Relative homological algebra and exact model structures

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Abstract

This thesis is concerned with relative homological algebra and exact model categories. The word “relative” indicates that a certain choice of short exact sequences has been made. Then one considers (relative) approximations of objects in a category (all categories are additive in this thesis), with respect to the chosen class of short exact sequences, in a way analogous to the fibrant/cofibrant approximations of objects in model categories.

There are two examples of such relative homological theories which are of interest in this thesis. One example comes from commutative algebra in the study of maximal Cohen–Macaulay approximations and its generalizations in Gorenstein homological algebra. Another example comes from the theory of purity in finitely accessible additive categories.

The thesis consists of an expository text, which consists of an introduction and three chapters, and three papers (two of them are already published and the third one is submitted),

A. *Quillen equivalences for stable categories* (joint with S.Estrada and H.Holm). Journal of Algebra 501, 130-149 (2018)

B. *A note on homotopy categories of FP-Injectives.* Homology, Homotopy and Applications 21(1) 95-105, (2019)

C. *Abelian model structures on categories of quiver representations.* https://arxiv.org/abs/1810.12684

Declaration

I declare that this PhD thesis, entitled "Relative homological algebra and exact model structures", is submitted in partial fulfillment of the requirements for the double degree PhD programme in Mathematics between the University of Murcia and the University of Copenhagen, which was carried out during the period February 2016–February 2019.

The research was carried out under the supervision of Sergio Estrada (Murcia) and Henrik Holm (Copenhagen), according to the PhD regulations of both Universities.

I ensure that the content in all three papers of this thesis is original, and, to the best of my knowledge, does not breach copyright law. All other sources used in this thesis have been cited and acknowledged within the text.

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Contents

Chapter 1

Introduction

Classical homological algebra is largely concerned with resolutions, or better "replacements", of modules in terms of simpler and more tractable objects, like projective or injective modules. Such resolutions measure how far an object is from having a certain property, e.g. a projective resolution of a module, and its associated projective dimension, measures how far a module is from being projective. Moreover, such resolutions are isomorphisms in a certain "localized" category where, by construction, the chain maps inducing isomorphic homology groups are isomorphisms. This classical theory is part of Quillen's motivation to develop his theory of model categories [67]. For any object M in a model category $\mathcal{M}$, there exist replacements $QM \to M$ and $M \to RM$ which become isomorphisms in the localized (or "homotopy") category $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$; where $\mathcal{W}$ is the class of morphisms we want to invert (the so-called weak equivalences in the model).

Relative homological algebra is concerned with resolutions with respect to a chosen class of objects of interest (in an abelian or more generally in an exact category). It was initially developed by Hochschild (1956) [45], Butler and Horrocks [19] and Eilenberg and Moore [29]. We refer to the books of Enochs and Jenda [35] and also Göbel and Trlifaj [41] for a modern exposition. The benefit from switching from the classical (absolute) case to the relative one is immense. For example, in categories of modules over non-regular rings, where there exist modules of infinite projective dimension, we may work with resolutions by maximal Cohen-Macaulay modules (see also paper A below), or in categories of quasi-coherent sheaves where we don't have enough projectives we may resolve by flats [33].

The theory of model categories is quite general and can cover also the relative homological algebra case. The precise framework for this to work was given by Hovey [50] and also by Beligiannis and Reiten [12]. For an abelian category $\mathcal{M}$, Hovey gives an one-to-one correspondence between the so-called *abelian model structures*[1] and certain *cotorsion pairs* in $\mathcal{M}$. A co-

[1] An abelian model structure on an abelian category $\mathcal{M}$ is a Quillen model structure

torsion pair in an abelian category $\mathcal{M}$ is a pair of $\mathrm{Ext}^1_{\mathcal{M}}(-,-)$–orthogonal to each other subcategories[2]. The basic idea behind Hovey's results is that, for an abelian model category $\mathcal{M}$, the various lifting properties in the model $\mathcal{M}$ can be interpreted as certain $\mathrm{Ext}^1_{\mathcal{M}}(-,-)$–orthogonality relations. Thus in order to give a model structure on an abelian category, it suffices to find certain cotorsion pairs and then use the correspondence of Hovey. Hovey's results were extended by Gillespie [40] to the realm of (weakly idempotent complete) exact categories. Classically, cotorsion pairs are used to encode information on right/left approximations in module or abelian categories, see [41]. Under Hovey's correspondence the right/left approximations correspond to the fibrant/cofibrant replacements in the model. The benefit of thinking in terms of model structures than in terms of cotorsion pairs is that in the first case we obtain a great deal of information on the homotopy category of the model.

In this thesis we provide applications of relative homological algebra and exact model structures in the context of (non)commutative ring theory.

Paper A

An excellent example of a relative homological theory is the theory of maximal Cohen-Macaulay approximations, as founded in the work of Auslander in the 60's [2, 3]. The starting point is a theorem of Serre-Auslander-Buchsbaum (1956) [4, 71] who characterised the regular local rings[3] as the commutative Noetherian local rings where all modules have finite projective dimension. Hence in order to understand the modules over a singular variety, such as $k[x]/(x^n)$ or $k[x,y]/(x^2,xy,y^2)$, one needs to understand its modules of infinite projective dimension. In this direction, a result of Eisenbud (1980) [30] is quite enlighting. Eisenbud [30, Thm. 6.1] proves that over a hypersurface ring $S = R/(f)$, where R is regular, any finitely generated S-module admits a projective resolution which becomes periodic after finitely many steps. The syzygies[4] in the periodic part of the resolution are therefore understood as "infinite syzygies"; indeed they are syzygies in an exact complex of projectives which has infinite left and right tails[5]. Necessarily, any such "infinite syzygy" Ω will satisfy the condition $\operatorname{depth}\Omega = \operatorname{depth} R$,

on $\mathcal{M}$, where,

- the (trivial) cofibrations are monomorphisms with (trivially) cofibrant cokernel,
- the (trivial) fibrations are epimorphisms with (trivially) fibrant kernel.

[2]In more detail, a pair of subcategories $(\mathcal{A},\mathcal{B})$ in an abelian category $\mathcal{M}$ is called a cotorsion pair if $\mathcal{B} = \mathcal{A}^{\perp} := \{M \mid \mathrm{Ext}^1_{\mathcal{M}}(A,M) = 0;\ \forall A \in \mathcal{A}\}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$.

[3]A commutative noeherian local ring $(R,\mathfrak{m},k)$ is called *regular* if its Krull dimension equals $\dim_k \mathfrak{m}/\mathfrak{m}^2$.

[4]The *syzygies* in a projective resolution $\cdots \xrightarrow{\theta_1} P_1 \xrightarrow{\theta_0} P_0 \to M \to 0$ are by definition the kernels of the maps θ_i in the resolution.

[5]Such complexes are usually called *complete projective resolutions*.

in other words, Ω is a *maximal Cohen-Macaulay* R-module [17, I 2.1.1]. We denote this class of modules (over a Cohen-Macaulay ring R) by $\mathrm{MCM}(R)$.

Buchweitz (1980) [18] singles out this class of modules, over a Gorenstein ring R[6], and proves that for any finitely generated R–module M, there exists a short exact sequence $0 \to P \to X \to M \to 0$, where $X \in \mathrm{MCM}(R)$ and P has finite projective dimension. Moreover, this short exact sequence extents to an exact complex $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$, where the X_i's are maximal Cohen-Macaulay. Thus even if a module has infinite projective dimension, it has finite MCM-dimension. Buchweitz in [18] explains the connection between (non-projective) maximal Cohen-Macaulay R-modules, their associated complete projective resolutions, and the singularities of the ring by providing equivalences of triangulated categories[7]

$$\mathbf{D}_{\mathrm{sg}}(R) \overset{\cong}{\longleftarrow} \underline{\mathrm{MCM}}(R) \overset{\cong}{\longrightarrow} \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)).$$

Over a general (not necessarily commutative) ring, we call (after Enochs and Jenda [34]) the modules that are "infinite syzygies", *Gorenstein projective* modules[8]. The dual notion of Gorenstein injective modules is also defined in [34]. In fact most of the results of Buchweitz can be stated using injectives (an article of Krause [54] discusses this in detail). Therefore, the analogue of the stable category $\underline{\mathrm{MCM}}(R)$, in the general ring case, is the stable category $\underline{\mathrm{GProj}}(R)$, and its injective counterpart is $\underline{\mathrm{GInj}}(R)$.

In paper **A** of this thesis, joint with Estrada and Holm [26], the motivating question is when the categories $\underline{\mathrm{GProj}}(R)$ and $\underline{\mathrm{GInj}}(R)$ are equivalent[9]. To approximate this question, we realize these stable categories as Quillen homotopy categories of certain exact model structures, and then we seek for suitable Quillen equivalences. In more detail, in [26, Thm. 3.7] we prove that the (exact) category $\mathcal{A}(R) := \{M \mid \mathrm{Gpd}_R\, M < \infty\}$[10] admits an exact model structure, where the cofibrant objects are the Gorenstein projective modules, and the trivial objects (i.e. weekly isomorphic to zero) are the modules with finite projective dimension. Moreover, the homotopy category of this

[6] A two-sided noetherian ring is called *Gorenstein* if it has finite injective dimension over itself on both sides. For the commutative case, see the survey of Bass (1963) [10].

[7] Here $\underline{\mathrm{MCM}}(R)$ denotes the stable category of maximal Cohen-Macaualy modules, i.e. the category $\mathrm{MCM}(R)$ "modulo projectives", which is a triangulated category [43]. Moreover, $\mathbf{D}_{\mathrm{sg}}(R)$ denotes the bounded derived category $\mathbf{D}^b(\mathrm{mod}(R))$ modulo the subcategory of complexes of finite projective dimension ($\mathbf{D}_{\mathrm{sg}}(R)$ is also called the *singularity category* of R), and $\mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R))$ denotes the homotopy category of acyclic complexes of finitely generated projective R–modules.

[8] To be more precise, an R-module M is called Gorenstein projective if it is a syzygy in an exact complex of projectives $P_\bullet$, which remains exact after applying functors of the form $\mathrm{Hom}_R(-, \mathrm{Proj}(R))$. Note that the last condition is automatically satisfied if R is a Gorenstein ring.

[9] The results in [26] are stated for bicomplete abelian categories with enough projectives and enough injectives, but for this presentation we stay with the ring case.

[10] We denote by $\mathcal{A}(R)$ (resp., $\mathcal{B}(R)$) the subcategory of all R–modules of finite Gorenstein projective (resp., injective) dimension.

model is non else than $\underline{\mathrm{GProj}}(R)$. What is important here is the following: a cofibrant replacement in this model will be a Gorenstein projective approximation (i.e. an epimorphism with source a Gorenstein projective module) with kernel a module of finite projective dimension, so this is really modeled on the MCM approximations of Buchweitz [18] and puts them in a homotopical context, therefore it seems to be an honest non-trivial model for the category $\underline{\mathrm{GProj}}(R)$. Its dual counterpart, as given in [26, Thm. 3.9] is a model on the exact category $\mathcal{B}(R) := \{M \mid \operatorname{Gid}_R M < \infty\}$, where the fibrant replacements are monomorphisms with target a Gorenstein injective and cokernel a module of finite injective dimension. Both of these model structures are obtained using work of Holm [46].

Results of Sharp [72] and Foxby [37], both published in 1972, tell us that over a Cohen–Macaulay local ring with a dualizing module D, the adjunction $D \otimes_R - : R\text{-Mod} \rightleftarrows R\text{-Mod} : \operatorname{Hom}_R(D, -)$ restricts to an equivalence $\mathcal{A}(R) \rightleftarrows \mathcal{B}(R)$ and also to an equivalence $\mathcal{P}(R) \rightleftarrows \mathcal{I}(R)$[11]. In paper **A** we formally call an adjunction (S, T) on R-Mod a *Sharp-Foxby adjunction* if it induces equivalences $\mathcal{A}(R) \rightleftarrows \mathcal{B}(R)$ and $\mathcal{P}(R) \rightleftarrows \mathcal{I}(R)$. This is the source of inspiration for the first main result of this thesis,

Theorem A. ([26, Thm. 3.11]) *Let R be a ring. A Sharp–Foxby adjunction (S, T) on $\operatorname{Mod}(R)$ induces a Quillen equivalence between the model categories $\mathcal{A}(R)$ and $\mathcal{B}(R)$. Thus the total (left/right) derived functors of S and T yield an adjoint equivalence of the corresponding homotopy categories,*

$$\underline{\mathrm{GProj}}(R) \simeq \operatorname{Ho}(\mathcal{A}(R)) \underset{\mathbf{R}T}{\overset{\mathbf{L}S}{\rightleftarrows}} \operatorname{Ho}(\mathcal{B}(R)) \simeq \underline{\mathrm{GInj}}(R)\,.$$

In fact, this is an equivalence of triangulated categories.

Variations of this result for categories of chain complexes and also categories of quiver representations are included in [26].

Paper B

We now explain the content of paper **B** of this thesis [25]. The aforementioned results of Sharp and Foxby, have some analogues in "bigger" categories. Indeed, Iyengar and Krause (2006) [51] prove that over a Noetherian ring with a dualizing complex D, the composite map $\mathbf{K}(\operatorname{Proj}(R)) \xrightarrow{i} \mathbf{K}(\operatorname{Flat}(R)) \xrightarrow{D\otimes_R -} \mathbf{K}(\operatorname{Inj}(R))$ is an equivalence of triangulated categories. When restricted to compact objects[12] this equivalence induces Grothendieck's duality $\mathbf{R}\operatorname{Hom}_R(-, D) : \mathbf{D}^b(R) \to \mathbf{D}^b(R)$.

[11] Here $\mathcal{P}(R)$ (resp. $\mathcal{I}(R)$) denote the subcategories of R–modules of finite projective (resp. injective) dimension.

[12] We know from Jørgensen [53] that $\mathbf{K}(\operatorname{Proj}(R))$ is compactly generated (over rings than are even more general than Noetherian with a dualizing complex), and we also know from Krause [54] that $\mathbf{K}(\operatorname{Inj}(R))$ is compactly generated over right Noetherian rings.

Neeman in [63] studies the embedding $\mathbf{K}(\mathrm{Proj}(R)) \hookrightarrow \mathbf{K}(\mathrm{Flat}(R))$ for a general ring R. In the case R is a Noetherian ring with a dualizing complex, since $\mathbf{K}(\mathrm{Proj}(R))$ is compactly ($\aleph_0$) generated, the inclusion will have a right adjoint from Neeman's Brown representability theorem [61] (this adjoint is used explicitly in Iyengar-Krause [51]). In the general case, Neeman obtains a right adjoint i_ρ of this inclusion by showing that $\mathbf{K}(\mathrm{Proj}(R))$ is $\aleph_1$–compactly generated, whatever that means (it's a generalization of compact generation). Moreover, Neeman identifies the kernel of this right adjoint, which as a kernel of a right adjoint is easily seen to be isomorphic to

$$\mathbf{K}(\mathrm{Proj}(R))^{\perp} := \{X \mid \mathrm{Hom}_{\mathbf{K}(R)}(P, X) = 0; \forall P \in \mathrm{Ch}(\mathrm{Proj}(R))\},$$

with the subcategory of $\mathbf{K}(\mathrm{Flat}(R))$ which consists of the exact complexes with flat syzygies, denoted by $\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}(R))$. This results to a diagram,

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Proj}(R)) \arrow[r, shift right, "\mathrm{inc}"'] & \mathbf{K}(\mathrm{Flat}(R)) \arrow[l, shift right, "i_\rho"'] \arrow[d, "\mathrm{can}"] & & \ker(i_\rho) \cong \mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}(R)) . \arrow[ll, "\mathrm{inc}"'] \\
& \mathbf{D}(\mathrm{Flat}(R)) := \frac{\mathbf{K}(\mathrm{Flat}(R)}{\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}(R))} \arrow[ul, "\cong"] & &
\end{tikzcd}$$

where the equivalence is obtained by the universal property of localization. In particular this diagram implies that the composite functor

$$\mathbf{K}(\mathrm{Proj}(R) \rightarrowtail \mathbf{K}(\mathrm{Flat}(R)) \xrightarrow{\mathrm{can}} \mathbf{D}(\mathrm{Flat}(R))$$

is an equivalence of triangulated categories.

The point of paper **B** of this thesis [25] is that we look at Neeman's result from a relative homological algebra perspective, and we dualize it. Neeman's result is a statement on the homotopy category of flat modules; it provides a certain localization of the latter category. The dual notion of flatness is called *fp-injectivity.* The passage from flatness to fp-injectivity is only understood using the notion of purity; a classical topic in relative homological algebra. We explain this in some detail now.

Purity was introduced by Cohn [22] in his study of products of rings and direct sum decompositions[13]. A submodule $A \leq B$ is called pure if any finite system of linear equations with constants from A and a solution in B, has a solution in A. This condition can be expressed diagrammatically, and is equivalent to asking for the sequence $A \rightarrowtail B \twoheadrightarrow B/A$ to remain exact after applying, for any finitely presented module F, the functor $\mathrm{Hom}_R(F, -)$, or equivalently the functor $F \otimes_R -$. Such sequences are called pure exact and they are of interest since they form the smallest class of short exact sequences

[13]For survey articles on purity, we suggest the articles of Zimmermann and Prest in [56]. See also the monograph of Prest [66].

which is closed under filtered colimits. It follows from this discussion that a module M is flat if and only if any epimorphism with target M is pure. Thus flatness can be defined in any additive category which has an appropriate notion of finitely presented objects, namely locally finitely presented additive categories [16, 23]. If $\mathcal{A}$ is such a category, it is well known that the relation between purity and flatness can be given formally via the equivalence $\mathcal{A} \cong \operatorname{Flat}(\operatorname{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$; $A \mapsto \operatorname{Hom}_{\mathcal{A}}(-, A)|_{\operatorname{fp}(\mathcal{A})}$[14], see [23, 1.4]. Thus, roughly speaking, the study of purity can be reduced to the study of flat (left exact) functors, and Neeman's results have analogues in the context of purity, see Emmanouil [32], Krause [55], Simson [74] and Šťovíček [80].

The dual notion of flatness, in a locally finitely presented Grothendieck category $\mathcal{A}$, is that of FP-injectivity. Namely, an object A in $\mathcal{A}$ is called FP-injective if any monomorphism with source A is pure. We denote the class of FP-injective objects by $\operatorname{FPI}(\mathcal{A})$. FP-injective modules were studied first by Stenström in [77]. One reason why they are of importance is because over (non-Noetherian) rings where injectives fail to be closed under coproducts, one can work with FP-injectives which are always closed under coproducts. Moreover, a ring is coherent if and only if the class of FP-injective modules is closed under filtered colimits [77, 3.2], in strong analogy with the dual situation, where coherent rings are characterized by the closure of flat modules under products.

In paper **B** we provide duals to the above mentioned results of Neeman, that is, we obtain analogous results for the homotopy category of FP-injectives. For this we look at the tensor embedding functor of a module category to FP-injective (right exact) functors, that is, the functor $\text{Mod-}R \to \mathcal{A} := (R\text{-mod}, \mathrm{Ab})$; $M \mapsto (M \otimes_R -)|_{R\text{-mod}}$, which identifies pure exact sequences in Mod-R with short exact sequences of FP-injective (right exact) functors, and induces an equivalence $\text{Mod-}R \cong \operatorname{FPI}(\mathcal{A})$ [42, §1]. It is easy to observe that under this equivalence, the pure projective modules (the projectives with respect to the pure exact sequences) correspond to functors in the class $\operatorname{FPI}(\mathcal{A}) \cap {}^{\perp}\operatorname{FPI}(\mathcal{A})$[15]. We point out that by work of Eklof and Trlifaj [31], we know that this class consists of those FP-injectives which are (summands of) transfinite extensions of finitely presented objects, see [41, 3.2]. The main result of Paper **B** is the following:

Theorem B. ([25, Thm. 3.5]) *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and denote by* $\operatorname{FPI}(\mathcal{A})$ *the class of FP-injective objects in $\mathcal{A}$. Then the homotopy category* $\mathbf{K}(\operatorname{FPI}(\mathcal{A}) \cap {}^{\perp}\operatorname{FPI}(\mathcal{A}))$ *is compactly generated. Moreover, if $\mathcal{A}$ is locally coherent, the composite functor*

$$\mathbf{K}(\operatorname{FPI}(\mathcal{A}) \cap {}^{\perp}\operatorname{FPI}(\mathcal{A})) \rightarrowtail \mathbf{K}(\operatorname{FPI}(\mathcal{A})) \xrightarrow{\text{can}} \mathbf{D}(\operatorname{FPI}(\mathcal{A}))$$

[14] $\operatorname{fp}(\mathcal{A})$ denotes a set of isomorphism classes of finitely presented objects in $\mathcal{A}$.

[15] ${}^{\perp}\operatorname{FPI}(\mathcal{A})$ denotes the left orthogonal to the class of FP-injectives with respect to the $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$ functor.

is an equivalence of triangulated categories.

It is possible to reformulate (parts of) this result in the language of model categories.

Theorem B'. ([25, Thm. 3.7]) *Let $\mathcal{A}$ be a locally coherent Grothendieck category and let* $\mathrm{Ch}(\mathrm{FPI}(\mathcal{A}))$ *denote the category of chain complexes with components FP-injective objects. Then there exists an (exact) model structure on* $\mathrm{Ch}(\mathrm{FPI}(\mathcal{A}))$*, where*

- *the cofibrant objects are the chain complexes in the category* $\mathrm{Ch}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A}))$.
- *every chain complex in* $\mathrm{Ch}(\mathrm{FPI}(\mathcal{A}))$ *is fibrant.*
- *the trivial objects are the pure acyclic complexes with FP-injective components.*

The homotopy category of this model structure is equivalent to $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$.

We point out that in the Noetherian case, Theorem **B** in particular implies that the category $\mathbf{K}(\mathrm{Inj}(R))$ is compactly generated, which was first proved by Krause (2005) [54]. Neeman in [65] studied the category $\mathbf{K}(\mathrm{Inj}(R))$ for a general ring, and proved that it is μ–compactly generated (for some cardinal μ). In the locally coherent case, Šťovíček [80] proves that $\mathbf{K}(\mathrm{Inj}(R))$ is canonically isomorphic to $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$ and also proves that the latter is compactly generated. Therefore combining this result of Šťovíček with Theorem **B**, for a locally coherent Grothendieck category $\mathcal{A}$, we obtain equivalences

$$\begin{tikzcd}
& \mathbf{D}(\mathrm{FPI}(\mathcal{A})) \arrow[dl, "{[80]}"', "\cong"] \arrow[dr, "\text{Thm. }\mathbf{B}", "\cong"'] & \\
\mathbf{K}(\mathrm{Inj}(\mathcal{A})) & & \mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A}))
\end{tikzcd}$$

Therefore, in terms of compact generation of homotopy categories of complexes, it seems like the homotopy category $\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A}))$ is the correct category to look at in the general (not necessarily coherent) case. Indeed, it has the advantage over $\mathbf{K}(\mathrm{Inj}(\mathcal{A}))$, that it's always compactly generated, and also identifies with the latter in the locally coherent case.

Paper C

The last item of this thesis is paper **C** [24]. It is concerned with abelian model structures on certain categories of diagrams. The motivating question is when an abelian model structure on a given abelian category $\mathcal{M}$ can be transferred to an abelian model structure on the diagram category

$\mathcal{M}^Q := \mathrm{Rep}_Q(\mathcal{M})$, where Q is a quiver (a directed graph). The latter (functor) category, in the representation theory community, is usually called the category of $\mathcal{M}$-valued representations of the quiver Q[16]. We should point out two things: Firstly, in the general theory of model categories, it is well-known that for a given model category $\mathcal{M}$, a model structure on a functor category $\mathcal{M}^I$ exists under certain assumptions, either on $\mathcal{M}$ or on I, see for instace the book of Hirschhorn [44, Ch. 11,15]; the point in [24] is that we are interested in *abelian* model structures. Secondly, the cotorsion pairs needed in order to construct abelian model structures on $\mathrm{Rep}_Q(\mathcal{M})$ are essentially given by Holm and Jørgnensen in their work on cotorsion pairs in categories of quiver representations [47], although they don't discuss model structures in their work.

In paper **C** we describe projective (resp. injective) model structures on the category $\mathcal{M}^Q := \mathrm{Rep}_Q(\mathcal{M})$ where $\mathcal{M}$ is a given abelian model structure and Q is left (resp. right) rooted quiver[17]. Assume for example that Q is left rooted and that $\mathcal{M}$ is a given abelian model category with cofibrant objects $\mathcal{C}$, trivial objects $\mathcal{W}$, and with all objects being fibrant. Then from [24, Thm. 3.5] there exists an induced abelian model structure on $\mathcal{M}^Q$, where the trivial (resp. fibrant) representations are precisely those which are vertexwise trivial (resp. fibrant) in the "ground" model $\mathcal{M}$. Moreover, to describe the cofibrant objects in $\mathcal{M}^Q$, consider for any representation X and any vertex $i \in Q_0$, the map

$$\bigoplus_{\alpha: j \to i} X(j) \xrightarrow{\phi_i^X} X(i).$$

Then the cofibrant objects in the category $\mathcal{M}^Q$ are given by the class

$$\Phi(\mathcal{C}) := \{X \mid \forall i \in Q_0,\ \phi_i^X \text{ is monic with } X(i) \in \mathcal{C},\ \operatorname{coker} \phi_i^X \in \mathcal{C}\}.$$

In other words, the input is a (projective) abelian model structure $\mathcal{M}$ and the output is a (projective) model structure on $\mathcal{M}^Q$. In our applications, we are interested in the case where the input is some model structure coming from relative homological algebra. For example, over a Gorenstein ring R, there exists an abelian model structure where the cofibrants are the Gorenstein projective R–modules, $\mathrm{GProj}(R)$, and the trivial objects $\mathcal{W}$ are the modules of finite projective dimension. Or even more generally over a Ding-Chen ring R[18], there exists an abelian model structure where the

[16] See for instance the book of Auslander-Reiten-Smalø [6] for the use of quivers in the representation theory of associative algebras.

[17] A quiver Q is called *left rooted* if it does not contain any subquiver of the form $\cdots \to \bullet \to \bullet \to \bullet$. Dually, Q is called *right rooted* if Q^{op} is left rooted.

[18] A ring is called Ding-Chen if it is left and right coherent with $\mathrm{FPI}-\dim_R R$ and $\mathrm{FPI}-\dim R_R$ both finite. Here $\mathrm{FPI}-\dim$ denotes the fp-injective dimension. Note that if R is two-sided Noetherian then this definition recovers the Iwanaga-Gorenstein rings.

cofibrants are the Ding projective R–modules, $\mathrm{DProj}(R)$, and the trivial objects $\mathcal{W}$ are the modules of finite projective dimension, see Gillepsie [39].

We now state the main result of manuscript **C**. The reader who is unfamiliar with Ding-Chen rings, should consider Gorenstein rings and replace the letter "D'' with the letter "G'' in the following statement.

Theorem C. ([24, Thm. 4.8]) *Let R be a Ding-Chen ring and Q a left rooted quiver. Consider the associated abelian model structure* $(\mathrm{DProj(R)}, \mathcal{W}, \mathrm{Mod}(R))$ *on the category* $\mathrm{Mod}(R)$*. Then there exists an induced abelian model structure on the category of quiver representations of right R–modules,* $\mathrm{Rep}_Q(R)$*, where*

- *The cofibrant representations are the (globally) Ding projective representations, i.e. the category* $\mathrm{DProj}(\mathrm{Rep}_Q(R))$*.*
- *The trivial representations are the vertexwise trivial representations.*
- *All representations are fibrant.*

The homotopy category of this model structure is

$$\mathrm{Ho}(\mathrm{Rep}_Q(R)) \cong \underline{\mathrm{DProj}}(\mathrm{Rep}_Q(R)),$$

the *stable category of Ding projective representations.*

The non-trivial part in this result is to identify the cofibrant objects in $\mathrm{Rep}_Q\,(R)$ (which is the class $\Phi(\mathrm{DProj}(R))$ as we discussed above), with the "globally" Ding projective objects in the functor category, i.e. the category $\mathrm{DProj}(\mathrm{Rep}_Q(R))$. The dual of this statement for Ding injectives is as follows.

Theorem C'. ([24, Thm. 4.9]) *Let R be a Ding-Chen ring and Q a right rooted quiver. Consider the associated abelian model structure* $(\mathrm{Mod}(R), \mathcal{W}, \mathrm{DInj(R)})$ *on the category* $\mathrm{Mod}(R)$*. Then there exists an induced abelian model structure on the category of quiver representations of right R–modules,* $\mathrm{Rep}_Q(R)$*, where*

- *All representations are cofibrant.*
- *The trivial representations are the vertexwise trivial representations.*
- *The fibrant representations are the (globally) Ding injective representations, i.e. the category* $\mathrm{DInj}(\mathrm{Rep}_Q(R))$*.*

The homotopy category of this model structure is

$$\mathrm{Ho}(\mathrm{Rep}_Q(R)) \cong \underline{\mathrm{DInj}}(\mathrm{Rep}_Q(R)),$$

the *stable category of Ding injective representations.*

Chapter 2

Preliminaries

This chapter is a recollection of well-known results from the literature. For most of what follows we refer to the monographs of Borceux [15], Stenström [78], and Prest [66].

2.1 Additive categories

Definition 2.1.1. An **abelian category** is a preadditive category, which has a zero object, finite coproducts, and such that every morphism has a kernel and monomorphisms are kernels, and dually every morphism has a cokernel and epimorphisms are cokernels. The opposite of an abelian category is an abelian category (and thus they also have finite products).

A **set of generators** for an abelian category $\mathcal{A}$, is **a set** $\mathcal{G} = \{G_i\}_{i\in I}$ of objects in $\mathcal{A}$, such that for any nonzero $f : A \to B$, there exists $G_i \in \mathcal{G}$ and $\phi : G_i \to A$ such that $f \circ \phi \neq 0$. If $\mathcal{G} = \{G\}$ we call G a **generator**. If $\mathcal{A}$ has coproducts and a set of generators $\{G_i\}_{i\in I}$, then $\oplus G_i$ is a generator. By definition, G is a generator if and only if $\mathrm{Hom}_{\mathcal{A}}(G, -)$ is faithful. Also note that if G is a generator then for all $A \in \mathcal{A}$, there exists an epimorphism $G^{(I)} \twoheadrightarrow A$, where I is a set. Indeed, let $I = \mathrm{Hom}_{\mathcal{A}}(G, A) \neq \emptyset$ and consider the canonical map $G^{(I)} \xrightarrow{\epsilon} A$. Then ϵ is an epimorphism since for all $\phi : A \to E$ such that $\phi \circ \epsilon = 0$, we may find nonzero $f : G \to A$ with $\phi \circ f = 0$, thus $\phi = 0$.

An abelian category $\mathcal{A}$ is called **Grothendieck** if:

- It has (small) coproducts.
- Colimits of directed systems are exact.
- It has a generator.

Gabriel-Popescu theorem: Every Grothendieck category is the localization of a module category.

Definition 2.1.2. A small category $\mathcal{I}$ is called **filtered** if it is non-empty, for every two objects i, j in $\mathcal{I}$ there exists an object k and morphisms $i \to k \leftarrow j$, and for any two parallel morphisms $f, g : i \rightrightarrows j$ there exists a morphism $h : j \to k$ such that $h \circ f = h \circ g$. A colimit of a functor with source a filtered category is called filtered.

Remark 2.1.3. [1, Thm. 1.5] A category has filtered colimits if and only if it has direct limits. A functor with source such a category preserves filtered colimits if and only if it preserves direct limits.

Definition 2.1.4. Let $\mathcal{A}$ be an additive category. An object F in $\mathcal{A}$ is called

- **finitely generated** if the functor $\mathrm{Hom}_{\mathcal{A}}(F, -)$ preserves direct unions,
- **finitely presented** if the functor $\mathrm{Hom}_{\mathcal{A}}(F, -)$ preserves direct limits.

Definition 2.1.5. An additive category $\mathcal{A}$ is called **locally finitely presented** if it is cocomplete, the isomorphism classes of finitely presented objects in $\mathcal{A}$ form a set $\mathrm{fp}(\mathcal{A})$, and every object in $\mathcal{A}$ is isomorphic to a filtered colimit of finitely presented objects.

Exact categories

Definition 2.1.6. Let $\mathcal{A}$ be an additive category. A pair (i, p) of composable maps

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

is called a *kernel-cokernel pair* if $\ker p = i$ and $\operatorname{coker} i = p$. We say that we have an isomorphism between two composable pairs when the relevant diagram commutes (via isomorphisms). If a class $\mathcal{E}$ of kernel-cokernel pairs is fixed, we call a morphism i an *inflation* (or an *admissible monomorphism*) if there exists a morphism p such that $(i, p) \in \mathcal{E}$. Dually, a morphism p is called a *deflation* (or an *admissible epimorphism*) if there exists a morphism i such that $(i, p) \in \mathcal{E}$. We depict inflations by $\rightarrowtail$ and deflations by $\twoheadrightarrow$.

Definition 2.1.7. Let $\mathcal{A}$ be an additive category. *An exact structure on* $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

(EX 0) For all A in $\mathcal{A}$, the identity 1_A is an inflation.

(EX 0)$^{\mathrm{op}}$ For all A in $\mathcal{A}$, the identity 1_A is a conflation.

(EX 1) The class of inflations is closed under composition.

(EX 1)$^{\mathrm{op}}$ The class of conflations is closed under composition.

(EX 2) Pushouts of inflations along arbitrary morphisms exist, and they are inflations.

(EX 2)$^{\mathrm{op}}$ Pullbacks of conflations along arbitrary morphisms exist, and they are conflations.

We define an *exact category* $(\mathcal{A}, \mathcal{E})$ as an additive category $\mathcal{A}$, with an exact structure $\mathcal{E}$. We might just write $\mathcal{A}$ when the class $\mathcal{E}$ is clear from the context. From the above axioms it is clear that $(\mathcal{A}, \mathcal{E})$ is an exact category if and only if $(\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$ is.

Definition 2.1.8. An exact category $\mathcal{F}$ is called **Frobenius** if the projective objects in $\mathcal{F}$ coincide with the injective objects in $\mathcal{F}$.

Example 2.1.9. A module category A-Mod is Frobenius if and only if the ring A is **quasi-Frobenius**, i.e if it is left and right Noetherian and self-injective (as a left A-module). Examples of this kind include semisimple rings and quotients of PIDs by nonzero proper ideals.

Of course a module category is not always Frobenius, but one can restrict to several important Frobenius subcategories.

Example 2.1.10. Over an Iwanaga-Gorenstein ring A, that is, a ring which is two-sided Noetherian and has finite injective dimension on both sides, the category of **maximal Cohen–Macaulay modules,**

$$\mathrm{MCM}(A) := \{M \in A\text{-Mod} \mid \mathrm{Ext}_A^{\geq 1}(M, R) = 0\},$$

is Frobenius.

Definition 2.1.11. If $\mathcal{F}$ is a Frobenius exact category, then its **stable category** $\underline{\mathcal{F}}$ is defined to have the same objects as $\mathcal{F}$, and parallel arrows identified if their difference factors through a projective object:

$$\mathrm{Hom}_{\underline{\mathcal{F}}}(X, Y) := \mathrm{Hom}_{\mathcal{F}}(X, Y)/\mathcal{P}(X, Y)$$

where $P(X, Y)$ is the equivalence relation where $f \sim g$ if and only of there exists a projective P such that $f - g$ factors through P.

It is well known that the stable category of a Frobenius category is a triangulated category, see for instance Happel [43, I.2].

2.2 Purity and functor categories

Purity, flatness, and fp-injectivity

Definition 2.2.1. Let R be a ring. A short exact sequence of right R–modules, $0 \to A \to B \to C \to 0$ is called **pure** if for any left R–module N, the induced sequence $0 \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is an exact sequence of abelian groups.

Fact 2.2.2. Let R be a ring. A right R–module M is flat if and only if any short exact sequence of the form $0 \to A \to B \to M \to 0$ is pure.

For the following fact we refer for instance to the book of Lam [57, Thm. 4.89].

Fact 2.2.3. Let R be a ring. Then the following are equivalent for a short exact sequence $0 \to A \xrightarrow{i} B \to C \to 0$ of right R–modules.

(i) The sequence is pure.

(ii) If $\{a_j\}_{j=1}^n$ are elements in A and $\{b_i\}_{i=1}^m$ are elements in B such that, for all $j = 1, ..., n$; $a_j = \sum_i b_i r_{ij}$, for some r_{ij} in R, then there exist elements $b'_1, ..., b'_m$ in A such that $a_j = \sum_i b'_i r_{ij}$.

(iii) For any commutative diagram of right R–modules,

$$\begin{tikzcd} R^n \arrow[r, "\lambda"] \arrow[d, "\phi"'] & R^m \arrow[d, "\psi"] \\ A \arrow[r, "i"] & B \end{tikzcd}.$$

there exists a map $\delta : R^m \to A$ such that $\delta \circ \lambda = \phi$.

(iv) The sequence remains exact after applying functors of the form $\mathrm{Hom}_R(\mathrm{fp}(R), -)$[1].

(v) The sequence is the direct limit of a direct system of split short exact sequences $0 \to A \xrightarrow{i} B_j \to C_j \to 0$, where for all j, C_j is a finitely presented right R–module.

The above result motivates the definition of purity and that of flatness in any locally finitely presented additive category.

Definition 2.2.4. Let $\mathcal{A}$ be a locally finitely presented additive category. A sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ is called *pure exact* if it is $\mathrm{Hom}_{\mathcal{A}}(\mathrm{fp}(\mathcal{A}), -)$–exact, that is, if for any $A \in \mathrm{fp}(\mathcal{A})$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, X) \to \mathrm{Hom}_{\mathcal{A}}(A, Y) \to \mathrm{Hom}_{\mathcal{A}}(A, Z) \to 0$$

is an exact sequence of abelian groups. An object $X \in \mathcal{A}$ is called *pure projective* if any pure exact sequence of the form $0 \to Z \to Y \to X \to 0$ splits, and dually X is called *pure injective* if any pure exact sequence of the form $0 \to X \to Y \to Z \to 0$ splits. We will denote the class of pure projectives in $\mathcal{A}$ by $\mathrm{PProj}(\mathcal{A})$ and the class of pure injectives in $\mathcal{A}$ by $\mathrm{PInj}(\mathcal{A})$.

The category $\mathcal{A}$ equipped with the class of pure exact sequences is an exact category. We denote this exact category by $\mathcal{A}_{\mathrm{pure}}$ or $(\mathcal{A}, \mathrm{pure})$.

[1] $\mathrm{fp}(R)$ denotes the class of finitely presented right R–modules.

Definition 2.2.5. Let $\mathcal{A}$ be a locally finitely presented additive category. An object $A \in \mathcal{A}$ is called

- **flat** if any epimorphism with target A is pure,
- **FP-injective** if any monomorphism with source A is pure. We denote the class of fp-injective objects in $\mathcal{A}$ by $\mathrm{fp}(\mathcal{A})$.

Fact 2.2.6. Let $\mathcal{A}$ be a locally finitely presented additive category and let M be an object in $\mathcal{A}$. Then the following are equivalent.

(i) $M \in \mathrm{FPI}(\mathcal{A})$.

(ii) $\mathrm{Ext}^1_{\mathcal{A}}(F, M) = 0$, for any $F \in \mathrm{fp}(\mathcal{A})$.

Proof. $(i) \Rightarrow (ii)$ Consider a short exact sequence $0 \to M \to X \to F \to 0$ with $F \in \mathrm{fp}(\mathcal{A})$. Since M is fp-injective this sequence is pure, and since F is finitely presented this sequence splits.
$(ii) \Rightarrow (i)$ Consider a short exact sequence $0 \to M \to A \to B \to 0$ in $\mathcal{A}$. If F is a finitely presented object in $\mathcal{A}$, then by assumption we have that this sequence is $\mathrm{Hom}_{\mathcal{A}}(F, -)$–exact, thus $M \in \mathrm{FPI}(\mathcal{A})$. $\square$

Next we want to illustrate how the study of purity is related to functor categories, via left exact or right exact functors. We first make a small detour on functor categories.

Proposition 2.2.7. *Let $\mathcal{A}$ be a skeletally small preadditive category. Then the representable functors form a set of finitely generated projective generators for the functor category $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$. If $\mathcal{A}$ has split idempotents*[2] *and finite coproducts, then the representables are precisely the finitely generated projective objects in $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. The fact that the representable functors are finitely generated and projective objects in $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ is an easy consequence of Yoneda's lemma. To see that they generate, let F be a functor in $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ and for the rest of the proof assume that we work with a set of isomorphism classes of objects in $\mathcal{A}$. Recall that from Yoneda's lemma we have, for all A in $\mathcal{A}$, a bijection $FA \cong \mathrm{Nat}(h_A, F)$. Then consider the morphism $\oplus_{(a, a \in FA)} h_A \xrightarrow{\epsilon} F$ which is defined, at a component indexed by $(a, a \in FA)$, by the natural transformation $h_A \to F$ which Yoneda–corresponds to $a \in FA$. The map ϵ is a well defined natural transformation which is an epimorphism by construction.

Now, assuming that $\mathcal{A}$ has finite coproducts, any finitely generated functor F is the epimorphic image of a representable, i.e. there is an epimorphism $\epsilon : h_A \twoheadrightarrow F$. Choose a splitting $\epsilon \circ \iota \cong 1_F$. Then $\gamma := \epsilon \circ \iota$ is an idempotent in $\mathrm{Nat}(h_A, h_A)$ which by the Yoneda lemma corresponds to an idempotent $e : A \to A$ (with $A \cong \ker(e) \oplus \ker(1 - e)$), and it follows that $F \cong h_{\ker(1-e)}$. $\square$

[2]We say that a preadditive category has split idempotents if any idempotent $e = e^2 : A \to A$ in $\mathcal{A}$ has a kernel and the natural map $\ker(e) \oplus \ker(1-e) \to \mathcal{A}$ is an isomorphism.

Next, we characterise the finitely presented functors in $(\mathcal{C}^{\text{op}}, \text{Ab})$, where $\mathcal{C}$ is a skeletally small preadditive category. First we state a standard lemma.

Lemma 2.2.8. *Let $\mathcal{A}$ be an abelian category with a set of finitely presented generators and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Then we have the following.*

- *Let B be finitely presented. Then A is finitely generated if and only if C is finitely presented.*

- *If C is finitely presented and B is finitely generated then A is finitely generated.*

Proposition 2.2.9. *Let $\mathcal{C}$ be a skeletally small preadditive category. Then a functor F in $(\mathcal{C}^{\text{op}}, \text{Ab})$ is finitely presented if and only if it isomorphic to a functor of the form $\operatorname{coker}(\phi^* : \operatorname{Hom}_{\mathcal{C}}(-, A)) \to \operatorname{Hom}_{\mathcal{C}}(-, B))$, for some $\phi : B \to A$ in $\mathcal{C}$.*

Proof. If $F \in (\mathcal{C}^{\text{op}}, \text{Ab})$ is a cokernel between representables, then it easy to see that the functor $\text{Nat}(F, -)$ commutes with filtered colimits since this is the case for the representables. To prove the converse implication, let F be a finitely presented functor in $(\mathcal{C}^{\text{op}}, \text{Ab})$. In particular F is finitely generated, thus invoking 2.2.7 we can find a short exact sequence $0 \to K \to h_A \to F \to 0$. Then, by 2.2.8 K is finitely generated, hence we obtain an exact sequence $h_B \to h_A \to F \to 0$. □

Theorem 2.2.10. *An abelian category $\mathcal{A}$ is locally finitely presented if and only if it is a Grothendieck category with a set of finitely presented generators.*

Proof. Assume that $\mathcal{A}$ is locally finitely presented abelian. Then $\mathcal{A}$ has set-indexed coproducts since for any set $\{X_i\}_{i \in \mathcal{I}}$ of objects in $\mathcal{A}$, the set $\coprod X_i$ can be written as a direct limit of its finite subsets, and these belong in $\mathcal{A}$. Next, since $\mathcal{A}$ is cocomplete abelian, the exactness of direct limits in $\mathcal{A}$ is equivalent to $\mathcal{A}$ satysfying AB5 (see [78, V,§1]), and this is clearly the case since direct limits in $\mathcal{A}$ (which exist by assumption) commute with finite limits. Last, we show that $\mathcal{A}$ has a set of finitely presented generators. Indeed, since for any object $A \in \mathcal{A}$, we may write $A \cong \varinjlim C_i$ with the C_i's in $\text{fp}(\mathcal{A})$; we obtain an epimorphism $\coprod C_i \to \varinjlim C_i = A$.

To prove the converse, assume that $\mathcal{A}$ is Grothendieck with a set $\{C_i\}_{i \in I}$ of finitely presented generators. Let F be an object in $\mathcal{A}$ and consider an epimorphism $G = \coprod_{i \in I} C_i \to F$. Now consider the set of all pairs (Y, X) where Y is a finitely generated subobject of G and X is a finitely generated subobject of Y. It is a standard argument due to Lazard that F can be reconstructed as a filtered colimit indexed on this set, see for instance [69, 5.39] or [78, IV,8.9]. □

Corollary 2.2.11. *If $\mathcal{A}$ be a locally finitely presented additive category then so is the functor category* $(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$*. In fact, the category* $(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ *is locally coherent (i.e. the subcategory of finitely presented objects in* $(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ *is abelian).*

Proof. By assumption $\mathrm{fp}(\mathcal{A})$ is skeletally small, and it also has split idempotents. Thus by 2.2.7 the representable functors form a set of (finitely presented) generators in the Grothendieck category $(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$. Thus by 2.2.10 we have that the latter category is locally finitely presented. For the fact that it is locally coherent we refer to [66, II.10.2.1]. □

Left exact functors

Let $\mathcal{A}$ be a locally finitely presented additive category. Then from 2.2.11 we have that the functor category $(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ is locally finitely presented. Hence flatness can be defined in the functor category using definition 2.2.5, that is, a functor F is flat if any epimorphism $G \twoheadrightarrow F$ in the functor category $(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ is $\mathrm{Hom}(\mathrm{fp}, -)$–exact. The functor category supports also a notion of tensor product of functors, and one can define a notion of flatness using this tensor product. The two resulting notions are equivalent, as shown by Stenström [76, Thm. 3].

Flat functors can be defined more generally, as the following result shows.

Theorem 2.2.12. *[15, 6.3.7/6.3.8] Let $F : \mathcal{A} \to \mathrm{Set}$ be a functor defined on a small category $\mathcal{A}$ with finite limits. Then the following are equivalent.*

(i) F preserves finite products ("left exactness").

(ii) The category of elements of F is cofiltered.

(iii) F is a filtered colimit of representable functors.

(iv) The left Kan extension $\mathrm{Lan}_Y F$ of F along the Yoneda embeding $Y : \mathcal{A} \to (\mathcal{A}^{\mathrm{op}}, \mathrm{Set})$ preserves finite products.

In the context of locally finitely presented additive categories all the notions of flatness encountered so far are equivalent, see for instance Stenström [76] or Crawley-Boevey [23].

Theorem 2.2.13. *[15, 6.7.3] Let $\mathcal{A}$ be a locally finitely presented additive category. Then the restricted Yoneda embedding $\mathcal{A} \to (\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$; $A \to h_A|_{\mathrm{fp}(\mathcal{A})}$ induces an equivalence of categories $\mathcal{A} \cong \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$. This is an equivalence of exact categories (where $\mathcal{A}$ is equipped with the pure exact structure).*

Proof. Consider the restricted Yoneda embedding. Since from 2.2.12 every flat functor F is a filtered colimit $F \cong \mathrm{colim}(h_{A_i})$, it is easy to see that the functor $F \to \mathrm{colim} A_i$ provides an inverse for $\tilde{Y}$. □

Remark 2.2.14. For a locally finitely presented additive category, denote by $\mathcal{E} := (\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$. Under the equivalence of Theorem 2.2.13, the class of pure projective objects in $\mathcal{A}$ corresponds to projective functors in $\mathcal{E}$ and the class of pure injective objects in $\mathcal{A}$ corresponds to functors in the class $\mathrm{Flat}\mathcal{E} \cap (\mathrm{Flat}\mathcal{E})^{\perp}$.

Right exact functors

In the case of a module category Mod-R, there is a "dual" version of Theorem 2.2.13 which is concerned with right exact functors (fp-injective) functors. We consider the tensor embedding functor

$$\text{Mod-}R \xrightarrow{\tau} (R\text{-mod}, \mathrm{Ab});\ M \mapsto (M \otimes_R -)|_{R\text{-mod}}$$

which by construction sends pure exact sequences of right R–modules to short exact sequences in the functor category $(R\text{-mod}, \mathrm{Ab})$. The functor τ has a right adjoint which is given by $F \mapsto F(R)$. The following result is due to Gruson and Jensen [42].

Theorem 2.2.15. *[42, §1] The tensor embedding functor induces an equivalence* Mod-$R \cong \mathrm{FPI}(R\text{-mod}, \mathrm{Ab})$ *of exact categories (where* Mod-R *is equipped with the pure exact structure).*

Remark 2.2.16. Under the equivalence of Theorem 2.2.15, the class of pure injective right R–modules corresponds to injective functors and the class of pure projective right R–modules corresponds to functors in the class $\mathrm{FPI}(R\text{-mod}, \mathrm{Ab}) \cap {}^{\perp}\,\mathrm{FPI}(R\text{-mod}, \mathrm{Ab})$.

2.3 Triangulated categories

In this section we make a small survey of some fundamental results on triangulated categories. The material here is mostly taken from [48] and from the book of Neeman [62].

Definition 2.3.1. Let $\mathcal{T}$ be an additive category equipped with an automorphism $\Sigma : \mathcal{T} \cong \mathcal{T}$. We call a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ a *triangle* in $\mathcal{T}$. A morphism between two triangles is given by morphisms (α, β, γ) which induce a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'.
\end{tikzcd}$$

A morphism (α, β, γ) between triangles as above is called an isomorphism if α, β, γ are isomorphisms in $\mathcal{T}$.

$\mathcal{T}$ is called a *triangulated category* if it admits a collection of triangles (called *distinguished*) such that the following axioms are satisfied:

(TR0) Any triangle isomorphic to a distinguished triangle is distinguished.

(TR1) For all $X \in \mathcal{T}$ the triangle $X = X \to 0 \to \Sigma X$ is distinguished.

(TR2) Every morphism $f : X \to Y$ in $\mathcal{T}$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.

(TR3) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle then $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is a distinguished triangle.

(TR4) Given distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$, any commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'
\end{tikzcd}$$

can be completed to a morphism of triangles (not necessarily uniquely).

(TR4) Given distinguished triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists a triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, equal] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & Z \arrow[r, "\alpha_3"] \arrow[d, "\delta_1"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] \arrow[d, "\delta_2"] & \Sigma X \arrow[d, "\Sigma\alpha_1"] \\
 & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, "\delta_3"] & \Sigma Y \\
 & \Sigma Y \arrow[r, "\Sigma\alpha_2"] & \Sigma U. &
\end{tikzcd}$$

Definition 2.3.2. Let $\mathcal{T}$ be a triangulated category. A non-empty full subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *thick* if it closed under retracts. $\mathcal{S}$ is called *localizing* (resp. *colocalizing*) if it is thick and closed under coproducts (resp. products).

Given a non-empty full subcategory $\mathcal{S}$ of $\mathcal{T}$, the intersection of all thick subcategories of $\mathcal{T}$ containing $\mathcal{S}$ is a thick subcategory denoted by $\mathrm{Thick}(\mathcal{S})$. Similarly, the intersection of all localizing (resp. colocalizing) subcategories of $\mathcal{T}$ containing $\mathcal{S}$ is a localizing (resp. colocalizing) subcategory denoted by $\mathrm{Loc}(\mathcal{S})$ (resp. $\mathrm{CoLoc}(\mathcal{S})$).

The following definition introduces an important finiteness condition.

Definition 2.3.3. Let $\mathcal{T}$ be a triangulated category with (small) coproducts. An object $X \in \mathcal{T}$ is called *compact*, if for any family $\{Y_i\}_{i\in\mathcal{I}}$ of objects in $\mathcal{T}$, any morphism

$$X \to \coprod_{i\in\mathcal{I}} Y_i$$

factors through a coproduct $\coprod_{j\in J} Y_j$, for a finite subset $J \subseteq I$.

The triangulated category $\mathcal{T}$ is said to be *compactly generated* if there exists a set of compact objects $\mathcal{S}$, such that for any non-zero $X \in T$ there exists a non-zero morphism $C \to X$ with $C \in \mathcal{C}$. In this case we say that $\mathcal{T}$ is compactly generated by $\mathcal{S}$.

Proposition 2.3.4. *Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{R}$ be a set of compact objects in $\mathcal{T}$ which is closed under suspensions. Then the category* $\mathrm{Loc}(\mathcal{R})$ *is compactly generated by the set $\mathcal{R}$.*

Proof. Assume that $L \in \mathrm{Loc}(\mathcal{R})$ is such that $\mathrm{Hom}_{\mathcal{T}}(R, L) = 0$ for all $R \in \mathcal{R}$. We wish to show that $L = 0$. For this consider the subcategory

$$^{\perp}L := \{T \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\Sigma^i T, L) = 0, \text{ for all } i\}.$$

It is easy to see that that is a localizing subcategory of $\mathcal{T}$ which contains $\mathcal{R}$, therefore $\mathrm{Loc}(\mathcal{R}) \subseteq^{\perp} L$. But then the identity map on L is zero. $\square$

The following theorem which is due to Neeman is a milestone in the theory of triangulated categories.

Theorem 2.3.5. [61, Thm. 4.1] *Let $\mathcal{S}$ be a compactly generated triangulated category, $\mathcal{T}$ any triangulated category and $F : \mathcal{S} \to \mathcal{T}$ a triangulated functor. If F respects coproducts then it admits a right adjoint.*

In fact Neeman proves this result as a corollary of the following:

Theorem 2.3.6. [61, Thm. 3.1](Brown representability) *Let $\mathcal{T}$ be a compactly generated triangulated category and let $F : \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ be a cohomological functor which sends coproducts to products. Then F is representable.*

Definition 2.3.7. We say that a diagram of triangulated categories

$$\mathcal{T}' \xrightarrow{I} \mathcal{T} \xrightarrow{Q} \mathcal{T}''$$

is a *localization sequence* if the following hold:

(i) $\forall T \in \mathcal{T};\ QT = 0 \Leftrightarrow T \cong I(T')$ for some $T' \in \mathcal{T}'$.

(ii) Q has a right adjoint R such that $Q \circ R \cong \mathrm{id}_{\mathcal{T}''}$.

(iii) I has a right adjoint R' such that $R' \circ I \cong \mathrm{id}_{\mathcal{T}'}$.

It follows that in a localization sequence $\mathcal{T}' \xrightarrow{I} \mathcal{T} \xrightarrow{Q} \mathcal{T}''$ the functors I and R are fully faithful and that $\mathcal{T}/\mathcal{T}' \cong \mathcal{T}''$. See for instance Verdier [79, II.2]. Therefore a localization sequence is basically a diagram of the form

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift right, "i"'] & \mathcal{T} \arrow[l, shift right, "R'"'] \arrow[r, shift right, "Q"'] & \mathcal{T}/\mathcal{S}, \arrow[l, shift right, "R"'] \end{tikzcd}.$$

where $\mathcal{S}$ is a thick subcategory of $\mathcal{T}$, i is the natural inclusion, Q is the canonical Verdier quotient map and R (resp., R') is right adjoint to the functor Q (resp., i).

We have the following result on localization sequences. For a proof see Verdier [79] or the article of Krause in [48].

Proposition 2.3.8. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a thick subcategory. Then the following are equivalent.*

(i) There exists a pair (L, η) where $L : \mathcal{T} \to \mathcal{T}$ is an exact functor with $\ker(L) = \mathcal{S}$ *and $\eta : 1_{\mathcal{T}} \Rightarrow L$ is a natural transformation such that $L(\eta)$ is invertible and $L(\eta) = \eta_L$.*

(ii) The quotient $Q : \mathcal{T} \twoheadrightarrow \mathcal{T}/\mathcal{S}$ admits a right adjoint.

(iii) The inclusion $i : \mathcal{S} \rightarrowtail T$ admits a right adjoint.

(iv) The natural map $\mathcal{S}^{\perp} \rightarrowtail \mathcal{T} \xrightarrow{q} \mathcal{T}/\mathcal{S}$ is an equivalence.

(v) The inclusion $j : \mathcal{S}^{\perp} \rightarrowtail \mathcal{T}$ admits a left adjoint and ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$.

Remark 2.3.9. (Orthogonality in a localization sequence) Given a localization sequence

$$\begin{tikzcd} \mathcal{T}' \arrow[r, shift right, "i"'] & \mathcal{T} \arrow[l, shift right, "R'"'] \arrow[r, shift right, "Q"'] & \mathcal{T}''. \arrow[l, shift right, "R"'] \end{tikzcd}$$

It follows from 2.3.8 that if we identify $\mathcal{T}'$ with the essential image of i and $\mathcal{T}''$ with the essential image of R, we have that $\mathcal{T}'' \cong (\mathcal{T}')^{\perp}$ and $\mathcal{T}' \cong {}^{\perp}(\mathcal{T}'')$.

There is the following neat characterization of compact generation.

Proposition 2.3.10. *Let $\mathcal{T}$ be a triangulated category with coproducts and let $\mathcal{S}$ be a set of compact objects in $\mathcal{T}$. Then the following are equivalent:*

(i) $\mathcal{T}$ is compactly generated by $\mathcal{S}$.

(ii) $\mathrm{Loc}(\mathcal{S}) = \mathcal{T}$.

Proof. (i)⇒(ii) From 2.3.4 we have that $\mathrm{Loc}(\mathcal{S})$ is compactly generated (by the set $\mathcal{S}$). Therefore by Theorem 2.3.5 we obtain a localization sequence

$$\mathrm{Loc}(\mathcal{S}) \underset{\iota}{\overset{\lambda}{\leftrightarrows}} \mathcal{T} \underset{\pi}{\overset{l}{\leftrightarrows}} \mathcal{T}/\mathrm{Loc}(\mathcal{S})\,.$$

We know that the right orthogonal of $\mathrm{Loc}(\mathcal{S})$ inside $\mathcal{T}$ are the $\mathrm{Loc}(\mathcal{S})$–local objects, i.e. $\mathrm{Loc}(\mathcal{S})^{\perp}$ consists of the objects $T \in T$ such that $l\pi(T) \cong T$. Since $\mathcal{S}$ generates, we have that this class of objects is zero. Therefore for all objects $T \in \mathcal{T}$ we obtain that $l\pi T = 0$. But then the identity map on πT, which can be factored as $\pi T \to (\pi l \pi)T \to \pi T$, is zero. Thus $\mathcal{T}/\mathrm{Loc}(\mathcal{S}) = 0$.

The assertion (ii)⇒(i) follows from 2.3.4. □

We give some examples of localization sequences which are of interest in this thesis.

Example 2.3.11. Let A be a ring and consider the Verdier quotient

$$\mathbf{K}_{\mathrm{ac}}(A) \rightarrowtail \mathbf{K}(A) \xrightarrow{q} \mathbf{D}(A).$$

By Bökstedt and Neeman [14] this is a localization sequence. In more detail, construction [14, 2.4] shows that for any complex in $qX \in \mathbf{D}(A)$ there exists a complex of injectives RqX and a morphism $\eta_X : X \to RqX$ in $\mathbf{K}(A)$ which is natural in X (and becomes a natural isomorphism after applying q). The construction is such that any other map $g : X \to RZ$ in $\mathbf{K}(A)$ there exists a unique up to homotopy map $f : qX \to Z$ such that $g = R(f) \circ \eta$. Therefore q admits a (fully faithful) right adjoint $R : \mathbf{D}(A) \to \mathbf{K}(A)$. From the specifics of the construction in [14] we actually have that the essential image of R is $\mathrm{CoLoc}(\mathbf{K}^{-}(\mathrm{Inj}(A)))$. The complexes in this class are called K-injective complexes. Thus combining with 2.3.9 we obtain equivalences

$$\mathbf{K}_{\mathrm{ac}}(A)^{\perp} \cong \mathbf{D}(A) \cong \mathrm{CoLoc}(\mathbf{K}^{-}(\mathrm{Inj}(A))).$$

By putting all of this together we obtain a diagram

$$\mathbf{K}_{\mathrm{ac}}(A) \underset{i}{\overset{R'}{\leftrightarrows}} \mathbf{K}(A) \underset{Q}{\overset{R}{\leftrightarrows}} \mathbf{D}(A) \cong \mathrm{CoLoc}(\mathbf{K}^{-}(\mathrm{Inj}(R)))\,.$$

Dually, from [14] we have that the canonical map $Q : \mathbf{K}(A) \to \mathbf{D}(A)$ admits a left adjoint L which takes a complex X to its K-projective resolution. The construction provides us with isomorphisms

$$^{\perp}\mathbf{K}_{\mathrm{ac}}(A) \cong \mathbf{D}(A) \cong \mathrm{Loc}(\mathbf{K}^{-}(\mathrm{Proj}(A)))$$

We point out that it is possible to obtain these adjoints in a more abstract way using Brown representability instead of the construction of K-projective resolutions. Indeed, consider the inclusion $i : \mathrm{Loc}(A) \to \mathbf{K}(A)$. From 2.3.10

we know that $\mathrm{Loc}(A)$ is compactly generated by the set $\{A\}$, therefore Theorem 2.3.5 provides us with a right adjoint R to the inclusion i. The kernel of this right adjoint is $\ker(R) = \mathrm{Loc}(A)^{\perp}$, which by a straightforward computation is isomorphic to $\mathbf{K}_{\mathrm{ac}}(A)$. From 2.3.8 we have that the composite $\mathrm{Loc}(A)^{\perp} \hookrightarrow \mathbf{K}(A) \xrightarrow{q} \mathbf{D}(A)$ is an equivalence. Hence from 2.3.8 we obtain a (co)localization sequence,

$$\mathbf{K}_{\mathrm{ac}}(A) \underset{i}{\overset{L'}{\leftrightarrows}} \mathbf{K}(A) \underset{Q}{\overset{L}{\leftrightarrows}} \mathrm{Loc}(A) \cong \mathbf{D}(R)\,.$$

The existence of left and right adjoints for the map $\mathbf{K}(A) \to \mathbf{D}(A)$ in the above example are usually depicted in the form of a recollement

$$\begin{tikzcd} \mathbf{K}_{\mathrm{ac}}(A) \arrow[r, "i" description] & \mathbf{K}(A) \arrow[l, shift right=2, "\mathrm{L}'"'] \arrow[l, shift left=2, "R'"] \arrow[r, "Q" description] & \mathbf{D}(A) \arrow[l, shift right=2, "L"'] \arrow[l, shift left=2, "R"] \end{tikzcd},$$

where the essential images of the functors L and R give us equivalences

$${}^{\perp}\mathbf{K}_{\mathrm{ac}}(A) \cong \mathbf{D}(A) \cong \mathbf{K}_{\mathrm{ac}}(A)^{\perp}.$$

The pure derived category

We now look at recollements in the context of purity. Let $\mathcal{A}$ be locally finitely presented additive category equipped with the pure exact structure. Consider the canonical map $Q : \mathbf{K}(\mathcal{A}) \to \mathbf{D}_{\mathrm{pure}}(\mathcal{A})$. There are many proofs providing left and right adjoints of Q. Krause in [55, Cor. 3/Thm. 7] proves the existence of a right adjoint, and also the existence of a left adjoint [55, Ex. 6]. This induces a recollement of the form

$$\begin{tikzcd} \mathbf{K}_{\mathrm{pac}}(\mathcal{A}) \arrow[r, "i" description] & \mathbf{K}_{\mathrm{pure}}(\mathcal{A}) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, "Q" description] & \mathbf{D}_{\mathrm{pure}}(\mathcal{A}) \arrow[l, shift right=2, "L"'] \arrow[l, shift left=2, "R"] \end{tikzcd}. \tag{2.1}$$

Šťovíček in [80, Cor. 5.8] identifies the essential image of L with $\mathbf{K}(\mathrm{PProj}\,\mathcal{A})$ and the essential image of R with $\mathbf{K}(\mathrm{PInj}\,\mathcal{A})$. Hence we obtain equivalences of triangulated categories

$${}^{\perp}\mathbf{K}_{\mathrm{pac}}(\mathcal{A}) \cong \mathbf{K}(\mathrm{PProj}\,\mathcal{A}) \cong \mathbf{D}(\mathcal{A}) \cong \mathbf{K}(\mathrm{PInj}\,\mathcal{A}) \cong \mathbf{K}_{\mathrm{pac}}(\mathcal{A})^{\perp}.$$

Note that we could obtain (parts of) the recollement in 2.1 if we knew that the corresponding results after passing to the category of flat functors $\mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ hold. We explain this in more detail. Recall from 2.2.13 the equivalence of (exact) categories

$$\mathcal{A} \cong \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab});\ A \mapsto \mathrm{Hom}_{\mathcal{A}}(-, A)|_{\mathrm{fp}(\mathcal{A})}. \tag{2.2}$$

Now, for a locally finitely presented additive category $\mathcal{B}$ consider the canonical map $Q : \mathbf{K}(\mathrm{Flat}\mathcal{B}) \to \mathbf{D}(\mathrm{Flat}\mathcal{B})$. Neeman in [63] proves the existence of left adjoint of Q and identifies its essential image with $\mathbf{K}(\mathrm{Proj}\,\mathcal{B})$.[3] Since the equivalence 2.2 identifies $\mathrm{PProj}(R)$ with the projective objects in the functor category, we see that half of the recollement in 2.1, and also the fact that $\mathbf{K}(\mathrm{PProj}\,R) \cong \mathbf{D}(R)$ follow from Neeman's result. In a subsequent paper [64] Neeman also proves the existence of a right adjoint of Q. We state these results in the form of a recollement.

Theorem 2.3.12. (Neeman [63, 64]) *Let $\mathcal{B}$ be a locally finitely presented additive category. Then $\mathbf{K}(\mathrm{Proj}\,\mathcal{B})$ is $\aleph_1$–compactly generated, and there exists a recollement*

$$\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}\mathcal{B}) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;i\;}}} \mathbf{K}(\mathrm{Flat}\mathcal{B}) \underset{\overset{\longleftarrow}{R}}{\overset{\overset{L}{\longleftarrow}}{\xrightarrow{\;Q\;}}} \mathbf{D}(\mathrm{Flat}\mathcal{B})\,.$$

where the essential image of the left adjoint L is $\mathbf{K}(\mathrm{Proj}\,\mathcal{B})$.

Remark 2.3.13. By now we also know the existence of the essential image of the right adjoint R. Indeed, it follows from [11] that we have equivalences

$$^{\perp}\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}\mathcal{B}) \cong \mathbf{D}(\mathrm{Flat}\mathcal{B}) \cong \mathbf{K}(\mathrm{Flat}\mathcal{B} \cap (\mathrm{Flat}\mathcal{B})^{\perp}).$$

Note that from the fact that R is a right adjoint of Q we already know the equivalence on the left hand side (for this recall 2.3.9). The point is to obtain an equivalence

$$^{\perp}\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}\mathcal{B}) \cong \mathbf{K}(\mathrm{Flat}\mathcal{B} \cap (\mathrm{Flat}\mathcal{B})^{\perp}).$$

This is proved in [11, Theorem 4.3]. This also implies that the category $\mathbf{K}(\mathrm{Flat}\mathcal{B} \cap (\mathrm{Flat}\mathcal{B})^{\perp})$ admits coproducts (inhereted from the aforementioned equivalence).

Let us get back to the recollement 2.1. In the module case, over a ring A, this is the recollement

$$\mathbf{K}_{\mathrm{pac}}(A) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;i\;}}} \mathbf{K}_{\mathrm{pure}}(A) \underset{\overset{\longleftarrow}{R}}{\overset{\overset{L}{\longleftarrow}}{\xrightarrow{\;Q\;}}} \mathbf{D}_{\mathrm{pure}}(A)\,. \tag{2.3}$$

(where we consider right A–modules.)

Recall that this recollement follows from the more general recollement involving flat functors in 2.3.12. On the other hand, note that we could obtain (parts of) the recollement in 2.3 if we knew that the corresponding

[3]Neeman's results are actually stated for module categories, but their proofs are valid in any locally finitely presented additive category.

results after passing to the category of right exact functors hold. We explain this in more detail. Recall from 2.2.15 the equivalence of categories

$$\text{Mod-}A \xrightarrow{\mathcal{T}} \mathcal{B} := (A\text{-mod}, \mathrm{Ab});\ M \mapsto (M \otimes_A -)|_{A\text{-mod}}$$

which identifies $\mathrm{PInj}(A)$ with $\mathrm{Inj}(\mathcal{B})$ and $\mathrm{PProj}(A)$ with $\mathrm{FPI}(\mathcal{B}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{B})$.

Parts of our main results in [25] give us the following.

Theorem 2.3.14. [25] *Let $\mathcal{A}$ be a locally coherent Grothendieck category. Then there exists a colocalization sequence*

$$\mathbf{K}_{\mathrm{pac}}(\mathrm{FPI}\,\mathcal{A}) \underset{i}{\overset{}{\rightleftarrows}} \mathbf{K}(\mathrm{FPI}\,\mathcal{A}) \underset{Q}{\overset{L}{\rightleftarrows}} \mathbf{D}(\mathrm{FPI}\,\mathcal{A}) \tag{2.4}$$

where the essential image of the left adjoint L is $\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A}))$.

Remark 2.3.15. For a localy coherent Grothendieck category $\mathcal{A}$, a localization sequence

$$\mathbf{K}_{\mathrm{pac}}(\mathrm{FPI}\,\mathcal{A}) \underset{i}{\overset{}{\rightleftarrows}} \mathbf{K}(\mathrm{FPI}\,\mathcal{A}) \underset{Q}{\overset{R}{\rightleftarrows}} \mathbf{D}(\mathrm{FPI}\,\mathcal{A}) \tag{2.5}$$

has been obtained by Šťovíček [80], who identified the essential image of R with $\mathbf{K}(\mathrm{Inj}\,\mathcal{A})$. Therefore combining this with 2.4 we obtain, in the locally coherent case, a complete picture of the recollement with middle term $\mathbf{K}(\mathrm{FPI}(\mathcal{A}))$,

$$\begin{tikzcd} \mathbf{K}_{\mathrm{pac}}(\mathrm{FPI}\,\mathcal{A}) \arrow[r, "i" description] & \mathbf{K}(\mathrm{FPI}\,\mathcal{A}) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, "Q" description] & \mathbf{D}(\mathrm{FPI}\,\mathcal{A}) \arrow[l, shift right=2, "L"'] \arrow[l, shift left=2, "R"] \end{tikzcd}.$$

which induces equivalences of triangulated categories

$$\begin{tikzcd} & \mathbf{D}(\mathrm{FPI}\,\mathcal{A}) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A})) & & \mathbf{K}(\mathrm{Inj}\,R), \end{tikzcd}$$

where $\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A})) \cong {}^{\perp}\mathbf{K}_{\mathrm{pac}}(\mathrm{FPI}\,\mathcal{A})$ and $\mathbf{K}_{\mathrm{pac}}(R)^{\perp} \cong \mathbf{K}(\mathrm{Inj}\,R)$.

Chapter 3

Relative Homological Algebra

3.1 Cotorsion pairs and approximation theory

The general references for the material in this section are the book of Enochs and Jenda [35] and the book of Göbel and Trlifaj [41].

Definition 3.1.1. A class $\mathcal{X}$ of objects in an abelian category $\mathcal{A}$ is called **precovering** if for all $A \in \mathcal{A}$ there exists a morphism $f : X \to A$ with $X \in \mathcal{X}$ such that, for any $X' \in \mathcal{X}$, the induced map $\mathrm{Hom}_{\mathcal{A}}(X', f) : \mathrm{Hom}_{\mathcal{A}}(X', X) \to \mathrm{Hom}_{\mathcal{A}}(X', A)$ is surjective.

Definition 3.1.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ be a class of objects in $\mathcal{A}$. A **proper left** $\mathcal{X}$**–resolution** of an object $A \in \mathcal{A}$ is a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact sequence

$$\cdots \to X_1 \to X_0 \to A \to 0.$$

Remark 3.1.3. Note that we can dualize the above definitions to obtain the notion of a **preenveloping class** and that of a **proper right** $\mathcal{X}$**–resolution**.

Remark 3.1.4. Let $\mathcal{A}$ be an abelian category. We observe the following:

(i) By definition, a proper left $\mathcal{X}$–resolution is not necessarily an exact complex.

(ii) If $\mathcal{X} = \mathrm{Proj}(\mathcal{A})$, then a proper $\mathrm{Proj}(\mathcal{A})$-resolution is just a projective resolution.

(iii) The importance of considering proper resolutions (of some fixed object) instead of arbitrary resolutions is that, any two proper resolutions will be equivalent up to homotopy.

(iv) If $\mathcal{X}$ is a precovering class in $\mathcal{A}$, any object has a proper left $\mathcal{X}$-resolution, which can be constructed inductively in the usual fashion.

Precovering classes usually arise as left hand sides of the so called "cotorsion pairs", that we now define.

Definition 3.1.5. ([70], see also [41]) Let $\mathcal{X}$ be a class of objects in an exact category $\mathcal{A}$. Put

$$\mathcal{X}^{\perp} := \{A \in \mathcal{A} \,|\, \forall X \in \mathcal{X},\ \mathrm{Ext}^1_{\mathcal{A}}(X, A) = 0\}$$

and define ${}^{\perp}\mathcal{X}$ analogously. A pair $(\mathcal{X}, \mathcal{Y})$ of classes in $\mathcal{A}$ is called a **cotorsion pair** if $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called **complete** if for every object A in $\mathcal{A}$ there exists a short exact sequence $0 \to Y \to X \to A \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and also a short exact sequence $0 \to A \to Y' \to X' \to 0$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$. It is called **hereditary** if $\mathcal{X}$ is closed under kernels of epimorphisms and $\mathcal{Y}$ is closed under cokernels of monomorphisms. Note that being hereditary is equivalent to having for all $X \in \mathcal{X}, Y \in \mathcal{Y}$ and $i \geq 1$, $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$.

Example 3.1.6. (i) In any abelian category $\mathcal{A}$ there exist cotorsion pairs $(\mathrm{Proj}(\mathcal{A}), \mathcal{A})$ and $(\mathcal{A}, \mathrm{Inj}(\mathcal{A}))$. When $(\mathrm{Proj}(\mathcal{A}), \mathcal{A})$ is complete we say that the category $\mathcal{A}$ has "enough projectives". Dually when $(\mathcal{A}, \mathrm{Inj}(\mathcal{A}))$ is complete we say that $\mathcal{A}$ has "enough injectives".

(ii) Let $\mathcal{A}_{\mathrm{pure}}$ be a locally finitely presented additive category equipped with the pure exact structure, see 2.2.4. Consider the class $\mathrm{PProj}(\mathcal{A})$ of pure projective objects in $\mathcal{A}_{\mathrm{pure}}$. From [23, Lemma 3.1] we have that $(\mathrm{PProj}(\mathcal{A}), \mathcal{A})$ is a complete cotorsion pair in $\mathcal{A}_{\mathrm{pure}}$. In case $\mathcal{A}$ is locally finitely presented Grothendieck, we also have that the cotorsion pair $(\mathcal{A}, \mathrm{PInj}(\mathcal{A}))$ is complete, see Simson [73, Thm. 4.1].

(iii) Let R be a Gorenstein ring and denote by $\mathrm{MCM}(R)$ the class of maximal Cohen–Macaulay right R–modules and by $\mathcal{P}(R)$ the class of right R–modules of finite projective dimension. Then from Buchweitz [18] we have that the pair $(\mathrm{MCM}(R), \mathcal{P}(R))$ is a complete cotorsion pair in the category of finitely generated right R–modules.

Definition 3.1.7. We say that a cotorsion pair in an exact category $\mathcal{A}$ is **generated by a set** if it is of the form $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ for a set $\mathcal{S}$ of objects in $\mathcal{A}$.

Example 3.1.8. (i) Let $\mathcal{A} := \text{Mod-}R$ be a module category over a ring R. The pair $(\mathcal{A}, \mathrm{Inj}(\mathcal{A}))$ is generated by the set $\{R/I \,|\, I \trianglelefteq R\}$. Indeed, by Baer's criterion we have that a module M is injective if and only if for all ideals $I \trianglelefteq R$ we have $\mathrm{Ext}^1_R(R/I, M) = 0$.

(ii) Let $\mathcal{A}$ be a Grothendieck category with a generator G, then the cotorsion pair $(\mathcal{A}, \mathrm{Inj}(\mathcal{A}))$ is generated by the set $\{G/S \mid S \leqslant G\}$. This follows from a generalized version of Baer's criterion.

(iii) Let $\mathcal{A}$ be a locally finitely presented additive category and denote by $\mathcal{S}$ a set of isomorphism classes of finitely presented objects in $\mathcal{A}$. Then we automatically obtain a cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp}) = ({}^{\perp}\mathrm{FPI}(\mathcal{A}), \mathrm{FPI}(\mathcal{A}))$. This cotorsion pair is of interest in paper B of this thesis [25], where we lift it to the category of chain complexes.

Next we recall an important fact concerning cotorsion pairs generated by sets (in Grothendieck categories). First a definition.

Definition 3.1.9. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a class of objects in $\mathcal{A}$. An object A in $\mathcal{A}$ is called **$\mathcal{S}$-filtered** if there exists a chain of subobjects

$$0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq \bigcup_{\alpha<\sigma} A_\alpha = A$$

where σ is an ordinal, $A_\lambda = \cup_{\beta<\lambda} A_\beta$ for all limit ordinals λ, and $A_{\alpha+1}/A_\alpha \in \mathcal{S}$ for all $\alpha < \sigma$. The class of $\mathcal{S}$-filtered objects will be denoted by $\mathrm{Filt}(\mathcal{S})$. Classes of $\mathcal{S}$–filtered objects are also called **deconstructible**.

Fact 3.1.10. ([31], see also [41, 3.2]) Let $\mathcal{S}$ be a (small) set of objects in a Grothendieck category and assume that $\mathcal{S}$ contains a generator. Then the following hold:

(i) The cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is complete.

(ii) The class ${}^{\perp}(\mathcal{S}^{\perp})$ consists of direct summands of $\mathcal{S}$-filtered objects, that is, for all $X \in {}^{\perp}(\mathcal{S}^{\perp})$ we have $P \cong X \oplus K$ where $P \in \mathrm{Filt}(S)$. Moreover, in this decomposition K can be chosen in ${}^{\perp}(\mathcal{S}^{\perp}) \cap \mathcal{S}^{\perp}$.

Corollary 3.1.11. *Let $\mathcal{A}$ be a Grothendieck category. Then the cotorsion pair $(\mathcal{A}, \mathrm{Inj}(\mathcal{A}))$ is complete, i.e. $\mathcal{A}$ has enough injectives.*

The following is known as the "flat cover conjecture".

Corollary 3.1.12. (Enochs [13]) *Let $\mathcal{A} := \mathrm{Mod}\text{-}R$ be the category of (right) R–modules over a ring R. Then the pair $(\mathrm{Flat}(R), \mathrm{Flat}(R)^{\perp})$ is complete.*

Proof. Let $\lambda > |R| + \omega$. For any R–module F there exists a filtration $0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq \bigcup\limits_{\alpha<\sigma} F_\alpha = F$, where all the inclusions are pure and, for all $\alpha < \sigma$, we have $|F_{\alpha+1}/F_\alpha| \leq \lambda$, see [13, Lemma 1].

Now, M belongs in the class $\mathrm{Flat}(R)^{\perp}$ if and only if it satisfies $\mathrm{Ext}^1_R(F, M) = 0$, for any flat F, or equivalently, if and only if it satisfies $\mathrm{Ext}^1_R(\bigcup\limits_{\alpha<\sigma} F_\alpha, M) = 0$, where we used the aforementioned filtration for a flat module F. Then using Eklof's lemma [31], we have that $M \in \mathrm{Flat}(R)^{\perp}$ if and only if $M \in \mathcal{F}^{\perp}$, where $\mathcal{F}$ is a set of representatives of the flat modules with cardinality $\leq \lambda$. Hence the result follows from 3.1.10. $\square$

3.2 Gorenstein homological algebra (over commutative rings)

In this section we give a small survey of Gorenstein homological algebra. We focus on the historical development of the subject in the context of commutative Noetherian local rings. A big survey can be found in [21], see also Christensen's book [20].

Homological characterizations of rings is a favourite topic with a vast number of applications. Roughly speaking, the idea is to characterize a ring via global properties of its category of left/right modules. One of the most classical and important examples in this direction is a theorem due to Serre, Auslander, and Buchsbaum. Recall that if R is a ring (from now on assumed commutative) we define its global dimension as

$$\text{gl.dim}(R) := \sup\{n \mid \text{pd}_R(M) \leq n\,;\, M \in Mod(R) \text{ with } \text{pd}_R(M) < \infty\}.$$

A theorem of Hilbert states that $\text{gl.dim}(\mathbb{C}[x_1, ..., x_n]) = n$. The next celebrated theorem can be seen as a generalization of Hilbert's theorem.

Theorem 3.2.1. (Serre-Auslander-Buchsbaum) *Let $(R, \mathfrak{m}, k)$ be a local commutative Noetherian ring. Then the following are equivalent:*

(i) R is regular (i.e. $\dim(R) = \dim_k \mathfrak{m}/\mathfrak{m}^2$)

(ii) $\text{pd}_R k < \infty$

(iii) $\text{gl.dim}(R) < \infty$.

Much of the progress in homological commutative algebra is based on attempts to obtain analogous theorems for more general classes of rings than regular local rings. To this end one has to invent new dimensions, which should be refinements of the projective dimension. We recall that the typical hierarchy of local rings is the following:

$$\text{Regular} \subseteq \text{Complete intersection} \subseteq \text{Gorenstein} \subseteq \text{Cohen–Macaulay}.$$

We briefly recall the definitions:

Definition 3.2.2. Let $(R, \mathfrak{m}, k)$ be a local commutative Noetherian ring. Then R is called

- *a complete intersection*, if it is (up to completion) a quotient of a regular local ring R modulo an R–regular sequence.

- *Gorenstein*, if $\text{inj.dim}_R R < \infty$.

- *Cohen–Macaulay*, if $\text{depth} R = \dim(R)$.

Proofs of the containments in the aforementioned hierarchy can be found in classical texts [17, 58], see also [52].

A study of modules over a non-regular local rings will have to include the study of modules of infinite projective dimension. An important result in this direction is the following.

Theorem 3.2.3. (Eisenbud (1980) [30]) *Let $(R, \mathfrak{m}, k)$ be a regular local ring and $f \in \mathfrak{m}$ a non-zero divisor. Let $S := R/(f)$ and N be a finitely generated S-module. Then N admits a free resolution (as an S-module) which becomes periodic after finitely many steps.*

Proof. Consider a projective resolution of N over S:

$$\cdots \to S^{n_2} \to S^{n_1} \to S^{n_0} \to N \to 0.$$

Since the depth of syzygies increases as we go along the resolution of N, there exists some integer k and a syzygy $\Omega := \Omega_k(N)$ in this resolution such that $\operatorname{depth}_S \Omega = \operatorname{depth} S$.

Now, over the regular ring R, from the Auslander-Buchsbaum formula we have $\operatorname{pd}_R \Omega = \operatorname{depth} R - \operatorname{depth}_R \Omega$. Therefore, $\operatorname{pd}_R \Omega = \operatorname{depth} R - \operatorname{depth} S = 1$. This means that we have a presentation of Ω as an R-module:

$$0 \to R^n \xrightarrow{\Phi} R^m \to \Omega \to 0.$$

Note that here $n = m$ since the rank of Ω over R is zero. Moreover, the fact that $f \cdot \Omega = 0$ provides as with a commutative diagram of R-modules,

$$\begin{tikzcd}
R^n \arrow[r, "\Phi"] \arrow[d, "\cdot f"'] & R^n \arrow[d, "\cdot f"] \arrow[dl, dashed, "\Psi"'] & \\
R^n \arrow[r, "\Phi"] & R^n \arrow[r, two heads] & \Omega.
\end{tikzcd}$$

Reducing this diagram to $S = R/(f)$ we have

$$\begin{tikzcd}
\Omega \arrow[r, tail] & R^n \arrow[r, "\overline{\Phi}"] \arrow[d, "0"'] & R^n \arrow[d, "0"] \arrow[dl, dashed, "\overline{\Psi}"'] & \\
& R^n \arrow[r, "\overline{\Phi}"] & R^n \arrow[r, two heads] & \Omega.
\end{tikzcd}$$

Unravelling this diagram we obtain an acyclic complex of projectives of the form

$$\begin{tikzcd}
\cdots \arrow[r] & S^n \arrow[r, "\overline{\Psi}"] & S^n \arrow[rr, "\overline{\Phi}"] \arrow[dr, two heads] & & S^n \arrow[r] & \cdots \\
& & & \Omega \arrow[ur, hook] & &
\end{tikzcd}$$

which concludes the proof. ∎

Eisenbud's result tells us that over a hypersurface ring S we can quite easily find modules M that are "infinite syzygies", in the sense that they are syzygies in exact complexes of projectives with infinite left and right tales:

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[rr] \arrow[dr] & & P_{-1} \arrow[r] & \cdots \\ & & & \Omega \arrow[ur] & & \end{tikzcd}$$

We give a general definition for the class of modules that are infinite syzygies. The terminology used in the definition will be soon explained.

Definition 3.2.4. Let R be a ring. A left R-module M is called *Gorenstein projective* if there exists an exact complex of projective left R-modules, which has M as a syzygy and remains exact after applying functors of the form $\mathrm{Hom}_R(-, \mathrm{Proj}(R))$. We denote this class of modules by $\mathrm{GProj}(R)$.

Buchweitz in his seminar paper [18] singles out this class of modules over a Gorenstein ring R and proves a number of interesting facts. One important result is [18, Section 5] which states that a ring is Gorenstein if and only if every finitely generated R-module admits a resolution by finitely generated Gorenstein-projective modules

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0.$$

Of course "resolution" here means that this complex has homology concentrated in degree zero and isomorphic to M. This justifies the terminology in the definition 3.2.4. Note that this is an analogue for Gorenstein rings of the theorem of Serre-Auslander-Buchbaum 3.2.1.

We state a well–known characterization of finitely generated Gorenstein-projective modules over a Gorenstein ring. See [9, Section 2] for a proof.

Proposition 3.2.5. *Let R be a Gorenstein local ring and M a finitely generated R-module. Then the following are equivalent:*

(i) M is an "infinite syzygy", i.e. M is Gorenstein projective.

(ii) M is maximal Cohen-Macaulay, i.e. $\mathrm{depth} M = \mathrm{dim} M = \mathrm{dim} R$.

(iv) M is totally reflexive, i.e. the natural map

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R);\ m \mapsto [\phi \mapsto \phi(m)]$$

is an isomorphism and $\mathrm{Ext}^{>0}_R(M, R) = 0 = \mathrm{Ext}^{>0}_{R^o}(\mathrm{Hom}_R(M, R), R)$.

Therefore the finitely generated Gorenstein-projective modules over a Gorenstein local commutative ring coincide with the class of maximal Cohen-Macaulay modules $\mathrm{MCM}(R)$. The following result explains in categorical terms the connection between maximal Cohen–Macaulay modules, complete resolutions and singularities over a Gorenstein ring.

Theorem 3.2.6. (Buchweitz [18]) *Let R be a Gorenstein ring. Denote by $\mathbf{D}_{\mathrm{sg}}(R)$ its singularity category, i.e. the Verdier quotient of the bounded derived category of finitely generated R-modules modulo the subcategory of perfect complexes. Then there exist equivalences of triangulated categories:*

$$\mathbf{D}_{\mathrm{sg}}(R) \overset{\cong}{\longleftarrow} \underline{\mathrm{MCM}}(R) \overset{\cong}{\longrightarrow} \mathbf{K}_{\mathrm{ac}}(\mathrm{proj}(R)).$$

The following result concerns maximal Cohen-Macaulay approximations over Gorenstein rings.

Theorem 3.2.7. (Buchweitz [18, Thm. 5.1.2]) *Let R be a Gorenstein ring. Then any finitely generated R-module M admits a short exact sequence $0 \to P \to X \to M \to 0$ where X is maximal Cohen-Macaulay and P has finite projective dimension. Moreover, M admits a short exact sequence $0 \to M \to Y \to K \to 0$ where Y has finite projective dimension and K is maximal Cohen-Macaulay.*

We point out that this result has been generalized by Auslander and Buchweitz in their *homological theory of maximal Cohen-Macaulay approximations* [5], where they prove the following:

Theorem 3.2.8. *Let R be a Cohen-Macaulay ring with a dualizing module. Then any finitely generated R-module M admits a short exact sequence $0 \to I \to X \to M \to 0$ where X is maximal Cohen-Macaulay*[1] *and I has finite injective dimension. Moreover, M admits a short exact sequence $0 \to M \to Y \to K \to 0$ where Y has finite injective dimension and K is maximal Cohen-Macaulay.*

Remark 3.2.9. In the language of relative homological algebra, Theorem 3.2.8 says that over a Cohen-Macaulay local ring with a dualizing module, the pair $(\mathrm{MCM}(R), \mathcal{I}(R))$ is a complete cotorsion pair in the category of finitely generated R-modules.

We close this section with some remarks. We discussed how Buchweitz [18] and Auslander-Buchweitz [5] characterized Gorenstein rings via the Gorenstein projective dimension, in a complete analogy with the theorem of Serre-Auslander-Buchsbaum 3.2.1 which characterizes regular local rings via the projective dimension. There is a similar story for complete intersections, which makes use of a "complete intersection dimension", see Avramov et al. [8].

Some of the aforementioned results (e.g. the ones from Buchweitz [18]) are actually valid for non-commutative analogues of Gorenstein rings. We recall that a ring is called *Iwanaga-Gorentein* if it is noetherian on both sides and has finite injective dimension on both sides, see Auslander-Bridger [3] and Zaks [81].

[1] In the sense that depth$M = \dim M = \dim R$.

Some non-noetherian analogues of Iwanaga-Gorenstein rings have also been studied, see Ding-Chen [27, 28] . For this theory to work, the word "noetherian" is replaced by "coherent" and the word "injective" is replaced by "fp-injective". Then most of the formal aspects of the theory are still valid, see for instance Gillespie [39].

Finally, we point out that over a general ring, an analogue of maximal Cohen-Macaulay approximations has been studied by Holm [46]. More precisely, the notion of Gorenstein projective modules, as in 3.2.4, can be defined over a general ring, thus Gorenstein projective resolutions and Gorenstein projective dimensions make sense. The same holds for the duals of these notions, i.e. Gorenstein injective modules, resolutions, and dimensions, see [46]. Having these notions at hand, for a general ring R, consider the classes $\mathcal{A}(R) := \{M \mid \mathrm{Gpd}_R\, M < \infty\}$ and $\mathcal{B}(R) := \{M \mid \mathrm{Gid}_R\, M < \infty\}$, and also the classes $\mathcal{P}(R) := \{M \mid \mathrm{pd}_R\, M < \infty\}$ and $\mathcal{I}(R) := \{M \mid \mathrm{id}_R\, M < \infty\}$. One of main results of Holm in [46] is that the pair $(\mathrm{GProj}(R), \mathcal{P}(R))$ is a complete and hereditary cotorsion pair in the exact category $\mathcal{A}(R)$. Dually, the pair $(\mathcal{I}(R), \mathrm{GInj}(R))$ is a complete and hereditary cotorsion pair in the exact category $\mathcal{B}(R)$. These results are worked out further in the context of exact model structures in Paper A of this thesis [26].

Dualizing complexes

Most of the nice results on Gorenstein rings that we saw in the previous section in one way or another depend on the finiteness of its injective dimension. A dualizing complex over a ring, when it exists, serves as the right tool in order to systematically study these properties. For the facts stated here we refer to Christensen's book [20].

Definition 3.2.10. Let R be a commutative ring. A complex D which is homologically bounded and has finitely generated homology is called a dualizing complex if it has finite injective dimension over R and the canonical homothety morphism $\chi_D : R \to \mathbf{R}\mathrm{Hom}_R(D, D)$ is a quasi–isomorphism.

Example 3.2.11. (1) If R is Gorenstein then R is a dualizing complex over itself.

(2) If R is Artinian then $E_R(k)$ is a dualizing complex for R.

(3) Quotients of Gorenstein rings have dualizing complexes as the next theorem shows.

Theorem 3.2.12. *Let R be a local commutative Noetherian ring with a dualizing complex D and let $\phi : R \to S$ be a local homomorphism. Then* $\mathrm{RHom}_R(S, D)$ *is a dualizing complex over S.*

Corollary 3.2.13. *Every complete local ring has a dualizing complex.*

Proof. By Cohen's structure theorem [58, 29.4] R is a quotient of a regular local ring and every such ring is Gorenstein [17, 3.12]. Hence from (3.2.12) R admits a dualizing complex. □

We now list some of the most important properties of dualizing complexes. For a ring R, we denote by $\mathbf{D}^b_f(R)$ the bounded derived category of all R–complexes having finitely generated homology.

Proposition 3.2.14. *Let $(R,\mathfrak{m},k)$ be a local commutative Noetherian ring with a dualizing complex D. Then the following hold:*

(i) (Biduality) *The functor* $\mathrm{RHom}(-,D) : \mathbf{D}^b_f(R) \to \mathbf{D}^b_f(R^o)$ *is an equivalence.*

(ii) (Localization) $\mathrm{Supp}(D) = \mathrm{Spec}(R)$ *and for all* $\mathfrak{p} \in \mathrm{Spec}(R)$, $D_\mathfrak{p}$ *is a dualizing complex over* $R_\mathfrak{p}$.

(iii) (Uniqueness) *If D and D' are dualizing complexes over a ring R, then they are isomorphic (up to a shift) in $\mathbf{D}(R)$.*

(iv) (Completion) *The complex $D \otimes^{\mathbf{L}}_R \hat{R}$ is a dualizing complex over the $\mathfrak{m}$–adic completion $\hat{R}$.*

The following result, is a theorem of Foxby [37] which in turn generalizes a theorem of Sharp [72]. It comes from the early days of Gorenstein homological algebra. Let $(R,\mathfrak{m},k)$ be a Cohen–Macaulay local ring with a dualizing complex D. To fix some notation, consider the adjunction

$$D \otimes_R - : Mod(R) \rightleftarrows Mod(R) : \mathrm{Hom}_R(D,-)$$

with unit maps $\eta_M \colon M \to \mathrm{Hom}_R(D, D \otimes_R M)$ and counit maps $\varepsilon_M \colon D \otimes_R \mathrm{Hom}_A(D,M) \to M$. Consider the classes[2]

- $\mathcal{A}(R) := \{M \mid \mathrm{Tor}^R_{i>0}(D,M) = 0 = \mathrm{Ext}^{i>0}_R(D, D\otimes_R M),\ \eta_M \text{ is an iso}\}$.
- $\mathcal{B}(R) := \{M \mid \mathrm{Ext}^{i>0}_R(D,M) = 0 = \mathrm{Tor}^R_{i>0}(D, \mathrm{Hom}_A(D,M)),\ \epsilon_M \text{ is an iso}\}$.

Denote by $\mathcal{P}(R)$ (resp $\mathcal{I}(R)$) the class of modules of finite projective (resp. injective) dimension.

Remark 3.2.15. Let $(R,\mathfrak{m},k)$ be a Cohen–Macaulay local ring. By Enochs, Jenda, and Xu [36, Cor. 2.4 and 2.6] an R-module belongs to $\mathbf{A}(R)$, (resp., $\mathbf{B}(R)$), if and only if it has finite Gorenstein projective (resp., injective) dimension.

Theorem 3.2.16. (Foxby [37]) *Let $(R,\mathfrak{m},k)$ be a Cohen–Macaulay local ring with a dualizing module D. Then the adjunction $D \otimes_R - : Mod(R) \rightleftarrows Mod(R) : \mathrm{Hom}_R(D,-)$ restricts to an equivalence $\mathcal{A}(R) \rightleftarrows \mathcal{B}(R)$ and also to an equivalence $\mathcal{P}(R) \rightleftarrows \mathcal{I}(R)$.*

[2]Some authors use the terms "Auslander class" for $\mathcal{A}(R)$ and "Bass class" for $\mathcal{B}(R)$.

The following result is a version of Theorem 3.2.16 in the context of derived categories.

Theorem 3.2.17. [7] *Let $(R, \mathfrak{m}, k)$ be a local commutative Noetherian ring with a dualizing complex D. Consider the following subcategories of $\mathbf{D}^b(R)$:*

- *$\mathcal{A}(R)$ consists of all complexes X in $\mathbf{D}^b(R)$ such that $D \otimes^{\mathbf{L}}_R X \in \mathbf{D}^b(R)$ and the canonical morphism $\eta_X : X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ induced by $m \mapsto (d \mapsto (-1)^{|m||d|} m \otimes d))$ is an isomorphism.*
- *$\mathcal{B}(R)$ consists of all complexes X in $\mathbf{D}^b(R)$ such that $\mathbf{R}\mathrm{Hom}_R(D, X) \in \mathbf{D}^b(R)$ and the canonical morphism $\epsilon_X : D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, X) \to X$ induced by $d \otimes \alpha \mapsto ((-1)^{|d||\alpha|}\alpha(d)))$ is an isomorphism.*

Then the adjunction $D \otimes^{\mathbf{L}}_R - : \mathbf{D}^b(R) \rightleftarrows \mathbf{D}^b(R) : \mathbf{R}\mathrm{Hom}_R(D, -)$ restricts to an equivalence $D \otimes^{\mathbf{L}}_R - : \mathcal{A}(R) \rightleftarrows \mathcal{B}(R) : \mathbf{R}\mathrm{Hom}_R(D, -)$, and also to an equivalence between the complexes with finite projective dimension $\mathcal{P}(R) \subseteq \mathcal{A}(R)$ and the complexes of finite injective dimension $\mathcal{I}(R) \subseteq \mathcal{B}(R)$.

Further developments in this direction can be found in the article of Iyengar and Krause [51]. They prove the following:

Theorem 3.2.18. [51] *Let R be a Noetherian ring with a dualizing complex. Then the composite functor $\mathbf{K}(\mathrm{Proj}(R)) \xrightarrow{i} \mathbf{K}(\mathrm{Flat}(R)) \xrightarrow{D \otimes_R -} \mathbf{K}(\mathrm{Inj}(R))$ is an equivalence of categories. When restricted to compact objects [3] it induces a duality $\mathbf{R}\mathrm{Hom}_R(-, D) : \mathbf{D}^b(R) \to \mathbf{D}^b(R)$.*

This result inspired Neeman's work on the homotopy category of flat modules [63]. Also, a version of Theorem 3.2.18 for Noetherian schemes was obtained by Murfet and Salarian [59].

[3] We know from Jørgensen [53] that $\mathbf{K}(\mathrm{Proj}(R))$ is compactly generated (over more general rings than the assumption of this theorem) and from Krause [54] that $\mathbf{K}(\mathrm{Inj}(R))$ is compactly generated over right Noetherian rings.

Chapter 4

Exact model structures

4.1 Quillen's model categories

The main sources we use in this chapter are the books of Hovey [49] and Riehl [68]. Originally model categories were introduced by Quillen [67].

The idea of replacing or approximating objects (such as spaces or varieties or chain complexes), up to a notion of weak equivalence, by more well–understood objects appears often throughout mathematics. Examples of this kind include:

- If X is a topological space there exists another space $Q(X)$ (which is a CW complex) and a surjective map $Q(X) \twoheadrightarrow X$ which is a *weak homotopy equivalence* (i.e. it induces isomorphisms at the level of homotopy groups). The map $Q(X) \to X$ is called a CW–approximation of X.

- Let X be a right bounded complex in an abelian category with enough projectives. Then there exists another right bounded complex $Q(X)$, which consists of projective objects, and an epimorhism $P \twoheadrightarrow X$ which is a *homology isomorphism* (i.e. it induces isomorphisms at the level of homology). The map $P \to X$ is called a projective resolution of X. This is old and standard. A version of this for unbounded complexes was obtained by Spaltenstein [75] and also by Bökstedt and Neeman [14]; recall Example 2.3.11.

The theory of model categories arises as a necessary formalism in order to conceptualize the above phenomena and put them in a categorical context.

To formalize the above constructions, one would start from a category $\mathcal{M}$ and a class of morphisms, denoted by `weak`, that we want to think of as "weak equivalences". For any object X we would like a notion of "left approximation" $Q(X) \xrightarrow{f} X$ where f belongs to a certain class of morphisms `fib` (the "fibrations"). Then f should become an isomorphism in the localized "category" $\mathcal{M}[\mathtt{weak}^{-1}]$ (which we will assume that it is a known construction). Dually, we might consider a class of morphisms `cof` (the "cofibrations"), such that for every object in X in $\mathcal{M}$ there exists a cofibration $X \to RX$ which becomes an isomorphism in $\mathcal{M}[\mathtt{weak}^{-1}]$.

The axioms of a model category are better understood if we have in mind the following topological fact: any continuous map $X \xrightarrow{g} Y$ between topological spaces has a factorization through the mapping cylinder:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, hook, "i"'] & & Y \\ & \mathrm{Cyl}(g) \arrow[ur, two heads, "f"'] & \end{tikzcd}$$

where i is an inclusion and a weak homotopy equivalence and f is surjective (in fact, a Serre fibration). Dually, using the mapping cone, g admits a factorization as a Serre fibration followed by a map which is a fibration and a weak homotopy equivalence. Similar factorizations also exist in categories of chain complexes.

Definition 4.1.1. Let $\mathcal{C}$ be a category equipped with two classes of morphisms $\mathcal{A}, \mathcal{B}$. We say that the pair $(\mathcal{A}, \mathcal{B})$ is a **weak factorization system** if

(i) Any map $f : X \to Y$ in $\mathcal{C}$ admits a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & Y \\ & F \arrow[ur, "\beta"'] & \end{tikzcd}$$

where $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.

(ii) $\mathcal{A}$ is precisely the class of maps that has the **left lifting property** (LLP for short) with respect to $\mathcal{B}$, meaning that whenever we have maps $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, for any solid diagram as below there exists a dotted arrow,

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\alpha"'] & \cdot \arrow[d, "\beta"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd}$$

making the diagram commutative. Also, $\mathcal{B}$ is precisely the class of maps that has the **right lifting property** (RLP for short) with respect to $\mathcal{A}$. A factorization system $(\mathcal{A}, \mathcal{B})$ is called functorial if it is natural in $\mathcal{A}$ and $\mathcal{B}$.

Definition 4.1.2. A **model category** is a complete and cocomplete category $\mathcal{M}$ equipped with three classes of morphisms `fib`, `cof` and `weak`, called fibrations, cofibrations and weak equivalences respectively, such that the following axioms are satisfied:

(i) The classes $\mathtt{fib}, \mathtt{cof}$ and $\mathtt{weak}$ are closed under retracts.

(ii) If $h = f \circ g$ in $\mathcal{M}$, then if any two of h, f, g belong in $\mathtt{fib}, \mathtt{cof}$ or $\mathtt{weak}$, so does the third.

(iii) The pairs $(\mathtt{cof}, \mathtt{fib} \cap \mathtt{weak})$ and $(\mathtt{cof} \cap \mathtt{weak}, \mathtt{fib})$ are functorial weak factorization systems.

We will denote the morphisms in the classes $\mathtt{cof}, \mathtt{fib}, \mathtt{weak}$, by $\hookrightarrow, \twoheadrightarrow$, and $\xrightarrow{\sim}$ respectively.

The definition well formalizes the idea of left/right approximations we discussed in the introduction of this section. Indeed, by the third axiom we obtain, for any object M in a model category $\mathcal{M}$, two factorizations:

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, hook] & & M & M \arrow[rr] \arrow[dr, hook, "\sim"'] & & 0\,. \\ & QM \arrow[ur, two heads, "\sim"'] & & & RM \arrow[ur, two heads] & \end{tikzcd}$$

Denote by $\mathcal{C} := \{M \in \mathcal{M} \,|\, 0 \hookrightarrow M \ \text{ cofibration}\}$ (the **cofibrant objects**) and by $\mathcal{M}_f := \{M \in \mathcal{M} \,|\, M \twoheadrightarrow 0 \ \text{ fibration}\}$ (the **fibrant objects**). Then the functor $Q : \mathcal{M} \to \mathcal{C}$ (resp., $R : \mathcal{M} \to \mathcal{F}$) is called the cofibrant (resp., fibrant) replacement functor.

The following fundamental theorem says that $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathtt{weak}^{-1}]$ is a category with small Hom-sets. In fact $\mathrm{Ho}(\mathcal{M})$ is a quotient $\mathcal{M}/\sim$ where $\sim$ is some sort of homotopy relation that resembles the homotopy relation of continuous maps between topological spaces.

Definition 4.1.3. Let $\mathcal{M}$ be a model category and let M be an object in $\mathcal{M}$. A **cylinder object** for M, denoted by $\mathrm{Cyl}(M)$, is given by a factorization of the fold (coproduct) map $M \sqcup M \xrightarrow{\triangledown} M$ as a cofibration $M \sqcup M \xrightarrow{i_0 + i_1} \mathrm{Cyl}(M)$ followed by a weak equivalence $\mathrm{Cyl}(M) \xrightarrow{\sim} M$.

Two parallel maps $f, g : X \rightrightarrows Y$ are called **left homotopic**, if there exists a map $H : \mathrm{Cyl}(X) \to Y$, where $\mathrm{Cyl}(X)$ is some cylinder of X, such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Path objects and **right homotopic** maps are defined as cylinder objects and left homotopic maps in the dual model category. Two parallel morphisms that are left and right homotopic are called homotopic.

Theorem 4.1.4. (Fundamental theorem of model categories) *Let $\mathcal{M}$ be a model category and denote by $q : \mathcal{M} \to \mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathtt{weak}^{-1}]$ the canonical functor. Let $\mathcal{C}$ (resp., $\mathcal{F}$) denote the cofibrant (resp., fibrant) objects in $\mathcal{M}$. If Q (resp., R) denotes the cofibrant (resp., fibrant) replacement functor, then the following hold:*

- *The inclusion $\mathcal{C} \cap \mathcal{F} \hookrightarrow \mathcal{M}$ induces an equivalence of categories*

$$\mathrm{Ho}(\mathcal{C} \cap \mathcal{F}) \cong \mathrm{Ho}(\mathcal{M}),$$

with inverse $\mathrm{Ho}(Q \circ R)$*.*

(ii) The canonical map $\mathcal{C} \cap \mathcal{F} \to \mathrm{Ho}(\mathcal{C} \cap \mathcal{F})$ induces an equivalence of categories $\mathcal{C} \cap \mathcal{F}/\sim \cong \mathrm{Ho}(\mathcal{C} \cap \mathcal{F})$, where $\sim$ denotes the homotopy relation defined in 4.1.3.

4.2 Abelian and exact model structures

The following definition is due to Hovey [50].

Definition 4.2.1. Let $\mathcal{M}$ be an abelian category. We say that $\mathcal{M}$ admits an **abelian model structure** (or that $\mathcal{M}$ is an **abelian model category**) if it admits a Quillen model structure where,

- the (trivial) cofibrations in $\mathcal{M}$ are the momomorhisms with (trivially) cofibrant cokernel.
- the (trivial) fibrations in $\mathcal{M}$ are the epimorphisms with (trivially) fibrant kenrel.

(Where an object in $\mathcal{M}$ is called trivial if it weakly equivalent to the zero object in $\mathcal{M}$).

If $\mathcal{M}$ is an abelian model category with cofibrant objects $\mathcal{C}$, fibrant objects $\mathcal{F}$ and trivial objects $\mathcal{W}$, we abbreviate by saying that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a **Hovey triple** on $\mathcal{M}$.

Hovey's main result in [50] gives an one-to-one correspondence between certain complete cotorsion pairs in $\mathcal{M}$ and abelian model structures on $\mathcal{M}$.

Theorem 4.2.2. [50, Thm. 2.2] (Hovey's correspondence) *Let $\mathcal{M}$ be an abelian category. Assume that $\mathcal{M}$ admits an abelian model structure and denote by $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$ the classes of cofibrant, fibrant and trivial (i.e. weakly isomorphic to zero) objects in $\mathcal{M}$. Then there exist two functorially complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ in $\mathcal{M}$.*

Conversely, if we are given classes of objects $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$ in $\mathcal{M}$, where $\mathcal{W}$ is thick[1]*, then any two functorially complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ give rise to an abelian model structure on $\mathcal{M}$, where $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ stand for the cofibrant, fibrant and trivial objects respectively.*

Proof. (sketch) We give a few arguments of the proof.
($\Rightarrow$) We show first that $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a cotorsion pair, i.e. that $\mathcal{C}^{\perp} = \mathcal{W} \cap \mathcal{F}$ and that ${}^{\perp}(\mathcal{W} \cap \mathcal{F}) = \mathcal{C}$.

- $\mathcal{C}^{\perp} \subseteq \mathcal{F} \cap \mathcal{W}$. Let $X \in \mathcal{C}^{\perp}$. To prove the assertion, it suffices to show that the map $X \to 0$ has the RLP with respect to the class of cofibrations. Since $\mathcal{M}$ admits an abelian model structure, by Definition

[1] A subcategory $\mathcal{W}$ of an abelian category $\mathcal{A}$ is called thick if for any short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ we have $B \in \mathcal{W} \Leftrightarrow A \in \mathcal{W}$ and $B \in \mathcal{W}$.

4.2.1, any cofibration $i : A \hookrightarrow B$ is a monomorphism with cofibrant $C \in \mathcal{C}$. We consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] & 0 \end{tikzcd}$$

and observe that that the condition $X \in \mathcal{C}^{\perp}$ implies a lifting $\delta : B \to X$ making the relevant diagram commutative.

- $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{C}^{\perp}$. For any $F \in \mathcal{F} \cap \mathcal{W}$, we need to show that $\mathrm{Ext}^1_{\mathcal{M}}(C, F) = 0$ for all cofibrant objects C. For this consider a short exact sequence $F \hookrightarrow Z \twoheadrightarrow C$. Since $\mathcal{M}$ admits an abelian model structure we have that $F \hookrightarrow Z$ is a cofibration. Thus we obtain a commutative diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, hook] & F \arrow[d, two heads, "\sim"] \\ Z \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd}$$

which provides a splitting of $F \hookrightarrow Z$.

- To show that $\mathcal{C} \subseteq {}^{\perp}(\mathcal{F} \cap \mathcal{W})$ note that $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{C}^{\perp} \Rightarrow {}^{\perp}(\mathcal{F} \cap \mathcal{W}) \supseteq {}^{\perp}(\mathcal{C}^{\perp}) \supseteq \mathcal{C}$.

- ${}^{\perp}(\mathcal{F} \cap W) \subseteq \mathcal{C}$. Let $X \in {}^{\perp}(\mathcal{F} \cap W)$. We want a solution of the lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & A \arrow[d, "\pi"] \\ X \arrow[r, "f"'] \arrow[ur, dotted, "?"] & B \end{tikzcd}$$

where π is a trivial fibration, i.e π is surjective and $\ker(\pi) \in \mathcal{F} \cap \mathcal{W}$. Note that the condition $X \in {}^{\perp}(\mathcal{F} \cap W)$ implies that the map $\mathrm{Hom}_{\mathcal{M}}(X, \pi) : \mathrm{Hom}_{\mathcal{M}}(X, A) \xrightarrow{\pi \circ -} \mathrm{Hom}_{\mathcal{M}}(X, B)$ is surjective, hence the solution of the lifting problem.

Hence $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a cotorsion pair in $\mathcal{M}$. Its completeness can be easily seen since, for any $X \in \mathcal{M}$, we may the factor the map $0 \to X$ as a cofibration followed by a trivial fibration, and we may also factor the map $X \to 0$ by a trivial cofibration followed by a fibration and use the fact that $\mathcal{M}$ has an abelian model structure. In a similar way one can deduce that the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a complete cotorsion pair in $\mathcal{M}$.
($\Leftarrow$) Assuming we are given the cotorsion pairs as in the statement, we call a cofibration (resp., trivial cofibration) a monomorphism with cokernel in $\mathcal{C}$ (resp., in $\mathcal{C} \cap \mathcal{W}$) and a fibration (resp., trivial fibration) an epimorphism

with kernel in $\mathcal{F}$ (resp., in $\mathcal{F} \cap \mathcal{W}$), and we call a weak equivalence a morphism which factors as a trivial cofibration followed by a trivial fibration. Then one needs to show that with these notions of (co)fibrations and weak equivalences, $\mathcal{M}$ satisfies the axioms of a model category.

We only show how to obtain the (functorial) weak factorization system $(\texttt{cof}, \texttt{weak} \cap \texttt{fib})$. Let $X \to Y$ be a monomorphism in $\mathcal{M}$, with cokernel $\mathcal{C}$. Since we are given a complete cotorsion pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, there exists a short exact sequence $W \rightarrowtail Q \twoheadrightarrow C$ with $Q \in \mathcal{C}$ and $W \in \mathcal{F} \cap W$. Consider the following pullback diagram:

$$
\begin{tikzcd}
 & W \arrow[r, equal] \arrow[d] & W \arrow[d] \\
X \arrow[r, tail] \arrow[d, equal] & P \arrow[r, two heads] \arrow[d, two heads, "\sim"'] & Q \arrow[d, two heads, "\sim"'] \\
X \arrow[r, tail, "f"] & Y \arrow[r, two heads] & C ,
\end{tikzcd}
$$

which by definition gives a factorization of f as a cofibration followed by a trivial fibration. If we are given an epimorphism f, one can show similarly that f can factor as a trivial cofibration followed by a fibration (using the completeness of the given cotorsion pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. Given an arbitrary morphism $f : X \to Y$ in $\mathcal{M}$, we may write f as $A \xrightarrow{i_1} A \oplus B \xrightarrow{f+1_B} B$, where i_1 is a split mono and $f + 1_B$ is a split epi. Therefore we can factor $f + 1_B = j' \circ q'$ where j' is a cofibration and q' is a trivial fibration. Then $g := j' \circ i_1$ is a monomorphism and we may factor it as $g = \epsilon \circ l$ where l is a cofibration and ϵ is a trivial fibration. Thus we obtain $f = (q' \circ \epsilon) \circ l$ where $q' \circ \epsilon$ is a trivial fibration and l is a cofibration. $\square$

Gillespie in [40] extended Hovey's work to the realm of exact categories. The definition of model categories 4.1.2 asks for certain classes of morphisms to be closed under retracts. To extend Hovey's correspondence from abelian model categories to the realm of exact categories, we need to work with weakly idempotent complete exact categories.

Definition 4.2.3. Let $\mathcal{A}$ be an exact category. We say that $\mathcal{A}$ is **weakly idempotent complete**, if every split monomorphism has a cokenrel and every split epimorphism has a kernel.

An exact category $\mathcal{A}$ is weakly idempotent complete if and only if admissible monomorphisms are closed under retracts if and only if admissible epimorphisms are closed under retracts, see for instance [40, Prop. 2.4].

Definition 4.2.4. Let $\mathcal{M}$ be an exact category. We say that $\mathcal{M}$ admits an **exact model structure** (or that $\mathcal{M}$ is an **exact model category**) if it admits a Quillen model structure where,

- the (trivial) cofibrations in $\mathcal{M}$ are the admissible monos with (trivially) cofibrant cokernel.

- the (trivial) fibrations in $\mathcal{M}$ are the admissible epis with (trivially) fibrant kernel.

Theorem 4.2.5. *[40, Thm. 3.3] Let $\mathcal{M}$ be an exact category. Assume that $\mathcal{M}$ admits an exact model structure and denote by $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$ the classes of cofibrant, fibrant and trivial (i.e. weakly isomorphic to zero) objects in $\mathcal{M}$. Then there exist two functorially complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ in $\mathcal{M}$.*

Conversely, assuming that $\mathcal{M}$ is weakly idempotent complete, if we are given classes of objects $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$, where $\mathcal{W}$ is thick[2], then any two functorially complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ give rise to an exact model structure on $\mathcal{M}$, where $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ stand for the cofibrant, fibrant and trivial objects respectively.

For an exact model category $\mathcal{M}$ the homotopy category $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathtt{weak}^{-1}]$ has an interesting description. We recall from 4.1.4 that $\mathrm{Ho}(\mathcal{M})$ is equivalent to the category $\mathcal{C} \cap \mathcal{F}/\sim$ where $\sim$ is a certain homotopy relation. Gillespie in [40, Prop. 4.4] shows that two parallel maps between objects in $\mathcal{C} \cap \mathcal{F}$ are homotopic if and only if their difference factors through an object in $\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}$. In addition, from [40, Lemma 4.7] we have that the category $\mathcal{C} \cap \mathcal{F}$ is a Frobenius exact category (recall 2.1.8), with projective–injective objects $\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}$. Hence Theorem 4.1.4 tells us that $\mathrm{Ho}(\mathcal{M}) \cong \underline{\mathcal{C} \cap \mathcal{F}}$; where the latter category is the stable category of the Frobenious category $\mathcal{C} \cap \mathcal{F}$ (recall 2.1.11). We summarize this discussion with the following result.

Theorem 4.2.6. [40](Ho($\mathcal{M}$) is an algebraic triangulated category) *Let $\mathcal{M}$ be an exact model structure and denote by $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$ the classes of cofibrant, fibrant and trivial objects respectively. Then the homotopy category* $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathtt{weak}^{-1}]$ *is equivalent to the stable category of the Frobenius category $\mathcal{C} \cap \mathcal{F}$. Via this equivalence the category* $\mathrm{Ho}(\mathcal{M})$ *admits a triangulated structure.*

4.3 Examples

Models for Frobenius rings and Gorenstein rings

Let R be a Frobenius ring (recall Example 2.1.9). There is a standard (trivial) model structure on the category $\mathcal{A} := \text{Mod-}R$, with homotopy category the stable category $\underline{\mathcal{A}}$. Indeed we define as trivial objects the projective–injective objects in $\mathcal{A}$ and we obtain a hereditary Hovey triple $(\mathcal{A}, \mathcal{W}, \mathcal{A})$.

[2] A subcategory $\mathcal{W}$ of an exact category $\mathcal{M}$ is called thick if it is closed under retracts and if, for any short exact sequence (conflation) in $\mathcal{A}$, if two of its terms are in $\mathcal{W}$ then so does the third.

Hence from 4.2.2 we obtain an exact model structure with homotopy category $\mathrm{Ho}(\mathcal{A}) \cong \mathcal{A}/\mathcal{W} \cong \underline{\mathcal{A}}$.

A generalization of this model structure is obtained in paper A [26]. If R is a Gorenstein ring, recall the Auslander and Bass classes, $\mathcal{A}(R) := \{M \mid \mathrm{Gpd}_R M < \infty\}$ and $\mathcal{B}(R) := \{M \mid \mathrm{Gid}_R M < \infty\}$, and also denote $\mathcal{P}(R) := \{M \mid \mathrm{pd}_R M < \infty\}$ and $\mathcal{I}(R) := \{M \mid \mathrm{id}_R M < \infty\}$.

Theorem 4.3.1. ([26, Thm. 3.7/3.9]) *Let R be a Gorenstein ring. The category $\mathcal{A}(R)$ is a weakly idempotent complete exact category and there exists a hereditary Hovey triple $(\mathrm{GProj}(R), \mathcal{P}(R), \mathcal{A}(R))$ on $\mathcal{A}(R)$. Thus from 4.2.5 there exists an exact model structure on $\mathcal{A}(R)$ with homotopy category $\mathrm{Ho}(\mathcal{A}(R)) \cong \underline{\mathrm{GProj}}(R)$.*

Dually, the category $\mathcal{B}(R)$ is a weakly idempotent complete exact category and there exists a hereditary Hovey triple $(\mathcal{B}(R), \mathcal{I}(R), \mathrm{GInj}(R))$ on $\mathcal{B}(R)$. Thus from 4.2.5 there exists an exact model structure on $\mathcal{B}(R)$ with homotopy category $\mathrm{Ho}(\mathcal{B}(R)) \cong \underline{\mathrm{GInj}}(R)$.

Models on categories of chain complexes

The following result of Gillespie gives a recipe in order to obtain model structures on categories of chain complexes in an abelian category. It starts with a complete cotorsion pair on a given "ground" abelian category $\mathcal{M}$ and produces a Hovey triple on the category of chain complexes $\mathrm{Ch}(\mathcal{M})$.

For a class of objects $\mathcal{X}$ in an abelian category $\mathcal{M}$, we will use the following notation:

- $\tilde{X}$ denotes the class of acyclic complexes in $\mathcal{M}$ with cycles in $\mathcal{X}$,
- $\mathrm{dg}\tilde{\mathcal{X}}$ denotes the class of all complexes X in $\mathcal{M}$ such that any chain map from X to a complex in $\tilde{X}$ is null–homotopic.

Theorem 4.3.2. (Gillespie [38, Cor. 3.8] and Yang-Ding [Cor. 2.8])
Let $\mathcal{M}$ be an abelian category with enough projectives and injectives and let $(\mathcal{A}, \mathcal{B})$ be a complete and hereditary cotorsion pair in $\mathcal{M}$. Denote by $\mathcal{W}$ the class of acyclic chain complexes in $\mathcal{M}$. Then there exists a hereditary Hovey triple

$$(\mathrm{dg}\tilde{\mathcal{A}}, \mathcal{W}, \mathrm{dg}\tilde{\mathcal{B}}),$$

in the category of chain complexes $\mathrm{Ch}(\mathcal{M})$.

Example 4.3.3. (Projective and Injective models for the derived category) Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the hereditary complete cotorsion pair $(\mathrm{Proj}\,\mathcal{A}, \mathcal{A})$ induces, after applying Theorem 4.3.2, a hereditary Hovey triple

$$(\mathrm{dg}\widetilde{\mathrm{Proj}\,\mathcal{A}}, \mathrm{Ch}_{\mathrm{ac}}(\mathcal{A}), \mathrm{Ch}(\mathcal{A}))$$

on the category of chain complexes. This gives a (projective) model for the derived category $\mathbf{D}(\mathcal{A})$.

An injective model for $\mathbf{D}(\mathcal{A})$ can be obtained after applying Theorem 4.3.2 to the complete cotorsion pair $(\mathcal{A}, \operatorname{Inj}\mathcal{A})$.

Example 4.3.4. (Flat model for the derived category) Let R be a ring, put $\mathcal{A} := \text{Mod-}R$ and consider the hereditary cotorsion pair $(\operatorname{Flat}(R), \operatorname{Cot}(R))$. Recall that this cotorsion pair is complete by Enochs 3.1.12. Thus Theorem 4.3.2 induces a hereditary Hovey triple

$$(\operatorname{dg}\widetilde{\operatorname{Flat}\mathcal{A}}, \operatorname{Ch}_{\mathrm{ac}}\mathcal{A}, \operatorname{dg}\widetilde{\operatorname{Cot}\mathcal{A}}).$$

Example 4.3.5. (Model for the homotopy category of complexes) Let R be a ring and consider the exact category $(\operatorname{Ch}(R), \mathrm{dw})$, which is the category $\operatorname{Ch}(R)$ equipped with the degreewise split exact structure, i.e. an admissible mono $X \to Y$ (resp., epi) in this exact structure is a degreewise split mono (resp., epi). We are interested in an (exact) model structure with homotopy category $\mathbf{K}(R)$. For this we consider as $\mathcal{W}$ the contractible complexes. Then we observe that there exist in $(\operatorname{Ch}(R), \mathrm{dw})$ two hereditary cotorsion pairs, $(\operatorname{Ch}(R), \mathcal{W})$ and $(\mathcal{W}, \operatorname{Ch}(R))$. Indeed, for any chain complex X and any contractible chain complex Z we have $\operatorname{Ext}^1_{(\operatorname{Ch}(R),\mathrm{dw})}(X, Z) \cong \operatorname{Hom}_{\mathbf{K}(R)}(X, \Sigma Z) = 0$.

It is not hard to see that $\mathcal{W}$ is a thick subcategory which consists of the projective–injective objects in the exact structure $(\operatorname{Ch}(R), \mathrm{dw})$. Completeness of the cotorsion pairs follows since for any chain complex X we may construct degreewise split short exact sequences $X \rightarrowtail \operatorname{Cone}(1_X) \twoheadrightarrow \Sigma X$ and $\Sigma^{-1}X \rightarrowtail \operatorname{Cone}(1_{\Sigma^{-1}X}) \twoheadrightarrow X$. Hence via Theorem 4.2.5 we obtain an exact model structure with Hovey triple $(\operatorname{Ch}(R), \mathcal{W}, \operatorname{Ch}(R))$ and homotopy category $\operatorname{Ho}(\operatorname{Ch}(R)) \cong \operatorname{Ch}(R)/\mathcal{W} \cong \mathbf{K}(R)$.

Models for pure derived categories

Let $(\mathcal{A}, \mathrm{pure})$ be a locally finitely presented Grothendieck category equipped with the pure exact structure (as in 2.2.4), and let $(\operatorname{Ch}(\mathcal{A}), \mathrm{pure})$ denote the "induced" pure exact structure[3] on the category $\operatorname{Ch}(\mathcal{A})$, i.e. a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in this exact structure is degreewise a conflation in $(\mathcal{A}, \mathrm{pure})$.

We are interested in a model structure on the exact category $(\operatorname{Ch}(\mathcal{A}), \mathrm{pure})$ with homotopy category $\mathbf{D}_{\mathrm{pure}}(\mathcal{A})$; the localization of $\mathbf{K}(\mathcal{A})$ with respect to the pure quasi–isomorphisms. For this we set as $\mathcal{W}$ the class of pure acyclic complexes, i.e. the complexes X where $Z_iX \rightarrowtail X \twoheadrightarrow Z_{i+1}X$ is for all $i \in \mathbb{Z}$ a pure exact sequence in $\mathcal{A}$. We need to understand the classes

- ${}^{\perp}\mathcal{W} := \{X \mid \operatorname{Ext}^1_{(\operatorname{Ch}(A),\mathrm{pure})}(X, W) = 0; \text{ for all } W \in \mathcal{W}\}$,

[3]The terminology is according to Neeman [60].

- $\mathcal{W}^\perp := \{X \mid \mathrm{Ext}^1_{(\mathrm{Ch}(A),\mathrm{pure})}(W, X) = 0; \text{ for all } W \in \mathcal{W}\}$.

In order to do this one can make use of the equivalence of exact categories $(\mathcal{A}, \mathrm{pure}) \cong \mathcal{B} := \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ from 2.2.13. So passing to flat functors, the above classes transfer to[4]

- ${}^\perp\widetilde{\mathrm{Flat}\mathcal{B}} := \{X \mid \mathrm{Ext}^1_{\mathrm{Ch}(\mathrm{Flat}\,\mathcal{B})}(X, W) = 0; \text{ for all } W \in \widetilde{\mathrm{Flat}\mathcal{B}}\}$,

- $(\widetilde{\mathrm{Flat}\mathcal{B}})^\perp := \{X \mid \mathrm{Ext}^1_{\mathrm{Ch}(\mathrm{Flat}\,\mathcal{B})}(W, X) = 0; \text{ for all } W \in \widetilde{\mathrm{Flat}\mathcal{B}}\}$.

Neeman [63] shows that ${}^\perp\widetilde{\mathrm{Flat}\mathcal{B}} = \mathrm{Ch}(\mathrm{Proj}\,\mathcal{B})$ while Bazzoni-Estrada-Izurdiaga [11] show that $(\widetilde{\mathrm{Flat}\mathcal{B}})^\perp = \mathrm{Ch}(\mathrm{Flat}\,\mathcal{B} \cap \mathrm{Cot}\,\mathcal{B})$. Here $\mathrm{Cot}\,\mathcal{B} := (\mathrm{Flat}\,\mathcal{B})^\perp$. These identifications are not trivial to prove. Combining these results with some standard results on lifting of model structures to categories of chain complexes (such as 4.3.2 for example), we obtain the following:

Theorem 4.3.6. (Neeman [63] and Bazzoni-Estrada-Izurdiaga [11]) *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then there exists a projective exact model structure on the exact category* $\mathrm{Ch}(\mathrm{Flat}\,\mathcal{A})$*, where the cofibrant objects are the complexes in* $\mathrm{Ch}(\mathrm{Proj}\,\mathcal{A})$*, the trivial objects are the acyclic complexes with flat cycles and all objects are fibrant. The homotopy category of this model structure is* $\mathbf{D}(\mathrm{Flat}\,\mathcal{A}) \cong \mathbf{K}(\mathrm{Proj}\,\mathcal{A})$.

Dually, there exists an injective exact model structure on the exact category $\mathrm{Ch}(\mathrm{Flat}\,\mathcal{A})$*, where all objects are cofibrant, the trivial objects are the acyclic complexes with flat cycles and the fibrant objects are the complexes in* $\mathrm{Ch}(\mathrm{Flat}\,\mathcal{A} \cap \mathrm{Cot}\,\mathcal{A})$*. The homotopy category of this model structure is* $\mathbf{D}(\mathrm{Flat}\,\mathcal{A}) \cong \mathbf{K}(\mathrm{Flat}\,\mathcal{A} \cap \mathrm{Cot}\,\mathcal{A})$.

As a corollary, via the equivalence of exact categories $(\mathcal{A}, \mathrm{pure}) \cong \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ from 2.2.13, we obtain the following models for the pure derived category.

Corollary 4.3.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then there exists a projective exact model structure on the exact category* $(\mathrm{Ch}(\mathcal{A}), \mathrm{pure})$*, where the cofibrant objects are the complexes in* $\mathrm{Ch}(\mathrm{PProj}\,\mathcal{A})$*, the trivial objects are the pure acyclic complexes and all objects are fibrant. The homotopy category of this model structure is* $\mathbf{D}_{\mathrm{pure}}(\mathcal{A})$.

Dually, there exists an injective exact model structure on the exact category $(\mathrm{Ch}(\mathcal{A}), \mathrm{pure})$*, where all objects are cofibrant, the trivial objects are the pure acyclic complexes and the fibrant objects are the complexes in* $\mathrm{Ch}(\mathrm{PInj}\,\mathcal{A})$*. The homotopy category of this model structure is* $\mathbf{D}_{\mathrm{pure}}(\mathcal{A})$.

[4] Recall the notation: $\widetilde{\mathrm{Flat}\mathcal{B}}$ denotes the class of acyclic complexes in $\mathrm{Ch}(\mathcal{B})$ with flat cycles.

In the module case $\mathcal{A} := \text{Mod-}R$, Corollary 4.3.7 also follows after applying the equivalence of exact categories $\text{Mod-}R \cong \text{FPI}(R\text{-mod}, \text{Ab})$ to the following result.

Theorem 4.3.8. (D. [25] and Šťovíček [80]) *Let $\mathcal{A}$ be a locally coherent Grothendieck category. Then there exists a projective exact model structure on the exact category* $\text{Ch}(\text{FPI}\,\mathcal{A})$*, where the cofibrant objects are the complexes in* $\text{Ch}(\text{FPI}\,\mathcal{A} \cap {}^{\perp}\,\text{FPI}\,\mathcal{A})$*, the trivial objects are the pure acyclic complexes of fp-injectives and all objects are fibrant. The homotopy category of this model structure is* $\mathbf{D}(\text{FPI}\,\mathcal{A}) \cong \mathbf{K}(\text{FPI}\,\mathcal{A} \cap {}^{\perp}\,\text{FPI}\,\mathcal{A})$.

Dually, there exists an injective exact model structure on the exact category $\text{Ch}(\text{FPI}\,\mathcal{A})$*, where all objects are cofibrant, the trivial objects are the pure acyclic complexes of fp-injectives and the fibrant objects are the complexes in* $\text{Ch}(\text{Inj}\,\mathcal{A})$*. The homotopy category of this model structure is* $\mathbf{D}(\text{FPI}\,\mathcal{A}) \cong \mathbf{K}(\text{Inj}\,\mathcal{A})$.

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Quillen equivalences for stable categories ☆

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For an abelian category $\mathcal{A}$ we investigate when the stable categories $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$ are triangulated equivalent. To this end, we realize these stable categories as homotopy categories of certain (non-trivial) model categories and give conditions on $\mathcal{A}$ that ensure the existence of a Quillen equivalence between the model categories in question. We also study when such a Quillen equivalence transfers from $\mathcal{A}$ to the category of chain complexes in $\mathcal{A}$.

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1. Introduction

Over an Iwanaga–Gorenstein ring A, that is, a ring which is noetherian and has finite injective dimension from both sides, the category $\mathrm{MCM}(A)$ of (finitely generated) maximal Cohen–Macaulay A-modules[1] is a Frobenius category in which the projective–injective objects are precisely the finitely generated projective A-modules. The associated stable category $\underline{\mathrm{MCM}}(A)$ is therefore triangulated, and a classic result of Buchweitz [8, Thm. 4.4.1] shows that $\underline{\mathrm{MCM}}(A)$ is triangulated equivalent to the singularity category[2] $\mathcal{D}_{\mathrm{sg}}(A)$, which is an important mathematical object that has been studied by many authors; see [5,6,27,33].

If A is not Iwanaga–Gorenstein, then the category $\mathrm{MCM}(A)$ is, in general, not Frobenius. However, over any ring A one can always consider the category $\mathrm{GProj}(A)$ of so-called Gorenstein projective modules (which are not assumed to be finitely generated); this category is always Frobenius and the associated stable category $\underline{\mathrm{GProj}}(A)$ is triangulated. In the case where A is Iwanaga–Gorenstein, an A-module is maximal Cohen–Macaulay if and only if it is finitely generated and Gorenstein projective, and hence $\underline{\mathrm{MCM}}(A)$ can be identified with the finitely generated modules in $\underline{\mathrm{GProj}}(A)$. This explains the interest in the category $\underline{\mathrm{GProj}}(A)$ for general ring A. Its injective counterpart $\underline{\mathrm{GInj}}(A)$, the stable category of Gorenstein injective A-modules, is equally important and has been studied in e.g. [7,26].

Our work is motivated by a recent result of Zheng and Huang [37] which asserts that for many rings A, the categories $\underline{\mathrm{GProj}}(A)$ and $\underline{\mathrm{GInj}}(A)$ are equivalent as triangulated categories. As it makes sense to consider the stable categories $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$ for any bicomplete abelian category $\mathcal{A}$ with enough projectives and injectives (see Section 2 for details), the following question naturally arises:

Question. For which abelian categories $\mathcal{A}$ (assumed to be bicomplete with enough projectives and injectives) are $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$ equivalent as triangulated categories?

Every Frobenius category $\mathcal{E}$, in particular, $\mathrm{GProj}(\mathcal{A})$ and $\mathrm{GInj}(\mathcal{A})$, can be equipped with a canonical model structure which has the property that the associated homotopy category $\mathrm{Ho}(\mathcal{E})$ is equivalent to the stable category $\underline{\mathcal{E}}$; see e.g. [18, Prop. 4.1]. Thus, if the Frobenius categories $\mathrm{GProj}(\mathcal{A})$ and $\mathrm{GInj}(\mathcal{A})$, equipped with these canonical model structures, happen to be Quillen equivalent, then we get an affirmative answer to the

[1] In the important special case where A is a quasi-Frobenius ring, for example, if $A = kG$ is the group algebra of a finite group G with coefficients in a field k, the category $\mathrm{MCM}(A)$ is just the category $\mathrm{mod}(A)$ of all finitely generated A-modules.

[2] The singularity category $\mathcal{D}_{\mathrm{sg}}(A)$ is defined to be the Verdier quotient $\mathcal{D}^{\mathrm{b}}(A)/\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(A)$ of the bounded derived category $\mathcal{D}^{\mathrm{b}}(A)$, whose objects are complexes of A-modules with bounded and finitely generated homology, by the subcategory $\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(A)$, whose objects are isomorphic (in $\mathcal{D}^{\mathrm{b}}(A)$) to a perfect complex, that is, to a bounded complex of finitely generated projective A-modules. The name *singularity category* and the symbol $\mathcal{D}_{\mathrm{sg}}(A)$ seem to be the popular choices nowadays, however, in the work of Buchweitz [8, Def. 1.2.2], this category is called the *stabilized derived category* and denoted by $\underline{\mathcal{D}^{\mathrm{b}}(A)}$, and in the work of Orlov [29], it is called the *triangulated category of singularities* and denoted by $\overline{\mathcal{D}_{\mathrm{sg}}(A)}$.

question above. However, the model categories $\mathrm{GProj}(\mathcal{A})$ and $\mathrm{GInj}(\mathcal{A})$, and even the underlying ordinary categories, will rarely be (Quillen) equivalent. In this paper, we consider instead the categories

$$\mathcal{U}^{\pi} = \{M \in \mathcal{A} \mid \mathrm{Gpd}_{\mathcal{A}}(M) < \infty\} \quad \text{and} \quad \mathcal{U}^{\iota} = \{N \in \mathcal{A} \mid \mathrm{Gid}_{\mathcal{A}}(N) < \infty\}$$

and show in Theorems 3.7 and 3.9 that $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ can be equipped with model structures for which the associated homotopy categories $\mathrm{Ho}(\mathcal{U}^{\pi})$ and $\mathrm{Ho}(\mathcal{U}^{\iota})$ are the stable categories $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$. The advantage of having these realizations of the stable categories is that in several cases the model categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ will be Quillen equivalent—even though $\mathrm{GProj}(\mathcal{A})$ and $\mathrm{GInj}(\mathcal{A})$ are not—and in such cases we therefore get an affirmative answer (for a strong reason) to the question above.[3] To investigate when $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ will be Quillen equivalent, we introduce the notion of a Sharp–Foxby adjunction (Definition 3.4). We prove in Theorem 3.11 and Corollary 3.12 that if $\mathcal{A}$ admits such an adjunction, then $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ will be Quillen equivalent:

Theorem A. *A Sharp–Foxby adjunction (S,T) on $\mathcal{A}$ induces a Quillen equivalence between the model categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$. Thus the total (left/right) derived functors of S and T yield an adjoint equivalence of the corresponding homotopy categories,*

$$\underline{\mathrm{GProj}}(\mathcal{A}) \simeq \mathrm{Ho}(\mathcal{U}^{\pi}) \underset{\mathbf{R}T}{\overset{\mathbf{L}S}{\rightleftarrows}} \mathrm{Ho}(\mathcal{U}^{\iota}) \simeq \underline{\mathrm{GInj}}(\mathcal{A})\,.$$

In fact, this is an equivalence of triangulated categories.

The choice to work with the categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ is historically motivated by classic results in commutative algebra by Sharp [31] and Foxby [14]. In the language of this paper, the results can be phrased as follows: If A is a Cohen–Macaulay ring with a dualizing module D, then the functors $S = D \otimes_A -$ and $T = \mathrm{Hom}_A(D,-)$ constitute a Sharp–Foxby adjunction on $\mathcal{A} = \mathrm{Mod}(A)$; see Example 3.6 for details. Thus, for such rings Theorem A improves the previously mentioned result of Zheng and Huang [37] to a triangulated equivalence between $\underline{\mathrm{GProj}}(A)$ and $\underline{\mathrm{GInj}}(A)$ induced by a Quillen equivalence.

In Section 4 we investigate to what extend a Sharp–Foxby adjunction on a category $\mathcal{A}$ (and hence also a Quillen equivalence between the model categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$, see Theorem A) transfers to the category of chain complexes in $\mathcal{A}$. In 4.5 we obtain the following.

[3] In general, we do not expect every (triangulated) equivalence between $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$, if such an equivalence even exists, to be induced from a Quillen equivalence between model categories. Indeed, it is well-known that there are examples of non Quillen equivalent model categories with equivalent homotopy categories.

Theorem B. *Assume that* (S,T) *is a Sharp–Foxby adjunction on* $\mathcal{A}$*; in particular,* $\underline{\mathrm{GProj}}(\mathcal{A})$ *and* $\underline{\mathrm{GInj}}(\mathcal{A})$ *are equivalent as triangulated categories by Theorem A. Assume furthermore that the finitistic projective and the finitistic injective dimensions of* $\mathcal{A}$ *are finite.*

If $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$*, then degreewise application of* S *and* T *yields a Sharp–Foxby adjunction on* $\mathcal{B}$*; in particular,* $\underline{\mathrm{GProj}}(\mathcal{B})$ *and* $\underline{\mathrm{GInj}}(\mathcal{B})$ *are equivalent as triangulated categories.*

2. Preliminaries

Throughout this paper, $\mathcal{A}$ denotes any bicomplete abelian category with enough projectives and enough injectives.

Gorenstein projective and Gorenstein injective modules (over any ring) were defined by Enochs and Jenda [10, §2], but the definition works for objects in any abelian category:

Definition 2.1. An acyclic (= exact) complex $P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ of projective objects in $\mathcal{A}$ is called *totally acyclic* if for any projective object Q in $\mathcal{A}$ the complex

$$\mathrm{Hom}_{\mathcal{A}}(P,Q) = \ \cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_{-1},Q) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_0,Q) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_1,Q) \longrightarrow \cdots$$

is acyclic. An object G in $\mathcal{A}$ is called *Gorenstein projective* if it is a cycle of such a totally acyclic complex of projectives, that is, if $G = \mathrm{Z}_j(P)$ for some integer j. We write $\mathrm{GProj}(\mathcal{A})$ for the full subcategory of $\mathcal{A}$ consisting of all Gorenstein projective objects.

Dually, an acyclic complex $I = \cdots \to I_1 \to I_0 \to I_{-1} \to \cdots$ of injective objects in $\mathcal{A}$ is called *totally acyclic* if for any injective object E in $\mathcal{A}$ the complex

$$\mathrm{Hom}_{\mathcal{A}}(E,I) = \ \cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E,I_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E,I_0) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E,I_{-1}) \longrightarrow \cdots$$

is acyclic. An object H in $\mathcal{A}$ is called *Gorenstein projective* if it is a cycle of such a totally acyclic complex of injectives, that is, if $H = \mathrm{Z}_j(I)$ for some integer j. We write $\mathrm{GInj}(\mathcal{A})$ for the full subcategory of $\mathcal{A}$ consisting of all Gorenstein injective objects.

The *Gorenstein projective dimension*, $\mathrm{Gpd}_{\mathcal{A}}(M)$, of an object M in $\mathcal{A}$ is defined by declaring that one has $\mathrm{Gpd}_{\mathcal{A}}(M) \leqslant n$ (for $n \in \mathbb{N}_0$) if and only if there exists an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ in $\mathcal{A}$ with $G_0, \ldots, G_n \in \mathrm{GProj}(\mathcal{A})$. The *Gorenstein injective dimension*, $\mathrm{Gid}_{\mathcal{A}}(M)$, of M is defined analogously.

Recall that a *Frobenius category* is an exact category $\mathcal{E}$ with enough (relative) projectives and enough (relative) injectives and where the classes of projectives and injectives coincide; such objects are called *projective–injective* (or just *pro-injective*) objects. The *stable category* $\underline{\mathcal{E}}$ is the quotient category $\mathcal{E}/{\sim}$ where the relation "$\sim$" is defined by

$f \sim g$ (here f and g are parallel morphisms in $\mathcal{E}$) if $f - g$ factors through a projective–injective object. The category $\underline{\mathcal{E}}$ is triangulated as described in Happel [21, Chap. I§2] (see also 2.5).

The following result is well-known, but for completeness we include a short proof.

Proposition 2.2. *The category* $\mathrm{GProj}(\mathcal{A})$ *is Frobenius and the projective–injective objects herein are the projective objects in* $\mathcal{A}$*. Thus, the stable category* $\underline{\mathrm{GProj}}(\mathcal{A})$ *is triangulated.*

The category $\mathrm{GInj}(\mathcal{A})$ *is Frobenius and the projective–injective objects herein are the injective objects in* $\mathcal{A}$*. Thus, the stable category* $\underline{\mathrm{GInj}}(\mathcal{A})$ *is triangulated.*

Proof. We only show the claims about the category $\mathrm{GProj}(\mathcal{A})$, as the claims about $\mathrm{GInj}(\mathcal{A})$ are proved similarly. The proof only uses basic properties of Gorenstein projective objects. In the case of modules, that is, if $\mathcal{A} = \mathrm{Mod}(A)$ for a ring A, these properties are recorded in [23], however, the reader easily verifies that the same properties hold for Gorenstein projective objects in any abelian category $\mathcal{A}$ with enough projectives.

First of all, by [23, Thm. 2.5] the class $\mathrm{GProj}(\mathcal{A})$ is an additive extension-closed subcategory of the abelian category $\mathcal{A}$, and thus $\mathrm{GProj}(\mathcal{A})$ is an exact category. Clearly, every (categorical) projective object P in $\mathcal{A}$ is a (relative) projective object in $\mathrm{GProj}(\mathcal{A})$, but it is also (relative) injective since every short exact sequence $0 \to P \to G \to G' \to 0$ in $\mathcal{A}$ with $G, G' \in \mathrm{GProj}(\mathcal{A})$ splits; indeed by [23, Prop. 2.3] one has $\mathrm{Ext}^1_{\mathcal{A}}(G', P) = 0$. By the definition of Gorenstein projective objects, every $G \in \mathrm{GProj}(\mathcal{A})$ fits into short exact sequences $0 \to H \to P \to G \to 0$ and $0 \to G \to P' \to H' \to 0$ in $\mathcal{A}$ where P, P' are (categorical) projective and H, H' are Gorenstein projective. It follows that if G is (relative) projective or (relative) injective, then G is a direct summand of a (categorical) projective object, P or P', and hence G is (categorical) projective. It also follows that $\mathrm{GProj}(\mathcal{A})$ has enough (relative) projectives and enough (relative) injectives. $\square$

In Theorems 3.7 and 3.9 we construct certain model categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ for which the associated homotopy categories $\mathrm{Ho}(\mathcal{U}^{\pi})$ and $\mathrm{Ho}(\mathcal{U}^{\iota})$ are $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$.

The standard references for the theory of cotorsion pairs are Enochs and Jenda [11] and Göbel and Trlifaj [20]. Below we recall a few notions that we need.

2.3. A pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects in $\mathcal{A}$ is a *cotorsion pair* if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Here, given a class $\mathcal{C}$ of objects in $\mathcal{A}$, the right orthogonal $\mathcal{C}^{\perp}$ is defined to be the class of all $Y \in \mathcal{A}$ such that $\mathrm{Ext}^1_{\mathcal{A}}(C, Y) = 0$ for all $C \in \mathcal{C}$. The left orthogonal ${}^{\perp}\mathcal{C}$ is defined similarly. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *hereditary* if $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and $i \geqslant 1$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *complete* if it has *enough projectives* and *enough injectives*, i.e. for each $A \in \mathcal{A}$ there exist short exact sequences $0 \to Y \to X \to A \to 0$ (enough projectives) and $0 \to A \to Y' \to X' \to 0$ (enough injectives) with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

In order for the above to make sense, the category $\mathcal{A}$ only needs to be exact (not necessarily abelian), so that one has a notion of “short exact sequences” (often called *conflations*) and hence also of (Yoneda) $\mathrm{Ext}_{\mathcal{A}}$.

Cotorsion pairs are related to relative homological algebra, see [11], and due to work of Hovey [25] they are also related to abelian (or exact) model category structures.

2.4. An *abelian model structure* on $\mathcal{A}$, that is, a model structure on $\mathcal{A}$ which is compatible with the abelian structure in the sense of [25, Def. 2.1], corresponds by Thm. 2.2 in [25] to a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of objects in $\mathcal{A}$ for which $\mathcal{W}$ is thick[4] and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$. Such a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is called a *Hovey triple* in $\mathcal{A}$. In the model structure on $\mathcal{A}$ determined by such a Hovey triple, $\mathcal{C}$ is precisely the class of cofibrant objects, $\mathcal{F}$ is precisely the class of fibrant objects, and $\mathcal{W}$ is precisely the class of trivial objects (that is, objects weakly equivalent to zero). A *hereditary Hovey triple* is a Hovey triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ for which the associated complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are both hereditary (as defined in 2.3).

Gillespie extends in [17, Thm. 3.3] Hovey's correspondence, mentioned above, from the realm of abelian categories to the realm of weakly idempotent complete exact categories. More precisely, if $\mathcal{A}$ is just an exact category (not necessarily abelian), then an *exact model structure* on $\mathcal{A}$ is a model structure on $\mathcal{A}$ which is compatible with the exact structure in the sense of [17, Def. 3.1]. If, in addition, $\mathcal{A}$ is weakly idempotent complete ([17, Def. 2.2]), then exact model structures on $\mathcal{A}$ correspond precisely to Hovey triples $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in $\mathcal{A}$.

Recall from [24, Cor. 1.2.7 and Thm. 1.2.10(i)] that if $\mathcal{C}$ is any model category, then the inclusion $\mathcal{C}_{\mathrm{cf}} \to \mathcal{C}$ induces an equivalence $\mathcal{C}_{\mathrm{cf}}/\!\sim\; \to \mathrm{Ho}(\mathcal{C})$. Here $\mathcal{C}_{\mathrm{cf}}$ is the full subcategory of $\mathcal{C}$ whose objects are both cofibrant and fibrant, "$\sim$" is the (abstract) homotopy relation from [24, Def. 1.2.4], and $\mathrm{Ho}(\mathcal{C})$ is the homotopy category of the model category $\mathcal{C}$ (that is, the localization of $\mathcal{C}$ with respect to the collection of weak equivalences).

2.5. Let $\mathcal{A}$ be a weakly idempotent complete exact category equipped with an exact model structure coming from a *hereditary* Hovery triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in $\mathcal{A}$. As explained in 2.4, one has $\mathcal{A}_{\mathrm{cf}} = \mathcal{C} \cap \mathcal{F}$, which by [17, Prop. 5.2(4)] / [32, Thm. 6.21(1)] is a Frobenius category with $\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ as the class of projective–injective objects. By [17, Prop. 4.4(5)] / [32, Lem. 6.16(3)] two parallel morphisms in $\mathcal{A}_{\mathrm{cf}} = \mathcal{C} \cap \mathcal{F}$ are homotopic, in the (abstract) model categorical sense, if and only their difference factors through an object in $\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$. Thus, $\mathcal{A}_{\mathrm{cf}}/\!\sim$ is nothing but the stable category $\underline{\mathcal{A}_{\mathrm{cf}}}$ of the Frobenius category $\mathcal{A}_{\mathrm{cf}}$ (see the remarks preceding Proposition 2.2), so the category $\mathcal{A}_{\mathrm{cf}}/\!\sim$ carries a natural triangulated structure. As mentioned above, one has an equivalence of categories $\mathrm{Ho}(\mathcal{A}) \simeq \mathcal{A}_{\mathrm{cf}}/\!\sim$, and via this equivalence the homotopy category $\mathrm{Ho}(\mathcal{A})$ inherits a triangulated structure from $\mathcal{A}_{\mathrm{cf}}/\!\sim$. More precisely, the distinguished triangles in $\mathrm{Ho}(\mathcal{A})$ are, up to isomorphism, the images in $\mathrm{Ho}(\mathcal{A})$ of distinguished triangles in $\underline{\mathcal{A}_{\mathrm{cf}}} = \mathcal{A}_{\mathrm{cf}}/\!\sim$ under the equivalence $\mathcal{A}_{\mathrm{cf}}/\!\sim\; \to \mathrm{Ho}(\mathcal{A})$. It is evident that when $\mathrm{Ho}(\mathcal{A})$

[4] Recall that a class $\mathcal{W}$ in an abelian (or, more generally, in an exact) category $\mathcal{A}$ is *thick* if it is closed under retracts and satisfies that whenever two out of three terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

is equipped with this triangulated structure, then the equivalence $\mathrm{Ho}(\mathcal{A}) \simeq \mathcal{A}_{\mathrm{cf}}/\!\sim$ (of ordinary categories *a priori*) becomes an equivalence of triangulated categories, that is, the functors $\mathrm{Ho}(\mathcal{A}) \leftrightarrows \mathcal{A}_{\mathrm{cf}}/\!\sim$ are triangulated.

3. Sharp–Foxby adjunctions

Recall from the beginning of Section 2 that $\mathcal{A}$ always denotes any bicomplete abelian category with enough projectives and enough injectives. In this section, we give conditions on $\mathcal{A}$ which ensure that $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\underline{\mathrm{GInj}}(\mathcal{A})$ are equivalent as triangulated categories.

Definition 3.1. Let $\mathcal{U}^{\pi}$ be the full subcategory of $\mathcal{A}$ whose objects are given by

$$\mathcal{U}^{\pi} = \{M \in \mathcal{A} \mid \mathrm{Gpd}_{\mathcal{A}}(M) < \infty\}\ .$$

Let and $\mathcal{C}^{\pi}$, $\mathcal{W}^{\pi}$, and $\mathcal{F}^{\pi}$ be the following subclasses of $\mathcal{U}^{\pi}$:

$$\mathcal{C}^{\pi} = \mathrm{GProj}(\mathcal{A})\,, \quad \mathcal{W}^{\pi} = \{M \in \mathcal{A} \mid \mathrm{pd}_{\mathcal{A}}(M) < \infty\}\,, \quad \text{and} \quad \mathcal{F}^{\pi} = \mathcal{U}^{\pi}\,.$$

The classes $\mathcal{U}^{\pi}$, $\mathcal{C}^{\pi}$, $\mathcal{W}^{\pi}$, and $\mathcal{F}^{\pi}$ depend on $\mathcal{A}$, and if necessary we use the more detailed notation $\mathcal{U}^{\pi}_{\mathcal{A}}$, $\mathcal{C}^{\pi}_{\mathcal{A}}$, $\mathcal{W}^{\pi}_{\mathcal{A}}$, and $\mathcal{F}^{\pi}_{\mathcal{A}}$ instead. (The superscript "π" is supposed to give the reader associations to the word "projective".)

Definition 3.2. Let $\mathcal{U}^{\iota}$ be the full subcategory of $\mathcal{A}$ whose objects are given by

$$\mathcal{U}^{\iota} = \{N \in \mathcal{A} \mid \mathrm{Gid}_{\mathcal{A}}(N) < \infty\}\ .$$

Let and $\mathcal{C}^{\iota}$, $\mathcal{W}^{\iota}$, and $\mathcal{F}^{\iota}$ be the following subclasses of $\mathcal{U}^{\iota}$:

$$\mathcal{C}^{\iota} = \mathcal{U}^{\iota}\,, \quad \mathcal{W}^{\iota} = \{N \in \mathcal{A} \mid \mathrm{id}_{\mathcal{A}}(N) < \infty\}\,, \quad \text{and} \quad \mathcal{F}^{\iota} = \mathrm{GInj}(\mathcal{A})\,.$$

The classes $\mathcal{U}^{\iota}$, $\mathcal{C}^{\iota}$, $\mathcal{W}^{\iota}$, and $\mathcal{F}^{\iota}$ depend on $\mathcal{A}$, and if necessary we use the more detailed notation $\mathcal{U}^{\iota}_{\mathcal{A}}$, $\mathcal{C}^{\iota}_{\mathcal{A}}$, $\mathcal{W}^{\iota}_{\mathcal{A}}$, and $\mathcal{F}^{\iota}_{\mathcal{A}}$ instead. (The superscript "ι" is supposed to give the reader associations to the word "injective".)

Lemma 3.3. *The categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ are additive and extension-closed subcategories of the abelian category $\mathcal{A}$; hence they are exact categories. Furthermore, $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ are closed under direct summands in $\mathcal{A}$; hence they are idempotent complete.*

Proof. In the case where $\mathcal{A} = \mathrm{Mod}(A)$ for a ring A, the assertions follow from [23, Prop. 2.19 and Thm. 2.24] (and the dual statements about Gorenstein injective modules). By inspection, one verifies that the same proofs work in any bicomplete abelian category $\mathcal{A}$ with enough projectives and enough injectives. $\square$

We show in Theorems 3.7 and 3.9 that $(\mathcal{C}^\pi, \mathcal{W}^\pi, \mathcal{F}^\pi)$ and $(\mathcal{C}^\iota, \mathcal{W}^\iota, \mathcal{F}^\iota)$ are Hovey triples (see 2.4) in the idempotent complete exact categories $\mathcal{U}^\pi$ and $\mathcal{U}^\iota$.

Definition 3.4. A *Sharp–Foxby adjunction* on $\mathcal{A}$ is an adjunction (S, T) of endofunctors on $\mathcal{A}$ for which the following properties hold:

(SF1) S maps $\mathcal{U}^\pi$ to $\mathcal{U}^\iota$ and it maps $\mathcal{W}^\pi$ to $\mathcal{W}^\iota$.
(SF2) The restriction of S to $\mathcal{U}^\pi$ is exact: if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{A}$ with $X', X, X'' \in \mathcal{U}^\pi$, then the sequence $0 \to SX' \to SX \to SX'' \to 0$ is exact.
(SF3) T maps $\mathcal{U}^\iota$ to $\mathcal{U}^\pi$ and it maps $\mathcal{W}^\iota$ to $\mathcal{W}^\pi$.
(SF4) The restriction of T to $\mathcal{U}^\iota$ is exact: if $0 \to Y' \to Y \to Y'' \to 0$ is an exact sequence in $\mathcal{A}$ with $Y', Y, Y'' \in \mathcal{U}^\iota$, then the sequence $0 \to TY' \to TY \to TY'' \to 0$ is exact.
(SF5) The unit of adjunction $\eta_X \colon X \to TSX$ is an isomorphism for every $X \in \mathcal{U}^\pi$.
(SF6) The counit of adjunction $\varepsilon_Y \colon STY \to Y$ is an isomorphism for every $Y \in \mathcal{U}^\iota$.

Remark 3.5. By (SF1), (SF3), (SF5), and (SF6) a Sharp–Foxby adjunction $S \colon \mathcal{A} \rightleftarrows \mathcal{A} : T$ restricts to adjoint equivalences of categories $\mathcal{U}^\pi \rightleftarrows \mathcal{U}^\iota$ and $\mathcal{W}^\pi \rightleftarrows \mathcal{W}^\iota$. By Lemma 3.3 the categories $\mathcal{U}^\pi$ and $\mathcal{U}^\iota$ have natural exact structures. Conditions (SF2) and (SF4) imply that the induced adjoint equivalence $\mathcal{U}^\pi \rightleftarrows \mathcal{U}^\iota$ preserves the exact structure, i.e. the functors are exact; thus it is an adjoint equivalence of exact categories.[5]

The following example explains the terminology in Definition 3.4.

Example 3.6. Let A be a commutative noetherian local Cohen–Macaulay ring with a dualizing module D. Foxby considered in [14, §1] two classes $\mathbf{A}(A)$ and $\mathbf{B}(A)$ of A-modules[6]:

A module M is in $\mathbf{A}(A)$ if and only if $\operatorname{Tor}_i^A(D, M) = 0$ and $\operatorname{Ext}_A^i(D, D \otimes_A M) = 0$ for all $i > 0$ and the natural homomorphism $\eta_M \colon M \to \operatorname{Hom}_A(D, D \otimes_A M)$ is an isomorphism.

A module N is in $\mathbf{B}(A)$ if and only if $\operatorname{Ext}_A^i(D, N) = 0$ and $\operatorname{Tor}_i^A(D, \operatorname{Hom}_A(D, N)) = 0$ for all $i > 0$ and the natural homomorphism $\varepsilon_N \colon D \otimes_A \operatorname{Hom}_A(D, N) \to N$ is an isomorphism.

Foxby [14] proved that the adjunction $(D \otimes_A -, \operatorname{Hom}_A(D, -))$ on $\operatorname{Mod}(A)$ restricts to an adjoint equivalence $\mathbf{A}(A) \rightleftarrows \mathbf{B}(A)$ and further to an adjoint equivalence

[5] If $\mathcal{E}$ and $\mathcal{E}'$ are exact categories and $F \colon \mathcal{E} \rightleftarrows \mathcal{E}' : G$ is an adjoint equivalence of the underlying (ordinary) categories, then it does not automatically follow that the functors F and G are exact. Indeed, if $\mathcal{E}$ and $\mathcal{E}'$ have the same underlying category and the exact structure on $\mathcal{E}$ is coarser than that on $\mathcal{E}'$ (that is, every sequence which is exact in $\mathcal{E}$ is also exact in $\mathcal{E}'$ — for example, $\mathcal{E}$ could have the trivial exact structure, in which the only "exact" sequences are the split exact ones, whereas $\mathcal{E}'$ could have any exact structure), then the identity functors $\mathcal{E} \rightleftarrows \mathcal{E}'$ constitute an adjoint equivalence of the underlying categories where only $\mathcal{E} \to \mathcal{E}'$ is exact (but $\mathcal{E} \leftarrow \mathcal{E}'$ is not).

[6] In the literature, the classes $\mathbf{A}(A)$ and $\mathbf{B}(A)$ are referred to as *Foxby classes*. Sometimes, $\mathbf{A}(A)$ is called the *Auslander class* and $\mathbf{B}(A)$ is called the *Bass class*. Foxby himself [14] used the symbols Φ_D and Ψ_D for these classes, but in the paper [12] by Enochs, Jenda, and Xu they are denoted by $\mathcal{G}_0$ and $\mathcal{J}_0$. We have adopted the symbols $\mathbf{A}(A)$ and $\mathbf{B}(A)$ from the joint work of Avramov and Foxby; see for example [1, §3].

$\mathcal{W}^{\pi}_{\mathrm{Mod}(A)} \rightleftarrows \mathcal{W}^{\iota}_{\mathrm{Mod}(A)}$ (see Definitions 3.1 and 3.2). The latter is an extension of a result [31, Thm. (2.9)] by Sharp, which asserts that $D \otimes_A -$ and $\mathrm{Hom}_A(D,-)$ restrict to an adjoint equivalence between the categories of *finitely generated* A-modules with finite projective dimension and *finitely generated* A-modules with finite injective dimension. Note that it is evident from the definitions that the restriction of $D \otimes_A -$ to $\mathbf{A}(A)$ and of $\mathrm{Hom}_A(D,-)$ to $\mathbf{B}(A)$ are exact functors.

By Enochs, Jenda, and Xu [12, Cor. 2.4 and 2.6] an A-module belongs to $\mathbf{A}(A)$, respectively, $\mathbf{B}(A)$, if and only if it has finite Gorenstein projective dimension, respectively, finite Gorenstein injective dimension. Thus, in the notation from 3.1 and 3.2 we have:

$$\mathbf{A}(A) = \mathcal{U}^{\pi}_{\mathrm{Mod}(A)} \qquad \text{and} \qquad \mathbf{B}(A) = \mathcal{U}^{\iota}_{\mathrm{Mod}(A)}.$$

Consequently, $(S,T) = (D \otimes_A -, \mathrm{Hom}_A(D,-))$ is a Sharp–Foxby adjunction on $\mathrm{Mod}(A)$. In view of [9, Thms. 4.1 and 4.4] this remains to be true if A is any two-sided noetherian ring with a dualizing module D, that is, a dualizing complex concentrated in degree zero.

Theorem 3.7. *Consider the idempotent complete exact category $\mathcal{U}^{\pi}$ from Lemma 3.3. The triple $(\mathcal{C}^{\pi}, \mathcal{W}^{\pi}, \mathcal{F}^{\pi})$ from Definition 3.1 is a hereditary Hovey triple in $\mathcal{U}^{\pi}$ (see 2.4). In particular, $\mathcal{U}^{\pi}$ has an exact model structure for which:*

- *The cofibrant objects in $\mathcal{U}^{\pi}$ are the Gorenstein projective objects in $\mathcal{A}$.*
- *The trivial objects in $\mathcal{U}^{\pi}$ are the objects in $\mathcal{A}$ with finite projective dimension.*
- *All objects in $\mathcal{U}^{\pi}$ are fibrant.*

The homotopy category of this model category is equivalent, as a triangulated category, to the stable category of Gorenstein projective objects in $\mathcal{A}$; in symbols:

$$\mathrm{Ho}(\mathcal{U}^{\pi}) \simeq \underline{\mathrm{GProj}}(\mathcal{A})\,.$$

Remark 3.8. A number of fundamental properties of Gorenstein projective modules, i.e. Gorenstein projective objects in the category $\mathcal{A} = \mathrm{Mod}(A)$ where A is a ring, are recorded in e.g. [9,23]. The results we need about Gorenstein projective objects in a general abelian category (still bicomplete with enough projectives and enough injectives) can be proved as it is done for modules. We leave it to the reader to inspect the relevant proofs.

Proof of Theorem 3.7. It is well-known that $\mathcal{W}^{\pi}$ is a thick subcategory of $\mathcal{A}$ (and hence also of $\mathcal{U}^{\pi}$). By [23, Prop. 2.27] the intersection $\mathcal{C}^{\pi} \cap \mathcal{W}^{\pi}$ equals the class $\mathrm{Proj}\,\mathcal{A}$ of projective objects in $\mathcal{A}$. Thus the pair $(\mathcal{C}^{\pi} \cap \mathcal{W}^{\pi}, \mathcal{F}^{\pi})$ is equal to $(\mathrm{Proj}\,\mathcal{A}, \mathcal{U}^{\pi})$, which we now argue is a complete hereditary cotorsion pair in $\mathcal{U}^{\pi}$. As $\mathrm{Ext}^{\geqslant 1}_{\mathcal{A}}(P,A) = 0$ for all $P \in \mathrm{Proj}\,\mathcal{A}$ and all $A \in \mathcal{U}^{\pi}$ (even all $A \in \mathcal{A}$), we get that $(\mathrm{Proj}\,\mathcal{A})^{\perp} = \mathcal{U}^{\pi}$ (as the "$\perp$" is only calculated inside of $\mathcal{U}^{\pi}$) and that $\mathrm{Proj}\,\mathcal{A} \subseteq {}^{\perp}\mathcal{U}^{\pi}$. To show that $\mathrm{Proj}\,\mathcal{A} \supseteq {}^{\perp}\mathcal{U}^{\pi}$ let

$M \in {}^{\perp}\mathcal{U}^{\pi}$ $(\subseteq \mathcal{U}^{\pi})$. By assumption, $\mathcal{A}$ has enough projectives, and hence there exists a short exact sequence in $\mathcal{A}$,

$$0 \longrightarrow A \longrightarrow P \longrightarrow M \longrightarrow 0\,, \tag{$\sharp 1$}$$

where P is projective. As M belongs to $\mathcal{U}^{\pi}$, so does A by [23, Thm. 2.24]. By assumption, $\operatorname{Ext}^1_{\mathcal{A}}(M,A)=0$, so ($\sharp$1) splits and hence $M \in \operatorname{Proj}\mathcal{A}$. This shows that $(\operatorname{Proj}\mathcal{A}, \mathcal{U}^{\pi})$ is a hereditary cotorsion pair. For completeness of this cotorsion pair, the sequence ($\sharp$1) shows that the pair has enough projectives. The trivial exact sequence $0 \to M \to M \to 0 \to 0$ (for any M in $\mathcal{U}^{\pi}$) shows that the pair has enough injectives.

Next we show that $(\mathcal{C}^{\pi}, \mathcal{W}^{\pi} \cap \mathcal{F}^{\pi}) = (\operatorname{GProj}\mathcal{A}, \mathcal{W}^{\pi})$ is a complete hereditary cotorsion pair in $\mathcal{U}^{\pi}$. By [23, Thm. 2.20] we have $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(G,A) = 0$ for all $G \in \operatorname{GProj}\mathcal{A}$ and $A \in \mathcal{W}^{\pi}$, and hence we get $\operatorname{GProj}\mathcal{A} \subseteq {}^{\perp}\mathcal{W}^{\pi}$ and $(\operatorname{GProj}\mathcal{A})^{\perp} \supseteq \mathcal{W}^{\pi}$. To show that $\operatorname{GProj}\mathcal{A} \supseteq {}^{\perp}\mathcal{W}^{\pi}$, let $M \in {}^{\perp}\mathcal{W}^{\pi}$ $(\subseteq \mathcal{U}^{\pi})$. By [23, Thm 2.10] there exists a short exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow M \longrightarrow 0 \tag{$\sharp 2$}$$

with $G \in \operatorname{GProj}\mathcal{A}$ and $A \in \mathcal{W}^{\pi}$. By assumption, $\operatorname{Ext}^1_{\mathcal{A}}(M,A) = 0$, so ($\sharp$2) splits and hence M is a direct summand in G. By [23, Thm 2.5] (see also Prop. 1.4 in [23]) the class $\operatorname{GProj}\mathcal{A}$ is closed under direct summands (here we use our assumption that $\mathcal{A}$ is cocomplete, or at least that $\mathcal{A}$ has countable coproducts), and it follows that M itself belongs to $\operatorname{GProj}\mathcal{A}$. To show $(\operatorname{GProj}\mathcal{A})^{\perp} \subseteq \mathcal{W}^{\pi}$, assume that $M \in (\operatorname{GProj}\mathcal{A})^{\perp}$ $(\subseteq \mathcal{U}^{\pi})$. By [9, Lem. 2.17] there is a short exact sequence

$$0 \longrightarrow M \longrightarrow A' \longrightarrow G' \longrightarrow 0 \tag{$\sharp 3$}$$

where $G' \in \operatorname{GProj}\mathcal{A}$ and $\operatorname{pd}_{\mathcal{A}}(A') = \operatorname{Gpd}_{\mathcal{A}}(M) < \infty$, that is, A' is in $\mathcal{W}^{\pi}$. By assumption, $\operatorname{Ext}^1_{\mathcal{A}}(G',M) = 0$, so ($\sharp$3) splits and hence M also belongs to $\mathcal{W}^{\pi}$ (which is thick). Thus $(\operatorname{GProj}\mathcal{A}, \mathcal{W}^{\pi})$ is a hereditary cotorsion pair in $\mathcal{U}^{\pi}$, and the existence of the sequences ($\sharp$2) and ($\sharp$3) shows that this cotorsion pair is complete.

These arguments prove that $(\mathcal{C}^{\pi}, \mathcal{W}^{\pi}, \mathcal{F}^{\pi})$ is a hereditary Hovey triple in $\mathcal{U}^{\pi}$. In view of the equalities $\mathcal{C}^{\pi} \cap \mathcal{F}^{\pi} = \operatorname{GProj}\mathcal{A}$ and $\mathcal{C}^{\pi} \cap \mathcal{W}^{\pi} \cap \mathcal{F}^{\pi} = \operatorname{Proj}\mathcal{A}$, where the latter is by [23, Prop 2.27], the rest of the theorem now follows from 2.4 and 2.5 (and Proposition 2.2). $\square$

Theorem 3.9. *Consider the idempotent complete exact category* $\mathcal{U}^{\iota}$ *from Lemma 3.3. The triple* $(\mathcal{C}^{\iota}, \mathcal{W}^{\iota}, \mathcal{F}^{\iota})$ *from Definition 3.2 is a hereditary Hovey triple in* $\mathcal{U}^{\iota}$ *(see 2.4). In particular,* $\mathcal{U}^{\iota}$ *has an exact model structure for which:*

- *All objects in* $\mathcal{U}^{\iota}$ *are cofibrant.*
- *The trivial objects in* $\mathcal{U}^{\iota}$ *are the objects in* $\mathcal{A}$ *with finite injective dimension.*
- *The fibrant objects in* $\mathcal{U}^{\iota}$ *are the Gorenstein injective objects in* $\mathcal{A}$.

The homotopy category of this model category is equivalent, as a triangulated category, to the stable category of Gorenstein injective objects in $\mathcal{A}$; in symbols:

$$\mathrm{Ho}(\mathcal{U}^{\iota}) \simeq \underline{\mathrm{GInj}}(\mathcal{A}) .$$

Proof. Dual to the proof of Theorem 3.1. □

Our next goal is to show that a Sharp–Foxby adjunction on $\mathcal{A}$ induces a Quillen equivalence between the model categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$. To this end, the next result will be useful.

Proposition 3.10. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two weakly idempotent complete exact model categories with associated Hovey triples $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}', \mathcal{W}', \mathcal{F}')$; see 2.4. Assume that (F, G) is a Quillen adjunction $\mathcal{M} \rightleftarrows \mathcal{M}'$ where the functors F and G are exact and satisfy $F(\mathcal{W}) \subseteq \mathcal{W}'$ and $G(\mathcal{W}') \subseteq \mathcal{W}$. Then (F, G) is a Quillen equivalence if and only if the unit $\eta_X : X \to GFX$ is a weak equivalence for every $X \in \mathcal{C}$ and the counit $\varepsilon_Y : FGY \to Y$ is a weak equivalence for every $Y \in \mathcal{F}'$.*

Proof. Write Q for the cofibrant replacement functor in $\mathcal{M}$ and $q_X : QX \to X$ for the natural trivial fibration ($X \in \mathcal{M}$). Similarly, write R for the fibrant replacement functor in $\mathcal{M}'$ and $r_Y : Y \to RY$ for the natural trivial cofibration ($Y \in \mathcal{M}'$). By [24, Prop. 1.3.13] we have that (F, G) is a Quillen equivalence if and only if the composite

$$X \xrightarrow{\eta_X} GFX \xrightarrow{Gr_{FX}} GRFX$$

is a weak equivalence for all $X \in \mathcal{C}$ and the composite

$$FQGY \xrightarrow{Fq_{GY}} FGY \xrightarrow{\varepsilon_Y} Y$$

is a weak equivalence for all $Y \in \mathcal{F}'$. We claim that the morphisms Gr_{FX} and Fq_{GY} are always weak equivalences for every $X \in \mathcal{M}$ and $Y \in \mathcal{M}'$ (which proves the assertion by the 2-out-of-3 property for weak equivalences). We only show that Gr_{FX} is a weak equivalence. The fact that $r_{FX} : FX \to RFX$ is a trivial cofibration means, by definition [17, Def. 3.1] of an exact model structure, that r_{FX} is an admissible monomorphism with a trivially cofibrant cokernel, that is, one has a conflation (a short exact sequence)

$$FX \underset{\simeq}{\overset{r_{FX}}{\rightarrowtail}} RFX \overset{\pi}{\twoheadrightarrow} C$$

in $\mathcal{M}'$ where C is trivially cofibrant, that is, $C \in \mathcal{C}' \cap \mathcal{W}'$ (and RFX is of course fibrant). By applying the exact functor G to the sequence above, we get a conflation in $\mathcal{M}$, which is the bottom row of the following pullback diagram:

$$\begin{tikzcd}
GFX \arrow[r, tail, "\iota", "\simeq"'] \arrow[d, equal] & T \arrow[r, two heads, "\varrho"] \arrow[d, two heads, "\varphi", "\simeq"'] & QGC \arrow[d, two heads, "q_{GC}", "\simeq"'] \\
GFX \arrow[r, tail, "Gr_{FX}"] & GRFX \arrow[r, two heads, "G\pi"] & GC\,.
\end{tikzcd}$$

Note that this pullback diagram really exists; indeed, by definition of an exact category, any pullback of an admissible epimorphism exists and admissible epimorphisms are stable under pullbacks. In particular, ϱ is an admissible epimorphism (and ϱ has the same kernel as $G\pi$; cf. Freyd [15, Thm. 2.52]). Since $C \in \mathcal{W}'$ we have $GC \in \mathcal{W}$ by assumption. Since one always has $Q(\mathcal{W}) \subseteq \mathcal{W}$, it follows that $QGC \in \mathcal{W}$, and hence $QGC \in \mathcal{C} \cap \mathcal{W}$ (as QY is always cofibrant). This means that ι is a trivial cofibration. In any model category, the class of trivial fibrations is stable under pullbacks by [25, Cor. 1.1.11]; thus the fact that q_{GC} is a trivial fibration forces φ to be the same. As ι and φ are, in particular, weak equivalences, so is their composite $Gr_{FX} = \varphi \circ \iota$, as desired. □

Theorem 3.11. *A Sharp–Foxby adjunction (S,T) on $\mathcal{A}$ induces a Quillen equivalence between the model categories $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ constructed in Theorems 3.7 and 3.9. Thus the total (left/right) derived functors of S and T yield an adjoint equivalence of the corresponding homotopy categories,*

$$\mathrm{Ho}(\mathcal{U}^{\pi}) \underset{\mathbf{R}T}{\overset{\mathbf{L}S}{\rightleftarrows}} \mathrm{Ho}(\mathcal{U}^{\iota})\,. \tag{$\sharp 4$}$$

In fact, this is an equivalence of triangulated categories.

Proof. As mentioned in Remark 3.5, a Sharp–Foxby adjunction (S,T) on $\mathcal{A}$ induces an exact adjoint equivalence between $\mathcal{U}^{\pi}$ and $\mathcal{U}^{\iota}$ with $S(\mathcal{W}^{\pi}) \subseteq \mathcal{W}^{\iota}$ and $T(\mathcal{W}^{\iota}) \subseteq \mathcal{W}^{\pi}$. Hence the unit $\eta_X \colon X \to TSX$ is an isomorphism, and hence also a weak equivalence, for all $X \in \mathcal{U}^{\pi}$ (in particular for $X \in \mathcal{C}^{\pi}$); and the counit $\varepsilon_Y \colon STY \to Y$ is an isomorphism, and hence also a weak equivalence, for all $Y \in \mathcal{U}^{\iota}$ (in particular for $Y \in \mathcal{F}^{\iota}$). Thus, if we can show that (S,T) is a Quillen adjunction $\mathcal{U}^{\pi} \rightleftarrows \mathcal{U}^{\iota}$, then Proposition 3.10 will imply that it is in fact a Quillen equivalence (as claimed). To show this, it must be argued that $S\colon \mathcal{U}^{\pi} \to \mathcal{U}^{\iota}$ is a left Quillen functor (see [24, Def. 1.3.1]), that is, we must argue that S maps (trivial) cofibrations in $\mathcal{U}^{\pi}$ to (trivial) cofibrations in $\mathcal{U}^{\iota}$. Let f be a (trivial) cofibration in $\mathcal{U}^{\pi}$, that is, f is an admissible monomorphism with a (trivially) cofibrant cokernel C (see [17, Def. 3.1]). Since S is exact, it follows that Sf is an admissible monomorphism in $\mathcal{U}^{\iota}$ with cokernel SC. Hence, we only need to prove that S maps (trivially) cofibrant objects in $\mathcal{U}^{\pi}$ to (trivially) cofibrant objects in $\mathcal{U}^{\iota}$. However, this is clear as every object in $\mathcal{U}^{\iota}$ is cofibrant, see Theorem 3.9, and since we have $S(\mathcal{W}^{\pi}) \subseteq \mathcal{W}^{\iota}$.

Having established that (S,T) yields a Quillen equivalence $\mathcal{U}^{\pi} \rightleftarrows \mathcal{U}^{\iota}$, the adjoint equivalence of homotopy categories displayed in ($\sharp$4) follows from [24, Prop. 1.3.13].

It remains to see that the functors $\mathbf{L}S$ and $\mathbf{R}T$ are triangulated. By [28, Lem. 5.3.6] it suffices to prove that $\mathbf{L}S$ is triangulated, because then its right adjoint $\mathbf{R}T$ will automatically be triangulated as well. Recall from 2.5 that the distinguished triangles in $\mathrm{Ho}(\mathcal{U}^\pi)$ are, up to isomorphism, the images in $\mathrm{Ho}(\mathcal{U}^\pi)$ of distinguished triangles in $\underline{\mathrm{GProj}}(\mathcal{A})$ under the equivalence $\underline{\mathrm{GProj}}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{U}^\pi)$ (see also Theorem 3.7).

At this point we need to recall from [21, Chap. I§2.5] how the triangulated structure on the stable category $\underline{\mathrm{GProj}}(\mathcal{A})$ is defined. For every morphism $u\colon G \to G'$ in the Frobenius category $\mathrm{GProj}(\mathcal{A})$ choose a short exact sequence (a conflation) $G \overset{i}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} \tilde{G}$ in $\mathrm{GProj}(\mathcal{A})$ where P is a projective–injective object, that is, $P \in \mathrm{Proj}(\mathcal{A})$. The object $\tilde{G}$ is the suspension of G; in symbols, $\tilde{G} = \Sigma G$ (the assignment $G \mapsto \tilde{G} = \Sigma G$ is not functorial on $\mathrm{GProj}(\mathcal{A})$, but it is functorial on $\underline{\mathrm{GProj}}(\mathcal{A})$). Then consider the pushout diagram in $\mathrm{GProj}(\mathcal{A})$,

$$\begin{tikzcd} G \arrow[r, tail, "i"] \arrow[d, "u"'] \arrow[dr, phantom, "\text{pushout}"] & P \arrow[r, two heads, "p"] \arrow[d, "t"] & \tilde{G} \arrow[d, equal] \\ G' \arrow[r, tail, "v"] & G'' \arrow[r, two heads, "w"] & \tilde{G}\,. \end{tikzcd} \tag{$\sharp$5}$$

The diagram

$$G \xrightarrow{\;u\;} G' \xrightarrow{\;v\;} G'' \xrightarrow{\;w\;} \tilde{G}\,, \tag{$\sharp$6}$$

considered as a diagram in $\underline{\mathrm{GProj}}(\mathcal{A})$, is called a *standard triangle*. By definition, a distinguished triangle in $\underline{\mathrm{GProj}}(\mathcal{A})$ is a diagram in this category which is isomorphic to some standard triangle. The triangulated structure on $\underline{\mathrm{GInj}}(\mathcal{A})$ is defined similarly.

We must show that the functor $\mathbf{L}S$ maps every distinguished triangle Δ in $\mathrm{Ho}(\mathcal{U}^\pi)$ to a distinguished triangle in $\mathrm{Ho}(\mathcal{U}^\iota)$. By the considerations above, we may assume that Δ is the image in $\mathrm{Ho}(\mathcal{U}^\iota)$ of a standard triangle ($\sharp$6) in $\underline{\mathrm{GProj}}(\mathcal{A})$. By definition, see [24, Def. 1.3.6], the action of the functor $\mathbf{L}S$ on an object X in $\mathrm{Ho}(\mathcal{U}^\pi)$ is $\mathbf{L}S(X) = SQX$ where QX is a cofibrant replacement of X. As the objects in ($\sharp$6) are already cofibrant in $\mathcal{U}^\pi$, see Theorem 3.7, the diagram $\mathbf{L}S(\Delta)$ is nothing but

$$SG \xrightarrow{\;Su\;} SG' \xrightarrow{\;Sv\;} SG'' \xrightarrow{\;Sw\;} S\tilde{G}\,, \tag{$\sharp$7}$$

which we must show is a distinguished triangle in $\mathrm{Ho}(\mathcal{U}^\iota)$. Since the pair $(\mathcal{C}^\iota \cap \mathcal{W}^\iota, \mathcal{F}^\iota) = (\mathcal{W}^\iota, \mathrm{GInj}\,\mathcal{A})$ is a hereditary cotorsion pair in $\mathcal{U}^\iota$, see Theorem 3.9 and Definition 3.2, it follows from [32, Lem. 6.20] that we can find a diagram in $\mathcal{U}^\iota$,

$$
\begin{tikzcd}
SG \arrow[r, tail, "Si"] \arrow[d, tail, "h"] & SP \arrow[r, two heads, "Sp"] \arrow[d, tail, "e"] & S\tilde{G} \arrow[d, tail, "\tilde{h}"] \\
H \arrow[r, tail, "i_0"] \arrow[d, two heads] & E \arrow[r, two heads, "p_0"] \arrow[d, two heads] & \tilde{H} \arrow[d, two heads] \\
J \arrow[r, tail] & I \arrow[r, two heads] & \tilde{J}
\end{tikzcd}
\tag{$\sharp 8$}
$$

whose rows and columns are conflations, where $H, E, \tilde{H}$ are Gorenstein injective, and where $J, I, \tilde{J}$ have finite injective dimension. As $P \in \operatorname{Proj}\mathcal{A} \subseteq \mathcal{W}^{\pi}$ we have $SP \in \mathcal{W}^{\iota}$, that is, SP has finite injective dimension. It follows from the middle column in ($\sharp$8) that E has finite injective dimension, and since E is also Gorenstein injective it must be injective (this is immediate from the definition, 2.1, of Gorenstein injective objects). Let $SG' \overset{h'}{\rightarrowtail} H' \twoheadrightarrow J'$ be a short exact sequence with $H' \in \operatorname{GInj}\mathcal{A}$ and $J' \in \mathcal{W}^{\iota}$. The morphism $h\colon SG \to H$ is a (special) Gorenstein injective preenvelope of SG since it is monic and its cokernel $J \in \mathcal{W}^{\iota}$ satisfies $\operatorname{Ext}^1_{\mathcal{A}}(J, X) = 0$ for all $X \in \operatorname{GInj}(\mathcal{A})$; see [34, Prop. 2.1.4]. Thus, the morphism $h'Su\colon SG \to H' \in \operatorname{GInj}(\mathcal{A})$ lifts to a morphism $u_0\colon H \to H'$ such that $u_0 h = h'Su$. This gives commutativity of the left wall in the following diagram:

$$
\begin{tikzcd}
& H \arrow[rr, tail, "i_0"] \arrow[dd, "u_0"] & & E \arrow[rr, two heads, "p_0"] \arrow[dd, "t_0"] & & \tilde{H} \arrow[dd, equal] \\
SG \arrow[ur, tail, "h"] \arrow[rr, tail, "Si"] \arrow[dd, "Su"] & & SP \arrow[ur, tail, "e"] \arrow[rr, two heads, "Sp"] \arrow[dd, "St"] & & S\tilde{G} \arrow[ur, tail, "\tilde{h}"] \arrow[dd, equal] & \\
& H' \arrow[rr, tail, "v_0"] & & H'' \arrow[rr, two heads, "w_0"] & & \tilde{H} \\
SG' \arrow[ur, tail, "h'"] \arrow[rr, tail, "Sv"'] & & SG'' \arrow[ur, dotted, tail, "h''"] \arrow[rr, two heads, "Sw"'] & & S\tilde{G} \arrow[ur, tail, "\tilde{h}"'] &
\end{tikzcd}
\tag{$\sharp 9$}
$$

The top wall in ($\sharp$9) is just the upper half of the commutative diagram ($\sharp$8). The back wall is the (commutative) pushout diagram of the morphisms $H' \overset{u_0}{\leftarrow} H \overset{i_0}{\rightarrowtail} E$. The right wall is evidently commutative. The front wall in ($\sharp$9) is obtained by applying the exact functor S to the diagram ($\sharp$5). Since S is a left adjoint functor, it preserves colimits, so the front wall in ($\sharp$9) is (still) a pushout diagram. As $(v_0 h')Su = v_0 u_0 h = t_0 i_0 h = (t_0 e)Si$ and since SG'' is the pushout of $SG' \xleftarrow{Su} SG \xrightarrow{Si} SP$, there exists a (unique) morphism $h''\colon SG'' \to H''$ such that $h''Sv = v_0 h'$ and $h''St = t_0 e$. The first of these identities show that the left square in the bottom wall in ($\sharp$9) is commutative. It follows from the universal property of the pushout SG'' that the right square in the bottom wall is commutative as well. By applying the Snake Lemma to this bottom wall, we see that h'' is monic (as h' and $\tilde{h}$ are so) and that the cokernel J'' of h'' sits in a short exact sequence $0 \to J' \to J'' \to \tilde{J} \to 0$. Since $J', \tilde{J} \in \mathcal{W}^{\iota}$ it follows that $J'' \in \mathcal{W}^{\iota}$. Since h, h',

h'', and $\tilde{h}$ are (admissible) monomorphisms in $\mathcal{U}^\iota$ whose cokernels belong to $\mathcal{W}^\iota$ (which are the trivially cofibrant objects in $\mathcal{U}^\iota$), they are trivial cofibrations in the exact model structure on $\mathcal{U}^\iota$; see [17, Def. 3.1]. In particular, h, h', h'', and $\tilde{h}$ are weak equivalences in $\mathcal{U}^\iota$ and therefore isomorphisms in $\mathrm{Ho}(\mathcal{U}^\iota)$. The commutative diagram (♯9) now shows that in the homotopy category $\mathrm{Ho}(\mathcal{U}^\iota)$, the diagram (♯7) is isomorphic to

$$H \xrightarrow{u_0} H' \xrightarrow{v_0} H'' \xrightarrow{w_0} \tilde{H}\,. \tag{♯10}$$

By definition, and by commutativity of the back wall in (♯9), the diagram (♯10) is a standard triangle in $\underline{\mathrm{GInj}}(\mathcal{A})$, and consequently, (♯7) is a distinguished triangle in $\mathrm{Ho}(\mathcal{U}^\iota)$. □

Corollary 3.12. *If there exists a Sharp–Foxby adjunction (S,T) on $\mathcal{A}$, then there is an equivalence of triangulated categories, $\underline{\mathrm{GProj}}(\mathcal{A}) \simeq \underline{\mathrm{GInj}}(\mathcal{A})$.*

Proof. By Theorems 3.7, 3.11, and 3.9 there are the following equivalences of triangulated categories, $\underline{\mathrm{GProj}}(\mathcal{A}) \simeq \mathrm{Ho}(\mathcal{U}^\pi) \simeq \mathrm{Ho}(\mathcal{U}^\iota) \simeq \underline{\mathrm{GInj}}(\mathcal{A})$. □

Remark 3.13. Before closing this section, we record a biproduct of Proposition 3.10 concerning virtually Gorenstein rings, which should be well known. We recall from [3,4] that an Artin algebra A is called virtually Gorenstein if $(\mathrm{GProj}(A))^\perp = {}^\perp(\mathrm{GInj}(A))$. The same notion for commutative rings has also been studied in [36]. In what follows, assume that A is an Artin algebra or a commutative noetherian ring with finite Krull dimension. In both cases, it is well known [4,19,26] that there are Hovey triples

$$(\mathrm{GProj}(A), (\mathrm{GProj}(A))^\perp, \mathrm{Mod}(A)) \quad \text{and} \quad (\mathrm{Mod}(A), {}^\perp(\mathrm{GInj}(A)), \mathrm{GInj}(A)).$$

Applying Proposition 3.10 in the case where $F = G = I_{\mathrm{Mod}(A)}$, we obtain that virtually Gorensteiness of A implies that the identity is a Quillen equivalence between the two model structures. Therefore the homotopy categories of these two models are, in fact, isomorphic. In case A is, in addition, commutative Gorenstein we recover the analogous statement for Gorenstein rings (see the comments after Theorem 8.6 in [25]).

4. The case of chain complexes

Recall from the beginning of Section 2 that $\mathcal{A}$ always denotes any bicomplete abelian category with enough projectives and enough injectives. In this section, we consider the abelian category $\mathrm{Ch}(\mathcal{A})$ of unbounded chain complexes in $\mathcal{A}$ and prove that, under suitable conditions, a Sharp–Foxby adjunction (S,T) on $\mathcal{A}$ induces a Sharp–Foxby adjunction on $\mathrm{Ch}(\mathcal{A})$ by degreewise application of the functors S and T. First we recall the following.

4.1. The *finitistic projective dimension*, $\mathrm{FPD}(\mathcal{A})$, of $\mathcal{A}$ is defined as

$$\mathrm{FPD}(\mathcal{A}) = \sup\{\mathrm{pd}_{\mathcal{A}} M \mid M \text{ is an object in } \mathcal{A} \text{ with finite projective dimension}\}.$$

Dually, the *finitistic injective dimension*, $\mathrm{FID}(\mathcal{A})$, of $\mathcal{A}$ is

$$\mathrm{FID}(\mathcal{A}) = \sup\{\mathrm{id}_{\mathcal{A}} M \mid M \text{ is an object in } \mathcal{A} \text{ with finite injective dimension}\}.$$

The *finitistic Gorenstein projective dimension*, $\mathrm{FGPD}(\mathcal{A})$, and the *finitistic Gorenstein injective dimension*, $\mathrm{FGID}(\mathcal{A})$, are defined similarly.

For most abelian categories that appear in applications, the finitistic dimensions defined above turn out to be finite. As in [23, (proofs of) Thms. 2.28 and 2.29] one easily proves:

Lemma 4.2. *There are equalities* $\mathrm{FGPD}(\mathcal{A}) = \mathrm{FPD}(\mathcal{A})$ *and* $\mathrm{FGID}(\mathcal{A}) = \mathrm{FID}(\mathcal{A})$*. Thus, if* $\mathrm{FPD}(\mathcal{A})$*, respectively,* $\mathrm{FID}(\mathcal{A})$*, is finite, then so is* $\mathrm{FGPD}(\mathcal{A})$*, respectively,* $\mathrm{FGID}(\mathcal{A})$*.* $\square$

In $\mathcal{A}$ we have the subcategories $\mathcal{U}^{\pi}_{\mathcal{A}}$, $\mathcal{C}^{\pi}_{\mathcal{A}}$, $\mathcal{W}^{\pi}_{\mathcal{A}}$ and $\mathcal{F}^{\pi}_{\mathcal{A}}$ from Definition 3.1. Similarly, in $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$ we have the subcategories $\mathcal{U}^{\pi}_{\mathcal{B}}$, $\mathcal{C}^{\pi}_{\mathcal{B}}$, $\mathcal{W}^{\pi}_{\mathcal{B}}$ and $\mathcal{F}^{\pi}_{\mathcal{B}}$. The following result explains the relation between all these subcategories.

Proposition 4.3. *Assume that* $\mathrm{FPD}(\mathcal{A}) < \infty$ *and let* $X = \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$ *be an object in* $\mathcal{B} := \mathrm{Ch}(\mathcal{A})$*. The following conclusions hold.*

(i) *X belongs to $\mathcal{U}^{\pi}_{\mathcal{B}}$ if and only if every X_n belongs to $\mathcal{U}^{\pi}_{\mathcal{A}}$.*
(ii) *X belongs to $\mathcal{C}^{\pi}_{\mathcal{B}}$ if and only if every X_n belongs to $\mathcal{C}^{\pi}_{\mathcal{A}}$.*
(iii) *X belongs to $\mathcal{W}^{\pi}_{\mathcal{B}}$ if and only if X is exact and every cycle $\mathrm{Z}_n(X)$ belongs to $\mathcal{W}^{\pi}_{\mathcal{A}}$.*
(iv) *X belongs to $\mathcal{F}^{\pi}_{\mathcal{B}}$ if and only if every X_n belongs to $\mathcal{F}^{\pi}_{\mathcal{A}}$.*

Proof. Part (ii) is proved in [35, Thm. 2.2] in the case $\mathcal{A} = \mathrm{Mod}(A)$ where A is any ring, but the proof works in any abelian category (with enough projectives).

In view of (ii), the "only if" part in (i) is clear. To prove the "if" part in (i), assume that every X_n is in $\mathcal{U}^{\pi}_{\mathcal{A}}$, that is, $\mathrm{Gpd}_{\mathcal{A}}(X_n) < \infty$. By our assumption $\mathrm{FPD}(\mathcal{A}) < \infty$ and by Lemma 4.2, it follows that $s = \sup\{\mathrm{Gpd}_{\mathcal{A}}(X_n) \,|\, n \in \mathbb{Z}\}$ belongs to $\mathbb{N}_0$. The proof is now by induction on s. If $s = 0$, then X is even in $\mathcal{C}^{\pi}_{\mathcal{B}} \subseteq \mathcal{U}^{\pi}_{\mathcal{B}}$ by part (ii). Now assume that $s > 0$. Choose any exact sequence

$$0 \longrightarrow K \longrightarrow P^{s-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$ where $P^0, \ldots, P^{s-1}$ are complexes consisting of projective objects in $\mathcal{A}$. For each $n \in \mathbb{Z}$ we have an exact sequence $0 \to K_n \to P^{s-1}_n \to P^1_n \to P^0_n \to X_n \to 0$ in $\mathcal{A}$, and since $P^0_n, \ldots, P^{s-1}_n$ are projectives and $\mathrm{Gpd}_{\mathcal{A}}(X_n) \leqslant s$, it follows that K_n is

Gorenstein projective; cf. [23, (proof of) Prop. 2.7]. Thus, K is a complex of Gorenstein projective objects in $\mathcal{A}$, which by (ii) means that K is a Gorenstein projective object in $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$. So the exact sequence displayed above shows that $\mathrm{Gpd}_{\mathcal{B}}(X) \leqslant s < \infty$, that is, $X \in \mathcal{U}^{\pi}_{\mathcal{B}}$.

To prove (iii), let $X \in \mathcal{W}^{\pi}_{\mathcal{B}}$, which means that we have an exact sequence

$$0 \longrightarrow P^m \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0 \tag{$\sharp$11}$$

in $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$ where $P^0, \dots, P^m$ are projective objects; i.e. each P^i is a split exact complex of projective objects in $\mathcal{A}$, and thus each cycle $\mathrm{Z}_n(P^i)$ is also projective in $\mathcal{A}$. As the complexes $P^0, \dots, P^m$ are, in particular, exact, so is X (and the same are all the kernel and cokernel complexes of the chain maps that appear in ($\sharp$11)). This implies that the functor $\mathrm{Z}_n(-)$ leaves the sequence ($\sharp$11) exact, and the hereby obtained exact sequence

$$0 \longrightarrow \mathrm{Z}_n(P^m) \longrightarrow \cdots \longrightarrow \mathrm{Z}_n(P^1) \longrightarrow \mathrm{Z}_n(P^0) \longrightarrow \mathrm{Z}_n(X) \longrightarrow 0$$

shows that $\mathrm{Z}_n(X)$ has finite projective dimension in $\mathcal{A}$, that is, $\mathrm{Z}_n(X)$ belongs to $\mathcal{W}^{\pi}_{\mathcal{A}}$.

The proof of the "if" part in (iii) is based on a standard construction; see (the dual of) [16, Thm. 3.1.3] (for this argument to work we make use the hypothesis $\mathrm{FPD}(\mathcal{A}) < \infty$).

Part (iv) is just a repetition of part (i) since $\mathcal{F}^{\pi}_{\mathcal{B}} = \mathcal{U}^{\pi}_{\mathcal{B}}$ and $\mathcal{F}^{\pi}_{\mathcal{A}} = \mathcal{U}^{\pi}_{\mathcal{A}}$. $\square$

In $\mathcal{A}$ we also have the subcategories $\mathcal{U}^{\iota}_{\mathcal{A}}$, $\mathcal{C}^{\iota}_{\mathcal{A}}$, $\mathcal{W}^{\iota}_{\mathcal{A}}$ and $\mathcal{F}^{\iota}_{\mathcal{A}}$ from Definition 3.2. Similarly, in $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$ we have the subcategories $\mathcal{U}^{\iota}_{\mathcal{B}}$, $\mathcal{C}^{\iota}_{\mathcal{B}}$, $\mathcal{W}^{\iota}_{\mathcal{B}}$ and $\mathcal{F}^{\iota}_{\mathcal{B}}$. By an argument dual to the proof of Proposition 4.3, one shows the following result.

Proposition 4.4. *Assume that* $\mathrm{FID}(\mathcal{A}) < \infty$ *and let* $Y = \cdots \to Y_{n+1} \to Y_n \to Y_{n-1} \to \cdots$ *be an object in* $\mathcal{B} := \mathrm{Ch}(\mathcal{A})$. *The following conclusions hold.*

(i) *Y belongs to $\mathcal{U}^{\iota}_{\mathcal{B}}$ if and only if every Y_n belongs to $\mathcal{U}^{\iota}_{\mathcal{A}}$.*
(ii) *Y belongs to $\mathcal{C}^{\iota}_{\mathcal{B}}$ if and only if every Y_n belongs to $\mathcal{C}^{\iota}_{\mathcal{A}}$.*
(iii) *Y belongs to $\mathcal{W}^{\iota}_{\mathcal{B}}$ if and only if Y is exact and every cycle $\mathrm{Z}_n(Y)$ belongs to $\mathcal{W}^{\iota}_{\mathcal{A}}$.*
(iv) *Y belongs to $\mathcal{F}^{\iota}_{\mathcal{B}}$ if and only if every Y_n belongs to $\mathcal{F}^{\iota}_{\mathcal{A}}$.* $\square$

We can now prove the main result of this section.

Theorem 4.5. *Let (S, T) be a Sharp–Foxby adjunction on $\mathcal{A}$, in particular, $\underline{\mathrm{GProj}}(\mathcal{A})$ and $\overline{\mathrm{GInj}}(\mathcal{A})$ are equivalent as triangulated categories by Corollary 3.12. If $\mathrm{FPD}(\mathcal{A}) < \infty$ and $\mathrm{FID}(\mathcal{A}) < \infty$, then degreewise application of S and T yields a Sharp–Foxby adjunction on $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$, and hence $\underline{\mathrm{GProj}}(\mathcal{B})$ and $\overline{\mathrm{GInj}}(\mathcal{B})$ are equivalent as triangulated categories.*

Proof. Write $\bar{S}$ and $\bar{T}$ for the endofunctors on $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$ that are given by degreewise application of S and T, and let η and ε be the unit and counit of the adjunction

(S,T) on $\mathcal{A}$. It is straightforward to verify that $(\bar{S},\bar{T})$ is an adjunction on $\mathcal{B}$ with unit $\bar{\eta}$ and counit $\bar{\varepsilon}$ given by $(\bar{\eta}_X)_n = \eta_{X_n}$ and $(\bar{\varepsilon}_X)_n = \varepsilon_{X_n}$, where X is a chain complex and n is an integer.

By assumption, S restricts to an exact functor $S\colon \mathcal{U}^{\pi}_{\mathcal{A}} \to \mathcal{U}^{\iota}_{\mathcal{A}}$ which maps $\mathcal{W}^{\pi}_{\mathcal{A}}$ to $\mathcal{W}^{\iota}_{\mathcal{A}}$; see (SF1) and (SF2) in Definition 3.4. It therefore follows from Propositions 4.3 and 4.4 that $\bar{S}$ restricts to an exact functor $\bar{S}\colon \mathcal{U}^{\pi}_{\mathcal{B}} \to \mathcal{U}^{\iota}_{\mathcal{B}}$ which maps $\mathcal{W}^{\pi}_{\mathcal{B}}$ to $\mathcal{W}^{\iota}_{\mathcal{B}}$, that is, the adjunction $(\bar{S},\bar{T})$ also satisfies conditions (SF1) and (SF2). A similar argument shows that this adjunction satisfies (SF3) and (SF4) as well. By (SF5) in Definition 3.4 we know that the unit $\eta_A\colon A \to TSA$ of (S,T) is an isomorphism for $A \in \mathcal{U}^{\pi}_{\mathcal{A}}$. From the definition of $\bar{\eta}$ and from Proposition 4.3 it now follows that $\bar{\eta}_X\colon X \to \bar{T}\bar{S}X$ is an isomorphism for $X \in \mathcal{U}^{\pi}_{\mathcal{B}}$, that is, $(\bar{S},\bar{T})$ satisfies (SF5). Similarly, $(\bar{S},\bar{T})$ also satisfies condition (SF6). $\square$

Corollary 4.6. *Let (S,T) be a Sharp–Foxby adjunction on $\mathcal{A}$ for which* $\mathrm{FPD}(\mathcal{A}) < \infty$ *and* $\mathrm{FID}(\mathcal{A}) < \infty$. *Then degreewise application of S and T yields a Sharp–Foxby adjunction on the category* $\mathrm{Ch}^2(\mathcal{A})$ *of double complexes (also called bicomplexes) in $\mathcal{A}$.*

Proof. The category $\mathrm{Ch}^2(\mathcal{A})$ of double complexes in $\mathcal{A}$ is naturally identified with the category $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$. Thus, the desired conclusion follows by applying Theorem 4.5 to the category $\mathrm{Ch}(\mathcal{A})$ (in place of $\mathcal{A}$). However, to do this we must first argue that the theorem's hypothesis is satisfied, i.e. that the numbers $\mathrm{FPD}(\mathrm{Ch}(\mathcal{A}))$ and $\mathrm{FID}(\mathrm{Ch}(\mathcal{A}))$ are finite. But it is immediate from (the proofs of) Propositions 4.3(iii) and 4.4(iii) that these numbers agree with $\mathrm{FPD}(\mathcal{A})$ and $\mathrm{FID}(\mathcal{A})$, which are finite by assumption. $\square$

Example 4.7. Let A be a commutative noetherian ring with a dualizing module. By Example 3.6 there exists a Sharp–Foxby adjunction on $\mathrm{Mod}(A)$. The finitistic projective/injective dimensions of $\mathrm{Mod}(A)$ are usually referred to as the finitistic projective/injective dimensions of the ring A, and they are denoted by $\mathrm{FPD}(A)$ and $\mathrm{FID}(A)$. These numbers are finite, indeed, one has $\mathrm{FPD}(A) = \dim A \geqslant \mathrm{FID}(A)$ by [30, Thm. II.(3.2.6) p. 84] and [2, Cor. 5.5], and $\dim A$ is finite by [22, Cor. V.7.2].

Theorem 4.5 and Corollary 4.6 now imply that the category $\mathrm{Ch}(A)$ of chain complexes and the category $\mathrm{Ch}^2(A)$ of double complexes of A-modules both have Sharp–Foxby adjunctions. In particular, there are by Corollary 3.12 equivalences of triangulated categories,

$$\underline{\mathrm{GProj}}(\mathrm{Ch}(A)) \simeq \underline{\mathrm{GInj}}(\mathrm{Ch}(A)) \qquad \text{and} \qquad \underline{\mathrm{GProj}}(\mathrm{Ch}^2(A)) \simeq \underline{\mathrm{GInj}}(\mathrm{Ch}^2(A))\,.$$

The key ingredient in the proof of Theorem 4.5 is that in the category $\mathcal{B} = \mathrm{Ch}(\mathcal{A})$ the Gorenstein projective/injective objects can be suitably described in terms of the Gorenstein projective/injective objects in $\mathcal{A}$ (as recorded in Propositions 4.3 and 4.4). This is also the case for the category $\mathcal{B} = \mathrm{Rep}(Q,\mathcal{A})$ of $\mathcal{A}$-valued representations of a left and right rooted quiver Q; see [13, Thm. 3.5.1]; thus by using the same methods as above one can prove:

Theorem 4.8. *Let (S, T) be a Sharp–Foxby adjunction on $\mathcal{A}$. If one has* $\mathrm{FPD}(\mathcal{A}) < \infty$ *and* $\mathrm{FID}(\mathcal{A}) < \infty$*, then vertexwise application of S and T yields a Sharp–Foxby adjunction on* $\mathcal{B} = \mathrm{Rep}(Q, \mathcal{A})$*, so* $\underline{\mathrm{GProj}}(\mathcal{B})$ *and* $\underline{\mathrm{GInj}}(\mathcal{B})$ *are equivalent as triangulated categories.* $\square$

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References

[1] Luchezar L. Avramov, Hans-Bjørn Foxby, Ring homomorphisms and finite Gorenstein dimension, Proc. Lond. Math. Soc. (3) 75 (2) (1997) 241–270, MR1455856.
[2] Hyman Bass, Injective dimension in Noetherian rings, Trans. Amer. Math. Soc. 102 (1962) 18–29, MR0138644.
[3] Apostolos Beligiannis, Henning Krause, Thick subcategories and virtually Gorenstein algebras, Illinois J. Math. 52 (2) (2008) 551–562, MR2524651.
[4] Apostolos Beligiannis, Idun Reiten, Homological and homotopical aspects of torsion theories, Mem. Amer. Math. Soc. 188 (883) (2007), viii+207 pp., MR2327478.
[5] David J. Benson, Jon F. Carlson, Jeremy Rickard, Thick subcategories of the stable module category, Fund. Math. 153 (1) (1997) 59–80, MR1450996.
[6] David J. Benson, Srikanth B. Iyengar, Henning Krause, Stratifying modular representations of finite groups, Ann. of Math. (2) 174 (3) (2011) 1643–1684, MR2846489.
[7] David J. Benson, Henning Krause, Complexes of injective kG-modules, Algebra Number Theory 2 (1) (2008) 1–30, MR2377361.
[8] Ragnar-Olaf Buchweitz, Maximal Cohen–Macaulay Modules and Tate-Cohomology over Gorenstein Rings, University of Hannover, 1986, available at http://hdl.handle.net/1807/16682.
[9] Lars W. Christensen, Anders Frankild, Henrik Holm, On Gorenstein projective, injective and flat dimensions—a functorial description with applications, J. Algebra 302 (1) (2006) 231–279, MR2236602.
[10] Edgar E. Enochs, Overtoun M.G. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (4) (1995) 611–633, MR1363858.
[11] Edgar E. Enochs, Overtoun M.G. Jenda, Relative Homological Algebra, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000, MR1753146.
[12] Edgar E. Enochs, Overtoun M.G. Jenda, Jinzhong Xu, Foxby duality and Gorenstein injective and projective modules, Trans. Amer. Math. Soc. 348 (8) (1996) 3223–3234, MR1355071.
[13] Hossein Eshraghi, Rasool Hafezi, Shokrollah Salarian, Total acyclicity for complexes of representations of quivers, Comm. Algebra 41 (12) (2013) 4425–4441, MR3169529.
[14] Hans-Bjørn Foxby, Gorenstein modules and related modules, Math. Scand. 31 (1972) 267–284, MR0327752.
[15] Peter Freyd, Abelian Categories. An Introduction to the Theory of Functors, Harper's Series in Modern Mathematics, Harper & Row Publishers, New York, 1964, MR0166240.
[16] Juan Ramon García Rozas, Covers and Envelopes in the Category of Complexes of Modules, Chapman & Hall/CRC Research Notes in Mathematics, vol. 407, Chapman & Hall/CRC, Boca Raton, FL, 1999, MR1693036.
[17] James Gillespie, Model structures on exact categories, J. Pure Appl. Algebra 215 (12) (2011) 2892–2902, MR2811572.
[18] James Gillespie, Exact model structures and recollements, J. Algebra 458 (2016) 265–306, MR3500779.
[19] James Gillespie, Gorenstein complexes and recollements from cotorsion pairs, Adv. Math. 291 (2016) 859–911, MR3459032.
[20] Rüdiger Göbel, Jan Trlifaj, Approximations and Endomorphism Algebras of Modules, de Gruyter Expositions in Mathematics, vol. 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2006, MR2251271.
[21] Dieter Happel, Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988, MR935124.

[22] Robin Hartshorne, Residues and Duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne, Lecture Notes in Mathematics, vol. 20, Springer-Verlag, Berlin, 1966, MR0222093.

[23] Henrik Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189 (1–3) (2004) 167–193, MR2038564.

[24] Mark Hovey, Model Categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999, MR1650134.

[25] Mark Hovey, Cotorsion pairs, model category structures, and representation theory, Math. Z. 241 (3) (2002) 553–592, MR1938704.

[26] Henning Krause, The stable derived category of a Noetherian scheme, Compos. Math. 141 (5) (2005) 1128–1162, MR2157133.

[27] Henning Krause, Greg Stevenson, A note on thick subcategories of stable derived categories, Nagoya Math. J. 212 (2013) 87–96, MR3290681.

[28] Amnon Neeman, Triangulated Categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001, MR1812507.

[29] Dmitri O. Orlov, Triangulated categories of singularities and D-branes in Landau–Ginzburg models, in: Algebr. Geom. Metody, Svyazi i Prilozh., Tr. Mat. Inst. Steklova 246 (2004) 240–262 (in Russian); English translation in Proc. Steklov Inst. Math. 246 (3) (2004) 227–248, MR2101296.

[30] Michel Raynaud, Laurent Gruson, Critères de platitude et de projectivité. Techniques de "platification" d'un module, Invent. Math. 13 (1971) 1–89, MR0308104.

[31] Rodney Y. Sharp, Finitely generated modules of finite injective dimension over certain Cohen–Macaulay rings, Proc. Lond. Math. Soc. (3) 25 (1972) 303–328, MR0306188.

[32] Jan Šťovíček, Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves, in: Advances in Representation Theory of Algebras, in: EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013, pp. 297–367, MR3220541.

[33] Ryo Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, Adv. Math. 225 (4) (2010) 2076–2116, MR2680200.

[34] Jinzhong Xu, Flat Covers of Modules, Lecture Notes in Mathematics, vol. 1634, Springer-Verlag, Berlin, 1996, MR1438789.

[35] Xiaoyan Yang, Zhongkui Liu, Gorenstein projective, injective, and flat complexes, Comm. Algebra 39 (5) (2011) 1705–1721, MR2821502.

[36] Fatemeh Zareh-Khoshchehreh, Mohsen Asgharzadeh, Kamran Divaani-Aazar, Gorenstein homology, relative pure homology and virtually Gorenstein rings, J. Pure Appl. Algebra 218 (12) (2014) 2356–2366, MR3227307.

[37] Yuefei Zheng, Zhaoyong Huang, Triangulated equivalences involving Gorenstein projective modules, Canad. Math. Bull. 60 (4) (2017) 879–890, MR3710670.

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A NOTE ON HOMOTOPY CATEGORIES OF FP-INJECTIVES

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(*communicated by J.P.C. Greenlees*)

Abstract

For a locally finitely presented Grothendieck category $\mathcal{A}$, we consider a certain subcategory of the homotopy category of FP-injectives in $\mathcal{A}$ which we show is compactly generated. In the case where $\mathcal{A}$ is locally coherent, we identify this subcategory with the derived category of FP-injectives in $\mathcal{A}$. Our results are, in a sense, dual to the ones obtained by Neeman on the homotopy category of flat modules. Our proof is based on extending a characterization of the pure acyclic complexes which is due to Emmanouil.

1. Introduction

The work of Neeman [21] on the homotopy category of flat modules has led to interesting advances in ring theory and homological algebra. Neeman was inspired by work of Iyengar and Krause [16], who proved that over a Noetherian ring R with a dualizing complex D, the composite $\mathbf{K}(\operatorname{Proj} R) \rightarrowtail \mathbf{K}(\operatorname{Flat} R) \xrightarrow{D\otimes_R -} \mathbf{K}(\operatorname{Inj} R)$ is an equivalence, which restricts to Grothendieck duality. Neeman in [21] focuses on the embedding $\mathbf{K}(\operatorname{Proj} R) \rightarrowtail \mathbf{K}(\operatorname{Flat} R)$ for a general ring R and shows that the category $\mathbf{K}(\operatorname{Proj} R)$ is $\aleph_1$–compactly generated and that the composite of canonical maps $\mathbf{K}(\operatorname{Proj} R) \rightarrowtail \mathbf{K}(\operatorname{Flat} R) \to \mathbf{D}(\operatorname{Flat} R)$ is an equivalence. Related work on homotopy categories and the existence of adjoints between them is done by Krause [17], Murfet and Salarian [18], Saorín and Šťovíček [23], and others.

Closely related to the notion of flatness, is the notion of purity [4]. A submodule $A \leqslant B$ is called pure if any finite system of linear equations with constants from A and a solution in B, has a solution in A. This condition can be expressed diagrammatically, and is equivalent to asking for the sequence $A \rightarrowtail B \twoheadrightarrow B/A$ to remain exact after applying, for any finitely presented module F, the functor $\operatorname{Hom}_R(F,-)$, or equivalently the functor $F \otimes_R -$. Such sequences are called pure exact and they are of interest since they form the smallest class of short exact sequences which is closed under filtered colimits. It follows from this discussion that a module M is flat if and only if any epimorphism with target M is pure. Thus flatness can be defined in any additive category which has an appropriate notion of finitely presented objects, namely locally finitely presented additive categories [2, 5]. If $\mathcal{A}$ is such

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a category, it is well known that the relation between purity and flatness can be given formally via the equivalence $\mathcal{A} \cong \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$; $A \mapsto \mathrm{Hom}_{\mathcal{A}}(-, A)|_{\mathrm{fp}(\mathcal{A})}$[1], see [5, 1.4]. Thus, roughly speaking, the study of purity can be reduced to the study of flat (left exact) functors, and Neeman's results have analogues in the context of purity, see Emmanouil [7], Krause [17], Simson [25] and Šťovíček [28].

The dual notion of flatness, in a locally finitely presented Grothendieck category $\mathcal{A}$, is that of FP-injectivity. Namely, an object A in $\mathcal{A}$ is called FP-injective if any monomorphism with source A is pure. We denote the class of FP-injective objects by $\mathrm{FPI}(\mathcal{A})$. FP-injective modules were studied first by Stenström in [26]. One reason why they are of importance is because over (non-Noetherian) rings where injectives fail to be closed under coproducts, one can work with FP-injectives which are always closed under coproducts. Moreover, a ring is coherent if and only if the class of FP-injective modules is closed under filtered colimits [26, 3.2], in strong analogy with the dual situation, where coherent rings are characterized by the closure of flat modules under products.

In this note our goal is to provide, in a sense, duals of the above mentioned results of Neeman, that is, to obtain analogous results for the homotopy category of FP-injectives. For this we look at the tensor embedding functor of a module category to FP-injective (right exact) functors, that is, the functor $\text{Mod-}R \to \mathcal{A} := (R\text{-mod}, \mathrm{Ab})$; $M \mapsto (M \otimes_R -)|_{R\text{-mod}}$, which identifies pure exact sequences in Mod-R with short exact sequences of FP-injective (right exact) functors, and induces an equivalence $\text{Mod-}R \cong \mathrm{FPI}(\mathcal{A})$ [13, §1]. It is easy to observe that under this equivalence, the pure projective modules (the projectives with respect to the pure exact sequences) correspond to functors in the class $\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A})$[2]. We point out that by work of Eklof and Trlifaj [6], we know that this class consists of those FP-injectives which are (summands of) transfinite extensions of finitely presented objects (see 2.2). We can now state our main result (proved in 3.5).

Theorem. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and denote by $\mathrm{FPI}(\mathcal{A})$ the class of FP-injective objects in $\mathcal{A}$. Then the homotopy category $\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A}))$ is compactly generated. Moreover, if $\mathcal{A}$ is locally coherent, the composite functor*

$$\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A})) \rightarrowtail \mathbf{K}(\mathrm{FPI}(\mathcal{A})) \xrightarrow{\mathrm{can}} \mathbf{D}(\mathrm{FPI}(\mathcal{A}))$$

is an equivalence of triangulated categories.

Our proof is based on Neeman's strategy. Let $\mathcal{C} := \mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A})$. Since $\mathbf{K}(\mathcal{C})$ is compactly generated and admits coproducts, we obtain a right adjoint of the inclusion $\mathbf{K}(\mathcal{C}) \rightarrowtail \mathbf{K}(\mathrm{FPI}(\mathcal{A}))$, and in the case where $\mathcal{A}$ is locally coherent, we identify its kernel with the pure acyclic complexes of FP-injective objects in $\mathcal{A}$. From this it follows that any chain map from a complex in $\mathbf{K}(\mathcal{C})$ to a pure acyclic complex of FP-injectives is null homotopic. In fact, in 3.2 we prove something more general, namely that for any locally finitely presented Grothendieck category $\mathcal{A}$, any chain map from a complex in $\mathbf{K}({}^{\perp}\mathrm{FPI}(\mathcal{A}))$ to a pure acyclic complex is null homotopic. This extends a result of Emmanouil [7].

[1] $\mathrm{fp}(\mathcal{A})$ denotes a set of isomorphism classes of finitely presented objects in $\mathcal{A}$, see section 2.
[2] ${}^{\perp}\mathrm{FPI}(\mathcal{A})$ denotes the left orthogonal class of FP-injectives with respect to the $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$ functor, see section 2.

Finally, we point out that Šťovíček [28] has also studied the category $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$ and in the locally coherent case has proved the existence of a model category structure with homotopy category $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$. Using our main result we can identify the cofibrant objects in this model structure with the category $\mathbf{K}(\mathcal{C})$, and combining with one of the main results of [28] we also obtain that for $\mathcal{A}$ locally coherent $\mathbf{K}(\mathrm{Inj}(\mathcal{A}))$ is compactly generated (see 3.7, 3.8).

2. Preliminaries

Locally finitely presented additive categories [2, 5]. In an additive category $\mathcal{A}$, an object X is called *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)\colon \mathcal{A} \to \mathrm{Ab}$ preserves filtered colimits. $\mathcal{A}$ is called *locally finitely presented* if it is cocomplete, the isomorphism classes of finitely presented objects in $\mathcal{A}$ form a set $\mathrm{fp}(\mathcal{A})$, and every object in $\mathcal{A}$ is a filtered colimit of objects in $\mathrm{fp}(\mathcal{A})$. An abelian category $\mathcal{A}$ is locally finitely presented if and only if it is a Grothendieck category with a generating set of finitely presented objects [2, Satz 1.5]. A locally finitely presented Grothendieck category $\mathcal{A}$ is called *locally coherent* if the subcategory $\mathrm{fp}(\mathcal{A})$ is abelian.

Purity. If $\mathcal{A}$ is a locally finitely presented additive category, a sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ is called *pure exact* if it is $\mathrm{Hom}_{\mathcal{A}}(\mathrm{fp}(\mathcal{A}), -)$–exact, that is, if for any $A \in \mathrm{fp}(\mathcal{A})$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, X) \to \mathrm{Hom}_{\mathcal{A}}(A, Y) \to \mathrm{Hom}_{\mathcal{A}}(A, Z) \to 0$$

is an exact sequence of abelian groups. An object $X \in \mathcal{A}$ is called *pure projective* if any pure exact sequence of the form $0 \to Z \to Y \to X \to 0$ splits, and dually X is called *pure injective* if any pure exact sequence of the form $0 \to X \to Y \to Z \to 0$ splits. We will denote the class of pure projective objects in $\mathcal{A}$ by $\mathrm{PProj}(\mathcal{A})$. Pure exact sequences induce on the category $\mathcal{A}$ the structure of a (Quillen) exact category, i.e., we equip $\mathcal{A}$ with an exact structure [3, Dfn. 2.1] where the conflations are the pure exact sequences; see Crawley-Boevey [5, 3.1]. We will refer to this exact structure as the pure exact structure on $\mathcal{A}$.

Cotorsion Pairs [12, 22]. Let $\mathcal{X}$ be a class of objects in an exact category $\mathcal{A}$. Put

$$\mathcal{X}^{\perp} := \{A \in \mathcal{A} \,|\, \forall X \in \mathcal{X},\ \mathrm{Ext}^1_{\mathcal{A}}(X, A) = 0\}$$

and define ${}^{\perp}\mathcal{X}$ analogously. A pair $(\mathcal{X}, \mathcal{Y})$ of classes in $\mathcal{A}$ is called a *cotorsion pair* if $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$. A cotorsion pair is said to be *generated by a set*[3] if it is of the form $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ where $\mathcal{S}$ is a set of objects in $\mathcal{A}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if for every object A in $\mathcal{A}$ there exists a short exact sequence $0 \to Y \to X \to A \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and also a short exact sequence $0 \to A \to Y' \to X' \to 0$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$. It is called *hereditary* if $\mathcal{X}$ is closed under kernels of epimorphisms and $\mathcal{Y}$ is closed under cokernels of monomorphisms. Note that being hereditary is equivalent to having for all $X \in \mathcal{X}, Y \in \mathcal{Y}$ and $i \geqslant 1$, $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$.

We recall a fundamental result on cotorsion pairs generated by a set from the work of Eklof and Trlifaj [6]. First a definition.

[3] This terminology is in accordance with Göbel and Trlifaj [12, Dfn. 2.2.1].

Definition 2.1. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a class of objects in $\mathcal{A}$. An object A in $\mathcal{A}$ is called *$\mathcal{S}$-filtered* if there exists a chain of subobjects

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq \bigcup_{\alpha<\sigma} A_\alpha = A,$$

where σ is an ordinal, $A_\lambda = \cup_{\beta<\lambda} A_\beta$ for all limit ordinals λ, and $A_{\alpha+1}/A_\alpha \in \mathcal{S}$ for all $\alpha < \sigma$. The class of $\mathcal{S}$-filtered objects will be denoted by $\mathrm{Filt}(\mathcal{S})$.

Proposition 2.2. ([6], see also [12, 3.2]) *Let $\mathcal{S}$ be a (small) set of objects in a Grothendieck category and assume that $\mathcal{S}$ contains a generator. Then the following hold:*

(i) The cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is complete.

(ii) The class ${}^\perp(\mathcal{S}^\perp)$ consists of direct summands of $\mathcal{S}$-filtered objects, that is, for all $X \in {}^\perp(\mathcal{S}^\perp)$ we have $P \cong X \oplus K$ where $P \in \mathrm{Filt}(S)$. Moreover, in this decomposition K can be chosen in ${}^\perp(\mathcal{S}^\perp) \cap \mathcal{S}^\perp$.

FP-Injectives [26]. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. The objects in the class

$$\mathrm{fp}(\mathcal{A})^\perp = \{A \in \mathcal{A} \,|\, \forall F \in \mathrm{fp}(\mathcal{A}),\ \mathrm{Ext}^1_{\mathcal{A}}(F, A) = 0\}$$

are called *FP-injective objects*. We will denote this class by $\mathrm{FPI}(\mathcal{A})$. Note that Prop. 2.2, applied on the cotorsion pair $({}^\perp\,\mathrm{FPI}(\mathcal{A}), \mathrm{FPI}(\mathcal{A}))$, tells us that the class ${}^\perp\,\mathrm{FPI}(\mathcal{A})$ consists of direct summands of $\mathrm{fp}(\mathcal{A})$-filtered modules.

We recall the following well known facts for the class of FP-injectives.

Proposition 2.3. ([26], see also [28, App. B]) *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following hold:*

(i) The class $\mathrm{FPI}(\mathcal{A})$ is closed under extensions, direct unions, products, coproducts, and pure subobjects.

(ii) An object $A \in \mathcal{A}$ belongs to $\mathrm{FPI}(\mathcal{A})$ if and only if any monomorphism with source A is pure.

Moreover, from [26, 3.2] (also [28, B.3]) the category $\mathcal{A}$ is locally coherent if and only if the class $\mathrm{FPI}(\mathcal{A})$ is closed under filtered colimits if and only if the class $\mathrm{FPI}(\mathcal{A})$ is closed under cokernels of monomorphisms.

The derived category of an exact category [19]. Let $\mathcal{E}$ be a (Quillen) exact category and denote by $\mathbf{C}(\mathcal{E})$ the corresponding category of chain complexes. $\mathbf{C}(\mathcal{E})$ has a canonical exact structure in which a diagram $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathbf{C}(\mathcal{E})$ is a conflation if and only if $X_n \rightarrowtail Y_n \twoheadrightarrow Z_n$ is a conflation in $\mathcal{E}$ for every $n \in \mathbb{Z}$; see Bühler [3, Lem. 9.1]. We refer to this exact structure as the *induced exact structure* on $\mathbf{C}(\mathcal{E})$.

A complex $X = \cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$ in $\mathbf{C}(\mathcal{E})$ is called *acyclic* (with respect to the exact structure of $\mathcal{E}$) if each map d_n decomposes in $\mathcal{E}$ as a deflation $X_n \twoheadrightarrow Z_{n-1}(X)$ followed by an inflation $Z_{n-1}(X) \rightarrowtail X_{n-1}$ and such that the induced sequence $Z_n(X) \rightarrowtail X_n \twoheadrightarrow Z_{n-1}(X)$ is a conflation in $\mathcal{E}$. Denote by $\mathbf{K}_{\mathrm{ac}}(\mathcal{E})$ the homotopy category of acyclic complexes. If the exact category $\mathcal{E}$ has split idempotents, then $\mathbf{K}_{\mathrm{ac}}(\mathcal{E})$ is a thick (épaisse) subcategory of $\mathbf{K}(\mathcal{E})$ [19, 1.2] and then by definition [19, 1.5] the *derived category* of $\mathcal{E}$ is the Verdier quotient $\mathbf{D}(\mathcal{E}) := \mathbf{K}(\mathcal{E})/\mathbf{K}_{\mathrm{ac}}(\mathcal{E})$.

3. On the homotopy category of FP-injectives

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category, viewed as an exact category with the pure exact structure, as in section 2. Then the acyclic complexes in $\mathcal{A}$ (with respect to this exact structure) are called *pure acyclic complexes* and are denoted by $\mathbf{C}_{\mathrm{pac}}(\mathcal{A})$. Moreover, the subcategory of FP-injective objects in $\mathcal{A}$ is closed under extensions, therefore it is an exact category. It is easy to see that the acyclic complexes in $\mathbf{C}(\mathrm{FPI}(\mathcal{A}))$ (with respect to the exact category $\mathrm{FPI}(\mathcal{A})$) are the acyclic complexes (in the usual sense) with cycles in $\mathrm{FPI}(\mathcal{A})$. Equivalently, since the class $\mathrm{FPI}(\mathcal{A})$ is closed under pure subobjects (by 2.3), they are the pure acyclic complexes with components FP-injectives. Thus we will denote them by $\mathbf{C}_{\mathrm{pac}}(\mathrm{FPI}(\mathcal{A}))$. Then by definition we have

$$\mathbf{D}(\mathrm{FPI}(\mathcal{A})) := \mathbf{K}(\mathrm{FPI}(\mathcal{A}))/\mathbf{K}_{\mathrm{pac}}(\mathrm{FPI}(\mathcal{A})).$$

In Theorem 3.5, we identify $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$ with a certain homotopy category. The key ingredient is to extend a characterization of the pure acyclic complexes which is due to Emmanouil. In [**7**, Thm. 3.6] Emmanouil proves that a complex X is pure acyclic if and only if any chain map from a complex of pure projectives to X is null homotopic. Emmanouil's proof is self-contained, while Simson [**25**] and also Šťovíček [**28**] give a functorial proof of this result by reducing it to Neeman's [**21**, Thm. 8.6].

We first recall a useful and well known lemma we will need.

Lemma 3.1. *Let $\mathcal{A}$ be an exact category and consider $\mathbf{C}(\mathcal{A})$ with the induced exact structure (as in section 2). If $X, Y \in \mathbf{C}(\mathcal{A})$, denote by $\mathrm{Ext}^1_{\mathbf{C}(\mathcal{A})}(X,Y)$ the abelian group of (Yoneda) extensions with respect to the induced exact structure, and by $\mathrm{Ext}^1_{\mathrm{dw}(\mathcal{A})}(X,Y)$ the subgroup consisting of extensions $Y \rightarrowtail T \twoheadrightarrow X$ which are degree-wise split. Then we have natural isomorphisms*

$$\mathrm{Ext}^1_{\mathrm{dw}(\mathcal{A})}(X, \Sigma^{-(n+1)}Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, \Sigma^{-n}Y) \cong \mathrm{H}_n \,\mathrm{Hom}_{\mathcal{A}}(X,Y),$$

where $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ denotes the Hom-complex.

Proof. For the first isomorphism, see for instance [**11**, Cor. 5.5 (4)]. The second follows from a direct computation of the homology of the Hom-complex. □

Proposition 3.2. (Compare with [**7**]) *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and let X be a chain complex in $\mathcal{A}$. Then the following are equivalent:*

(i) X is a pure acyclic complex.

(ii) Any chain map from a complex in $\mathbf{C}(\mathrm{PProj}(\mathcal{A}))$ to X is null-homotopic.

(iii) Any chain map from a complex in $\mathbf{C}({}^{\perp}\mathrm{FPI}(\mathcal{A}))$ to X is null-homotopic.

In particular, any pure acyclic complex with components in ${}^{\perp}\mathrm{FPI}(\mathcal{A})$ is contractible.

Proof. As we discussed above the assertions $(i) \Leftrightarrow (ii)$ have been proved in [**7**]. Moreover, $(iii) \Rightarrow (ii)$ is trivial, thus we are left with $(ii) \Rightarrow (iii)$. First consider the case where we are given a chain map $Y \to X$, with Y having components in $\mathrm{Filt}(\mathrm{fp}(\mathcal{A}))$. From the fact that each component of Y is $\mathrm{fp}(\mathcal{A})$-filtered, a result of Šťovíček [**27**, Prop. 4.3] implies that Y itself is $\mathbf{C}^-(\mathrm{fp}(\mathcal{A}))$-filtered, that is, Y is given as a continuous chain of subcomplexes

$$0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq \bigcup_{\alpha<\sigma} Y_\alpha = Y,$$

where σ is an ordinal, $Y_\lambda = \cup_{\beta<\lambda} Y_\beta$ for all limit ordinals λ, and for all $\alpha < \sigma$ the quotient $Y_{\alpha+1}/Y_\alpha$ is a bounded below complex with components finitely presented objects. Now, denote by $\mathbf{C}(\mathcal{A})_{\text{pure}}$ the exact category of chain complexes with the induced pure exact structure (as in section 2). For all ordinals $\alpha < \sigma$ we have

$$\begin{aligned}\operatorname{Ext}^1_{\mathbf{C}(\mathcal{A})_{\text{pure}}}(Y_{\alpha+1}/Y_\alpha, X) &= \operatorname{Ext}^1_{\text{dw}(\mathcal{A})}(Y_{\alpha+1}/Y_\alpha, X)\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(Y_{\alpha+1}/Y_\alpha, \Sigma^1 X)\\ &= 0,\end{aligned}$$

where the first equality holds because each degreewise pure extension of a complex with pure projective components is degreewise split exact, the isomorphism is obtained by Lemma 3.1 and the last equality follows by assumption. Hence Eklof's lemma [**6**, Lemma 1], in its version for exact categories [**23**, Prop. 2.12], gives the result. Now consider the case where Y has components in ${}^\perp\operatorname{FPI}(\mathcal{A})$. Then from 2.2 we know that for all $n \in \mathbb{Z}$ there exists J_n such that $Y_n \oplus J_n \cong F_n$, where F_n is $\operatorname{fp}(\mathcal{A})$-filtered. Consider for each n the disc complex $D_n(J_n) = 0 \to J_n = J_n \to 0$, which is concentrated in homological degrees n and $n-1$. Then the complex

$$Y' := Y \oplus \left(\bigoplus_{n\in\mathbb{Z}} D_n(J_n)\right) \oplus \Sigma^{-1} Y$$

has components of the form $F_n \oplus F_{n+1}$, and these are $\operatorname{fp}(\mathcal{A})$-filtered. Then from the previous treated case we have that $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(Y', X) = 0$, thus $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(Y, X) = 0$ too. Finally, by what we have proved, if X is a pure acyclic complex with components in the class ${}^\perp\operatorname{FPI}(\mathcal{A})$, then the identity map on X is null homotopic, in other words X is contractible. □

As a corollary we obtain a result on pure periodicity which extends the following fact: if M is a module fitting into a pure exact sequence $0 \to M \to P \to M \to 0$ with P pure projective, then M is pure projective as well. In other words, every $\operatorname{PProj}(\mathcal{A})$–pure periodic module is pure projective. This result was first proved by Simson [**24**, Thm. 1.3] and recently by Emmanouil [**7**, Cor. 3.8]. We point out that in [**1**] the authors provide a proof of this result and also a proof of the dual statement.

Our version below extends the case of $\operatorname{PProj}(\mathcal{A})$–pure periodicity to the case of ${}^\perp\operatorname{FPI}(\mathcal{A})$–pure periodicity.

Corollary 3.3. (Compare with [**7**, **24**]) *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and let M be an object in $\mathcal{A}$ admitting a pure short exact sequence of the form*

$$0 \to M \to F \to M \to 0,$$

where $F \in {}^\perp\operatorname{FPI}(\mathcal{A})$. Then $M \in {}^\perp\operatorname{FPI}(\mathcal{A})$. In other words, any ${}^\perp\operatorname{FPI}(\mathcal{A})$–pure periodic object belongs to the class ${}^\perp\operatorname{FPI}(\mathcal{A})$.

Proof. The argument is identical as in [**7**, Cor. 3.8], but invoking 3.2. Namely, we may splice copies of the given short exact sequence to obtain a pure acyclic complex with components in ${}^\perp\operatorname{FPI}(\mathcal{A})$, thus a contractible complex. Hence M is a summand of $F \in {}^\perp\operatorname{FPI}(\mathcal{A})$ and the assertion follows since the class ${}^\perp\operatorname{FPI}(\mathcal{A})$ is closed under summands. □

We now relate Proposition 3.2 with the theory of cotorsion pairs.

Lemma 3.4. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and let $\mathcal{C} := \mathbf{C}(\operatorname{FPI}(\mathcal{A}) \cap {}^{\perp}\operatorname{FPI}(\mathcal{A}))$ and $\mathcal{W} := \mathbf{C}_{\mathrm{pac}}(\operatorname{FPI}(\mathcal{A}))$. Then the following hold.*

(i) $(\mathbf{C}({}^{\perp}\operatorname{FPI}(\mathcal{A})), \mathcal{W})$ *is a cotorsion pair in* $\mathbf{C}(\mathcal{A})$.

(ii) If $\mathcal{A}$ is locally coherent, then the cotorsion pair of (i) is complete. Moreover, in this case the pair $(\mathcal{C}, \mathcal{W})$ is a complete and hereditary cotorsion pair in $\mathbf{C}(\operatorname{FPI}(\mathcal{A}))$.

Proof. (i) Recall that by definition $({}^{\perp}\operatorname{FPI}(\mathcal{A}), \operatorname{FPI}(\mathcal{A}))$ is a cotorsion pair which is generated by a set, therefore by 2.2 it is complete. Thus, from work of Gillespie [**9**, Prop. 3.6], there exists an induced cotorsion pair $({}^{\perp}\mathcal{W}, \mathcal{W})$ in the abelian category $\mathbf{C}(\mathcal{A})$ where the class ${}^{\perp}\mathcal{W}$ can be identified with

$${}^{\perp}\mathcal{W} = \{X \in \mathbf{C}({}^{\perp}\operatorname{FPI}(\mathcal{A})) \mid \forall W \in \mathcal{W},\ \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, W) = 0\}.$$

Since every complex in $\mathcal{W}$ is pure acyclic, 3.2 implies that ${}^{\perp}\mathcal{W} = \mathbf{C}({}^{\perp}\operatorname{FPI}(\mathcal{A}))$, which proves the claim.

(ii) Assume that $\mathcal{A}$ is locally coherent. In this case, from 2.3, we obtain that the complete cotorsion pair $({}^{\perp}\operatorname{FPI}(\mathcal{A}), \operatorname{FPI}(\mathcal{A}))$ is also hereditary. Thus, [**10**, Cor. 3.7] implies that the cotorsion pair of (i) is complete.

Now, we prove that $(\mathcal{C}, \mathcal{W})$ is a cotorsion pair in $\mathbf{C}(\operatorname{FPI}(\mathcal{A}))$. Let $C \in \mathcal{C}$ and $W \in \mathcal{W}$. Invoking Lemma 3.1 we have

$$\operatorname{Ext}^1_{\mathbf{C}(\operatorname{FPI}(\mathcal{A}))}(C, W) = \operatorname{Ext}^1_{\mathrm{dw}(\mathcal{A})}(C, W) \cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(C, \Sigma^1 W).$$

Since W is pure acyclic, from part (i) we obtain ${}^{\perp}\mathcal{W} = \mathcal{C}$ and $\mathcal{W} \subseteq \mathcal{C}^{\perp}$.

To prove the inclusion $\mathcal{C}^{\perp} \subseteq \mathcal{W}$, let $X \in \mathbf{C}(\operatorname{FPI}(\mathcal{A}))$ be such that, for all $C \in \mathcal{C}$, $\operatorname{Ext}^1_{\mathbf{C}(\operatorname{FPI}(\mathcal{A}))}(C, X) = 0$. We need to show that $X \in \mathcal{W}$. Since the cotorsion pair $(\mathbf{C}({}^{\perp}\operatorname{FPI}(\mathcal{A})), \mathcal{W})$ is complete, there exists a short exact sequence $X \rightarrowtail W \twoheadrightarrow C$ with $W \in \mathcal{W}$ and $C \in \mathbf{C}({}^{\perp}\operatorname{FPI}(\mathcal{A}))$. Since $\mathcal{A}$ is locally coherent, from 2.3 we obtain that the complex C has components in $\operatorname{FPI}(\mathcal{A})$. By the assumption on X this short exact sequence splits, therefore the fact that $\mathcal{W}$ is closed under direct summands implies that $X \in \mathcal{W}$.

Completeness of the cotorsion pair $(\mathcal{C}, \mathcal{W})$ in $\mathbf{C}(\operatorname{FPI}(\mathcal{A}))$ follows easily from the completeness of the cotorsion pair in (i). We show that $(\mathcal{C}, \mathcal{W})$ is hereditary. The category $\mathcal{C}$, as a subcategory of $\mathbf{C}(\operatorname{FPI}(\mathcal{A}))$, is easily seen to be closed under kernels of epimorphisms. To see that the class $\mathcal{W}$ is closed under cokernels of monomorphisms let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathbf{C}(\operatorname{FPI}(\mathcal{A}))$ with $A, B \in \mathcal{W}$. Then C is an exact complex and using the fact that in the coherent case $\operatorname{FPI}(\mathcal{A})$ is closed under cokernels of monomorphisms 2.3, we obtain that C has cycles in $\operatorname{FPI}(\mathcal{A})$, thus $C \in \mathcal{W}$. □

Recall that if $\mathcal{T}$ is a triangulated category with set-indexed coproducts, an object $S \in \mathcal{T}$ is called *compact* if for any family $\{X_i\}_{i \in I}$ of objects in $\mathcal{T}$, the natural map $\coprod_{i \in I} \operatorname{Hom}_{\mathcal{T}}(S, X_i) \to \operatorname{Hom}_{\mathcal{T}}(S, \coprod_{i \in I} X_i)$ is an isomorphism. $\mathcal{T}$ is called *compactly generated* if there exists a set $\mathcal{S}$ of compact objects in $\mathcal{T}$, such that for any non-zero $T \in \mathcal{T}$ there exists a non-zero morphism $S \to T$ for some $S \in \mathcal{S}$.

Theorem 3.5. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the homotopy category $\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A}))$ is compactly generated. Moreover, if $\mathcal{A}$ is locally coherent, the composite functor*

$$\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A})) \rightarrowtail \mathbf{K}(\mathrm{FPI}(\mathcal{A})) \xrightarrow{\text{can}} \mathbf{D}(\mathrm{FPI}(\mathcal{A}))$$

is an equivalence of triangulated categories.

Proof. Put $\mathcal{C} := \mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A})$. We will make use of [**14**, Thm. 3.1], which asserts that for any class of objects $\mathcal{C}$ which is closed under (set indexed) coproducts and direct summands, the homotopy category $\mathbf{K}(\mathcal{C})$ is compactly generated provided the following hold:

(i) Every finitely presented object A has a *right $\mathcal{C}$-resolution* [**8**, Dfn. 8.1.2], which by definition means that there exists a sequence $0 \to A \to C_0 \to C_1 \to \cdots$ with $C_i \in \mathcal{C}$ which is exact after applying functors of the form $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})$.

(ii) Every pure exact sequence consisting of objects in $\mathcal{C}$ is split exact.

Their result is stated for modules over associative rings, but in fact their proof only uses properties shared by locally finitely presented Grothendieck categories.

To check condition (i), recall that the cotorsion pair $({}^{\perp}\mathrm{FPI}(\mathcal{A}), \mathrm{FPI}(\mathcal{A}))$ is complete, therefore for any $A \in \mathrm{fp}(\mathcal{A})$, we can construct an exact sequence

$$C(A) := 0 \to A \xrightarrow{\partial^{-1}} C_0 \xrightarrow{\partial^0} C_1 \xrightarrow{\partial^1} C_2 \to \cdots,$$

where ∂^{-1} is a (special) FP-injective preenvelope of A with cokernel $\epsilon_0 \colon C_0 \twoheadrightarrow Z_0 \in {}^{\perp}\mathrm{FPI}(\mathcal{A})$, $\partial^0 = d_0 \circ \epsilon_0$ where d_0 is an FP-injective envelope of Z_0 with cokernel $C_1 \twoheadrightarrow Z_1 \in {}^{\perp}\mathrm{FPI}(\mathcal{A})$ etc. Since $\mathrm{fp}(\mathcal{A})$ is contained in ${}^{\perp}\mathrm{FPI}(\mathcal{A})$ and the latter class is closed under extensions, we deduce that for all $i = 0, 1, \ldots$; $C_i \in {}^{\perp}\mathrm{FPI}(\mathcal{A})$. The sequence constructed has all the C_i's in $\mathcal{C}$ and clearly is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})$–exact, thus it is a right $\mathcal{C}$–resolution of A.

To check condition (ii), let $C := \cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$ be a pure exact sequence consisting of objects in $\mathcal{C}$. In particular, C is a pure acyclic complex with components in ${}^{\perp}\mathrm{FPI}(\mathcal{A})$, hence by 3.2 it is contractible. Thus employing [**14**, Thm. 3.1] we obtain that $\mathbf{K}(\mathcal{C})$ is compactly generated by the set $\{\Sigma^i C(A) \mid A \in \mathrm{fp}(\mathcal{A}),\ i \in \mathbb{Z}\}$.

We now assume that $\mathcal{A}$ is locally coherent. Since $\mathbf{K}(\mathcal{C})$ is compactly generated and the inclusion $j_! \colon \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathrm{FPI}(\mathcal{A}))$ preserves coproducts (which exist because $\mathrm{FPI}(\mathcal{A})$ is closed under coproducts), by Neeman's Brown representability theorem [**20**, Thm. 4.1], the functor $j_!$ admits a right adjoint $j^* \colon \mathbf{K}(\mathrm{FPI}(\mathcal{A})) \to \mathbf{K}(\mathcal{C})$. The kernel of this right adjoint is $\ker(j^*) = \{Y \mid \forall X \in \mathbf{K}(\mathcal{C}),\ \mathrm{Hom}_{\mathbf{K}(\mathrm{FPI}(\mathcal{A}))}(X, Y) = 0\}$, which by 3.4 (ii) is precisely the category $\mathbf{K}_{\mathrm{pac}}(\mathrm{FPI}(\mathcal{A}))$.

Therefore, well known arguments (see for instance [**21**, Remark 2.12]) imply that the composite $\mathbf{K}(\mathcal{C}) \xrightarrow{j_!} \mathbf{K}(\mathrm{FPI}(\mathcal{A})) \xrightarrow{\text{can}} \mathbf{D}(\mathrm{FPI}(\mathcal{A}))$ is an equivalence of triangulated categories and that the canonical map $\mathbf{K}(\mathrm{FPI}(\mathcal{A})) \to \mathbf{D}(\mathrm{FPI}(\mathcal{A}))$ is equivalent (up to natural isomorphism) with j^*. □

Remark 3.6. For any locally finitely presented Grothendieck category $\mathcal{A}$, Krause in [**17**, Example 7] shows the existence of a left adjoint of the canonical map $\mathbf{K}(\mathrm{FPI}(\mathcal{A})) \to \mathbf{D}(\mathrm{FPI}(\mathcal{A}))$. In the proof of 3.5 above, we obtain such a left adjoint after restricting ourselves to the case where $\mathcal{A}$ is locally coherent, and we identify its essential image with $\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\mathrm{FPI}(\mathcal{A}))$.

Before closing this note, we mention that our theorem 3.5 has an interpretation in the language of (Quillen) model categories. By the work of Hovey [**15**] (resp., Gillespie [**11**]) we know that certain cotorsion pairs on an abelian (resp., exact) category $\mathcal{A}$ correspond bijectively to the so-called abelian (resp., exact) model structures on the category $\mathcal{A}$. If $\mathcal{A}$ is a locally coherent Grothendieck category, it is not hard to see that the cotorsion pair on the category $\mathbf{C}(\mathrm{FPI}(\mathcal{A}))$ we obtained in 3.4, corresponds (via the aforementioned Hovey–Gillespie theory) to an exact model structure on the category $\mathbf{C}(\mathrm{FPI}(\mathcal{A}))$ with Quillen homotopy category $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$. The precise statement is as follows.

Theorem 3.7. *Let $\mathcal{A}$ be a locally coherent Grothendieck category and let $\mathbf{C}(\mathrm{FPI}(\mathcal{A}))$ denote the category of chain complexes with components FP-injective objects. Then there exists an (exact) model structure on $\mathbf{C}(\mathrm{FPI}(\mathcal{A}))$, where*

– *the cofibrant objects are the chain complexes in $\mathbf{C}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A}))$,*

– *every chain complex in $\mathbf{C}(\mathrm{FPI}(\mathcal{A}))$ is fibrant,*

– *the trivial objects are the pure acyclic complexes with FP-injective components.*

The homotopy category of this model structure is equivalent to $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$.

Remark 3.8. Let $\mathcal{A}$ be a locally coherent Grothendieck category. Šťovíček in [**28**, Thm. 6.12] shows the existence of a model structure with Quillen homotopy category $\mathbf{D}(\mathrm{FPI}(\mathcal{A}))$ and also proves an equivalence $\mathbf{D}(\mathrm{FPI}(\mathcal{A})) \cong \mathbf{K}(\mathrm{Inj}(\mathcal{A}))$. In 3.7 we identify the cofibrant objects of this model structure with the category $\mathbf{C}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A}))$. Thus, combining 3.7 with [**28**, Thm. 6.12] we obtain equivalences

$$\mathbf{K}(\mathrm{FPI}(\mathcal{A}) \cap {}^{\perp}\,\mathrm{FPI}(\mathcal{A})) \cong \mathbf{D}(\mathrm{FPI}(\mathcal{A})) \cong \mathbf{K}(\mathrm{Inj}(\mathcal{A})).$$

From this and our theorem 3.5 we obtain that the homotopy category of injective objects $\mathbf{K}(\mathrm{Inj}(\mathcal{A}))$ is compactly generated, which is one of the main results in [**28**].

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References

[1] S. Bazzoni, M. Cortés Izurdiaga, and S. Estrada, *Periodic modules and acyclic complexes*, 2017. https://arxiv.org/abs/1704.06672.

[2] S. Breitsprecher, *Lokal endlich präsentierbare Grothendieck-Kategorien*, Mitt. Math., Semin. Giessen **85** (1970), 1–25.

[3] T. Bühler, *Exact categories*, Expo. Math. **28** (2010), no. 1, 1–69.

[4] P.M. Cohn, *On the free product of associative rings*, Math. Z. **71** (1959), 380–398.

[5] W. Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), no. 5, 1641–1674.

[6] P.C. Eklof and J. Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), no. 1, 41–51.

[7] I. Emmanouil, *On pure acyclic complexes*, J. Algebra **465** (2016), 190–213.

[8] E.E. Enochs and O.M.G. Jenda, *Relative Homological Algebra*, de Gruyter Exp. Math., vol. 30, Walter de Gruyter & Co., Berlin, 2000.

[9] J. Gillespie, *The flat model structure on* $\mathrm{Ch}(R)$, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390.

[10] ———, *The flat model structure on complexes of sheaves*, Trans. Amer. Math. Soc. **358** (2006), no. 7, 2855–2874.

[11] ———, *Model structures on exact categories*, J. Pure Appl. Algebra **215** (2011), no. 12, 2892–2902.

[12] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, de Gruyter Exp. Math., vol. 41, Walter de Gruyter & Co., Berlin, 2006.

[13] L. Gruson and C.U. Jensen, *Dimensions cohomologiques reliées aux foncteurs* $\varprojlim^{(i)}$, Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), 1981, pp. 234–294.

[14] H. Holm and P. Jørgensen, *Compactly generated homotopy categories*, Homol. Homotopy Appl. **9** (2007), no. 1, 257–274.

[15] M. Hovey, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002), no. 3, 553–592.

[16] S. Iyengar and H. Krause, *Acyclicity versus total acyclicity for complexes over Noetherian rings*, Doc. Math. **11** (2006), 207–240.

[17] H. Krause, *Approximations and adjoints in homotopy categories*, Math. Ann. **353** (2012), no. 3, 765–781.

[18] D. Murfet and S. Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. **226** (2011), no. 2, 1096–1133.

[19] A. Neeman, *The derived category of an exact category*, J. Algebra **135** (1990), no. 2, 388–394.

[20] ———, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

[21] ———, *The homotopy category of flat modules, and Grothendieck duality*, Invent. Math. **174** (2008), no. 2, 255–308.

[22] L. Salce, *Cotorsion theories for abelian groups*, Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977), 1979, pp. 11–32.

[23] M. Saorín and J. Šťovíček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. **228** (2011), no. 2, 968–1007.

[24] D. Simson, *Pure-periodic modules and a structure of pure-projective resolutions*, Pacific J. Math. **207** (2002), no. 1, 235–256.

[25] ———, *Flat complexes, pure periodicity and pure acyclic complexes*, J. Algebra **480** (2017), 298–308.

[26] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329.

[27] J. Šťovíček, *Deconstructibility and the Hill lemma in Grothendieck categories*, Forum Math. **25** (2013), no. 1, 193–219.

[28] ______, *On purity and applications to coderived and singularity categories*, 2014. https://arxiv.org/abs/1412.1615.

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ABELIAN MODEL STRUCTURES ON CATEGORIES OF QUIVER REPRESENTATIONS

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Abstract. Let $\mathcal{M}$ be an abelian model category (in the sense of Hovey). For a large class of quivers, we describe associated abelian model structures on categories of quiver representations with values in $\mathcal{M}$. This is based on recent work of Holm and Jørgensen on cotorsion pairs in categories of quiver representations. An application on Ding projective and Ding injective representations of quivers over Ding-Chen rings is given.

1. Introduction

Model structures on abelian categories have been studied extensively by Hovey [23], who introduced the general notion of an abelian model structure on an abelian category $\mathcal{M}$, and gave a correspondence between such models and certain cotorsion pairs in $\mathcal{M}$. A cotorsion pair in an abelian category $\mathcal{M}$ is a pair of $\operatorname{Ext}^1_{\mathcal{M}}(-,-)$–orthogonal to each other subcategories. The basic idea behind Hovey's results is that, for an abelian model category $\mathcal{M}$, the various lifting properties in the model $\mathcal{M}$ can be interpreted as certain $\operatorname{Ext}^1_{\mathcal{M}}(-,-)$–orthogonality relations. Thus in order to give a model structure on an abelian category, it suffices to find certain cotorsion pairs and then use the correspondence of Hovey.

Given an abelian model structure on an abelian category $\mathcal{M}$ and a quiver (a directed graph) Q, we consider the category of quiver representations $\operatorname{Rep}_Q \mathcal{M}$, that is, diagrams of shape Q in $\mathcal{M}$, and study how the given model on $\mathcal{M}$ transfers to the abelian category $\operatorname{Rep}_Q \mathcal{M}$. Representations of quivers in module categories are of interest in the representation theory of finite dimensional algebras [4]. Moreover, derived categories of the category $\operatorname{Rep}_Q \mathcal{M}$ are usually thought of as enhancements of the derived category of $\mathcal{M}$ and have recently attracted much attention, see for instance [3, 18].

In general, for a given model category $\mathcal{M}$ and a small category I, a model on the functor category $\mathcal{M}^I$ might exist or not, depending on conditions on either $\mathcal{M}$ or I, see [19, Chapters 11, 15]. In Theorems 3.5/3.6 we give a description of certain projective (resp. injective) model structures on categories of quiver representations, based on cotorsion pairs in such categories as obtained by Holm and Jørgensen [21]. The examples we are interested in here are of a ring-theoretic flavour. In 3.8/3.9 we provide examples which realize stable categories of Gorenstein projective (resp. injective) representations of left (resp. right) rooted quivers, over certain rings, as Quillen homotopy categories.

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The last section is concerned with quiver representations over Ding-Chen rings, a generalization of Gorenstein rings studied by Gillespie [13]. In Theorems 4.8/4.9 we provide abelian model structures for Ding projective and Ding injective representations over such rings, which generalize the analogous statements for Gorenstein rings from 3.8/3.9.

2. Preliminaries

In this section we briefly summarize some known facts on cotorsion pairs, abelian model structures and quiver representations.

2.1. Cotorsion pairs. Let $\mathcal{M}$ be an abelian category. For a class $\mathcal{C}$ of objects in $\mathcal{M}$ the right orthogonal $\mathcal{C}^{\perp}$ is defined to be the class of all $M \in \mathcal{M}$ such that $\mathrm{Ext}^1_{\mathcal{M}}(C, M) = 0$ for all $C \in \mathcal{C}$. The left orthogonal ${}^{\perp}\mathcal{C}$ is defined analogously. We say that a pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects in $\mathcal{M}$ is a *cotorsion pair* if $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if for every object M in $\mathcal{M}$ there exists a short exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and also a short exact sequence $0 \to M \to Y' \to X' \to 0$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$. It is called *hereditary* if for all $X \in \mathcal{X}, Y \in \mathcal{Y}$ and $i \geq 1$, $\mathrm{Ext}^i_{\mathcal{M}}(X, Y) = 0$. We refer to [17] for the theory of cotorsion pairs. Following the terminology of [17, Dfn. 2.2.1], a cotorsion pair is said to be *generated*, respectively *cogenerated*, by a set of objects $\mathcal{S}$, if it is of the form $({}^{\perp}(\mathcal{S})^{\perp}, \mathcal{S}^{\perp})$, respectively $({}^{\perp}\mathcal{S}, ({}^{\perp}\mathcal{S})^{\perp})$.

2.2. Abelian model structures. Let $\mathcal{M}$ be an abelian category. Following Hovey [23, Dfn. 2.1] we say that $\mathcal{M}$ admits an *abelian model structure* (or that $\mathcal{M}$ is an abelian model category), if it admits a Quillen model structure [22] where the (trivial) cofibrations are the monomorphisms with (trivially) cofibrant kernel, and the (trivial) fibrations are the epimorphisms with (trivially) fibrant kernel. If we denote by $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$ the classes of cofibrant, fibrant and trivial (i.e. weakly isomorphic to zero) objects in this model category, we obtain from [23, Thm. 2.2] two (functorially) complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ in the category $\mathcal{M}$. Conversely, for classes of objects $\mathcal{C}, \mathcal{W}$ and $\mathcal{F}$, where $\mathcal{W}$ is thick, any two complete cotorsion pairs of the above form give rise to an abelian model structure on $\mathcal{M}$, see again [23, Thm. 2.2]. We abbreviate by saying that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a *Hovey triple* on the category $\mathcal{M}$.

2.3. A Hovey triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on an abelian category $\mathcal{M}$ is called *hereditary* if the corresponding complete cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are hereditary. In this case, the category $\mathcal{C} \cap \mathcal{F}$ is Frobenius [14, Prop. 5.2.(4)] (where $\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}$ is the class projective–injective objects) and the homotopy category of the model category $\mathcal{M}$ (in the classical sense) is canonically equivalent to the stable category of the Frobenius category $\mathcal{C} \cap \mathcal{F}$, we refer to [14, Section 4.2] for the details.

2.4. Setup. Throughout the text $\mathcal{M}$ denotes an abelian category with enough projectives and injectives which satisfies the axioms AB4 and AB4*, that is, $\mathcal{M}$ is bicomplete and such that any coproduct of monomorphisms in $\mathcal{M}$ is a monomorphism, and dually any product of epimorphisms in $\mathcal{M}$ is an epimorphism.

2.5. Quivers. We recall that a quiver $Q = (Q_0, Q_1)$ is a directed graph Q with set of vertices Q_0 and set of arrows Q_1. For $\alpha \in Q_1$ we denote by $s(\alpha)$ its source and by $t(\alpha)$ its target. If Q is a quiver and $\mathcal{X}$ is a class of objects in $\mathcal{M}$, then viewing Q as a small category, we consider the category $\mathrm{Rep}_Q\, \mathcal{X}$ of diagrams of shape Q in

$\mathcal{X}$. The objects of $\operatorname{Rep}_Q \mathcal{X}$ are also called $\mathcal{X}$-valued representations of Q. For any such representation X and any vertex $i \in Q_0$, there exist two natural maps

$$\bigoplus_{\alpha:j\to i} X(j) \xrightarrow{\phi_i^X} X(i) \qquad \text{and} \qquad X(i) \xrightarrow{\psi_i^X} \prod_{\alpha:i\to j} X(j).$$

For a quiver Q, consider, as in [9, Section 4], a sequence of subsets of Q_0 defined by transfinite recursion as follows: Put $W_0 := \emptyset$, for a successor $\alpha = \beta + 1$, put

$$W_\alpha := \{i \in Q_0 \mid i \text{ is not the source of any arrow that has target outside of } W_\beta\}$$

and for a limit ordinal α put $W_\alpha := \cup_{\beta<\alpha} W_\beta$.

A quiver is called *right rooted* if for some ordinal λ we have $W_\lambda = Q_0$. From [9, Section 4] we have that Q is right rooted if and only if it does not contain any path of the form $\bullet \to \bullet \to \bullet \to \cdots$. A dual definition and a (dual) characterization holds for *left rooted* quivers, see [8].

2.6. Adjoints of evaluation functors. Let Q be a quiver, $i \in Q_0$ a vertex and let $\mathcal{A}$ be a category that admits finite products and finite coproducts. We recall, for instance from [21, 3.7], that the evaluation at i functor $(-)(i) : \operatorname{Rep}_Q \mathcal{A} \to \mathcal{A}$; $X \mapsto X(i)$, admits a left adjoint f_i and a right adjoint g_i which are defined, on a vertex j, by the rules $f_i(M)(j) := \coprod_{\alpha:i\to j} M$ and $g_i(M)(j) = \prod_{\alpha:j\to i} M$ respectively. We refer to [21, Section 3] for the full definition and properties of these functors.

We start by recalling some of the main results of [21].

Fact 2.7. [21, Thm. A] Let Q be a left rooted quiver and let $\mathcal{M}$ be abelian category as in setup 2.4. If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{M}$, then there is an induced cotorsion pair $(\Phi(\mathcal{A}), \operatorname{Rep}_Q B)$ in the category $\operatorname{Rep}_Q \mathcal{M}$, where

$$\Phi(\mathcal{A}) := \{X \mid \forall i \in Q_0,\ \phi_i^X \text{ is monic with } X(i) \in \mathcal{A},\ \operatorname{coker} \phi_i^X \in \mathcal{A}\}.$$

In addition, if $(\mathcal{A}, \mathcal{B})$ is hereditary or generated by a set, then so is $(\Phi(\mathcal{A}), \operatorname{Rep}_Q \mathcal{B})$.

The dual of this statement is as follows:

Fact 2.8. [21, Thm. B] Let Q be a right rooted quiver and let $\mathcal{M}$ be abelian category as in setup 2.4. If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{M}$, then there is an induced cotorsion pair $(\operatorname{Rep}_Q \mathcal{A}, \Psi(\mathcal{B}))$ in the category $\operatorname{Rep}_Q \mathcal{M}$, where

$$\Psi(\mathcal{B}) := \{X \mid \forall i \in Q_0,\ \psi_i^X \text{ is epic with } X(i) \in \mathcal{B},\ \ker \psi_i^X \in \mathcal{B}\}.$$

In addition, if $(\mathcal{A}, \mathcal{B})$ is hereditary or generated by a set, then so is $(\operatorname{Rep}_Q \mathcal{A}, \Psi(\mathcal{B}))$.

The following is the main result of [26] and addresses the question of when the cotorsion pairs found in 2.7 and 2.8 are complete.

Fact 2.9. [26, Thm. 4.1.3] Let $\mathcal{M}$ be an abelian category as in setup 2.4[1] and let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in $\mathcal{M}$. Then the following hold:

- If Q is a left rooted quiver, then the induced cotorsion pair $(\Phi(\mathcal{A}), \operatorname{Rep}_Q \mathcal{B})$ in $\operatorname{Rep}_Q \mathcal{A}$ (which exists by 2.7) is complete.
- If Q is a right rooted quiver, then the induced cotorsion pair $(\operatorname{Rep}_Q \mathcal{A}, \Psi(\mathcal{B}))$ in $\operatorname{Rep}_Q \mathcal{A}$ (which exists by 2.8) is complete.

[1] In fact, as it follows from [26], even less assumptions might be considered, see [26, 3.11, 3.12].

3. Abelian model structures on categories of quiver representations

Based on the results stated in the previous section, we describe here a general recipe in order to produce abelian model structures on the category $\operatorname{Rep}_Q \mathcal{M}$, where $\mathcal{M}$ is as in the setup 2.4, Q is left rooted and $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a hereditary Hovey triple on the "ground category" $\mathcal{M}$. The associated complete hereditary cotorsion pairs in $\mathcal{M}$ are $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$. Using 2.7 and 2.9 we obtain two hereditary and complete cotorsion pairs in $\operatorname{Rep}_Q \mathcal{M}$,

$$(1) \qquad (\widetilde{\mathcal{Q}} = \Phi(\mathcal{C} \cap \mathcal{W}), \mathcal{R} = \operatorname{Rep}_Q \mathcal{F}) \quad \text{and} \quad (\mathcal{Q} = \Phi(\mathcal{C}), \widetilde{\mathcal{R}} = \operatorname{Rep}_Q(\mathcal{F} \cap \mathcal{W})).$$

We want to check if the above cotorsion pairs induce an abelian model structure on $\operatorname{Rep}_Q \mathcal{M}$. The following result of [15] gives conditions on two complete cotorsion pairs in an abelian category $\mathcal{A}$ in order for them to constitute a Hovey triple.

Fact 3.1. [15, Thm. 1.1] Let $\mathcal{M}$ be an abelian category and assume that $(\widetilde{\mathcal{Q}}, \mathcal{R})$ and $(\mathcal{Q}, \widetilde{\mathcal{R}})$ are two hereditary complete cotorsion pairs on $\mathcal{M}$ such that

(i) $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$ and $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$.
(ii) $\widetilde{\mathcal{R}} \cap \mathcal{Q} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$.

Then there is a unique thick class $\mathcal{T}$ for which $(\mathcal{Q}, \mathcal{T}, \mathcal{R})$ is a Hovey triple. Moreover, this class can be described as follows:

$$\begin{aligned} \mathcal{T} &= \{X \in \mathcal{M} \mid \text{there exists a s.e.s. } X \rightarrowtail R \twoheadrightarrow Q \text{ with } R \in \widetilde{\mathcal{R}},\, Q \in \widetilde{\mathcal{Q}}\} \\ &= \{X \in \mathcal{M} \mid \text{there exists a s.e.s. } R' \rightarrowtail Q' \twoheadrightarrow X \text{ with } R' \in \widetilde{\mathcal{R}},\, Q' \in \widetilde{\mathcal{Q}}\}. \end{aligned}$$

For the cotorsion pairs in (1), the only nontrivial relation is $\widetilde{\mathcal{R}} \cap \mathcal{Q} \subseteq \widetilde{\mathcal{Q}} \cap \mathcal{R}$. If $X \in \widetilde{\mathcal{R}} \cap \mathcal{Q}$, there is a short exact sequence

$$0 \to \bigoplus_{\alpha: j \to i} X(j) \xrightarrow{\phi_i^X} X(i) \to \operatorname{coker} \phi_i^X \to 0$$

where $\operatorname{coker} \phi_i^X \in \mathcal{C}$ and $X(i) \in \mathcal{C} \cap \mathcal{F} \cap \mathcal{W}$. Note that $X \in \mathcal{R} = \operatorname{Rep}_Q \mathcal{F}$ trivially and that $X \in \widetilde{\mathcal{Q}} = \Phi(\mathcal{C} \cap \mathcal{W})$ if and only if $\operatorname{coker} \phi_i^X \in \mathcal{W}$. Since $\mathcal{W}$ is a thick subcategory of $\mathcal{M}$, by the short exact sequence above, we see that $\operatorname{coker} \phi_i^X \in \mathcal{W}$ if $\mathcal{W}$ is closed under (small) coproducts. We point out that this condition will be automatically satisfied for all finite and also many infinite quivers. For example, for quivers Q such that for all $i \in Q_0$, the set $\{s(\alpha) \mid \alpha \in Q_1 \text{ with } t(\alpha) = i\}$ is finite.

The above discussion proves the following:

Proposition 3.2. *Let $\mathcal{M}$ be an abelian category as in setup 2.4 with a hereditary Hovey triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ and let Q be a left rooted quiver. Then if $\mathcal{W}$ is closed under (small) coproducts, there exists an induced hereditary Hovey triple $(\Phi(\mathcal{C}), \mathcal{T}, \operatorname{Rep}_Q \mathcal{F})$ on the category of representations $\operatorname{Rep}_Q \mathcal{M}$, where $\mathcal{T}$ is defined as in 3.1. In particular, for all $i \in Q_0$ and $X \in \mathcal{T}$ we have $X(i) \in \mathcal{W}$.*

Using duals of the above arguments we easily obtain the following:

Proposition 3.3. *Let $\mathcal{M}$ be an abelian category as in setup 2.4 with a hereditary Hovey triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ and let Q be a right rooted quiver. Then if $\mathcal{W}$ is closed under (small) products, there exists an induced hereditary Hovey triple $(\operatorname{Rep}_Q \mathcal{C}, \mathcal{T}, \Psi(\mathcal{F}))$ on the category of representations $\operatorname{Rep}_Q \mathcal{M}$, where $\mathcal{T}$ is defined as in 3.1. In particular, for all $i \in Q_0$ and $X \in \mathcal{T}$ we have $X(i) \in \mathcal{W}$.*

In the model structures 3.2 and 3.3, the class of trivial objects $\mathcal{T}$ is contained in the class of "vertexwise trivial" representations, $\operatorname{Rep}_Q \mathcal{W}$. For computational purposes we are interested in knowing when these two classes coincide. For this we will restrict to more special types of model structures (although still abundant).

A priori one needs two suitable complete cotorsion pairs in an abelian category $\mathcal{M}$ in order to define an abelian model structure on $\mathcal{M}$ (as we recalled in 2.2/2.3). However, it is possible to obtain quite naturally a model structure starting with only one cotorsion pair. We recall the following from [16].

Definition 3.4. [16, Dfn. 3.4] Let $\mathcal{M}$ be an abelian category with enough projectives. A complete cotorsion pair $(\mathcal{C}, \mathcal{W})$ in $\mathcal{M}$ is called *projective* if $\mathcal{C} \cap \mathcal{W} = \operatorname{Proj} \mathcal{M}$ and $\mathcal{W}$ is thick. In this case $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is a Hovey triple on $\mathcal{M}$. Dually, if $\mathcal{M}$ has enough injectives, a complete cotorsion pair $(\mathcal{W}, \mathcal{F})$ in $\mathcal{M}$ is called *injective* if $\mathcal{W} \cap \mathcal{F} = \operatorname{Inj} \mathcal{M}$ and $\mathcal{W}$ is thick. In this case $(\mathcal{M}, \mathcal{W}, \mathcal{F})$ is a Hovey triple on $\mathcal{M}$.

In this paper, we call an abelian model structure projective (resp. injective) if its associated Hovey triple is induced by a projective (resp. injective) cotorsion pair.

The following two results provide us with a large class of projective (resp. injective) model structures for categories of quiver representations.

Theorem 3.5. *Let $\mathcal{M}$ be an abelian category as in setup 2.4 and let Q be a left rooted quiver. Let $(\mathcal{C}, \mathcal{W})$ be a hereditary projective cotorsion pair in $\mathcal{M}$ with $\mathcal{W}$ closed under (small) coproducts. Then there exists a* ***projective model structure*** *on the category of representations $\operatorname{Rep}_Q \mathcal{M}$ with hereditary Hovey triple*

$$(\Phi(\mathcal{C}), \operatorname{Rep}_Q \mathcal{W}, \operatorname{Rep}_Q \mathcal{M}).$$

Proof. By assumption, $(\mathcal{C}, \mathcal{W})$ is a hereditary cotorsion pair, thus from 3.2 we know that $(\Phi(\mathcal{C}), \mathcal{T}, \operatorname{Rep}_Q \mathcal{M})$ is a hereditary Hovey triple on $\mathcal{M}$. In particular $(\Phi(\mathcal{C}), \mathcal{T})$ is a cotorsion pair in $\mathcal{M}$. Moreover, from 2.7 we have that $(\Phi(\mathcal{C}), \operatorname{Rep}_Q \mathcal{W})$ is a cotorsion pair in $\mathcal{M}$. Hence we have $\operatorname{Rep}_Q \mathcal{W} = \mathcal{T}$. The cotorsion pair obtained is projective since

$$\Phi(\mathcal{C}) \cap \mathcal{W} = \Phi(\mathcal{C} \cap \mathcal{W}) = \Phi(\operatorname{Proj} \mathcal{M}) = \operatorname{Proj}(\operatorname{Rep}_Q \mathcal{M}),$$

where the last equality is given by 2.7. □

Dually, we have the following:

Theorem 3.6. *Let $\mathcal{M}$ be an abelian category as in setup 2.4 and let Q be a right rooted quiver. Let $(\mathcal{W}, \mathcal{F})$ be an hereditary injective cotorsion pair in $\mathcal{M}$ with $\mathcal{W}$ closed under (small) products. Then there exists an* ***injective model structure*** *on the category of representations $\operatorname{Rep}_Q \mathcal{M}$ with hereditary Hovey triple*

$$(\operatorname{Rep}_Q \mathcal{M}, \operatorname{Rep}_Q \mathcal{W}, \Psi(\mathcal{F})).$$

Proof. The proof of this follows the same lines as the proof of 3.5, where one instead makes use of Proposition 3.3 in order to obtain a hereditary Hovey triple $(\operatorname{Rep}_Q \mathcal{M}, \mathcal{T}, \Psi(\mathcal{F}))$, and then argues that $\mathcal{T} = \operatorname{Rep}_Q \mathcal{W}$ and that the cotorsion pair $(\mathcal{W}, \Psi(\mathcal{F}))$ is injective. □

Remark 3.7. We should explain how Theorems 3.5 and 3.6 connect with some classical results from the theory of model categories. Given a small category Q and a cofibrantly generated model category $\mathcal{M}$, it is well known that there exists an induced cofibrantly generated model structure on the functor category $\mathcal{M}^Q$, see for instance [19, Thm. 11.6.1]. In the language of this paper, [19, Thm. 11.6.1] when restricted to abelian model structures says the following:

Let Q be a small category and let $\mathcal{M}$ be an abelian model category with Hovey triple $(\mathcal{C},\mathcal{W},\mathcal{F})$, where the associated cotorsion pairs $(\mathcal{C},\mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ are each generated by a set (and so $\mathcal{M}$ is cofibrantly generated by [23, Lemma 6.7]*). Say $(\mathcal{C},\mathcal{W}\cap\mathcal{F}) = ({}^{\perp}(\mathcal{S}^{\perp}),\mathcal{S}^{\perp})$ for a set $\mathcal{S}\subseteq\mathcal{C}$. Then there exists an abelian model structure on the functor category $\mathcal{M}^Q := \operatorname{Rep}_Q\mathcal{M}$ with Hovey triple*

$$(\Sigma,\operatorname{Rep}_Q\mathcal{W},\operatorname{Rep}_Q\mathcal{F}),$$

where the cotorsion pair $(\Sigma,\operatorname{Rep}_Q(\mathcal{W}\cap\mathcal{F}))$ is generated by the set

$$f_*(\mathcal{S}) := \{f_iS \mid i\in Q_0\,, S\in\mathcal{S}\}.$$

We point out that this theorem agrees, for certain cotorsion pairs, with results of Holm and Jørgensen, cf. [21, Thm. 7.4(a)]. Moreover, note that the class Σ is the left hand side of a complete cotorsion pair which is generated by the set $f_*(\mathcal{S})$. Hence it consists of summands of transfinite extensions of objects in $f_*(\mathcal{S})$, [17, 3.2]. For this reason it is not very computable in general. Note that 3.5 identifies this class with $\Phi(\mathcal{C})$ in case the given model on $\mathcal{M}$ is projective and Q is a left rooted quiver.

Next, we provide some examples modelling stable categories of Gorenstein projective and Gorenstein injective representations of quivers. For a definition of these classes, we refer for instance to [20]. If R is a ring, $\operatorname{GProj}(R)$ (resp. $\operatorname{GInj}(R)$) denotes the class of Gorenstein projective (resp. Gorenstein injective) right R-modules.

Example 3.8. Let R be a Noetherian commutative ring with a dualizing complex or a left-coherent and right-noetherian k-algebra (with k a field) admitting a dualizing complex (in the sense of [24, Setup 1.4']). Then from [24, Thm. 1.10] the pair $(\operatorname{GProj}(R),\operatorname{GProj}(R)^{\perp})$ is hereditary projective cotorsion pair[2]. Let Q be a left rooted quiver. Assuming that Q is such that for all $i\in Q_0$ the set $\{s(\alpha)\mid \alpha\in Q_1 \text{ with } t(\alpha)=i\}$ is finite or assuming that $\operatorname{GProj}(R)^{\perp}$ is closed under coproducts[3] from Theorem 3.5 we obtain a hereditary Hovey triple

$$(\Phi(\operatorname{GProj}(R)),\operatorname{Rep}_Q(\operatorname{GProj}(R)^{\perp}),\operatorname{Rep}_Q(R)),$$

on the category of quiver representations of right R-modules, $\operatorname{Rep}_Q(R)$.

From [11, Thm 3.5.1] we have that $\Phi(\operatorname{GProj}(R)) = \operatorname{GProj}(\operatorname{Rep}_Q(R))$, thus the above Hovey triple is

$$(\operatorname{GProj}(\operatorname{Rep}_Q(R),\operatorname{Rep}_Q(\operatorname{GProj}(R)^{\perp}),\operatorname{Rep}_Q(R)).$$

The homotopy category of this model category is

$$\operatorname{Ho}(\operatorname{Rep}_Q(R)) \cong \underline{\operatorname{GProj}}(\operatorname{Rep}_Q(R)),$$

the *stable category of Gorenstein projective representations.*

[2] It is also proved in [12] that the pair $(\operatorname{GProj}(R),\operatorname{GProj}(R)^{\perp})$ is hereditary projective cotorsion pair over right coherent and left n-perfect rings.

[3] This holds for example if R is Iwanaga-Gorenstein, since in this case $\operatorname{GProj}(R)^{\perp}$ is the class of modules of finite projective dimension [20, Thm. 2.20].

Example 3.9. Let R be a right Noetherian ring. Then the pair $({}^{\perp}\mathrm{GInj}(R), \mathrm{GInj}(R))$ is a hereditary injective cotorsion pair [25, 7.3][4]. Assuming that Q is a quiver such that for all $i \in Q_0$ the set $\{t(\alpha) \mid \alpha \in Q_1 \text{ with } s(\alpha) = i\}$ is finite or assuming that ${}^{\perp}\,\mathrm{GInj}(R)$ is closed under products[5], then by Theorem 3.6 we obtain a hereditary Hovey triple

$$(\mathrm{Rep}_Q(R), \mathrm{Rep}_Q({}^{\perp}\mathrm{GInj}(R)), \Psi(\mathrm{GInj}(R))),$$

on the category of quiver representations of right R-modules, $\mathrm{Rep}_Q(R)$.

From [11, Thm 3.5.1] we have that $\Psi(\mathrm{GInj}(R)) = \mathrm{GInj}(\mathrm{Rep}_Q(R))$, thus the above Hovey triple is

$$(\mathrm{Rep}_Q(R), \mathrm{Rep}_Q({}^{\perp}\mathrm{GInj}(R)), \mathrm{GInj}(\mathrm{Rep}_Q(R))).$$

The homotopy category of this model category is

$$\mathrm{Ho}(\mathrm{Rep}_Q(R)) \cong \underline{\mathrm{GInj}}(\mathrm{Rep}_Q(R)),$$

the *stable category of Gorenstein injective representations.*

4. Quiver representations over Ding-Chen rings

The examples 3.8 and 3.9 admit generalizations which are worth mentioning. Gillespie in [13] based on work of Ding and Chen [7] defines Ding-Chen rings as a generalization of Gorenstein rings. A ring is called Ding-Chen if it is left and right coherent with $\mathrm{FPI}-\mathrm{dim}_R R$ and $\mathrm{FPI}-\mathrm{dim} R_R$ both finite[6]. In this case from [6] we necessarily have $\mathrm{FPI}-\mathrm{dim}_R R = n = \mathrm{FPI}-\mathrm{dim} R_R$ for some $n \in \mathbb{N}$. Note that if R is two-sided Noetherian then this definition recovers the Iwanaga-Gorenstein rings.

Gillespie studies in [13] Ding projective, injective and flat modules which stand for generalizations of Gorenstein projective, injective and flat modules respectively.

Remark 4.1. The definition of Ding projective and Ding injective modules over a ring involves the concepts of flat and fp-injective modules respectively [13, Dfn. 3.2/3.7]. Since we are interested in Ding projective and Ding injective representations of quivers, we need to make sense of flatness and fp-injectivity in a more general context than module categories. The appropriate setup to define such notions is that of a locally finitely presented additive (usually Grothendieck) category, see [1, 5]. In this context an object M is called flat if any epimorphism with target M is pure, and dually, M is called fp-injective if any monomorphism with source M is pure. For a locally finitely presented Grothendieck category $\mathcal{A}$ and a quiver Q, the category of quiver representations $\mathrm{Rep}_Q\,\mathcal{A}$ is again locally finitely presented Grothendieck [1, Cor. 1.54].

Definition 4.2. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. An object M in $\mathcal{A}$ is called *Ding projective* if there exists an exact complex of projective objects in $\mathcal{A}$ which has M as a syzygy and remains exact after applying functors of the form $\mathrm{Hom}_{\mathcal{A}}(-, F)$, for F a flat object in $\mathcal{A}$. We denote the class of Ding projective objects in $\mathcal{A}$ by $\mathrm{DProj}(\mathcal{A})$.

[4]In the recent work [28] the authors prove that $({}^{\perp}\mathrm{GInj}(R), \mathrm{GInj}(R))$ is a hereditary injective cotorsion pair over any ring.

[5]Again, this holds if R is Iwanaga-Gorenstein, since in this case ${}^{\perp}\,\mathrm{GInj}(R)$ is the class of modules of finite injective dimension [20, Thm. 2.22].

[6]Here $\mathrm{FPI}-\mathrm{dim}$ denotes the fp-injective dimension. We recall that an R-module M is called fp-injective if for any finitely presented module F we have $\mathrm{Ext}^1_R(F, M) = 0$, see [27]. These modules define a (relative) homological dimension, see [10, Ch. 8].

Definition 4.3. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. An object M in $\mathcal{A}$ is called *Ding injective* if there exists an exact complex of injectives in $\mathcal{A}$ which has M as a syzygy and remains exact after applying functors of the form $\mathrm{Hom}_{\mathcal{A}}(F, -)$, for F an fp-injective object in $\mathcal{A}$. We denote the class of Ding injective objects in $\mathcal{A}$ by $\mathrm{DInj}(\mathcal{A})$.

We will make use of the following facts which concern flat and fp-injective representations, i.e. flat and fp-injective objects in the category of quiver representations of right R-modules, $\mathrm{Rep}_Q(R)$.

Fact 4.4. Let R be a ring, Q a quiver and let $X \in \mathrm{Rep}_Q(R)$. Then we have:

(i) [8, 3.4/3.7] If X is a flat representation, then for each vertex $v \in Q_0$, the natural map ϕ_v^X as in 2.5 is a pure monomorphism and $X(v)$ is flat. The converse holds in case Q is left rooted.

(ii) [2, 4.4/4.10] If X is an fp-injective representation, then for each vertex $v \in Q_0$, the natural map ψ_v^X as in 2.5 is a pure epimorphism and $X(v)$ is fp-injective. The converse holds in case Q is right rooted and R is right coherent.

The proofs of the following two results are based on techniques developed in Eshraghi et al. [11], although some modifications are needed. We keep the presentation as concise as possible.

Lemma 4.5. (cf. [11, 3.1.5]) *Let R be a ring, Q a quiver and let $X \in \mathrm{Rep}_Q R$. Then the following hold:*

(i) *Assuming that Q is left rooted, if for all $v \in Q_0$, $X(v)$ is flat, then* $\mathrm{Flat}-\dim(X) \leq 1$.

(ii) *Assuming that R is right coherent and Q is right rooted, if for all $v \in Q_0$, $X(v)$ is fp-injective, then* $\mathrm{FPI}-\dim(X) \leq 1$.

Proof. (ii) Keeping the notation as in 2.6, from [11, 3.1(1)] there exists a short exact sequence

$$0 \to X \to \prod_{v \in Q_0} g_v X(v) \to \prod_{\alpha \in Q_1} g_{s(\alpha)} X(t(\alpha)) \to 0$$

in the category $\mathrm{Rep}_Q(R)$. For the representation $g_v X(v)$ and for all $w \in Q_0$, the natural map $\psi^w_{g_v X(v)}$, is a split epimorphism. Moreover, by assumption, vertexwise the representation $g_v X(v)$ consists of fp-injective modules, thus by 4.4(ii) we obtain that $g_v X(v)$ is an fp-injective representation. Hence the middle term in the above short exact sequence is an fp-injective representation (since the class of fp-injective representations is closed under products [29, App. B]). To prove that the term on the right hand side is fp-injective, in order to simplify the notation, denote $Y := \prod_{v \in Q_0} g_v X(v)$ and $W := \prod_{\alpha \in Q_1} g_{s(\alpha)} X(t(\alpha))$, so the displayed short exact sequence above is $0 \to X \to Y \to W \to 0$.

Consider for each vertex $v \in Q_0$ the commutative diagram of R-modules

$$\begin{tikzcd} Y(v) \arrow[r, two heads] \arrow[d, "\psi_v^Y"'] & W(v) \arrow[d, "\psi_v^W"] \\ \prod\limits_{\alpha: v \to t(\alpha)} Y(t(\alpha)) \arrow[r, two heads] & \prod\limits_{\alpha: v \to t(\alpha)} W(t(\alpha)). \end{tikzcd}$$

Then observe that the top map is a pure epimorphism (by the assumption that its kernel, which is $X(v)$, is an fp-injective module), hence also the bottom map is a pure epimorphism. Moreover, the map on the left hand side is a split epimorphism, hence ψ_v^W is a pure epimorphism. Since W vertexwise consists of fp-injectives, in view of 4.4(ii) we obtain the desired result.

The proof of (i) is dual where one uses instead a short exact sequence of representations ending in X, see [11, 3.1(2)], and makes use of 4.4(i). $\square$

Proposition 4.6. (cf. [11, 3.5.1]) *Let R be a ring and let Q be a quiver. Then the following hold:*

(i) *If Q is left rooted then $\Phi(\mathrm{DProj}(R)) = \mathrm{DProj}(\mathrm{Rep}_Q(R))$.*
(ii) *If R is right coherent and Q is right rooted then $\Psi(\mathrm{DInj}(R)) = \mathrm{DInj}(\mathrm{Rep}_Q(R))$.*

Proof. (ii) Assume that $D \in \mathrm{DInj}(\mathrm{Rep}_Q(R))$, which by definition means that there exists an exact complex of injective representations,

$$X^\bullet = \cdots \to X^{-1} \to X^0 \to X^1 \to \cdots,$$

which has $D = \ker(X^0 \to X^1) := Z^0(X^\bullet)$ and remains exact after applying functors of the form $\mathrm{Hom}_{\mathrm{Rep}_Q(R)}(\mathrm{FPI}, -)$. We need to prove that $D \in \Psi(\mathrm{DInj}(R))$.

We first prove that for each vertex $v \in Q_0$ we have $D(v) \in \mathrm{DInj}(R)$. Indeed, the complex of R-modules $X^\bullet(v)$ is exact, consists of injective modules [9, 2.1], has $D(v)$ as a syzygy, and it remains to check that, for any fp-injective module F, the complex $\mathrm{Hom}_R(F, X^\bullet(v))$ is exact. Now, the functor f_vM, as in 2.6, from [21, 5.2(a)] is such that the complex $\mathrm{Hom}_R(F, X^\bullet(v))$ is exact if and only if the complex of representations $\mathrm{Hom}_{\mathrm{Rep}_Q(R)}(f_vF, X^\bullet)$ is exact. Note that f_vF vertexwise consists of fp-injectives (since $\mathrm{FPI}(R)$ is closed under coproducts [27]), hence Lemma 4.5(ii) implies that f_vF is a representation with fp-injective dimension ≤ 1. This means that there exists a short exact sequence of representations $0 \to f_vF \to Y_0 \to Y_1 \to 0$ with Y_0, Y_1 fp-injectives. Hence we obtain a short exact sequence of complexes of representations

$$0 \to \mathrm{Hom}(Y_1, X^\bullet) \to \mathrm{Hom}(Y_0, X^\bullet) \to \mathrm{Hom}(f_vF, X^\bullet) \to 0,$$

where by the assumption on X the two leftmost terms are exact. Hence so is $\mathrm{Hom}(f_vF, X^\bullet)$. Thus $D(v) \in \mathrm{DInj}(R)$.

Next, we show that for all $v \in Q_0$ the natural map ψ_v^D of 2.5 is an epimorphism with kernel in $\mathrm{DInj}(R)$. For this fix a vertex $v \in Q_0$, then in the commutative diagram

$$\begin{tikzcd}
X^0(v) \arrow[r, two heads, "\partial_v^0"] \arrow[d, "\psi_v^{X^0}"'] & D(v) \arrow[d, "\psi_v^D"] \\
\prod_{\alpha: v \to t(\alpha)} X^0(t(\alpha)) \arrow[r, two heads] & \prod_{\alpha: v \to t(\alpha)} D(t(\alpha)),
\end{tikzcd}$$

the map on the left is an epimorphism (by the characterization of injective representations in [9, 2.1]), hence ψ_D^v is also an epimorphism. Consider the degreewise split short exact sequence of complexes of R-modules

$$0 \to \ker \psi_{X^\bullet}^v \to X^\bullet(v) \to \prod_{\alpha: v \to t(\alpha)} X^\bullet(t(\alpha)) \to 0.$$

By a two-out-of-three argument we see that the complex on the left is an exact complex of injectives which stays exact after applying functors of the form $\mathrm{Hom}_R(\mathrm{FPI}(R), -)$, and moreover has $\ker(\psi_D^v)$ as a syzygy. Thus $\ker(\psi_D^v) \in \mathrm{DInj}(R)$ which concludes the proof that $D \in \Psi(\mathrm{DInj}(R))$.

Conversely, let $D \in \Psi(\mathrm{DInj}(R))$. We want to prove that $D \in \mathrm{DInj}(\mathrm{Rep}_Q(R))$, i.e. we want to find an exact complex of injective representations $E^\bullet$ which has D as a syzygy and remains exact after applying functors of the form $\mathrm{Hom}(\mathrm{FPI}, -)$. Recall the transfinite sequence (W_λ) as defined in 2.5. Following the proof of [11, Thm. 3.5.1(a)], we will see how to construct recursively, for each ordinal λ, an exact complex $E_\lambda^\bullet$ of injective representations of the subquiver $Q^\lambda := \{v \in Q_0 \mid v \in W_\lambda\}$, which is such that for all $v \in Q^\lambda$, the complex $E_\lambda^\bullet(v)$ is $\mathrm{Hom}(\mathrm{FPI}(R), -)$–exact and has $D(v)$ as a syzygy. We give the first step of this construction:

For each vertex $v \in W_1 := \{i \in Q_0 \,|\, i \text{ is not the target of any arrow in Q}\}$, by the assumption that $D(v) \in \mathrm{DInj}(R)$, there exists an exact complex of injectives $I_v^\bullet$ with $Z^0(I_v^\bullet) = D(v)$. Thus we may use the right adjoint g_v from 2.6 to obtain an exact complex $E_1^\bullet := g_v I_v^\bullet$ of injective representations of the subquiver $Q^1 := \{v \in Q_0 \mid v \in W_1\}$. Then note that for all $v \in Q_0$, the exact complex of injectives $E_1^\bullet(v)$ has $D(v)$ as a syzygy and is $\mathrm{Hom}(\mathrm{FPI}(R), -)$–exact.

Then one can follow verbatim the rest of the argument in [11, Thm. 3.5.1(a)] to obtain an exact complex $E^\bullet = \cup_\lambda E_\lambda^\bullet$ of injective representations which has D as a syzygy, and is such that for all $v \in Q_0$ the exact complex of injectives $E^\bullet(v)$ is $\mathrm{Hom}_R(\mathrm{FPI}(R), -)$–exact. The proof will be finished if we show that for any fp-injective representation F, the complex $\mathrm{Hom}(F, E^\bullet)$ is exact. This follows after considering the degreewise split short exact sequence

$$0 \to E^\bullet \to \prod_{v \in Q_0} g_v E^\bullet(v) \to \prod_{\alpha \in Q_1} g_{s(\alpha)} E^\bullet(t(\alpha)) \to 0$$

and observing that, for any fp-injective representation F, the two rightmost terms are $\mathrm{Hom}(F, -)$–exact.

The proof of (i) is completely dual, one just needs to make use of the duals of the arguments in the proof of (i), which are provided in our previous results. □

We now give a projective model structure for Ding projective representations over a Ding-Chen ring. We will need the following result of Ding and Chen.

Fact 4.7. [6, Prop. 3.16] Let R be a Ding-Chen ring with $\mathrm{FPI}-\dim_R R \leq n$ and $\mathrm{FPI}-\dim R_R \leq n$ for some integer $n \in \mathbb{N}$. Then for any right R-module M we have bi–implications:

$$\begin{aligned} \mathrm{Flat}-\dim(M) < \infty &\Leftrightarrow \mathrm{Flat}-\dim(M) \leq n \\ &\Leftrightarrow \mathrm{FPI}-\dim(M) < \infty \\ &\Leftrightarrow \mathrm{FPI}-\dim(M) \leq n. \end{aligned}$$

Theorem 4.8. *Let R be a Ding-Chen ring and let Q be a left rooted quiver. Then there exists a hereditary Hovey triple* $(\mathrm{DProj}(\mathrm{Rep}_Q(R)), \mathrm{Rep}_Q \mathcal{W}, \mathrm{Rep}_Q(R))$ *on the category of quiver representations of right R-modules,* $\mathrm{Rep}_Q(R)$. *The homotopy category of this model structure is*

$$\mathrm{Ho}(\mathrm{Rep}_Q(R)) \cong \underline{\mathrm{DProj}}(\mathrm{Rep}_Q(R)),$$

the stable category of Ding projective representations.

Proof. From [13, Thm. 4.7] we have that over a Ding-Chen ring R there exists a hereditary complete cotorsion pair $(\mathrm{DProj}(R), \mathcal{W})$ in the category of right R-modules, where $\mathcal{W} \cap \mathrm{DProj}(R) = \mathrm{Proj}(R)$. Moreover, from [13, Thm 4.2] the class $\mathcal{W}$ consists of all modules that have finite flat dimension, hence $\mathcal{W}$ is closed under coproducts by 4.7. Hence, for a left rooted quiver Q, Theorem 3.5 provides us with a hereditary Hovey triple

$$(\Phi(\mathrm{DProj}(R)), \mathrm{Rep}_Q\,\mathcal{W}, \mathrm{Rep}_Q(R)),$$

in the category of quiver representation of right R-modules, $\mathrm{Rep}_Q(R)$. Now the result follows after applying 4.6(i). □

Our last result concerns Ding injective representations over a Ding-Chen ring.

Theorem 4.9. *Let R be a Ding-Chen ring and Q a right rooted quiver. Then there exists a hereditary Hovey triple* $(\mathrm{Rep}_Q(R), \mathrm{Rep}_Q\,\mathcal{W}, \mathrm{DInj}(\mathrm{Rep}_Q(R)))$ *on the category of quiver representations of right R-modules,* $\mathrm{Rep}_Q(R)$*. The homotopy category of this model structure is*

$$\mathrm{Ho}(\mathrm{Rep}_Q(R)) \cong \underline{\mathrm{DInj}}(\mathrm{Rep}_Q(R)),$$

the stable category of Ding injective representations.

Proof. Dual to that of 4.8 where one instead makes use of the hereditary injective cotorsion pair $(\mathcal{W}, \mathrm{DInj}(R))$ of [13, Thm. 4.7], where $\mathcal{W}$ consists of all modules of finite fp-injective dimension [13, Thm 4.2]. Note that by 4.7 $\mathcal{W}$ is closed under products. Then one applies Theorem 3.6 and 4.6(ii). □

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References

1. Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. MR1294136
2. Mansour Aghasi and Hamidreza Nemati, *Absolutely pure representations of quivers*, J. Korean Math. Soc. **51** (2014), no. 6, 1177–1187. MR3275710
3. Benjamin Antieau and Greg Stevenson, *Derived categories of representations of small categories over commutative Noetherian rings*, Pacific J. Math. **283** (2016), no. 1, 21–42. MR3513838
4. Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995. MR1314422
5. William Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), no. 5, 1641–1674. MR1264733
6. Nan Qing Ding and Jian Long Chen, *The flat dimensions of injective modules*, Manuscripta Math. **78** (1993), no. 2, 165–177. MR1202159
7. Nanqing Ding and Jianlong Chen, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra **24** (1996), no. 9, 2963–2980. MR1396867
8. Edgar Enochs, Luis Oyonarte, and Blas Torrecillas, *Flat covers and flat representations of quivers*, Comm. Algebra **32** (2004), no. 4, 1319–1338. MR2100360
9. Edgar E. Enochs, Sergio Estrada, and Juan R. García Rozas, *Injective representations of infinite quivers. Applications*, Canad. J. Math. **61** (2009), no. 2, 315–335. MR2504018
10. Edgar E. Enochs and Overtoun M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR1753146

11. Hossein Eshraghi, Rasool Hafezi, and Shokrollah Salarian, *Total acyclicity for complexes of representations of quivers*, Comm. Algebra **41** (2013), no. 12, 4425–4441. MR3169529
12. Sergio Estrada, Alina Iacob, and Sinem Odabasi, *Gorenstein flat and projective (pre)covers*, Publ. Math. Debrecen **91** (2017), no. 1-2, 111–121. MR3690524
13. James Gillespie, *Model structures on modules over Ding-Chen rings*, Homology Homotopy Appl. **12** (2010), no. 1, 61–73. MR2607410
14. ______, *Model structures on exact categories*, J. Pure Appl. Algebra **215** (2011), no. 12, 2892–2902. MR2811572
15. ______, *How to construct a Hovey triple from two cotorsion pairs*, Fund. Math. **230** (2015), no. 3, 281–289. MR3351474
16. ______, *Gorenstein complexes and recollements from cotorsion pairs*, Adv. Math. **291** (2016), 859–911. MR3459032
17. Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules*, de Gruyter Expositions in Mathematics, vol. 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2006. MR2251271
18. Moritz Groth and Jan Šťovíček, *Abstract representation theory of Dynkin quivers of type A*, Adv. Math. **293** (2016), 856–941. MR3474336
19. Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041
20. Henrik Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. MR2038564
21. Henrik Holm and Peter Jørgensen, *Cotorsion pairs in categories of quiver representations*, Kyoto J. Math (to appear).
22. Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134
23. ______, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002), no. 3, 553–592. MR1938704
24. Peter Jørgensen, *Existence of Gorenstein projective resolutions and Tate cohomology*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 59–76. MR2283103
25. Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162. MR2157133
26. Sinem Odabaşi, *Completeness of the induced cotorsion pairs in categories of quiver representations*, preprint, https://arxiv.org/abs/1711.00559.
27. Bo Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329. MR0258888
28. Jan Šaroch and Jan Šťovíček, *Singular compactness and definability for Σ-cotorsion and Gorenstein modules*, preprint, https://arxiv.org/abs/1804.09080.
29. Jan Šťovíček, *On purity and applications to coderived and singularity categories*, preprint, 2014, https://arxiv.org/abs/1412.1615.

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Bibliography

[1] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. MR1294136

[2] Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles. Available at http://www.numdam.org. MR0225844

[3] Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR0269685

[4] Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405. MR0086822

[5] Maurice Auslander and Ragnar-Olaf Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 5–37, Colloque en l'honneur de Pierre Samuel (Orsay, 1987). MR1044344

[6] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995. MR1314422

[7] Luchezar L. Avramov and Hans-Bjørn Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR1455856

[8] Luchezar L. Avramov, Vesselin N. Gasharov, and Irena V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 86, 67–114 (1998). MR1608565

[9] Luchezar L. Avramov and Alex Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), no. 2, 393–440. MR1912056

[10] Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. MR0153708

[11] Silvana Bazzoni, Manuel Cortés Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes*, preprint, 2017, https://arxiv.org/abs/1704.06672.

[12] Apostolos Beligiannis and Idun Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207. MR2327478

[13] Ladislav Bican, Robert El Bashir, and Edgar E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390. MR1832549

[14] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234. MR1214458

[15] Francis Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994, Basic category theory. 1291599

[16] Siegfried Breitsprecher, *Lokal endlich präsentierbare Grothendieck-Kategorien*, Mitt. Math. Sem. Giessen Heft **85** (1970), 1–25. MR0262330

[17] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956

[18] Ragnar-Olaf Buchweitz, *Maximal cohen-macaulay modules and tate-cohomology over gorenstein rings*, University of Hannover **21** (1986).

[19] M. C. R. Butler and G. Horrocks, *Classes of extensions and resolutions*, Philos. Trans. Roy. Soc. London Ser. A **254** (1961/1962), 155–222. MR0188267

[20] Lars W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000. MR1799866

[21] Lars Winther Christensen, Hans-Bjørn Foxby, and Henrik Holm, *Beyond totally reflexive modules and back: a survey on Gorenstein dimensions*, Commutative algebra—Noetherian and non-Noetherian perspectives, Springer, New York, 2011, pp. 101–143. MR2762509

[22] P.M. Cohn, *On the free product of associative rings*, Math. Z. **71** (1959), 380–398. MR0106918

[23] William Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), no. 5, 1641–1674. MR1264733

[24] Georgios Dalezios, *Abelian model structures on categories of quiver representations*, preprint, 2018, https://arxiv.org/abs/1810.12684.

[25] ______, *A note on homotopy categories of fp-injectives*, Homology, Homotopy Appl. **21** (2019), no. 1, 95–105.

[26] Georgios Dalezios, Sergio Estrada, and Henrik Holm, *Quillen equivalences for stable categories*, J. Algebra **501** (2018), 130–149. MR3768129

[27] Nan Qing Ding and Jian Long Chen, *The flat dimensions of injective modules*, Manuscripta Math. **78** (1993), no. 2, 165–177. MR1202159

[28] Nanqing Ding and Jianlong Chen, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra **24** (1996), no. 9, 2963–2980. MR1396867

[29] Samuel Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. No. **55** (1965), 39. 0178036

[30] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64. MR570778

[31] Paul C. Eklof and Jan Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), no. 1, 41–51. MR1798574

[32] Ioannis Emmanouil, *On pure acyclic complexes*, J. Algebra **465** (2016), 190–213. MR3537821

[33] Edgar Enochs and Sergio Estrada, *Relative homological algebra in the category of quasi-coherent sheaves*, Adv. Math. **194** (2005), no. 2, 284–295. MR2139915

[34] Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. MR1363858

[35] ______, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR1753146

[36] Edgar E. Enochs, Overtoun M. G. Jenda, and Jinzhong Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3223–3234. MR1355071

[37] Hans-Bjørn Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973). MR0327752

[38] James Gillespie, *The flat model structure on* Ch(R), Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390. MR2052954

[39] ———, *Model structures on modules over Ding-Chen rings*, Homology Homotopy Appl. **12** (2010), no. 1, 61–73. MR2607410

[40] ———, *Model structures on exact categories*, J. Pure Appl. Algebra **215** (2011), no. 12, 2892–2902. MR2811572

[41] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules*, de Gruyter Expositions in Mathematics, vol. 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2006. MR2251271

[42] Laurent Gruson and Christian U. Jensen, *Dimensions cohomologiques reliées aux foncteurs* $\varprojlim^{(i)}$, Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), Lecture Notes in Math., vol. 867, Springer, Berlin, 1981, pp. 234–294. MR633523

[43] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988. MR935124

[44] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041

[45] G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc. **82** (1956), 246–269. MR0080654

[46] Henrik Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. MR2038564

[47] Henrik Holm and Peter Jørgensen, *Cotorsion pairs in categories of quiver representations*, Kyoto J. Math (to appear).

[48] Thorsten Holm, Peter Jø rgensen, and Raphaël Rouquier (eds.), *Triangulated categories*, London Mathematical Society Lecture Note Series, vol. 375, Cambridge University Press, Cambridge, 2010. 2723238

[49] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134

[50] ———, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002), no. 3, 553–592. MR1938704

[51] Srikanth Iyengar and Henning Krause, *Acyclicity versus total acyclicity for complexes over Noetherian rings*, Doc. Math. **11** (2006), 207–240. MR2262932

[52] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007. MR2355715

[53] Peter Jørgensen, *The homotopy category of complexes of projective modules*, Adv. Math. **193** (2005), no. 1, 223–232. MR2132765

[54] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162. MR2157133

[55] ______, *Approximations and adjoints in homotopy categories*, Math. Ann. **353** (2012), no. 3, 765–781. MR2923949

[56] Henning Krause and Claus Michael Ringel (eds.), *Infinite length modules*, Trends in Mathematics, Birkhäuser Verlag, Basel, 2000, Invited lectures from the conference held in Bielefeld, September 7–11, 1998. MR1789208

[57] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR1653294

[58] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR1011461

[59] Daniel Murfet and Shokrollah Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. **226** (2011), no. 2, 1096–1133. MR2737778

[60] Amnon Neeman, *The derived category of an exact category*, J. Algebra **135** (1990), no. 2, 388–394. MR1080854

[61] ______, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236. MR1308405

[62] ______, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR1812507

[63] ______, *The homotopy category of flat modules, and Grothendieck duality*, Invent. Math. **174** (2008), no. 2, 255–308. MR2439608

[64] ———, *Some adjoints in homotopy categories*, Ann. of Math. (2) **171** (2010), no. 3, 2143–2155. MR2680406

[65] ———, *The homotopy category of injectives*, Algebra Number Theory **8** (2014), no. 2, 429–456. MR3212862

[66] Mike Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, vol. 121, Cambridge University Press, Cambridge, 2009. 2530988

[67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967. 0223432

[68] Emily Riehl, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014. 3221774

[69] Joseph J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009. 2455920 (2009i:18011)

[70] Luigi Salce, *Cotorsion theories for abelian groups*, (1979), 11–32. MR565595

[71] Jean-Pierre Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955 (Tokyo), Science Council of Japan, 1956, pp. 175–189. MR0086071

[72] Rodney Y. Sharp, *Finitely generated modules of finite injective dimension over certain Cohen-Macaulay rings*, Proc. London Math. Soc. (3) **25** (1972), 303–328. MR46 #5315

[73] Daniel Simson, *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. **96** (1977), no. 2, 91–116. MR0480690

[74] ———, *Flat complexes, pure periodicity and pure acyclic complexes*, J. Algebra **480** (2017), 298–308. MR3633309

[75] Nicolas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154. MR0932640

[76] Bo Stenström, *Purity in functor categories*, J. Algebra **8** (1968), 352–361. MR0229697

[77] Bo Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329. MR0258888

[78] Bo Stenström, *Rings of quotients*, Springer-Verlag, New York-Heidelberg, 1975, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory. 0389953

[79] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. 98c:18007

[80] Jan Šťovíček, *On purity and applications to coderived and singularity categories*, preprint, 2014, https://arxiv.org/abs/1412.1615.

[81] Abraham Zaks, *Injective dimension of semi-primary rings*, J. Algebra **13** (1969), 73–86. MR0244325