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ON K-THEORY, GROUPS, AND TOPOLOGICAL DYNAMICS

PhD thesis • Department of Mathematical Sciences • University of Copenhagen

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ON K-THEORY, GROUPS, AND TOPOLOGICAL DYNAMICS

Abstract. — This thesis studies the K-theory of groupoid C^*-algebras and its applications to topological dynamics and index theory.

Chapter 1 introduces a homology theory for groupoids admitting an open "computable" subgroupoid. This is part of a work-in-progress project whose objective is computing the K-groups of C^*-algebras associated to hyperbolic dynamics.

Paper A (joint work with Jens Kaad) focuses on the assembly map for principal bundles with fiber a countable discrete group. We derive Atiyah's L^2-index theorem in the general context of flat C^*-module bundles over compact Hausdorff spaces. Our approach does not rely on geometric K-homology but rather on a Chern character construction for Alexander-Spanier cohomology.

Paper B deals with the homology groups for Smale spaces defined by Putnam. We introduce a simplicial framework by which the various complexes attached to this theory can be understood as suitable "symmetric" Moore complexes. We prove they are all quasi-isomorphic and discuss a parallel with sheaf cohomology by computing the projective cover of a Smale space.

Appendix A contains an induction-restriction adjunction result in the setting of equivariant Kasparov categories. As a consequence, the KK^G-category is described through a complementary pair of subcategories, and a general formulation of the strong Baum-Connes conjecture for étale groupoids is given.

Résumé. — Denne afhandling studerer K-teorien af gruppoid-C^*-algebraer og dens applikationer til topologisk dynamik og indeksteori.

Kapitel 1 introducerer en homologiteori for gruppoider der indeholder en åben beregnelig undergruppoid. Dette er en del af et igangværende projekt, hvis mål er at beregne K-grupperne på C^*-algebraer forbundet med hyperbolsk dynamik.

Artikel A (fælles arbejde med Jens Kaad) fokuserer på den såkaldte "assembly map" for principalbundter med fiber en tællelig diskret gruppe. Vi udleder Atiyahs L^2-indekssætning i den generelle kontekst af flade C^*-modulbundter over kompakte Hausdorffrum. Vores metode er ikke afhængig af geometrisk K-homologi men snarer af en Chern-karakterkonstruktion for Alexander-Spanier-kohomologi.

Artikel B omhandler homologigrupperne for Smalerum defineret af Putnam. Vi introducerer en simpliciel opsætning, hvormed de forskellige komplekser forbundet til denne teori kan forstås som passende "symmetriske" Moore komplekser. Vi beviser, at de alle er quasi-isomorfe og diskuterer en parallel med knippekohomologi ved at beregne det projektive overlegning af et Smalerum.

Appendiks A indeholder et resultat om en induktionrestriktionsadjungering i kontekst af ækvivariante Kasparovkategorier. Som en følge heraf beskrives KK^G-kategorien gennem et komplementært par af underkategorier, og en generel formulering af den stærke Baum-Connes-formodning for étale gruppoider er givet.

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CONTENTS

Introduction 1
The contents of this thesis, in one example 1
The contents of this thesis, reprise 8

1. Homology and topological dynamics 13
1.1. Smale spaces 13
1.2. Groupoids and C^*-algebras 17
1.3. K-theory 22
1.4. Maps and inclusions 24
1.5. A homology theory for open inclusions 27
1.6. Future outlook: conjectures and computations 32

Appendix A. The strong Baum-Connes conjecture 37
A.1. Preliminaries 37
A.2. Induction-restriction adjunction 40
A.3. Complementary pairs of subcategories and localization 44
A.4. The Baum-Connes conjecture 50

Bibliography 53

Paper A. Index theory on the Miščenko bundle 59

Paper B. A note on homology for Smale spaces 89

INTRODUCTION

The first introduction provides some basic intuition for the topics of this thesis and more generally for a field of mathematics called *noncommutative topology*. It has been written with the non-technical reader in mind and it includes a number of references.

The second introduction assumes familiarity with groupoids, C^*-algebras, and K-theory. It gives a summary for each part of this thesis and briefly explains the main mathematical results and their interrelations.

The contents of this thesis, in one example

The analysis of *symmetry* is one of the fundamental motivations for the development of mathematics. In essence, whenever a system is left structurally *invariant* under a given transformation, we are in the presence of symmetry. As an example, the triangle in Figure 1 stays the same after reflection across the vertical axis.

The operations that do not change a given system, i.e., the *symmetries*, enjoy two important features: they are reversible and they can be composed with each other, giving rise to further symmetries. In mathematical terms, we say these operations form a *group*. Groups are among the simplest mathematical objects that are used to capture symmetry.

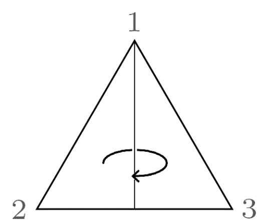

FIGURE 1. Reflection across the vertical axis.

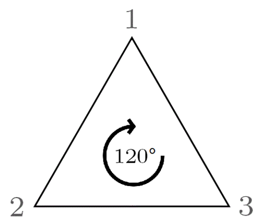

Figure 2. Clockwise rotation of 120 degrees.

Because symmetry can occur in a wide variety of contexts, the general structure of a group can be quite complicated. Nonetheless, by imposing certain constraints on the system we wish to study, we can import techniques from other areas of mathematics and extract information on the group of interest.

The common mathematical model for physical space is a *vector space* equipped with an *inner product*. The self-transformations of this system are called *orthogonal operators* and it is natural to attempt to realize any given group as a subgroup of the orthogonal operators. This idea belongs to a field called *representation theory*.

As illustration of this technique, let us consider again the equilateral triangle of Figure 1. Its group of symmetries consists of all possible permutations of the set $\{1, 2, 3\}$. For instance, the reflection of Figure 1 corresponds to swapping 2 and 3, while the clockwise rotation described in Figure 2 corresponds to moving 1 in place of 3, 3 in place of 2, and finally 2 in place of 1.

There are in total six permutations of the three-element set, and together they form the so-called *symmetric group*, denoted S_3. Consider a three-dimensional vector space with basis e_1, e_2, e_3. Let the symmetric group permute the basis vectors, and consider the induced action on the vector space. This is a three-dimensional representation as orthogonal operators.

The two-dimensional subspace of all vectors of the form $x_1e_1+x_2e_2+x_3e_3$, where $x_1+x_2+x_3=0$, gives a two-dimensional sub-representation, which precisely corresponds to the symmetries of the triangle in the figures. For each of these symmetries, we can record whether it preserves the standard orientation of the plane, and discover in this way the *sign representation*.

It turns out that S_3, along with many other groups, can be completely reconstructed from the information encoded in these representations. This can be considered a success of the basic idea underlying representation theory, i.e., analyzing groups through the tools of *linear algebra*.

Aside from linear algebra, is there another field of mathematics from which we can import techniques which aid in the study of groups? The answer is "yes", and this brings us to the branch of mathematics known as *algebraic topology*. In this field, the interest is in classifying geometrical shapes "up to continuous deformation". The

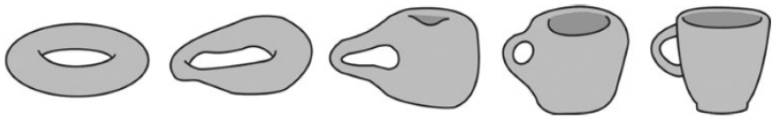

FIGURE 3. This is the basis for a geeky joke that a topologist does not know the difference between a doughnut and a coffee mug.

intuition for this notion is provided in Figure 3, where a coffee mug is morphed into a toroidal shape.

The formal counterpart of the idea of "deformation class" is known as *homotopy type* and it encompasses shapes as well as *mappings* between them. Indeed, it is a central idea in mathematics that, in order to extract information about a given object X, it is often useful to understand the "morphisms out of X", or in other words the maps from X into a predetermined target space Y. In algebraic topology, one takes this a step further by defining a *cohomology theory*, i.e., a family of "targets" $\{Y_n\}_{n\in\mathbb{N}}$ satisfying certain desirable properties, inasmuch as the collection of homotopy types of mappings $X \to Y_n$ is supposed to encode the n-dimensional stable-under-deformation geometric information about X.

To see this idea in action, we take X to be a circle and consider the target space $Y = S^\infty/\sim$, obtained by identifying the nonzero opposite vectors in an infinite-dimensional "punctured" vector space. In mathematical terms, we are considering the set of homotopy classes of maps $\gamma \in [S^1, S^\infty/\sim]$. Each γ can be viewed as a "loop" in the target space and admits a "lift" to a path $\tilde{\gamma}$ on S^∞ whose endpoints belong to the same $\sim$-equivalence class, inducing a *fiber-preserving* self-transformation of the sphere.

In other words, each element in $[S^1, S^\infty/\sim]$ expresses a certain symmetry of S^∞. It can be shown there is only one (nontrivial) symmetry of this kind, namely the one given by interchanging the antipodal points of the sphere. We have outlined the following computation:

$$[S^1, S^\infty/\sim] \xrightarrow{\cong} \mathrm{Sym}(\mathrm{S}^\infty, \sim) \xrightarrow{\cong} O(1) = \{\pm 1\}\ . \tag{1}$$

In geometric terms this means there are only two topologically distinguishable shapes which fiber over the circle *in a linear fashion*, or more precisely there are exactly two non-isomorphic *line bundles* having the circle as a base space. The line bundle corresponding to the unique non-trivial symmetry is represented by the *Möbius band*, depicted in Figure 4 along with the cylinder (i.e., the trivial bundle).

The space Y in the previous example is part of a parametrized family of spaces denoted $BO(n), n \in \mathbb{N}$. By piecing together these spaces we get a "universal" object which serves as a basis for a cohomology theory called (real) *K-theory*. By expanding on the calculation in (1), one can prove that $\tilde{K}^0(S^1) \cong O(1)$, i.e., the zeroth K-theory group of the circle is cyclic of order two.

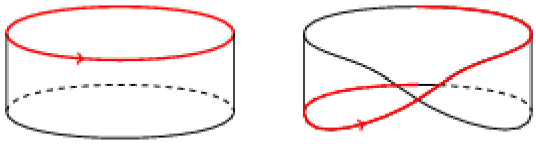

FIGURE 4. Line bundles over the circle: the cylinder is given by juxtaposing vertical lines going through each point in the circle, the Möbius band is obtained by cutting the cylinder along a vertical line, performing a half-twist, and gluing back along the cut.

$$A = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

FIGURE 5. A Toeplitz matrix has constant diagonals.

After this *excursus* in the realm of algebraic topology, we go back to the original problem of studying the structure of groups. We seek for a mathematical construct which encapsulates the information about a given group G, while being amenable to analysis through the combined lenses of representation theory and K-theory.

The solution to this problem is found in the field of *operator algebras* and it's called the *group C^*-algebra* construction, which associates an object denoted $C^*(G)$ to our group of interest G.

The defining axioms of C^*-algebras make them a suitable mathematical model for the phase space observables of a physical system. In particular, the *commutative* algebra of coordinates of a topological shape naturally recovers its geometric information, while more general C^*-algebras model the finer structure of a *quantized system*, in alignment with the predictions of quantum mechanics.

For instance, the observables for a simple spin system consisting of a single electron generate a noncommutative C^*-algebra isomorphic to $M_2(\mathbb{C})$. A more sophisticated example is obtained by the *quantum disk*, obtained by introducing a *deformation parameter* in the defining relations for the coordinate algebra of the unit disk:

$$zz^* = qz^*z + (1-q)\cdot 1 \qquad -1 \leq q \leq 1.$$

When $-1 < q < 1$, the associated algebra $\mathcal{T}_q$ is called the *Toeplitz algebra* and is concretely realized as bi-infinite matrices in which descending diagonals are constant (see Figure 5). The entries along these diagonals can be interpreted as *Fourier*

FIGURE 6. Two water particles at the top of each water spring are considered stably equivalent because they will eventually flow down into the stream and become indistinguishable.

coefficients, i.e., they describe the decomposition of a periodic signal (e.g., a continuous function of S^1) into simple oscillating waves. The algebra $\mathcal{T}_q$ can thus be considered as the result of a *quantization* process applied to the circle.

Furthermore, the algebra $\mathcal{T}_{-1}$ is described by a noncommutative analogue of the familiar equation $x^2 + y^2 = 1$, representing a kind of "noncommutative circle" whose fine structure can be carefully analyzed, and ultimately leads to a proof of a deep theorem known as *Bott periodicity.* This roughly states that *complex* K-theory is a cohomology theory defined by a 2-*periodic* family of target spaces, i.e., $Y_n = Y_{n+2}$.

We have argued that C^*-algebras are sufficiently close to topological spaces, so it should be hardly surprising that K-theory admits an alternative definition which can be readily extended to cover this class of algebras. In essence, this means the group C^*-algebra $C^*(G)$ can be studied through the methods of algebraic topology.

In addition, one has a recipe to "upgrade" each representation of G to a linear representation of $C^*(G)$, and this process is reversible, yielding a one-to-one correspondence between the representation theory of G and of its associated C^*-algebra:

$$g \mapsto \pi(g) \qquad \longleftrightarrow \qquad f \mapsto \tilde{\pi}(f) = \int_G f(g)\pi(g)\, dg.$$

There are situations where only *partial* symmetry is observed, therefore it is desirable to have a mathematical gadget which is more "flexible" than a group, being able to capture more general "long-range order" phenomena. A case in point is the theory of *dynamical systems*, which is concerned with studying configurations of points which move according to an evolution law, and often requires keeping track of various pseudo-symmetric relations between points.

Figure 6 depicts a real-life example of a dynamical system: the laws of physics prescribe how the masses of water progress down the river step. In this thesis we consider a simple *equivalence relation* which encodes a key property of many dynamical

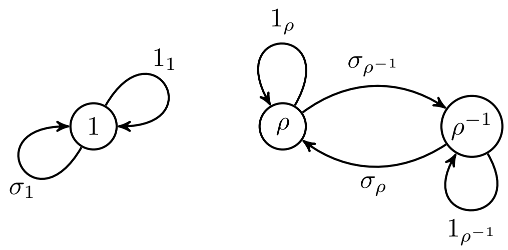

FIGURE 7. The groupoid $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$ associated to S_3.

systems called *stable equivalence*, and is used to categorize configurations of points which get closer and closer under the evolution rule.

A *groupoid* is a mathematical structure which generalizes groups and equivalence relations, providing a unified perspective in the study of symmetry. In essence, a groupoid is formed by a a collection of points (the *objects*) together with *invertible* arrows (the *morphisms*) between them. From this standpoint, a group corresponds to a groupoid with only one object, while an equivalence relation corresponds to a groupoid where the morphisms are completely determined by their source and range.

By changing the viewpoint on the group S_3, we can give a simple example of the usage of groupoids in modeling dynamical systems. If we denote by σ the operation of Figure 1 and by ρ the rotation in Figure 2, then it is easy to verify that $\sigma^{-1}\rho\sigma = \rho^{-1}$. In words, this amounts to saying that if we perform a horizontal reflection followed by a rotation and finally a reflection in the opposite direction, we end up rotating counterclockwise the original triangle.

In mathematical language we say S_3 is a *semidirect product*, which simply means that σ prescribes a rule by which the configuration $(1, \rho, \rho^{-1})$ is transformed into $(1, \rho^{-1}, \rho)$. We thus obtain an elementary dynamical system whose associated groupoid is given in Figure 7.

In complete analogy with the case of groups, we can construct a C^*-algebra starting from a given groupoid G. Let us construct $C^*(G)$ where $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ as in Figure 7. Evidently, G can be decomposed in two subgroupoids, G_1 and G_2, where G_1 consists of one point and two loops, while G_2 is made of two points and four arrows in total. By the previous description, we see that G_1 is a group and G_2 is an equivalence relation.

Since G_1 is a commutative 2-element group, its associated C^*-algebra simply records the possible coordinate values of two isolated points, thus it is given by two copies of the complex numbers, i.e., $C^*(G_1) \cong \mathbb{C} \oplus \mathbb{C}$. The algebra associated to G_2 should express the nontrivial interaction between two states, which should remind the reader of the simple spin-system (see Figure 8) whose associated algebra is $M_2(\mathbb{C})$.

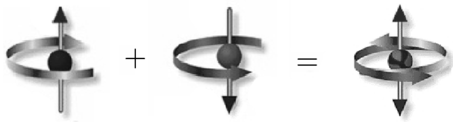

FIGURE 8. A superposition of “spin up” and “spin down” states.

In summary, we obtain the following expression for the groupoid C^*-algebra associated to G:

$$C^*(G) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}). \tag{2}$$

Notice how the formula for $C^*(G)$ retains information about the representation theory of S_3. The first copy of $\mathbb{C}$ corresponds to the trivial representation, the second copy of $\mathbb{C}$ corresponds to the sign representation, and $M_2(\mathbb{C})$ stands for the two-dimensional representation as symmetries of the regular triangle.

We conclude this introduction by computing the complex K-theory groups of $C^*(G)$. The group $K_0(C^*(G))$ is obtained by considering the free abelian group on the “symbols” appearing in (2). Thus we get $K_0(C^*(G)) \cong \mathbb{Z}^3$, and we see that K-theory recognizes the number of (irreducible) representations of S_3. Finally, it can be proved that $K_1(C^*(G))$ vanishes, intuitively because the system we are considering is discrete, hence zero-dimensional, but as we previously explained the K_1-group exclusively keeps track of higher-dimensional data.

At its core, this thesis is concerned with the K-theoretical analysis of groups and groupoids associated to (topological) dynamical systems. It is our hope that this introduction has shed some light on the basic motivations underlying this area of mathematics, as well as on the methods and problems which characterize its development.

Some references. — The reader who wishes to expand his knowledge on the topics discussed above (and much more) may find the following books and papers useful. This is not to be considered a complete list of references.

For groups and representation theory see [**20, 49**]. For a general introduction to algebraic topology and homotopy theory we suggest [**22, 40**]. Topological K-theory is treated in [**4, 26**], while an operator-theoretic approach can be found in [**6, 59**]. For the general theory of C^*-algebras, and constructions related to groups and group actions, consult [**17, 74**]. The quantum disk and the noncommutative circle are discussed in [**45, 46**]. For K-homology and index theory, including a proof of Bott periodicity using the Toeplitz algebra, see [**24**]. For the applications of noncommutative geometry to quantum physics, consult [**33, 64**]. Groupoids and C^*-algebras are treated in [**1, 57**]. Concerning topological dynamics and operator algebras we suggest [**55, 67**].

The contents of this thesis, reprise

This thesis consists of the following parts:

1. Chapter 1. Homology and topological dynamics;
2. Appendix A. The strong Baum-Connes conjecture;
3. Paper A. Index theory on the Miščenko bundle;
4. Paper B. A note on homology for Smale spaces.

A summary of each item is given in the next paragraph. The *leitmotif* underlying each part is the study of K-theory groups associated to group and groupoid C^*-algebras. This is done mainly through the lenses of homological algebra and index theory. The groupoids of interest originate from topological dynamics, but many of our methods are applicable to a wider range of objects, including groupoid crossed products. The Baum-Connes conjecture fundamentally permeates this circle of ideas. It will be discussed from two angles: in the abstract framework of triangulated categories, and in the concrete analytical picture of Kasparov's bivariant K-theory.

Chapter 1 contains material from a work-in-progress project with the objective of computing K-theory and homology groups associated to a class of hyperbolic dynamical systems called Smale spaces. These are compact metric spaces endowed with two transversal foliations and a self-map whose local structure along the leaves is the product of a contraction and a dilation.

Our main results are in the context of Smale spaces with a totally disconnected leaf. Examples include tiling systems and generalized solenoids. We define a chain complex by combining Putnam's lifting result for factor maps [**53**] and the induction-restriction adjunction proved in the appendix (Theorem A.2).

Theorem. — *Let (X, ϕ) be an irreducible Smale space with totally disconnected stable sets. There is an irreducible shift of finite type (Σ, σ) and an s-bijective map $f\colon (\Sigma, \sigma) \to (X, \phi)$. Denote by G and H the unstable equivalence relations affiliated to X and Σ, respectively. Then $f \times f\colon H \to G$ is an open inclusion inducing a comonad L_f acting on the KK^G-category. We obtain an augmented simplicial object $L_f^\bullet X \to X$ and an associated chain complex $K_*(L_f^\bullet X \rtimes G)$.*

The triangulated structure of the equivariant Kasparov category KK^G can be exploited to import machinery from homological algebra and homotopy theory. This allows defining cellular approximations with respect to a given homological ideal. In our setting, this leads to a spectral sequence of the following type:

Theorem. — *There exists a homological spectral sequence converging towards the crossed product of G and the cellular approximation,*

$$E^2_{pq} = H_p(K_q(L_f^\bullet X \rtimes G)) \Rightarrow K_{p+q}(PX \rtimes G).$$

Moreover, there exists a natural comparison map $K_*(PX \rtimes G) \to K_*(C^*(G))$*, which is an isomorphism whenever* G *satisfies the* strong *Baum-Connes conjecture.*

The chapter also presents some partial computations and conjectures which we believe should work as guiding principles in future research on this topic.

Appendix A studies the Baum-Connes conjecture for groupoids in the setting of triangulated categories and localization, partly generalizing results from [**38**], where this program was initiated in the case of locally compact groups. The main result is an induction-restriction statement of the following form:

Theorem. — *Let* G *be a groupoid with a Haar system and* H *an open subgroupoid. There is a natural restriction functor* $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$ *between the associated equivariant Kasparov categories. A corresponding induction functor, defined on objects as*

$$\begin{aligned} \mathrm{Ind}_H^G\colon \mathrm{KK}^H &\longrightarrow \mathrm{KK}^G \\ A &\longmapsto (C_0(G) \otimes A) \rtimes H, \end{aligned}$$

gives rise to an adjunction

$$(\epsilon, \eta)\colon \mathrm{Ind}_H^G \dashv \mathrm{Res}_G^H$$

with explicitly described unit η *and counit* ϵ*.*

A number of corollaries are derived, including a structural theorem for the KK^G-category in terms of complementary subcategories. Given a proper G-algebra over $\underline{\mathrm{E}}G$, we can locally express properness through a slice theorem, allowing to identify a class of "compact actions" $Q \subseteq G$ which we collectively denote by $\mathcal{F}$.

Define the full subcategory of *compactly induced objects*,

$$\mathcal{CI} = \{\mathrm{Ind}_Q^G(A) \mid A \in \mathrm{KK}^Q, Q \in \mathcal{F}\}.$$

There is a homological ideal $\mathcal{I}$ defined as the kernel of a single functor

$$\begin{aligned} F\colon \mathrm{KK}^G &\to \prod_{Q \in \mathcal{F}} \mathrm{KK}^Q \\ A &\mapsto (\mathrm{Res}_G^Q(A))_{Q \in \mathcal{F}}. \end{aligned}$$

Theorem. — *The projective objects for* $\mathcal{I}$ *are the retracts of direct sums of objects in* $\mathcal{CI}$ *and the ideal* $\mathcal{I}$ *has enough projective objects. If* $N_{\mathcal{I}}$ *denotes the class of* $\mathcal{I}$*-contractible objects, we have that* $(\langle \mathcal{CI} \rangle, N_{\mathcal{I}})$ *is a pair of complementary subcategories.*

This leads to defining the *strong* Baum-Connes conjecture as follows: for a locally compact Hausdorff groupoid G, the localizing subcategory $\langle \mathcal{CI} \rangle$ equals KK^G.

Paper A is in joint work with Jens Kaad. It discusses a generalization of Atiyah's L^2-index theorem for G-coverings $\tilde{X} \to X$. In this summary G is assumed discrete and torsion-free. The Miščenko line bundle is defined through the associated bundle construction $\tilde{X} \times_G C_r^*(G) \to X$ with fiber the group C^*-algebra. Our main results are summarized in the following theorem.

Theorem. — *Let $p \in M_N(C(X) \otimes C_r^*(G))$ denote the projection which represents the Miščenko line bundle. Set M to be the associated module of sections.*

- *There exists a group H, containing G, such that $\mathbb{L}(H)$ is a von Neumann II_1-factor and the equality $[p] = [1]$ holds in $K_0(C(X) \otimes \mathbb{L}(H))$.*
- *There is a tensor product presentation*

$$M \cong Y^* \mathbin{\widehat{\otimes}}_{C_0(\tilde{X}) \rtimes_r G} Z \rtimes_r G,$$

where Y is Rieffel's imprimitivity bimodule [**58**]*, and Z is a $C_0(\tilde{X})$-$C(X)$ G-C^*-correspondence such that*

$$[Z] \mathbin{\widehat{\otimes}} -\colon \ \mathrm{KK}(C(X), \mathbb{C}) \to \mathrm{KK}^G_*(C_0(\tilde{X}), \mathbb{C})$$

provides an inverse to the dual Green-Julg isomorphism.

The equality of projection classes involves a "diagonal-localization" technique in the spirit of Teleman's results in cyclic homology [**65**], and relies on an explicit description of the Chern character for tracial algebras with values in Alexander-Spanier cohomology. Our methods involve minimal analysis and do not appeal to smoothness.

In particular, the index theorem below can be expressed completely K-theoretically and holds in the general context of compact Hausdorff spaces and flat bundles, which was not known before. This also provides an alternative proof of the idempotent conjecture for groups whose K-theory is in the image of the assembly map.

Corollary. — *After the identification given by the dual Green-Julg isomorphism, the Baum-Connes assembly map coincides with the Miščenko-Fomenko index map*

$$\mathrm{RKK}_*(C_0(BG), \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C_r^*(G)).$$

The L^2-index of an elliptic differential operator on $\tilde{X}$, commuting with G, coincides with the ordinary index of the operator on the base space X (i.e., Atiyah's L^2-index theorem holds). If G satisfies the Baum-Connes conjecture, then $C_r^*(G)$ contains no nontrivial idempotents, i.e., it satisfies the Kadison-Kaplansky idempotent conjecture.

Paper B deals with the homology theory for Smale spaces defined by Putnam [**54**]. Smale spaces were introduced by Ruelle as a purely topological description of the basic sets of Axiom A diffeomorphisms [**63**].

There are many interesting and open questions concerning Putnam's homology, e.g., the search for machinery such as long exact sequences, excision, homotopy invariance, etc. The results of this paper represent a first step in the direction of a more conceptual definition which could shed light on these issues.

Theorem. — *The s/u-bijective pair* [**54**, Section 2.6] *associated to a Smale space gives rise to a bisimplicial shift of finite type. Krieger's dimension group is applied to functorially map such bisimplicial object to a simplicial group, and the chain complex defining Putnam's homology is obtained through the Dold-Kan correspondence. Moreover, the bisimplicial group admits a symmetric structure, and the associated "symmetric" Moore complexes yield all of Putnam's complexes.*

Corollary. — *There are spectral sequences which can be used to simplify many arguments from* [**54**]*. In particular, the complex $C^s(\pi)$ is quasi-isomorphic to the remaining three double complexes, solving a conjecture posed in* [**54**, page 90].

Finally, we prove a characterization of Krieger's invariant [**31**] and introduce a discussion on projective resolutions.

Theorem. — *Given a shift of finite type Σ, there exists a category $\mathcal{C}(\Sigma)$ whose K-theory (in the sense of Grothendieck) is isomorphic to Krieger's dimension group $D^s(\Sigma)$. The projective cover of a Smale space X is obtained by taking the limit over the projective system of shift spaces and factor maps onto X.*

CHAPTER 1

HOMOLOGY AND TOPOLOGICAL DYNAMICS

This chapter introduces a homology theory for groupoids admitting an open "computable" subgroupoid, e.g., an AF-equivalence relation [**42**]. These homology groups approximate the K-groups for the associated groupoid C^*-algebras in a sense made precise by a spectral sequence (see Theorem 1.22).

Our main motivation for the development of this machinery is an application to topological dynamics. However, many of the arguments hold in greater generality and ultimately derive from the triangulated structure of the equivariant Kasparov category. On that basis some proofs have been postponed to Appendix A where they are presented in a more general context.

Here we focus on a class of hyperbolic dynamical systems known as Smale spaces. Sections 1.1 to 1.4 provide background on these objects and cover their associated groupoids, C^*-algebras, and K-theory groups. The chain complex which defines the homology groups is finally introduced in Section 1.5. The last section includes partial computations and conjectures which are meant to clarify the extent to which our theory is useful in studying Smale spaces. As this is part of an ongoing project, the reader will find some missing details; we plan on filling these gaps in future investigations.

1.1. Smale spaces

A *Smale space* (X, ϕ) is a dynamical system consisting of a homeomorphism ϕ on a compact metric space (X, d) such that the space is locally the product of a coordinate that contracts under the action of ϕ and a coordinate that expands under the action of ϕ. The precise definition requires the definition of a bracket map satisfying certain axioms [**54, 60**].

Definition 1.1. — A Smale space (X, ϕ) consists of a compact metric space X with metric d, along with a homeomorphism $\phi : X \to X$ such that there exist constants

$\epsilon_X > 0, 0 < \lambda < 1$, and a continuous map

$$\{(x,y) \in X \times X \mid d(x,y) \leq \epsilon_X\} \mapsto [x,y] \in X$$

satisfying the bracket axioms for any x, y, z in X when both sides are defined:

- $[x,x] = x$;
- $[x,[y,z]] = [x,z]$;
- $[[x,y],z] = [x,z]$;
- $\phi([x,y]) = [\phi(x), \phi(y)]$.

In addition, (X, ϕ) is required to satisfy the contraction axioms:

- for $x, y \in X$ such that $[x,y] = y$, we have $d(\phi(x), \phi(y)) \leq \lambda d(x,y)$;
- for $x, y \in X$ such that $[x,y] = x$, we have $d(\phi^{-1}(x), \phi^{-1}(y)) \leq \lambda d(x,y)$.

Suppose $x \in X$ and $0 < \epsilon \leq \epsilon_X$. We define

$$X^s(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon, [y,x] = x\}$$
$$X^u(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon, [x,y] = x\}.$$

The set $X^s(x,\epsilon)$ is called the *local stable set* and the set $X^u(x,\epsilon)$ is called the *local unstable set*. For $x, y \in X$ such that $d(x,y) < \epsilon_X/2$, the bracket map $[x,y]$ is the unique point where the local stable set of x intersects the local unstable set of y and vice versa, as in Figure 1. This means that, locally, we can choose coordinates. For $\epsilon \in (0, \epsilon_X/2)$ and $x \in X$,

$$[\cdot,\cdot] \colon X^u(x,\epsilon) \times X^s(x,\epsilon) \to X$$

is a homeomorphism onto an open neighborhood of $x \in X$.

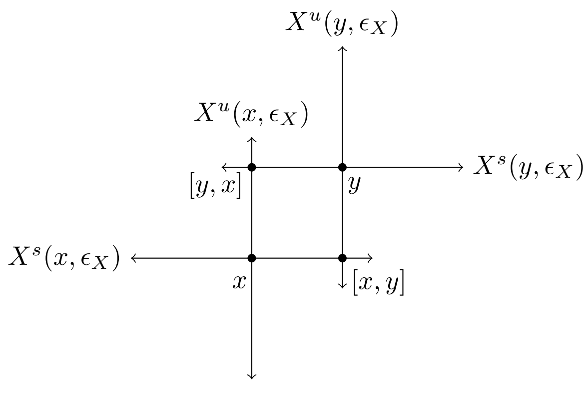

FIGURE 1. The local coordinates of $x, y \in X$ and their bracket maps.

It is worth noting that if a bracket map exists on (X, ϕ), then it is unique.

The most essential feature of Smale spaces is given by the definition of two equivalence relations, named respectively *stable* and *unstable*, which reads as follows:

– given $x, y \in X$, we say they are *stably equivalent* if
$$\lim_{n\to\infty} d(\phi^n(x), \phi^n(y)) = 0;$$
– given $x, y \in X$, we say they are *unstably equivalent* if
$$\lim_{n\to\infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0.$$

The orbit of $x \in X$ under the stable (respectively unstable) equivalence relation is called the *global stable* (resp. *unstable*) *set* and is denoted $X^s(x)$ (resp. $X^u(x)$). These satisfy the following identities:
$$X^s(x) = \bigcup_{n\geq 0} \phi^{-n}(X^s(\phi^n(x), \epsilon)) \tag{3}$$
$$X^u(x) = \bigcup_{n\geq 0} \phi^{n}(X^s(\phi^{-n}(x), \epsilon)).$$

A point $x \in X$ is called *non-wandering* if for all nonempty opens $U \subseteq X$, containing x, there exists $N \in \mathbb{N}$ with $U \cap f^N(U) \neq \emptyset$. We will assume that each point in a Smale space is non-wandering. This condition holds in virtually all interesting examples.

We can make a further simplification by focusing on *irreducible* spaces. This means that for every (ordered) pair U, V of nonempty open sets in X, there exists $N \in \mathbb{N}$ such that $U \cap f^n(V) \neq \emptyset$, $n \geq N$. It can be shown that any non-wandering Smale space (X, ϕ) can be partitioned in a finite number of ϕ-invariant clopens $X_1, \dots, X_n$, in a unique way, such that $(X_k, \phi|_{X_k})$ is irreducible for $k = 1, \dots, n$ [**52**]. In view of this we shall work almost exclusively with irreducible Smale spaces.

Example 1.2. — A *directed graph* $G = (G^0, G^1, i, t)$ consists of finite sets G^0 and G^1, called vertices and edges, such that each edge $e \in G^1$ is given by a directed arrow from $i(e) \in G^0$ to $t(e) \in G^0$.

The standard definition of a *shift of finite type* is given in [**35**, Definition 2.1.1]. However, an equivalent and more convenient definition is to start out with a finite directed graph G. Then a shift of finite type (Σ_G, σ) is defined as the space of bi-infinite sequences of paths
$$\Sigma_G = \{e = (e_k)_{k\in\mathbb{Z}} \in (G^1)^{\mathbb{Z}} \mid t(e_k) = i(e_{k+1})\},$$
together with the left shift map $\sigma(e)_k = e_{k+1}$. The metric is such that $d(e, f) \leq 2^{-n-1}$ if e, f coincide on the interval $[-n, n]$. In particular, $d(e, f) = 2^{-1}$ means that e, f share the central edge, i.e., $e_0 = f_0$. Then we can define
$$[e, f] = (\dots, f_{-2}, f_{-1}, e_0, e_1, e_2, \dots).$$
The pair (Σ_G, σ) is a Smale space with constant $\epsilon = 1/2$.

For instance, if G is as is in Figure 2 then any $e \in \Sigma_G$ is such that either $e_0 = a$ or $e_0 = b$. Therefore, Σ_G is the disjoint union of two clopens, $\{e_0 = a\} \sqcup \{e_0 = b\}$. Each of these is homeomorphic (via the bracket map) to a product space, as is shown in

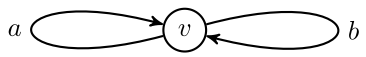

FIGURE 2. The graph consisting of one vertex and two edges is simply depicted as two loops. The associated shift of finite type is the *full* 2*-shift*, i.e., the shift of bi-infinite sequences on two symbols.

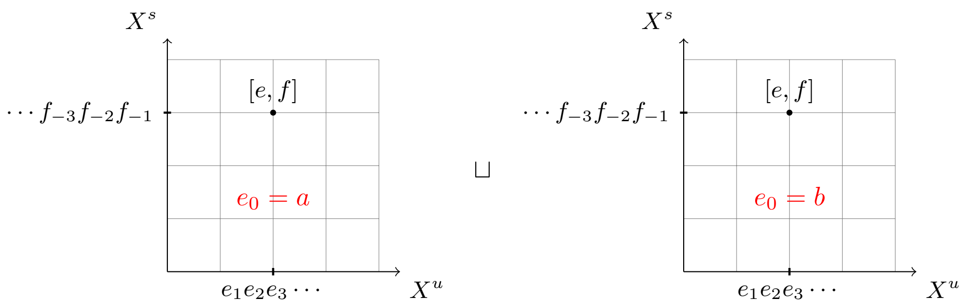

FIGURE 3. The local product structure of the shift of finite type associated to the graph in Figure 2

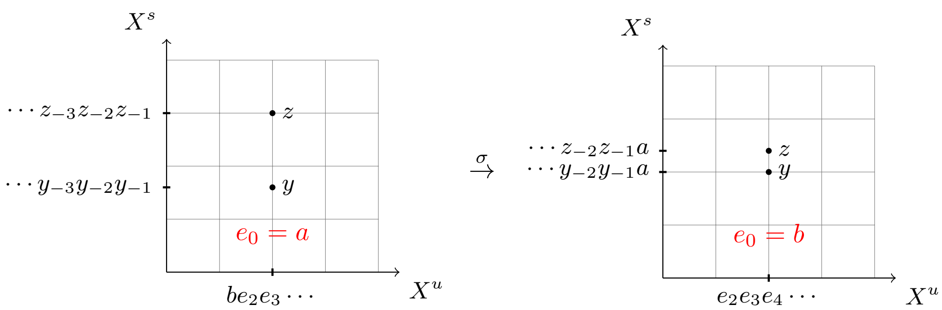

FIGURE 4. After applying σ the paths agree on a larger interval and are therefore closer, i.e., their distance contracts.

Figure 3. The contracting dynamical behavior for points lying on a local stable set is represented in Figure 4.

Example 1.3. — This next example belongs to a class of systems known as *one-dimensional solenoids*. These were firstly introduced in [**15**] and later generalized by Williams in [**75**]. They can also be obtained as a special case of a general construction involving branched manifolds and inverse limits [**73**].

Consider the doubling map on the circle,

$$S^1 \xleftarrow{z^2 \leftarrow\!\shortmid z} S^1 \xleftarrow{z^2 \leftarrow\!\shortmid z} S^1 \xleftarrow{z^2 \leftarrow\!\shortmid z} \cdots \tag{4}$$

and define X to be the projective limit of (4). The map $\phi\colon X \to X$ sending $(z_0, z_1, \dots)$ to $(z_0^2, z_1^2, \dots)$ is a homeomorphism. Indeed notice that the k-th coordinate z_k of any point $z \in X$ satisfies $z_k^2 = z_{k-1}$, so that the shift map provides a continuous inverse to ϕ, as follows:

$$(z_0^2, z_1^2, z_2^2, \dots) = (z_0^2, z_0, z_1, \dots) \mapsto (z_0, z_1, z_2, \dots).$$

Denote by π the canonical projection $X \to S^1$ on the first factor. For any $z_0 \in S^1$ we see that

$$\pi^{-1}(z_0) \cong \prod_{n=1}^{\infty} \{0, 1\} = \Sigma,$$

where Σ is our notation for the Cantor set. Moreover, π is a fiber bundle: each $z = (z_k)_{k=0}^{\infty} \in X$ admits a neighborhood which is homeomorphic to a product space of the form

$$\{z \in S^1 \mid |z - z_0| < \epsilon\} \times \Sigma$$

for a small enough $\epsilon > 0$. If we identify $x = (x_0, (a_n)_{n=1}^{\infty})$ with the coordinate representation above, then we can define local stable and unstable sets

$$\begin{aligned} X^s(x, \epsilon) &= \{x_0\} \times \Sigma \\ X^u(z, \epsilon) &= \{z \in S^1 \mid |z - x_0| < \epsilon\} \times \{(a_n)_{n=1}^{\infty}\}. \end{aligned} \tag{5}$$

More precisely, $X^s(x, \epsilon)$ and $X^u(z, \epsilon)$ are the sets in X indentified with the ones given in (5). The contractive dynamics within local stable sets is clear: if z and z' share the same 0-th coordinate, then in the product metric $d(\phi(z), (\phi(z'))) = 2^{-1} \cdot d(z, z')$, because applying ϕ has the same effect of shifting a sequence to the right and inserting z_0^2 in the 0-th place.

Other examples of Smale spaces include *hyperbolic toral automorphisms* [**9, 18**] and more generally Anosov diffeomorphisms [**3**] (see in particular [**27**, Exercises 6.4.1 and 6.4.2]). *Substitution tiling systems* are also examples of Smale spaces. This is proved in [**2**, Theorem 3.3]. The K-theory of C^*-algebras associated with tiling systems has been subject of extensive study in the literature (see [**61**] for an overview).

1.2. Groupoids and C^*-algebras

We begin this section by briefly recalling the groupoid C^*-algebra construction. Details in full generality are found in [**57**], here we shall focus on the setting which arises when considering Smale spaces.

Let G be a locally compact Hausdorff groupoid with Hausdorff unit space X. We will assume that G is *étale*, i.e., the source and range maps are local homeomorphisms. In this case X is open in G and the Haar system is given by the counting measure.

To such a groupoid G, we associate its *reduced* groupoid C^*-algebra as follows. Let $C_c(G)$ denote the (vector space of) compactly supported continuous functions on G. For $f_1, f_2, f \in C_c(G)$ we define multiplication and involution by

$$(f_1 \cdot f_2)(\gamma) = \sum_{\mu\nu=\gamma} f_1(\mu) f_2(\nu)$$
$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

for all elements $\gamma \in G$. With these operations $C_c(G)$ is a $*$-algebra. For every $x \in X$, let $\ell^2(s^{-1}(x))$ denote the Hilbert space of square-summable functions on $s^{-1}(x)$.

From this we can define a *-representation

$$\pi_x : C_c(G) \to \mathcal{B}(\ell^2(s^{-1}(x)))$$
$$(\pi_x(f)g)(\gamma) = \sum_{\mu\nu=\gamma} f(\mu) g(\nu)$$

for $f \in C_c(G)$, $g \in \ell^2(s^{-1}(x))$, $\gamma \in s^{-1}(x)$.

The reduced groupoid C^*-algebra, denoted $C_r^*(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\|.$$

One may also define a full groupoid C^*-algebra $C^*(G)$. In the case that the groupoid is amenable [**1**], the two coincide. See also Remark 1.6.

The groupoids associated to a Smale space are given by its equivalence relations. More precisely, given a Smale space (X, ϕ), we define

$$R^s(X, \phi) = \{(x, y) \in X \times X \mid \lim_{n\to\infty} d(\phi^n(x), \phi^n(y)) = 0\} \tag{6}$$
$$R^u(X, \phi) = \{(x, y) \in X \times X \mid \lim_{n\to\infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0\}.$$

Occasionally we write $x \sim_s y$ meaning that $(x, y) \in R^s(X, f)$, and $x \sim_u y$ meaning that $(x, y) \in R^u(X, f)$.

These are locally compact Hausdorff groupoids (in the subspace topology of the product) with the following definitions for source, range, and composition:

$$s(x, y) = y \qquad r(x, y) = x$$
$$(x, y) \circ (w, z) = (x, z) \quad \text{if } y = w.$$

Although these groupoids are *not* étale, one can construct and analyze their associated C^*-algebras following [**51**]. We now turn to explaining a technique for turning the groupoids in (6) into equivalent étale groupoids. This method consists in finding an *abstract transversal* $T \subseteq X$ and considering the restriction groupoid G_T^T given by all arrows with source and range in T.

A "retopologizing" procedure is applied to T and G_T^T, resulting in a new groupoid, say G', which can be shown to be "Morita equivalent" (more accurately, equivalent in the sense of [**43**]) to the original groupoid G. Note that K-theory does *not* detect the difference between G and G'. The details of this construction are spelled out in [**56**].

If G is an equivalence relation, the relevant topologies are also discussed in [**66**]. The basic idea is as follows: once a suitable topology for T has been chosen (cf. the *Wagoner topology* [**66**, Section 4.1]), the groupoid G_T^T can be endowed with a topology of "local conjugacies". In details, given $(x, y) \in G_T^T$, a neighborhood base is given by triples (U, V, γ) where U, V are open neighborhoods of x, y in T, respectively, and $\gamma\colon V \to U$ is a homeomorphism such that $\gamma(y) = x$ and $(\gamma(z), z)$ is in G' for all $z \in V$.

Let (X, ϕ) be an irreducible Smale space. Hereafter we consider $X^s(x)$ and $X^u(s)$ (for any $x \in X$) as topological spaces with respect to the inductive limit topology associated to the increasing unions in (3). In addition, for a subset $P \subseteq X$, we write $X^s(P)$ meaning the union of all $X^s(x)$'s for $x \in P$, with the disjoint union topology. Analogously we define $X^u(P) = \cup_{x\in P} X^u(x)$.

Definition 1.4. — We define the following groupoids associated to the stable and unstable equivalence relations. Let P and Q be finite ϕ-invariant sets of periodic points of (X, ϕ).

$$R^s(X, P) = \{(x, y) \in X \times X \mid x \sim_s y \text{ and } x, y \in X^u(P)\}$$
$$R^u(X, Q) = \{(x, y) \in X \times X \mid x \sim_u y \text{ and } x, y \in X^s(Q)\}.$$

To define a topology on $R^s(X, P)$ we follow the strategy outlined above. It turns out that the local conjugacies are determined by the bracket map.

Let $(x, y) \in R^s(X, P)$. Then there exists $N > 0$ such that $d(\phi^N(x), \phi^N(y)) < \epsilon_X/2$. Now by continuity there is $0 < \delta < \epsilon_X/2$ such that, for $0 \le n \le N$, we have

$$\phi^n(X^u(x, \delta)) \subseteq X^u\left(\phi^n(x), \epsilon_X/2\right)$$
$$\phi^n(X^u(y, \delta)) \subseteq X^u(\phi^n(y), \epsilon_X/2).$$

From this we define a map $\gamma : X^u(y, \delta) \to X^u(x, \delta)$ via

$$z \mapsto \phi^{-N}([\phi^N(z), \phi^N(x)]).$$

The results in [**56**] show that $\gamma : X^u(y, \delta) \to X^u(x, \delta)$ is a local homeomorphism, mapping $X^u(y, \delta)$ homeomorphically to a neighborhood of x. For such a 5-tuple $(x, y, \delta, \gamma, N)$ as above, we define an open set

$$V(x, y, \delta, \gamma, N) = \{(\gamma(z), z) \mid z \in X^u(y, \delta)\} \subseteq R^s(X, P).$$

Such sets are the basic sets generating the topology for $R^s(X, P)$.

Proposition 1.5 **(**[**29**, Theorem 2.17]**).** — *We have the following properties of $R^s(X, P)$ and the basic sets introduced in the previous paragraph.*

– *The map γ is a local homeomorphism;*

– *The $V(x,y,\delta,\gamma,N)$'s form a neighborhood base for a topology on $R^s(X,P)$;*
– *$R^s(X,P)$ is an étale groupoid in this topology;*
– *the unit space of $R^s(X,P)$ is $X^u(P)$; it is locally compact, but not compact.*

The topology on $R^u(X,Q)$ is completely analogous and hence the details are omitted. Since we have chosen ϕ-invariant sets P and Q, it is clear that $\phi\times\phi : X\times X \to X\times X$ defines an automorphism on each of $R^s(X,P)$ and $R^u(X,Q)$.

Remark 1.6. — It is shown in [**56**, Theorem 1.1] that each of these groupoids is amenable. In this case, the completion of any faithful $*$-representation of the compactly supported functions on the groupoid will be $*$-isomorphic to the reduced and full groupoid C^*-algebras. The choice of the set P (or Q) only affects the C^*-algebra up to stable isomorphism.

Example 1.7. — The construction presented here is well known and can be found for example in [**50**] or [**66**]. Let G be a strongly connected graph, i.e., a graph where every vertex is reachable form any other vertex. This condition ensures that the associated shift of finite type Σ_G is irreducible. We are going to assume, for simplicity, that there is a loop $\bar{e}$ at a vertex $\bar{v}$. By repeating $\bar{e}$ over and over, we get a fixed point, which we keep denoting $\bar{e}$, in Σ_G. Let R_n denote the set of pairs of paths (ξ,η) of length $2n$ satisfying $i(\xi)=i(\eta)=\bar{v}$ and $t(\xi)=t(\eta)$. In words, these are pairs of paths starting at $\bar{v}$ and terminating at some common vertex.

For $(\xi,\eta)\in R_n$, we define

$$\begin{aligned}E(\xi,\eta)=\{(e,f)\in R^s(\Sigma_G,\sigma)\mid e_k=f_k=\bar{e}\quad \forall k\le -n,\\ (e_{-n+1},\dots,e_n)=\xi,\ (f_{-n+1},\dots,f_n)=\eta,\\ e_k=f_k\quad \forall k>n\}.\end{aligned}$$

It is easy to see that each set $E(\xi,\eta)$ is a compact open subset of $R^s(\Sigma_G,\bar{e})$. The collection of all sets $E(\xi,\eta)$ with $n\ge 1$ and $(\xi,\eta)\in R_n$ is a base for the topology of $R^s(\Sigma_G,\bar{e})$. It is useful to think of $E(\xi,\eta)$ as of the form $(\gamma(x),x)$, where γ is the local conjugacy operation given by replacing η with ξ.

Define $e(\xi,\eta)$ to be the indicator function of $E(\xi,\eta)$. For a vertex $v\in G^0$, the set

$$S_n(v)=\operatorname{span}\{e(\xi,\eta)\mid(\xi,\eta)\in R_n,\ t(\xi)=v\}$$

is a finite-dimensional C^*-subalgebra of $C_c(R^s(\Sigma_G,\bar{e}))$ and is isomorphic to $M_{k(n,v)}(\mathbb{C})$, where $k(n,v)$ is the number of paths of length $2n$ with source in $\bar{v}$ and target in v. Then it is a simple matter to see that

$$S_n=\operatorname{span}\{e(\xi,\eta)\mid(\xi,\eta)\in R_n\}=\bigoplus_{v\in G^0} S_n(v).$$

Now notice that below is a union of pairwise disjoint sets,

$$E(\xi,\eta) = \bigcup_{\substack{e\in G^1\\ i(e)=t(\xi)}} E(\bar{e}\xi e, \bar{e}\eta e). \tag{7}$$

In particular we have $S_n \subseteq S_{n+1}$ and the union $\cup_{n\geq 1} S_n$ is dense in $C^*(R^s(\Sigma_G, \bar{e}))$, therefore we obtain an AF-algebra.

Example 1.8. — Let (X,ϕ) be the dyadic solenoid from Example 1.3. From the description of stable orbits as increasing unions, it is easy to see that $x \sim_s y$ in X if and only if there exists $N \in \mathbb{N}$ with $x_0^{2^N} = y_0^{2^N}$. Define the subset of the circle

$$D = \{\exp\bigl(2\pi i n 2^{-k}\bigr) \mid n \in \mathbb{Z},\, k \in \mathbb{N}\},$$

called the *dyadic roots of unity.* If $\pi\colon X \to S^1$ denotes the projection onto the 0-th coordinate, then we can rephrase stable equivalence by saying $x \sim_s y$ if and only if $\pi(x) = \pi(y)d$ for some $d \in D$. This quickly leads to a proof that

$$C^*(R^s(X,\phi)) \cong (C(S^1) \otimes \mathcal{K}(L^2(\Sigma))) \rtimes D \cong (C(S^1) \rtimes D) \otimes \mathcal{K}. \tag{8}$$

This observation has been made before in [**51**]. The indexing by the natural numbers for elements in D naturally gives an inductive limit presentation $D = \varinjlim_{k\in\mathbb{N}} D_k$, where D_k is a cyclic group of order 2^k. This suggest an inductive limit structure for the crossed product $C(S^1) \rtimes D$. This approach allows identifying $C(S^1) \rtimes D$ with the so-called *Bunce-Deddens algebra* of type 2^∞ [**13**]; we simply state the result since we won't need this computation:

$$C(S^1) \rtimes D \cong \varinjlim_{k\in\mathbb{N}} (M_{2^k}(C(S^1)), \iota_{2^k}),$$

where $\iota_n\colon M_n(C(S^1)) \to M_{2n}(C(S^1))$ is the *standard twice-around embedding* given by

$$\iota_n(f)(t) = u_t \cdot \mathrm{diag}(f\Bigl(\frac{t}{2}\Bigr), f\Bigl(\frac{t+1}{2}\Bigr)) \cdot u_t^*,$$

and the conjugation action of u_t is induced by the unitary matrix

$$\begin{bmatrix} \cos\bigl(\frac{\pi}{2}t\bigr) & -\sin\bigl(\frac{\pi}{2}t\bigr) \\ \sin\bigl(\frac{\pi}{2}t\bigr) & \cos\bigl(\frac{\pi}{2}t\bigr) \end{bmatrix}.$$

The structure of the unstable C^*-algebra associated to the dyadic solenoid is explained in [**76**]. It is not hard to see that Fourier transform gives a $*$-isomorphism

$$C(S^1) \rtimes D \cong C(\Sigma) \rtimes \mathbb{Z}. \tag{9}$$

It turns out that $C^*(R^u(X,\phi))$ is stably isomorphic to the crossed product on the right of (9). Let us briefly explain this structure from a dynamical viewpoint. There is a natural flow F_t on X given by

$$F_t(z_0, z_1, z_2, \dots) = (e^{2\pi i t} z_0, e^{2\pi i t 2^{-1}} z_1, e^{2\pi i t 2^{-2}} z_2, \dots)$$

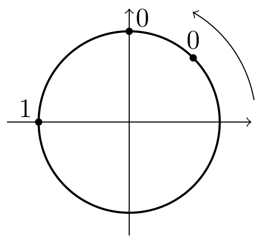

FIGURE 5. The point $y \in X$ with $(y_1, y_2, y_3, \dots) = (-1, e^{\frac{\pi}{2}i}, e^{\frac{\pi}{4}i}, \dots)$ is turned into the sequence of bits $(1, 0, 0, \dots)$.

and its orbits exactly coincide with the unstable equivalence classes. As a result,

$$C^*(R^u(X, \phi)) \cong C(X) \rtimes_F \mathbb{R}.$$

Evidently the first return map F_1 admits a natural cross-section given by $\pi^{-1}(1, 1, \dots) \cong \Sigma$ and it can be described by the so-called 2^∞-*odometer*, i.e., the translation action of $\mathbb{Z}$ on the 2-adic numbers. In other words, the action groupoid $X \rtimes_F \mathbb{R}$ is Morita equivalent to $\Sigma \rtimes_{F_1} \mathbb{Z}$, and this explains why $C^*(R^u(X, \phi))$ is stably isomorphic to the C^*-algebra on the right-hand side of (9).

Let us take a moment to understand the odometer action. We start by restricting the groupoid $R^u(X, \phi)$ to the transversal given by $X^s(x, \epsilon)$, where x is the constant sequence $(1, 1, \dots) \in X$. This means the unit space of the restricted groupoid is homeomorphic to Σ. In this representation, the point x is given by the constant sequence $(0, 0, \dots)$. More generally, if $y \in X^s(x, \epsilon)$, we need to explain a rule to convert each y_k, $k \geq 1$ into a bit $a_k \in \{0, 1\}$. This will give the representation of y in the coordinates of Σ. Note that y_{k+1} is one of the two square roots of y_k. The number a_{k+1} will determine which square root is picked, based on the following recipe: if the point y_{k+1} is met before the other possible square root of y_k, when moving counter-clockwise from $1 \in S^1$, then a_{k+1} equals 0, otherwise it equals 1. Figure 5 may help visualizing this rule.

Now the odometer action is simply given as "addition by 1" in Σ. We want to see, at least in one example, that this is the same as applying the first return map F_1. Indeed, consider

$$F_1(1, 1, 1, \dots) = (1, -1, e^{\frac{\pi}{2}i}, e^{\frac{\pi}{4}i}, \dots). \tag{10}$$

As is shown in Figure 5, the point on the right of (10) is given in Σ as $(1, 0, 0, \dots) = 1 + (0, 0, 0, \dots)$, which is what we wanted to check.

1.3. K-theory

In this short section we compute the K-theory groups for Examples 1.7 and 1.8. For a groupoid G we write $K_*(G)$ as as a shorthand for $K_*(C_r^*(G))$.

Example 1.9. — Let (Σ_G, σ) be a shift of finite type as in Example 1.7. We have shown that $C^*(R^s(\Sigma_G, \bar{e}))$ is an approximately finite dimensional C^*-algebra (in particular the odd K-group vanishes). The connecting morphisms in the inductive limit which represents $C^*(R^s(\Sigma_G, \bar{e}))$ are unital $*$-homomorphisms between direct sums of matrix algebras.

It is well known that homomorphisms between two such algebras are completely determined, up to inner automorphisms on both sides, by the multiplicity number between matrix algebra components. Thus an injective homomorphism of $\oplus_{k=1}^{i} M_{n_k}(\mathbb{C})$ into $\oplus_{h=1}^{j} M_{m_h}(\mathbb{C})$ may be represented by a collection of positive numbers $a_{k,h}$ satisfying $\sum n_k a_{kh} = m_h$. This information is often packaged in a *Bratelli diagram* [**10**].

In our setting the matrix algebra components are indexed by the vertices of the graph and the numbers a_{kh} are given precisely by the transpose of the adjacency matrix of the graph, say A_G. This is evident from equation (7).

Since the K-functor is continuous we know that $K_*(R^s(\Sigma_G, \bar{e}))$ is an inductive limit of groups. All in all, if N denotes the number of vertices in G, since we know $K_0(M_n(\mathbb{C})) = \mathbb{Z}, 0$ for any $n \geq 1$, we obtain

$$K_0(R^s(\Sigma_G, \bar{e})) \cong \ \mathbb{Z}^n \xrightarrow{A_G^t} \mathbb{Z}^n \xrightarrow{A_G^t} \mathbb{Z}^n \xrightarrow{A_G^t} \cdots .$$

As an example, let us consider the C^*-algebra associated to the full 2-shift. The graph is given in Figure 2 with edges a and b. A natural choice for the transversal, as indicated in the construction of Example 1.7, would be $\Sigma_G^s(\bar{a})$.

However we can do with an even smaller subset, namely $\Sigma_G^s(\bar{a}, 1)$. Hence the unit space of the associated groupoid is simply given by sequences $(c_n)_{n \geq 1}$ with $c_n \in \{a, b\}$. This results in a slightly different topological base when compared to Example 1.7. A generic clopen in $R^s(\Sigma_G, \Sigma_G^u(\bar{a}, 1))$ is now given by

$$\begin{aligned} E(\xi, \eta) = \{(e, f) \in R^s(\Sigma_G, \sigma) \mid & e_k = f_k = a \quad \forall k \leq 0, \\ & (e_1, \ldots, e_n) = \xi, \, (f_1, \ldots, f_n) = \eta, \\ & e_k = f_k \quad \forall k > n\}. \end{aligned}$$

Once again, it is useful to understand $E(\xi, \eta)$ as the basic open affiliated to the local conjugacy operation which replaces η by ξ in a given sequence.

If $\iota_n \colon M_n(\mathbb{C}) \to M_{2n}(\mathbb{C})$ denotes the map $\iota_n(x) = \mathrm{diag}(x, x)$, then the C^*-algebra of the stable groupoid $R^s(\Sigma_G, \Sigma_G^u(\bar{a}, 1))$ is given by

$$\mathbb{C} \xrightarrow{\iota_1} M_2(\mathbb{C}) \xrightarrow{\iota_2} M_4(\mathbb{C}) \xrightarrow{\iota_4} \cdots ,$$

which means the K_0-group is isomorphic to

$$\varinjlim_{n \in \mathbb{N}} (\mathbb{Z}, k \mapsto 2k) \cong \mathbb{Z}\left[\frac{1}{2}\right],$$

the group of rational numbers whose denominator is a power of 2.

We now turn to the dyadic solenoid.

Example 1.10. — As we have seen, we can understand the structure of this system via two Morita equivalent groupoids, namely $X \rtimes_F \mathbb{R}$ and $\Sigma \rtimes_{F_1} \mathbb{Z}$. The latter suggests a possible computation strategy making use of the Pimsner-Voiculescu exact sequence. Alternatively, one can make use of Connes's Thom isomorphism [**19**], which says

$$K_*(X \rtimes \mathbb{R}) \cong K_{*+1}(X). \tag{11}$$

It is worth pointing out that (11) also follows from Paschke's result on the K-theory of the mapping torus of an automorphism [**48**]. Since Σ is the cross-section for the first return map, it follows by construction that X is homeomorphic to the space of the suspended action, which is precisely the mapping torus

$$X \cong \Sigma \times_{\mathbb{Z}} \mathbb{R}.$$

In conclusion, we have reduced ourselves to computing the topological K-theory of the space X. Since it is given as an inverse limit and the K-functor is continuous, we easily get

$$K_0(X) \cong \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{\text{id}} \cdots$$

$$K_1(X) \cong \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \cdots ,$$

because the even K-group of the circle is generated by $z \mapsto 1$, while the odd group is generated by $z \mapsto z$. Hence the K-theory of the solenoid is given by $\mathbb{Z}\left[\frac{1}{2}\right]$ in even degree and $\mathbb{Z}$ in odd degree.

1.4. Maps and inclusions

A continuous and surjective map $f\colon (X, \phi) \to (Y, \psi)$ between Smale spaces is called a *factor map* if it intertwines the respective self-maps, i.e.,

$$f \circ \phi = \psi \circ f. \tag{12}$$

It is rather surprising, at least at first thought, that (12) is enough to guarantee that f preserves the local product structure.

Proposition 1.11 **([50, Theorem 5.1.4]).** — *Let $f\colon (X, \phi) \to (Y, \psi)$ be a factor map. Then there exists $\epsilon_f > 0$ such that, for all x_1, x_2 with $d(x_1, x_2) < \epsilon_f$, it holds that both $[x_1, x_1]$ and $[f(x_1), f(x_2)]$ are defined and*

$$f([x_1, x_2]) = [f(x_1), f(x_2)].$$

Definition 1.12. — Let $f\colon (X, \phi) \to (Y, \psi)$ be a factor map. Consider for each $x \in X$ the restrictions

$$f\colon X^s(x) \to Y^s(f(x)) \tag{13}$$

$$f\colon X^u(x) \to Y^u(f(x)). \tag{14}$$

– If (13) is injective, we say that f is *s-resolving*. If it is injective and surjective, then we say f is *s-bijective*.
– If (14) is injective, we say that f is *u-resolving*. If it is injective and surjective, then we say f is *u-bijective*.

A first indication that resolving maps are an interesting (and useful) class of maps to consider is given by the following proposition. In the sequel we will state results exclusively in the s-resolving case, but completely analogous theorems hold for u-resolving maps.

Proposition 1.13 **([50, Theorem 5.2.5]).** — *Let $f\colon (X,\phi) \to (Y,\psi)$ be an s-resolving map. There is a constant $N \geq 1$ such that*

– *for any $y \in Y$ there exists $x_1, \ldots, x_n$ in X, with $n \leq N$, satisfying*

$$f^{-1}(Y^u(y)) = \bigcup_{k=1}^{n} X^u(x_k).$$

– *for any $y \in Y$, the cardinality of the fiber $f^{-1}(y)$ is less than or equal to N.*

Example 1.14. — Let (Σ_G, σ) be the shift associated to the graph of Figure 2, with edges relabeled to $a = 0$ and $b = 1$. Denote by (Y, ψ) the solenoid of Example 1.3. We have a factor map $f\colon (\Sigma_G, \sigma) \to (Y, \psi)$, given quite explicitly by

$$f(e)_n = \exp\left(2\pi i \sum_{k\geq 0} 2^{-k-1} e_{k-n}\right). \tag{15}$$

This map can be constructed as follows. Let $\zeta\colon \prod_{n\geq 0}\{0,1\} \to [0,1]$ be the binary expansion of a number in the unit interval. By considering $\bar{f} = \exp(2\pi i \zeta(\cdot))$ we parametrize the unit circle. Now we "take inverse limits" on both sides.

More precisely, note that as a space $\Sigma_G \cong \prod_{n\in\mathbb{Z}}\{0,1\}$, and the latter can be obtained as a projective limit of spaces $X_k = \prod_{n\geq -k}\{0,1\}$ for $k \in \mathbb{N}$. The map $d_k\colon X_{k+1} \to X_k$ simply deletes the first entry in a binary sequence. There is a commuting square $\bar{f}((a_n))^2 = \bar{f}(d_k(a_n))$ which induces the mapping in (15).

It is clear from this construction that f is s-bijective and 2-to-1 (at most).

The essential result concerning resolving maps is that they are automatically bijective under non-wandering dynamics.

Theorem 1.15 **([50, Theorem 5.2.9]).** — *Let $f\colon (X,\phi) \to (Y,\psi)$ be an s-resolving map. Suppose that each point in Y is non-wandering. Then f is s-bijective.*

Theorem 1.16 **([52, Theorem 3.4]).** — *Let $f\colon (X,\phi) \to (Y,\psi)$ be an s-resolving map. For each $x \in X$, the map*

$$f\colon X^s(x) \to Y^s(f(x)) \tag{16}$$

is continuous and proper, where the sets above are given the inductive limit topology with respect to (3)*. In particular, if* Y *is non-wandering then* (16) *is a homeomorphism.*

The fundamental corollary of the previous theorem is an open inclusion result at the level of unstable equivalence relations.

Corollary 1.17. — *Let* $f\colon (X,\phi)\to(Y,\psi)$ *be an* s*-bijective map between irreducible Smale spaces. Suppose that* P *is a finite (or countable) subset of* X *such that no two points of* P *are stably equivalent after applying* f*. Then*

$$f\times f\colon R^u(X,P)\to R^u(Y,f(P))$$

is a homeomorphism onto an open subgroupoid of $R^u(Y,f(P))$*.*

Proof. — It is enough to prove the result where $P=\{x\}$. This corresponds to the case where X and Y are *mixing* dynamical systems (cf. [**52**, Corollary 3.5]).

This means the unit space of $R^u(X,x)$ is given by $X^s(x)$. Since f is a homeomorphism when restricted to that subset, it is clear that $f\times f$ is injective. To see that $f\times f$ is open (and continuous), suppose (U,V,γ) is a local conjugacy in X. Then $f(U)$ and $f(V)$ are open sets in $Y^s(f(x))$ and $f\circ\gamma\circ f^{-1}$ is a homeomorphism $f(U)\to f(V)$. In other words the triple $(f(U),f(V),f\circ\gamma\circ f^{-1})$ is a local conjugacy in Y, therefore the open set associated to (U,V,γ) in $R^u(X,x)$ gets sent to an open set in $R^s(Y,f(x))$. □

Example 1.18. — Let us analyze further the inclusion of groupoids induced by the mapping f from Example 1.14. The morphism is given as

$$f\times f\colon R^u(\Sigma_G,\Sigma_G^s(\bar a,1))\to R^u(Y,Y^s((1,1,1,\dots),\epsilon)), \tag{17}$$

where G is the graph with one vertex and two edges labeled $a=0$ and $b=1$. On the right-hand side, we have chosen as a transversal the local stable set through the point $(1,1,1,\cdots)\in Y$. Both transversal are homeomorphic to Σ and the map f becomes the identity after this identification.

In Example 1.8 we explained that $R^u(Y)=R^u(Y,Y^s((1,1,1,\dots),\epsilon)$ has the structure of an action groupoid, namely the translation action of $\mathbb{Z}$ on the group of binary sequences. In practice, this means that each arrow in this groupoid can be represented as a pair $\gamma=((c_n)_{n\ge1},m)$ with $m\in\mathbb{Z}$, $r(\gamma)=(c_n)$, and $s(\gamma)=(c_n)-m$.

Our aim is recognizing which arrows among the γ's actually come from the groupoid on the left of (17). We can categorize all arrows in two types: the ones which modify a *finite* string of bits, and those which affect an *infinite* number of bits. The following equations represent respectively both kinds:

$$1+(0,0,0,\dots)=(1,0,0,\dots) \tag{18}$$

$$1+(1,1,1,\dots)=(0,0,0,\dots). \tag{19}$$

We have explained in Example 1.9 that the arrows coming from the full 2-shift can be understood as "the operation of replacing a path of finite length by another one of the same length". Therefore Equation (18) represents an arrow in the image of $f \times f$. In the notation of Example 1.9, the basic open $E(1,0)$ is also an open in $R^u(Y)$. On the other hand, Equation (19) shows a "new" kind of arrow, which cannot possibly come from the unstable relation of the shift.

The following straightforward observation is fundamental: these "new" arrows in the solenoid can only occur between sequences which are definitely constant. This is because of the amount carried over after the addition. Let γ_0 be the arrow associated to the operation in (19). Any other arrow in the solenoid's groupoid can be written in the form $\mu\gamma\nu$, where $\mu, \nu \in (f \times f)(R^u(\Sigma_G))$, and γ is either in $\{\gamma_0, \gamma_0^{-1}\}$ or in the unit space (this representation is not unique when γ is a unit, but the interesting case is the first one). This is best understood with an example: the pair $((1,0,0,\dots),(0,0,1,1,\dots))$ evidently belongs to $R^u(Y)$. Indeed, in the action groupoid picture, that pair corresponds to $((1,0,0,\dots),+5)$. Now consider the expression

$$(0,0,1,1,\dots)+3= \tag{20}$$
$$(1,1,1,1,\dots)+1=$$
$$(0,0,0,0,\dots)+1= \tag{21}$$
$$(1,0,0,0,\dots),$$

and set "$\nu =$(20)" and "$\mu =$(21)".

A technical result by Putnam [**53**] allows us to find open subgroupoids inside the unstable equivalence relation of any Smale space with totally disconnected stable sets. Putnam's result is actually more general than the form presented here, however the statement below is the one we are going to use.

Theorem 1.19 **(**[**53**, Corollary 3]**).** — *Let (Y,ψ) be an irreducible Smale space such that $Y^s(y,\epsilon)$ is totally disconnected for every $y \in Y$ and $0<\epsilon<\epsilon_Y$. Then there is an irreducible shift of finite type (Σ_G,σ) and an s-bijective map*

$$f\colon (\Sigma_G,\sigma) \to (Y,\psi).$$

Note that solenoids and tiling spaces are examples of Smale spaces to which the previous theorem can be applied. Example 1.14 is an instance of this result.

1.5. A homology theory for open inclusions

In view of the previous section, we start by considering a second countable, locally compact, Hausdorff groupoid G along with an open subgroupoid $H \subseteq G$, over the same second countable, Hausdorff unit space X. We assume the existence of a (left) Haar system $\{\lambda^x\}_{x\in X}$ on G [**57**].

Under these assumptions there is an induction-restriction adjunction in the associated equivariant Kasparov categories (see [**38, 39**] and Appendix A)

$$(\epsilon, \eta)\colon\ \mathrm{Ind}_H^G \dashv \mathrm{Res}_G^H$$

with counit and unit

$$\epsilon\colon\ \mathrm{Ind}_H^G \mathrm{Res}_G^H \to 1_{\mathrm{KK}^G}$$
$$\eta\colon\ 1_{\mathrm{KK}^H} \to \mathrm{Res}_G^H \mathrm{Ind}_H^G .$$

As a result, the composition $L = \mathrm{Res}_G^H \mathrm{Ind}_H^G\colon\ \mathrm{KK}^G \to \mathrm{KK}^G$ is an endofunctor with the structure of a comonad. In particular, the counit of L is given by ϵ (the counit of the adjunction), and the comultiplication can be written as

$$\delta_A = \mathrm{Ind}_H^G(\eta_{\mathrm{Res}_G^H(B)})\colon L(A) \mapsto L(L(A)),$$

where η is the unit of the adjunction. Details on the comonad identities can be found in [**72**, Paragraph 8.6.2].

We are going to use L to associate a simplicial structure to the groupoid G. This is done by repeatedly applying the comonad to the C^*-algebra of continuous functions (vanishing at infinity) of the space X. Let us write X in place of $C_0(X)$. Define face and degeneracy maps

$$d_i^n = L^i \epsilon L^{n-i}\colon L^{n+1}A \to L^n A$$
$$s_i^n = L^i \delta L^{n-i}\colon L^{n+1}A \to L^{n+2}A.$$

Then [**72**, Paragraph 8.6.4] shows that

$$L^\bullet X = \cdots \ \vdots\ L^3X \ \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \ L^2X \ \underset{d_1^1}{\overset{d_0^1}{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}}} \ LX \xrightarrow{\epsilon} X \tag{22}$$

is an augemented simplicial object in KK^G. The expression in (22) can be understood as a “resolution” of X, see A.11. By choosing a functor with values in abelian groups, for example $A \mapsto K_*(A \rtimes G)$, we can map $L^\bullet X$ to a simplicial $\mathbb{Z}/(2)$-graded abelian group $K_*(L^\bullet X \rtimes G)$.

As is well-known, the Dold-Kan correspondence [**72**, Theorem 8.4.1] establishes an equivalence of categories between simplicial groups and chain complexes, hence it can be used to convert $K_*(L^\bullet X \rtimes G)$ to a chain complex. The *unnormalized* version of this complex looks like

$$\cdots \longrightarrow K_*(L^3X \rtimes G) \xrightarrow{\delta_2} K_*(L^2X \rtimes G) \xrightarrow{\delta_1} K_*(LX \rtimes G) \longrightarrow 0, \tag{23}$$

where the boundary map is simply given by the alternating sum of the induced face maps,

$$\delta_n = \sum_{i=0}^{n-1} (-1)^i d_i^n\colon K_*(L^{n+1}X \rtimes G) \to K_*(L^nX \rtimes G). \tag{24}$$

There is a normalized version of this complex, which is quasi-isomorphic to (23), making use of the degeneracy maps of the simplicial object (see [**72**, Theorem 8.3.8]).

Definition 1.20. — The homology groups of the pair (G, H) are the homology groups of the complex in (23). More precisely, we set $H_p^{(q)}(G, H) = H_p(K_q(L^\bullet X \rtimes G))$.

In homological algebra one says that Definition 1.20 is an instance of *comonadic homology* (sometimes *cotriple homology*) [**72**, Section 8.7]. Because of Bott periodicity, the superscript in $H_p^{(q)}$ can be understood modulo 2.

Example 1.21. — Let us see more concretely how the first boundary map in (23) looks like. First of all, we need to write a groupoid picture for LX and L^2X. Since the general formula for induction is given by (see Appendix A)

$$\mathrm{Ind}_H^G(A) = (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} A) \rtimes_{\mathrm{diag}} H,$$

we see that L^1X is simply represented by the right translation groupoid $G \rtimes_{\mathrm{rt}} H$. Note that this is a G-object because the groupoid G acts via left translation on it. In order to compute L^2X we look once again at the general formula

$$\mathrm{Ind}_H^G \mathrm{Res}_G^H(\mathrm{Ind}_H^G \mathrm{Res}_G^H(A)) = [C_0(G) \underset{C_0(X)}{{}_s\otimes_r} (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} A) \rtimes H] \rtimes_{\mathrm{diag}} H. \qquad (25)$$

By using the G-action on $\mathrm{Ind}_H^G \mathrm{Res}_G^H(A)$, we can balance the first tensor product over the range map, rather than the source map. This also changes the G-action of (25): when s is present G acts via left translation on the leftmost $C_0(G)$ factor, when r is used G acts diagonally via left translation on both $C_0(G)$ and $\mathrm{Ind}_H^G \mathrm{Res}_G^H(A)$. All in all, the groupoid picture for L^2X is

$$(G\, {}_r\!\times_r G) \rtimes_{\mathrm{rt}\times\mathrm{rt}} (H \times H).$$

Since $K_*(- \rtimes G)$ is actually the composition of two functors, namely the Kasparov descent functor (see Section A.1) and the K-functor, it is useful to analyze the formulas after descent has been applied. We have

$$\begin{aligned} \jmath_G(LX) &= G \ltimes (G \rtimes H) \\ \jmath_G(L^2X) &= G \ltimes (G\, {}_r\!\times_r G) \rtimes_{\mathrm{rt}\times\mathrm{rt}} (H \times H). \end{aligned} \qquad (26)$$

It turns out that both of these formulas can be simplified. The reasoning is the same: a free and proper groupoid is equivalent (in the sense of [**43**]) to its quotient (in particular KK-equivalent, see Section A.2). Since the left and right actions commute, we can rewrite (26) as

$$\begin{aligned} \jmath_G(LX) &= (G \ltimes G) \rtimes H \\ \jmath_G(L^2X) &= (G \ltimes (G\, {}_r\!\times_r G)) \rtimes (H \times H) \end{aligned}$$

and proceed by replacing the free and proper actions by their respective quotients, namely X and G respectively:

$$\begin{aligned} \jmath_G(LX) &\cong H \\ \jmath_G(L^2X) &\cong G \ltimes_\alpha (H \times H). \end{aligned} \tag{27}$$

Note that, after the equivalence, the action α in (27) has changed accordingly. It is given by

$$\alpha_{(\eta_1,\eta_2)}(\gamma) = \eta_1\gamma\eta_2^{-1}$$

for $\gamma \in G$ and $\eta_1, \eta_2 \in H$ with $s(\eta_1) = r(\gamma)$ and $s(\gamma) = s(\eta_2)$.

We now turn to the description of the boundary map $\delta_1 = d_0^1 - d_1^1$. We have

$$\begin{aligned} d_1^0 &= \epsilon_{LX} = \epsilon_{\mathrm{Ind}_H^G \mathrm{Res}_G^H(X)} \\ d_1^1 &= L(\epsilon) = \mathrm{Ind}_H^G \mathrm{Res}_G^H(\epsilon_X). \end{aligned}$$

Evidently we need to make the counit explicit to get a concrete picture. Given any object $A \in \mathrm{KK}^G$, denote by

$$\kappa_A \colon \mathrm{Ind}_H^G \mathrm{Res}_G^H(A) = (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} A) \rtimes H \to (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} A) \rtimes G$$

the natural inclusion induced by H as open subgroupoid of G. Note that the algebra on the left coincides with $\mathrm{Ind}_G^G \mathrm{Res}_G^G(A)$ and is therefore KK^G-equivalent to A itself (see Section A.1). This equivalence is denoted by $X_A^G \in \mathrm{KK}^G(\mathrm{Ind}_G^G \mathrm{Res}_G^G(A), A)$. With this information at hand, we can write

$$\epsilon_A = X_A^G \circ \kappa_A,$$

so that for example, when $A = C_0(X)$,

$$d_1^1 = L(G \ltimes H \to \underset{\substack{\wr\wr \\ X}}{G \ltimes G}) = L(\underset{\substack{\wr\wr \\ L^2X}}{G \ltimes H}) \to LX.$$

Already at this level of generality the homology groups defined above appear on the second table of a spectral sequence. The general convergence properties of this spectral sequence are discussed in [**37**]. In our context, it turns out that the limiting sheet computes the K-theory groups associated to a certain object PX, which we call the $\mathcal{H}$-cellular approximation of X. The notation "$\mathcal{H}$" is used as a reminder that PX is determined (up to homotopy) by the inclusion of H in G.

***Theorem 1.22* (cf. Theorem A.12).** — *There exists a spectral sequence of the following type,*

$$E_{pq}^2 = H_p^{(q)}(G, H) \Rightarrow K_{p+q}(PX \rtimes G).$$

Moreover, we have a natural comparison map

$$K_*(PX \rtimes G) \to K_*(G). \tag{28}$$

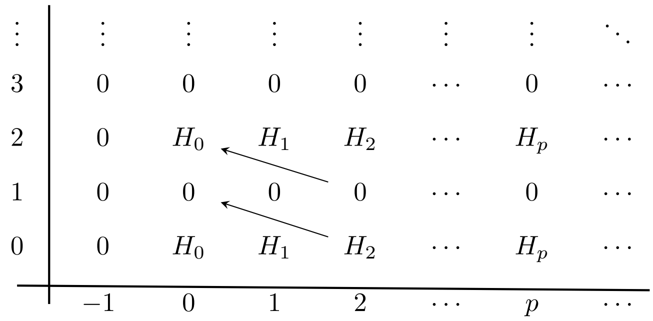

FIGURE 6. The E^2-page of the spectral sequence in Theorem 1.22 under the assumptions of Remark 1.23. The differentials have bidegree $(-2, 1)$.

The comparison map is actually induced from a morphism at the KK^G-level $PX \to X$. This is ultimately a consequence of the fact that the localizing triangulated subcategory generated by H-induced objects is part of a complementary pair of subcategories (Section A.3). In the next section (Section 1.6) we discuss a strategy for proving that (28) is an isomorphism, based on the celebrated result of Tu [**69**] for groupoids with the Haagerup property.

The spectral sequence above was first introduced, in a slightly different context, by D. Christensen in [**14**]. The existence of such sequence *per se* is not too surprising: the triangulated category axioms allow embedding the projective objects of a given resolution recursively into a sequence of exact triangles, which can be turned into an exact couple after applying a functor to abelian groups. The convergence issue is more delicate and it is analyzed in details in [**37**].

On a related note, the reader who's familiar with algebraic topology may understand the spectral sequence above as an analogue of the Bousfield-Kan spectral sequence for *homotopy colimits* [**8**]. In that setting, the spectral sequence is obtained by filtering the simplicial category which is "indexing" the simplicial object of interest. Moreover, it is true in our case too that the $\mathcal{H}$-cellular approximation is isomorphic to a homotopy colimit, namely the one associated to the *cellular approximation tower* induced by the resolution $L^\bullet X$ (the details of this cosntruction can be found in [**37**, Section 3.2]).

Remark 1.23. — This is a good moment to discuss what could be a good definition of "computable" for the subgroupoid H. A reasonable approach is requiring the C^*-algebras associated with $L^\bullet X$ to be made of "basic building blocks". More precisely we might ask for $C^*_r(L^\bullet X \rtimes G)$ to be an approximately finite-dimensional C^*-algebra.

In particular, since AF-algebras have trivial odd K-groups, we get $H^{(1)}_*(G, H) = 0$ so we can simply set $H^{(0)}_* = H_*$. Then as one can see from Figure 6 the odd rows of the E^2-page vanish and all d^2-type differentials are zero, because they either start or end at a trivial group. However note that there will be non-trivial d^3-type differential going between the homology groups, e.g., $d^3 \colon H_3(G,H) \to H_0(G,H)$, therefore we cannot conclude that the spectral sequence degenerates at the E^2-page.

Nonetheless in "low homological dimension", i.e., when we have $H_p(G,H) = 0, p \geq 3$, then all the relevant differentials have their targets on the left of the q-axis, hence we do get a collapsing spectral sequence.

1.5.1. Smale spaces with totally disconnected stable sets. — In light of the previous section, the way to use Theorem 1.22 in the setting of Smale spaces is quite clear: start with an irreducible Smale space (X,ϕ) whose local stable sets are totally disconnected. Apply Theorem 1.19 to get a shift of finite type (Σ,σ) along with an s-bijective map $f \colon \Sigma \to X$. Set $H = R^s(\Sigma, P)$ for some set $P \subseteq \Sigma$ such that $X^s(\phi(P))$ meets all unstable orbits in X (it suffices that $\phi(P)$ meets every mixing component of X). Use Corollary 1.17 to identify H with an open subgroupoid of $G = R^u(X, \phi(P))$. We are now in the setup outlined at the beginning of this section. Note that the unit space of G is not X, but rather $X^u(\phi(P))$; to avoid confusion, in case G comes from a Smale space, we write G^0 to indicate its unit space. In this setting, Theorem 1.22 is likely to hold in a stronger form, i.e., with (28) being an isomorphism; see Conjecture 2 for more details.

Let us make a few comments about the existing K-theory computations on tiling systems. These can be found for example in [**2**] and more recently in [**21**]. They have all been obtained in dimension one or two, which seems to correspond to the case discussed in Remark 1.23 where the spectral sequence of Theorem 1.22 collapses. Therefore our approach could provide a conceptual reason for why the computations are more accessible in lower dimensions.

It is worth pointing out that a certain spectral sequence, suitable specifically for tiling spaces (beyond the substitution type), has been found in [**62**]. Our standpoint is that, in many cases, one does not need to use anything specific about the nature of the dynamical system in order to establish a computational procedure, because all the systems falling under the net of Smale spaces admit an open subgroupoid structure (as per Section 1.4), which can be exploited as indicated in Section 1.5.

1.6. Future outlook: conjectures and computations

Let us discuss the strong Baum-Connes conjecture (see Definition A.18). We are going to use notation from Appendix A. Our starting point is the following result proved by J.-L. Tu.

Theorem 1.24 **([69]).** — *Suppose G is a second countable, locally compact, Hausdorff groupoid with second countable, Hausdorff unit space X. If G acts properly on a continuous field of affine Euclidean spaces, then there exists a proper G-C^*-algebra P such that $P \cong C_0(X)$ in KK^G.*

As a consequence, given any other algebra $A \in \mathrm{KK}^G$, we have that $A \otimes_{C_0(X)} P$ is proper and KK^G-equivalent to A. In the notation of Section A.4, we have the equality of categories $\langle \mathcal{P}r \rangle = \mathrm{KK}^G$.

Our objective is proving the strong Baum-Connes conjecture for all groupoids satisfying the conclusion of Theorem 1.24. From Definition A.18, it is clear that it suffices to show that the localizing subcategory generated by compactly induced objects equals the one generated by proper objects.

Conjecture 1. — *Suppose G satisfies the conclusion of Theorem 1.24. Then we have the equality of categories $\langle \mathcal{CI} \rangle = \langle \mathcal{P}r \rangle = \mathrm{KK}^G$. In particular, G satisfies the strong Baum-Connes conjecture.*

Idea of proof. — Suppose A is proper G-C^*-algebra over $\underline{E}G$. Recall from the appendix that $\mathcal{V}$ is a countable open cover of $\underline{E}G$ such that $A|_V \in \langle \mathcal{CI} \rangle$ for any $V \in \mathcal{V}$.

Given $V, V' \in \mathcal{V}$, we wish to prove that $A|_{V \cup V'} = A|_V + A|_{V'}$ belongs to $\langle \mathcal{CI} \rangle$. Notice that $I_0 = A|_V$ and $I_1 = A|_{V'}$ are ideals inside $B = A|_{V \cup V'}$. Indeed they correspond to $C_0(V)B$ and $C_0(V')B$ respectively. This suggests using the Mayer-Vietoris exact triangle for the sum of two ideals. Define an auxiliary algebra Q as

$$Q = \{f \in C([0,1], B) \mid f(0) \in I_0, f(1) \in I_1\}.$$

The map $f \mapsto (f(0), f(1))$ gives a short exact sequence

$$0 \longrightarrow \Sigma B \longrightarrow Q \longrightarrow I_0 \oplus I_1 \longrightarrow 0\ . \tag{29}$$

There is an obvious equivariant completely positive splitting given by taking the convex combination of $(x, y) \in I_0 \oplus I_1$. Hence (29) gives rise to an exact triangle. If Q is in $\langle \mathcal{CI} \rangle$, then we get $\Sigma B \in \langle \mathcal{CI} \rangle$, thus $B \in \langle \mathcal{CI} \rangle$. One way to accomplish this would be showing that Q is KK^G-equivalent to $I_0 \cap I_1 = A|_{V \cap V'} \in \langle \mathcal{CI} \rangle$. Consider the short exact sequence

$$0 \longrightarrow C([0,1], I_0 \cap I_1) \xrightarrow{\ \psi\ } Q \longrightarrow C_0 \oplus C_1 \longrightarrow 0\ , \tag{30}$$

where C_i denotes the mapping cone C^*-algebra associated to the identity morphism of $I_i/(I_0 \cap I_1)$, and $Q \to C_i$ is given by

$$f \mapsto (f(1-t) \bmod I_1, f(t) \bmod I_0).$$

Proving $Q \cong C([0,1], I_0 \cap I_1)$ is of course equivalent to showing that the generalized mapping cone of ψ is contractible. This seems plausible because $C_0 \oplus C_1$ is contractible, so if (30) was an extension triangle, we could conclude $\mathrm{cone}(\psi) \cong \Sigma C_0 \oplus \Sigma C_1$, simply by uniqueness of the mapping cone (and a rotation). Hence we reduced ourselves to showing that (30) is (equivariantly) semi-split. At this point we don't know how to proceed, therefore we assume $B \in \langle \mathcal{CI} \rangle$.

Since B is in $\langle \mathcal{CI} \rangle$, we can rearrange (by taking finite unions) the open cover of $\underline{E}G$ to an increasing cover $\{V_n\}_{n \in \mathbb{N}}$. The direct limit $A_\infty = \varinjlim_{n \in \mathbb{N}} (A|_{V_n}, \iota_n)$, where ι_n denotes the inclusion $A|_{V_n} \hookrightarrow A|_{V_{n+1}}$, is easily seen to be isomorphic to A. Hence our objective is showing that A_∞ belongs to $\langle \mathcal{CI} \rangle$. We can do this by showing that it is isomorphic to the direct *homotopy* colimit, i.e., to the object $A_\infty^h \in \mathrm{KK}^G$ which fits in an exact triangle

$$\Sigma A_\infty^h \longrightarrow \bigoplus_{n\in\mathbb{N}} A|_{V_n} \xrightarrow{D} \bigoplus_{n\in\mathbb{N}} A|_{V_n} \longrightarrow A_\infty^h \ ,$$

where D is defined by $(D(a_n))_{m+1} = a_{m+1} - \iota_m(a_m)$. Let $\alpha_n\colon A|_{V_n} \to A_\infty$ be the canonical map. By [**38**, Lemma 2.7], it is sufficient to show the existence of equivariant completely positive contractions $\beta_n\colon A_\infty \to A|_{V_n}$ such that the composition $\alpha_n \circ \beta_n$ converges in the point-norm topology to the identity. Since each V_n is G-invariant, there exists an invariant partition of unity $\{\rho_n\}_{n\in\mathbb{N}}$ subordinate to this cover. We can use this to write the map $\beta_n(a) = \sum_{k\leq n} \rho_n a$. □

We have already emphasized that [**56**, Theorem 1.1] shows that groupoids arising from Smale spaces are amenable. In particular, they act properly on a continuous field of affine Euclidean spaces [**69**, Lemma 3.5]. This is often stated by saying that (topological) amenability implies the Haagerup property.

Conjecture 2. — *Suppose (G, H) is a pair of groupoids originating from Smale spaces (X, Σ), as in Subsection 1.5.1. Then the comparison map* (28) *of Theorem 1.22 is an isomorphism. In other words, $H_*^{(i)}(G, H)$ provides a "good approximation" of the K-theory associated to X.*

Proof, assuming Conjecture 1. — Since G is an equivalence relation, it has trivial isotropy groups. Hence the subcategory of compactly induced objects is simply reduced to the objects induced from G^0, i.e., the objects of the form $\mathrm{Ind}_{G^0}^G(B), B \in \mathrm{KK}^{G^0}$. Here we are viewing G^0 as an open subgroupoid of G (recall G is étale). Since $\mathrm{Ind}_{G^0}^G(B) \cong \mathrm{Ind}_H^G(\mathrm{Ind}_{G^0}^H(B))$, we have in particular that the H-induced objects entirely generate KK^G, that is $\mathcal{H} = \mathrm{KK}^G$ in the notation of the appendix (cf. Corollary A.11 and Theorem A.17). Therefore the $\mathcal{H}$-cellular approximation of any object is KK^G-equivalent to the object itself. □

I. F. Putnam has introduced in [**54**] a homology theory for non-wandering Smale spaces. The details of this construction are discussed in greater detail in Paper B. In the case of a Smale space X with totally disconnected stable sets, the starting point of Putnam's theory is exactly the same as ours, namely a shift of finite type and an s-bijective factor map $f\colon \Sigma \to X$. From this data one defines a chain complex $C_\bullet(f)$ and associated homology groups $H_n(X) = H_n(C_\bullet(f))$.

Conjecture 3. — *Suppose (G, H) is a pair of groupoids originating from Smale spaces (X, Σ) as in Subsection 1.5.1. Then the associated chain complex (of even K-groups) in Equation* (23) *is quasi-isomorphic to the chain complex $C_\bullet(f)$ which defines Putnam's homology groups.*

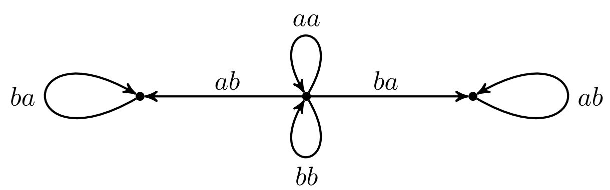

FIGURE 7. The graph associated to Σ_1.

A few comments are in order about the conjecture above. First of all, Putnam's homology groups do *not* depend on the choice of (Σ, f) [**54**, Section 5.5], whereas at least in general the choice of $H \subseteq G$ *does* affect $H_*^{(i)}(G, H)$, for example $H = G$ is a possible choice which yields vanishing higher homology.

Hence Conjecture 3 would imply that if (G, H') is pair of groupoids originating from Smale spaces (X, Σ'), then the associated chain complex of Equation (23), with $L = \operatorname{Ind}_{H'}^{G} \operatorname{Res}_{G}^{H'}$, is quasi-isomorphic to the one associated to (G, H). This is because they would be both quasi-isomorphic to Putnam's complex.

A first step towards a comparison between $H_*(X)$ and $H_*^{(0)}(G, H)$ is to obtain a K-theoretical description of Putnam's chain complex. Set $\Sigma = \Sigma_0$ and denote by Σ_n the closed subspace of $\Sigma^{\times(n+1)}$ consisting of tuples $(e^0, e^1, \dots, e^n)$ such that $f(e^0) = f(e^1) = \dots = f(e^n)$. It is easy to see that $(\Sigma_n, \sigma \times \dots \times \sigma)$ is a Smale space. By definition, we have $C_n(f) = K_0(R^u(\Sigma_n, \sigma \times \dots \times \sigma))$.

Unfortunately, describing the boundary maps appearing in $C_\bullet(f)$ requires choosing many different transversals. This is due to the fact that Σ_n, $n \geq 1$, is *not* irreducible. For simplicity, the sequel will focus on the groups $\{C_n(f)\}$.

Suppose (G, H) is the pair of groupoids corresponding to (Σ, f) according to Subsection 1.5.1. We want to show an example where we have $C_n(f) \cong K_0(L^{n+1}G^0 \rtimes G)$. Notice that, if we were able to show these isomorphisms commute with the respective boundary maps, then we would prove something stronger than Conjecture 3, namely that $C_\bullet(f)$ and $K_0(L^\bullet G^0 \rtimes G)$ are isomorphic as complexes.

Example 1.25. — As the reader might have guessed, our example is based on the inclusion of groupoids explained in Example 1.18. Let (Σ_0, σ) be the full 2-shift and (Y, ϕ) denote the dyadic solenoid. The map $f\colon (\Sigma_0, \sigma) \to (Y, \phi)$ from Example 1.14 is s-bijective and induces and open inclusion

$$(f \times f)(R^u(\Sigma_0, \Sigma_0^s(\bar{a}, 1))) \subseteq R^u(Y, Y^s((1, 1, 1, \dots), \epsilon)). \tag{31}$$

First we compute the groups in Putnam's complex, namely (omitting transversals) $K_0(R^u(\Sigma_0))$ and $K_0(R^u(\Sigma_1))$. We already know the answer for the first one from Example 1.9, $K_0(R^u(\Sigma_0)) \cong \mathbb{Z}\left[\frac{1}{2}\right]$. In order to compute the K-group for Σ_1 we need to find an *edge shift presentation* for Σ_1, i.e., we need a graph G such that $\Sigma_1 \cong \Sigma_G$ as shifts of finite type. It is easy to verify that the graph in Figure 7 furnishes a suitable presentation: each path on this graph describes a pair $(e^0, e^1) \in \Sigma_0 \times \Sigma_0$ for which $f(e^0) = f(e^1)$. Note how the graph in Figure 7 contains a copy of the "two loops on one vertex" graph of Figure 2. Deviating from this subgraph forces the pair of binary sequences to be definitely constant, one being the flipped version of the other, according to the two ways of expanding 1 in binary form, i.e., $1 \cdot 2^0 = 0 \cdot 2^0 + \sum_{k\geq 1} 1 \cdot 2^{-k}$. The groupoid $R^u(\Sigma_1)$ can be restricted to the stable orbits of $\overline{ab}, \overline{ba} \in \Sigma_1$, namely the fixed points given by the loops on the two sides of Figure 7. Since these loops are "sinks", it is evident that $\overline{ab}$ is unstably equivalent only to itself, and similarly for $\overline{ba}$. All in all, $R^u(\Sigma_1)$ contains an open sub-equivalence relation isomorphic to $R^u(\Sigma_0)$ and two isolated points. This implies that $K_0(R^u(\Sigma_1)) \cong \mathbb{Z}\big[\frac{1}{2}\big] \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Let us turn to the computation of the corresponding groups for the homology defined in Section 1.5. Denote the inclusion in (31) by $H \subseteq G$. The groups involved have been already studied in Example 1.21, and they are given by

$$K_0(G \ltimes (G \rtimes H)) \cong K_0(H)$$
$$K_0(G \ltimes (G \,{}_r\!\times_r G) \rtimes (H \times H)) \cong K_0(G \rtimes (H \times H)).$$

We see immediately that the first group is exactly the same as the one considered above, therefore we focus on the second one. First we observe that we have an open inclusion

$$H \rtimes (H \times H) \hookrightarrow G \rtimes (H \times H).$$

This gives rise to an extension of C^*-algebras

$$0 \longrightarrow C^*(H \rtimes (H \times H)) \xrightarrow{f_*} C^*(G \rtimes (H \times H)) \longrightarrow C_0(G \setminus H) \rtimes (H \times H) \longrightarrow 0 .$$

A rigorous proof of the isomorphism

$$C^*(H \rtimes (H \times H)) \cong (C^*(H) \rtimes H) \rtimes H \tag{32}$$

is given in [**12**]. Since we know the translation action of H on itself is free and proper, the C^*-algebra on the right of (32) is Morita equivalent to $C^*(H)$.

Recall that H has an inductive limit structure $H \cong \varinjlim_{n\in\mathbb{N}} R_n$. From the description of Example 1.18, where an arrow in $G \setminus H$ is represented as $\mu\gamma_0\nu$ or $\mu\gamma_0^{-1}\nu$, we can write $(G \setminus H) \rtimes (H \times H)$ as the union over $n \in \mathbb{N}$ of

$$R_n \ltimes (R_n \cdot \gamma_0 \cdot R_n) \rtimes R_n \cup R_n \ltimes (R_n \cdot \gamma_0^{-1} \cdot R_n) \rtimes R_n. \tag{33}$$

Using Morita equivalence, and replacing free and proper actions with their respective quotients, we see the K_0-group for (33) is the free abelian group on $\{\gamma_0, \gamma_0^{-1}\}$. Overall we have $K_0(G \rtimes (H \times H)) \cong K_0(R^u(\Sigma_1))$ by direct inspection.

By using techniques from Paper B, one can show that the homology groups in degree higher than 1 are all zero in this example. The main idea is replacing the defining complexes with quasi-isomorphic "reduced" versions, for which the groups in degree ≥ 2 vanish already at the level of the complex (see also [**54**, Section 7.3]).

APPENDIX A

THE STRONG BAUM-CONNES CONJECTURE

We start by giving an overview of the results contained in this appendix.

Let G be a second countable, locally compact, Hausdorff groupoid with second countable, Hausdorff unit space X. We let $s, r\colon G \to X$ denote respectively the source and range maps. We assume the existence of a (left) Haar system $\{\lambda^x\}_{x\in X}$ on G [**57**].

Let KK^G be the equivariant KK-category of separable and trivially graded C^*-algebras equipped with an action of G [**34**]. In particular, all objects in this category are $C_0(X)$-algebras [**7**]. Let $H \subseteq G$ be an open subgroupoid.

We have a natural restriction functor $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$. There is a corresponding induction functor which will be defined in Section A.1. Section A.2 is devoted to proving that these functors satisfy an adjunction relationship. In the less general setting of action groupoids this result is stated without proof in [**38**].

Section A.3 explains how to derive a structure theorem for the KK^G-category, namely a complementary pair of subcategories, from the induction-restriction theorem. The main consequence of this result is the existence of cellular approximations along with a spectral sequence converging to the localization (at the subcategory of contractible objects) of a given homological functor. Most results in this section are simple applications of the material contained in [**37, 39**].

Section A.4 discusses proper G-C^*-algebras and provides a formulation of the strong Baum-Connes conjecture for étale groupoids in the style of [**38**].

A.1. Preliminaries

Given an algebra $B \in \mathrm{KK}^H$ with moment map $\rho\colon C_0(X) \to Z(M(B))$, we consider the tensor product, balanced over the source and moment map,

$$s^*B = C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} B$$

with its associated diagonal action of H, given by right translation on $C_0(G)$. Notice that G also acts on $s^*(B)$, by left translation on $C_0(G)$. These actions commute and therefore we get a G-algebra by considering the crossed product

$$s^*B \rtimes H = (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} B) \rtimes_{\mathrm{diag}} H.$$

The crossed product is understood to be reduced *throughout this appendix.* We set $\mathrm{Ind}_H^G(B) = s^*B \rtimes H$. Details on the construction of groupoid crossed product C^*-algebras can be found in [**28, 44**].

Consider a right Hilbert module E. By using an approximate unit in B, we can equip E with a compatible $C_0(X)$-action. We can form the module

$$s^*E = C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} E,$$

which comes with a right action of s^*B by pointwise "multiplication".

We now describe a crossed product construction which, when applied to s^*E, will yield a Hilbert module over $\mathrm{Ind}_H^G(B) = s^*B \rtimes H$.

Let D be a G-algebra, F a right Hilbert D-module, and $\xi \in F$. Recall we also have a (left) $C_0(X)$-action on F. We consider projection maps

$$p_x \colon F \to \frac{F}{C_0(X \smallsetminus \{x\}) \cdot F}$$

and, given $f \in C_c(G), \xi \in F$, we denote by $f \otimes \xi$ the function sending $\gamma \in G$ to $f(\gamma) \cdot p_{r(\gamma)}(\xi)$. Let $\Gamma_e(G, r^*F)$ denote the linear span of the $f \otimes \xi$'s.

A general *compactly supported* section is an element in the completion of $\Gamma_e(G, r^*F)$ with respect to nets $\{f_i\}$ such that $S = \cup\,\mathrm{supp}(f_i)$ is compact and $\{f_i|_S\}$ is Cauchy in the uniform norm. We write $\Gamma_c(G, r^*F)$ for the set of compactly supported sections. Notice that $\Gamma_c(G, r^*D)$ makes sense equally well.

Let us denote by α the G-action on D. Given $h \in \Gamma_c(G, r^*F), f \in \Gamma_c(G, r^*D)$, we define

$$(hf)(\gamma) = \int_G h(\eta)\alpha_\eta(f(\eta^{-1}\gamma))\, d\lambda^{r(\gamma)}(\eta).$$

Define a $\Gamma_c(G, r^*D)$-valued inner product by

$$\langle h|g\rangle\,(\gamma) = \int_G \alpha_{\eta^{-1}}(\langle h(\eta)|g(\eta\gamma)\rangle)\, d\lambda_{r(\gamma)}(\eta).$$

The completion of $\Gamma_c(G, r^*F)$ with respect to the inner product above is a right Hilbert $D \rtimes_\alpha H$-module, denoted $F \rtimes G$.

A G-action β on F is a family of $C_0(X)$-linear operators

$$\beta_\gamma \colon p_{s(\gamma)}(F) \to p_{r(\gamma)}(F)$$

which is compatible with the algebraic structure of G in the obvious sense, and for which it holds

$$\beta_\gamma(\xi d) = \beta_\gamma(\xi)\alpha_\gamma(d)$$

for all $\gamma \in G, \xi \in F, d \in D$.

Let A be a G-algebra and suppose $\pi\colon A \to \mathrm{End}^*_D(F)$ is a non-degenerate representation as adjointable operators. Since π can be extended to $M(A)$, for $f \in C_0(X), \xi \in F$ the equality $\pi(f)\xi = f \cdot \xi$ makes sense and will be assumed.

If π is equivariant with respect to the adjoint action of β, we can "integrate" it to a representation $\tilde{\pi}\colon A \rtimes G \to \mathrm{End}^*_{D\rtimes G}(F \rtimes G)$.

Let $f \in \Gamma_c(G, r^*A)$ and $h \in \Gamma_c(G, r^*F)$ and define

$$\tilde{\pi}(f)h(\gamma) = \int_G \pi(f(\eta))\beta_\eta(h(\eta^{-1}\gamma))\, d\lambda^{r(\gamma)}(\eta).$$

We write $F \rtimes_\beta G$ (rather than $F \rtimes G$) to emphasize the presence of a representation acting on the left. If (π, F, T) is an equivariant Kasparov module, i.e., T is an adjointable operator commuting with the left $C_0(X)$-action and satisfying the usual compactness conditions [**34**], then we define

$$(\tilde{T}h)(\gamma) = T(h(\gamma))$$

for $f \in \Gamma_c(G, r^*F), \gamma \in G$. With this definition, $\tilde{T}$ can be extended to an adjointable operator of $F \rtimes_\beta G$ and the assignment

$$(\pi, F, T) \mapsto (\tilde{\pi}, F \rtimes_\beta G, \tilde{T})$$

is well-defined between $(A\text{-}D)\text{-}G$-Kasparov modules and $(A \rtimes G\text{-}D \rtimes G)$-Kasparov modules. In particular, it descends to a group homomorphism

$$\jmath_G\colon \mathrm{KK}^G(A, D) \to \mathrm{KK}(A \rtimes G, D \rtimes G)$$

which generalizes the familiar *Kasparov descent* in the context of groups, and inherits all its functorial properties (cf. [**32**]). Depending on the completion, this construction works for full and reduced crossed products.

Going back to the B-module E, we assume it carries an action of H (call it β) along with a non-degenerate equivariant representation $\pi\colon B_0 \to \mathrm{End}^*_B(E)$ of an H-algebra B_0 as adjointable operators of E. Then we get a representation $s^*\pi\colon s^*B_0 \to \mathrm{End}^*_{s^*B}(s^*E)$ defined by applying π_0 pointwise.

Notice that s^*E carries a diagonal H-action combining *right* translation and β. It is easy to see that $s^*\pi$ is equivariant with respect to this action. So $s^*E \rtimes_{\mathrm{rt}\otimes\beta} H$ now makes sense and is endowed with a left G-action by translation. We set

$$\mathrm{Ind}_H^G(\pi_0, E) = (\widetilde{s^*\pi}, s^*E \rtimes_{\mathrm{rt}\otimes\beta} H).$$

Moreover, if (π, E, T) is an equivariant Kasparov module then (s^*, s^*E, s^*T) is also one, provided s^*T is defined as $1 \otimes T$, i.e., T is applied pointwise. In this case we set

$$\mathrm{Ind}_H^G(\pi_0, E, T) = (\widetilde{s^*\pi}, s^*E \rtimes_{\mathrm{rt}\otimes\beta} H, \widetilde{s^*T}) = \jmath_H(s^*\pi, s^*E, s^*T).$$

It can be checked that the description so far gives a functor $\mathrm{Ind}_H^G\colon \mathrm{KK}^H \to \mathrm{KK}^G$.

A.2. Induction-restriction adjunction

Recall that if G acts freely and properly on a second countable, locally compact, Hausdorff space Y, then the C^*-algebra $C_0(Y) \rtimes G \cong C^*(Y \rtimes G)$ is strongly Morita equivalent to $C_0(Y/G)$ [**11**].

Remark A.1. — In the setting above, $Y \rtimes G$ is an amenable groupoid so that the reduced and full crossed products are isomorphic. See for example [**1**, Corollary 2.1.17 & Proposition 6.1.10].

In particular, when Y equals G itself and the action is given by right translation, the imprimitivity bimodule gives a $*$-isomorphism $C_0(G) \rtimes_{\mathrm{rt}} G \cong \mathcal{K}(L^2_s(G))$, where we denoted by $L^2_s(G)$ the standard continuous field of Hilbert spaces associated to G, fibered over the source map.

Moreover, if A is a $C_0(X)$-algebra (with moment map ρ) and $\alpha\colon s^*A \cong r^*A$ is a G-action, then there is an isomorphism of C^*-dynamical systems

$$(s^*A, G, \mathrm{rt} \otimes \alpha) \to (r^*A, G, \mathrm{rt} \otimes 1)$$

where the intertwining map is given precisely by α [**34**]. As a consequence we get an isomorphism

$$\mathrm{Ind}_G^G \mathrm{Res}_G^G(A) \cong (C_0(G) \rtimes_{\mathrm{rt}} G) \underset{C_0(X)}{{}_r\otimes_\rho} A \cong \mathcal{K}(L^2_s(G)) \underset{C_0(X)}{{}_r\otimes_\rho} A.$$

After the first isomorphism above, the left G-action goes from left translation to diagonal, considering the original action of A on the second tensor factor. More generally, the same argument gives an isomorphism

$$\phi\colon \mathrm{Ind}_H^G \mathrm{Res}_G^H A \cong (C_0(G) \rtimes H) \underset{C_0(X)}{{}_r\otimes_\rho} A,$$

where G acts diagonally on the algebra on the right.

Denote by

$$\kappa\colon \mathrm{Ind}_H^G \mathrm{Res}_G^H(A) = (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} A) \rtimes H \to (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} A) \rtimes G$$

the natural inclusion induced by H as open subgroupoid of G.

From the previous discussion we have a KK^G-equivalence

$$X_A^G \in \mathrm{KK}^G(\mathrm{Ind}_G^G \mathrm{Res}_G^G(A), A)$$

given by the right A-module $L^2_s(G) \,{}_r\otimes_\rho A$, where A acts pointwise as "constant functions". The representation of the crossed product $r^*A \rtimes G \cong \mathrm{Ind}_G^G \mathrm{Res}_G^G(A)$ is the integrated form of the covariant pair given by the right regular representation of G, and pointwise multiplication of functions in r^*A. We will denote this by $M_A \rtimes R_G$.

It is worth pointing out that the *left* action of G on $L^2_s(G)\,{}_r\otimes_\rho A$ is not decomposable along the field, i.e., it comes from the range map, because the moment map for the left action of $r^*A \rtimes G$ is given by $r^* \otimes \rho\colon C_0(X) \to M(C_0(G) \,{}_r\otimes_\rho A)$.

Let $B \in \mathrm{KK}^H$ and denote by

$$\iota\colon (C_0(H) \underset{C_0(X)}{{}_s\otimes_\rho} B) \rtimes H \to (C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} B) \rtimes H = \mathrm{Res}^H_G \mathrm{Ind}^G_H(B)$$

the map induced by the ideal inclusion $C_0(H) \subseteq C_0(G)$.

Theorem A.2. — *There is an adjunction*

$$(\epsilon, \eta)\colon \ \mathrm{Ind}^G_H \dashv \mathrm{Res}^H_G$$

with counit and unit

$$\epsilon\colon \ \mathrm{Ind}^G_H \mathrm{Res}^H_G \to 1_{\mathrm{KK}^G}$$
$$\eta\colon 1_{\mathrm{KK}^H} \to \mathrm{Res}^H_G \mathrm{Ind}^G_H$$

described as follows:

$$\epsilon_A = X^G_A \circ \kappa$$
$$\eta_B = \iota \circ \left(X^H_B\right)^{op}.$$

Proof. — We need to verify the counit-unit equations. We start by proving that the following composition equals the identity:

$$\mathrm{Res}^H_G(B) \xrightarrow{\eta_{\mathrm{Res}^H_G(B)}} \mathrm{Res}^H_G \mathrm{Ind}^G_H \mathrm{Res}^H_G(B) \xrightarrow{\mathrm{Res}^H_G(\epsilon_B)} \mathrm{Res}^H_G(B).$$

After the isomorphisms

$$(C_0(H) \underset{C_0(X)}{{}_s\otimes_\rho} \mathrm{Res}^H_G(H)) \rtimes H \cong \mathcal{K}(L^2_s(H)) \underset{C_0(X)}{{}_r\otimes_\rho} \mathrm{Res}^H_G(B)$$
$$(C_0(G) \underset{C_0(X)}{{}_s\otimes_\rho} \mathrm{Res}^H_G(B)) \rtimes G \cong \mathcal{K}(L^2_s(G)) \underset{C_0(X)}{{}_r\otimes_\rho} \mathrm{Res}^H_G(B)$$

the map $\kappa \circ \iota$ gives an inclusion

$$i\colon \mathcal{K}(L^2_s(H)) \to \mathcal{K}(L^2_s(G))$$

and the required verification is easily seen to be reduced to showing that the (interior) Kasparov product

$$[(X^H_{C_0(X)})^{\mathrm{op}}] \mathbin{\widehat{\otimes}}_{\mathcal{K}(L^2_s(H))} i^*[X^G_{C_0(X)}]$$

equals the class of 1 in $\mathrm{KK}^H(C_0(X), C_0(X))$.

Let us focus on the second factor. The representation of $\mathcal{K}(L^2_s(H))$ on $L^2_s(G)$ is induced by i and it fails to be non-degenerate. Its non-degenerate closure can be equally used to represent the KK-class $i^*[X^G_{C_0(X)}]$, and it is easily seen to be (isomorphic to) $L^2_s(H)$. Therefore $i^*[X^G_{C_0(X)}] = [(X^H_{C_0(X)})]$. Now a routine computation shows that

$$[(X^H_{C_0(X)})^{\mathrm{op}}] \mathbin{\widehat{\otimes}}_{\mathcal{K}(L^2_s(H))}[(X^H_{C_0(X)})] = 1 \in \mathrm{KK}^H(C_0(X), C_0(X)).$$

The next verification in order regards the composition

$$\mathrm{Ind}^G_H(A) \xrightarrow{\mathrm{Ind}^G_H(\eta_A)} \mathrm{Ind}^G_H \mathrm{Res}^H_G \mathrm{Ind}^G_H(A) \xrightarrow{\epsilon_{\mathrm{Ind}^G_H(A)}} \mathrm{Ind}^G_H(A).$$

The map $\kappa \circ \mathrm{Ind}_H^G(\iota)$ gives an inclusion

$$\begin{array}{c}\Big(C_0(G)\ {}_s\!\otimes_r \Big[\Big(C_0(H)\ {}_s\!\otimes_\rho A\Big) \rtimes H\Big]\Big) \rtimes H \\ \Big\downarrow \\ \Big(C_0(G)\ {}_s\!\otimes_r \Big[\Big(C_0(G)\ {}_s\!\otimes_\rho A\Big) \rtimes H\Big]\Big) \rtimes G.\end{array}$$

By using the isomorphism ϕ introduced above, we can turn the previous inclusion into the more convenient

$$\begin{array}{c}\Big(C_0(G)\ {}_s\!\otimes_{r\otimes\rho} \Big[\Big(C_0(H)\ {}_r\!\otimes_\rho A\Big) \rtimes_{\mathrm{rt}\otimes 1} H\Big]\Big) \rtimes H \\ \Big\downarrow i \\ \Big(C_0(\overset{\gamma}{G})\ {}_r\!\otimes_r \Big[\Big(C_0(\overset{\nu}{G})\ {}_s\!\otimes_\rho A\Big) \rtimes \overset{\mu}{H}\Big]\Big) \rtimes_{\mathrm{rt}\otimes\mathrm{lt}} \overset{\eta}{G}.\end{array}$$

Above, the greek letters indicate our choice of notation for the variable on the given groupoid. These will be useful in a moment.

Recall the action on A is denoted by α. Suppressing notation for the inclusions $H \subseteq G$ and $C_0(H) \subseteq C_0(G)$, the map i can be understood by

$$i(f)(\eta,\gamma,\mu,\nu) = \alpha_{\nu^{-1}\gamma}(f(\eta,\gamma,\mu,\gamma^{-1}\nu)),$$

where f is in $\Gamma_c(H, r^*(C_0(G)\ {}_s\!\otimes_{r\otimes\rho}(C_0(H)\ {}_r\!\otimes_\rho A) \rtimes_{\mathrm{rt}\otimes 1} H))$. Note that the right-hand side is zero unless $\gamma^{-1}\nu \in H$ and $\eta \in H$.

The composition that we aim to analyze equals the (interior) Kasparov product (over the domain of i)

$$[\mathrm{Ind}_H^G\big((X_A^H)^{\mathrm{op}}\big)] \mathbin{\widehat{\otimes}} i^*[X^G_{\mathrm{Ind}_H^G(A)}].$$

The class on the right is represented by the data

$$\Big((M_{(C_0(G)\otimes A)\rtimes H} \rtimes R_G)\circ i, L^2_s(\overset{\gamma}{G}) \underset{C_0(X)}{{}_r\!\otimes_r} (C_0(\overset{\nu}{G}) \underset{C_0(X)}{{}_s\!\otimes_\rho} A) \rtimes \overset{\zeta}{H}\Big).$$

Note that the representation is only non-degenerate on the submodule $F_{\gamma^{-1}\mu}$ of functions which vanish whenever $\gamma^{-1}\nu$ is not in H (it is easy to check this set is preserved by the right action and left group action). Passing to this submodule leaves the KK-class invariant, therefore it is sufficient to show that we have an isomorphism of Hilbert modules, intertwining the representations and the left group action,

$$\Phi\colon\ \mathrm{Ind}_H^G\big(X_A^H\big) \to F_{\gamma^{-1}\mu}.$$

The class $[\mathrm{Ind}_H^G\big(X_A^H\big)]$ is given by

$$\Big(\widetilde{s^*(M_A \rtimes R_H)}, \Big(C_0(\overset{\gamma}{G}) \underset{C_0(X)}{{}_s\!\otimes_{r\otimes\rho}} \Big[L^2_s(\overset{\nu}{H}) \underset{C_0(X)}{{}_r\!\otimes_\rho} A\Big]\Big) \rtimes_{\mathrm{rt}\otimes(\mathrm{lt}\otimes\alpha)} \overset{\zeta}{H}\Big).$$

The map Φ is defined as follows:

$$\Phi(h)(\gamma,\zeta,\nu)=\alpha_{\nu^{-1}\gamma}(h(\gamma,\gamma^{-1}\nu\zeta,\gamma^{-1}\nu))$$

for h in $\Gamma_c(\overset{\zeta}{H},r^*(C_0(\overset{\gamma}{G})_s\otimes_{r\otimes\rho}(C_c(\overset{\nu}{H})_r\otimes_\rho A)))$. We notice from the start that the image of Φ is dense, so it will be sufficient to show that it is isometric.

Let us begin by showing that Φ intertwines the representations. We are going to use left module notation (for the representation) for brevity.

For $h\in\Gamma_c(H,r^*(C_0(G)_s\otimes_{r\otimes\rho}(C_c(H)_r\otimes_\rho A)))$ and $f\in\Gamma_c(H,r^*(C_0(G)\ {}_s\otimes_r (C_0(H)\ {}_s\otimes_\rho A)\rtimes H))$, we have

$$f\Phi(h)(\gamma,\zeta,\nu)=\int_H d\lambda^{s(\gamma)}(\eta)\int_H d\lambda^{s(\nu)}(\mu)\,i(f)(\eta,\gamma,\mu,\nu)\alpha_\mu(\Phi(h)(\gamma\eta,\eta^{-1}\zeta,\eta^{-1}\nu\mu))$$

$$=\int_H d\lambda^{s(\gamma)}(\eta)\int_H d\lambda^{s(\nu)}(\mu)\,\alpha_{\nu^{-1}\gamma}(f(\eta,\gamma,\mu,\gamma^{-1}\nu))\alpha_{\nu^{-1}\gamma\eta}(h(\gamma\eta,\eta^{-1}\gamma^{-1}\nu\zeta,\eta^{-1}\gamma^{-1}\nu\mu)).$$

The representation on $\mathrm{Ind}_H^G(X_A^H)$ is given by

$$fh(\gamma,\zeta,\nu)=\int_H d\lambda^{r(\zeta)}(\eta)\int_H d\lambda^{s(\nu)}(\mu)\,f(\eta,\gamma,\mu,\nu)\alpha_\eta(h(\gamma\eta,\eta^{-1}\zeta,\eta^{-1}\nu\mu)),$$

note that $r(\zeta)=s(\nu)$. Therefore we compute

$$\Phi(fh)(\gamma,\zeta,\nu)=\alpha_{\nu^{-1}\gamma}(fh(\gamma,\gamma^{-1}\nu\zeta,\gamma^{-1}\nu))$$

$$=\int_H d\lambda^{s(\gamma)}(\eta)\int_H d\lambda^{s(\nu)}(\mu)\,\alpha_{\nu^{-1}\gamma}(f(\eta,\gamma,\mu,\gamma^{-1}\nu))\alpha_{\nu^{-1}\gamma\eta}(h(\gamma\eta,\eta^{-1}\gamma^{-1}\nu\zeta,\eta^{-1}\gamma^{-1}\nu\mu)),$$

from which we see that Φ intertwines the representations.

Let us check that Φ is a right module map.

For h as before and $f\in\Gamma_c(H,r^*(C_0(H)\ {}_s\otimes_\rho A))$ we have

$$\Phi(h)f(\gamma,\zeta,\nu)=\int_H d\lambda^{s(\nu)}(\mu)\,\alpha_\gamma(h(\gamma,\gamma^{-1}\nu\mu,\gamma^{-1}\nu))\alpha_\mu(f(\mu^{-1}\zeta,\nu\mu)),$$

$$\Phi(hf)(\gamma,\zeta,\nu)=\int_H d\lambda^{s(\gamma)}(\mu)\,\alpha_{\nu^{-1}\gamma}(h(\gamma,\mu,\gamma^{-1}\nu))\alpha_{\nu^{-1}\gamma\nu}(f(\mu^{-1}\gamma^{-1}\nu\zeta,\gamma\mu)).$$

We can perform the change of coordinate $\mu\mapsto\gamma^{-1}\nu\mu$ in the second integral above, changing the measure from $\lambda^{s(\gamma)}$ to $\lambda^{s(\nu)}$, so Φ commutes with the right action.

We turn to verifying that Φ is an isometry.

Given $h_1,h_2\in\Gamma_c(H,r^*(C_0(G)_s\otimes_{r\otimes\rho}(C_c(H)_r\otimes_\rho A)))$, we compute

$$\langle\Phi(h_1)|\Phi(h_2)\rangle(\zeta,\nu)=$$

$$\int_G d\lambda^{r(\nu)}(\gamma)\int_H d\lambda^{s(\nu)}(\mu)\,\alpha_{\nu^{-1}\gamma}(h_1(\gamma,\gamma^{-1}\nu,\gamma^{-1}\nu\mu)^*h_2(\gamma,\gamma^{-1}\nu\zeta,\gamma^{-1}\nu\mu)).$$

It is convenient to exchange order of integration and introduce new coordinates

$$\mu'=\nu^{-1}\gamma,\qquad\gamma'=(\mu')^{-1}\mu.$$

We can assume $\mu'\in H$ and therefore $\gamma'\in H$ too. Note $r(\gamma')=s(\mu')$.

Let us rewrite the previous inner product with these new positions and compare it to the inner product on $\mathrm{Ind}_H^G\big(X_A^H\big)$:

$$\begin{aligned}
&\langle\Phi(h_1)|\Phi(h_2)\rangle(\zeta,\nu) = \\
&\qquad\int_H d\lambda^{s(\nu)}(\mu') \int_H d\lambda^{s(\mu')}(\gamma')\, \alpha_{\mu'}(h_1(\nu\mu',\mu'^{-1},\gamma')^* h_2(\nu\mu',\mu'^{-1}\zeta,\gamma')),\\
&\langle h_1|h_2\rangle(\zeta,\nu) = \\
&\qquad\int_H d\lambda^{s(\nu)}(\mu) \int_H d\lambda^{s(\mu)}(\gamma)\, \alpha_{\mu}(h_1(\nu\mu,\mu^{-1},\gamma)^* h_2(\nu\mu,\mu^{-1}\zeta,\gamma)).
\end{aligned}$$

All is left to show is that the left action of G commutes with Φ. Let us take $\eta\in G$ with $r(\eta)=r(\gamma)$ and compute

$$\begin{aligned}
\eta\Phi(h)(\gamma,\zeta,\nu) &= \Phi(h)(\eta^{-1}\gamma,\zeta,\eta^{-1}\nu) = \alpha_{\nu^{-1}\gamma}(h(\eta^{-1}\gamma,\gamma^{-1}\nu\zeta,\gamma^{-1}\nu)),\\
\Phi(\eta h)(\gamma,\zeta,\nu) &= \alpha_{\nu^{-1}\gamma}(\eta h(\gamma,\gamma^{-1}\nu\zeta,\gamma^{-1}\nu)) = \alpha_{\nu^{-1}\gamma}(h(\eta^{-1}\gamma,\gamma^{-1}\nu\zeta,\gamma^{-1}\nu)).
\end{aligned}$$

The proof is complete. □

A.3. Complementary pairs of subcategories and localization

In this section we draw consequences on the structure of the KK^G-category from the induction-restriction theorem that has just been proved. Many of the results presented here belong to the general context of *triangulated categories*. We will not recall this notion and instead refer the reader to [**5, 14, 47**].

The framework of triangulated categories is ideal to extend basic constructions from homotopy theory to categories of C^*-algebras. Much work in this direction has been carried out by R. Meyer and R. Nest in [**37, 38, 39**]. Most results in this section are simple applications of the material contained in these papers.

Proposition A.3. — *The equivariant Kasparov category* KK^G *is triangulated.*

The most natural triangulated structure lives on the opposite category $(\mathrm{KK}^G)^{\mathrm{op}}$. Fortunately the opposite category of a triangulated category inherits a canonical triangulated category structure, which has "the same" exact triangles. The passage to opposite categories exchanges suspensions and desuspensions and modifies some sign conventions. Thus the functor $\Sigma\colon A\to C_0(\mathbb{R},A)$ becomes the *desuspension* functor in KK^G. Note that Bott periodicity implies $\Sigma^2\cong\mathrm{id}$, so that Σ and Σ^{-1} agree.

Moreover, depending on the definition of triangulated category, one may want the suspension to be an equivalence or an isomorphism of categories. In the latter case KK^G should be replaced by an equivalent category (see [**38**, Section 2.1]); this is not important and will be ignored in the sequel.

The triangulated category axioms are discussed in greater detail in [**47, 71**]. Most of them amount to formal properties of mapping cones and mapping cylinders, which can be shown in analogy with classical topology. The fundamental axiom requires that any morphism $A \to B$ should be part of an exact triangle. In our setting this can be proved as a consequence of the generalization of [**36**] to groupoid-equivariant KK-theory. Having done that, the rest of the proof follows the same outline of [**38**, Appendix A], where the triangulated structure is established in the case of action groupoids. There is an alternative, perhaps more conceptual path which consists in *defining* the Kasparov category as a certain localization of the Spanier-Whitehead category associated to the standard tensor category of G-C^*-algebras and $*$-homomorphisms [**16**, Appendix A].

The triangulated structure of the Spanier-Whithead category is proved in [**16**, Theorem A.5.3]. The argument given there can be directly used to show that KK^G is triangulated, because it makes use of only two facts, which we prove below.

Proposition A.4. — *Let C be the standard tensor category of separable G-C^*-algebras (with the minimal tensor product) and $*$-homomorphisms. Denote by F the canonical functor from C to KK^G. The following hold:*

- *up to an isomorphism of morphisms in KK^G, each morphism of KK^G is in the image of F;*
- *up to an isomorphism of diagrams $Q \to K \to D$ in KK^G, each composable pair of morphisms of KK^G is in the image of F.*

Proof. — In order to show the lifting properties above we make use of extension triangles. Let $f \in \mathrm{KK}_0^G(Q, K)$ be a morphism and denote by $\tilde{f}$ the corresponding element $\tilde{f} \in \mathrm{KK}_1^G(\Sigma Q, K)$. By applying [**32**, Lemma A.3.4] we can represent $\tilde{f}$ by a Kasparov module where the operator T is G-equivariant. Then the proof of [**32**, Lemma A.3.2] gives that $\tilde{f}$ is represented by an equivariant semi-split extension, whose associated triangle (see [**38**, Section 2.3]) fits a diagram as follows:

$$\begin{array}{ccccccc}
\Sigma^2 Q & \xrightarrow{f\beta_Q^{-1}} & K & \longrightarrow & E & \xrightarrow{p_f} & \Sigma Q \\
\| & & \downarrow{\scriptstyle \epsilon_K} & & \| & & \| \\
\Sigma^2 Q & \xrightarrow{\iota_f} & \mathrm{cone}(p_f) & \longrightarrow & E & \xrightarrow{p_f} & \Sigma Q,
\end{array}$$

where β_Q is the Bott isomorphism and ϵ_K is an equivalence. Hence we have that $F(\iota_f) \cong f$. Notice how this argument automatically shows that f is contained into an exact triangle (up to equivalence).

Now given $g \in \mathrm{KK}_0^G(K, D)$, set $h = g \circ \epsilon_K^{-1}$, $C_f = \mathrm{cone}(p_f)$ and consider the diagram

$$\begin{array}{ccccc} Q & \xrightarrow{f} & K & \xrightarrow{g} & D \\ \downarrow{\scriptstyle \beta_Q} & & \downarrow{\scriptstyle \epsilon_K} & & \| \\ \Sigma^2 Q & \xrightarrow{\iota_f} & C_f & \xrightarrow{h} & D \\ \downarrow{\scriptstyle \beta_{\Sigma^2 Q}} & & \downarrow{\scriptstyle \beta_{C_f}} & & \downarrow{\scriptstyle \epsilon_D} \\ \Sigma^4 Q & \xrightarrow{\Sigma^2 \iota_f} & \Sigma^2 C_f & \xrightarrow{\iota_h} & C_h. \end{array}$$

This shows that the pair (f, g) can be lifted to a composable pair $(\Sigma^2 \iota_f, \iota_h)$. □

Let $F\colon \mathcal{T} \to \mathcal{S}$ be an exact functor between triangulated categories. This means that F intertwines suspensions and preserves exact triangles. The kernel of F, denoted $\mathcal{I} = \ker F$, will be called a *homological ideal* (see [**39**, Remark 19]). We say that $\mathcal{I}$ is *compatible with direct sums* if F commutes with directs countable direct sums (see [**37**, Proposition 3.14]). Note that triangulated categories involving KK-theory have no more than countable direct sums, because separability assumptions are needed for certain analytical results in the background.

An object $P \in \mathcal{T}$ is called $\mathcal{I}$-*projective* if $\mathcal{I}(P, A) = 0$ for all objects $A \in \mathcal{T}$. An object $N \in \mathcal{T}$ is called $\mathcal{I}$-*contractible* if id_N belongs to $\mathcal{I}(N, N)$. Reference to $\mathcal{I}$ is often omitted in the sequel. Let $P_{\mathcal{I}}, N_{\mathcal{I}} \subseteq \mathcal{T}$ be the full subcategories of projective and contractible objects, respectively.

We denote by $\langle P_I \rangle$ the *localizing* subcategory generated by the projective objects, i.e., the smallest triangulated subcategory that is closed under countable direct sums and contains $P_{\mathcal{I}}$. In particular, $\langle P_{\mathcal{I}} \rangle$ is closed under isomorphisms, suspensions, and if

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

is an exact triangle in $\mathcal{T}$ where any two of the objects A, B, C are in $\langle P_{\mathcal{I}} \rangle$, so is the third. Note that $N_{\mathcal{I}}$ is localizing, and any localizing subcategory is *thick*, that is closed under direct summands (see [**47**]).

We are going to prove that $(\langle P_{\mathcal{I}} \rangle, N_{\mathcal{I}})$ is a pair of *complementary* subcategories. To fully understand what this means and how it is proved, we need a few more preliminaries.

Definition A.5. — Given an object $A \in \mathcal{T}$ and a chain complex

$$\cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A \tag{34}$$

we say that (34) is a *projective resolution* of A if

- all the P_n's are projective;
- for all $B \in \mathcal{S}$, the chain complex of abelian groups

$$\mathcal{S}(B, F(P_\bullet)) \xrightarrow{(\delta_0)_*} \mathcal{S}(B, A) \longrightarrow 0$$

is a resolution in the classical sense, i.e., it is exact.

We say that $\mathcal{T}$ has *enough projectives* if any object admits a projective resolution.

Remark A.6. — The map $\delta_0\colon P_0 \to A$ in a projective resolution is $\mathcal{I}$-*epic*, i.e., when embedded in a triangle

$$P_0 \xrightarrow{\delta_0} A \xrightarrow{f} C \longrightarrow \Sigma P_0 \tag{35}$$

we find that f belongs to $\mathcal{I}(A, C)$. Therefore, if A is projective then $f_0 = 0$. In this situation it is easy to show (see [**47**, Corollary 1.2.7]) that A is a retract of P_0, that is A is a direct summand of P_0 up to isomorphism (split triangles are isomorphic to direct sum triangles).

To prove that δ_0 is $\mathcal{I}$-epic, note that for each $B \in \mathcal{S}$ the functor $A \mapsto \mathcal{S}(B, F(A))$ is *homological* [**39**, Proposition 11], that is it sends exact triangles to exact-in-the-middle sequences. Hence the triangle in (35) gets sent to

$$\mathcal{S}(B, F(P_0)) \xrightarrow{(\delta_0)_*} \mathcal{S}(B, A) \xrightarrow{0} \mathcal{S}(B, C).$$

We conlclude $F(f) = 0$, i.e., $f \in \mathcal{I}(A, C)$.

Proposition A.7 **(**[**39**, Proposition 44]**).** — *The construction of projective resolutions yields a functor $\mathcal{T} \to \mathrm{Ho}(\mathcal{T})$. In particular, two projective resolutions of the same object are chain homotopy equivalent.*

Definition A.8. — We call two thick triangulated subcategories $\mathcal{P}, \mathcal{N}$ of $\mathcal{T}$ *complementary* if $\mathcal{T}(P, N) = 0$ for all $P \in \mathcal{P}, N \in \mathcal{N}$ and, for any $A \in \mathcal{T}$, there is an exact triangle

$$P \longrightarrow A \longrightarrow N \longrightarrow \Sigma P$$

where $P \in \mathcal{P}$ and $N \in \mathcal{N}$.

Proposition A.9 **(**[**38**, Proposition 2.9]**).** — *Let $(\mathcal{P}, \mathcal{N})$ be a pair of complementary subcategories of $\mathcal{T}$.*

- *The exact triangle $P \to A \to N \to \Sigma P$ with $P \in \mathcal{P}$ and $N \in \mathcal{N}$ is uniquely determined up to isomorphism and depends functorially on A. In particular, its entries define functors*

$$P\colon \mathcal{T} \to \mathcal{P} \qquad\qquad N\colon \mathcal{T} \to \mathcal{N}$$
$$A \mapsto P \qquad\qquad A \to N.$$

- *The functors P and N are respectively left adjoint to the embedding functor $\mathcal{P} \to \mathcal{T}$ and right adjoint to the embedding functor $\mathcal{N} \to \mathcal{T}$.*
- *The localizations $\mathcal{T}/\mathcal{N}$ and $\mathcal{T}/\mathcal{P}$ exist and the compositions*

$$\mathcal{P} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N}$$
$$\mathcal{N} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{P}$$

are equivalences of triangulated categories (see [**30**] *for localization).*
- *If* $K\colon \mathcal{T} \to \mathcal{C}$ *is a covariant functor, then its* localization *with respect to* $\mathcal{N}$ *is defined by* $\mathbb{L}K = K \circ P$ *and the natural maps* $P(A) \to A$ *provide a natural transformation* $\mathbb{L}K \Rightarrow K$.

We come to the key result of this section.

Theorem A.10 **(**[**37**, Theorem 3.16]**).** — *Let* $\mathcal{T}$ *be a triangulated category with countable direct sums, and let* $\mathcal{I}$ *a be a homological ideal with enough projective objects. Suppose that* $\mathcal{I}$ *is compatible with countable direct sums. Then the pair of localizing subcategories* $(\langle P_{\mathcal{I}}\rangle, N_{\mathcal{I}})$ *in* $\mathcal{T}$ *is complementary.*

Corollary A.11. — *Consider the restriction functor* $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$ *and set* $\mathcal{I} = \ker \mathrm{Res}_G^H$. *Let* $\mathcal{H} \subseteq \mathcal{T}$ *be the full subcategory of objects* $A \in \mathrm{KK}^G$ *of the form* $A = \mathrm{Ind}_H^G(B)$ *for some* $B \in \mathrm{KK}^H$. *Then* $\langle P_{\mathcal{I}}\rangle = \langle \mathcal{H}\rangle$ *and* $(\mathcal{H}, N_{\mathcal{I}})$ *is a complementary pair of localizing subcategories.*

Proof. — The restriction functor Res_G^H is exact and commutes with countable direct sums, therefore $\mathcal{I}$ is a homological ideal (see for example [**37**, Section 7]). We are going to show that each $A \in \mathrm{KK}^G$ admits a projective resolution $P_\bullet \to A$ where each P_n belongs to $\mathcal{H}$. Then Remark A.6 shows that $\langle P_{\mathcal{I}}\rangle = \langle \mathcal{H}\rangle$ and the previous theorem gives the complementarity property.

The composition $L = \mathrm{Res}_G^H \mathrm{Ind}_H^G\colon \mathrm{KK}^G \to \mathrm{KK}^G$ is a *comonad* with counit ϵ (the counit of the adjunction) and comultiplication

$$\delta_A = \mathrm{Ind}_H^G(\eta_{\mathrm{Res}_G^H(B)})\colon L(A) \mapsto L(L(A)),$$

where η is the unit of the adjunction. Details on the comonad identities can be found in [**72**, Paragraph 8.6.2]. Given $A \in \mathrm{KK}^G$ define *face* and *degeneracy* maps

$$\begin{aligned} d_i^n &= L^i \epsilon L^{n-i}\colon L^{n+1}A \to L^n A \\ s_i^n &= L^i \delta L^{n-i}\colon L^{n+1}A \to L^{n+2}A. \end{aligned}$$

Then it is a simple matter to show that

$$L^\bullet A = \cdots \begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix} L^3 A \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; L^2 A \;\substack{d_0^1\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ d_1^1}\; LA \xrightarrow{\epsilon} A$$

is a simplicial object in KK^G (see [**72**, Paragraph 8.6.4]). Setting $P_n = L^{n+1}(A)$ and $\delta_0 = \epsilon, \delta_n = \sum_i (-1)^i d_i^n$ transforms $L^\bullet A$ into a chain complex. To see that the P_n's are projective consider the isomorphism

$$\begin{aligned} \mathrm{KK}^G((\mathrm{Ind}_H^G \mathrm{Res}_G^H)^{n+1}(A), B) &\xrightarrow{\cong} \mathrm{KK}^H(\mathrm{Res}_G^H(\mathrm{Ind}_H^G \mathrm{Res}_G^H)^n(A), \mathrm{Res}_G^H B) \\ f &\longmapsto \mathrm{Res}_G^H(f) \circ \eta_{\mathrm{Res}_G^H(\mathrm{Ind}_H^G \mathrm{Res}_G^H)^n(A)} \end{aligned}$$

and note that it sends $\mathcal{I}(P_n, B)$ to zero, which implies $\mathcal{I}(P_n, B) = 0$.

Now by [**72**, Proposition 8.6.10] the augmented simplicial object $L^\bullet A \to A$ is sent via Res_G^H to a (left) contractible object. In particular, the augmented chain complex

$$\mathrm{KK}^H(B, \mathrm{Res}_G^H(L^\bullet A)) \longrightarrow \mathrm{KK}^H(B, \mathrm{Res}_G^H(A)) \longrightarrow 0$$

is split exact [**72**, Paragraph 8.4.6]. □

A complementary pair of subcategories helps clarify the degree to which a projective resolution "computes" a homological functor into the category of abelian groups.

Theorem A.12 **([37**, Theorem 4.3 & Theorem 5.1]**).** — *Let $(\langle P_{\mathcal{I}}\rangle, N_{\mathcal{I}})$ be a pair of complementary subcategories of $\mathcal{T}$. Let $K\colon \mathcal{T} \to \mathcal{A}b$ be a homological functor with values in abelian groups, commuting with countable direct sums. Given $A \in \mathcal{T}$, suppose $P_\bullet \to A$ is an* odd *projective resolution. Then there is a convergent spectral sequence*

$$E^2_{pq} = \mathbb{L}_p K_q(A) \Rightarrow \mathbb{L}K_{p+q}(A).$$

A few comments are in order about the theorem above. An *odd* projective resolution is a projective resolution where the boundary maps of positive index have degree one, i.e., the morphism $\delta_n\colon P_n \to P_{n-1}$ gets replaced, for $n \geq 1$, by a morphism $\delta_n\colon P_n \to \Sigma P_{n-1}$. Evidently, if (P_n, δ_n) is an odd projective resolution, then (P'_n, δ'_n) is an even resolution, where $P'_n = \Sigma^{-n} P_n$, $\delta'_n = \Sigma^{-n}\delta_n$, and $\delta'_0 = \delta_0$.

The convention is that $K_n = K\Sigma_{-n}$. The derived functor appearing on the left is to be understood (in degree p) as the p-th homology group of the chain complex $K_q(P_\bullet)$. Since this complex is

$$\cdots \longrightarrow K_{q+2}(P_2) \longrightarrow K_{q+1}(P_1) \longrightarrow K_q(P_0) \longrightarrow 0,$$

we see that by reverting to even resolutions we obtain

$$\cdots \longrightarrow K_q(P'_2) \longrightarrow K_q(P'_1) \longrightarrow K_q(P'_0) \longrightarrow 0,$$

so that $\mathbb{L}_p K_q(A) = H_p(K_q(P'_\bullet))$.

The limit of the spectral sequence is the localization of K with respect to $N_{\mathcal{I}}$, therefore $\mathbb{L}K(A) = (K \circ P)(A)$. The object $P(A)$ is called the *$P_{\mathcal{I}}$-cellular approximation* of A and it can be computed as the homotopy colimit of an inductive system (P_n, ϕ_n) with $P_n \in P^n_{\mathcal{I}}$ (this is proved in [**37**, Proposition 3.18]). Here, an object P_n belongs to $P^n_{\mathcal{I}}$ if it is the retract of an object $A \in \mathcal{T}$ for which

$$P_{n-1} \longrightarrow A \longrightarrow P_1 \longrightarrow \Sigma P_{n-1}$$

is an exact triangle where $P_{n-1} \in P^{n-1}_{\mathcal{I}}$ and $P_1 \in P_{\mathcal{I}}$.

Corollary A.13. — *In the setting of Corollary A.11, there is a convergent spectral sequence*

$$E^2_{pq} = H_p(K_q(L^\bullet A \rtimes G)) \Rightarrow K_{p+q}(P(A) \rtimes G),$$

where $L^n A = (\mathrm{Ind}_H^G \mathrm{Res}_G^H)^n(A)$ and $P(A)$ is the $\mathcal{H}$-cellular approximation of A.

Proof. — The functor "K-theory of the crossed product", namely

$$K_0(- \rtimes G)\colon \mathrm{KK}^G \to \mathcal{A}b,$$

is homological because $\jmath_G\colon \mathrm{KK}^G \to \mathrm{KK}$ is exact (it preserves mapping cone triangles, see [**39**, Example 13 & Example 15]). □

A.4. The Baum-Connes conjecture

In most cases, one is interested in computing $K_*(A \rtimes G)$ rather than $K_*(P(A) \rtimes G)$. Obviously, the KK^G-cellular approximation of A is isomorphic to A, therefore it is desirable to have a measure of the difference between $\langle \mathcal{H} \rangle$ and KK^G. A natural idea is identifying a "probing" class of objects $\mathcal{P}r \subseteq \mathcal{T}$, that we understand somewhat better than a generic object of KK^G, and for which we can prove $\langle \mathcal{P}r \rangle = \mathrm{KK}^G$.

Definition A.14. — We say that G is *proper* if the *anchor map* $(s, r)\colon G \to X \times X$ is proper. Furthermore, if Z is a a second countable, locally compact, Hausdorff G-space, we say that G *acts properly* on Z if $Z \rtimes G$ is proper. A G-algebra A is called proper if there is a proper G-space Z such that A is a $Z \rtimes G$-algebra.

We let $\mathcal{P}r$ denote the class of proper objects in KK^G.

Evidently, a commutative G-C^*-algebra is proper if and only if its spectrum is a proper G-space. Recall that G is called *étale* if its source and range maps are local homeomorphisms. A *bisection* is an open $W \subseteq G$ such that $s|_W, r|_W$ are homeomorphisms onto an open in X. Hereafter it is assumed that G is étale.

The following proposition clarifies the local picture of proper actions (cf. [**41**, Theorem 4.1.1] and [**70**, Proposition 2.42]).

Proposition A.15. — *Suppose G acts properly on Z and denote by $\rho\colon Z \to X$ the moment map. Then for each $z \in Z$ there are open neighborhoods U^ρ, U, respectively of $z \in Z$ and $\rho(z) \in X$, satisfying:*

- *the isotropy group $G_z = G_{\rho(z)}$ acts on U;*
- *$G|_U$ contains an open copy of $U \rtimes G_z$;*
- *the G-action restricted to U^ρ is induced from $U \rtimes G_z$.*

Proof. — Since the G-action on Z is proper, the isotropy group G_z is finite. Choose a bisection W_g for each $\gamma \in G_z$. For any two $\gamma, \eta \in G_z$, there is an open neighborhood V of $\rho(z)$ such that $W_{\gamma\eta}|_V$ and $(W_\gamma W_\eta)|_V$ are defined and equal, because both are bisections containing $\gamma\eta$. Likewise, for each γ in G_z there is an open neighborhood V of $\rho(z)$ where $W_{\gamma^{-1}}|_V$ and $(W_\gamma)^{-1}|_V$ are defined and equal. Ranging over the group G_z, we collect a finite number of V's whose intersection we denote by U. Notice U is an open neighborhood of $\rho(z)$. We now restrict all the W_γ's to U. Their union, say W, is an open copy of $U \rtimes G_z$ inside $G|_U$. Define $U' = \rho^{-1}(U)$. By construction $G|_{U'} = (Z \rtimes G)|_U$ contains a copy of $U' \rtimes W$. The anchor map is proper at z, so we can find a neighborhood $U^\rho \subseteq U'$ such that $G|_{U^\rho}$ is $U^\rho \rtimes W$, for example the complement of $(s,r)(G|_{U'} \smallsetminus U' \rtimes W)$. □

Remark A.16. — As a simple corollary of Proposition A.15, the range map $r\colon s^{-1}(U^\rho) \to Z$ descends to a G-equivariant homeomorphism

$$G \times_{G|_U} U^\rho \to G \cdot U^\rho = V. \tag{36}$$

Moreover, the space $s^{-1}(U^\rho)$ provides a principal bibundle implementing a $(G|_{U^\rho}, G|_V)$-equivalence in the sense of [**43**] (cf. [**25**]). Hence, as is suggested by (36), the induction functor $\mathrm{KK}^{G|_{U^\rho}} \to \mathrm{KK}^{G|_V}$ is essentially surjective, i.e., if A is a G-algebra over Z then $A|_V$ is isomorphic to $\mathrm{Ind}_{G|_{U^\rho}}^{G|_V}(A|_U) = \mathrm{Ind}_{G|_U}^{G}(A|_U)$.

In Definition A.14 for a proper G-algebra we can always assume Z to be a realization of $\underline{\mathrm{E}}G$, the classifying space for proper actions of G. Indeed if $\phi\colon Z \to \underline{\mathrm{E}}G$ is a G-equivariant continuous map, then $\phi^*\colon C_0(\underline{\mathrm{E}}G) \to M(C_0(X))$ can be precomposed with the structure map $\Phi\colon C_0(Z) \to ZM(A)$, making A into an $\underline{\mathrm{E}}G \rtimes G$-algebra.

Note that if G is locally compact, σ-compact, Hausdorff, $\underline{\mathrm{E}}G$ always exists and is locally compact, σ-compact, and Hausdorff; in our case G is second countable hence $\underline{\mathrm{E}}G$ is too [**68**, Proposition 6.15].

A subgroupoid of the form $U \rtimes G_z \subseteq G$, as in Proposition A.15, will be called a *compact action* around $\rho(z)$. Given a proper G-algebra over $Z = \underline{\mathrm{E}}G$, for any $z \in Z$ we can find a neighborhood V as in (36). These opens cover Z and we can extract a countable subcover $\mathcal{V}$ (being second countable, Z is a Lindelöf space). Corresponding to this subcover we get a countable collection of compact actions which we denote by $\mathcal{F}$. Define the full subcategory of *compactly induced objects*,

$$\mathcal{CI} = \{\mathrm{Ind}_Q^G(B) \mid B \in \mathrm{KK}^Q, Q \in \mathcal{F}\}.$$

Following [**37**, page 27], we define a homological ideal $\mathcal{I}$ as the kernel of a single functor

$$F\colon \mathrm{KK}^G \to \prod_{Q \in \mathcal{F}} \mathrm{KK}^Q$$
$$A \mapsto (\mathrm{Res}_G^Q(A))_{Q \in \mathcal{F}}$$

The functor F commutes with direct sums because each restriction functor clearly does. Hence $\mathcal{I}$ is compatible with countable direct sums.

Theorem A.17. — *The projective objects for $\mathcal{I}$ are the retracts of direct sums of objects in $\mathcal{CI}$ and the ideal $\mathcal{I}$ has enough projective objects. Therefore the pair of subcategories $(\langle \mathcal{CI} \rangle, N_{\mathcal{I}})$ is complementary.*

Proof. — Since each $Q \in \mathcal{F}$ is open in G, the functor Ind_Q^G is left adjoint to Res_G^Q. Thus we may take

$$F^\dagger((A_Q)_{Q \in \mathcal{F}}) = \bigoplus_{Q \in \mathcal{F}} \mathrm{Ind}_Q^G(A_Q).$$

Since $\mathcal{F}$ is countable this definition is legitimate. Then in complete analogy with the proof of Corollary A.11, we get that $\mathcal{I}$ has enough projective objects, and they are all direct summands of $\oplus_{Q\in\mathcal{F}} \mathrm{Ind}_Q^G(A_Q)$ for suitable families $(A_Q)_{Q\in\mathcal{F}}$. □

Notice that if G is *free*, i.e., its isotropy groups are all trivial, then a compactly induced object is simply an object induced from the unit space X (recall that $X \subseteq G$ is open because G is étale). In particular, setting $H = X$, the above theorem becomes exactly Corollary A.11.

We denote by $P(A)$ the $\mathcal{CI}$-cellular approximation of A.

Definition A.18. — We say that G satisfies the *strong* Baum-Connes conjecture if the natural map $\jmath_G(P(A) \to A) = P(A) \rtimes G \to A \rtimes G$ is a KK-equivalence.

A stronger variant of the formulation above is requiring $P(A) \to A$ to be an isomorphism in KK^G. However it is known that even a weaker form of the conjecture in Definition A.18 admits counterexamples [**23**].

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PAPER A

INDEX THEORY ON THE MIŠČENKO BUNDLE

INDEX THEORY ON THE MIŠČENKO BUNDLE

by

Jens Kaad & Valerio Proietti

Abstract. — We consider the assembly map for principal bundles with fiber a countable discrete group. We obtain an index-theoretic interpretation of this homomorphism by providing a tensor-product presentation for the module of sections associated to the Miščenko line bundle. In addition, we give a proof of Atiyah's L^2-index theorem in the general context of principal bundles over compact Hausdorff spaces. We thereby also reestablish that the surjectivity of the Baum-Connes assembly map implies the Kadison-Kaplansky idempotent conjecture in the torsion-free case. Our approach does not rely on geometric K-homology but rather on an explicit construction of Alexander-Spanier cohomology classes coming from a Chern character for tracial function algebras.

Contents

1. Preliminaries and main results 3
2. Miščenko module — Proof of Theorem A 8
3. Chern characters and flat bundles 14
   3.1. Reminders on Alexander-Spanier cohomology 14
   3.2. Construction of the Chern character 15
   3.3. Multiplicative properties 16
   3.4. Flat bundles 18
4. Index theorem — Proof of Theorem B 24
References 26

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Introduction

One of the important applications of the Baum-Connes conjecture is to the Kadison-Kaplansky idempotent conjecture, which asserts that the reduced group C^*-algebra of a countable discrete torsion-free group contains no non-trivial idempotents. Indeed, it holds that surjectivity of the Baum-Connes assembly map implies the idempotent conjecture [**28**]. The proof of this implication uses two main ingredients, namely the computation of analytic K-homology for finite CW complexes using geometric K-homology [**7**], in combination with Atiyah's L^2-index theorem [**4**]. In this paper a proof of this implication is provided which avoids the description of analytic K-homology using Baum-Douglas geometric K-cycles.

The aim of the present paper is thus twofold: on one hand we wish to clarify the index-theoretic interpretation of the assembly map for torsion-free groups, on the other we intend to show Atiyah's L^2-index theorem (Theorem B below) by means of a purely topological argument, involving nothing more than K-theory and the Chern character with values in Alexander-Spanier cohomology of the base space.

It is well-known, though maybe not well-documented, that the Miščenko-Fomenko index map coincides with the Baum-Connes assembly map, once the relevant K-homology groups are identified (Corollary 1.4 below). To our knowledge, the only published proof of this result is in [**19**]. The argument there makes use of propositions on fixed-point algebras from [**11**], combined with a clever argument involving dual coactions on crossed products.

The main obstacle towards a more direct proof, we think, seems to be the usual description of the module of sections associated to the Miščenko line bundle, which is not at first glance amenable to be analyzed through the standard tools of KK-theory, e.g., the Kasparov product and the descent homomorphism.

In the first part of this paper we provide a structure theorem for the Miščenko module in terms of tensor products and crossed products of Hilbert C^*-modules (Theorem A below). This presentation is compatible with the basic functorial properties of KK-theory, and it allows for a different proof of the main theorem in [**19**]. It turns out that this structure theorem can also be derived from more general results on weakly proper actions, which can be found in [**10, 11**].

The second part of this paper is devoted to the L^2-index theorem of Atiyah. We give a proof of this theorem, which works in the setting of principal bundles defined over any second-countable compact Hausdorff space. All the proofs known to the authors are set in a smooth setting, ([**2, 3, 4, 12, 25**]), where the manifold structure is used to get a description of the Chern character in terms of connections (i.e., Chern-Weil theory). This approach can then be used in combination with the Baum-Douglas picture of K-homology (see [**6, 7**]) since this describes the entire analytic K-homology using data of geometric origin. Remark that geometric K-homology is only known to be isomorphic to analytic K-homology for (locally) finite CW complexes.

In particular, the use of Baum-Douglas geometric K-cycles has been crucial in deriving the Kadison-Kaplansky idempotent conjecture (in the context of torsion-free groups satisfying the Baum-Connes conjecture) as a corollary of the L^2-index theorem (see for example [**28**, Section 6.3]).

Our main motivation for writing this paper was to provide a "self-contained and topological" proof of the fact that the surjectivity of the Baum-Connes assembly map implies the Kadison-Kaplansky idempotent conjecture for countable discrete torsion-free groups. By this we mean that the proof should not rely on the geometric K-homology description of analytic K-homology, but rather be based on the original Kasparov picture of KK-theory, together with purely topological considerations. In particular, our proof does not involve differential geometric entities such as connections and differential operators on manifolds.

Instead, our goals are achieved by virtue of Alexander-Spanier cohomology, whose definition incorporates a "diagonal localization" feature which we exploit to compute the index pairing with the Miščenko line bundle. An important ingredient is therefore supplied by the explicit description of the Chern character for compact Hausdorff spaces with values in Alexander-Spanier cohomology [**16**].

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1. Preliminaries and main results

Let G be a countable discrete group and let us fix a second-countable, locally compact, Hausdorff space $\tilde{X}$, equipped with a free and proper action of G. We will moreover assume that the quotient space $\tilde{X}/G = X$ is compact and we note that the quotient map $p : \tilde{X} \to X$ forms a principal G-bundle. The action of G on $\tilde{X}$ induces an action on the C^*-algebra $C_0(\tilde{X})$, which we denote by

$$\alpha\colon G \to \mathrm{Aut}(C_0(\tilde{X})).$$

We use the convention that G acts on $\tilde{X}$ *from the right* and the induced action on the C^*-algebra is therefore given by $\alpha_g(f)(\tilde{x}) = f(\tilde{x} \cdot g)$.

Let us turn to the description of the Baum-Connes assembly map

$$\mu_{\tilde{X}} : \mathrm{KK}^G_*(C_0(\tilde{X}), \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C^*_r(G))$$

associated to $p : \tilde{X} \to X$. The left-hand side is the G-equivariant K-homology of the G-space $\tilde{X}$ whereas the right-hand side is the K-theory of the reduced group C^*-algebra $C^*_r(G)$.

First of all we recall the construction of Rieffel's imprimitivity bimodule. We merely sketch the proof, mostly to set up notational conventions, and refer the reader to [**22, 23**] for more details (and more general results).

Proposition 1.1. — *There exists a C^*-correspondence Y, implementing a strong Morita equivalence between the reduced crossed product $C_0(\tilde{X}) \rtimes_r G$ and $C(X)$.*

Sketch of proof. — The C^*-correspondence Y is defined as a completion of $C_c(\tilde{X})$. The unital C^*-algebra $C(X)$ acts from the right as bounded continuous functions using the pullback along the quotient map $p : \tilde{X} \to X$. The full $C(X)$-valued inner product on $C_c(\tilde{X})$ is defined as

$$\langle \xi | \eta \rangle (x) = \sum_{p(y)=x} (\bar{\xi} \cdot \eta)(y), \tag{1}$$

where $\xi, \eta \in C_c(\tilde{X})$ and $x \in X$. Define Y to be the completion of $C_c(\tilde{X})$ with respect to the induced norm. The left action on Y of the reduced crossed product is given by

$$f \cdot \xi = \sum_{g \in G} f(g) \alpha_g(\xi), \tag{2}$$

where $f \in C_c(G, C_0(\tilde{X}))$ and $\xi \in C_c(\tilde{X})$. The assignment

$$\Phi(\Theta_{\xi,\eta})(g) = \xi \alpha_g(\bar{\eta}) \tag{3}$$

mapping from rank-one operators (i.e., $\Theta_{\xi,\eta}(\zeta) = \xi \langle \eta | \zeta \rangle$ with $\xi, \eta \in C_c(\tilde{X})$) to $C_c(G, C_0(\tilde{X}))$, extends to a $*$-isomorphism $\Phi\colon \mathcal{K}(Y) \to C_0(\tilde{X}) \rtimes_r G$ from the compact operators of the Hilbert C^*-module Y to the reduced crossed product. □

Remark 1.2. — *It follows from* [**1**, Proposition 2.2] *that the action of G is amenable and then by* [**1**, Theorem 5.3] *that the* full *crossed product $C_0(\tilde{X}) \rtimes G$ is isomorphic to the* reduced *crossed product $C_0(\tilde{X}) \rtimes_r G$. In particular, any covariant pair of representations for $C_0(\tilde{X})$ and G gives rise to a representation of the (reduced) crossed product $C_0(\tilde{X}) \rtimes_r G$, namely the integrated form.*

The Baum-Connes assembly map $\mu_{\tilde{X}}$ is defined as the composition of the following maps:

$$\begin{array}{ccc} \mathrm{KK}_*^G(C_0(\tilde{X}),\mathbb{C}) \xrightarrow{j_r^G} \mathrm{KK}_*(C_0(\tilde{X}) \rtimes_r G, C_r^*(G)) & & \\ \Big\downarrow [Y^*]\widehat{\otimes}_{C_0(\tilde{X})\rtimes_r G} - & & \\ \mathrm{KK}_*(C(X), C_r^*(G)) \xrightarrow{\iota^*} \mathrm{KK}_*(\mathbb{C}, C_r^*(G)), & & \end{array}$$

where we specify that

- the upper horizontal map is the reduced version of Kasparov's descent homomorphism ([**18**, page 172]);
- the vertical map is given by interior Kasparov product with the class $[Y^*] \in \mathrm{KK}_0(C(X), C_0(\tilde{X}) \rtimes_r G)$, induced by the dual of Rieffel's imprimitivity bimodule;
- the lower horizontal map is the pullback along the inclusion $\iota\colon \mathbb{C} \hookrightarrow C(X)$.

Remark 1.3. — *The Baum-Connes assembly map is defined more generally for proper actions of G with cocompact quotient* [**5**]. *In this paper we focus on free and proper actions since we are interested in the link to the Miščenko-Fomenko index map and the Kadison-Kaplansky idempotent conjecture.*

We now turn to the description of the Miščenko-Fomenko index map

$$\eta_{\tilde{X}} : \mathrm{KK}_*(C(X),\mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C_r^*(G)).$$

This homomorphism is defined as the composition of the following maps:

$$\begin{array}{c} \mathrm{KK}_*(C(X),\mathbb{C}) \xrightarrow{\tau_{C_r^*(G)}} \mathrm{KK}_*(C(X)\,\widehat{\otimes}\, C_r^*(G), C_r^*(G)) \\ \Big\downarrow [M]\widehat{\otimes}_{C(X)\widehat{\otimes} C_r^*(G)} - \\ \mathrm{KK}_*(\mathbb{C}, C_r^*(G)). \end{array}$$

We specify that the homomorphism $\tau_{C_r^*(G)}$ is defined on Kasparov modules as

$$(E,F) \mapsto (E\widehat{\otimes} C_r^*(G), F\widehat{\otimes} 1),$$

using the exterior tensor product of C^*-correspondences, see [**9**, Section 17.8.5]. The second homomorphism is given by interior Kasparov product with a class $[M] \in \mathrm{KK}_0(\mathbb{C}, C(X)\,\widehat{\otimes}\, C_r^*(G))$, induced by a finitely generated projective Hilbert C^*-module M. More precisely, M can be identified with the module of sections associated to the *Miščenko line bundle*, i.e., the hermitian bundle of C^*-algebras obtained from the associated bundle construction

$$\tilde{X} \times_G C_r^*(G) \to X,$$

where G acts diagonally, acting on the reduced group C^*-algebra via the left regular representation [**21**].

The link between the Baum-Connes assembly map and the Miščenko-Fomenko index map is furnished by the *dual Green-Julg isomorphism*:

$$J_{\tilde{X}}: \mathrm{KK}_*^G(C_0(\tilde{X}), \mathbb{C}) \to \mathrm{KK}_*(C(X), \mathbb{C}),$$

which we will describe in details in Section 2.

Let us recall that any G-C^*-correspondence E between two G-C^*-algebras A and B gives rise to a reduced crossed product $E \rtimes_r G$, which is then a C^*-correspondence between the corresponding reduced crossed product C^*-algebras, thus from $A \rtimes_r G$ to $B \rtimes_r G$, see [**18**, page 170-171], [**13**], and Section 2 for more details.

Our first result is the following:

Theorem A. — *There exists a G-C^*-correspondence Z from $C_0(\tilde{X})$ to $C(X)$, inducing a class $[Z] \in \mathrm{KK}_0^G(C_0(\tilde{X}), C(X))$, such that the inverse of the dual Green-Julg isomorphism is given by*

$$J_{\tilde{X}}^{-1} = [Z]\widehat{\otimes}_{C(X)} - \colon \mathrm{KK}_*(C(X), \mathbb{C}) \to \mathrm{KK}_*^G(C_0(\tilde{X}), \mathbb{C}).$$

Moreover, there is an isomorphism of Hilbert C^*-modules over $C(X)\widehat{\otimes} C_r^*(G)$:

$$M \cong Y^* \,\widehat{\otimes}_{C_0(\tilde{X}) \rtimes_r G} \, Z \rtimes_r G.$$

In particular, we have the KK*-theoretic identity*

$$[M] = \iota^*[Y^*] \,\widehat{\otimes}_{C_0(\tilde{X}) \rtimes_r G} \, j_r^G[Z] \in \mathrm{KK}_0(\mathbb{C}, C(X) \,\widehat{\otimes}\, C_r^*(G)).$$

Corollary 1.4. — *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{KK}_*^G(C_0(\tilde{X}), \mathbb{C}) & \xrightarrow{\mu_{\tilde{X}}} & \mathrm{KK}_*(\mathbb{C}, C_r^*(G)) \\ \downarrow J_{\tilde{X}} & \nearrow \eta_{\tilde{X}} & \\ \mathrm{KK}_*(C(X), \mathbb{C}). & & \end{array} \tag{4}$$

The previous corollary is well-known to experts working on the Baum-Connes conjecture and index theory. It has been proved in [**19**] with a different method.

We now turn to our second result. Let $\phi\colon C_r^*(G) \to \mathbb{C}$ denote the canonical tracial state and denote by $\phi_* : K_0(C_r^*(G)) \to \mathbb{R}$ the induced map on even K-theory. Then the homomorphism

$$\phi_* \circ \eta_{\tilde{X}} \colon \mathrm{KK}_0(C(X), \mathbb{C}) \to \mathbb{R} \tag{5}$$

can be interpreted as an index, namely the L^2-index of Atiyah [**4**] and, equivalently, the Miščenko-Fomenko index [**21**]. These identifications are explained in [**25**, Theorem 5.15 & Theorem 5.22]. In view of this, we denote the map in (5) by

$$\mathrm{ind}_{C_r^*(G)} \colon \mathrm{KK}_0(C(X), \mathbb{C}) \to \mathbb{R}.$$

On the other hand, there is a simple index map

$$\mathrm{ind}: \mathrm{KK}_0(C(X), \mathbb{C}) \to \mathbb{Z},$$

obtained by pairing with the class $[1_{C(X)}] \in K_0(C(X))$, or equivalently by applying the pullback via the unital $*$-homomorphism $\iota\colon \mathbb{C} \hookrightarrow C(X)$.

Theorem B. — *There is an equality of index maps:*

$$\mathrm{ind}_{C_r^*(G)} = \mathrm{ind}: \mathrm{KK}_0(C(X), \mathbb{C}) \to \mathbb{R}.$$

In particular,

$$\mathrm{ind}_{C_r^*(G)}(x) \in \mathbb{Z} \qquad \text{for all } x \in \mathrm{KK}_0(C(X), \mathbb{C}).$$

We emphasize once more that the space X need not be a CW complex but merely a second-countable, compact Hausdorff space. This provides a generalization of the already known L^2-index theorem.

In [**4**] the previous theorem is proved when $\tilde{X}$ and X are smooth manifolds and without reference to KK-theory. More precisely, in the smooth setting, the KK-classes whose index we consider here, are concretely realized in [**4**] as coming from elliptic differential operators, acting on sections of a bundle over X, and their lifts to equivariant differential operators acting on the corresponding sections of the pullback bundle. A generalization of Atiyah's L^2-index theorem is also proved in [**20**], using the universal center-valued trace instead of the standard trace.

The application of the L^2-index theorem to the Kadison-Kaplansky idempotent conjecture is based on the following well-known argument [**24**, Corollary 6.3.13].

Proposition 1.5. — *Let A be a C^*-algebra with a unit 1, and let ϕ be a faithful tracial state on A. If $\phi_*\colon K_0(A) \to \mathbb{R}$ only takes integer values, then A contains no idempotents other than 0 and 1.*

Proof. — Let $e \in A$ be an idempotent. There is a projection $p \in A$ which is similar to e, in particular $\phi(e) = \phi(p)$. Since $1 - p \geq 0$, we have $\phi(1) - \phi(p) \geq 0$, and therefore $0 \leq \phi(p) = \phi(e) \leq 1$. Now, if $\phi(p)$ is an integer we must have that $\phi(p) \in \{0, 1\}$ and therefore since ϕ is faithful and $1 \geq p \geq 0$ we conclude that $p \in \{0, 1\}$. Thus, since e is similar to p, e is equal to either 0 or 1. □

Suppose for a little while that the group G is torsion-free. In this case, Corollary 1.4 implies that we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{RKK}_*^G(C_0(EG), \mathbb{C}) & \xrightarrow{\ \mu\ } & \mathrm{KK}_*(\mathbb{C}, C_r^*(G)) \\ \big\downarrow J & \nearrow_{\eta} & \\ \mathrm{RKK}_*(C_0(BG), \mathbb{C}) & & \end{array} \qquad (6)$$

Above, $EG \to BG$ is the universal principal G-bundle. It is known that the classifying space for proper actions, usually denoted $\underline{E}G$, coincides (up to equivariant homotopy) with EG when G is torsion-free, because in this case proper actions are automatically free. The groups on the left are K-homology groups *with compact support*, and are defined as

$$\begin{aligned} \mathrm{RKK}_*^G(C_0(EG), \mathbb{C}) &= \varinjlim_{\tilde{X} \subseteq EG} \mathrm{KK}_*^G(C_0(\tilde{X}), \mathbb{C}) \\ \mathrm{RKK}_*(C_0(BG), \mathbb{C}) &= \varinjlim_{X \subseteq BG} \mathrm{KK}_*(C(X), \mathbb{C}), \end{aligned}$$

where $\tilde{X}$ ranges over locally compact Hausdorff proper G-spaces with compact quotient X. The various homomorphisms are induced on the direct limits by their "localized" counterparts, namely $J_{\tilde{X}}$, $\mu_{\tilde{X}}$ and $\eta_{\tilde{X}}$.

The mentioned application to the Kadison-Kaplansky conjecture is now an immediate consequence of the commutative diagram in (6), Theorem B, and Proposition 1.5.

Corollary 1.6. — *Suppose that G is a discrete countable torsion-free group such that the assembly map $\mu : \mathrm{RKK}_*^G(C_0(EG), \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C_r^*(G))$ is surjective. Then every idempotent $e \in C_r^*(G)$ is either equal to 0 or 1.*

2. Miščenko module — Proof of Theorem A

Recall that G is assumed to be a discrete countable group acting freely, properly, and cocompactly on a second-countable, locally compact Hausdorff space $\tilde{X}$.

We start this section by taking a closer look at the dual Green-Julg isomorphism

$$J_{\tilde{X}} : \mathrm{KK}_*^G\big(C_0(\tilde{X}), \mathbb{C}\big) \to \mathrm{KK}_*(C(X), \mathbb{C}).$$

This map is defined as the composition of two isomorphisms:

$$\begin{aligned} \psi : \mathrm{KK}_*^G\big(C_0(\tilde{X}), \mathbb{C}\big) &\to \mathrm{KK}_*\big(C_0(\tilde{X}) \rtimes_r G, \mathbb{C}\big) \qquad \text{and} \qquad (7) \\ [Y^*]\widehat{\otimes}_{C_0(\tilde{X})\rtimes_r G} - : \mathrm{KK}_*\big(C_0(\tilde{X}) \rtimes_r G, \mathbb{C}\big) &\to \mathrm{KK}_*\big(C(X), \mathbb{C}\big). \end{aligned}$$

The second of these isomorphisms is given by taking interior Kasparov product with the C^*-correspondence Y^* from $C(X)$ to $C_0(\tilde{X}) \rtimes_r G$. This C^*-correspondence provides the Morita equivalence between the C^*-algebras $C(X)$ and $C_0(\tilde{X}) \rtimes_r G$ and the homomorphism $[Y^*]\widehat{\otimes}_{C_0(\tilde{X})\rtimes_r G} -$ is thus an isomorphism with inverse given by the interior Kasparov product with the C^*-correspondence Y from $C_0(\tilde{X}) \rtimes_r G$ to $C(X)$:

$$[Y]\widehat{\otimes}_{C(X)} - : \mathrm{KK}_*\big(C(X), \mathbb{C}\big) \to \mathrm{KK}_*\big(C_0(\tilde{X}) \rtimes_r G, \mathbb{C}\big),$$

see Proposition 1.1.

We now explain the first of the two isomorphisms in (7).

Suppose that $\mathcal{H}$ is a countably generated and non-degenerate G-C^*-correspondence from $C_0(\tilde{X})$ to $\mathbb{C}$. Thus, $\mathcal{H}$ is a separable Hilbert space equipped with a unitary G-action $U\colon G \to \mathcal{U}(\mathcal{H})$ and a non-degenerate G-equivariant $*$-homomorphism $\pi\colon C_0(\tilde{X}) \to B(\mathcal{H})$. The left actions of $C_0(\tilde{X})$ and U combine into a non-degenerate left action of the crossed product $C_0(\tilde{X}) \rtimes_r G$ on $\mathcal{H}$ in the following way:

$$\widetilde{\pi}(f\lambda_g)\xi = (\pi(f) \circ U(g))\xi, \qquad f \in C_0(\tilde{X})\,,\ \xi \in \mathcal{H}. \tag{8}$$

Remark that the $*$-homomorphism $\widetilde{\pi} : C_0(\tilde{X}) \rtimes_r G \to B(\mathcal{H})$ is indeed well-defined since the reduced crossed product $C_0(\tilde{X}) \rtimes_r G$ agrees with the full crossed product $C_0(\tilde{X}) \rtimes G$ in our setting, see Remark 1.2. Hence we get a C^*-correspondence $\widetilde{\mathcal{H}}$ from $C_0(\tilde{X}) \rtimes_r G$ to $\mathbb{C}$.

The isomorphism

$$\psi : \mathrm{KK}^G_* \big(C_0(\tilde{X}), \mathbb{C}\big) \to \mathrm{KK}_* \big(C_0(\tilde{X}) \rtimes_r G, \mathbb{C}\big)$$

is now defined by the formula $\psi\big([\mathcal{H}, F]\big) = [\widetilde{\mathcal{H}}, F]$ (the representation is omitted in this notation for Kasparov modules). We need to explain why ψ is an isomorphism and this is most conveniently done by providing an explicit inverse.

Indeed, suppose on the other hand that $\mathcal{K}$ is a countably generated and non-degenerate C^*-correspondence from $C_0(\tilde{X}) \rtimes_r G$ to $\mathbb{C}$. We denote the left action by $\rho : C_0(\tilde{X}) \rtimes_r G \to B(\mathcal{K})$. Since G is a countable discrete group we have the inclusion $i : C_0(\tilde{X}) \to C_0(\tilde{X}) \rtimes_r G$ and we thus obtain the non-degenerate left action $\widehat{\rho} = \rho \circ i : C_0(\tilde{X}) \to B(\mathcal{K})$. Moreover, we obtain a group homomorphism $V : G \to \mathcal{U}(\mathcal{K})$ by defining

$$V(g)(\xi) = \lim_{n\to\infty} \rho(f_n\lambda_g)(\xi), \qquad g \in G\,,\ \xi \in \mathcal{K}$$

for some countable approximate identity $\{f_n\}_{n\in\mathbb{N}}$ for the σ-unital C^*-algebra $C_0(\tilde{X})$. This data provides us with a G-C^*-correspondence $\widehat{\mathcal{K}}$ from $C_0(\tilde{X})$ to $\mathbb{C}$.

The inverse

$$\psi^{-1} : \mathrm{KK}_* \big(C_0(\tilde{X}) \rtimes_r G, \mathbb{C}\big) \to \mathrm{KK}^G_* \big(C_0(\tilde{X}), \mathbb{C}\big)$$

is then defined by the formula $\psi^{-1}\big([\mathcal{K}, F]\big) = [\widehat{\mathcal{K}}, F]$.

We are now going to provide a slightly better description of the inverse

$$J^{-1}_{\tilde{X}} = \psi^{-1} \circ \big([Y]\widehat{\otimes}_{C(X)} -\big) : \mathrm{KK}_* \big(C(X), \mathbb{C}\big) \to \mathrm{KK}^G_* \big(C_0(\tilde{X}), \mathbb{C}\big)$$

to the dual Green-Julg isomorphism. As above, using that the left action of $C_0(\tilde{X}) \rtimes_r G$ on Y is non-degenerate and that the elements in $C(X)$ are G-invariant, we obtain a non-degenerate G-C^*-correspondence

$$Z = \widehat{Y}$$

from $C_0(\tilde{X})$ to $C(X)$. In fact, recalling that Y is obtained as a completion of $C_c(\tilde{X})$, we see that the left action of $C_0(\tilde{X})$ on Z comes from the multiplication operation in

$C_0(\tilde{X})$ and that the G-action on Z is induced by

$$V(g)(\xi)(x) = \xi(x \cdot g), \qquad g \in G\,,\ \xi \in C_c(\tilde{X})\,,\ x \in \tilde{X}. \tag{9}$$

When considered as right Hilbert C^*-modules over $C(X)$, Z and Y agree. Remark in particular that $C_0(\tilde{X})$ acts as compact operators on Z so that Z determines a class $[Z] = [Z, 0] \in \mathrm{KK}_0^G(C_0(\tilde{X}), C(X))$.

Proposition 2.1. — *We have the formula*

$$J_{\tilde{X}}^{-1} = [Z]\widehat{\otimes}_{C(X)} - : \mathrm{KK}_*\left(C(X), \mathbb{C}\right) \to \mathrm{KK}_*^G\left(C_0(\tilde{X}), \mathbb{C}\right)$$

for the inverse to the dual Green-Julg isomorphism

$$J_{\tilde{X}} : \mathrm{KK}_*^G\left(C_0(\tilde{X}), \mathbb{C}\right) \to \mathrm{KK}_*\left(C(X), \mathbb{C}\right).$$

Proof. — This follows immediately by noting that

$$Z\widehat{\otimes}_{C(X)}\mathcal{H} = \widehat{Y}\widehat{\otimes}_{C(X)}\mathcal{H} = \widehat{Y\widehat{\otimes}_{C(X)}\mathcal{H}},$$

whenever $\mathcal{H}$ is a ($\mathbb{Z}/2\mathbb{Z}$-graded) countably generated C^*-correspondence from $C(X)$ to $\mathbb{C}$. Indeed, for a Kasparov module $(\mathcal{H}, F_2)$ from $C(X)$ to $\mathbb{C}$, the interior Kasparov product

$$[Y, 0]\widehat{\otimes}_{C(X)}[\mathcal{H}, F_2] \in \mathrm{KK}_*(C_0(\tilde{X}) \rtimes_r G, \mathbb{C})$$

is represented by any Kasparov module $(Y\widehat{\otimes}_{C(X)}\mathcal{H}, F)$ from $C_0(\tilde{X}) \rtimes_r G$ to $\mathbb{C}$, where F is an F_2-connection [**9**, Definition 18.3.1]. We thus have that

$$J_{\tilde{X}}^{-1}\big([\mathcal{H}, F_2]\big) = \big[\widehat{Y\widehat{\otimes}_{C(X)}\mathcal{H}}, F\big] = \big[Z\widehat{\otimes}_{C(X)}\mathcal{H}, F\big].$$

But the G-equivariant Kasparov module $(Z\widehat{\otimes}_{C(X)}\mathcal{H}, F)$ from $C_0(\tilde{X})$ to $\mathbb{C}$ clearly represents the G-equivariant interior Kasparov product $[Z, 0]\widehat{\otimes}_{C(X)}[\mathcal{H}, F_2]$ since F is still an F_2-connection. □

The proposition above establishes the first half of Theorem A. In order to proceed with the second half, we need a more concrete description of the module of sections associated to the Miščenko line bundle. To this end, we use a proposition found in [**14**, page 102]. We present the details here since they are omitted in [**14**].

Let us choose a finite open cover $\{V_i\}_{i=1}^N$ of X together with a local trivialization $\phi_i : p^{-1}(V_i) \to V_i \times G$ for each $i \in \{1, 2, \ldots, N\}$. The transition map $\phi_i \circ \phi_j^{-1}$ can then be identified with a continuous map $g_{ij} : V_i \cap V_j \to G$ for each $i, j \in \{1, 2, \ldots, N\}$. Notice that since G is discrete each g_{ij} is in fact locally constant and we may thus make sense of the element $\lambda_{g_{ij}} \in C(V_i \cap V_j, C_r^*(G))$.

Proposition 2.2. — *The Miščenko module M is the finitely generated projective Hilbert C^*-module, described as the completion of $C_c(\tilde{X})$ with respect to the norm* *induced by the following $C(X)\,\widehat{\otimes}\,C_r^*(G)$-valued inner product:*

$$\langle \xi | \zeta \rangle (t)(x) = \sum_{p(y)=x} \bar{\xi}(y)\zeta(y\cdot t), \tag{10}$$

where $\xi, \zeta \in C_c(\tilde{X})$, $t \in G$, $x \in X$ and $p\colon \tilde{X} \to X$ is the quotient map. The right action of $C(X)\,\widehat{\otimes}\,C_r^*(G)$ on M is defined by

$$(\xi \cdot f)(y) = \sum_{g\in G} f(g)(p(y))\cdot \xi(y\cdot g^{-1}), \tag{11}$$

where $\xi \in C_c(\tilde{X})$, $f \in C_c(G, C(X))$ and $y \in \tilde{X}$.

Proof. — Choose a partition of unity $\{\chi_i\}_{i=1}^N$ such that $\mathrm{supp}(\chi_i) \subseteq V_i$ for all $i \in \{1,2,\ldots,N\}$. For each $i \in \{1,2,\ldots,N\}$ we then define the compactly supported continuous function

$$\rho_i(y) = \begin{cases} (\chi_i \circ p)(y) & \text{for} \quad p(y) \in V_i \quad \text{and} \quad \phi_i(y) = (p(y), e) \\ 0 & \text{elsewhere} \end{cases}$$

on $\tilde{X}$. The following computation shows that the elements $\{\sqrt{\rho_i}\}_{i=1}^N$ form a finite frame for M, see [**15**, Theorem 4.1]. Indeed, for each $j \in \{1,2,\ldots,N\}$ and $y \in p^{-1}(V_j)$ with $\phi_j(y) = (x,h)$ (for some $x \in V_j$ and $h \in G$) we have that

$$\begin{aligned} \sqrt{\rho_j}\left\langle \sqrt{\rho_j} \middle| \xi \right\rangle (y) &= \sum_{g\in G} \left\langle \sqrt{\rho_j} \middle| \xi \right\rangle (g)(x) \cdot \sqrt{\rho_j}(y\cdot g^{-1}) \\ &= \left\langle \sqrt{\rho_j} \middle| \xi \right\rangle (h)(x) \cdot \sqrt{\chi_j}(x) = \chi_j(x)\xi(y). \end{aligned}$$

From this we see that $\xi = \sum_{i=1}^N \sqrt{\rho_i}\left\langle \sqrt{\rho_i} \middle| \xi \right\rangle$ for all $\xi \in C_c(\tilde{X})$.

But then the projection associated to M takes the form $\big(p_{C_r^*(G)}\big)_{ij} = \left\langle \sqrt{\rho_i} \middle| \sqrt{\rho_j} \right\rangle$ and we compute

$$\begin{aligned} \left\langle \sqrt{\rho_i} \middle| \sqrt{\rho_j} \right\rangle (g)(x) &= \sqrt{\rho_i}(\phi_i^{-1}(x,e)) \cdot \sqrt{\rho_j}(\phi_i^{-1}(x,e)\cdot g) \\ &= \begin{cases} \sqrt{\chi_i \cdot \chi_j}(x) & \text{for} \quad g = g_{ij}(x) \\ 0 & \text{for} \quad g \neq g_{ij}(x) \end{cases}, \end{aligned}$$

whenever $x \in V_i \cap V_j$ and $g \in G$. We thus obtain that

$$\left\langle \sqrt{\rho_i} \middle| \sqrt{\rho_j} \right\rangle = \sqrt{\chi_i \chi_j} \cdot \lambda_{g_{ij}}.$$

It is now clear that the projection $p_{C_r^*(G)} \in M_N\big(C(X, C_r^*(G))\big)$ describes the module of sections of the hermitian bundle of C^*-algebras $\tilde{X} \times_G C_r^*(G) \to X$. □

We recall that $Z = \widehat{Y}$ is a non-degenerate G-C^*-correspondence from $C_0(\tilde{X})$ to $C(X)$. Remark that the action of G on $C(X)$ is the trival action and the reduced crossed product $Z \rtimes_r G$ is therefore a C^*-correspondence from $C_0(\tilde{X}) \rtimes_r G$ to $C(X)\widehat{\otimes} C_r^*(G)$. Before proving the second half of Theorem A we recall the formulae for the inner product and the left and right actions on the reduced crossed product $Z \rtimes_r G$. The $C(X)\widehat{\otimes}C_r^*(G)$-valued inner product is defined by

$$\langle \xi | \zeta \rangle = \sum_{g \in G} \langle \xi(g) | \zeta(gt) \rangle \,, \tag{12}$$

where $\xi, \zeta \in C_c(G, Z)$ and $t \in G$. The right action of $C(X)\widehat{\otimes}C_r^*(G)$ is determined by the formula

$$(\xi \cdot f)(t) = \sum_{g \in G} \xi(g) \cdot f(g^{-1}t),$$

where $\xi \in C_c(G, Z), f \in C_c(G, C(X))$ and $t \in G$. The left action of $C_0(\tilde{X}) \rtimes_r G$ on $Z \rtimes_r G$ is determined by

$$(f \cdot \xi)(t) = \sum_{g \in G} f(g) \cdot V(g)(\xi(g^{-1}t)), \tag{13}$$

where $f \in C_c(G, C_0(\tilde{X})), \xi \in C_c(G, Z)$ and $t \in G$.

Proposition 2.3. — *The following map extends to an isomorphism of Hilbert C^*-modules,*

$$\begin{aligned} Y^* \,\widehat{\otimes}_{C_0(\tilde{X}) \rtimes_r G} \,(Z \rtimes_r G) &\xrightarrow{\Phi} M \\ \langle \xi | \otimes \zeta &\mapsto \sum_{g \in G} V(g^{-1})\big(\overline{\xi} \cdot \zeta(g)\big), \end{aligned} \tag{14}$$

where $\xi \in C_c(\tilde{X}), \zeta \in C_c(G, C_c(\tilde{X}))$.

In particular, we have the KK*-theoretic identity*

$$[M] = \iota^*[Y^*] \,\widehat{\otimes}_{C_0(\tilde{X}) \rtimes_r G} \, \jmath_r^G[Z] \in \mathrm{KK}_0(\mathbb{C}, C(X) \,\widehat{\otimes}\, C_r^*(G)).$$

Proof. — The result at the level of KK-theory follows in a straightforward way from the isomorphism in (14), hence we turn to the proof of this isomorphism.

It suffices to check that $\Phi : C_c(\tilde{X})^* \otimes C_c(G, C_c(\tilde{X})) \to M$ (defined on the algebraic tensor product over $\mathbb{C}$) has dense image and preserves the relevant inner products.

The fact that Φ has dense image follows since for any $\phi \in C_c(\tilde{X}) \subseteq M$ we may find a $\psi \in C_c(\tilde{X})$ such that $\psi \cdot \phi = \phi$ (using the pointwise product here). We then have

$$\Phi(\langle \overline{\psi} | \otimes \phi \cdot \lambda_e) = \psi \cdot \phi = \phi.$$

To check that Φ preserves the inner products we let $\xi_1, \xi_2 \in C_c(\tilde{X})$ and $\zeta_1, \zeta_2 \in C_c(G, C_c(\tilde{X}))$ and compute, for each $x \in X$ and $t \in G$,

$$\begin{aligned} \langle \langle \xi_1 | \otimes \zeta_1 | \langle \xi_2 | \otimes \zeta_2 \rangle \,(t)(x) &= \sum_{g,h \in G} \big\langle \zeta_1(g) \big| \xi_1 \cdot V(h)\big(\overline{\xi_2} \cdot \zeta_2(h^{-1}gt)\big) \big\rangle \,(x) \\ &= \sum_{g,h \in G} \sum_{p(y)=x} \big(\overline{\zeta_1(g)} \cdot \xi_1\big)(y) \cdot \big(\overline{\xi_2} \cdot \zeta_2(h^{-1}gt)\big)(y \cdot h), \end{aligned}$$

where we are using Equation (1), (3), (12) and (13). On the other hand, we have that

$$\begin{aligned}
&\langle \Phi(\langle \xi_1| \otimes \zeta_1)|\Phi(\langle \xi_2| \otimes \zeta_2)\rangle\,(t)(x)\\
&\quad= \sum_{s,r\in G} \big\langle V(s^{-1})(\overline{\xi_1}\cdot\zeta_1(s))\big|V(r^{-1})(\overline{\xi_2}\cdot\zeta_2(r))\big\rangle\,(t)(x)\\
&\quad= \sum_{s,r\in G}\sum_{p(z)=x} (\xi_1\cdot\overline{\zeta_1(s)})(z\cdot s^{-1})\cdot(\overline{\xi_2}\cdot\zeta_2(r))(z\cdot t\cdot r^{-1}),
\end{aligned}$$

where we are using Equation (10). After a few changes of variables, we obtain that

$$\langle\langle \xi_1| \otimes \zeta_1|\langle \xi_2| \otimes \zeta_2\rangle\,(t)(x) = \langle \Phi(\langle \xi_1| \otimes \zeta_1)|\Phi(\langle \xi_2| \otimes \zeta_2)\rangle\,(t)(x)$$

and this ends the proof of the proposition. □

Remark 2.4. — *The previous proposition can be interpreted as a particular case of* [**11**, Proposition 3.6].

Corollary 2.5. — *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathrm{KK}_*(C_0(\tilde{X}),\mathbb{C}) & \xrightarrow{\mu_{\tilde{X}}} & \mathrm{KK}_*(\mathbb{C},C_r^*(G))\\
\Big\downarrow{\scriptstyle J_{\tilde{X}}} & \nearrow{\scriptstyle \eta_{\tilde{X}}} & \\
\mathrm{KK}_*(C(X),\mathbb{C}). & &
\end{array} \tag{15}$$

Proof. — Suppose x is in $\mathrm{KK}_*^G(C_0(\tilde{X}),\mathbb{C})$. Then by Proposition 2.1 there is a $y\in \mathrm{KK}_*(C(X),\mathbb{C})$ with

$$x = J_{\tilde{X}}^{-1}(y) = [Z]\,\widehat{\otimes}_{C(X)}\,y.$$

By functoriality of descent [**18**, page 172], we obtain that

$$\jmath_r^G(x) = \jmath_r^G([Z]\,\widehat{\otimes}_{C(X)}\,y) = \jmath_r^G([Z])\widehat{\otimes}_{C(X)\widehat{\otimes}C_r^*(G)}\,\jmath_r^G(y).$$

Note that, since G acts trivially on both $C(X)$ and $\mathbb{C}$, we have $C(X)\rtimes_r G\cong C(X)\,\widehat{\otimes}\,C_r^*(G)$ and

$$\jmath_r^G(y) = \tau_{C_r^*(G)}(y).$$

We thus see that,

$$\begin{aligned}
\mu_{\tilde{X}}(x) &= \iota^*[Y^*]\widehat{\otimes}_{C_0(\tilde{X})\rtimes_r G}\,\jmath_r^G(x)\\
&= \iota^*[Y^*]\,\widehat{\otimes}_{C_0(\tilde{X})\rtimes_r G}\,\jmath_r^G([Z])\,\widehat{\otimes}_{C(X)\,\widehat{\otimes}\,C_r^*(G)}\,\tau_{C_r^*(G)}(y).
\end{aligned}$$

Applying Proposition 2.3, the expression above simplifies to

$$\mu_{\tilde{X}}(x) = [M]\,\widehat{\otimes}_{C(X)\widehat{\otimes}C_r^*(G)}\,\tau_{C_r^*(G)}(y) = \eta_{\tilde{X}}(y).$$

Hence we have the identity $\mu_{\tilde{X}}(x) = \eta_{\tilde{X}}(J_{\tilde{X}}(x))$ and this proves the corollary. □

3. Chern characters and flat bundles

Throughout this section A will be a unital C^*-algebra equipped with a faithful tracial state $\phi : A \to \mathbb{C}$ and X will be a compact Hausdorff space.

We consider the unital C^*-algebra $C(X, A) \cong C(X)\widehat{\otimes} A$ of continuous A-valued maps on X.

For every positive integer $n \geq 0$, we will construct an explicit Chern character

$$\mathrm{Ch}_\phi^{2n} : K_0\big(C(X, A)\big) \to H^{2n}(X, \mathbb{R})$$

with values in the Alexander-Spanier cohomology of X. In the case where $A = \mathbb{C}$, we recover the explicit version of the usual Chern character

$$\mathrm{Ch}^{2n} : K_0(C(X)) \to H^{2n}(X, \mathbb{R})$$

discovered in [**16**].

3.1. Reminders on Alexander-Spanier cohomology.

— Here is a short summary of how Alexander-Spanier cohomology is defined. For more details, we point the reader to [**27**, Chapter 6].

Let X be a compact Hausdorff space. Let $\mathrm{Cov}(X)$ denote the set of all finite open coverings of X, and let $\mathfrak{U} \in \mathrm{Cov}(X)$. For each $k \in \mathbb{N} \cup \{0\}$, let $\mathfrak{U}^k$ denote the open neighborhood of the diagonal in X^k given by $\cup_{U\in\mathfrak{U}} U^k$, where the superscript k indicates the k^{th} Cartesian power.

The real vector space of Alexander-Spanier k-cocycles (corresponding to the finite open cover $\mathfrak{U}$) is denoted by $C^k(X, \mathfrak{U})$ and is made of continuous real valued functions on $\mathfrak{U}^{k+1}$. The coboundary map $\partial : C^k(X, \mathfrak{U}) \to C^{k+1}(X, \mathfrak{U})$ is defined by the formula

$$\partial f(x_0, x_1, \ldots, x_{k+1}) = \sum_{j=0}^{k+1} (-1)^j f(x_0, \ldots, \widehat{x_j}, \ldots, x_{k+1}), \tag{16}$$

where $f \in C^k(X, \mathfrak{U})$ and the notation $\widehat{\cdot}$ means the term has been omitted.

It can be shown that $\partial^2 = 0$ and the cohomology of the cochain complex $(C^*(X, \mathfrak{U}), \partial)$ is called the Alexander-Spanier cohomology of the covering $\mathfrak{U}$. It is denoted $H^*(X, \mathfrak{U})$.

If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, there is an obvious cochain map (restriction) from $C^*(X, \mathfrak{U})$ to $C^*(X, \mathfrak{V})$, which defines a map $H^*(X, \mathfrak{U}) \to H^*(X, \mathfrak{V})$.

The (real-valued) Alexander-Spanier cohomology of X is defined as a direct limit over finite open covers:

$$H^*(X, \mathbb{R}) = \varinjlim H^*(X, \mathfrak{U}).$$

These cohomology groups are vector spaces over the real numbers.

3.2. Construction of the Chern character. — The construction outlined here is entirely based on [**16**]. We simply provide a minor generalization of those ideas incorporating the faithful tracial state $\phi : A \to \mathbb{C}$.

Let us fix a positive integer $n \geq 0$. Let $p \in M_m(C(X, A))$ be a projection for some positive integer $m \geq 0$ and choose a finite open cover $\mathfrak{U}$ of X such that

$$\|p(x) - p(x')\| \leq 1/4 \qquad \forall U \in \mathfrak{U}\,,\ x, x' \in U.$$

We now construct an Alexander-Spanier $2n$-cocycle

$$\mathrm{Ch}^{2n}_{\phi}(p) \in C(\mathfrak{U}^{2n+1}, \mathbb{R}) = C^{2n}(X, \mathfrak{U}),$$

which will represent the Chern character in degree $2n$.

For $n = 0$ we put $\mathrm{Ch}^0_{\phi}(p)(x) = \phi(p(x))$ for all $x \in X$, where the trace

$$\phi : M_m(A) \to \mathbb{C}$$

is given by the formula $\phi(a) = \sum_{i=1}^m \phi(a_{ii})$ for all $a \in M_m(A)$. We remark that $\mathrm{Ch}^0_{\phi}(p)$ is constant on every $U \in \mathfrak{U}$ since $p(x)$ and $p(x')$ are similar for $x, x' \in U$. In particular, we see that the continuous map $\mathrm{Ch}^0_{\phi}(p) : X \to \mathbb{R}$ defines an Alexander-Spanier 0-cocycle.

We now consider the case where $n \geq 1$. For every integer $k \geq 1$, we let Δ^k denote the k-simplex

$$\Delta^k = \big\{(t_1, t_2, \ldots, t_k) \in [0,1]^k \mid \sum_{i=1}^k t_i \leq 1\big\}.$$

Let $x = (x_0, x_1, \ldots, x_{2n}) \in \mathfrak{U}^{2n+1}$. For every $t \in \Delta^{2n}$, we define

$$a_p(x,t) = p(x_0) + \sum_{i=1}^{2n} t_i(p(x_i) - p(x_0))$$

and remark that $\|a_p(x,t) - p(x_0)\| \leq 1/4$, in particular we have a well-defined spectral projection

$$e_p(x,t) = \frac{1}{2\pi i} \int_{|\lambda - 1| = 1/2} (\lambda - a_p(x,t))^{-1}\, d\lambda \in M_m(A), \tag{17}$$

where the circle of radius $1/2$ appearing in the formula is given the usual counterclockwise orientation.

The Alexander-Spanier $2n$-cocycle

$$\mathrm{Ch}^{2n}_{\phi}(p) \in C(\mathfrak{U}^{2n+1}, \mathbb{R})$$

is then defined by the explicit formula

$$\mathrm{Ch}^{2n}_{\phi}(p)(x) = \frac{(-1)^n}{n!} \int_{\Delta^{2n}} \phi\big(e_p(x,t) d(e_p(x,t)) \wedge \ldots \wedge d(e_p(x,t))\big) \qquad x \in \mathfrak{U}^{2n+1},$$

where the $2n$-simplex $\Delta^{2n} \subseteq \mathbb{R}^{2n}$ is given the orientation coming from the form $dt_1 \wedge dt_2 \wedge \ldots \wedge dt_{2n}$. Notice that the exterior derivative d appearing in the above expression only differentiates in the direction of the standard simplex Δ^{2n}.

The proof of the next lemma is almost identical to the proofs of [**16**, Lemma 8 & Lemma 9] and will not be given here.

Lemma 3.1. — *The cochain* $\mathrm{Ch}_\phi^{2n}(p) \in C(\mathfrak{U}^{2n+1}, \mathbb{R})$ *is an Alexander-Spanier cocycle and the class* $\big[\mathrm{Ch}_\phi^{2n}(p)\big] \in H^{2n}(X, \mathbb{R})$ *in Alexander-Spanier cohomology only depends on the class of* p *in the abelian semigroup* $V(C(X,A))$*, whose Grothendieck completion gives* $K_0(C(X,A))$*.*

It follows from the above lemmas that we have a well-defined map

$$\mathrm{Ch}_\phi^{2n} : V(C(X,A)) \to H^{2n}(X, \mathbb{R})$$

and it can be verified that this map is a homomorphism, thus that

$$[\mathrm{Ch}_\phi^{2n}(p \oplus q)] = [\mathrm{Ch}_\phi^{2n}(p)] + [\mathrm{Ch}_\phi^{2n}(q)],$$

whenever $p \in M_m(C(X,A))$ and $q \in M_{m'}(C(X,A))$ are projections.

In particular, we have the following:

Definition 3.2. — The Chern character in degree $2n$ associated to the faithful tracial state $\phi : A \to \mathbb{C}$ and the compact Hausdorff space X is the homomorphism of abelian groups

$$\mathrm{Ch}_\phi^{2n} : K_0(C(X,A)) \to H^{2n}(X, \mathbb{R}), \qquad \mathrm{Ch}_\phi([p] - [q]) = [\mathrm{Ch}_\phi^{2n}(p)] - [\mathrm{Ch}_\phi^{2n}(q)].$$

3.3. Multiplicative properties. — We let

$$\times : K_0(C(X)) \otimes_{\mathbb{Z}} K_0(A) \to K_0\big(C(X,A)\big)$$

denote the exterior product. Recall that for projections $p \in M_m(C(X))$ and $q \in M_{m'}(A)$, the exterior product

$$[p] \times [q] \in K_0\big(C(X,A)\big)$$

is represented by the projection $p \otimes q \in M_{m \cdot m'}(C(X,A)) \cong M_{m'}\big(M_m(C(X,A))\big)$ given by the block-matrix

$$(p \otimes q)_{ij} = p \cdot q_{ij} \qquad i, j \in \{1, \ldots, m'\}.$$

We recall that the faithful tracial state $\phi : A \to \mathbb{C}$ induces a homomorphism

$$\phi_* : K_0(A) \to \mathbb{R}, \qquad \phi_*([p] - [q]) = \phi(p - q).$$

Lemma 3.3. — *For every positive integer $n \geq 0$, we have the commutative diagram*

$$\begin{array}{ccc} K_0(C(X)) \otimes_{\mathbb{Z}} K_0(A) & \xrightarrow{\times} & K_0\big(C(X,A)\big) \\ {\scriptstyle 1\otimes\phi_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ch}^{2n}_{\phi}} \\ K_0(C(X)) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{(\mathrm{Ch}^{2n}\otimes 1)} & H^{2n}(X,\mathbb{R}) \end{array}$$

Proof. — Given $p \in M_m(C(X))$ and $q \in M_{m'}(A)$, the commutativity of the diagram follows from the identity

$$[\mathrm{Ch}^{2n}(p)] \cdot \phi(q) = [\mathrm{Ch}^{2n}_{\phi}(p \otimes q)].$$

We shall in fact see that this identity holds at the level of cochains. We focus on the case where $n \geq 1$. In this situation, it suffices to show that

$$e_{p\otimes q}(x,t) = e_p(x,t) \otimes q, \tag{18}$$

for all $x \in \mathfrak{U}^{2n+1}$ and all $t \in \Delta^{2n}$. Indeed, if Equation (18) were true, then from Equation (17) we would have that

$$\begin{aligned} \mathrm{Ch}^{2n}_{\phi}(p \otimes q)(x) &= \frac{(-1)^n}{n!} \int_{\Delta^{2n}} \mathrm{Tr}\big(e_p(x,t) d(e_p(x,t)) \wedge \ldots \wedge d(e_p(x,t))\big) \cdot \phi(q) \\ &= \mathrm{Ch}^{2n}(p)(x) \cdot \phi(q), \end{aligned}$$

for all $x \in \mathfrak{U}^{2n+1}$, where $\mathrm{Tr} : M_m(\mathbb{C}) \to \mathbb{C}$ denotes the matrix trace (without normalization). Now, for each $\lambda \in \mathbb{C}$ with $|\lambda - 1| = 1/2$, it is easily verified that

$$(\lambda - a_{p\otimes q}(x,t))^{-1} = (\lambda - a_p(x,t))^{-1} \otimes q + \frac{1}{\lambda} \otimes (1-q).$$

The identity above is exactly what we need, since the function $\frac{1}{\lambda} \otimes (1-q)$ is analytic on an open set containing $\{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq 1/2\}$, and therefore its contour integral along the boundary of that disk is zero. □

Proposition 3.4. — *Suppose that A is a II_1-factor. Then the Chern character*

$$\mathrm{Ch}_{\phi} : K_0\big(C(X,A)\big) \to \oplus_{n=0}^{\infty} H^{2n}(X,\mathbb{R}) \qquad \mathrm{Ch}_{\phi}(x) = \{\mathrm{Ch}^{2n}_{\phi}(x)\}$$

is an isomorphism.

The previous proposition is proved in [**25**, Theorem 5.7] in the smooth setting by using Chern-Weil theory, see also [**2**, Diagram 3.7].

Proof. — Since A is a II_1-factor, it follows as in [**8**, III.1.7.9, page 242], that the faithful tracial state $\phi : A \to \mathbb{C}$ induces an isomorphism $K_0(A) \cong \mathbb{R}$ of abelian groups. Moreover, since A is a von Neumann algebra, we have that $K_1(A) \cong \{0\}$. Since $\mathbb{R}$ is torsion-free, the exterior product

$$\times : K_0(C(X)) \otimes_{\mathbb{Z}} K_0(A) \to K_0(C(X,A))$$

is an isomorphism by the Künneth theorem, see [**26**, Proposition 2.11]. Therefore it suffices to show that the composition

$$\mathrm{Ch}_\phi \circ \times : K_0(C(X)) \otimes_{\mathbb{Z}} K_0(A) \to \oplus_{n=0}^{\infty} H^{2n}(X, \mathbb{R})$$

is an isomorphism. However, by Lemma 3.3 we have that

$$(\mathrm{Ch}_\phi \circ \times) = (\mathrm{Ch} \otimes 1) \circ (1 \otimes \phi_*),$$

where $\mathrm{Ch} : K_0(C(X)) \to \oplus_{n=0}^{\infty} H^{2n}(X, \mathbb{R})$ is the usual Chern character with values in Alexander-Spanier cohomology. This ends the proof of the proposition since Ch becomes an isomorphism after tensorizing with $\mathbb{R}$, see [**17**]. □

3.4. Flat bundles. — We now consider a flat bundle over the compact Hausdorff space X with fiber a finitely generated projective module qA^m over the unital C^*-algebra A, thus $q \in M_m(A)$ is a projection. We thus fix an open cover $\{V_i\}_{i=1}^N$ of the compact Hausdorff space X together with locally constant maps

$$g_{ij} : V_i \cap V_j \to U(qA^m) \qquad i, j \in \{1, \ldots, N\},$$

for some fixed $m \in \mathbb{N}$, where $U(qA^m)$ denotes the group of unitary transformations of the Hilbert C^*-module qA^m. We identify $U(qA^m)$ with the group of $(m \times m)$-matrices u satisfying the conditions

$$qu = u = uq \qquad \text{and} \qquad u^* u = uu^* = q.$$

Our locally constant maps are supposed to satisfy the cocycle condition:

$$\begin{aligned} &g_{ii} = q \qquad \text{and} \\ &g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x) \qquad \forall x \in V_i \cap V_j \cap V_k, \end{aligned}$$

whenever $i, j, k \in \{1, \ldots, N\}$.

Let us choose a partition of unity $\{\chi_i \mid i = 1, \ldots, N\}$ for X with $\mathrm{supp}(\chi_i) \subseteq V_i$ for all $i \in \{1, \ldots, N\}$. Our cocycle then gives rise to a projection $p_A \in M_{N \cdot m}(C(X, A)) \cong M_N(M_m(C(X, A)))$ defined as the block-matrix

$$(p_A)_{ij} = \sqrt{\chi_i \chi_j} \cdot g_{ij} \qquad i, j \in \{1, \ldots, N\}.$$

Finally, we have the projection $p \in M_N(C(X, A))$ defined as the matrix

$$p_{ij} = \sqrt{\chi_i \chi_j} \qquad i, j \in \{1, \ldots, N\}.$$

We are going to prove the following:

Theorem 3.1. — *We have the identity*

$$\mathrm{Ch}_\phi^{2n}([p_A]) = \mathrm{Ch}_\phi^{2n}([q]) = \begin{cases} 0 & \text{for} \quad n > 0 \\ [\phi(q)] & \text{for} \quad n = 0 \end{cases},$$

where $[\phi(q)] \in H^0(X, \mathbb{R})$ refers to the class in Alexander-Spanier cohomology associated to the constant function on X equal to $\phi(q) \in [0, \infty)$ at every point.

When the base space is a compact manifold without boundary, the previous result is proved in [**25**, Theorem 5.8] and [**2**, Section 4].

The more general case where the base space is just a compact Hausdorff space requires extra care. We start with a technical lemma.

Lemma 3.5. — *Suppose that $\mathfrak{K} = \{K_1, K_2, \ldots, K_l\}$ is a finite set of closed subsets of the compact Hausdorff space X. Then there exists a finite open cover $\mathfrak{U}$ of X such that the implication*

$$\Big((U \cap K_i) \neq \emptyset \ \forall i \in I\Big) \Rightarrow \Big(\bigcap_{i\in I} K_i \neq \emptyset\Big)$$

holds for all subsets $I \subseteq \{1, 2, \ldots, l\}$ and all $U \in \mathfrak{U}$.

Proof. — The case where $\mathfrak{K}$ is empty is trivial, so we suppose that $l = \sharp\mathfrak{K} \geq 1$.

Define the set

$$\mathfrak{A} = \big\{ \cap_{i\in I} K_i \mid I \subseteq \{1, 2, \ldots, l\}\,,\ I \neq \emptyset \big\} \cup \{\emptyset\}.$$

Let $n \in \mathbb{N}$ and suppose that $C_1, C_2, \ldots, C_n \in \mathfrak{A}$ and that $U_1, U_2, \ldots, U_n \subseteq X$ are open subsets such that

1. $C_1 = U_1 = \emptyset$;
2. $C_j \in \mathfrak{A} \setminus \{C_1, C_2, \ldots, C_{j-1}\}$ for all $j \in \{2, 3, \ldots, n\}$;
3. it holds for all $K \in \mathfrak{K}$ and all $j \in \{2, 3, \ldots, n\}$ that
$$C_j \cap K = C_j \qquad \text{or} \qquad C_j \cap K \in \{C_1, C_2, \ldots, C_{j-1}\};$$
4. $C_j \cap (X \setminus \cup_{i=1}^{j-1} U_i) \subseteq U_j$ for all $j \in \{2, 3, \ldots, n\}$;
5. the implication
$$\Big(C_j \cap K \in \{C_1, C_2, \ldots, C_{j-1}\}\Big) \Rightarrow \Big(U_j \cap K = \emptyset\Big)$$
holds for all $K \in \mathfrak{K}$ and all $j \in \{2, 3, \ldots, n\}$.

We remark that $\cup_{i=1}^{j} C_i \subseteq \cup_{i=1}^{j} U_i$ for all $j \in \{1, 2, \ldots, n\}$. Indeed, to see this it suffices to check that $C_j \subseteq \cup_{i=1}^{j} U_i$ for $j \in \{2, 3, \ldots, n\}$. But this is clear since $C_j \cap (X \setminus \cup_{i=1}^{j-1} U_i) \subseteq U_j$ by construction and obviously $C_j \cap (\cup_{i=1}^{j-1} U_i) \subseteq \cup_{i=1}^{j-1} U_i$.

Suppose now that $\mathfrak{A} \setminus \{C_1, C_2, \ldots, C_n\} \neq \emptyset$. We may then choose $C_{n+1} \in \mathfrak{A} \setminus \{C_1, C_2, \ldots, C_n\}$ such that it holds for all $K \in \mathfrak{K}$ that

$$C_{n+1} \cap K = C_{n+1} \qquad \text{or} \qquad C_{n+1} \cap K \in \{C_1, C_2, \ldots, C_n\}.$$

We define

$$L_{n+1} = C_{n+1} \cap \big(X \setminus \cup_{i=1}^{n} U_i\big)$$

and claim that it holds for all $K \in \mathfrak{K}$ that

$$\big(C_{n+1} \cap K \in \{C_1, C_2, \ldots, C_n\}\big) \Rightarrow \big(L_{n+1} \cap K = \emptyset\big)$$

Indeed, if $C_{n+1} \cap K \in \{C_1, C_2, \dots, C_n\}$, then $C_{n+1} \cap K \subseteq \cup_{i=1}^n U_i$ so that

$$K \cap L_{n+1} = K \cap C_{n+1} \cap \big(X \setminus \cup_{i=1}^n U_i\big) = \emptyset.$$

Since X is compact Hausdorff, we may choose an open subset $U_{n+1} \subseteq X$ such that

$$L_{n+1} \subseteq U_{n+1}$$

and such that the implication

$$\big(C_{n+1} \cap K \in \{C_1, C_2, \dots, C_n\}\big) \Rightarrow \big(U_{n+1} \cap K = \emptyset\big)$$

holds for all $K \in \mathfrak{K}$.

Since the set $\mathfrak{A}$ is finite, we may thus inductively construct $C_1, C_2, \dots, C_m \in \mathfrak{A}$ and open subsets $U_1, U_2, \dots, U_m \subseteq X$ satisfying (1) – (5) from above and such that $\mathfrak{A} = \{C_1, C_2, \dots, C_m\}$. We define

$$U_{m+1} = X \setminus \cup_{i=1}^l K_i$$

and claim that $\mathfrak{U} = \{U_i\}_{j=1}^{m+1}$ is the desired open cover.

First of all, we prove that $\mathfrak{U}$ is indeed a cover. To this end, we just need to show that $\cup_{i=1}^l K_i \subseteq \cup_{j=1}^m U_j$, but this is clear since $\cup_{i=1}^l K_i = \cup_{j=1}^m C_j \subseteq \cup_{j=1}^m U_j$.

Now suppose that $I \subseteq \{1, 2, \dots, l\}$ is a non-empty subset, that $U \in \mathfrak{U}$ and that $U \cap K_i \neq \emptyset$ for all $i \in I$. By the definition of U_1 and U_{m+1}, we must have that $U = U_j$ for some $j \in \{2, \dots, m\}$. By property (3) and (5), it thus holds that $C_j \cap K_i = C_j$ for all $i \in I$. But this implies that

$$C_j = C_j \cap \big(\cap_{i \in I} K_i\big) \subseteq \cap_{i \in I} K_i$$

and hence since $C_j \neq \emptyset$ that $\cap_{i \in I} K_i \neq \emptyset$. This proves the lemma. □

For each $i, j \in \{1, \dots, N\}$ with $i \neq j$ define the closed subset

$$K_{ij} = \operatorname{supp}(\chi_i) \cap \operatorname{supp}(\chi_j) \subseteq X$$

and define

$$\mathfrak{K} = \big\{K_{ij} \mid i, j \in \{1, 2, \dots, N\}\big\}.$$

Let $\mathfrak{U}$ be a finite open cover of X satisfying the conclusion of Lemma 3.5. By passing to a refinement we may also arrange that:

- $\|p(x) - p(x')\|$ and $\|p_A(x) - p_A(x')\| \leq 1/4$ for all $U \in \mathfrak{U}$ and $x, x' \in U$;
- for all $U \in \mathfrak{U}$, whenever $V_i \cap V_j \cap U \neq \emptyset$, the map $g_{ij} \colon V_i \cap V_j \cap U \to U(qA^m)$ is constant.

The following lemma is exactly what we need to prove Theorem 3.1.

Lemma 3.6. — *Let $n \geq 1$, let $x = (x_0, x_1, \ldots, x_{2n}) \in \mathfrak{U}^{2n+1}$ and let $t \in \Delta^{2n}$ be given. We have the identity,*

$$\Big(e_{p_A}(x,t)d\big(e_{p_A}(x,t)\big)^{\wedge 2n}\Big)_{ij} = e_p(x,t)d\big(e_p(x,t)\big)^{\wedge 2n} \cdot g_{ij}(x_0),$$

for each $i, j \in \{1, \ldots, N\}$ indexing the $(m \times m)$-block matrices.

Proof. — Let us choose a $U \in \mathfrak{U}$, such that $x = (x_0, x_1, \ldots, x_{2n}) \in U^{2n+1}$. We remark that $g_{ij}(x_s) = g_{ij}(x_0)$ for all $s \in \{0, 1, 2, \ldots, 2n\}$ and all $i, j \in \{1, 2, \ldots, N\}$.

For $\lambda \in \mathbb{C}$ with $|\lambda - 1| = 1/2$, we define

$$\gamma_{p_A}(\lambda, x_0) = p_A(x_0)/(\lambda - 1) + (1 - p_A(x_0))/\lambda \qquad \text{and}$$

$$\delta_{p_A}(x,t) = \sum_{s=1}^{2n} t_s \cdot (p_A(x_s) - p_A(x_0)),$$

so that $a_{p_A}(x,t) = p_A(x_0) + \delta_{p_A}(x,t)$ and

$$(\lambda - p_A(x_0)) \cdot \gamma_{p_A}(\lambda, x_0) = 1.$$

In particular, we have the power-series expansion

$$(\lambda - a_{p_A})^{-1} = \sum_{k=0}^{\infty} (\gamma_{p_A}(\lambda) \cdot \delta_{p_A})^k \gamma_{p_A}(\lambda) \qquad |\lambda - 1| = 1/2, \tag{19}$$

which converges absolutely since $\|\delta_{p_A}\| \leq \frac{1}{4}$. Remark that we are suppressing the point $(x,t) \in U^{2n+1} \times \Delta^{2n}$ and the point $x_0 \in U$ from the notation (and we will often do so below as well).

Notice that the exterior derivative of $(\lambda - a_{p_A})^{-1}$ (again in the direction of the simplex Δ^{2n}) can be easily computed:

$$d\big((\lambda - a_{p_A})^{-1}\big) = \sum_{s=1}^{2n} (\lambda - a_{p_A})^{-1}(p_A(x_s) - p_A(x_0))(\lambda - a_{p_A})^{-1} dt_s.$$

We thus have that

$$d(e_{p_A}) = \frac{1}{2\pi i} \sum_{s=1}^{2n} \int_{|\lambda - 1| = 1/2} (\lambda - a_{p_A})^{-1}(p_A(x_s) - p_A(x_0))(\lambda - a_{p_A})^{-1} \, d\lambda \, dt_s$$

and hence that

$$e_{p_A} d(e_{p_A})^{\wedge 2n} = \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma) \cdot \frac{1}{(2\pi i)^{2n+1}} \int_{|\lambda_0 - 1| = 1/2} \cdots \int_{|\lambda_{2n} - 1| = 1/2}$$
$$(\lambda_0 - a_{p_A})^{-1} \prod_{r=1}^{2n} (\lambda_r - a_{p_A})^{-1}(p_A(x_{\sigma(r)}) - p_A(x_0))(\lambda_r - a_{p_A})^{-1}$$
$$d\lambda_0 \ldots d\lambda_{2n} \, dt_1 \wedge \ldots \wedge dt_{2n},$$

where S_{2n} denotes the group of all permutations σ of $2n$ letters.

Since a similar expression holds when p_A is replaced by p, we may focus on proving the identity

$$\Big((\lambda_0 - a_{p_A})^{-1} \prod_{r=1}^{2n} (\lambda_r - a_{p_A})^{-1} (p_A(x_{\sigma(r)}) - p_A(x_0))(\lambda_r - a_{p_A})^{-1}\Big)_{ij}$$
$$= \Big((\lambda_0 - a_p)^{-1} \prod_{r=1}^{2n} (\lambda_r - a_p)^{-1} (p(x_{\sigma(r)}) - p(x_0))(\lambda_r - a_p)^{-1}\Big)_{ij} \cdot g_{ij}(x_0),$$

for each fixed $i, j \in \{1, \dots, N\}$, each permutation $\sigma \in S_{2n}$ and each $\lambda_0, \dots, \lambda_{2n} \in \mathbb{C}$ with $|\lambda_r - 1| = 1/2$ for all $r \in \{0, \dots, 2n\}$.

We now apply the power-series expansion from Equation (19) (both for a_{p_A} and a_p), so we reduce the lemma to proving that

$$\Big((\gamma_{p_A}(\lambda_0) \cdot \delta_{p_A})^{k_0} \gamma_{p_A}(\lambda_0)$$
$$\cdot \prod_{r=1}^{2n} (\gamma_{p_A}(\lambda_r) \cdot \delta_{p_A})^{k_r} \gamma_{p_A}(\lambda_r) \cdot (p_A(x_{\sigma(r)}) - p_A(x_0))(\gamma_{p_A}(\lambda_r) \cdot \delta_{p_A})^{l_r} \gamma_{p_A}(\lambda_r)\Big)_{ij} \qquad (20)$$

is equal to

$$\Big((\gamma_p(\lambda_0) \cdot \delta_p)^{k_0} \gamma_p(\lambda_0)$$
$$\cdot \prod_{r=1}^{2n} (\gamma_p(\lambda_r) \cdot \delta_p)^{k_r} \gamma_p(\lambda_r) \cdot (p(x_{\sigma(r)}) - p(x_0))(\gamma_p(\lambda_r) \cdot \delta_p)^{l_r} \gamma_p(\lambda_r)\Big)_{ij} \cdot g_{ij}(x_0),$$

for every $(k_0, \dots, k_{2n}) \in (\mathbb{N} \cup \{0\})^{2n+1}$ and every $(l_1, \dots, l_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$.

We are going to reduce our task even further. Thus, let us fix $(k_0, \dots, k_{2n}) \in (\mathbb{N} \cup \{0\})^{2n+1}$ and $(l_1, \dots, l_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$. Using indices $\alpha, \beta \in \{1, \dots, N\}$ for the $(m \times m)$-blocks we notice that

$$(\gamma_{p_A})_{\alpha\beta} \cdot q = (\gamma_p)_{\alpha\beta} \cdot g_{\alpha\beta}(x_0)$$
$$(\delta_{p_A})_{\alpha\beta} = (\delta_p)_{\alpha\beta} \cdot g_{\alpha\beta}(x_0)$$
$$(p_A)_{\alpha\beta}(x_s) = p_{\alpha\beta}(x_s) \cdot g_{\alpha\beta}(x_0),$$

for all $s \in \{0, 1, \dots, 2n\}$. Letting

$$M = 1 + 6n + 2k_0 + 2\sum_{r=1}^{2n} (k_r + l_r)$$

denote the number of multiplicative factors involved in the operator in Equation (20), using block-multiplication, again with $(m \times m)$-blocks, we may rewrite this operator as

$$\sum_{i_0, \dots, i_{M-2}=1}^{N} C_{ii_0i_1 \dots i_{M-2}j} \cdot (g_{ii_0} g_{i_0i_1} \cdot \dots \cdot g_{i_{M-3}i_{M-2}} g_{i_{M-2}j})(x_0),$$

where

$$\sum_{i_0,\dots,i_{M-2}=1}^{N} C_{ii_0i_1\dots i_{M-2}j} = \Big((\gamma_p(\lambda_0)\cdot\delta_p)^{k_0}\gamma_p(\lambda_0)$$

$$\cdot \prod_{r=1}^{2n}(\gamma_p(\lambda_r)\cdot\delta_p)^{k_r}\gamma_p(\lambda_r)\cdot(p(x_{\sigma(r)})-p(x_0))(\gamma_p(\lambda_r)\cdot\delta_p)^{l_r}\gamma_p(\lambda_r)\Big)_{ij}.$$

Let us now fix indices $i_0,\dots,i_{M-2}\in\{1,\dots,N\}$. To ease the notation, we put

$$i_{-1}=i \qquad \text{and} \qquad i_{M-1}=j.$$

It suffices to show that

$$C_{i_{-1}i_0\dots i_{M-1}}\cdot(g_{i_{-1}i_0}\cdot\ldots\cdot g_{i_{M-2}i_{M-1}})(x_0)=C_{i_{-1}i_0\dots i_{M-1}}\cdot g_{i_{-1}i_{M-1}}(x_0).$$

We claim that if $C_{i_{-1}i_0\dots i_{M-1}}$ is nonzero, then

$$K_{i_\alpha i_{\alpha+1}}\cap U\neq\emptyset \qquad \forall\alpha\in\{-1,0,\dots,M-2\}\text{ with } i_\alpha\neq i_{\alpha+1}. \tag{21}$$

If Equation (21) holds, then by virtue of Lemma 3.5 we must have that

$$\bigcap_{\alpha=-1}^{M-2}K_{i_\alpha i_{\alpha+1}}\neq\emptyset,$$

which in turn means $\bigcap_{\alpha=-1}^{M-2}V_{i_\alpha i_{\alpha+1}}\neq\emptyset$, therefore the cocycle relations hold and

$$(g_{i_{-1}i_0}\cdot\ldots\cdot g_{i_{M-2}i_{M-1}})(x_0)=g_{i_{-1}i_{M-1}}(x_0),$$

which is what we set out to prove.

So let us suppose that $K_{i_\alpha i_{\alpha+1}}\cap U=\emptyset$ for some $\alpha\in\{-1,0,\dots,M-2\}$ with $i_\alpha\neq i_{\alpha+1}$. We are going to show that $C_{i_{-1}i_0\dots i_{M-1}}=0$. There are three cases: the pair $i_\alpha i_{\alpha+1}$ can appear in a term of the form $\gamma_p(\lambda,x_0)_{i_\alpha i_{\alpha+1}}$, or $(\delta_p(x,t))_{i_\alpha i_{\alpha+1}}$, or $(p(x_s)-p(x_0))_{i_\alpha i_{\alpha+1}}$. The expression for $C_{i_{-1}i_0\dots i_{M-1}}$ involves products of terms of the previous three forms, so that if one is zero, then $C_{i_{-1}i_0\dots i_{M-1}}=0$.

In the first case, since $i_\alpha\neq i_{\alpha+1}$, $x_0\in U$ and $\operatorname{supp}(\chi_{i_\alpha})\cap\operatorname{supp}(\chi_{i_{\alpha+1}})=K_{i_\alpha i_{\alpha+1}}$, we have that

$$\gamma_p(\lambda,x_0)_{i_\alpha i_{\alpha+1}}=\frac{\sqrt{\chi_{i_\alpha}\chi_{i_{\alpha+1}}}(x_0)}{\lambda-1}-\frac{\sqrt{\chi_{i_\alpha}\chi_{i_{\alpha+1}}}(x_0)}{\lambda}=0.$$

The second case follow from the third case, which is obvious since $x_s,x_0\in U$ so that

$$(p(x_s)-p(x_0))_{i_\alpha i_{\alpha+1}}=\sqrt{\chi_{i_\alpha}\chi_{i_{\alpha+1}}}(x_s)-\sqrt{\chi_{i_\alpha}\chi_{i_{\alpha+1}}}(x_0)=0. \qquad \square$$

Proof of Theorem 3.1. — When $n=0$, we have that

$$\operatorname{Ch}^0_\phi(p_A)(x)=\phi(p_A(x))=\sum_{i=1}^{N}\chi_i(x)\phi(g_{ii}(x))=\phi(q),$$

for all $x \in X$. When $n > 0$, we obtain from Lemma 3.6 that

$$\sum_{i=1}^{N} \phi\Big(\Big(e_{p_A}(x,t)d\big(e_{p_A}(x,t)\big)^{\wedge 2n}\Big)_{ii}\Big) = \sum_{i=1}^{N} \mathrm{Tr}\Big(e_p(x,t)d\big(e_p(x,t)\big)^{\wedge 2n}\Big) \cdot \phi(g_{ii}(x_0)),$$

for all $x \in \mathfrak{U}^{2n+1}$ and all $t \in \Delta^{2n}$. Hence, since $\phi(g_{ii}(x_0)) = \phi(q)$ and the projection p is Murray-von Neumann equivalent to $1 \in C(X)$, we obtain that

$$\mathrm{Ch}^{2n}_{\phi}([p_A]) = \mathrm{Ch}^{2n}([1]) \cdot \phi(q) = 0. \qquad \square$$

4. Index theorem — Proof of Theorem B

In this section we are going to prove the following K-theoretic version of Atiyah's L^2-index theorem:

Theorem 4.1. — *Suppose that G is a countable discrete group and let $\phi\colon C^*_r(G) \to \mathbb{C}$ denote the canonical faithful tracial state. Suppose that $p : \tilde{X} \to X$ is a principal G-bundle, where $\tilde{X}$ is a second-countable, locally compact, Hausdorff space and X is compact and Hausdorff. Then*

$$\phi_* \circ \eta_{\tilde{X}} = \mathrm{ind}_{C^*_r(G)} = \mathrm{ind} : \mathrm{KK}_0(C(X), \mathbb{C}) \to \mathbb{R}.$$

In particular,

$$\mathrm{ind}_{C^*_r(G)}(x) \in \mathbb{Z} \qquad \text{for all } x \in \mathrm{KK}_0(C(X), \mathbb{C}).$$

For a countable discrete group H we let $\mathbb{L}(H) \subseteq B(\ell^2(H))$ denote the group von Neumann algebra of H and we let $\phi : \mathbb{L}(H) \to \mathbb{C}$ denote the faithful tracial state on $\mathbb{L}(H)$ given by $\phi(x) = \langle \delta_e, x \cdot \delta_e \rangle$, $x \in \mathbb{L}(H)$.

When H contains G as a subgroup we let $\iota : C^*_r(G) \to \mathbb{L}(H)$ denote the injective $*$-homomorphism coming from the inclusion $G \subseteq H$ via the functoriality of the reduced group C^*-algebra and the inclusion $C^*_r(H) \subseteq \mathbb{L}(H)$. We remark that $\phi(\iota(x)) = \phi(x)$ for all $x \in C^*_r(G)$.

Let $p_{C^*_r(G)}$ be the projection coming from the Miščenko line bundle associated to the principal bundle $\tilde{X} \to X$ and set $p_{\mathbb{L}(H)} = \iota(p_{C^*_r(G)})$.

Remark 4.2. — *If $\{V_i\}_{i=1}^N$ is a finite open cover of X, such that $\{p^{-1}(V_i)\}_{i=1}^N$ is a trivializing cover for $p : \tilde{X} \to X$, then we get locally constant transition functions*

$$g_{ij} \colon V_i \cap V_j \to G,$$

satisfying the cocycle relations $g_{ii} = e$ and $g_{ij}g_{jk} = g_{ik}$ whenever $V_i \cap V_j \cap V_k$ is non-empty. If we compose with the left regular representation, we get locally constant maps into the unitary group of $C^*_r(G)$

$$\lambda_{g_{ij}} \colon V_i \cap V_j \to U(C^*_r(G)),$$

which fit the setup outlined in Section 3.4.

We then have the von Neumann algebraic index map $\eta_{\mathbb{L}(H)}$ defined by

$$\begin{array}{ccc} \mathrm{KK}_0(C(X),\mathbb{C}) & & \\ \Big\downarrow{\scriptstyle \tau_{\mathbb{L}(H)}} & & \\ \mathrm{KK}_0(C(X,\mathbb{L}(H)),\mathbb{L}(H)) & \xrightarrow{[p_{\mathbb{L}(H)}]\widehat{\otimes}_{C(X,\mathbb{L}(H))}-} & \mathrm{KK}_0(\mathbb{C},\mathbb{L}(H)), \end{array}$$

which we may compose with the character $\phi_* : \mathrm{KK}_0(\mathbb{C},\mathbb{L}(H)) \to \mathbb{R}$, obtaining the index map

$$\mathrm{ind}_{\mathbb{L}(H)} = \phi_* \circ \eta_{\mathbb{L}(H)} : \mathrm{KK}_0(C(X),\mathbb{C}) \to \mathbb{R}.$$

The following two simple lemmas link the various index maps.

Lemma 4.3. — *For every countable discrete group H containing the group G we have the identity*

$$\mathrm{ind}_{C_r^*(G)} = \mathrm{ind}_{\mathbb{L}(H)} : \mathrm{KK}_0(C(X),\mathbb{C}) \to \mathbb{R}.$$

Proof. — We put $f = [p_{C_r^*(G)}]\widehat{\otimes}_{C(X,C_r^*(G))}-$ and notice that each subdiagram in the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{KK}_0(C(X),\mathbb{C}) & \xrightarrow{\tau_{\mathbb{L}(H)}} & \mathrm{KK}_0(C(X,\mathbb{L}(H)),\mathbb{L}(H)) & \xrightarrow{[p_{\mathbb{L}(H)}]\widehat{\otimes}_{C(X,\mathbb{L}(H))}-} & K_0(\mathbb{L}(H)) \\ \Big\downarrow{\scriptstyle \tau_{C_r^*(G)}} & & \Big\downarrow{\scriptstyle \iota^*} & & \\ \mathrm{KK}_0(C(X,C_r^*(G)),C_r^*(G)) & \xrightarrow{\iota_*} & \mathrm{KK}_0(C(X,C_r^*(G)),\mathbb{L}(H)) & \xrightarrow{f} & K_0(\mathbb{L}(H)) \\ & \searrow{\scriptstyle f} & & \nearrow{\scriptstyle \iota_*} & \Big\downarrow{\scriptstyle \phi_*} \\ & & K_0(C_r^*(G)) & \xrightarrow{\phi_*} & \mathbb{R}. \end{array}$$

This proves the lemma. □

Lemma 4.4. — *Let H be a countable discrete group. The index map* $\mathrm{ind} = \iota^*$: $\mathrm{KK}_0(C(X),\mathbb{C}) \to \mathbb{Z} \subseteq \mathbb{R}$ *agrees with the composition*

$$\begin{array}{ccccc} \mathrm{KK}_0(C(X),\mathbb{C}) & \xrightarrow{\tau_{\mathbb{L}(H)}} & \mathrm{KK}_0(C(X,\mathbb{L}(H)),\mathbb{L}(H)) & & \\ & & \Big\downarrow{\scriptstyle [1]\widehat{\otimes}_{C(X,\mathbb{L}(H))}-} & & \\ & & \mathrm{KK}_0(\mathbb{C},\mathbb{L}(H)) & \xrightarrow{\phi_*} & \mathbb{R}. \end{array}$$

Proof. — We record that the interior Kasparov product $[1]\widehat{\otimes}_{C(X,\mathbb{L}(H))}-$ agrees with the pullback via the inclusion $\iota : \mathbb{C} \to C(X,\mathbb{L}(H))$. The result of the lemma now follows by noting that the diagram here below is commutative

$$\begin{array}{ccccc}
\mathrm{KK}_0(C(X),\mathbb{C}) & \xrightarrow{\tau_{\mathbb{L}(H)}} & \mathrm{KK}_0(C(X,\mathbb{L}(H)),\mathbb{L}(H)) & & \\
\big\downarrow{\scriptstyle \iota^*} & & \big\downarrow{\scriptstyle \iota^*} & & \\
\mathrm{KK}_0(\mathbb{C},\mathbb{C}) & \xrightarrow{\tau_{\mathbb{L}(H)}} & \mathrm{KK}_0(\mathbb{L}(H),\mathbb{L}(H)) & & \\
\big\downarrow{\scriptstyle \cong} & & & \searrow{\scriptstyle \iota^*} & \\
\mathbb{Z} & \xrightarrow[\subseteq]{} & \mathbb{R} & \xleftarrow[\phi_*]{} & \mathrm{KK}_0(\mathbb{C},\mathbb{L}(H)),
\end{array}$$

where the right vertical $*$-homomorphism $\iota : \mathbb{L}(H) \to C(X, \mathbb{L}(H))$ sends operators to constant maps. □

Proposition 4.5. — *There exists a countable discrete group H containing G and having infinite conjugacy classes. In particular $\mathbb{L}(H)$ is a II_1-factor.*

Proof. — We can choose $H = G * \mathbb{F}_2$, the free product of G with the free group on two generators. The statement about the associated group von Neumann algebra is proved in [**8**, III.3.3.7, page 289]. □

Proof of Theorem 4.1. — By Proposition 4.5 we may choose a countable discrete group H containing G such that the group von Neumann algebra $\mathbb{L}(H)$ is a II_1-factor. By Proposition 3.4 and Theorem 3.1 we have that $[p_{\mathbb{L}(H)}] = [1] \in \mathrm{KK}_0(\mathbb{C}, C(X, \mathbb{L}(H)))$ and thus by Lemma 4.3 and Lemma 4.4 that

$$\begin{aligned}
\mathrm{ind}_{C_r^*(G)}(x) = \mathrm{ind}_{\mathbb{L}(H)}(x) &= \phi_*\big([p_{\mathbb{L}(H)}]\widehat{\otimes}_{C(X,\mathbb{L}(H))}\tau_{\mathbb{L}(H)}(x)\big) \\
&= \phi_*\big([1]\widehat{\otimes}_{C(X,\mathbb{L}(H))}\tau_{\mathbb{L}(H)}(x)\big) = \iota^*(x) = \mathrm{ind}(x)\,.
\end{aligned}$$

This proves the theorem. □

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PAPER B

A NOTE ON HOMOLOGY FOR SMALE SPACES

A NOTE ON HOMOLOGY FOR SMALE SPACES

by

Valerio Proietti

Abstract. — We collect three observations on the homology for Smale spaces defined by Putnam. The definition of such homology groups involves four complexes. It is shown here that a simple convergence theorem for spectral sequences can be used to prove that all complexes yield the same homology. Furthermore, we introduce a simplicial framework by which the various complexes can be understood as suitable "symmetric" Moore complexes associated to the simplicial structure. The last section discusses projective resolutions in the context of dynamical systems. It is shown that the projective cover of a Smale space is realized by the system of shift spaces and factor maps onto it.

Contents

1. Preliminaries . . . 3
   1.1. Dimension groups . . . 5
   1.2. Complexes . . . 8
2. $C^s_{,\mathcal{A}}(\pi)$ is quasi-isomorphic to $C^s(\pi)$ . . . 10
   2.1. Filtrations . . . 11
   2.2. Spectral sequences . . . 11
3. Simplicial viewpoint . . . 13
   3.1. Symmetric simplicial groups . . . 15
   3.2. Symmetric cosimplicial groups . . . 18
4. Projective covers . . . 18

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References.. 22

Introduction and main results

When Steven Smale initiated his study of smooth maps on manifolds, he defined the notion of Axiom A diffeomorphism [**19**]. The main condition is that the map, restricted to its set of non-wandering points, has a hyperbolic structure. The non-wandering set of these systems can be canonically decomposed into finitely many disjoints sets, called *basic sets*, each of which is irreducible in a certain sense.

One of Smale's great insights was that, even though one began with a smooth system, the non-wandering set itself would not usually be a submanifold, but rather an object of fractal-like nature. This can be taken as a motivation for moving from the smooth category to the topological one.

Smale spaces were introduced by Ruelle as a purely topological description of the basic sets of Smale's Axiom A diffeomorphisms [**17**].

In this paper we consider the homology theory for Smale spaces introduced by Putnam in [**15**]. This can be viewed as a solution to Smale's problem of classifying Axiom A systems by relatively simple combinatorial data, in the same fashion that Morse-Smale systems could be described.

Shifts of finite type are the zero dimensional examples of Smale spaces and are the basic building blocks of the theory. Putnam's homology can be viewed as a far-reaching generalization of Krieger's dimension groups for shifts of finite type [**9**]. In the preliminaries of this paper, we review the notion of Krieger's invariant and explain its connection to K-theory by examining the stable and unstable equivalence relations which define the associated C^*-algebras (this is a well-known result, here it is simply expressed in a slightly unusual form, see Theorem 1.10).

There are many interesting and open questions concerning Putnam's homology for Smale spaces. In the literature, computations of the homology groups have been done mostly by resorting to the definition, e.g., [**15**, Chapter 7]. It is desirable to have some machinery, as it occurs with algebraic topology, which would aid in these calculations by appealing to techniques such as long exact sequences, excision, etc.

Exact analogues are at the moment not so clear, but it is reasonable that an alternative, perhaps more conceptual definition of the homology could shed some light on these issues. Moreover, this could also lead to clarifying the relations with Čech cohomology and K-theory (beyond the case of shifts of finite type). More on these questions can be found in [**15**, Chapter 8]. This paper started as an effort to research in this direction.

The technical definition of Putnam's homology groups involves four bicomplexes [**15**, Chapter 5]. Only three of these are shown to be quasi-isomorphic, leaving out the largest (but perhaps most natural) double complex, which has a clear connection to K-theory. The first result of this paper fills this gap by showing, thanks to a simple convergence theorem for spectral sequences, that this double complex also yields the same homology groups. The formal statement is given in Corollary 2.7.

Section 3 is concerned with proving a collection of results that are already proved in Putnam's memoir, by taking a slightly different and somewhat more unified perspective. The key observation stems from the simplicial nature of the homology theory for Smale spaces: a given Smale space is suitably "replaced" by a bisimplicial shift of finite type, to which Krieger's invariant is applied to get (in conjunction with the Dold-Kan correspondence) a bicomplex which defines the homology groups of interest.

From this viewpoint, the different variants of this bicomplex appear as the associated Moore complexes (i.e., the normalized chain complexes). There is also an action of the symmetric group which is exploited to obtain all of Putnam's complexes as "reduced" complexes with respect to this mixed simplicial-symmetric structure. The main result in this section, proved as application of these methods, is Theorem 3.9.

The last section introduces the concept of projectivity for dynamical systems and attempts to justify the definition of the homology theory for Smale spaces by drawing a parallel with sheaf cohomology. The main result here is that the projective cover of a Smale space can be defined as a certain projective limit over the symbolic presentations for the given space. The rigorous statement is found in Theorem 4.3.

Most of the conventions and notations in this paper are taken directly from [**15**]. No attempt is made to put the results in broader context or expand on detail. For these reasons the reader is advised to have a copy of Putnam's *A Homology Theory for Smale Spaces* [**15**] handy.

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1. Preliminaries

A Smale space (X, ϕ) is a dynamical system consisting of a homeomorphism ϕ on a compact metric space (X, d) such that the space is locally the product of a coordinate that contracts under the action of ϕ and a coordinate that expands under the action of ϕ. The precise definition requires the definition of a bracket map satisfying certain axioms [**15, 17**].

The most essential feature of Smale spaces is given by the definition of two equivalence relations, named respectively *stable* and *unstable*, which reads as follows:

– given $x, y \in X$, we say they are *stably equivalent* if
$$\lim_{n\to\infty} d(\phi^n(x), \phi^n(y)) = 0;$$
– given $x, y \in X$, we say they are *unstably equivalent* if
$$\lim_{n\to\infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0.$$

The orbit of $x \in X$ under the stable (respectively unstable) equivalence relation is called the *global stable* (resp. *unstable*) set and is denoted $X^s(x)$ (resp. $X^u(x)$). Given a small enough $\epsilon > 0$, *local* stable and unstable sets are also defined, and they are denoted respectively $X^s(x, \epsilon)$ and $X^u(x, \epsilon)$.

These satisfy the following identities:
$$X^s(x) = \bigcup_{n\geq 0} \phi^{-n}(X^s(\phi^n(x), \epsilon))$$
$$X^u(x) = \bigcup_{n\geq 0} \phi^{n}(X^s(\phi^{-n}(x), \epsilon)).$$

Let (X, ϕ) be a Smale space. We will assume that (X, ϕ) is *non-wandering*, so that there exists an *s/u-bijective pair* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ (see [**15**, Section 2.6] for this notion). Recall from [**15**, Sections 2.5 and 2.6] that we can assume Y and Z to be non-wandering, and also ψ and ζ to be finite-to-one.

We define a subshift of finite type for each $L, M \geq 0$,
$$\Sigma_{L,M}(\pi) = \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \,|y_l \in Y, z_m \in Z, \\ \pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L, 0 \leq m \leq M\}.$$

We have maps
$$\delta_l \colon \Sigma_{L,M} \to \Sigma_{L-1,M} \tag{1}$$
$$\delta_{,m} \colon \Sigma_{L,M+1} \to \Sigma_{L,M}$$
which delete respectively entries y_l and z_m. Theorem 2.6.13 in [**15**] asserts that the maps δ_l are s-bijective and the maps $\delta_{,m}$ are u-bijective (these will be defined shortly).

Given a subshift of finite type Σ, we can associate to it an abelian group, denoted $D^s(\Sigma)$, defined in [**15**, Chapter 3] (see also [**9**]). It will be called the *(stable) dimension group* of Σ. This construction is covariant for s-bijective maps and contravariant for u-bijective maps [**15**, Sections 3.4 and 3.5]. We summarize here these definitions:

Definition 1.1. — Let $f \colon (X, \phi) \to (Y, \psi)$ be a map of Smale spaces. Consider for each $x \in X$ the restrictions
$$f \colon X^s(x) \to Y^s(f(x)) \tag{2}$$
$$f \colon X^u(x) \to Y^u(f(x)). \tag{3}$$

– If (2) is injective, we say that f is *s-resolving*. If it is injective and surjective, then we say f is *s-bijective*.
– If (3) is injective, we say that f is *u-resolving*. If it is injective and surjective, then we say f is *u-bijective*.

1.1. Dimension groups. — Let us start with the definition of Krieger's dimension groups.

Definition 1.2. — Let (Σ, σ) be a subshift of finite type. For $e \in \Sigma$, consider the family of compact open subsets in the stable orbit $\Sigma^s(e)$ and denote it by $CO^s(\Sigma, \sigma, e)$. Define $CO^s(\Sigma, \sigma) = \cup_{e \in \Sigma} CO^s(\Sigma, \sigma, e)$. Let $\sim$ be the smallest equivalence relation such that, for $E, F \in CO^s(\Sigma, \sigma)$, we have

– $E \sim F$ if $[E, F] = E, [F, E] = F$, assuming both sets are defined;
– $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$.

We define $D^s(\Sigma, \sigma)$ (abbreviated $D^s(\Sigma)$) to be the free abelian group on the $\sim$-equivalences $[E]$, modulo the subgroup generated by $[E \cup F] - [E] - [F]$, where E, F belong to $CO^s(\Sigma, \sigma)$ and $E \cap F = \emptyset$.

There is a definition of $D^u(\Sigma, \sigma)$, which is left to the imagination of the reader, since it won't be used in the rest of this paper.

It is easy to see that, in the construction above, it is sufficient to consider clopens lying in the *local* stable sets.

Lemma 1.3. — *Define a family of sets $CO^s_\epsilon(\Sigma, \sigma)$, composed of clopens $E \subseteq \Sigma^s(e, \epsilon)$ for some $e \in \Sigma$ and $\epsilon < 1/4$. Consider the abelian group $D^s_\epsilon(\Sigma, \sigma)$, defined as in Definition 1.2, but replacing $CO^s(\Sigma, \sigma)$ with $CO^s_\epsilon(\Sigma, \sigma)$.*

Then we have $D^s(\Sigma, \sigma) \cong D^s_\epsilon(\Sigma, \sigma)$.

Proof. — Given $E \in CO^s(\Sigma, \sigma), E \subseteq \Sigma^s(f)$, there is a well-defined function $E \to \mathbb{N}$, defined assigning to $e \in E$ the minimum number $N(e)$ such that $e_n = f_n$ whenever $n \geq N(e)$. In other words, $N(e)$ is the minimum natural number such that

$$e \in \sigma^{-N(e)}(\Sigma^s(\sigma^{N(e)}(f), \epsilon)).$$

By definition $E \cap \sigma^{-n}(\Sigma^s(\sigma^n(f), \epsilon))$ is clopen for each $n \in \mathbb{N}$, which implies the assignment $e \mapsto N(e)$ is continuous. Since E is compact, there is $N(E) \in \mathbb{N}$ such that

$$E \subseteq \sigma^{-N(E)}(\Sigma^s(\sigma^{N(E)}(f), \epsilon)).$$

Therefore, E can be partitioned in a finite number of disjoint clopens E_i with $E_i \in CO^s_\epsilon(\Sigma, \sigma)$. We conclude $[E] \in D^s_\epsilon(\Sigma, \sigma)$. All is left to show is the equivalence relation defining $D^s(\Sigma, \sigma)$ is determined within the clopens in $CO^s_\epsilon(\Sigma, \sigma)$. Let $E \sim F$ be sets in $CO^s(\Sigma, \sigma)$ and take N to be the maximum between $N(E)$ and $N(F)$. By definition $E \sim F$ if and only if $\sigma^N(E) \sim \sigma^N(F)$, and of course $\sigma^N(E)$ and $\sigma^N(F)$ belong to local stable sets. This completes the proof. □

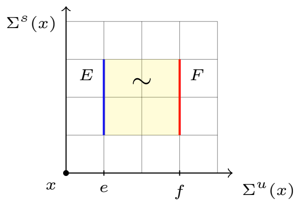

FIGURE 1. In this figure, E and F are compact opens in $CO^s_\epsilon(\Sigma,\sigma)$, with $E \subseteq \Sigma^s(e,\epsilon)$ and $F \subseteq \Sigma^s(f,\epsilon)$. The shaded area in yellow indicates that $[E,F]=E, [F,E]=F$ and therefore E and F are equivalent sets.

A consequence of the previous lemma is that we can illustrate the definition of dimension group by a simple figure (Figure 1).

When (Σ,σ) is non-wandering we can simplify the computation of the dimension group even further, because we can decompose Σ in basic pieces as follows (see [**15**, Theorem 2.1.13]).

Theorem 1.4. — *Given a non-wandering Smale space (X,ϕ), there are closed pairwise disjoint sets $X_1,\dots,X_n$ and a permutation $\alpha \in S_n$ such that $\phi(X_i)=X_{\alpha(i)}$ for all $i=1,\dots,n$. Moreover, for any i and k such that $\alpha^k(i)=i$, the system (X_i,ϕ^k) is a mixing Smale space.*

Since the stable and unstable orbits are the same for (X,ϕ) and (X,ϕ^k), it is a simple matter to see that, applying the previous theorem to (Σ,σ), we get a decomposition

$$D^s(\Sigma) \cong D^s(\Sigma_1) \oplus \cdots \oplus D^s(\Sigma_n), \tag{4}$$

(see also [**13**, Section 2]).

Remark 1.5. — In this paper we consider the dimension group merely as a group-invariant, without keeping track of the positive cone and of the induced automorphism (for more details, see [**10**, Chapter 7]). Since the decomposition in (4) holds at the level of C^*-algebras, the positive cones decompose along the same shape. The induced automorphism (which also exists at the C^*-level) permutes the summands according to α as in Theorem 1.4.

In view of the preceding discussion, for the rest of this subsection we assume that (Σ,σ) is mixing, in particular the global stable sets are dense.

Lemma 1.6. — *Let $f \in \Sigma$ and define $CO^s_f(\Sigma,\sigma) = \{E \in CO^s_\epsilon(\Sigma,\sigma) \mid E \subseteq \Sigma^s(f)\}$. Consider the abelian group $D^s_f(\Sigma,\sigma)$, defined as in Definition 1.2, but replacing $CO^s(\Sigma,\sigma)$ with $CO^s_f(\Sigma,\sigma)$. Then we have $D^s(\Sigma,\sigma) \cong D^s_f(\Sigma,\sigma)$.*

Proof. — Given $E \in CO^s_\epsilon(\Sigma,\sigma)$, it is sufficient to prove $[E]=[F]$ for some $F \in CO^s_f(\Sigma,\sigma)$. Suppose $E \subseteq \Sigma^s(e,\epsilon)$ and let $\Sigma(e,\epsilon)$ denote the open ball centered at e of radius ϵ. Note that $\Sigma^s(e,\epsilon) \subseteq \Sigma(e,\epsilon)$. Since $\Sigma^s(f)$ is dense, we can find

$f' \in \Sigma^s(f) \cap \Sigma(e, \epsilon)$ and define $F = [f', E]$. The basic properties of the bracket imply $[E, F] = E, [F, E] = F$. □

Remark 1.7. — It is clear that $CO^s_f(\Sigma, \sigma)$ is a basis for the topology of $\Sigma^s(f)$. Thus $D^s(\Sigma)$ is generated by equivalence classes of basic clopens in some global stable set.

Let $R^u(\Sigma, f)$ be the set of pairs of unstably equivalent points which belong to the stable orbit through $f \in \Sigma$. This is an amenable and étale groupoid when endowed with the topology as in [**20**, Section 1.2] (see also [**16**, Theorem 3.6]).

Remark 1.8. — In [**12**, page 14] the question arises if stable and unstable equivalence relations of any mixing Smale space are locally compact amenable groupoids. The answer is positive and the proof is as follows: by [**4**, Corollary 3.8], in the equivalence class (in the sense of [**11**]) of such equivalence relations we can find étale amenable groupoids, because their corresponding C^*-algebras have finite nuclear dimension (see [**3**, Theorem 5.6.18]). Amenability is invariant under this sort of equivalence by [**1**, Theorem 2.2.17].

A subbase for the topology on $R^u(\Sigma, f)$ is given by triples (E, F, γ) where E, F are basic clopens of the unit space $\Sigma^s(f)$ and $\gamma\colon E \to F$ is homeomorphism such that $(e, \gamma(e)) \in R^u(\Sigma, f)$ for all $e \in E$. We consider the following "categorification" of $R^u(\Sigma, f)$: define a category $\mathcal{C}(\Sigma, f)$ whose objects are the clopens in $CO^s_f(\Sigma, \sigma)$ and morphisms $E \to F$ are inclusions $E \hookrightarrow F$ and triples (E, F, γ) as above.

Recall that the K-theory $K_0(\mathcal{C})$ of an additive category $(\mathcal{C}, \oplus)$ is the abelian group generated by isomorphism classes $[E]$ of objects $E \in \mathcal{C}$ subject to the relation $[E \oplus F] = E + F$. If we interpret isomorphism classes as (E, F, γ)-orbits in $\mathcal{C}(\Sigma, f)$, and we take $E \oplus F$ to mean $E \cup F, E \cap F = \emptyset$, then we obtain a well-defined abelian group $K_0(\mathcal{C}(\Sigma, f))$.

Remark 1.9. — Note that the condition $E \oplus F$ is completely determined by inclusions. Indeed unions and intersections are specific colimits and limits in $\mathcal{C}(\Sigma, f)$.

Theorem 1.10. — *There is an isomorphism $D^s(\Sigma) \cong K_0(\mathcal{C}(\Sigma, f))$ for any $f \in \Sigma$.*

Proof. — Let us take E and F such that $[E] = [F]$. In particular there is $n \in \mathbb{N}$ and $f \in \sigma^{-n}(F)$ such that $[f, \sigma^{-n}(E)] = \sigma^{-n}(F)$. It is easy to see that the map

$$\gamma(e) = \sigma^n([f, \sigma^{-n}(e)]) \tag{5}$$

is a homeomorphism of E onto F, and obvioulsy $\sigma^{-n}(e)$ belongs to the local unstable set of $\sigma^{-n}(\gamma(e))$, therefore $(e, \gamma(e)) \in R^u(\Sigma, f)$.

Conversely, if (E, F, γ) is an isomorphism in $\mathcal{C}(\Sigma, f)$, then by [**20**, Lemma 4.14] (and compactness), we can partitition E in a finite number of clopens $E_1, \dots, E_n$, and correspondingly F in $F_1, \dots, F_n$, where E_i is homeomorphic to F_i through a map in the form of (5). Therefore $[E_i] = [F_i]$ and by the defining relation $[E] = [F]$. □

If χ_E is the indicator function of the clopen E inside the groupoid C^*-algebra $C^*(R^u(\Sigma, f))$, then a little thinking over the assignment $E \mapsto \chi_E$ gives the following well-known result (for more details see [**20**, Section 4.3]).

Corollary 1.11. — *There is an isomorphism*

$$K_0(C^*(R^u(\Sigma, f))) \cong K_0(\mathcal{C}(\Sigma, f)) \cong D^s(\Sigma)$$

for any $f \in \Sigma$.

Remark 1.12. — As was already implicitly noted in Remark 1.8, the reason why the choice of $f \in \Sigma$ doesn't affect the K-theory group is to be found in the statement that reducing a groupoid to a transversal preserves its equivalence class, as explained in more detail in [**11**, Example 2.7].

1.2. Complexes. — The maps in (1) will induce group morphisms denoted respectively $\delta^s_l, \delta^{s*}_{,m}$. For each $L, M \geq 0$, we consider maps

$$\partial^s_{L,M} \colon D^s(\Sigma_{L,M}(\pi)) \to D^s(\Sigma_{L-1,M}(\pi)) \tag{6}$$
$$\partial^s_{L,M} = \sum_{0 \leq l \leq L} (-1)^l \delta^s_l$$
$$\partial^{s*}_{L,M} \colon D^s(\Sigma_{L,M}(\pi)) \to D^s(\Sigma_{L,M+1}(\pi)) \tag{7}$$
$$\partial^s_{L,M} = \sum_{0 \leq m \leq M+1} (-1)^{L+m} \delta^{s*}_{,m}.$$

It is clear from the definition that

$$\partial^s_{L,M+1} \circ \partial^{s*}_{L,M} = \partial^{s*}_{L-1,M} \circ \partial^s_{L,M}.$$

Furthermore, by applying [**15**, Theorem 2.6.11,2.6.12,4.1.14], we have that

- for each $M \geq 0$, (6) is a chain complex;
- for each $L \geq 0$, (7) is a cochain complex.

Altogether, we have a double complex $(C^s(\pi)_{\bullet,\bullet}, \partial^s, \partial^{s*})$, where

$$C^s(\pi)_{L,M} = \begin{cases} D^s(\Sigma_{L,M}(\pi)) & \text{if } L \geq 0 \text{ and } M \geq 0 \\ 0 & \text{else.} \end{cases}$$

The *totalization* of this complex is the chain complex $(\mathrm{Tot}^{\oplus}(C^s(\pi))_\bullet, d^s)$, where

$$\mathrm{Tot}^{\oplus}(C^s(\pi))_N = \bigoplus_{L-M=N} C^s(\pi)_{L,M}$$
$$d^s_{L,M} = \partial^s_{L,M} + \partial^{s*}_{L,M}$$
$$d^s_N = \bigoplus_{L-M=N} d^s_{L,M} \colon \mathrm{Tot}^{\oplus}(C^s(\pi))_N \to \mathrm{Tot}^{\oplus}(C^s(\pi))_{N-1}.$$

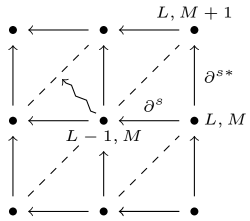

FIGURE 2. A representation of the complexes $C^s(\pi)_{\bullet,\bullet}$ and $\mathrm{Tot}^{\oplus}(C^s(\pi))_\bullet$. The direct sums of the groups lying on the dashed diagonals give $\mathrm{Tot}^{\oplus}(C^s(\pi))_\bullet$. The differentials d^s (e.g., the zigzag arrow in the top-left square) run from south-east to north-west, decreasing degree by 1.

By slightly modifying the invariant $\Sigma \mapsto D^s(\Sigma)$, we can introduce a cochain complex which is related to (7), and will give rise to another double complex. We summarize the details of this construction (see [**15**, Definition 4.1.5]):

- For any $L \geq 0$, the symmetric group S_{M+1} acts by automorphisms (in particular, s-bijective maps) on $\Sigma_{L,M}(\pi)$. Define the group
$$D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi)) = \{a \in D^s(\Sigma_{L,M}(\pi)) \mid \ a = \mathrm{sgn}(\beta)\beta(a) \quad \forall \beta \in S_{M+1}\};$$
- By [**15**, Lemma 5.1.6], we have
$$\partial^s_{L,M} D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi)) \subseteq D^s_{,\mathcal{A}}(\Sigma_{L-1,M}(\pi))$$
$$\partial^{s*}_{L,M} D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi)) \subseteq D^s_{,\mathcal{A}}(\Sigma_{L,M+1}(\pi));$$
- Define a bicomplex $(C^s_{,\mathcal{A}}(\pi)_{\bullet,\bullet}, \partial^s, \partial^{s*})$ by setting
$$C^s_{,\mathcal{A}}(\pi)_{L,M} = D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi));$$
- The inclusion map $J\colon D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi)) \to D^s(\Sigma_{L,M}(\pi))$ induces chain maps
$$(C^s_{,\mathcal{A}}(\pi)_{\bullet,\bullet}, \partial^s, \partial^{s*}) \to (C^s(\pi)_{\bullet,\bullet}, \partial^s, \partial^{s*})$$
$$(\mathrm{Tot}^{\oplus}(C^s_{,\mathcal{A}}(\pi))_\bullet, d^s) \to (\mathrm{Tot}^{\oplus}(C^s(\pi))_\bullet, d^s),$$
and in particular, for each $L \geq 0$, a cochain map
$$(C^s_{,\mathcal{A}}(\pi)_{L,\bullet}, \partial^{s*}) \to (C^s(\pi)_{L,\bullet}, \partial^{s*}).$$

The advantage in using the complex just defined lies in the following propositions, proved in [**15**, Theorem 4.2.12, Theorem 4.3.1]

Proposition 1.13. — *There is $N \geq 0$ such that $C^s_{,\mathcal{A}}(\pi)_{L,M} = 0$ whenever $M \geq N$.*

Proposition 1.14. — *For each, $L \geq 0$, the cochain map*
$$J\colon (C^s_{,\mathcal{A}}(\pi)_{L,\bullet}, \partial^{s*}) \to (C^s(\pi)_{L,\bullet}, \partial^{s*})$$
is a quasi-isomorphism, i.e., for each $N \in \mathbb{Z}$ there are induced isomorphisms
$$J_*\colon H_N(C^s_{,\mathcal{A}}(\pi)_{L,\bullet}, \partial^{s*}) \cong H_N(C^s(\pi)_{L,\bullet}, \partial^{s*}).$$

Definition 1.15. — By considering the symmetric group action S_{L+1} on $\Sigma_{L,M}(\pi)$, one can introduce another invariant

$$D^s_{\mathcal{Q}}(\Sigma_{L,M}(\pi)) = \frac{D^s(\Sigma_{L,M}(\pi))}{D^s_{\mathcal{B}}(\Sigma_{L,M}(\pi))},$$

where $D^s_{\mathcal{B}}(\Sigma_{L,M}(\pi))$ is the subgroup of $D^s(\Sigma)$ generated by

- all elements a satisfying $\alpha(a) = a$ for some non-trivial transposition α in the symmetric group S_{L+1};
- all elements of the form $a - \mathrm{sgn}(\alpha)\alpha(a)$, where $\alpha \in S_{L+1}$.

Associated to this invariant is the bicomplex denoted $C^s_{\mathcal{Q}}(\pi)$ in [**15**, Chapter 5]. This complex enjoys the analogous property of Proposition 1.13, i.e., it is zero outside a bounded region in the L-direction.

By combining both approaches, one can also introduce a fourth bicomplex, denoted $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ and based on the following "dimension group":

$$D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi)) = \frac{D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))}{D^s_{\mathcal{B}}(\Sigma_{L,M}(\pi)) \cap D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))}.$$

This complex is zero outside a bounded rectangle of the first quadrant. Further details on these constructions are found in [**15**, Definition 5.1.7]. It is proved in [**15**, Section 5.3] that there are quasi-isomorphisms

$$C^s_{,\mathcal{A}}(\pi) \to C^s_{\mathcal{Q},\mathcal{A}}(\pi) \to C^s_{\mathcal{Q}}(\pi).$$

These results and constructions will be obtained through different methods in the next sections of this paper.

By definition, the *homology groups of* (X, ϕ) are given by $(N \in \mathbb{Z})$

$$H_N(X, \phi) = H_N(\mathrm{Tot}^{\oplus}(C^s_{\mathcal{Q},\mathcal{A}}(\pi))_\bullet, d^s).$$

It is proved in [**15**, Section 5.5] that this definition does not depend on the particular choice of s/u-bijective map π.

2. $C^s_{,\mathcal{A}}(\pi)$ is quasi-isomorphic to $C^s(\pi)$

We are going to prove in this section that $C^s_{,\mathcal{A}}(\pi)$ is quasi-isomorphic to $C^s(\pi)$. This is the missing (but conjectured) result from Putnam's memoir [**15**, page 90]. For brevity, we write $C = C^s(\pi)$ and $C_{\mathcal{A}} = C^s(\pi)_{,\mathcal{A}}$.

There are at least two reasons why this quasi-isomorphism is important: firstly, it is clear that C is the most straightforward among the definable complexes for the homology of Smale spaces, and therefore it is a basic fundamental result that it's computing the same invariants as the other complexes. Secondly, and maybe more importantly, C is also the complex with the most evident connection to K-theory for the associated C^*-algebras. Indeed, if we consider the C^*-morphisms induced by δ_l

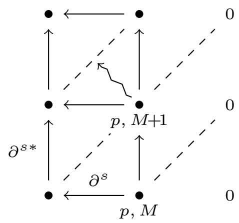

FIGURE 3. The vetical filtration.

and $\delta_{,m}$ as explained in [**13**], their corresponding K-theory maps agree with δ^s_l and $\delta^{s*}_{,m}$ after the identification given in Corollary 1.11.

2.1. Filtrations. — We proceed by defining the *vertical filtration* on $C_{\bullet,\bullet}$, i.e., the family of subcomplexes given by ($p \in \mathbb{Z}$)

$$F_pC_{L,M} = \begin{cases} C_{L,M} & \text{if } L \leq p \\ 0 & \text{else,} \end{cases}$$

in other words everything to the right of the vertical line $L = p$ is set to zero, see Figure 3.

The resulting family $\{\mathrm{Tot}^{\oplus}(F_pC)_\bullet \mid p \in \mathbb{Z}\}$ is a filtration of the totalization chain complex. Note there is a chain of inclusions

$$\cdots \subseteq \mathrm{Tot}^{\oplus}(F_pC)_\bullet \subseteq \mathrm{Tot}^{\oplus}(F_{p+1}C)_\bullet \subseteq \cdots . \tag{8}$$

In complete analogy, we get a filtration

$$\cdots \subseteq \mathrm{Tot}^{\oplus}(F_pC_{\mathcal{A}})_\bullet \subseteq \mathrm{Tot}^{\oplus}(F_{p+1}C_{\mathcal{A}})_\bullet \subseteq \cdots . \tag{9}$$

The following remarks will be important in the next subsection.

Remark 2.1. — The filtration in (8) is *exhaustive*, i.e., the union over all p of $\mathrm{Tot}^{\oplus}(F_pC)_\bullet$ is $\mathrm{Tot}^{\oplus}(C)_\bullet$. Note this implies the induced filtration on homology is also exhaustive. The same holds for (9).

Remark 2.2. — The filtration in (8) is *bounded below*, i.e., for each $N \in \mathbb{Z}$ there exists $s \in \mathbb{Z}$ such that $Tot(F_sC)_N = 0$. For $N \geq 0$ we can take $s = N - 1$; when $N < 0$ we take $s = -1$. Note this implies the induced filtration on homology is also bounded below. The same holds for (9).

2.2. Spectral sequences. — A filtration of a chain complex gives rise to a spectral sequence, see [**21**, Theorem 5.4.1] for a proof.

Proposition 2.3. — *A filtration $\mathcal{F}$ of a chain complex $\mathcal{C}$ determines a spectral sequence:*

$$E^0_{pq} = F_p\mathcal{C}_{p+q}/F_{p-1}\mathcal{C}_{p+q}$$
$$E^1_{pq} = H_{p+q}(E^0_{p\bullet}).$$

In order to discuss convergence for the spectral sequence in Proposition 2.3, we introduce a bit of terminology (we follow [**21**, Chapter 5]). The expert reader may skip to Corollary 2.7.

Recall that a (homology) spectral sequence is *bounded below* if for each n there is $s = s(n)$ such that the terms E^r_{pq} with $p + q = n$ vanish for all $p < s$. A spectral sequence is *regular* if for each p and q the differentials d^r_{pq} leaving E^r_{pq} are zero for all large r.

Remark 2.4. — Bounded below spectral sequences are regular. If $\mathcal{F}$ is a bounded below filtration (see Remark 2.2), then the spectral sequence in Proposition 2.3 is bounded below, hence regular.

For each $n \in \mathbb{Z}$, the homology group $H_n(\mathcal{C})$ receives an induced filtration

$$\cdots \subseteq \mathcal{F}_pH_n(\mathcal{C}) \subseteq \mathcal{F}_{p+1}H_n(\mathcal{C}) \subseteq \cdots \subseteq H_n(C).$$

We say the spectral sequence *abuts to* to $H_*(\mathcal{C})$ if, for all $p, q, n \in \mathbb{Z}$,

1. there are isomorphisms
$$\beta_{pq} \colon E^\infty_{pq} \cong \mathcal{F}_pH_{p+q}(\mathcal{C})/\mathcal{F}_{p-1}H_{p+q}(\mathcal{C}); \tag{10}$$
2. $H_n(\mathcal{C}) = \cup_p\mathcal{F}_pH_n(\mathcal{C})$;
3. $\cap\mathcal{F}_pH_n(\mathcal{C}) = 0$.

When $(\mathcal{F},\mathcal{C}) = (F, \mathrm{Tot}^\oplus(C))$ or $(\mathcal{F},\mathcal{C}) = (F, \mathrm{Tot}^\oplus(C_\mathcal{A}))$, items 2 and 3 above follow from Remarks 2.1 and 2.2 respectively.

We say the spectral sequence *converges to* $H_*(\mathcal{C})$ if it abuts to $H_*(\mathcal{C})$, it is regular, and it holds for each $n \in \mathbb{Z}$ that

$$H_n(\mathcal{C}) = \varprojlim_{p\in\mathbb{Z}} \frac{H_n(\mathcal{C})}{\mathcal{F}_pH_n(\mathcal{C})}.$$

Note that a bounded below (hence regular) spectral sequence always satisfies the condition above, therefore it converges to $H_*(\mathcal{C})$ as soon as the abutment condition holds. This applies to the spectral sequences associated to $(F, \mathrm{Tot}^\oplus(C))$ and $(F, \mathrm{Tot}^\oplus(C_\mathcal{A}))$, because of Remarks 2.2 and 2.4.

Suppose $\{E^r_{pq}\}$ and $\{E'^r_{pq}\}$ satisfy (10) with respect to H_* and H'_* respectively. We say that a map $h \colon H_* \to H'_*$ is *compatible* with a morphism $f \colon E \to E'$ if

- $h(\mathcal{F}_pH_n) \subseteq \mathcal{F}_pH'_n$ for all $n \in \mathbb{Z}$;
- the induced maps $\mathcal{F}_pH_n/\mathcal{F}_{p-1}H_n \to \mathcal{F}_pH'_n/\mathcal{F}_{p-1}H'_n$ correspond under β and β' to $f^\infty_{pq} \colon E^\infty_{pq} \to E'^\infty_{pq}$, $q = n - p$.

We recall the following result [**21**, Theorem 5.5.1].

Theorem 2.5. — *Condition* (10) *holds for bounded below spectral sequences.*

In particular, if $(\mathcal{F}, \mathcal{C})$ is a filtered chain complex where $\mathcal{F}$ is exhaustive and bounded below, then the associated spectral sequence is bounded below and converges to $H_*(\mathcal{C})$. Moreover, the convergence is natural: if $f\colon \mathcal{C} \to \mathcal{C}'$ is a map of filtered complexes, then the map $f_*\colon H_*(\mathcal{C}) \to H_*(\mathcal{C}')$ is compatible with the corresponding morphism of spectral sequences.

Corollary 2.6. — *With notations as above, if $f^r\colon E^r_{pq} \cong E'^r_{pq}$ is an isomorphism for all p, q and some r (hence for $r = \infty$, see* [**21**, Lemma 5.2.4]*), then $f_*\colon H_*(\mathcal{C}) \to H_*(\mathcal{C}')$ is an isomorphism.*

Corollary 2.7. — *There are convergent spectral sequences*

$$E^1_{pq} = H_q((C_{\mathcal{A}})_{p,\bullet}, \partial^{s*}) \Rightarrow H_{p+q}(\mathrm{Tot}^{\oplus}(C_{\mathcal{A}})_\bullet, d^s) \cong H_{p+q}(X, \phi)$$
$$E'^1_{pq} = H_q(C_{p,\bullet}, \partial^{s*}) \Rightarrow H_{p+q}(\mathrm{Tot}^{\oplus}(C)_\bullet, d^s).$$

Furthermore, the inclusion chain map

$$J\colon (\mathrm{Tot}^{\oplus}(C_{\mathcal{A}})_\bullet, d^s) \to (\mathrm{Tot}^{\oplus}(C)_\bullet, d^s)$$

is a quasi-isomorphism.

Proof. — The spectral sequences arise by applying Proposition 2.3 with $(\mathcal{F}, \mathcal{C}) = (F, \mathrm{Tot}^{\oplus}(C))$ and $(\mathcal{F}, \mathcal{C}) = (F, \mathrm{Tot}^{\oplus}(C_{\mathcal{A}}))$. Convergence follows from Theorem 2.5. The map J induces isomorphisms

$$J^1\colon E^1_{pq} = H_q((C_{\mathcal{A}})_{p,\bullet}, \partial^{s*}) \cong E'^1_{pq} = H_q(C_{p,\bullet}, \partial^{s*})$$

for all p, q by Proposition 1.14. The result follows from Corollary 2.6 above. □

Corollary 2.8. — *Homology groups for the Smale space (X, ϕ) can be equally defined as ($N \in \mathbb{Z}$)*

$$H_N(X, \phi) = H_N(\mathrm{Tot}^{\oplus}(C)_\bullet, d^s).$$

3. Simplicial viewpoint

The s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ for (X, ϕ) gives rise to a *bisimplicial* Smale space $(\Sigma_{L,M}(\pi))_{L,M\geq 0}$. We will drop the reference to π for brevity.

The face maps are given by (1). We stress that the δ_l's are s-bijective and the $\delta_{,m}$'s are u-bijective. The degeneracy maps are as follows:

$$s_l\colon \Sigma_{L,M} \to \Sigma_{L+1,M}$$
$$(y_0, \dots, y_l, \dots, y_L, z_0, \dots, z_M) \mapsto (y_0, \dots, y_l, y_l \dots, y_L, z_0, \dots, z_M)$$
$$s_{,m}\colon \Sigma_{L,M} \to \Sigma_{L,M+1}$$
$$(y_0, \dots, y_L, z_0, \dots, z_m, \dots, z_M) \mapsto (y_0, \dots, y_L, z_0, \dots, z_m, z_m, \dots, z_M),$$

for $l = 0, \cdots, L-1$ and $m = 0, \ldots, M-1$.

Remark 3.1. — Note that $s_l(\Sigma_{L,M}) \subseteq (\Sigma_{L+1,M})$ is a closed shift-invariant system, clearly isomorphic to $\Sigma_{L,M}$. The same holds for $s_{,m}(\Sigma_{L,M}) \subseteq (\Sigma_{L,M+1})$.

Remark 3.2. — It is not difficult to see that, for each l, the map s^l is s-bijective because its inverse is given by δ_l. The situation is different for the maps $s_{,m}$: they are only s-resolving.

Proposition 3.3. — *There are induced maps*

$$s_l^s \colon D^s(\Sigma_{L,M}) \to D^s(\Sigma_{L+1,M})$$
$$s_{,m}^{s*} \colon D^s(\Sigma_{L,M+1}) \to D^s(\Sigma_{L,M}).$$

Moreover, the map s_l^s is split-injective.

Proof. — Recall that (equivalence classes of) compact open sets inside stable orbits provide generators for the dimension groups. Given one of such classes $[E] \in D^s(\Sigma_{L,M})$, the assignment $[E] \mapsto [s_l(E)]$ is a well-defined group morphism because the map $s_l \colon \Sigma^s_{L,M}(e) \to \Sigma^s_{L+1,M}(s_l(e))$ is a homeomorphism. The splitting for the map s_l^s is given by δ_l^s. The definition of $s_{,m}^{s*}$ is given by $E \mapsto s_{,m}^{-1}(E)$. This preimage is compact because $s_{,m}$ is proper, as s-resolving maps are proper [**15**, Theorem 2.5.4]. Since $E \cap F = \emptyset$ implies $s_{,m}(E) \cap s_{,m}(F) = \emptyset$, the map respects the group operation. □

The theorem below follows easily from the discussion so far (and some simple verifications). See [**21**, Chapter 8] for the Dold-Kan correspondence.

Theorem 3.4. — *Applying the D^s-functor to the bisimplicial space $\Sigma_{\bullet,\bullet}$ results in a simplicial cosimplicial group $(D^s(\Sigma_{\bullet,\bullet}), \delta_l^s, s_l^s, \delta_{,m}^s, s_{,m}^{s*})$. Furthermore the* unnormalized *double complex associated to said group via the Dold-Kan correspondence is $(C_{L,M}, \partial^s, \partial^{s*})$, as defined in Section 1.*

Remark 3.5. — As was mentioned at the beginning of Section 2, the complex $(C_{L,M}, \partial^s, \partial^{s*})$ is also the result of applying the K-theory functor to $\Sigma_{\bullet,\bullet}$. The intermediate step in this case is constructing the associated C^*-algebras, which are AF [**20**, Section 4.3], so the odd K-groups vanish.

By considering the normalizations (sometimes called the *Moore complexes*) associated to $D^s(\Sigma_{\bullet,\bullet})$ we obtain simplicial versions of the bicomplexes $C_{\mathcal{A}}, C_{\mathcal{Q}}, C_{\mathcal{A},\mathcal{Q}}$ that were previously introduced. It is well-known that these all yield isomorphic homology groups (see [**21**, Theorem 8.3.8]). However, it should be noted that these complexes are not as useful as their "symmetric" counterpart (to be introduced in the next section), because they don't allow for computational simplifications as in Proposition 1.13.

3.1. Symmetric simplicial groups. — Fix $M \geq 0$ and consider the simplicial group $(\Sigma_{\bullet,M}, \delta_l^s, s_l^s)$. It carries an action of the symmetric group S_{L+1}. Recall that this group is generated by the *adjacent* transpositions $t_l = (l\, l+1)$, $l = 0, \dots, L-1$ (see [**8**, Section 5, Theorem 3]).

The functorial properties of the D^s-invariant easily give the theorem below. The notion of symmetric simplicial group is inspired by [**7**].

Theorem 3.6. — *The simplicial group* $(\Sigma_{\bullet,M}, \delta_l^s, s_l^s)$ *is a* symmetric object*, i.e., it carries an action of the transpositions* t_i*'s, subject to the defining relations of* S_{L+1} *and to the following* mixed relations*:*

$$\begin{array}{lll} \delta_j^s t_i = t_i \delta_j^s & s_j^s t_i = t_i s_j^s & (i < j-1) \\ \delta_i^s t_i = \delta_{i+1}^s & s_i^s t_i = t_{i+1} t_i s_{i+1}^s & \\ \delta_j^s t_i = t_{i-1} \delta_j^s & s_j^s t_i = t_{i+1} s_j^s & (i > j) \\ & t_i s_i^s = s_i^s. & \end{array}$$

For some l and $j = 1, \dots, L+1-l$ consider the cycle $\sigma_j = (l+j\; l+j-1\; \cdots\; l+1)$ in S_{L+1} and the compositions $\sigma_j s_l$. Note $\sigma_1 s_l = s_l$. In other words $\sigma_j s_l$ is an additional degeneracy map which repeats entry y_l at coordinate $l+j$:

$$(y_0, \dots, y_l, \dots, y_L, \dots) \xrightarrow{\sigma_j s_l} (y_0, \dots, y_l, \dots, y_{l+j-1}, y_l, y_{l+j}, \dots y_L, \dots) \in \Sigma_{L+1,M}.$$

As composition of s-bijective maps, the $\sigma_j s_l$'s induce group morphisms

$$D^s(\Sigma_{L,M}) \xrightarrow{(\sigma_j s_l)^s} D^s(\Sigma_{L+1,M}).$$

It is then natural to define the groups of *degenerate* chains,

$$\tilde{D}C_{L,M} = \sum_{l,j} (\sigma_j s_l)^s (C_{L-1,M}).$$

The subgroup $\sum_l (\sigma_1 s_l)^s (C_{L-1,M})$ is preserved by the differential ∂^s thanks to the simplicial identities, but when $j > 1$ the identities in Theorem 3.6 give the following relation:

$$\partial^s(\tilde{D}C_{L,M}) \subseteq \tilde{D}C_{L-1,M} + \langle \sigma_j(a) - \operatorname{sgn}(\sigma_j)(a) \mid a \in C_{L,M} \rangle, \tag{11}$$

because $\delta_l^s (\sigma_j s_l)^s(a) = \sigma_j(a)$ and $\delta_{l+j}^s (\sigma_j s_l)^s(a) = a$.

Lemma 3.7. — *There is an equality ($a \in C_{L,M}$)*

$$\begin{aligned} \langle \sigma_j(a) - sgn(\sigma_j)(a) \rangle &= \langle t_i(a) + a \mid i = 1, \dots, L-1 \rangle \\ &= \langle \alpha(a) - sgn(\alpha)(a) \mid \alpha \in S_{L+1} \rangle. \end{aligned}$$

Proof. — Since the t_i's are generators we can write $\alpha(a) = t_{i_1} \cdots t_{i_n}(a)$. Then we have

$$\begin{aligned}(t_{i_1} \cdots t_{i_n}(a) + t_{i_2} \cdots t_{i_n}(a)) - (t_{i_2} \cdots t_{i_n}(a) + t_{i_3} \cdots t_{i_n}(a))\\ + (t_{i_3} \cdots t_{i_n}(a) + t_{i_4} \cdots t_{i_n}(a)) - \cdots \pm (t_{i_n}(a) + a) = \alpha(a) \pm a.\end{aligned}$$

The sign is positive when n is odd and negative when n is even, i.e., it is in accordance with $-\mathrm{sgn}(\alpha)$. Note that our notation for σ_j does not make reference to the index l, so that $\sigma_j(a)$ for $j = 2$ includes all elements of the form $t_i(a)$. □

We can now "correct" our definition of degenerate chains by setting $DC_{L,M}$ to be the group generated by $\tilde{D}C_{L,M}$ and $\langle \alpha(a) - \mathrm{sgn}(\alpha)(a) \mid a \in D^s(\Sigma_{L,M}), \alpha \in S_{L+1}\rangle$.

Lemma 3.8. — *The complex $(DC_{\bullet,M}, \partial^s)$ is a well-defined subcomplex of $(C_{\bullet,M}, \partial^s)$.*

Proof. — In view of the remark in (11), we only need to check what happens to $\partial^s(t_i(a) + a)$. By looking at the identities in Theorem 3.6, we see that it suffices to check the expression $\delta_i^s(t_i(a) + a) - \delta_{i+1}^s(t_i(a) + a)$. It is easy to see that $\delta_{i+1}^s t_i = \delta_i^s$ so we get

$$\delta_i^s(t_i(a) + a) - \delta_{i+1}^s(t_i(a) + a) = \delta_{i+1}^s(a) + \delta_i^s(a) - \delta_i^s(a) - \delta_{i+1}^s(a) = 0.$$

Therefore ∂^s preserves the subgroup $\langle \alpha(a) - \mathrm{sgn}(\alpha)(a) \mid a \in D^s(\Sigma_{L,M}), \alpha \in S_{L+1}\rangle$. □

Theorem 3.9. — *Consider the short exact sequence*

$$0 \longrightarrow DC_{\bullet,M} \longrightarrow C_{\bullet,M} \longrightarrow \frac{C_{\bullet,M}}{DC_{\bullet,M}} \longrightarrow 0.$$

The complex $DC_{\bullet,M}$ is acyclic, hence the projection map is a quasi-isomorphism.

Proof. — Set $DC_L = DC_{L,M}$ for brevity. We filter $DC_{\bullet,M}$ by setting $F_0 DC_L = 0$ and

$$\begin{aligned}F_p DC_L = &\sum_{l=0}^{k} \sum_{j=1}^{L-l} (\sigma_j s_l)^s (C_{L-1,M})\\ &+ \sum_{j=1}^{n} (\sigma_j s_{k+1})^s (C_{L-1,M}) + \langle \alpha(a) - \mathrm{sgn}(\alpha)(a)\rangle\end{aligned}$$

when $p = L + (L-1) + \cdots + (L-k) + n$ and $0 \le n \le L-k-1$. When $p \ge L(L+1)/2$ we have $F_p DC_L = DC_L$. The simplicial (and mixed) identities show that each $F_p DC_\bullet$ is a subcomplex. This filtration F is bounded, so there is a convergent spectral sequence (see [**21**, Theorem 5.5.1])

$$E^1_{pq} = H_{p+q}(F_p DC_\bullet / F_{p-1} DC_\bullet) \Rightarrow H_{p+q}(DC_\bullet).$$

So we have reduced ourselves to showing that each F_pDC/F_{p-1} is acyclic. We take $x \in DC_{L-1}$ and compute in F_pDC/F_{p-1}:

$$\partial^s(\sigma_n s_{k+1})^s(x) = \sum_{i=k+n+2}^{L} (-1)^i (\sigma_n s_{k+1})^s(\delta^s_{i-1})(x)$$

$$\begin{aligned}\partial^s(\sigma_n s_{k+1})^s(\sigma_n s_{k+1})^s(x) + (\sigma_n s_{k+1})^s\partial^s(\sigma_n s_{k+1})^s(x) &\\ = \sum_{i=k+n+2}^{L+1} (-1)^i (\sigma_n s_{k+1})^s(\delta^s_{i-1})(\sigma_n s_{k+1})^s(x) &\\ - \sum_{i=k+n+2}^{L} (-1)^i (\sigma_n s_{k+1})^s(\sigma_n s_{k+1})^s(\delta^s_{i-1})(x) &\\ = (-1)^p(\sigma_n s_{k+1})^s(x). &\end{aligned}$$

Hence $\psi_L = (-1)^p(\sigma_n s_{k+1})^s$ is a chain contraction of the identity map which implies F_pDC/F_{p-1} is acyclic. □

Corollary 3.10. — *There is an isomorphism of double complexes:*

$$((C_{\mathcal{Q}})_{\bullet,\bullet}), \partial^s, \partial^{s*}) \cong \left(\frac{C_{\bullet,M}}{DC_{\bullet,M}}, \partial^s, \partial^{s*}\right).$$

In particular, for each $M \geq 0$ there is a quasi-isomorphism of chain complexes

$$((C_{\bullet,M}), \partial^s) \to ((C_{\mathcal{Q}})_{\bullet,M}, \partial^s). \tag{12}$$

Proof. — All we need to do is identifying $D^s_{\mathcal{B}}(\Sigma_{L,M})$ with $DC_{L,M}$. Obviously

$$\langle \alpha(a) - \operatorname{sgn}(\alpha)(a)\rangle = \langle a - \operatorname{sgn}(\alpha)\alpha(a)\rangle,$$

and elements in the image of the degeneracy maps are clearly left invariant by some non-trivial transposition. Now given $[E] \in D^s_{\mathcal{B}}(\Sigma_{L,M})$ such that $[E] = [\alpha(E)]$ for some transposition, we need to show $[E] = [F]$ for some clopen F in the image of a degeneracy map. Now suppose $\alpha = (i\, i+k)$ and define F to be $(\sigma_k s_i)\delta_{i+k}(E)$. Then the condition $[E, F] = E, [F, E] = F$ trivially holds separately on each coordinate y_l with $l \neq i + k$, and when $l = i + k$ we can check the condition replacing F by $\alpha(E)$, because the coordinate of index $i + k$ is pointwise the same in F and $\alpha(E)$. □

Note that (12) is a chain version of Proposition 1.14 and is proved in [**15**, Theorem 4.3.1]. We have used Proposition 1.14 in order to establish the quasi-isomorphism $(\mathrm{Tot}^{\oplus}(C_{\mathcal{A}})_\bullet, d^s) \to (\mathrm{Tot}^{\oplus}(C)_\bullet, d^s)$ given by inclusion. Dually, it is natural to seek a quasi-isomorphism $(\mathrm{Tot}^{\oplus}(C)_\bullet, d^s) \to (\mathrm{Tot}^{\oplus}(C_{\mathcal{Q}})_\bullet, d^s)$ induced by projection, which makes use of (12). Of course the strategy is completely similar to Corollary 2.7, but considering the *horizontal filtration* instead of the vertical one. We skip the details.

Corollary 3.11. — *The projection map in Theorem 3.9 induces a quasi-isomorphism*

$$(\mathrm{Tot}^{\oplus}(C)_\bullet, d^s) \to (\mathrm{Tot}^{\oplus}(C_{\mathcal{Q}})_\bullet, d^s).$$

To complete the picture, we also give the dual version of Proposition 1.13.

Proposition 3.12. — *There is* $N \geq 0$ *such that* $C_{L,M} = DC_{L,M}$ *whenever* $L \geq N$. *Therefore* $(C_Q)_{L,M} = 0$ *whenever* $L \geq N$.

Proof. — Recall that we can choose the s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ so that π_s is finite-to-one. Let $N - 1$ be the maximum cardinality of a fiber of π_s and $L \geq N$. A generic generator for $D^s(\Sigma_{L,M})$ is a compact open in some stable orbit $E \subseteq \Sigma^s_{L,M}(e)$. By the choice of L, there are i and k such that $e = (y_0, y_1, \dots, y_L, \dots)$ with $y_i = y_{i+k}$. Since $\delta_{i+k} \colon \Sigma^s_{L,M}(e) \to \Sigma^s_{L-1,M}(\delta_{i+k}(e))$ is a homeomorphism, we see that $E = (\sigma_k s_i)(E)$. □

3.2. Symmetric cosimplicial groups. — As was hinted at the end of the previous section, the methods so far can be promptly dualized by considering the symmetric cosimplicial group $(\Sigma_{L,\bullet}, \delta^{s*}_{,m}, (\sigma_i s_{,m})^{s*})$ for fixed $L \geq 0$.

We will omit most details since this is a standard argument, and simply outline how to define the cochain complex of degenerate chains, which is the essential object needed to define the relevant "symmetric" Moore complex.

Where we used quotients in the previous section, we now have subgroups; moreover, by interpreting "coinvariants" to mean equivalence classes modulo $\langle a - \operatorname{sgn}(\alpha)\alpha(a)\rangle$, we are led to consider the dual notion of "invariants". This brings to defining

$$\begin{aligned} CC_{L,M} = \{a \in C_{L,M} \mid (\sigma_j s_{,m})^{s*}(a) = 0,\ a - \operatorname{sgn}(\alpha)\alpha(a) = 0 \\ \text{for all } m = 0, \dots, M, j = 1, \dots, M+1-m, \alpha \in S_M\}, \end{aligned}$$

that is the invariant elements lying in intersection of kernels for all degeneracy maps.

By the (dual) argument of 3.9 one can proceed to show that the quotient complex $C_{L,\bullet}/CC_{L,\bullet}$ is acyclic, so the inclusion

$$CC_{L,\bullet} \to C_{L,\bullet}$$

is a quasi-isomorphism. Finally, we mention that the projection map in Theorem 3.9 clearly induces a chain map

$$CC_{\bullet,M} \to \frac{CC_{\bullet,M}}{DC_{\bullet,M} \cap CC_{\bullet,M}},$$

which is an isomorphism on homology. An explicit inverse for the map is constructed in [**15**, page 98].

4. Projective covers

Given a sheaf S over a paracompact Hausdorff space X, sheaf cohomology $H^*(X, S)$ is computed from the complex $\operatorname{Hom}_X(\mathbb{Z}, I^\bullet)$, where $I^\bullet$ is an injective resolution of S. It is true, but perhaps less well-known, that the same calculation can be performed by means of the complex $\mathrm{Hom}_X(E^\bullet X, S)$, where $E^\bullet X$ is a semi-simplicial resolution of X arising from a projective cover E of X (see [**5**]).

This alternative path to computing sheaf cohomology calls for an analogy with the homology theory for Smale spaces. Indeed, we have seen how the defining complex arises by applying Krieger's invariant to the bisimplicial space induced by a chosen s/u-bijective pair. So the role of the global section functor is played, in our context, by the dimension group construction for subshifts of finite type.

The analogy is stronger when we start with a Smale space with totally disconnected stable sets. In this case, the homology is computed by the complex $(C_{\mathcal{Q}}(\pi)_{\bullet,0}, \partial^s)$ and the s/u-bijective pair is reduced to a simple s-bijective map

$$\pi\colon \Sigma_0 \to X,$$

where Σ_0 is a subshift of finite type (see [**15**, Section 7.2]). Thus in this case the analogy calls for considering Σ_0 as a "projective" cover of X, together with its associated simplicial resolution $\Sigma_\bullet$ obtained by taking iterated fibered products over π.

While the usage of the term "resolution" is somewhat justified (since by definition $\Sigma_\bullet$ computes the "right" homology groups), the attribute *projective* requires further reasoning. This section contains a simple theorem in this direction.

In the category of compact Hausdorff spaces and continuous maps, a projective object is a space E such that, whenever we are given $f\colon E \to A$ and $g\colon B \twoheadrightarrow A$ (onto), there is $h\colon E \to B$ with $f = g \circ h$.

A *projective cover* of X is a pair (E, e) with E projective and $e\colon E \twoheadrightarrow X$ *irreducible*, i.e., mapping proper closed sets onto proper subsets.

Gleason [**6**] has proved that projective covers exist and are unique (up to a homeomorphism making the obvious diagram commute). Moreover he showed that a space is projective if and only if it is extremally disconnected, i.e., the closure of each of its open sets is open. Recall that Σ_0 is a compact, Hausdorff, totally disconnected space. In general extremally disconnected Hausdorff spaces are totally disconnected, but the converse does not hold.

Let (X, ϕ) be a non-wandering Smale space and (E, e) its projective cover. Note that ϕ induces a self-homeomorphism $\tilde{\phi}$ of E and e intertwines $\tilde{\phi}, \phi$. Consider the totally disconnected space $\Sigma_{0,0}(\pi)$ associated to a choice of s/u-bijective pair π (this is the correct analogue of Σ_0 when X is not totally disconnected along the stable direction). The difference between E and $\Sigma_{0,0}(\pi)$ can be recast in terms of the dependence of the latter space on π. This suggests that in order to make sense of projectivity in the context of Smale spaces we ought to consider *all* s/u-bijective pairs at the same time.

The discussion on projectivity will inevitably bring us outside the category of Smale spaces (e.g., extremally disconnected spaces are not metrizable, unless they are discrete), therefore the following setup is in the context of (invertible) dynamical systems. See also Remark 4.2 below.

An open set in a space X is called regular if it is the interior of its closure. A *regular partition* $\mathcal{P}$ of X is a finite collection of disjoint regular opens whose union is dense.

Let (X, ϕ) be an invertible dynamical system and $\mathcal{P}$ a regular partition of X. View $\mathcal{P}$ as an alphabet and let $a_1 a_2 \cdots a_n$ be a word. We say this word is *allowed* if $\cap_{i=1}^{n} \phi^{-i}(a_i) \neq \emptyset$ and let $L_{\mathcal{P}}$ be the family of allowed words. It can be checked [**10**, Section 6.5] that $L_{\mathcal{P}}$ is the language of a shift space that we denote $\Sigma_{\mathcal{P}}$. Note that for each $x \in \Sigma_{\mathcal{P}}$ and $n \in \mathbb{N}$, the set

$$D_n(x) = \bigcap_{i=-n}^{n} \phi^{-i}(x_i) \subseteq X$$

is nonempty.

Definition 4.1. — We say that $\mathcal{P}$ is a *symbolic presentation* of (X, ϕ) if for every $x \in \Sigma_{\mathcal{P}}$ the set $\cap_{n=0}^{\infty} \overline{D_n(x)}$ consists of exactly one point. We call $\mathcal{P}$ a *Markov partition* if $\Sigma_{\mathcal{P}}$ is a subshift of finite type.

Other definitions of Markov partitions are common in the literature, e.g., [**2**].

Notice that the set of regular partitions is directed: we write $\mathcal{P}_1 \leq \mathcal{P}_2$ if $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$, i.e., each member of $\mathcal{P}_2$ is contained in a member of $\mathcal{P}_1$. Given partitions $\mathcal{P}_1, \mathcal{P}_2$ we can define an upper bound $\mathcal{P}_1 \cap \mathcal{P}_2$, obtained by taking pairwise intersections of elements from each partition.

If (X, ϕ) admits a symbolic presentation $\mathcal{P}_1$, then given any regular partition $\mathcal{P}_2$ we have that $\mathcal{P}_1 \cap \mathcal{P}_2$ is again a symbolic presentation. In other words, once a symbolic presentation exists, we can guarantee that the family of symbolic presentations is cofinal among all regular partitions.

Associated to $\mathcal{P}_1$ we get a factor map (i.e., an equivariant surjection) $\pi_{\mathcal{P}_1} \colon \Sigma_{\mathcal{P}_1} \to X$ (see [**10**, Proposition 6.5.8]). If $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$, then $\pi_{\mathcal{P}_2} \colon \Sigma_{\mathcal{P}_2} \to X$ is a factor map which factors through $\Sigma_{\mathcal{P}_1}$. Indeed if we view $\mathcal{P}_1$ and $\mathcal{P}_2$ as alphabets, there is a code $\mu_{\mathcal{P}_1, \mathcal{P}_2}$ which assigns to each letter $a \in \mathcal{P}_2$ the unique letter $b \in \mathcal{P}_1$ such that $a \subseteq b$, and $\pi_{\mathcal{P}_2} = \pi_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_1, \mathcal{P}_2}$.

As a result, if I denotes the family of symbolic presentations of (X, ϕ) (assuming it is nonempty), then $(\Sigma_i, \mu_{ij}, \pi_i)_{i \leq j \in I}$ defines a projective system in the category of dynamical systems over X. Let E be the inverse limit of $(\Sigma_i, \mu_{ij}, \pi_i)_{i \leq j \in I}$. Since $E \subseteq \prod_i \Sigma_i$, the shift map σ applied componentwise turns E into a dynamical system. Given $\mathcal{P} \in I$, denote by $p_{\mathcal{P}}$ the canonical projection $E \to \Sigma_{\mathcal{P}}$.

The case of Smale spaces is as follows. A non-wandering Smale space (X, ϕ) always admits a Markov partition [**17**, Section 7]. If we denote such partition by $\mathcal{M}$, then $\Sigma_{\mathcal{M}}$ is a subshift of finite type endowed with an *almost one-to-one* factor map $\pi_{\mathcal{M}} \colon \Sigma_{\mathcal{M}} \to X$ (i.e., an equivariant surjection that is finite-to-one, and the set of points in X with single preimage is a dense G_δ). If $\mathcal{P}$ is a refinement of $\mathcal{M}$, then $\pi_{\mathcal{P}} \colon \Sigma_{\mathcal{P}} \to X$ is an almost one-to-one factor map which factors through $\Sigma_{\mathcal{M}}$. As a result, in this case we can take I to be the family of refinements of $\mathcal{M}$.

Remark 4.2. — It is worth noting that $\{\Sigma_i\}_{i\in I}$ is a collection of shift spaces that are not necessarily of finite type (in particular, they are not Smale spaces). That is because the refinement of a Markov partition is not a Markov partition (in general). It is unclear to the author if there are conditions under which a Smale space admits a cofinal collection of Markov partitions.

Theorem 4.3. — *Let (X,ϕ) be a dynamical system which admits a symbolic presentation $\mathcal{P}$. Suppose $(\Sigma_i,\mu_{ij},\pi_i)_{i\leq j\in I}$ is the projective system associated to the collection of symbolic presentations of (X,ϕ) and denote by (E,σ) the associated inverse limit. Then (E,σ) is a projective cover of (X,ϕ) and the map $e\colon E\to X$ is given by the composition*

$$E \xrightarrow{p_{\mathcal{P}}} \Sigma_{\mathcal{P}} \xrightarrow{\pi_{\mathcal{P}}} X\ .$$

Proof. — Let J be the family of regular partitions of X. Given $\mathcal{P}\in J$, denote by $X(\mathcal{P})$ the topological space given by the disjoint union $\cup_{Y\in\mathcal{P}}\overline{Y}$. Then by [**18**, Proposition 17] we have that

$$E' = \varprojlim_{j\in J}(X(j), f_{jk})$$

is a projective cover of X (here $f_{jk}\colon X(k)\twoheadrightarrow X(j)$ when $j\leq k$ is the obvious surjection induced by the refinement). First of all we notice that I is cofinal in J so that the limit can be taken over the index set I. Secondly, notice that for each $i\in I$ there is a natural surjection $p_i\colon X(i)\to X$. We claim that $\pi_i\colon\Sigma_i\to X$ factors through p_i. Indeed, note that if $x\in\Sigma_i$, then $\pi_i(x)$ belongs to $\overline{x_0}\in i$ and $\pi_i(x)$ admits a unique lift $\tilde{x}\in\overline{x_0}\subseteq X(i)$. Define $\tilde{\pi}_i(x)=\tilde{x}$ and by construction $\pi_i=p_i\circ\tilde{\pi}_i$.

It is easy to check that $(i\leq j)$

$$\begin{array}{ccc} \Sigma_j & \xrightarrow{\mu_{ij}} & \Sigma_i \\ \downarrow{\scriptstyle\tilde{\pi}_j} & & \downarrow{\scriptstyle\tilde{\pi}_i} \\ X(j) & \xrightarrow{f_{ij}} & X(i) \end{array}$$

is a commuting diagram so that $\{\tilde{\pi}_i\}_{i\in I}$ induces a (continuous) map of spaces $\tilde{\pi}\colon E\to E'$. Since $\tilde{\pi}$ is a map of compact Hausdorff spaces, we only need to show it is bijective in order to get the required homeomorphism $E\cong E'$. In fact, it is sufficient to show that it is one-to-one, because $\tilde{\pi}(E)\subseteq E'$ is a closed set mapping onto X, thus by irreducibility $\tilde{\pi}(E)=E'$.

Suppose $x,y\in E, x\neq y$, so there is $i\in I$ with $x_i\neq y_i$. Recall that x_i and y_i are bi-infinite sequences in Σ_i, let us denote their components by $(x_i^k)_{k\in\mathbb{Z}}, (y_i^k)_{k\in\mathbb{Z}}$.

There is $m\in\mathbb{Z}$ with $x_i^m\neq y_i^m$. Note that $\phi^{-m}(i)$ is also a symbolic presentation, and if we set $\alpha=i\cap\phi^{-m}(i)$ we have $i\leq\alpha$, thus there are elements $x_\alpha,y_\alpha\in\Sigma_\alpha$, appearing at the α-th component of respectively x,y, and satisfying $\mu_{i\alpha}(x_\alpha)=x_i, \mu_{i\alpha}(y_\alpha)=y_i$.

We claim $\tilde{\pi}_\alpha(x_\alpha) \neq \tilde{\pi}_\alpha(y_\alpha)$. In fact, there are $A_x, B_x, A_y, B_y \in i$ with

$$x_\alpha^0 = x_i^0 \cap \phi^{-m}(A_x) \qquad x_\alpha^m = x_i^m \cap \phi^{-m}(B_x)$$
$$y_\alpha^0 = x_i^0 \cap \phi^{-m}(A_y) \qquad x_\alpha^m = x_i^m \cap \phi^{-m}(B_y)$$
$$x_i^0 \cap \phi^{-m}(A_x) \cap \phi^{-m}(x_i^m) \cap \phi^{-2m}(B_x) \neq \emptyset$$
$$y_i^0 \cap \phi^{-m}(A_y) \cap \phi^{-m}(y_i^m) \cap \phi^{-2m}(B_y) \neq \emptyset.$$

From the above we derive $A_x = x_i^m, A_y = y_i^m$ and in particular $A_x \neq A_y$. But by definition

$$\tilde{\pi}_\alpha(x_\alpha) \in \overline{x_i^0 \cap \phi^{-m}(A_x)} \subseteq X(\alpha)$$
$$\tilde{\pi}_\alpha(y_\alpha) \in \overline{y_i^0 \cap \phi^{-m}(A_y)} \subseteq X(\alpha)$$

so $\tilde{\pi}_\alpha(x_\alpha)$ cannot be equal to $\tilde{\pi}_\alpha(y_\alpha)$. This proves injectivity of $\tilde{\pi}$ and concludes the proof. □

Remark 4.4. — At first sight, it it reasonable to view E as the "universal" version of the spaces of the form $\Sigma_{0,0}(\pi)$. In the same spirit, one could think of defining a "universal s/u-bijective pair" $\pi = (E_s, \tilde{\psi}, e_s, E_u, \tilde{\zeta}, e_u)$, where E_s and E_u would be projective with respect to s-bijective and u-bijective maps.

The first step towards this program would be applying Putnam's lifting theorem [**14**] to the projective system $\{\Sigma_i\}_{i\in I}$ of Theorem 4.3 (assuming the system, or a cofinal replacement, consists entirely of shifts of finite type). Unfortunately, in order to lift the entire (infinite) system, limits of spaces are necessary, thus we run once again into the problem that these limits are not Smale spaces, and the notions of s- and u-bijective maps don't work well in this context.

This suggests that, if one desires importing the machinery of homological algebra and studying the homology of Smale spaces under this light, the ambient category should be chosen with care. A good candidate for this category might be the equivariant (with respect to the stable or unstable equivalence relation) KK-category, but this idea will not be pursued in the present paper.

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