Some classification results for translating solitons and ancient mean curvature flows

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A Daniela, Fulvio, Emanuele e Michele

Abstract. In this PhD thesis we make some contributions to the study of translating solitons and ancient mean curvature flows.

In Paper A, which is joint work with my supervisor Niels Martin Møller, we focus on properly immersed translating solitons of the mean curvature flow, with compact (possibly empty) boundary. We prove that they cannot be contained in the intersection of two transverse vertical halfspaces. As an application, we classify their convex hull, up to an orthogonal projection. The proofs are crucially based on an Omori-Yau maximum principle.

In Paper B, also joint work with Niels Martin Møller, we extend the ideas contained in Paper A to the more general setting of ancient flows. We prove a parabolic Omori-Yau maximum principle for ancient flows, which is of independent interest, and we use it to show that properly immersed ancient mean curvature flows cannot be contained in the intersection of two transverse half-spaces. In particular we classify the convex hulls of their sets of reach.

In Paper C, we prove that 2-dimensional embedded simply connected translating solitons with entropy < 3 and which are contained in a slab, must be mean convex. In order to achieve this result, we provide a curvature estimate for 2-dimensional, simply connected translaters with finite entropy.

Dansk resumé. Denne ph.d.-afhandling består af bidrag til studiet af selv-translaterende solitoner og begyndelsesløse middelkrumningsflåd.

I Artikel A, som er udført i fællesskab med min ph.d.-vejleder Niels Martin Møller, fokuserer vi på ægte immerserede translaterende solitoner for middelkrumningsflåd, med kompakt (muligvis tom) rand. Vi beviser at sådanne ikke kan være indeholdt i snittet af to transverse vertikale halvrum. Som en anvendelse klassificerer vi deres konvekse hylstre op til en ortogonal projektion. Beviserne hviler på et Omori-Yau maksimumsprincip.

I Artikel B, som også er fælles med Niels Martin Møller, udvider vi idéerne fra Artikel A til det mere generelle tilfælde af begyndelsesløse middelkrumningsflåd. Vi viser her et parabolsk Omori-Yau maksimumsprincip for begyndelsesløse middelkrumningsflåd, som er af uafhængig interesse, og bruger dette til at vise at ægte immerserede begyndelsesløse middelkrumningsflåd ikke kan være indeholdt i to transverse halvrum. Specielt klassificerer vi de konvekse hylstre af de overstrøgne punktmængder.

I Artikel C beviser vi at 2-dimensionelle enkeltsammenhængende indlejrede translaterende solitioner med entropi mindre < 3, og som er indeholdt i en skive, nødvendigvis må være middelkonvekse. Til dette viser vi et krumningsestimat for indlejrede 2-dimensionelle enkeltsammenhængende translaterende solitoner af endelig entropi.

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Contents

Introduction

The interplay between geometry and analysis, also known as *geometric analysis*, has led to many spectacular results and geometric flows have been the key tool in many of the most important developments. Roughly speaking, geometric flows are evolutionary equations which deform geometric objects (e.g. maps between Riemannian manfiolds, Riemannian metrics on a smooth manifold, etc.) by geometric quantities such as curvature.

Inspired by the seminal work of Eells and Sampson [**ES64**], Richard Hamilton [**Ham82**] introduced a "heat equation" for Riemannian metrics: the *Ricci flow*. His work has lead to many outstanding results, such as the proof of Thurston's geometrization conjecture (and in particular the Poincaré conjecture) by Perelman [**Pe02**] [**Pe03**] [**Pe03**] and the differentiable sphere theorem proved by Brendle and Schoen [**BS09**].

Hamiltion's results on Ricci flow inspired the influential work of Huisken [**Hu84**] on mean curvature flow. From that moment, Ricci flow and mean curvature flow have been deeply influenced each others.

Mean curvature flow deforms hypersurfaces of the Euclidean space (or, more in general, of a Riemannian manifold) in such a way that the velocity at each point is given by the mean curvature vector. Geometrically, it is a very natural evolutionary equation for hypersurfaces because it is the L^2-gradient flow of the area functional on the space of closed hypersurfaces and minimal submanifolds are stationary solutions.

The history of mean curvature flow is actually older than the one of Ricci flow. In fact, in 1956, Mullins [**Mu56**] was probably the first one who wrote the mean curvature flow equation, using it as a model for the motion of grain boundaries in aluminum during the annealing process. Brakke [**Br78**] gave the first mathematical treatment of the subject, from a geometric measure theoretically approach, very different from the one took later by Huisken, who started a systematic treatment of the field from a differential geometric point of view and with geometric applications in mind.

Why studying mean curvature flow? Let us mention a few situations in which mean curvature flow has been proved to be a useful tool. We do not have the ambition to be exhaustive.

A part from the already mentioned paper by Mullins, it has been used to model other physical phenomena, such as the behavior of charged droplets [**HT13**]. Moreover, mean curvature flow and the closely related *inverse* *mean curvature flow* have found important applications in general relativity [**HY96**] [**HI01**]. Another motivation coming from physics for studying mean curvature flow is that it arises as limit of the Allen-Cahn equation [**Il93**] [**ESS92**].

On the other hand, mean curvature flow is interesting also for purely mathematical reasons. For example, it has been studied because of the connections with isoperimetric inequalities [**To98**] [**Wh09**]. Its 1-dimensional version, the curve shortening flow, has been employed to prove that any 2-dimensional sphere with a smooth Riemannian metric has at least three simple closed geodesics [**Gr89b**]. Mean curvature flow has been used to study isotopy of maps between spheres [**TW04**] and it has also been successfully applied to study the topology of closed hypersufaces of the Euclidean space which are 2-*convex* [**HS09**] [**BH16**] and their moduli space [**BHH16**] [**BHH19**] [**Mr18**]. Mean curvature flow is expected to find even more topological applications in the future. For example it is hoped to use it to give an alternative proof of Smale's conjecture (proved by Hatcher [**Hat83**] without using geometric flows techniques).

Overview. In Chapter 1 we briefly present some background material on mean curvature flow. The aim is to provide a quick overview of the field and to explain the role of translating solitons and more in general of ancient mean curvature flows.

In Chapter 2 we summarize our contributions to the field and we collect open problems and some possible future research directions.

Finally, the rest of the thesis consists of the following three papers:

- Paper A: *Bi-halfspace and convex hull theorems for translating solitons* [**ChMø19a**],
- Paper B: *Ancient mean curvature flows and their space time track* [**ChMø19b**],
- Paper C: *Simply connected translating solitons contained in slabs* [**Ch19**].

CHAPTER 1

Preliminaries

1. Notation

In this thesis we often denote by $\Sigma^n \subseteq \mathbb{R}^{n+1}$ a hypersurface immersed in $\mathbb{R}^{n+1}$. We are not interested in the parametrization, but sometimes it is useful to specify one. We usually denote by M^n a n-dimensional smooth manifold and by $F\colon M^n \to \mathbb{R}^{n+1}$ a smooth immersion. We are mainly interested in *properly* immersed hypersurfaces. Recall that an immersion $F\colon M^n \to \mathbb{R}^{n+1}$ is called *proper* if $F^{-1}(K)$ is compact in M, for any compact set $K \subseteq \mathbb{R}^{n+1}$.

- Usually ν denotes a smooth unit normal vector field on Σ, in the case when Σ is an orientable hypersurface.
- Given a point $p \in \Sigma$, we denote by $T_p\Sigma$ the tangent space of Σ at p.
- The second fundamental form is denoted by A and it is defined as $A_p(v, w) := \langle \nabla_v \nu, w\rangle$ for any $v, w \in T_p\Sigma$.
- The (scalar) mean curvature, namely the trace of A, is denoted by H.
- The mean curvature vector, denoted by $\vec{H}$ and sometimes by $\mathbf{H}$, is defined as $\vec{H} = -H\nu$.

Observe that the sign of the scalar mean curvature depends on the choice of ν, but the mean curvature vector does not depend on the choice of ν. In particular this implies that $\vec{H}$ is globally define also on nonorientable hypersurfaces.

We say that Σ is *mean convex* (respectively *strictly mean convex*) if it is orientable and if there exists a smooth unit normal vector field ν on Σ such that the mean curvature H w.r.t. ν satisfies $H \geq 0$ (respectively $H > 0$). We say that Σ is *convex* (respectively *strictly convex*) if Σ is orientable and if there exists a smooth unit normal vector field ν on Σ such that A is semidefinite (respectively definite).

2. Mean curvature flow

Let M^n be a smooth n-dimensional manifold and let $I \subseteq \mathbb{R}$ be a time interval. A *mean curvature flow* is a smooth map $F\colon M \times I \to \mathbb{R}^{n+1}$ which satisfies

$$\left(\frac{\partial}{\partial t}F(x,t)\right)^{\perp} = \vec{H}(F(x,t)) \tag{1}$$

for each $(x,t) \in M \times I$, where $\vec{H}(F(x,t))$ is the mean curvature vector of the hypersurface $F(\cdot,t)\colon M \to \mathbb{R}^{n+1}$ at the point $F(x,t)$. It is customary to use the notation

$$F_t(\cdot) \coloneqq F(\cdot,t) \qquad \text{and} \qquad M_t \coloneqq F_t(M) \subseteq \mathbb{R}^{n+1}.$$

2.1. Short time existence. Let $\widetilde{F}\colon M^n \to \mathbb{R}^{n+1}$ be a smooth immersion. It is natural to ask whether there exist $T > 0$ and a solution $F\colon M^n \times [0,T) \to \mathbb{R}^{n+1}$ of the following initial value problem

$$\begin{cases} \left(\frac{\partial}{\partial t}F(x,t)\right)^{\perp} = \vec{H}\left(F(x,t)\right) \\ F(x,0) = \widetilde{F}(x). \end{cases} \tag{2}$$

The answer is affirmative in several important cases, namely when the initial data is a smooth closed hypersurface (see Theorem 1.5.1 in [**Man11**], Section 2 in [**GH86**] or [**HP99**]) and when the initial data is a noncompact hypersurface with bounded second fundamental form (see Theorem 4.2 in [**EH91**]). The regularity of the initial data can be considerably relaxed (see for instance [**Wa04**] and [**He17**]).

2.2. Weak definitions of mean curvature flow. The above definition of mean curvature flow allows an elegant treatment using tools from differential geometry and parabolic PDEs. In this thesis we take this classical point of view. For a proper introduction to the subject we refer to [**Ec04**] and [**Man11**]. However, we have to mention that this approach can be too restrictive in several situations. In particular, if one wants to define a mean curvature flow with non-smooth initial data or continuing the flow after the occurrence of singularities, then a weaker notion of mean curvature flow is needed. Moreover, limit flows arising in the blow-up analysis of a singularity of a (smooth) mean curvature flow are usually defined only in a geometric measure sense (they are Brakke flows [**Il95**]).

The most important weak formulations are *Brakke's mean curvature flow* introduced in the seminal work by Brakke [**Br78**] (see also the enhanced version given by Ilmanen [**Il94**]) in the framework of geometric measure theory and the *level set* formulation developed independently by Evans and Spruck [**ES91**] and Chen, Giga and Goto [**CGG91**].

2.3. Comparison principle. From a PDE point of view, the mean curvature flow is a quasilinear parabolic equation. Namely, let $(M_t)_{t\in I}$ be a mean curvature flow. Assume that M_t is the graph of a smooth function $u(\cdot,t)\colon \Omega \to \mathbb{R}$, for some domain $\Omega \subseteq \mathbb{R}^n$, for all $t \in I$ (note that this is always true locally in space and time). Then $u\colon \Omega \times I \to \mathbb{R}$ satisfies the following quasilinear parabolic PDE

$$\frac{\partial u}{\partial t} = \sqrt{1+\|Du\|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+\|Du\|^2}}\right). \tag{3}$$

The local description (3) implies that the mean curvature flow satisfies the following parabolic maximum principle, known as *comparison principle*, *avoidance principle*, or *separating tangency principle*.

THEOREM 1 (Theorem 2.2.1 in [**Man11**]). *Let $F\colon M^n \times [0,T) \to \mathbb{R}^{n+1}$ and $G\colon N^n \times [0,T) \to \mathbb{R}^{n+1}$ be two mean curvature flows such that M_t is properly immersed for every $t \in [0,T)$ and N is compact.*

Then the distance between M_t and N_t is nondecreasing in time.

2.4. Ancient, immortal, eternal flows. Mean curvature flows which exist for an unbounded time interval, are particularly important. They are called

- *ancient flows* if $I = (-\infty, a)$, for some $a \in (-\infty, +\infty]$,
- *immortal flows* if $I = (a, \infty)$, for some $a \in [-\infty, +\infty)$,
- *eternal flows* if $I = \mathbb{R}$.

These special flows play a crucial role in understanding the asymptotic behaviour of mean curvature flows near singularities. We comment more on this below.

REMARK 2. Observe that ancient mean curvature flows are particularly interesting because they are the natural parabolic counterpart of minimal hypersurfaces. In fact, if we think about the mean curvature flow as a geometric heat equation, then minimal hypersurfaces play the role of harmonic functions and ancient flows play the role of ancient caloric functions. Many properties of harmonic functions hold more generally for ancient caloric functions (e.g. Liouville's theorem, see [**CM19**]). Likewise, it is natural to expect that many properties enjoyed by minimal hypersurfaces are satisfied by ancient flows.

Theorem 3 in Paper B confirms this general principle. Namely, we show that the wedge theorem for properly immersed minimal hypersurfaces holds more in general for properly immersed ancient mean curvature flows (see Section 1 of Chapter 2).

3. Singularities

Due to their nonlinear nature, geometric flows typically develop singularities in finite time. In view of the applications, it is extremely important to understand the behavior near the singularities. In fact, most of the research in this field is devoted to the study of singularities.

3.1. Existence of singularities. A mean curvature flow starting from a closed hypersurface becomes singular in finite time. An easy way to see this is by comparing the given mean curvature flow with the evolution of an enclosing sphere and by using Theorem 1. A sphere evolves homotetically under the mean curvature flow and develops a singularity in finite time. More precisely, let $R_0 > 0$, and let us consider the function

$$R(t) \coloneqq \sqrt{R_0 - 2nt},$$

for $t \in \left[0, \frac{R_0}{2n}\right)$. Let S_R denote the sphere of radius $R > 0$ centered at 0. It is easy to check that the 1-parameter family of spheres $(S_{R(t)})_{t \in I}$, where $I = \left[0, \frac{R_0}{2n}\right)$, is a mean curvature flow of spheres which shrink homotetically to the origin.

Let $(M_t)_{t \in [0,T)}$ be a mean curvature flow starting at a closed hypersurface M_0. Let R_0 be large enough such that the sphere S_{R_0} encloses M_0. From Theorem 1, the distance between M_t and $S_{R(t)}$ cannot decrease. Therefore M_t must develop a singularity at some time $0 < t^* \leq \frac{R_0}{2n}$, before the sphere disappears.

It is natural to ask whether, given a mean curvature flow of a closed hypersurface, the surface disappears at the singular time, as for the shrinking sphere. In his seminal paper [**Hu84**], Huisken proved that this is in fact the case, if the initial surface is convex.

THEOREM 3 (Huisken [**Hu84**]). *Let $\widetilde{F}\colon M^n \to \mathbb{R}^{n+1}$ be a smooth closed embedded hypersurface which is strictly convex.*

Then there exists $T > 0$ and a solution $F\colon M \times [0,T) \to \mathbb{R}^{n+1}$ of (2) *such that M_t is strictly convex for every $t \in [0,t)$ and such that M_t converges to a single point as $t \to T$. Moreover, M_t becomes asymptotically round before disappearing.*

It has been shown by Mramor and Payne [**MP19**] that the class of closed hypersurfaces which shrink to a point under the mean curvature flow is quite large. However, singularities can appear before the evolving surface disappears. This was proved first by Grayson [**Gr89a**]. Later Angenent [**Ang92**] and Topping [**To98**] gave alternative proofs. Observe that all these proofs are based on a barrier argument and thus on the avoidance principle, namely Theorem 1.

3.2. Type I and Type II singularities. Instead of using the comparison argument with shrinking spheres above, one can prove the existence of singularities in finite time for the evolution of closed hypersurfaces by looking at the evolution equation for $|A|^2$:

$$\frac{\partial}{\partial t}|A|^2 = \Delta^{\Sigma}|A|^2 - 2|\nabla^{\Sigma} A|^2 + 2|A|^4. \tag{4}$$

In fact, using (4), Huisken proved that $|A|^2$ blows up in finite time on a mean curvature flow with a closed hypersurfaces as initial data. Moreover the flow becomes singular precisely when $|A|^2$ blows up (see Theorem 8.1 in [**Hu84**] and Lemma 1.2 in [**Hu90**]).

THEOREM 4 (Theorem 2.4.11 in [**Man11**]). *Let* $\widetilde{F}\colon M^n \to \mathbb{R}^{n+1}$ *be a smooth immersion of a closed manifold* M^n*. Then there exists a maximal time* $T_{\max} > 0$ *and a unique mean curvature flow* $F\colon M^n \times [0, T_{\max}) \to \mathbb{R}^{n+1}$ *satisfying* (2).

Moreover $T_{\max}$ *is finite and*

$$\max_{M_t} |A| \geq \frac{1}{\sqrt{2\,(T_{\max} - t)}}. \tag{5}$$

Equation (5) gives a lower bound on the blow-up rate of the second fundamental form. In the literature, singularities of the mean curvature flow are classified according to the blow-up rate of A. More precisely, let $T_{\max}$ be the maximal time of existence of a mean curvature flow $(M_t)_{t\in[0,T_{\max})}$. Then we say that the flow develops a singularity of *type I* if

$$\max_{M_t} |A| \leq \frac{C}{\sqrt{2\,(T_{\max} - t)}}$$

for some constant $C \geq 1$ and it develops a singularity of *type II* otherwise, i.e. if

$$\limsup_{t\to T_{\max}} \max_{M_t} |A| \sqrt{(T_{\max} - t)} = +\infty.$$

As an example, observe that spheres (and cylinders) evolving under the mean curvature flow develop a singularity of type I in finite time. However, more exotic type I singularities exist [**AL86**] [**Ang92**] [**Mø11**] [**KKM14**] [**Ke16**] and Type II singularities exist as well [**Ang91**] [**AV97**] [**GS09**] [**IW19**] [**IWZ19**].

3.3. Monotonicity formula and Colding-Minicozzi's entropy. Let us consider the function

$$\Phi(x,t) := \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{\frac{\|x\|^2}{4t}}$$

for $x \in \mathbb{R}^{n+1}$ and $t < 0$. Moreover, let $\Phi_{(x_0,t_0)}$ be the spacetime translation of Φ at (x_0, t_0), namely

$$\Phi_{(x_0,t_0)}(x,t) := \Phi(x - x_0, t - t_0).$$

Huisken [**Hu90**] proved the following fundamental theorem.

THEOREM 5 (Monotonicity Formula). *Let* $(M_t)_{t\in I}$ *be a mean curvature flow such that* $\int_{M_t} \Phi_{(x_0,t_0)} < \infty$ *for all* $t < t_0$*.*

Then

$$\frac{d}{dt}\int_{M_t} \Phi_{(x_0,t_0)} = -\int_{M_t} \left\| \mathbf{H} - \frac{(x - x_0)^\perp}{2(t_0 - t)} \right\|^2 \Phi_{(x_0,t_0)}. \tag{6}$$

Let us now define the *entropy functional* which has been introduced by Colding and Minicozzi [**CM12**] (see also [**MM09**]).

Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a hypersurface. The *entropy* of Σ is defined as

$$\lambda(\Sigma) := \sup_{x_0 \in \mathbb{R}^{n+1}, t_0 > 0} \frac{1}{(4\pi t_0)^{\frac{n}{2}}} \int_\Sigma e^{-\frac{\|x - x_0\|}{4t_0}} \, d\mu(x), \tag{7}$$

where $d\mu$ is the area element on Σ. From Theorem 5 it easily follows that the entropy is monotonically nonincreasing along a mean curvature flow. Namely, if $(M_t)_{t\in I}$ is a mean curvature flow, then

$$\lambda(M_t) \geq \lambda(M_s)$$

for every $t \leq s$.

3.4. Tangent flows and limit flows. The mean curvature flow equation is invariant under spacetime translations and parabolic rescaling. More precisely, let $(M_t)_{t\in[0,T)}$ be a mean curvature flow, let $c > 0$ be a constant and let $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ be a spacetime point. Then the following

$$M_s^{(x_0,t_0),c} := c\left(M_{c^{-2}s - t_0} - x_0\right)$$

is a mean curvature flow, for $c^2 t_0 < s < c^2 (T + t_0)$.

Consider a sequence of spacetime points $(x_j, t_j) \in \mathbb{R}^{n+1} \times \mathbb{R}$ converging to a spacetime point (x_0, t_0) and a sequence $(c_j)_{j\in\mathbb{N}}$ such that $c_j \nearrow +\infty$. The sequence of mean curvature flows $M_s^{(x_j,t_j),c_j}$ is called *blow-up sequence* at (x_0, t_0). Any mean curvature flow M_s^∞ which is a subsequent limit of $M_s^{(x_j,t_j),c_j}$, is called *limit flow*. In the case when the sequence of spacetime points is constant, i.e. $(x_j, t_j) = (x_0, t_0)$ for $j \in \mathbb{N}$, any subsequent limit is called *tangent flow* (see also [**Wh03**]).

Tangent flows are shrinking solitons. One of the most important applications of the monotonicity formula is the study of tangent flows. In fact, using Theorem 5, one can prove the following theorem.

THEOREM 6 (Proposition 4.21 in [**Ec04**]). *Let $(M_t)_{t\in[0,T)}$ be a mean curvature flow and let us assume that there exists a blow-up sequence $M_s^{(x_0,T),c_j}$ at the spacetime point $(x_0, T) \in \mathbb{R}^{n+1} \times \mathbb{R}$ which converge smoothly on compact subsets of $\mathbb{R}^{n+1}$ to a smooth tangent flow $(M_s^\infty)_{s\in(-\infty,0)}$.*

Then $(M_s^\infty)_{s\in(-\infty,0)}$ is a shrinking soliton*, namely*

$$M_s^\infty = \sqrt{-s} M_{-1}^\infty \tag{8}$$

for every $s < 0$.

REMARK 7. In fact, if a mean curvature flow $(M_t)_{t\in[0,T)}$ develops a type I singularity at $(x_0, T) \in \mathbb{R}^{n+1} \times \mathbb{R}$, Huisken [**Hu90**] proved that any blow-up sequence $M_s^{(x_0,T),c_j}$ converges smoothly (up to a subsequence) to a smooth self-shrinking soliton (see Section 4 below for more details on solitons of the mean curvature flow).

If one drops the type I hypothesis, then it is still true that blow-up sequences converge, up to a subsequence, to self-shrinking solitons [**Il95**] [**Wh97**]. However, in general, the convergence is only in a measure-teoretically way and the tangent flow may be nonsmooth (namely is in general only a Brakke flow). This also follows crucially from Huisken's monotonicity formula, which holds also for Brakke flows.

We remark that Ilmanen [**Il95**] proved that if the mean curvature flow is 2-dimensional and each time-slice is properly immersed in $\mathbb{R}^3$ and it has finite genus, then each limit flow is a smooth shrinking soliton, but the convergence is not necessarily smooth.

Remark 8. Note that different blow-up sequences might converge to different tangent flows. The problem of uniqueness of tangent flows is a difficult one and it has been solved only in some special (but important) cases [**Sc14**] [**CM15**].

Limit flows are ancient flows (and sometimes translating solitons). Limit flows at a spacetime singularity $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ are always ancient mean curvature flows. In general it is not clear whether they must be self-similar or not (see Section 4).

If the singularity is of type II, it is possible to choose a suitable blow-up sequence which converge smoothly on compact subsets of $\mathbb{R}^{n+1}$ to a smooth eternal flow $(M_s^\infty)_{s\in\mathbb{R}}$. Moreover the second fundamental forms of M_s^∞ and all their covariant derivatives are uniformly bounded (see Proposition 4.1.4 in [**Man11**]). Huisken and Sinsestrari [**HS99**], using an earlier result by Hamilton [**Ham95**], showed that limit flows arising from a particular blow-up sequence at a type II singularity of a mean convex mean curvature flow, must be convex *translating solitons* (see Section 4 below for the definition of translating solitons).

In general, if the initial surface is a closed mean convex hypersurface, then every limit flow is a convex ancient flow [**Wh03**]. There has been an extensive study on singularities of mean convex mean curvature flows and it is still an active research field (see [**HK17**] and references therein). As we said, in some special situations it is known that limit flows arising from mean convex mean curvature flows are solitons, but this is not known in general (see Conjecture 1.1 in [**CHH19**]).

If we drop the mean convexity assumption, even less is known. However, very recently, 2-dimensional ancient mean curvature flows in $\mathbb{R}^3$ satisfying only a bound on the entropy, have been classified in [**CHH18**].

Remark 9. Observe that Colding-Minicozzi's entropy is invariant under rescaling. The entropy of a limit flow at a singularity is therefore bounded from above by the entropy of the initial data of the flow [**CM12**]. Since a closed hypersurface has finite entropy, it follows that limit flows of mean curvature flows of closed hypersurfaces are ancient mean curvature flows with finite entropy. Moreover, ancient flows with "low" entropy are "more likely" to arise in the blow-up analysis of singularities (see [**CM12**] for a rigorous explanation of these ideas).

4. Solitons of mean curvature flow

A hypersurface moving by mean curvature flow is called *self-similar*, or *soliton*[1], if it evolves along the flow of a conformal Killing vector field of $\mathbb{R}^{n+1}$.

The most important and well-studied solitons are:

- *homotetically shrinking soltions*,
- *homotetically expanding solitons*,
- *translating soltitons.*

Since solitons move in a self-similar way, their evolution is completely characterized by a time-slice, which satisfies a time independent equation of elliptic nature.

We mention that other kinds of solitons have received some attention as well (see for instance [**Is98**], [**HS00**], [**Hal12**], [**Hal13**], [**HR11**]).

4.1. Homotetic solitons. Shrinking solitons are particularly important because they arise as tangent flows in the blow-up analysis of singularities (see Theorem 6 and Remark 7). For this reasons they are the most studied solitons of the mean curvature flow. We refer to the survey [**DLN17**] and references therein for a brief introduction to shrinking solitons.

Expanders have received less attention, but they are also important because they are expected to model mean curvature flows flowing out of a conical singularity [**ACI95**] and they model the long time behaviour of certain immortal flows [**EH89**], [**St98**].

4.2. Translating solitons. An eternal mean curvature flow $F\colon M^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ is a *translating soliton*, or *translater*, if there exists an hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ and a nonzero vector $V \in \mathbb{R}^{n+1}$ such that

$$M_t = \Sigma + tV. \tag{9}$$

Note that $\Sigma = M_0$.

Equation (9) implies that the mean curvature vector $\vec{H}$ of Σ satisfies

$$\vec{H} = V^{\perp}. \tag{10}$$

The converse is also true. More precisely, if a hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ satisfies (10), then it is the timeslice of a translating soliton. Moreover, up to a rescaling, we can assume that V is a unit vector. In this thesis we follow the convention of choosing $V = e_{n+1}$ (the convention $V = -e_{n+1}$ is common as well [**HIMW19a**] [**HIMW19b**] [**HMW19**] [**HMW19b**]). For

[1] In the literature, these are sometimes called *conformal solitons* (e.g. [**AS13**]) and sometimes the term *soliton* is reserved for the case when the evolution is along the flow of a Killing vector field (e.g. [**HS00**]).

this reason, with a small abuse of language, we call *translating solitons*, or *translaters*, hypersurfaces $\Sigma^n \subseteq \mathbb{R}^{n+1}$ satisfying (10) with $V = e_{n+1}$.

Why studying translating solitons? Translating solitons are a very special instance of ancient mean curvature flows (actually they are eternal mean curvature flows). A part for their role in the analysis of singularities (Section 3), they are interesting because they often model the long time behavior of eternal mean curvature flows. For example, Altschuler and Wu [**AW94**] showed that graphical mean curvature flows over a compact convex domain of $\mathbb{R}^2$ satisfying a contact angle condition at the boundary, converge to a translating soliton. Moreover, Clutterbuck, Schnürer and Schulze [**CSS07**] showed that the evolution of small perturbations of the bowl translater converge to the evolution of the rotationally symmetric bowl translater.

Studying solitons can be very fruitful also because their simple evolution provides qualitative insight into the dynamic of mean curvature flows. For instance, Topping [**To98**] used two grim reaper cylinders to give an alternative proof of the existence of a singularity before the extinction time for the dumbbell. His elegant argument, that he called the "guillotine", is more elementary than the ones given by Angenent [**Ang92**] and Grayson [**Gr89a**]. Another example is given in [**CSS07**], where the authors used the rotationally symmetric wing-like translaters as barriers to study the dynamical stability of the bowl translater.

We refer to the Appendix of our Paper A for another example of the use of wing-like translater as barriers and the proof of Theorem 10 in Paper C for some more barrier arguments involving translating solitons.

CHAPTER 2

Presentation of our contributions

1. Summaries of the papers

This thesis consists of the following three papers

- Paper A: *Bi-halfspace and convex hull theorems for translating solitons* [**ChMø19a**],
- Paper B: *Ancient mean curvature flows and their space time track* [**ChMø19b**],
- Paper C: *Simply connected translating solitons contained in slabs* [**Ch19**].

The first two papers have been written in collaboration with my supervisor Niels Martin Møller.

All these three papers contain some classification results. A complete classification of ancient mean curvature flows is hopeless, even for translating solitons, as suggested by the zoo of known examples (we refer to the introductions of Paper A and Paper B for an overview on the known examples). However, in the literature there are several classification results for solitons and ancient mean curvature flows [**DHS10**] [**Wa11**] [**MSS15**] [**Has15**] [**He18**] [**MPSS18**] [**CHH18**] [**BC19**] (see the introduction to Paper B for more references) but they all assume some additional conditions, such as assumptions on the asymptotic behavior, on the curvature, or on the entropy. The main feature of the results contained in Paper A and Paper B is that we require the flows (translating solitons in Paper A and C and ancient flows in Paper B) only to be properly immersed. The price to pay, however, is that we do not obtain a classification of such flows, but instead we classify the convex hulls of their sets of reach.

In Paper C we classify 2-dimensional simply connected translating solitons with entropy < 3 which are contained in a slab. We show that such translaters must be mean convex, and thus convex, thanks to a result by Spruck and Xiao [**SX17**]. An interesting feature of this result is that the convexity is a consequence rather than an assumption.

Paper A ([ChMø19a]). In this work we prove that a properly immersed translating soliton of the mean curvature flow, with compact (possibly empty) boundary, cannot be contained in the intersection of two vertical transverse halfspaces, unless it is compact (Theorem 1.1 and Theorem 1.2 in Paper A). This implies the following classification.

THEOREM 10 (Theorem 1.3 in Paper A). *Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a properly immersed translater with (possibly empty) compact boundary. Let $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the orthogonal projection $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$ and let "*$\mathrm{Conv}(\cdot)$*" denote the (closed) convex hull. Then exactly one of the following holds:*

- $\mathrm{Conv}(\pi(\Sigma)) = \mathbb{R}^n$,
- $\mathrm{Conv}(\pi(\Sigma))$ *is a halfspace of* $\mathbb{R}^n$,
- $\mathrm{Conv}(\pi(\Sigma))$ *is a closed slab between two parallel hyperplanes of* $\mathbb{R}^n$,
- $\mathrm{Conv}(\pi(\Sigma))$ *is a hyperplane of* $\mathbb{R}^n$,
- $\mathrm{Conv}(\pi(\Sigma))$ *is compact and this case occurs precisely when* Σ *is compact.*

This classification has been inspired by the work of Hoffman and Meeks [**HM90**] in which they classified all the possible convex hulls of properly embedded minimal hypersurfaces of $\mathbb{R}^{n+1}$ with compact (possibly empty) boundary, obtaining morally the same list above. Their proof is based on a clever use of barriers. This argument is not readily available in our situation (see the Appendix of Paper A). Therefore our proofs are completely different and are instead based on an Omori-Yau maximum principle for translating solitons and are inspired by the work of Borbély [**Bo11**].

Paper B ([ChMø19b]). In this paper we extend the ideas from Paper A to the case of ancient mean curvature flows. The main difficulty is to adapt the Omori-Yau maximum principle to this parabolic setting. Ma [**Ma17**] proved an Omori-Yau maximum principle for mean curvature flows which exist on a compact time interval. We modified his proof in order to obtain an Omori-Yau maximum principle for ancient flows. Some complications arise from the fact that the time interval of ancient flows is not compact.

We use the Omori-Yau maximum principle for ancient flows to prove the following theorem.

THEOREM 11 (Theorem 3 in Paper B). *Given two transverse halfspaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{R}^{n+1}$, there does not exist any proper ancient mean curvature flow $F\colon M \times (-\infty, 0) \to \mathbb{R}^{n+1}$ such that the timeslice $F_t(M)$ is contained in $\mathcal{H}_1 \cap \mathcal{H}_2$ for all times $t \in (-\infty, 0)$.*

Observe that Theorem 11 was already known in the special case of properly immersed minimal hypersurfaces [**HM90**] [**Bo11**]. See Remark 2 for an heuristic discussion on why it is natural to expect such a result to hold.

As a corollary, we obtain the following classification.

THEOREM 12 (Theorem 5). *Let $F\colon M^n \times (-\infty, 0) \to \mathbb{R}^{n+1}$ be a properly immersed ancient mean curvature flow and let $\mathcal{R} \coloneqq \bigcup_{t\in(-\infty,0)} F_t(M)$ be its set of reach. Then* $\mathrm{Conv}(\mathcal{R})$ *is either a hyperplane, a slab, a halfspace or all of* $\mathbb{R}^{n+1}$.

Paper C ([Ch19]). The last paper included in this thesis focuses again on translating solitons. The main contribution is the following result.

THEOREM 13 (Theorem 1 in Paper C). *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a complete embedded translater which is simply connected, contained in a slab and such that $\lambda(\Sigma) < 3$. Then Σ is mean convex.*

We refer to Remark 3 in Paper C for a discussion about the hypothesis of Theorem 13.

As a corollary, using results in [**SX17**] and [**HIMW19a**], we obtain that a translating soliton satisfying the assumptions in Theorem 13 is, up to an ambient isometry, one of the following:

- a plane,
- a (tilted) grim reaper cylinder,
- a Δ-wing translater.

We refer to Paper C and references therein for more informations on these examples of translaters.

The proof of Theorem 13 is built on several auxiliary results, some of which are of independent interest. Namely we prove a curvature estimate for complete simply connected 2-dimensional translaters embedded in $\mathbb{R}^3$ (Proposition 7 in Paper C) and the following result, which provides information on the asymptotic behavior of translating solitons.

THEOREM 14 (Theorem 10 in Paper C). *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a properly immersed translater such that $\partial \operatorname{Conv}(\pi(\Sigma)) \neq \emptyset$.*

Then for every $q \in \partial \operatorname{Conv}(\pi(\Sigma))$ and for every $\varrho > 0$ we have that

$$\sup_{\Sigma \cap \pi^{-1}(B_\varrho(q))} x_3 = +\infty. \tag{11}$$

In the proof of this theorem, we make use both of the Omori-Yau maximum principle for translaters, as in Paper A, and also of some barriers arguments.

2. Future directions

We collect here some open questions that we find interesting and discuss some possible research directions.

Halfspace problems. It is known that all the cases in the classification of Theorem 10 (Theorem 1.3 in Paper A) can occur, except the second one. Namely, there is no known examples of a properly embedded translater $\Sigma^n \subseteq \mathbb{R}^{n+1}$ with compact (possibly empty) boundary such that $\operatorname{Conv}(\pi)$ is a halfspace of $\mathbb{R}^n$ (see Remark 2 in Paper A).

PROBLEM 1. *Prove the existence of a properly embedded translater $\Sigma^n \subseteq \mathbb{R}^{n+1}$ with compact (possibly empty) boundary such that $\operatorname{Conv}(\pi)$ is a halfspace of $\mathbb{R}^n$, or prove that such a translater does not exist (possibly under some extra assumptions such as finite entropy).*

This is related to a similar open question for minimal hypersurfaces.

Problem 2. *Prove the existence of a properly immersed minimal hypersurface* $\Sigma^n \subseteq \mathbb{R}^{n+1}$ *(without boundary) such that* $\mathrm{Conv}(\Sigma)$ *is a halfspace for* $n \geq 3$*, or prove that such a minimal hypersurface does not exist (possibly under some extra assumptions).*

Observe that Problem 2 is stated for $n \geq 3$ because, from the Halfspace Theorem by Hoffman and Meeks [**HM90**], we know that any properly immersed minimal surface contained in a halfspace must be a flat plane.

One can ask the same question, more in general, for ancient mean curvature flows.

Problem 3. *Prove the existence of a properly immersed ancient mean curvature flow* $F\colon M^n \times (-\infty, 0) \to \mathbb{R}^{n+1}$ *such that* $\mathrm{Conv}(\mathcal{R})$ *is a halfspace, where* $\mathcal{R} = \bigcup_{t\in(-\infty,0)} F_t(M)$ *is the set of reach.*

Classification of 2-dimensional translaters. As explained in Paper A,, a full classification of simply connected 2-dimensional translaters in $\mathbb{R}^3$ seems hopeless. However we believe that it is possible to classify simply connected 2-dimensional translaters with entropy < 3.

Problem 4 (Conjecture 4 in Paper C). *Let* $\Sigma^2 \subseteq \mathbb{R}^3$ *be an embedded, simply connected translater such that* $\lambda(\Sigma) < 3$*. Is it possible to prove that* Σ *must be mean convex? In other words, is it possible to remove the assumption* (iii) *from Theorem 1 in Paper C?*

We quickly sketch here a possible strategy to solve Problem 4. It is possible to show that any *blow-down* sequence of a 2-dimensional translating soliton with finite entropy and finite topology is a smooth shrinking soliton, possibly with multiplicity (see [**Il95**]). Such a shrinking soliton has the same entropy of the initial translating soliton and must split off a line.

Since 1-dimensional shrinking solitons have been completely classified [**AL86**], from the bound on the entropy, we have that $\lambda(\Sigma)$ must be 1 (in this case Σ is a plane), $\lambda(\mathbb{S}^1 \times \mathbb{R})$ (in this case Σ is the bowl translater) or 2 (see [**BS18**]). It is crucial, therefore, to understand what happens in the case $\lambda(\Sigma) = 2$. For example, if one can show that in this case Σ must be contained in a slab, then the solution of Problem 4 would follow from Theorem 1 in Paper C. The case $\lambda(\Sigma) = 2$ happens precisely when the blow-down of Σ is a plane with multiplicity 2, but it is not even clear whether the blow-down is unique. Known uniqueness results for tangent flows [**Sc14**] [**CM15**] do not apply here.

Problem 5. *Let* $\Sigma^2 \subseteq \mathbb{R}^3$ *be an immersed translater such that* $\lambda(\Sigma) = 2$*. Is the blow-down unique? If the answer is affirmative, is it true that* Σ *is contained in a slab?*

Bibliography

[AL86] U. Abresch, J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom. 23, 175-196 (1986).

[AW94] S.J. Altschuler, L.F. Wu, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations 2(1), 101–111 (1994).

[And12] *Noncollapsing in mean-convex mean curvature flow*, Geom. Topol. Volume 16, Number 3, 1413-1418 (2012).

[Ang91] S. Angenent, *On the formation of singularities in the curve shortening flow*, J. Diff. Geom 33, 601-633 (1991).

[Ang92] S. Angenent, *Shrinking doughnuts*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), 21–38, Progr. Nonlinear Differential Equations Appl. 7 (1992).

[ACI95] S.B. Angenent, D.L. Chopp, T. Ilmanen, *A computed example of non-uniqueness of mean curvature flow in* $\mathbb{R}^3$, Commun. in Partial Differential Equations 20, no. 11-12, 1937–1958 (1995).

[AV97] S. Angenent, J.J.L. Velázquez, *Degenerate neckpinches in mean curvature flow*, J. reine angew. Math. 482, 15–66 (1997).

[AS13] C. Arezzo, J. Sun, *Conformal solitons to the mean curvature flow and minimal submanifolds*, Math. Nachr. 286(8-9), 772 - 790 (2013).

[BS18] J. Baldauf, A. Sun, *Sharp entropy bounds for plane curves and dynamics of the curve shortening flow*, arXiv:1808.03936 (2018).

[Bo11] A. Borbély, *On minimal surfaces satisfying the Omori-Yau principle*, Bull. Aust. Math. Soc. 84, 33–39 (2011).

[Br78] K.A. Brakke, *The motion of a surface by its mean curvature flow*, Princeton University Press, NJ, (1978).

[BH16] S. Brendle, G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in* $\mathbb{R}^3$, Invent. Math. 203, 615–654 (2016).

[BS09] S. Brendle, R. Schoen, *Manifolds with* $1/4$*-pinched curvature are space forms*, J. Amer. Math. Soc. 22, no. 1, 287–307 (2009).

[BC19] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in* $\mathbb{R}^3$, Inventiones Mathematicae 217, 35–76 (2019).

[BHH16] R. Buzano, R. Haslhofer, O. Hershkovits, *The moduli space of two-convex embedded spheres*, arXiv:1607.05604 (2016).

[BHH19] R. Buzano, R. Haslhofer, O. Hershkovits, *The moduli space of two-convex embedded tori*, Int. Math. Res. Not., Issue 2, 392–406 (2019).

[CY75] S.Y. Cheng, S.T. Yau, *Diferential equations on Riemanninan manifolds and their geometric applications*, Comm. Pure Appl. Math. 28, 333–354 (1975).

[CGG91] Y.-G. Chen, Y. Giga, S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. 33 749-786 (1991).

[Ch19] F. Chini, *Simply connected translating solitons contained in slabs*, arXiv:1912.12426v2 (2019).

[ChMø19a] F. Chini, N.M. Møller, *Bi-Halfspace and Convex Hull Theorems for Translating Solitons*, Int. Math. Res. Not. (2019) (To appear).

[ChMø19b] F. Chini, N.M. Møller, *Ancient mean curvature flows and their spacetime tracks*, arXiv:1901.05481v2 (2019).

[CHH19] B. Choi, R. Haslhofer, O. Hershkovits, *A note on the selfsimilarity of limit flows*, arXiv:1910.02341 (2019).

[CHH18] K. Choi, R. Haslhofer, O. Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*, arXiv:1810.08467 (2018).

[CHHW19] K. Choi, R. Haslhofer, O. Hershkovits, B. White, *Ancient asymptotically cylindrical flows and applications* arXiv:1910.00639v2 (2019).

[CSS07] J. Clutterbuck, O. Schnürer, F. Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differ. Equ. 29, 281–293 (2007).

[CM11a] T.H. Colding, W.P. Minicozzi II, *A Course in Minimal Surfaces*, AMS (2011).

[CM11b] T.H. Colding, W.P. Minicozzi II, *Minimal surfaces and mean curvature flow*, Surveys in geometric analysis and relativity, 73143, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, (2011).

[CM12] T.H. Colding, W.P. Minicozzi II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. 175, 755-833 (2012).

[CM15] T.H Colding, W.P. Minicozzi II, *Uniqueness of blowups and Łojasiewicz inequalities*, Ann. of Math. 182, 221–285 (2015).

[CM19] T.H. Colding, W.P. Minicozzi II, *Optimal bounds for ancient caloric functions* arXiv:1902.01736v2 (2019).

[DHS10] *Classification of compact ancient solutions to the curve shortening flow*, J. Diff. Geometry, 84, 455-464 (2010).

[DLN17] G. Drugan, H. Lee, X.H. Nguyen, *A survey of closed self-shrinkers with symmetry*, arXiv:1708.09113v1 (2017).

[Ec04] K. Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, (2004).

[EH89] K. Ecker, G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) 130, no. 3, 453–471 (1989).

[EH91] K. Ecker, G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. MAth. 105, no. 3, 547-569 (1991).

[ES64] J. Eells, J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86, 109–160 (1964).

[ESS92] L.C. Evans, H.M. Soner, P.E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. 45(9), 1097–1123 (1992).

[ES91] L.C. Evans, J Spruck, *Motion of level sets by mean curvature*, J. Differential Geom. 33 635-681 (1991).

[GH86] M. Gage, R. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. 23, 69-95 (1986).

[GS09] Z. Gang, I.M. Sigal, *Neck pinching dynamics under mean curvature flow*, J. Geom. Anal. 19, no. 1, 36–80 (2009).

[GT83] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 2nd edition, (1983).

[Gr89a] M. Grayson, *A short note on the evolution of a surface by its mean curvature* Duke Mathematical Journal 58.3, pp. 555–558 (1989).

[Gr89b] M. Grayson, *Shortening embedded curves*, Annals of Mathematics, 129, 71-111 (1989).

[Hal12] H.P. Halldorsson, *Self-similar solutions to the curve shortening flow*, Trans. Am. Math. Soc. 364 (10), 5285–5309 (2012).

[Hal13] H.P. Halldorsson, *Helicoidal surfaces rotating/translating under the mean curvature flow*, Geom. Dedicata 162, 45–65 (2013).

[Ham82] R.S. Hamilton *Three manifolds with positive Ricci curvature*, J. Diff. Geom. 17, 255-306 (1982).

[Ham95] R. Hamilton, *Harnack estimates for the mean curvature flow*, J. Differential Geom., 41, pp. 215–226 (1995).

[HL11] Q. Han, F. Lin, *Elliptic partial differential equations*, Second edition, Courant Lecture Notes in Mathematics, vol. 1, New York University, AMS, (2011).

[Has15] R. Haslhofer, *Uniqueness of the bowl soliton*, Geom. Topol. 19, 2393–2406, (2015).

[HK17] R. Haslhofer, B. Kleiner, *Mean curvature flow of mean convex hypersurfaces*, Comm. Pure Appl. Math. 70 (3) (2017) 511–546.

[Hat83] A. Hatcher, *A Proof of the Smale Conjecture,* $\mathrm{diff}(S^3) \simeq O(4)$, Annals of Mathematics, (2) 117, no. 3, 553-607 (1983).

[HT13] S. Helmensdorfer, P. Topping, *Bouncing of charged droplets: an explanation using mean curvature flow*, Europhys. Lett. 104 (3), 34001, (2013).

[He17] O. Hershkovits, *Mean curvature flow of Reifenberg sets*, Geom. Topol. 21(1), 441–484 (2017).

[He18] O. Hershkovits, *Translators asymptotic to cylinders*, arXiv:1805.10553v1 (2018).

[HIMW19a] D. Hoffman, T. Ilmanen, F. Martín, B. White, *Graphical translators for mean curvature flow* Calc. Var. (2019) 58:117.

[HIMW19b] D. Hoffman, T. Ilmanen, F. Martín, B. White, *Notes on translating solitons for mean curvature flow* arXiv:1901.09101v2 (2019).

[HMW19] D. Hoffman, F. Martín, B. White, *Scherk-like translators for mean curvature flow*, arXiv:1903.04617v3 (2019).

[HMW19b] D. Hoffman, F. Martín, B. White, *Nguyen's tridents and the classification of semigraphical translators for mean curvature flow*, arXiv:1909.09241v1 (2019).

[HM90] D. Hoffman, W. H. Meeks, III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. 101, 373–377 (1990).

[Hu84] G. Huisken, *Flow by mean curvature of convex surfaces into spheres* G. Differential Geom., 20, 237-266 (1984).

[Hu90] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. 31, 285–299 (1990).

[HI01] G. Huisken, T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. 59, 353–437 (2001).

[HP99] G. Huisken, A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Springer-Verlag, Berlin, pp. 45-84 (1999).

[HS99] G. Huisken, C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math., 183, 45-70 (1999).

[HS09] G. Huisken, C. Sinestrari, *Mean curvature flow with surgeries of two–convex hypersurfaces*, Invent. math. 175, 137–221 (2009).

[HY96] G. Huisken, S.-T. Yau *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. 124 281–311 (1996).

[HS00] N. Hungerbühler, K. Smoczyk, *Soliton solutions for the mean curvature flow*, Differential Integral Equations 13 , no. 10–12, 1321–1345 (2000).

[HR11] N. Hungerbühler, B. Roost, *Mean curvature flow solitons*, Analytic aspects of problems in Riemannian geometry: elliptic PDEs, solitons and computer imaging, Sémin. Congr. vol. 22, pp. 129–158. Soc. Math. France, Paris (2011).

[Il93] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., 38, 417-461 (1993).

[Il94] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc., 108(520), AMS, (1994).

[Il95] T. Ilmanen, *Singularities of mean curvature flow of surfaces*, preprint (1995).

[IW19] J. Isenberg, H. Wu, *Mean curvature flow of noncompact hypersurfaces with Type-II curvature blow-up*, J. Reine Angew. Math. 754, 225–251 (2019).

[IWZ19] J. Isenberg, H. Wu, Z. Zhang, *Mean curvature flow of noncompact hypersurfaces with Type-II curvature blow-up. II*, arXiv:1911.07282 (2019).

[Is98] N. Ishimura, *Shape of spirals*, Tohoku Math. J. (2) 50, no. 2, 197-202 (1998).

[Ke16] D. Ketover, *Self-shrinking Platonic solids* arXiv:1602.07271 (2016).

[KKM14] S.J. Kleene, N.M. Møller, *Self-shrinkers with a rotational symmetry*, Trans. Amer. Math. Soc. 366, no. 8, 3943–3963 (2014).

[Ma17] J.M.S. Ma, *Parabolic Omori-Yau Maximum Principle for Mean Curvature Flow and Some Applications*, arXiv:1701.02004 (2017).

[MM09] C. Mantegazza and A. Magni, *Some remarks on Huisken's monotonicity formula for mean curvature flow*, in Singularities in Nonlinear Evolution Phenomena and Applications, CRM Ser. Center "Ennio De Giorgi", Pisa 9, pp. 157–169 (2009).

[Man11] C. Mantegazza, *Lecture notes on mean curvature flow*, Birkhäuser (2011).

[MSS15] F. Martín, A. Savas-Halilaj, K. Smoczyk, *On the topology of translating solitons of the mean curvature flow*, Calculus of Variations and PDE's, vol. 54(3), 2853 — 2882 (2015).

[MPSS18] F. Martín, J. Pérez-García, A. Savas-Halilaj, K. Smoczyk, *A characterization of the grim reaper cylinder*, J. Reine Angew. Math. (2018).

[Mø11] N.M. Møller, *Closed self-shrinking surfaces in $\mathbb{R}^3$ via the torus*, arXiv:1111.7318 (2011).

[Mr18] A. Mramor, *A Finiteness Theorem Via the Mean Curvature Flow with Surgery*, J. Geom. Analysis 28, 3348–3372 (2018).

[MP19] A. Mramor, A. Payne, *Nonconvex surfaces which flow to round points* arXiv:1901.02863v2 (2019).

[Mu56] W.W. Mullins, *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. 27, 900–904 (1956).

[Pe02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159v1 (2002).

[Pe03] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109v1 (2003).

[Pe03] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245v1 (2003).

[Sc14] F. Schulze, *Uniqueness of compact tangent flows in mean curvature flow*, J. Reine Angew. Math. 690, 163–172 (2014).

[SX17] J. Spruck, L Xiao, *Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature*, (to appear in Amer. J. Math) arXiv:1703.01003v2 (2017).

[St98] N. Stavrou, *Selfsimilar solutions to the mean curvature flow*, J. Reine Angew. Math. 499, 189–198, (1998).

[To98] P. M. Topping, *Mean Curvature Flow and Geometric Inequalities*, J. Reine Angew. Math. 503, 47–61 (1998).

[TW04] M.-P. Tsui, M.-T. Wang, *Mean curvature flows and isotopy of maps between spheres*, Comm. Pure Appl. Math. 57, no. 8, 1110–1126 (2004).

[Wa04] M.T. Wang, *The mean curvature flow smoothes lipschitz submanifolds*, Comm. Anal. Geom., 12:581–599, (2004).

[Wa11] X. Wang. *Convex solutions to the mean curvature flow*, Ann. of Math. (2), 173(3):1185–1239, (2011).

[Wh97] B. White, *Strati*
cation of minimal surfaces, mean curvature flows, and harmonic maps, J. Reine Angew. Math. 488 (1997).

[Wh03] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. 16, no. 1, 123–138 (2003).

[Wh09] B. White, *Which ambient spaces admit isoperimetric inequalities for submanifolds?*, J. Differential Geom. 83, no. 1, 213-228 (2009).

Paper A. Bi-halfspace and convex hull theorems for translating solitons

BI-HALFSPACE AND CONVEX HULL THEOREMS FOR TRANSLATING SOLITONS

FRANCESCO CHINI
AND
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Abstract. While it is well known from examples that no interesting "halfspace theorem" holds for properly immersed n-dimensional self-translating mean curvature flow solitons in Euclidean space $\mathbb{R}^{n+1}$, we show that they must all obey a general "bi-halfspace theorem" (aka "wedge theorem"): Two transverse vertical halfspaces can never contain the same such hypersurface. The same holds for any infinite end. The proofs avoid the typical methods of nonlinear barrier construction for the approach via distance functions and the Omori-Yau maximum principle.

As an application we classify the convex hulls of all properly immersed (possibly with compact boundary) n-dimensional mean curvature flow self-translating solitons Σ^n in $\mathbb{R}^{n+1}$, up to an orthogonal projection in the direction of translation. This list is short, coinciding with the one given by Hoffman-Meeks in 1989, for minimal submanifolds: All of $\mathbb{R}^n$, halfspaces, slabs, hyperplanes and convex compacts in $\mathbb{R}^n$.

1. Introduction

The mean curvature flow for hypersurfaces in Euclidean space has been studied systematically since the late 1970s (to name but a few, see [LT78], [Br78], [Hu84], [GH86], [Gr87], [Ha95], [Wh02], [CM11-2], [CM12], and for early work on curve shortening flow [Mu56]), with considerable emphasis on the singularity models for the flow: the self-similar solitons.

The oldest known nontrivial complete embedded soliton is Calabi's self-translating curve in $\mathbb{R}^2$, also sometimes called the "grim reaper" translating soliton (see Grayson [Gr87] and also [Mu56], where it seems to have been first found). For readers more familiar with the Ricci flow, the most analogous object there would be Hamilton's cigar soliton (see [Ha88], and recall G. Perelman's central "no cigar" theorem [Pe02]).

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Self-translaters arise in the study of the so-called "Type II" singularities of the mean curvature flow. Indeed, using a classical result of Hamilton contained in [Ha95], Huisken and Sinestrari [HS99a] showed that blow-up limit flows at Type II singularities of mean convex mean curvature flows are complete, self-translaters of the kind $\mathbb{R}^{n-k} \times \Sigma^k$, where Σ^k is a convex translater in $\mathbb{R}^{k+1}$, with $k = 1, \ldots, n$. For the mean convex case see also [HS99b], [Whi00], [Whi03] and [HK17]. If we remove the mean convexity hypothesis, it is known that blow-ups at Type II singularities must be eternal flows, but, to our knowledge, it is still not known whether these eternal flows are generally self-translaters. (See Chapter 4 in [Ma11].)

In the classical subject of minimal surfaces one of the cornerstones of the modern theory is the so-called "Halfspace Theorem" and convex hull classification, proven in 1989 by Hoffman and Meeks [HM90]. Numerous other authors have written about such halfspace theorems and convex hull properties, in various contexts: See f.ex. [Xa84], [MR90], [BJO01], [MR08], [HRS08], [NS10] and [RSS13].

In the literature, there are some results at the intersection of these two topics, of solitons and halfspace theorems. For instance in [WW09] (see also [PW03]) there are some results for f-minimal hypersurfaces for the case of $\mathrm{Ric}_f > 0$, including a halfspace theorem for one important class of mean curvature solitons, the self-shrinkers (see also [PR14]). The paper [CE16] also showed a halfspace theorem (by using the half-catenoid-like "self-shrinking trumpets" from [KM14] as barriers) and [IPR18] showed a "Frankel property" for self-shrinkers (meaning: when it so happens that all minimal surfaces in a space must intersect, as in [Fr66] and [PW03]). Additionally, for self-translaters, a few significant geometric classification and nonexistence results are now known, see [Wa11], [Sh11], [MSS14], [Mø14], [Ha15], [Pé16], [IR17], [Bu18] and [HIMW18-1], but these do not directly address the question of (bi-)halfspace and convex hull properties.

One good reason for the lack of results with a (bi-)halfspace theorem flavor in the case of self-translaters would likely be that the most naive results one might imagine are wrong: F.ex. vertical planes and grim reaper cylinders readily coexist as self-translating solitons without ever intersecting, so there is no easy general "halfspace theorem" nor any "Frankel property". Moreover the typical arguments employed often rely on constructing barriers. As discussed in the Appendix, a strategy using other exact solutions to the translater equation does not seem readily available here, except in the case of 2-dimensional surfaces in $\mathbb{R}^3$.

In the present paper we will present the following three main contributions on n-dimensional mean curvature flow self-translating solitons (also known as "translaters", "self-translaters", "translators" or "self-translators") in $\mathbb{R}^{n+1}$. We assume in the below that the translation direction is e_{n+1}.

Theorem 1 (Bi-Halfspace Theorem). *There does not exist any properly immersed self-translating n-dimensional hypersurface* $\Sigma^n \subseteq \mathbb{R}^{n+1}$*, without boundary, which is contained in two transverse vertical halfspaces of* $\mathbb{R}^{n+1}$*.*

Theorem 2 (Bi-Halfspace Theorem w/ Compact Boundary). *Suppose a properly immersed connected self-translating n-dimensional hypersurface* $(\Sigma^n, \partial\Sigma)$ *in* $\mathbb{R}^{n+1}$ *is contained in two transverse vertical halfspaces of* $\mathbb{R}^{n+1}$*. If* $\partial\Sigma$ *is compact then* Σ *is compact.*

In the next theorem we let $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection in the direction of translation $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$.

Theorem 3 (Convex Hull Classification). *Let* $(\Sigma^n, \partial\Sigma)$ *be a properly immersed connected self-translater in* $\mathbb{R}^{n+1}$*, with (possibly empty) compact boundary* $\partial\Sigma$*.*

Then exactly one of the following holds.

(1) $\mathrm{Conv}(\pi(\Sigma)) = \mathbb{R}^n$,
(2) $\mathrm{Conv}(\pi(\Sigma))$ *is a halfspace of* $\mathbb{R}^n$,
(3) $\mathrm{Conv}(\pi(\Sigma))$ *is a closed slab between two parallel hyperplanes of* $\mathbb{R}^n$,
(4) $\mathrm{Conv}(\pi(\Sigma))$ *is a hyperplane in* $\mathbb{R}^n$,
(5) $\mathrm{Conv}(\pi(\Sigma))$ *is a compact convex set. This case occurs precisely when* Σ *is compact.*

Remark 4. From examples (see below) there appears to be no hope of classifying any of the likely wild classes Σ, $\mathrm{Conv}(\Sigma)$ or $\pi(\Sigma)$: Only after applying *both* of the forgetful operations $\mathrm{Conv}(\cdot)$ and $\pi(\cdot)$ do we find a short list, which in fact can be thought of plainly as "vertical slabs" (including their three degenerate cases).

Note also that $\mathrm{Conv}(\cdot)$ and $\pi(\cdot)$ can be freely switched in the statement of Theorem 3, because for any subset $\Omega \subseteq \mathbb{R}^{n+1}$ they commute:

$$\mathrm{Conv}\left(\pi\left(\Omega\right)\right) = \pi\left(\mathrm{Conv}\left(\Omega\right)\right).$$

Remark 5. We note that each of the five cases of Theorem 3 can happen, when $n \geq 2$, except possibly for Case (2). Leaving the case $n = 1$ to the reader, let us list examples for each case, assuming $n \geq 2$ (see also the longer list of examples below at the end of Section 3):

(1) Take any rotationally symmetric Σ^n, e.g. the "bowl" translater.
(2) No examples appear to be known.
(3) Take as Σ^n a grim reaper cylinder or any in Ilmanen's Δ-wing family.
(4) Take as Σ^n any vertical hyperplane of $\mathbb{R}^{n+1}$.
(5) Take any compact subset of any of the known examples.

Observe that an immediate consequence of Theorem 2 is the following

Corollary 6. *(Ends) Any end of a properly immersed self-transating n-dimensional hypersurface* Σ *cannot be contained in two transverse vertical halfspaces of* $\mathbb{R}^{n+1}$*.*

Remark 7. The compact boundary version in Theorem 2 does not follow from any generally valid modification of the proof of Theorem 1: For other related ambient spaces it can happen that even a halfspace theorem is true and yet no bi-halfspace theorem holds for the compact boundary case. See f.ex. the halfspace theorem for self-shrinkers in [CE16], and note how the asymptotically conical self-shrinkers in [KM14] can easily be cut to get such examples which are noncompact with compact boundary.

Let us quickly note how this is (for $\partial\Sigma = \emptyset$) strictly stronger than the old Hoffman-Meeks result, so that in the process we get a new proof of this classical fact:

Corollary 8 (Hoffman-Meeks: [HM90])**.** *The classification (1)-(5) in Hoffman-Meeks's Theorem 2 (Theorem 25 below) holds true for properly immersed minimal hypersurfaces in* $\mathbb{R}^{n+1}$ *without boundary.*

Proof of Corollary 8. For $n \geq 2$, let $N^{n-1} \subseteq \mathbb{R}^n$ be a connected properly immersed minimal hypersurface. If $\partial N = \emptyset$, apply Theorem 3 to the self-translater $\Sigma^n = N^{n-1} \times \mathbb{R}$. Then note

$$\mathrm{Conv}(N^{n-1}) = \mathrm{Conv}(\pi(N^{n-1} \times \mathbb{R})) = \mathrm{Conv}(\pi(\Sigma)),$$

from which the conclusion follows. □

As immediate corollaries to Theorem 3, we also recover the following previously known result:

Corollary 9 (Corollary 2.2 [Wa11])**.** *Let* $\Sigma^n \subseteq \mathbb{R}^{n+1}$ *be a complete connected convex graphical self-translater. I.e. there exists a smooth function* $u : \Omega \to \mathbb{R}$*, where* $\Omega \subseteq \mathbb{R}^n$*, such that* $\mathrm{graph}\,(u) = \Sigma$.

Then exactly one of the following holds.

(1) $\Omega = \mathbb{R}^n$.
(2) Ω *is a halfspace in* $\mathbb{R}^n$.
(3) Ω *is a slab between two parallel hyperplanes of* $\mathbb{R}^n$.

Proof. Since Σ is convex and complete, from a theorem of Sacksteder (see [Sa60]), we have that $\Sigma = \partial C$, where $C \subseteq \mathbb{R}^{n+1}$ is a convex set. Therefore Σ is a closed set w.r.t. the ambient topology and thus is properly embedded.

Let $u \colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\Sigma = \mathrm{graph}(u)$. Then clearly Ω is convex (indeed it is the orthogonal projection of the convex set C onto $\mathbb{R}^n$) and u is a convex function. Therefore

$$\mathrm{Conv}(\pi(\Sigma)) = \mathrm{Conv}(\Omega) = \Omega.$$

We can now apply Theorem 3 in order to conclude the proof. □

Remark 10. X.-J. Wang proved more than Corollary 9: For convex graphs, Case (2) (graph over a halfspace) cannot happen.

In [SX17], Spruck and Xiao showed that any complete oriented immersed mean convex 2-dimensional self-translater is convex. In particular, any complete 2-dimensional graphical self-translater is convex. Therefore in the case

$n = 2$ one can improve Corollary 9 removing the convexity assumption. In particular we recover the following result.

Corollary 11 ([HIMW18-1] and [SX17])**.** *The domains for 2-dimensional graphical self-translaters belong to the Cases (1)-(3), respectively all* $\mathbb{R}^2$*, half-planes or slabs in* $\mathbb{R}^2$*. In particular, a properly immersed self-translating* 2*-dimensional hypersurface* $\Sigma^2 \subseteq \mathbb{R}^3$ *cannot be the graph over a wedge-shaped domain in* $\mathbb{R}^2$*.*

Remark 12. The above Corollary 11 is contained in the paper [HIMW18-1], where all complete 2-dimensional graphical self-translaters have very recently been fully classified (using [SX17]). Again, Case (2) in fact cannot happen for 2-dimensional graphs.

In [Sh11] and [Sh15], Shahriyari proved that there are no complete 2-dimensional translaters which are graphical over a bounded domain. This fact was later generalized by Møller in [Mø14] (see [MSS14] for the half-cylinder case), where he proved that there are no properly embedded without boundary n-dimensional self-translaters contained in a cylinder of the kind $\Omega \times \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ is bounded:

Corollary 13 ([Mø14])**.** *No noncompact properly immersed self-translating* n*-dimensional hypersurface* $(\Sigma^n, \partial\Sigma)$ *in* $\mathbb{R}^{n+1}$ *with compact boundary can be contained in a cylinder* $\Omega \times \mathbb{R}$ *with* $\Omega \subseteq \mathbb{R}^n$ *bounded.*

Proof. The proof follows easily from Theorem 2. Indeed note that given a bounded set $\Omega \subseteq \mathbb{R}^n$, the cylinder $\Omega \times \mathbb{R}$ is contained in the intersection of two transverse vertical halfspaces. □

Remark 14. The proof shows more than Corollary 13, namely that the conclusion holds assuming only boundedness in two directions: $\Sigma^n \subseteq \Omega_2 \times \mathbb{R}^{n-1}$ cannot happen for $\Omega_2 \subseteq \mathbb{R}^2$.

As will be clear below, most of the ideas that we will need were essentially in place as early as the 1960s, much earlier than the minimal surface and curvature flow papers cited above. Namely, in the original paper by Omori [Om67], he showed by quite similar methods that in Euclidean n-space, cones with angle $0 < \theta < \pi$ cannot contain properly embedded minimal surfaces.

Somewhat later, in 1989, contained within the proof of "Theorem 2" from [HM90] (which seems independent of Omori's ideas) is the fact that, while the Hoffman-Meeks "halfspace theorem" only works for minimal 2-surface immersions $\Sigma^2 \to \mathbb{R}^3$, one has a "bi-halfspace theorem" (stronger than the cone theorems) for minimal hypersurfaces $\Sigma^n \to \mathbb{R}^{n+1}$ for $n \geq 3$, even allowing compact boundary. Their proof used barriers from the nonlinear Dirichlet problem known as the n-dimensional Plateau problem for graphs. Some disadvantages of that approach are clear: For when do such barriers exist, and if they in fact do, what are their precise properties, as needed for a "separating tangency" argument to run?

It then appears that only within the last decade it was realized by Borbély [Bo11] that one can prove bi-halfspace theorems for minimal 2-surface immersions $\Sigma^2 \to \mathbb{R}^3$, under the assumption that the Omori-Yau principle (so named after [Om67]-[CY75]) is known to be available on the given Σ^2. This was also expanded by Bessa, de Lira and Medeiros in [BLM13] where they showed Borbély-style "wedge" theorems for stochastically complete minimal surfaces in Riemannian products $(M \times N, g_M \oplus g_N)$, where (N, g_N) is complete without boundary. Seeing as the Huisken-Ilmanen metric, in which self-translaters are the minimal surfaces, is not a Riemannian product[1] nor complete, and our surfaces can have boundaries, we will directly take Borbély's method as our point of departure.

Here, in our case of n-dimensional self-translaters $\Sigma^n \to \mathbb{R}^{n+1}$, the Omori-Yau maximum principle in turn works quite generally, which is a well-established fact that has previously been invoked by several authors for related problems: See [Xi15], [SX16]-[SX17] and [IR17]. Many other authors have written on the topic, see e.g. [SY94], [PRS03], [BF14]. For a general yet particularly easy to state result, let us mention this: The Omori-Yau maximum principle holds for every submanifold properly immersed with bounded mean curvature into a Riemannian space form (see [PRS05]). Here we will be using the formulation and short proof in [Xi15], so as to make the whole presentation quite elementary and essentially self-contained, including as a biproduct the proof of the Hoffman-Meeks results for $n \geq 3$ and empty boundary, in Corollary 8 below.

In a later work [CM19], we generalize the main ideas contained in the present paper to ancient mean curvature flows, providing a parabolic Omori-Yau principle and using it for proving a bi-halfspace theorem for ancient flows.

2. Overview

In Section 3 we introduce notation and list a few of the technical lemmas in the form that we will need them later, with (references to) short proofs.

In Section 4 we prove a new "Bi-Halfspace Theorem" for properly immersed self-translaters, which is Theorem 1. We also fully classify all the possible pairs of halfspaces such that their intersections contain a complete self-translater, in Corollary 19.

In Section 5 we study the convex hull of such hypersurfaces, both for compact self-translaters and for noncompact ones, but with compact (possibly empty) boundary. We observe a behavior very similar to the one of minimal submanifolds of the Euclidean space. The main result of the section is Theorem 3 and it was inspired by a result by Hoffman and Meeks in the context of minimal submanifolds of $\mathbb{R}^{n+1}$ (see [HM90]). The proof here is based on our "Bi-Halfspace" Theorem 1 and the compact boundary version Theorem 2 and hence diverges significantly from the proof of the theorem of Hoffman and

[1] Note however that [Sm01] showed that it can be seen as a warped Riemannian product.

Meeks, which relied on constructing barriers via certain nonlinear Dirichlet problems.

In the Appendix (Section 6) we will comment more on this point and we will provide an alternative proof of Theorem 3, which is closer in spirit to the one by Hoffman and Meeks, but which only works in the case $n = 2$.

3. Preliminaries and Notation

In what follows, $(x_1, x_2, \dots, x_n, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$ and $(e_1, e_2, \dots, e_n, e_{n+1})$ is the standard orthonormal basis of $\mathbb{R}^{n+1}$.

On $\mathbb{R}^{n+1}$ we will, with a slight abuse of notation, denote the coordinate vector fields by $\partial_i = \frac{\partial}{\partial x_i} = e_i$.

In this paper $\Sigma^n \subseteq \mathbb{R}^{n+1}$ will always denote a smooth properly immersed self-translater with velocity vector e_{n+1}. Recall that properly immersed hypersurfaces with boundary are geodesically complete with boundary in the induced Riemannian metric (the Heine-Borel property with Hopf-Rinow).

The evolution of Σ^n under the mean curvature flow is a unit speed translation in the direction of the positive x_{n+1}-axis. Therefore Σ^n satisfies the following equation

$$\boldsymbol{H} = \langle e_{n+1}, \nu \rangle \nu, \tag{1}$$

where $\boldsymbol{H} = -H\nu$ is the mean curvature vector of Σ^n and ν is the unit normal vector field on Σ^n.

Let us recall here two important tools that we will need for our work.

Lemma 15 (Comparison Principle for MCF). *Let $\varphi\colon M_1 \times [0, T) \to \mathbb{R}^{n+1}$ and $\psi\colon M_2 \times [0, T) \to \mathbb{R}^{n+1}$ be two hypersurfaces evolving by mean curvature flow and let us assume that M_1 is properly immersed while M_2 is compact. Then the distance between them is nondecreasing in time.*

Proof. See e.g. the proof of Theorem 2.2.1 in [Ma11]. □

Lemma 16 (Principle of Separating Tangency for Self-Translaters). *Let Σ_1^n and Σ_2^n be two connected (unit speed, same direction) self-translaters immersed into $\mathbb{R}^{n+1}$, with (possibly empty) boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$.*

Suppose that there exists a point $p \in \Sigma_1 \cap \Sigma_2$ such that it is an interior point for both the self-translaters. Let us assume that the corresponding tangent spaces $T_p\Sigma_1$ and $T_p\Sigma_2$ coincide and assume that, locally around p, Σ_1 lies on one side of Σ_2.

Then there are open neighborhoods $U_1 \subseteq \Sigma_1$ and $U_2 \subseteq \Sigma_2$ of p such that $U_1 = U_2$.

Proof. This uses the maximum principle and unique continuation. See Theorem 2.1.1 in [Pé16], Lemma 2.4 in [Mø14] and Theorem 2.1 in [MSS15]. □

Well-known Examples. We conclude this section by enumerating some of the most well-known examples of self-translaters.

(1) (Translating minimal hypersurfaces) Any hyperplane of $\mathbb{R}^{n+1}$ which is parallel to e_{n+1} is a self-translater. More generally, if $N^{n-1} \subseteq \mathbb{R}^n$ is a minimal submanifold, then we have that $\Sigma := N \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ is self-translating in the e_{n+1}-direction. This follows from the short computation $H_{N\times\mathbb{R}} = 0 = \langle(\nu_N, 0), (\mathbf{0}, 1)\rangle_{\mathbb{R}^{n+1}}$.

(2) (Grim reaper cylinder) Consider the function $f\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined as $f(x) := -\ln(\cos(x))$. Its graph $\Gamma := \operatorname{graph}(f)$ is called Calabi's *grim reaper* curve (first found in [Mu56]) and it is the only nonflat connected complete translating soliton for the curve shortening flow. The hypersurface $\Gamma^n := \mathbb{R}^{n-1} \times \Gamma \subseteq \mathbb{R}^{n+1}$ is called a *grim reaper cylinder* and it is a self-translater.

(3) (Rotationally symmetric self-translaters) In [CSS07], the authors classify all the self-translaters which are rotationally symmetric with respect to the x_{n+1}-axis. These are the so-called *bowl soliton* U which was already discovered in [AW94], and the family of *winglike self-translaters*, also known as *translating catenoids*. The bowl soliton is the graph of an entire convex function $u\colon \mathbb{R}^n \to \mathbb{R}$ and it is asymptotic to a paraboloid. Indeed it is also known as the *translating paraboloid*.

The wing-like self-translaters are all diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$, where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere. They roughly look like two bowl solitons, one above the other, glued together with a vertical neck. Both of the ends are asymptotic to U. For each $R > 0$ there exists a unique (up to a translation in the x_{n+1} direction) winglike self-translater W_R such that the size of its neck is $R > 0$.

(4) (Gluing constructions) The desingularization techniques, originally developed by Kapouleas (see [Ka90]) for building new examples of minimal and constant mean curvature hypersurfaces, have been applied by X.H. Nguyen and others, in order to prove the existence of new translating solitons, by "gluing together" already known examples. For more details, we refer to [Ng09], [Ng13], [Ng15], [DDPN17] and [Sm15]. See also [KKM11] (and [Ng11]) for the first gluing construction for mean curvature solitons with non-flat ends.

(5) (Delta-wing self-translaters) Recently, Bourni, Langford, and Tinaglia (Theorem 1 in [BLT18]), and independently Hoffman, Ilmanen, Martín and White (Theorems 4.1, 8.1 in [HIMW18-1]) have proved that for each $b > \frac{\pi}{2}$, there exists a strictly convex and complete self-translater which lies in the slab $(-b, b) \times \mathbb{R}^n$ and in no smaller slab.

Furthermore, also uniqueness was proven in [HIMW18-1]. They called this new family of self-translaters, which is parametrized by the width of the slab, the Δ-wings.

(6) (Annuli, helicoid and Scherk's) In an upcoming paper [HIMW18-2], the authors have announced that they will be constructing several

new families of properly embedded (nongraphical) translators (quoting the abstract for a talk at Stanford in July 2018): "[...] a two-parameter family of translating annuli, examples that resemble Scherk's minimal surfaces, and examples that resemble helicoids."

4. Bi-halfspace Theorems for Self-Translating Solitons

In this section we prove the "Bi-Halfspace" Theorem 1 and the case with boundary Theorem 2. Let us first make a few remarks:

Remark 17. In the theorems, the transversality can simply be defined via the unit normals to the boundary hypersurfaces (which are affine hyperplanes) of the halfspaces: They must not be (anti-)parallel as vectors in $\mathbb{R}^{n+1}$.

Note that these theorems are vacuously true for $n = 1$, as in $\mathbb{R}^2$ all vertical affine halfspaces are (anti-)parallel and hence never transverse. Thus, in the below we will throughout tacitly assume $n \geq 2$.

Note also that the statements and proofs of the "Bi-Halfspace" Theorem 1 and the case with boundary Theorem 2 can be either false or true, with an easy proof, if one or both of the two halfspaces are not vertical. See Corollary 19 at the end of this section for a clarification.

Let us state the version of the Omori-Yau lemma which we will be needing:

Lemma 18. *(Omori-Yau for Translating Solitons) Let* $(\Sigma^n, \partial\Sigma)$ *be a properly immersed self-translating soliton in* $\mathbb{R}^{n+1}$ *which is complete with boundary. Suppose that* $f : \Sigma^n \to \mathbb{R}$ *is a function which satisfies:*

(i) $\sup_\Sigma |f| < \infty, \quad \sup_{\partial\Sigma} f < \sup_\Sigma f$,
(ii) $f \in C^0(\Sigma)$,
(iii) $\exists \varepsilon_f > 0$ *s.t.* f *is* C^2 *on the set* $\{p \in \Sigma : f(p) > \sup_\Sigma f - \varepsilon_f\}$.

Then there exists a sequence $\{p_k\}$ *in* Σ^n *such that:*

$$\lim_{k\to\infty} f(p_k) = \sup_\Sigma f, \tag{2}$$

$$\lim_{k\to\infty} \nabla^\Sigma f(p_k) = 0, \tag{3}$$

$$\lim_{k\to\infty} \Delta_\Sigma f(p_k) \leq 0. \tag{4}$$

Proof of Lemma 18. A short direct proof can be found in [Xi15] (using that Σ^n is complete with boundary and properly immersed), which is easily adapted to the form stated here. For bounded $|f|$ the condition of Xin,

$$a_k \in \Sigma^n, \quad \|a_k\|_{\mathbb{R}^{n+1}} \to \infty \quad \Rightarrow \quad \lim_{k\to\infty} \frac{f(a_k)}{\|a_k\|_{\mathbb{R}^{n+1}}} = 0$$

is of course trivially satisfied. □

Proof of the "Bi-Halfspace" Theorem 1. Any affine halfspace $H \subseteq \mathbb{R}^{n+1}$ can be given by a pair of (offset and direction, resp.) vectors $(b, w) \in \mathbb{R}^{n+1} \times \mathbb{S}^n$, where we view $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Namely:

$$H = H_{(b,w)} := \left\{x \in \mathbb{R}^{n+1} : \langle x - b, w\rangle \geq 0\right\},$$
$$P := \partial H = \left\{x \in \mathbb{R}^{n+1} : \langle x - b, w\rangle = 0\right\}.$$

Note that w is unique but any $b \in \partial H$ works. Recall that such two n-planes P_1, P_2 have transverse intersection $P_1 \pitchfork P_2$ if and only if the corresponding unit normals $w_1 \nparallel w_2$ (so antiparallel is also forbidden). This is also what it means for two halfspaces H_1 and H_2 to be transverse.

What we call vertical halfspaces are those $H_{(b,w)}$ for which $w \perp e_{n+1}$, i.e. $w = (w^{(1)}, \ldots, w^{(n)}, 0) \in \mathbb{S}^n \times \{0\}$.

We now perform a couple of normalizations which are not essential but greatly simplify some of the computations: Suppose that an e_{n+1}-directed self-translating hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is contained in a pair of transverse vertical halfspaces, i.e. that $\Sigma^n \subseteq H_1 \cap H_2$. By simultaneously moving Σ^n and H_i, we may assume $b_1 = b_2 = 0$ (pick any $b \in H_1 \cap H_2$, then translate by $-b$). Note also that $\mathrm{span}(w_1, w_2)$ defines a 2-dimensional subspace in $\mathbb{R}^n \times \{0\}$.

We can then, by acting rigidly with $O(n)$ on the $\mathbb{R}^n$-factor (take an orthonormal basis for this 2-plane, fill out to an orthonormal basis of $\mathbb{R}^n$ finally compose with an $O(2)$-map in the two first coordinates), we can assume that there exists (ξ, η) such that $\xi, \eta > 0$ with $\|(\xi, \eta)\| = 1$ and:

$$w_1 = (\xi, \eta, 0, \ldots, 0), \quad w_2 = (\xi, -\eta, 0, \ldots, 0).$$

As explained in the introduction, we will now proceed with an adaptation of the method of Borbély to our situation of n-dimensional self-translaters. Consider for $R > 0$ the respective affine hyperplanes of equidistance: $P_i + Rw_i = \{x : \langle x, w_i\rangle = R\}$. Their intersection locus is an $(n-1)$-dimensional vertical affine subspace $\mathscr{L}_R := (P_1 + Rw_1) \cap (P_2 + Rw_2)$. Linear algebra reveals a simple explicit expression for this locus:

$$\mathscr{L}_R := \left\{\left(\tfrac{R}{\xi}, 0, x_3, \ldots, x_{n+1}\right) : (x_3, \ldots, x_{n+1}) \in \mathbb{R}^{n-1}\right\}. \tag{5}$$

We consider then the ambient Euclidean distance function from points $x \in \mathbb{R}^{n+1}$ to $\mathscr{L}_R$:

$$d(x) := d_R(x) := \mathrm{dist}_{\mathbb{R}^{n+1}}(x, \mathscr{L}_R) = \sqrt{\left(x_1 - \tfrac{R}{\xi}\right)^2 + x_2^2}, \quad x \in \mathbb{R}^{n+1}. \tag{6}$$

Clearly $\mathscr{L}_R = \{x \in \mathbb{R}^{n+1} : d_R(x) = 0\}$ and $\|\nabla^{\mathbb{R}^{n+1}} d\| = 1$ on $\mathbb{R}^{n+1} \setminus \mathscr{L}_R$. We define the cylindrical set by:

$$\mathscr{D}_R = \left\{x \in \mathbb{R}^{n+1} : d_R(x) \leq R\right\},$$

which is an $(n+1)$-dimensional solid with boundary. Then for any $R > 0$, explicitly

$$\mathscr{D}_R \cap P_i = \left\{\left(\tfrac{R\eta^2}{\xi}, (-1)^i R\eta, x_3, \ldots, x_{n+1}\right) : (x_3, \ldots, x_{n+1}) \in \mathbb{R}^{n-1}\right\},$$

which disconnects $\partial(H_1 \cap H_2)$ and the set $(H_1 \cap H_2) \setminus \mathscr{D}_R$ has exactly two connected components (both unbounded).

We label by $\mathcal{V}_R$ the connected component of $(H_1 \cap H_2) \setminus \mathscr{D}_R$ where d_R is bounded (the other component, where d_R is unbounded, we will not need to refer to directly). Notice that as $R \nearrow \infty$ we have $\mathcal{V}_R \nearrow H_1 \cap H_2$. From now on, we will pick a fixed $R > 0$ large enough so that $\Sigma \cap \mathcal{V}_R \neq \emptyset$.

In the below, we will at times drop the subscript and write $d(x) := d_R(x)$.

A couple of standard, elementary computations show that

$$\text{Hess}_{\mathbb{R}^{n+1}} d\left(\nabla^{\mathbb{R}^{n+1}} d_R, \nabla^{\mathbb{R}^{n+1}} d_R\right) = 0, \quad \text{on} \quad \mathbb{R}^{n+1} \setminus \mathscr{L}_R, \tag{7}$$

$$\Delta_{\mathbb{R}^{n+1}} d_R = \frac{1}{d_R}, \quad \text{on} \quad \mathbb{R}^{n+1} \setminus \mathscr{L}_R. \tag{8}$$

The first equation, giving an eigenvector field for the eigenvalue $\lambda = 0$, can also be deduced from $d_R(x)$ being linear in the gradient direction. Note also that as d_R does not depend on the last $n-1$ coordinates of $\mathbb{R}^{n+1}$, $\text{Hess}_{\mathbb{R}^{n+1}}$ has the $n-1$ orthonormal eigenvector fields with eigenvalue zero $e_3, \ldots, e_{n+1}$, all perpendicular to $\nabla^{\mathbb{R}^{n+1}} d_R$. The only nonzero eigenvalue is $\lambda = 1/d_R$ with unit length eigenvector field correspondingly given by e.g.

$$\chi = \left(-\tfrac{\partial d_R}{\partial x_2}, \tfrac{\partial d_R}{\partial x_1}, 0, \ldots, 0\right), \quad \text{on} \quad \mathbb{R}^{n+1} \setminus \mathscr{L}_R, \tag{9}$$

which together with the other listed eigenvector fields forms an orthonormal frame field on $\mathbb{R}^{n+1} \setminus \mathcal{L}_R$.

The following simple fact follows from a small exercise in linear algebra: Given a square symmetric matrix $A \in \text{Mat}_{n+1}(\mathbb{R})$ the trace over an n-dimensional hyperplane P_μ defined by a unit normal vector $\mu \in \mathbb{R}^{n+1}$ is:

$$\text{tr}_\mu(A) = \sum_{i=1}^{n+1} \lambda_i \left(1 - (\langle v_i, \mu\rangle_{\mathbb{R}^{n+1}})^2\right), \tag{10}$$

where the $(\lambda_1, \ldots, \lambda_{n+1})$ are the eigenvalues of A with multiplicity and $(v_i) \subseteq \mathbb{R}^{n+1}$ a corresponding orthonormal basis of eigenvectors. Thus in our case of a Hessian with only one nonzero eigenvalue and corresponding unit eigenvector field χ, we get the comparatively simple expression from tracing over $T_p\Sigma$ with the unit normal ν:

$$\text{tr}_\Sigma\left(\text{Hess}_{\mathbb{R}^{n+1}} d\right) = \frac{1 - (\langle \chi, \nu\rangle_{\mathbb{R}^{n+1}})^2}{d}, \quad \text{on} \quad \Sigma^n \setminus \mathscr{L}_R. \tag{11}$$

We now define the modified distance function $f : \Sigma^n \to \mathbb{R}$:

$$f(p) = \begin{cases} d_R(p), & p \in \Sigma \cap \mathcal{V}_R, \\ R, & p \in \Sigma^n \setminus \big(\mathcal{V}_R \cap \mathscr{D}_R\big). \end{cases} \tag{12}$$

This function is well-defined and continuous (as $d_{|\partial \mathscr{D}_R} = R$) and it is smooth on $\Sigma^n \setminus \mathscr{D}_R$. It is also bounded, namely note that explicitly we have (using for the first inequality that $R > 0$ was fixed large enough that $\Sigma \cap \mathcal{V}_R \neq \emptyset$, and recall also $0 < \xi < 1$):

$$R < \sup_{\Sigma} f \leq R/\xi < \infty. \tag{13}$$

At points $p \in \Sigma \cap \mathcal{V}_R$ (so that in particular $f = d_{|\Sigma}$ is smooth), we have that the gradient equals the tangential part of the ambient gradient:

(14)

$$\nabla^\Sigma f = \left(\nabla^{\mathbb{R}^{n+1}} d\right)^\top = \nabla^{\mathbb{R}^{n+1}} d - \left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp = \nabla^{\mathbb{R}^{n+1}} d - \langle \nabla^{\mathbb{R}^{n+1}} d, \nu\rangle_{\mathbb{R}^{n+1}} \nu,$$

with length computed using (9) to be (recall again $\|\nabla^{\mathbb{R}^{n+1}} d\|_{\mathbb{R}^{n+1}} = 1$):

$$\begin{aligned} \|\nabla^\Sigma f\| &= \sqrt{1 - \left(\left\langle \nabla^{\mathbb{R}^{n+1}} d, \nu\right\rangle_{\mathbb{R}^{n+1}}\right)^2} \\ &= \left|\langle \chi, \nu\rangle_{\mathbb{R}^{n+1}}\right|. \end{aligned} \tag{15}$$

So we can finally recast (11) as the following fundamental identity for the distance function to the locus $\mathcal{L}_R$:

$$\operatorname{tr}_\Sigma \left(\operatorname{Hess}_{\mathbb{R}^{n+1}} d_R\right) = \left(1 - \|\nabla^\Sigma f\|^2\right) \Delta_{\mathbb{R}^{n+1}} d_R, \quad \text{on} \quad \Sigma \cap \mathcal{V}_R. \tag{16}$$

We recall that the vector-valued second fundamental form is $A(X,Y) := (\nabla_X^{\mathbb{R}^{n+1}} Y)^\perp$. Now apply (14) and recall $\nabla^\Sigma_X Z = \left(\nabla_X^{\mathbb{R}^{n+1}} \overline{Z}\right)^\top$, for $\overline{Z}$ any extension of Z. Then for any $X, Y \in T_p\Sigma$:

$$\begin{aligned} \operatorname{Hess}_\Sigma f(X,Y) := \left\langle \nabla^\Sigma_X \nabla^\Sigma f, Y\right\rangle_\Sigma &= \left\langle \nabla^\Sigma_X \left[\nabla^{\mathbb{R}^{n+1}} d - \left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp\right], Y\right\rangle \\ &= \left\langle \nabla^{\mathbb{R}^{n+1}}_X \left[\nabla^{\mathbb{R}^{n+1}} d - \overline{\left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp}\right], Y\right\rangle \\ &= \operatorname{Hess}_{\mathbb{R}^{n+1}} d(X,Y) - \left\langle \nabla^{\mathbb{R}^{n+1}}_X \overline{\left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp}, Y\right\rangle \\ &= \operatorname{Hess}_{\mathbb{R}^{n+1}} d(X,Y) + \left\langle \nabla^{\mathbb{R}^{n+1}} d, A(X,Y)\right\rangle_{\mathbb{R}^{n+1}}, \end{aligned}$$

where the last step is seen by computing

$$X.\left\langle \overline{\left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp}, \overline{Y}\right\rangle = \left\langle \nabla^{\mathbb{R}^{n+1}}_X \overline{\left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp}, \overline{Y}\right\rangle + \left\langle \overline{\left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp}, \nabla^{\mathbb{R}^{n+1}}_X \overline{Y}\right\rangle,$$

and then evaluting on Σ to get:

$$0 = \left\langle \nabla^{\mathbb{R}^{n+1}}_X \overline{\left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp}, Y\right\rangle + \left\langle \left(\nabla^{\mathbb{R}^{n+1}} d\right)^\perp, A(X,Y)\right\rangle.$$

Taking now the trace over $T_p\Sigma$ we see:

$$\Delta_\Sigma f = \operatorname{tr}_\Sigma \left(\operatorname{Hess}_{\mathbb{R}^{n+1}} d\right) + \left\langle \nabla^{\mathbb{R}^{n+1}} d, \boldsymbol{H}\right\rangle_{\mathbb{R}^{n+1}} \tag{17}$$

Here we used that the mean curvature vector is $\boldsymbol{H} := \operatorname{tr}_\Sigma A = -H\nu$. Using now the self-translater equation $H = \langle e_{n+1}, \nu\rangle$, we get:

$$\Delta_\Sigma f = \operatorname{tr}_\Sigma \left(\operatorname{Hess}_{\mathbb{R}^{n+1}} d\right) - \langle \nabla^{\mathbb{R}^{n+1}} d, \nu\rangle \langle e_{n+1}, \nu\rangle. \tag{18}$$

Combining (16) and (18) we finally have shown:

$$\Delta_\Sigma f = \frac{1-\|\nabla^\Sigma f\|^2}{d} - \big\langle \nabla^{\mathbb{R}^{n+1}} d, \nu\big\rangle\big\langle e_{n+1}, \nu\big\rangle, \quad \text{on} \quad \Sigma\cap\mathcal{V}_R. \tag{19}$$

We will now apply the Omori-Yau principle in Lemma 18 to $f:\Sigma^n\to\mathbb{R}$, so we get a sequence of points $\{p_k\}$ on Σ^n with the Omori-Yau properties (2)-(4). To see that the Omori-Yau principle indeed applies here, we check that all the conditions in Lemma 18 hold. By construction $0<\sup_\Sigma f<\infty$, $f\in C^0(\Sigma)$ and f is C^2 where relevant. Recall also that since by (13) we know $\sup_\Sigma f > R$, and as $f|_{\Sigma\setminus\mathcal{V}_R}\leq R$ (note also that in principle $\Sigma\setminus\mathcal{V}_R=\emptyset$ is possible), we may assume that all $p_k\in\Sigma\cap\mathcal{V}_R$.

To proceed we now need to analyze the last "perturbation term" in (19), which came from the self-translater equation. Notice first that by the triangle inequality

$$\big|\big\langle e_{n+1},\nu\big\rangle\big| \leq \big|\big\langle e_{n+1}, \nabla^{\mathbb{R}^{n+1}} d\big\rangle\big| + \big|\big\langle e_{n+1}, \nu - \nabla^{\mathbb{R}^{n+1}} d\big\rangle\big| \leq \big\|\nu - \nabla^{\mathbb{R}^{n+1}} d\big\|, \tag{20}$$

using also the fact that $\langle e_{n+1}, \nabla^{\mathbb{R}^{n+1}} d\rangle = 0$ and finally applying the Cauchy-Schwarz inequality.

We know from the property (3) combined with Equation (15) that the limit

$$\big|\big\langle \nabla^{\mathbb{R}^{n+1}} d, \nu\big\rangle\big|(p_k)\to 1, \quad \text{as} \quad k\to\infty. \tag{21}$$

holds, so from a certain stage the inner product has at each point a definite sign. By the Pigeon Hole Principle, there must then exist a sign $\sigma_\infty\in\{-1,1\}$ and a subsequence of points such that $\langle\nabla^{\mathbb{R}^{n+1}} d,\nu\rangle\to\sigma_\infty$. So by, if necessary, flipping orientations $\nu\leftrightarrow-\nu$ (a symmetry for the self-translater equation) we may assume that $0<\langle\nabla^{\mathbb{R}^{n+1}} d,\nu\rangle\to 1$ on the sequence of points. This also leads to:

$$\big\|\nu(p_k) - \nabla^{\mathbb{R}^{n+1}} d_R(p_k)\big\|^2_{\mathbb{R}^{n+1}} = 2\left[1-\langle\nabla^{\mathbb{R}^{n+1}} d,\nu\rangle\right]\to 0. \tag{22}$$

In consequence, we can use (20) to conclude that:

$$\left|\langle e_{n+1},\nu\rangle\right|(p_k)\to 0. \tag{23}$$

Now, from (23) with either (21) or simply $|\langle\nabla^{\mathbb{R}^{n+1}} d,\nu\rangle|\leq 1$, the last term in (18) tends to zero. Going to the limit in (19), we thus conclude that the limits exist in the following relation:

$$\lim_{k\to\infty}\Delta_\Sigma f(p_k) = \lim_{k\to\infty}\frac{1}{d(p_k)} \geq \frac{\xi}{R} > 0, \tag{24}$$

using again $0<\xi<1$. This violates Property (4) in the Omori-Yau maximum principle of Lemma 18, namely that $\lim_{k\to\infty}\Delta_\Sigma f(p_k)\leq 0$. This contradiction concludes the proof that there cannot exist any such self-translater. □

Proof of the Theorem 2. To proceed in the case of compact nonempty boundary, we will again assume that H_1 and H_2 are as in the proof of the "Bi-Halfspace" Theorem 1, while we now allow $(\Sigma^n, \partial\Sigma)$ to be complete with compact boundary and still properly immersed. We furthermore assume that Σ^n is connected. For every $R > 0$, let $\mathscr{L}_R$, $\mathscr{D}_R$ and $d = d_R$ be as in the proof of the Theorem 1. Recall that $\mathcal{V}_R$ denotes that connected component of $(H_1 \cap H_2) \setminus \mathscr{D}_R$ on which d is bounded. Let again f be the function defined in (12). Note that since $\partial\Sigma$ is compact, we can pick $R > 0$ large enough so that $\partial\Sigma \subseteq \mathcal{V}_R$.

We will now, for contradiction, assume that $(\Sigma, \partial\Sigma)$ is not compact. We will distinguish between two different cases and finally see that each of them leads to a contradiction.

- **Case (a)**: $\Sigma \cap \mathcal{V}_R$ is bounded in $\mathbb{R}^{n+1}$ for every $R > 0$.
- **Case (b)**: There exists $R > 0$ s.t. $\Sigma \cap \mathcal{V}_R$ is unbounded in $\mathbb{R}^{n+1}$.

Proof for Case (a): By the definition of $\mathscr{D}_R$, we can fix $R > 0$ large enough so that

$$\operatorname{dist}(\partial\Sigma, \mathscr{D}_R) > \pi. \tag{25}$$

Since $\mathscr{D}_R \subseteq \mathbb{R}^{n+1}$ has compact vertical projection, there exists an open vertical slab $S \subseteq \mathbb{R}^{n+1}$ between two parallel vertical hyperplanes at distance π separating $\partial\Sigma$ and $\mathscr{D}_R$. More precisely, we can arrange that $\partial\Sigma$ and $\mathscr{D}_R$ are contained in two different connected components of $\mathbb{R}^{n+1} \setminus \overline{S}$. Let now $\Gamma^n := \Gamma \times \mathbb{R}^{n-1} \subseteq S$ be a grim reaper cylinder. Let us consider the family $\{\Gamma^n_s\}_{s\in\mathbb{R}}$ defined via $\Gamma^n_s := \Gamma^n + se_{n+1}$. Note that $\cup_{s\in\mathbb{R}}\Gamma^n_s = S$.

Since in the present case, Σ^n is assumed noncompact and hence unbounded (using that it is properly immersed), while $\Sigma \cap \mathcal{V}_R$ is assumed bounded, we surely have $\Sigma \setminus \mathcal{V}_R \neq \emptyset$ regardless of how large we take $R > 0$. Seeing as Σ^n is connected, we therefore conclude that $\Sigma \cap S \neq \emptyset$. Therefore there also exists $s \in \mathbb{R}$ small enough so that $(\Sigma \cap \mathcal{V}_R) \cap \Gamma^n_s \neq \emptyset$.

On the other hand, since $\Sigma \cap \mathcal{V}_R$ is assumed bounded, then for $s \in \mathbb{R}$ large enough we have that $(\Sigma \cap \mathcal{V}_R) \cap \Gamma^n_s = \emptyset$. Because Γ^n is properly embedded, and since $\Sigma \cap \mathcal{V}_R$ is assumed bounded, there exists an extremal value s_0:

$$s_0 := \sup\{s \in \mathbb{R} \colon (\Sigma \cap \mathcal{V}_R) \cap \Gamma^n_s \neq \emptyset\} < \infty.$$

By compactness of $\overline{\Sigma \cap \mathcal{V}_R}$ hence of $\overline{\Sigma \cap S}$ and since Σ is properly immersed, this s_0 is attained at some $p_0 \in (\Sigma \cap \mathcal{V}_R) \cap \Gamma^n_{s_0}$, where we note that $p_0 \in \overline{S}$. Therefore p is a point of $\Sigma \cap \mathcal{V}_R$ which is interior relative to Σ. We can therefore apply Separating Tangency from Lemma 16, which by completeness, connectedness and compactness of the boundary implies that Σ and $\Gamma \times \mathbb{R}^{n-1}$ coincide outside some ambient ball, leading to a contradiction with f.ex. the assumption that $\Sigma \subseteq H_1 \cap H_2$ (or with the boundedness of $\Sigma \cap \mathcal{V}_R$).

Proof for Case (b): Let us summarize how we will now fix the setup throughout the rest of the proof: $R > 0$ will be taken large enough so that $\partial\Sigma \subseteq \mathcal{V}_R$ and, as we are in Case (b), also taken so large that $\Sigma \cap \mathcal{V}_R$ is unbounded (in particular nonempty).

The proof of Theorem 1 might not work here, because it could be that the function f approaches its supremum only by attaining it on the boundary $\partial\Sigma$. Therefore the idea is to modify f in a suitable way, so that the supremum of the new function is guaranteed to not be attained on $\partial\Sigma$ and also in such a way that the argument in the proof of the "Bi-Halfspace" Theorem 1 still goes through. The resulting argument, using the noncompactness to our advantage, is what we call an "adiabatic trick" since it involves tuning a certain length scale as slowly as needed together with estimates for the PDE.

To begin, recall that in the present case, $\Sigma \cap \mathcal{V}_R$ is now assumed to be an unbounded subset of $\mathbb{R}^{n+1}$, so the extrinsic distance to $0 \in \mathbb{R}^{n+1}$ is an unbounded function on $\Sigma \cap \mathcal{V}_R$:

$$\sup_{p \in \Sigma \cap \mathcal{V}_R} \|p\|_{\mathbb{R}^{n+1}} = \infty. \tag{26}$$

Since $\partial\Sigma$ is compact, there exists a radius $\rho > 0$ large enough so that $\partial\Sigma \subseteq B_\rho(0) = \{x \in \mathbb{R}^{n+1} \colon \|x\|_{\mathbb{R}^{n+1}} \leq \rho\}$. For every length scale $\ell > \rho > 0$ (which we soon plan to take as large as needed), let us define the $C^\infty(\mathbb{R}^{n+1})$ function $\chi_\ell \colon \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\chi_\ell(x) = \psi(\|x\|/\ell), \tag{27}$$

where $\psi : [0,\infty) \to \mathbb{R}$ is a standard C^∞ monotone increasing cut-off function $0 \leq \psi \leq 1$ such that $\psi|_{[0,1]} \equiv 0$ while $\psi|_{[2,\infty)} \equiv 1$. Thus since $\ell > \rho > 0$ we have that χ_ℓ vanishes inside the ball $B_\rho(0)$ and therefore also on $\partial\Sigma$. Furthermore, all ambient derivatives of χ_ℓ are uniformly bounded with upper bounds depending only on ℓ (and of course ψ, which we fix once and for all):

$$\sup_{x \in \mathbb{R}^{n+1}} \left\| \nabla^{\mathbb{R}^{n+1}} \chi_\ell(x) \right\|_{\mathbb{R}^{n+1}} \leq \frac{C}{\ell} \quad \text{and} \quad \sup_{x \in \mathbb{R}^{n+1}} |\Delta_{\mathbb{R}^{n+1}} \chi_\ell(x)| \leq \frac{C}{\ell^2}. \tag{28}$$

For every $\ell > 0$, let us define the new function $f_\ell \colon \Sigma^n \to \mathbb{R}$ as follows. With f as in Equation (12) let $M \coloneqq \sup_\Sigma f$ and define:

$$f_\ell(p) \coloneqq f(p) + M\chi_\ell(p), \quad p \in \Sigma. \tag{29}$$

Note that the continuity and smoothness of f_ℓ are no worse than of f. Recall from (13) that $f \leq \frac{R}{\xi}$ so that f_ℓ is also bounded:

$$\sup_\Sigma f_\ell \leq \tfrac{R}{\xi} + M < \infty. \tag{30}$$

Also, since $f > R$ on $\Sigma \cap \mathcal{V}_R$ we have by (26) and by the fact that $\chi_\ell|_{\mathbb{R}^{n+1} \setminus B_{2l}(0)} = 1$:

$$\forall \ell > \rho : \max_{\partial\Sigma} f_\ell \leq M < R + M < \sup_\Sigma f_\ell = \sup_{\Sigma \cap \mathcal{V}_R} f_\ell, \tag{31}$$

using for the first equality that $\chi_\ell|_{\partial\Sigma} = 0$ and for the last that $\sup_{\Sigma \setminus \mathcal{V}_R} f_\ell \leq R + M$. Thus we can now for each $\ell > \rho$ apply the Omori-Yau argument as in the proof of the "Bi-Halfspace" Theorem 1 to the function f_ℓ, this time in the boundary version, now that we by (31) have verified the condition in Lemma 18(i).

Suppose now that there exists $\ell_0 > 0$ such that there is at least one Omori-Yau sequence $p_k \in \Sigma \cap \mathcal{V}_R$ for $f_{\ell_0} : \Sigma \to \mathbb{R}$ with the property that $\|p_k\|_{\mathbb{R}^{n+1}} \to \infty$. Since χ_ℓ is constant outside a compact subset of $\mathbb{R}^{n+1}$, we see $\Delta_\Sigma f(p_k) = \Delta_\Sigma f_\ell(p_k)$ for all sufficiently large values of k, so that the argument in (24) from the case without boundary applies.

Assume now conversely that for every $\ell > 0$, none of the Omori-Yau sequences have unbounded Euclidean norm. Then in consequence f_ℓ attains its maximum at some point $q_\ell \in \Sigma \cap \mathcal{V}_R \backslash \partial\Sigma$ so that $f_\ell(q_\ell) = \sup_{\Sigma\cap\mathcal{V}_R} f_\ell$. Note that then in fact $\|q_\ell\| \geq \ell$ must be the case, as follows from Equation (31). Namely, inside $B_\ell(0)$ holds that $\chi_\ell = 0$, so we get $\sup_{B_\ell(0)} f_\ell \leq M < \sup_\Sigma f_\ell$ and thus the maximum must be attained outside of $B_\ell(0)$.

Now we do analysis on the sequence of maximum points $\{q_\ell\}$. By criticality we have $\nabla^\Sigma f_\ell(q_\ell) = 0$, so by (28) and $\nabla^\Sigma \chi_\ell = \frac{1}{\ell}\psi'(\|p\|/\ell)\nabla^\Sigma\|p\|$:

$$\left\|\nabla^\Sigma f(q_\ell)\right\| = \left\|\nabla^\Sigma f_\ell(q_\ell) - M\nabla^\Sigma \chi_\ell(q_\ell)\right\| = M\left\|\nabla^\Sigma \chi_\ell(q_\ell)\right\| \leq \frac{CM}{\ell}, \tag{32}$$

where we also used

$$\left\|\nabla^\Sigma \|p\|\right\| = \left\|\left(\nabla^{\mathbb{R}^{n+1}}\|p\|\right)^\top\right\| \leq \left\|\nabla^{\mathbb{R}^{n+1}}\|p\|\right\| = 1. \tag{33}$$

As for estimating the Laplacian, we can compute:

$$\begin{aligned}
\Delta_\Sigma\|p\| &= \operatorname{div}_\Sigma\left(\nabla^\Sigma\|p\|\right)\\
&= \operatorname{div}_\Sigma\left(\left(\nabla^{\mathbb{R}^{n+1}}\|p\|\right)^\top\right)\\
&= \operatorname{div}_\Sigma\left(\nabla^{\mathbb{R}^{n+1}}\|p\| - \left(\nabla^{\mathbb{R}^{n+1}}\|p\|\right)^\perp\right)\\
&= \frac{n}{\|p\|} + H\left\langle \nabla^{\mathbb{R}^{n+1}}\|p\|, \nu\right\rangle.
\end{aligned}$$

Therefore, since Σ is a self-translater and hence $|H| \leq 1$, we get by Cauchy-Schwarz:

$$|\Delta_\Sigma\|p\|| \leq \frac{n}{\|p\|} + 1, \quad p \in \Sigma. \tag{34}$$

We thus get, using (33) and (34) with $\|q_\ell\| \geq \ell$:

$$|\Delta_\Sigma \chi_\ell(q_\ell)| \leq \left[\frac{\psi'(\|p\|/\ell)}{\ell}|\Delta_\Sigma\|p\|| + \frac{|\psi''|(\|p\|/\ell)}{\ell^2}\|\nabla^\Sigma\|p\|\|^2\right]_{|q_\ell} \leq \frac{C'}{\ell}. \tag{35}$$

Thus, since $\Delta_\Sigma f_\ell(q_\ell) \leq 0$ we get:

$$\lim_{\ell\to\infty} \Delta_\Sigma f(q_\ell) = \lim_{\ell\to\infty} \Delta_\Sigma f_\ell(q_\ell) - \lim_{\ell\to\infty} \Delta_\Sigma \chi_\ell(q_\ell) \leq 0. \tag{36}$$

Therefore, by (32) and (36), we can plug the sequence of maximum points $\{q_\ell\}$ directly into the same identity (19) derived in the course of the proof of the "Bi-Halfspace" Theorem 1 for the $\partial\Sigma = \emptyset$ case, in order to get a contradiction.

Since, both in Case (1) and in Case (2), we have thus reached a contradiction, we conclude that the hypersurface $(\Sigma, \partial\Sigma)$ must in fact be compact. □

The following corollary completes the picture given by the "Bi-Halfspace" Theorem 1, providing a complete characterization of all the possible couples of hyperspaces such that their intersection contains a properly immersed self-translater. In particular it shows that the "Bi-Halfspace" Theorem 1 does not not hold anymore if we drop the assumption about the verticality of the halfspaces.

Corollary 19. *Let $w_1, w_2 \in \mathbb{S}^n$ and let $H_1 \coloneqq H_{(0,w_1)}$ and $H_2 \coloneqq H_{(0,w_2)}$.*

Then there exists a properly immersed self-translater without boundary contained in $H_1 \cap H_2$ if and only if one of the following conditions hold.

(1) $\langle w_1, e_{n+1}\rangle > 0$ and $\langle w_2, e_{n+1}\rangle > 0$;
(2) $\langle w_1, e_{n+1}\rangle > 0$ and $\langle w_2, e_{n+1}\rangle = 0$;
(3) $\langle w_1, e_{n+1}\rangle = 0$ and $\langle w_2, e_{n+1}\rangle > 0$;
(4) $\langle w_1, e_{n+1}\rangle = \langle w_2, e_{n+1}\rangle = 0$ and $w_1 \parallel w_2$.

Proof. Let us first assume that none of the conditions (1), (2), (3) and (4) are satisfied. This means that $\langle w_1, e_{n+1}\rangle = \langle w_2, e_{n+1}\rangle = 0$ and $w_1 \nparallel w_2$ or one of the two scalar products is strictly negative. In the first case, we know from the "Bi-Halfspace" Theorem 1 that there cannot be properly immersed self-translaters contained in $H_1 \cap H_2$.

Let us assume that one of the two scalar products is strictly negative, say $\langle w_1, e_{n+1}\rangle < 0$. Then we claim that H_1 cannot contain any properly immersed self-translater. This, in particular implies that $H_1 \cap H_2$ does not contained any properly immersed self-translater. Indeed, by contradiction, assume that there exists a properly immersed self-translater $\Sigma^n \subseteq H_1$. Then one can easily find a contradiction by using Lemma 15 and comparing the time evolution of Σ^n with the evolution of some suitably large sphere lying in $\mathbb{R}^{n+1} \setminus H_1$.

Let us now check that if any of (1), (2), (3) or (4) hold, then there exists a properly immersed self-translater contained in $H_1 \cap H_2$.

If (1) holds, then consider for instance the bowl self-translater U. Since U is asymptotic to a paraboloid at infinity, it is clear that, up to a translation in the e_{n+1} direction, $U \subseteq H_1 \cap H_2$.

Let us now assume that (2) or (3) hold. Without loss of generality, we can assume $H_1 = \{x_1 \geq 0\}$ and $\langle w_2, e_{n+1}\rangle > 0$. Since we are assuming $\langle w_2, e_{n+1}\rangle > 0$, we have that $P_2 \coloneqq \partial H_2$ is the graph of an affine function f defined over $\{x_{n+1} = 0\}$. More precisely, let $w_2 = (w_{2,1}, \ldots, w_{2,n}, w_{2,n+1})$. Then f is defined as

$$f(x_1, \ldots, x_n) \coloneqq -\frac{x_1 w_{2,1} + x_2 w_{2,n} \cdots + x_n w_{2,n}}{w_{2,n+1}}.$$

For any $L > 0$, let us define the slab $S_L \coloneqq (0, L) \times \mathbb{R}^{n-1}$. Note that on S_L the function $f|_{S_L}$ is bounded from above by the function

$$g_L(x_1, \dots x_n) \coloneqq L\frac{|w_{2,1}|}{w_{2,n+1}} - \frac{x_2 w_{2,2} \cdots + x_n w_{2,n}}{w_{2,n+1}}$$

and clearly $\nabla g_L = \frac{1}{w_{2,n+1}}(0, w_{2,2}, \dots, w_{2,n})$. Note that ∇g_L does not depend on L. Now take L large enough so that there exists a tilted grim reaper cylinder Σ which is the graph of a function defined on S_L and such that it grows linearly in the direction of ∇g_L and with the same slope of g_L (for a detailed description of tilted grim reaper cylinders, see [GM18] and [BLT18]). Then, since Σ is the graph of a function which is strictly convex w.r.t. the first variable x_1, it can be chosen in such a way that it lies above the graph of g_L and, in particular, inside H_2. Moreover, by construction, Σ is also contained in H_1.

If (4) holds, then observe that $P \coloneqq \partial H_1 = \partial H_2$ is a translater contained in $H_1 \cap H_2$. □

5. On the Convex Hulls of Self-Translaters

In this section we want to study the convex hulls of self-translaters. We will derive a sort of "convex hull property" for compact self-translaters and then we will discuss the classification of the convex hulls of (possibly non-compact) self-translaters with compact boundary, proving Theorem 3. Those two results have been inspired by the theory of classical minimal submanifolds of the Euclidean space. They both show that, up to projecting onto the hyperplane $\mathbb{R}^n \times \{0\}$, the convex hull of a self-translater behaves quite similarly to the convex hull of a minimal submanifold of $\mathbb{R}^{n+1}$.

5.1. **Convex Hulls of Compact Self-Translaters.** The first lemma is a well-known fact about self-translaters and can be proved in several different ways, but, at least to our knowledge, they are all based on some version of the maximum principle. For the sake of completeness we include a proof, close in spirit to an argument given in [Py16].

Lemma 20. *Let $(\Sigma^n, \partial\Sigma)$ be a compact e_{n+1}-directed self-translater in $\mathbb{R}^{n+1}$. Then $\partial\Sigma \neq \emptyset$ and*

$$\max_{\overline{\Sigma}} x_{n+1} = \max_{\partial\Sigma} x_{n+1}.$$

Proof. Recall that given a function $f \in C^1(\mathbb{R}^{n+1})$, the gradient $\nabla^\Sigma f|_\Sigma$ is given by

$$\nabla^\Sigma f|_\Sigma = (\nabla f)^\top, \tag{37}$$

where $(\nabla f)^\top$ is the projection of ∇f on the tangent bundle of Σ.

If we apply (37) to the coordinate function x_{n+1}, we get

$$\nabla^\Sigma x_{n+1} = e_{n+1}^\top. \tag{38}$$

Let $E_1, \dots, E_n$ be a orthonormal frame on Σ and let ν be a unit normal vector field.

Then, using (1), we have

$$\begin{aligned}\Delta_\Sigma x_{n+1} &= \mathrm{div}_\Sigma(e_{n+1}^\top) = \mathrm{div}_\Sigma(e_{n+1} - e_{n+1}^\perp)\\ &= -\Sigma_{j=1}^n \langle \nabla_{E_j} \langle e_{n+1}, \nu\rangle \nu, E_j\rangle\\ &= -\langle e_{n+1}, \nu\rangle \Sigma_{j=1}^n \langle \nabla_{E_j} \nu, E_j\rangle\\ &= H^2.\end{aligned}$$

Therefore x_{n+1} is a subharmonic function on Σ, and hence by the strong maximum principle it cannot have any interior maximum points. □

Now let us show a new "convex hull" property for self-translaters, in the same spirit as the classical one for minimal hypersurfaces. Let us first remind the reader of the minimal hypersurface case.

Proposition 21. *(See e.g. Proposition 1.9 in* [CM11-1]*). If $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is a compact minimal hypersurface with boundary, then $\Sigma \subseteq \mathrm{Conv}(\partial\Sigma)$, where $\mathrm{Conv}(\partial\Sigma)$ is the convex hull of $\partial\Sigma \subseteq \mathbb{R}^{n+1}$.*

Read verbatim, such a statement is ostensibly wrong for self-translaters, as e.g. seen by taking the (compact) pieces of the Altschuler-Wu bowl solution below planes perpendicular to e_{n+1}. Nonetheless, we do have the following modified version. We will by $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the standard orthogonal projection $\pi(x_1, \dots, x_n, x_{n+1}) \coloneqq (x_1, \dots, x_n)$.

Proposition 22. *Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a compact e_{n+1}-directed self-translater with boundary $\partial\Sigma \neq \emptyset$.*

Then

$$\Sigma \subseteq \mathrm{Conv}\left(\pi\left(\partial\Sigma\right)\right) \times (-\infty, \max_{\partial\Sigma} x_{n+1}],$$

where $\mathrm{Conv}\left(\pi\left(\partial\Sigma\right)\right)$ is the convex hull of $\pi(\partial\Sigma) \subseteq \mathbb{R}^n$.

Proof. Let $\tilde{\mathbb{R}}^{n+1} \coloneqq \left(\mathbb{R}^{n+1}, e^{\frac{2}{n}x_{n+1}}\delta_{ij}\right) = \left(\mathbb{R}^{n+1}, \tilde{h}\right)$ be the so-called Huisken-Ilmanen space. It plays an important role due to the following well-known correspondence: $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is a unit speed self-translating surface in the x_{n+1}-direction if and only if Σ is a minimal submanifold of $\tilde{\mathbb{R}}^{n+1}$. See for instance [Sh11] for a proof in the case $n = 2$ or [Pé16] for the general case.

Observe that given a function $f \in C^1\left(\mathbb{R}^{n+1}\right)$, the gradient $\tilde{\nabla} f$ of f w.r.t. the metric $\tilde{h}$ is given by

$$\tilde{\nabla} f = e^{-\frac{2}{n}x_{n+1}} \nabla f. \tag{39}$$

We can now compute $\Delta_{\tilde{\Sigma}} x_j$, for $j = 1, \ldots, n$, using (39) and (37).

$$\begin{aligned}
\Delta_{\tilde{\Sigma}} x_j &= \mathrm{div}_{\tilde{\Sigma}} \left(\nabla^{\tilde{\Sigma}} x_j \right) \\
&= \mathrm{div}_{\tilde{\Sigma}} \left(\left(\tilde{\nabla} x_j \right)^T \right) \\
&= \mathrm{div}_{\tilde{\Sigma}} \left(e^{-\frac{2}{n} x_{n+1}} e_j^\top \right) \\
&= -\frac{2}{n} e^{-\frac{2}{n} x_{n+1}} \tilde{h} \left(\nabla^{\tilde{\Sigma}} x_{n+1}, e_j^\top \right) + e^{-\frac{2}{n} x_{n+1}} \mathrm{div}_{\tilde{\Sigma}} \left(e_j^\top \right) \\
&= -\frac{2}{n} \tilde{h} \left(\nabla^{\tilde{\Sigma}} x_{n+1}, \nabla^{\tilde{\Sigma}} x_j \right) + e^{-\frac{2}{n} x_{n+1}} \mathrm{div}_{\tilde{\Sigma}} \left(e_j \right).
\end{aligned}$$

Note that $\mathrm{div}_{\tilde{\Sigma}} \left(e_j^\top \right) = \mathrm{div}_{\tilde{\Sigma}} \left(e_j \right)$ because $\tilde{\Sigma}$ is minimal in $\tilde{\mathbb{R}}^{n+1}$. Moreover note that $\mathrm{div}_{\tilde{\Sigma}} \left(e_j \right) = 0$ since e_j is a Killing field on $\tilde{\mathbb{R}}^{n+1}$, for every $j = 1, \ldots, n$. Indeed let $\mathcal{L}$ denote the Lie derivative. Then we have

$$\mathcal{L}_{e_j} \tilde{h} = \mathcal{L}_{e_j} \left(e^{\frac{2}{n} x_{n+1}} h \right) = e^{\frac{2}{n} x_{n+1}} \mathcal{L}_{e_j} h = 0. \tag{40}$$

Therefore for each $j = 1, \ldots, n$, the coordinate function x_j satisfies the following linear elliptic PDE:

$$\Delta_{\tilde{\Sigma}} x_j + \frac{2}{n} \tilde{h} \left(\nabla^{\tilde{\Sigma}} x_{n+1}, \nabla^{\tilde{\Sigma}} x_j \right) = 0, \quad j = 1, \ldots, n.$$

From the maximum principle we have that each x_j, for $j = 1, \ldots, n$, attains its maximum and minimum on $\partial\Sigma$. This, together with Lemma 20, concludes the proof. □

Remark 23. Observe that for the proof of Proposition 22 one could alternatively have proven by contradiction that x_j, for $j = 1, \ldots, n$ has no interior maxima and minima using the Lemma 16 and comparing with vertical translating planes. This is not surprising, since the Principle of Separating Tangency is another manifestation of the strong maximum principle for quasilinear elliptic equations.

Note also that only x_i when $i = 1, \ldots, n$ works, and that one could not use x_{n+1} in Proposition 22, as the similar computation as in (40) performed for e_{n+1} shows that e_{n+1} is not a Killing field of $\tilde{\mathbb{R}}^{n+1}$.

The "convex hull" property provides immediately the following monotonicity of topology for compact self-translaters.

Corollary 24. *Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a compact self-translater. Let $C \subseteq \mathbb{R}^n$ be a compact convex set such that $C \cap \pi(\partial\Sigma) = \emptyset$, where π is the usual projection $\pi\colon (x_1, \ldots, x_n, x_{n+1}) \to (x_1, \ldots, x_n)$.*

Then the inclusion map $i\colon (C \times \mathbb{R}) \cap \Sigma \hookrightarrow \Sigma$ induces an injection on the $(n-1)$-st homology group.

Proof. The proof is very similar to the one of Lemma 1.11 in [CM11-1]. □

5.2. **Convex Hulls of Noncompact Self-Translaters.** Note that the results in the preceding section were all about compact self-translaters. We will now study the convex hull property in the noncompact case (Theorem 3). Also, as mentioned in the introduction, this result was inspired by the classical result for minimal submanifolds in Euclidean space proved by Hoffman and Meeks in [HM90] that we recall here.

Theorem 25 (Hoffman-Meeks: Theorem 3 in [HM90])**.** *Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a properly immersed connected minimal submanifold whose (possibly empty) boundary $\partial\Sigma$ is compact. Then exactly one of the following holds:*

(1) $\mathrm{Conv}(\Sigma) = \mathbb{R}^{n+1}$*,*
(2) $\mathrm{Conv}(\Sigma)$ *is a halfspace,*
(3) $\mathrm{Conv}(\Sigma)$ *is a closed slab between two parallel hyperplanes,*
(4) $\mathrm{Conv}(\Sigma)$ *is a hyperplane,*
(5) $\mathrm{Conv}(\Sigma)$ *is a compact convex set. This case occurs precisely when Σ is compact.*

Moreover, when $n = 2$, $\partial\Sigma$ has nonempty intersection with each boundary component of $\mathrm{Conv}(\Sigma)$*.*

Recall again that from the known examples (see Section 3), we cannot hope to have the same characterization of the convex hulls of self-translaters. But we can characterize the convex hull of the projection onto the hyperplane $\mathbb{R}^n \times \{0\}$. This is the content of Theorem 3 and the proof is based on the "Bi-Halfspace" Theorem 1.

Remark 26. Note that the last statement of Theorem 25, which follows from the Halfspace Theorem (Theorem 1 in [HM90]), does not have a straightforward equivalent in the context of self-translaters. Indeed it is natural to ask if it is true or not that given a connected, properly immersed, 2-dimensional self-translater $\Sigma^2 \subseteq \mathbb{R}^3$ with compact boundary, $\pi(\partial\Sigma)$ has nonempty intersection with each topological boundary component of $\mathrm{Conv}(\pi(\Sigma))$. The answer is negative. Indeed one can easily build a counterexample by taking as Σ a grim reaper cylinder with a compact set removed.

Before giving the proof of Theorem 3, let us first prove the following simple characterizations of compact self-translaters.

Lemma 27 (Characterization of Compact Self-Translaters)**.** *Let $(\Sigma^n, \partial\Sigma)$ be a properly immersed, connected self-translater with compact boundary. Then the following are equivalent.*

(1) Σ *is compact.*
(2) $\sup_\Sigma x_{n+1} < \infty$.
(3) Σ *is contained in a cylinder of the kind $K \times \mathbb{R}$, where $K \subseteq \mathbb{R}^n$ is a compact set.*

Proof of Lemma 27. (1) ⇒ (2). If Σ is compact, then clearly $\sup_\Sigma x_{n+1} < \infty$.

(2) $\Rightarrow$ (3). Let us assume that $\sup_\Sigma x_{n+1} < \infty$. Let $R > 0$ be a radius large enough such that $\pi(\partial\Sigma) \subseteq B_R(0)$, where $B_R(0)$ is the ball of radius $R > 0$ in $\mathbb{R}^n \times \{0\}$, centered in 0.

Let us consider the winglike self-translaters W_R from [CSS07], which we translate so that $\inf_{p\in W_R} x_{n+1}(p) = 0$. Let us define the one-parameter family of wing-like self-translater $\{W_{R,s}\}_{s\in\mathbb{R}}$, where $W_{R,s} \coloneqq W_R + s\, e_{n+1}$. Clearly we have that

$$W_{R,s} \cap \Sigma = \emptyset, \tag{41}$$

for every $s > \sup_\Sigma x_{n+1}$. Assume by contradiction that there exists $s \in \mathbb{R}$ such that $W_{R,s} \cap \Sigma \neq \emptyset$. Since Σ is properly immersed, there exists

$$s_0 \coloneqq \max\{s \in \mathbb{R} \colon W_{R,s} \cap \Sigma \neq \emptyset\}.$$

This leads to a contradiction, thanks to Lemma 16. Therefore (41) holds for every $s \in \mathbb{R}$ and thus Σ is contained in the cylinder $B_R(0) \times \mathbb{R}$.

(3) $\Rightarrow$ (1) Let us assume that $\Sigma \subseteq K \times \mathbb{R}$, for some compact set $K \subseteq \mathbb{R}^n$. Let us assume by contradiction that Σ is not compact. This implies that $\sup_\Sigma x_{n+1} = \infty$ or $\inf_\Sigma x_{n+1} = -\infty$. Let us consider the first case (the other case is similar).

Since $\partial\Sigma$ is compact, we can assume w.l.o.g. that $\partial\Sigma \subseteq \{x_{n+1} \leq -1\}$. For every $R > 0$, let $W_{R,0}$ be the winglike self-translater with neck size $R > 0$ and such that $\min_{W_{R,0}} x_{n+1} = 0$. Let us consider the family $\{W_{R,0}\}_{R>0}$. Note the difference with the winglike self-translaters family above: now the "height" is fixed and $R > 0$ is a parameter.

Observe that $W_{R,0} \cap (K \times \mathbb{R}) = \emptyset$ for $R > 0$ large enough. Therefore $W_{R,0} \cap \Sigma = \emptyset$, for $R > 0$ large enough. On the other hand, since Σ is connected and since $\sup_\Sigma x_{n+1} = \infty$, there exists $r > 0$ small enough such that $W_{r,0} \cap \Sigma \neq \emptyset$. Since Σ is properly immersed, there exists

$$r_0 \coloneqq \max\{r > 0 \colon W_{r,0} \cap \Sigma \neq \emptyset\}.$$

Note that since $\partial\Sigma \subseteq \{x_{n+1} \leq -1\}$ every point in the intersection $W_{r_0,0} \cap \Sigma$ is an interior point. This contradicts Lemma 16. $\square$

Proof of Theorem 3. First of all, observe that the "if and only if" part in Theorem 3's Case (5) follows directly from Lemma 27.

Take $\Sigma^n \subseteq \mathbb{R}^{n+1}$ possibly with compact boundary $\partial\Sigma$. The vertical projection of the convex hull of Σ^n, or equivalently convex hull of the vertical projection, can be written as the intersection of all vertical halfspaces in $\mathbb{R}^{n+1}$ which contain it:

$$\mathrm{Conv}(\pi(\Sigma)) = \bigcap_{\{H \colon \Sigma\subseteq H \text{ vertical halfspace of } \mathbb{R}^{n+1}\}} \pi(H) \subseteq \mathbb{R}^n. \tag{42}$$

If the index set is empty we get $\mathrm{Conv}(\pi(\Sigma)) = \mathbb{R}^n$ and arrive at Case (1). So, we assume now that this is not the case.

We will now deduce that in the intersection (42) all the involved halfspaces $H \subseteq \mathbb{R}^{n+1}$, and hence all the $\pi(H) \subseteq \mathbb{R}^n$, are in fact (anti-)parallel halfspaces, unless we are in Case (5). Namely, let H_1 and H_2 be any two vertical closed halfspaces of $\mathbb{R}^{n+1}$, i.e. such that $P_1 := \partial H_1$ and $P_2 := \partial H_2$ are two hyperplanes both containing e_{n+1}, and with $\Sigma^n \subseteq H_1 \cap H_2$. Then if H_1 and H_2 were not (anti-)parallel, the compact boundary version of the "Bi-Halfspace" Theorem 2 would imply that Σ^n is compact (and note that necessarily $\partial\Sigma \neq \emptyset$ too), so that we would arrive at Case (5).

We may thus finally assume that we are not in Case (1) nor in Case (5). Since all vertical halfspaces in $\mathbb{R}^{n+1}$ which contain Σ^n are then mutually (anti-)parallel, so are all the $(n-1)$-dimensional hyperplanes $\pi(H)$ in $\mathbb{R}^n$ and the intersection in (42) is now easy to evaluate: One of the Cases (2), (3) or (4) must occur. This concludes the proof of Theorem 3. □

Remark 28. Even though Theorem 3 was inspired by Theorem 25, our proof is quite different from the original proof of Hoffman and Meeks in [HM90].

First of all, observe that the "if and only if" of point (5) in Theorem 25 is trivial, but one implication of the "if and only if" of point (5) in Theorem 3 is not completely obvious.

But the most important difference is that the proof of Hoffman and Meeks is an elaborate application of the maximum principle for the nonlinear minimal hypersurface equation, while our proof is based on the Omori-Yau maximum principle.

In the Appendix 6 we provide an alternative proof of Theorem 3 in the case $n = 2$ which is based on Lemma 16 and it is closer in spirit to the original proof of Hoffman and Meeks. We also explain why it is hard to extend it to higher dimension.

6. Appendix

In this appendix we present an alternative proof of Theorem 3, which works only in the case $n = 2$.

Before presenting the proof, let us recall the following simple property about winglike self-translaters.

Lemma 29. *Let $R > 0$ and let $W_R \subseteq \mathbb{R}^{n+1}$ be the wing-like self-translater as in* [CSS07] *and* [Mø14]*. Let us denote by $R^* > R$ the radius at which the coordinate function x_{n+1} attains the minimum on W_R.*

Then

$$R^* - R \leq \frac{\pi}{2}.$$

Proof. The proof of this lemma is contained in the proof of Lemma 2.1 in [Mø14]. □

Proof of the 2-dimensional version of Theorem 3. Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a properly immersed self-translater with compact boundary $\partial\Sigma$. In the theorem, let us assume that the Cases (1), (4) and (5) do not occur. We want to show that then Case (2) or Case (3) must occur. Let H_1 and H_2 be two closed halfspaces (here: halfplanes) in $\mathbb{R}^2$ such that $\mathrm{Conv}(\pi(\Sigma)) \subseteq H_1 \cap H_2$. Let

$P_1 \coloneqq \partial H_1$ and $P_2 \coloneqq \partial H_2$. In order to show that case (2) or case (3) must occur, it is sufficient to show that the lines P_1 and P_2 are parallel.

Let us assume by contradiction that P_1 and P_2 are not parallel. The idea is to show that Σ must be then contained in a halfspace of the kind $\{x_3 \leq K\}$ for K large enough. This will contradict Lemma 27.

Let us consider $\tilde{H}_1 \coloneqq \pi^{-1}(H_1) = H_1 \times \mathbb{R}$ and $\tilde{H}_2 \coloneqq \pi^{-1}(H_2) = H_2 \times \mathbb{R}$. Note that $\tilde{H}_1$ and $\tilde{H}_2$ are closed halfspaces of $\mathbb{R}^3$ and $\Sigma \subseteq \tilde{H}_1 \cap \tilde{H}_2$. Moreover we will denote $\tilde{P}_1 \coloneqq \pi^{-1}(P_1) = P_1 \times \mathbb{R}$ and $\tilde{P}_1 \coloneqq \pi^{-1}(P_1) = P_1 \times \mathbb{R}$. Note that $\tilde{P}_1$ and $\tilde{P}_1$ are affine planes in $\mathbb{R}^3$, both parallel to the x_3-axis. Without loss of generality, we may assume that $\tilde{P}_1 \cap \tilde{P}_2$ is the x_3-axis.

From Lemma 16, since $\tilde{P}_1$ and $\tilde{P}_2$ are both self-translaters, Σ does not have any interior point in common with them, i.e. $(\Sigma \setminus \partial\Sigma) \cap \left(\tilde{P}_1 \cup \tilde{P}_2\right) = \emptyset$. For every $R > 0$, let $S_R \subseteq H_1 \cap H_2 \subseteq \mathbb{R}^2$ be the unique circle of radius $R > 0$ and tangent to P_1 and P_2 and let $p_R \in H_1 \cap H_2$ be the center of S_R. Moreover let $\bar{B}_R(p_R)$ be the closed ball of center p_R and radius $R > 0$. Observe that since S_R is tangent to P_1 and P_2, $(H_1 \cap H_2) \setminus \bar{B}_R$ consists of two connected regions, one bounded and the other one unbounded. Let us denote by A_R the the closure of the bounded region. Observe that

$$\lim_{R \searrow 0} \operatorname{diam} A_R = 0.$$

For each $R > 0$, let W_R be the wing-like self-translater such that it is rotationally symmetric around $\{p_R\} \times \mathbb{R}$ and $\min_{W_R} x_3 = 0$ and $R > 0$ is the aperture of the "hole". Moreover, let R^* be the radius as in Lemma 29, i.e. $x_3 = 0$ on the circle $S_{R^*}(p_R)$ of radius R^* and centered in p_R.

$$\tilde{W}_R \coloneqq W_R \cap (A_R \times \mathbb{R}).$$

It is easy to check that $\tilde{W}_R \subseteq \tilde{H}_1 \cap \tilde{H}_2$ is compact and $\partial\tilde{W}_R \subseteq \tilde{P}_1 \cup \tilde{P}_2$.

Since $\partial\Sigma$ is compact, up to a translation in the x_3-direction, we can assume $\partial\Sigma \subseteq \{x_3 \leq -1\}$.

Moreover, since Σ is properly immersed, we have that there exists $r > 0$ small enough, such that

$$\tilde{W}_r \cap \Sigma = \emptyset.$$

Consider the 1-parameter family $\{\tilde{W}_R\}_{R>0}$. Using Lemma 16 and a standard argument, we have that $\tilde{W}_R \cap \Sigma = \emptyset$ for every $R > 0$.

From Lemma 29, we have that $S_{R^*}(p_R) \cap A_R \neq \emptyset$, for every $R > 0$ such that $\operatorname{dist}(p_R, 0) > \frac{\pi}{2}$. Moreover the family of compact sets $\{S_{R^*}(p_R) \cap A_R\}_{R>0}$ swipes out the whole plane $\mathbb{R}^2 \times \{0\}$, i.e.

$$\bigcup_{R>0} S_{R^*}(p_R) \cap A_R = \mathbb{R}^2 \times \{0\}.$$

Therefore we have that

$$\Sigma \subseteq \{x_3 \leq 0\}. \tag{43}$$

Recall that Σ is not compact, because we are assuming that (1), (4) and (5) do not hold. This generates a contradiction because from (43) and from Lemma 27, we have that Σ must be compact.

Therefore we showed that if (1), (4) and (5) do not hold, then (2) or (3) must occur. □

Observe that the above proof is quite similar to the proof in [HM90], but it works only for $n = 2$. Indeed note that it is not possible to naively generalize the above proof to higher dimension. The problem is that it is not possible to define the set A_R. Indeed let us assume that $n \geq 3$ and let H_1 and H_2 be halfspaces of $\mathbb{R}^n$ as in the proof above, and let P_1 and P_2 be their boundaries respectively. Then let B a closed ball such that $S = \partial B$ is tangent both to P_1 and to P_2 and such that $B \subseteq H_1 \cap H_2$. Then $(H_1 \cap H_2) \setminus B$ is connected. Therefore the argument of the proof above does not work.

However, with a straightforward generalization of the argument above, one can prove a weaker version of Theorem 2. More precisely, one can prove the following result.

Theorem 30. *Let $(\Sigma^n, \partial\Sigma)$ be a properly immersed connected self-translating n-dimensional hypersurface in $\mathbb{R}^{n+1}$. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a half-cone, i.e.*

$$\mathcal{C} = \{x \in \mathbb{R}^n \colon \operatorname{angle}(x, w) < \alpha\}$$

for some $w \in \mathbb{S}^{n-1}$ and some angle $\alpha \in (0, \frac{\pi}{2})$.

Then if $\Sigma^n \subseteq \mathcal{C} \times \mathbb{R}$ it must be compact.

Remark 31. The proof of Hoffman and Meeks works in any dimension because they used as barriers solutions of a Dirichlet problem for the minimal hypersurface equation.

Indeed it is known that for every bounded, convex, C^2 domain $\Omega \subseteq \mathbb{R}^n$, and for every $\varphi \in C^0(\partial\Omega)$ there exist a solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of the following Dirichlet problem.

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi & \text{on } \partial\Omega. \end{cases} \tag{44}$$

For more details, see Section 16.3 in [GT77].

In our case we would have needed to solve a Dirichlet problem of the kind (45). Indeed it is easy to verify that a self-translater which is graphical w.r.t. a direction orthogonal to the moving direction e_{n+1} is the graph of a function satisfying the PDE below in (45). Unfortunately in this case there is no general existence result, even assuming the initial data to be smooth. See Proposition 32 below. Therefore we firstly resorted to building barriers carefully from the known family of wing-like self-translaters, the drawback being that this procedure only works in the case $n = 2$, as we already explained. This motivated us to look for a different approach and led us to the proof of the "Bi-Halfspace" Theorems 1–2 and consequently to

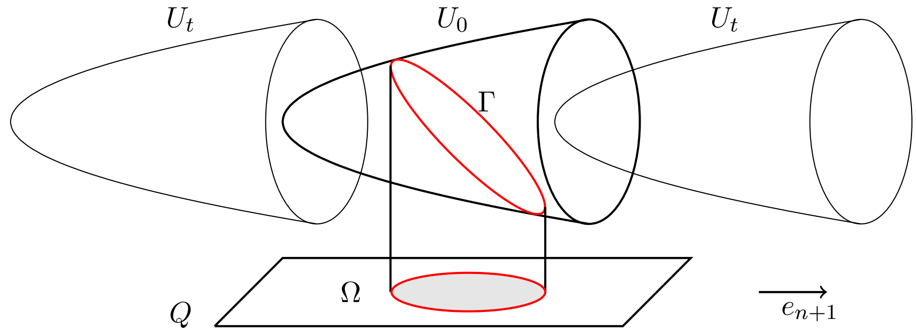

FIGURE 1.

the proof of Theorem 3, as presented in the main parts (see Section 5.2) of this paper.

Proposition 32. *There exists $\Omega \subseteq \mathbb{R}^n$ bounded, convex with smooth boundary $\partial\Omega$ and there exists $\varphi \in C^\infty(\partial\Omega)$ such that there exists no function $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $u = u(y_1, \dots, y_n)$, satisfying the following Dirichlet problem.*

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{u_{y_1}}{\sqrt{1+|Du|^2}} & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi & \text{on } \partial\Omega \end{cases} \tag{45}$$

Proof. Let $U \subseteq \mathbb{R}^{n+1}$ be the bowl self-translater. Let P be an affine hyperplane of $\mathbb{R}^{n+1}$ such that it is not parallel to e_{n+1} but not orthogonal to e_{n+1}. Let Q be another hyperplane parallel to e_{n+1} and such that P is graphical over Q.

Let $\Gamma \coloneqq U \cap P$. Observe that, up to translating P in the direction of e_{n+1}, we can assume $\Gamma \neq \emptyset$. Moreover, we can take P such that $\Gamma = \partial U_\Gamma$, where $U_\Gamma \subseteq U$ is a bounded subset of U which is not graphical over Q.

Let $\pi_Q \colon \mathbb{R}^{n+1} \to Q$ be the orthogonal projection onto Q.

Since U is a convex hypersurface, we have that $\pi(\Gamma)$ is the boundary of some bounded convex domain $\Omega \subseteq Q$ (see Figure 1). Since P is graphical over Q, we have that Γ is the graph of some function $\phi \colon \partial\Omega \to \mathbb{R}$.

Let $y_1, \dots, y_n$ be Cartesian coordinates on Q such that the coordinate y_1 coincides with x_{n+1}.

Now assume by contradiction that there exists a solution u for the Dirichlet problem (45).

Therefore $\operatorname{graph}(u)$ is a compact self-translater with unit velocity e_{n+1} with boundary Γ.

Now for every $t \in \mathbb{R}$ define $U_t \coloneqq U + te_{n+1}$. Observe that the family $\{U_t\}_{t\in\mathbb{R}}$ foliates $\mathbb{R}^{n+1}$.

Since $\operatorname{graph}(u)$ is compact and each U_t is properly immersed, there exist

$$t_{\min} \coloneqq \min\{t \in \mathbb{R} \colon U_t \cap \operatorname{graph}(u) \neq \emptyset\}$$

and

$$t_{\max} \coloneqq \max\{t \in \mathbb{R} \colon U_t \cap \operatorname{graph}(u) \neq \emptyset\}.$$

If $t_{\min} < 0$, then every point $p \in U_{t_{\min}} \cap \operatorname{graph}(u)$ would be an interior point of $\operatorname{graph}(u)$. From Lemma 16, we would have that $\operatorname{graph}(u) \subseteq U_{t_{\min}}$, and therefore $\Gamma = \partial(\operatorname{graph}(u)) \subseteq U_{t_{\min}}$. But this is a contradiction because $\Gamma \subseteq U_0 = U$. Therefore $t_{\min} = 0$.

With a similar argument one can show that $t_{\max} = 0$. Therefore $\operatorname{graph}(u) = U_\Gamma \subseteq U_0$. But this is a contradiction, because U_Γ is not graphical by construction.

□

References

[AL86] U. Abresch, J. Langer, *The normalised curve shortening flow and homothetic solutions*, J. Differential Geometry **23** (1986), 175–196.

[Al91] S. Altschuler,*Singularities for the curve shortening flow for space curves*, J. Differential Geometry **34** (1991), 491–514.

[AMR16] L.J. Alías, P. Mastrolia, M. Rigoli, *Maximum principles and geometric applications*, Springer Monographs in Mathematics (2016), 570 pp. ISBN: 978-3-319-24335-1.

[An91-1] S. Angenent, Parabolic equations for curves on surfaces (II). Intersections, blowup and generalised solutions, Annals of Math. **133** (1991), 171–215.

[An91-2] S. Angenent, On the formation of singularities in the curve shortening flow, J. Differential Geometry **33** (1991), 601–633.

[AW94] S.J. Altschuler, L.F. Wu, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations **2** (1994), no. 1, 101–111.

[ACI95] S. Angenent, D. L. Chopp, T. Ilmanen, *A computed example of nonuniqueness of mean curvature flow in* $\mathbb{R}^3$, Comm. in Partial Differential Equations **20** (1995) no. 11–12, 1937–1958.

[BF14] A.P. Barreto, F. Fontenele, *Some remarks on the Pigola-Rigoli-Setti version of the Omori-Yau maximum principle*, Bull. Aust. Math. Soc. **89** (2014), no. 2, 337–342.

[BJO01] G.P. Bessa, L.P. Jorge, G. Oliveira-Filho, *Half-space theorems for minimal surfaces with bounded curvature*, J. Differential Geom. **57** (2001), no. 3, 493–508.

[BLPS13] G.P. Bessa, J. H. de Lira, S. Pigola, A.G. Setti, *Curvature estimates for submanifolds immersed into horoballs and horocylinders*, arXiv:1308.5926v2.

[BLM13] G.P. Bessa, J.H. de Lira, A.A. Medeiros, *Comparison principle, stochastic completeness and half-space theorems*, arXiv:1307.2658v2.

[BLP15] G.P. Bessa, B.P. Lima, L.F. Pessoa, *Curvature estimates for properly immersed* ϕ_h*-bounded submanifolds*, Ann. Mat. Pura Appl. (4) **194** (2015), no. 1, 109–130.

[Bo11] A. Borbély, *On minimal surfaces satisfying the Omori-Yau principle*, Bull. Aust. Math. Soc. **84** (2011), 33–39.

[Bo17] A. Borbély, *Stochastic completeness and the Omori-Yau maximum principle*, J. Geom. Anal. **27** (2017), no. 4, 3228–3239.

[BLT18] T. Bourni, M. Langford, and G. Tinaglia, *On the existence of translating solutions of mean curvature flow in slab regions*, arXiv:1805.05173v3 (2018).

[Br78] K.A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, 20, Princeton University Press, Princeton, N.J., ISBN 0-691-08204-9, MR 0485012.

[Bu18] A. Bueno, *Translating solitons of the mean curvature flow in the space* $\mathbb{H}^2 \times \mathbb{R}$, arXiv:1803.02783v3.

[CE16] M.P. Cavalcante, J.M. Espinar, *Halfspace type theorems for self-shrinkers*, Bull. Lond. Math. Soc. **48** (2016), no. 2, 242–250.

[CM11-1] T. H. Colding and W.P. Minicozzi, *A Course in Minimal Surfaces*, AMS (2011).

[CM11-2] Colding, Tobias H.; Minicozzi, William P., II Minimal surfaces and mean curvature flow. Surveys in geometric analysis and relativity, 73–143, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, 2011.

[CM12] T.H. Colding, W.P. Minicozzi, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833.

[CM19] F. Chini, N.M. Møller, *Ancient mean curvature flows and their spacetime tracks*, arXiv:1901.05481.

[CSS07] J. Clutterbuck, O. C. Schnürer and F. Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 281–293.

[CY75] S.Y. Cheng, S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354.

[DDPN17] J. Dávila, M. Del Pino, X.H. Nguyen, *Finite topology self-translating surfaces for the mean curvature flow in* $\mathbb{R}^3$ Advances in Mathematics **320** (2017), 674–729.

[DLN18] G. Drugan, H. Lee, and X. H. Nguyen, *A survey of closed self-shrinkers with symmetry*, Results Math **73** (2018), no. 1, Art. 32, 32 pp.

[Fr66] T. Frankel, *On the fundamental group of a compact minimal submanifold*, Ann. of Math. (2) **83** (1966), 68–73.

[GH86] M. Gage, R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), no. 1, 69–96.

[Gr87] M.A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987), no. 2, 285–314.

[Gr89] M.A. Grayson, *Shortening embedded curves*, Annals of Math. **129** (1989), 71–111.

[GM18] E. S. Gama and F. Martín, *Translating Solitons of the Mean Curvature Flow Asymptotic to Hyperplanes of* $\mathbb{R}^{n+1}$, arXiv:1802.08468v2 (2018).

[GT77] D. Gilbarg and N.S. Trudinger (1977), *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag.

[Ha88] R.S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math. **71** (1988), 237–261.

[Ha95] R.S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Differential Geom. **41** (1995), no. 1, 215–226.

[Ha15] R. Haslhofer, *Uniqueness of the bowl soliton*, Geometry & Topology **19** (2015) 2393–2406.

[HK17] R. Haslhofer, B. Kleiner *Mean curvature flow of mean convex hypersurfaces*, Comm. Pure Appl. Math. **70** (2017), no. 3, 511–546.

[HIMW18-1] D. Hoffman, T. Ilmanen, F. Martín, and B. White, *Graphical Translators for the Mean Curvature Flow*, arXiv:1805.10860v2 (2018).

[HIMW18-2] D. Hoffman, T. Ilmanen, F. Martín, B. White, *Families of Translators for Mean Curvature Flow*, in preparation (2018).

[HM90] D. Hoffman and W. H. Meeks, III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990), 373–377.

[HRS08] L. Hauswirth, H. Rosenberg, J. Spruck, *On complete mean curvature* $\frac{1}{2}$ *surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Comm. Anal. Geom. **16** (2008), no. 5, 989–1005.

[Hu84] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984) 237–266.

[HS99a] G. Huisken, C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), 45–70.

[HS99b] G. Huisken, C. Sinestrari, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations **8** (1999), 1–14.

[IR17] D. Impera and M. Rimoldi, *Rigidity results and topology at infinity of translating solitons of the mean curvature flow*, Comm. Contemp. Math. **19** (2017), no. 6, 21 pp.

[IPR18] D. Impera, S. Pigola, M. Rimoldi, *The Frankel property for self-shrinkers from the viewpoint of elliptic PDEs*, arXiv:1803.02332.

[Ka90] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. (2) **131** (1990), 239–330.

[KKM11] N. Kapouleas, S.J. Kleene, N.M. Møller, *Mean curvature self-shrinkers of high genus: non-compact examples*, J. Reine Angew. Math. **739** (2018), 1–39. arXiv:1106.5454.

[KM14] S. Kleene, N.M. Møller, *Self-shrinkers with a rotational symmetry*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 3943–3963.

[LT78] A. Lichnewsky, R. Temam, *Pseudosolutions of the time-dependent minimal surface problem*, J. Differential Equations **30** (1978), no. 3, 340–364.

[Ma11] C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Birkhäuser (2011)

[MSS14] F. Martin, A. Savas-Halihaj, K. Smoczyk, *On the topology of translating solitons of the mean curvature flow*, arXiv:1404.6703.

[MR90] W.H. Meeks, H. Rosenberg, *The maximum principle at infinity for minimal surfaces in flat three manifolds*, Comment. Math. Helv. **65** (1990), no. 2, 255–270.

[MR08] W.H. Meeks, H. Rosenberg, *Maximum principles at infinity*, J. Differential Geom. **79** (2008), no. 1, 141–165.

[MSS15] F. Martín, A. Savas-Halilaj, and K. Smoczyk, *On the topology of translating solitons of the mean curvature flow*, Calc. Var. Partial Differential Equations **54** (2015), 2853–2882.

[Mø14] N. M. Møller, *Non-existence for self-translating solitons*, arXiv:1411.2319 (2014).

[Mu56] W.W. Mullins, *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. **27** (1956), 900–904.

[Na96] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*, Invent. Math. **126** (1996), no. 3, 457–465.

[Ng09] X.H. Nguyen, *Translating tridents*, Commun. Partial Differ. Equ. **34** (2009), 257–280.

[Ng11] X.H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow*, Duke Math. J. **163** (2014), no. 11, 2023–2056. arXiv: 1106.5272.

[Ng13] X.H. Nguyen, *Complete embedded self-translating surfaces under mean curvature flow*, J. Geom. Anal. **23**(3) (2013) 1379–1426.

[Ng15] X.H. Nguyen, *Doubly periodic self-translating surfaces for the mean curvature flow*, Geom. Dedicata **174** (2015) 177–185.

[NS10] B. Nelli, R. Sa Earp, *A halfspace theorem for mean curvature $H = \frac{1}{2}$ surfaces in* $\mathbb{H}^2 \times \mathbb{R}$., J. Math. Anal. Appl. **365** (2010), no. 1, 167–170.

[Om67] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967) 205–214.

[Pe02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:0211159.

[Pé16] J. Pérez-García, *Some results on Translating Solitons of the Mean Curvature Flow*, Doctoral thesis, University of Granada (2016).

[PW03] P. Petersen, F. Wilhelm, *On Frankel's theorem*, Canad. Math. Bull. **46** (2003), no. 1, 130–139.

[PR14] S. Pigola, M. Rimoldi, *Complete self-shrinkers confined into some regions of the space*, Ann. Global Anal. Geom. **45** (1), 47–65 (2014).

[PRS03] S. Pigola, M. Rigoli, A.G. Setti, *A remark on the maximum principle and stochastic completeness*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1283–1288.

[PRS05] S. Pigola, M. Rigoli, A.G. Setti, *Maximum principles on Riemannian manifolds and applications*, Mem. Amer. Math. Soc. **174** (2005), no. 822, x+99 pp.

[Py16] J. Pyo, *Compact translating solitons with non-empty planar boundary*, Differential Geom. Appl. **47** (2016), 79–85.

[RRS95] A. Ratto, M. Rigoli, A.G. Setti, *On the Omori-Yau maximum principle and its applications to differential equations and geometry*, J. Funct. Anal. **134** (1995), no. 2, 486–510.

[RSS13] H. Rosenberg, F. Schulze, J. Spruck, *The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$*, J. Differential Geom. **95** (2013), no. 2, 321–336.

[Sa60] Richard Sacksteder, *On hypersurfaces with no negative sectional curvatures*, Amer. J. Math. **82**, (1960), 609–630.

[Sh11] L. Shahriyari, *Translating graphs by mean curvature flow*, Doctoral thesis, Johns Hopkins University (2012).

[Sh15] L. Shahriyari, *Translating graphs by mean curvature flow*, Geom. Dedicata **175** (2015), 57–64.

[Sm15] G. Smith, *On complete embedded translating solitons of the mean curvature flow that are of finite genus*, arXiv: 1501.04149.

[Sm01] K. Smoczyk, *A relation between mean curvature flow solitons and minimal submanifolds*, Math. Nachr. **229** (2001), 175–186.

[SX17] J. Spruck, L. Xiao, *Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature*, arXiv:1703.01003.

[SX16] J. Spruck, L. Xiao, *Entire downward translating solitons to the mean curvature flow in Minkowski space*, Proc. Amer. Math. Soc. **144** (2016), 3517–3526.

[SY94] R. Schoen, S.-T. Yau, *Lectures on differential geometry*, International Press, Cambridge, MA (1994), ISBN: 1-57146-012-8.

[Xa84] F. Xavier, *Convex hulls of complete minimal surfaces*, Math. Ann. **269** (1984), no. 2, 179–182.

[Xi15] Y.L. Xin, *Translating solitons of the mean curvature flow*, Calc. Var. **54** (2015), 1995–2016.

[Wa11] X.-J. Wang, *Convex solutions to the mean curvature flow*, Ann. Math. **173** (2011), 1185–1239.

[Whi00] B. White. *The size of the singular set in mean curvature flow of mean convex sets*, J. Amer. Math. Soc., **13(3)** (2000), 665–695.

[Wh02] B. White *Evolution of curves and surfaces by mean curvature*, Proceeding of the ICM (2002), pp. 525-538.

[Whi03] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc., **16(1)** (2003), 123–138.

[WW09] G. Wei, W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405.

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Paper B. Ancient mean curvature flows and their spacetime tracks

ANCIENT MEAN CURVATURE FLOWS AND THEIR SPACETIME TRACKS

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Abstract. We study properly immersed ancient solutions of the codimension one mean curvature flow in n-dimensional Euclidean space, and classify the convex hulls of the subsets of space reached by any such flow.

In particular, it follows that any compact convex ancient mean curvature flow can only have a slab, a halfspace or all of space as the closure of its set of reach.

The proof proceeds via a bi-halfspace theorem (also known as a wedge theorem) for ancient solutions derived from a parabolic Omori-Yau maximum principle for ancient mean curvature flows.

1. Introduction

Ancient mean curvature flows show up naturally in the study of singularities, as tangent flows from blow-up analysis (for a basic discussion, see e.g. Chapter 4 in [Ma11]). Especially in recent years they have gained much attention and some partial classifications are now available.

Convex ancient solutions arise in the case of mean convex mean curvature flows [Wh00], [HS99a], [HS99b] and have also been investigated by [Wa11].

In the case of the curve shortening flow, Daskalopoulos, Hamilton and Sesum provided a complete classification [DHS10] of closed convex embedded ancient curve shortening flows. Note that there also exist nonconvex examples [AY18].

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Key words and phrases. Mean curvature flow, ancient solutions, minimal surfaces, bi-halfspace theorem, wedge theorem, Omori-Yau principle, maximum principles, nonlinear partial differential equations, parabolic equations.

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In [HS15], Huisken and Sinestrari again studied ancient mean curvature flow under several natural curvature assumptions, namely convexity and k-convexity, and provided some characterizations of the shrinking sphere, assuming convexity.

Haslhofer and Hershkovits [HH16] proved the existence of an ancient oval in dimensions $n > 1$, as conjectured by Angenent in [An13], building on an idea of White [Wh03]. Recently Angenent, Daskalopoulos and Sesum [ADS18] proved the uniqueness of this ancient mean curvature flow under some assumptions, based on their previous work [ADS15] (using the barriers from [KM14]). More precisely they prove that an ancient mean curvature flow which is compact, smooth, noncollapsed, not self-similar and uniformly 2-convex must be the solution constructed in [Wh03]-[An13]-[HH16].

In [BC17] Brendle and Choi classified convex, noncompact, noncollapsed ancient flows in $\mathbb{R}^3$, proving that they agree (up to isometry and up to scaling) with the self-translating bowl soliton. Recently, in [BC18], they extended their result to higher dimensions, under the extra assumption of uniform 2-convexity.

In [CHH18] Choi, Haslhofer, Hershkovits classified all 2-dimensional ancient mean curvature flows with low entropy in $\mathbb{R}^3$.

In this paper we generalize some of the results from [CM18], which classified the projected convex hulls of all proper self-translaters. This establishes the following string of generalizations: The below time-dependent Theorem 5 for ancient flows implies the time-independent self-translating hypersurfaces case [CM18], which again implies the minimal hypersurface case [HM90], which finally implies the Euclidean case of conically bounded minimal surfaces [Om67].

Note that in Theorem 3 and Theorem 5 below, we do not have any curvature or non-collapsing nor entropy assumptions. Here we only need to assume the flows to be properly immersed.

We expect these results to be useful in the future investigation of ancient solutions, both for problems of classification and construction of examples, and hence to the investigation of the set of possible singularities in the mean curvature flow.

Finally, also in regularity questions for mean curvature flow with boundary, bi-halfspace theorems (a.k.a. wedge theorems) are useful: In January 2019, Brian White posted a paper [Wh19] with a result on boundary regularity (announced some time ago, e.g. in [Wh09], see also [St96]), proved there using a new wedge theorem for self-shrinking Brakke flows. It would be interesting to understand its relation to our smooth results in [CM18] and in Theorem 3 and Theorem 5 below.

2. Preliminaries

Definition 1. Let M^n be a smooth, connected n-dimensional manifold without boundary and let $I \subseteq \mathbb{R}$ be a (time) interval. A mean curvature flow is a smooth map $F\colon M \times I \to \mathbb{R}^{n+1}$ such that $F_t : M \to \mathbb{R}^{n+1}$ is an immersion for every $t \in I$, where $F_t(x) := F(x,t)$, and F satisfies the following equation

$$\frac{\partial F}{\partial t} = \overrightarrow{H}. \tag{1}$$

The mean curvature flow is said to be an *ancient*, *immortal* or *eternal* solution, if respectively after a time translation $I = (-\infty, 0)$, $(0,\infty)$ or $\mathbb{R}$.

In what follows, we will denote by M_t the manifold M endowed with the pullback metric induced by F_t, i.e.

$$M_t := (M, F_t^*\langle\cdot,\cdot\rangle_{\mathbb{R}^{n+1}})\,.$$

The Levi-Civita connection and the Laplacian on M_t will be denoted by ∇^{M_t} and Δ^{M_t} respectively.

Moreover, we will always consider proper mean curvature flows, meaning that for every $t \in I$ the map $F_t : M \to \mathbb{R}^{n+1}$ is a proper immersion. We remind the reader that properly immersed hypersurfaces are geodesically complete w.r.t. the induced Riemannian metric (by the Heine-Borel property and Hopf-Rinow). As always, most of our results fail without the properness assumption, see e.g. the examples in [Na96] of minimal surfaces non-properly immersed into ambient balls.

3. Main results

Lemma 2 (Omori-Yau Maximum Principle for Ancient MCFs)**.** *Let $F\colon M \times (-\infty,0) \to \mathbb{R}^{n+1}$ be a proper ancient mean curvature flow. Let $f\colon M \times (-\infty,0) \to \mathbb{R}$ be a bounded and twice differentiable function.*

Then there is a sequence of points $(x_i,t_i) \in M \times (-\infty,0)$ such that

(i) $\lim_{i\to\infty} f(x_i,t_i) = \sup_{M\times(-\infty,0)} f$*,*
(ii) $\lim_{i\to\infty} |\nabla^{M_{t_i}} f(x_i,t_i)| = 0$*,*
(iii) $\liminf_{i\to\infty} \left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}}\right) f(x_i,t_i) \geq 0$*.*

Theorem 3 (Wedge Theorem for Ancient Mean Curvature Flows)**.** *Let H_1 and H_2 be two halfspaces of $\mathbb{R}^{n+1}$ such that the hyperplanes $P_1 := \partial H_1$ and $P_2 := \partial H_2$ are not parallel.*

Then $H_1 \cap H_2$ does not contain any proper ancient mean curvature flow. More precisely, there does not exist any proper ancient mean curvature flow $F\colon M \times (-\infty,0) \to \mathbb{R}^{n+1}$ such that $F_t(M) \subseteq H_1 \cap H_2$ for all times $t \in (-\infty,0)$.

Remark 4. Theorem 3 holds in particular for proper eternal mean curvature flows, i.e. $I = \mathbb{R}$. There are two particularly important subclasses of eternal mean curvature flows: self-translating solitons and minimal hypersurfaces. Therefore it generalizes Theorem 1 contained in the paper [CM18] by the present authors (see also the corollaries and discussion there), as well as classical theorems by Omori [Om67] (in the Euclidean case) and Hoffman-Meeks [HM90] (in the case without boundary). A third type of ancient solutions are the self-shrinking solitons (where there are many examples, see e.g. [KKM11]-[Mø11]), which even obey a halfspace theorem [CE16] (proved by Cavalcante-Espinar using the barriers from [KM14]).

As in [CM18], it is interesting to ask which of the cases can actually occur in Theorem 5. For each case we have respectively: Flat planes, reaper cylinders (plus "Angenent ovals" [An92] and "ancient pancakes" [BLT17]), which give slabs. No examples (to our knowledge) of halfspaces. Spheres, cylinders and the bowl soliton for all of $\mathbb{R}^{n+1}$. Note of course that by [CE16], self-shrinkers cannot provide examples in the halfspace case (see also the discussion in [CM18]).

Note also that Theorem 3 is the "bi-halfspace" result we can expect for ancient mean curvature flows. In fact a "halfspace theorem" version would be false. There are several counterexamples: for instance planes and grim reaper cylinders. Also, a "halfspace" statement would be false, even for those ancient mean curvature flows all of whose time-slices are compact: A counterexample to this is given by the so-called *ancient pancake* [BLT17] (or for $n = 1$, Angenent's ovals [An92]) which is contained for all its evolution in a slab between two parallel hyperplanes, and thus no general halfspace theorem could hold for all ancient solutions.

We remark that the statement of Theorem 3 is false for general immortal mean curvature flows, i.e. $I = (0, \infty)$. In fact there are self-expanding mean curvature flows such that they are contained for their entire evolution in the intersection of two halfspaces with nonparallel boundaries (see f.ex. [SS07]). Note that f.ex. Lemma 13 as stated, and hence the proof of Theorem 3, would have failed if we had instead taken $I = (0, \infty)$.

Theorem 5 (Classification of Sets of Reach of Ancient Flows)**.** *Consider a proper ancient mean curvature flow. Let*

$$\mathcal{R} := \bigcup_{t \in (-\infty, 0)} F_t(M) \subseteq \mathbb{R}^{n+1}$$

denote its set of reach. Then the convex hull $\mathrm{Conv}(\mathcal{R})$ *is either a hyperplane, a slab, a halfspace or all of* $\mathbb{R}^{n+1}$*.*

In the next corollary, we keep track of the time coordinate to get a spacetime track version, which is of course equivalent to Theorem 5.

Corollary 6 (Spacetime Tracks of Ancient Flows)**.** *Consider for a proper ancient mean curvature flow its spacetime track* $\mathcal{ST}$

$$\mathcal{ST} := \bigcup_{t\in(-\infty,0)} \{t\} \times F_t(M) \subseteq \mathbb{R} \times \mathbb{R}^{n+1}.$$

Then $\mathrm{Conv}(\pi_2(\mathcal{ST}))$ *is either a hyperplane, a slab, a halfspace or all of* $\mathbb{R}^{n+1}$*, where* π_2 *denotes the projection to the* $\mathbb{R}^{n+1}$*-factor.*

In the next corollary, the set $\mathcal{R} \cup \{p_\infty\}$ can also be thought of simply as the closure of the set of reach.

Corollary 7 (Sets of Reach of Compact Convex Ancient Flows)**.** *Consider any compact convex ancient mean curvature flow in* $\mathbb{R}^{n+1}$*, which at time* 0 *becomes extinct at a point* $p_\infty \in \mathbb{R}^{n+1}$*.*

Then $\mathcal{R} \cup \{p_\infty\}$ *(the set of points reached, with the singular point added in) is either a slab, a halfspace or all of* $\mathbb{R}^{n+1}$*.*

Remark 8. Note that Corollary 7 is in agreement with Corollary 6.3 in [Wa11] where blow-downs $(-t)^{-\frac{1}{2}}F_t(M)$ as $t \to -\infty$ for convex ancient solutions were classified: any $\mathbb{S}^k \times \mathbb{R}^{n-k}$, with $k = 1, \dots, n$ or multiplicity two hyperplanes. It is not clear to us whether the halfspace case of Corollary 7 could occur. For instance in the convex case it does not happen in the curve shortening flow (i.e. the case $n = 1$) because of the classification for closed curves in [DHS10], which shows that the only possible sets of reach are strips and $\mathbb{R}^2$.

Moreover Wang (Corollary 6.1 [Wa11]) showed that the set of reach of a (not necessarily compact) convex ancient mean curvature flow arising as a limit flow of a mean convex flow, is the entire $\mathbb{R}^{n+1}$. Note that the set of reach there is taken over the whole maximal time interval, which might be $(-\infty, \infty)$.

4. Proofs

The proof of Lemma 2 is based on the Omori-Yau maximum principle (tracing its roots back to [Om67]–[CY75]) proven by Ma in [Ma17]. The main difference is that here we are interested in ancient mean curvature flows and thus our time interval is not finite, which complicates slightly (but essentially) the proof. On the other hand, because of the applications we have in mind, we focus on the case where the ambient manifold is Euclidean space $\mathbb{R}^{n+1}$ and the codimension is 1, and the argument we give here is self-contained.

Proof of Lemma 2. Let $(\bar{x}_i, \bar{t}_i)$ be a sequence in $M \times (-\infty, 0)$ such that

$$\lim_{i\to\infty} f(\bar{x}_i, \bar{t}_i) = \sup_{M\times(-\infty,0)} f. \tag{2}$$

Consider the function $r\colon \mathbb{R}^{n+1} \to \mathbb{R}$ defined as $r(y) := \|y\|$. This defines a function ϱ on $M \times (-\infty, 0)$ by $\varrho(x,t) := r(F(x,t))$.

Now let $(\varepsilon_i)_{i\in\mathbb{N}}$ be the sequence of positive numbers (well-defined even if $\varrho(\bar{x}_i, \bar{t}_i) = 0$)

$$0 < \varepsilon_i := \min\left(\frac{1}{i}, \frac{1}{i}\frac{1}{\varrho(\bar{x}_i, \bar{t}_i)^2}\right) < \infty. \tag{3}$$

Note that $\lim_{i\to\infty} \varepsilon_i = 0$ and that for every $i \in \mathbb{N}$

$$\varepsilon_i \varrho(\bar{x}_i, \bar{t}_i)^2 \le \frac{1}{i}. \tag{4}$$

Let us now for $i = 1, 2, \ldots$ define $f_i : M \times (-\infty, 0) \to \mathbb{R}$ by

$$f_i(x,t) := f(x,t) - \varepsilon_i \left(\varrho(x,t)\right)^2. \tag{5}$$

Note that each f_i is bounded from above by $\sup_{M\times(-\infty,0)} f < \infty$.

Claim: Fix a time $t \in (-\infty, 0)$ and fix $i \in \mathbb{N}$. Then there exists a point $x_t^i \in M$ where the function $f_i(\cdot, t)$ attains its supremum over M. Furthermore, this is locally uniform in the sense that considering τ near t, all the points $x_\tau^i \in M$ can be chosen from a fixed compact subset $K \subseteq M$ (with $K = K_t$ possibly dependent on t and on the proximity of τ to t).

If M is compact, then the claim is trivial. If M is not compact, it follows from the crucial properness assumption. In fact, let $R > 0$ be large enough so that $F_t(M) \cap B_R \neq \emptyset$, where B_R is the ambient open ball of radius $R > 0$ in $\mathbb{R}^{n+1}$ centered at 0. Since f is bounded on $M \times (-\infty, 0)$, we can choose $S > R > 0$ so that

$$\sup_{M\times(-\infty,0)} f - \varepsilon_i S^2 < \inf_{M\times(-\infty,0)} f - \varepsilon_i R^2 \tag{6}$$

Equation (5) now shows that for points $p \in M \setminus F_t^{-1}(B_S)$ (which is nonempty, because from the properness of F_t and noncompactness of M follows that $F_t(M)$ cannot be contained in any finite radius ambient ball) holds:

$$f_i(p,t) \le f(p,t) - \varepsilon_i S^2 \le \sup_{M\times(-\infty,0)} f - \varepsilon_i S^2. \tag{7}$$

Therefore taking the supremum over $p \in M \setminus F_t^{-1}(B_S)$ yields

$$\sup_{M\setminus F_t^{-1}(B_S)} f_i(\cdot,t) \le \sup_{M\times(-\infty,0)} f - \varepsilon_i S^2 < \inf_{M\times(-\infty,0)} f - \varepsilon_i R^2, \tag{8}$$

using (6). Thus, finally, using $f - \varepsilon_i R^2 \le f_i$ on $F_t^{-1}(B_R)$:

$$\sup_{M\setminus F_t^{-1}(B_S)} f_i(\cdot,t) < \inf_{F_t^{-1}(B_R)} f_i(\cdot,t) \tag{9}$$

From continuity of F, we also have that there exists $\delta > 0$ such that, for every time $\tau \in (t-\delta, t+\delta)$, we have $F_\tau(M) \cap B_R \neq \emptyset$ and thus (9) still holds. Properness of the flow implies that each $F_\tau^{-1}(\bar{B}_S) \subseteq M$ is compact, therefore we have for every $\tau \in (t-\delta, t+\delta)$:

$$\sup_M f_i(\cdot,\tau) = \max_{F^{-1}(\bar{B}_S)} f_i(\cdot,\tau). \tag{10}$$

It only remains to find a uniform compact $K \subseteq M$ as claimed. This is also guaranteed by the properness of the immersion at each time, as follows.

Since $F_t^{-1}(\bar{B}_S)$ is compact, we can choose K to be any larger compact set such that $F_t^{-1}(\bar{B}_S) \subseteq K^\circ \subseteq M$, where K° denotes the interior of K. Consider the closed set $C := M \setminus K^\circ$. The two sets $F_t(C)$ and B_S are disjoint. Together with compactness of $\bar{B}_S$, and the assumption that F_t is proper (and hence a closed map), which ensures closedness of $F_t(C)$, it then implies $\mathrm{dist}_{\mathbb{R}^{n+1}}(F_t(C), \bar{B}_S) > 0$.

From the triangle inequality follows that $\mathrm{dist}_{\mathbb{R}^{n+1}}(F_\tau(C), B_S)$ is continuous in $\tau \in (t-\delta, t+\delta)$. Therefore after possibly taking $\delta > 0$ smaller holds $\mathrm{dist}_{\mathbb{R}^{n+1}}(F_\tau(C), B_S) > 0$ for all $\tau \in (t-\delta, t+\delta)$. But then as claimed $F_\tau^{-1}(B_S) \subseteq K$ and finally, using (10) we finish the proof of the final part of the claim:

$$\forall \tau \in I: \quad \sup_M f_i(\cdot,\tau) = \max_K f_i(\cdot,\tau). \tag{11}$$

For any time $t \in (-\infty, 0)$ and $i \in \mathbb{N}$, let us use the claim and denote

$$L_i(t) := \max_{x\in M} f_i(x,t) = f_i(x_t^i, t), \tag{12}$$

for some $x_t^i \in K_t$.

Note that the function $(-\infty, 0) \ni t \mapsto L_i(t)$ is bounded from above by $L := \sup_{M\times(-\infty,0)} f$.

By Hamilton's Trick [Ha86] (see f.ex. Lemma 2.1.3. in [Ma11], which is where we use the uniformicity property (11) and the K_t in the claim), each L_i is a locally Lipschitz function of t and therefore continuous. The function L_i is also differentiable almost everywhere, and at any of its differentiability times $t \in (-\infty, 0)$ we have as usual

$$\frac{dL_i}{dt}(t) = \frac{\partial f_i}{\partial t}(x^i_t, t).$$

This implies (using Lemma 9 in the Appendix) that there is a time $t_i \in (-\infty, 0)$ such that $\frac{dL_i}{dt}(t_i)$ exists and with $x_i := x^i_{t_i}$ there holds

$$\frac{\partial f_i}{\partial t}(x_i, t_i) = \frac{dL_i}{dt}(t_i) \geq -2\varepsilon_i, \tag{13}$$

and also

$$\left| L_i(t_i) - \sup_{(-\infty,0)} L_i \right| < \varepsilon_i, \tag{14}$$

or in other words

$$\left| f_i(x_i, t_i) - \sup_{M\times(-\infty,0)} f_i \right| < \varepsilon_i. \tag{15}$$

We have from (12) that $\Delta^{M_{t_i}} f_i(x_i, t_i) \leq 0$, therefore with (13)

$$\left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}} \right) f_i(x_i, t_i) \geq -2\varepsilon_i. \tag{16}$$

From standard computations and (1) we also have (as the only step in the proof where we use the mean curvature flow equation) the following

$$\left(\frac{\partial}{\partial t} - \Delta^{M_t} \right) (\varrho(x, t))^2 = -2n, \tag{17}$$

for every $(x, t) \in M \times (-\infty, 0)$.

Combining (16) and (17), we get with (5)

$$\left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}} \right) f(x_i, t_i) \geq -2(n+1)\varepsilon_i. \tag{18}$$

This shows Part *(iii)* of the Lemma. Let us now check that also *(i)* and *(ii)* hold.

From (5) and (15), and since $L_i(t_i) = f_i(x_i, t_i) = \max_M f_i(\cdot, t_i)$, we have (see (2) for the definition of $(\bar{x}_i, \bar{t}_i)$)

$$\begin{aligned} f(x_i, t_i) &\geq f_i(x_i, t_i) > \sup_{M\times(-\infty,0)} f_i - \varepsilon_i \\ &\geq f_i(\bar{x}_i, \bar{t}_i) - \varepsilon_i \geq f(\bar{x}_i, \bar{t}_i) - \frac{1}{i} - \varepsilon_i, \end{aligned} \tag{19}$$

where the final inequality made use of (4). This shows Part *(i)*, by taking the limit for $i \to \infty$ in the string of inequalities (19).

Let us now show Part *(ii)*. Observe that $\nabla^{M_{t_i}} f_i(x_i,t_i) = 0$, since $x_i \in M$ is a maximum point for $f_i(\cdot, t_i)$. Thus we have

$$\nabla^{M_{t_i}} f(x_i,t_i) = 2\varepsilon_i \varrho(x_i,t_i) \nabla^{M_{t_i}} \varrho(x_i,t_i) = 2\varepsilon_i \varrho(x_i,t_i) \left(\left(\nabla^{\mathbb{R}^{n+1}} r \right) (F(x_i,t_i) \right)^\top .$$

Noting

$$\left\| \left(\left(\nabla^{\mathbb{R}^{n+1}} r \right) (F(x_i,t_i) \right)^\top \right\| \le \left\| \left(\nabla^{\mathbb{R}^{n+1}} r \right) (F(x_i,t_i) \right\| \le 1, \tag{20}$$

it is enough to show that $\varepsilon_i \varrho(x_i,t_i) \xrightarrow[i\to\infty]{} 0$ (note that (4) concerns $(\bar{x}_i, \bar{t}_i)$). But from (19), we have $f_i(x_i,t_i) > f(\bar{x}_i,\bar{t}_i) - \frac{1}{i} - \varepsilon_i$, so that

$$\varepsilon_i \varrho(x_i,t_i)^2 = f(x_i,t_i) - f_i(x_i,t_i) < \sup_{M\times(-\infty,0)} f - f(\bar{x}_i,\bar{t}_i) + \frac{1}{i} + \varepsilon_i.$$

Therefore $\sqrt{\varepsilon_i}\varrho(x_i,t_i) \xrightarrow[i\to\infty]{} 0$ and therefore by (2), finally

$$\varepsilon_i \varrho(x_i,t_i) \xrightarrow[i\to\infty]{} 0.$$

□

Proof of Theorem 3. The rest of the proof is very similar to that of Theorem 1 in [CM18], in the case of self-translating solitons without boundary, the proof of which was in turn inspired by an idea for 2-dimensional minimal surfaces in $\mathbb{R}^3$ by Borbély in [Bo11]. In particular we are going to apply Lemma 2 to a function f which is constructed exactly in the same way as in [CM18].

Let $H_1, H_2 \subseteq \mathbb{R}^{n+1}$ be two halfspaces such that $P_1 := \partial H_1$ and $P_2 := \partial H_2$ are not parallel. Let us assume by contradiction that there exists a proper ancient mean curvature flow $F\colon M \times (-\infty,0) \to \mathbb{R}^{n+1}$ such that $F_t(M) \subseteq H_1 \cap H_2$ for every $t \in (-\infty,0)$.

We can assume without loss of generality that $0 \in P_1 \cap P_2$. Let $w_1, w_2 \in \mathbb{S}^n$ such that $H_i = \{x \in \mathbb{R}^{n+1} \colon \langle x, w_i\rangle \ge 0\}$.

For $R > 0$, let $\mathcal{L}_R \subseteq \mathbb{R}^{n+1}$ be the $(n-1)$-dimensional affine subspace obtained by translating $P_1 \cap P_2$ in the direction of $w_1 + w_2$ and such that the boundary of the solid cylinder

$$\mathcal{D}_R := \{x \in \mathbb{R}^{n+1} \colon \operatorname{dist}(x, \mathcal{L}_R) \le R\}$$

is tangent to P_1 and P_2.

Let $d_R \colon \mathbb{R}^{n+1} \to \mathbb{R}$ denote the distance function from $\mathcal{L}_R$, i.e. $d_R(x) := \operatorname{dist}(x, \mathcal{L}_R)$. Observe that $(H_1 \cap H_2) \setminus \mathcal{D}_R$ consists of two connected components. Let $\mathcal{V}_R$ be the one where d_R is bounded. Let us choose $R > 0$ large enough such that there exists $t \in (-\infty,0)$ such that $F_t(M) \cap \mathcal{V}_R \neq \emptyset$.

Let us now define a function $f\colon M \times (-\infty, 0) \to \mathbb{R}$ as follows

$$f(x,t) := \begin{cases} d_R(F(x,t)) & \text{if } F(x,t) \in \mathcal{V}_R \\ R & \text{otherwise.} \end{cases} \tag{21}$$

Observe that by construction f is continuous and bounded. Since we have chosen $R > 0$ in such a way that $F(x,t) \in \mathcal{V}_R$ for some $(x,t) \in M \times (-\infty, 0)$, we have that

$$0 < R < \sup_{F^{-1}(\mathcal{V}_R)} f = \sup_{M\times(-\infty,0)} f < \infty. \tag{22}$$

We want to apply the Omori-Yau maximum principle in the form of Lemma 2 to f. Note that f is smooth on the interior of $F^{-1}(\mathcal{V}_R)$ and, because of (22), this is actually enough in order to apply Lemma 2.

By standard computations, see e.g. [CM18], one can check that on $F^{-1}(\mathcal{V}_R)$ we have

$$\left(\frac{\partial}{\partial t} - \Delta^{M_t}\right) f = -\frac{1 - \|\nabla^{M_t} f\|^2}{d_R}. \tag{23}$$

Let $(x_i, t_i) \in M \times (-\infty, 0)$ be an Omori-Yau sequence given by Lemma 2. From (23), Part *(ii)* of Lemma 2 and (22), we have that the function f eventually becomes strictly subcaloric at points in the sequence:

$$\lim_{i\to\infty} \left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}}\right) f(x_i, t_i) = -\frac{1}{\sup_{M\times(-\infty,0)} f} < 0.$$

On the other hand, this is in contradiction with Part *(iii)* of Lemma 2, which concludes the proof. □

Proof of Theorem 5. This proof proceeds quite like in the case of minimal surfaces [HM90] and self-translaters [CM18]:

$$\operatorname{Conv}(\mathcal{R}) = \bigcap \{H \subseteq \mathbb{R}^{n+1} \colon H \text{ is a halfspace s.t. } \mathcal{R} \subseteq H\},$$

the intersection of all halfspaces containing the set of reach. If any such two halfspaces H_1 and H_2 were not parallel, we would conclude that for all times $t \in (-\infty, 0)$ the flow is contained in a non-halfspace wedge, $F_t(M) \subseteq H_1 \cap H_2$, violating Theorem 3. Hence the conclusion follows. □

Proof of Corollary 7. Let us first remind the reader that by a convex hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ we mean one where all principal curvatures $\kappa_i > 0, i = 1, \ldots, n$, and that by a theorem of Sacksteder [Sa60], this implies that $\Sigma = \partial\Omega$, for some strictly convex domain in $\mathbb{R}^{n+1}$. Knowing this we immediately rule out the "flat plane minus one point" as a possible set of reach.

Let now $F\colon M \times (-\infty, 0) \to \mathbb{R}^{n+1}$ be a mean curvature flow as in the statement. Following Huisken [Hu84], the flow will become extinct at a "round point" $p_\infty \in \mathbb{R}^{n+1}$ at time 0. Let Ω_t be the bounded convex body such that $\Sigma_t = \partial\Omega_t$. We have that the flow sweeps out the interior of each Ω_t. These facts easily imply that adding the singular point to $\mathcal{R}$ we get a convex set: Namely, suppose that $p_1, p_2 \in \mathcal{R}$ are given. Then there exist $t_1, t_2 \in (-\infty, 0)$ so that $p_i \in F_{t_i}(M)$, and with $t_0 := \min(t_1, t_2)$ we have, by the monotonicity of the domains Ω_t, that $p_1, p_2 \in \Omega_{t_0}$. Also, considering the line segment between p_1 and p_2 it is, by convexity of Ω_{t_0}, contained in Ω_{t_0}. Hence by the property that the flow sweeps the interior of Ω_{t_0}, the line segment is contained in $\mathcal{R} \cup \{p_\infty\}$. The case where one $p_i = p_\infty$ follows similarly (or by continuity).

Thus, having shown that $\mathrm{Conv}(\mathcal{R}) = \mathcal{R} \cup \{p_\infty\}$ under these extra assumptions, we apply Theorem 3 to finish the proof of Corollary 7. $\square$

5. Appendix

In this section, we state and prove the following elementary lemma, needed in the proofs in the paper's main sections:

Lemma 9. *Let $L\colon (-\infty,0)\to\mathbb{R}$ be a locally Lipschitz function bounded from above.*

Then for every $\varepsilon>0$ there exists some $t_0\in(-\infty,0)$ such that L is differentiable at t_0 and satisfies the following

(i) $L'(t_0)\geq-\varepsilon$,

(ii) $L(t_0)>\sup_{(-\infty,0)}L-\varepsilon$.

Proof of the Lemma. Recall that Lipschitz continuity implies absolute continuity. Let us fix $\varepsilon>0$. Let us first assume that there exists $t_0\in(-\infty,0)$ such that

$$L(t_0)=\sup_{(-\infty,0)}L. \tag{24}$$

If L is differentiable at t_1, then we are done. Let us assume it is not. Let $\delta>0$ be such that $|L(t)-L(t_1)|<\varepsilon$ for any $|t-t_1|<\delta$. Then

$$\int_{t_1-\delta}^{t_1}L'(t)\,dt=L(t_1)-L(t_1-\delta)\geq 0. \tag{25}$$

Therefore there exists $t_0\in(t_1-\delta,t_1)$ such that L is differentiable at t_0 and such that $L'(t_0)\geq 0$. Moreover $|L(t_0)-L(t_1)|<\varepsilon$.

Let us now assume that the supremum is not attained. The case where $\sup_{(-\infty,0)}L=\lim_{t\to 0^-}L(t)$ can be studied similarly to the above.

Therefore let us study the case where $\sup_{(-\infty,0)}L=\lim_{t\to-\infty}L(t)$. We can assume that there is an interval $I\coloneqq(-\infty,\tau)\subseteq(-\infty,0)$ such that $L|_I\geq\sup L-\varepsilon$ and such that there are no local maxima and no local minima in I. Namely, otherwise we could proceed as we did above. Note however that for the case of local minima we have to consider intervals of the kind $(t_1,t_1+\delta)$ instead.

Having no local extrema implies together with continuity that the function L is monotone on I. Since $\sup_{(-\infty,0)}L=\lim_{t\to-\infty}L(t)$, it must be monotonically decreasing and thus satisfy $L'\leq 0$ at all points of differentiability in I, so Lebesgue-almost everywhere. Moreover

$$\int_I L'=\int_{-\infty}^{\tau}L'(t)\,dt=L(\tau)-\sup L\geq-\varepsilon. \tag{26}$$

Therefore $L'|_I$ is summable. There also exists a differentiability point t_0 such that $L'(t_0)\geq-\varepsilon$, otherwise we would get a contradiction with summability from $L'(t_0)<-\varepsilon$ a.e in I. □

References

[An92] S. Angenent, *Shrinking doughnuts*, Progr. Nonlinear Differential Equations Appl. **7** (1992), Birkhäuser, Boston.

[An13] S. Angenent, *Formal asymptotic expansions for symmetric ancient ovals in mean curvature flow*, Netw. Heterog. Media **8** (2013), 1–8.

[ADS15] S. Angenent, P. Daskalopoulos, N. Sesum, *Unique asymptotics of ancient convex mean curvature flow solutions* arXiv:1503.01178, to appear in J. Diff. Geom.

[ADS18] S. Angenent, P. Daskalopoulos, N. Sesum, *Uniqueness of two-convex closed ancient solutions to the mean curvature flow*, arXiv:1804.07230v1.

[AY18] S. Angenent, Q. You, *Ancient solutions to curve shortening with finite total curvature*, arXiv:1803.01399v1.

[BC17] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in* $\mathbb{R}^3$, arXiv:1711.00823.

[BC18] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions*, arXiv:1804.00018v2.

[BLT17] T. Bourni, M. Langford, G. Tinaglia, *A collapsing ancient solution of mean curvature flow in* $\mathbb{R}^3$, arXiv:1705.06981.

[Bo11] A. Borbély, *On minimal surfaces satisfying the Omori-Yau principle*, Bull. Aust. Math. Soc. 84 (2011), 33–39.

[CE16] M.P. Cavalcante, J.M. Espinar, *Halfspace type theorems for self-shrinkers*, Bull. Lond. Math. Soc. **48** (2016), no. 2, 242–250.

[CHH18] K. Choi, R. Hashlofer, O. Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*, arXiv:1810.08467v1.

[CM18] F. Chini, N. M. Møller, *Bi-halfspace and convex hull theorems for translating solitons*, arXiv:1809.01069.

[CY75] S.Y. Cheng, S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354.

[DHS10] P. Daskalopoulos, R. Hamilton, N. Sesum, *Classification of compact ancient solutions to the curve shortening flow*, J. Diff. Geom. **84** (2010), 455–464.

[Ha86] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Diff. Geom. **24** (1986), no. 2, 153–179.

[HH16] R. Haslhofer, O. Hershkovits, *Ancient solutions of the mean curvature flow*, Comm. Anal. Geom. **24** (2016), no. 3, 593–604.

[HM90] D. Hoffman, W. H. Meeks, III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990), 373–377.

[Hu84] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differ. Geom. **20** (1984), 237–266.

[HS99a] G. Huisken, C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), 45–70.

[HS99b] G. Huisken, C. Sinestrari, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations **8** (1999), 1–14.

[HS15] G. Huisken, C. Sinestrari, *Convex ancient solutions of the mean curvature flow*, J. Diff. Geom. **101** (2015), 267–287.

[KKM11] N. Kapouleas, S.J. Kleene, N.M. Møller, *Mean curvature self-shrinkers of high genus: non-compact examples*, J. Reine Angew. Math. **739** (2018), 1–39. arXiv:1106.5454.

[KM14] S. Kleene, N.M. Møller, *Self-shrinkers with a rotational symmetry*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 3943–3963.

[Ma11] C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Birkhäuser (2011).

[Ma17] J. M. S. Ma, *Parabolic Omori-Yau Maximum Principle for Mean Curvature Flow and Some Applications*, arXiv:1701.02004.

[Mø11] N.M. Møller, *Closed self-shrinking surfaces in* $\mathbb{R}^3$ *via the torus*, arXiv:1111.7318.

[Na96] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*, Invent. Math. **126** (1996), no. 3, 457–465.

[Om67] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967) 205–214.

[Sa60] R. Sacksteder, *On hypersurfaces with no negative sectional curvatures*, Amer. J. Math. **82**, (1960), 609–630.

[SS07] O. Schnürer, F. Schulze, *Self-similarly expanding networks to curve shortening flow*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), no. 4, 511–528.

[St96] A. Stone, *A regularity theorem for mean curvature flow*, J. Differential Geom. **44** (1996), no. 2, 371–434. MR1425580

[Wa11] X.-J. Wang, *Convex solutions to the mean curvature flow*, Ann. of Math. (2) **173** (2011), no. 3, 1185–1239.

[Wh00] B. White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **13** (2000), 665–695.

[Wh03] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), no. 1, 123–138.

[Wh09] B. White, *Currents and flat chains associated to varifolds, with an application to mean curvature flow*, Duke Math. J. **148** (2009), no. 1, 41–62, DOI 10.1215/00127094-2009-019. MR2515099

[Wh19] B. White, *Mean Curvature Flow with Boundary*, arXiv: 1901.03008 (Jan 10th, 2019).

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Paper C. Simply connected translating solitons contained in slabs

SIMPLY CONNECTED TRANSLATING SOLITONS CONTAINED IN SLABS

FRANCESCO CHINI

Abstract. In this work we show that 2-dimensional, simply connected, translating solitons of the mean curvature flow embedded in a slab of $\mathbb{R}^3$ with entropy strictly less than 3 must be mean convex and thus, thanks to a result by J. Spruck and L. Xiao [SX17], are convex. Recently, such 2-dimensional convex translating solitons have been completely classified in [HIMW19a], up to an ambient isometry, as vertical plane, (tilted) grim reaper cylinders, Δ-wings and bowl translater. These are all contained in a slab, except for the rotationally symmetric bowl translater. New examples by [HMW19a] show that the bound on the entropy is necessary.

Introduction

In [Br16], Brendle proved that any properly embedded 2-dimensional mean curvature self-shrinker in $\mathbb{R}^3$ which is homeomorphic to an open subset of the sphere must be a round sphere, or a cylinder or a plane, solving two problems posed by Ilmanen (see 14 and 15 in [Il03]). In particular, it follows that the round sphere is the only closed, embedded shrinker with genus 0. The main step in Brendle's paper was to first prove that any shrinker satisfying such a topological assumption must be mean convex and with polynomial area growth (his argument was partially inspired by [Ro95]). Then the conclusion follows from Theorem 10.1 in [CM12], which is a refinement of a theorem by Huisken [Hu90b] (see also [Hu90a] for the closed case).

One cannot expect such a strong result for translating solitons, even under the more restrictive topological assumption of being simply connected. In fact Hoffman, Martín and White [HMW19a] recently constructed new examples of properly embedded translaters which are simply connected but are not mean convex. The most surprising one is the so-called *pitchfork* translater which has entropy equal to 3 and is contained in a slab.

In these notes we will consider smooth 2-dimensional translating solitons of the mean curvature flow in $\mathbb{R}^3$, i.e. smooth surfaces immersed in $\mathbb{R}^3$

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satisfying the equation

$$H = -\langle \nu, e_3 \rangle, \tag{1}$$

where ν is a smooth unit normal vector field on Σ and H denotes the mean curvature. Note that Σ satisfies (1) if and only if the 1-parameter family $\Sigma + te_3$ is a mean curvature flow, with $t \in \mathbb{R}$.

Following [CM12] denote the entropy of Σ by $\lambda(\Sigma)$ (see Appendix A for details).

The main contribution of this work is the following result.

Theorem 1. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a complete, embedded, translater satisfying the following assumptions:*

(i) Σ is simply connected,
(ii) $\lambda(\Sigma) < 3$,
(iii) Σ is contained in a slab.

Then Σ is mean convex.

Spruck and Xiao proved that 2-dimensional, mean convex translaters are actually convex (Theorem 1.1 in [SX17]). Therefore their result together with the classification of Hoffman, Ilmanen, Martín and White [HIMW19a], yields the following corollary.

Corollary 2. *Let Σ be as in Theorem 1. Then Σ is, up to an ambient isometry, one of the following translating solitons:*

(i) a vertical plane,
(ii) a grim reaper cylinder (possibly tilted),
(iii) a Δ-wing translater.

Remark 3. As mentioned above, the pitchfork example shows that the bound on the entropy in the assumptions of Theorem 1 is necessary and cannot be relaxed.

On the other hand, there are currently no known examples of complete translaters contained in a slab with entropy strictly less than 3 which are not simply connected. So it is not clear whether the topological assumption is necessary. Hershkovits [He18] classified translaters with entropy less or equal than the entropy of a cylinder without any further assumptions. More precisely, he proved that a translater $\Sigma^2 \subseteq \mathbb{R}^3$ satisfying the following entropy bound

$$\lambda(\Sigma) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}) = \sqrt{\frac{2\pi}{e}} \approx 1.52 \tag{2}$$

must be either a plane ($\lambda(\Sigma) = 1$) or the rotationally symmetric bowl translater ($\lambda(\Sigma) = \sqrt{\frac{2\pi}{e}}$). However, even though Hershkovits does not need any toplogical assumption, his bound (2) is much more restrictive than the entropy bound in our Theorem 1. In a later work, Hershkovits, Haslhofer and Choi [HHC18] completely classified 2-dimensional ancient mean curvature flows with entropy less or equal to $\lambda(\mathbb{S}^1 \times \mathbb{R})$ and they used this classification to prove the mean convex neighborhood conjecture (see also the very recent paper [HHCW19] for a higher dimensional analog).

Moreover, we believe that the assumption (*iii*) in Theorem 1 is purely technical and can be removed.

Conjecture 4. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be an embedded simply connected translater such that $\lambda(\Sigma) < 3$.*

Then Σ is mean convex.

Remark 5. Simply connected translating solitons are particularly interesting because it is known (see [IR17], [IR19], [KS18]) that complete 2-dimensional stable translaters in $\mathbb{R}^3$ must be simply connected. By *stable translaters* we mean translaters which are linearly stable as minimal surfaces w.r.t. to the conformally flat metric $e^{x_3}\delta_{ij}^{\mathrm{Eucl}}$ (for more details see the survey [HIMW19b]).

Observe that for shrinking solitons there is a connection between *stability* and *mean convexity*. More precisely, Colding and Minicozzi [CM12] proved that entropy stable shrinkers (with polynomial volume growth) are mean convex. Entropy stability is intimately related with the index of the stability operator of shrinkers as minimal surfaces in the gaussian metric $e^{-\frac{|x|^2}{4}}\delta_{ij}^{\mathrm{Eucl}}$. For these reasons and motivated by Theorem 1 , we are tempted to state the following conjecture.

Conjecture 6. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a complete embedded stable translater. Then Σ is mean convex and therefore, thanks to* [SX17] *and* [HIMW19a], *up to an ambient isometry, one of the following translating solitons:*

(i) a vertical plane,
(ii) a grim reaper cylinder (possibly tilted),
(iii) a Δ-wing translater,
(iv) the rotationally symmetric bowl translater.

Organization of the paper. In Section 1 we derive a curvature estimate for embedded, simply connected translating solitons with finite entropy, which allows us to use a compactness theorem (based on a standard Arzelà-Ascoli argument) in a crucial step of the proof of Theorem 1. The curvature estimate is a consequence of an estimate by Schoen and Simon [SS83].

In Section 2, which is the longest section of this work, we prove Theorem 10, which is a refinement of results contained in the paper by Møller and the author [CM19a]. The proofs are based on a combination of the Omori-Yau maximum principle and barrier arguments. As a byproduct of Theorem 10, we obtain a Bernstein type theorem for 1-periodic properly immersed translaters.

Section 3 is devoted to the study of the structure of the intersection $Z := \Sigma \cap T_p\Sigma$, where $T_p\Sigma$ denotes the geometric tangent space of Σ at some point $p \in \Sigma$ where $H(p) = 0$. This is done by observing that Z is the nodal set of a function $f\colon \Sigma \to \mathbb{R}$ solving an elliptic PDE of the kind $\Delta^\Sigma f = hf$ for some function $h \in C^\infty(\Sigma)$ and applying a result by [Ch76]. Under the assumption of Σ being simply connected, we study the topology of Z. Namely, we show, by using a maximum principle argument, that each connected component of Z is contractible.

In Section 4 we study the structure of $\{H = 0\}$ and show that on a translater the unit normal vector cannot be constant along $\{H = 0\}$ unless the translater is mean convex.

In Section 5 we finally prove Theorem 1. The proof proceeds by contradiction. We assume that Σ is not mean convex and we carefully study the intersection $Z = \Sigma \cap T_p\Sigma$, where $p \in \Sigma$ is some point such that $H(p) = 0$. We distinguish two different cases and we see that one case contradicts the entropy bound and the other one contradicts the topological assumption of Σ being simply connected.

In the Appendix A we recall the definition and some basic well-known properties of the entropy functional.

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1. Curvature estimate

In this section we derive a curvature estimate for simply connected translating solitons with finite entropy, which is of independent interest. Similar results have already been obtained for 2-dimensional translating solitons, but under different assumptions. See for instance Theorem 3.2 in [Sh15], Theorem 4.8 in [Gu16], Theorem 2.8 in [SX17] and Theorem A.3 in [HMW19a].

Proposition 7. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a complete, embedded, simply connected translater such that $\lambda(\Sigma) < \infty$.*

Then there exists a constant $C > 0$ such that $|A| \leq C$.

Proof. Remark 23 in the Appendix A, implies that there exists a constant $C_1 = C_1(\lambda(\Sigma)) > 0$ such that

$$\text{Area}(\Sigma \cap \mathcal{B}_R(x)) \leq C_1 R^2$$

for any radius $R > 0$, for any point $x \in \mathbb{R}^3$, where $\mathcal{B}_R(x)$ is the open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ of radius $R > 0$.

Recall that $\Sigma^2 \subseteq \mathbb{R}^3$ is said to have (γ_1, γ_2)-quasiconformal Gauss map, with $\gamma_1, \gamma_2 \geq 0$, if

$$|A|^2(p) \leq -\gamma_1 K(p) + \gamma_2, \qquad p \in \Sigma, \tag{3}$$

where K denotes the Gauss curvature. Since Σ is a translater, i.e. satisfies (1), then it has $(2,1)$-quasiconformal Gauss map, namely

$$|A|^2(p) = -2K(p) + H^2(p) \leq -2K(p) + 1, \qquad p \in \Sigma.$$

We can apply the estimate for embedded simply connected surfaces with quasiconformal Gauss map of Schoen and Simon [SS83]. More precisely, let us fix $R > 0$. Theorem 1 in [SS83] implies that there exist constants $C_2 = C_2(R, \lambda(\Sigma)) > 0$ and $\alpha = \alpha(R, \lambda(\Sigma)) \in (0,1)$ such that for any $p \in \Sigma$ we have

$$\|\nu(x) - \nu(\tilde{x})\| \leq C_2 \|x - \tilde{x}\|^\alpha, \tag{4}$$

for any $x, \tilde{x} \in \Sigma'$, where Σ' is the connected component of $\Sigma \cap \mathcal{B}_R(p)$ containing p.

Equation (4) implies that there exists $\varrho = \varrho(\lambda(\Sigma)) > 0$, such that for any $p \in \Sigma$, the connected component of $\Sigma \cap \mathcal{B}_\varrho(p)$ containing p is the graph of a smooth function u over an open domain Ω of $T_p\Sigma$ such that $\|\nabla u\| < 1$. Note that $T_p\Sigma$ denotes the geometric tangent plane of Σ at p. It is easy to see that the 2-dimensional disk $B_{\frac{\varrho}{\sqrt{2}}}(p) \subseteq T_p\Sigma$ is contained in Ω. With a small abuse of notation, we keep denoting the restriction $u|_{B_{\frac{\varrho}{\sqrt{2}}}(p)}$ by u. Note that we have $\sup_{B_{\frac{\varrho}{\sqrt{2}}}(p)} |u| \leq \frac{\varrho}{\sqrt{2}}$. We summarize the above observations as follows

$$\|u\|_{C^1\left(B_{\frac{\varrho}{\sqrt{2}}}(p)\right)} \leq 1 + \frac{\varrho}{\sqrt{2}}. \tag{5}$$

Observe that equation (1) implies that u solves the following elliptic equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + \|Du\|^2}}\right) = F, \tag{6}$$

where $F(y) \coloneqq \langle \nu(y, u(y)), e_3 \rangle$ is a smooth function and $y = (y_1, y_2)$ are Cartesian coordinates on the plane $T_p\Sigma$. Observe that $|F| \leq 1$ and from (4), we have a uniform estimate of the α-Hölder norm of F. Namely, given $y, \tilde{y} \in B_{\frac{\varrho}{\sqrt{2}}}(p) \subseteq T_p\Sigma$, we have

$$\begin{aligned}
\frac{|F(y) - F(\tilde{y})|}{\|y - \tilde{y}\|^\alpha} &\leq \frac{\|\nu(y, u(y)) - \nu(\tilde{y}, u(\tilde{y}))\|}{\|y - \tilde{y}\|^\alpha} \\
&\leq 2^\alpha \frac{\|\nu(y, u(y)) - \nu(\tilde{y}, u(\tilde{y}))\|}{\|(y, u(y)) - (\tilde{y}, u(\tilde{y})\|^\alpha} \\
&\leq 2^\alpha \frac{C_2}{R^\alpha} \eqqcolon C_3.
\end{aligned}$$

We can think of (6) as a linear elliptic equation in u where the coefficients depend on Du. The uniform C^1 estimate (5) implies uniform ellipticity and a uniform bound on C^1-norms of the coefficients. This, together with the uniform estimate of the α-Hölder norm of F, allow us to apply standard

Schauder estimates (see for instance Corollary 6.3 in [GT83]). Therefore for every $\delta \in (0, \frac{\varrho}{\sqrt{2}})$ there exists a constant $C_4 > 0$ such that

$$\|u\|_{C^2(B_\delta(p))} \leq C_4. \tag{7}$$

The constant C_4 depends only on δ and on the bounds on the C^1-norm of u and the α-Hölder norm of F. Observe that none of those bounds depend on the point $p \in \Sigma$. In fact, they ultimately depend on the value of the entropy $\lambda(\Sigma)$. This concludes the proof, since $|A|^2(p) = |\operatorname{Hess} u|^2(p)$. □

Remark 8. Note that in the proof of Proposition 7, after using [SS83] to show that Σ can be locally described as a graph with a uniform control on its C^1 norm, we could have obtained a uniform estimate for $|A|$ by applying the local curvature estimate by Ecker and Huisken, i.e. Theorem 3.1 in [EH91].

2. Asymptotic behavior of properly immersed translaters

In this section, $\Sigma^2 \subseteq \mathbb{R}^3$ is a properly immersed translater. We do not assume any bound on the entropy and we do not put any restriction on the topology of Σ.

Let us fix some notation.

- $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ denotes the projection $\pi(x_1, x_2, x_3) = (x_1, x_2)$.
- $\operatorname{Conv}(\cdot)$ denotes the (closed) convex hul.
- $B_\varrho(q)$ denotes the open ball in $\mathbb{R}^2$ centered at a point $q \in \mathbb{R}^2$, with radius $\varrho > 0$.
- We say that a plane $P \subseteq \mathbb{R}^3$ is *vertical* if $P \parallel e_3$.
- We say that a halfspace $\mathcal{H} \subseteq \mathbb{R}^3$ is *vertical* if the plane $\partial\mathcal{H}$ is vertical.

Remark 9. From [CM19a] (see also the more general case of ancient flows [CM19b]) it is known that $\operatorname{Conv}(\pi(\Sigma))$ is either a line, a strip, a half-plane or the whole $\mathbb{R}^2$. Therefore $\pi^{-1}(\partial \operatorname{Conv}(\pi(\Sigma)))$ can be, respectively, only one of the following

(i) a vertical plane,
(ii) two parallel vertical planes,
(iii) the empty set.

We will see in this section that, if we are in Case (i) or (ii), Σ is (in some weak sense) asymptotic to $\pi^{-1}(\partial \operatorname{Conv}(\pi(\Sigma)))$ as $x_3 \to \infty$. See the Theorem 10 and Corollary 11 below for a precise statement.

Theorem 10. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a properly immersed translater such that $\partial \operatorname{Conv}(\pi(\Sigma)) \neq \emptyset$.*

Then for every $q \in \partial \operatorname{Conv}(\pi(\Sigma))$ and for every $\varrho > 0$ we have that

$$\sup_{\Sigma \cap \pi^{-1}(B_\varrho(q))} x_3 = +\infty. \tag{8}$$

Proof. Let us assume for contradiction that there exists $q^* \in \partial \operatorname{Conv}(\pi(\Sigma))$ and a radius $\varrho^* > 0$ such that

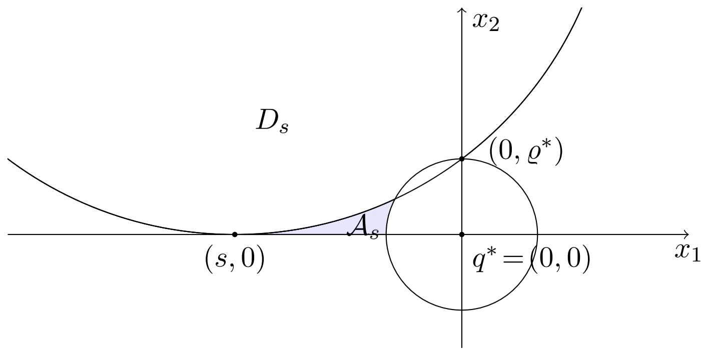

FIGURE 1.

$$\sup_{\Sigma \cap \pi^{-1}(B_{2\varrho^*}(q^*))} x_3 < +\infty.$$

Up to a translation in the e_3 direction, we can assume that

$$\sup_{\Sigma \cap \pi^{-1}(\overline{B_{\varrho^*}(q^*)})} x_3 < 0, \tag{9}$$

where $\overline{B_{\varrho^*}(q^*)}$ is the closure of $B_{\varrho^*}(q^*)$.

W.l.o.g. we can assume that $\pi(\Sigma)$ is contained in the upper half-plane of $\mathbb{R}^2$, i.e. $\pi(\Sigma) \subseteq \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 \geq 0\}$ and let us assume that the x_1-axis $\{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 = 0\}$ is a connected component of $\partial \operatorname{Conv}(\pi(\Sigma))$. Let us also assume $q^* = (0, 0)$.

The rest of the proof will be divided into three steps.

(i) By using the Omori-Yau maximum principle for properly immersed translaters, we are going to prove that x_3 is bounded from above on $\Sigma \cap \pi^{-1}(\mathcal{K})$, for every compact set $\mathcal{K} \subseteq \{(x_1, x_2) \in \mathbb{R}^2 \colon 0 \leq x_2 < \varrho^*\}$.

(ii) By using a family of grim reaper cylinders as barriers, we will prove that x_3 is uniformly bounded from above on

$$\Sigma_\delta \coloneqq \Sigma \cap \{x \in \mathbb{R}^3 \colon 0 \leq x_2 \leq \delta\},$$

for every $\delta < \varrho^*$.

(iii) By using a family of Δ-wing translaters as barriers, we will finally get a contradiction by proving that $\Sigma_\delta = \emptyset$, for every $\delta < \varrho^*$.

Step (i): Observe that for every $s \in \mathbb{R}$ such that $|s| > \varrho^*$, there exists a unique closed disk $D_s \subseteq \{(x_1, x_2) \colon x_2 \geq 0\}$ such that D_s is tangent to the x_1-axis at the point $(s, 0)$, i.e.

$$D_s \cap \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 = 0\} = \{(s, 0)\}$$

and such that $(0, \varrho^*) \in \partial D_s$. See Figure 1.

Observe that $\{(x_1,x_2)\in\mathbb{R}^2\colon x_2\ge 0\}\setminus(D_s\cup B_{\varrho^*}(0))$ has two connected components a bounded one and an unbounded one. Let us call $\mathcal{A}_s$ the bounded one. Observe that the family $(\mathcal{A}_s)_{|s|>\varrho^*}$ together with $B_{\varrho^*}(0)$, cover the strip $\{(x_1,x_2)\colon 0\le x_2<\varrho^*\}$. Namely,

$$\left(B_{\varrho^*}(0)\cup\bigcup_{s\in\mathbb{R}\colon |s|>\varrho^*}\mathcal{A}_s\right)\supseteq\{(x_1,x_2)\colon 0\le x_2<\varrho^*\}. \tag{10}$$

We are now going to prove that x_3 is bounded on $\Sigma\cap\pi^{-1}(\mathcal{A}_s)$ for every $s\in\mathbb{R}$ such that $|s|>\varrho^*$. This will finish the proof of Step (i), because of (10). We will do this by using the Omori-Yau maximum principle for properly immersed translaters and we refer to [CM19a] and to [Xi15] for details.

Let us assume for contradiction that there exists $s^*\in\mathbb{R}$ such that $|s^*|>\varrho^*$ and such that

$$\sup_{\Sigma\cap\pi^{-1}(\mathcal{A}_{s^*})} x_3=+\infty.$$

Let $c\in\{(x_1,x_2)\colon x_2\ge 0\}$ be the center of the disk D_{s_0} and let $R>0$ be its radius. Let $\mathcal{L}$ be the vertical line passing through the center c, i.e. $\mathcal{L}\coloneqq\pi^{-1}(\{c\})=\{c\}\times\mathbb{R}$. Let us define a function $f\colon\mathbb{R}^3\to\mathbb{R}$ as follows:

$$f(x)\coloneqq\begin{cases}\operatorname{dist}(x,\mathcal{L}) & \text{if } \pi(x)\in\mathcal{A}_{s^*}\\ R & \text{if } \pi(x)\notin\mathcal{A}_{s^*}.\end{cases} \tag{11}$$

Since $\mathcal{A}_{s^*}$ is bounded, f is bounded. Observe that the the set of points where $f|_\Sigma$ may be discontinuous is $\pi^{-1}\left(\partial B_{\varrho^*}(q^*)\cap\partial\mathcal{A}_{s^*}\right)\cap\Sigma$, which is contained in $\pi^{-1}\left(\partial B_{\varrho^*}(q^*)\right)\cap\Sigma$. Let us consider the translater with boundary

$$\widetilde{\Sigma}\coloneqq\Sigma\cap\{x\in\mathbb{R}^3\colon x_3\ge 0\}.$$

From (9), we have that

$$\pi^{-1}\left(\partial B_{\varrho^*}(q^*)\right)\cap\widetilde{\Sigma}=\pi^{-1}\left(\partial B_{\varrho^*}(q^*)\right)\cap\Sigma\cap\{x\in\mathbb{R}^3\colon x_3\ge 0\}=\emptyset,$$

therefore $f|_{\widetilde{\Sigma}}$ is continuous. Moreover, $f|_{\Sigma\cap\pi^{-1}(\mathcal{A}_{s^*})}$ is smooth. From standard computations (see [CM19a]) and using equation (1), one can easily see that on $\Sigma\cap\pi^{-1}(\mathcal{A}_{s^*})$

$$\Delta^\Sigma f=\frac{1-\|\nabla^\Sigma f\|^2}{f}-\langle\nabla^{\mathbb{R}^3}f,\nu\rangle\langle\nu,e_3\rangle. \tag{12}$$

As in the proof of Theorem 1.2 in [CM19a], we will use the Omori-Yau maximum principle combined with an "adiabatic trick". More precisely, we would like to apply the Omori-Yau maximum principle to the function $f|_{\widetilde{\Sigma}}$ defined on the translater with boundary $\widetilde{\Sigma}$. But we need to employ the adiabatic trick because the maximum might be reached on the boundary $\partial\widetilde{\Sigma}=\Sigma\cap\{x_3=0\}$.

Let $\psi\colon\mathbb{R}\to\mathbb{R}$ be a smooth cut-off function such that

- $0\le\psi\le 1$,
- $\psi|_{(-\infty,0]}\equiv 0$,

- $\psi|_{[1,\infty)} \equiv 1$.

For every $l > 0$, let $\chi_l \colon \mathbb{R}^3 \to \mathbb{R}$ be a function defined as follows:

$$\chi_l(x) \coloneqq \psi\left(\frac{x_3}{l}\right).$$

Observe that there exists a constant C, which does not depend on l, such that

$$\sup_{x\in\mathbb{R}^3} \|\nabla^{\mathbb{R}^3}\chi(x)\| \leq \frac{C}{l}, \qquad \sup_{x\in\mathbb{R}^3} \|\operatorname{Hess}^{\mathbb{R}^3}\chi(x)\| \leq \frac{C}{l^2}. \tag{13}$$

Now let us define the function $f_l \colon \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$f_l(x) \coloneqq f(x) + M\chi_l(x),$$

where $M \coloneqq \sup f$.

Observe that f_l is bounded. In fact,

$$R \leq f_l \leq 2M. \tag{14}$$

Moreover, observe that

$$\sup_{\widetilde{\Sigma}\cap\pi^{-1}(\mathcal{A}_{s^*})} f_l > R + M = \sup_{\widetilde{\Sigma}\setminus\pi^{-1}(\mathcal{A}_{s^*})} f_l \tag{15}$$

and also note that f_l is smooth on $\widetilde{\Sigma} \cap \pi^{-1}(\mathcal{A}_{s^*})$ away from $\partial\widetilde{\Sigma}$. The Omori-Yau maximum principle yields the existence of a sequence $(p_k) \subseteq \widetilde{\Sigma} \cap \pi^{-1}(\mathcal{A}_{s^*})$ satisfying the following properties:

(i) $\lim_{k\to\infty} f_l(p_k) = \sup_\Sigma f_l$,
(ii) $\lim_{k\to\infty} \nabla^\Sigma f_l(p_k) = 0$,
(iii) $\lim_{k\to\infty} \Delta^\Sigma f_l(p_k) \leq 0$.

Such a sequence (p_k) is said to be an *Omori-Yau sequence* for f_l.

We now distinguish two cases and we will see that they both lead to a contradiction. Let us assume first that there exists $l_0 > 0$ for which f_{l_0} admits an Omori-Yau sequence $(p_k) \subseteq \widetilde{\Sigma} \cap \pi^{-1}(\mathcal{A}_{s^*})$ with $x_3(p_k)$ unbounded in the $+\infty$ direction. Therefore for k large enough, we have that $x_3(p_k) \geq l_0$ and thus $f_{l_0}(p_k) = f(p_k) + M$, $\nabla^\Sigma f_{l_0}(p_k) = \nabla^\Sigma f(p_k)$ and $\Delta^\Sigma f_{l_0}(p_k) = \Delta^\Sigma f(p_k)$. Therefore we have that

$$\lim_{k\to\infty} \nabla^\Sigma f(p_k) = 0 \tag{16}$$

and

$$\lim_{k\to\infty} \Delta^\Sigma f(p_k) \leq 0. \tag{17}$$

Note that on $\pi^{-1}(\mathcal{A}_{s^*})$, we have that $\nabla^{\mathbb{R}^3} f$ is a unit vector field, since it is the gradient of a distance function. Observe that from (16), and from the decomposition

$$\|\nabla^\Sigma f\|^2 = \left\|\nabla^{\mathbb{R}^3} f\right\|^2 - \left\|\left(\nabla^{\mathbb{R}^3} f\right)^\perp\right\|^2,$$

we have that $\lim_{k\to\infty} |\langle\nabla^{\mathbb{R}^3} f(p_k), \nu(p_k)\rangle| = 1$.

Since $\nabla^{\mathbb{R}^3} f \perp e_3$, this implies

$$\lim_{k\to\infty} \langle \nu(p_k), e_3 \rangle = 0. \tag{18}$$

From (12), (18) and (16), we obtain

$$\lim_{k\to\infty} \Delta^{\Sigma} f(p_k) = \frac{1}{\lim_{k\to\infty} f(p_k)} = \frac{1}{\sup_{\Sigma} f_{l_0} - M} > 0 \tag{19}$$

and this is in contradiction with (17).

Let us now assume that for every $l > 0$, every Omori-Yau sequence has bounded x_3-coordinate. This implies, since Σ is proper, that f_l attains its maximum at some point $q_l \in \widetilde{\Sigma} \cap \pi^{-1}(\mathcal{A}_{s^*})$. Therefore we have

(i) $f_l(q_l) = \sup_{\Sigma} f_l$,
(ii) $\nabla^{\Sigma} f_l(q_l) = 0$,
(iii) $\Delta^{\Sigma} f_l(q_l) \leq 0$.

From the estimates (13), we can estimate the gradient of f at q_l,

$$\begin{aligned} \|\nabla^{\Sigma} f(q_l)\| &= \|\nabla^{\Sigma} f_l(q_l) - M\nabla^{\Sigma} \chi_l(q_l)\| \\ &= \|M\nabla^{\Sigma} \chi_l(q_l)\| \leq \|M\nabla^{\mathbb{R}^3} \chi_l(q_l)\| \leq \frac{C}{l}. \end{aligned}$$

Taking the limit for $l \to \infty$, we have

$$\lim_{l\to\infty} \|\nabla^{\Sigma} f(q_l)\| = 0. \tag{20}$$

Note that from (13) we can estimate the Laplacian $\Delta^{\Sigma}\chi$ as follows:

$$\begin{aligned} |\Delta^{\Sigma}\chi_l| &= \left| \Delta^{\mathbb{R}^3} \chi_l - \mathrm{Hess}^{\mathbb{R}^3} \chi_l(\nu,\nu) + H\langle \nu, \nabla^{\mathbb{R}^3} \chi_l \rangle \right| \\ &\leq \frac{C}{l^2} + \frac{C}{l}. \end{aligned}$$

Therefore, we obtain

$$\limsup_{l\to\infty} \Delta^{\Sigma} f(q_l) \leq 0. \tag{21}$$

On the other hand, if we evaluate (12) at points q_l, by using (20), we have

$$\lim_{l\to\infty} \Delta^{\Sigma} f(q_l) > 0$$

and this is in contradiction with (21). This completes the proof of Step (i).

Step (ii): Let us now prove that x_3 is uniformly bounded from above on

$$\Sigma_\delta \coloneqq \Sigma \cap \{x \in \mathbb{R}^3 \colon 0 \leq x_2 \leq \delta\}$$

for every $0 < \delta < \varrho^*$.

Let us decompose Σ_δ as

$$\Sigma_\delta = \Sigma_+ \cup \Sigma_-,$$

where $\Sigma_+ \coloneqq \Sigma_\delta \cap \{x \in \mathbb{R}^3 \colon x_1 \geq 0\}$, similarly $\Sigma_- \coloneqq \Sigma_\delta \cap \{x \in \mathbb{R}^3 \colon x_1 \leq 0\}$. We are going to show that x_3 is bounded from above separately on Σ_+ and Σ_-.

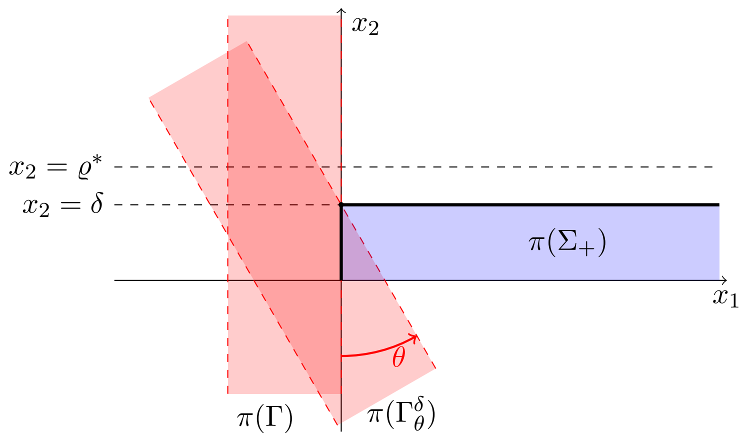

FIGURE 2.

We prove the claim for Σ_+ only, since the considerations for Σ_- are analogous. Let us consider the grim reaper cylinder

$$\Gamma := \left\{(x_1, x_2, x_3) \colon x_3 = -\log\left(\cos\left(x_1 + \frac{\pi}{2}\right)\right),\ -\pi < x_1 < 0\right\}.$$

Observe that $\Gamma \cap \Sigma_+ = \emptyset$.

Let $F_\theta^\delta \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle $\theta \in [0, \frac{\pi}{2})$ around the vertical line $\{(0, \delta)\} \times \mathbb{R}$. Let us define the following 1-parameter family

$$\Gamma_\theta^\delta := F_\theta^\delta(\Gamma).$$

Note that

(22)
$$\partial\Sigma_+ = \{x \in \Sigma \colon x_1 = 0 \text{ and } 0 \le x_2 \le \delta\} \cup \{x \in \Sigma \colon x_2 = \delta \text{ and } x_1 \ge 0\}.$$

Because of assumption (9), we have that for every $\theta \in [0, \frac{\pi}{2})$,

$$\Gamma_\theta^\delta \cap \Sigma \cap \{(0, x_2, x_3) \in \mathbb{R}^3 \colon 0 \le x_2 \le \delta\} = \emptyset. \tag{23}$$

In fact, the grim reaper cylinder Γ is the graph of a convex and nonnegative function, therefore the x_3-coordinate function is nonnegative on each Γ_θ^δ. Moreover from the construction of the family Γ_θ^δ, we have that for every $\theta \in [0, \frac{\pi}{2})$,

$$\Gamma_\theta^\delta \cap \Sigma \cap \{(x_1, \delta, x_3) \colon x_1 \ge 0\} = \emptyset. \tag{24}$$

Therefore, combing (23) and (24) with (22), we conclude that

$$\Gamma_\theta^\delta \cap \partial\Sigma_+ = \emptyset \tag{25}$$

for every $\theta \in [0, \frac{\pi}{2})$.

We want to prove that $\Gamma^\delta_\theta \cap \Sigma_+ = \emptyset$ for every $\theta \in [0, \frac{\pi}{2})$. Recall that $\Gamma^\delta_0 \cap \Sigma_+ = \Gamma \cap \Sigma_+ = \emptyset$. Consider the function

$$\theta \mapsto \operatorname{dist}(\Gamma^\delta_\theta, \Sigma_+) = \operatorname{dist}(F^\delta_\theta(\Gamma), \Sigma_+).$$

It is clearly a continuous and nonnegative on $[0, \frac{\pi}{2})$, since it is the composition of two continuous functions. We want to prove that it is actually strictly positive on $[0, \frac{\pi}{2})$. Assume for contradiction that this is not the case and let

$$\theta^* := \min\left\{\theta \in \left[0, \frac{\pi}{2}\right) : \operatorname{dist}(\Gamma^\delta_\theta, \Sigma_+) = 0\right\}.$$

Observe that $\pi(\Gamma^\delta_\theta) \cap \pi(\Sigma_+)$ is a triangle for each $\theta \in [0, \frac{\pi}{2})$ (see Figure 2). From Step (i), we have that the x_3-coordinate is bounded from above on $\pi^{-1}\left(\pi(\Gamma^\delta_\theta) \cap \pi(\Sigma_+)\right) \cap \Sigma_+$ and the x_3-coordinate is bounded from belove (is nonnegative) on Γ^δ_θ. Thus, since Σ_+ and Γ^δ_θ are properly immersed, the distance between Γ^δ_θ and Σ_+ is always attained. In particular we have

$$\operatorname{dist}(F^\delta_\theta(\Gamma), \Sigma_+) = 0 \Leftrightarrow F^\delta_\theta(\Gamma) \cap \Sigma_+ = \emptyset,$$

thus there exists $p \in \Gamma^\delta_{\theta^*} \cap \Sigma_+$. From (25), we have that $p \in (\Sigma_+ \setminus \partial\Sigma_+)$. But this is in contradiction with the separating tangency principle (see Lemma 2.4 in [Mø14]).

Similarly, one can show that $\Gamma^\delta_\theta \cap \Sigma_- = \emptyset$ for $\theta \in (-\frac{\pi}{2}, 0]$. This implies that

$$\Sigma_\delta \cap \Gamma_{\frac{\pi}{2}} = \emptyset. \tag{26}$$

Note that

$$\Gamma_{\frac{\pi}{2}} = \Gamma_{-\frac{\pi}{2}} = \left\{\left(x_1, x_2, -\log\left(\cos\left(x_2 - \delta + \frac{\pi}{2}\right)\right)\right), \delta - \pi < x_2 < \delta\right\}.$$

In other words, the grim reaper cylinder $\Gamma_{\frac{\pi}{2}}$ lies "above" Σ_δ. Observe that (26) holds for every $0 < \delta < \varrho^*$ (note that the domain of $\Gamma_{\frac{\pi}{2}}$ depends on δ). This finishes the proof of Step (ii).

Step (iii): We will now finally show that $\Sigma_\delta = \emptyset$ for every $\delta < \varrho^*$. Thanks to Step (ii), we can assume w.l.o.g.

$$\sup_{\Sigma_\delta} x_3 < 0. \tag{27}$$

Let $S \subseteq \mathbb{R}^3$ be a Δ-wing translater (see [BLT18] and [HIMW19a]) such that it is the graph of a convex function $u \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, where Ω is the strip

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \colon -\gamma < x_2 < \delta\}$$

for some $\gamma > 0$ such that $\gamma + \delta > \pi$. Let us now define a one parameter family of translaters with boundary $\widetilde{S}_t$ as follows:

$$\widetilde{S}_t := (S + te_3) \cap \{x \in \mathbb{R}^3 \colon x_3 \leq 0\}.$$

Note that $\widetilde{S}_t$ is compact and $\partial(\widetilde{S}_t) = (S + te_3) \cap \{x_3 = 0\}$. Observe that

$$\bigcup_{t \in \mathbb{R}} \widetilde{S}_t = \Omega \times (-\infty, 0]. \tag{28}$$

From the way we chose Ω, we have that $\Sigma \cap (\Omega \times (-\infty, 0]) \neq \emptyset$.

Since $\widetilde{S}_t$ is compact for every $t \in \mathbb{R}$ and since Σ is properly immersed, there exists $t^* \in \mathbb{R}$ such that $\widetilde{S}_{t^*} \cap \Sigma \neq \emptyset$ and such that $\widetilde{S}_t \cap \Sigma = \emptyset$ for $t > t^*$. From (27), we have that any intersection point $p \in \widetilde{S}_{t^*} \cap \Sigma$ is an interior point for $\widetilde{S}_{t^*}$. We can therefore apply the separating tangency principle and get that $\Sigma = S + t^* e_3$, which is a contradiction because we assumed $\pi(\Sigma) \subseteq \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 \geq 0\}$. □

Corollary 11. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a properly immersed translater contained in a slab. W.l.o.g. let us assume*

$$\operatorname{Conv}(\pi(\Sigma)) = \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_2| < \delta\},$$

for some $\delta > 0$. Thus $\{x \in \mathbb{R}^3 \colon |x_2| = \delta\} = \pi^{-1}(\partial \operatorname{Conv}(\pi(\Sigma)))$. Let P be a vertical plane such that $P \nparallel \{x \in \mathbb{R}^3 \colon |x_2| = \delta\}$.

Then there exist two distinct sequences $(p_k^1), (p_k^2) \subseteq \Sigma \cap P$ satisfying the following properties:

(i) $\lim_{k\to\infty} x_3(p_k^1) = \lim_{k\to\infty} x_3(p_k^2) = \infty$,

(ii) $\lim_{k\to\infty} \operatorname{dist}(p_k^1, L_1) = \lim_{k\to\infty} \operatorname{dist}(p_k^2, L_2) = 0$,

where L_1 and L_2 are the two vertical lines $L_1 = \{x \in \mathbb{R}^3 \colon x_2 = \delta\} \cap P$ and $L_2 = \{x \in \mathbb{R}^3 \colon x_2 = -\delta\} \cap P$.

Proof. Assume by contradiction that the statement is not true. For instance, let us assume that there is no sequence (p_k^1) satisfying (i) and (ii). Then this means that x_3 is bounded from above on $\{x \in \Sigma \cap P \colon \operatorname{dist}(x, L_1) \leq \varepsilon\}$ for some $\varepsilon > 0$. W.l.o.g. we can assume that

$$x_3 < 0 \tag{29}$$

for every $x = (x_1, x_2, x_3) \in \Sigma \cap P$ such that $\operatorname{dist}(x, L_1) < \varepsilon$.

Let $\mathcal{H}$ be one of the two halfspaces such that $\partial\mathcal{H} = P$. Note that from Theorem 10, we can assume that $\mathcal{H} \cap \Sigma$ contains a sequence of points $(q_k) \subseteq \mathcal{H} \cap \Sigma$ such that $x_3(q_k) \nearrow \infty$ and $\operatorname{dist}(q_k, L_1) \to 0$. Let $\mathcal{C}$ be a vertical cylinder such that $\mathcal{C} \subseteq \mathcal{H} \cap \{x \in \mathbb{R}^3 \colon x_2 \leq \delta\}$ and such that $\mathcal{C}$ is tangent to P and to $\{x \in \mathbb{R}^3 \colon x_2 = \delta\}$. Observe that $\pi\left(\left(\mathcal{H} \cap \{x \in \mathbb{R}^3 \colon x_2 \leq \delta\}\right) \setminus \mathcal{C}\right)$ consists of two connected components, one bounded and another one unbounded. Let $\mathcal{A}$ be the bounded one. Moreover let $\mathcal{L}$ be the axis of the cylinder. Then we define the function $f \colon \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_2| < \delta\} \to \mathbb{R}$ as follows

$$f(x) := \begin{cases} \operatorname{dist}(x, \mathcal{L}) & \text{if } \pi(x) \in \mathcal{A} \\ R & \text{if } \pi(x) \notin \mathcal{A}. \end{cases}$$

where R is the radius of $\mathcal{C}$. Let us consider the restriction $f|_{\widetilde{\Sigma}}$, where $\widetilde{\Sigma}$ is the translater with boundary defined as

$$\widetilde{\Sigma} := \Sigma \cap \{x \in \mathbb{R}^3 \colon x_3 \geq 0\}.$$

Note that, because of the existence of the sequence (q_k), we have that

$$\sup_{\widetilde{\Sigma}}(f) = \operatorname{dist}(\mathcal{L}, L) > R$$

and from (29) follows that $f|_{\widetilde{\Sigma}}$ is smooth on the set

$$\{x \in \widetilde{\Sigma} \colon f|_{\widetilde{\Sigma}}(x) > \sup_{\widetilde{\Sigma}} f - \varepsilon\}.$$

We can therefore apply the Omori-Yau principle directly, without the need of the "adiabatic trick", in order to get a contradiction. The computations are similar (and simpler, since we do not need the cut-off function here) to the ones in the proof of Theorem 10. □

We include here a Bernstein type theorem for 1-periodic translaters, which will not be used in the proof of Theorem 1 but it is worth mentioning. It is a simple consequence of Theorem 10.

Corollary 12 (Bernstein type theorem for 1-periodic translaters)**.** *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a properly immersed translater such that $\Sigma \subset \mathcal{H}$, where $\mathcal{H}$ is a vertical halfspace. Let us assume that Σ is 1-periodic in the e_3-direction, i.e. there exists $a > 0$ such that*

$$\Sigma = \Sigma + ae_3. \tag{30}$$

Then Σ is a vertical hyperplane.

Proof. Let us assume $\partial\mathcal{H} \subseteq \pi^{-1}(\partial \operatorname{Conv}(\pi(\Sigma)))$. From Theorem 10 and the 1-periodicity assumption (30), it follows that $\Sigma \cap \partial\mathcal{H}$. The conclusion follows from the separating tangency principle. □

Remark 13. Observe that nontrivial periodic translaters do exist but the known examples are in line with Corollary 12, because their period is a vector orthogonal to e_3 (see [Ng09] and the very recent paper [HMW19b]).

3. The structure of the set Z.

In this section we assume Σ to be a properly embedded translating soliton. We want to study the structure of the intersection of Σ with a vertical plane P and we denote such intersection as

$$Z \coloneqq \Sigma \cap P.$$

Note that Z can be described as the zero set of a function defined on Σ as follows. Let $p \in Z$ and let $V \in \mathbb{S}^2$ be a unit vector orthogonal to P. Then Z is the zero set of the function

$$x \mapsto \langle V, x - p\rangle, \qquad x \in \Sigma.$$

The structure of Z is described by the following lemma, which is inspired by Lemma 6 in [Br16] and [Ro95].

Lemma 14. *Let us assume that Σ is not flat, i.e. is not a vertical plane. Then for each point $x \in Z$ there exists an open neighborhood $x \in U \subseteq \Sigma$, such that $Z \cap U$ is a union of finitely many C^2-arcs $\Gamma_1, \ldots, \Gamma_m$ which intersect transversally at x. The number m is the vanishing order of the function $x \mapsto \langle V, x - p\rangle$ at p.*

Globally, the set Z is the union of countably many 1-dimensional properly immersed C^2-submanifolds without boundary of $\mathbb{R}^3$ and they may intersect pairwise only at isolated points.

Proof. Let $f(x) := \langle V, x-p\rangle$. Observe that $\nabla^\Sigma f = V^\top$. Moreover, using the translater equation (1) and the fact that $V \perp e_3$, we have

$$\begin{aligned}\Delta^\Sigma f &= \operatorname{div}^\Sigma(V^\top) = \operatorname{div}^\Sigma(V) - \operatorname{div}^\Sigma(V^\perp)\\ &= -\langle V,\nu\rangle H = \langle V, e_3^\perp\rangle = -\langle V, e_3^\top\rangle = -\langle \nabla^\Sigma f, e_3\rangle.\end{aligned}$$

Thus f satisfies the following elliptic equation

$$\Delta^\Sigma f + \langle \nabla^\Sigma f, e_3\rangle = 0. \tag{31}$$

Therefore,

$$\begin{aligned}\Delta^\Sigma(e^{\frac{x_3}{2}} f) &= \operatorname{div}^\Sigma(\nabla^\Sigma(e^{\frac{x_3}{2}} f))\\ &= \operatorname{div}^\Sigma\left(e^{\frac{x_3}{2}}\frac{f}{2}e_3^T + e^{\frac{x_3}{2}}\nabla^\Sigma f\right)\\ &= e^{\frac{x_3}{2}}\left(\frac{f}{4}|e_3^T|^2 + \frac{f}{2}\operatorname{div}^\Sigma(e_3^\top) + \langle\nabla^\Sigma f, e_3\rangle + \Delta^\Sigma f\right)\\ &= \left(e^{\frac{x_3}{2}} f\right)\left(\frac{|e_3^T|^2}{4} + \frac{\operatorname{div}^\Sigma(e_3^\top)}{2}\right).\end{aligned}$$

The conclusion of the first part of the statement follows from applying Theorem 2.5 in [Ch76] to the function $x \mapsto e^{\frac{x_3}{2}} f(x)$ and observing that its zero set coincides with the zero set of f.

The second part of the statement follows immediately from the first part and from the properness of Σ. □

Remark 15. We are mainly interested in the special case where

$$H(p) = 0, \quad P = T_p\Sigma, \quad V = \nu(p),$$

where $T_p\Sigma$ denotes, with a little abuse of notation, the geometric tangent plane of Σ at p. Observe that from equation (1), $H(p) = 0$ if and only if $T_p\Sigma$ is a vertical plane. Note that in this case $f\colon x \mapsto \langle V, x-p\rangle$ has vanishing order $m \geq 2$ at p, because $\nabla^\Sigma f = V^\top$ and $V = \nu(p)$, we have that $\nabla^\Sigma f|_p = 0$. Therefore there exists a neighborhood U of p such that $Z \cap U$ consists of at least two C^2-curves intersecting transversally at p.

We have also the following information about Z.

Lemma 16. *Under the same assumptions as Lemma 14, if we further assume Σ to be simply connected, then each connected component of Z is simply connected. In particular, Z is the union of the images of countably many, C^2-embeddings $\gamma_j\colon \mathbb{R} \to \Sigma$ which may intersect pairwise at most at one point.*

Proof. Assume by contradiction that there exists a continuous and injective loop $\delta\colon \mathbb{S}^1 \to Z$ which is not homotopically trivial. Then δ is a Jordan curve in Σ, and since we are assuming Σ to be homeomorphic to the plane, from the Jordan theorem, the image of δ is the boundary of a nonempty, bounded open set $\Omega \subseteq \Sigma$. This means that the function $f(x) = \langle V, x - p\rangle$ satisfies the following boundary problem:

$$\begin{cases} \Delta^\Sigma f + \langle \nabla^\Sigma f, e_3\rangle = 0 & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases}$$

From the maximum principle, it follows that f is identically zero in Ω, which means $\Omega \subseteq Z$. But this contradicts Lemma 14. □

Lemma 17. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a simply connected, properly embedded translater. Let $\mathcal{H} \subseteq \mathbb{R}^3$ be a vertical halfspace.*

Then each connected component of $\Sigma \cap \mathcal{H}$ is simply connected.

Proof. Let P be the vertical plane $P = \partial\mathcal{H}$, $p \in Z = P \cap \Sigma$ and let V be the orthogonal unit vector to P pointing outside $\mathcal{H}$. Let us assume, for contradiction, that there exists an embedding $\gamma\colon \mathbb{S}^1 \to \Sigma \cap \mathcal{H}$ which is not homotopically trivial in $\Sigma \cap \mathcal{H}$. Since Σ is simply connected, there exists $\Omega \subseteq \Sigma$ such that $\partial\Omega = \gamma(\mathbb{S}^1)$. Let $f\colon \Sigma \to \mathbb{R}$ be defined as $f(x) = \langle V, x - p\rangle$ as above. Observe that

$$f|_{\partial\Omega} \leq 0.$$

Since we are assuming γ is not homotopically trivial in $\Sigma \cap \mathcal{H}$, we have that $\Omega \nsubseteq \Sigma \cap \mathcal{H}$. This implies

$$\max_{\Omega} f > 0 \geq \max_{\partial\Omega} f. \tag{32}$$

On the other hand f satisfies the elliptic equation (31), therefore (32) violates the maximum principle.

□

4. The structure of $\{H = 0\}$

In this section we study the zero set of the mean curvature of Σ.

Remark 18. On a translater Σ, the mean curvature H solves the following equation:

$$\Delta^\Sigma H + \langle \nabla^\Sigma H, e_3\rangle + |A|^2 H = 0, \tag{33}$$

see for instance Lemma 2.1 in [MSS15]. As in the proof of Lemma 14, one can readily check that $e^{\frac{x_3}{2}} H$ satisfies the equation (without first order term):

$$\Delta^\Sigma (e^{\frac{x_3}{2}} H) = \left(e^{\frac{x_3}{2}} H\right) h,$$

for some smooth function h. Observe that the zero set of $e^{\frac{x_3}{2}} H$ coincides with $\{H = 0\}$. If Σ is not flat, from Theorem 2.5 in [Ch76], we have that it is a union of 1-dimensional C^2-manifolds and the singular points (namely the intersection points of such 1-dimensional manifolds) are isolated.

Lemma 19. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a complete translater, such that the unit normal vector field ν is constant along $\{H = 0\}$.*

Then Σ is mean convex.

Proof. We can assume $\{H = 0\} \neq \emptyset$ and that Σ is not flat, otherwise the statement is trivially true. Let us assume that ν is constant along $\{H = 0\}$. Let $V \in \mathbb{S}^2$ be such that $\nu|_{\{H=0\}} \equiv V$. Note that from (1) we have that $V \perp e_3$. From Remark 18 we have that $\{H = 0\}$ is a 1-dimensional smooth manifold away from a set of isolated points.

Let $p \in \{H = 0\}$ be a regular point and let $\gamma\colon (-\varepsilon, \varepsilon) \to \{H = 0\}$ be a regular curve such that $\gamma(0) = p$. Since ν is constant along $\{H = 0\}$, we have that $T_{\gamma(s)}\Sigma = T_p\Sigma$ and $\gamma(s) \in T_p\Sigma \cap \Sigma = Z$, for every $s \in (-\varepsilon, \varepsilon)$.

From Lemma 14 we have that there exists a neighborhood $p \in U \subseteq \Sigma$ such that $Z \cap U$ is the union of finitely many C^2-arcs intersecting transversally at p. Moreover, we can assume that p is the only singular point of the 1-dimensional C^2-manifold $Z \cap U$.

Observe that the function $x \mapsto \langle V, x - p\rangle$ has vanishing order $m \geq 2$ at $\gamma(s)$ for every $s \in (-\varepsilon, \varepsilon)$. Therefore, from Lemma 14, we have that each point $\gamma(s)$ is a singular point of $\{H = 0\}$ and this is in contradiction with the fact that p is an isolated singular point. □

We conclude this section with the following proposition which will not be used in the proof of Theorem 1, but is a stand-alone observation.

Proposition 20. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a complete translater with only one end and assume $\{H = 0\}$ to be compact.*

Then $\{H = 0\}$ is empty. Namely, Σ is strictly mean convex.

Proof. Let us assume by contradiction that $\{H = 0\}$ is compact and non-empty. From Remark 18, we have that it is a 1-dimensional smooth manifold away from a closed set of isolated points. Therefore, since we are assuming $\{H = 0\}$ to be compact, the singular set is a union of finitely many points.

Since Σ has one end, we have that either $\{H \geq 0\}$ or $\{H \leq 0\}$ is compact. Let us assume without loss of generality that $\Omega := \{H \geq 0\}$ is compact. Since H solves the elliptic equation (33), as an application of the strong maximum principle applied to H, we have that the interior $\{H > 0\}$ is non-empty, unless Σ is flat. Observe that Ω is a compact translater with boundary $\partial\Omega = \{H = 0\}$.

Let $V \in \mathbb{R}^3$ be a vector such that $\langle V, e_3\rangle = 0$. Let $P_V := \{x \in \mathbb{R}^3 \colon \langle V, x\rangle = 0\}$ and let us consider the one parameter family of planes $P_{V,t} := P_V + tV$, with $t \in \mathbb{R}$. Since Ω is compact, there exists $t^* = t^*(V)$ such that $P_{V,t} \cap \Omega = \emptyset$ for every $t < t^*$ and $P_{V,t^*} \cap \Omega \neq \emptyset$. Let $p \in P_{V,t^*} \cap \Omega$. Observe that P_{V,t^*} is also a translater. Therefore, if $p \in \Omega \setminus \partial\Omega$, we get a contradiction from the separating tangency principle for translaters (Lemma 2.4 in [Mø14]). Thus $P_{V,t^*} \cap \Omega \subseteq \partial\Omega$.

Since $\partial\Omega$ has at most finitely many singular points, we can choose V, such that there exists $p \in P_{V,t^*} \cap \Omega \subseteq \partial\Omega$ which is not a singular point. From the translater equation (1), we have that the geometric tangent space $T_p\Sigma$ coincides with P_{V,t^*}. Since $\partial\Omega$ is regular at p, we get a contradiction also in this case from the boundary version of the separating tangency principle (see for instance Theorem 2.1.1 in [Pé16]). $\square$

Remark 21. Observe that if Σ has more than one end, then $\{H = 0\}$ can be non-empty and compact. Consider for example the wing-like translater introduced in [CSS07].

5. Proof of Theorem 1

Proof. The proof proceeds by contradiction. Let Σ be as in the assumptions of Theorem 1 and let us assume for contradiction that Σ is not mean convex.

Since Σ has finite entropy and $|H| \leq 1$, Lemma 24 in the Appendix implies that Σ is properly embedded. Therefore, from the results in [CM19a] (see Remark 9) we have that $\mathrm{Conv}(\pi(\Sigma))$ is a strip. Let $\mathcal{S}$ be the slab $\mathcal{S} := \pi^{-1}(\mathrm{Conv}(\pi(\Sigma)))$.

From Lemma 19, we can find a point $p \in \{H = 0\}$, such that $T_p\Sigma$ is not parallel to $\partial\mathcal{S}$. Note that $T_p\Sigma$ is a vertical plane, because of (1). Observe that $\mathcal{S} \cap T_p\Sigma$ is a vertical strip on which x_1 and x_2 are bounded and x_3 is unbounded. From Lemma 14 and Lemma 16, the set $Z = \Sigma \cap T_p\Sigma$ is the union of the images of countably many (possibly finitely many) C^2-embeddings $\gamma_j : \mathbb{R} \to \Sigma$. Each of these 1-dimensional submanifolds is properly embedded in $\mathbb{R}^3$ and since the coordinates x_1 and x_2 are bounded on Z, we have that for each j the two limits $\lim_{t\to+\infty} x_3(\gamma_j(t))$ and $\lim_{t\to-\infty} x_3(\gamma_j(t))$ exist and each of them is equal to $+\infty$ or $-\infty$.

In what follows, we use the term "ray" to denote a half curve, i.e. to denote $\gamma_j^+ := \gamma_j|_{[0,\infty)}$ or $\gamma_j^- := \gamma_j|_{(-\infty,0]}$.

Case 1: Let us assume that there are at least 3 rays in Z for which their x_3 coordinates goes to $+\infty$. We will find a contradiction with the bound on the entropy. This implies that there are at least three distinct sequences of points $(q_k^1), (q_k^2), (q_k^3) \subseteq Z$ such that

$$x_3(q_k^1) = x_3(q_k^2) = x_3(q_k^3) = k$$

for every sufficiently large $k \in \mathbb{N}$. From Corollary 11, we can assume

$$\mathrm{dist}(q_k^1, L_1) \xrightarrow[k\to 0]{} 0, \qquad \mathrm{dist}(q_k^2, L_2) \xrightarrow[k\to 0]{} 0, \tag{34}$$

where L_1 and L_2 are the two vertical parallel lines such that $L_1 \cup L_2 = P \cap \partial\mathcal{S}$. Moreover, since $\pi(P \cap \mathcal{S})$ is compact, we can assume, up to extracting a subsequence, that

$$\pi(q_k^3) \xrightarrow[k\to\infty]{} q \tag{35}$$

for some $q \in \pi(P \cap \mathcal{S})$. Let us consider the sequence of translaters (Σ_k), defined as

$$\Sigma_k := \Sigma - ke_3.$$

Let us define the sequences $(\tilde{q}_k^i) \subseteq \Sigma_k$, for $i = 1, 2, 3$ as follows:

$$\tilde{q}_k^i := q_k^i - ke_3.$$

From Proposition 7, we know that the norm of the second fundamental form of Σ_k is uniformly bounded by a constant. Moreover, from (34) and from (35), we have that

$$\tilde{q}_k^1 \xrightarrow[k\to\infty]{} \pi(L_1), \qquad \tilde{q}_k^2 \xrightarrow[k\to\infty]{} \pi(L_2), \qquad \tilde{q}_k^3 \xrightarrow[k\to\infty]{} q.$$

Therefore, by employing a standard Arzelà-Ascoli argument (see for instance Theorem 2.14 in [BGM19]), we have that there exists a properly embedded, not necessarily connected, smooth translater Σ_∞, such that, up to a subsequence, we have

$$\Sigma_k \xrightarrow[k\to\infty]{C^\infty_{loc}} \Sigma_\infty.$$

Moreover, we have that $\Sigma_\infty \subseteq \mathcal{S}$, $L_1 \cap \Sigma_\infty \neq \emptyset$ and $L_2 \cap \Sigma_\infty \neq \emptyset$. Therefore, from the separating tangency principle, we can conclude that Σ_∞ is the following disjoint union

$$\Sigma_\infty = P_1 \cup P_2 \cup \Sigma^{'},$$

where P_1 and P_2 are the two vertical parallel planes such that $P_1 \cup P_2 = \partial\mathcal{S}$ and $\Sigma^{'}$ is a complete translater passing through q. Corollary 26 and Remark 22 in the Appendix implies that

$$\lambda(\Sigma_\infty) = \lambda(P_1) + \lambda(P_2) + \lambda(\Sigma') \geq 3.$$

Observe that q might coincide with $\pi(L_1)$ or $\pi(L_2)$ and in that case $\Sigma^{'}$ would coincide with P_1 or P_2, respectively. This is not a problem because in this situation, the convergence of Σ_k to P_1 or P_2 would be of multiplicity at least 2.

Let $\mathcal{B}_R$ denote the ball in $\mathbb{R}^3$ of radius $R > 0$ centered at 0. Observe that, for any $x_0 \in \mathbb{R}^3$ and $t_0 > 0$ we have

$$\lambda(\Sigma) = \lambda(\Sigma_k) \geq F_{x_0,t_0}(\Sigma_k) \geq F_{x_0,t_0}\left(\Sigma_k \cap \mathcal{B}_R\right). \tag{36}$$

The first equality in (36) follows from the translation invariance of the entropy. Taking the limit for $k \to \infty$ in (36) and using the fact that $\lim_{k\to\infty} F_{x_0,t_0}\left(\Sigma_k \cap \mathcal{B}_R\right) = F_{x_0,t_0}\left(\Sigma_\infty \cap \mathcal{B}_R\right)$, we obtain

$$\lambda(\Sigma) \geq F_{x_0,t_0}(\Sigma_\infty \cap \mathcal{B}_R). \tag{37}$$

Inequality (37) holds for every $R > 0$, thus $\lambda(\Sigma) \geq F_{x_0,t_0}(\Sigma_\infty)$. After taking the supremum over $x_0 \in \mathbb{R}^3$ and $t_0 > 0$, we finally obtain the following contradiction

$$3 > \lambda(\Sigma) \geq \lambda(\Sigma_\infty) \geq 3.$$

Case 2: Let us now assume that there are at most 2 rays such that their x_3 coordinate goes to $+\infty$. From Corollary 11, we know that x_3 can not be bounded from above on Z. Therefore there is at least one ray in Z on which x_3 goes to $+\infty$.

In what follows $\mathcal{H}^+$ and $\mathcal{H}^-$ are the two open halfspaces with boundary $T_p\Sigma$, namely

$$\mathcal{H}^+ := \{x \in \mathbb{R}^3 \colon \langle x - p, \nu(p)\rangle > 0\},$$
$$\mathcal{H}^- := \{x \in \mathbb{R}^3 \colon \langle x - p, \nu(p)\rangle < 0\}.$$

Moreover, $\overline{\mathcal{H}}^+$ and $\overline{\mathcal{H}}^-$ will denote the closure of $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively.

Let U be a neighborhood of p in Σ as in Lemma 14. Therefore,

$$Z \cap U = \bigcup_{j=1}^{m} \Gamma_j,$$

where Γ_j are C^2-arcs meeting transversally at p and $m \geq 2$ (see Remark 15). We can choose U such that each Γ_j divides U into two connected components. From Lemma 17, the arcs Γ_j intersect pair-wise only at p.

Moreover, we can assume U to be the graph of a function $u \colon B \to \mathbb{R}$ for some ball $B \subseteq T_p\Sigma$. From the discussion above and from the separating tangency principle, $U \setminus Z$ is the union of $2m$ connected components $U_1^+, \dots, U_m^+, U_1^-, \dots, U_m^-$, where $U_j^+ \subseteq \mathcal{H}^+$ and $U_j^- \subseteq \mathcal{H}^-$.

We denote by Ω_j^+ the connected component of $\Sigma \cap \mathcal{H}^+$ containing U_j^+ and similarly, we denote by Ω_j^- the connected component of $\Sigma \cap \mathcal{H}^-$ containing U_j^-.

Observe that from Lemma 16 and Lemma 17, it follows that if $j \neq k$, $U_j^\pm$ and $U_k^\pm$ belong to two distinct connected components of $\Sigma \cap \mathcal{H}^\pm$. In other words $\Omega_1^+, \dots, \Omega_m^+, \Omega_1^-, \dots, \Omega_m^-$ are all distinct. Moreover observe that from Lemma 16 we have that

$$\partial\Omega_j^+ \cap \partial\Omega_k^+ = \{p\}, \qquad \partial\Omega_j^- \cap \partial\Omega_k^- = \{p\}, \tag{38}$$

for $j \neq k$. Moreover, from Corollary 11, we have

$$\sup_{\partial\Omega_j^+} x_3 = +\infty, \qquad \sup_{\partial\Omega_j^-} x_3 = +\infty, \tag{39}$$

for $j = 1, \dots, m$.

Let $\widetilde{Z}$ be the connected component of Z containing p. We will now distinguish the following subcases.

(a) The x_3 coordinate is bounded from above on $\widetilde{Z}$.
(b) $\widetilde{Z}$ contains one ray such that the x_3 coordinate goes to $+\infty$.
(c) $\widetilde{Z}$ contains two rays such that the x_3 coordinate goes to $+\infty$.

(a) Let us assume the coordinate x_3 to be bounded from above on $\widetilde{Z}$. Since we are in **Case 2**, there can be at most 2 connected components of Z for which x_3 goes to $+\infty$. Note that (38) and (39), together with the fact that $m \geq 2$, imply that, in fact, $m = 2$ and there are exactly 2 distinct connected components Z_1 and Z_2 of Z on which x_3 goes to $+\infty$ and such that

$$Z_1 \subseteq \partial\Omega_1^+ \quad \text{and} \qquad Z_1 \subseteq \partial\Omega_1^- \tag{40}$$

and

$$Z_2 \subseteq \partial\Omega_2^+ \quad \text{and} \qquad Z_2 \subseteq \partial\Omega_2^- . \tag{41}$$

But this is in contradiction with the fact that Σ is simply connected. Indeed, we can construct a loop in Σ with base point p which is not homotopically trivial as follows: let $\delta_1\colon [0, l_1] \to \Sigma$ be a regular curve such that $\delta_1(0) = p$ and $\delta_1(l_1) \in Z_1$ and $\delta_1(t) \in \Omega_1^+$ for $0 < t < l_2$. Let $\delta_2\colon [0, l_2] \to \Sigma$ be another regular curve connecting Z_1 and $\{p\}$, such that $\delta_2(0) = \delta_1(l_1)$, $\delta_2(l_2) = p$ and such that $\delta_2(t) \in \Omega_1^-$. Let $\delta = \delta_1 * \delta_2$ be the concatenation of δ_1 and δ_2. Observe that the existence of δ_1 and δ_2 is guaranteed by (40). It is immediate to see that δ is not homotopically trivial, because Z_1 and $\widetilde{Z}$ are two distinct connected components of Z.

(b) Let us assume that $\widetilde{Z}$ contains one ray, such that the x_3 coordinate goes to $+\infty$. One can find again a contradiction with a similar argument as in the subcase (a).

(c) Let us assume $\widetilde{Z}$ contains two rays r_1 and r_2, such that the x_3-coordinate goes to $+\infty$. Since we are in **Case 2**, this implies that x_3 is bounded on all the other connected components of Z. For the sake of clarity, let us assume that both rays are emanating from p (it is easy to deal with the general case). Namely, let us assume that $r_i\colon [0, \infty) \to Z$ and $r_i(0) = p$ for $i = 1, 2$. Note that there cannot be any other ray "between" them, otherwise its x_3-coordinate would have to go to $+\infty$, violating the hypothesis of subcase (c). Therefore, one of the connected components $U_1^+, \dots, U_m^+, U_1^-, \dots, U_m^-$ must have $r_1 \cap U$ and $r_2 \cap U$ as boundary in U. W.l.o.g., let us assume $\partial U_1^+ = (r_1 \cup r_2) \cap U$. Observe that (38) implies $\partial\Omega_j^+ \cap (r_1 \cup r_2) = \{p\}$ for every $j = 2, \dots, m$. Therefore, x_3 is bounded from above on $\partial\Omega_j^+$ for $j = 2, \dots, m$ but this contradicts (39).

□

Appendix A. Colding-Minicozzi's entropy

Let $\Sigma^n \subseteq \mathbb{R}^{n+k}$ be a submanifold. Following [CM12], given $x_0 \in \mathbb{R}^{n+k}$ and $t_0 > 0$, the functional F_{x_0,t_0} is defined as follows

$$F_{x_0,t_0}(\Sigma) \coloneqq \frac{1}{(4\pi t_0)^{\frac{n}{2}}} \int_\Sigma e^{-\frac{\|x-x_0\|^2}{4t_0}} \, d\mu(x). \tag{42}$$

Then the *entropy* functional $\lambda(\Sigma)$ is defined as follows (see also [MM09]):

$$\lambda(\Sigma) \coloneqq \sup_{x_0 \in \mathbb{R}^{n+k},\, t_0 > 0} F_{x_0,t_0}(\Sigma). \tag{43}$$

The functionals $F_{(x_0,t_0)}$ and the entropy functional, naturally extend to Radon measures.

Remark 22. Observe that for any n-dimensional submanifold $\Sigma^n \subseteq \mathbb{R}^{n+k}$ we have the bound $\lambda(\Sigma) \geq 1$. The equality is reached if Σ is a flat n-plane.

An important feature of the entropy functional is that it is monotonically nonincreasing along a mean curvature flow. This is a consequence of Huisken's monotonicity formula [Hu90a].

Remark 23. For any submanifold $\Sigma^n \subseteq \mathbb{R}^{n+k}$, having finite entropy is equivalent to having bounds on area growth. See for instance Theorem 2.2 in [Su18]. In particular there exists a constant C such that for every $x \in \mathbb{R}^{n+k}$ and for every $R > 0$, we have

$$\operatorname{Area}\left(\Sigma \cap \mathcal{B}_R(x)\right) \leq C\lambda(\Sigma)R^n, \tag{44}$$

where $\mathcal{B}_R(x)$ is the open ball in $\mathbb{R}^{n+k}$ of radius $R > 0$ centered at x.

Lemma 24. *Let $\Sigma^n \subseteq \mathbb{R}^{n+k}$ be a complete, noncompact, immersed and oriented submanifold. Let us assume that it has finite entropy $\lambda(\Sigma) < \infty$ and that the mean curvature H is bounded, namely $|H| \leq C$ for some constant $C > 0$.*

Then Σ is properly immersed.

This result in particular applies to translating solitons, since they have bounded mean curvature. Note that we do not put any restriction on the codimension k. The proof is essentially a corollary of Theorem 2.1 in [CL98].

Proof of Lemma 24. Let $\Sigma^k \subseteq \mathbb{R}^{n+k}$ be a complete, immersed and oriented k-dimensional submanifold and let us assume that it is not properly immersed. This implies that there exist $x \in \mathbb{R}^{n+k}$ and a sequence $(p_j)_j \subseteq \Sigma$ such that

$$\|p_j - x\|_{\mathbb{R}^{n+k}} \xrightarrow[j\to\infty]{} x$$

and such that there exists $\delta > 0$ such that

$$\operatorname{dist}^{\Sigma}(p_j, p_i) \geq 2\delta, \qquad j \neq i,$$

where $\operatorname{dist}^{\Sigma}(\cdot,\cdot)$ denotes the intrinsic distance of Σ.

Let $B^{\Sigma}_{\delta}(p_j)$ denote the intrinsic geodesic ball of Σ of radius δ, centered at p_j. From Theorem 2.1 in [CL98], we have that there exists a constant $\beta > 0$ such that

$$\mathcal{H}^n\left(B^{\Sigma}_{\delta}(p_j)\right) \geq \beta\delta$$

for every $j \in \mathbb{N}$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. Let $\mathcal{B}_R(x)$ be the ball in $\mathbb{R}^{n+k}$ of radius $R > 0$ centered at x. Take R large enough such that $B^{\Sigma}_{\delta}(p_j) \subseteq \mathcal{B}_R(x)$ for every j. Then we have

$$\mathcal{H}^n\left(\Sigma \cap \mathcal{B}_R(x)\right) \geq \sum_{j=1}^{\infty} \mathcal{H}^n\left(B^{\Sigma}_{\delta}(p_j)\right) \geq \sum_{j=1}^{\infty} \beta\delta = +\infty.$$

Therefore

$$\begin{aligned}\lambda(\Sigma) \geq F_{x,1}(\Sigma) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_\Sigma e^{-\frac{\|y-x\|^2}{4}}\, d\mu(y) \\ &\geq \frac{e^{-\frac{R^2}{4}}}{(4\pi t)^{\frac{n}{2}}} \mathcal{H}^n\left(\Sigma \cap \mathcal{B}_R(x)\right) = +\infty.\end{aligned}$$

□

The entropy of a translater is determined by its asymptotic behavior. More precisely, we have the following explicit way for computing the entropy.

Lemma 25. *Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a translater with finite entropy. Then*

$$\lambda(\Sigma) = \lim_{\tau\to\infty} F_{(0,1)}\left(\frac{1}{\tau}\Sigma - \tau e_{n+1}\right).$$

Proof. Let $(y,t) \in \mathbb{R}^{n+1}\times\mathbb{R}$. From Huisken's monotonicity formula we have that

$$F_{(y,t)}\left(\Sigma\right) \leq F_{(y+\tau e_{n+1},t+\tau)}\left(\Sigma\right), \tag{45}$$

for any $\tau > 0$ (see equation (1.9) in [CM12] and Lemma 4.2 in [Gu16]). Therefore there exists

$$\lim_{\tau\to\infty} F_{(y+\tau e_{n+1},t+\tau)}\left(\Sigma\right) =: \mu(y,t).$$

Let $\varepsilon > 0$ and let $(y_0,t_0) \in \mathbb{R}^{n+1}\times\mathbb{R}$ such that $F_{(y_0,t_0)}(\Sigma) \geq \lambda(\Sigma) - \varepsilon$. Clearly we have that

$$\lambda(\Sigma) - \varepsilon \leq \mu(y_0,t_0) \leq \lambda(\Sigma). \tag{46}$$

Moreover it is easy to check that the limit $\mu(y,t)$ actually is a constant, namely it does not depend on (y,t). Therefore (46) implies that $\mu = \lambda(\Sigma)$. □

Corollary 26. *Let $\Sigma_1^n, \Sigma_2^n \subseteq \mathbb{R}^{n+1}$ be translaters with finite entropy. Then*

$$\lambda(\Sigma_1 + \Sigma_2) = \lambda(\Sigma_1) + \lambda(\Sigma_2), \tag{47}$$

where "$\Sigma_1 + \Sigma_2$" denotes the sum of Radon measures naturally induced by Σ_1 and Σ_2.

Remark 27. Observe that (47) does not hold in general for hypersurfaces which are not translating solitons. For instance take a hypersurface Σ for which the function $(x_0,t_0) \mapsto F_{x_0,t_0}(\Sigma)$ achieves a strict global maximum. This holds true, for instance, for shrinking solitons with polynomial volume growth which do not split off a line isometrically (see Section 7 in [CM12]). Let $V \in \mathbb{R}^{n+1}$ be a nonzero vector and define $\widetilde{\Sigma} := \Sigma + V$. Note that the function $(x_0,t_0) \mapsto F_{x_0,t_0}(\Sigma + \widetilde{\Sigma})$ achieves a strict global maximum as well and $\lambda(\Sigma + \widetilde{\Sigma}) < \lambda(\Sigma) + \lambda(\widetilde{\Sigma})$.

References

[Br16] S. Brendle, *Embedded self-similar shrinkers of genus* 0, Ann. of Math. 183 (2), 715-728 (2016).

[BLT18] T. Bourni, M. Langford, G. Tinaglia, *On the existence of translating solutions of mean curvature flow in slab regions*, arXiv:1805.05173v3 (2018).

[BGM19] A. Bueno, J. A. Gálvez, P. Mira, *The global geometry of surfaces with prescribed mean curvature in* $\mathbb{R}^3$, arXiv:1802.08146v2 (2019).

[Ch76] S. Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv. 51, 43-55 (1976).

[CL98] L.-F. Cheung, P.-F. Leung, *The mean curvature and volume growth of complete noncompact submanifolds*, Differential Geom. Appl. 8, no. 3, 251 – 256 (1998).

[CM19a] F. Chini, N.M. Møller, *Bi-halfspace and convex hull theorems for translating solitons*, Int. Math. Res. Not. `https://doi.org/10.1093/imrn/rnz183` (2019).

[CM19b] F. Chini, N.M. Møller, *Ancient mean curvature flows and their spacetime tracks*, arXiv:1901.05481v2 (2019).

[CSS07] J. Clutterbuck, O. Schnürer, F. Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differ. Equ. 29, 281–293 (2007).

[CM04] T.H. Colding, W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks*, Ann. of MAth., (2) 160, no. 1, 69-92 (2004).

[CM11] T.H. Colding, W.P. Minicozzi II, *A Course in Minimal Surfaces*, AMS (2011).

[CM12] T.H. Colding, W.P. Minicozzi II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. 175, 755-833 (2012).

[EH91] K. Ecker, G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. 105, no. 3, 547–569 (1991).

[Gu16] Q. Guang, *Volume growth, entropy and stability for translating solitons*, arXiv:1612.05312v1 (2016).

[GT83] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 2nd edition, (1983).

[Ha15] R. Haslhofer, *Uniqueness of the bowl soliton*, Geom. Topol. Volume 19, Number 4, 2393-2406, (2015).

[HHC18] K. Choi, R. Haslhofer, O. Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*, arXiv:1810.08467v1 (2018).

[HHCW19] K. Choi, R. Haslhofer, O. Hershkovits, B. White, *Ancient asymptotically cylindrical flows and applications* arXiv:1910.00639v2 (2019).

[HK17] R. Haslhofer, B. Kleiner, *Mean curvature flow of mean convex hypersurfaces*, Comm. Pure Appl. Math. 70 (3), 511–546 (2017).

[He18] O. Hershkovits, *Translators asymptotic to cylinders*, arXiv:1805.10553v1 (2018).

[HIMW19a] D. Hoffman, T. Ilmanen, F. Martín, B. White, *Graphical translators for mean curvature flow* Calc. Var., 58:117 (2019).

[HIMW19b] D. Hoffman, T. Ilmanen, F. Martín, B. White, *Notes on translating solitons for mean curvature flow* arXiv:1901.09101v2 (2019).

[HMW19a] D. Hoffman, F. Martín, B. White, *Scherk-like translators for mean curvature flow*, arXiv:1903.04617v3 (2019).

[HMW19b] D. Hoffman, F. Martín, B. White, *Nguyen's Tridents and the Classification of Semigraphical Translators for Mean Curvature Flow*, arXiv:1909.09241v1 (2019).

[Hu90a] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. 31, 285–299 (1990).

[Hu90b] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA), Proc. Sympos. Pure Math. 54, Amer. Math. Soc. (1990).

[Il95] T. Ilmanen, *Singularities of mean curvature flow of surfaces*, preprint (1995).

[Il03] T. Ilmanen *Problems in mean curvature flow*, available at `http://people.math.ethz.ch/~ilmanen/classes/eil03/problems03.ps` (2003).

[IR17] D. Impera, M. Rimoldi, *Rigidity results and topology at infinity of translating solitons of the mean curvature flow*, Commun. Contemp. Math. 19(6), 21 (2017).

[IR19] D. Impera, M. Rimoldi, *Quantitative index bounds for translators via topology*, arXiv:1804.07709v4 (2019).

[KS18] K. Kunikawa, S. Saito, *Remarks on topology of stable translating solitons*, Geom. Dedicata (2018).

[Ma11] C. Mantegazza, *Lecture notes on mean curvature flow*, Birkhäuser (2011).

[MM09] C. Mantegazza, A. Magni, *Some remarks on Huisken's monotonicity formula for mean curvature flow*, Singularities in Nonlinear Evolution Phenomena and Applications, CRM Ser. Center "Ennio De Giorgi", Pisa 9, 157-169, (2009).

[MSS15] F. Martín, A. Savas-Halilaj, K. Smoczyk, *On the topology of translating solitons of the mean curvature flow*, Calculus of Variations and PDE's, vol. 54(3), 2853 - 2882, (2015).

[Mø14] N.M. Møller, *Non-existence for self-translating solitons*, arXiv:1411.2319 (2014).

[Ng09] X.H. Nguyen, *Translating tridents*, Comm. Partial Differential Equations 34, no. 1-3, 257–280, (2009).

[Pé16] J. Pérez-García, *Some results on Translating Solitons of the Mean Curvature Flow*, Doctoral thesis, University of Granada (2016).

[Ro95] A. Ros, *A two-piece property for compact minimal surfaces in a three-sphere*, Indiana Univ. Math. J. 44, 841–849 (1995).

[SS83] R. Schoen, L. Simon, *Regularity of simply connected surfaces with quasi-conformal Gauss map*, Seminar on Minimal Submanifolds, Annals of Math. Studies, vol. 103, 127-145, Princeton University Press, Princeton, N.J. (1983).

[Sh15] L. Shahriyari, *Translating graphs by mean curvature flow*, Geom. Dedicata, 175(1):57–64, (2015).

[SX17] J. Spruck, L Xiao, *Complete translating solitons to the mean curvature flow in* $\mathbb{R}^3$ *with nonnegative mean curvature*, (to appear in Amer. J. Math) arXiv:1703.01003v2 (2017).

[Su18] A. Sun, *Singularities of mean curvature flow of surfaces with additional forces*, arXiv:1808.03937v1 (2018).

[Xi15] Y.L. Xin, *Translating solitons of the mean curvature flow*, Calc. Var. 54, 1995–2016 (2015).

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