On the algebraic K-theory of coordinate axes and truncated polynomial algebras

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For Christine and Carl

Abstract. This thesis is concerned with computations of algebraic K-theory using the cyclotomic trace map. We use the framework for cyclotomic spectra due to Nikolaus and Scholze, which avoids the use of genuine equivariant homotopy theory. The thesis contains an introduction followed by two papers.

The first paper computes the K-theory of the coordinate axes in affine d-space over perfect fields of positive characteristic. This extends work by Hesselholt in the case $d = 2$. The analogous results for fields of characteristic zero were found by Geller, Reid and Weibel in 1989. We also extend their computations to base rings which are smooth $\mathbb{Q}$-algebras.

In the second paper we revisit the computation, due to Hesselholt and Madsen, of the K-theory of truncated polynomial algebras for perfect fields of positive characteristic. The original proof relied on an understanding of cyclic polytopes in order to determine the genuine equivariant homotopy type of the cyclic bar construction for a suitable monoid. Using the Nikolaus-Scholze framework we achieve the same result using only the homology of said cyclic bar construction, as well as the action of Connes' operator.

Resumé. Hovedemnet i denne afhandling er beregninger af algebraisk K-teori ved brug af den cyklotomiske sporafbildning. Vi gør brug af det framework for cyklotomiske spektra, udviklet af Nikolaus og Scholze, som undgår brugen af genuin ækvivariant homotopiteori. Afhandlingen består af en introduktion efterfulgt af to artikler.

Den første artikel beregner K-teorien af koordinatakserne i det affine d-rum over perfekte legemer af positiv karakteristik. Dette generaliserer et resultat af Hesselholt i tilfældet $d = 2$. Det tilsvarende resultat for legemer af karakteristik nul blev fundet af Geller, Reid og Weibel i 1989. Vi udvider også deres beregninger til glatte $\mathbb{Q}$-algebraer.

I den anden artikel genbeviser vi en sætning, først bevist af Hesselholt og Madsen, af K-teorien for trunkerede polynomiumsalgebraer over perfekte legemer af positiv karakteristik. Det oprindelige argument beror på en forståelse af cykliske polytoper for dermed at bestemme den ækvivariante homotopitype af den cykliske barkonstruktion for en passende monoid. Igennem Nikolaus og Scholzes framework opnår vi det samme resultat ved kun af gøre brug af homologien af førnævnte cykliske barkonstruktion, samt virkningen af Connes' operator.

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Contents

Part 1

Introduction

This thesis comprises two papers. Both of them are concerned with computing algebraic K-theory using the cyclotomic trace map to topological cyclic homology.

Paper A. *On the K-theory of coordinate axes in affine space.*

Paper B. *On the K-theory of truncated polynomial algebras, revisited.*

Apart from providing new computations of algebraic K-theory (in Paper A) the main contribution of these papers is to show how to use the framework of Nikolaus and Scholze [31] to compute topological cyclic homology in the cases at hand. This framework removes the use of genuine equivariant homotopy theory from the study of topological cyclic homology. In Paper B I revisit a result due to Hesselholt and Madsen [18] using the framework of [31] and substantially reduce the amount of work needed to carry out the computation. In particular my calculation avoids the use of genuine equivariant homotopy theory. I expect that this method may be applied more generally to give new K-theory computations. Hesselholt and Nikolaus [23] have work in progress which applies these methods to give new calculations of the K-theory of planar cuspidal curves.

Before proceeding to the papers I wish to introduce the objects studied, give a short historical survey of the methods, and sketch the approach which has been developed. This will take up most of the introductory chapter. At the end I will describe some future perspectives and research projects, arising from the work presented in these papers.

1. Algebraic K-theory and trace methods

Algebraic K-theory is a fundamental invariant. It connects and informs a wide range of mathematical disciplines including such diverse subjects as geometric topology and number theory. As an example of the latter, suppose $\mathcal{O}_F$ is the ring of integers in a number field F. The torsion subgroups of $K_0(\mathcal{O}_F)$ and $K_1(\mathcal{O}_F)$ are given by the ideal class group of $\mathcal{O}_F$ and the roots of unity μ_F in F, respectively. It follows from the class number formula and the functional equation for ζ_F that

$$\lim_{s\to 0} s^{-r}\zeta_F(s) = -R\frac{\#K_0(\mathcal{O}_F)_{tors}}{\#K_1(\mathcal{O}_F)_{tors}}$$

where r is the rank of $K_1(\mathcal{O}_F)$ and R is the regulator of F. Lichtenbaum conjectures [27] that the higher K-theory groups similarly determine other special values of the zeta function ζ_F for many number fields F.

Algebraic K-theory functorially associates to a ring A a spectrum $K(A)$. The homotopy groups $K_n(A) = \pi_n K(A)$ are called the K-theory groups of A. The first three groups $K_0(A), K_1(A)$ and $K_2(A)$ were intensly studied in the 1950s and 1960s. They were defined purely algebraically. Quillen constructed the full spectrum $K(A)$ in the early 1970s [33]. He also presented the first complete calculation of $K_*(A)$, in the case where A is a finite field [32].

With Quillen's construction of the higher K-theory groups arose the question of computing them. This turned out to be a very difficult problem and for many years Quillen's results for finite fields remained the only known complete computations.

In mid 1970s Dennis constructed a map from K-theory to Hochschild homology, thereby initiating a new approach to computing K-theory. Goodwillie [15] further refined this approach by factoring the Dennis trace map through the negative version of cyclic homology. This led to several new computations of rational K-theory. Indeed, Goodwillie [15] showed that, rationally, the relative K-theory agrees with relative negative cyclic homology for nilpotent extensions of rings.

With the development of topological cyclic homology TC, by Bökstedt-Hsiang-Madsen [6] in the early 1990s it became possible to access the integral K-theory groups for certain rings. In *op. cit.* the authors show that the Dennis trace map factors through the map from TC, to topological Hochschild homology[1] THH. The resulting natural transformation

$$\mathrm{trc} : K \longrightarrow \mathrm{TC}$$

is called the cyclotomic trace map. In [29] McCarthy, using the work of Goodwillie, proves that for a surjective map of rings $A \to \bar{A}$ with nilpotent kernel, the square

$$\begin{array}{ccc} K(A) & \xrightarrow{trc} & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(\bar{A}) & \xrightarrow{trc} & \mathrm{TC}(\bar{A}) \end{array}$$

becomes homotopy cartesian after completion at any prime p. This result was later extended by Dundas [12] to the case of ring spectra. This led to a flurry of comparison results between K-theory and TC. For example Hesselholt and Madsen [21] used McCarthy's result to show that the cyclotomic trace $K(A) \to \mathrm{TC}(A)$ becomes an equivalence after p-adic completion when A is a finite algebra over the Witt vectors $W(k)$ of a perfect field of characteristic $p > 0$. Clausen, Mathew and Morrow [8]

[1] I will describe THH and TC in more detail below.

have recently extended the work of Dundas-Goodwillie-McCarthy, in the commutative case, by providing comparison results for any henselian pair (A, I).

One can restate McCarthy's result in terms of the corresponding relative theories. Given a surjective ring homomorphism $f : A \to \bar{A}$ with kernel I we call the fiber $K(A, I) = \mathrm{hofib}(K(A) \to K(\bar{A}))$ the relative K-theory of f. The same terminology applies to TC and all other spectrum-valued functors defined on the category of rings. McCarthy's theorem says that the relative cyclotomic trace map

$$K(A, I) \to \mathrm{TC}(A, I)$$

is an equivalence, after p-completion for any prime p.

1.1. Topological Hochschild homology. Topological Hochschild homology is a homology theory for associative algebras. It is defined in analogy with the ordinary Hochschild homology, originally defined by Hochschild in 1940s [24]. Let k be a commutative ring and A a unital associative k-algebra. Assume for simplicity that A is flat as a k-module. Consider the simplicial abelian group $\mathrm{B^{cy}}_{\otimes}(A/k)[-]$ with $\mathrm{B^{cy}}_{\otimes}(A/k)[n] = A^{\otimes n+1}$ where $\otimes = \otimes_k$ and with structure maps

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_n) &= \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 \le i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases} \\ s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \end{aligned}$$

We may additionally consider the cyclic permutation maps

$$t_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

thus endowing $\mathrm{B^{cy}}(A/k)[-]$ with the structure of a cyclic object, in the sense of Connes (see Loday [28] or [31, Appendix T] for more on cyclic theory). We now define Hochschild homology

$$\mathrm{HH}(A/k) = |\,\mathrm{B^{cy}}_{\otimes}(A/k)[-]|$$

as the geometric realization of this simplicial object. The Hochschild homology groups are the homotopy groups of this space.

By Connes' theory of cyclic objects, the Hochschild homology space $\mathrm{HH}(A/k)$ admits an action of the group $\mathbb{T}$ of complex numbers of unit modulus. Thus we may further define the cyclic homology, negative cyclic homology and periodic cyclic homology spaces,

$$\begin{aligned}\mathrm{HC}(A/k) &= \mathrm{HH}(A/k)_{h\mathbb{T}}\\ \mathrm{HC}^-(A/k) &= \mathrm{HH}(A/k)^{h\mathbb{T}}\\ \mathrm{HP}(A/k) &= (\Sigma^\infty \mathrm{HH}(A/k))^{t\mathbb{T}}\end{aligned}$$

by applying the homotopy orbits, homotopy fixed points, and $\mathbb{T}$-Tate construction, respectively. The $\mathbb{T}$-Tate construction for a $\mathbb{T}$-space X sits in a cofiber sequence

$$X^{h\mathbb{T}} \xrightarrow{\text{can}} X^{t\mathbb{T}} \longrightarrow \Sigma^2 X_{h\mathbb{T}}$$

generalizaing the SBI-sequence from the Connes-Tsygan cyclic theory.

All three of the theories $\mathrm{HC}, \mathrm{HC}^-$, and HP may also be defined in terms of explicit bicomplexes. Hoyois [25] provides a proof that the definitions agree. In particular these invariants are calculable in terms of homological algebra.

Topological Hochschild homology is, as the name suggests, a topological refinement of Hochschild homology. Bökstedt [4] was the first to construct THH, following ideas of Goodwillie and Waldhausen and Breen [7]. The original definition of THH in [4] was complicated by the lack of good symmetric monoidal models of the stable homotopy category. Since then the modern approaches allow for a straightforward definition using the cyclic bar construction. We follow [31] for the construction. In particular we work with the symmetric monoidal ∞-category of spectra, Sp with smash product $\otimes = \otimes_{\mathbb{S}}$ and unit given by the sphere spectrum $\mathbb{S}$. Given an $\mathbb{E}_1$-algebra in spectra A, form the simplicial spectrum $\mathrm{B}^{\mathrm{cy}}(A)[-]$ with

$$\mathrm{B}^{\mathrm{cy}}(A)[n] = A^{\otimes n+1}.$$

The spectrum $\mathrm{THH}(A)$ is by definition the geometric realization

$$\mathrm{THH}(A) = |\,\mathrm{B}^{\mathrm{cy}}(A)[-]|.$$

As before this acquires a circle action so $\mathrm{THH}(A)$ is an object in the category $\mathrm{Fun}(B\mathbb{T}, \mathrm{Sp})$ of spectra with a $\mathbb{T}$-action. In [*op. cit.* Theorem III.6.1.] the authors show that this construction agrees with Bökstedt's original definition from [4].

Furthermore one may equip $\mathrm{THH}(A)$ with the structure of a *cyclotomic spectrum*. In general for X a spectrum with $\mathbb{T}$-action, a cyclotomic structure on X is given by a $\mathbb{T}$-equivariant "Frobenius" map

$$\varphi_p : X \to X^{tC_p}$$

for each prime p. The codomain X^{tC_p} carries a $\mathbb{T}/C_p$-action which is identified with a $\mathbb{T}$-action using the p'th power map $\rho : \mathbb{T}/C_p \to \mathbb{T}$. If the underlying spectrum X is p-complete then $X^{tC_l} \simeq 0$ for any prime l different from p, so that it suffices to provide φ_p in order to give X a cyclotomic structure. This is the case for example if $X = \mathrm{THH}(A)$ where A is an $\mathbb{F}_p$-algebra.

1.2. Topological cyclic homology. Given a spectrum X with a $\mathbb{T}$-action we define topological versions of negative cyclic and periodic cyclic homology using the same constructions as before

$$\mathrm{TC}^-(X) = X^{h\mathbb{T}}$$
$$\mathrm{TP}(X) = X^{t\mathbb{T}}.$$

There is always a map from the homotopy orbits to the Tate construction

$$\mathrm{can} : \mathrm{TC}^-(X) \to \mathrm{TP}(X)$$

called the canonical map. If X is a p-complete bounded below cyclotomic spectrum then the Frobenius map gives rise to another map

$$\mathrm{TC}^-(X) = X^{h\mathbb{T}} \longrightarrow (X^{tC_p})^{h(\mathbb{T}/C_p)} \simeq X^{t\mathbb{T}} = \mathrm{TP}(X)$$

which is again denoted by φ_p. We now define $\mathrm{TC}(X)$ by the fiber sequence

$$\mathrm{TC}(X) \to \mathrm{TC}^-(X) \xrightarrow{\varphi_p - can} \mathrm{TP}(X).$$

When $X = \mathrm{THH}(A)$ for a ring A we write $\mathrm{TC}(A) = \mathrm{TC}(\mathrm{THH}(A))$, similarly for TC^- and TP. In [31] it is shown that $\mathrm{TC}(X)$ agrees with the classical Bökstedt-Hsiang-Madsen construction of [6] when X is bounded below. This is the case for example for THH of a connective ring spectrum A.

2. TC of pointed monoid algebras

In their seminal paper on topological cyclic homology [6], Bökstedt, Hsiang and Madsen study TC of spherical group rings $\mathbb{S}(\Gamma_+)$, where Γ is a group. They give a simple formula for $\mathrm{TC}(\mathbb{S}(\Gamma_+))$ in terms of the free loop space $\mathrm{Map}(S^1, B\Gamma)$. Dropping the assumption that Γ has inverses, i.e. supposing it is a monoid, it is often the case that $\mathrm{TC}(\mathbb{S}(\Gamma_+))$ is still amenable to computations.

In both papers of this thesis I consider rings which are (pointed) monoid algebras. A pointed monoid Π is a monoid object in the symmetric monoidal category of pointed sets with monoidal structure given by the smash product. Suppose k is some ring and $A = k(\Pi)$ is the pointed monoid algebra of Π over k. What gets the TC computation off the ground is the observation that there is a splitting

$$\mathrm{THH}(k(\Pi)) \simeq \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi)$$

of cyclotomic spectra. Here $\mathrm{B}^{\mathrm{cy}}(\Pi)$ is the cyclic bar construction of Π. The suspension spectrum $\Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi)$ is cyclotomic with Frobenius map which factors over the homotopy $\mathbb{T}$-fixed points. It arises from a space-level "unstable Frobenius" $\psi_p : \mathrm{B}^{\mathrm{cy}}(\Pi) \to \mathrm{B}^{\mathrm{cy}}(\Pi)^{C_p}$ arising from the diagonal map $\tilde{\Delta}_p : \Pi^{\wedge n} \to \Pi^{\wedge pn}$. At this point the specifics of Paper A and Paper B differ, but the overall strategy is the same, as I will now explain. I follow closely Paper B, which is notationally simpler, but the same ideas are present in Paper A.

One starts by finding a $\mathbb{T}$-equivariant splitting

$$\mathrm{B}^{\mathrm{cy}}(\Pi) \simeq \bigvee B(m)$$

with two key properties

(1) Each sub $\mathbb{T}$-space $B(m)$ has non-zero reduced homology in only two consecutive degrees $d(m)$ and $d(m)+1$, depending on m and with $d(m) \to \infty$ as $m \to \infty$.

(2) The unstable Frobenius ψ_p restricts to $\mathbb{T}$-equivariant homeomorphisms $\psi_p|_{B(m)} : B(m) \to B(pm)^{C_p}$.

Property (1) implies that the Atiyah-Hirzebruch spectral sequence computing $\pi_*(\mathrm{THH}(k) \otimes B(m))$ degenerates allowing one to directly read off these groups from the E^2-page. This also uses the fundamental *Bökstedt periodicity*

$$\pi_* \mathrm{THH}(k) = k[x] \quad \text{where} \quad |x| = 2$$

due to Bökstedt for $k = \mathbb{F}_p$ in [5] and extended to general perfectoid rings in [3].

Property (2) allows sufficient understanding of the cyclotomic Frobenius $\varphi_p : \mathrm{THH}(k(\Pi)) \to \mathrm{THH}(k(\Pi))^{tC_p}$ to conclude that its restriction

$$\varphi_p : \mathrm{THH}(k) \otimes B(m) \longrightarrow (\mathrm{THH}(k) \otimes B(pm))^{tC_p}$$

induces isomorphisms on homotopy groups in degrees $\geq d(m)+1$. One can now determine $\mathrm{TC}^-_*(\mathrm{THH}(k) \otimes B(m))$ and $\mathrm{TP}_*(\mathrm{THH}(k) \otimes B(m))$ using the Tate spectral sequence

$$E^2 = \pi_*(\mathrm{THH}(k) \otimes B(m)) \otimes_k k[t^{\pm 1}] \quad \Rightarrow \quad \pi_* \mathrm{TP}(\mathrm{THH}(k) \otimes B(m))$$

and the homotopy $\mathbb{T}$-fixed point spectral sequence

$$E^2 = \pi_*(\mathrm{THH}(k) \otimes B(m)) \otimes_k k[t] \quad \Rightarrow \quad \pi_* \mathrm{TC}^-(\mathrm{THH}(k) \otimes B(m)).$$

This is done by induction on m with the Frobenius (which induces an isomorphism on high enough homotopy groups)

$$\pi_* \varphi_p : \pi_* \operatorname{TC}^-(\operatorname{THH}(k) \otimes B(m)) \longrightarrow \pi_* \operatorname{TP}(\operatorname{THH}(k) \otimes B(pm))$$

providing the necessary input for the induction step. Here again one uses the Bökstedt periodicity theorem in the form of the calculation

$$\operatorname{TP}_*(k) = W(k)[t^{\pm 1}]$$

to express $\operatorname{TC}^-_*(\operatorname{THH}(k) \otimes B(m))$ and $\operatorname{TP}_*(\operatorname{THH}(k) \otimes B(pm))$ in terms of Witt vectors. By this point one has all the pieces needed in order to determine $\operatorname{TC}(k(\Pi))$ using the fiber sequence

$$\operatorname{TC}(k(\Pi)) \longrightarrow \operatorname{TC}^-(k(\Pi)) \xrightarrow{\varphi_p - \mathrm{can}} \operatorname{TP}(k(\Pi))$$

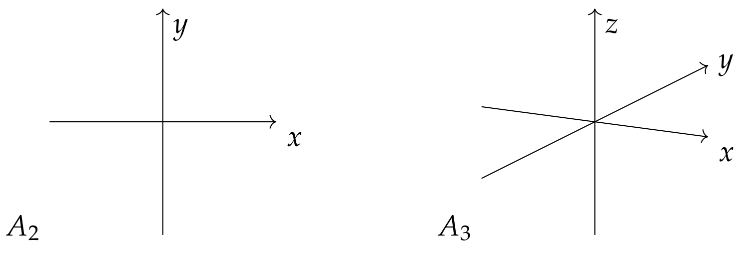
from [31].

3. Paper A – On the K-theory of coordinate axes in affine space

This goal of this paper is to compute the K-theory of the rings

$$\begin{aligned}
A_2 &= k[x,y]/(xy) \\
A_3 &= k[x,y,z]/(xy,xz,yz) \\
A_4 &= k[x,y,z,w]/(xy,xz,xw,yz,yw,zw) \\
&\dots
\end{aligned}$$

For each dimension $d \geq 2$ the ring $A_d = k[x_1,\dots,x_d]/(x_i x_j)_{i \neq j}$ is the ring of functions on the coordinate axes in affine d-space.



I take k to be a perfect field of characteristic $p > 0$. The main theorem describes the K-groups of A_d relative to the ideal $I_d = (x_1, \dots, x_d)$ defining the singular point at the origin. These relative K-groups are expressed in terms of the p-typical Witt vectors $W_t(k)$ of the field k. If, for example $k = \mathbb{F}_p$, then $W_t(k) = \mathbb{Z}/p^t\mathbb{Z}$.

Suppose $m' \geq 2$ and let $t_{ev} = t_{ev}(p,r,m')$ be the unique positive integer such that $p^{t_{ev}-1}m' \leq 2r < p^{t_{ev}}m'$, or zero if such t_{ev} does not exist. Let $t_{od} = t_{od}(p,r,m')$ be the unique positive integer such that $p^{t_{od}-1}m' \leq 2r+1 < p^{t_{od}}m'$, or zero if such t_{od} does not exist. Let J_p denote the set of positive integers $m' \geq 2$ which are coprime to p. The result is stated using a function $\mathrm{cyc}_d(s)$, which counts the number of cyclic words of in d letters, of length s, period s, having no cyclic repetitions. This function is studied in an appendix to the paper.

THEOREM 3.1. *Let k be a perfect field of characteristic $p > 0$. Let A_d be the ring $k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$ of coordinate axes, and let $I_d = (x_1, \ldots, x_d)$. Then if $p > 2$*

$$K_q(A_d, I_d) \cong \begin{cases} \prod_{\substack{m' \in J_p \\ \text{even}}} \prod_{\substack{s' \mid m' \\ \text{even}}} \prod_{u \leq t_{ev}} W_{t_{ev}-u}(k)^{\oplus \mathrm{cyc}_d(p^u s')} & q = 2r \\ \prod_{\substack{m' \in J_p \\ \text{odd}}} \prod_{\substack{s' \mid m' \\ \text{odd}}} \prod_{u \leq t_{od}} W_{t_{od}-u}(k)^{\oplus \mathrm{cyc}_d(p^u s')} & q = 2r+1 \end{cases}$$

If $p = 2$ then

$$K_q(A_d, I_d) \cong \begin{cases} \prod_{\substack{m' \geq 1 \\ \text{odd}}} \prod_{s' \mid m'} \prod_{1 \leq u \leq t_{ev}} W_{t_{ev}-u}(k)^{\oplus \mathrm{cyc}_d(2^u s')} & q = 2r \\ \prod_{\substack{m' \geq 1 \\ \text{odd}}} \prod_{s' \mid m'} \prod_{0 \leq v \leq t_{ev}} k^{\oplus \mathrm{cyc}_d(s')} & q = 2r+1 \end{cases}$$

The two-dimensional case of Theorem 3.1 is due to Hesselholt [19]. If $q = 2$ then the calculation is due to Dennis and Krusemeyer in 1979 [11]. In the analogous case of coordinate axes over fields k of characteristic zero the K-theory was calculated by Geller, Reid, and Weibel in 1989. I extend their calculation to the ind-smooth $\mathbb{Q}$-algebras.

THEOREM 3.2. *Suppose k is an ind-smooth $\mathbb{Q}$-algebra. For $d \geq 2$ consider the ring $A_d = k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$, and let $I_d = (x_1, \ldots, x_d)$. Then*

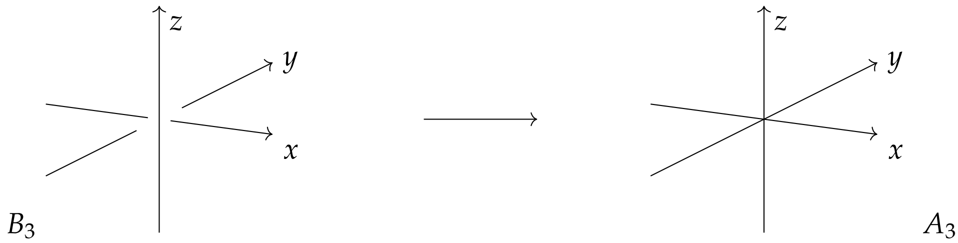
$$K_q(A_d, I_d) \cong k^{\oplus c_d(q)} \oplus (\Omega^1_{k/\mathbb{Q}})^{\oplus c_d(q-1)} \oplus \cdots \oplus (\Omega^{q-2}_{k/\mathbb{Q}})^{\oplus c_d(2)}.$$

Here $\Omega^*_k/\mathbb{Q}$ is the algebraic de Rham complex and $c_d(q)$ is a counting function closely related to the one appearing in Theorem 3.1. The proof of Theorem 3.2 uses the result, due to Cortiñas [9], that the obstruction to excision in rational K-theory and rational negative cyclic homology are the same.

3.1. Sketch of proof. In this subsection I will sketch the proof of Theorem 3.1. A salient feature of the rings A_d is the singularity at the origin

$$x = y = z = \cdots = 0$$

Since K-theory of non-singular varieties is better understood, it is a good idea to remove this singularity. This is quite simple using the normalization map $A_d \to B_d$ where $B_d = k[x_1] \times \cdots \times k[x_d]$ is the disjoint union of d lines.



The K-theory of B_d is very simple using the homotopy invariance of K-theory for regular rings. The result is that

$$K(B_d) \simeq K(k)^{\times d}$$

It remains to figure out how K-theory interacts with "gluing at the origin" as encoded in the normalization map $A_d \to B_d$. This is an example of the *excision problem* in K-theory. More generally, given a ring homomorphism $f : A \to B$ which maps an ideal $I \subseteq A$ isomorphically to an ideal $f(I) \cong I \subseteq B$ one obtains a pullback diagram of rings

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A/I \arrow[d] \\ B \arrow[r] & B/I \end{tikzcd}$$

The excision problem asks whether the associated diagram induced on K-theory

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & K(A/I) \arrow[d] \\ K(B) \arrow[r] & K(B/I) \end{tikzcd}$$

is a pullback diagram in spectra. This is not true in general. However, the discrepancy for it to be true for K-theory agrees with the discrepancy for it to be true for TC. More precisely, in analogy with the result by Cortiñas [9], Geisser and Hesselholt [14] show that the cyclotomic trace between the bi-relative K-theory and bi-relative TC

$$K_q(A, B, I, \mathbb{Z}/p^\nu) \longrightarrow \mathrm{TC}_q(A, B, I; p, \mathbb{Z}/p^\nu)$$

is an isomorphism on homotopy groups with $\mathbb{Z}/p^\nu\mathbb{Z}$ coefficients. Returning to the rings A_d and B_d we may now translate the K-theory problem into a TC-problem. The result is that Theorem 3.1 follows from the following theorem.

THEOREM 3.3. *Let k be a perfect field of characteristic $p > 0$. Let A_d be the ring $k[x_1, \dots, x_d]/(x_i x_j)_{i \neq j}$ of coordinate axes, $B_d = k[x_1] \times \cdots \times k[x_d]$ and let $I_d = (x_1, \dots, x_d)$. Then if $p > 2$*

$$\mathrm{TC}_q(A_d, B_d, I_d) \cong \begin{cases} \prod_{\substack{m' \in J_p \\ even}} \prod_{\substack{s' | m' \\ even}} \prod_{u \leq t_{ev}} W_{t_{ev}-u}(k)^{\oplus \operatorname{cyc}_d(p^u s')} & q = 2r \\ \prod_{\substack{m' \in J_p \\ odd}} \prod_{s' | m'} \prod_{u \leq t_{od}} W_{t_{od}-u}(k)^{\oplus \operatorname{cyc}_d(p^u s')} & q = 2r+1 \end{cases}$$

If $p = 2$ then

$$\mathrm{TC}_q(A_d, B_d, I_d) \cong \begin{cases} \prod_{\substack{m' \geq 1 \\ odd}} \prod_{s' | m'} \prod_{1 \leq u \leq t_{ev}} W_{t_{ev}-u}(k)^{\oplus \operatorname{cyc}_d(2^u s')} & q = 2r \\ \prod_{\substack{m' \geq 1 \\ odd}} \prod_{s' | m'} \prod_{0 \leq v \leq t_{ev}} k^{\oplus \operatorname{cyc}_d(s')} & q = 2r+1. \end{cases}$$

To prove this I apply the method sketched above in Section 2 using the monoid $\Pi^d = \{0, 1, x_1, x_1^2, \dots, x_2, x_2^2, \dots, x_d, x_d^2, \dots\}$. The required splitting of $\mathrm{B}^{\mathrm{cy}}(\Pi^d)$ is achieved through the use of cyclic words without cyclic repetitions. The function $\operatorname{cyc}_d(s)$ counts such words.

4. Paper B – On the K-theory of truncated polynomial algebras, revisited

In this paper I compute the K-theory of truncated polynomial algebras

$$A = k[x]/(x^e), \qquad e \geq 2$$

over perfect fields k of positive characteristic. This was done already in the mid 1990s by Hesselholt and Madsen [20], and the aim of Paper B is to display the strength of the new approach to cyclotomic spectra from [31]. Indeed using the method sketched in Section 2 it is possible to substantially reduce the difficulty of this computation.

THEOREM 4.1 ([20]). *Let k be a perfect field of positive characteristic. Then there is an isomorphism*

$$K_{2r-1}(k[x]/(x^e), (x)) \simeq \mathbb{W}_{re}(k)/V_e \mathbb{W}_r(k)$$

and the groups in even degrees are zero.

To give a sense of the intricacies inherent in the original proof I will briefly sketch the ideas involved. The ring A is a pointed monoid algebra on the monoid $\Pi_e = \{0, 1, x, \dots, x^{e-1}\}$ determined by $x^e = 0$. In [20] the authors consider the $\mathbb{T}$-equivariant splitting

$$\mathrm{B}^{\mathrm{cy}}(\Pi_e) \simeq \bigvee_{m \geq 0} B(m)$$

where $B(m)$ is the geometric realization of the sub-cyclic set generated by the $m-1$-simplex $x \wedge \cdots \wedge x$ with m factors. In order to compute $\mathrm{TC}(A)$

they determine the $\mathbb{T}$-equivariant homotopy type of the spaces $B(m)$. This turns out to be difficult task. Indeed, it is fairly simple to obtain an equivariant homeomorphism

$$B(m) \cong \mathbb{T}_+ \wedge_{C_m} (\Delta^{m-1}/C_m \cdot \Delta^{m-e})$$

where C_m acts on Δ^{m-1} by permuting the vertices, and where Δ^{m-e} is the face spanned by the first $m-e+1$ vertices. In order to determine the $\mathbb{T}$-equivariant homotopy type of $\mathbb{T}_+ \wedge_{C_m} (\Delta^{m-1}/C_m \cdot \Delta^{m-e})$ the authors investigate the facet structure of regular cyclic polytopes. The regular cyclic m-polytope $P_{m,d}$ in $2d$-space is defined as the convex hull in $\mathbb{R}^{2d}$ of the m points $x(2\pi j/m)$, (for $j = 0, \dots, m-1$) on the trigonometric moment curve

$$x(t) = (\cos t, \sin t, cos2t, sin2t, \dots, \cos dt, \sin dt) \in \mathbb{R}^{2d}.$$

Gale's evenness criterion [13] (known by Carathéodory already in 1911!) states that a facet of $P_{m,d}$ is given by the convex hull of the d points $x_{i_1}, x_{i_1+1}, \dots, x_{i_d}, x_{i_d+1}$. The regular C_m-representation $\mathbb{R}C_m$ has a projection

$$\pi_d : \mathbb{R}C_m \to \lambda_d$$

whenever[2] $d \leq \lfloor \frac{m-1}{2} \rfloor$. Under this projection, $\pi_d(\Delta^{m-1}) = P_{m,d}$ and one lets $Q_{m,d} = \pi_d(C_m \cdot \Delta^{m-e})$. In [20, Theorem 3.1.2] the authors use Gale's evenness criterion to show that $0 \notin Q_{m,d}$ and $\partial P_{m,d} \subseteq Q_{m,d}$ whenever $de < m < (d+1)e$ (the case where $m = (d+1)e$ is similar but a bit more complicated). This allows them to consider the radial projection away from 0 to get a C_m-equivariant map

$$\Delta^{m-1}/(C_m \cdot \Delta^{m-e}) \xrightarrow{\pi_d} P_{m,d}/Q_{m,d} \xrightarrow{r} P_{m,d}/\partial P_{m,d} = S^{\lambda_d}$$

Now the authors show that the resulting map

$$\mathbb{T}_+ \wedge_{C_m} \Delta^{m-1}/(C_m \cdot \Delta^{m-e}) \longrightarrow \mathbb{T}_+ \wedge_{C_m} S^{\lambda_d}$$

is a homotopy equivalence. With this at hand the methods of [21] now apply.

I show in Paper B how one can make do with only the reduced homology of the spaces $B(m)$, together with the action of Connes' operator. Determining this is much easier than computing the homotopy type, since the reduced homology of $\mathrm{B}^{\mathrm{cy}}(\Pi_e)$ is the Hochschild homology of the k-algebra A, which is known by work of the Buenos Aires cyclic homology group [16]. One may then read off the homology of the sub-spaces from a corresponding splitting of the Hochschild homology. From here one

[2] See Lemma 3.4. of Paper A for notation and a proof

follows the strategy outlined in Section 2. In particular this avoids any recourse to the theory of cyclic polytopes.

5. Future perspectives

The work done in Papers A and B suggests several exciting research problems. There are at least two different types of questions that I believe are both interesting and accessible.

Firstly, it would be very interesting to use these methods to make new computations in K-theory. The literature on Hochschild and cyclic homology abounds with computations, [17, 26, 30, 34], though most computations are only carried out in the characteristic zero context. Two concrete examples that I have in mind are the following. Let k be a perfect field of characteristic $p > 0$.

(1) Let $1 \leq r \leq d$ and consider the ring $A = k[x_1, \ldots, x_d]/(x_1 \ldots x_r)$ defining an intersection of coordinate hyperplanes in affine d-space over k. This is the local form for a strict normal crossings divisor, so computing $K(A)$ could yield applications to algebraic geometry.
(2) The K-theory of the cone $z^2 = xy$ was computed in [10, Theorem 4.3.] when the base ring is a field of characteristic zero. The result is quite similar to that of Theorem 3.2 above, suggesting that the methods we develop could be used to make similar computations in the positive characteristic case.

A second direction for future work consists in extending the results of Papers A and B. In particular I would like to study the extent to which the assumptions on the base ring k may be relaxed. There are at least two plausible directions here.

(1) In [2] the authors make partial computations of the K-theory of truncated polynomial algebras over the integers. The same is done in [1] for the case A_2 of coordinate axes in two variables. It seems likely that the methods of *op. cit.* should be applicable also to higher dimensions $A_d = \mathbb{Z}[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$.
(2) Sticking to the case of $\mathbb{F}_p$-algebras a clear deficiency of the method outlined in Section 2 is the reliance on the assumption that k is a perfect field. Using the classical approach to TC, Hesselholt-Madsen [22] extend their methods to the case of regular $\mathbb{F}_p$-algebras, a huge leap of generality. They express their results in terms of the

de Rham-Witt complex. This should be extended to the Nikolaus-Scholze setup, and potentially requires the use results of from [3]. A first step would be to generalize the computations for general perfectoid rings. Indeed, the authors of *op. cit.* generalize Bökstedt periodicity to this setting, indicating that such a generalization should be possible.

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Part 2

Paper A

ON THE K-THEORY OF COORDINATE AXES IN AFFINE SPACE

MARTIN SPEIRS

1. Introduction

Let k be a perfect field of characteristic $p > 0$ and let A_d denote the k-algebra $k[x_1, \dots, x_d]/(x_i x_j)_{i \neq j}$, which is the coordinate ring of the coordinate $(x_1, \dots, x_d)$-axes in affine d-space $\mathbb{A}^d = \operatorname{Spec}(k[x_1, \dots, x_d])$ over k. This is an affine curve with a singularity at the origin, the singularity being defined by the ideal $I_d = (x_1, \dots, x_d)$. Our main result is the computation of the relative algebraic K-theory, $K(A_d, I_d)$ of the pair (A_d, I_d). The relative K-theory is defined to be the mapping fiber of the map $K(A_d) \to K(k)$ induced by the projection onto $k = A_d/I_d$.

To state the result we introduce some notation. We consider words in d letters $x_1, \dots, x_d$, i.e. a finite string $\omega = w_1 w_2 \dots w_m$ where each w_i is one of the letters $x_1, \dots, x_d$. A word $\omega = w_1 w_2 \dots w_m$ has *no cyclic repetitions* if $w_i \neq w_{i+1}$ for all $i = 1, \dots, m-1$ and if $w_m \neq w_1$. For $d \geq 1$ and $s \geq 1$ let $\operatorname{cyc}_d(s)$ denote the number of cyclic words in d letters, of length s, period s, having no cyclic repetitions. In Section 7 we give a formula for $\operatorname{cyc}_d(s)$. Suppose $m' \geq 2$ and let $t_{ev} = t_{ev}(p, r, m')$ be the unique positive integer such that $p^{t_{ev}-1} m' \leq 2r < p^{t_{ev}} m'$, or zero if such t_{ev} does not exist. Let $t_{od} = t_{od}(p, r, m')$ be the unique positive integer such that $p^{t_{od}-1} m' \leq 2r + 1 < p^{t_{od}} m'$, or zero if such t_{od} does not exist. Let J_p denote the set of positive integers $m' \geq 2$ which are coprime to p.

Theorem 1.1. *Let k be a perfect field of characteristic $p > 0$. Let A_d be the ring $k[x_1, \dots, x_d]/(x_i x_j)_{i \neq j}$ of coordinate axes, and let $I_d = (x_1, \dots, x_d)$. Then if $p > 2$*

$$K_q(A_d, I_d) \cong \begin{cases} \prod_{\substack{m' \in J_p \\ \text{even}}} \prod_{\substack{s' \mid m' \\ \text{even}}} \prod_{u \leq t_{ev}} W_{t_{ev}-u}(k)^{\oplus \operatorname{cyc}_d(p^u s')} & q = 2r \\ \prod_{\substack{m' \in J_p \\ \text{odd}}} \prod_{s' \mid m'} \prod_{u \leq t_{od}} W_{t_{od}-u}(k)^{\oplus \operatorname{cyc}_d(p^u s')} & q = 2r+1 \end{cases}$$

If $p = 2$ then

$$K_q(A_d, I_d) \cong \begin{cases} \prod_{\substack{m' \geq 1 \\ odd}} \prod_{s'|m'} \prod_{1 \leq u \leq t_{ev}} W_{t_{ev}-u}(k)^{\oplus \operatorname{cyc}_d(2^u s')} & q = 2r \\ \prod_{\substack{m' \geq 1 \\ odd}} \prod_{s'|m'} \prod_{0 \leq v \leq t_{ev}} k^{\oplus \operatorname{cyc}_d(s')} & q = 2r+1 \end{cases}$$

In both cases $t_{ev} = t_{ev}(p, r, m')$ and $t_{od} = t_{od}(p, r, m')$ are as defined above.

Note that the products appearing in the statement are all finite since for m' large enough $t_{ev} = t_{od} = 0$.

The result extends the calculation by Dennis and Krusemeyer when $q = 2$, [5, Theorem 4.9]. Hesselholt carried out the computation in the case $d = 2$ in [13]. Our strategy follows the one in [13], except that we use the framework for TC set up by Nikolaus and Scholze [20] to which we refer for background on TC and cyclotomic spectra. The computation is achieved through the use of the cyclotomic trace map from K-theory to topological cyclic homology, TC. See also [14] for background on cyclotomic spectra and for similar calculations. Recently, Hesselholt and Nikolaus [16] have completed a calculation for K-theory of cuspidal curves using similar methods as this paper.

1.1. **Overview.** In Section 2 we reduce the K-theory computation to a TC computation. Section 3 carries out the requisite THH computation. Then Section 4 and Section 5 assemble this to complete the proof. Section 4 contains a new method for computing TP, which makes crucial use of the Nikolaus-Scholze framework, see also [16]. In Section 6 we consider the characteristic zero situation and extend the computation of [8, Theorem 7.1.]. Finally in Section 7 we derived the necessary counting formula for cyclic words.

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2. Bi-relative K-theory and topological cyclic homology

The normalization of A_d is just d disjoint lines, whose coordinate ring is

$$B_d = k[x_1] \times \cdots \times k[x_d].$$

Gluing these lines together at $x_1 = x_2 = \cdots = x_d = 0$ one obtains $\mathrm{Spec}(A_d)$. Algebraically this is the statement that the following square is a pullback of rings

$$\begin{tikzcd} A_d \arrow[r] \arrow[d] & k \arrow[d] \\ B_d \arrow[r] & k^{\times d} \end{tikzcd}$$

Here the horizontal maps take the variables x_i to zero. The left-vertical map is the normalization map. It maps x_i to $(0, \ldots, x_i \ldots, 0)$ where x_i is in the i'th position. The right-vertical map is the diagonal. If K-theory preserved pullbacks then the diagram would give a computation of $K(A_d)$ in terms of $K(B_d)$ and $K(k)$. Using the fundamental theorem of K-theory (since k is regular) one would get a formula for $K(A_d)$ purely in terms of $K(k)$. But K-theory does not preserve pullbacks. However, there is still something to be done. We can form the bi-relative K-theory, $K(A_d, B_d, I_d)$ as the iterated mapping fiber of the diagram

$$\begin{tikzcd} K(A_d) \arrow[r] \arrow[d] & K(k) \arrow[d] \\ K(B_d) \arrow[r] & K(k^{\times d}) \end{tikzcd}$$

Again, if K-theory preserved pullbacks then $K(A_d, B_d, I_d)$ would be trivial, but (as we shall see) it is not. Since k is regular the fundamental theorem of algebraic K-theory [21, Section 6], and the fact that K-theory *does* preserve products, shows that the canonical map

$$K(A_d, B_d, I_d) \longrightarrow K(A_d, I_d)$$

is an equivalence. Here $K(A_d, I_d)$ is the relative K-theory spectrum, i.e. the mapping fiber of the map $K(A_d) \to K(A_d/I_d)$ induced by the quotient. Any splitting of the quotient map $A_d \to A_d/I_d$ provides a splitting of

K-groups

$$K_q(A_d) \cong K_q(k) \oplus K_q(A_d, I_d).$$

So if we can compute $K_q(A_d, I_d) = K_q(A_d, B_d, I_d)$ we have a computation of $K_q(A_d)$. This is what we will do.

Geisser and Hesselholt [6] have shown that the cyclotomic trace induces an isomorphism

$$K_q(A, B, I, \mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TC}_q(A, B, I; p, \mathbb{Z}/p^v)$$

between the bi-relative K-theory and the bi-relative topological cyclic homology, when both a considered with $\mathbb{Z}/p^v$ coefficients. The corresponding statement with rational coefficients (using Connes-Tsygan negative cyclic homology and the Chern character) was proven by Cortiñas in [3].

In fact it suffices to compute $\mathrm{TC}(A_d, B_d, I_d, p; \mathbb{Z}_p)$ since the trace map

$$K(A, B, I) \to \mathrm{TC}(A, B, I, p; \mathbb{Z}_p)$$

is an equivalence whenever p is nilpotent in A, as shown in [7, Theorem C]. Furthermore, since $\mathrm{THH}(A_d, B_d, I_d; p)$ is an Hk-module, it is in particular p-complete, and so $\mathrm{TC}(A, B, I, p; \mathbb{Z}_p) \simeq \mathrm{TC}(A, B, I, p)$ (see [20, Section II.4]). Thus, to prove Theorem 1.1 it suffices to prove the following result.

Theorem 2.1. *Let k be a perfect field of characteristic $p > 0$. Let A_d be the ring $k[x_1, \dots, x_d]/(x_i x_j)_{i \neq j}$ of coordinate axes, $B_d = k[x_1] \times \cdots \times k[x_d]$ and let $I_d = (x_1, \dots, x_d)$. Then if $p > 2$*

$$\mathrm{TC}_q(A_d, B_d, I_d) \cong \begin{cases} \prod_{\substack{m' \in J_p \\ even}} \prod_{\substack{s' \mid m' \\ even}} \prod_{u \le t_{ev}} W_{t_{ev}-u}(k)^{\oplus \mathrm{cyc}_d(p^u s')} & q = 2r \\ \prod_{\substack{m' \in J_p \\ odd}} \prod_{s' \mid m'} \prod_{u \le t_{od}} W_{t_{od}-u}(k)^{\oplus \mathrm{cyc}_d(p^u s')} & q = 2r+1 \end{cases}$$

If $p = 2$ then

$$\mathrm{TC}_q(A_d, B_d, I_d) \cong \begin{cases} \prod_{\substack{m' \ge 1 \\ odd}} \prod_{s' \mid m'} \prod_{1 \le u \le t_{ev}} W_{t_{ev}-u}(k)^{\oplus \mathrm{cyc}_d(2^u s')} & q = 2r \\ \prod_{\substack{m' \ge 1 \\ odd}} \prod_{s' \mid m'} \prod_{0 \le v \le t_{ev}} k^{\oplus \mathrm{cyc}_d(s')} & q = 2r+1. \end{cases}$$

3. THH and the cyclic bar construction

The spectrum $\mathrm{TC}(A_d, B_d, I_d)$ is defined using topological Hochschild homology, so that is where we start. In this section we drop the subscript d, so that $A = A_d$, $B = B_d$ and $I = I_d$. The bi-relative topological Hochschild homology is the spectrum $\mathrm{THH}(A, B, I)$ defined as the iterated mapping fiber of the following diagram.

$$\begin{array}{ccc} \mathrm{THH}(A) & \longrightarrow & \mathrm{THH}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{THH}(B) & \longrightarrow & \mathrm{THH}(B/I) \end{array} \tag{1}$$

The ring A is a pointed monoid ring and, by Lemma 3.1 below, we may compute $\mathrm{THH}(A)$ in terms of the cyclic bar construction of the defining pointed monoid.

3.1. **Unstable cyclotomic structure on the cyclic bar construction.** Let Π be a pointed monoid, that is a monoid object in the symmetric monoidal category of based spaces and smash product. The cyclic bar construction of Π is the cyclic space $\mathrm{B}^{\mathrm{cy}}(\Pi)[-]$ with

$$\mathrm{B}^{\mathrm{cy}}(\Pi)[k] = \Pi^{\wedge(k+1)}$$

and with the usual Hochschild-type structure maps.

$$\begin{aligned} d_i(\pi_0 \wedge \cdots \wedge \pi_m) &= \begin{cases} \pi_0 \wedge \cdots \wedge \pi_i\pi_{i+1} \wedge \cdots \wedge \pi_m & 0 \leq i < m \\ \pi_m\pi_0 \wedge \pi_1 \wedge \cdots \wedge \pi_{m-1} & i = m \end{cases} \\ s_i(\pi_0 \wedge \cdots \wedge \pi_m) &= \pi_0 \wedge \cdots \wedge \pi_i \wedge 1 \wedge \pi_{i+1} \wedge \cdots \wedge \pi_m \\ t_m(\pi_0 \wedge \cdots \wedge \pi_m) &= \pi_m \wedge \pi_0 \wedge \cdots \wedge \pi_{m-1} \end{aligned}$$

We write $\mathrm{B}^{\mathrm{cy}}(\Pi)$ for the geometric realization of $\mathrm{B}^{\mathrm{cy}}(\Pi)[-]$. It is a space with $\mathbb{T}$-action where $\mathbb{T}$ is the circle group (for a proof see for example [19, Theorem 7.1.4.]). Furthermore it is an unstable cyclotomic space, i.e. there is a map

$$\psi_p : \mathrm{B}^{\mathrm{cy}}(\Pi) \to \mathrm{B}^{\mathrm{cy}}(\Pi)^{C_p}$$

which is equivariant when the domain is given the natural $\mathbb{T}/C_p$-action. This map goes back to [2] section 2. We briefly sketch the construction. The C_p-action on $\mathrm{B}^{\mathrm{cy}}(\Pi)$ is not simplicial, but we can make it so by using the edge-wise subdivision functor $\mathrm{sd}_p : \Delta \to \Delta$ which is given by the p-fold concatenation, $\mathrm{sd}_p[m-1] = [m-1] \amalg \cdots \amalg [m-1]$ and $\mathrm{sd}_p(\theta) = \theta \amalg \cdots \amalg \theta$ for morphisms $\theta : [m-1] \to [n-1]$. Given a simplicial set $X[-]$ we let $\mathrm{sd}_p X[-] = X[-] \circ \mathrm{sd}_p^{op}$. For the topological simplex $\Delta^{m-1} \subset \mathbb{R}^m$ let $d_p : \Delta^{m-1} \to \Delta^{pm-1}$ be the diagonal embedding

$$d_p(z) = \frac{1}{p} z \oplus \cdots \oplus \frac{1}{p} z.$$

This induces a (non-simplicial) map on geometric realization

$$D_p : |(\mathrm{sd}_p X)[-]| \to |X[-]|$$

by $id \times d_p : X[pm-1] \times \Delta^{m-1} \to X[pm-1] \times \Delta^{pm-1}$. Then [2, Lemma 1.1.] shows that D_p is a homeomorphism – one need only check on the representables $\Delta^k[-]$ where it follows by explicit calculations. In the case $X[-] = \mathrm{B}^{\mathrm{cy}}(\Pi)[-]$ one has available the simplicial diagonal

$$\tilde{\Delta_p} : \mathrm{B}^{\mathrm{cy}}(\Pi)[-] \to \mathrm{sd}_p \mathrm{B}^{\mathrm{cy}}(\Pi)[-]$$

given by

$$\vec{\pi} = \pi_0 \wedge \cdots \wedge \pi_m \quad \longmapsto \quad \vec{\pi} \wedge \cdots \wedge \vec{\pi} \qquad (p \text{ copies})$$

It clearly lands in the C_p-fixed points of $\mathrm{sd}_p \mathrm{B}^{\mathrm{cy}}(\Pi)[-]$ and induces an isomorphism $\tilde{\Delta_p} : \mathrm{B}^{\mathrm{cy}}(\Pi)[-] \to \mathrm{sd}_p \mathrm{B}^{\mathrm{cy}}(\Pi)[-]^{C_p}$. We define

$$\psi_p : \mathrm{B}^{\mathrm{cy}}(\Pi) \to \mathrm{B}^{\mathrm{cy}}(\Pi)^{C_p}$$

to be the composite

$$\begin{aligned} \mathrm{B}^{\mathrm{cy}}(\Pi) = |\,\mathrm{B}^{\mathrm{cy}}(\Pi)[-]| \quad & \xrightarrow[\cong]{\tilde{\Delta}_p} \quad |\,\mathrm{sd}_p \mathrm{B}^{\mathrm{cy}}(\Pi)[-]^{C_p}| \\ & \xrightarrow[\cong]{} \quad |\,\mathrm{sd}_p \mathrm{B}^{\mathrm{cy}}(\Pi)[-]|^{C_p} \\ & \xrightarrow[\cong]{D_p} \quad |\,\mathrm{B}^{\mathrm{cy}}(\Pi)[-]|^{C_p} = \mathrm{B}^{\mathrm{cy}}(\Pi)^{C_p} \end{aligned}$$

where the middle map is the canonical equivalence witnessing the fact that geometric realization commutes with finite limits.

Passing to suspension spectra now gives us a cyclotomic structure on $\mathrm{THH}(\mathbb{S}(\Pi))$ where $\mathbb{S}(\Pi) = \Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi)$. Indeed, there is always a $\mathbb{T}/C_p$-equivariant map $\mathrm{B}^{\mathrm{cy}}(\Pi)^{C_p} \to \mathrm{B}^{\mathrm{cy}}(\Pi)^{hC_p}$. Composing with this map gives a $\mathbb{T} \simeq \mathbb{T}/C_p$-equivariant map $\psi^h : \mathrm{B}^{\mathrm{cy}}(\Pi) \to \mathrm{B}^{\mathrm{cy}}(\Pi)^{hC_p}$. Thus, we obtain a

$\mathbb{T} \simeq \mathbb{T}/C_p$-equivariant map

$$\begin{aligned}
\mathrm{THH}(\mathbb{S}(\Pi)) = \Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi) &\xrightarrow{\psi^h} \Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi)^{hC_p} \\
&\longrightarrow (\Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi))^{hC_p} \\
&\xrightarrow{can} (\Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi))^{tC_p} = \mathrm{THH}(\mathbb{S}(\Pi))^{tC_p}
\end{aligned}$$

where the middle map is the canonical map from the suspension spectrum of a homotopy limit, to the homotopy limit of the suspension spectrum and the last map is the canonical map from the homotopy fixed points to the Tate construction.

3.2. THH **of monoid algebras.**

Both of the algebras A and B (and their quotients by I) are pointed monoid algebras, enabling us to use the following splitting result.

Lemma 3.1. [14, Theorem 5.1.] *Let k be a ring, Π a pointed monoid, and $k(\Pi)$ the pointed monoid algebra. Let $\mathrm{B}^{\mathrm{cy}}(\Pi)$ be the cyclic bar-construction on Π. Then there is a $\mathbb{T}$-equivariant equivalence*

$$\mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi) \xrightarrow{\sim} \mathrm{THH}(k(\Pi)).$$

Under this equivalence, the Frobenius on $\mathrm{THH}(k(\Pi))$ is identified with the tensor product of the Frobenius on $\mathrm{THH}(k)$ with the one on $\mathrm{THH}(\mathbb{S}(\Pi))$.

Proof. Since THH is symmetric monoidal [20, Section IV.2.] we obtain

$$\mathrm{THH}(k(\Pi)) = \mathrm{THH}(k \otimes \mathbb{S}(\Pi)) \simeq \mathrm{THH}(k) \otimes \mathrm{THH}(\mathbb{S}(\Pi)).$$

Since $\mathrm{THH}(\mathbb{S}(\Pi)) = \mathbb{S}(\mathrm{B}^{\mathrm{cy}}(\Pi))$ we obtain

$$\mathrm{THH}(k(\Pi)) \simeq \mathrm{THH}(k) \otimes \mathbb{S}(\mathrm{B}^{\mathrm{cy}}(\Pi))$$

as claimed. □

Let $\Pi^d = \{0, 1, x_1, x_1^2, \ldots, x_2, x_2^2, \ldots, x_d, x_d^2, \ldots\}$ be the multiplicative monoid with base-point 0 and multiplication determined by $x_i x_j = 0$ when $i \neq j$. Then $A_d \cong k(\Pi^d)$. Let $\Pi^1 = \{0, 1, t, t^2, \ldots\}$ and $\Pi^0 = \{0, 1\}$, so $B = k(\Pi^1) \times \cdots \times k(\Pi^1)$ (with d products) and $k = k(\Pi^0)$. The diagram of cyclotomic spectra (1) is induced by the diagram of pointed monoids

(2)
$$\begin{tikzcd}
\Pi^d \arrow[rr, "\varepsilon"] \arrow[d, "(\varphi_1, \dots, \varphi_d)"] & & \Pi^0 \arrow[d, "\Delta"] \\
\Pi^1 \times \cdots \times \Pi^1 \arrow[rr, "\varepsilon^{\times d}"] & & \Pi^0 \times \cdots \times \Pi^0
\end{tikzcd}$$

The map φ_j takes x_i and to 0 when $i \neq j$ and takes x_j to t. The map ε takes the variables $x_1, \dots, x_d$ and t to 0. The map Δ is the diagonal.

The cyclic bar-construction sometimes decomposes, as a pointed $\mathbb{T}$-space, into a wedge of spaces simple enough that one can understand their $\mathbb{T}$-homotopy type. To do this for $\mathrm{B}^{\mathrm{cy}}(\Pi^d)$ we need the notion of cyclic words.

We consider the alphabet $S = \{x_1, \dots, x_d\}$ and words

$$\omega : \{1, 2, \dots, m\} \to S.$$

Here m is the *length* of ω. The cyclic group C_m acts on the set of words of length m. An orbit for this action is called a *cyclic word*. Such a cyclic word $\overline{\omega}$ has a *period* namely the cardinality of the orbit. Words may be concatenated to give new, longer words, though concatenation is not well-defined for cyclic words. The empty word $\varnothing \to S$ is the unit for concatenation. It has length 0 and period 1.

We can associate words to non-zero elements of $\mathrm{B}^{\mathrm{cy}}(\Pi^d)[m]$ as follows. If $\pi \in \Pi^d = \mathrm{B}^{\mathrm{cy}}(\Pi^d)[0]$ is non-zero then it is of the form $\pi = x_j^l$ for some $1 \leq j \leq d$ and $l \geq 0$. Let $\omega(\pi)$ be the unique word of length l all of whose letters are x_j. For example $\omega(x_1^2) = x_1 x_1$ and $\omega(1) = \varnothing$. Now for a non-zero element $\pi_0 \wedge \cdots \wedge \pi_m \in \mathrm{B}^{\mathrm{cy}}(\Pi^d)[m]$ we let

$$\omega(\pi_0 \wedge \cdots \wedge \pi_m) = \omega(\pi_0) \star \cdots \star \omega(\pi_m)$$

be the concatenation of each of the words $\omega(\pi_j)$. Note that $\omega(1)$ is the empty word, which is the unit for concatenation. For a cyclic word $\overline{\omega}$ we define

$$\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})[m] \subseteq \mathrm{B}^{\mathrm{cy}}(\Pi^d)[m]$$

to be the subset consisting of the base-point and all elements $\pi_0 \wedge \cdots \wedge \pi_m$ such that

$$\omega(\pi_0 \wedge \cdots \wedge \pi_m) \in \overline{\omega}.$$

The cyclic structure maps preserve this property and so, as $m \geq 0$ varies, this defines a cyclic subset

$$\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})[-] \subseteq \mathrm{B}^{\mathrm{cy}}(\Pi^d)[-].$$

Denote by $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})$ the geometric realization of this subset. We will also often abbreviate $B(\overline{\omega}) = \mathrm{B}^{\mathrm{cy}}(\Pi^b, \overline{\omega})$. As $\overline{\omega}$ ranges over all cyclic words every non-zero m-simplex $\pi_0 \wedge \cdots \wedge \pi_m$ appears in exactly one such cyclic subset $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})$. Thus we get a decomposition

$$\mathrm{B}^{\mathrm{cy}}(\Pi^d) = \bigvee \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})$$

indexed on the set of all cyclic words with letters in $S = \{x_1, \dots, x_d\}$.

Lemma 3.2. *There is a canonical $\mathbb{T}$-equivariant equivalence*

$$\bigoplus \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}) \xrightarrow{\sim} \mathrm{THH}(A, B, I)$$

where the sum on the left-hand side is indexed over all cyclic words whose period is ≥ 2.

Proof. We consider the diagram induced from (1) using Lemma 3.1,

$$(3) \qquad \begin{array}{ccc} \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi^d) & \xrightarrow{\varepsilon} & \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi^0) \\ \downarrow{\scriptstyle (\varphi_1, \ldots, \varphi_d)} & & \downarrow{\scriptstyle \Delta} \\ (\mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi^1))^{\times d} & \xrightarrow{\varepsilon^{\times d}} & (\mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi^0))^{\times d} \end{array}$$

$\mathrm{THH}(A, B, I)$ is the iterated mapping fiber of this diagram. The mapping fiber of the left-hand vertical map consists of two parts; the part of the wedge sum indexed on cyclic words containing at least two different letters from S and a part on which ε is an equivalence. The map ε is trivial on this part indexed on cyclic words containing at least two different letters, finishing the claim. □

3.3. **Homotopy type of $\mathrm{B}^{\mathrm{cy}}(\Pi^d; \overline{\omega})$.** In this section we determine the homotopy type of the subspaces $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}) \subseteq \mathrm{B}^{\mathrm{cy}}(\Pi^d)$.

Definition 3.1. We say that a word $\omega = w_1 w_2 \dots w_m$ has *no cyclic repetitions* if $w_i \neq w_{i+1}$ for all $i = 0, 1, \dots, m-1$ and if $w_m \neq w_1$. If on the other hand this is not satisfied, we say ω (or $\overline{\omega}$) *has cyclic repetitions*.

Lemma 3.3. *Let $\overline{\omega}$ be a cyclic word of period $s \geq 2$, with letters in the alphabet $S = \{x_1, \ldots, x_d\}$. The homotopy type of the pointed $\mathbb{T}$-space $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})$ is given as follows.*

(1) If $\overline{\omega}$ has length $m = si$ and has no cyclic repetitions then a choice of representative word $\omega \in \overline{\omega}$ determines a $\mathbb{T}$-equivariant homeomorphism

$$S^{\mathbb{R}[C_m]-1} \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\sim} \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}),$$

where $\mathbb{R}[C_m] - 1$ is the reduced regular representation of C_m.

(2) If $\overline{\omega}$ has cyclic repetitions, then $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})$ is $\mathbb{T}$-equivariantly contractible.

Proof. This proof follows closely that of [13, Lemma 1.6]. Choose $\omega \in \overline{\omega}$ and let $(\pi_0, \ldots, \pi_{m-1})$ be the unique m-tuple of non-zero elements in Π^d such that $\omega(\pi_0 \wedge \cdots \wedge \pi_{m-1}) = \omega$. The pointed cyclic set $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})[-]$ is generated by the $(m-1)$-simplex $\pi_0 \wedge \cdots \wedge \pi_{m-1}$ and so by the Yoneda lemma there is a unique surjective map of pointed cyclic sets

$$f_\omega : \Lambda^{m-1}[-] \to \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})[-]$$

mapping the generator of $\Lambda^{m-1}[-]$ to the generator $\pi_0 \wedge \cdots \wedge \pi_{m-1}$. Since ω has period s, the generator $\pi_0 \wedge \cdots \wedge \pi_{m-1}$ is fixed by the cyclic operator t_m^s and so f_ω factors over the quotient subgroup of order $i = m/s$,

$$f_\omega : (\Lambda^{m-1}[-]/C_i) \to \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})[-].$$

From [14, Section 7.2.] we have a $\mathbb{T}$-equivariant homeomorphism

$$|\Lambda^{m-1}[-]| \xrightarrow{\sim} \Delta^{m-1} \times \mathbb{T}$$

(where $\mathbb{T}$ acts on the second factor) where the dual cyclic operator acts on Δ^{m-1} by the affine map that cyclicly permutes the vertices and on $\mathbb{T}$ by rotation through $2\pi/m$. Thus f_ω gives a continuous $\mathbb{T}$-equivariant surjection

$$f_\omega : (\Delta^{m-1} \times_{C_i} \mathbb{T})_+ \to \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}).$$

If $\overline{\omega}$ has no cyclic repetitions then all the faces of the generator

$$\pi_0 \wedge \cdots \wedge \pi_{m-1}$$

are the base point 0, and so f_ω collapses the entire boundary $\partial\Delta^{m-1}$ of Δ^{m-1} to the base-point. There are no other relations, and so we have a $\mathbb{T}$-equivariant homeomorphism

$$f_\omega : (\Delta^{m-1}/\partial\Delta^{m-1}) \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\sim} \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}).$$

Now we use the identification of the C_m-space $\Delta^{m-1}/\partial\Delta^{m-1}$ with the one-point compactification of the reduced regular representation to finish the proof of part *(1)*.

In case $\overline{\omega}$ *does* have a cyclic repetition, then $\pi_0 \wedge \cdots \wedge \pi_{m-1}$ will have a non-base point face. So f_ω collapses at least one codimension 1 face (and its $\mathbb{T}$-orbit) to the base-point, and leaves at least one codimension 1 face, say $F \subseteq \Delta^{m-1}$, un-collapsed. The cone on F is canonically homeomorphic to Δ^{m-1} and has a canonical null-homotopy given by shrinking down to the basepoint of the cone. This null-homotopy then induces a null-homotopy on $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega})$. However it may not be a $\mathbb{T}$-equivariant null-homotopy. To get this we note that there is a C_i-equivariant homeomorphism

$$\Delta^{s-1} * \cdots * \Delta^{s-1} \to \Delta^{m-1}$$

where C_i acts on the left by cyclically permuting the factors of the join. Again f_ω will collapse at least one codimension 1 face of Δ^{s-1} and leave at least one un-collapsed, say $F \subseteq \Delta^{s-1}$. Now the null-homotopy of $cone(F)$ will induce a $\mathbb{T}$-equivariant null-homotopy

$$\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}) \wedge [0,1]_+ \to \mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{\omega}).$$

This completes the proof. □

Let $\mathbb{C}(i)$ denote the 1-dimensional complex $\mathbb{T}$-representation where $z \in \mathbb{T} \subseteq \mathbb{C}$ acts through multiplication with the i'th power z^i. For $i \geq 1$ let $\lambda_i = \mathbb{C}(1) \oplus \cdots \oplus \mathbb{C}(i)$.

Lemma 3.4. *The regular representation $\mathbb{R}[C_m]$ is isomorphic to $\mathbb{R} \oplus \lambda_{\frac{m-2}{2}} \oplus \mathbb{R}_-$ if m is even, and $\mathbb{R} \oplus \lambda_{\frac{m-1}{2}}$ if m is odd.*

Proof. By Maschke's theorem $\mathbb{R}[C_m]$ is a semisimple ring. In particular it decomposes as a sum of irreducible sub-representations. Furthermore it contains a copy of every irreducible C_m-representation (since for any such V and any non-zero $v \in V$, there is a surjection $\mathbb{R}[C_m] \to V$ given by $\sum \lambda_g g \mapsto \sum \lambda_g g v$ so, by semisimplicity, V embeds into $\mathbb{R}[C_m]$). Since $\mathbb{R}$ and $\mathbb{C}(i)$, for $1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$, (and $\mathbb{R}_-$ in case m is even) are irreducible, pair-wise non-isomorphic, and have real dimensions summing to $\dim_{\mathbb{R}} \mathbb{R}[C_m] = m$ this completes the proof. □

Lemma 3.5. *Let $\overline{\omega}$ be a cyclic word with no cyclic repetitions, of length m, period s and with $i = m/s$ blocks.*

(1) If s is even, then there is a $\mathbb{T}$-equivariant homeomorphism

$$\Sigma B(\overline{\omega}) \simeq S^{\lambda_{m/2}} \otimes (\mathbb{T}/C_i)_+$$

(2) If both s and i are odd, then there is a $\mathbb{T}$-equivariant homeomorphism

$$B(\overline{\omega}) \simeq S^{\lambda_{(m-1)/2}} \otimes (\mathbb{T}/C_i)_+$$

(3) If s is odd, and i is even, then there is a $\mathbb{T}$-equivariant homeomorphism

$$B(\overline{\omega}) \simeq S^{\lambda_{(m-2)/2}} \otimes \mathbb{R}P^2(i)$$

where $\mathbb{R}P^2(i)$ is by definition the mapping cone of the quotient map $\mathbb{T}/C_{i/2+} \to \mathbb{T}/C_{i+}$.

Proof. We use Lemma 3.4 in each case.

(1) If $s = 2k$ is even then so is m and so

$$\mathbb{R}[C_m] - 1 \cong \lambda_{(m-2)/2} \oplus \mathbb{R}_-.$$

The restriction along the inclusion $C_i \subseteq C_m$ (given by $\sigma_i \mapsto \sigma_m^s$ where σ_j is a generator) turns the sign representation $\mathbb{R}_-$ into a trivial representation. Since $\mathbb{C}(\frac{m}{2}) = \mathbb{C}(ki) = \mathbb{R} \oplus \mathbb{R}$ as C_i-representations, we have

$$S^{\mathbb{R}[C_m]-1} \wedge_{C_i} \mathbb{T}_+ \cong S^{-1} \wedge S^{\lambda_{m/2}} \wedge_{C_i} \mathbb{T}_+ \cong S^{-1} \wedge S^{\lambda_{m/2}} \wedge (\mathbb{T}/C_i)_+$$

where the last isomorphism uses that $\lambda_{m/2}$ extends to a representation of $\mathbb{T}$ and so allows the $\mathbb{T}$-equivariant untwisting map $(x, z) \mapsto (xz, zC_i)$.

(2) If both s and i are odd then so is $m = si$ and so

$$\mathbb{R}[C_m] - 1 = \lambda_{(m-1)/2}.$$

Then we proceed as above, using the untwisting map.

(3) If s is odd and i is even, then m is even and so

$$\mathbb{R}[C_m] - 1 = \lambda_{(m-2)/2} \oplus \mathbb{R}_-.$$

The restriction along the inclusion $C_i \subseteq C_m$ leaves the sign representation unchanged. Now repeat the argument of [14, Cor. 7.2.].

□

3.4. Homology of $B^{cy}(\Pi^d;\overline{\omega})$.

Proposition 3.6. *Let R be any commutative ring. Let $\overline{\omega}$ be a cyclic word with no cyclic repetitions, of length m, period s and with $i = m/s$ blocks. The singular homology $\tilde{H}_*(B(\overline{\omega});R)$ is concentrated in degrees $m-1$ and m. If either s is even, or both s and i are odd, then the R-modules in degree $m-1$ and m are free of rank 1. If s is odd and i is even, then the R-module in degree $m-1$ is isomorphic to $R/2R$, and the R-module in degree m is isomorphic to ${}_2R$.*

Furthermore, when m and s have the same parity, Connes' operator takes a generator $y_{\overline{\omega}}$ of the R-module $\tilde{H}_{m-1}(B(\overline{\omega});R)$ to i times a generator $z_{\overline{\omega}}$ of $\tilde{H}_m(B(\overline{\omega});R)$, that is $d(y_{\overline{\omega}}) = iz_{\overline{\omega}}$. When s is odd and i is even, Connes' operator acts trivially.

Proof. The homology computations follow directly from Lemma 3.5. We also deduce the behaviour of Connes' operator from Lemma 3.5. In general for a pointed $\mathbb{T}$-space X, Connes' operator $d : \tilde{H}_*(X;k) \to \tilde{H}_{*+1}(K;k)$ is given by taking the cross-product with the fundamental class $[\mathbb{T}]$ and then applying the map induced by the action $\mu : \mathbb{T}_+ \wedge X \to X$ on reduced homology. We consider two cases.

(1) When $X = \mathbb{T}/C_{i+}$ we claim that $d : \tilde{H}_0(X;k) \to \tilde{H}_1(X;k)$ is multiplication by i (up to a unit). This follows from the fact that the map
$$S^1 \to \mathbb{T} \times \mathbb{T}/C_i \to \mathbb{T}/C_i$$
is a degree i map.

(2) When $X = \mathbb{R}P^2(i)$ we claim that $d : \tilde{H}_1(X;k) \to \tilde{H}_2(X;k)$ is trivial. Consider the diagram
$$\begin{tikzcd} \mathbb{T}_+ \wedge (\mathbb{T}/C_i)_+ \arrow[r] \arrow[d, "id\wedge q"] & (\mathbb{T}/C_i)_+ \arrow[d, "q"] \\ \mathbb{T}_+ \wedge \mathbb{R}P^2(i) \arrow[r] & \mathbb{R}P^2(i) \end{tikzcd}$$
Now $H_1((\mathbb{T}/C_i)_+)$ surjects onto $H_1(\mathbb{R}P^2(i))$ and the above diagram commutes. Since $H_2((\mathbb{T}/C_i)_+) = 0$ the claim follows.

□

Remark 3.1. From this proposition we see that if the characteristic of k is different 2 then $\tilde{H}_*(B(\overline{\omega};k)$ is trivial when s is odd and i is even. On the other hand, if k has characteristic 2 then $k/2k = {}_2k = k$. This explains why the combinatorics of Geller, Reid, and Weibel in [8], avoids cyclic words whose period is not congruent (mod 2) to the length, [8, Remark 3.9.1.], since they work over characteristic zero fields.

3.5. THH **of the coordinate axes.** We put together the various results of the previous sections to describe $\mathrm{THH}(A, B, I)$ as a cyclotomic spectrum. Again, let $A = A_d, B = B_d$ and $I = I_d$.

We will use the Segal conjecture for C_p. This is the statement that the map

$$\mathbb{S} \to \mathbb{S}^{tC_p}$$

identifies the codomain as the p-completion of the domain. For a proof of this see [20, Theorem III.1.7], though the result was originally proved by Lin [18] (for $p = 2$) and Gunawardena [10] (for p odd) in 1980.

Lemma 3.7. *Let T be a bounded below spectrum with C_p-action and X a finite pointed C_p-CW-complex. Then the lax symmetric monoidal structure map*

$$T^{tC_p} \otimes (\Sigma^\infty X)^{tC_p} \longrightarrow (T \otimes \Sigma^\infty X)^{tC_p}$$

is an equivalence.

Proof. Since both $T^{tC_p} \otimes (-)^{tC_p}$ and $(T \otimes -)^{tC_p}$ are exact functors we may reduce to checking the statement for the C_p-spectra $\mathbb{S}$ and $\mathbb{S} \otimes C_{p+}$. This is because $\Sigma^\infty X$ may be constructed from $\mathbb{S}$ and $\mathbb{S} \otimes C_{p+}$ using finitely many cofiber sequences (since X is built by attaching finitely many C_p-cells). Replacing $\Sigma^\infty X$ by $\mathbb{S}$ the map in question reduces to

$$T^{tC_p} \otimes \hat{\mathbb{S}_p} \longrightarrow T^{tC_p}$$

where we use the Segal conjecture to identify $\mathbb{S}^{tC_p} \simeq \hat{\mathbb{S}_p}$. Since T is bounded below it follows from [20, Lemma I.2.9]) that T^{tC_p} is p-complete and so the map is an equivalence. Replacing $\Sigma^\infty X$ by $\mathbb{S} \otimes C_{p+}$ instead we see that both the domain and codomain of the map are zero, since $(-)^{tC_p}$ kills induced spectra as well as spectra of the form $T \otimes Z$ where Z is induced (cf. [20, Lemma I.3.8. (i) and (ii)]). $\square$

We will describe the Frobenius map on

$$\mathrm{THH}(k(\Pi^d)) \simeq \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi^d)$$

in terms of the splitting of $\mathrm{B}^{\mathrm{cy}}(\Pi^d)$ into the $\mathbb{T}$-equivariant subspaces $B(\overline{\omega})$. The unstable Frobenius $\psi_p : \mathrm{B}^{\mathrm{cy}}(\Pi^d) \to \mathrm{B}^{\mathrm{cy}}(\Pi^d)^{C_p}$ (defined in Section 3.1) restricts to a homeomorphism

$$\psi_p : B(\overline{\omega}) \to B(\overline{\omega^{\star p}})^{C_p}$$

landing in the subspace $B(\overline{\omega^{\star p}})$ corresponding to the cyclic word $\overline{\omega^{\star p}}$ which has length pm and period s (if $\overline{\omega}$ has length m and period s).

Proposition 3.8. *There is a $\mathbb{T}$-equivariant equivalence of spectra*

$$\mathrm{THH}(A, B, I) \simeq \bigoplus \mathrm{THH}(k) \otimes B(\overline{\omega})$$

where the sum is indexed over cyclic words of length $m \geq 2$, having no cyclic repetitions. Under this equivalence the Frobenius map restricts to the map

$$\mathrm{THH}(k) \otimes B(\overline{\omega}) \xrightarrow{\varphi_p \otimes \tilde{\varphi}_p} \mathrm{THH}(k)^{tC_p} \otimes B(\overline{\omega^{\star p}})^{tC_p} \longrightarrow (\mathrm{THH}(k) \otimes B(\overline{\omega^{\star p}}))^{tC_p}$$

where the second map is the lax symmetric monoidal structure of the Tate-C_p-construction. This second map is an equivalence, while the restricted Frobenius $\tilde{\varphi} : \Sigma^\infty B(\overline{\omega}) \to (\Sigma^\infty B(\overline{\omega^{\star p}}))^{tC_p}$ is a p-adic equivalence.

Proof. Applying Lemma 3.7 with $T = \mathrm{THH}(k)$ and $X = B(\overline{\omega^{\star p}})$ we get that the map

$$\mathrm{THH}(k)^{tC_p} \otimes B(\overline{\omega^{\star p}})^{tC_p} \longrightarrow (\mathrm{THH}(k) \otimes B(\overline{\omega^{\star p}}))^{tC_p}$$

is an equivalence. It remains to show that

$$\tilde{\varphi}_p : \mathbb{S} \otimes B(\overline{\omega}) \to (\mathbb{S} \otimes B(\overline{\omega^{\star p}}))^{tC_p}$$

is a p-adic equivalence. To do this we factor it as follows. To ease notation, let $Y = |\operatorname{sd}_p B(\overline{\omega^{\star p}})|$. By definition $\tilde{\varphi}_p$ factors as

$$\mathbb{S} \otimes B(\overline{\omega}) \xrightarrow{\tilde{\Delta}_p} \mathbb{S} \otimes Y^{C_p} \longrightarrow (\mathbb{S} \otimes Y)^{tC_p} \xrightarrow{D_p} (\mathbb{S} \otimes B(\overline{\omega^{\star p}}))^{tC_p}$$

where $\tilde{\Delta_p}$ (the space-level diagonal) and D_p are a homeomorphisms as remarked in Section 3.1. The middle map is the composition

$$\tilde{\varphi}'_p : \mathbb{S} \otimes Y^{C_p} \xrightarrow{\gamma} \mathbb{S} \otimes Y^{hC_p} \longrightarrow (\mathbb{S} \otimes Y)^{hC_p} \xrightarrow{\text{can}} (\mathbb{S} \otimes Y)^{tC_p}$$

This map fits into the following commutative diagram

$$
\begin{tikzcd}
\mathbb{S} \otimes Y^{C_p} \arrow[rr, "\tilde{\varphi}_p'"] \arrow[dd, "\Delta_p \otimes id"'] \arrow[dr] & & (\mathbb{S} \otimes Y)^{tC_p} \\
& (\mathbb{S} \otimes Y^{C_p})^{hC_p} \arrow[ur, "\mathrm{can} \circ i"] \arrow[d, "\mathrm{can}"] & \\
\mathbb{S}^{tC_p} \otimes Y^{C_p} \arrow[r, "(1.)"] & (\mathbb{S} \otimes Y^{C_p})^{tC_p} \arrow[uur, bend right, "(2.)"'] &
\end{tikzcd}
$$

Here $\Delta_p : \mathbb{S} \to \mathbb{S}^{tC_p}$ is a p-adic equivalence by the Segal conjecture. The map labelled (1.) is the equivalence witnessing that $(-)^{tC_p}$ is an exact functor, and $\mathrm{sd}_p\, B(\overline{\omega^{\star p}})^{C_p}$ is finite and has trivial C_p-action. The left-most square commutes by construction of the map (1.). Indeed by exactness in the variable Y^{C_p} (a spectrum with trivial C_p-action) one reduces to the case $Y^{C_p} = \mathbb{S}$ where the square becomes

$$
\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d, "\Delta_p"] & \mathbb{S}^{hC_p} \arrow[d, "\mathrm{can}"] \\
\mathbb{S}^{tC_p} \arrow[r, "\mathrm{id}"] & \mathbb{S}^{tC_p}
\end{tikzcd}
$$

The map labelled (2.) is the equivalence witnessing that the inclusion of the C_p-singular set $B^{C_p} \subseteq B$ induces an equivalence

$$(T \otimes B^{C_p})^{tC_p} \to (T \otimes B)^{tC_p}$$

for any C_p-spectrum T and finite C_p-CW-complex B, cf. [14, Lemma 9.1.]. The right-most triangle (with (2.) as a side) commutes since can is natural.

Finally, in the top triangle, the map $\mathbb{S} \otimes Y^{C_p} \to (\mathbb{S} \otimes Y^{C_p})^{hC_p}$ arises since $\mathbb{S} \otimes Y^{C_p}$ has trivial C_p-action. By the universal property of $(\mathbb{S} \otimes Y)^{hC_p}$ it follows that the following diagram commutes.

$$
\begin{tikzcd}
\mathbb{S} \otimes Y^{C_p} \arrow[r] \arrow[dr] & \mathbb{S} \otimes Y^{hC_p} \arrow[r] & (\mathbb{S} \otimes Y)^{hC_p} \\
& (\mathbb{S} \otimes Y^{C_p})^{hC_p} \arrow[ur] &
\end{tikzcd}
$$

This completes the proof. $\square$

I thank Malte Leip for abundant help with the above argument.

Corollary 3.1. *Let $\overline{\omega}$ be a cyclic word having no cyclic repetitions and length m. The map*

$$\varphi : \mathrm{THH}(k) \otimes B(\overline{\omega}) \to (\mathrm{THH}(k) \otimes B(\overline{\omega^{\star p}}))^{tC_p}$$

induces an isomorphism on homotopy groups in degrees greater than or equal to m

Proof. The Frobenius map $\varphi : \mathrm{THH}(k) \to \mathrm{THH}(k)^{tC_p}$ induces an isomorphism on non-negative homotopy groups. This is shown in [20, Prop. IV. 4.13] for $k = \mathbb{F}_p$, and in [14, Section 5.5] for k a perfect field of characteristic p. The result now follows from Proposition 3.6 and Proposition 3.8 as can be shown for example with the Atiyah-Hirzebruch spectral sequence. $\square$

By Proposition 3.6 the homology of $B(\overline{\omega})$ depends only on the length and period of $\overline{\omega}$. For $\overline{\omega}$ of length m and period s we therefore introduce the notation $B(m,s) = B(\overline{\omega})$. The function $\mathrm{cyc}_d(s)$ counts how many cyclic words with no cyclic repetitions, of length s, and period s, there are. For a fixed m there are $\mathrm{cyc}_d(s)$ many cyclic words with no cyclic repetitions, of length m, and period s. Since the homology of $B(m,s)$ depends only on the parity of m and s (by Proposition 3.6) we may rewrite Proposition 3.8 as follows; if $p > 2$ then

$$\mathrm{THH}(A,B,I) \simeq \bigoplus_{\substack{m\geq 2\\ \text{even}}} \bigoplus_{\substack{s|m\\ \text{even}}} (\mathrm{THH}(k) \otimes B(m,s))^{\oplus \mathrm{cyc}_d(s)} \tag{4}$$

$$\oplus \bigoplus_{\substack{m\geq 2\\ \text{odd}}} \bigoplus_{\substack{s|m\\ \text{odd}}} (\mathrm{THH}(k) \otimes B(m,s))^{\oplus \mathrm{cyc}_d(s)}. \tag{5}$$

If $p = 2$ then we add the similar double sum indexed over m even and s odd, i.e.

$$\begin{aligned}\mathrm{THH}(A,B,I) \simeq & \bigoplus_{\substack{m\geq 2\\ \text{even}}} \bigoplus_{\substack{s|m\\ \text{even}}} (\mathrm{THH}(k) \otimes B(m,s))^{\oplus \mathrm{cyc}_d(s)} \\ & \oplus \bigoplus_{\substack{m\geq 2\\ \text{odd}}} \bigoplus_{\substack{s|m\\ \text{odd}}} (\mathrm{THH}(k) \otimes B(m,s))^{\oplus \mathrm{cyc}_d(s)} \\ & \oplus \bigoplus_{\substack{m\geq 2\\ \text{even}}} \bigoplus_{\substack{s|m\\ \text{odd}}} (\mathrm{THH}(k) \otimes B(m,s))^{\oplus \mathrm{cyc}_d(s)}.\end{aligned}$$

Note that we do not need to know the homotopy-type of $B(m,s)$ for the above equivalences, since the homotopy type of $\mathrm{THH}(k) \otimes B(m,s)$ is determined – using the Atiyah-Hirzebruch spectral sequence – from the homology of $B(m,s)$. It is the simplicity of the homology of $B(m,s)$ that makes it possible for us to compute the Atiyah-Hirzebruch spectral sequence.

4. Negative - and periodic topological cyclic homology

In this section we compute $\mathrm{TC}^-(A,B,I)$ and $\mathrm{TP}(A,B,I)$. We will need the following general lemma about the Tate $\mathbb{T}$-construction.

Lemma 4.1. *Let $\{X_i\}_{i\geq 0}$ be a sequence of spectra with a $\mathbb{T}$-action, such that the connectivity of (the underlying spectrum) X_i is unbounded, as i grows. Then the canonical map $(\bigoplus_i X_i)^{t\mathbb{T}} \to \prod (X_i)^{t\mathbb{T}}$ is an equivalence.*

Proof. Because of the connectivity assumption the canonical map $\bigoplus X_i \to \prod X_i$ is an equivalence. Consider the norm cofiber sequence

$$(\Sigma X)_{h\mathbb{T}} \to X^{h\mathbb{T}} \to X^{t\mathbb{T}}$$

The first term commutes with colimits, the second with limits, so we get the cofiber sequence

$$\bigoplus (\Sigma X_i)_{h\mathbb{T}} \to \prod X_i^{h\mathbb{T}} \to (\bigoplus X_i)^{t\mathbb{T}}$$

Now $(-)_{hG}$ preserves connectivity for any group G. To see this one may use the homotopy orbit spectral sequence whose E^2-page consists of ordinary group homology. Thus the cofiber sequence becomes

$$\prod (\Sigma X_i)_{h\mathbb{T}} \to \prod X_i^{h\mathbb{T}} \to (\bigoplus X_i)^{t\mathbb{T}}$$

and we see that $(\bigoplus_i X_i)^{t\mathbb{T}} \to \prod (X_i)^{t\mathbb{T}}$ is an equivalence. □

4.1. The Tate spectral sequence.

Let X be a $\mathbb{T}$-spectrum. The Tate construction $\mathrm{TP}(X) = X^{t\mathbb{T}}$ is the target of the Tate spectral sequence for the circle group $\mathbb{T}$, see [15, end of section 4.4], and also [1, section 3]. This spectral sequence has the form

$$E^2(\mathbb{T}, X) = S\{t^{\pm 1}\} \otimes \pi_* X \quad \Rightarrow \quad \pi_* \mathrm{TP}(X)$$

Here t has bi-degree $(-2,0)$ and is a generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$. This spectral sequence is conditionally convergent. When $X = \mathrm{THH}(R)$ is an E_∞ algebra the spectral sequence is multiplicative, and if R is a k-algebra then

$E^r(\mathbb{T}, \mathrm{THH}(R))$ is a module over $E^r(\mathbb{T}, \mathrm{THH}(k))$. By Bökstedt periodicity, when k is a perfect field of characteristic $p > 0$ there is an isomorphism

$$\mathrm{THH}_*(k) \simeq k[x]$$

with x in degree 2. In particular, in this case $\mathrm{TP}_*(R)$ is periodic. The following is well-known.

Proposition 4.2. *Let k be a perfect field of characteristic $p > 0$, then there is an isomorphism*

$$\mathrm{TP}_*(k) \simeq W(k)[t^{\pm 1}]$$

Proof. By Bökstedt periodicity the E^2-term of the spectral sequence takes the form $k[t^{\pm 1}] \otimes k[x]$, and since all classes are in even total degree there are no non-trivial differentials on any page. Thus $E^2 = E^\infty$ and it remains to determine the extensions. Again by periodicity it suffices to show that $\mathrm{TP}_0(k) = W(k)$. For $k = \mathbb{F}_p$ this is done in [20, Cor. IV. 4.8.]. To conclude the result for k we use functoriality along the map $\mathbb{F}_p \to k$. More precisely we argue as follows. From the Tate spectral sequence we have a complete descending multiplicative filtration

$$\ldots \subseteq \mathrm{Fil}^{i+1}(k) \subseteq \mathrm{Fil}^i(k) \subseteq \ldots \mathrm{Fil}^1(k) \subseteq \mathrm{Fil}^0 = \mathrm{TP}_0(k)$$

with associated graded $\mathrm{gr}^i(k) \simeq \mathrm{THH}_{2i}(k) \simeq k$. By the universal property of the p-typical Witt vectors $W(k)$ [22, Chapter II, paragraph 5] we get a unique multiplicative continous map $W(k) \to \mathrm{TP}_0(k)$. By functoriality we have a commuting diagram

$$\begin{tikzcd} W(\mathbb{F}_p) \arrow[r, "\simeq"] \arrow[d] & \mathrm{TP}_0(\mathbb{F}_p) \arrow[d] \\ W(k) \arrow[r] & \mathrm{TP}_0(k) \end{tikzcd}$$

of ring homomorphisms. In particular the map on associated graded induced by $W(k) \to \mathrm{TP}(k)$ maps 1 to 1 and so must be an isomorphism. It follows that $W(k) \to \mathrm{TP}(k)$ is itself an isomorphism. $\square$

Proposition 4.3. *Let k be a perfect field of characteristic $p > 0$. The elements x and $t^\pm$ from the E^2-page of the Tate spectral sequence are infinite cycles.*

Proof. This is true for degree reasons, by Bökstedt periodicity. $\square$

Lemma 4.4. *Let X be a $\mathbb{T}$-spectrum such that the underlying spectrum is an $H\mathbb{Z}$-module . The d^2 differential of the $\mathbb{T}$-Tate spectral sequence is given by $d^2(\alpha) = td(\alpha)$ where d is Connes' operator.*

Proof. See [11, Lemma 1.4.2] □

Following Proposition 3.6 we now choose some generators for the homology of the spaces $B(m,s)$. When m and s have the same parity let $z_{(m,s)}$ be a generator for $\tilde{H}_m(B(m,s);k)$ and let $y_{(m,s)}$ a generator for $\tilde{H}_{m-1}(B(m,s);k)$. When m is even and s is odd let $z_{(m,s)}$ be a generator of $\tilde{H}_{m-1}(B(m,s);k)$. Finally, when $p = 2$ (so $\tilde{H}_m(B(m,s);k)$ is free of rank 1 over k) let $w_{(m,s)}$ be a generator.

Lemma 4.5. *As an element of the E^2-page of the Tate spectral sequence, $z_{(m,s)}$ is an infinite cycle.*

Proof. By [14, Theorem B] for any perfect field k of positive characteristic, there is an equivalence $\tau_{\geq 0}\,\mathrm{TC}(k) \simeq \mathbb{Z}_p$. As a result we obtain a map

$$\mathbb{Z}_p \simeq \tau_{\geq 0}\,\mathrm{TC}(k) \to \mathrm{TC}(k) \to \mathrm{TC}^-(k) \to \mathrm{THH}(k).$$

which is $\mathbb{T}$-equivariant, for the trivial $\mathbb{T}$-action on the domain. It thus induces a map of Tate spectral sequences,

$$\begin{tikzcd}
E^2 = \mathbb{Z}_p[t^{\pm 1}]\{y_{(m,s)}, z_{(m,s)}\} \arrow[r, Rightarrow] \arrow[d] & \pi_*(\mathbb{Z}_p \otimes B(m,s))^{t\mathbb{T}} \arrow[d] \\
E^2 = k[t^{\pm 1}, x]\{y_{(m,s)}, z_{(m,s)}\} \arrow[r, Rightarrow] & \pi_*(\mathrm{THH}(k) \otimes B(m,s))^{t\mathbb{T}}
\end{tikzcd}$$

For degree reasons $z_{(m,s)}$ is an infinite cycle in the top spectral sequence. It follows that $z_{(m,s)}$, viewed as a class in the bottom spectral sequence, is an infinite cycle. □

4.2. TC^- **and** TP **of coordinate axes.** In this section we compute TC^- and TP using the Tate spectral sequence. When $p > 2$. By Lemma 4.1 and Section 3.5 we have

$$\begin{aligned}
\mathrm{TP}(A,B,I) \simeq & \prod_{\substack{m\geq 2 \\ \text{even}}} \prod_{\substack{s\mid m \\ \text{even}}} \Big((\mathrm{THH}(k) \otimes B(m,s))^{t\mathbb{T}} \Big)^{\oplus\, \mathrm{cyc}_d(s)} \\
& \oplus \prod_{\substack{m\geq 2 \\ \text{odd}}} \prod_{\substack{s\mid m \\ \text{odd}}} \Big((\mathrm{THH}(k) \otimes B(m,s))^{t\mathbb{T}} \Big)^{\oplus\, \mathrm{cyc}_d(s)}
\end{aligned}$$

and likewise for negative topological cyclic homology, we have

$$\begin{aligned}\mathrm{TC}^-(A,B,I) \simeq & \prod_{\substack{m\geq 2\\ \text{even}}}\prod_{\substack{s\mid m\\ \text{even}}} \Big((\mathrm{THH}(k)\otimes B(m,s))^{h\mathbb{T}}\Big)^{\oplus\,\mathrm{cyc}_d(s)} \\ & \oplus \prod_{\substack{m\geq 2\\ \text{odd}}}\prod_{\substack{s\mid m\\ \text{odd}}} \Big((\mathrm{THH}(k)\otimes B(m,s))^{h\mathbb{T}}\Big)^{\oplus\,\mathrm{cyc}_d(s)}\end{aligned}$$

When $p=2$ there is an extra double product indexed over $m\geq 2$ even and $s\mid m$ odd (cf. Proposition 3.6 and Remark 3.1). See Section 4.3 below.

Since $\mathrm{THH}(k)$ is p-complete, it follows from [20, Lemma II. 4.2.] that we may identify the homotopy $\mathbb{T}$-fixed points of the Frobenius morphism for $\mathrm{THH}(A,B,I)$ with the map induced by the product of the maps

$$\varphi(m,s) : (\mathrm{THH}(k)\otimes B(m,s))^{h\mathbb{T}} \to (\mathrm{THH}(k)\otimes B(pm,s))^{t\mathbb{T}}.$$

Since the homotopy fixed point functor preserves co-connectivity, it follows from Corollary 3.1 that $\pi_*\varphi(m,s)$ is an isomorphism when $*\geq m$. Indeed, the fiber of

$$(\mathrm{THH}(k)\otimes B(m,s)) \to (\mathrm{THH}(k)\otimes B(pm,s))^{tC_p}$$

has no homotopy groups above degree $m-1$, hence the same is true of the $\mathbb{T}$-homotopy fixed point spectrum.

Proposition 4.6. *Let k be a perfect field of characteristic $p>0$. Let $m\geq 2$ and $s\mid m$. Write $m=p^v m'$ and $s=p^u s'$ with m' and s' coprime to p.*

(1) If both m and s are even then $\pi_*((\mathrm{THH}(k)\otimes B(m,s))^{t\mathbb{T}})$ as well as $\pi_*((\mathrm{THH}(k)\otimes B(m,s))^{h\mathbb{T}})$ are concentrated in even degrees and given by

$$\pi_{2r}((\mathrm{THH}(k)\otimes B(m,s))^{t\mathbb{T}}) \simeq W_{v-u}(k)$$

and

$$\pi_{2r}((\mathrm{THH}(k)\otimes B(m,s))^{h\mathbb{T}}) \simeq \begin{cases} W_{v-u+1}(k) & 2r\geq m \\ W_{v-u}(k) & 2r<m \end{cases}.$$

(2) If both m and s are odd, then $\pi_*((\mathrm{THH}(k)\otimes B(m,s))^{t\mathbb{T}})$ as well as $\pi_*((\mathrm{THH}(k)\otimes B(m,s))^{h\mathbb{T}})$ are concentrated in odd degrees and given by

$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B(m,s))^{t\mathbb{T}}) \simeq W_{v-u}(k)$$

and

$$\pi_{2r+1}((\mathrm{THH}(k) \otimes B(m,s))^{h\mathbb{T}}) \simeq \begin{cases} W_{v-u+1}(k) & 2r \geq m \\ W_{v-u}(k) & 2r < m \end{cases}.$$

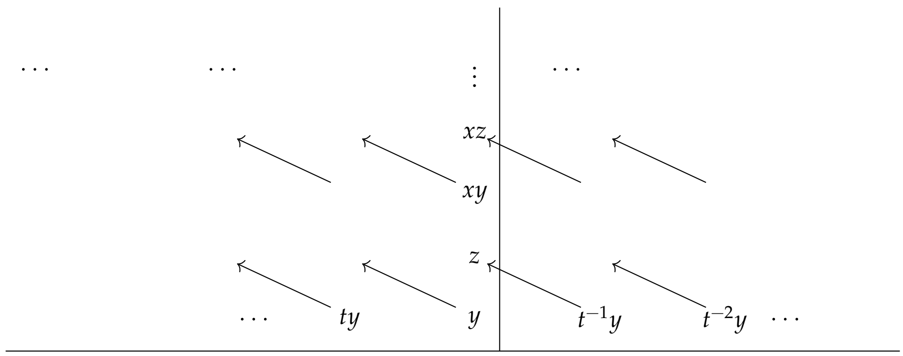
Proof. Suppose first that m and s are even. We proceed by induction on $v \geq 0$. Suppose $v = 0$, so $m = m'$ and $s = s'$. Consider the Tate spectral sequence (cf. Section 4.1)

$$E^2 = k[t^{\pm 1}, x]\{y_{(m',s')}, z_{(m',s')}\} \quad \Rightarrow \quad \pi_*(\mathrm{THH}(k) \otimes B(m',s'))^{t\mathbb{T}}$$

By Lemma 4.5 and Proposition 4.3 the differential structure is determined by the differentials on $y_{(m',s')}$. Furthermore

$$d^2(y_{(m',s')}) \doteq td(y_{(m',s')}) \doteq tiz_{(m',s')}$$

by Lemma 4.4 and Proposition 3.6, where $i = m'/s'$. Here we use the characterization of the d^2-differential from Lemma 4.4. Since i is a unit in k, d^2 is an isomorphism. In summary, the E^2-page looks as follows (where we have dropped the indices for clarity).



All the arrows indicate isomorphisms. Thus E^3, hence E^∞, is trivial as claimed. To determine the $\mathbb{T}$-homotopy fixed points, we truncate the Tate spectral sequence, removing the first quadrant. Thus the class $z_{(m',s')}$ and its multiples by x^n, are no longer hit by differentials and so

$$E^3 = E^\infty = k[x]\{z_{(m',s')}\}$$

where $z_{(m',s')}$ has degree m. This proves the claim for $v = 0$. The same argument works for $v > 0$ and $u = v$, since in this case i is again coprime to p.

Suppose the claim is known for all integers less than or equal to v and all $u \leq v$. As we saw above, the homotopy fixed points of the Frobenius map

$$\pi_*(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}} \to \pi_*(\mathrm{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{t\mathbb{T}}$$

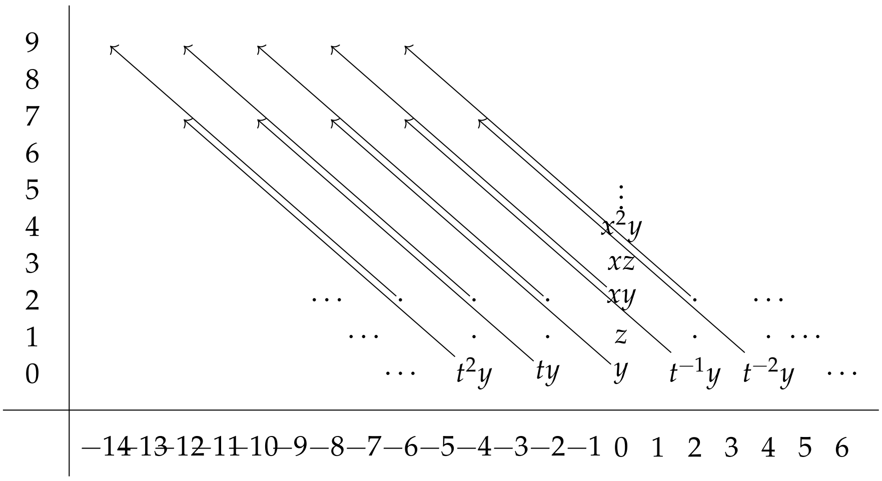
is an isomorphism when $* \geq p^v m'$. The induction hypothesis implies that the domain is isomorphic to $W_{v-u+1}(k)$ when $* = 2r \geq p^v m'$. By periodicity we conclude that

$$\pi_{2r}(\mathrm{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{t\mathbb{T}} \simeq W_{v-u+1}(k)$$

for any $r \in \mathbb{Z}$. Considering again the Tate spectral sequence we see that we must have

$$d^{2(v-u)+2}(y_{(p^{v+1}m', p^u s')}) \doteq t z_{(p^{v+1}m', p^u s')} (xt)^{v-u}$$

(see the figure, which is page E^8 when $v = 2$ and $u = 0$) and so we conclude that $E^{2(v-u)+3} = E^\infty$.



Truncating the spectral sequence we now see that

$$\pi_{2r}(\mathrm{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{h\mathbb{T}} \simeq \begin{cases} W_{v-u+2}(k) & \text{if } 2r \geq p^{v+1} m' \\ W_{v-u+1}(k) & \text{if } 2r < p^{v+1} m' \end{cases}$$

This completes the proof of *(1)*.

The arguments in case *(2)*, where m and s are both odd, are very similar. □

Remark 4.1. In particular this shows that $\mathrm{TP}(A, B, I)$ is non-trivial. This may be contrasted with the Cuntz-Quillen result [4], that $\mathrm{HP}((A, B, I)/\mathbb{Q})$

is trivial, i.e. that rational periodic cyclic homology satisfies excision. Similarly it is shown in [17] that TP does not satisfy nil-invariance. Here again it is a result of Goodwillie [9] that rational periodic cyclic homology *does* satisfy nil-invariance.

Suppose m' is even and let $t_{ev} = t_{ev}(p, r, m')$ be the unique positive integer such that $p^{t_{ev}-1}m' \leq 2r < p^{t_{ev}}m'$ (or zero if such t_{ev} does not exist, i.e. if m' is too big). Then we may restate the TC^- calculations as

$$\pi_{2r}((\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}}) \simeq \begin{cases} W_{v-u+1}(k) & v < t_{ev} \\ W_{v-u}(k) & v \geq t_{ev} \end{cases}.$$

Similarly if m' and s' are both odd, let $t_{od} = t_{od}(p, r, m')$ be the unique positive integer such that $p^{t_{od}-1}m' \leq 2r+1 < p^{t_{od}}m'$ (or zero if such t_{od} does not exist, i.e. if m' is too big). Then

$$\pi_{2r+1}((\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}}) \simeq \begin{cases} W_{v-u+1}(k) & v < t_{od} \\ W_{v-u}(k) & v \geq t_{od} \end{cases}.$$

4.3. **The case** $p = 2$**.** We now deal with the case when k has characteristic two. From Section 3.5 we see that the case m even and s odd, is missing from Proposition 4.6.

Proposition 4.7. *Assume $p = 2$, m even and s odd. Write $m = 2^v m'$ with m' odd, and let $s' \mid m'$. Then the homotopy groups of $(\mathrm{THH}(k) \otimes B(m, s))^{t\mathbb{T}}$ and $(\mathrm{THH}(k) \otimes B(m, s))^{h\mathbb{T}}$ are concentrated in odd degrees where they are isomorphic to k.*

Proof. When $v = 0$ we are in the case *(2)* of Proposition 4.6 so

$$\pi_*(\mathrm{THH}(k) \otimes B(m', s'))^{t\mathbb{T}} = 0$$

and

$$\pi_*(\mathrm{THH}(k) \otimes B(m', s'))^{h\mathbb{T}} = k \quad \text{when} \quad * = 2r \geq m'.$$

From the high co-connectivity of the Frobenius we conclude that

$$\pi_*(\mathrm{THH}(k) \otimes B(2m', s'))^{t\mathbb{T}} = k$$

when $*$ is odd, and zero otherwise. Now the Tate spectral sequence

$$E^2 = k[t^{\pm 1}, x]\{z, w\} \quad \Rightarrow \quad \pi_*(\mathrm{THH}(k) \otimes B(2m', s'))^{t\mathbb{T}}$$

must collapse on the E^2-page, from which we conclude that $d^2(w) \doteq (tx)z$. Truncating the spectral sequence we see the homotopy fixed point spectral sequence has $E^3 = E^\infty = (k[t,x]/(tx))\{z\}$ so

$$\pi_*(\mathrm{THH}(k) \otimes B(2m', s'))^{h\mathbb{T}} = k$$

in every odd degree. Now an induction argument shows that this pattern continues for all $v > 1$. $\square$

5. Topological cyclic homology of coordinate axes

In this section we complete the proof of Theorem 2.1, and thus Theorem 1.1. We start with the case where $p > 2$. For $2 \leq m'$ and $s' \mid m'$ let

$$\mathrm{TP}(m', s') := \prod_{0 \leq v} \prod_{u \leq v} \left((\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{t\mathbb{T}} \right)^{\mathrm{cyc}_d(p^u s')}$$

and

$$\mathrm{TC}^-(m', s') := \prod_{0 \leq v} \prod_{u \leq v} \left((\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}} \right)^{\mathrm{cyc}_d(p^u s')}$$

Thus, $\mathrm{TC}(A, B, I)$ is identified with the product of the equalizer of the maps $\varphi, can : \mathrm{TC}^-(m', s') \to \mathrm{TP}(m', s')$ as m' and s' vary accordingly. Let us denote by $\mathrm{TC}(m', s')$ the equalizer

$$\mathrm{TC}(m', s') \to \mathrm{TC}^-(m', s') \underset{can}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(m', s')$$

A priori the homotopy groups of $\mathrm{TC}(m', s')$ sit in a long exact sequence with those of $\mathrm{TC}^-(m', s')$ and $\mathrm{TP}(m', s')$. However this sequence splits into short exact sequences since $\mathrm{TC}^-(m', s')$ and $\mathrm{TP}(m', s')$ are concentrated in either even or odd degrees, depending on the parity of m' and s', cf. Proposition 4.6.

If m' and s' are even then the Frobenius map

$$\pi_{2r}(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}} \longrightarrow \pi_{2r}(\mathrm{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{t\mathbb{T}}$$

is an isomorphism for $0 \leq v < t_{ev}$, and the canonical map

$$\pi_{2r}(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}} \longrightarrow \pi_{2r}(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{t\mathbb{T}}$$

is an isomorphism for $t_{ev} \leq v$. Thus we have a map of short exact sequences

$$\begin{tikzcd}
\prod_{t_{ev} \leq v} \prod_{u \leq v} (W_{v-u}(k))^{\mathrm{cyc}_d(p^u s')} \arrow[r, "\varphi - can"] \arrow[d, hook] & \prod_{t_{ev} \leq v} \prod_{u \leq v} (W_{v-u}(k)))^{\mathrm{cyc}_d(p^u s')} \arrow[d, hook] \\
\mathrm{TC}^-_{2r}(m', s') \arrow[r, "\varphi - can"] \arrow[d, two heads] & \mathrm{TP}_{2r}(m', s') \arrow[d, two heads] \\
\prod_{0 \leq v < t_{ev}} \prod_{u \leq v} (W_{v-u+1}(k))^{\mathrm{cyc}_d(p^u s')} \arrow[r, "\overline{\varphi - can}"] & \prod_{0 \leq v < t_{ev}} \prod_{u \leq v} (W_{v-u}(k)))^{\mathrm{cyc}_d(p^u s')}
\end{tikzcd}$$

The top horizontal map is an isomorphism, and the bottom horizontal map is an epimorphism so, by the snake lemma, we conclude that

$$\mathrm{TC}_{2r}(m', s') \simeq \prod_{u \leq t_{ev}} (W_{t_{ev}-u}(k))^{\mathrm{cyc}_d(p^u s')}.$$

Now suppose m' and s' are odd. Then the Frobenius map

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}} \to \pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{t\mathbb{T}}$$

is an isomorphism for $0 \leq v < t_{od}$, and the canonical map

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{h\mathbb{T}} \to \pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^v m', p^u s'))^{t\mathbb{T}}$$

is an isomorphism for $t_{od} \leq v$. Thus we have a map of short exact sequences

$$\begin{tikzcd}
\prod_{t_{od} \leq v} \prod_{u \leq v} (W_{v-u}(k))^{\mathrm{cyc}_d(p^u s')} \arrow[r, "\varphi - can"] \arrow[d, hook] & \prod_{t_{od} \leq v} \prod_{u \leq v} (W_{v-u}(k)))^{\mathrm{cyc}_d(p^u s')} \arrow[d, hook] \\
\mathrm{TC}^-_{2r+1}(m', s') \arrow[r, "\varphi - can"] \arrow[d, two heads] & \mathrm{TP}_{2r+1}(m', s') \arrow[d, two heads] \\
\prod_{0 \leq v < t_{od}} \prod_{u \leq v} (W_{v-u+1}(k))^{\mathrm{cyc}_d(p^u s')} \arrow[r, "\overline{\varphi - can}"] & \prod_{0 \leq v < t_{od}} \prod_{u \leq v} (W_{v-u}(k)))^{\mathrm{cyc}_d(p^u s')}
\end{tikzcd}$$

The top horizontal map is an isomorphism, and the bottom horizontal map is en epimorphism so, by the snake lemma, we conclude that

$$\mathrm{TC}(m', s') \simeq \prod_{u \leq t_{od}} (W_{t_{od}-u}(k))^{\mathrm{cyc}_d(p^u s')}.$$

This finishes the proof of Theorem 2.1 in the case $p > 2$.

5.1. **The case** $p = 2$. If $p = 2$ then we must correct slightly the definition of $\mathrm{TP}(m', s')$ and $\mathrm{TC}^-(m', s')$. Suppose m' and s' are odd. To deal with the case $m = p^v m'$ even and $s = p^u s'$ even, we let

$$\mathrm{TP}(m', s')_{ev} := \prod_{1 \le v} \prod_{1 \le u \le v} \left((\mathrm{THH}(k) \otimes B(2^v m', 2^u s'))^{t\mathbb{T}} \right)^{\mathrm{cyc}_d(2^u s')}$$

and

$$\mathrm{TC}^-(m', s')_{ev} := \prod_{1 \le v} \prod_{1 \le u \le v} \left((\mathrm{THH}(k) \otimes B(2^v m', 2^u s'))^{h\mathbb{T}} \right)^{\mathrm{cyc}_d(2^u s')}$$

Both of these spectra are concentrated in even degrees, with their homotopy groups given by Proposition 4.6. As a result we see that, for $m' \geq 1$ odd, and $s' \mid m'$ odd,

$$\mathrm{TC}_{2r}(m', s')_{ev} \simeq \prod_{1 \le u \le t_{ev}} W_{t_{ev}-u}(k)^{\oplus \mathrm{cyc}_d(2^u s')}.$$

To deal with the case where m is even and $s \mid m$ is odd, let

$$\mathrm{TP}(m', s')_{ev,od} := \prod_{0 \le v} \left((\mathrm{THH}(k) \otimes B(2^v m', s'))^{t\mathbb{T}} \right)^{\mathrm{cyc}_d(s')}$$

and

$$\mathrm{TC}^-(m', s')_{ev,od} := \prod_{0 \le v} \left((\mathrm{THH}(k) \otimes B(2^v m', s'))^{h\mathbb{T}} \right)^{\mathrm{cyc}_d(s')}$$

(note that when $v = 0$, we have $m = m'$ odd, but we must include this case since the Frobenius connects it with the case $v = 1$) By Proposition 4.7 $\mathrm{TP}(m', s')_{ev,od}$ and $\mathrm{TC}^-(m', s')_{ev,od}$ are concentrated in odd degrees, where they are isomorphic to k. Thus

$$\mathrm{TC}_{2r+1}(m', s')_{ev,od} \simeq \prod_{0 \le v \le t_{ev}} k^{\oplus \mathrm{cyc}_d(s')}.$$

This completes the proof of Theorem 2.1 when $p = 2$.

6. The characteristic zero case

In this section we compute the relative cyclic homology and the bi-relative K-theory of $A_d = k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$ in the case that k is an ind-smooth $\mathbb{Q}$-algebra. In the case where $k/\mathbb{Q}$ is a field extension, the results in this section were found, by different means, already in 1989 by Geller, Reid and Weibel [8, Theorem 7.1.].

We proceed as in [12, Section 3.9]. We will compute the relative groups

$$\mathrm{HC}_q((A,I)/\mathbb{Z}) \otimes \mathbb{Q} \simeq \mathrm{HC}_q((A,I)/\mathbb{Q})$$

using our understanding of the $\mathbb{T}$-homotopy type of the cyclic bar construction for Π^d, as found in Lemma 3.5. First we need the following analogue of Lemma 3.1

Lemma 6.1. *Let k be a ring, Π a pointed monoid, and $k(\Pi)$ the pointed monoid algebra. There is a natural equivalence of $\mathbb{T}$-spectra*

$$\mathrm{HH}(k(\Pi)) \xleftarrow{\sim} \mathrm{HH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi)$$

So the arguments from Lemma 3.2 still work, yielding a description of the relative (and bi-relative) Hochshild homology spectrum of (A_d, I_d) (and (A_d, B_d, I_d)). First we note that the relative Hochschild homology differs only slightly from the absolute version. The sole difference is that we "cut out" the part of the space $\mathrm{B}^{\mathrm{cy}}(\Pi^d)$ given by $\mathrm{B}^{\mathrm{cy}}(\Pi^d;\overline{\varnothing})$, i.e. the part corresponding to the unique cyclic word of length zero. Since the spaces $B(m,s)$ for m even and s odd have torsion integral homology they disappear in the rational case. So we conclude that

$$\begin{aligned}
\mathrm{HH}(A,I) \simeq \quad & \bigoplus_{\substack{m\geq 2\\ even}} \bigoplus_{\substack{s|m\\ even}} (\mathrm{HH}(k)\otimes B(m,s))^{\oplus\,\mathrm{cyc}_d(s)} \\
\oplus \quad & \bigoplus_{\substack{m\geq 2\\ odd}} \bigoplus_{\substack{s|m\\ odd}} (\mathrm{HH}(k)\otimes B(m,s))^{\oplus\,\mathrm{cyc}_d(s)} \\
\oplus \quad & \bigoplus_{i\geq 1} (\mathrm{HH}(k)\otimes (\mathbb{T}/C_i)_+)^{\oplus d}
\end{aligned}$$

where $B(m,s)$ is given by Lemma 3.5. The bottom summands, with terms $\mathrm{HH}(k)\otimes(\mathbb{T}/C_i)_+$, corresponds to the spaces $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{x_1^i})$, $\mathrm{B}^{\mathrm{cy}}(\Pi^d, \overline{x_2^i})$ etc. These bottom summands disappear when looking at the bi-relative theory, $\mathrm{HH}(A,B,I)$.

The following theorem is due to Geller, Reid, Weibel, [8, Corollary 3.12.]) when k is a field extension of $\mathbb{Q}$.

Theorem 6.2. *Let k be any commutative unital ring. There is an isomorphism*

$$\mathrm{HC}_0(A_d/\mathbb{Q}) \simeq \mathrm{HC}_0(k/\mathbb{Q})$$

and, for $q \geq 1$, an isomorphism

$$\begin{aligned}
\mathrm{HC}_q((A_d, I_d)/\mathbb{Q}) \simeq & \bigoplus_{\substack{m\geq 2\\ even}} \bigoplus_{\substack{s\mid m\\ even}} \left(\mathrm{HH}_{q+1-m}(k) \otimes \mathbb{Q}\right)^{\oplus \mathrm{cyc}_d(s)} \\
\oplus & \bigoplus_{\substack{m\geq 2\\ odd}} \bigoplus_{\substack{s\mid m\\ odd}} \left(\mathrm{HH}_{q+1-m}(k) \otimes \mathbb{Q}\right)^{\oplus \mathrm{cyc}_d(s)} \\
\oplus & \bigoplus_{i\geq 1} \left(\mathrm{HH}_q(k) \otimes \mathbb{Q}\right)^{\oplus d}
\end{aligned}$$

Proof. We must compute $\pi_q(X_{h\mathbb{T}})$ where X ranges over the summands in the above decomposition of $\mathrm{B}^{\mathrm{cy}}(\Pi^d)$. If $X = \mathrm{HH}(k) \otimes S^{\lambda_j} \otimes (\mathbb{T}/C_i)_+$ for some complex $\mathbb{T}$-representation λ_j of complex dimension j, then since,

$$\left(\mathrm{HH}(k) \otimes S^{\lambda_j} \otimes \mathbb{T}/C_i\right)_{h\mathbb{T}} \simeq \left(\mathrm{HH}(k) \otimes S^{\lambda_j}\right)_{hC_i},$$

we may regard the C_i-homology spectral sequence

$$E^2_{s,t} = H_s(C_i, \pi_t(\mathrm{HH}(k) \otimes S^{\lambda_j}) \otimes \mathbb{Q}) \Rightarrow \pi_{s+t}(\mathrm{HH}(k) \otimes S^{\lambda_j})_{hC_i} \otimes \mathbb{Q}$$

Since the rational group homology of C_i is concentrated in degree 0, the edge homomorphism

$$H_0(C_i, \pi_q(\mathrm{HH}(k) \otimes S^{\lambda_j}) \otimes \mathbb{Q}) \xrightarrow{\simeq} \pi_q\left(\left(\mathrm{HH}(k) \otimes S^{\lambda_j}\right)_{hC_i}\right) \otimes \mathbb{Q}$$

is an isomorphism. Furthermore, since the C_i-action on $\mathrm{HH}(k) \otimes S^{\lambda_j}$ extends to a $\mathbb{T}$-action, the induced action on homotopy groups is trivial. We conclude that

$$\pi_q\left(\left(\mathrm{HH}(k) \otimes S^{\lambda_j}\right)_{h\mathbb{T}}\right) \cong \mathrm{HH}_{q-2j}(k)$$

The result follows. □

The following theorem is due to Geller, Reid, Weibel, [8, Theorem 7.1.]) when k is a field extension of $\mathbb{Q}$. We use their counting function $c_d(q)$ which we recall in Section 7, Eq. (6).

Theorem 6.3. *Suppose k is an ind-smooth $\mathbb{Q}$-algebra. Let $d \geq 2$ and consider the ring $A_d = k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$, and let $I_d = (x_1, \ldots, x_d)$. Then*

$$K_q(A_d, I_d) \cong k^{\oplus c_d(q)} \oplus (\Omega^1_{k/\mathbb{Q}})^{\oplus c_d(q-1)} \oplus \cdots \oplus (\Omega^{q-2}_{k/\mathbb{Q}})^{\oplus c_d(2)}.$$

Proof. By [3, Corollary 0.2] we have $K_n(A_d, I_d) \simeq \mathrm{HC}_{n-1}(A_d, B_d, I_d)$. Here we use that $K_q(A_d, I_d)$ is a rational vector space [3, Theorem 0.1], so no further rationalization is necessary. By Theorem 6.2 we are reduced to

Hochschild homology calculations. By the Hochschild-Kostant-Rosenberg theorem [23, Theorem 9.4.7.] we have

$$\mathrm{HH}_n(k/\mathbb{Q}) \simeq \Omega^n_{k/\mathbb{Q}}.$$

Thus,

$$\begin{aligned} K_q(A_d, I_d) &\simeq \mathrm{HC}_{q-1}(A_d, B_d, I_d) \\ &\simeq k^{\oplus c_d(q)} \oplus (\Omega^1_{k/\mathbb{Q}})^{\oplus c_d(q-1)} \oplus \cdots \oplus (\Omega^{q-2}_{k/\mathbb{Q}})^{\oplus c_d(2)} \end{aligned}$$

This completes the proof. □

Remark 6.1. If k is a field extension of $\mathbb{Q}$ for which we know the transcendence degree of k over $\mathbb{Q}$ then this result completely determines the relative K-theory, since $\dim_k \Omega^1_{k/\mathbb{Q}} = tr.deg(k/\mathbb{Q})$. For example if $k/\mathbb{Q}$ is algebraic then $\Omega^1_{k/\mathbb{Q}} = 0$ and so the calculation reduces to

$$K_q(A_d, I_d) \simeq k^{\oplus c_d(q)}.$$

7. Appendix: counting cyclic words

In this section we use some counting techniques inspired by [8]. We first deal with all words, then with cyclic words. Let $A_d(m)$ denote the number of words in d letters of length m having no cyclic repetitions.

Lemma 7.1. *For all $d \geq 1$ and all $m \geq 1$ we have*

$$A_d(m) = (d-1)^m + (-1)^m(d-1).$$

Proof. Define the auxiliary counter, $B_d(m)$ counting words $\omega = w_1 w_2 \ldots w_m$ with no allowed repetitions, except that we require $w_m = w_1$. Then

$$A_d(m) + B_d(m) = d(d-1)^{m-1}.$$

To see this, consider a word $\omega = w_1 w_2 \ldots w_m$. There are d choices for w_1, and $d-1$ choices for $w_2, w_3, \ldots$ and w_{m-1}. Finally for w_m there are again d choices since choosing any letter different from w_{m-1} gives either a type A ($d-2$ possibly choices) or a type B word (one choice, namely $w_m = w_1$). The formula for $A_d(m)$ now follows by induction using the equation $B_d(m) = A_d(m-1)$ (for $m \geq 2$). This last equality is true since deleting the last letter of a type B word yields a word with no cyclic repetitions. □

Let $\widetilde{\text{cyc}}_d(s)$ denote the number of words in d letters of length s, period s and having no cyclic repetitions (see Definition 3.1) and let $\text{cyc}_d(s)$ denote the number of *cyclic* words in d letters of length s, period s and having no cyclic repetitions. Then $\text{cyc}_d(s) = \frac{1}{s}\widetilde{\text{cyc}}_d(s)$. Also we have

$$A_d(m) = \sum_{s|m} \widetilde{\text{cyc}}_d(s)$$

So using Möbius inversion we have

$$\widetilde{\text{cyc}}_d(s) = \sum_{j|s} \mu(\frac{s}{j}) A_d(j).$$

hence we obtain a formula for $\text{cyc}_d(s)$,

$$\text{cyc}_d(s) = \frac{1}{s} \sum_{j|s} \mu(\frac{s}{j})((d-1)^j + (-1)^j(d-1)).$$

Below is a table of the first few values.

s	$\text{cyc}_d(s)$	$d=3$	$d=4$
1	0	0	0
2	$\frac{1}{2}((d-1)^2+(d-1))$	3	6
3	$\frac{1}{3}((d-1)^3-(d-1))$	2	8
4	$\frac{1}{4}((d-1)^4-(d-1)^2)$	3	18
5	$\frac{1}{5}((d-1)^5-(d-1))$	6	48
6	$\frac{1}{6}((d-1)^6-(d-1)^3-(d-1)^2+(d-1))$	9	116
7	$\frac{1}{7}((d-1)^7-(d-1))$	18	312
8	$\frac{1}{8}((d-1)^8-(d-1)^4)$	30	810
9	$\frac{1}{9}((d-1)^9-(d-1)^3)$	56	2184
10	$\frac{1}{10}((d-1)^{10}-(d-1)^5-(d-1)^2+(d-1))$	99	5880
11	$\frac{1}{11}((d-1)^{11}-(d-1))$	186	16104
12	$\frac{1}{12}((d-1)^{12}-(d-1)^6-(d-1)^4+(d-1)^2)$	335	44220

If we want to count the number of cyclic words without cyclic repetitions, with length m but with no restrictions on period then we can just sum $\mathrm{cyc}_d(s)$ over all divisors s of m. For example there are $3+2+9=14$ cyclic words in $d=3$ letter, with no repetitions having length $m=6$.

We may describe the function $c_{d-1}(m)$ that Geller, Reid, and Weibel introduce as follows

$$c_{d-1}(m) = \sum_{\substack{s|m \\ s\equiv m \mod 2}} \mathrm{cyc}_{d-1}(s). \tag{6}$$

That is, $c_{d-1}(m)$ is the number of cyclic words without repetitions, having length m and period s where s and m have the same parity.

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Part 3

Paper B

ON THE K-THEORY OF TRUNCATED POLYNOMIAL ALGEBRAS, REVISITED

MARTIN SPEIRS

1. Introduction

The algebraic K-theory of truncated polynomial algebras over perfect fields of positive characteristic was first evaluated by Hesselholt and Madsen [6]. Their proof relied on a delicate analysis of the facet structure of regular cyclic polytopes. We present a new proof that only uses the homology of the cyclic bar construction together with Connes' operator.

Theorem 1.1. *[6, Theorem A] Let k be a perfect field of positive characteristic. Then there is an isomorphism*

$$K_{2r-1}(k[x]/(x^e),(x)) \simeq \mathbb{W}_{re}(k)/V_e\mathbb{W}_r(k)$$

and the groups in even degrees are zero.

We briefly summarize the method. Let k be a perfect field of characteristic $p > 0$ and let $A = k[x]/(x^e)$ and $I = (x)$ the ideal generated by the variable. The k-algebra A is the pointed monoid algebra for the pointed monoid $\Pi_e = \{0, 1, x, \ldots, x^{e-1}\}$ determined by $x^e = 0$. There is a canonical equivalence of cyclotomic spectra

$$\mathrm{THH}(A) \simeq \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi_e)$$

where the Frobenius morphism on the right is the tensor product of the usual Frobenius and the unstable Frobenius on the cyclic bar construction of Π_e. Using the theory of cyclic sets one obtains a $\mathbb{T}$-equivariant splitting of the cyclic bar construction,

$$\mathrm{B}^{\mathrm{cy}}(\Pi_e) \simeq \bigvee_{m\geq 0} B(m)$$

into simpler $\mathbb{T}$-spaces $B(m)$. The singular homology and Connes' operator of these $\mathbb{T}$-spaces is easily determined and reduces to computations of the Hochschild homology of A first carried out in [2] and [9]. The answer is simple enough that the Atiyah-Hirzebruch spectral sequence degenerates allowing us to directly determine the homotopy groups of $\mathrm{THH}(k) \otimes B(m)$. From [11] the topological cyclic homology of A is given by the equalizer

$$\mathrm{TC}(A;p) \to \mathrm{TC}^{-}(A) \xrightarrow{\varphi_p - \mathrm{can}} \mathrm{TP}(A)$$

so using the above splitting this reduces to computing $(\mathrm{THH}(k) \otimes B(m))^{h\mathbb{T}}$ and $(\mathrm{THH}(k) \otimes B(m))^{t\mathbb{T}}$. We achieve this by an inductive procedure, making use of the highly co-connective Frobenius map

$$\varphi : (\mathrm{THH}(k) \otimes B(m))^{h\mathbb{T}} \to (\mathrm{THH}(k) \otimes B(pm))^{t\mathbb{T}}$$

and the periodicity of $(\mathrm{THH}(k) \otimes B(m))^{t\mathbb{T}}$. Assembling the answers for varying m then yields the TC-calculation. Applying McCarthy's theorem one obtains the result.

We note that the method used here has recently been applied by Hesselholt and Nikolaus [8] to compute the K-theory of cuspidal curves over k, thereby verifying the conjectural result from [4]. We consider this method a first step towards making topological cyclic homology as easy to compute as Connes' cyclic homology HC.

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2. Witt vectors, big and small

The purpose of this short section is to show the following well-known splitting. Let $s = s(p,r,d)$ be the unique positive integer such that

$$p^{s-1}d \leq r < p^s d$$

if it exists, or else $s = 0$. Let $e = p^u e'$ with $(p, e') = 1$.

Lemma 2.1. *Let k be a perfect field of characteristic $p > 0$. There is an isomorphism*

$$\mathbb{W}_{re}(k)/V_e\mathbb{W}_r(k) \simeq \prod W_h(k)$$

where the product is indexed over $1 \leq m' \leq re$ with $(p, m') = 1$ and with $h = h(p, r, e, m')$ given by

$$h = \begin{cases} s & \text{if } e' \nmid m' \\ \min\{u, s\} & \text{if } e' \mid m' \end{cases}$$

where $s = s(p, re, m')$ is the function defined above.

Proof. We use the isomorphism

$$\mathbb{W}_r(k) \xrightarrow{\langle I_d \rangle} \prod W_s(k)$$

(natural with respect to $\mathbb{Z}_{(p)}$-algebras) where the product runs over d such that $(p, d) = 1$ and $1 \leq d \leq r$ and where $s = s(p, r, d)$, see for example [5, Prop. 1.10 and Example 1.11]. The d'th component of this map is the composite

$$I_d : \mathbb{W}_r(k) \xrightarrow{F_d} \mathbb{W}_{\lfloor r/d \rfloor}(k) \xrightarrow{pr} W_s(k)$$

where F_d is the Frobenius map. If $m' = e'd$ with $d \leq r$ then one readily checks that $s(p, re, m') = s(p, r, d) + u$ and that the following diagram commutes

$$\begin{tikzcd} \mathbb{W}_r(k) \arrow[r, "I_d"] \arrow[d, "V_e"] & W_s(k) \arrow[d, "e'V_{p^u}"] \\ \mathbb{W}_{re}(k) \arrow[r, "I_{m'}"] & W_{s+u}(k) \end{tikzcd}$$

This corresponds to the case $u \leq s$. Since $(p, e') = 1$ we have

$$W_{s+u}(k)/(e'V_{p^u}W_s(k)) \cong W_u(k).$$

Thus, we get an isomorphism

$$\mathbb{W}_{re}(k)/V_e\mathbb{W}_r(k) \xrightarrow{\cong} \prod W_u(k) \times \prod W_s(k) \times \prod W_s(k) \xrightarrow{\cong} \prod W_h(k)$$

where in the middle term, the first product is indexed over $1 \leq d \leq r$ with $(p, d) = 1$, the second product is indexed over $1 \leq m' \leq re$ with $e' \mid m'$ and with $u > s$, the third product is indexed over $1 \leq m' \leq re$ with $e' \nmid m'$ and with $(p, m') = 1$. In the last term, the product is indexed over $1 \leq m' \leq re$ with $(p, m') = 1$. $\square$

3. Hochschild homology of truncated polynomial algebras

In this section we review the results of [2] and [9] on cyclic homology of algebras of the form $A = k[x]/f(x)$. We work over a general commutative unital base ring k. The Hochschild homology of A over k is the homology of the associated chain complex for the cyclic k-module

$$\mathrm{B}^{\mathrm{cy}}(A/k)[n] = A^{\otimes n+1}$$

where the tensor product is over k. The cyclic structure maps are given as follows

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_n) &= \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 \leq i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases} \\ s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \\ t_n(a_0 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The Hochschild homology then is the homology $\mathrm{HH}_*(A/k)$ of the associated chain complex with differential given by the alternating sum of the face maps.

Proposition 3.1. *Let $A = k[x]/(x^e)$ where k is a commutative unital ring. There is an isomorphism*

$$\mathrm{HH}_*(A/k) = \begin{cases} A & \text{if } * = 0 \\ {}_e k\{1\} \oplus k\{x, \ldots, x^{e-1}\} & \text{if } * > 0 \text{ even} \\ k\{1, x, \ldots, x^{e-1}\} \oplus k/ek\{x^{e-1}\} & \text{if } * > 0 \text{ odd} \end{cases}$$

where ${}_e k$ denotes the e-torsion elements of k.

The proof uses a common technique for such rings, namely the construction of a small and computable complex. The task is then to show that this complex is quasi-isomorphic to the Hochschild complex. For a k-algebra A of the form $A = k[x]/(f(x))$, assuming it is flat as an k-module then the Hochschild homology may be calculated as $\mathrm{Tor}_*^{A^e}(A, A)$ where $A^e = A \otimes A^{op}$. So it suffices to find a small $A{-}A$-bimodule resolution of A. Given such a resolution $R(A)_* \to A$ one now tensors over A^e with A to get a complex, $\overline{R}(A)_*$ computing $\mathrm{HH}_*(A/k)$. For an appropriate choice of resolution the corresponding complex $\overline{R}(A)_*$ has the following form

$$0 \leftarrow A \overset{0}{\leftarrow} A \overset{f'(x)}{\leftarrow} A \overset{0}{\leftarrow} A \overset{f'(x)}{\leftarrow} A \leftarrow \cdots$$

from which the result readily follows.

We now introduce a splitting of the Hochschild homology of the k-algebra $A = k[x]/(x^e)$. Equip A with a "weight" grading by declaring x^m have weight m. This induces a grading on the tensor powers of A and we let

$$\mathrm{B}^{\mathrm{cy}}(A/k;m)[n] \subseteq \mathrm{B}^{\mathrm{cy}}(A/k)[n] = A^{\otimes n+1}$$

be the sub k-module of weight m. It is generated by those tensor monomials whose weight is equal to m. This forms a sub cyclic k-module of $\mathrm{B}^{\mathrm{cy}}(A/k)[-]$ and so we obtain a splitting

$$\mathrm{B}^{\mathrm{cy}}(A/k)[-] \simeq \bigoplus_{m\geq 0} \mathrm{B}^{\mathrm{cy}}(A/k;m)[-]$$

of cyclic k-modules, and of the associated chain complexes. Taking homology then gives a splitting as well,

$$\mathrm{HH}_*(A/k) \simeq \bigoplus_{m\geq 0} \mathrm{HH}_*(A/k;m).$$

In the following lemma, let $d = d(e,m) = \lfloor \frac{m-1}{e} \rfloor$ be the largest integer less that $(m-1)/e$.

Lemma 3.1. *Let k and A be as in Proposition 3.1. If m is not a multiple of e then $\mathrm{HH}_*(A/k;m)$ is concentrated in degrees $2d$ and $2d+1$ where it is free of rank 1 as a k-module. In this case Connes' B-operator takes the generator in degree $2d$ to m times the generator in degree $2d+1$, up to a sign. If m is a multiple of e then $\mathrm{HH}_*(A/k;m)$ is concentrated in degree $2d+1$ and $2d+2$. The group in degree $2d+1$ is isomorphic to k/ek while the group in degree $2d+2$ is isomorphic to ${}_ek$. In this case Connes' operator acts trivially.*

Proof. First we prove that the groups are as stated. We follow the proof given in [7, Section 7.3.]. Consider the resolution of A as an $A\otimes A$-module constructed by [2], denoted $R(A)_*$ having the form

$$\cdots \xrightarrow{\Delta} A\otimes A \xrightarrow{\delta} A\otimes A \xrightarrow{\Delta} A\otimes A \xrightarrow{\delta} A\otimes A \xrightarrow{\mu} A \to 0$$

where

$$\Delta = \frac{x^e\otimes 1 - 1\otimes x^e}{x\otimes 1 - 1\otimes x} \quad \text{and} \quad \delta = 1\otimes x - x\otimes 1$$

In [2] a quasi-isomorphism $\psi : R(A/k)_* \longrightarrow B(A/k)_*$ with the bar-resolution is constructed. Since Δ increases the weight by $e-1$ and δ by 1, and since the differential b' of the bar resolution preserves weight, we see (by induction on j) that ψ_{2j} increases weight by je, whereas ψ_{2j+1} increases weight by $je+1$. Tensoring over A^e with A gives a quasi-isomorphism $\overline{\psi} : \overline{R}(A/k)_* \longrightarrow \mathrm{B}^{\mathrm{cy}}(A/k)_*$ which has the same weight shift. The result now follows from Proposition 3.1

For the statements about Connes' operator, this follows by an explicit choice of a quasi-isomorphism ψ (and its inverse). This is done in [2, Section 1] and in [2, Proposition 2.1.] the computation of Connes' operator is given. □

4. Topological Hochschild homology and the cyclic bar construction

Let $\Pi_e = \{0, 1, x, \ldots, x^{e-1}\}$ be the pointed monoid determined by setting $x^e = 0$. Then the truncated polynomial algebra A is the pointed monoid ring $k(\Pi_e) = k[\Pi_e]/k[0]$. The cyclic bar construction of Π_e is the cyclic set $\mathrm{B}^{\mathrm{cy}}(\Pi_e)[-]$ with

$$\mathrm{B}^{\mathrm{cy}}(\Pi_e)[k] = \Pi_e^{\wedge(k+1)}$$

and with the usual Hochschild-type structure maps. We write $\mathrm{B}^{\mathrm{cy}}(\Pi_e)$ for the geometric realization of $\mathrm{B}^{\mathrm{cy}}(\Pi_e)[-]$. The space $\mathrm{B}^{\mathrm{cy}}(\Pi_e)$ admits a natural $\mathbb{T}$-action where $\mathbb{T}$ is the circle group, as does the geometric realization of any cyclic set. Furthermore it is an unstable cyclotomic space, i.e. there is a map

$$\psi_p : \mathrm{B}^{\mathrm{cy}}(\Pi) \to \mathrm{B}^{\mathrm{cy}}(\Pi)^{C_p}$$

which is equivariant when the domain is given the natural $\mathbb{T}/C_p$-action. For a construction of this map see [1, Section 2] or, for a review in our setup, see [12, Section on cyclic bar construction].

To every non-zero n-simplex $\pi_0 \wedge \cdots \wedge \pi_n \in \mathrm{B}^{\mathrm{cy}}(\Pi_e)[n]$ we associate its *weight* as follows, each π_i is equal to x^{m_i} for some $0 \leq m_i \leq e-1$. Let

$$w(\pi_0 \wedge \cdots \wedge \pi_n) = \sum_{i=0}^{n} m_i.$$

The weight is preserved by the cyclic structure maps and so we obtain a splitting of pointed cyclic sets

$$\mathrm{B}^{\mathrm{cy}}(\Pi_e)[-] = \bigvee_{m \geq 0} \mathrm{B}^{\mathrm{cy}}(\Pi_e; m)[-]$$

where $\mathrm{B}^{\mathrm{cy}}(\Pi_e; m)[-] \subseteq \mathrm{B}^{\mathrm{cy}}(\Pi_e)[-]$ consists of all simplicies with weight m. Let $B(m)$ denote the geometric realization of $\mathrm{B}^{\mathrm{cy}}(\Pi_e; m)[-]$. So we have a splitting of pointed $\mathbb{T}$-spaces

$$\mathrm{B}^{\mathrm{cy}}(\Pi_e) \simeq \bigvee_{m \geq 0} B(m).$$

By [12, Splitting lemma] we have $\mathrm{THH}(k(\Pi_e)) \simeq \mathrm{THH}(k) \otimes \mathrm{B}^{\mathrm{cy}}(\Pi_e)$ as cyclotomic spectra. Here the Frobenius on the right hand side is the tensor product of the usual Frobenius on $\mathrm{THH}(k)$ (as constructed in [11, Section III.2]) and the Frobenius on $\Sigma^\infty \mathrm{B}^{\mathrm{cy}}(\Pi_e)$ arising from the unstable Frobenius (see [12, Section on cyclic bar construction]). The relative THH corresponds to simply cutting out the weight zero part, i.e. we have an equivalence of $\mathbb{T}$-spectra

$$\mathrm{THH}(A, I) \simeq \bigoplus_{m \geq 1} \mathrm{THH}(k) \otimes B(m)$$

where $I = (x)$ is the ideal generated by the variable.

Given any pointed monoid Π there is an isomorphism of cyclic k-modules

$$w : k(\mathrm{B}^{\mathrm{cy}}(\Pi)[-]) \longrightarrow \mathrm{B}^{\mathrm{cy}}(k(\Pi)/k)[-]$$

which map $\pi_0 \wedge \cdots \wedge \pi_n$ to $\pi_0 \otimes \cdots \otimes \pi_n$. Note that $k(\mathrm{B}^{\mathrm{cy}}(\Pi)[-])$ is the cellular complex for the space $\mathrm{B}^{\mathrm{cy}}(\Pi)$. In particular the associated homology $H_*(k(\mathrm{B}^{\mathrm{cy}}(\Pi)[-]))$ computes the cellular homology of $\mathrm{B}^{\mathrm{cy}}(\Pi)$.

In the following lemma, we let $d = d(e, m) = \lfloor \frac{m-1}{e} \rfloor$ for any $m \geq 1$.

Lemma 4.1. ([7, Lemma 7.3]) Let k and A be as in Proposition 3.1 and let $B(m) \subseteq \mathrm{B}^{\mathrm{cy}}(\Pi_e)$ be as described above.

(1) If $e \nmid m$ then $\tilde{H}_*(B(m); \mathbb{Z})$ is free of rank 1 if $* = 2d, 2d+1$ and zero, otherwise. The Connes' operator takes a generator in degree $2d$ to m times a generator in degree $2d+1$.

(2) If $e \mid m$ then $\tilde{H}_*(B(m); \mathbb{Z})$ is isomorphic to k/ek if $* = 2d+1$, to ${}_e k$ if $* = 2d+2$, and zero otherwise.

Proof. We use the isomorphism of cyclic k-modules

$$w : k(\mathrm{B}^{\mathrm{cy}}(\Pi_e)[-]) \to \mathrm{B}^{\mathrm{cy}}(A/k)[-]$$

This map preserves the weight decomposition, mapping $k(B(m)[-])$ isomorphically to $\mathrm{B}^{\mathrm{cy}}(A/k;m)[-]$. Furthermore the map commutes with the Connes operator, as shown in the proof of [3, Proposition 1.4.5.]. Now by Lemma 3.1 we can read off what

$$\mathrm{HH}_*(A/k;m) = \tilde{H}_*(B(m);k)$$

is and how Connes' operator acts. $\square$

Note that in particular $\tilde{H}_{2d+2}(B(m);k)$ is free of rank 1 over k, when e is zero in k. Thus there is room for a non-trivial Connes' operator in this case. However, it follows again from Lemma 3.1 that it is trivial in this case.

Lemma 4.2. *Let T be a bounded below C_p-spectrum and X a finite pointed C_p-CW-complex. Then the lax symmetric monoidal structure map*

$$T^{tC_p} \otimes (\Sigma^\infty X)^{tC_p} \longrightarrow (T \otimes \Sigma^\infty X)^{tC_p}$$

is an equivalence.

Proof. See [12, Lemma 3.5.1]. $\square$

Proposition 4.1. *Let $A = k[x]/(x^e)$. There is a $\mathbb{T}$-equivariant equivalence of spectra*

$$\mathrm{THH}(A) \simeq \bigoplus_{m \geq 0} \mathrm{THH}(k) \otimes B(m).$$

Under this equivalence the Frobenius morphism $\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ restricts to the map

$$\mathrm{THH}(k) \otimes B(m) \longrightarrow \mathrm{THH}(k)^{tC_p} \otimes B(pm)^{tC_p} \longrightarrow (\mathrm{THH}(k) \otimes B(pm))^{tC_p}$$

where the second map is the lax symmetric monoidal structure on the Tate-C_p-construction. This second map is an equivalence, while the restricted Frobenius $\tilde{\varphi} : \Sigma^\infty B(m) \to (\Sigma B(pm))^{tC_p}$ is a p-adic equivalence.

Proof. The proof follows that of the similar statement in [12, Proposition 3.5.1]. Taking $T = \mathrm{THH}(k)$ and $X = B(m)$ in Lemma 4.2 yields the claim about the lax symmetric monoidal structure map. To see that the restricted Frobenius is a p-adic equivalence, one factors it accordingly as follows.

$$\begin{tikzcd}
\mathbb{S}\otimes B(m) \arrow[rrr, "\tilde{\varphi}"] \arrow[d, "\Delta_p\otimes\tilde{\Delta}_p"] & & & (\mathbb{S}\otimes B(pm))^{tC_p} \\
\mathbb{S}^{tC_p}\otimes(\mathrm{sd}_p\, B(pm))^{C_p} \arrow[r, "(1.)"] & (\mathbb{S}\otimes \mathrm{sd}_p\, B(pm)_p^{C})^{tC_p} \arrow[rr, "D_p"] & & (\mathbb{S}\otimes B(pm)^{C_p})^{tC_p} \arrow[u, "(2.)"']
\end{tikzcd}$$

Now the Segal conjecture says that $\Delta_p : \mathbb{S} \to \mathbb{S}^{tC_p}$ is a p-adic equivalence. The map labelled (1.) is an equivalence since $\mathrm{sd}_p\, B(pm)^{C_p}$ is a finite C_p-CW-complex with trivial C_p-action, and since $(-)^{tC_p}$ is exact. The map D_p is the equivalence from the p-subdivision of $B(pm)$ to $B(pm)$ itself. Finally, the map labelled (2.) is an equivalence since $(-)^{tC_p}$ is trivial on free C_p-CW-complexes, cf. [7, Lemma 9.1.]. □

Corollary 4.1. *The restricted Frobenius map*

$$\varphi(m) : \mathrm{THH}(k)\otimes B(m) \to (\mathrm{THH}(k)\otimes B(pm))^{tC_p}$$

induces an isomorphism in degrees $\geq 2d+1$ when $e \nmid m$, and induces an isomorphism in degrees $\geq 2d+2$ when $e \mid m$.

Proof. This follows readily from Proposition 4.1 and Lemma 4.1 using the Atiyah-Hirzebruch spectral sequence. □

5. Negative and periodic topological cyclic homolgy

We compute TP and TC^{-} using an inductive procedure based on the p-adic valuation of the integer m in indexing the $\mathbb{T}$-space $B(m)$. We choose generators for the homology of the spaces $B(m)$. If $e \nmid m$ let y_m and z_m be generators for the homology in degree $2d$ and $2d+1$, respectively. If $e \mid m$ and $p \mid e$ then we let z_m and w_m be generators of the homology in degree $2d+1$ and $2d+2$, respectively.

Lemma 5.1. *In the Tate spectral sequence for $\pi_*(\mathrm{THH}(k)\otimes B(m))$ the class z_m is an infinite cycle for all m.*

Proof. Although the statement does not seem to require it, we must deal with the cases $e \mid m$ and $e \nmid m$ separately. In both cases we use the $\mathbb{T}$-equivariant map $H\mathbb{Z}_p \to \mathrm{THH}(k)$. One way of getting such a map is by using the calculation $\tau_{\geq 0}\,\mathrm{TC}(k) = H\mathbb{Z}_p$ since then we have the $\mathbb{T}$-equivariant map

$$H\mathbb{Z}_p \simeq \tau_{\geq 0} \operatorname{TC}(k) \to \operatorname{TC}(k) \to \operatorname{TC}^-(k) \to \operatorname{THH}(k).$$

This map induces a map of Tate spectral sequences from $\pi_*(H\mathbb{Z}_p \otimes B(m))$ to $\pi_*(\operatorname{THH}(k) \otimes B(m))$.

Suppose first that $e \nmid m$. Then from Lemma 4.1 we may compute the E^2-page of the Tate spectral sequence for $H\mathbb{Z}_p \otimes B(m)$ to be

$$E^2 = \mathbb{Z}_p[t^{\pm 1}]\{y_m, z_m\} \quad \Rightarrow \quad \pi_*(H\mathbb{Z}_p \otimes B(m))^{t\mathbb{T}}$$

where $|y_m| = (0, m-1)$ and $|z_m| = (0, m)$. The differential structure is determined by $d^2(y_m) = mtz_m$, and so $E^3 = E^\infty = \mathbb{Z}_p/m\mathbb{Z}_p[t^{\pm 1}]\{z_m\}$ so z_m is an infinite cycle. It follows that $z_m \in k[t^{\pm 1}, x]\{y_m, z_m\}$ (the E^2 page for the target spectral sequence) is an infinite cycle.

Now suppose $e \mid m$. Then from Lemma 4.1 we may compute the E^2-page of the Tate spectral sequence for $H\mathbb{Z}_p \otimes B(m)$ to be $\mathbb{Z}_p[t^{\pm 1}]\{z_m\}$ with $|z_m| = (0, m)$ from which it follows immediately that z_m is an infinite cycle. □

Lemma 5.2. *Let X be a $\mathbb{T}$-spectrum such that the underlying spectrum is an $H\mathbb{Z}$-module . The d^2 differential of the $\mathbb{T}$-Tate spectral sequence is given by $d^2(\alpha) = td(\alpha)$ where d is Connes' operator.*

Proof. See [3, Lemma 1.4.2] □

Proposition 5.1. *If $e \nmid m$ then*

$$\pi_{2r+1}(\operatorname{THH}(k) \otimes B(p^v m'))^{t\mathbb{T}} \simeq W_v(k)$$

for all $r \in \mathbb{Z}$, and

$$\pi_{2r+1}(\operatorname{THH}(k) \otimes B(p^v m'))^{h\mathbb{T}} \simeq \begin{cases} W_{v+1}(k) & \text{if } d \leq r \\ W_v(k) & \text{if } r < d \end{cases}$$

The even homotopy groups are trivial.

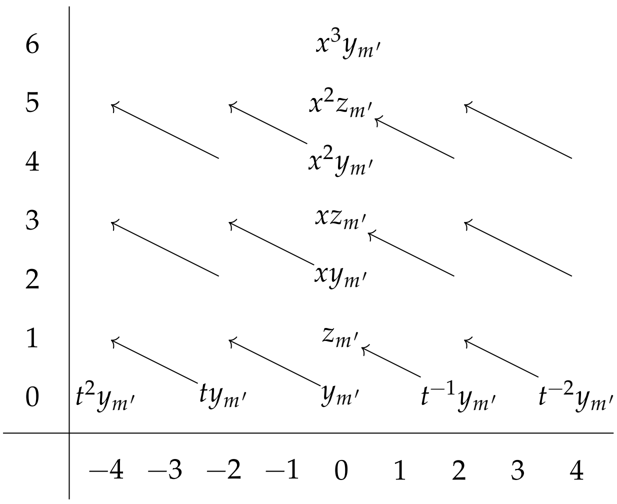
Proof. We proceed by induction on $v \geq 0$. Suppose $v = 0$, so $m = m'$, and consider the Tate spectral sequence

$$E^2 = k[t^{\pm 1}, x]\{y_{m'}, z_{m'}\} \quad \Rightarrow \quad \pi_*(\operatorname{THH}(k) \otimes B(m'))^{t\mathbb{T}}$$

By Lemma 5.1 the only possible non-zero differentials are those beginning at $y_{m'}$. Furthermore

$$d^2(y_{m'}) \doteq m' t d(y_{m'}) \doteq m' t z_{m'}$$

by Lemma 5.2 and Lemma 4.1. Since m' is a unit in k, d^2 is an isomorphism. In summary, the E^2-page looks as follows (shifted up by $2d$ in the horizontal direction).



Thus $E^3 = E^\infty = 0$ is trivial, as claimed. To determine the $\mathbb{T}$-homotopy fixed points, we truncate the Tate spectral sequence, removing the first quadrant. The classes $z_{m'}x^n$ are no longer hit by differentials and so $E^3 = E^\infty = k[x]\{z_{m'}\}$ where $z_{m'}$ has degree $2d+1$. This proves the claim for $v = 0$.

Suppose the claim is known for all integers less than or equal to v. By Proposition 4.1 the Frobenius

$$\varphi(p^v m') : \pi_*(\mathrm{THH}(k) \otimes B(p^v m'))^{h\mathbb{T}} \to \pi_*(\mathrm{THH}(k) \otimes B(p^{v+1} m'))^{t\mathbb{T}}$$

is an isomorphism in high degrees. The induction hypothesis then implies that the domain is isomorphic to $W_{v+1}(k)$ when $* = 2r+1 \geq 2d+1$. By periodicity we conclude that $\pi_*(\mathrm{THH}(k) \otimes B(p^{v+1} m'))^{t\mathbb{T}}$ is concentrated in odd degrees where,

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^{v+1} m'))^{t\mathbb{T}} \simeq W_{v+1}(k)$$

for any $r \in \mathbb{Z}$. Considering again the Tate spectral sequence we see that we must have

$$d^{2v+2}(y_{p^{v+1}m'}) \doteq tz_{p^{v+1}m'}(xt)^v$$

and so $E^{2v+3} = E^\infty$. Truncating the spectral sequence to obtain the homotopy fixed-point spectral sequence, we now see that

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^{v+1}m'))^{h\mathbb{T}} \simeq \begin{cases} W_{v+2}(k) & \text{if } d \leq r \\ W_{v+1}(k) & \text{if } r < d \end{cases}$$

This completes the proof. □

To deal with the case where e does divide m we factor $e = p^u e'$ where $(p, e') = 1$. Thus $e \mid p^v m'$ if and only if $v \geq u$ and $e' \mid m'$.

Proposition 5.2. *If $e \mid m$ then*

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^v m'))^{t\mathbb{T}} \simeq W_u(k)$$

and

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^v m'))^{h\mathbb{T}} \simeq W_u(k)$$

for all $r \in \mathbb{Z}$.

Proof. We use induction on $v \geq u$. Suppose $v = u$. Then

$$\pi_*(\mathrm{THH}(k) \otimes B(p^{v-1}m'))^{h\mathbb{T}} \xrightarrow{\varphi(p^{v-1}m')} \pi_*(\mathrm{THH}(k) \otimes B(p^v m'))^{t\mathbb{T}}$$

is an isomorphism in high enough degrees. The domain was evaluated in Proposition 5.1, it is $W_u(k)$ in odd degrees greater than $2d+1$. By periodicity we conclude the result of the codomain. Now suppose the result has been verified for all integers greater than u and strictly less than v. Again using the Frobenius we conclude that

$$\pi_{2r+1}(\mathrm{THH}(k) \otimes B(p^v m'))^{t\mathbb{T}} \simeq W_u(k)$$

for all $r \in \mathbb{Z}$.

Consider the Tate spectral sequence with E^2-page $k[t^{\pm 1}, x]\{z_m, w_m\}$. Since z_m is an infinite cycle the only possible way that this sequence collapses to yield the correct result is if

$$d^{2u}(w_m) \doteq (tx)^u z_m$$

Thus $E^{2u+1} = E^{\infty}$. As before, by truncating the first quadrant, we get the spectral sequence for the homotopy $\mathbb{T}$-fixed points whose E^{2u}-page clearly shows the result. □

6. Topological cyclic homology

We now prove Theorem 1.1. By McCarthy's result [10, Main Theorem] it suffices to prove the following.

Theorem 6.1. *Let k be a perfect field of positive characteristic. Then there is an isomorphism*

$$\mathrm{TC}_{2r-1}(k[x]/(x^e),(x)) \simeq \mathbb{W}_{re}(k)/V_e\mathbb{W}_r(k)$$

and the groups in even degrees are zero.

Proof. In view of Lemma 2.1 it suffices to give an isomorphism

$$\mathrm{TC}_{2r-1}(k[x]/(x^e),(x)) \simeq \prod W_h(k)$$

where the product is indexed over $1 \leq m' \leq re$ with $(p,m') = 1$ and with $h = h(p,r,e,m')$ given by

$$h = \begin{cases} s & \text{if } e' \nmid m' \\ \min\{u,s\} & \text{if } e' \mid m' \end{cases}$$

where $s = s(p,re,m')$ is such that $p^{s-1}m' \leq re < p^s m'$. Now $\mathrm{TC}(A,I)$ is given as the equalizer of $\mathrm{TC}^-(A,I) \xrightarrow{\varphi - can} \mathrm{TP}(A,I)$. This map splits as

$$\prod_{\substack{m' \geq 1 \\ (p,m')=1}} \prod_{v \geq 0} \mathrm{TC}^-(p^v m') \xrightarrow{\varphi - can} \prod_{\substack{m' \geq 1 \\ (p,m')=1}} \prod_{v \geq 0} \mathrm{TP}(p^v m')$$

By Proposition 5.1 and Proposition 5.2 both $\mathrm{TC}^-(p^v m')$ and $\mathrm{TP}(p^v m')$ are concentrated in odd degrees, so the long exact sequence calculating TC splits into short exact sequences

$$0 \to \mathrm{TC}_*(m') \to \prod_{v \geq 0} \mathrm{TC}^-_*(p^v m') \xrightarrow{\varphi - can} \prod_{v \geq 0} \mathrm{TP}_*(p^v m') \to 0$$

Now if $e' \nmid m'$ then from Proposition 5.1 we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_{v\geq s} W_v(k) & \longrightarrow & \mathrm{TC}^-_{2r+1}(p^v m') & \longrightarrow & \prod_{0\leq v<s} W_{v+1}(k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi - can} & & \downarrow{\scriptstyle \varphi - can} & & \downarrow{\scriptstyle \overline{\varphi - can}} & & \\ 0 & \longrightarrow & \prod_{v\geq s} W_v(k) & \longrightarrow & \mathrm{TP}_{2r+1}(p^v m') & \longrightarrow & \prod_{0\leq v<s} W_v(k) & \longrightarrow & 0 \end{array}$$

where $s = s(p, r, d(p^v m'))$. The left hand vertical map is an isomorphism (since in this range *can* is an isomorphism and φ is divisible by powers of p) and the right hand vertical map is an epimorphism with kernel $W_s(k)$. Thus $\mathrm{TC}(p^v m') = W_s(k)$. Note that in this case $h = s$.

If $e' \mid m'$ then we must distinguish between two cases. First, if $s < u$ then again we get a map of short exact sequences

$$\begin{array}{ccccc} \prod_{s\leq v<u} W_v(k) \times \prod_{u\leq v} W_u(k) & \longrightarrow & \mathrm{TC}^-_{2r+1}(p^v m') & \longrightarrow & \prod_{0\leq v<s} W_{v+1}(k) \\ \downarrow{\scriptstyle \varphi - can} & & \downarrow{\scriptstyle \varphi - can} & & \downarrow{\scriptstyle \overline{\varphi - can}} \\ \prod_{s\leq v<u} W_v(k) \times \prod_{u\leq v} W_u(k) & \longrightarrow & \mathrm{TP}_{2r+1}(p^v m') & \longrightarrow & \prod_{0\leq v<s} W_v(k) \end{array}$$

so in this case $\mathrm{TC}_{2r+1}(m') = W_s(k)$ Since $u > s$ we have $h = s$ as claimed. If instead, $u \leq s$ then the map of short exact sequences looks as follows

$$\begin{array}{ccccc} \prod_{v\geq s} W_v(k) & \longrightarrow & \mathrm{TC}^-_{2r+1}(p^v m') & \longrightarrow & \prod_{0\leq v<u} W_{v+1}(k) \times \prod_{u\leq v<s} W_u(k) \\ \downarrow{\scriptstyle \varphi - can} & & \downarrow{\scriptstyle \varphi - can} & & \downarrow{\scriptstyle \overline{\varphi - can}} \\ \prod_{v\geq s} W_v(k) & \longrightarrow & \mathrm{TP}_{2r+1}(p^v m') & \longrightarrow & \prod_{0\leq v<u} W_v(k) \times \prod_{u\leq v<s} W_u(k) \end{array}$$

so in this case $\mathrm{TC}_{2r+1}(m') = W_u(k)$. Since $u \leq s$ we see that $u = h$ in this case. This completes the proof. $\square$

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And you may ask yourself, how do I work this?