infty$. In particular, c is a continuity constant for h relative to ℓ. We would now like to find conditions on the block maps which ensure that the induced map on the path spaces is bijective.

DEFINITION C.3.6. An (ℓ, c)-block map $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ satisfies the *surjectivity condition* if for every $k \geqslant \ell$ and every $\beta \in F^{1+k}$, there exists $\alpha \in E^{1+k+c}$ such that $\psi(\alpha) = \beta$. On the other hand, ψ satisfies the *injectivity condition* if for every $k \geqslant \ell$ there is $K \in \mathbb{N}$ such that

$$\psi(x_{[0,k+K]}) = \psi(x'_{[0,k+K]}) \implies x_{[0,k]} = x'_{[0,k]}$$

for every $x, x' \in E^\infty$. We say that ψ satisfies the *bijectivity condition* if it satisfies both the injectivity and the surjectivity conditions.

REMARK C.3.7. Note that for the case $c = 0$, the surjectivity condition reduces to surjectivity of the block map while the injectivity condition is, in general, weaker than injectivity of the block map. The term *bijectivity condition* is only meant to reflect the fact that the induced map on the infinite path spaces is bijective (see the proof below). If $\psi\colon E^{1+\ell} \longrightarrow F^{1+\ell}$ is in fact a *bijective* block map, then the inverse homeomorphism ${h_\psi}^{-1}\colon E^\infty \longrightarrow F^\infty$ is induced by ψ^{-1} and has vanishing continuity constant.

We now arrive at the main result of this section.

THEOREM C.3.8. *Let E and F be finite graphs with no sinks and let $\ell, c \in \mathbb{N}$. There is a one-to-one correspondence between the collection of (ℓ, c)-block maps $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ and the collection of (ℓ, c)-sliding block codes $h\colon E^\infty \longrightarrow F^\infty$. In addition, ψ satisfies the injectivitiy condition or the surjectivity condition if and only if h is injective or surjective, respectively.*

PROOF. Suppose $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ is an (ℓ, c)-block map. Let $h = h_\psi\colon E^\infty \longrightarrow F^\infty$ be the continuous extension to the infinite path spaces given by

$$h(x)_{[0,k]} = \psi(x_{[0,k+c]})$$

for $x \in E^\infty$ and $k \geqslant \ell$. Note that c is a continuity constant for h relative to ℓ. If $ax \in E^\infty$, then

$$\begin{aligned}
\sigma_F^{\ell+1}(h(ax)) &= \sigma_F^{\ell+1}\Big(\psi((ax)_{[0,\ell+c]})\psi((ax)_{[1,\ell+c+1]})_\ell \cdots \psi((ax)_{[i,\ell+c+i]})_\ell \cdots\Big) \\
&= \psi((ax)_{[1,1+\ell+c]})_\ell \cdots \psi((ax)_{[i,\ell+c+i]})_\ell \cdots \\
&= \psi(x_{[0,\ell+c]})_\ell \cdots \psi(x_{[i,\ell+c+i]})_\ell \cdots \\
&= \sigma_F^\ell\Big(\psi(x_{[0,\ell+c]}) \cdots \psi(x_{[i,\ell+c+i]})_\ell \cdots\Big) \\
&= \sigma_F^\ell(h(x)).
\end{aligned}$$

Hence h is an (ℓ, c)-sliding block code.

Suppose ψ satisfies the surjectivity condition and let $y \in F^\infty$. For each $k \geqslant \ell$ choose $\alpha^{(k)} \in E^{1+k+c}$ such that $\psi(\alpha^{(k)}) = y_{[0,k]}$. Pick $x^{(k)} \in Z_E(\alpha^{(k)})$. Since E^∞ is compact the sequence $(x^{(k)})_k$ has a convergent subsequence; let $x \in E^\infty$ be its limit. Then $h(x) = y$ and h is surjective.

Next, assume ψ satisfies the injectivity condition and suppose $h(x) = h(x')$ for some $x, x' \in E^\infty$. Let $k \geqslant \ell$ and choose $K \geqslant c$ in accordance with the injectivity condition. Then

$$\psi(x_{[0,k+K]}) = h(x)_{[0,k+K-c]} = h(x')_{[0,k+K-c]} = \psi(x'_{[0,k+K]}),$$

from which it follows that $x_{[0,k]} = x'_{[0,k]}$. As this is the case for every $k \geqslant \ell$, we see that $x = x'$ and that h is injective.

For the other direction, let $h\colon E^\infty \longrightarrow F^\infty$ be an (ℓ, c)-sliding block code. Define $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ by

$$\psi(x_{[0,\ell+c]}) = h(x)_{[0,\ell]}$$

for $x \in E^\infty$. This is well-defined by the choice of c and ψ is compatible with E and F. Hence ψ is an (ℓ, c)-block map. Furthermore, the (ℓ, c)-sliding block code h_ψ induced by ψ coincides with h.

Fix $\beta \in F^{1+k}$ for some $k \geqslant \ell$ and let $\beta y \in Z_F(\beta)$. If h is surjective, we may choose $x \in E^\infty$ such that $h(x) = \beta y$. Note that $\psi(x_{[0,k+c]}) = h(x)_{[0,k]} = \beta$. Hence ψ satisfies the surjectivity condition.

If h is injective, then it is a homeomorphism onto its image; let $\phi\colon h(E^\infty) \longrightarrow E^\infty$ be the inverse homeomorphism. Fix $k \geqslant \ell$. Since ϕ is continuous there is $c_\phi \in \mathbb{N}$ such that $\phi(h(x)_{[0,k+c_\phi]}) = \phi(h(x')_{[0,k+c_\phi]})$ implies $x_{[0,k]} = x'_{[0,k]}$ for $x, x' \in E^\infty$. If $\psi(x_{[0,k+c+c_\phi]}) = \psi(x'_{[0,k+c+c_\phi]})$, for some $x, x' \in E^\infty$, then

$$h(x)_{[0,k+c_\phi]} = h(x')_{[0,k+c_\phi]}$$

by the choice of c and

$$x_{[0,k]} = (\phi h(x))_{[0,k]} = (\phi h(x'))_{[0,k]} = x'_{[0,k]}$$

by the choice of c_ϕ. Hence ψ satisfies the injectivity condition.

Finally, it is straightforward to verify that if $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ is an (ℓ, c)-block map and h is the (ℓ, c)-sliding block code induced from ψ, then the (ℓ, c)-block map ψ_h induced from h coincides with ψ. □

We record the following immediate corollary.

Corollary C.3.9. *Let $\ell, c \in \mathbb{N}$. There is a one-to-one correspondence between the collection of (ℓ, c)-block maps $\psi\colon E^{1+\ell+c} \longrightarrow F^{1+\ell}$ satisfying the* bijectivity condition *and the collection of (ℓ, c)-conjugacies $h\colon E^\infty \longrightarrow F^\infty$. For $c = 0$, there is a one-to-one correspondence between the collection of* bijective *block maps $\psi\colon E^{1+\ell} \longrightarrow F^{1+\ell}$ and the collection of $(\ell, 0)$-conjugacies $h\colon E^\infty \longrightarrow F^\infty$ whose inverses are also $(\ell, 0)$-conjugacies.*

Remark C.3.10. At the level of block maps, the index (ℓ, c) can be interpreted as memory and anticipation, respectively. For one-sided conjugacies, *memory is not allowed* (cf. [**68**, Section 13.8]), however, we have seen that memory of the block map exactly corresponds to eventual conjugacies. At the level of path spaces, the ℓ indicates the lag of the eventual conjugacy while c indicates the continuity complexity of the associated homeomorphism.

C.4. Moves on graphs

In this section, we relate eventual conjugacies with moves on the graph. We first recall the definition of the out-split and in-split of [**116**] modified for graphs (cf. [**68, 4**]). For simplicity, we only define these moves for finite graphs with no sinks.

Definition C.4.1 (Move (O)). Let $G = (G^0, G^1, r_G, s_G)$ be a finite graph with no sinks. Let $v \in G^0$ be a vertex, let $n \in \mathbb{N}_+$ and partition $s_G^{-1}(v)$ into finitely many nonempty sets

$$s_G^{-1}(v) = \mathcal{E}_v^1 \amalg \cdots \amalg \mathcal{E}_v^n. \tag{C.2}$$

The *out-split graph* E of G at v with respect to the partition (C.2) is given by

$$E^0 = \{v_1, \dots, v_n\} \cup \{w_1 \mid w \in G^0 \setminus \{v\}\},$$
$$E^1 = \{e_1, \dots, e_n \mid r_G(e) = v\} \cup \{f_1 \mid r_G(f) \neq v\},$$

with source and range maps $r_E, s_E \colon E^1 \longrightarrow E^0$ given by

$$r_E(e_i) = \begin{cases} r_G(e)_1 & \text{if } s_G(e) \neq v, \\ v_i & \text{if } s_G(e) = v \end{cases}$$

$$s_E(e_i) = \begin{cases} s_G(e)_1 & \text{if } s_G(e) \neq v, \\ v_j & \text{if } s_G(e) = v, e \in \mathcal{E}_v^j, \end{cases}$$

for $e_i \in E^1$.

A directed graph is conjugate to its out-split graph, see [**15**, Corollary 6.2]. Next, we introduce a slight modification of the classical in-split graph, cf. [**43**].

Definition C.4.2 (Move (I-)). Let $G = (G^0, G^1, r_G, s_G)$ be a finite graph with no sinks. Let $v \in G^0$ be a vertex and partition $r_G^{-1}(v)$ into $n \in \mathbb{N}_+$ possibly empty sets

$$\mathcal{E}^v : r^{-1}(v) = \mathcal{E}_1^v \amalg \cdots \amalg \mathcal{E}_n^v. \tag{C.3}$$

The *in-split graph* E of G at v with respect to the partition $\mathcal{E}$ is given by

$$E^0 = \{v^1, \dots, v^n\} \cup \{w^1 \mid w \in G^0 \setminus \{v\}\},$$
$$E^1 = \{e^1, \dots, e^n \mid s_G(e) = v\} \cup \{f^1 \mid s_G(f) \neq v\}$$

with source and range maps $s_E, r_E \colon E^1 \longrightarrow E^0$ given by

$$s_E(e^i) = \begin{cases} s_G(e)^1 & \text{if } s_G(e) \neq v, \\ v^i & \text{if } s_G(e) = v, \end{cases}$$

$$r_E(e^i) = \begin{cases} r_G(e)^1 & \text{if } r_G(e) \neq v, \\ v^j & \text{if } r_G(e) = v, e \in \mathcal{E}_j^v, \end{cases}$$

for $e^i \in E^1$.

Remark C.4.3. The above definition is a modification of [**4**] in that we allow the partition sets to be empty, this is called Move (I-) in [**43**]. Note that this in-split introduces new sources, one for each empty partition set. The out-split introduces no new sources.

Definition C.4.4 (Move (I+)). Let G be a finite graph with no sinks. An *elementary balanced in-split* of G is a pair of in-split graphs E and F of G at the same vertex using the same number of partition sets.

Suppose E and F are elementary balanced in-split graphs of G at $v \in G^0$. We shall see in Proposition C.4.6 that E and F are eventually conjugate. The labelings on the vertices and edges (as in Definition C.4.2) define canonical bijections $\phi \colon E^0 \longrightarrow F^0$ and $\psi^{(0)} \colon E^1 \longrightarrow F^1$ given by $\phi(v_E^i) = v_F^i$ for $v_E^i \in E^0$ and $\psi^{(0)}(e_E^i) = e_F^i$ for $e_E^i \in E^1$, respectively. In general, $\psi^{(0)}$ is not a block map since it is not compatible with E and F. *We shall identify the vertices and edges of E and F via these bijections.*

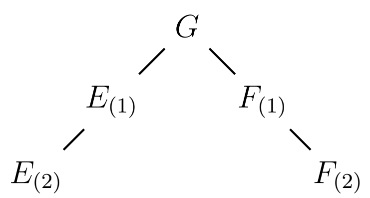
Given $v_E^i \in E^0$ we can perform an in-split of E at v_E^i using n partition elements to obtain a graph $E_{(2)}$. Similarily, we obtain a graph $F_{(2)}$ as an in-split of F at $v_F^i \in F^0$ using n partition elements. Again there is a canonical identification of the vertices and edges in the two graphs. Proposition C.4.6 below shows that $E_{(2)}$ and $F_{(2)}$ are eventually conjugate and we say that they are a *2-step balanced in-split of* G. Iterating this process, we define an ℓ-step balanced in-split recursively.

Definition C.4.5 (Iterated (I+)). Let G be a finite graph with no sinks and let $\ell \in \mathbb{N}$. For $\ell \geqslant 2$, two graphs $E_{(\ell)}$ and $F_{(\ell)}$ are *ℓ-step balanced in-split graphs of* G if the following holds:

- There are graphs $E_{(\ell-1)}$ and $F_{(\ell-1)}$ which are $(\ell-1)$-step balanced in-splits of G;
- $E_{(\ell)}$ is the in-split graph of $E_{(\ell-1)}$ at a vertex $v^i_{E_{(\ell-1)}} \in E^0_{(\ell-1)}$ using m partition elements; and
- $F_{(\ell)}$ is the in-split graph of $F_{(\ell-1)}$ at the identified vertex $v^i_{F_{(\ell-1)}} \in F^0_{(\ell-1)}$ using m partition elements.

For $\ell = 0$, we require that $E_{(0)} = G = F_{(0)}$ and for $\ell = 1$ the graphs $E_{(1)}$ and $F_{(1)}$ should be elementary balanced in-splits of G.

For $\ell = 2$, we depict this as



Proposition C.4.6. *Let G be a finite graph with no sinks and let $\ell \in \mathbb{N}$. If $E_{(\ell)}$ and $F_{(\ell)}$ are ℓ-step balanced in-split graphs of G, then $E_{(\ell)}$ and $F_{(\ell)}$ are ℓ-conjugate. In particular, if E and F are elementary balanced in-split graphs of G, then E and F are 1-conjugate.*

Proof. Suppose E and F are in-split graphs of G at $v \in G^0$ using n sets in the partitions

$$\mathcal{E}^v : r_G^{-1}(v) = \mathcal{E}_1^v \amalg \cdots \amalg \mathcal{E}_n^v, \qquad \mathcal{F}^v : r_G^{-1}(v) = \mathcal{F}_1^v \amalg \cdots \amalg \mathcal{F}_n^v.$$

There is a canonical surjection $q_E \colon E^2 \longrightarrow G^2$ given by

$$q_E(e^i f^j) = ef,$$

for $e^i f^j \in E^2$. The map simply forgets the indeces of the edges. Similarly, there is a canonical surjection $q_F \colon F^2 \longrightarrow G^2$. We will construct a map $\psi^{(1)} \colon E^2 \longrightarrow F^2$ satisfying $q_E = q_F \circ \psi^{(1)}$.
If $ef \in G^2$ and $s(e) = v$ then there is a unique bijection $\theta \colon q_E^{-1}(ef) \longrightarrow q_F^{-1}(ef)$ which preserves the label of the first edge, that is,

$$\theta(e^i f^j) = e^i f^{j'}$$

for $e^if^j \in q_E^{-1}(ef)$. If instead $s(e) \neq v$, then the preimage of ef under q_E is a singleton. We may therefore define $\psi^{(1)}\colon E^2 \longrightarrow F^2$ as

$$\psi^{(1)}(e^if^j) = \begin{cases} q_F^{-1}(ef) & \text{if } s(e) \neq v, \\ \theta(e^if^j) & \text{if } s(e) = v, \end{cases}$$

for $e^if^j \in E^2$. This is bijective and compatible and satisfies $q_E = q_F \circ \psi^{(1)}$. It therefore induces a 1-conjugacy between E^∞ and F^∞.

Step 2. Set $E_{(1)} = E$ and $F_{(1)} = F$ and suppose $E_{(2)}$ is an in-split graph of $E_{(1)}$ at $w_E \in E^0_{(1)}$ while $F_{(2)}$ is an in-split graph of $F_{(1)}$ at an identified vertex $w_F \in F^0_{(1)}$ using m sets in the partitions

$$\mathcal{E}^w : r_E^{-1}(w) = \mathcal{E}^w_1 \amalg \cdots \amalg \mathcal{E}^w_m, \qquad \mathcal{F}^w : r_F^{-1}(w) = \mathcal{F}^w_1 \amalg \cdots \amalg \mathcal{F}^w_m.$$

We will use the block map $\psi^{(1)}\colon E^2_{(1)} \longrightarrow F^2_{(1)}$ constructed in *Step 1* to define a 2-block map $\psi^{(2)}\colon E^3_{(2)} \longrightarrow F^3_{(2)}$. As before there is a canonical surjection $q_{E_{(2)}}\colon E^3_{(2)} \longrightarrow F^3_{(1)}$ given by

$$q_{E_{(2)}}(e^if^jg^k) = efg$$

for $e^if^jg^k \in E_{(2)}$. The map simply forgets the indeces of the edges. Similarily, there is a canonical surjection $q_{F_{(2)}}\colon F^3_{(2)} \longrightarrow F^3_{(1)}$. Since $\psi^{(1)}$ is compatible, we can extend it to a 2-block map $\bar{\psi}^{(1)}\colon E^3_{(1)} \longrightarrow F^3_{(1)}$. We will define a map $\psi^{(2)}\colon E^3_{(2)} \longrightarrow F^3_{(2)}$ such that $\bar{\psi}^{(1)} \circ q_{E_{(2)}} = q_{F_{(2)}} \circ \psi^{(2)}$.

If $e^if^jg^k \in E^3_{(2)}$ and $s(e) = w_E$, then there is a unique bijection $\theta\colon q^{-1}_{E_{(2)}}(efg) \longrightarrow q^{-1}_{F_{(2)}}(\bar{\psi}^{(1)}(efg))$ which preserves the label of the first index, that is,

$$\theta(e^if^jg^k) = e^if^{j'}g^{k'},$$

for $e^if^jg^k \in E^3_{(2)}$. If instead $s(e) \neq w_E$, then $s(\bar{\psi}^{(1)})(efg) \neq w_F$, and the preimage of $efg \in E_{(1)}$ under $q_{E_{(2)}}$ is a singleton, and the preimage of $\bar{\psi}^{(1)}(efg)$ under $q_{F_{(2)}}$ is a singleton. We may therefore define $\psi^{(2)}\colon E^3_{(2)} \longrightarrow F^3_{(2)}$ by

$$\psi^{(2)}(e^if^jg^k) = \begin{cases} q^{-1}_{F_{(2)}}(\bar{\psi}^{(1)}(efg)) & \text{if } s(e) \neq w_E, \\ \theta(e^if^jg^k) & \text{if } s(e) = w_E, \end{cases}$$

for $e^if^jg^k \in E^3_{(2)}$. Then $\psi^{(2)}$ is bijective and compatible and satisfies $\bar{\psi}^{(1)} \circ q_{E_{(2)}} = q_{F_{(2)}} \circ \psi^{(2)}$. It induces a 2-conjugacy between $E^\infty_{(2)}$ and $F^\infty_{(2)}$.

Step ℓ: Assume now that $E_{(\ell)}$ and $F_{(\ell)}$ are ℓ-step balanced in-split graphs of G. In particular, there is an $(\ell-1)$-block map $\psi^{(\ell-1)}\colon E^\ell_{(\ell-1)} \longrightarrow F^\ell_{(\ell-1)}$. This may be extended to a block map $\bar{\psi}^{(\ell-1)}\colon E^{\ell+1}_{(\ell-1)} \longrightarrow F^{\ell+1}_{(\ell-1)}$, and there are canonical surjections $q_{E_{(\ell)}}\colon E^{\ell+1}_{(\ell)} \longrightarrow E^{\ell+1}_{(\ell-1)}$ and $q_{F_{(\ell)}}\colon F^{\ell+1}_{(\ell)} \longrightarrow F^{\ell+1}_{(\ell-1)}$ as in Step 2. Using the same strategy as above, we can define a block map $\psi^{(\ell)}\colon E^{\ell+1}_{(\ell)} \longrightarrow F^{\ell+1}_{(\ell)}$ which is bijective and compatible and satisfies $\bar{\psi}^{(\ell-1)} \circ q_{E_{(\ell)}} = q_{F_{(\ell)}} \circ \psi^{(\ell)}$. It therefore induces an ℓ-conjugacy between $E^\infty_{(\ell)}$ and $F^\infty_{(\ell)}$. $\square$

Next, we prove a converse to Proposition C.4.6. The proof uses ideas from [**8**] (see also [**62**]).

THEOREM C.4.7. *Let E and F be finite directed graphs with no sinks. Then E and F are eventually conjugate if and only if there exists a graph G such that E and F are (conjugate to) the iterated balanced in-split graphs of G.*

PROOF. If E and F are the iterated balanced in-split graphs of G, then they are eventually conjugate by Proposition C.4.6. For the converse implication suppose that E and F are eventually conjugate. We may assume that there is a homeomorphism $h\colon E^\infty \longrightarrow F^\infty$ which is an ℓ-conjugacy and whose inverse $h^{-1}\colon F^\infty \longrightarrow E^\infty$ is also an ℓ-conjugacy and that $c \in \mathbb{N}$ is a continuity constant for h and h^{-1}.

We start by constructing the graph G as follows: The set of vertices is given by

$$G^0 = \{\big(x_{[\ell,\ell+2c]}, y_{[\ell,\ell+2c]}\big) \mid x \in E^\infty, y \in F^\infty, h(x) = y\}$$

with the following transition rule: There is an edge from $\big(x_{[\ell,\ell+c]}, y_{[\ell,\ell+c]}\big)$ to $\big(z_{[\ell,\ell+c]}, w_{[\ell,\ell+c]}\big)$ if and only if $x_{(\ell,\ell+c]} = z_{[\ell,\ell+c)}$ and $y_{(\ell,\ell+c]} = w_{[\ell,\ell+c)}$.

For each $j = 0, \ldots, \ell$, define the graph $E_{(\ell-j)}$ with the following set of vertices

$$E^0_{(j)} = \{\big(x_{[\ell-j,\ell+2c]}, y_{[\ell,\ell+2c]}\big) \mid x \in E^\infty, y \in F^\infty, h(x) = y\}$$

and a transition rule similar to the above. Define also the graph $F_{(\ell-j)}$ with vertices

$$F^0_{(j)} = \{\big(x_{[\ell,\ell+2c]}, y_{[\ell-j,\ell+2c]}\big) \mid x \in E^\infty, y \in F^\infty, h(x) = y\}$$

and a similar transition rule. Then $E_{(0)} = G = F_{(0)}$. We will show that $E_{(\ell)}$ and $F_{(\ell)}$ are iterated balanced in-splits of G.

Whenever $\big(x_{[\ell-j,\ell+2c]}, y_{[\ell,\ell+2c]}\big)$ is a vertex in $E_{(j)}$ and

$$\big(ax_{[\ell-j,\ell+2c]}, y_{[\ell,\ell+2c]}\big), \big(a'x_{[\ell-j,\ell+2c]}, y_{[\ell,\ell+2c]}\big),$$

are distinct vertices in $E_{(j+1)}$, then the vertices have the same future and distinct pasts. It follows that $E_{(j+1)}$ in an in-split graph of $E_{(j)}$ for $j = 0, \ldots, \ell - 1$. Fix a vertex $\big(x_{[\ell,\ell+2c]}, y_{[\ell,\ell+2c]}\big) \in G^0$ and consider the set

$$A = \{z \in E^\infty \mid z_{[\ell,\ell+2c]} = x_{[\ell,\ell+2c]}, h(z)_{[\ell,\ell+2c]} = y_{[\ell,\ell+2c]}\}.$$

It follows from the choice of $c \in \mathbb{N}$ that if $z, z' \in A$ and $h(z)_{[0,\ell)} = h(z')_{[0,\ell)}$ then $z_{[0,\ell)} = z'_{[0,\ell)}$. By a symmetric argument using h^{-1}, there is a bijection between the words of length ℓ which can be appended to $x_{[\ell,\ell+2c]}$ and the words of length ℓ which can be appended to $y_{[\ell,\ell+2c]}$. In particular, if $\ell = 1$ then $E_{(1)}$ and $F_{(1)}$ are balanced in-splits of G.

Assume now that $\ell > 0$ and take $z, z' \in A$ with $z \in Z_E(a_{[0,\ell)})$ and $z' \in Z_E(a_{[0,\ell)})$, for some words $a_{[0,\ell)}, a'_{[0,\ell)} \in E^\ell$ with $a_{[1,\ell)} \neq a'_{[1,\ell)}$. By the choice of c, we see that $h(\sigma_E(z))_{[0,\ell)} \neq h(\sigma_E(z'))_{[0,\ell)}$. It follows that there is a bijection between the set of words of length $\ell-1$ which can be appended to $x_{[\ell,\ell+2c]}$ and the set of words of length $\ell-1$ which can be appended to $y_{[\ell,\ell+2c]}$. Continuing this process we see that for each $j = 0, \ldots, \ell$, the graphs $E_{(\ell-j)}$ and $F_{(\ell-j)}$ are iterated balanced in-splits of G. In particular, $E_{(\ell)}$ and

$F_{(\ell)}$ are ℓ-step balanced in-splits of G.

It remains to verify that $E_{(\ell)}$ can be reached from E by a finite sequence of out-splits, and that $F_{(\ell)}$ can be reached from F by a finite sequence of out-splits.

For each $i = 0, \ldots, 2c$, consider the graph $E^{(\ell+i)}$ with the set of vertices

$$(E^{(\ell+i)})^0 = \big\{ \big(x_{[0,\ell+2c]}, y_{[\ell,\ell+i]}\big) \mid h(x) = y \big\}$$

and an overlapping transition rule similar to the one described above. Then $E^{(\ell)}$ and $E^{[\ell+2c+1]}$ are graph isomorphic, and it is well-known that $E^{[\ell+2c+1]}$ is conjugate to E via a sequence of out-splits, see, e.g., [**68**, Section 2.4]. Furthermore, $E^{(\ell+(i+1))}$ is constructed from $E^{(\ell+i)}$ by a sequence of out-splits. Indeed, if $\big(x_{(0,\ell+2c]}z, w_{[\ell,\ell+i]}\big)$ and $\big(x_{(0,\ell+2c]}z', w'_{[\ell,\ell+i]}\big)$ are distinct vertices which follow $\big(x_{[0,\ell+2c]}, y_{[\ell,\ell+i]}\big)$ in $E^{(\ell)}$, then the latter vertex splits into distinct vertices $\big(x_{[0,\ell+2c]}z, y_{[\ell,\ell+i]}w_{\ell+i}\big)$ and $\big(x_{[0,\ell+2c]}, y_{[\ell,\ell+i]}w'_{\ell+i}\big)$ in $E^{(\ell+1)}$ with identical pasts but distinct futures. A similar argument shows that $F_{(\ell)}$ can be reached from F via a sequence of out-splits.
Hence E and F are conjugate to graphs which are the iterated balanced in-split of the graph G. □

For $\ell = 0$, we have an immediate consequence.

Corollary C.4.8. *One-sided conjugacy among finite graphs with no sinks is generated by out-splits (Move (*`O`*)).*

Finally, we will show that ℓ-step balanced in-split graphs can be connected by a sequence of elementary balanced in-splits.

Theorem C.4.9. *Eventual conjugacy of finite graphs with no sinks is generated by out-splits (Move (*`O`*)) and elementary balanced in-splits (Move (*`I+`*)).*

Proof. Suppose E and F are eventually conjugate graphs. We know from Theorem C.4.7 that E and F are conjugate (via a sequence of out-splits) to graphs which are ℓ-step balanced in-splits of a graph G. We may assume that $\ell \geqslant 2$. If E and F are ℓ-step balanced in-splits of G, we will show that they can be connected by a finite sequence of elementary balanced in-splits.

Let $(E_{(1)}, \ldots, E_{(\ell)})$ and $(F_{(1)}, \ldots, F_{(\ell)})$ be an ℓ-step balanced in-split connecting $E_{(\ell)} = E$ and $F_{(\ell)} = F$. More specifically, for $i = 0, \ldots, \ell-1$, the graph $E_{(i+1)}$ is an in-split of $E_{(i)}$ at the vertex $v_{(i)}$ using $n_{(i)}$ sets. By a slight abuse of notation, we use the same symbol $v_{(i)}$ for the vertex in $F_{(i)}$ which is being in-split using $n_{(i)}$ sets to construct $F_{(i+1)}$. We let $q_{E_{(i+1)}} \colon E^0_{(i+1)} \longrightarrow E^0_{(i)}$ denote the canonical surjection of vertices which simply forgets the labeling.

Construct a graph $E_{(\ell-1,\ell-2)}$ as an in-split of $E_{(\ell-1)}$ at $v_{(\ell-1)}$ using $n_{(\ell-1)}$ sets where every edge is placed in the first partition set. This introduces $n_{(\ell-1)} - 1$ sources

$$\{v^2_{(\ell-1)}, \ldots, v^{n_{(\ell-1)}}_{(\ell-1)}\} \subseteq E^0_{(\ell-1,\ell-2)},$$

and the graphs $E_{(\ell-1,\ell-2)}$ and $E_{(\ell)}$ are elementary balanced in-splits of $E_{(\ell-1)}$, by construction. Next we construct a graph $E'_{(\ell-2)}$ as $E_{(\ell-2)}$ with sources attached. More precisely,

$$(E'_{(\ell-2)})^0 = E^0_{(\ell-2)} \cup \{v^j_{(\ell-1)} \mid j = 2, \dots, n_{(\ell-1)}\},$$
$$(E'_{(\ell-2)})^1 = E^1_{(\ell-2)} \cup \{e^j \mid j = 2, \dots, n_{(\ell-2)}, e \in s^{-1}_{E_{(\ell-1)}}(v_{(\ell-1)}),$$

with $s_{E'_{(\ell-2)}}(e^j) = v^j_{(\ell-1)}$ and $r_{E'_{(\ell-2)}}(e^j) = q_{E_{(\ell-1)}}(r_{E_{(\ell-1)}}(e))$. The notation reflects the fact that $E_{(\ell-1,\ell-2)}$ is an in-split of $E_{(\ell-1)}$ (at $v_{(\ell-1)}$) and an in-split of $E'_{(\ell-2)}$ (at $v_{(\ell-2)}$).

If $\ell = 2$, we apply a similar procedure to $F_{(\ell)}$ to obtain a graph $F_{(\ell-2,\ell-1)} = F_{(0,1)}$, and $F_{(0,1)}$ and $F_{(2)}$ are elementary balanced in-splits of $F_{(1)}$. Furthermore, $E_{(1,0)}$ and $F_{(0,1)}$ are elementary balanced in-splits of $G' = E'_{(0)} = F'_{(0)}$ at $v_{(0)}$ using $n_{(0)}$ sets.

If $\ell > 2$, construct the graph $E_{(\ell-2,\ell-3)}$ as the in-split of $E'_{(\ell-2)}$ at $v_{(\ell-2)}$ using $n_{(\ell-2)}$ sets where every edge is placed in the first partition set. This introduces $n_{(\ell-2)} - 1$ sources, and $E_{(\ell-1,\ell-2)}$ and $E_{(\ell-2,\ell-3)}$ are elementary balanced in-splits of $E'_{(\ell-2)}$. Next construct the graph $E'_{(\ell-3)}$ as $E_{(\ell-3)}$ with additional sources attached as follows:

$$(E'_{\ell-3})^0 = E^0_{\ell-3} \cup \{v^j_{\ell-1} \mid j = 2, \dots, n_{(\ell-1)}\} \cup \{v^k_{(\ell-2)} \mid k = 2, \dots, n_{(\ell-2)},$$
$$E_{(\ell-3)'}{}^1 = E^1_{(\ell-3)} \cup \{e^j \mid j = 2, \dots, n_{(\ell-1)}, e \in s^{-1}_{E_{(\ell-1)}}(v_{(\ell-1)})\}$$
$$\cup \{f^k \mid k = 2, \dots, n_{(\ell-2)}, f \in s^{-1}_{E_{(\ell-2)}}(v_{(\ell-2)})$$

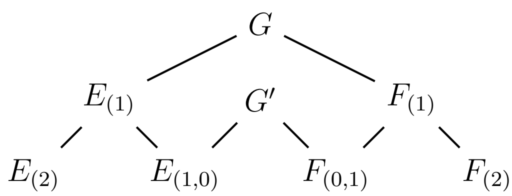
with $s(e^j) = v^j_{(\ell-1)}$ and $s(f^k) = v^k_{(\ell-2)}$, and

$$r(e^j) = q_{E_{(\ell-2)}} \circ q_{E_{(\ell-1)}}(r_{E_{(\ell-1)}}(e)), \quad r(f^k) = q_{E_{(\ell-2)}}(r_{E_{(\ell-2)}}(f)).$$

Note that $E_{(\ell-2,\ell-3)}$ is an in-split of $E'_{(\ell-3)}$ (at $v_{(\ell-3)}$).

Applying the same procedure starting in $F_{(\ell)}$ and iterating the process, we finally obtain a graph $G' = E'_{(0)} = F'_{(0)}$ with $\sum_{i=0}^{\ell-1}(n_{(i)}-1)$ sources attached, and graphs $E_{(1,0)}$ and $F_{(0,1)}$ which are elementary balanced in-splits of G'. Therefore, $E_{(\ell)}$ and $F_{(\ell)}$ are connected by $2\ell - 1$ elementary balanced in-splits. $\square$

Remark C.4.10. For $\ell = 2$, the procedure in the proof above looks like this



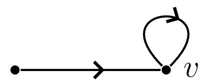
We illustrate this in Example C.5.2.

Remark C.4.11. In the proof of Theorem C.4.9 it is crucial to make use of graphs with sources in order to connect ℓ-step balanced in-split graphs by elementary balanced in-splits. This clearifies the utility of considering the class of finite graphs with no sinks (but potentially with sources) instead of the smaller class of essential finite graphs. It is not clear if a similar result can be proved without the use of sources.

C.5. Examples

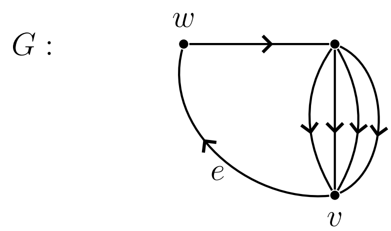
EXAMPLE C.5.1. This example shows that one-sided conjugacy and eventual conjugacy can be distinguished among graphs with finite path spaces. Consider the graphs

The adjacency matrices of the graphs have different total column amalgamations, so they are not conjugate. However, we can construct E and F as a balanced in-split of the graph

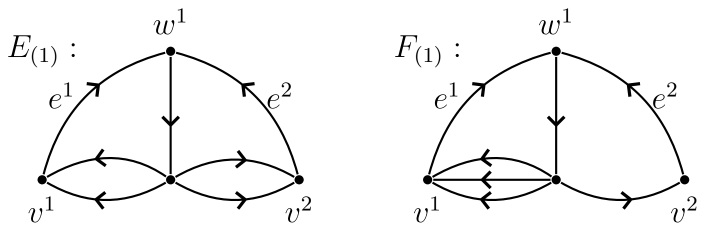


at the vertex v. Hence E and F are 1-conjugate. In fact, any bijection of the path spaces which maps the loop in E to the loop in F is an explicit 1-conjugacy.

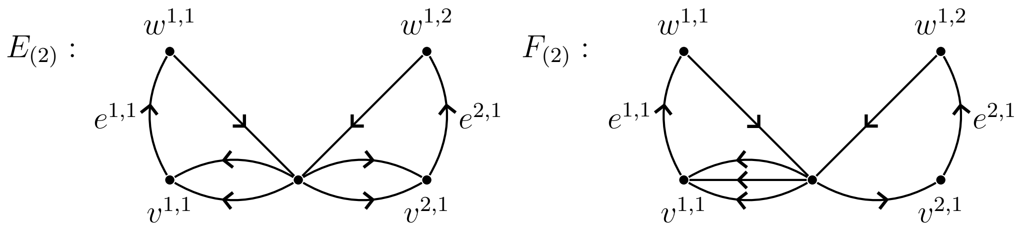
EXAMPLE C.5.2. In this example we generate ℓ-conjugate graphs. Consider the graph



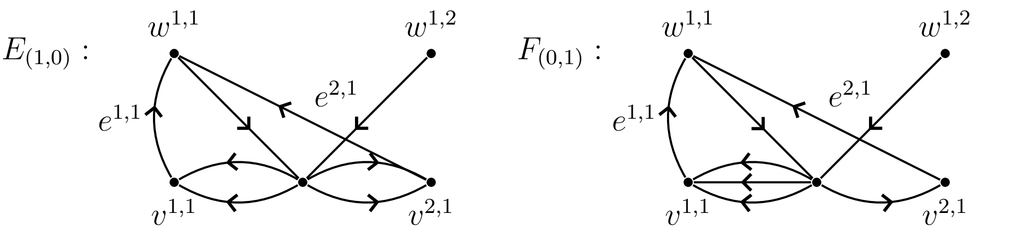
where e is a path of length $\ell - 1$. If we perform balanced in-splits along the vertices of the path e ending at w we will obtain ℓ-conjugate graphs. The case with $\ell = 1$ and $v = w$ appeared in [**12**].
We illustrate the procedure when $\ell = 2$ and e is a path of length one. A balanced in-split at v is given by



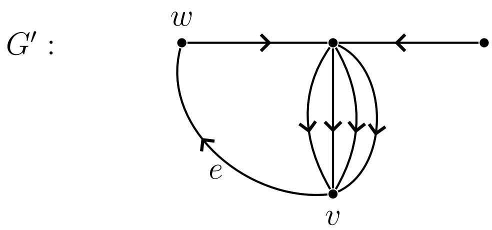
Next, we perform a balanced in-split at w^1 to get



and the graphs $E_{(2)}$ and $F_{(2)}$ are 2-conjugate.
Finally, we use the procedure of the proof of Theorem C.4.9 for this example, see the picture of Remark C.4.10. Observe that the graphs $E_{(1,0)}$ and $F_{(0,1)}$ below



are elementary balanced in-splits of



Hence $E_{(2)}$ and $F_{(2)}$ are connected by three elementary balanced in-splits.

ARTICLE D

C*-simplicity and representations of topological full groups of groupoids

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Abstract

Given an ample groupoid G with compact unit space, we study the canonical representation of the topological full group $[[G]]$ in the full groupoid C^*-algebra $\mathrm{C}^*(G)$. In particular, we show that the image of this representation generates $\mathrm{C}^*(G)$ if and only if $\mathrm{C}^*(G)$ admits no tracial state. The techniques that we use include the notion of groups covering groupoids.

As an application, we provide sufficient conditions for C^*-simplicity of certain topological full groups, including those associated with topologically free and minimal actions of non-amenable and countable groups on the Cantor set.

D.1. Introduction

Topological full groups associated to group actions on the Cantor set have given rise to examples of groups with interesting new properties. See, e.g., [**56**] and [**92**] for recent developments. In the context of groupoids, the topological full group was introduced by H. Matui in [**87**], who investigated their relation with homology groups of groupoids.

Following a slightly different approach, V. Nekrashevych ([**93**]) defined the topological full group $[[G]]$ of an ample groupoid G with compact unit space to consist of the clopen bisections $U \subseteq G$ such that $r(U) = s(U) = G^{(0)}$. In this paper, we study the unitary representation $\pi\colon [[G]] \longrightarrow \mathrm{C}^*(G)$ given by $\pi(U) := 1_U$, for every $U \in [[G]]$. Let $\mathrm{C}^*_\pi([[G]])$ denote the C^*-algebra generated by $\pi([[G]])$ in $\mathrm{C}^*(G)$.

Our main result is as follows:

Theorem (Theorems D.4.3 and D.4.6). Let G be an ample groupoid with compact unit space such that the orbit of each $x \in G^{(0)}$ has at least three points. Then $\overline{\mathrm{span}}\{1 - 1_U \in \mathrm{C}^*(G) \mid U \in [[G]]\}$ is a hereditary C^*-subalgebra of $\mathrm{C}^*(G)$. Moreover, $\mathrm{C}^*(G)$ admits no tracial state if and only if $\mathrm{C}^*_\pi([[G]]) = \mathrm{C}^*(G)$.

This generalizes part of [**53**, Proposition 5.3] (see Remark D.4.7). If, in addition, G is second countable, essentially principal and minimal, then $\mathrm{C}^*_r(G)$ is stably isomorphic to $\overline{\mathrm{span}}\{1 - 1_U \in \mathrm{C}^*_r(G) \mid U \in [[G]]\}$ (Corollary D.4.4).

Given an ample groupoid G with compact unit space, let π_r denote the canonical representation of $[[G]]$ in $\mathrm{C}^*_r(G)$.

Recall that a group is said to be C*-simple if its reduced C*-algebra is simple. Recently, there has been a lot of progress in understanding this notion, and new characterizations of C*-simplicity have been obtained (see [**10**], [**57**], [**59**]). In [**66**], A. Le Boudec and N. Matte Bon showed that a countable group of homeomorphisms on a Hausdorff space X is C*-simple if the rigid stabilizers of non-empty and open subsets of X are non-amenable. By using this result, we show the following:

Theorem (Theorem D.5.2). Let G be a second countable, essentially principal, minimal and ample groupoid with compact unit space. If

(i) G is not amenable, or
(ii) π_r does not weakly contain the trivial representation,

then $[[G]]$ is C*-simple.

Consequently, the topological full group associated with a topologically free and minimal action of a countable and non-amenable group on the Cantor set is C*-simple (Corollary D.5.4). For free actions, this was shown to be true in [**66**].

The paper is organized as follows. In Section D.2, we collect basic definitions about groupoids, establish notation and present some relevant examples.
In Section D.3, we study groups covering groupoids. Given an ample groupoid G with compact unit space, a subgroup $\Gamma \leqslant [[G]]$ is said to cover G if $G = \bigcup_{U\in\Gamma} U$. We investigate under which conditions $[[G]]$ covers G and show that $C^*_\pi([[G]])$ admits a character if and only if $G^{(0)}$ admits a G-invariant probability measure (Corollary D.3.7).
In Section D.4, we analyze the representation of the topological full group in the full and the reduced groupoid C*-algebras to reach the main theorem above.
In Section D.5, we apply the results of Sections D.3 and D.4 in order to study C*-simplicity of the topological full group.

D.2. Preliminaries

In this section we introduce relevant concepts and establish notation. Throughout the paper, we let $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ denote the non-negative integers.

D.2.1. Ample groupoids. A topological groupoid G is *ample* if G is locally compact, Hausdorff, étale (in the sense that the range and source maps $r, s\colon G \longrightarrow G$ are local homeomorphisms onto $G^{(0)}$) and the unit space $G^{(0)}$ is totally disconnected. The *orbit* of a point $x \in G^{(0)}$ is the set $G(x) := r(s^{-1}(x))$, and G is said to be *minimal* if $\overline{G(x)} = G^{(0)}$, for every $x \in G^{(0)}$.

A *bisection* is a subset $S \subseteq G$ such that $r|_S$ and $s|_S$ are injective. Note that, if S is open, then $r|_S$ and $s|_S$ are homeomorphisms onto their images. We will denote by $\mathcal{S}$ the inverse semigroup of open bisections of G, and by $\mathcal{C} \subseteq \mathcal{S}$ the sub-inverse semigroup of compact open bisections. There is a homomorphism θ from $\mathcal{S}$ to the inverse semigroup of homeomorphisms between open subsets of $G^{(0)}$, given by $\theta_U := r \circ (s|_U)^{-1}\colon s(U) \longrightarrow r(U)$. As observed in [**102**], θ is injective if and only if G is *essentially principal* (that is, $\mathrm{Int}\{g \in G : r(g) = s(g)\} = G^{(0)}$).

In the following we let $C_c(G)$ be the collection of complex valued, continuous and compactly supported functions on G. This is a *-algebra with the convolution product

$$f \star g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta),$$

for $f, g \in C_c(G)$ and $\gamma \in G$, and *-involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$, for $f \in C_c(G)$ and $\gamma \in G$.

Let $\mathrm{C}^*_r(G)$ and $\mathrm{C}^*(G)$ denote the reduced and full groupoid C*-algebras, respectively. For an introduction to (étale) groupoids and their C^*-algebras, the reader is refered to, e.g., [**99**] or [**109**].
If G is minimal and essentially principal, then $\mathrm{C}^*_r(G)$ is simple (see, e.g., [**109**, Proposition 4.3.7]).

A regular Borel measure μ on $G^{(0)}$ is G-*invariant* if $\mu(r(S)) = \mu(s(S))$, for each $S \in \mathcal{S}$. Clearly, μ is G-invariant if and only if $\mu(r(U)) = \mu(s(U))$, for each $U \in \mathcal{C}$. The following proposition is well-known.

Proposition D.2.1. *Let G be an ample groupoid with compact unit space. The following conditions are equivalent:*

(i) $G^{(0)}$ *admits a G-invariant probability measure;*
(ii) $\mathrm{C}^*_r(G)$ *admits a tracial state;*
(iii) $\mathrm{C}^*(G)$ *admits a tracial state.*

Proof. The proof of the implications (i) $\implies$ (ii) $\implies$ (iii) can be found in [**99**, Theorem 3.4.4].
(iii) $\implies$ (i): Let τ be a tracial state on $\mathrm{C}^*(G)$. Given $U \in \mathcal{C}$, we have

$$\tau(1_{r(U)}) = \tau(1_U 1_{U^{-1}}) = \tau(1_{U^{-1}} 1_U) = \tau(1_{s(U)}).$$

Thus, the probability measure on $G^{(0)}$ induced by $\tau|_{C(G^{(0)})}$ is G-invariant. □

Suppose $G^{(0)}$ admits a G-invariant measure μ. Then there is a representation $\rho\colon C_c(G) \longrightarrow B(L^2(G^{(0)}, \mu))$ given by

$$(\rho(f)(\xi))(x) := \sum_{g \in r^{-1}(x)} f(g)\xi(s(g)), \tag{D.1}$$

for $f \in C_c(G)$, $\xi \in L^2(G^{(0)}, \mu)$ and $x \in G^{(0)}$.

Note that $\rho|_{C(G^{(0)})}$ is the representation by multiplication operators. Moreover, if U is a compact open bisection, then

$$(\rho(1_U)(\xi))(x) = \begin{cases} \xi(\theta_U^{-1}(x)), & x \in s(U), \\ 0, & x \notin s(U), \end{cases}$$

for $\xi \in L^2(G^{(0)}, \mu)$ and $x \in G^{(0)}$.

D.2.2. Topological full groups. Given an ample groupoid G with compact unit space, the *topological full group* of G is

$$[[G]] := \{U \in \mathcal{C} \mid r(U) = s(U) = G^{(0)}\}.$$

This definition coincides with the one from **[93]**. In **[87]**, however, H. Matui defines the topological full group of G as $\theta([[G]])$. Therefore, if G is essentially principal then θ is injective and the two definitions coincide.
Two examples to have in mind are as follows.

EXAMPLE D.2.2. Let φ be an action of a group Γ on a compact Hausdorff space X. As a space, the *transformation groupoid* associated with φ is $G_\varphi := \Gamma \times X$ equipped with the product topology. The product of two elements $(h, y), (g, x) \in G_\varphi$ is defined if and only if $y = gx$ in which case $(h, gx)(g, x) := (hg, x)$. Inversion is given by $(g, x)^{-1} := (g^{-1}, gx)$. The unit space $G^{(0)}$ is naturally identified with X and G_φ is ample if X is totally disconnected.
The topological full group $[[G_\varphi]]$ consists of sets of the form $\bigcup_{i=1}^n \{g_i\} \times A_i$, where $g_1, \ldots, g_n \in \Gamma$ and $A_1, \ldots, A_n \subseteq X$ are clopen sets such that

$$X = \bigsqcup_{i=1}^n A_i = \bigsqcup_{i=1}^n g_i A_i.$$

In particular, there is a canonical injective homomorphism $\Gamma \longrightarrow [[G_\varphi]]$ sending $g \longmapsto \{g\} \times X$.

EXAMPLE D.2.3. Let $X := \{0,1\}^{\mathbb{N}}$ be the full one-sided 2-shift and consider the Deaconu-Renault groupoid

$$G_{[2]} := \{(y, n, x) \in X \times \mathbb{Z} \times X \mid \exists l, k \in \mathbb{N} : n = l - k, y_{l+i} = x_{k+i} \ \forall i \in \mathbb{N}\}.$$

The product of $(z, n, y'), (y, m, x) \in G_{[2]}$ is well-defined if and only if $y' = y$ in which case $(z, n, y)(y, m, x) := (z, n + m, x)$. Inversion is given by $(y, n, x)^{-1} := (x, -n, y)$.
Let X_f be the set of finite words (including the empty word) on the alphabet $\{0, 1\}$. Given $\alpha \in X_f$, let $|\alpha|$ denote its length and let $\overline{\alpha} := \{x \in X \mid x_i = \alpha_i, 0 \leqslant i < |\alpha|\}$ be the cylinder set of α. The topology on $G_{[2]}$ is generated by sets of the form

$$Z(\beta, \alpha) := \{(y, |\beta| - |\alpha|, x) \in G_{[2]} \mid y \in \overline{\beta}, x \in \overline{\alpha}, y_{|\beta|+i} = x_{|\alpha|+i} \ \forall i \in \mathbb{N}\},$$

for $\alpha, \beta \in X_f$. This topology is strictly finer than the one inherited from the product topology and $G_{[2]}$ is ample with compact unit space. Note as well that $G_{[2]}$ is minimal.
The topological full group $[[G_{[2]}]]$ consists of sets of the form

$$\bigcup_{j=1}^n Z(\beta^j, \alpha^j), \tag{D.2}$$

with $X = \bigsqcup_{j=1}^n \overline{\alpha^j} = \bigsqcup_{j=1}^n \overline{\beta^j}$.
We would now like to recall the isomorphism between Thompson's group V and $[[G_{[2]}]]$, observed in **[88]** (see also **[90]** and **[91]**).
Thompson's group V consists of piecewise linear, right continuous bijections on $[0, 1)$ which have finitely many points of non-differentiability, all being dyadic rationals, and have a derivative which is a power of 2 at each point of differentiability.

Given $\alpha, \beta \in X_f$, let $\psi(\alpha) := \sum_i \alpha_i 2^{-i} \in [0,1)$ and $I(\alpha) := [\psi(\alpha), \psi(\alpha) + 2^{-|\alpha|})$. The isomorphism from $[[G_{[2]}]]$ to V takes $\bigcup_j Z(\beta^j, \alpha^j)$ as in (D.2) and sends it to the bijection on $[0,1)$ which, restricted to $I(\alpha^j)$, is linear, increasing and onto $I(\beta^j)$, for every j.

The next example shows that the short exact sequence induced by the quotient $\theta\colon [[G]] \longrightarrow \theta([[G]])$ is not always split. Since we are interested in studying the canonical representation of $[[G]]$ in $\mathrm{C}^*(G)$, this illustrates why we have chosen to treat the topological full group as bisections, rather than homeomorphisms on the unit space.

Example D.2.4. Let $X := \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of $\mathbb{Z}$ and define an action $\varphi\colon \mathbb{Z} \curvearrowright X$ by

$$\varphi_n(x) := \begin{cases} (-1)^n x, & x \in \mathbb{Z}, \\ \infty, & x = \infty, \end{cases}$$

for $n \in \mathbb{Z}$. Note that $\{1\} \times X$ is a compact open bisection in the transformation groupoid G_φ and that the homeomorphism

$$\theta_{\{1\}\times X}(x) = \begin{cases} -x, & x \in \mathbb{Z}, \\ \infty, & x = \infty, \end{cases}$$

for $x \in X$, has order 2.
Moreover, for any $U \in [[G_\varphi]]$ satisfying $\theta_U = \theta_{\{1\}\times X}$, there is an odd integer n such that $(n, \infty) \in U$. In particular, U has infinite order. Therefore, the short exact sequence induced by $\theta\colon [[G_\varphi]] \longrightarrow \theta([[G_\varphi]])$ is not split.

D.2.3. Unitary representations. Let G be an an ample groupoid with compact unit space. There is a unitary representation

$$\begin{aligned} \pi\colon [[G]] &\longrightarrow \mathrm{C}^*(G) \\ U &\longmapsto 1_U, \end{aligned}$$

We will denote the analogous representation of $[[G]]$ in $\mathrm{C}^*_r(G)$ by π_r.

If σ and η are unitary representations of a group Γ on unital C^*-algebras, then σ is said to *weakly contain* η if

$$\left\| \sum_i \alpha_i \eta(g_i) \right\| \leqslant \left\| \sum_i \alpha_i \sigma(g_i) \right\|,$$

for every $\sum_i \alpha_i g_i \in \mathbb{C}\Gamma$. The *trivial representation* $\Gamma \longrightarrow \mathbb{C}$ satisfies $g \longmapsto 1$, for every $g \in \Gamma$.

Given a unitary representation η of Γ on a unital C^*-algebra A, we denote by $\mathrm{C}^*_\eta(\Gamma)$ the C^*-algebra generated by the image of η. Note that if η weakly contains the trivial representation, then $\mathrm{C}^*_\eta(\Gamma)$ admits a character whose kernel is $\overline{\mathrm{span}}\{1_A - \eta(g) \mid g \in \Gamma\}$.

Proposition D.2.5. *Let η be a unitary representation of a group Γ on a unital C^*-algebra A. Then η weakly contains the trivial representation if and only if $1_A \notin \overline{\mathrm{span}}\{1_A - \eta(g) \mid g \in \Gamma\}$.*

PROOF. The forward implication is evident, so we only prove the backward one. Let $B := \overline{\operatorname{span}}\{1_A - \eta(g) : g \in \Gamma\}$. If $1_A \notin B$, then, since B is a C*-algebra, $\operatorname{dist}(1_A, B) = 1$. Hence, for every $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $g_1, \dots, g_n \in \Gamma$, we have that

$$\left\| \sum \alpha_i \eta(g_i) \right\| = \left\| \left(\sum \alpha_i \right) .1_A - \sum \alpha_i (1_A - \eta(g_i)) \right\| \geqslant \left| \sum \alpha_i \right|,$$

thus showing that η weakly contains the trivial representation. ☐

D.3. Groups covering groupoids

An ample groupoid G can always be covered by compact open bisections. We investigate to which degree G can be covered by compact open bisections U which satisfy $r(U) = s(U) = G^{(0)}$. We show that if $\Gamma \leqslant [[G]]$ covers G and μ is a Γ-invariant probability measure on $G^{(0)}$, then μ is also G-invariant.

DEFINITION D.3.1. Given an ample groupoid G with compact unit space, we say that a subgroup $\Gamma \leqslant [[G]]$ *covers* G if $G = \bigcup_{U \in \Gamma} U$.

The idea of covering a groupoid G by compact open bisections U such that $r(U) = s(U) = G^{(0)}$ has already appeared in H. Matui's study of automorphisms of G, cf. [**87**, Proposition 5.7].
If G is essentially principal, then a subgroup $\Gamma \leqslant [[G]]$ covers G if and only if, for each open bisection S and $x \in s(S)$, there are $U \in \Gamma$ and a neighborhood $W \subseteq s(S)$ of x such that $\theta_U|_W = \theta_S|_W$.

EXAMPLE D.3.2. If φ is an action of a group Γ on a compact Hausdorff and totally disconnected space, then the copy of Γ in $[[G_\varphi]]$ covers G_φ.

EXAMPLE D.3.3. Recall that Thompson's group $T < V$ consists of the elements of Thompson's group V (see Example D.2.3) which have at most one point of discontinuity. Let $G_{[2]}$ be the groupoid of Example D.2.3. Under the identification of V with $[[G_{[2]}]]$, T covers $G_{[2]}$. This follows from the fact that if $I, J \subseteq [0, 1)$ are left-closed and right-open intervals with endpoints in $\mathbb{Z}[1/2]$, then there exists a piecewise linear homeomorphism $f\colon I \longrightarrow J$ with a derivative which is a power of 2 at each point of differentiability and with finitely many points of non-differentiability, all of which belong to $\mathbb{Z}[1/2]$.

LEMMA D.3.4. *Let G be an ample groupoid with compact unit space. If $|G(x)| \geqslant 2$ for every $x \in G^{(0)}$, then $[[G]]$ covers G.*

PROOF. Let $g \in G$. If $r(g) \neq s(g)$, then there is a compact open bisection V containing g and such that $s(V) \cap r(V) = \emptyset$. Let $U := V \cup V^{-1} \cup (G^{(0)} \setminus (s(V) \cup r(V)))$. Then $g \in U \in [[G]]$.
If $r(g) = s(g)$, then there is $h \in s^{-1}(r(g))$ such that $r(hg) = r(h) \neq s(h) = s(hg)$ since $|G(r(g))| \geqslant 2$. As before, there are $U, U' \in [[G]]$ such that $h \in U$ and $hg \in U'$. Hence, $g \in U^{-1}U' \in [[G]]$. ☐

The purpose of the next example is to show that the above result may fail if one does not make any assumption on the orbits.

EXAMPLE D.3.5. Consider $X := \mathbb{Z} \cup \{\pm\infty\}$ equipped with the order topology and let $\varphi\colon \mathbb{Z} \curvearrowright X$ be the action given by $\varphi_t(x) := t + x$, for $t \in \mathbb{Z}$ and $x \in X$. The transformation groupoid G_φ is ample with compact unit space.

Given $x, z \in X$ we put $[x, z] := \{y \in X : x \leqslant y \leqslant z\}$. Then

$$H := \{(t, x) \in \mathbb{Z} \times [0, +\infty] : -t \leqslant x\}$$

is an ample subgroupoid of G_φ. Incidentally, this is the groupoid of the partial action obtained by restricting φ to $[0, +\infty]$ (see **[46]** and **[67]** for more details). Observe that $|H((0, +\infty))| = 1$.

We claim that if $U \in [[H]]$, then $(1, +\infty) \notin U$. Otherwise, there is $t \in \mathbb{N}$ such that $S := \{1\} \times [t, +\infty] \subseteq U$ and $U \setminus S \in \mathcal{C}$. But then $s(U \setminus S) = [0, t-1]$ and $r(U \setminus S) = [0, t]$ contradicting the fact that r and s are injective on $U \setminus S$. Hence, $(1, +\infty) \notin U$ and $[[H]]$ does not cover H.

Recall that a probability measure μ on $G^{(0)}$ is G-*invariant* if $\mu(s(S)) = \mu(r(S))$ for every $S \in \mathcal{S}$. Moreover, if $\Gamma \leqslant [[G]]$, then we say μ is Γ-*invariant* if it is invariant with respect to the action θ.

Proposition D.3.6. *Let G be an ample groupoid with compact unit space and Γ a subgroup of $[[G]]$. Consider the following conditions:*

(i) *$G^{(0)}$ admits a G-invariant probability measure;*
(ii) *$\pi|_\Gamma$ weakly contains the trivial representation;*
(iii) *$C^*_\pi(\Gamma)$ admits a character;*
(iv) *$G^{(0)}$ admits a Γ-invariant probability measure.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). If Γ covers G, then (iv) $\Longrightarrow$ (i) and all conditions are equivalent.

Proof. (i) $\Longrightarrow$ (ii): Suppose μ is a G-invariant measure on $G^{(0)}$ and let $\rho\colon C_c(G) \longrightarrow B(L^2(G^{(0)}, \mu))$ be the representation given by (D.1). The vector $1_{G^{(0)}} \in L^2(G^{(0)}, \mu)$ is invariant for the representation $\rho \circ \pi|_\Gamma \colon \Gamma \longrightarrow B(L^2(G^{(0)}, \mu))$. Hence, $\pi|_\Gamma$ weakly contains the trivial representation.

The implication (ii) $\Longrightarrow$ (iii) is evident.

(iii) $\Longrightarrow$ (iv): Let φ be a character on $\mathrm{C}^*_\pi(\Gamma)$ and τ a state on $\mathrm{C}^*(G)$ which is an extension of φ. Then $\mathrm{C}^*_\pi(\Gamma)$ is in the multiplicative domain of τ. Clearly, $\tau|_{C(G^{(0)})}$ induces a Γ-invariant probability measure on $G^{(0)}$.

Now, suppose Γ covers $[[G]]$ and let us show that (iv) $\Longrightarrow$ (i). Let μ be a Γ-invariant probability measure on $G^{(0)}$. We claim that μ is also G-invariant. Indeed, since Γ covers G, given $S \in \mathcal{C}$, we have that $S = \bigcup_{U \in \Gamma}(S \cap U)$. As S is compact, there are $S_1, \ldots, S_n \in \mathcal{C}$ and $U_1, \ldots, U_n \in \Gamma$ such that $S = \bigsqcup_i S_i$ and $S_i \subseteq U_i$ for $1 \leqslant i \leqslant n$. In particular, $\theta_{U_i}(s(S_i)) = r(S_i)$ for every i. It follows that

$$\mu(r(S)) = \sum_{i=1}^{n} \mu(r(S_i)) = \sum_{i=1}^{n} \mu(s(S_i)) = \mu(s(S)).$$

Therefore, μ is a G-invariant probability measure on $G^{(0)}$. $\square$

Corollary D.3.7. *Let G be an ample groupoid with compact unit space. The following conditions are equivalent:*

(i) $G^{(0)}$ *admits a* G*-invariant probability measure;*
(ii) π *weakly contains the trivial representation;*
(iii) $C^*_\pi([[G]])$ *admits a character;*
(iv) $G^{(0)}$ *admits a* $[[G]]$*-invariant probability measure.*

PROOF. The implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) follow from Proposition D.3.6. (iv) $\implies$ (i): If, for each $x \in G^{(0)}$, $|G(x)| \geqslant 2$, then the result follows from Lemma D.3.4 and Proposition D.3.6.
If there is $x \in X$ such that $|G(x)| = 1$, then point evaluation at x is a G-invariant probability measure. □

D.4. Representations of topological full groups

In this section, we prove the main results of the article. We start with two technical lemmas.

LEMMA D.4.1. *Let G be an ample groupoid with compact unit space. If $S, T \in [[G]]$ and $W \subseteq G^{(0)}$ is a clopen subset such that $\theta_S(W), W, \theta_T^{-1}(W)$ are mutually disjoint, then $(1 - 1_S)1_W(1 - 1_T) \in \operatorname{span}\{1 - 1_U \in C_c(G) \mid U \in [[G]]\}$.*

PROOF. We have

$$\begin{aligned}(1 - 1_S)1_W(1 - 1_T) &= 1_{SWT} + 1_{T^{-1}WS^{-1}} + 1_W + 1_{G^{(0)}\setminus(\theta_S(W)\cup W\cup\theta_T^{-1}(W))} \\ &\quad - (1_{T^{-1}WS^{-1}} + 1_{G^{(0)}\setminus(\theta_S(W)\cup W\cup\theta_T^{-1}(W))} + 1_{SW} + 1_{WT}).\end{aligned}$$

The sets $SWT, T^{-1}WS^{-1}, W$ and $G^{(0)} \setminus (\theta_T^{-1}(W) \cup W \cup \theta_S(W))$ are mutually disjoint and their union is in $[[G]]$. This is also the case for the sets $T^{-1}WS^{-1}, SW, WT$ and $G^{(0)} \setminus (\theta_S(W) \cup W \cup \theta_T^{-1}(W))$ and so the result follows. □

In order to employ Lemma D.4.1, the following result will be useful.

LEMMA D.4.2. *Let G be an ample groupoid with compact unit space. If $x \in G^{(0)}$ and $y \in G(x) \setminus \{x\}$, then*

$$\begin{aligned}\operatorname{span}\{1 - 1_U \mid U \in [[G]]\} &= \operatorname{span}\{1_L(1 - 1_S) \mid S, L \in [[G]], \theta_S(x) = y\} &\text{(D.3)}\\ &= \operatorname{span}\{(1 - 1_T)1_R \mid T, R \in [[G]], \theta_T^{-1}(x) = y\}. &\text{(D.4)}\end{aligned}$$

PROOF. Let

$$B := \operatorname{span}\{1_L(1 - 1_S) \mid S, L \in [[G]], \theta_S(x) = y\}$$

and take $U \in [[G]]$. We will show that $1 - 1_U \in B$.
If $\theta_U(x) = x$, we take $L \in [[G]]$ such that $\theta_{LU}(x) = \theta_L(x) = y$. Then $1 - 1_U = 1_{L^{-1}}(1_L - 1) + 1_{L^{-1}}(1 - 1_{LU}) \in B$.

On the other hand, if $\theta_U(x) \neq x$, we take $L \in [[G]]$ such that $\theta_L(x) = x$ and $\theta_{LU}(x) = y$. Then $\theta_{L^{-1}}(x) = x$ so $1 - 1_{L^{-1}} \in B$ by the above. Hence $1 - 1_U = (1 - 1_{L^{-1}}) + 1_{L^{-1}}(1 - 1_{LU}) \in B$ proving (D.3).

By taking adjoints and interchanging x and y, the equality in (D.4) follows from (D.3). □

The next result generalizes [**107**, Theorem 3.7], which was obtained in the setting of Cantor minimal $\mathbb{Z}$-systems.

THEOREM D.4.3. *Let G be an ample groupoid with compact unit space. If $|G(x)| \geqslant 3$ for every $x \in G^{(0)}$, then $\overline{\text{span}}\{1 - 1_U \in \mathrm{C}^*(G) \mid U \in [[G]]\}$ is a hereditary C^*-subalgebra of $\mathrm{C}^*(G)$.*

PROOF. Let $B := \overline{\text{span}}\{1 - 1_U \in \mathrm{C}^*(G) \mid U \in [[G]]\}$. We will first show that

$$BC(G^{(0)})B \subseteq B. \tag{D.5}$$

It suffices to prove that, given $U, V \in [[G]]$, there is a basis $\mathcal{W}$ for $G^{(0)}$ consisting of compact open sets satisfying $(1 - 1_U)1_W(1 - 1_V) \in B$, for each $W \in \mathcal{W}$. Take $x \in G^{(0)}$ and let y and z be distinct elements in $G(x)\setminus\{x\}$. By Lemma D.4.2, there are $n \in \mathbb{N}$ and $L_1, \ldots, L_n$,$U_1, \ldots, U_n$, $V_1, \ldots, V_n$ and $R_1, \ldots, R_n$ in $[[G]]$ and $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n \in \mathbb{C}$ such that

$$1 - 1_U = \sum_{i=1}^{n} \alpha_i 1_{L_i}(1 - U_i), \qquad 1 - 1_V = \sum_{i=1}^{n} \beta_i (1 - 1_{V_i}) R_i$$

with $\theta_{U_i}(x) = y$ and $\theta_{V_i}^{-1}(x) = z$ for every $i = 1, \ldots, n$. By Lemma D.4.1, we see that $(1 - 1_U)1_W(1 - 1_V) \in B$ for every sufficiently small compact open neighborhood W of x. This proves (D.5).

Next we show that $BC^*(G)B \subseteq B$. It suffices to prove that $B1_W B \subseteq B$, for every W in a basis for G consisting of compact open sets. Given $g \in G$, take $U \in [[G]]$ such that $\theta_U(r(g)) \neq s(g)$. Then, for $W \subseteq G^{(0)}$ sufficiently small compact open neighborhood of g, we have that $\theta_U(r(W)) \cap s(W) = \emptyset$. Let

$$V := UW \cup (UW)^{-1} \cup (G^{(0)} \setminus (\theta_U(r(W)) \cup s(W))) \in [[G]].$$

Since $\theta_U(r(W)) \cap s(W) = \emptyset$, we have $UWV = \theta_U(r(W)) \subseteq G^{(0)}$ and, finally,

$$B1_W B = B(1_U 1_W 1_V)B = B1_{\theta_U(r(W))}B \subseteq B$$

by (D.5). □

COROLLARY D.4.4. *Let G be an ample groupoid with compact unit space. If $|G(x)| \geqslant 3$ for every $x \in G^{(0)}$, then $\overline{\text{span}}\{1 - 1_U \in \mathrm{C}_r^*(G) \mid U \in [[G]]\}$ is a hereditary C^*-subalgebra of $\mathrm{C}_r^*(G)$. If, in addition, G is second countable, essentially principal and minimal, then $\overline{\text{span}}\{1 - 1_U \in \mathrm{C}_r^*(G) \mid U \in [[G]]\}$ is stably isomorphic to $\mathrm{C}_r^*(G)$.*

PROOF. The first assertion follows directly from Theorem D.4.3 while the second follows from simplicity of $\mathrm{C}_r^*(G)$ and Brown's theorem [**16**, Theorem 2.8]. □

The next example shows that Theorem D.4.3 does not hold without the hypothesis on orbits.

EXAMPLE D.4.5. Let $X := \mathbb{Z} \cup \{\pm\infty\}$ with the order topology and let φ be the action of the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ on X given by $\varphi_{(n,j)}(x) := n + (-1)^j x$, for $(n, j) \in \mathbb{Z} \rtimes \mathbb{Z}_2$ and $x \in X$. Then $|G_\varphi(x)| \geqslant 2$ for every $x \in G_\varphi^{(0)}$.
By arguing as in Example D.3.5, one concludes that, given $U \in [[G_\varphi]]$, there is $(n, j) \in \mathbb{Z} \rtimes \mathbb{Z}_2$ such that $((n, j), \pm\infty) \in U$.

Let $E\colon \mathrm{C}^*(G_\varphi) \longrightarrow C(G_\varphi^{(0)})$ be the canonical conditional expectation and let $\delta_{+\infty}$ and $\delta_{-\infty}$ be the two states on $C(G_\varphi^{(0)})$ given by point-evaluations at $+\infty$ and $-\infty$, respectively. Then $\delta_{+\infty} \circ E$ and $\delta_{-\infty} \circ E$ are two distinct states on $\mathrm{C}^*(G_\varphi)$ whose restrictions to $B = \overline{\mathrm{span}}\{1 - 1_U \in \mathrm{C}^*(G_\varphi) \mid U \in [[G_\varphi]]\}$ agree. Hence, B is not a hereditary C^*-subalgebra of $\mathrm{C}^*(G_\varphi)$.

By combining Theorem D.4.3 with the results of the previous section, we obtain the following:

THEOREM D.4.6. *Let G be an ample groupoid with compact unit space. Assume that $|G(x)| \geqslant 3$ for every $x \in G^{(0)}$. The following conditions are equivalent.*

(i) $\mathrm{C}^*(G)$ *admits no tracial state;*
(ii) $\mathrm{C}^*_\pi([[G]])$ *admits no character;*
(iii) π *does not weakly contain the trivial representation;*
(iv) $\mathrm{C}^*_\pi([[G]]) = \mathrm{C}^*(G)$.

PROOF. The equivalences (i) $\iff$ (ii) $\iff$ (iii) follow from Proposition D.2.1 and Corollary D.3.7.

(iii) $\implies$ (iv): By Proposition D.2.5 and Theorem D.4.3, $B := \overline{\mathrm{span}}\{1 - 1_U \mid U \in [[G]]\}$ is a hereditary C^*-subalgebra of $\mathrm{C}^*(G)$ and $1_{\mathrm{C}^*(G)} \in B$. Hence, $B = \mathrm{C}^*(G)$. Since $B \subseteq \mathrm{C}^*_\pi([[G]])$ the result follows.

(iv) $\implies$ (i): If $\mathrm{C}^*(G)$ has a tracial state, then $G^{(0)}$ admits an invariant probability measure μ, cf. Proposition D.2.1. Since $|G(x)| > 1$ for each $x \in G^{(0)}$, μ cannot be a point-evaluation. Let ρ be the representation of $C_c(G)$ in $B(L^2(G^{(0)}, \mu))$ as in (D.1). Then ρ extends to a representation of $\mathrm{C}^*(G)$ and of $\mathrm{C}^*_\pi([[G]])$. Note that the vector $1_{G^{(0)}} \in L^2(G^{(0)}, \mu)$ is invariant under $\rho(\pi([[G]]))$ and thus under $\rho|_{\mathrm{C}^*_\pi([[G]])}$. Now, if $\mathrm{C}^*_\pi([[G]]) = \mathrm{C}^*(G)$, then $C(G^{(0)}) \subseteq \mathrm{C}^*_\pi([[G]])$ but $\mathbb{C}1_{G^{(0)}}$ is not invariant under $\rho|_{C(G^{(0)})}$. Indeed, if $X \subseteq G^{(0)}$ is any proper, non-empty subset which is compact and open, then $\rho(1_X)(1_{G^{(0)}}) = 1_X$. Therefore $\mathrm{C}^*_\pi([[G]]) \neq \mathrm{C}^*(G)$. □

REMARK D.4.7. In [**53**, Proposition 5.3], U. Haagerup and K. Olesen considered a certain representation σ of Thompson's group V in the Cuntz algebra $\mathcal{O}_2$ and showed that $\mathrm{C}^*_\sigma(V) = \mathcal{O}_2$. Under the identifications of V with $[[G_{[2]}]]$ (see Example D.2.3) and $\mathcal{O}_2$ with $\mathrm{C}^*(G_{[2]})$, one can check that σ and π coincide. Hence, Theorem D.4.6 recovers part of U. Haagerup and K. Olesen's result.

We now state and prove a version of Theorem D.4.6 regarding $\mathrm{C}^*_r(G)$.

THEOREM D.4.8. *Let G be an ample groupoid with compact unit space. Assume that $|G(x)| \geqslant 3$ for each $x \in G^{(0)}$ and consider the following conditions:*

(i) $\mathrm{C}^*_r(G)$ *admits no tracial state;*
(ii) $\mathrm{C}^*_{\pi_r}([[G]])$ *admits no character;*
(iii) π_r *does not weakly contain the trivial representation;*
(iv) $\mathrm{C}^*_{\pi_r}([[G]]) = \mathrm{C}^*_r(G)$.

Then (i) $\implies$ (ii) $\iff$ (iii) $\iff$ (iv).

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are done as in the full case. (iv) $\Longrightarrow$ (ii). If $C^*_{\pi_r}(G) = C^*_r(G)$ admits a character τ, then $\tau|_{C(G^{(0)})}$ is a point evaluation at some $x \in G^{(0)}$. As τ is a tracial state, it follows that for each compact and open bisection S with $x \in s(S)$, we have $\theta_S(x) = x$. This contradicts the hypothesis that $|G(x)| > 1$. □

The next example shows that the implication from (ii) to (i) in the above theorem fails in general, even in the case when G is a principal, minimal and ample groupoid with unit space homeomorphic to the Cantor set.

Example D.4.9. Let Γ be a non-amenable, countable and residually finite group. There is a descending sequence $(\Gamma_n)_n$ of finite-index normal subgroups of Γ such that the canonical map $j\colon \Gamma \longrightarrow \prod \frac{\Gamma}{\Gamma_n}$ is injective. Then $X := \overline{j(\Gamma)}$ is a topological group homeomorphic to the Cantor set. Furthermore, the action φ by multiplication of Γ on X is free, minimal and the Haar measure on X is Γ-invariant (actions of this sort were studied in detail in [**32**]).
Then $C^*_r(G_\varphi)$ admits a tracial state, whereas $C^*_{\pi_r}([[G_\varphi]])$ does not admit a character, since $C^*_r(\Gamma)$ embeds unitally in it and Γ is non-amenable.

D.5. C*-simplicity of topological full groups

As an application of the above results, we provide conditions which ensure that the topological full group of an ample groupoid is C*-simple.

Recall that an ample groupoid G is *amenable* if there exists a net $(\mu_i)_i$ in $C_c(G)$ of non-negative functions such that

$$\sum_{h\in s^{-1}(r(g))} \mu_i(h) \longrightarrow 1 \quad \text{and} \quad \sum_{h\in s^{-1}(r(g))} |\mu_i(h) - \mu_i(hg)| \longrightarrow 0, \tag{D.6}$$

for $g \in G$, uniformly on compact subsets of G. Amenability of G is equivalent to nuclearity of $C^*_r(G)$, and it implies that $C^*(G)$ and $C^*_r(G)$ are canonically isomorphic. For a proof of these facts, see, e.g., [**17**] and [**104**]. R. Willett constructed in [**115**] an example of non-amenable groupoid G such that $C^*(G)$ is canonically isomorphic to $C^*_r(G)$.

Lemma D.5.1. *Let G be an ample groupoid with compact unit space. If $\Gamma \leqslant [[G]]$ is an amenable subgroup which covers G, then G is amenable.*

Proof. We are going to construct functions satisfying (D.6). Let $K \subseteq G$ be a compact subset and let $\epsilon > 0$. As Γ covers G and is amenable, there are $V_1, \ldots, V_n \in \Gamma$ such that $K \subseteq \bigcup_{i=1}^n V_i$ and a finite subset $F \subseteq \Gamma$ such that

$$\frac{|F \triangle F V_i|}{|F|} < \epsilon,$$

for $1 \leqslant i \leqslant n$. Let $\mu := \frac{1}{|F|} \sum_{U \in F} 1_U$. For $x \in G^{(0)}$ we have $\sum_{h \in s^{-1}(x)} \mu(h) = 1$. Given $g \in K$, take V_i such that $g \in V_i$. Then, for $h \in s^{-1}(r(g))$,

$$\begin{aligned}
|F||\mu(h) - \mu(hg)| &= \left| \sum_{U \in F} 1_U(h) - 1_{UV_i^{-1}}(h) \right| \\
&\leqslant \left| \sum_{U \in F \setminus (FV_i^{-1})} 1_U(h) - \sum_{U \in F \setminus (FV_i)} 1_{UV_i^{-1}}(h) \right| \\
&\quad + \left| \sum_{U \in F \cap FV_i^{-1}} 1_U(h) - \sum_{U \in F \cap FV_i} 1_{UV_i^{-1}}(h) \right| \\
&\leqslant \sum_{U \in F \setminus (FV_i^{-1})} 1_U(h) + \sum_{U \in F \setminus (FV_i)} 1_{UV_i^{-1}}(h).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{h \in s^{-1}(r(g))} |\mu(h) - \mu(hg)| &\leqslant \frac{1}{|F|} \sum_h \left(\sum_{U \in F \setminus (FV_i^{-1})} 1_U(h) + \sum_{U \in F \setminus (FV_i)} 1_{UV_i^{-1}}(h) \right) \\
&= \frac{|F \setminus (FV_i^{-1})| + |F \setminus (FV_i)|}{|F|} < \epsilon.
\end{aligned}$$

Therefore, G is amenable. □

The converse implication is not true in general, see, e.g., Remark D.5.5.

Suppose a group Γ is acting on a set X and let $U \subseteq X$ be a subset. The *rigid stabilizer* of U with respect to the action is the subgroup $\Gamma_U \leqslant \Gamma$ of the elements which pointwise fix the complement $X \setminus U$. Let $W \subseteq G^{(0)}$ be non-empty and clopen and let $G_W = r^{-1}(W) \cap s^{-1}(W)$ be the restricted groupoid. If $[[G]]_W$ is the rigid stabilizer of W with respect to the action $\theta\colon [[G]] \curvearrowright G^{(0)}$, then there is a surjective homomorphism $[[G]]_W \longrightarrow [[G_W]]$ given by restriction. If G is essentially principal this map is an isomorphism.

THEOREM D.5.2. *Let G be a second countable, essentially principal, minimal and ample groupoid with compact unit space. If*

(i) *G is not amenable, or*
(ii) *π_r does not weakly contain the trivial representation,*

then $[[G]]$ is C**-simple.*

PROOF. Assume $[[G]]$ is not C*-simple. By [**66**, Theorem 3.7], there exists a non-empty and clopen $W \subseteq G^{(0)}$ such that the rigid stabilizer $[[G]]_W \cong [[G_W]]$ is amenable. Clearly, G_W is an essentially principal, minimal and ample groupoid with compact unit space. By Lemma D.5.1, G_W is thus amenable and $C^*_r(G_W)$ is nuclear.

Since $C^*_r(G)$ is simple, the projection $1_W \in C^*_r(G)$ is full. It therefore follows from [**87**, Lemma 5.2] and Brown's theorem ([**16**, Theorem 2.8]) that the full corner $C^*_r(G_W) =$

$1_W C_r^*(G) 1_W$ is stably isomorphic to $C_r^*(G)$. Consequently, $C_r^*(G)$ is nuclear, and G is amenable.

Furthermore, amenability of $[[G_W]]$ implies that W admits a $[[G_W]]$-invariant probability measure. Corollary D.3.7 and Proposition D.2.1 then imply that $C_r^*(G_W)$ admits a tracial state. As $C_r^*(G_W)$ is simple, the tracial state is faithful. Hence, $C_r^*(G_W)$ is stably finite and, consequently, so is $C_r^*(G)$.

Now, [**100**, Theorem 6.5] (or [**9**, Theorem 5.14]) implies that $C_r^*(G)$ admits a tracial state. Since $C_r^*(G) = C^*(G)$, we conclude from Corollary D.3.7 again that $\pi = \pi_r$ weakly contains the trivial representation. □

The next corollary is an immediate consequence of Theorems D.4.8 and D.5.2.

Corollary D.5.3. *Let G be a second countable, essentially principal, minimal and ample groupoid with compact unit space. If $C_r^*(G)$ admits no tracial state, then $[[G]]$ is C^*-simple.*

Recall that an action of a group Γ on a topological space X is *topologically free* if $\mathrm{Int}\{x \in X \mid gx = x\} = \emptyset$, for each $g \in \Gamma \setminus \{e\}$. The following result generalizes [**66**, Theorem 4.38], which assumed freeness of the action.

Corollary D.5.4. *Let φ be a topologically free and minimal action of a countable and non-amenable group Γ on the Cantor set. Then $[[G_\varphi]]$ is C^*-simple.*

Proof. Since $C_r^*(\Gamma)$ embeds unitally in $C_{\pi_r}^*([[G_\varphi]])$, non-amenability of Γ implies that π_r does not weakly contain the trivial representation. □

Remark D.5.5. In [**44**], G. Elek and N. Monod constructed a free and minimal action φ of $\mathbb{Z}^2$ on the Cantor set such that $[[G_\varphi]]$ is not amenable. This example is not covered by Theorem D.5.2, and we do not know whether $[[G_\varphi]]$ is C^*-simple.

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