On characteristic classes of manifold bundles

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Abstract. This thesis consists of three articles in which we make several contributions to the study of characteristic classes of manifold bundles and closely related topics.

In Article A, we compare the ring of characteristic classes of smooth bundles with fibre a closed simply connected manifold M of dimension $2n \neq 4$ to the respective ring resulting from replacing M by the connected sum $M \sharp \Sigma$ with an exotic sphere Σ. We show that, after inverting the order of Σ in the group of homotopy spheres, the two rings in question are isomorphic in a range of degrees. Furthermore, we construct infinite families of examples witnessing that inverting the order of Σ is necessary.

In Article B, which is joint with Jens Reinhold, we study smooth bundles over surfaces with highly connected almost parallelisable fibre M of even dimension. We provide necessary conditions for a manifold to be bordant to the total space of such a bundle and show that, in most cases, these conditions are also sufficient. Using this, we determine the characteristic numbers realised by total spaces of bundles of this type, deduce divisibility constraints on their signatures and $\hat{A}$-genera, and compute the second integral cohomology of $\mathrm{BDiff}^+(M)$ up to torsion in terms of generalised Miller–Morita–Mumford classes.

In Article C, we introduce a framework to study homological stability properties of E_2-algebras and their modules, generalising work of Randal-Williams and Wahl in the case of discrete groups. As an application, we prove twisted homological stability results for various families of topological moduli spaces, such as configuration spaces and moduli spaces of manifolds, and explain how these results imply representation stability for related sequences of spaces.

Resumé. Denne afhandling består af tre artikler hver med bidrag til studiet af karakteristiske klasser af mangfoldighedsbundter og tæt relaterede emner.

I Article A sammenligner vi ringen af karakteristiske klasser af glatte bundter hvis fiber er en lukket enkeltsammenhængende mangfoldighed M af dimension $2n \neq 4$, med ringen der fås ved at erstatte M med den sammenhængende sum $M\sharp\Sigma$ hvor Σ er en eksotisk sfære. Vi viser, at hvis ordnen af Σ inverteres som element i gruppen af homotopisfærer, så er de to ringe isomorfe i et interval af grader. Desuden konstruerer vi uendelige familier af eksempler som bevidner af det er nødvendigt at inverterer ordnen af Σ.

I Article B, udført sammen med Jens Reinhold, studerer vi glatte bundter på flader med fiber M af høj konnektivitet, næsten paralleliserbar og af lige dimension. Vi giver nødvendige betingelser for at en mangfoldighed er kobordent med totalrummet for et sådan bundt, og viser, at i de fleste tilfælde, er disse betingelser også tilstrækkelige. Ved brug af dette, bestemmer vi de karakteristiske tal som realiseres af totalrummet for bundter af denne type, deducerer divisibilitetsbegrænsninger på deres signaturer og $\hat{A}$-genera og beregner den anden heltallige kohomologi af $\mathrm{BDiff}^+(M)$ op til torsion i termer af generaliserede Miller-Morita-Mumford klasser.

I Article C introducerer vi et framework for studiet af homologisk stabilitetsegenskaber for E_2-algebraer og deres moduler. Dette generaliserer arbejde af Randal-Williams og Wahl i tilfældet af diskrete grupper. Som anvendelse bevises homologisk stabilitet med twistede koefficienter for en række familier af topologiske modulirum, så som konfigurationsrum, og modulirum af mangfoldigheder, og vi forklarer hvorledes disse resultater implicerer repræsentationsstabilitet for relaterede følger af rum.

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Manuel Krannich
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Contents

Introduction

Overview

This thesis consists of the following three articles.

Article A. On characteristic classes of exotic manifold bundles
Article B. Characteristic numbers of manifold bundles over surfaces with highly connected fibers (joint with Jens Reinhold)
Article C. Homological stability of topological moduli spaces

After providing some background, we briefly summarise each of them.

Background

The classifying space $\mathrm{BDiff}(M)$ of the topological group of diffeomorphisms of a closed manifold M classifies smooth fibre bundles with fibre M, so it is hardly surprising that its homotopy type has been a longstanding object of interest to algebraic and geometric topologists. Over time, two predominant approaches to studying $\mathrm{BDiff}(M)$ have emerged.

The first and more classical strategy stems from the deep connections between geometric topology and algebraic K- and L-theory. It is based on the idea of introducing a larger space $\widetilde{\mathrm{Diff}}(M)$ of so-called *block diffeomorphisms* whose deviation from the space of diffeomorphisms $\mathrm{Diff}(M)$ is measured in terms of Waldhausen's algebraic K-theory of spaces, at least in a range depending only on the dimension of M (see e.g. [WW88]). Studying the space of block diffeomorphisms instead of $\mathrm{Diff}(M)$ is advantageous, since the difference between $\widetilde{\mathrm{Diff}}(M)$ and the more accessible space $\mathrm{hAut}(M)$ of homotopy equivalences can be understood by means of surgery theory (see e.g. [Qui70]).

In the course of the last two decades, a new approach for the investigation of $\mathrm{BDiff}(M)$ arose, grounded in Madsen, Tillmann, and Weiss' [Til97; MT01; MW07] work on Mumford's conjecture [Mum83] on the moduli space of Riemann surfaces. This new strategy primarily targets the ring $\mathrm{H}^*(\mathrm{BDiff}(M))$ of characteristic classes of smooth bundles with fibre M and has so far been developed the furthest for manifolds of even dimension $2n$. At first sight, this set of techniques is, like the previous method, restricted to a range of degrees[1]—the so-called *stable range*—but instead of the dimension, this range depends on the *genus* of M, defined as

$$g(M) = \max\{g \geq 0 \mid \text{there exists a manifold } N \text{ such that } M \cong N \sharp (S^n \times S^n)^{\sharp g}\}.$$

Turning towards the trailblazing achievement of this line of investigation, we assume M to be oriented, restrict to the subgroup $\mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms, and define $\mathbf{MTSO}(2n)$ as the Thom spectrum $\mathbf{Th}(-\gamma_{2n})$ of the inverse of the universal bundle over $\mathrm{BSO}(2n)$. A parametrised version of the Pontryagin–Thom collapse map results in a canonical homotopy class of maps

$$\mathrm{BDiff}^+(M) \longrightarrow \Omega^\infty_M \mathbf{MTSO}(2n)$$

[1] By way of an insight due to Weiss [Wei15] based on comparing diffeomorphisms to self-embeddings, one can obtain information outside the stable range as well. This was recently taken up by Kupers [Kup16].

to a certain path component $\Omega^\infty_M\mathbf{MTSO}(2n)$ of the infinite loop space of $\mathbf{MTSO}(2n)$. Despite the purely homotopy theoretical nature of its target, it turns out that the parametrised Pontryagin–Thom map knows a surprising amount about $\mathrm{BDiff}^+(M)$ when M is a surface: it induces an isomorphism in homology in a range of degrees growing with the genus. This remarkable fact for surfaces is a combination of a classical stability result of Harer [Har85], saying that the homology of $\mathrm{BDiff}^+(M)$ is independent of $g(M)$ in small degrees relative to $g(M)$, and the celebrated theorem of Madsen–Weiss [MW07], showing that the parametrised Pontryagin–Thom map becomes an isomorphism in homology in the limit $g(M)\to\infty$. The latter statement was formerly known as the *generalised* Mumford conjecture, as it has the classical formulation as a direct consequence, since the rational cohomology ring of $\Omega^\infty_M\mathbf{MTSO}(2)$ can, by fairly standard methods, be computed as a polynomial ring in the so-called *Miller–Morita–Mumford classes* κ_i of degrees $2i > 0$.

It is thanks to Galatius and Randal-Williams that the success of parametrised Pontryagin–Thom theory is by no means restricted to the case of surfaces anymore. Their seminal sequence of articles [GR14; GR17; GR18], building on earlier work of Galatius–Madsen–Tillmann–Weiss [GMTW09], culminated in a homotopy theoretical formula for the ring of characteristic classes $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ in a range of degrees growing with the genus $g(M)$ for any simply connected manifold M of dimension $2n\geq 6$. More specifically, after establishing higher dimensional analogues of Harer's stability result and the Madsen–Weiss theorem, they proved that the parametrised Pontryagin–Thom map induces an isomorphism on homology in a range of degrees for any manifold of the described type, after replacing its target $\Omega^\infty\mathbf{MTSO}(2n)$ with a certain refinement depending on M. Since this refinement is still defined entirely in homotopy theoretical terms and is thus amenable to calculation, their work has paved the way for a variety of applications, for instance, to the study of spaces of homotopy equivalences [BM14], topological Pontryagin classes [Wei15], mapping class groups [GR16], finiteness properties of automorphism spaces [Kup16], and spaces of metrics of positive scalar curvature [BER17; ER17].

Articles A and B of the present thesis join the ranks of applications of Galatius and Randal-Williams' programme. In Article A, we use their work to investigate the behaviour of the ring of characteristic classes $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ under changes to the smooth structure of M, whereas in Article B, we study bundles over surfaces with highly connected fibres. Article C, on the other hand, is more foundational in nature and focusses on extending and conceptualising stability results such as Harer's.

Article A. On characteristic classes of exotic manifold bundles

In Article A, we study the question of how the ring $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ of characteristic classes of smooth oriented bundles with fibre a closed oriented d-manifold M behaves under changes to the smooth structure of M on an embedded d-disc, or, equivalently, when replacing M by the connected sum $M\sharp\Sigma$ with an exotic sphere Σ in Θ_d. Here Θ_d denotes the finite group of homotopy spheres up to h-cobordism, classically studied by Kervaire and Milnor [KM63]. In the first part of the article, we utilise the work of Galatius and Randal-Williams to provide an answer to this question for simply connected manifolds M of dimension $2n\neq 4$ in a range of degrees.

Theorem (Article A, Thm A). *Let M be a closed, oriented, simply connected manifold of dimension $2n\neq 4$ and $\Sigma\in\Theta_{2n}$. There is a zig-zag of maps of spaces inducing an isomorphism*

$$\mathrm{H}_*(\mathrm{BDiff}^+(M);\mathbf{Z}[\tfrac{1}{k}])\cong\mathrm{H}_*(\mathrm{BDiff}^+(M\sharp\Sigma);\mathbf{Z}[\tfrac{1}{k}])$$

in degrees $*\leq\frac{g(M)-3}{2}$, where k denotes the order of $\Sigma\in\Theta_{2n}$.

In the second part of Article A, we focus on the family of manifolds $W_g=\sharp^g(S^d\times S^d)$ to examine whether inverting the order of Σ is necessary for the previous theorem to be valid. Inspired by work of Kreck [Kre79], we show that, in small degrees, the possible difference between the homology of $\mathrm{BDiff}^+(W_g)$ and $\mathrm{BDiff}^+(W_g\sharp\Sigma)$ can be detected in a suitable bordism theory. Combining this with known computations in stable homotopy theory, we find families of homotopy spheres Σ for which the integral homology of $\mathrm{BDiff}^+(W_g)$ differs from that of $\mathrm{BDiff}^+(W_g\sharp\Sigma)$ for all $g \geq 0$. We refer the reader to Article A for details, but state two particular consequences of our considerations.

Theorem (Article A, Cor. C, E)**.** *There are $\Sigma \in \Theta_{2n}$ for infinitely many values of n such that*

$$\mathrm{H}_1(\mathrm{BDiff}^+(W_g);\mathbf{Z}) \quad \textit{and} \quad \mathrm{H}_1(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z})$$

are not isomorphic for $g \geq 0$. Furthermore, for every $\Sigma \in \Theta_{10} \cong \mathbf{Z}/6\mathbf{Z}$ of order 3 or 6,

$$\mathrm{H}_3(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\tfrac{1}{2}]) \quad \textit{and} \quad \mathrm{H}_3(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\tfrac{1}{2}])$$

are not isomorphic for $g \geq 0$.

Remark. It would be interesting to know whether the rings of characteristic classes $\mathrm{H}^*(\mathrm{BDiff}^+(M);\mathbf{Z}[\frac{1}{k}])$ and $\mathrm{H}^*(\mathrm{BDiff}^+(M\sharp\Sigma);\mathbf{Z}[\frac{1}{k}])$ are isomorphic without assumptions on the degree. It follows from work by Dwyer–Szczarba [DS83] that this is indeed the case if one restricts to the unit component $\mathrm{Diff}_0(M)$ of $\mathrm{Diff}^+(M)$. Passing from $\mathrm{BDiff}^+(M)$ to $\mathrm{BDiff}_0(M)$, however, usually changes the (co)homology significantly, so one might hope to find an example for which $\mathrm{H}^*(\mathrm{BDiff}^+(M);\mathbf{Z}[\frac{1}{k}])$ and $\mathrm{H}^*(\mathrm{BDiff}^+(M\sharp\Sigma);\mathbf{Z}[\frac{1}{k}])$ are not isomorphic in all degrees. In the light of the theorem above, such a difference can only occur outside the stable range of Galatius–Randal-Williams.

Article B. Characteristic numbers of manifold bundles over surfaces with highly connected fibres (joint with Jens Reinhold)

Article B is concerned with the problem of determining which manifolds arise, up to bordism, as total spaces of bundles of oriented closed manifolds over surfaces with fibre a fixed manifold M of dimension d. Equivalently, this asks for the image of the morphism

(1) $$\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M)) \longrightarrow \Omega_{d+2}^{\mathrm{SO}},$$

defined on the bordism group of oriented M-bundles over oriented closed surfaces, which assigns to a bundle its total space. Work of Meyer [Mey72; Mey73] shows that the signature $\sigma\colon \Omega_*^{\mathrm{SO}} \to \mathbf{Z}$ of classes in this image has to be divisible by 4. Refining Meyer's result, we provide a complete answer to the posed question in the case of highly connected, almost parallelisable fibres M of even dimension, assuming a mild additional condition on M (see Article B, Prop. 1.9). Our results in particular apply to the iterated connected sums $W_g = \sharp(S^n \times S^n)^{\sharp g}$ to which we restrict our attention in this summary. Using parametrised Pontryagin–Thom theory, we show that, for this family of manifolds, classes in the image of the morphism (1) lift to the bordism group $\Omega_{2n+2}^{\langle n\rangle}$ of highly connected (i.e. n-connected) manifolds, and that this property, when combined with the divisibility of the signature by 4, detects such bordism classes for $g \geq 5$.

Theorem (Article B, Thm A)**.** *The image of the morphism*

$$\Omega_2^{\mathrm{SO}}\left(\mathrm{BDiff}^+(W_g)\right) \longrightarrow \Omega_{2n+2}^{\mathrm{SO}}$$

is contained in the subgroup

$$\mathrm{im}(\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}) \cap \sigma^{-1}(4\cdot\mathbf{Z}).$$

Moreover, equality holds if $g \geq 5$. For $2n = 2$, requiring $g \geq 3$ is sufficient.

This reduces the initial problem to understanding the image of the canonical morphism

(2) $$\Omega_{2n+2}^{\langle n\rangle} \longrightarrow \Omega_{2n+2}^{\mathrm{SO}},$$

a task to which the second part of Article B is devoted. We combine work of Kervaire–Milnor [KM63] and Wall [Wal62] with enhancements due to Brumfiel [Bru68] and Stolz [Sto85; Sto87] to describe this image concretely, both, in terms of explicitly constructed manifolds, as well as expressed in characteristic numbers (see Article B, Sect. 2). These descriptions, however, depend on one unknown: the order $\mathrm{ord}([\Sigma_Q])$ of the class $[\Sigma_Q]$ in $\mathrm{coker}(J)_{4m-1}$ of a certain homotopy sphere Σ_Q in Θ_{4m-1}. Although key to the classification of highly connected manifolds, this class is still only known in very special cases (see Article B, Sect. 2.3). Combining our computations of the image of (2) with the previous theorem, we derive divisibility constraints on the characteristic numbers, signatures, and $\hat{A}$-genera of total spaces of W_g-bundles over surfaces, and determine these invariants completely for $g \geq 5$. We point the reader to Article B for the statements of the general results, but mention one concrete consequence for the divisibility of the signature. In order to do so, we denote the 2-adic evaluation of an integer by $\nu_2(-)$.

Theorem (Article B, Cor. C)**.** *There is an oriented W_g-bundle over a closed oriented surface with $4m$-dimensional total space of signature 4 if and only if $m = 1, 2, 4$. For $m \neq 1, 2, 4$, the signature of such a total space is divisible by 2^{2m+2} for m odd and by $2^{2m-2\nu_2(m)-3}$ for m even.*

In the final part of the work, we explain how our results can be used to derive a basis of the torsion free quotient $\mathrm{H}^2(\mathrm{BDiff}^+(M);\mathbf{Z})_{\mathrm{free}}$ for most closed, highly connected, almost parallelisable manifolds in terms of generalised Miller–Morita–Mumford classes κ_c in $\mathrm{H}^2(\mathrm{BDiff}^+(M))$ associated to classes c in $\mathrm{H}^{2+2n}(\mathrm{BSO})$ (see Article B, Thm 3.5). This extends the corresponding rational computation, which can be obtained by a direct application of the work of Galatius–Randal-Williams (see Article B, Sect. 3.2). To state our integral refinement in the case of W_g, we denote the ith Bernoulli number by B_i and refer the reader to the introduction of Article B for the definition of the other variables appearing in the statement; all of them are explicit, aside from the order $\mathrm{ord}([\Sigma_Q])$ discussed earlier.

Theorem (Article B, Thm D)**.** *Let $2n \geq 6$ and $g \geq 7$. The group $\mathrm{H}^2(\mathrm{BDiff}^+(W_g);\mathbf{Z})_{\mathrm{free}}$ is trivial for $2n \equiv 0 \pmod 4$; of rank 1 generated by*

$$\frac{\kappa_{p_m}}{2(2m-1)!j_m}$$

for $2n \equiv 2 \pmod 8$, where $m = (n+1)/2$; and of rank 2 generated by

$$\frac{\kappa_{p_k^2}}{2\mu_k a_k^2\,\mathrm{ord}([\Sigma_Q])(2k-1)!^2} \quad \textit{and} \quad \frac{2\kappa_{p_{2k}} - \kappa_{p_k^2}}{2(4k-1)!j_{2k}} - \frac{\frac{|B_{2k}|}{4k}\left(c_{2k}\frac{|B_{2k}|}{4k} + 2d_{2k}(-1)^k\right)\kappa_{p_k^2}}{2(2k-1)!^2},$$

for $2n \equiv 6 \pmod 8$, where $k = (n+1)/4$.

Remark. Article B leaves open the question of whether the additional assumption on a highly connected almost parallelisable $2n$-manifold M that we impose (see Article B, Prop. 1.9) is necessary. This is equivalent to asking whether the images of the bordism groups $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M, D^{2n}))$ and $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ in $\Omega_{2n+2}^{\mathrm{SO}}$ agree, which is, in turn, closely related to the torsion in $\mathrm{H}_2(\mathrm{BDiff}^+(M);\mathbf{Z})$ (see Article B, Rem. 1.11). It would be desirable to understand this torsion subgroup better; to date, very little is known about it, even in the presence of high genus.

Article C. Homological stability of topological moduli spaces

Harer's stability theorem [Har85] and Galatius–Randal-Williams' higher dimensional analogue [GR18] are special instances of a general phenomenon known as *homological stability*—the topic of Article C. A sequence of spaces

$$\ldots \to \mathcal{M}_{g-1} \to \mathcal{M}_g \to \mathcal{M}_{g+1} \to \ldots$$

is said to satisfy *homological stability* if the induced maps in homology are isomorphisms in a range of degrees increasing with g. Most proofs of homological stability trace back to a classical argument of Quillen, and in Article C, we conceptualise this pattern to provide a general framework for homological stability results. It is based on the observation that the majority of the families $\mathcal{M}_g$ known to stabilise homologically assemble into a graded

E_1-module $\mathcal{M} = \coprod_{g\geq 0} \mathcal{M}_g$ over an E_2-algebra $\mathcal{A}$—the homotopy theoretical analogue of a graded module over a braided monoidal category (see Article C, Sect. 2.1). In particular, they come equipped with a homotopy associative multiplication $\oplus\colon \mathcal{M}\times\mathcal{A} \to \mathcal{M}$ and a grading $g_{\mathcal{M}}\colon \mathcal{M} \to \mathbf{N}_0$. We introduce the *canonical resolution* $R_\bullet(\mathcal{M}) \to \mathcal{M}$ associated to such a module $\mathcal{M}$ with a *stabilising object* $X \in \mathcal{A}$, i.e. an element of degree 1. Vaguely speaking, this is an augmented semi-simplicial space up to higher coherent homotopy whose connectivity controls the stability behaviour of the sequence $(-\oplus X)\colon \mathcal{M}_g \to \mathcal{M}_{g+1}$ of subspaces $\mathcal{M}_g = g_{\mathcal{M}}^{-1}(\{g\})$ of fixed degree. To state one of the results we prove in this direction, we call $\mathcal{M}$ *graded $\varphi(g_{\mathcal{M}})$-connected* for a function $\varphi\colon \mathbf{N} \to \mathbf{Q}$ if its realisation $|R_\bullet(\mathcal{M})|_g \to \mathcal{M}_g$, restricted to degree g, is $\lfloor\varphi(g)\rfloor$-connected for all $g \geq 1$.

Theorem (Article C, Thm A)**.** *If the canonical resolution of a graded E_1-module over an E_2-algebra with stabilising object X is graded $(\frac{g_{\mathcal{M}}-2+k}{k})$-connected for some $k \geq 2$, then*

$$(-\oplus X)_*\colon \mathrm{H}_i(\mathcal{M}_g) \longrightarrow \mathrm{H}_i(\mathcal{M}_{g+1})$$

is an isomorphism for $i \leq \frac{g-1}{k}$ and an epimorphism for $i \leq \frac{g-2+k}{k}$.

Furthermore, we show that under the same conditions as in the preceding theorem, the sequence $\mathcal{M}_g$ stabilises in a stronger, *twisted* sense, namely with respect to *abelian* and *finite degree* coefficients (see Article C, Thms A and C).

Examples. In many cases, the canonical resolution recovers semi-simplicial spaces of geometric nature that have been studied before and are known to be sufficiently connected for the previous theorem to apply. As a consequence, we derive various new homological stability results and improve many known ones by means of more general coefficients, better ranges, or fewer assumptions. The following is a selection of the examples presented in Article C, generalising results contained in [GR18; McD75; Pal18; Seg73; Til16] and confirming a conjecture by Palmer [Pal18].

Theorem (Article C, Thms D and H, Cor. F)**.** *The following sequences of spaces satisfy homological stability with abelian as well as finite degree coefficients.*

- *(i)* $\mathrm{BDiff}_\partial(M\sharp(S^n\times S^n)^{\sharp g})$*, the moduli space of manifolds diffeomorphic to $M\sharp(S^n\times S^n)^{\sharp g}$ relative to the boundary, for a compact simply connected manifold M of dimension $2n \geq 6$ with non-empty boundary.*
- *(ii) $C_g^\pi(M)$, the unordered configuration space of g points in a manifold M with labels in a fibration $\pi\colon E \to M$ with path connected fibres, where M is connected, has dimension at least 2, and has non-empty boundary.*
- *(iii) $C_g^k(M)$, the configuration space of g unordered embedded k-discs for a fixed $k \geq 0$ in a manifold M satisfying the assumptions of (ii).*
- *(iv)* $\mathrm{BDiff}_{g,\partial}^k(M)$*, the moduli space of manifolds diffeomorphic to a manifold M relative to the boundary, together with g unordered embedded k-discs for a fixed $k \geq 0$, where M satisfies the assumptions of (ii).*

Representation stability. By relating twisted homological stability and *representation stability* in the sense of Church and Farb [CF13], we also conclude that our twisted stability results imply representation stability for related families of moduli spaces equipped with compatible group actions (see Article C, Sect. 5.3.2).

Applications to group homology. The classifying space of a module over a braided monoidal category forms an E_1-module over an E_2-algebra, so our theory can be applied to the study of homological stability for families of groups or monoids as well. This enhances prior work by Randal-Williams and Wahl [RW17] on the stability behaviour of families of discrete groups forming a braided monoidal category in various ways (see Article C, Sect. 7). In particular, we improve their ranges and remove all assumptions that they impose on the braided monoidal groupoid.

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Article A

On characteristic classes of exotic manifold bundles

ON CHARACTERISTIC CLASSES OF EXOTIC MANIFOLD BUNDLES

MANUEL KRANNICH

Abstract. Given a closed simply connected manifold M of dimension $2n \neq 4$, we compare the ring of characteristic classes of smooth oriented bundles with fibre M to the respective ring resulting from replacing M by the connected sum $M\sharp\Sigma$ with an exotic sphere Σ. We show that, after inverting the order of Σ in the group of homotopy spheres, the two rings in question are isomorphic in a range of degrees. Furthermore, we construct infinite families of examples witnessing that inverting the order of Σ is necessary.

The classifying space $\mathrm{BDiff}^+(M)$ of the topological group of orientation-preserving diffeomorphisms of a closed oriented manifold M in the smooth Whitney-topology classifies smooth oriented fibre bundles with fibre M. Its cohomology $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ is the ring of characteristic classes of such bundles and is thus of great interest from the point of view of geometric topology. Initiated by Madsen–Weiss' solution of the Mumford conjecture on the moduli space of Riemann surfaces [MW07], there has been significant progress in the study of $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ in recent years, including in high dimensions. A programme of Galatius–Randal-Williams [GR14; GR17; GR18] culminated in an identification of the cohomology in consideration in a range of degrees in purely homotopy theoretical terms for all simply connected manifolds M of dimension $2n \geq 6$. Analogous to the case of surfaces, this range depends on the *genus* of M, defined as

$$g(M) = \max\{g \geq 0 \mid \text{there exists a manifold } N \text{ such that } M \cong N\sharp(S^n \times S^n)^{\sharp g}\}.$$

The present work is concerned with the behaviour of the cohomology $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ when changing the smooth structure of the underlying d-manifold M on an embedded disc of codimension zero; that is, when replacing M by the connected sum $M\sharp\Sigma$ with an exotic sphere Σ in Θ_d. Here Θ_d denotes the finite abelian group of oriented homotopy spheres, classically studied by Kervaire–Milnor [KM63]. In the first part, we use the work of Galatius–Randal-Williams to show that for closed simply connected manifolds M of dimension $2n \neq 4$, the cohomology $\mathrm{H}^*(\mathrm{BDiff}^+(M))$ is insensitive to replacing M by $M\sharp\Sigma$ in a range of degrees, at least after inverting the order of Σ. As our methods of proof are more of homological than of cohomological nature, we state our results in homology.

Theorem A. *Let M be a closed, oriented, simply connected manifold of dimension $2n \neq 4$ and $\Sigma \in \Theta_{2n}$ an exotic sphere. There is a zig-zag of maps of spaces inducing an isomorphism*

$$\mathrm{H}_*(\mathrm{BDiff}^+(M); \mathbf{Z}[\tfrac{1}{k}]) \cong \mathrm{H}_*(\mathrm{BDiff}^+(M\sharp\Sigma); \mathbf{Z}[\tfrac{1}{k}])$$

in degrees $* \leq \frac{g(M)-3}{2}$, where k denotes the order of $\Sigma \in \Theta_{2n}$.

Remark. Using recent work of Friedrich [Fri17], one can enhance Theorem A to all oriented, closed, connected manifolds M of dimension $2n \neq 4$ whose associated group ring $\mathbf{Z}[\pi_1(M)]$ has finite unitary stable rank, as defined e.g. in [KM02, Def. 6.3]. This applies for instance if the fundamental group is finite or finitely generated and abelian.

Remark. By means of parametrized smoothing theory, Dwyer–Szczarba [DS83] compared the homotopy types of the unit components $\mathrm{Diff}_0(M) \subseteq \mathrm{Diff}^+(M)$ for different smooth structures of M. In particular, their work implies that for a closed manifold M of dimension $d \neq 4$ and a homotopy sphere $\Sigma \in \Theta_d$, the spaces $\mathrm{BDiff}_0(M)$ and $\mathrm{BDiff}_0(M\sharp\Sigma)$ have the

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same $\mathbf{Z}[\frac{1}{k}]$-homotopy type for k the order of Σ in Θ_d, which results in an analogue of Theorem A for $\mathrm{BDiff}_0(M)$ without assumptions on the degree. However, the homology of $\mathrm{BDiff}^+(M)$ and $\mathrm{BDiff}_0(M)$ is usually very different and we do not know whether Theorem A holds outside the stable range of Galatius–Randal-Williams, except in a very special case: specialised to degree 1, the assumption on degree in Theorem A is $g(M) \geq 5$ and from work of Kreck [Kre79], one can conclude that Theorem A holds in this case without assuming $g(M) \geq 5$ if M is $(n-1)$-connected, n-parallelisable, and of dimension $2n \neq 4$.

In the second part of this work, we focus on the family of manifolds $W_g = \sharp^g(S^n \times S^n)$ to address the question of whether Theorem A fails without inverting the order of the homotopy sphere, i.e. with integral coefficients. To state our first result in that direction, we denote by $\mathbf{MO}\langle n\rangle$ the Thom-spectrum of the n-connected cover $\mathrm{BO}\langle n\rangle \to \mathrm{BO}$.[1] By the classical theorem of Pontryagin–Thom, the ring of homotopy groups $\pi_*\mathbf{MO}\langle n\rangle$ is isomorphic to the ring $\Omega_*^{\langle n\rangle}$ of bordism classes of closed manifolds equipped with a lift of their stable normal bundle along $\mathrm{BO}\langle n\rangle \to \mathrm{BO}$. An oriented homotopy sphere Σ of dimension $2n$ has, up to homotopy, a unique such lift that is compatible with its orientation and hence defines a canonical class $[\Sigma]$ in the bordism group $\Omega_{2n}^{\langle n\rangle}$. As is common, we denote by $\eta \in \pi_1\mathbf{S}$ the generator of the first stable homotopy group of the sphere spectrum.

Theorem B. *For $g \geq 0$ and an exotic sphere $\Sigma \in \Theta_{2n}$ with $2n \neq 4$, there is an exact sequence*

$$\mathbf{Z}/2 \longrightarrow \mathrm{H}_1(\mathrm{BDiff}^+(W_g);\mathbf{Z}) \longrightarrow \mathrm{H}_1(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}) \longrightarrow 0.$$

If the product $\eta\cdot[\Sigma] \in \pi_{2n+1}\mathbf{MO}\langle n\rangle$ does not vanish, then the first map is nontrivial. Moreover, the converse holds for $g \geq 5$.

Note that the first homology group $\mathrm{H}_1(\mathrm{BDiff}^+(W_g);\mathbf{Z})$ agrees with the abelianisation of the group $\pi_0\,\mathrm{Diff}^+(W_g)$ of isotopy classes of diffeomorphisms of M. It follows from a result of Kreck [Kre79, Thm. 2] that this group is finitely generated for $2n \geq 6$, so the previous theorem implies that $\mathrm{H}_1(\mathrm{BDiff}^+(W_g);\mathbf{Z})$ and $\mathrm{H}_1(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z})$ cannot be isomorphic if the product $\eta\cdot[\Sigma]$ is nontrivial in $\pi_{2n+1}\mathbf{MO}\langle n\rangle$. From computations in stable homotopy theory, we derive the existence of infinite families of homotopy spheres for which this product does not vanish, hence for which the integral version of Theorem A fails in degree 1. Such examples exist already in dimension 8—the first possible dimension. Combining Theorem B with work of Kreck [Kre79], we also find Σ such that $\pi_0\,\mathrm{Diff}^+(W_g)$ and $\pi_0\,\mathrm{Diff}^+(W_g\sharp\Sigma)$ are nonisomorphic, but become isomorphic after abelianisation.

Corollary C. *There are exotic spheres $\Sigma \in \Theta_{2n}$ in infinitely many dimensions $2n$ such that*

$$\mathrm{H}_1(\mathrm{BDiff}^+(W_g);\mathbf{Z}) \quad \text{and} \quad \mathrm{H}_1(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z})$$

are not isomorphic for $g \geq 0$. Furthermore, there are $\Sigma \in \Theta_{8k+2}$ for all $k \geq 1$ such that

$$\pi_1\,\mathrm{BDiff}^+(W_g) \quad \text{and} \quad \pi_1\,\mathrm{BDiff}^+(W_g\sharp\Sigma)$$

are not isomorphic for $g \geq 0$, but become isomorphic after abelianisation for $g \geq 5$.

By Theorem B, the first homology $\mathrm{BDiff}^+(W_g)$ and $\mathrm{BDiff}^+(W_g\sharp\Sigma)$ is isomorphic after inverting 2. More generally, this holds with W_g replaced by any simply connected manifold M, which raises the question of whether the failure for Theorem A to hold integrally is only 2-primary. We answer this question in the negative by proving the following.

Theorem D. *For $g \geq 0$ and an exotic sphere $\Sigma \in \Theta_{2n}$ with $2n \neq 4$, there is an isomorphism*

$$\mathrm{H}_*(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\tfrac{1}{2}]) \cong \mathrm{H}_*(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\tfrac{1}{2}])$$

in degrees $* \leq 2$ and furthermore an exact sequence

$$\mathbf{Z}[\tfrac{1}{2}] \longrightarrow \mathrm{H}_3(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\tfrac{1}{2}]) \longrightarrow \mathrm{H}_3(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\tfrac{1}{2}]) \longrightarrow 0.$$

If $2n = 10$ and the order of Σ is a positive multiple of 3, then the leftmost map is nontrivial.

[1] Note that, instead of $X\langle n\rangle$, some authors write $X\langle n+1\rangle$ for the n-connected cover of a space X.

It follows from a recent result of Kupers [Kup16, Cor. C] that $\mathrm{H}_3(\mathrm{BDiff}^+(W_g);\mathbf{Z})$ is finitely generated for $2n \neq 4$, so we conclude that the two groups $\mathrm{H}_3(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\frac{1}{2}])$ and $\mathrm{H}_3(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\frac{1}{2}])$ cannot be isomorphic if the first morphism in the exact sequence of Theorem D is nontrivial. As Θ_{10} is cyclic of order 6, this holds for four of the six homotopy spheres in dimension 10 by the second part of Theorem D.

Corollary E. *For $g \geq 0$ and $\Sigma \in \Theta_{10}$ with order a positive multiple of 3, the groups*

$$\mathrm{H}_3(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\tfrac{1}{2}]) \quad \textit{and} \quad \mathrm{H}_3(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\tfrac{1}{2}])$$

are not isomorphic.

Remark. Our methods of proof also show that the analogous statements of of Corollary E and the first part of Corollary C hold for homotopy groups instead of homology groups as well. This implies the existence of homotopy spheres $\Sigma \in \Theta_n$ for which $\pi_i(\mathrm{BDiff}^+(W_g\sharp S^n))$ and $\pi_i(\mathrm{BDiff}^+(W_g\sharp\Sigma))$ are not isomorphic for some i and all g. In the case of $g = 0$, such examples have been constructed earlier by Habegger–Szczarba [HS87].

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1. Exotic spheres and parametrised Pontryagin–Thom theory

We recall high-dimensional parametrised Pontryagin–Thom theory à la Galatius–Randal-Williams and prove Theorem A.

1.1. **Recollection on bordism theory.** Let $\theta\colon B \to \mathrm{BO}$ be a fibration. A *tangential*, respectively *normal*, θ-structure of a manifold M is a lift $\ell_M\colon M \to B$ of its stable tangent, respectively normal, bundle $M \to \mathrm{BO}$ along θ, up to homotopy over BO. The collection of bordism classes of closed d-manifolds equipped with a normal θ-structure forms an abelian group Ω_d^θ under disjoint union (see e.g. [Sto68, Ch. 2] for details). By the classical Pontryagin–Thom theorem, this group is isomorphic to the dth homotopy group $\pi_d\mathbf{M}\theta$ of the Thom spectrum $\mathbf{M}\theta$ associated to θ. Normal θ-structures of a manifold are in natural bijection to tangential $\theta^\perp$-structures, where $\theta^\perp\colon B^\perp \to \mathrm{BO}$ is the pullback of θ along the canonical involution $-1\colon \mathrm{BO} \to \mathrm{BO}$. Justified by this, we do not distinguish between normal θ-structures and tangential $\theta^\perp$-structures.

1.2. **Parametrised Pontryagin–Thom theory.** Let M be a closed, connected, oriented manifold of dimension $2n$. Choose a Moore–Postnikov n-factorisation

$$M \xrightarrow{\ell_M} B \xrightarrow{\theta} \mathrm{BSO}$$

of its stable oriented tangent bundle; that is, a factorisation into an n-connected cofibration ℓ_M followed by an n-co-connected fibration θ. This factorisation is unique up to weak equivalence under M and over BSO, and we call it the *tangential n-type of M*. Using this, the manifold M naturally defines a class $[M, \ell_M]$ in the bordism group $\Omega_{2n}^{\theta^\perp}$ associated to its tangential n-type. We denote by $\theta_d\colon B_d \to \mathrm{BSO}(d)$ the pullback of $\theta\colon B \to \mathrm{BSO}$ along the canonical map $\mathrm{BSO}(d) \to \mathrm{BSO}$. For $d = 2n$, this pullback induces a factorisation

$$M \longrightarrow B_{2n} \xrightarrow{\theta_{2n}} \mathrm{BSO}(2n)$$

of the unstable oriented tangent bundle of M, which can be seen to again be a Moore–Postnikov n-factorisation. We define $\mathbf{MT}\theta_d$ to be the Thom spectrum $\mathbf{Th}(-\theta_d{}^*\gamma_d)$ of the inverse of the pullback of the canonical vector bundle γ_d over $\mathrm{BSO}(d)$ along θ_d. The topological monoid $\mathrm{hAut}(\theta_d)$ of weak equivalences $B_d \to B_d$ over $\mathrm{BSO}(d)$ acts on ${\theta_d}^*\gamma_d$ by bundle automorphisms, inducing an action on the spectrum $\mathbf{MT}\theta_d$ and hence on its associated infinite loop space. The homotopy quotient $\Omega^\infty\mathbf{MT}\theta_d/\!\!/\mathrm{hAut}(\theta_d)$ of this action is the target of the parametrised Pontryagin–Thom map with which the following result of Galatius–Randal-Williams deals (see [GR17, Cor. 1.9]).

Theorem 1.1 (Galatius–Randal-Williams)**.** *Let M be a simply connected, closed, oriented manifold of dimension $2n \geq 6$. There is a parametrised Pontryagin–Thom map*

$$\mathrm{BDiff}^+(M) \longrightarrow \Omega^\infty\mathbf{MT}\theta_{2n}/\!\!/\mathrm{hAut}(\theta_{2n})$$

inducing an isomorphism on homology in degrees $* \leq \frac{g(M)-3}{2}$ onto the path component hit.

Here $g(M)$ denotes the genus of the manifold M, as defined in the introduction.

Remark 1.2. Theorem 1.1 is the higher-dimensional analogue of a pioneering result for surfaces obtained by combining a classical homological stability result due to Harer [Har85] with the celebrated theorem of Madsen–Weiss [MW07].

Remark 1.3. Recent work of Friedrich [Fri17] can be used to strengthen Theorem 1.1 to manifolds that are not simply connected, but whose associated group ring $\mathbf{Z}[\pi_1(M)]$ has finite unitary stable rank (compare the remark in the introduction).

1.3. **The path components of $\mathbf{MT}\theta_{2n}$.** Recall from [GTMW09, Ch. 5] that there is a cofibre sequence of spectra

$$\mathbf{MT}\theta_{d+1} \longrightarrow \Sigma^\infty_+ B_{d+1} \longrightarrow \mathbf{MT}\theta_d.$$

The induced maps $\pi_0\mathbf{MT}\theta_{2n} \to \pi_0\Sigma^\infty_+ B_{2n} \cong \mathbf{Z}$ and $\pi_0\mathbf{MT}\theta_{2n} \to \pi_{-1}\mathbf{MT}\theta_{2n+1}$ assemble into an isomorphism of groups

$$\pi_0\mathbf{MT}\theta_{2n} \xrightarrow{\cong} \{(\chi, y) \in \mathbf{Z} \times \pi_{-1}\mathbf{MT}\theta_{2n+1} \mid \chi \bmod 2 = w_{2n}(y)\}, \tag{1}$$

where $w_{2n}(y)$ denotes the value of the cup product $\theta^*_{2n+1}w_{2n} \cup u_{-\theta_{2n+1}}$ of the $2n$th Stiefel–Whitney class of $\theta^*_{2n+1}\gamma_{2n+1}$ with the Thom class of $-\theta^*_{2n+1}\gamma_{2n+1}$ on the Hurewicz image of $y \in \pi_{-1}\mathbf{MT}\theta_{2n+1}$ (see [GR16b, Ch. 10]). There is a stabilisation map

$$\Sigma^{2n+1}\mathbf{MT}\theta_{2n+1} \longrightarrow \mathbf{M}\theta^\perp, \tag{2}$$

which is $(2n+1)$-connected (cf. [GTMW09, Ch. 3]) and hence identifies $\pi_{-1}\mathbf{MT}\theta_{2n+1}$ with the bordism group $\Omega^{\theta^\perp}_{2n}$. Using this, together with the equivalence (1), we regard $\pi_0\mathbf{MT}\theta_{2n}$ as a subgroup of $\mathbf{Z} \times \Omega^{\theta^\perp}_{2n}$, and with respect to this identification, $\pi_0\,\mathrm{hAut}(\theta_{2n})$ acts on $\pi_0\mathbf{MT}\theta_{2n}$ by changing the θ-structure of the $\Omega^{\theta^\perp}_{2n}$-coordinate while fixing the $\mathbf{Z}$-coordinate. The path components of $\Omega^\infty\mathbf{MT}\theta_{2n}/\!\!/\mathrm{hAut}(\theta_{2n})$ are given by the quotient

$$\pi_0\mathbf{MT}\theta_{2n}/\pi_0\,\mathrm{hAut}(\theta_{2n}) \subseteq \mathbf{Z} \times \Omega^{\theta^\perp}_{2n}/\pi_0\,\mathrm{hAut}(\theta_{2n}),$$

and the path component hit by the parametrised Pontryagin–Thom map of Theorem 1.1

$$\mathrm{BDiff}^+(M) \longrightarrow \Omega^\infty\mathbf{MT}\theta_{2n}/\!\!/\mathrm{hAut}(\theta_{2n})$$

is the class represented by $(\chi(M), [M, \ell_M]) \in \pi_0\mathbf{MT}\theta_{2n}$, where $\chi(M) \in \mathbf{Z}$ is the Euler characterstictic of M (cf. [GTMW09, Ch. 3]). We denote this path component of the homotopy quotient by $\left(\Omega^\infty\mathbf{MT}\theta_{2n}/\!\!/\mathrm{hAut}(\theta_{2n})\right)_M$ and the one of $\Omega^\infty\mathbf{MT}\theta_{2n}$ corresponding to $(\chi(M), [M, \ell_M])$ by $\Omega^\infty_{(M,\ell_M)}\mathbf{MT}\theta_{2n}$. Note that the inclusion $\Omega^\infty_{(M,\ell_M)}\mathbf{MT}\theta_{2n} \subseteq \Omega^\infty\mathbf{MT}\theta_{2n}$ induces a weak equivalence

$$\Omega^\infty_{(M,\ell_M)}\mathbf{MT}\theta_{2n}/\!\!/\mathrm{Stab}(M, \ell_M) \simeq \left(\Omega^\infty\mathbf{MT}\theta_{2n}/\!\!/\mathrm{hAut}(\theta_{2n})\right)_M, \tag{3}$$

where $\mathrm{Stab}(M, \ell_M) \subseteq \mathrm{hAut}(\theta_{2n})$ is the submonoid defined as the union of the path components of $\mathrm{hAut}(\theta_{2n})$ that fix $[M, \ell_M] \in \Omega^{\theta^\perp}_{2n}$.

1.4. **Exotic spheres and Theorem A.** Recall from [KM63] that taking connected sum turns the collection Θ_d of h-cobordism classes of closed oriented d-manifolds with the homotopy type of a d-sphere into a finite abelian group. For $d \neq 4$, this group can be described equivalently as the group of exotic spheres, i.e. the group of oriented d-manifolds homeomorphic to the d-sphere, modulo orientation-preserving diffeomorphism.

By obstruction theory, the tangential n-type of a closed oriented $2n$-manifold M depends only on the manifold $M \backslash \operatorname{int}(D^{2n})$ obtained by cutting out an embedded disc $D^{2n} \subseteq M$.[2] This implies that the tangential n-type of the connected sum $M\sharp\Sigma$ of M with an oriented homotopy sphere Σ has the form

$$M\sharp\Sigma \xrightarrow{\ell_{M\sharp\Sigma}} B \xrightarrow{\theta} \mathrm{BSO}$$

for the same B and θ as for M. Here $\ell_{M\sharp\Sigma}$ is the unique (up to homotopy) extension to $M\sharp\Sigma$ of the restriction of ℓ_M to $M \backslash \operatorname{int}(D^{2n})$, where $D^{2n} \subseteq M$ is the disc at which the connected sum was taken. Consequently, the targets of the two parametrised Pontryagin–Thom maps of Theorem 1.1 for M and $M\sharp\Sigma$ agree,

$$\mathrm{BDiff}^+(M) \longrightarrow \Omega^\infty \mathbf{MT}\theta_{2n} /\!\!/ \mathrm{hAut}(\theta_{2n}) \longleftarrow \mathrm{BDiff}^+(M\sharp\Sigma),$$

and both maps induce an isomorphism on homology in a range of degrees onto the respective path components hit. However, different path components of the target space will usually have non-isomorphic homology groups; nonetheless, we have the following lemma comparing the two components in question.

Lemma 1.4. *For a closed, oriented $2n$-manifold M and $\Sigma \in \Theta_{2n}$, there is a zig-zag between*

$$\big(\Omega^\infty \mathbf{MT}\theta_{2n} /\!\!/ \mathrm{hAut}(\theta_{2n})\big)_M \quad \textit{and} \quad \big(\Omega^\infty \mathbf{MT}\theta_{2n} /\!\!/ \mathrm{hAut}(\theta_{2n})\big)_{M\sharp\Sigma}$$

inducing a homology isomorphism with $\mathbf{Z}[\frac{1}{k}]$-coefficients, k being the order of Σ.

Proof. The stable oriented tangent bundle $\Sigma \to \mathrm{BSO}$ of Σ lifts to a unique tangential θ-structure ℓ_Σ of Σ since θ is n-co-connected. By obstruction theory, the standard bordism between the connected sum $M\sharp\Sigma$ and the disjoint union of M and Σ extends to a bordism that respects the θ-structure. This gives the relation $[M\sharp\Sigma, \ell_{M\sharp\Sigma}] = [M, \ell_M] + [\Sigma, \ell_\Sigma]$ in the group $\Omega_{2n}^{\theta^\perp}$. By the same argument, we obtain a morphism $\Theta_{2n} \to \Omega_{2n}^{\theta^\perp}$ by sending $\Sigma \in \Theta_{2n}$ to $[\Sigma, \ell_\Sigma]$, from which we conclude that the order of $[\Sigma, \ell_\Sigma]$ in $\Omega_{2n}^{\theta^\perp}$ divides k. As the Euler characteristics of M and $M\sharp\Sigma$ evidently agree, we have the relation

$$k \cdot (\chi(M\sharp\Sigma), [M\sharp\Sigma, \ell_M\sharp\Sigma]) = k \cdot (\chi(M), [M, \ell_M])$$

in the abelian group $\pi_0 \mathbf{MT}\theta_{2n}$, using the identification (1). Consequently, multiplication by k in the infinite loop space $\Omega^\infty \mathbf{MT}\theta_{2n}$ maps the path components $(\chi(M), [M, \ell_M])$ and $(\chi(M\sharp\Sigma), [M\sharp\Sigma, \ell_{M\sharp\Sigma}])$ to the same component, denoted by $\Omega^\infty_{k\cdot(M,\ell_M)} \mathbf{MT}\theta_{2n}$. As observed above, the homotopy sphere Σ has a unique θ-structure compatible with its orientation, hence the class $[\Sigma, \ell_\Sigma]$ in $\Omega_{2n}^{\theta^\perp}$ is fixed by the action of $\mathrm{hAut}(\theta)$, which in turn implies $\mathrm{Stab}(M, \ell_M) = \mathrm{Stab}(M\sharp\Sigma, \ell_{M\sharp\Sigma}) \subseteq \mathrm{hAut}(\theta_{2n})$. Since multiplication by k in $\Omega^\infty \mathbf{MT}\theta_{2n}$ is $\mathrm{hAut}(\theta_{2n})$-equivariant, we obtain an induced zig-zag of homotopy quotients

$$\begin{array}{ccccc} & & \Omega^\infty_{k\cdot(M,\ell_M)} \mathbf{MT}\theta_{2n} /\!\!/ \mathrm{Stab}(M, \ell_M) & & \\ & \overset{k\cdot -}{\nearrow} & & \overset{k\cdot -}{\nwarrow} & \\ \Omega^\infty_{(M,\ell_M)} \mathbf{MT}\theta_{2n} /\!\!/ \mathrm{Stab}(M, \ell_M) & & & & \Omega^\infty_{(M\sharp\Sigma,\ell_{M\sharp\Sigma})} \mathbf{MT}\theta_{2n} /\!\!/ \mathrm{Stab}(M\sharp\Sigma, \ell_{M\sharp\Sigma}). \end{array}$$

The corresponding zig-zag between the respective path components of $\Omega^\infty \mathbf{MT}\theta_{2n}$ before taking homotopy quotients is given by multiplication by k, so induces an isomorphism on

[2] In fact, it only depends on the n-skeleton of the manifold.

homology with $\mathbf{Z}[\frac{1}{k}]$-coefficients. The claim now follows from a comparison of the Serre spectral sequences of the homotopy quotients, together with the equivalences (3). ◻

By taking connected sums with the homotopy sphere Σ and its inverse, one sees that the genera of M and $M\sharp\Sigma$ agree. Since the group Θ_d is trivial in dimensions $d < 7$ by [KM63], Theorem A now follows from Theorem 1.1 and Lemma 1.4.

2. The collar twisting

In this section, we examine the homotopy fibre sequence

$$\mathrm{SO}(d) \xrightarrow{t} \mathrm{BDiff}(M, D^d) \longrightarrow \mathrm{BDiff}^+(M, *) \tag{4}$$

for a closed oriented d-manifold M, induced by the inclusion $\mathrm{Diff}(M, D^d) \subseteq \mathrm{Diff}^+(M, *)$ of the subgroups of diffeomorphisms that pointwise fix an embedded disc $D^d \subseteq M$ or its centre $* \in D^d$, respectively. In particular, we study the effect of the fibre inclusion t on homotopy groups. After fixing a collar $c\colon [0,1] \times S^{d-1} \to M\backslash\mathrm{int}(D^d)$ satisfying $c^{-1}(\partial D^d) = \{1\} \times S^{d-1}$, the map t can be described geometrically as the delooping of the map $\Omega\,\mathrm{SO}(d) \to \mathrm{Diff}(M, D^d)$ sending a smooth loop $\gamma \in \Omega\,\mathrm{SO}(d)$ to the diffeomorphism of M that is the identity on D^d as well as outside the collar, and the twist

$$\begin{array}{ccc} [0,1] \times S^{d-1} & \longrightarrow & [0,1] \times S^{d-1} \\ (t, x) & \longmapsto & (t, \gamma(t)x) \end{array}$$

on the collar. Inspired by this geometric description, we call the map

$$t\colon \mathrm{SO}(d) \longrightarrow \mathrm{BDiff}(M, D^d)$$

the *collar twisting of M*.

Remark 2.1. In dimension $d = 2$, the collar twisting $\mathrm{SO}(2) \to \mathrm{BDiff}(M, D^2)$ is trivial on homotopy groups, except on fundamental groups, on which the induced map $\mathbf{Z} \to \pi_0\,\mathrm{BDiff}(M, D^2)$ is given by a Dehn twist on $[0,1] \times S^1 \subseteq M$.

2.1. Triviality of the collar twisting.

As indicated by the following lemma, the collar twisting serves as a measure for the degree of linear symmetry of the underlying manifold.

Lemma 2.2. *Let M be a closed oriented d-manifold that admits a smooth orientation-preserving action of $\mathrm{SO}(k)$ with $k \le d$. If the action has a fixed point whose tangential representation is the restriction of the standard representation of $\mathrm{SO}(d)$ to $\mathrm{SO}(k)$, then the long exact sequence induced by the fibre sequence (4) reduces to split short exact sequences*

$$0 \longrightarrow \pi_i\,\mathrm{BDiff}(M, D^d) \longrightarrow \pi_i\,\mathrm{BDiff}^+(M, *) \longrightarrow \pi_{i-1}\,\mathrm{SO}(d) \longrightarrow 0$$

for $i \le k-1$. In particular, the collar twisting is trivial on homotopy groups in this range.

Proof. The map $d\colon \mathrm{Diff}^+(M, *) \to \mathrm{SO}(d)$ resulting from looping the fibre sequence (4) once is induced by taking the differential at $* \in M$. On the subgroup $\mathrm{SO}(k) \subseteq \mathrm{SO}(d)$, the action on M provides a left-inverse to d. As the inclusion $\mathrm{SO}(k) \subseteq \mathrm{SO}(d)$ is $(k-1)$-connected, we conclude that the map d is surjective on homotopy groups in degree $k-1$ and split surjective in lower degrees, which implies the result. ◻

The action of $SO(d)$ on the standard sphere S^d by rotation along an axis satisfies the assumption of the lemma, so the collar twisting of S^d is trivial on homotopy groups up to degree $d-1$. In fact, one can show that, in this case, it is even nullhomotopic. Another family of manifolds that admit a smooth action of $\mathrm{SO}(k)$ as in the lemma is given by the g-fold connected sums

$$W_g = \sharp^g(S^n \times S^n).$$

Indeed, by [GGR17, Prop. 4.3], there is a smooth $\mathrm{SO}(n) \times \mathrm{SO}(n)$-action on W_g whose restriction to a factor can be seen to provide an action of $\mathrm{SO}(n)$ as in the lemma. However, we give an alternative description of this action, kindly pointed out to us by Jens Reinhold.

Consider the action of $\mathrm{SO}(n)$ on $S^n \times S^n$ by rotating the first factor around the vertical axis. Both, the product of the two upper hemispheres of the two factors and the product of the two lower ones are preserved by the action and are, after smoothing corners, diffeomorphic to a disc D^{2n} acted upon by $\mathrm{SO}(n)$ via the inclusion $\mathrm{SO}(n) \subseteq \mathrm{SO}(2n)$, followed by the standard action of $\mathrm{SO}(2n)$ on D^{2n}. Taking the g-fold equivariant connected sum of $S^n \times S^n$ using these discs results in an action of $\mathrm{SO}(n)$ on W_g as in Lemma 2.2 and thus has the following as a consequence.

Corollary 2.3. *The collar twisting $\pi_i\,\mathrm{SO}(2n) \to \pi_i\,\mathrm{BDiff}(W_g, D^{2n})$ is trivial for $i \leq n-1$.*

2.2. **Detecting the collar twisting in bordism.** Recall that a manifold M is (stably) n-parallelisable if it admits a tangential $\langle n\rangle$-structure for the n-connected cover $\mathrm{BO}\langle n\rangle \to \mathrm{BO}$. This map factors for $n \geq 1$ over BSO and obstruction theory shows that there is a unique (up to homotopy) equivalence $\mathrm{BO}\langle n\rangle^{\perp} \simeq \mathrm{BO}\langle n\rangle$ over BSO, so tangential and normal $\langle n\rangle$-structures of a manifold M are naturally equivalent. For oriented manifolds M, we require that $\langle n\rangle$-structures on M are compatible with the orientation; that is, they lift the oriented stable tangent bundle $M \to \mathrm{BSO}$ of M. Another application of obstruction theory shows that the map $\mathrm{BSO}\times\mathrm{BSO} \to \mathrm{BSO}$ classifying the external sum of oriented stable vector bundles is, up to homotopy, uniquely covered by a map $\mathrm{BO}\langle n\rangle \times \mathrm{BO}\langle n\rangle \to \mathrm{BO}\langle n\rangle$ turning $\mathbf{MO}\langle n\rangle$ into a homotopy commutative ring spectrum.

In the following, we describe a method to detect the nontriviality of certain collar twistings. For this, we restrict our attention to oriented manifolds M that are $(n-1)$-connected, n-parallelisable, and $2n$-dimensional. These manifolds have by obstruction theory a unique $\langle n\rangle$-structure $\ell_M\colon M \to \mathrm{BO}\langle n\rangle$, so they naturally determine a class $[M, \ell_M]$ in the bordism group $\Omega_{2n}^{\langle n\rangle}$. The examples we have in mind are the connected sums $W_g\sharp\Sigma$ of the manifolds $W_g = \sharp^g(S^n \times S^n)$ with homotopy spheres Σ.

Remark 2.4. In fact, all $(n-1)$-connected n-parallelisable $2n$-manifolds M with positive genus and vanishing signature are of the form $W_g\sharp\Sigma$ for a homotopy sphere Σ, except possibly those in the Kervaire invariant one dimensions and in dimension 4. Indeed, the vanishing of the signature implies that M is stably parallelisable and therefore by the work of Kervaire–Milnor [KM63] framed bordant to a homotopy sphere. From this, an application of Kreck's modified surgery [Kre99, Thm. C–D] shows the claim.

For an oriented $(n-1)$-connected n-parallelisable $2n$-manifold M, we define morphisms

$$\Phi_i\colon \pi_i\,\mathrm{BDiff}(M, D^{2n}) \longrightarrow \Omega_{2n+i}^{\langle n\rangle} \tag{5}$$

as follows. A homotopy class φ in $\pi_i\,\mathrm{BDiff}(M, D^{2n})$ classifies a smooth fibre bundle

$$M \to E_\varphi \to S^i,$$

together with the choice of a trivialised D^{2n}-subbundle $S^i \times D^{2n} \subseteq E_\varphi$. The latter induces a trivialisation of the normal bundle of $S^i = S^i \times \{0\}$ in E_φ. Using this, a given $\langle n\rangle$-structure on E_φ induces an $\langle n\rangle$-structure on the embedded S^i. Conversely, obstruction theory shows that every $\langle n\rangle$-structure on S^i is induced by a unique $\langle n\rangle$-structure on E_φ in this manner. We can hence define Φ_i by sending a homotopy class φ to the total space E_φ of the associated bundle, together with the unique $\langle n\rangle$-structure extending the canonical $\langle n\rangle$-structure on S^i induced by the standard stable framing of S^i.

Remark 2.5. The morphisms Φ_i are induced by a map of spaces

$$\mathrm{BDiff}(M, D^{2n}) \longrightarrow \Omega^\infty\Sigma^{-2n}\mathbf{MO}\langle n\rangle,$$

which results from composing the parametrised Pontryagin–Thom map $\mathrm{BDiff}(M, D^{2n}) \to \Omega^\infty\mathbf{MTO}(2n)\langle n\rangle$ (see e.g. [GR14, Thm. 1.2]) with the $2n$-fold desuspension $\mathbf{MTO}(2n)\langle n\rangle \to \Sigma^{-2n}\mathbf{MO}\langle n\rangle$ of the stabilisation map (2) for $\mathbf{MTO}(2n)\langle n\rangle$.

Lemma 2.6. *The following diagram is commutative*

$$\begin{array}{ccc} \pi_i\,\mathrm{SO}(2n) & \xrightarrow{t_*} & \pi_i\,\mathrm{BDiff}(M, D^{2n}) \\ {\scriptstyle J}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_i} \\ \Omega_i^{\langle n\rangle} & \xrightarrow{-\times[M,\ell_M]} & \Omega_{2n+i}^{\langle n\rangle}, \end{array}$$

the left vertical map being the J-homomorphism followed by the natural morphism from framed to $\langle n\rangle$-bordism, and the bottom map the multiplication by $[M, \ell_M] \in \Omega_{2n}^{\langle n\rangle}$.

Proof. Recall from the beginning of the chapter that the map t is given by twisting a collar $[0,1] \times S^{2n-1} \subseteq M\backslash \operatorname{int}(D^{2n})$. The composition of t_* with Φ_i maps a class in $\pi_i\,\mathrm{SO}(2n)$, represented by a smooth map $\varphi\colon S^i \to \mathrm{SO}(2n)$, to the bordism class of a certain manifold $E_{t(\varphi)}$, equipped with a particular $\langle n\rangle$-structure. The manifold $E_{t(\varphi)}$ is constructed using a clutching function that twists the collar using φ and is constant outside of it. Its associated $\langle n\rangle$-structure is the unique one that extends the canonical one on S^i via the given trivialisation of its normal bundle. Untwisting the collar using the standard $\mathrm{SO}(2n)$-action on the disc D^{2n} yields a diffeomorphism $E_{t(\varphi)} \cong S^i \times M$, which coincides with the twist

$$\begin{array}{ccc} D^{2n} \times S^i & \longrightarrow & D^{2n} \times S^i \\ (x, t) & \longmapsto & (\varphi(t)x, t) \end{array}$$

on the canonically embedded $S^i \times D^{2n}$ and is the identity on the other component of the complement of the twisted collar $S^i \widetilde{\times}([0,1] \times S^{2n-1}) \subseteq E_\varphi$. Hence, the $\langle n\rangle$-structure on $S^i \times M$ induced from the one on $E_{t(\varphi)}$ via this diffeomorphism coincides with the product of the twist of the canonical $\langle n\rangle$-structure on S^i by φ with the unique one ℓ_M on M. By the bordism theoretic description of the J-homomorphism, this implies the claim. □

Lemma 2.6 serves us to detect the nontriviality of the collar twisting

$$\mathrm{SO}(2n) \longrightarrow \mathrm{BDiff}(M, D^{2n}).$$

Indeed, if there is a nontrivial element in the subgroup $\operatorname{im}(J)_i \cdot [M, \ell_M]$ of $\pi_{2n+i}\mathbf{MO}\langle n\rangle$, then Lemma 2.6 implies that the collar twisting of M is nontrivial on homotopy groups in degree i. The g-fold connected sum $W_g = \sharp^g(S^n \times S^n)$ is the boundary of the parallelisable handlebody $\natural^g D^{n+1} \times S^n$ and is thus trivial in framed bordism, and so also in $\langle n\rangle$-bordism. From this, we obtain the relation

$$[W_g \sharp\Sigma, \ell_{W_g\sharp\Sigma}] = [W_g, \ell_{W_g}] + [\Sigma, \ell_\Sigma] = [\Sigma, \ell_\Sigma]$$

in $\Omega_{2n}^{\langle n\rangle}$ for all Σ in Θ_{2n}. This proves the first part of the following proposition.

Proposition 2.7. *Let $g \geq 0$ and let $\Sigma \in \Theta_{2n}$ be a homotopy sphere. If the subgroup*

$$\operatorname{im}(J)_i \cdot [\Sigma, \ell_\Sigma] \subseteq \pi_{2n+i}\mathbf{MO}\langle n\rangle$$

is nontrivial for some $i \geq 1$, then the abelianisation of the morphism

$$t_*\colon \pi_i\,\mathrm{SO}(2n) \longrightarrow \pi_i\,\mathrm{BDiff}(W_g\sharp\Sigma, D^{2n})$$

is nontrivial. Assuming $2n \neq 4$, the converse holds for $i = 1$ and $g \geq 5$.

Proof. We are left to prove the second claim. For $i = 1$, the group $\pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma, D^{2n})$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $W_g\sharp\Sigma$ that fix the embedded disc D^{2n} pointwise. Letting such diffeomorphisms act on the middle-dimensional homology group $\mathrm{H}_n(W_g\sharp\Sigma; \mathbf{Z})$ provides a morphism $\pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma, D^{2n}) \to \operatorname{Aut}(Q_{W_g\sharp\Sigma})$ to the subgroup $\operatorname{Aut}(Q_{W_g\sharp\Sigma}) \subseteq \mathrm{GL}(\mathrm{H}_n(W_g\sharp\Sigma; \mathbf{Z}))$ of automorphisms that preserve Wall's quadratic form $Q_{W_g\sharp\Sigma}$ associated to the $(n-1)$-connected $2n$-manifold $W_g\sharp\Sigma$ (see [Wal62a]). Together with the morphism (5) in degree 1, this combines to a morphism

$$\pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma, D^{2n}) \longrightarrow \operatorname{Aut}(Q_{W_g\sharp\Sigma}) \oplus \Omega_{2n+1}^{\langle n\rangle},$$

which is the abelianisation of $\pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n})$ by [GR16a, Thm. 1.3], assuming $g \geq 5$ and $2n \neq 4$.[3] The isotopy classes in the image of $\pi_1 \operatorname{SO}(2n) \to \pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n})$ are supported in a disc and hence act on homology as the identity. Consequently, on fundamental groups, the collar twisting is for $g \geq 5$ nontrivial in the abelianisation if and only if it is nontrivial after composition with $\pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n}) \to \Omega^{\langle n \rangle}_{2n+1}$. The claim hence follows from Lemma 2.6 and the discussion before Proposition 2.7. □

Remark 2.8. (i) Since the image of the J-homomorphism is cyclic, the condition in Proposition 2.7 is equivalent to the nonvanishing of a single element, e.g. $\eta \cdot [\Sigma, \ell_\Sigma]$ in $\pi_{2n+1}\mathbf{MO}\langle n \rangle$ for $i = 1$ or $\nu \cdot [\Sigma, \ell_\Sigma]$ in $\pi_{2n+3}\mathbf{MO}\langle n \rangle$ for $i = 3$.

(ii) By obstruction theory, the sphere S^i has a unique $\langle n \rangle$-structure for $i \geq n+1$, so the left vertical morphism in the diagram of Lemma 2.6 is trivial in this range. Hence, the morphism $\pi_i \operatorname{BDiff}(M, D^{2n}) \to \Omega^{\langle n \rangle}_{2n+i}$ can only detect the possible nontriviality of the collar twisting in low degrees relative to the dimension. Another consequence of this uniqueness is that the image $\operatorname{im}(J)_i$ of the J-homomorphism is contained in the kernel of the canonical morphism from framed to $\langle n \rangle$-bordism for $i \geq n+1$.

Remark 2.9. (i) Combining Lemma 2.2 and 2.6, we see that the subgroups $\operatorname{im}(J)_i \cdot [M]$ of $\pi_{2n+i}\mathbf{MO}\langle n \rangle$ obstruct smooth $\operatorname{SO}(k)$-actions on M with a certain fixed point.

(ii) In the case $g = 0$, the first part of Proposition 2.7 is closely related to a result obtained by Schultz [Sch71, Thm. 1.2]. In combination with methods developed by Hsiang–Hsiang (see e.g. [HH69]), Schultz used this to bound the degree of symmetry of certain homotopy spheres, i.e. the maximum of the dimensions of compact Lie groups which act effectively.

In the following, we attempt to detect nontrivial elements in the subgroup

$$\operatorname{im}(J)_i \cdot [\Sigma, \ell_\Sigma] \subseteq \pi_{2n+i}\mathbf{MO}\langle n \rangle \tag{6}$$

for homotopy spheres Σ in Θ_{2n} by mapping $\mathbf{MO}\langle n \rangle$ to ring spectra whose rings of homotopy groups are better understood. A natural choice of such spectra are $\mathbf{MO}\langle m \rangle$ for small $m \leq n$ via the canonical map $\mathbf{MO}\langle n \rangle \to \mathbf{MO}\langle m \rangle$. As homotopy spheres are stably parallelisable, the elements $[\Sigma, \ell_\Sigma]$ are in the image of the unit $\mathbf{S} \to \mathbf{MO}\langle n \rangle$. Conversely, outside of the Kervaire invariant one dimensions, the work of Kervaire–Milnor [KM63] implies that all elements in this image are represented by homotopy spheres. Since the image of the unit $\mathbf{S} \to \mathbf{MO}\langle 2 \rangle = \mathbf{MSpin}$ lies in degrees $8k+1$ and $8k+2$ for $k \geq 0$ [ABP67, Cor. 2.7], we cannot detect nontrivial elements in (6) by relying on spin bordism. Consequently, since $\mathbf{MO}\langle 2 \rangle = \mathbf{MO}\langle 3 \rangle$, the smallest value of m such that $\mathbf{MO}\langle m \rangle$ might possibly detect nontrivial such elements is 4. Via the string orientation $\mathbf{MString} \to \mathbf{tmf}$ (see e.g. [DFHH14, Ch. 10]), the Thom spectrum $\mathbf{MO}\langle 4 \rangle = \mathbf{MString}$ maps to the spectrum $\mathbf{tmf}$ of topological modular forms, whose ring of homotopy groups has been computed (see e.g. [DFHH14, Ch. 13]). It is concentrated at the primes 2 and 3 and has a certain periodicity of degree 192 at the prime 2 and of degree 72 at the prime 3. The Hurewicz image of $\mathbf{tmf}$, i.e. the image of the composition

$$\mathbf{S} \longrightarrow \mathbf{MSO}\langle n \rangle \longrightarrow \mathbf{MString} \longrightarrow \mathbf{tmf}$$

on homotopy groups, is known at the prime 3 (see e.g. [DFHH14, Ch. 13]) and determined in work in progress by Behrens–Mahowald at the prime 2. It is known to contain many periodic families of nontrivial products with $\eta \in \pi_1\mathbf{S}$ and $\nu \in \pi_3\mathbf{S}$, which thus provide infinite families of homotopy spheres Σ for which the subgroup (6) is nontrivial. The following lemma carries out this strategy in the lowest dimensions possible.

Proposition 2.10. *In the following cases, there exists a homotopy sphere Σ in Θ_{2n} for which the subgroup $\operatorname{im}(J)_i \cdot [\Sigma, \ell_\Sigma]$ of $\pi_{2n+i}\mathbf{MO}\langle n \rangle$ is nontrivial at the prime p:*

[3] Although Theorem 1.3 of [GR16a] is stated for W_g, the given proof goes through for $W_g \sharp \Sigma$ and even more generally for all $(n-1)$-connected n-parallelisable $2n$-manifolds M, replacing g with the genus $g(M)$ of M.

(i) for $p = 2$ *in degree* $i = 1$ *and all dimensions* $2n \equiv 8 \pmod{192}$*,*
(ii) for $p = 2$ *in degree* $i = 3$ *and all dimensions* $2n \equiv 14 \pmod{192}$*, and*
(iii) for $p = 3$ *in degree* $i = 3$ *and all dimensions* $2n \equiv 10 \pmod{72}$*.*

Consequently, in the respective cases, the abelianisation of the morphism

$$t_*\colon \pi_i \operatorname{SO}(2n) \longrightarrow \pi_i \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n})$$

is nontrivial at the prime p *for all* $g \geq 0$*.*

Proof. By work of Browder [Bro69], all Kervaire invariant one dimensions have the form $2^k - 2$ for $k \geq 0$, so all elements in $\pi_{2n}\mathbf{S}$ for $2n$ as in one of the three cases can be represented by homotopy spheres. The elements $\varepsilon \in \pi_8\mathbf{S}$ and $\kappa \in \pi_{14}\mathbf{S}$ give rise to two 192-periodic families in $\pi_*\mathbf{S}$ at the prime 2 whose elements are nontrivial when multiplied with $\eta \in \operatorname{im}(J)_1$ and $\nu \in \operatorname{im}(J)_3$, respectively (see e.g. [HM14, Ch. 11]). All of these products are detected in **tmf** and hence are also nontrivial in $\mathbf{MO}\langle n\rangle$. This proves the first and the second case. To prove the third one, we make use of a 72-periodic family in $\pi_*\mathbf{S}$ at the prime 3 generated by $\beta \in \pi_{10}\mathbf{S}$. The products of the elements in this family with $\nu \in \pi_3\mathbf{S}$ are nontrivial and are detected in **tmf** (see e.g. [DFHH14, Ch. 13]). This proves the first part of the proposition. The second part is implied by the first by means of Proposition 2.7. □

From work of Kreck [Kre79], we also derive the existence of homotopy spheres Σ for which the morphism $t_*\colon \pi_1 \operatorname{SO}(2n) \to \pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n})$ induced by the collar twisting is nontrivial, but becomes trivial in the abelianisation.

Proposition 2.11. *For all* $k \geq 1$*, there are homotopy spheres* Σ *in* Θ_{8k+2} *for which*

$$t_*\colon \pi_1 \operatorname{SO}(8k+2) \longrightarrow \pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{8k+2})$$

is nontrivial for $g \geq 0$*, but vanishes in the abelianisation for* $g \geq 5$*.*

Proof. Let Σ be the homotopy sphere that maps via the canonical morphism $\Theta_{8k+2} \to \operatorname{coker}(J)_{8k+2}$ to the class of Adams' element μ_{8k+2} in $\pi_{8k+2}\mathbf{S}$ (see [Ada66, Thm. 1.8]). By [Kre79, Lem. 4], the morphism in question is trivial if and only if a certain homotopy sphere $\Sigma_{W_g \sharp \Sigma}$ vanishes in Θ_{8k+3}. Combining [Kre79, Cor. 3] with the fact that W_g bounds the parallelisable manifold $\natural^g D^{4k+2} \times S^{4k+1}$, it follows that the element $\Sigma_{W_g \sharp \Sigma}$ does not vanish. However, the product $\eta \cdot \mu_{8k+2}$ is known to be contained in the image of J (see e.g. [Rav86, Thm. 5.3.7]), so the class $\eta \cdot [\Sigma, \ell_\Sigma]$ is trivial in $\pi_{8k+3}\mathbf{MT}\langle 4k+1\rangle$ by the second part of Remark 2.8. The claim now follows from the second part of Proposition 2.7. □

Remark 2.12. Note that, since $\pi_1(\operatorname{BDiff}(\Sigma, D^{2n}))$ is abelian, the conclusion of the second part of Proposition 2.11 fails for $g = 0$.

Remark 2.13. Recall that the fundamental group $\pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n})$ is the group of isotopy classes of diffeomorphisms of $W_g \sharp \Sigma$ that fix an embedded disc D^{2n} pointwise. The image of $\pi_1 \operatorname{SO}(2n) \to \pi_1 \operatorname{BDiff}(W_g \sharp \Sigma, D^{2n})$ is generated by a single diffeomorphism $t(\eta)$, which is the higher-dimensional analogue of a Dehn twist for surfaces. It twists a collar around D^{2n} by a generator in $\pi_1 \operatorname{SO}(2n)$ and is the identity elsewhere. Consequently, the morphism in consideration is trivial if and only if $t(\eta)$ is isotopic to the identity, and it is trivial in the abelianisation if and only $t(\eta)$ is isotopic to a product of commutators.

3. Detecting exotic smooth structures in diffeomorphism groups

Evaluating diffeomorphisms of a closed oriented d-manifold M at a fixed point induces a homotopy fibre sequence

$$M \longrightarrow \operatorname{BDiff}^+(M, *) \longrightarrow \operatorname{BDiff}^+(M),$$

from which we see that for highly connected manifolds M, the map $\mathrm{BDiff}^+(M,*) \to \mathrm{BDiff}^+(M)$ is also highly connected. Studying the space $\mathrm{BDiff}^+(M,*)$ instead of $\mathrm{BDiff}^+(M)$ is advantageous as $\mathrm{BDiff}^+(M,*)$ fits into a homotopy fibre sequence

$$\mathrm{BDiff}(M,D^d) \longrightarrow \mathrm{BDiff}^+(M,*) \longrightarrow \mathrm{BSO}(d), \tag{7}$$

obtained by delooping the sequence (4). Evidently, the rightmost space in (7) is not affected by replacing M by $M\sharp\Sigma$ for homotopy spheres Σ, and the following lemma shows that the same holds for the leftmost space as well.

Lemma 3.1. *For an oriented manifold M of dimension $d \neq 4$ and a homotopy sphere Σ in Θ_d, there is an isomorphism of topological groups of the form*

$$\mathrm{Diff}(M,D^d) \cong \mathrm{Diff}(M\sharp\Sigma,D^d).$$

Proof. The group $\mathrm{Diff}(M\sharp\Sigma,D^d)$ can be equivalently described as the group of diffeomorphisms of $M\sharp\Sigma\backslash \mathrm{int}(D^d)$ that extend over the disc $D^d \subseteq M\sharp\Sigma$ by the identity. Manifolds of dimension $d \neq 4$ with nonempty boundary are insensitive to taking the connected sum with a homotopy sphere, so there is a diffeomorphism $M\sharp\Sigma\backslash\mathrm{int}(D^d) \cong M\backslash \mathrm{int}(D^d)$. This diffeomorphism does not necessarily preserve the boundary, but conjugation with it nevertheless induces an isomorphism of topological groups as claimed. □

Remark 3.2. By Lemma 3.1, the total space of the homotopy fibre sequence

$$\mathrm{Fr}^+(M) \longrightarrow \mathrm{BDiff}(M,D^d) \longrightarrow \mathrm{BDiff}^+(M)$$

involving the frame bundle $\mathrm{Fr}^+(M)$ of M does not change when replacing M by $M\sharp\Sigma$ for homotopy spheres Σ and surprisingly, the same holds for $\mathrm{Fr}^+(M)$ (see e.g. the proof of [CZ08, Thm. 1.2]). However, we will not make use of this fact.

Proof of Theorem B and Corollary C. Since the group Θ_d is trivial for $d < 7$, we may restrict our attention to dimensions $2n \geq 8$. The fibration (7) induces an exact sequence

$$\mathbf{Z}/2 \cong \pi_1\,\mathrm{SO}(2n) \xrightarrow{t_*} \pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n}) \longrightarrow \pi_1\,\mathrm{BDiff}^+(W_g\sharp\Sigma) \longrightarrow 0.$$

The first map is trivial in the case of the standard sphere $\Sigma = S^{2n}$ by Corollary 2.3—a fact which, when combined with Lemma 3.1, gives isomorphisms

$$\pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n}) \cong \pi_1\,\mathrm{BDiff}(W_g,D^{2n}) \cong \pi_1\,\mathrm{BDiff}^+(W_g).$$

We arrive at an exact sequence of the shape

$$\mathbf{Z}/2 \longrightarrow \pi_1\,\mathrm{BDiff}(W_g) \longrightarrow \pi_1\,\mathrm{BDiff}^+(W_g\sharp\Sigma) \longrightarrow 0,$$

whose first map agrees with the collar twisting of $W_g\sharp\Sigma$ on fundamental groups. Abelianising this exact sequence implies Theorem B by virtue of Proposition 2.7. From a result of Sullivan [Sul77, Thm. 13.3], we know that the mapping class group $\pi_0\,\mathrm{Diff}^+(W_g)$ is commensurable to an arithmetic group and hence finitely generated and residually finite. This implies that it is Hopfian; that is, every surjective endomorphism is an isomorphism. Therefore, the two groups $\pi_1\,\mathrm{BDiff}(W_g)$ and $\pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma)$ are isomorphic if and only if the morphism $\pi_1\,\mathrm{SO}(2n) \to \pi_1\,\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n})$ is trivial, and the two respective abelianisations are isomorphic if and only if the morphism in consideration is trivial after abelianisation. Thus, combining Proposition 2.10 and 2.11 proves Corollary C. □

Remark. It follows from Corollary C that there are infinitely many dimensions in which the inertia group $I(W_g)$ of W_g is a proper subgroup of Θ_{2n}. Results of Kosinski [Kos67, Thm. 3.1] and Wall [Wal62b] show that $I(W_g)$ is in fact trivial in all dimensions.

Proof of Theorem D. As in the foregoing proof, we may assume $2n \geq 8$. In this range, the table below displays the homology of $\mathrm{BSO}(2n)$ in low degrees (see e.g. [Bro82]).

	$i=1$	$i=2$	$i=3$	$i=4$
$\mathrm{H}_i(\mathrm{BSO}(2n);\mathbf{Z})$	0	$\mathbf{Z}/2$	0	$\mathbf{Z}\oplus\mathbf{Z}/2$

Consequently, the homology $\mathrm{H}_*(\mathrm{BSO}(2n);\mathbf{Z}[\frac{1}{2}])$ vanishes in degrees $*\leq 3$ and is isomorphic to $\mathbf{Z}[\frac{1}{2}]$ in degree 4. From the Serre exact sequence with $\mathbf{Z}[\frac{1}{2}]$-coefficients of the homotopy fibre sequence (7), one sees that the induced map

$$\mathrm{H}_*(\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n});\mathbf{Z}[\tfrac{1}{2}])\longrightarrow \mathrm{H}_*(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\tfrac{1}{2}])$$

is an isomorphism in degrees $*\leq 2$, which, together with Lemma 3.1, proves the first part of the theorem. To show the second, we use the Hurewicz homomorphism to map the long exact sequence on homotopy groups of the homotopy fibre sequence in consideration to its Serre exact sequence. This yields a commutative diagram with exact rows as follows.

$$\begin{array}{ccccccc}
\pi_4\,\mathrm{BSO}(2n)\otimes\mathbf{Z}[\frac{1}{2}] & \xrightarrow{t_*} & \pi_3\,\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n})\otimes\mathbf{Z}[\frac{1}{2}] & \longrightarrow & \pi_3\,\mathrm{BDiff}^+(W_g\sharp\Sigma)\otimes\mathbf{Z}[\frac{1}{2}] & \longrightarrow & 0\\
\cong\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}_4(\mathrm{BSO}(2n);\mathbf{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{H}_3(\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n});\mathbf{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{H}_3(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\frac{1}{2}]) & \longrightarrow & 0
\end{array}$$

The groups in the left column are isomorphic to $\mathbf{Z}[\frac{1}{2}]$ and the upper left morphism t_* identifies with the induced map $\pi_3\,\mathrm{SO}(2n)\to\pi_3\,\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n})$ of the collar twisting, tensored with $\mathbf{Z}[\frac{1}{2}]$. By Corollary 2.3, this morphism is trivial in the case of the standard sphere $\Sigma=S^{2n}$. We therefore obtain an isomorphism of the form

$$\mathrm{H}_3(\mathrm{BDiff}(W_g);\mathbf{Z}[\tfrac{1}{2}])\cong\mathrm{H}_3(\mathrm{BDiff}(W_g,D^{2n});\mathbf{Z}[\tfrac{1}{2}]).$$

The latter group is in turn isomorphic to $\mathrm{H}_3(\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n});\mathbf{Z}[\frac{1}{2}])$ by Lemma 3.1, so the lower row of the diagram provides an exact sequence as claimed. To prove the second part of the theorem, we use the stabilised Pontryagin–Thom map $\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n})\to\Omega^\infty_\bullet\Sigma^{-2n}\mathbf{MO}\langle n\rangle$ with a path component of $\Omega^\infty\Sigma^{-2n}\mathbf{MO}\langle n\rangle$ as its target (see Remark 2.5). Using this map, we arrive at a commutative diagram of the form

$$\begin{array}{ccccc}
\pi_4\,\mathrm{BSO}(2n)\otimes\mathbf{Z}[\frac{1}{2}] & \xrightarrow{t_*} & \pi_3\,\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n})\otimes\mathbf{Z}[\frac{1}{2}] & \longrightarrow & \pi_3\Omega^\infty_\bullet\Sigma^{-2n}\mathbf{MO}\langle n\rangle\otimes\mathbf{Z}[\frac{1}{2}]\\
\cong\downarrow & & \downarrow & & \downarrow\\
\mathrm{H}_4(\mathrm{BSO}(2n);\mathbf{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{H}_3(\mathrm{BDiff}(W_g\sharp\Sigma,D^{2n});\mathbf{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{H}_3(\Omega^\infty_\bullet\Sigma^{-2n}\mathbf{MO}\langle n\rangle;\mathbf{Z}[\frac{1}{2}]).
\end{array}$$

By the discussion leading to Proposition 2.7, the upper composition is nontrivial for all homotopy spheres Σ in Θ_{2n} for which the subgroup $\mathrm{im}(J)_3\cdot[\Sigma,\ell_\Sigma]$ of $\pi_{2n+3}\mathbf{MO}\langle n\rangle$ is nontrivial away from the prime 2. In the case $2n=10$, the group Θ_{10} is cyclic of order 6, so by Proposition 2.10, the upper composition is nontrivial for all 10-dimensional homotopy spheres Σ with order a positive multiple of 3. Consequently, in order to finish the proof, it suffices to show that the Hurewicz homomorphism

$$\pi_3\Omega^\infty_\bullet\Sigma^{-10}\mathbf{MString}\otimes\mathbf{Z}[\tfrac{1}{2}]\longrightarrow\mathrm{H}_3(\Omega^\infty_\bullet\Sigma^{-10}\mathbf{MString};\mathbf{Z})\otimes\mathbf{Z}[\tfrac{1}{2}]$$

of the desuspended Thom spectrum $\Sigma^{-10}\mathbf{MString}=\Sigma^{-10}\mathbf{MO}\langle 5\rangle$ is injective. This follows from the subsequent lemma, which concludes the proof. □

Lemma 3.3. *The Hurewicz homomorphism*

$$\pi_3\Omega^\infty_\bullet\Sigma^{-10}\mathbf{MString}\longrightarrow\mathrm{H}_3(\Omega^\infty_\bullet\Sigma^{-10}\mathbf{MString};\mathbf{Z})$$

is an isomorphism.

Proof. The component $\Omega^\infty_\bullet\Sigma^{-10}\mathbf{MString}$ is the infinite loop space of the 0-connected cover of the spectrum $\Sigma^{-10}\mathbf{MString}$. By [Gia71], its first homotopy groups are given by the table

	$i=11$	$i=12$	$i=13$
$\pi_i\mathbf{MString}$	0	$\mathbf{Z}$	$\mathbf{Z}/3$,

from which we see that the first possibly nontrivial k-invariant of the 0-connected cover of $\Sigma^{-10}\mathbf{MString}$ lies in $\mathrm{H}^2(\mathbf{HZ};\mathbf{Z}/3)$. But this group is isomorphic to the unstable group $\mathrm{H}^5(K(\mathbf{Z},3);\mathbf{Z}/3)$, which can be seen to vanish by employing the Serre spectral sequence of the homotopy fibre sequence $K(\mathbf{Z},2)\to *\to K(\mathbf{Z},3)$. Consequently, the 3-truncation of the 0-connected cover of $\Sigma^{-10}\mathbf{MString}$ splits into $\Sigma^2\mathbf{HZ}$ and $\Sigma^3\mathbf{HZ}/3$. The result is now implied by combining the induced splitting of the associated infinite loop space with the classical Hurewicz theorem for spaces. □

Remark 3.4. By Corollary E, the groups

$$\mathrm{H}_3(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\tfrac{1}{2}]) \quad \text{and} \quad \mathrm{H}_3(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\tfrac{1}{2}])$$

are not isomorphic for $g\geq 0$ and any homotopy sphere Σ in Θ_{10} with order a positive multiple of 3. In fact, in this dimension, one can explicitly calculate the homology of $\mathrm{BDiff}^+(W_g\sharp\Sigma)$ in low degrees for large g, at least after inverting 2. One starts by computing $\mathrm{H}_*(\mathrm{BDiff}(W_g,D^{10});\mathbf{Z}[\frac{1}{2}])$ in low degrees for large g. This can be achieved by combining computations of the homotopy groups of **MString** [Gia71; HR95] with a calculation of the first possibly nontrivial k-invariant of the 0-connected cover of $\mathbf{MTString}(10)$ and an extension of a method of Galatius–Randal-Williams [GR16a, Ch. 5]. From this, the groups in question can be derived using the ideas of the proof of Theorem D. The following table describes the ith homology with $\mathbf{Z}[\frac{1}{2}]$-coefficients of $\mathrm{BDiff}^+(W_g\sharp\Sigma)$ for $g\geq 2i+3$. The first row displays the respective homology groups for the two homotopy spheres Σ of order 0 and 2 in $\Theta_{10}\cong\mathbf{Z}/6$, whereas the second row shows the respective groups for the other four homotopy spheres in Θ_{10}.

	$i=1$	$i=2$	$i=3$
$\mathrm{H}_i(\mathrm{BDiff}^+(W_g);\mathbf{Z}[\frac{1}{2}])$	0	$\mathbf{Z}[\frac{1}{2}]$	$(\mathbf{Z}/3)^2$
$\mathrm{H}_i(\mathrm{BDiff}^+(W_g\sharp\Sigma);\mathbf{Z}[\frac{1}{2}])$	0	$\mathbf{Z}[\frac{1}{2}]$	$\mathbf{Z}/3$

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Article B

Characteristic numbers of manifold bundles over surfaces with highly connected fibers

CHARACTERISTIC NUMBERS OF MANIFOLD BUNDLES OVER SURFACES WITH HIGHLY CONNECTED FIBERS

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Abstract. We study smooth bundles over surfaces with highly connected almost parallelizable fiber M of even dimension, providing necessary conditions for a manifold to be bordant to the total space of such a bundle and showing that, in most cases, these conditions are also sufficient. Using this, we determine the characteristic numbers realized by total spaces of bundles of this type, deduce divisibility constraints on their signatures and $\hat{A}$-genera, and compute the second integral cohomology of $\mathrm{BDiff}^+(M)$ up to torsion in terms of generalized Miller–Morita–Mumford classes. Along the way, we identify the lattices of characteristic numbers of highly connected manifolds and give an alternative proof of a result of Meyer on the divisibility of the signature.

By work of Chern–Hirzebruch–Serre [CHS57], the signature of a closed oriented manifold is multiplicative in fiber bundles as long as the fundamental group of the base acts trivially on the rational cohomology of the fiber. The necessity of this assumption was illustrated by Kodaira [Kod67], Atiyah [Ati69], and Hirzebruch [Hir69], who constructed manifolds of nontrivial signature fibering over surfaces, whereupon Meyer [Mey72; Mey73] computed the minimal positive signature arising in this way to be 4. This is in line with a more recent result of Hambleton–Korzeniewski–Ranicki [HKR07], which establishes the multiplicativity of the signature modulo 4 for bundles over general bases. The divisibility of the signature $\sigma\colon \Omega_*^{\mathrm{SO}} \to \mathbf{Z}$ by 4 is therefore a necessary condition for a manifold to fiber over a surface up to bordism, which, when combined with the vanishing of a certain Stiefel–Whitney number, is also sufficient (see [AK80, Thm 3]).

A more refined problem is to decide which manifolds fiber over a surface up to bordism with prescribed d-dimensional fiber M, or equivalently, to determine the image of the map

$$\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M)) \to \Omega_{d+2}^{\mathrm{SO}},$$

defined on the bordism group of oriented M-bundles over oriented surfaces, which assigns to a bundle its total space. The main objective of this work is to provide a solution to this problem for highly connected, almost parallelizable, even dimensional manifolds M satisfying a mild additional condition (see Proposition 1.9). Although we prove versions of all our results for this class of manifolds, for simplicity's sake we restrict our attention in this introduction to the family of examples given by the g-fold connected sums $W_g = \sharp^g(S^n \times S^n)$.

In the first part of this work, we use parametrized Pontryagin–Thom theory as developed by Galatius, Madsen, Randal-Williams, Tillmann, and Weiss to show that the bordism class of a manifold fibering over a surface with fiber W_g lifts to the bordism group $\Omega_{2n+2}^{\langle n\rangle}$ of highly-connected (i.e. n-connected) manifolds, and that this property, together with the divisibility of the signature by 4, detects such bordism classes for $g \geq 5$.

Theorem A. *The image of the morphism*

$$\Omega_2^{\mathrm{SO}}\left(\mathrm{BDiff}^+(W_g)\right) \to \Omega_{2n+2}^{\mathrm{SO}}$$

is contained in the subgroup

$$\mathrm{im}(\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}) \cap \sigma^{-1}(4 \cdot \mathbf{Z}).$$

Moreover, equality holds if $g \geq 5$. For $2n = 2$, requiring $g \geq 3$ is sufficient.

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We refer to Theorem 1.14 for the more general version of Theorem A.

Remark. For $4m \neq 4, 8, 16$, the intersection form of a highly connected $4m$-manifold is even. Therefore, in these cases, the signature is divisible by 8, so the second condition in the definition of the subgroup in Theorem A is vacuous. This is in contrast to the other three cases in which there exist manifolds of signature 1, such as $\mathbf{C}P^2$, $\mathbf{H}P^2$, and $\mathbf{O}P^2$.

Remark. Theorem A was already known for small dimensions. For $2n = 2$, it follows from Meyer's aforementioned results; for $2n = 4$ the statement is trivial since Ω_6^{SO} is trivial; and for $2n = 6$, it was observed by Randal-Williams in an unpublished note [Ran17].

Highly connected bordism. Section 2 is devoted to a closer study of the image of the morphism $\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}$ appearing in Theorem A. Note that this morphism factors over the quotient of $\Omega_{2n+2}^{\langle n\rangle}$ by Kervaire–Milnor's [KM63] group Θ_{2n+2} of homotopy spheres. A characteristic class argument shows furthermore that the morphism is trivial unless $2n+2 = 4m$, leaving us with the need to understand the image of the morphism

$$\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m} \longrightarrow \Omega_{4m}^{\mathrm{SO}}. \tag{1}$$

To this end, we combine Wall's work on the classification of highly connected manifolds [Wal62; Wal67] with a theorem due to Brumfiel [Bru68] and enhancements by Stolz [Sto85; Sto87] to derive a concrete description of $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$ (see Theorem 2.9), which, however, depends on one unknown: the order $\mathrm{ord}([\Sigma_Q])$ of the class

$$[\Sigma_Q] \in \mathrm{coker}(J)_{4m-1}$$

of a certain homotopy sphere $\Sigma_Q \in \Theta_{4m-1}$. Although central to the classification of highly connected manifolds, this class is only known in special cases (see Theorem 2.4); Galatius–Randal-Williams [GR16, Conj. A] conjectured it to be trivial. Nevertheless, our description of $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$ is explicit enough to compute the Pontryagin numbers, signatures, and $\hat{A}$-genera realized by highly connected manifolds in terms of $\mathrm{ord}([\Sigma_Q])$ (see Proposition 2.13 and 2.14), resulting in various descriptions of the image of (1) expressed in these invariants.

Divisibility of the signature. Combining these computations with Theorem A in Section 3, we derive divisibility constraints for characteristic numbers of total spaces of W_g-bundles over surfaces and we determine these numbers completely for $g \geq 5$. To state the consequences for the signature, we remind the reader of the minimal positive signature

$$\sigma_m = a_m 2^{2m+1}(2^{2m-1}-1)\,\mathrm{num}\left(\frac{|B_{2m}|}{4m}\right)$$

realized by an almost parallelizable $4m$-manifold (see [MK60]). Here B_i is the ith Bernoulli number, and a_m is 1 if m is even and 2 otherwise. The 2-adic valuation is denoted by $\nu_2(-)$.

Theorem B. *Let $\pi : E \longrightarrow S$ be a smooth oriented bundle over a closed oriented surface S with $(4m-2)$-dimensional fiber W_g. The signature of E is divisible by*

$$\begin{cases} 4 & \text{for } m = 1, 2, 4 \\ \sigma_m & \text{for } m \neq 1 \text{ odd} \\ 2^{i_m}\gcd(\sigma_{m/2}^2, \sigma_m) & \text{for } m \neq 2, 4 \text{ even,} \end{cases}$$

where

$$i_m = \min\left(0, \nu_2(\mathrm{ord}([\Sigma_Q])) - 2\nu_2(m) - 4 + 2\nu_2(a_{m/2})\right).$$

For $m \geq 2$, the $\hat{A}$-genus of E is integral and divisible by

$$\begin{cases} 2\,\mathrm{num}(\frac{|B_{2m}|}{4m}) & \text{for } m \text{ odd} \\ \gcd(\mathrm{num}(\frac{|B_{2m}|}{4m}), \mathrm{num}(\frac{|B_m|}{2m})^2) & \text{for } m \text{ even.} \end{cases}$$

Moreover, if $g \geq 5$, then these numbers are realized as signatures and $\hat{A}$-genera of total spaces of bundles of the above type. For $m = 1$, requiring $g \geq 3$ is sufficient.

Corollary C. *There exists a smooth oriented W_g-bundle over a closed oriented surface with $4m$-dimensional total space of signature 4 if and only if $4m = 4, 8, 16$. For $4m \neq 4, 8, 16$, the signature of such a total space is divisible by 2^{2m+2} for m odd and by $2^{2m-2\nu_2(m)-3}$ for m even.*

Theorem B and Corollary C are special cases of more general results, stated as Theorem 3.3 and Corollary 3.4. These generalizations imply in particular that the divisibility statements of Theorem B and Corollary C remain valid if W_g is replaced by any closed highly connected almost parallelizable manifold, but at the cost of losing a factor of 2.

Remark. Rovi [Rov18] identified the non-multiplicativity of the signature modulo 8 as an Arf–Kervaire invariant. As a consequence of Corollary C and its generalization Corollary 3.4, her invariant vanishes for bundles over surfaces with highly connected almost parallelizable fibers, except possibly for those with total space of dimensions 4, 8, or 16.

Remark. Hanke–Schick–Steimle [HSS14, Thm 1.4] constructed manifolds of nontrivial $\hat{A}$-genus that fiber over spheres, illustrating the non-multiplicativity of the $\hat{A}$-genus. Their construction does not yield an explicit description of the fiber, whereas Theorem B provides bundles of nontrivial $\hat{A}$-genus with fiber W_g, but over (non simply connected) surfaces.

Generalized Miller–Morita–Mumford classes. Recall the *vertical tangent bundle* $T_\pi E$ of a smooth oriented bundle $\pi\colon E \longrightarrow B$ with closed d-dimensional fiber M over an l-dimensional base, defined as the kernel $T_\pi E = \ker(d\pi\colon TE \longrightarrow \pi^* TB)$ of the differential. The *generalized Miller–Morita–Mumford class* κ_c associated to a class $c \in \mathrm{H}^{*+d}(\mathrm{BSO}; k)$ with coefficients in an abelian group k is obtained by integrating $c(T_\pi E)$ along the fibers,

$$\kappa_c(\pi) = \int_M c(T_\pi E) \in \mathrm{H}^*(B; k).$$

In the universal case, this gives rise to $\kappa_c \in \mathrm{H}^*(\mathrm{BDiff}^+(M); k)$. If B is stably parallelizable, the bundles TE and $T_\pi E$ are stably isomorphic, so for $c \in \mathrm{H}^{d+l}(\mathrm{BSO}; k)$, the two characteristic numbers obtained by integrating $\kappa_c(\pi) \in \mathrm{H}^l(B; k)$ over B and $c(TE) \in \mathrm{H}^{d+l}(E; k)$ over E coincide. For B a surface, this is expressed in the commutativity of the diagram

$$\begin{array}{ccc} \Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M)) & \longrightarrow & \Omega_{d+2}^{\mathrm{SO}} \\ & \searrow^{\kappa_c} & \downarrow c \\ & & k. \end{array}$$

All our results on characteristic numbers of total spaces of bundles over surfaces with a fixed fiber M can thus be expressed in terms of values of classes $\kappa_c \in \mathrm{H}^2(\mathrm{BDiff}^+(M); k)$ for various c. To exemplify this, note that Theorem B computes the divisibility of the classes $\kappa_{\mathcal{L}_{4m}}$ and $\kappa_{\hat{\mathcal{A}}_{4m}}$ in the torsion free quotient $\mathrm{H}^2(\mathrm{BDiff}^+(W_g); \mathbf{Z})_{\mathrm{free}}$ for $g \geq 5$.

Exploiting this alternative viewpoint, we conclude this work in Section 3 by determining an explicit basis of $\mathrm{H}^2(\mathrm{BDiff}^+(M); \mathbf{Z})_{\mathrm{free}}$ for most highly connected almost parallelizable $2n$-manifolds M (see Theorem 3.5). To explain this result in the special case of W_g, we recall that $\mathrm{H}^*(\mathrm{BDiff}^+(W_g); \mathbf{Z})$ was identified by Galatius–Randal-Williams [GR14; GR17; GR18] in a range of degrees growing with g, which for $2n \geq 6$ and $g \geq 7$ in particular gives

$$\mathrm{H}^2\left(\mathrm{BDiff}^+(W_g); \mathbf{Q}\right) = \begin{cases} 0 & \text{if } 2n \equiv 0 \pmod 4 \\ \mathbf{Q}\kappa_{p_{(n+1)/2}} & \text{if } 2n \equiv 2 \pmod 8 \\ \mathbf{Q}\kappa_{p_{(n+1)/2}} \oplus \mathbf{Q}\kappa_{p^2_{(n+1)/4}} & \text{if } 2n \equiv 6 \pmod 8. \end{cases}$$

Having computed the values of these classes on bundles over surfaces enables us to lift this rational basis to a basis of $\mathrm{H}^2(\mathrm{BDiff}^+(W_g); \mathbf{Z})_{\mathrm{free}}$. To state the outcome, we define

$$j_m = \mathrm{denom}\left(\frac{|B_{2m}|}{4m}\right), \quad a_m = \begin{cases} 2 & \text{for } m \text{ odd} \\ 1 & \text{for } m \text{ even,} \end{cases} \quad \text{and} \quad \mu_m = \begin{cases} 2 & \text{if } m = 1, 2 \\ 1 & \text{else,} \end{cases}$$

and fix Bézout coefficients c_m and d_m for the numerator and denominator of $\frac{|B_{2m}|}{4m}$, i.e.,

$$c_m \operatorname{num}\left(\frac{|B_{2m}|}{4m}\right) + d_m \operatorname{denom}\left(\frac{|B_{2m}|}{4m}\right) = 1.$$

Theorem D. *Let $2n \geq 6$ and $g \geq 7$. The group $\mathrm{H}^2(\mathrm{BDiff}^+(W_g);\mathbf{Z})_{\mathrm{free}}$ is generated by*

$$\frac{\kappa_{p_m}}{2(2m-1)!j_m}$$

for $2n \equiv 2 \pmod 8$, where $m = (n+1)/2$, and by

$$\frac{\kappa_{p_k^2}}{2\mu_k a_k^2 \operatorname{ord}([\Sigma_Q])(2k-1)!^2} \quad \textit{and} \quad \frac{2\kappa_{p_{2k}} - \kappa_{p_k^2}}{2(4k-1)!j_{2k}} - \frac{\frac{|B_{2k}|}{4k}\left(c_{2k}\frac{|B_{2k}|}{4k} + 2d_{2k}(-1)^k\right)\kappa_{p_k^2}}{2(2k-1)!^2}$$

for $2n \equiv 6 \pmod 8$, where $k = (n+1)/4$.

Remark.

(i) In the case $2n = 2$, the group $\mathrm{H}^2(\mathrm{BDiff}^+(W_g);\mathbf{Z})$ is torsion-free for $g \geq 3$ (see [Pow78]) and generated by $(1/12)\cdot\kappa_{p_1}$ for $g \geq 5$ (see [Har83; MT01, p.537]). Since $\kappa_{p_1} = 3\kappa_{\mathcal{L}_1}$, we recover the computation of the divisibility of κ_{p_1} as part of Theorem B.

(ii) For $2n \geq 6, g \geq 5$, the torsion in $\mathrm{H}^2(\mathrm{BDiff}^+(W_g);\mathbf{Z})$ has been computed by Galatius–Randal-Williams [GR16]. It is nontrivial, except when $2n = 6$. Therefore, in this case, the basis of $\mathrm{H}^2(\mathrm{BDiff}^+(W_g);\mathbf{Z})_{\mathrm{free}}$ in Theorem D is also a basis of $\mathrm{H}^2(\mathrm{BDiff}^+(W_g);\mathbf{Z})$.

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1. Bordism classes of manifolds fibering over surfaces

Let M be an oriented closed $2n$-manifold. As noted in the introduction, assigning to an M-bundle over a surface its total space yields a morphism of the form

$$\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M)) \longrightarrow \Omega_{2n+2}^{\mathrm{SO}}. \tag{2}$$

This section begins our study of its image. After a recollection on parametrized Pontryagin–Thom theory, we give an alternative proof of the fact that the signature of classes in the image of (2) is divisible by 4—a result originally due to Meyer [Mey72; Mey73]. This is then used to identify the image of (2) precomposed with the natural morphism $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M, D^{2n})) \to \Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ for highly connected, almost parallelizable manifolds M satisfying an assumption on their *genus*

$$g(M) = \max\{g \geq 0 \mid \text{there is a manifold } N \text{ such that } M \cong \sharp^g(S^n \times S^n)\sharp N\}.$$

Here $\mathrm{Diff}(M, D^{2n}) \subseteq \mathrm{Diff}^+(M)$ denotes the subgroup of diffeomorphisms of M that fix an embedded disc $D^{2n} \subseteq M$. Hence, the homology group $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M, D^{2n}))$ can be described as the bordism group of oriented M-bundles over closed oriented surfaces together with a trivialized D^{2n}-subbundle. We finish this section by arguing that, in our situation, the images of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M, D^{2n}))$ and $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ in $\Omega_{2n+2}^{\mathrm{SO}}$ agree in most cases, proving Theorem A as part of the main result of the section—Theorem 1.14.

1.1. **Parametrized Pontryagin–Thom theory.** Given a fibration $\theta\colon B \to \mathrm{BSO}(2n)$, a θ*-structure* on an oriented vector bundle $E \to \mathrm{BSO}(2n)$ is a lift $E \to B$ along θ. Define the spectrum $\mathbf{MT}\theta$ as the Thom spectrum $\mathbf{Th}(-\theta^*\gamma_{2n})$ of the pullback of the canonical bundle γ_{2n} over $\mathrm{BSO}(2n)$. These spectra—in particular the case $\theta = \mathrm{id}$, commonly denoted as $\mathbf{MT}\theta = \mathbf{MTSO}(2n)$—are natural recipients of parametrized Pontryagin–Thom constructions. More precisely, for a smooth bundle $\pi\colon E \to W$ with closed $2n$-dimensional fibers and a θ-structure on its vertical tangent bundle, a parametrized version of the Pontryagin–Thom collapse map gives a canonical homotopy class $W \to \Omega^\infty\mathbf{MT}\theta$ (see e.g. [GMTW09]). In particular, for an oriented closed $2n$-manifold M, this results in a map of the form

$$\mathrm{BDiff}^+(M) \to \Omega^\infty\mathbf{MTSO}(2n), \tag{3}$$

called the *parametrized Pontryagin–Thom map*. For M a surface, the celebrated theorem of Madsen–Weiss [MW07], combined with classical stability results of Harer [Har85] (improved by Boldsen [Bol12] and Randal-Williams [Ran16]), implies that, depending on the genus of M, the map (3) provides a good homological approximation of $\mathrm{BDiff}^+(M)$. It follows from recent work by Galatius–Randal-Williams [GR14; GR18; GR17] that this holds for simply connected manifolds M in higher dimensions as well, after replacing $\Omega^\infty\mathbf{MTSO}(2n)$ with a refinement depending on M. To explain their program in the special case needed for our purposes, we assume that M is $(n-1)$-connected and n-parallelizable; that is, its tangent bundle $M \to \mathrm{BSO}(2n)$ admits a θ_n-structure for the n-connected cover

$$\theta_n\colon\ \mathrm{BSO}(2n)\langle n\rangle \to \mathrm{BSO}(2n).$$

Fix an embedded disc $D^{2n} \subseteq M$ and note that the orientation on D^{2n} extends uniquely to a θ_n-structure $\ell_{D^{2n}}$ on the tangent bundle of D^{2n}. For every smooth M-bundle $\pi\colon E \to W$ with a trivialized D^{2n}-subbundle, the θ_n-structure on the vertical tangent bundle of the D^{2n}-subbundle induced by $\ell_{D^{2n}}$ extends uniquely to a θ_n-structure on the vertical tangent bundle of π by obstruction theory. In the universal case, this results in a map of the form

$$\mathrm{BDiff}(M, D^{2n}) \to \Omega^\infty\mathbf{MT}\theta_n,$$

which hits a particular component of $\Omega^\infty\mathbf{MT}\theta_n$, denoted by $\Omega^\infty_M\mathbf{MT}\theta_n$.

Theorem 1.1 (Boldsen, Galatius–Randal-Williams, Harer, Madsen–Weiss)**.** *For a closed, $(n-1)$-connected, n-parallelizable $2n$-manifold M, the parametrized Pontryagin–Thom map*

$$\mathrm{BDiff}(M, D^{2n}) \to \Omega^\infty_M\mathbf{MT}\theta_n$$

induces an isomorphism on homology in degrees $3* \leq 2g(M) - 2$ if $2n = 2$, and for $2* \leq g(M) - 3$ if $2n \geq 6$. Furthermore, it induces an epimorphism in degrees $3* \leq 2g(M)$ if $2n = 2$, and for $2* \leq g(M) - 1$ if $2n \geq 6$.

1.2. **Signatures of bundles over surfaces.** Using the theory recalled above, we give an alternative proof of the following result of Meyer, more in the spirit of this work.

Theorem 1.2 (Meyer)**.** *The signature of the total space of a smooth bundle of oriented closed manifolds over a surface is divisible by* 4.

There is a stabilization map $\mathbf{MTSO}(2n) \to \Sigma^{-2n}\mathbf{MSO}$ (cf. [GMTW09, Ch. 3]), which—combined with the parametrized Pontryagin–Thom map (3), the counit $\Sigma^\infty_+\Omega^\infty\mathbf{MTSO}(2n) \to \mathbf{MTSO}(2n)$, and the multiplication $\mathbf{MSO} \wedge \mathbf{MSO} \to \mathbf{MSO}$—induces a factorization

$$\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M)) \to \Omega_2^{\mathrm{SO}}(\Omega^\infty\mathbf{MTSO}(2n)) \to \Omega_2^{\mathrm{SO}}(\mathbf{MTSO}(2n)) \to \Omega^{\mathrm{SO}}_{2n+2} \tag{4}$$

of (2). To prove Theorem 1.2, it is thus sufficient to show that the signatures of classes in the image of $\Omega_2^{\mathrm{SO}}(\Omega^\infty\mathbf{MTSO}(2n))$ are divisible by 4. As a result of the lemma below, it is enough to test the image of the Hurewicz homomorphism $\pi_2\mathbf{MTSO}(2n) \to \Omega_2^{\mathrm{SO}}(\Omega^\infty\mathbf{MTSO}(2n))$.

Lemma 1.3. *Let* $\Omega^\infty_\bullet \mathbf{MTSO}(2n) \subseteq \Omega^\infty \mathbf{MTSO}(2n)$ *be a path component. The images of*

$$\pi_2 \mathbf{MTSO}(2n) \to \Omega^{\mathrm{SO}}_{2n+2} \quad \textit{and} \quad \Omega^{\mathrm{SO}}_2 \left(\Omega^\infty_\bullet \mathbf{MTSO}(2n)\right) \to \Omega^{\mathrm{SO}}_{2n+2}$$

agree.

Proof. We first show that the image of the second morphism does not depend on the chosen path component. For this, note that for an oriented surface S, the composition of the natural map from the group of homotopy classes $[S, \Omega^\infty \mathbf{MTSO}(2n)]$ to $\Omega^{\mathrm{SO}}_2(\Omega^\infty \mathbf{MTSO}(2n))$ with $\Omega^{\mathrm{SO}}_2(\Omega^\infty \mathbf{MTSO}(2n)) \to \Omega^{\mathrm{SO}}_{2n+2}$ is $\pi_0 \mathbf{MTSO}(2n)$-equivariant, where $\pi_0 \mathbf{MTSO}(2n)$ acts on the domain in the obvious way and on the codomain via the composition $\pi_0 \mathbf{MTSO}(2n) \to \Omega^{\mathrm{SO}}_{2n} \to \Omega^{\mathrm{SO}}_{2n+2}$, the first map being induced by the stabilization map $\mathbf{MTSO}(2n) \to \Sigma^{-2n}\mathbf{MSO}$ and the second one by taking products with S. This equivariance can, for instance, be seen by using the geometric description of $[S, \Omega^\infty \mathbf{MTSO}(2n)]$ provided by classical Pontryagin–Thom theory. Since Ω^{SO}_2 is trivial, the action of $\pi_0 \mathbf{MTSO}(2n)$ on $\Omega^{\mathrm{SO}}_{2n+2}$ is trivial, which implies that it suffices to show the claim for the unit component $\Omega^\infty_0 \mathbf{MTSO}(2n)$. By comparing $\mathbf{MTSO}(2n)$ to its connected cover and using that the unit $\mathbf{S} \to \mathbf{MSO}$ is 1-connected, one concludes that the images of $\pi_2 \mathbf{MTSO}(2n)$ and $\Omega^{\mathrm{SO}}_2(\Omega^\infty_0 \mathbf{MTSO}(2n))$ in $\Omega^{\mathrm{SO}}_2(\mathbf{MTSO}(2n))$ agree, which, given the factorization (4), implies the claim. □

In the light of Lemma 1.3, Theorem 1.2 follows from the computation of the signatures realized by classes in $\pi_2 \mathbf{MTSO}(2n)$, which we learnt from W. Gollinger [Gol16, Thm 2.0.10].

Theorem 1.4 (Gollinger)**.** *The composition of* $\pi_2 \mathbf{MTSO}(4m-2) \to \Omega^{\mathrm{SO}}_{4m}$ *with the signature* $\sigma\colon \Omega^{\mathrm{SO}}_{4m} \to \mathbf{Z}$ *has image* $4 \cdot \mathbf{Z}$.

Proof sketch. By classical Pontryagin–Thom theory, the group $\pi_2 \mathbf{MTSO}(4m-2)$ can be described geometrically as the bordism group of closed $4m$-manifolds N together with an oriented $(4m-2)$-dimensional vector bundle $E \to N$ and a stable isomorphism $\varphi\colon E \oplus \varepsilon^2 \cong_s TN$. The morphism $\pi_2 \mathbf{MTSO}(4m-2) \to \Omega^{\mathrm{SO}}_{4m}$ is given by assigning to such a triple the bordism class of N. Since stably, a trivial plane bundle splits off from TN, the top-dimensional Stiefel–Whitney class of N vanishes, so the Euler characteristic $\chi(N)$ is even. Therefore, by taking connected sums with suitable products of spheres, we can assume that $\chi(N)$ vanishes. In this case, both of the bundles $E \oplus \varepsilon^2 \cong_s TN$ have trivial Euler class and hence, they are (unstably) isomorphic. Consequently, N admits two pointwise linearly independent vector fields from which the classical relation between the signature and the vector field problem implies that the signature of N is a multiple of 4 (see e.g. [LM89, Thm IV.2.7]). We are left to show that there is a class in $\pi_2(\mathbf{MTSO}(4m-2))$ with signature 4. Using K-theory, one can show that a trivial plane bundle splits off stably from $T\sharp^4\mathbf{C}P^2$ [Gol16, Cor. 2.2.10], which gives a class as desired for $m = 1$. But this also provides suitable classes for all m by taking products with $\mathbf{C}P^2$, finishing the proof. □

Remark 1.5. Theorem 1.2 implies that the signature of the total space of a surface bundle over a surface is divisible by 4. Meyer [Mey73] proved in addition that there exist such bundles of signature 4 for any fiber genus $g \geq 3$. This can also be derived from our proof of Theorem 1.2, since the Madsen–Weiss theorem (see Theorem 1.1) implies that, in this case, the first morphism in (4) is surjective onto $\Omega^{\mathrm{SO}}_2(\Omega^\infty_M \mathbf{MTSO}(2))$.

1.3. **Formal bundles with highly connected fibers.** We now specialize to the case of M being $(n-1)$-connected and n-parallelizable. For such M, the factorization (4) can with the help of the natural map $\mathbf{MT}\theta_n \to \mathbf{MTSO}(2n)$ be extended to a commutative diagram

$$\begin{array}{ccccccc} \Omega^{\mathrm{SO}}_2\left(\mathrm{BDiff}(M, D^{2n})\right) & \longrightarrow & \Omega^{\mathrm{SO}}_2\left(\Omega^\infty \mathbf{MT}\theta_n\right) & \longrightarrow & \Omega^{\mathrm{SO}}_2(\mathbf{MT}\theta_n) & \searrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \Omega^{\mathrm{SO}}_2\left(\mathrm{BDiff}^+(M)\right) & \longrightarrow & \Omega^{\mathrm{SO}}_2\left(\Omega^\infty \mathbf{MTSO}(2n)\right) & \longrightarrow & \Omega^{\mathrm{SO}}_2\left(\mathbf{MTSO}(2n)\right) & \longrightarrow & \Omega^{\mathrm{SO}}_{2n+2}. \end{array} \tag{5}$$

Analogous to $\mathbf{MTSO}(2n)$, there is a stabilization map $s\colon \mathbf{MT}\theta_n \to \Sigma^{-2n}\mathbf{MSO}\langle n\rangle$ for $\mathbf{MT}\theta_n$. The induced morphism on second homotopy groups fits into a commutative square

$$\begin{array}{ccc} \pi_2\mathbf{MT}\theta_n & \xrightarrow{s_*} & \Omega_{2n+2}^{\langle n\rangle} \\ \downarrow & & \downarrow \\ \Omega_2^{\mathrm{SO}}(\Omega^\infty\mathbf{MT}\theta_n) & \longrightarrow & \Omega_{2n+2}^{\mathrm{SO}}. \end{array}$$

Its cokernel can be derived from work of Galatius–Randal-Williams [GR16, Lem. 5.2,5.5,5.6].

Lemma 1.6 (Galatius–Randal-Williams). *Let $n \geq 2$. The cokernel of the morphism*

$$s_*\colon \pi_2\mathbf{MT}\theta_n \to \Omega_{2n+2}^{\langle n\rangle}$$

is trivial for $n \neq 3, 7$ and isomorphic to $\mathbf{Z}/4$ otherwise.

Remark 1.7. Since the signature $\sigma\colon \Omega_*^{\mathrm{SO}} \to \mathbf{Z}$ is an isomorphism in dimension $* = 4$, the cokernel considered in Lemma 1.6 is by Theorem 1.4 for $n = 1$ isomorphic to $\mathbf{Z}/4$ as well.

Lemma 1.6 leads to a description of the image of $\Omega_2^{\mathrm{SO}}(\Omega^\infty\mathbf{MT}\theta_n) \to \Omega_{2n+2}^{\mathrm{SO}}$ as follows.

Proposition 1.8. *Let $\Omega_\bullet^\infty\mathbf{MT}\theta_n \subseteq \Omega^\infty\mathbf{MT}\theta_n$ be a path component. The image of the morphism $\Omega_2^{\mathrm{SO}}(\Omega_\bullet^\infty\mathbf{MT}\theta_n) \to \Omega_{2n+2}^{\mathrm{SO}}$ agrees with the subgroup*

$$\operatorname{im}\left(\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}\right) \cap \sigma^{-1}(4\cdot\mathbf{Z}).$$

Proof. By the same argument as in the proof of Lemma 1.3, the image of the map in consideration coincides with the image of the composition $\pi_2\mathbf{MT}\theta_n \to \Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}$. From this, one inclusion follows by observing that the latter composition factors through $\pi_2\mathbf{MTSO}(2n) \to \Omega_{2n+2}^{\mathrm{SO}}$, which has image in $\sigma^{-1}(4\cdot\mathbf{Z})$ by Theorem 1.4. Since $\pi_2\mathbf{MT}\theta_n \to \Omega_{2n+2}^{\langle n\rangle}$ is surjective if $n \neq 1, 3, 7$ by Lemma 1.6, there is nothing left to prove in these cases. For $n = 1, 3, 7$, the group $\Omega_{2n+2}^{\langle n\rangle}$ contains a class of signature 1, such as $\mathbf{C}P^2$, $\mathbf{H}P^2$, and $\mathbf{O}P^2$—a fact which, together with Lemma 1.6 and Remark 1.7, implies that the sequence

$$\pi_2\mathbf{MT}\theta_n \to \Omega_{2n+2}^{\langle n\rangle} \to \mathbf{Z}/4 \to 0,$$

where the last map is induced by the signature, is exact. The assertion follows. □

By virtue of Theorem 1.1, Proposition 1.8 has an analogue of Theorem A for bundles over surfaces with a trivialized disc-subbundle and fixed highly connected almost parallelizable fiber of even dimension as a corollary. This is stated as part of Theorem 1.14 below.

1.4. **A reduction of the structure group.** To discuss conditions on M for which the images of $\Omega^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ and $\Omega^{\mathrm{SO}}(\mathrm{BDiff}(M, D^{2n}))$ in $\Omega_{2n+2}^{\mathrm{SO}}$ agree, we recall the canonical ring isomorphisms between Ω_*^{fr} and π_*^s, and between $\Omega_*^{\langle n\rangle}$ and $\pi_*\mathbf{MSO}\langle n\rangle$, given by the Pontryagin–Thom construction. Here Ω_*^{fr} is the bordism ring of stably framed manifolds, π_*^s the stable homotopy groups of spheres, $\mathbf{MSO}\langle n\rangle$ the Thom ring spectrum of $\mathrm{BSO}\langle n\rangle$, and $\Omega_*^{\langle n\rangle}$ the bordism ring of oriented manifolds equipped with a $\mathrm{BSO}\langle n\rangle$-structure on their stable normal bundle. By standard surgery techniques, the group $\Omega_k^{\langle n\rangle}$ is for $k \geq 2n+2$ canonically isomorphic to the bordism group of oriented n-connected k-manifolds, and we use these two descriptions interchangeably. The stable normal bundle of an oriented $(n-1)$-connected n-parallelizable manifold M admits a unique $\mathrm{BSO}\langle n\rangle$-structure by obstruction theory, so M canonically defines a class $[M]$ in $\pi_{2n}\mathbf{MSO}\langle n\rangle$. Note furthermore that the chain of inclusions $\mathrm{Diff}(M, D^{2n}) \subseteq \mathrm{Diff}^+(M, *) \subseteq \mathrm{Diff}^+(M)$ of subgroups of diffeomorphisms fixing a chosen point or disc, respectively, induces homotopy fiber sequences

$$M \to \mathrm{BDiff}^+(M,*) \to \mathrm{BDiff}^+(M) \quad\text{and}\quad \mathrm{BDiff}(M, D^{2n}) \to \mathrm{BDiff}^+(M,*) \xrightarrow{d} \mathrm{BSO}(2n), \tag{6}$$

where d is the delooping of the map given by taking the differential at the fixed point. Finally, as is common, $\eta \in \pi_1^s \cong \mathbf{Z}/2$ denotes the nontrivial element in the first stable stem.

Proposition 1.9. *Let M be a closed, oriented, $(n-1)$-connected, n-parallelizable manifold. If M satisfies one of the two conditions*

(i) $\eta \cdot [M] \in \pi_{2n+1}\mathbf{MSO}\langle n\rangle$ is not trivial,

(ii) $\eta \cdot [M,\alpha] \in \pi^s_{2n+1}$ is trivial for a stable framing α of M,

then the images of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M,D^{2n}))$ and $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ in $\Omega^{\mathrm{SO}}_{2n+2}$ agree.

Proof. For $2n=2$, the conclusion of the statement is always valid by Theorem 1.2 and Remark 1.5. Since Ω_6^{SO} is trivial, we can assume $2n \geq 6$, so M is in particular 2-connected. The first fiber sequence of (6) implies that $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M,*)) \to \Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ is surjective. It therefore suffices to show the claim for $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M,*))$ instead of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$. Mapping the long exact sequence in homotopy groups of the second fiber sequence of (6) to the corresponding Serre exact sequence yields a commutative ladder with exact rows

$$
\begin{array}{ccccccc}
\pi_2\,\mathrm{BDiff}(M,D^{2n}) & \longrightarrow & \pi_2\,\mathrm{BDiff}^+(M,*) & \longrightarrow & \mathbf{Z}/2 & \xrightarrow{\pi_1(t)} & \pi_1(\mathrm{BDiff}(M,D^{2n})) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\Omega_2^{\mathrm{SO}}\left(\mathrm{BDiff}(M,D^{2n})\right) & \to & \Omega_2^{\mathrm{SO}}\left(\mathrm{BDiff}^+(M,*)\right) & \to & \mathbf{Z}/2 & \xrightarrow{\Omega_1^{\mathrm{SO}}(t)} & \Omega_1^{\mathrm{SO}}\left(\mathrm{BDiff}(M,D^{2n})\right),
\end{array} \tag{7}
$$

where $t\colon \mathrm{SO}(2n) \to \mathrm{BDiff}^+(M,D^{2n})$ is the map obtained by looping the fiber sequence once. By [Kra18, Lem. 2.6], the first condition implies that $\Omega_1^{\mathrm{SO}}(t)$ is nontrivial, so the morphism $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M,D^{2n})) \to \Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ is surjective, which proves the claim. The second condition implies that $\pi_1(t)$ is trivial [Kre79, Lem. 4,Thm 3 c)], so there is an oriented bundle $\pi\colon E \to S^2$ with fiber M whose class maps nontrivially to $\mathbf{Z}/2$. The argument at the beginning of the proof of [HSS14, Prop. 1.9] shows that the Pontryagin and Stiefel–Whitney numbers of E vanish, so E is nullbordant. Since $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M,*))$ is generated by the class of this bundle and the image of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M,D^{2n}))$, the images of these two groups in $\Omega^{\mathrm{SO}}_{2n+2}$ agree, as claimed. □

Remark 1.10. Proposition 1.9 leaves open the cases where $\eta \cdot [M,\alpha]$ defines a nontrivial element in the kernel of the Hurewicz map $\pi^s_{2n+1} \to \pi_{2n+1}\mathbf{MSO}\langle n\rangle$. This morphism naturally factors over the cokernel of the J-homomorphism,

$$\pi^s_{2n+1} \to \mathrm{coker}(J)_{2n+1} \to \pi_{2n+1}\mathbf{MSO}\langle n\rangle,$$

and work by Stolz implies that the second morphism in this composition is often bijective (see e.g. [GR16, Thm 1.4]). This can be used to further narrow down the manifolds for which the images of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M,D^{2n}))$ and $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ in $\Omega^{\mathrm{SO}}_{2n+2}$ might disagree, but some cases remain, for instance, $\Sigma_{8k+2}\sharp(S^{4k+1}\times S^{4k+1})^{\sharp g}$ for $g,k \geq 1$, where Σ_{8k+2} is the homotopy sphere corresponding to Adams' element $\mu_{8k+2} \in \pi^s_{8k+2}$ (see [Ada66]). This is because $\eta \cdot \mu_{8k+2}$ is nontrivial and contained in $\mathrm{im}(J)_{8k+3}$ (see e.g. [Rav86, Thm 5.3.7]).

Remark 1.11. Assuming $g(M) \geq 7$, one can show that the images of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M,D^{2n}))$ and $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ in $\Omega^{\mathrm{SO}}_{2n+2}$ agree if and only if $\eta \cdot [M] \in \pi_{2n+1}\mathbf{MSO}\langle n\rangle$ vanishes or $d^*w_2\colon \mathrm{Tor}(\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M,*))) \to \mathbf{Z}/2$ is nontrivial. Here d^*w_2 is induced by the pullback of $w_2 \in \mathrm{H}^2(\mathrm{BSO}(2n);\mathbf{Z}/2)$ along $d\colon \mathrm{BDiff}^+(M,*) \to \mathrm{BSO}(2n)$.

Remark 1.12. From the exactness of the second row of (7), we see that every 2-divisible class in $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ is contained in the image of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M,D^{2n}))$.

Remark 1.13. For a homotopy d-sphere Σ, the image of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(\Sigma))$ in $\Omega^{\mathrm{SO}}_{d+2}$ is trivial. More generally, the total space of any Σ-bundle is nullbordant. Indeed, coning off the fibers shows that there exists a topological nullbordism and hence, by the topological invariance of Stiefel–Whitney and Pontryagin numbers, also a smooth one.

Propositions 1.8 and 1.9, together with Theorem 1.1, imply the following result. It includes Theorem A, since W_g satisfies the second condition of Proposition 1.9, being the boundary of the parallelizable handlebody $\natural^g(D^{n+1}\times S^n)$.

Theorem 1.14. *Let M be a closed, $(n-1)$-connected, n-parallelizable $2n$-manifold. The image of the morphism $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M, D^{2n})) \to \Omega_{2n+2}^{\mathrm{SO}}$ is contained in the subgroup*

$$\mathrm{im}\left(\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}\right) \cap \sigma^{-1}(4 \cdot \mathbf{Z}).$$

Moreover, equality holds for $g(M) \geq 5$, and for $g(M) \geq 3$ if $2n = 2$. If M satisfies one of the conditions of Proposition 1.9, then the same conclusions apply to $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$.

2. Bordism classes of highly connected manifolds

This section is concerned with the image of the natural map

$$\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}} \tag{8}$$

from highly connected to oriented bordism, which is by Theorem 1.14 closely related to smooth bundles over surfaces with highly connected almost parallelizable fibers. As noted in the introduction, this morphism is trivial unless $2n+2 = 4m$, and factors over the quotient $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$ by the group of homotopy spheres. In the first part of this section, we combine work of Brumfiel, Kervaire–Milnor, Stolz, and Wall to give an explicit description of this quotient, which we use thereafter to describe the image of (8) in terms of characteristic numbers and to derive divisibility constraints for the signature and the $\hat{A}$-genera of highly connected manifolds. We assume $m \geq 2$ throughout the section.

2.1. Wall's classification of highly connected almost closed manifolds.

A compact manifold is *almost closed* if its boundary is a homotopy sphere. We denote by $A_{4m}^{\langle 2m-1\rangle}$ the group of oriented, almost closed, $(2m-1)$-connected $4m$-manifolds, up to $(2m-1)$-connected oriented bordism restricting to an h-cobordism on the boundary; the group structure is induced by boundary connected sum. This group receives a map from $\Omega_{4m}^{\langle 2m-1\rangle}$ given by cutting out the interior of an embedded $4m$-disk. This fits into an exact sequence

$$\Theta_{4m} \to \Omega_{4m}^{\langle 2m-1\rangle} \to A_{4m}^{\langle 2m-1\rangle} \xrightarrow{\partial} \Theta_{4m-1}. \tag{9}$$

The first morphism maps a homotopy sphere to its bordism class, and the last one assigns to an almost closed manifold its boundary. From his pioneering work on the classification of highly connected manifolds, Wall [Wal62] derived a complete description of the groups $A_{4m}^{\langle 2m-1\rangle}$. For us, the outcome is most conveniently phrased in terms of two particular manifolds P and Q, which play a key role in the remainder of this section.

Milnor's E_8-plumbing P. We denote by *P Milnor's E_8-plumbing*, i.e., the parallelizable manifold of dimension $4m$ arising from plumbing eight copies of the disk bundle of the tangent bundle of the $2m$-sphere, such that the intersection form of P is isomorphic to the E_8-form (see e.g. [Bro72, Ch. V.2]). Since the latter has signature 8, so does the manifold P.

The plumbing Q. Let Q be the plumbing of two copies of a $2m$-dimensional linear disk bundle over the $2m$-sphere that generates the image of $\pi_{2m}\,\mathrm{BSO}(2m-1)$ in $\pi_{2m}\,\mathrm{BSO}(2m)$. This bundle can be characterized equivalently as having vanishing Euler number and representing a generator of $\pi_{2m}\,\mathrm{BSO}$ for $m \neq 2, 4$, and twice a generator for $m = 2, 4$ (cf. [Lev85, §1A)]). Via the isomorphism $\mathrm{H}^{2m}(S^{2m}) \oplus \mathrm{H}^{2m}(S^{2m}) \cong \mathrm{H}^{2m}(Q)$ induced by the inclusion of the $2m$-skeleton, the intersection form of Q is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so it has vanishing signature. For $m = 2k$ even, the kth Pontryagin class of a generator of $\pi_{2m}\,\mathrm{BSO}$ is $a_k(2k-1)![S^{4k}]^* \in \mathrm{H}^{4k}(S^{4k})$ (see e.g. [Lev85, Thm 3.8]), where a_k equals 1 for k even and 2 otherwise, and $[S^{4k}]^*$ denotes the Poincaré dual to $1 \in \mathrm{H}_0(S^{4k})$. From this, we compute the square of the kth Pontryagin class of Q as

$$p_k^2(Q, \partial Q) = 2\lambda_k^2 a_k^2 (2k-1)!^2 \cdot [Q, \partial Q]^* \in \mathrm{H}^{8k}(Q, \partial Q), \tag{10}$$

with λ_k being 1 if $k \neq 1, 2$ and 2 elsewise.

The boundaries of both plumbings P and Q are homotopy spheres (see e.g. [Bro72, p. V.2.7]). We denote them by Σ_P and Σ_Q, respectively.

Theorem 2.1 (Wall). *The bordism group $A_{4m}^{\langle 2m-1\rangle}$ satisfies*

$$A_{4m}^{\langle 2m-1\rangle} \cong \begin{cases} \mathbf{Z}\oplus\mathbf{Z}/2 & \text{if } m\equiv 1 \pmod 4\\ \mathbf{Z} & \text{if } m\equiv 3 \pmod 4\\ \mathbf{Z}\oplus\mathbf{Z} & \text{if } m\equiv 0 \pmod 2,\end{cases}$$

where the first summand is generated by P, except for $m=2,4$ where it is generated by $\mathbf{H}P^2$ and $\mathbf{O}P^2$, respectively. The second summand in the cases $m\not\equiv 3 \pmod 4$ is generated by Q.

Proof. The statement regarding the isomorphism type of the group $A_{4m}^{\langle 2m-1\rangle}$ follows from [Wal62, Thm 2; Wal67, Thm 11]. Denoting by $\langle Q\rangle\subseteq A_{4m}^{\langle 2m-1\rangle}$ the subgroup generated by Q, it follows from the discussion in [Wal67, p. 295] that there is an exact sequence

$$0\longrightarrow\langle Q\rangle\longrightarrow A_{4m}^{\langle 2m-1\rangle}\longrightarrow\mathbf{Z}$$

in which the last morphism is induced by the signature. As $\mathbf{H}P^2$ and $\mathbf{O}P^2$ have signature 1, the cases $m=2,4$ follow. The other cases are concluded by observing that the intersection form associated to a manifold representing a class in $A_{4m}^{\langle 2m-1\rangle}$ is even for $m\neq 2,4$, so it has signature divisible by 8—the signature of the E_8-plumbing. □

To treat the different cases for m even in a uniform manner, it is convenient to use a basis of $A_{4m}^{\langle 2m-1\rangle}$ that is different from the one described in the previous theorem.

Lemma 2.2.

(i) *Two almost closed $8k$-manifolds M and N define the same class in $A_{8k}^{\langle 4k-1\rangle}$ if and only if $\sigma(M)=\sigma(N)$ and $p_k^2(M)=p_k^2(N)$.*

(ii) *The classes $8\cdot\mathbf{H}P^2$ and $8\cdot\mathbf{O}P^2$ in $A_{8k}^{\langle 4k-1\rangle}$ equal $P+Q$ for $k=1,2$, respectively.*

(iii) *The group $A_{8k}^{\langle 4k-1\rangle}\cong\mathbf{Z}\oplus\mathbf{Z}$ is generated by P and Q for $k\neq 1,2$, by P and $\mathbf{H}P^2$ for $k=1$, and by P and $\mathbf{O}P^2$ for $k=2$.*

Proof. As opposed to the plumbing Q, the manifolds $\mathbf{H}P^2$, $\mathbf{O}P^2$, and P have nontrivial signature. Since we computed the Pontryagin number $p_k^2(Q)$ to be nontrivial in (10), the first claim follows from Theorem 2.1, as $A_{8k}^{\langle 4k-1\rangle}$ is free abelian of rank 2. The Pontryagin numbers $p_k^2(\mathbf{H}P^2)$ and $p_k^2(\mathbf{O}P^2)$ can be computed as $a_k^2(2k-1)!^2$ for $k=1,2$, respectively, which agrees with $1/8\cdot p_k^2(Q)$ by (10). The second claim follows from the first, remembering that $\sigma(P)=8$ and $\sigma(Q)=0$. The third claim follows from the second using Theorem 2.1. □

2.2. **Homotopy $(4m-1)$-spheres.** Recall from [KM63] that the group Θ_{4m-1} of h-cobordism classes of oriented homotopy spheres fits into an exact sequence

$$0\longrightarrow \mathrm{bP}_{4m}\longrightarrow\Theta_{4m-1}\longrightarrow\operatorname{coker}(J)_{4m-1}\longrightarrow 0 \tag{11}$$

involving the subgroup $\mathrm{bP}_{4m}\subseteq\Theta_{4m-1}$ of homotopy spheres bounding parallelizable manifolds and the cokernel of the stable J-homomorphism in degree $(4m-1)$. The subgroup bP_{4m} is generated by the *Milnor sphere* $\Sigma_P=\partial P$. It is of order $\sigma_m/8$ with

$$\sigma_m=a_m2^{2m+1}(2^{2m-1}-1)\operatorname{num}\left(\frac{|B_{2m}|}{4m}\right)$$

as defined in the introduction (see e.g. [Lev85, Cor. 3.20, Lem. 3.5(2)]). Brumfiel [Bru68] has shown that every homotopy sphere $\Sigma\in\Theta_{4m-1}$ bounds a spin manifold W_Σ with vanishing decomposable Pontryagin numbers and that the signature $\sigma(W_\Sigma)$ of such a manifold is divisible by 8. This enabled him to establish a decomposition

$$\Theta_{4m-1}\cong\mathrm{bP}_{4m}\oplus\operatorname{coker}(J)_{4m-1} \tag{12}$$

via a splitting $s_B\colon \Theta_{4m-1} \to \mathrm{bP}_{4m}$ of the exact sequence (11), defined by mapping a homotopy sphere Σ to $(\sigma(W_\Sigma)/8)\cdot\Sigma_P$. Refining Brumfiel's definition, Stolz [Sto87] gave a formula for $s_B(\Sigma)$ in terms of invariants of any spin manifold that bounds Σ, without assumptions on its characteristic numbers. To state his result, fix integers c_m and d_m with

$$c_m \operatorname{num}\left(\frac{|B_{2m}|}{4m}\right) + d_m \operatorname{denom}\left(\frac{|B_{2m}|}{4m}\right) = 1,$$

and define a rational polynomial $\mathcal{S}_m \in \mathrm{H}^{4m}(\mathrm{BSO};\mathbf{Q})$ in Pontryagin classes as

$$\mathcal{S}_m = \mathcal{L}_m + \frac{\sigma_m}{a_m}\left(c_m\hat{\mathcal{A}}_m + (-1)^m d_m(\hat{\mathcal{A}}\,\mathrm{ph})_m\right),$$

which involves the $\mathcal{L}$- and $\hat{\mathcal{A}}$-class, as well as product of the $\hat{\mathcal{A}}$-class with the reduced Pontryagin character ph. Here *reduced* refers to the triviality of ph in degree 0. The polynomial $\mathcal{S}_m$ has no contributions from the mth Pontryagin class (see [Sto87, p. 2]), so its evaluation on an oriented almost closed manifold M can be considered as a relative class $\mathcal{S}_m(M) \in \mathrm{H}^{4m}(M,\partial M;\mathbf{Q})$.

Theorem 2.3 (Stolz). *For an almost closed spin manifold M of dimension $4m$, the invariant*

$$s(M) = \frac{1}{8}\Big(\sigma(M) - \langle \mathcal{S}_m(M), [M,\partial M]\rangle\Big)$$

is integral and computes the value of Brumfiel's splitting on the boundary of M, i.e.,

$$s_B(\partial M) = s(M)\cdot\Sigma_P.$$

2.3. **Closing almost closed manifolds.** By the exactness of the sequence (9), the bordism group $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$ is naturally isomorphic to the kernel of the morphism

$$\partial\colon A_{4m}^{\langle 2m-1\rangle} \to \Theta_{4m-1},$$

which leads us to identify these two groups henceforth. Since $A_{4m}^{\langle 2m-1\rangle}$ is generated by the classes of P and Q for $m \neq 2,4$ by Theorem 2.1, we need to examine their boundaries Σ_P and Σ_Q in Θ_{4m-1} in order to determine the kernel in question. As mentioned earlier, the Milnor sphere Σ_P is well understood; it generates the subgroup bP_{4m}. Regarding Σ_Q, we use Stolz's invariant to compute its image under the projection onto bP_{4m} with respect to the decomposition (12) (see Lemma 2.7). Concerning its image $[\Sigma_Q]$ in $\operatorname{coker}(J)_{4m-1}$, there are partial results due to Anderson and Stolz. To state the ones relevant for us, denote by

$$j_n = \operatorname{denom}\left(\frac{|B_{2n}|}{4n}\right)$$

the size of the image of the stable J-homomorphism in degree $4n-1$ (see [Ada66; Qui71]).

Theorem 2.4 (Anderson, Stolz). *The class $[\Sigma_Q]$ in $\operatorname{coker}(J)_{4m-1}$ satisfies*

(i) $j_{m/2}^2\cdot[\Sigma_Q] = 0$ *for $m \neq 2,4$ even, and*
(ii) $[\Sigma_Q] = 0$ *for $m \neq 5$ odd.*

Proof. The first claim follows from the beginning of the proof of [And69, Lem. 1.5]. To prove the second, observe that the homotopy sphere Σ_Q is trivial if $m \equiv 3 \pmod 4$ by Theorem 2.1. A result by Stolz [Sto85, Thm B (i)] settles the remaining case. □

Despite Anderson's bound on its order, very little is known about the class $[\Sigma_Q]$ in $\operatorname{coker}(J)_{4m-1}$ for m even. Galatius–Randal-Williams [GR16, Conj. A] conjectured that it is trivial. A weaker version of this conjecture appeared independently in work of Bowden–Crowley–Stipsicz [BCS14, Conj. 5.9].

Conjecture 2.5 (Galatius–Randal-Williams). *For m even, $[\Sigma_Q]$ is trivial in* $\operatorname{coker}(J)_{4m-1}$.

Remark 2.6. The conjecture is known in the first two cases $m = 2,4$ (cf. [GR16, Ch. 6]) and is, to the author's knowledge, already unknown for $m = 6$.

To state our formula for the image of Σ_Q under the projection onto bP_{4m}, we denote by
$$T_n = 2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n}$$
the nth *tangent number*, which is known to be integral (see e.g. [AIK14, Rem. 1.18]).

Lemma 2.7. *Stolz' invariant $s(Q)$ of Q vanishes if m is odd. For $m=2k$ even, it satisfies*
$$s(Q) = -\frac{\lambda_k^2}{8j_k^2}\left(\sigma_k^2 + a_k^2\sigma_{2k}\,\mathrm{num}\left(\frac{|B_{2k}|}{4k}\right)\left(c_{2k}\,\mathrm{num}\left(\frac{|B_{2k}|}{4k}\right)+2(-1)^k d_{2k}j_k\right)\right),$$
as well as
$$s(Q) = \frac{\lambda_k^2 a_k^2}{4}\left(\sigma_{2k}d_{2k}\frac{|B_{2k}|}{4k}\left(\frac{|B_{2k}|}{|B_{4k}|}+(-1)^{k+1}\right)-\frac{T_k^2}{4}\right),$$
where $\lambda_k = 1$ if $k \neq 1,2$ and $\lambda_k = 2$ otherwise.

We postpone the proof of this lemma to the next subsection and continue by elaborating on some of its consequences instead.

Corollary 2.8. *For m even, the homotopy spheres Σ_P and Σ_Q in Θ_{4m-1} satisfy*
(i) $j_{m/2}^2\cdot\Sigma_Q = (\sigma_{m/2}^2/8)\cdot\Sigma_P$ for $m\neq 2,4$, and
(ii) $\Sigma_Q = -\Sigma_P$ for $m = 2,4$.

Proof. We write $m=2k$. By Theorem 2.4, the homotopy sphere $j_k^2\cdot\Sigma_Q$ lies in bP_{8k} for $2k\neq 2,4$, so Theorem 2.3 gives the relation $j_k^2\cdot\Sigma_Q = j_k^2 s(Q)\cdot\Sigma_P$. From the first formula of Lemma 2.7, we see that $j_k^2 s(Q)$ is congruent to $\sigma_k^2/8$ modulo σ_{2k}. This has the first claim as a consequence. Since $\mathrm{num}(|B_4|/8)=\mathrm{num}(|B_8|/16)=1$, we conclude $d_2=d_4=0$. Thus, the second formula of the lemma gives $s(Q)=-T_1^2$ for $k=1$ and $s(Q)=-T_2^2/4$ for $k=2$. As $T_1=1$ and $T_2=2$, we have $s(Q)=-1$ for $k=1,2$, which, together with Theorem 2.3 and Remark 2.6, implies the second claim. □

Combining Wall's classification, Stolz' invariant, and Theorem 2.4, we determine the kernel of the boundary map in the sequence (9), and hence the bordism group $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$, as follows. The order of the class $[\Sigma_Q]$ in $\mathrm{coker}(J)_{4m-1}$ is denoted by $\mathrm{ord}([\Sigma_Q])$.

Theorem 2.9. *The bordism group $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$ satisfies*
$$\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m} \cong \begin{cases} \mathbf{Z}\oplus\mathbf{Z}/2 & \text{if } m\equiv 1 \pmod 4, m\neq 5\\ \mathbf{Z} & \text{if } m\equiv 3 \pmod 4\\ (\mathbf{Z}\oplus\mathbf{Z}/2) \text{ or } \mathbf{Z} & \text{if } m=5\\ \mathbf{Z}\oplus\mathbf{Z} & \text{if } m\equiv 0 \pmod 2.\end{cases}$$
The first summand is generated by $(\sigma_m/8)\cdot P$ in all cases. The $\mathbf{Z}/2$-summands in the respective cases are generated by Q. For m even, the second summand is generated by $\mathbf{H}P^2$ if $m=2$, by $\mathbf{O}P^2$ if $m=4$, and by $\mathrm{ord}([\Sigma_Q])\big(Q - s(Q)\cdot P\big)$ otherwise.

Proof. By Theorem 2.1 and Lemma 2.2, the group $A_{4m}^{\langle 2m-1\rangle}$ is isomorphic to a direct sum $\mathbf{Z}\oplus C$ for a cyclic group C, where the first summand is generated by P, and the second summand by Q for $m\neq 2,4$, by $\mathbf{H}P^2$ for $m=2$, and by $\mathbf{O}P^2$ for $m=4$. Recall that the Milnor sphere Σ_P generates the cyclic group bP_{4m} of order $\sigma_m/8$. By exactness of (11), the homotopy sphere $\mathrm{ord}([\Sigma_Q])\cdot\Sigma_Q$ is contained in bP_{4m}, so it coincides with $\mathrm{ord}([\Sigma_Q])s(Q)\cdot\Sigma_P$ by Theorem 2.3. Using this, it follows from elementary algebraic considerations that the classes $(\sigma_m/8)\cdot P$ and $\mathrm{ord}([\Sigma_Q])\big(Q-s(Q)\cdot P\big)$ generate the kernel for $m\neq 2,4$. As Q has infinite order for m even, this settles the case for $m\neq 2,4$ even. The class of Q has order 2 for $m\equiv 1 \pmod 4$ and is trivial for $m\equiv 3 \pmod 4$ by Theorem 2.1. Together with the fact that, for m odd, we have $s(Q)=0$ by Lemma 2.7 and $\mathrm{ord}([\Sigma_Q])=1$ as long as $m\neq 5$ by Theorem 2.4, this concludes the proof for m odd. The cases $m=2,4$ are immediate. □

Remark 2.10. The preceding proof also shows that for $m = 5$, the kernel in question is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}/2$ if and only if the class $[\Sigma_Q] \in \operatorname{coker}(J)_{19}$ is trivial.

Remark 2.11. Note that, for $m = 2k \neq 2, 4$, Corollary 2.8 implies that $(\sigma_k^2/8) \cdot P - j_k^2 \cdot Q$ is contained in the kernel under consideration.

Remark 2.12. After replacing $\mathbf{H}P^2$ by $4 \cdot \mathbf{H}P^2$ and $\mathbf{O}P^2$ by $4 \cdot \mathbf{O}P^2$, the statement of Theorem 2.9 remains valid if $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m}$ is replaced by the subgroup $\Omega_{4m}^{\langle 2m-1\rangle}/\Theta_{4m} \cap \sigma^{-1}(4\cdot\mathbf{Z})$. This is because the signatures of P and Q are divisible by 8, whereas $\sigma(\mathbf{H}P^2) = \sigma(\mathbf{O}P^2) = 1$.

2.4. **Characteristic numbers of highly connected manifolds.** This subsection serves to prove Lemma 2.7 and to use it in combination with Theorem 2.9 to compute the lattices of characteristic numbers realized by closed $(2m-1)$-connected $4m$-manifolds. Note that for such manifolds, all Pontryagin classes vanish, except possibly for p_m, and $p_{m/2}$ if m is even. As a result of this, the $\mathcal{L}$-class, the $\hat{\mathcal{A}}$-class, the reduced Pontryagin character ph, and the product $(\hat{\mathcal{A}}\,\mathrm{ph})$ of the latter two, have the form (cf. [Hir66, Ch. 1,3])

$$\mathcal{L}_m = \begin{cases} s_m p_m & \text{if } m \text{ is odd} \\ \frac{1}{2}(s_k^2 - s_{2k})p_k^2 + s_{2k}p_{2k} & \text{if } m = 2k \text{ is even,} \end{cases}$$

$$\hat{\mathcal{A}}_m = \begin{cases} \hat{s}_m p_m & \text{if } m \text{ is odd} \\ \frac{1}{2}(\hat{s}_k^2 - \hat{s}_{2k})p_k^2 + \hat{s}_{2k}p_{2k} & \text{if } m = 2k \text{ is even,} \end{cases}$$

$$\mathrm{ph}_m = \begin{cases} \frac{(-1)^{m+1}}{(2m-1)!}p_m & \text{if } m \text{ is odd} \\ \frac{1}{2(4k-1)!}p_k^2 - \frac{1}{(4k-1)!}p_{2k} & \text{if } m = 2k \text{ is even,} \end{cases} \quad \text{and}$$

$$(\hat{\mathcal{A}}\,\mathrm{ph})_m = \begin{cases} \frac{(-1)^{m+1}}{(2m-1)!}p_m & \text{if } m \text{ is odd} \\ \frac{(-1)^{k+1}\hat{s}_k}{(2k-1)!}p_k^2 + \frac{1}{2(4k-1)!}p_k^2 - \frac{1}{(4k-1)!}p_{2k} & \text{if } m = 2k \text{ is even,} \end{cases}$$

where $\hat{s}_n$ and s_n are given by

$$\hat{s}_n = \frac{-1}{(2n-1)!}\frac{|B_{2n}|}{4n} \quad \text{and} \quad s_n = -2^{2n+1}(2^{2n-1}-1)\hat{s}_n = \frac{\sigma_n}{a_n(2n-1)!j_n} \tag{13}$$

for $n \geq 1$. Solving $\mathcal{L}_{2k}$ for p_{2k} and expressing $\hat{\mathcal{A}}_{2k}$ in terms of $\mathcal{L}_{2k}$ and p_k^2, we obtain

$$p_{2k} = \frac{1}{s_{2k}}\mathcal{L}_{2k} - \frac{s_k^2 - s_{2k}}{2s_{2k}}p_k^2 \quad \text{and} \quad \hat{\mathcal{A}}_{2k} = \frac{s_{2k}\hat{s}_k^2 - \hat{s}_{2k}s_k^2}{2s_{2k}}p_k^2 + \frac{\hat{s}_{2k}}{s_{2k}}\mathcal{L}_{2k}. \tag{14}$$

Substituting the variables with their values, the last equation becomes

$$\hat{\mathcal{A}}_{2k} = \frac{T_k^2}{(2k-1)!^2 2^{4k+3}(2^{4k-1}-1)}p_k^2 - \frac{1}{2^{4k+1}(2^{4k-1}-1)}\mathcal{L}_{2k}. \tag{15}$$

Proof of Lemma 2.7. As Q is $(2m-1)$-connected, all its decomposable Pontryagin numbers vanish for m odd, and hence so does the characteristic number $\langle \mathcal{S}_m(Q), [Q, \partial Q]\rangle$, since $\mathcal{S}_m$ does not involve p_m (see [Sto87, p. 2]). The first part of the lemma follows therefore from the triviality of the signature of Q. For $m = 2k$, we use the formulas above to calculate

$$\mathcal{S}_{2k}(Q) = \left(\frac{1}{2}(s_k^2 - s_{2k}) + \sigma_{2k}c_{2k}\frac{1}{2}(\hat{s}_k^2 - \hat{s}_{2k}) + \sigma_{2k}d_{2k}\hat{s}_k\frac{(-1)^{k+1}}{(2k-1)!} + \sigma_{2k}d_{2k}\frac{1}{2(4k-1)!}\right)p_k^2(Q).$$

Using the second description of s_{2k} in (13), one obtains the identity

$$\frac{1}{2}s_{2k} = \sigma_{2k}d_{2k}\frac{1}{2(4k-1)!} - \sigma_{2k}c_{2k}\frac{1}{2}\hat{s}_{2k},$$

which simplifies the above formula for $\mathcal{S}_{2k}(Q)$ to

$$\mathcal{S}_{2k}(Q) = \left(\frac{1}{2}s_k^2 + \sigma_{2k}c_{2k}\frac{1}{2}\hat{s}_k^2 + \sigma_{2k}d_{2k}\hat{s}_k\frac{(-1)^{k+1}}{(2k-1)!}\right)p_k^2(Q).$$

Substituting $\hat{s}_k$ with its value and using (10) as well as the last identity in (13), we arrive at

$$\mathcal{S}_{2k}(Q) = \left(\frac{\lambda_k^2\sigma_k^2}{j_k^2} + \lambda_k^2\sigma_{2k}\left(a_k^2 c_{2k}(\frac{|B_{2k}|}{4k})^2 + 2(-1)^k d_{2k}\frac{|B_{2k}|}{4k}\right)\right)[Q,\partial Q]^*,$$

from which we obtain the first formula of the statement, since $s(Q) = -1/8\langle \mathcal{S}_{2k}(Q), [Q,\partial Q]\rangle$ as $\sigma(Q) = 0$. The second formula follows from the first together with the identity

$$\sigma_{2k}c_{2k} = 2^{4k+1}(2^{4k-1}-1)\left(1-\operatorname{denom}\left(\frac{|B_{4k}|}{8k}\right)d_{2k}\right)$$

by combining two of the summands to $-a_k^2\lambda_k^2T_k^2/16$ and simplifying the expressions. □

To determine the combinations of Pontryagin numbers, signatures, and $\hat{A}$-genera that are realized by closed highly connected $4m$-manifolds, we recall that, even for almost closed $4m$-manifolds M, these invariants are additive bordism invariants, where the top-dimensional Pontryagin number $p_m(M)$ is defined such that the Hirzebruch signature formula $\sigma(M) = \langle \mathcal{L}_m(M), [M]\rangle$ holds. Using this description of p_m, the $\hat{A}$-genus of an almost closed $4m$-manifold is defined by $\hat{A}(M) = \langle \hat{\mathcal{A}}_m(M), [M]\rangle$.

For m odd, Theorem 2.9 shows that the torsion free quotient of $\Omega_{4m}^{\langle 2m-1\rangle}$ is generated by the class $(\sigma_m/8)\cdot P$, whose invariants can be easily computed from $\sigma(P) = 8$ and $p_k^2(P) = 0$ using the formulas recalled at the beginning of this subsection.

Proposition 2.13. *For $m \neq 1$ odd, the torsion free quotient of $\Omega_{4m}^{\langle 2m-1\rangle}$ is generated by $(\sigma_m/8)\cdot P$, whose signature, $\hat{A}$-genus and Pontryagin numbers are*

$$\sigma\big((\sigma_m/8)\cdot P\big) = \sigma_m, \quad \hat{A}\big((\sigma_m/8)\cdot P\big) = -2\operatorname{num}\left(\frac{|B_{2m}|}{4m}\right), \quad \text{and}$$

$$p_m\big((\sigma_m/8)\cdot P\big) = 2(2m-1)!j_m.$$

For m even, we compute the occurring characteristic numbers as follows.

Proposition 2.14. *The torsion free quotient of $\Omega_{8k}^{\langle 4k-1\rangle}$ agrees with $\Omega_{8k}^{\langle 4k-1\rangle}/\Theta_{8k}$, and the morphisms induced by the respective characteristic numbers*

$$\sigma, \hat{A}, p_{2k}, p_k^2\colon \Omega_{8k}^{\langle 4k-1\rangle}/\Theta_{8k} \longrightarrow \mathbf{Z}$$

are, with respect to the ordered basis described in Theorem 2.9, given by

$$\sigma = \begin{pmatrix} \sigma_{2k} \\ \operatorname{ord}([\Sigma_Q])\frac{a_k^2}{\mu_k}\left(\frac{T_k{}^2}{2} - 2\sigma_{2k}d_{2k}\frac{|B_{2k}|}{4k}\left(\frac{|B_{2k}|}{|B_{4k}|} + (-1)^{k+1}\right)\right)\end{pmatrix},$$

$$\hat{A} = \begin{pmatrix} -\operatorname{num}\left(\frac{|B_{4k}|}{8k}\right) \\ 2\operatorname{ord}([\Sigma_Q])\frac{a_k^2}{\mu_k}\operatorname{num}\left(\frac{|B_{4k}|}{8k}\right)d_{2k}\frac{|B_{2k}|}{4k}\left(\frac{|B_{2k}|}{|B_{4k}|} + (-1)^{k+1}\right)\end{pmatrix},$$

$$p_{2k} = \begin{pmatrix} (4k-1)!j_{2k} \\ \operatorname{ord}([\Sigma_Q])\frac{a_k^2}{\mu_k}\left((2k-1)!^2 + (4k-1)!j_{2k}\frac{|B_{2k}|}{4k}\left(c_{2k}\frac{|B_{2k}|}{4k} + 2d_{2k}(-1)^k\right)\right)\end{pmatrix}, \quad \text{and}$$

$$p_k^2 = \begin{pmatrix} 0 \\ 2\operatorname{ord}([\Sigma_Q])\frac{a_k^2}{\mu_k}(2k-1)!^2\end{pmatrix},$$

where $\mu_k = 2$ if $k = 1,2$ and $\mu_k = 1$ otherwise. Furthermore, after replacing μ_k with its reciprocal, the same formulas hold for the subgroup $\Omega_{8k}^{\langle 4k-1\rangle}/\Theta_{8k} \cap \sigma^{-1}(4\cdot\mathbf{Z})$ using the ordered basis described in Remark 2.12.

Proof. The first statement follows from Theorem 2.9 and the computation of the invariants for $(\sigma_m/8)\cdot P$, analogous to Proposition 2.13, using the fact that $p_k^2(P)$ vanishes since P is parallelizable. The second part of Corollary 2.8 and Lemma 2.2 imply that $\mathrm{ord}([\Sigma_Q])\big(Q - s(Q)\cdot P\big)$ equals $8\cdot \mathbf{H}P^2$ for $k=1$ and $8\cdot \mathbf{O}P^2$ for $k=2$. Since $\mu_k\lambda_k^2$ is 8 if $k=1,2$ and 1 otherwise, it is sufficient to show that the invariants of the class $\mathrm{ord}([\Sigma_Q])\big(Q-s(Q)\cdot P\big)$ are in all cases given by the product of $\mu_k\lambda_k^2$ with the claimed invariants of the second basis vector. For p_k^2, this is implied by $p_k^2(P)=0$ and (10). The signature of $\mathrm{ord}([\Sigma_Q])\big(Q-s(Q)\cdot P\big)$ is obtained using the second formula in Lemma 2.7, together with $\sigma(P)=8$ and $\sigma(Q)=0$. Using (15), the values of these signatures, together with $p_k^2(P)=0$ and (10), result in

$$\hat{A}\left(\mathrm{ord}([\Sigma_Q])\,(Q-s(Q)\cdot P)\right) = \mathrm{ord}([\Sigma_Q])\left(\frac{\lambda_k^2 a_k^2 T_k^2}{2^{4k+2}(2^{4k-1}-1)} + \frac{s(Q)}{2^{4k-2}(2^{4k-1}-1)}\right),$$

which, using the second formula of Lemma 2.7, gives the desired value. The calculation of p_{2m} is obtained by combining (14) with the first formula for $s(Q)$ of Lemma 2.7 and

$$\frac{\lambda_k^2 s_k^2}{s_{2k}} a_k^2 (2k-1)!^2 = \frac{\lambda_k^2\sigma_k^2}{s_{2k} j_k^2}.$$

The last part of the statement follows from Remark 2.12. □

We proceed by computing the signatures and $\hat{A}$-genera realized by highly connected $4m$-manifolds. The following consequences of the von Staudt–Clausen theorem will be useful. A proof of the latter can be derived, for instance, from [AIK14, Ch. 3].

Theorem 2.15 (von Staudt–Clausen). *The prime factor decomposition of* $\mathrm{denom}(\frac{|B_{2n}|}{n})$ *is*

$$\mathrm{denom}\left(\frac{|B_{2n}|}{n}\right) = \prod_{p-1\mid 2n} p^{1+\nu_p(n)}.$$

In particular, $j_{2n} = \mathrm{denom}(\frac{|B_{4n}|}{8n})$ *is divisible by* $j_n = \mathrm{denom}(\frac{|B_{2n}|}{4n})$*, and* $\nu_2(j_n) = \nu_2(n)+3$.

Proposition 2.16. *There exists a closed* $(2m-1)$*-connected* $4m$*-manifold with signature*

$$\begin{cases} \sigma_m & \text{if } m \neq 1 \text{ is odd} \\ 2^{i_m}\gcd(\sigma_m, \sigma_{m/2}^2) & \text{if } m\neq 2,4 \text{ is even} \\ 1 & \text{if } m = 1,2,4, \end{cases}$$

where

$$i_m = \min\left(0, \nu_2(\mathrm{ord}([\Sigma_Q])) - 2\nu_2(m) - 4 + 2\nu_2(a_{m/2})\right),$$

and the signature of any closed $(2m-1)$*-connected* $4m$*-manifold is a multiple of this number.*

Remark 2.17. Note that we obtain $\nu_2(\mathrm{ord}([\Sigma_Q])) \leq 2\nu_2(m)+4$ from Theorem 2.4 and 2.15.

Remark 2.18. Proposition 2.16 should be compared with the analogous result for $(2m-1)$-connected $4m$-manifolds that are assumed to be almost parallelizable. It follows from work of Milnor–Kervaire [MK60, p. 457] that, under this additional assumption, the minimal positive signature that occurs is σ_m, independent of the parity of m. Since $i_m \leq 0$, one gets

$$2^{i_m}\gcd(\sigma_m, \sigma_{m/2}^2) < \frac{\sigma_m}{2^{m-\nu_2(m)-8}}$$

by [ABK72, Thm 1.5.2(c)], so the minimal positive signature becomes, for m even, significantly larger if one restricts to manifolds that are almost parallelizable.

The proof of Proposition 2.16 owes a significant intellectual debt to a computation due to Lampe [Lam81, Satz 1.3], to whom the authors would like to express their gratitude for sending them a copy of his diploma thesis.

Proof of Proposition 2.16. For $m \neq 1$ odd, the claim is a consequence of Proposition 2.13, and for $m = 1, 2, 4$, it follows from the existence of the manifolds $\mathbf{C}P^2$, $\mathbf{H}P^2$, and $\mathbf{O}P^2$ of signature 1. For $m = 2k \neq 2, 4$, it is, by Proposition 2.14, sufficient to show that the integer described in the statement is the greatest common divisor of

$$\sigma\left((\sigma_{2k}/8)\cdot P\right) = \sigma_{2k} \quad \text{and} \quad \sigma\left(\operatorname{ord}([\Sigma_Q])\big(Q - s(Q)\cdot P\big)\right) = -8\operatorname{ord}([\Sigma_Q])s(Q).$$

From the first formula of Lemma 2.7, one sees that there is an odd integer b such that

$$-8\operatorname{ord}([\Sigma_Q])s(Q) = \frac{1}{j_k^2}\left(\operatorname{ord}([\Sigma_Q])(\sigma_k^2 + \sigma_{2k}a_k^2 b)\right).$$

Using $\nu_2(\sigma_k) = \nu_2(a_k) + 2k + 1$ and $\nu_2(j_k) = \nu_2(k) + 3$, we compute

$$\nu_2\left(\gcd\left(\sigma_{2k}, 8\operatorname{ord}([\Sigma_Q])s(Q)\right)\right) = \min\left(4k+1, \nu_2\left(\operatorname{ord}([\Sigma_Q])\right) + 4k + 2\nu_2(a_k) - 2\nu_2(k) - 5\right).$$

To determine the odd part $\gcd(\sigma_{2k}, 8\operatorname{ord}([\Sigma_Q])s(Q))_{\mathrm{odd}}$ of the greatest common divisor, we note that, since T_k is an even integer for $k > 1$, the number j_k divides $2^{2k-1}(2^{2k}-1)$. This, together with the fact that $(2^{2k}-1)$ and $(2^{4k-1}-1)$ are coprime, implies that j_k and $(2^{4k-1}-1)$ are coprime. But since j_k is also coprime to $\operatorname{num}(\frac{|B_{4k}|}{8k})$ by the von Staudten–Clausen theorem, it cannot share odd prime divisors with σ_{2m}. As $\operatorname{ord}([\Sigma_Q])$ divides j_k^2 by Theorem 2.4, the conclusion also holds for $\operatorname{ord}([\Sigma_Q])$ and σ_{2m}. This leads to

$$\gcd\left(\sigma_{2k}, 8\operatorname{ord}([\Sigma_Q])s(Q)\right)_{\mathrm{odd}} = \gcd\left(\sigma_{2k}, \operatorname{ord}([\Sigma_Q])(\sigma_k^2 + \sigma_{2k}a_k^2 b)\right)_{\mathrm{odd}} = \gcd(\sigma_{2k}, \sigma_k^2)_{\mathrm{odd}},$$

which implies the statement, because $\nu_2(\gcd(\sigma_{2k}, \sigma_k^2)) = 4k + 1$. □

Since $\nu_2(\sigma_m) = 2m+1+\nu_2(a_m)$ and $i_m \geq -2\nu_2(m)-4$, we obtain the following divisibility result for the signature as an immediate corollary of Proposition 2.16.

Corollary 2.19. *The signature of a closed $(2m-1)$-connected manifold of dimension $4m$ for $m \neq 1, 2, 4$ is divisible by 2^{2m+2} if m is odd and by $2^{2m-2\nu_2(m)-3}$ if m is even.*

For $m \geq 2$, closed $(2m-1)$-connected $4m$-manifolds admit a spin structure, so their $\hat{A}$-genus is integral. We compute it as follows.

Proposition 2.20. *There exists a closed $(2m-1)$-connected $4m$-manifold with $\hat{A}$-genus*

$$\begin{cases} 2\operatorname{num}(\frac{|B_{2m}|}{4m}) & \text{if } m \neq 1 \text{ is odd} \\ \gcd(\operatorname{num}(\frac{|B_{2m}|}{4m}), \operatorname{num}(\frac{|B_m|}{2m})^2) & \text{if } m \text{ is even,} \end{cases}$$

and the $\hat{A}$-genus of any closed $(2m-1)$-connected $4m$-manifold is a multiple of this number.

Proof. Arguing similarly as in the proof of Proposition 2.16, the claim for m odd follows from Proposition 2.13. To prove the remaining case of m even using Proposition 2.14, we need to compute the greatest common divisor of $\operatorname{num}(\frac{|B_{4k}|}{8k})$ and

$$\begin{aligned} &2\operatorname{ord}([\Sigma_Q])\frac{a_k^2}{\mu_k}\operatorname{num}\left(\frac{|B_{4k}|}{8k}\right)d_{2k}\frac{|B_{2k}|}{4k}\left(\frac{|B_{2k}|}{|B_{4k}|} + (-1)^{k+1}\right) \\ &\quad = \frac{1}{j_k^2\mu_k}\operatorname{ord}([\Sigma_Q])a_k^2 d_{2k}\operatorname{num}\left(\frac{|B_{2k}|}{4k}\right)\left(\operatorname{num}\left(\frac{|B_{2k}|}{4k}\right)j_{2k} + 2(-1)^{k+1}j_k\operatorname{num}\left(\frac{|B_{4k}|}{8k}\right)\right) \end{aligned}$$

As j_k, j_{2k}, a_k, μ_k, and d_{2k} are coprime to $\operatorname{num}(\frac{|B_{4k}|}{8k})$, the number in question agrees with

$$\gcd\left(\operatorname{num}\left(\frac{|B_{4k}|}{8k}\right), \operatorname{ord}([\Sigma_Q])\operatorname{num}\left(\frac{|B_{2k}|}{4k}\right)^2\right).$$

But $\operatorname{ord}([\Sigma_Q])$ divides j_k^2 by Theorem 2.4, so the von Staudt–Clausen theorem implies that it has no common divisors with $\operatorname{num}(\frac{|B_{4k}|}{8k})$. This yields the result. □

Remark 2.21. Computer calculations show that for $m < 2678$ even, the greatest common divisor of σ_m and $\sigma_{m/2}^2$ is a power of 2, whereas it is $2^{2\cdot 2678+1} \cdot 34511$ for $m = 2678$. Since $\nu_2(\gcd(\sigma_m, \sigma_{m/2}^2)) = 2m+1$, the minimal positive signature of a closed highly connected $4m$-manifold for $m \neq 2, 4$ even, $m < 2678$, equals 2^{i_m+2m+1} by Proposition 2.16. Similar computations show that $\mathrm{num}(\frac{|B_{2m}|}{4m})$ and $\mathrm{num}(\frac{|B_m|}{2m})^2$ are coprime for $m < 44000$ even, and hence Proposition 2.20 implies that in these dimensions, there exists a closed $(2m-1)$-connected $4m$-manifold of $\hat{A}$-genus 1. We do not know whether $\mathrm{num}(\frac{|B_{2m}|}{4m})$ and $\mathrm{num}(\frac{|B_m|}{2m})^2$ are coprime for all m even.

3. Characteristic numbers of bundles over surfaces with highly connected fiber

Harvesting the fruits of our labor, we derive consequences for characteristic numbers of bundles over surfaces, as well as for $\mathrm{H}^2(\mathrm{BDiff}^+(M); \mathbf{Z})$. This leads to proofs of Theorem B, Corollary C, and Theorem D. Throughout the section, we require M to be closed, $(n-1)$-connected, $2n$-dimensional, and almost parallelizable (or, equivalently, n-parallelizable).

3.1. Characteristic numbers of total spaces.

To treat various cases in a uniform manner, the following auxiliary definition turns out to be convenient.

Definition 3.1. A smooth oriented M-bundle over an oriented surface is called *admissible* if its image under $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M)) \to \Omega_{4m}^{\mathrm{SO}}$ is contained in the subgroup

$$\mathrm{im}\left(\Omega_{2n+2}^{\langle n\rangle} \to \Omega_{2n+2}^{\mathrm{SO}}\right) \cap \sigma^{-1}(4\cdot\mathbf{Z}). \tag{16}$$

Theorem 1.14 and Remark 1.12 show that most bundles are admissible, as summarized in the following. See Remark 1.10 for a discussion of which bundles might not be admissible.

Lemma 3.2. *Let $\pi\colon E \to S$ an oriented M-bundle over an oriented closed surface. If one of the following conditions is satisfied, then π is admissible.*

(i) M satisfies one of the conditions of Proposition 1.9, for instance, if $M = W_g$.
(ii) π admits a trivial D^{2n}-subbundle for an embedded disc $D^{2n} \subseteq M$.
(iii) The class of π in $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ is divisible by 2.

For $g(M) \geq 5$, every class in the subgroup (16) *is represented by a bundle of type (ii).*

In Proposition 2.13 and 2.14, we determined the lattices of Pontryagin numbers, signatures and $\hat{A}$-genera realized by classes in the subgroup (16), which hence also computes the respective invariants for total spaces of admissible M-bundles over surfaces. As an example, note that Proposition 2.16 and Corollary 2.19 imply the following two results. They include Theorem B and Corollary C as special cases.

Theorem 3.3. *Let M be a closed, highly connected, almost parallelizable $(4m-2)$-manifold. For an admissible M-bundle $\pi\colon E \to S$ over a closed oriented surface, $\sigma(E)$ is divisible by*

$$\begin{cases} 4 & \text{for } m = 1,2,4 \\ \sigma_m & \text{for } m \neq 1 \text{ odd} \\ 2^{i_m}\gcd(\sigma_{m/2}^2, \sigma_m) & \text{for } m \neq 2,4 \text{ even,} \end{cases}$$

where

$$i_m = \min\left(0, \nu_2(\mathrm{ord}([\Sigma_Q]) - 2\nu_2(m) - 4 + 2\nu_2(a_{m/2})\right).$$

For $m \geq 2$, the $\hat{A}$-genus of E is integral and divisible by

$$\begin{cases} 2\,\mathrm{num}(\frac{|B_{2m}|}{4m}) & \text{for } m \text{ odd} \\ \gcd(\mathrm{num}(\frac{|B_{2m}|}{4m}), \mathrm{num}(\frac{|B_m|}{2m})^2) & \text{for } m \text{ even.} \end{cases}$$

Moreover, if $g(M) \geq 5$, then these numbers are realized as signatures and $\hat{A}$-genera of total spaces of bundles of the above type. For $m = 1$, requiring $g \geq 3$ is sufficient.

Corollary 3.4. *Let M be a closed, highly connected, almost parallelizable $(4m-2)$-manifold. There exists a smooth M-bundle over a closed oriented surface with total space of signature 4 if and only if $m = 1, 2, 4$. If $m \neq 1, 2, 4$, then the signature of the total space of an admissible M-bundle over a surface is divisible by 2^{2m+2} for m odd and by $2^{2m-2\nu_2(m)-3}$ for m even.*

Since every 2-divisible class in $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M))$ is admissible, the admissibility assumption in the divisibility statements of Theorem 3.3 and Corollary 3.4 can be dropped, but at the cost of losing a factor of 2.

3.2. **Generalized Miller–Morita–Mumford classes.** We conclude with a computation of the torsion free quotient $\mathrm{H}^2(\mathrm{BDiff}^+(M, D^{2n}); \mathbf{Z})_{\mathrm{free}}$ for $g(M) \geq 7$ and $2n \geq 6$ in terms of generalized Miller–Miller–Mumford classes (as recalled in the introduction). This also applies to $\mathrm{H}^2(\mathrm{BDiff}^+(M); \mathbf{Z})_{\mathrm{free}}$ as long as M satisfies one of the conditions of Proposition 1.9. The root of the computation is the analogous result for rational cohomology due to Galatius–Randal-Williams: their high dimensional analogue of the Madsen–Weiss theorem (see Theorem 1.1) provides an isomorphism

$$\mathrm{H}^*\left(\mathrm{BDiff}(M, D^{2n}); \mathbf{Z}\right) \cong \mathrm{H}^*(\Omega^\infty_M \mathbf{MT}\theta_n; \mathbf{Z})$$

for $2n \geq 6$, $g(M) \geq 7$, and $* \leq 2$. The cohomology groups $\mathrm{H}^*(\Omega^\infty_M \mathbf{MT}\theta_n; \mathbf{Z})$ are finitely generated and can be computed rationally to be the free graded commutative algebra

$$\mathrm{H}^*(\Omega^\infty_M \mathbf{MT}\theta_n; \mathbf{Q}) \cong \Lambda\left(\mathrm{H}^{*+2n>0}(\mathrm{BSO}(2n)\langle n\rangle; \mathbf{Q})\right),$$

see [GR14, Ch. 2.5]. As $\mathrm{H}^*(\mathrm{BSO}(2n)\langle n\rangle; \mathbf{Q})$ is a polynomial ring in the Pontryagin classes p_i of degree $4i > n$, we arrive at

$$\mathrm{H}^2\left(\mathrm{BDiff}(M, D^{2n}); \mathbf{Q}\right) = \begin{cases} 0 & \text{if } 2n \equiv 0 \pmod 4 \\ \mathbf{Q}\kappa_{p_{(n+1)/2}} & \text{if } 2n \equiv 2 \pmod 8 \\ \mathbf{Q}\kappa_{p_{(n+1)/2}} \oplus \mathbf{Q}\kappa_{p^2_{(n+1)/4}} & \text{if } 2n \equiv 6 \pmod 8 \end{cases}$$

for $2n \geq 6$ and $g \geq 7$. Making use of the Serre exact sequences of the two homotopy fiber sequences in (6), one sees that this formula, as well as the finite generation property, holds equally well for $\mathrm{BDiff}^+(M)$ instead of $\mathrm{BDiff}(M, D^{2n})$, an observation which incidentally yields the description of $\mathrm{H}^2(\mathrm{BDiff}^+(W_g); \mathbf{Q})$ mentioned in the introduction. Excluding the trivial case, we assume $4m = 2n + 2$. Since classes in $\Omega_{4m}^{\langle 2m-1\rangle}$ have vanishing Stiefel–Whitney numbers and are, up to torsion, detected by Pontryagin numbers (see e.g. Proposition 2.13 and 2.14), the induced morphism $(\Omega_{4m}^{\langle 2m-1\rangle})_{\mathrm{free}} \to \Omega_{4m}^{\mathrm{SO}}$, defined on the torsion free quotient, is injective, and the rank of its image agrees with that of $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}(M, D^{2n}))_{\mathrm{free}}$. From this, using Theorem 1.14, we conclude that the morphism $\Omega_2^{\mathrm{SO}}(\mathrm{BDiff}^+(M, D^{4m-2})) \to \Omega_{4m}^{\mathrm{SO}}$ induces an isomorphism of the form

$$\Omega_2^{\mathrm{SO}}\left(\mathrm{BDiff}^+(M, D^{4m-2})\right)_{\mathrm{free}} \cong \mathrm{im}\left(\Omega_{4m}^{\langle 2m-1\rangle} \to \Omega_{4m}^{\mathrm{SO}}\right) \cap \sigma^{-1}(4 \cdot \mathbf{Z}).$$

In view of the commutative diagram displayed in the introduction, the functional on the left hand side induced by a class κ_c for $c \in \mathrm{H}^{4m}(\mathrm{BSO}; \mathbf{Z})$ corresponds via this isomorphism to the usual characteristic number defined by c on the right hand side. Propositions 2.13 and 2.14 provide an explicit basis for the right hand side and represent the functionals induced by Pontryagin numbers in terms of this basis. Computing the appropriate change of basis matrix, the integral dual of this basis can be expressed in terms of Pontryagin numbers. Ultimately, this results in the following basis of $\mathrm{H}^2(\mathrm{BDiff}^+(M, D^{4m-2}); \mathbf{Z})_{\mathrm{free}}$, and in particular proves Theorem D. The various variables are defined as in the introduction.

Theorem 3.5. *Let M a closed, highly connected, almost parallelizable $(4m-2)$-manifold. If $m \geq 2$ and $g(M) \geq 7$, then the group* $\mathrm{H}^2(\mathrm{BDiff}^+(M, D^{4m-2}); \mathbf{Z})_{\mathrm{free}}$ *is generated by*

$$\frac{\kappa_{p_m}}{2(2m-1)!j_m}$$

for m odd, and by

$$\frac{\kappa_{p_k^2}}{2\mu_k a_k^2\,\mathrm{ord}([\Sigma_Q])(2k-1)!^2} \quad \text{and} \quad \frac{2\kappa_{p_{2k}}-\kappa_{p_k^2}}{2(4k-1)!j_{2k}} - \frac{\frac{|B_{2k}|}{4k}\left(c_{2k}\frac{|B_{2k}|}{4k}+2d_{2k}(-1)^k\right)\kappa_{p_k^2}}{2(2k-1)!^2}$$

for $m = 2k$ even. Moreover, if M satisfies one of the assumptions of Proposition 1.9, then the same conclusions apply to the group $\mathrm{H}^2(\mathrm{BDiff}^+(M); \mathbf{Z})_{\mathrm{free}}$.

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Article C

Homological stability of topological moduli spaces

HOMOLOGICAL STABILITY OF TOPOLOGICAL MODULI SPACES

MANUEL KRANNICH

Abstract. Given a graded E_1-module over an E_2-algebra in spaces, we construct an augmented semi-simplicial space up to higher coherent homotopy over it, called its canonical resolution, whose graded connectivity yields homological stability for the graded pieces of the module with respect to constant and abelian coefficients. We furthermore introduce a notion of coefficient systems of finite degree in this context and show that, without further assumptions, the corresponding twisted homology groups stabilise as well. This generalises a framework of Randal-Williams and Wahl for families of discrete groups.

In many examples, the canonical resolution recovers geometric resolutions with known connectivity bounds. As a consequence, we derive new twisted homological stability results for e.g. moduli spaces of high-dimensional manifolds, unordered configuration spaces of manifolds with labels in a fibration, and moduli spaces of manifolds equipped with unordered embedded discs. This in turn implies representation stability for the ordered variants of the latter examples.

A sequence of spaces

$$\ldots \longrightarrow \mathcal{M}_{n-1} \longrightarrow \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1} \longrightarrow \ldots$$

is said to satisfy *homological stability* if the induced maps in homology are isomorphisms in degrees that are small relative to n. There is a well-established strategy for proving homological stability that traces back to an argument by Quillen for the classifying spaces of a sequence of inclusions of groups G_n. Given simplicial complexes whose connectivity increases with n and on which the groups G_n act simplicially, transitively on simplices, and with stabilisers isomorphic to groups G_{n-k} prior in the sequence, stability can often be derived by employing a spectral sequence relating the different stabilisers. In [RW17], Randal-Williams and Wahl axiomatised this strategy of proof, resulting in a convenient categorical framework for proving homological stability for families of discrete groups that form a braided monoidal groupoid. Their work unifies and improves many classical stability results and has led to a number of applications since its introduction [Fri17; GW16; PW16; Ran18; SW14].

However, homological stability phenomena have been proved to occur not only in the context of discrete groups, but also in numerous non-aspherical situations, many of them of a moduli space flavor, such as unordered configuration spaces of manifolds [McD75; Seg73; Seg79], the most classical example, or moduli spaces of high-dimensional manifolds [GR17; GR18] to emphasise a more recent one. The majority of the stability proofs in this context resemble the original line of argument for discrete groups, and one of the objectives of the present work is to provide a conceptualisation of this pattern.

Instead of considering the single spaces $\mathcal{M}_n$ and the maps $\mathcal{M}_n \to \mathcal{M}_{n+1}$ between them one at a time, it is beneficial to treat them as a single space $\mathcal{M} = \coprod_{n\geq 0} \mathcal{M}_n$ together with a *grading* $g_{\mathcal{M}}\colon \mathcal{M} \to \mathbf{N}_0$ to the nonnegative integers, capturing the decomposition of $\mathcal{M}$ into the pieces $\mathcal{M}_n$, and a *stabilisation map* $s\colon \mathcal{M} \to \mathcal{M}$ that restricts to the maps $\mathcal{M}_n \to \mathcal{M}_{n+1}$, so it increases the degree by one. From the perspective of homotopy theory, such $\mathcal{M}$ that result from families $\mathcal{M}_n$ that are known to stabilise homologically usually share the characteristic of forming a *(graded) E_1-module over an E_2-algebra*—the homotopy theoretical analogue of a module over a braided monoidal category. This observation is the driving force behind the present work.

Referring to Section 2.1 for a precise definition, we encourage the reader to think of a graded E_1-module $\mathcal{M}$ over an E_2-algebra $\mathcal{A}$ as a pair of spaces $(\mathcal{M}, \mathcal{A})$ together with gradings $g_{\mathcal{M}}\colon \mathcal{M} \to \mathbf{N}_0$ and $g_{\mathcal{A}}\colon \mathcal{A} \to \mathbf{N}_0$, a homotopy-commutative multiplication $\oplus\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a homotopy-associative action-map $\oplus\colon \mathcal{M} \times \mathcal{A} \to \mathcal{M}$. These are required to satisfy various axioms, among them additivity with respect to $g_{\mathcal{M}}$ and $g_{\mathcal{A}}$ (see Definition 2.3). Given such $\mathcal{M}$ and $\mathcal{A}$, the choice of a *stabilising object* $X \in \mathcal{A}$, meaning an element of degree 1, results in a *stabilisation map*

$$s \coloneqq (- \oplus X)\colon \mathcal{M} \to \mathcal{M}$$

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that increases the degree by 1 and hence gives rise to a sequence

$$\ldots \longrightarrow \mathcal{M}_{n-1} \xrightarrow{s} \mathcal{M}_n \xrightarrow{s} \mathcal{M}_{n+1} \longrightarrow \ldots$$

of the subspaces $\mathcal{M}_n = g_{\mathcal{M}}^{-1}(n)$ of a fixed degree. The sequences of spaces arising in this fashion are the ones whose homological stability behaviour the present work is concerned with.

The key construction of this work is introduced in Section 2.2. We assign to $\mathcal{M}$ its *canonical resolution*

$$R_\bullet(\mathcal{M}) \longrightarrow \mathcal{M}, \tag{1}$$

which is an augmented semi-simplicial space up to higher coherent homotopy—a notion made precise in Section 1.5, but which can be thought of as an augmented semi-simplicial space in the usual sense. The fibre $W_\bullet(A)$ of the canonical resolution at a point $A \in \mathcal{M}$ is an analogue of the simplicial complex in Quillen's argument; it is a semi-simplicial space up to higher coherent homotopy whose space of p-simplices $W_p(A)$ is the homotopy fibre at A of the $(p+1)$st iterated stabilisation map $s^{p+1}\colon \mathcal{M} \to \mathcal{M}$. Thus $W_\bullet(A)$ should be thought of as the *space of destabilisations of A*—a terminology that suggests that the canonical resolution controls the stability behaviour of $\mathcal{M}$, justified by Theorem A and C.

To state our main theorems, we call the canonical resolution of $\mathcal{M}$ *graded $\varphi(g_{\mathcal{M}})$-connected in degrees* $\geq m$ for a function $\varphi\colon \mathbf{N}_0 \to \mathbf{Q}$ if the restriction $|R_\bullet(\mathcal{M})|_n \to \mathcal{M}_n$ of the geometric realisation of (1) to the preimage of $\mathcal{M}_n$ is $\lfloor \varphi(n) \rfloor$-connected in the usual sense for all $n \geq m$. The first theorem, proved in Section 3, treats homological stability with constant and abelian coefficients, the latter being local systems on which the commutator subgroups of the fundamental groups at all basepoints act trivially.

Theorem A. *Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilising object X and L a local system on $\mathcal{M}$. If the canonical resolution of $\mathcal{M}$ is graded $(\frac{g_{\mathcal{M}}-2+k}{k})$-connected in degrees ≥ 1 for some $k \geq 2$, then*

$$s_*\colon \mathrm{H}_i(\mathcal{M}_n; s^*L) \longrightarrow \mathrm{H}_i(\mathcal{M}_{n+1}; L)$$

(i) is an isomorphism for $i \leq \frac{n-1}{k}$ and an epimorphism for $i \leq \frac{n-2+k}{k}$, if L is constant, and
(ii) is an isomorphism for $i \leq \frac{n+1-k}{k}$ and an epimorphism for $i \leq \frac{n}{k}$, if L is abelian and $k \geq 3$.

Remark. In certain cases, discussed in Remark 3.3, the ranges of Theorem A can be improved marginally.

Restricting to homological degree 0, the theorem has the following cancellation result as a consequence.

Corollary B. *Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilising object X. If the connectivity assumption of Theorem A is satisfied, then the fundamental groupoid of $\mathcal{M}$ is X-cancellative for objects of positive degree, i.e. for objects A and A' of $\mathcal{M}$ of positive degree, $A \oplus X \cong A' \oplus X$ in $\Pi(\mathcal{M})$ implies $A \cong A'$.*

To cover more general coefficients, we note that the fundamental groupoid of an E_2-algebra $\mathcal{A}$ naturally carries the structure of a braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$ and the fundamental groupoid of an E_1-module $\mathcal{M}$ over $\mathcal{A}$ becomes a right-module $(\Pi(\mathcal{M}), \oplus)$ over it (see Section 2.1). In terms of this, we define in Section 4.1 a *coefficient system F for $\mathcal{M}$* with stabilising object X as an abelian group-valued functor F on $\Pi(\mathcal{M})$, together with a natural transformation $\sigma^F\colon F \to F(-\oplus X)$ for which the image of the canonical morphism $B_m \to \mathrm{Aut}_{\mathcal{A}}(X^{\oplus m})$ from the braid group on m strands acts trivially on the image of $(\sigma^F)^m\colon F \to F(-\oplus X^{\oplus m})$ for all n and m. Such a coefficient system enhances the stabilisation map to a map of spaces with local systems

$$(s; \sigma^F)\colon (\mathcal{M}_n; F) \to (\mathcal{M}_{n+1}; F)$$

by restricting F to subspaces of homogenous degree. A coefficient system F induces a new one $\Sigma F = F(-\oplus X)$, called its *suspension*, which comes with a morphism $F \to \Sigma F$, named the *suspension map* (see Definition 4.3). The coefficient system F is inductively said to be *of degree r* if the kernel of the suspension map vanishes and the cokernel has degree $(r-1)$; the zero coefficient system having degree -1. In fact, we define a more general notion of being of *(split) degree r at N* such that F is of degree r in the sense just described if it is of degree r at 0 (see Definition 4.1). This notion of a coefficient system of finite (split) degree generalises the one introduced by Randal-Williams and Wahl [RW17] for braided monoidal groupoids (see Remarks 4.11 and 4.12), which was itself inspired by work of Dwyer [Dwy80] and van der Kallen [Kal80] on general linear groups, and work of Ivanov [Iva93] on mapping class groups of surfaces.

Remark. There is an alternative point of view on coefficient systems for $\mathcal{M}$, namely as abelian-group valued functors on a category $\langle \mathcal{M}, \mathcal{B} \rangle$ constructed from the action of $\Pi(\mathcal{A})$ on $\Pi(\mathcal{M})$ (see Remark 4.12).

Our second main theorem, demonstrated in Section 4.2, addresses homological stability of $\mathcal{M}$ with coefficients in a coefficient system of finite degree.

Theorem C. *Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilising object X and F a coefficient system for $\mathcal{M}$ of degree r at $N \geq 0$. If the canonical resolution of $\mathcal{M}$ is graded $(\frac{g_{\mathcal{M}}-2+k}{k})$-connected in degrees ≥ 1 for some $k \geq 2$, then the map induced by stabilisation*

$$(s;\sigma^F)_*\colon \ \mathrm{H}_i(\mathcal{M}_n;F) \longrightarrow \mathrm{H}_i(\mathcal{M}_{n+1};F)$$

is an isomorphism for $i \leq \frac{n-rk-k}{k}$ and an epimorphism for $i \leq \frac{n-rk}{k}$, when $n > N$. If F is of split degree r at $N \geq 0$ then $(s;\sigma^F)_*$ is an isomorphism for $i \leq \frac{n-r-k}{k}$ and an epimorphism for $i \leq \frac{n-r}{k}$, when $n > N$.

As a proof of concept, we apply the developed theory to three main classes of examples to which we devote the remainder of this introduction.

Configuration spaces. The *unordered configuration space* $C_n^\pi(W)$ of a manifold with boundary W with labels in a Serre fibration $\pi\colon E \longrightarrow W$ is the quotient of the *ordered configuration space*

$$F_n^\pi(W) = \{(e_1,\ldots,e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j \text{ and } \pi(e_i) \in W \backslash \partial W\}$$

by the apparent action of the symmetric group Σ_n. If W is of dimension $d \geq 2$ and has nonempty boundary, then the union of its configuration spaces $\mathcal{M} = \coprod_{n\geq 0} C_n^\pi(W)$ admits the structure of an E_1-module over the E_2-algebra $\mathcal{A} = \coprod_{n\geq 0} C_n(D^d)$ of configurations in a d-disc, graded by the number of points (see Lemma 5.1). In Section 5.1, we identify its canonical resolution with the *resolution by arcs*—an augmented semi-simplicial space of geometric nature that has already been considered in the context of homological stability (see e.g. [MW16; KM14]) and is known to be sufficiently connected to apply Theorem A and C.

Theorem D. *Let W be a connected manifold of dimension at least 2 with nonempty boundary and let $\pi\colon E \longrightarrow W$ be a Serre fibration with path-connected fibres.*

(i) For a local system L on $C_{n+1}^\pi(W)$, the stabilisation map

$$s_*\colon \ \mathrm{H}_i(C_n^\pi(W);s^*L) \longrightarrow \mathrm{H}_i(C_{n+1}^\pi(W);L)$$

is an isomorphism for $i \leq \frac{n-1}{2}$ and an epimorphism for $i \leq \frac{n}{2}$, if L is constant. It is an isomorphism for $i \leq \frac{n-2}{3}$ and an epimorphism for $i \leq \frac{n}{3}$, if L is abelian.

(ii) If F is a coefficient system of degree r at $N \geq 0$, then the stabilisation map

$$(s;\sigma^F)_*\colon \ \mathrm{H}_i(C_n^\pi(W);F) \longrightarrow \mathrm{H}_i(C_{n+1}^\pi(W);F)$$

is an isomorphism for $i \leq \frac{n-2r-2}{2}$ and an epimorphism for $i \leq \frac{n-2r}{2}$, when $n > N$. If F is of split degree r at $N \geq 0$, then it is an isomorphism for $i \leq \frac{n-r-2}{2}$ and an epimorphism for $i \leq \frac{n-r}{2}$, when $n > N$.

Remark. Employing the improvement of Remark 3.3, one obtains a slightly better isomorphism range of $i \leq \frac{n}{2}$ than the one stated in Theorem D.

Configuration spaces have a longstanding history in the context of homological stability, starting with work of Arnold [Arn68], who established stability for $C_n(D^2)$ with constant coefficients. McDuff and Segal [McD75; Seg73; Seg79] observed that this behaviour is not restricted to the 2-disc and proved stability for more general $C_n^\pi(W)$ with constant coefficients and $\pi = \mathrm{id}_W$, which can be extended to general π, e.g. by adapting the proof for a trivial fibration presented in [Ran13] (see [CP15; KM14] for alternative proofs).

As proved for example in [Ran13], the stabilisation map for configuration spaces is in fact split injective in homology with constant coefficients in all degrees—a phenomenon special to configuration spaces, not captured by our general approach.

For a trivial fibration, stability of $C_n^\pi(W)$ with respect to a nontrivial coefficient system F was studied by Palmer [Pal18], building on work of Betley [Bet02] on symmetric groups. The second part of Theorem D extends his result to nontrivial fibrations and a significantly larger class of coefficient systems, partly conjectured by Palmer [Pal18, Rem. 1.5] (see Remark 5.12 for a more detailed comparison to his work). In the case of surfaces and a trivial fibration, a result similar to Theorem D, but with respect to a slightly smaller class of coefficient systems, is contained in work by Randal-Williams and Wahl [RW17, Thm D].

In Section 5.2, we provide a discussion of coefficient systems for configuration spaces by relating them, for instance, to the theory of $\mathcal{FI}$-modules as introduced by Church, Ellenberg, and Farb [CEF15] or to coefficient systems studied in [RW17]. These considerations provide numerous nontrivial coefficient systems F with respect to which the homology of $C_n^\pi(W)$ stabilises.

To our knowledge, stability with abelian coefficients for configuration spaces of manifolds of dimensions greater than two has not been considered so far. We next discuss a direct consequence of stability with respect to this class of coefficients as the first item in a series of applications exploiting Theorem D.

Oriented configuration spaces. The *oriented configuration space* $C_n^{\pi,\mathrm{or}}(W)$ with labels in a Serre fibration π over W is the double cover of $C_n^\pi(W)$ given as the quotient of the ordered configuration space $F_n^\pi(W)$ by the action of the alternating group A_n, or equivalently, the space of labelled configurations ordered up to even permutations. By the space version of Shapiro's lemma, the homology of $C_n^{\pi,\mathrm{or}}(W)$ is isomorphic to $\mathrm{H}_*(C_n^\pi(W);\mathbf{Z}[\mathbf{Z}/2\mathbf{Z}])$, with the action of $\pi_1(C_n^\pi(W))$ on the group ring $\mathbf{Z}[\mathbf{Z}/2\mathbf{Z}]$ being induced by the composition of the sign homomorphism with the morphism $\pi_1(C_n^\pi(W))\rightarrow\Sigma_n$, obtained by choosing an ordering of a basepoint. These local systems are abelian and are preserved by pulling back along the stabilisation map, hence homological stability for $C_n^{\pi,\mathrm{or}}(W)$ follows as a by-product of Theorem D.

Corollary E. *Let W and π be as in Theorem D. The map induced by stabilisation*

$$s_*\colon\ \mathrm{H}_i(C_n^{\pi,\mathrm{or}}(W);\mathbf{Z})\longrightarrow\mathrm{H}_i(C_{n+1}^{\pi,\mathrm{or}}(W);\mathbf{Z})$$

is an isomorphism for $i\leq\frac{n-2}{3}$ and an epimorphism for $i\leq\frac{n}{3}$.

Stability for oriented configuration spaces of connected orientable surfaces with nonempty boundary and without labels was proved by Guest, Kozlowsky, and Yamaguchi [GKY96] using computations due to Bödigheimer, Cohen, Taylor, and Milgram [BCT89; BCM93]. Palmer [Pal13] extended this to manifolds of higher dimensions with nonempty boundary and labels in a trivial fibration. Corollary E gives an alternative proof of his result and enhances it by means of general labels and an improved stability range.

Configuration spaces of embedded discs. The *configuration space $C_n^k(W)$ of unordered k-discs* in a connected d-manifold W is the quotient by the action of Σ_n on the *configuration space of ordered k-discs*

$$F_n^k(W)=\mathrm{Emb}(\textstyle\coprod^n D^k,W\backslash\partial W),$$

equipped with the C^∞-topology. For $k=d$ and oriented W, there are variants $F_n^{d,+}(W)$ and $C_n^{d,+}(W)$ by restricting to orientation preserving embeddings. Mapping an embedding of a k-disc to its centre point, labelled with the k-frame induced by standard framing of D^k at the origin, results in a map $C_n^k(W)\rightarrow C_n^{\pi_k}(W)$, where π_k is the bundle of k-frames in M. This map can be seen to be a weak equivalence by choosing a metric and exponentiating frames. For $k<d$, the fibre of π_k is path-connected, so the homological stability results of Theorem D carry over to $C_n^k(W)$, comprising part of Corollary F below. Using the bundle π_d^+ of oriented d-framings, the argument for $C_n^{d,+}(W)$ is analogous, since the orientability condition ensures that the fibres of π_k^+ are path-connected.

The topological group of diffeomorphisms $\mathrm{Diff}_\partial(W)$ fixing a neighbourhood of the boundary in the C^∞-topology naturally acts on the configuration spaces $F_n^k(W)$ and $C_n^k(W)$, and the resulting homotopy quotients $F_n^k(W)/\!\!/\mathrm{Diff}_\partial(W)$ and $C_n^k(W)/\!\!/\mathrm{Diff}_\partial(W)$ model the classifying spaces of the subgroups

$$\mathrm{PDiff}_{\partial,n}^k(W)\subseteq\mathrm{Diff}_{\partial,n}^k(W)\subseteq\mathrm{Diff}_\partial(W),$$

where $\mathrm{PDiff}_{\partial,n}^k(W)$ are the diffeomorphisms that fix n chosen embedded k-discs in W and $\mathrm{Diff}_{\partial,n}^k(W)$ are the ones permuting them (see Lemma 5.13). If W is orientable, the (sub)groups of orientation preserving diffeomorphisms are denoted with a (+)-superscript. In Example 2.21, we explain how the canonical resolution of a graded E_1-module $\mathcal{M}$ over an E_2-algebra $\mathcal{A}$ relates to that of the E_1-module $EG\times_G\mathcal{M}$ over $\mathcal{A}$ in the presence of a graded action of a group G on $\mathcal{M}$ that commutes with the action of $\mathcal{A}$. An application of this consideration to the situation at hand implies the following, carried out in Section 5.3.1.

Corollary F. *Let W be a d-dimensional manifold as in Theorem D and let $0\leq k<d$.*

(i) For a local system L, the stabilisation maps

$$\mathrm{H}_i(C_n^k(W);s^*L)\rightarrow\mathrm{H}_i(C_{n+1}^k(W);L)\quad\textit{and}\quad\mathrm{H}_i(B\,\mathrm{Diff}_{\partial,n}^k(W);s^*L)\rightarrow\mathrm{H}_i(B\,\mathrm{Diff}_{\partial,n+1}^k(W);L)$$

are isomorphisms for $i\leq\frac{n-1}{2}$ and epimorphisms for $i\leq\frac{n}{2}$, if L is constant. If L is abelian, then they are isomorphisms for $i\leq\frac{n-2}{3}$ and epimorphisms for $i\leq\frac{n}{3}$.

(ii) If F is a coefficient system of degree r at $N \geq 0$, then the maps induced by the stabilisation $(s;\sigma^F)$

$$\mathrm{H}_i(C^k_n(W);F) \to \mathrm{H}_i(C^k_{n+1}(W);F) \quad \text{and} \quad \mathrm{H}_i(B\operatorname{Diff}^k_{\partial,n}(W);F) \to \mathrm{H}_i(B\operatorname{Diff}^k_{\partial,n+1}(W);F)$$

are isomorphisms for $i \leq \frac{n-2r-2}{2}$ and epimorphisms for $i \leq \frac{n-2r}{2}$, when $n > N$. If F is of split degree r at $N \geq 0$, then they are isomorphisms for $i \leq \frac{n-r-2}{2}$ and epimorphisms for $i \leq \frac{n-r}{2}$, when $n > N$.

If W is oriented, then the analogous statements hold for the variants $C^{d,+}_n(W)$ and $B\operatorname{Diff}^{d,+}_{\partial,n}(W)$.

Remark. The isomorphism range for constant coefficients in the previous theorem can be improved to $i \leq \frac{n}{2}$ by virtue of Remark 3.3.

For compact manifolds W, Tillmann [Til16] has proved homological stability with constant coefficients for variants of $B\operatorname{Diff}^0_{\partial,n}(W)$ and $B\operatorname{Diff}^{d,+}_{\partial,n}(W)$ involving diffeomorphisms that are only required to fix a disc in the boundary instead of the whole boundary. A Serre spectral sequence argument shows that stability for these variants follows from stability of the spaces $B\operatorname{Diff}^0_{\partial,n}(W)$ and $B\operatorname{Diff}^{d,+}_{\partial,n}(W)$. Hatcher and Wahl [HW10, Prop. 1.5] have shown stability with constant coefficients for the mapping class groups $\pi_0(\operatorname{Diff}^0_{\partial,n}(W))$, which can be seen to be equivalent to $\operatorname{Diff}^0_{\partial,n}(W)$ for compact 2-dimensional W as a result of the homotopy discreteness of the space of diffeomorphisms of a compact surface [EE67; Gra73]. In this case, stability with respect to some of the twisted coefficients systems Corollary F deals with is contained in work by Randal-Williams and Wahl [RW17, Thm 5.22].

Representation stability. The first rational homology group of the ordered configuration space of the 2-disc

$$\mathrm{H}_1(F_n(D^2);\mathbf{Q}) \cong \mathbf{Q}^{\binom{n}{2}},$$

as e.g. computed in [Arn69], exemplifies that—in contrast to unordered configuration spaces—the homology of the ordered variant does not stabilise. However, by incorporating the action of the symmetric groups Σ_n, it does stabilise in a more refined, representation theoretic sense. To make this precise, recall the correspondence between irreducible representations of Σ_n and partitions of n [FH91, Ch. 4]. We denote the irreducible $\Sigma_{|\lambda|}$-module corresponding to a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k) \vdash |\lambda|$ of length $|\lambda|$ by V_λ and define for $n \geq |\lambda| + \lambda_1$, the *padded partition* $\lambda[n] = (n - |\lambda| \geq \lambda_1 \geq \ldots \geq \lambda_k) \vdash n$. Using the Totaro spectral sequence [Tot96], Church [Chu12] has shown that for a connected orientable manifold of dimension at least two with finite-dimensional rational cohomology, the groups $\mathrm{H}^i(F_n(W);\mathbf{Q})$ are *uniformly representation stable*—a concept introduced by Church and Farb [CF13]. This implies the existence of a constant $N(i)$, depending solely on i, such that the multiplicity of $V_{\lambda[n]}$ in the Σ_n-module $\mathrm{H}^i(F_n(W);\mathbf{Q})$ is independent of n for $n \geq N(i)$. Church's result has been extended in several directions [CEF15; MW16; Pet17; Tos16].

A twisted Serre spectral sequence argument (see Lemma 5.14) shows that the multiplicity of an irreducible Σ_n-module V_λ in $\mathrm{H}^i(F^\pi_n(W);\mathbf{Q})$ agrees with the dimension of $\mathrm{H}_i(C^\pi_n(W);V_\lambda)$, where $\pi_1(C^\pi_n(W))$ acts on V_λ via the morphism $\pi_1(C^\pi_n(W)) \to \Sigma_n$. This fact allows us to derive the stability of these multiplicities from Theorem D, at least for all manifolds to which the latter theorem applies (see Section 5.3.2).

Corollary G. *Let W and π be as in Theorem D and let Z_n be one of the following sequences of Σ_n-spaces:*

(i) $F^\pi_n(W)$,
(ii) $F^k_n(W)$ for $0 \leq k < d$,
(iii) $F^{d,+}_n(W)$ if W is oriented,
(iv) $B\operatorname{PDiff}^k_{\partial,n}(W)$ for $0 \leq k < d$,
(v) $B\operatorname{PDiff}^{d,+}_{\partial,n}(W)$ if W is orientable.

The $V_{\lambda[n]}$-multiplicity in $\mathrm{H}^i(Z_n;\mathbf{Q})$ for a fixed partition λ is independent of n for n large relative to i.

In Remark 5.16, we discuss explicit ranges for Corollary G and compare them to Church's. Let us at this juncture record that our approach leads to ranges that depend on $|\lambda|$, so we do not recover *uniform* representation stability. On the other hand, in contrast to Church's result, we neither require W to be orientable nor to have finite dimensional rational cohomology or π to be the identity.

Jiménez Rolland [Jim11; Jim15] has shown uniform representation stability for the cohomology groups $\mathrm{H}^i(B\operatorname{PDiff}^0_{\partial,n}(W);\mathbf{Q})$ for compact orientable surfaces and for compact connected manifolds W of dimension $d \geq 3$, assuming that $B\operatorname{Diff}_\partial(W)$ has the homotopy type of a CW-complex of finite type. Furthermore, she proved uniform representation stability for $\pi_0(\operatorname{PDiff}^0_{\partial,n}(W))$ for compact orientable surfaces, as well as for higher-dimensional manifolds under some further assumptions.

Moduli spaces of manifolds. The moduli space $\mathcal{M}$ of compact d-dimensional smooth manifolds with a fixed boundary P forms an E_1-module over the E_d-algebra $\mathcal{A}$ given by the moduli space of compact d-manifolds with a sphere as boundary (see Lemma 6.1). The homotopy types of $\mathcal{M}$ and $\mathcal{A}$ are

$$\mathcal{M} \simeq \coprod_{[W]} B\operatorname{Diff}_\partial(W) \quad \text{and} \quad \mathcal{A} \simeq \coprod_{[N]} B\operatorname{Diff}_\partial(N),$$

where $[W]$ runs over diffeomorphism classes relative to P of compact d-manifolds with P-boundary and $[N]$ over the ones of compact d-manifolds with a sphere as boundary. Acting with a manifold $X \in \mathcal{A}$ on $\mathcal{M}$ corresponds to taking the boundary connected sum $(-\natural X)$ with X, so the resulting stabilisation map thus restricts on path components to a map of the form

$$s\colon B\operatorname{Diff}_\partial(W) \longrightarrow B\operatorname{Diff}_\partial(W\natural X), \tag{2}$$

which models the map on classifying spaces induced by extending diffeomorphisms by the identity.

As shown in Section 6.1, the canonical resolution of $\mathcal{M}$ with respect to a choice of a stabilising manifold X is equivalent to the *resolution by embeddings*—an augmented semi-simplicial space of submanifolds $W \in \mathcal{M}$, together with embeddings of X with a fixed behaviour near their boundary. For specific manifolds X and W, this resolution and its connectivity has been studied to prove homological stability of (2), first by Galatius and Randal-Williams in their work [GR18] for $X \cong D^{2p}\sharp(S^p \times S^p)$ and simply-connected $2p$-dimensional W with $p \geq 3$. Their results extend the classical stability result for mapping class groups of surfaces by Harer [Har85] to higher dimensions. As in Harer's theorem, the known connectivity of the resolution by embeddings, and hence the resulting stability ranges, depend on the X*-genus* of W,

$$g^X(W) = \max\{k \geq 0 \mid \text{there exists } M \in \mathcal{M} \text{ such that } M\natural X^{\natural k} \cong W \text{ relative to } P\},$$

which incidentally provides a method of grading E_1-modules $\mathcal{M}$ in general (see Section 2.3). Perlmutter [Per16a] succeeded in carrying out this strategy for $X \cong D^{p+q}\sharp(S^p \times S^q)$ with certain $p \neq q$ depending on which W is required to satisfy a connectivity assumption. Recently, Friedrich [Fri17] extended the work of Galatius and Randal-Williams to manifolds W with nontrivial fundamental group in terms of the *unitary stable rank* [KM02, Def. 6.3] of the group ring $\mathbf{Z}[\pi_1(W)]$. These connectivity results can be restated in our context as graded connectivity for the canonical resolution of $\mathcal{M}$ with respect to different gradings (see Corollary 6.7), allowing us to apply Theorem A and C.

Employing the improvement of Remark 3.3, the ranges with constant and abelian coefficients obtained from Theorem A agree with the ones established in [Fri17; GR18; Per16a] (after extending [Per16a] to abelian coefficients by adapting the methods of [GR18]). The cancellation result for connected sums of manifolds that we derive from Corollary B coincides with their cancellation results as well. Our main contribution with respect moduli spaces of manifolds lies in the application of Theorem C, i.e. homological stability with respect to a large class of nontrivial coefficient systems, which has not yet been considered in the context of moduli spaces of high-dimensional manifolds. On path components, it reads as follows.

Theorem H. *Let W be a compact $(p+q)$-manifold with nonempty boundary and F a coefficient system of degree r. Denote by $g(W)$ the $(S^p \times S^q)$-genus of W, and set u to be 1 if W is simply connected and to be the unitary stable rank of $\mathbf{Z}[\pi_1(W)]$ otherwise. The stabilisation map*

$$(s,\sigma^F)_*\colon \mathrm{H}_i(B\operatorname{Diff}_\partial(W);F) \longrightarrow \mathrm{H}_i(B\operatorname{Diff}_\partial(W\sharp(S^p\times S^q));F)$$

(i) is an isomorphism for $i \leq \frac{g(W)-2r-u-3}{2}$ and an epimorphism for $i \leq \frac{g(W)-2r-u-1}{2}$, if $p = q \geq 3$, and
(ii) an isomorphism for $i \leq \frac{g(W)-2r-m-4}{2}$ and an epimorphism for $i \leq \frac{g(W)-2r-m-2}{2}$, if W is $(q-p+2)$-connected and $0 < p < q < 2p-2$ with $m = \min\{i \in \mathbf{N}_0 \mid$ there exists an epimorphism $\mathbf{Z}^i \to \pi_q(S^p)\}$.

If F is (split) of degree r at a number $N \geq 0$, the ranges in Theorem H change as per Theorem C.

Remark. The unitary stable rank [KM02, Def. 6.3] of a group ring $\mathbf{Z}[G]$ need not be finite. To provide a class of examples of finite unitary stable rank, recall that G is called *virtually polycyclic* if there is a series $1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$ such that G_i is normal in G_{i+1} and the quotients G_{i+1}/G_i are either finite or cyclic. Its *Hirsch length* $h(G)$ is the number of infinite cyclic factors. Crowley and Sixt [CS11, Thm 7.3] showed $\operatorname{usr}(\mathbf{Z}[G]) \leq h(G)+3$ for virtually polycyclic groups G. In particular, we have $\operatorname{usr}(\mathbf{Z}[G]) \leq 3$ for finite groups and $\operatorname{usr}(\mathbf{Z}[G]) \leq \operatorname{rank}(G)+3$ for finitely generated abelian groups.

In Remark 6.8, we briefly elaborate on how to include the case of orientable surfaces in this picture by utilising high-connectivity of the *complex of tethered chains*—a result of Hatcher and Vogtmann [HV17]. For constant coefficients, this implies Harer's classical stability theorem [Har85] with a better, but not optimal range (see [Bol12; Ran16]). For twisted coefficients, it extends a result by Ivanov [Iva93] to more general coefficient systems. However, in the case of surfaces, stability with respect to most of these more general coefficient systems was already known by [RW17].

In Section 6.2, we show that coefficient systems for $\mathcal{M}$ are equivalent to certain families of modules over the mapping class groups $\pi_0(\mathrm{Diff}_\partial(W)) \cong \pi_1(B\,\mathrm{Diff}_\partial(W))$ and explain how the action of the mapping class groups on the homology of the manifolds gives rise to a coefficient system of degree 1 for $\mathcal{M}$. This yields the following corollary.

Corollary I. *Let W be a compact $(p+q)$-manifold with nonempty boundary and $k \geq 0$. The stabilisation*

$$\mathrm{H}_i\left(B\,\mathrm{Diff}_\partial(W);\mathrm{H}_k(W)\right) \longrightarrow \mathrm{H}_i(B\,\mathrm{Diff}_\partial(W\sharp(S^p\times S^q));\mathrm{H}_k(W\sharp(S^p\times S^q)))$$

is an epi- and isomorphism for the same W as in Theorem H and with the same ranges, after replacing r by 1.

Furthermore, in Section 6.3, we provide a short discussion of how our methods can be applied to the case of certain stably parallelisable $(2n-1)$-connected $(4n+1)$-manifolds X and 2-connected W, extending stability results by Perlmutter [Per16b]. Similarly, we also briefly explain how to enhance work of Kupers [Kup15] on homeomorphisms of topological manifolds and automorphisms of piecewise linear manifolds.

Modules over braided monoidal categories. We close in Section 7 by explaining applicability of our results to discrete situations, such as groups or monoids, and by drawing a comparison to [RW17].

The classifying space $B\mathcal{M}$ of a graded module $\mathcal{M}$ over a braided monoidal category is a graded E_1-module over an E_2-algebra (see Lemma 7.2), so forms a suitable input for Theorem A and C. In Lemma 7.6, we identify the space of destabilisations $W_\bullet(A)$ of $A \in \mathcal{M}$ with a semi-simplicial set $W_\bullet^{\mathrm{RW}}(A)$ in the case of $\mathcal{M}$ being a groupoid satisfying an injectivity condition. This identification gives rise to a framework for homological stability for modules over braided monoidal categories, phrased entirely in terms of $\mathcal{M}$ and semi-simplicial sets instead of semi-simplicial spaces up to higher coherent homotopy (see Remark 7.8).

Using this, it can, for instance, be concluded that work of Hepworth on homological stability for Coxeter groups [Hep16] with constant coefficients implies their stability with respect to a large class of nontrivial coefficient systems without further effort, as well as stability of their commutator subgroups.

In the case of a braided monoidal groupoid acting on itself, the semi-simplicial sets $W_\bullet^{\mathrm{RW}}(A)$ were introduced by Randal-Williams–Wahl in [RW17] as part of their stability results for the automorphisms of a braided monoidal groupoid, which this work enhances in various ways. We generalise from braided monoidal groupoids to modules over such, remove all hypotheses on the categories they impose, improve the stability ranges in certain cases (see Remark 7.10), and enlarge the class of coefficients systems (see Remark 7.9). We refer to Section 7.3 for a more detailed comparison of our results in the discrete setting to [RW17] and also for an analysis of their assumptions on the braided monoidal groupoid.

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Contents

1. Preliminaries

This section is devoted to fix conventions and collect general techniques. We work in the category of compactly generated spaces, use Moore paths throughout, and denote the endpoint of a path μ by $\omega(\mu)$.

1.1. **Graded spaces and categories.** We denote by $(\bar{\mathbf{N}}, +)$ the discrete abelian monoid obtained by extending the non-negative integers $(\mathbf{N}_0, +)$ by an element ∞ satisfying $k + \infty = \infty$ for all $k \geq 0$.

A *graded space* is a space X together with a continuous map $g_X : X \longrightarrow \bar{\mathbf{N}}$. A *map of graded spaces* is a map that preserves the grading and a *map of degree k between graded spaces* for a number $k \geq 0$ is a map that increases the degree by k. The category of graded spaces is symmetric monoidal with the monoidal product of two graded spaces (X, g_X) and (Y, g_Y) given by $(X \times Y, g_X + g_Y)$. The subspace of elements of degree $n \in \bar{\mathbf{N}}$ is denoted by $X_n = g_X^{-1}(\{n\}) \subseteq X$. By restricting the grading, subspaces of graded spaces are implicitly considered as being graded. A graded space (X, g_X), or a map $(Y, g_Y) \longrightarrow (X, g_X)$ of graded spaces, is $\varphi(g_X)$*-connected in degrees* $\geq m$ for a function $\varphi\colon \bar{\mathbf{N}} \longrightarrow \mathbf{Q} \cup \{\infty\}$ satisfying $\phi(\infty) = \infty$ if X_n or $Y_n \longrightarrow Z_n$, respectively, is $\lfloor \varphi(n) \rfloor$-connected for all $m \leq n < \infty$ in the usual sense. Note that we do not require anything on X_∞ or $Y_\infty \longrightarrow X_\infty$.

A *graded set* X is a graded space that is discrete. A *graded category* C is a category internal to graded sets, i.e. a category C with a function $g_C\colon \operatorname{ob} C \longrightarrow \bar{\mathbf{N}}$ whose value on objects that are connected by morphisms is constant. This is equivalent to a grading on the classifying space BC. A *graded monoidal category* is a monoid internal to graded categories with the monoidal product $(C, g_C) \times (\mathcal{D}, g_{\mathcal{D}}) = (C \times \mathcal{D}, g_C + g_{\mathcal{D}})$, i.e. a monoidal category $(\mathcal{A}, \oplus, 0)$ together with a grading $g_{\mathcal{A}}$ on $\mathcal{A}$ that satisfies $g_{\mathcal{A}}(0) = 0$ and $g_{\mathcal{A}}(X \oplus Y) = g_{\mathcal{A}}(X) + g_{\mathcal{A}}(Y)$. A *graded right-module* $(\mathcal{M}, \oplus)$ over a graded monoidal category $(\mathcal{A}, \oplus, 0)$ is a graded category $\mathcal{M}$ together with a right-action of $(\mathcal{A}, \oplus, 0)$ on $\mathcal{D}$ internal to graded categories, i.e. a functor $\oplus\colon \mathcal{M} \times \mathcal{A} \longrightarrow \mathcal{M}$ which is unital and associative up to coherent isomorphisms, and satisfies $g_{\mathcal{M}}(A \oplus X) = g_{\mathcal{M}}(A) + g_{\mathcal{M}}(X)$.

1.2. **Homology with local coefficients.** We adopt the convention of [Whi78, Ch. VI]: for points x and y in a space X, a morphism in the fundamental groupoid $\Pi(X)$ from x to y is a homotopy class of paths from y to x, resulting in the fundamental group $\pi_1(X,x)$ being a subgroupoid of $\Pi(X)$. A *local system* on a pair of spaces (X,A) with $A \subseteq X$ is a functor F from the fundamental groupoid $\Pi(X)$ of X to the category of abelian groups. It is *constant* if it is constant as a functor. For a path-connected space X, local systems can equivalently be described as modules over $\pi_1(X,x)$, since the fundamental groupoid $\Pi(X)$ is equivalent to the one-object groupoid $\pi_1(X,x)$. Subspaces of spaces with local systems are implicitly equipped with the local system obtained by restriction along the inclusion. When we write (X,A) for a map $A \to X$ that is not necessarily an inclusion, we implicitly replace X by the mapping cylinder of $A \to X$. A *morphism* $(f;\eta)$ *between pairs with local systems* $(X,A;F)$ and $(Y,B;G)$ is a map of pairs $f\colon (X,A) \to (Y,B)$ with a natural transformation $\eta\colon F \to f^*G$ of functors on $\Pi(X)$. A *homotopy* between $(f_0;\eta_0)$ and $(f_1;\eta_1)$ from $(X,A;F)$ to $(Y,B;G)$ consists of a homotopy of pairs $H_t\colon (X,A) \to (Y,B)$ from f_0 and f_1 such that

$$\begin{array}{ccc} & \xrightarrow{\ \eta_1\ } & G(f_1(-)) \\ F(-) & & \big\downarrow{\scriptstyle G(H_t(-))} \\ & \xrightarrow[\ \eta_0\ ]{} & G(f_0(-)) \end{array}$$

commutes. Taking singular chains with coefficients in a local system provides a homotopy invariant functor $\mathrm{C}_*(-)$ from pairs with local systems to chain complexes. The homology $\mathrm{H}_*(X,A;F)$ of $\mathrm{C}_*(X,A;F)$ is the *homology of the pair* (X,A) *with coefficients in the local system* F. A grading on X results in an additional grading $\bigoplus_{n\in\bar{\mathbb{N}}} \mathrm{H}_*(X_n,A_n;F)$ on $\mathrm{H}_*(X,A;F)$. For a morphism $(X,A;F) \to (Y,B;G)$, the homology of the mapping cone of $\mathrm{C}_*(X,A;F) \to \mathrm{C}_*(Y,B;G)$ is denoted by $\mathrm{H}_*((Y,B;G),(X,A;F))$. If X and Y are graded and the underlying map $X \to Y$ is of degree k, then $\mathrm{H}_*((Y,B;G),(X,A;F))$ inherits an extra grading

$$\mathrm{H}_*\big((Y,B;G),(X,A;F)\big) = \textstyle\bigoplus_{n\in\bar{\mathbb{N}}} \mathrm{H}_*\big((Y_{n+k},B_{n+k};G),(X_n,A_n;F)\big).$$

We refer to [Whi78, Ch. VI] for more details on homology with local coefficients.

1.3. **Augmented semi-simplicial spaces.** Denoting by $[p]$ the ordered set $\{0,1,\ldots,p\}$, the *semi-simplicial category* is the category Δ_{inj} with objects $[0],[1],\ldots$ and order-preserving injections between them. A *semi-simplicial space* $X_\bullet$ is a space-valued functor on $\Delta_{\mathrm{inj}}^{\mathrm{op}}$, or equivalently, a collection of spaces X_p for $p \geq 0$, together with *face maps* $d_i\colon X_p \to X_{p-1}$ for $0 \leq i \leq p$ that satisfy the *face relations* $d_i d_j = d_{j-1} d_i$ for $i<j$. An *augmented semi-simplicial space* $X_\bullet \to X_{-1}$ is a semi-simplicial space $X_\bullet$ with maps $X_p \to X_{-1}$ for $p \geq 0$ that commute with the face maps. As for simplicial spaces, augmented semi-simplicial spaces $X_\bullet \to X_{-1}$ have a *geometric realisation*—a space over X_{-1}, denoted by $|X_\bullet| \to X_{-1}$ (see [ER17, Sect. 1.2]).

Given an augmented semi-simplicial space $X_\bullet \to X_{-1}$ and a local system F on X_{-1}, we obtain local systems on the spaces of p-simplices X_p and on the realisation $|X_\bullet|$ by pulling back F along the augmentation. Filtering $|X_\bullet|$ by skeleta induces a strongly convergent homologically graded spectral sequence

$$E^1_{p,q} \cong \mathrm{H}_q(X_p;F) \implies \mathrm{H}_{p+q+1}(X_{-1},|X_\bullet|;F), \tag{3}$$

defined for $q \geq 0$ and $p \geq -1$ (see [ER17, Sect. 1.4; MP15, Lem. 2.7]). The differential $d^1\colon \mathrm{H}_q(X_p;F) \to \mathrm{H}_q(X_{p-1};F)$ is the alternating sum $\sum_{i=0}^p (-1)^i (d_i;\mathrm{id})_*$ of the morphisms induced by the face maps for $p>0$, and induced by the augmentation for $p=0$. Given a morphism of augmented semi-simplicial spaces $(f_\bullet,f_{-1})\colon (X_\bullet \to X_{-1}) \to (Y_\bullet \to Y_{-1})$, local systems F on X_{-1} and G on Y_{-1}, and a morphism of local systems $F \to {f_{-1}}^*G$, we obtain a morphism of augmented semi-simplicial objects in spaces with local systems, resulting in a relative version of the spectral sequence (3),

$$E^1_{p,q} \cong \mathrm{H}_q\big((Y_p;G),(X_p;F)\big) \implies \mathrm{H}_{p+q+1}\big((Y_{-1},|Y_\bullet|;G),(X_{-1},|X_\bullet|;F)\big). \tag{4}$$

If X_{-1} is graded, all spaces X_p and $|X_\bullet|$ inherit a grading by pulling back $g_{X_{-1}}$ along the augmentation. This results in a third grading of the spectral sequence (3), but since the differentials preserve the additional grading, it is just a sum of spectral sequences, one for each $n \in \bar{\mathbb{N}}$. Analogously, if the map f_{-1} of $(f_\bullet,f_{-1})\colon (X_\bullet \to X_{-1}) \to (Y_\bullet \to Y_{-1})$ is a map of degree k for gradings on X_{-1} and Y_{-1}, the spectral sequence (4) splits as a sum, with the nth summand of the E_1-page being $E^1_{p,q,n} \cong \mathrm{H}_q((Y_{p,n+k};G),(X_{p,n};F))$.

1.4. **C-spaces and their rectification.** We set up an ad-hoc theory of spaces parametrised by a topologically enriched category, serving us as a convenient language in the body of this work.

We call an enriched space-valued functor $X_\bullet$ on a topologically enriched category C a *C-space*, and write X_C for its value at an object C. An *augmentation* $f_\bullet\colon X_\bullet \to X_{-1}$ of a C-space $X_\bullet$ over a space X_{-1} is a lift of $X_\bullet$ to a functor with values in the overcategory $\mathcal{Top}/X_{-1}$, and an *augmented C-space* is a C-space together with an augmentation. We denote the value of an augmented C-space $f_\bullet\colon X_\bullet \to X_{-1}$ at an object C by $f_C\colon X_C \to X_{-1}$. A *morphism of augmented C-spaces* is a natural transformation of functors $C \to \mathcal{Top}/X_{-1}$, and it is called a *weak equivalence* if it is a weak equivalence objectwise. A morphism between a C-space $X_\bullet$ augmented over X_{-1} and a C-space $Y_\bullet$ over Y_{-1} consists of a map $h\colon X_{-1} \to Y_{-1}$ and a morphism $h_\bullet\colon h_*(X_\bullet) \to Y_\bullet$ of C-spaces augmented over Y_{-1}, where $h_*(X_\bullet)$ denotes $X_\bullet$ considered augmented over Y_{-1} via h. Such a morphism is a *weak equivalence* if h is a weak equivalence of spaces and $h_\bullet$ is one of C-spaces over Y_{-1}. An augmented C-space $f_\bullet$ is *fibrant* if all maps f_C are Serre fibrations.

Example 1.1. For C being the opposite of the semi-simplicial category, the notion of a C-space agrees with the one of a semi-simplicial spaces (see Section 1.3). This example motivated our choice of notation.

Definition 1.2. The *fibrant replacement* of an augmented C-space $X_\bullet \to X_{-1}$ is the augmented C-space $X_\bullet^{\mathrm{fib}} \to X_{-1}$ obtained by applying the path-space construction objectwise,

$$X_C^{\mathrm{fib}} = \{(x,\mu) \in X_C \times \operatorname{Path} X_{-1} \mid \omega(\mu) = f_C(x)\},$$

considered as a space over X_{-1} by evaluating paths at zero. It is fibrant and admits a canonical weak equivalence $X_\bullet \to X_\bullet^{\mathrm{fib}}$ of augmented C-spaces, given by mapping $x \in X_C$ to $(x, \mathrm{const}_{f_C(x)}) \in X_C^{\mathrm{fib}}$.

The *fibre* $X_{x,\bullet}$ of an augmented C-space $f_\bullet\colon X_\bullet \to X_{-1}$ at $x \in X_{-1}$ is the C-space that assigns to an object C the fibre $X_{x,C} = f_C^{-1}(x)$. Its *homotopy fibre* $\operatorname{hofib}_x(X_\bullet)$ at x is the fibre of $X_\bullet^{\mathrm{fib}} \to X_{-1}$ at x. If $X_\bullet \to X_{-1}$ is fibrant, then the weak equivalence $X_\bullet \to X_\bullet^{\mathrm{fib}}$ induces a weak equivalence $X_{x,\bullet} \to \operatorname{hofib}_x(X_\bullet)$.

Definition 1.3. Let C be a small topologically enriched category.

(i) The *bar construction* $B(Y_\bullet, C, X_\bullet)$ of a pair of C-spaces $(X_\bullet, Y_\bullet)$, where $X_\bullet$ is co- and $Y_\bullet$ is contravariant, is the realisation of the semi-simplicial space $B_\blacksquare(Y_\bullet, C, X_\bullet)$ with p-simplices

$$\textstyle\coprod_{C_0,\ldots,C_p \in \mathrm{ob}\, C} X_{C_0} \times C(C_0,C_1) \times \ldots \times C(C_{p-1},C_p) \times Y_{C_p}.$$

The ith face map is induced by composing morphisms in $C(C_{i-1},C_i)$ and $C(C_i,C_{i+1})$ for $1 \le i \le p-1$, and by the evaluations $X_{C_0} \times C(C_0,C_1) \to X_{C_1}$ and $C(C_{p-1},C_p) \times Y_{C_p} \to X_{C_{p-1}}$ for $i = p-1$ and $i = p$, respectively. An augmentation $X_\bullet \to X_{-1}$ naturally induces a map $B(Y_\bullet, C, X_\bullet) \to X_{-1}$.

(ii) The *homotopy colimit*

$$\operatorname{hocolim}_C X_\bullet \longrightarrow X_{-1}$$

of an augmented C-space $X_\bullet \to X_{-1}$ is the bar construction $B(*, C, X_\bullet) \to X_{-1}$.

A C-space is *k-connected* for a number $k \geq 0$ if its homotopy colimit is so. If the base X_{-1} of an augmented C-space $X_\bullet \to X_{-1}$ is graded, then its values X_C and its homotopy colimit inherit gradings by pulling back $g_{X_{-1}}$ from X_{-1}. It is *graded $\varphi(g_{X_{-1}})$-connected in degrees $\geq m$* for $\varphi\colon \bar{\mathbf{N}} \to \mathbf{Q} \cup \{\infty\}$ if $\operatorname{hocolim}_C X_\bullet \to X_{-1}$ is. A functor between topologically enriched categories is a *weak equivalence* if it induces weak equivalences on morphism spaces and a bijection on the set of objects. Note that this notion of weak equivalence is slightly stronger than the usual one. With this choice, it is immediate to see that the map on bar constructions induced by a weak equivalence $(X_\bullet, C, Y_\bullet) \to (X'_\bullet, C', Y'_\bullet)$ of triples, defined in the appropriate sense, is a weak equivalence, since levelwise weak equivalences of semi-simplicial spaces realise to weak equivalences (see e.g. [ER17, Thm 2.2]). In particular, taking homotopy colimits turns weak equivalences of C-spaces augmented over X_{-1} into weak equivalences of spaces over X_{-1}.

Lemma 1.4. *Let $X_\bullet \to X_{-1}$ be an augmented C-space and $x \in X_{-1}$. The canonical map*

$$\operatorname{hocolim}_C(\operatorname{hofib}_x(X_\bullet \to X_{-1})) \to \operatorname{hofib}_x(\operatorname{hocolim}_C X_\bullet \to X_{-1})$$

is a weak equivalence.

Proof. We show that the map in consideration is even a homeomorphism, provided X_{-1} is a weak Hausdorff space. This implies the claim, since the two functors in comparison both preserve weak equivalences of augmented C-spaces and every augmented C-space $X_\bullet \to X_{-1}$ can be replaced, up to weak equivalence, by one over a weak Hausdorff space, for instance by pulling back the fibrant replacement of $X_\bullet \to X_{-1}$ along a CW-approximation of X_{-1}. We have $\mathrm{hocolim}_C(\mathrm{hofib}_x(X_\bullet \to X_{-1})) = |B_\blacksquare(*, C, \mathrm{hofib}_x(X_\bullet \to X_{-1}))|$ and $\mathrm{hofib}_x(\mathrm{hocolim}_C X_\bullet \to X_{-1}) = \mathrm{hofib}_x(|B_\blacksquare(*, C, X_\bullet \to X_{-1})|)$, so the statement follows from proving that both the bar construction $B_\blacksquare(*, C, -)$ as well as the geometric realisation $|-|$ commute with taking homotopy fibres $\mathrm{hofib}_x(-)$. Unwrapping the definitions of $B_\blacksquare(*, C, -)$ and $|-|$, these two claims are implied by the fact that the functor $\mathrm{hofib}_x(-)\colon \mathcal{T}op/X_{-1} \to \mathcal{T}op$ commutes with colimits and also with taking products $- \times Z$ with a fixed space Z. The latter is clear, and the former follows from the fact that the functor $\omega^*\colon \mathcal{T}op/X_{-1} \to \mathcal{T}op/(\mathrm{Path}_x X_{-1})$ given by pulling back the path fibration $\omega\colon \mathrm{Path}_x X_{-1} \to X_{-1}$ is a left adjoint [MS06, Prop. 2.1.3], so preserves colimits, together with the observation that the forgetful functor from $\mathcal{T}op/X_{-1}$ to $\mathcal{T}op$ is colimit-preserving as well. □

For an augmented C-space $X_\bullet \to X_{-1}$, the composition in C and the evaluation maps $X'_C \times C(C', C) \to X_C$ combine to augmentations $B_\bullet(C(\blacksquare, C), C, X_\blacksquare) \to X_C$ for each C in C, which realise to weak equivalences as they admit extra degeneracies by inserting the identity (see e.g. [ER17, Thm 2.2]). These equivalences are natural in C and compatible with the augmentation to X_{-1}, so assemble to a weak equivalence

$$B\big(C(\blacksquare, \bullet), C, X_\blacksquare\big) \to X_\bullet$$

of augmented C-spaces—the *bar resolution* of $X_\bullet \to X_{-1}$.

Lemma 1.5. *Let $p\colon C \to \mathcal{D}$ be a weak equivalence of topologically enriched categories. There is a functor*

$$p_*\colon (\mathcal{T}op/X_{-1})^C \longrightarrow (\mathcal{T}op/X_{-1})^{\mathcal{D}}$$

that fits into a zig-zag of natural transformations between endofunctors on $(\mathcal{T}op/X_{-1})^C$,

$$p^* p_* \longleftarrow \cdot \longrightarrow \mathrm{id}_{(\mathcal{T}op/X_{-1})^C},$$

where $p^*\colon (\mathcal{T}op/X_{-1})^{\mathcal{D}} \to (\mathcal{T}op/X_{-1})^C$ is given by precomposition with p. When evaluated at an augmented C-space, the zig-zag consists of weak equivalences of augmented C-spaces.

Proof. The value $p_* X_\bullet$ for $X_\bullet \in (\mathcal{T}op/X_{-1})^C$ is the homotopy left Kan-extension of $X_\bullet$ along p, mapping an object D in $\mathcal{D}$ to $B(\mathcal{D}(p(\blacksquare), D), C, X_\blacksquare)$. Its pullback $p^* p_* X_\bullet$ fits into a zig-zag of augmented C-spaces

$$p^* p_* X_\bullet = B\big(\mathcal{D}(p(\blacksquare), p(\bullet)), C, X_\blacksquare\big) \longleftarrow B\big(C(\blacksquare, \bullet), C, X_\blacksquare\big) \longrightarrow X_\bullet,$$

in which the left arrow is induced by p and the right one is the bar resolution of $X_\bullet$, so both are weak equivalences and compatible with the augmentation. As the zig-zag is natural in $X_\bullet$, the claim follows. □

Lemma 1.6. *The homotopy colimit of an augmented semi-simplicial space $X_\bullet \to X_{-1}$ and its geometric realisation are weakly equivalent as spaces over X_{-1}.*

Proof. The classifying space of the overcategory $\Delta_{\mathrm{inj}}/[p]$ is isomorphic to the pth topological standard simplex Δ^p, since the nerve of $\Delta_{\mathrm{inj}}/[p]$ is the barycentric subdivision of the pth simplicial standard simplex. This extends to an isomorphism $\Delta^\bullet \cong B(\Delta_{\mathrm{inj}}/\bullet)$ of co-semi-simplicial spaces from which [Rie14, Thm 6.6.1] implies that, given an augmented semi-simplicial space $X_\bullet \to X_{-1}$, the thin realisation (see [ER17, Sect. 1.2]) of $B_\bullet(*, \Delta_{\mathrm{inj}}^{\mathrm{op}}, X_\bullet)$, considered as a simplicial space, is homeomorphic over X_{-1} to the realisation of $X_\bullet$. But for augmented C-spaces $X_\bullet \to X_{-1}$ on a discrete category C, the fat and the thin geometric realisation of $B_\bullet(*, C, X)$ are weakly equivalent over X_{-1}, because $B_\bullet(*, C, X)$ is *good* in the sense of [Seg74, Prop. A.1]. □

1.5. **Semi-simplicial spaces up to higher coherent homotopy.** In the course of this work, a number of constructions that are key to the theory require choices of contractible ambiguity. To deal with such, we are led to consider objects that are as good as semi-simplicial spaces, but only in a homotopical sense. To model those, let us define an *(augmented) semi-simplicial space up to higher coherent homotopy* as an (augmented) $\widetilde{\Delta}_{\mathrm{inj}}$-space X, defined on any topologically enriched category $\widetilde{\Delta}_{\mathrm{inj}}$ that comes with a weak equivalence $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$. Roughly speaking, these are categories with the same objects as Δ_{inj} and a (weakly) contractible space of choices for all morphisms in Δ_{inj}. In particular, a $\widetilde{\Delta}_{\mathrm{inj}}$-space $X_\bullet$ includes spaces X_p for $p \geq 0$, together with face maps $\tilde{d}_i\colon X_p \to X_{p-1}$, unique up to homotopy. By precomposing with $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$, every semi-simplicial space is a $\widetilde{\Delta}_{\mathrm{inj}}$-space and in the light of Lemma 1.5, every $\widetilde{\Delta}_{\mathrm{inj}}$-space is equivalent to one arising in this way. By virtue of this rectification result and Lemma 1.6, all homotopy invariant constructions for semi-simplicial spaces carry over to $\widetilde{\Delta}_{\mathrm{inj}}$-spaces, so in particular, we have analogues of the spectral sequences (3) and (4), the differentials being the alternating sum $\sum_{i=0}^{p}(-1)^i(\tilde{d}_i)_*$ of morphisms induced by (weakly) contractible choices $\tilde{d}_i$ of face maps. A $\widetilde{\Delta}_{\mathrm{inj}}$-space $X_\bullet$ induces a simplicial set $\pi_0(X_\bullet)$ by taking path components, together with a morphism $X_\bullet \to \pi_0(X_\bullet)$ of $\widetilde{\Delta}_{\mathrm{inj}}$-spaces, which is a weak equivalence if and only if $X_\bullet$ is homotopy discrete, i.e. takes values in homotopy discrete spaces. To emphasise similarities and by abuse of notation justified by Lemma 1.6, we call the homotopy colimit of an augmented $\widetilde{\Delta}_{\mathrm{inj}}$-space $X_\bullet \to X_{-1}$ its *realisation*, and denote it by $|X_\bullet| \to X_{-1}$.

2. The canonical resolution of an E_1-module over an E_2-algebra

2.1. **E_1-modules over E_n-algebras and their fundamental groupoids.** We recall the notion of an E_1-module over an E_n-algebra and explain its relation to modules over monoidal categories.

By an *operad*, we mean a symmetric coloured operad in spaces (see e.g. [BM07, Sect. 1.1]), and an *algebra* over such is understood in the usual sense (see e.g. [BM07, Sect. 1.1]). For a subspace $X \subseteq \mathbf{R}^n$, we let $\mathcal{D}^k(X)$ be the space of tuples of k embeddings of the closed disc D^n into X that have disjoint interiors and are compositions of scalings and translations. Recall the one-coloured operad $\mathcal{D}^\bullet(D^n)$ of little n-discs [BV73; May72] with k-operations $\mathcal{D}^k(D^n)$ and operadic composition induced by composing embeddings.

Definition 2.1. Let $\mathcal{SC}_n$ be the coloured operad with colours $\mathfrak{m}$ and $\mathfrak{a}$ whose space of operations $\mathcal{SC}_n(\mathfrak{m}^k, \mathfrak{a}^l; \mathfrak{m})$ is empty for $k \neq 1$ and for $k = 1$ the space of pairs $(s, \phi) \in [0, \infty) \times \mathcal{D}^l(\mathbf{R}^n)$ such that $\phi \in \mathcal{D}^l((0, s) \times (-1, 1)^{n-1})$, allowing $(0, \emptyset) \in [0, \infty) \times \mathcal{D}^0(\mathbf{R}^n)$ as a valid element of $\mathcal{SC}_n(\mathfrak{m}^k, \mathfrak{a}^0; \mathfrak{m})$. The space $\mathcal{SC}_n(\mathfrak{m}^k, \mathfrak{a}^l, \mathfrak{a})$ is empty for $k \neq 0$ and equals $\mathcal{D}^l(D^n)$ otherwise. The composition restricted to the $\mathfrak{a}$-colour is given by the composition in $\mathcal{D}^\bullet(D^n)$ and the composition

$$\gamma\colon \mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m}) \times \left(\mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \times \mathcal{SC}_n(\mathfrak{a}^{i_1}; \mathfrak{a}) \times \ldots \times \mathcal{SC}_n(\mathfrak{a}^{i_l}; \mathfrak{a})\right) \longrightarrow \mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^{k+i}; \mathfrak{m})$$

for $i = \sum_j i_j$ by mapping an element $((s, \phi), ((s', \psi), (\varphi^1, \ldots, \varphi^l)))$ in the codomain to $(s' + s, (\psi, (\phi_1 \circ \varphi^1) + s', \ldots, (\phi_l \circ \varphi^l) + s')) \in \mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^{k+i}; \mathfrak{m})$, where $(- + s')$ denotes the translation by s' in the first coordinate. In words, it is defined by adding the parameters, putting the discs of $\mathcal{SC}_n(m, a^k; m)$ to the left of the ones of $\mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m})$, and composing the embeddings of discs of the $\mathcal{SC}_n(\mathfrak{a}^{i_j}; \mathfrak{a})$-factors with the ones of $\mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m})$ as in the operad of little n-discs. See Figure 1 for an illustration.

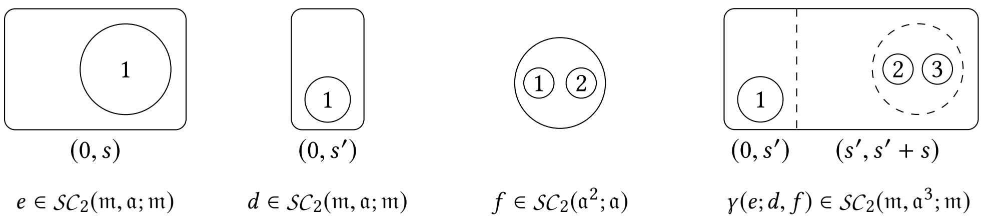


Figure 1. The operadic composition of $\mathcal{SC}_n$

The canonical embedding $\mathcal{D}^\bullet(D^n) \to \mathcal{D}^\bullet(D^{n+1})$ of little discs operads (see e.g. [Fre17, Sect. 4.1.5]), extends to an embedding of two-coloured operads $\mathcal{SC}_n \to \mathcal{SC}_{n+1}$ by taking products with $(-1, 1)$ from the right. Consequently, any algebra over $\mathcal{SC}_{n+1}$ is also one over $\mathcal{SC}_n$.

We call two coloured operads *weakly equivalent* if there is a zig-zag between them that consists of morphisms of operads that are weak homotopy equivalences on all spaces of operations.

Remark 2.2. The operad $\mathcal{SC}_n$ is weakly equivalent to a suboperad of the n-dimensional version of the Swiss-Cheese operad of [Vor99], motivating the notation.

Definition 2.3. An $E_{1,n}$*-operad* is an operad $\mathcal{O}$ that is weakly equivalent to $\mathcal{SC}_n$. A graded E_1*-module* $\mathcal{M}$ *over an* E_n*-algebra* $\mathcal{A}$ is an algebra $(\mathcal{M}, \mathcal{A})$ over an $E_{1,n}$-operad $\mathcal{O}$, considered as an operad in graded spaces, where $\mathcal{M}$ corresponds to the $\mathfrak{m}$- and $\mathcal{A}$ to the $\mathfrak{a}$-colour. That is, it consists of two graded spaces $(\mathcal{M}, g_{\mathcal{M}})$ and $(\mathcal{A}, g_{\mathcal{A}})$, together with multiplication maps for $l \geq 0$ of the form

$$\theta\colon \mathcal{O}(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m}) \times \mathcal{M} \times \mathcal{A}^l \longrightarrow \mathcal{M} \quad \text{and} \quad \theta\colon \mathcal{O}(\mathfrak{a}^l; \mathfrak{a}) \times \mathcal{A}^l \longrightarrow \mathcal{A},$$

which are graded, where $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m})$ and $\mathcal{O}(\mathfrak{a}^l; \mathfrak{a})$ are equipped with the grading that is constant at 0, i.e. the degree of a multiplication of points is the sum of their degrees. These structure maps are required to satisfy the usual associativity, unitality, and equivariance axioms for an algebra over an coloured operad.

The fundamental groupoid of an algebra over the little 2-discs operad has a braided monoidal groupoid structure; the multiplication is induced by the choice of a 2-operation [Fre17, Ch. 5–6]. Similarly, for a graded algebra $(\mathcal{M}, \mathcal{A})$ over an $E_{1,2}$-operad $\mathcal{O}$ and operations $c \in \mathcal{O}(\mathfrak{m}, \mathfrak{a}; \mathfrak{m})$ and $d \in \mathcal{O}(\mathfrak{a}^2; \mathfrak{a})$, the fundamental groupoid $\Pi(\mathcal{A})$ is a graded braided monoidal groupoid with multiplication induced by d, and $\Pi(\mathcal{M})$ becomes a graded right-module over $\Pi(\mathcal{A})$ with the action induced by c. In other words, the functor $\oplus\colon \Pi(\mathcal{M}) \times \Pi(\mathcal{A}) \to \Pi(\mathcal{M})$ induced by $\theta(c; -, -)$ is associative, unital up to coherent natural isomorphisms, and compatible with the grading on $\Pi(\mathcal{M})$ and $\Pi(\mathcal{A})$ induced by the grading on $\mathcal{M}$ and $\mathcal{A}$.

Remark 2.4. Since the path components of a space coincide with the path components of its fundamental groupoid in the categorical sense, a grading on an E_1-module over an E_n-algebra is equivalent to a grading of the induced right-module $(\Pi(\mathcal{M}), \oplus)$ over the braided monoidal groupoid $(\Pi(\mathcal{A}), \oplus, b, 0)$.

2.2. **The canonical resolution.** Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra $\mathcal{A}$ with underlying $E_{1,2}$-operad $\mathcal{O}$ and structure maps θ. We call a point $X \in \mathcal{A}$ of degree 1 a *stabilising object* for $\mathcal{M}$, and define the *stabilisation map* with respect to a stabilising point X,

$$s\colon \mathcal{M} \longrightarrow \mathcal{M},$$

as the multiplication $\theta(c; -, X)$ by X, using an operation $c \in \mathcal{O}(\mathfrak{m}, \mathfrak{a}, \mathfrak{m})$, which we fix once and for all. As X has degree 1, so does the stabilisation map, which hence restricts to maps $s\colon \mathcal{M}_n \to \mathcal{M}_{n+1}$ between the subspaces of consecutive degrees for all $n \geq 0$. It will be convenient to denote the stabilisation map also by $(- \oplus X)\colon \mathcal{M} \to \mathcal{M}$ and we use the two notations interchangeably.

Remark 2.5. We chose to restrict to stabilising objects of degree 1 to simplify the exposition. However, by keeping track of the gradings, the developed theory generalises to stabilising objects of arbitrary degree.

In the following, we assign to a graded E_1-module $\mathcal{M}$ over an E_2-algebra with stabilising object X an augmented semi-simplicial space $R_\bullet(\mathcal{M}) \to \mathcal{M}$ up to higher coherent homotopy, called the *canonical resolution*. It will be defined as an augmented $\widetilde{\Delta}_{\mathrm{inj}}$-space for a topologically enriched category $\widetilde{\Delta}_{\mathrm{inj}}$ weakly equivalent to the semi-simplicial category, constructed from the underlying $E_{1,2}$-operad $\mathcal{O}$. We begin by recalling the braided analogue of the category of finite sets and injections, as introduced in [RW17].

Definition 2.6. Define the category $U\mathcal{B}$ with objects $[0], [1], \ldots$ as in Δ_{inj}, no morphisms from $[q]$ to $[p]$ for $q > p$ and $U\mathcal{B}([q],[p])$ for $q \leq p$ given by the cosets B_{p+1}/B_{p-q}, where B_i denotes the braid group on i strands and B_{p-q} acts on B_{p+1} from the right as the first $(p-q)$ strands. The composition is defined as

$$\begin{aligned} U\mathcal{B}([l],[q]) \times U\mathcal{B}([q],[p]) &\longrightarrow U\mathcal{B}([l],[p]) \\ [b],[b'] &\longmapsto [b'(1^{p-q} \oplus b)], \end{aligned}$$

where $1^{p-q} \oplus b$ is the braid obtained by inserting $(p-q)$ trivial strands to the left of b (see Figure 2).

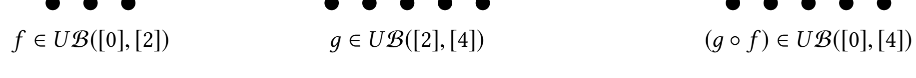


Figure 2. The categorical composition of $U\mathcal{B}$

The category $U\mathcal{B}$ admits a canonical functor to the category $\mathcal{FI}$ of finite sets and injections by sending a class in B_{p+1}/B_{p-q} to the injection obtained by following the last $(q+1)$ strands of a representing braid. Visualising $U\mathcal{B}$ as indicated by Figure 2, two braids represent the same morphism if and only if they differ by a braid of the ∘-ends. Following the braids of the upper •-ends to the lower ends gives the induced injections. This functor admits a section on the subcategory $\Delta_{\mathrm{inj}} \subseteq \mathcal{FI}$, as shown by the following lemma.

Lemma 2.7. *There is a unique functor $\Delta_{\mathrm{inj}} \longrightarrow U\mathcal{B}$ that maps the face map $d_i \in \Delta_{\mathrm{inj}}([p-1],[p])$ to*

$$[b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}] \in U\mathcal{B}([p-1],[p]),$$

where $\coprod_{n\geq 0} B_n$ is considered as the free braided monoidal category on one object X. The composition of this functor with the functor $U\mathcal{B} \longrightarrow \mathcal{FI}$ described above agrees with the inclusion $\Delta_{\mathrm{inj}} \subseteq \mathcal{FI}$.

Proof. To prove the first part, it is sufficient to check the face relations

$$[b^{-1}_{X^{\oplus j},X} \oplus X^{\oplus p+1-j}] \circ [b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}] = [b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p+1-i}] \circ [b^{-1}_{X^{\oplus j-1},X} \oplus X^{\oplus p-j+1}]$$

for $i < j$ in $U\mathcal{B}([p-1,p+1])$. The left hand side agrees with the class of the braid

$$(b^{-1}_{X^{\oplus j},X} \oplus X^{\oplus p+1-j})(X \oplus b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}),$$

which, by applying braid relations, can be seen to agree with the braid

$$(b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p+1-i})(X \oplus b^{-1}_{X^{\oplus j-1},X} \oplus X^{\oplus p-j+1})(b^{-1}_{X,X} \oplus X^{\oplus p}),$$

whose class in $U\mathcal{B}([p-1,p+1]) = B_{p+2}/B_2$ coincides with the right hand side of the claimed equation. The proof is concluded by observing that the two functors $\Delta_{\mathrm{inj}} \longrightarrow \mathcal{FI}$ in question agree on the face maps by construction, and thus on all of Δ_{inj}. □

Remark 2.8. In the language of [RW17], the category $U\mathcal{B}$ is the *free pre-braided monoidal category on one object* [RW17, Sect. 1.2]. Unwinding the definitions, their semi-simplicial set $W_n(A,X)$ associated to objects A and X of a pre-braided monoidal category $\mathcal{D}$ (see [RW17, Sect. 2]) agrees with the composition

$$\Delta^{\mathrm{op}}_{\mathrm{inj}} \longrightarrow U\mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{S}ets,$$

in which the first arrow is the described section, the second is induced by X, and the third is $\mathcal{D}(-, A\oplus X^{\oplus n})$.

In the following, we introduce topological analogues of $U\mathcal{B}$ and Δ_{inj} for any $E_{1,2}$-operad $\mathcal{O}$. To that end, we denote by $\mathcal{O}(k)$ the space obtained from $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m})$ by quotienting out the action of the symmetric group Σ_k on the $\mathfrak{a}$-inputs. To simplify the construction, we assume that the quotient maps $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \to \mathcal{O}(k)$ are covering spaces, although this is not strictly necessary (see Remark 2.22). As the operadic composition γ on $\mathcal{O}$ is equivariant, it induces composition maps $\gamma(-;-,1^k_{\mathfrak{a}})\colon \mathcal{O}(k)\times\mathcal{O}(l) \to \mathcal{O}(k+l)$. The fixed operation $c \in \mathcal{O}(1)$, used to define the stabilisation map, yields iterated operations $c_k \in \mathcal{O}(k)$ by setting c_0 as the unit $1_{\mathfrak{m}}$ and c_{k+1} inductively as $\gamma(c; c_k, 1_{\mathfrak{a}})$. As a last preparotary step before defining the category $U\mathcal{B}$, we recall that we denote the endpoint of a Moore path μ by $\omega(\mu)$.

Definition 2.9. Define a topologically enriched category $U\mathcal{O} = U(\mathcal{O}, c)$ with objects $[0],[1],\ldots$ and

$$U\mathcal{O}([q],[p]) = \{(d,\mu) \in \mathcal{O}(p-q) \times \mathrm{Path}_{c_{p+1}} \mathcal{O}(p+1) \mid \omega(\mu) = \gamma(c_{q+1}; d, 1^{q+1}_{\mathfrak{a}})\},$$

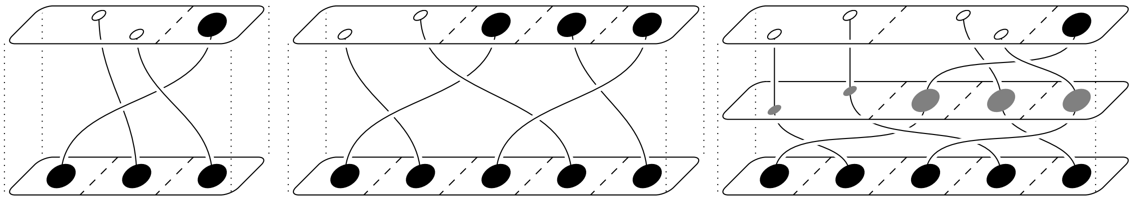
where $\mathrm{Path}_{c_{p+1}} \mathcal{O}(p+1)$ is the space of Moore paths in $\mathcal{O}(p+1)$ starting at c_{p+1}. The composition is

$$\begin{array}{ccc} U\mathcal{O}([l],[q]) \times U\mathcal{O}([q],[p]) & \longrightarrow & U\mathcal{O}([l],[p]) \\ \big((e,\zeta),(d,\mu)\big) & \longmapsto & \big(\gamma(e; d, 1^{q-l}_{\mathfrak{a}}), \mu \cdot \gamma(\zeta; d, 1^{q+1}_{\mathfrak{a}})\big), \end{array}$$

as visualised by Figure 3. Since we are using Moore paths, associativity and unitality follow from the respective properties of the operadic composition.

The construction $U(-)$ is functorial in $(\mathcal{O}, c)$ and preserves weak equivalences, since $U\mathcal{O}([q],[p])$ agrees with the homotopy fibre at c_{p+1} of the map $\gamma(c_{q+1}; -, 1^{q+1}_{\mathfrak{a}})\colon \mathcal{O}(p-q) \to \mathcal{O}(p+1)$.

Remark 2.10. Using Quillen's bracket-construction $\langle -,-\rangle$ for modules over monoidal categories (see [Gra76, p. 219]), the category $U\mathcal{B}$ is given by $\langle \mathcal{B},\mathcal{B}\rangle$, where $\mathcal{B} = \coprod_{n\geq 0} B_n$ is the free braided monoidal category acting on itself. Similarly, $U\mathcal{O}$ can be obtained via an analogue of Quillen's construction for monoidal categories internal to spaces, applied to the path-category of the monoid $\coprod_{n\geq 0} \mathcal{O}(n)$.



$(e, \zeta) \in UO([0], [2])$ $(d, \mu) \in UO([2], [4])$ $(d, \mu)(e, \zeta) \in UO([0], [4])$

FIGURE 3. The categorical composition of UO

Lemma 2.11. *The category UO is homotopy discrete and satisfies $\pi_0(UO) \cong U\mathcal{B}$.*

Before turning to the proof of Lemma 2.11, we suggest the reader to compare Figure 2 with Figure 3.

Proof. As $U(-)$ preserves weak equivalences, it suffices to prove the claim for $O = \mathcal{SC}_2$. Mapping embeddings of discs to their centre yields a homotopy equivalence from the space of operations $\mathcal{SC}_2(n)$ to the unordered configuration space $C_n(\mathbf{R}^2)$ of the plane, which is an Eilenberg–MacLane space $K(B_n, 1)$ for the braid group B_n. On fundamental groups, the map $\gamma(c_{q+1}; -, 1_{\mathfrak{a}}^{q+1})\colon O(p-q) \to O(p+1)$ is injective, since it is given by including B_{p-q} in B_{p+1} as the first $(p-q)$ strands. From this, one concludes that its homotopy fibre $\operatorname{hofib}_{c_{p+1}}(\gamma(c_{q+1}; -, 1_{\mathfrak{a}}^{q+1})) = UO([q], [p])$ is homotopy discrete with path components B_{p+1}/B_{p-q} and that, via this equivalence, the composition coincides with that of $U\mathcal{B}$ proving the claim. □

Equipped with Lemma 2.11, we fix an isomorphism $\pi_0(UO) \cong U\mathcal{B}$ once and for all, which we use, for instance, to identify $\pi_1(O(p+1), c_{p+1}) \cong \pi_0(UO([p], [p]))$ with the braid group B_{p+1}.

Definition 2.12. The *thickening of the semisimplicial category* associated to an $E_{1,2}$-operad O is the subcategory $\widetilde{\Delta}_{\text{inj}} \subseteq UO$ obtained by restricting UO to the path components hit by the section $\Delta_{\text{inj}} \to U\mathcal{B} \cong \pi_0(UO)$ of Lemma 2.7. It comes with a weak equivalence to Δ_{inj}, induced by the functor $UO \to \mathcal{FI}$.

Before proceeding to the central definitions of this section, we remind the reader of the theory of augmented C-spaces for a topologically enriched category C, set up in Section 1.4.

Definition 2.13. Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with structure maps θ and stabilising object X. Define the contravariant UO-space $B_\bullet(\mathcal{M})$ by sending $[p]$ to the path-space construction of s^{p+1},

$$B_p(\mathcal{M}) = \{(A, \zeta) \in \mathcal{M} \times \operatorname{Path} \mathcal{M} \mid \omega(\zeta) = s^{p+1}(A)\},$$

and by

$$\begin{array}{rcl} UO([q], [p]) \times B_p(\mathcal{M}) & \longrightarrow & B_q(\mathcal{M}) \\ \big((d, \mu), (A, \zeta)\big) & \longmapsto & \big(\theta(d; A, X^{p-q}), \zeta \cdot \theta(\mu; A, X^{p+1})\big). \end{array}$$

Functoriality follows from the associativity of the module-structure θ and the composition of Moore paths. Evaluating paths at zero defines an augmentation $B_\bullet(\mathcal{M}) \to \mathcal{M}$, which is a levelwise fibration.

Definition 2.14. Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilising object X.

(i) The *canonical resolution* of $\mathcal{M}$ is the fibrant augmented $\widetilde{\Delta}_{\text{inj}}$-space

$$R_\bullet(\mathcal{M}) \longrightarrow \mathcal{M}$$

obtained by restricting the augmented UO-space $B_\bullet(\mathcal{M})$ to the semi-simplicial thickening $\widetilde{\Delta}_{\text{inj}} \subseteq UO$.

(ii) The *space of destabilisations* of a point $A \in \mathcal{M}$ is the $\widetilde{\Delta}_{\text{inj}}$-space $W_\bullet(A)$ defined as the fibre of the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ at A.

Unwrapping the definition, the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is an augmented semi-simplicial space up to higher coherent homotopy with p-simplices

$$R_p(\mathcal{M}) = \{(A, \zeta) \in \mathcal{M} \times \operatorname{Path} \mathcal{M} \mid \omega(\zeta) = s^{p+1}(A)\},$$

augmented over $\mathcal{M}$ by evaluating paths at zero. There is a contractible space of ith face maps, but the following lemma provides a particularly convenient one after choosing a loop $\mu_i \in \Omega_{c_{p+1}} O(p+1)$ corresponding to the braid $b^{-1}_{X^{\oplus i}, X} \oplus X^{\oplus p-i}$ via the fixed isomorphism $B_{p+1} \cong \pi_1(O(p+1), c_{p+1})$.

Lemma 2.15. *The morphism $(c, \mu_i) \in UO([p-1],[p])$ lies in the path component of the image of the ith face map $d_i \in \Delta_{\mathrm{inj}}([p-1],[p])$ in $U\mathcal{B}([p-1],[p]) \cong \pi_0 UO([p-1],[p])$ via the section of Lemma 2.7, so the map*

$$\tilde{d}_i \colon \begin{array}{ccc} R_p(\mathcal{M}) & \longrightarrow & R_{p-1}(\mathcal{M}) \\ (A,\zeta) & \longmapsto & \big(s(A), \zeta \cdot \theta(\mu_i; A, X^{p+1})\big) \end{array}$$

is an ith face map of the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$.

Proof. The choice of μ_i ensures that, via the isomorphism $\pi_0(UO([p-1],[p])) \cong U\mathcal{B}([p-1],[p])$, the element (c, μ_i) is in the component of the class $[b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}]$ in $U\mathcal{B}([p-1],[p])$. This is exactly the image of $d_i \in \Delta_{\mathrm{inj}}([p-1],[p])$ in $U\mathcal{B}([p-1],[p])$, as claimed. □

Remark 2.16. We borrowed the term *space of destabilisations* from [RW17], where it stands for certain semi-simplicial sets $W_n(A, X)$ associated to a braided monoidal groupoid. In Section 7.3, it is explained that these semi-simplicial sets are special cases of the spaces of destabilisations in our sense.

Remark 2.17. As $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is fibrant, its fibre $W_\bullet(A)$ is equivalent to its homotopy fibre $\mathrm{hofib}_A(R_\bullet(\mathcal{M}))$, so by virtue of Lemma 1.4, the homotopy fibre at A of the realisation $|R_\bullet(\mathcal{M})| \to \mathcal{M}$ is equivalent to $|W_\bullet(A)|$. In particular, the canonical resolution of $\mathcal{M}$ is graded $\varphi(g_\mathcal{M})$-connected in degree $\geq m$ for a function $\varphi\colon \bar{\mathbb{N}} \to \mathbb{Q} \cup \{\infty\}$ satisfying $\phi(\infty) = \infty$ if and only if the spaces of destabilisations $W_\bullet(A)$ are $(\lfloor \varphi(g_\mathcal{M}(A)) \rfloor - 1)$-connected for all points $A \in \mathcal{M}$ with finite degree $g_\mathcal{M}(A) \geq m$. As points in the same component have equivalent homotopy fibres, it is sufficient to check one point in each component.

Example 2.18. Recall the *free E_2-algebra on a point* $O^\mathfrak{a} = \coprod_{n\geq 0} O(\mathfrak{a}^n; \mathfrak{a})/\Sigma_n$, graded in the evident way, with the *free E_1-module on a point* $O^\mathfrak{m} = \coprod_{n\geq 0} O(\mathfrak{m}; \mathfrak{a}^n; \mathfrak{m})/\Sigma_n$ as a graded E_1-module over it. Choosing the unit $1_\mathfrak{a} \in O(\mathfrak{a}; \mathfrak{a})$ as the stabilising object, the space of destabilisations $W_\bullet(c_{p+1})$ is the $\widetilde{\Delta}_{\mathrm{inj}}$-space obtained by restricting the UO-space $UO(\bullet, [p])$ to $\widetilde{\Delta}_{\mathrm{inj}}$. As the category UO is homotopy discrete with $\pi_0(UO) \cong U\mathcal{B}$ by Lemma 2.11, the $\widetilde{\Delta}_{\mathrm{inj}}$-space $W_\bullet(c_{p+1})$ is equivalent to the semi-simplicial set given as the composition of the section $\Delta^{\mathrm{op}}_{\mathrm{inj}} \to U\mathcal{B}^{\mathrm{op}}$ of Lemma 2.7 with $U\mathcal{B}(\bullet, [p])$. Using [HV17, Prop. 3.2], the realisation of this semi-simplicial set can be seen to be contractible, but we do not go into details, since the consequences of Theorem A and C regarding the twisted homological stability of $K(B_n, 1) \simeq O(\mathfrak{m}; \mathfrak{a}^n; \mathfrak{m})/\Sigma_n$ correspond to the case $M = D^2$ and $\pi = \mathrm{id}$ of Theorem D, which is proved in Section 5.

Remark 2.19. The choice of a stabilising object $X \in \mathcal{A}$ for a graded E_1-module $\mathcal{M}$ over an E_2-algebra $\mathcal{A}$ induces a graded E_1-module structure on $\mathcal{M}$ over $O^\mathfrak{a}$. The two canonical resolutions of $\mathcal{M}$ when considered as an module over $\mathcal{A}$ or over $O^\mathfrak{a}$ are identical. In fact, all our constructions and results solely depend on the induced module structure of $\mathcal{M}$ over $O^\mathfrak{a}$ and are in that sense independent of $\mathcal{A}$.

Remark 2.20. Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilising object X and consider $\mathcal{M}$ as a graded E_1-module over $O^\mathfrak{a}$ (see Remark 2.19). For a union of path components $\mathcal{M}' \subseteq \mathcal{M}$ that is closed under the multiplication by X, we define a new grading on $\mathcal{M}$ as an E_1-module over $O^\mathfrak{a}$ by modifying the original grading on $\mathcal{M}'$ by assigning the complement of $\mathcal{M}'$ degree ∞ and leaving the grading on $\mathcal{M}'$ unchanged. We call $\mathcal{M}$ with this new grading the *localisation at $\mathcal{M}'$*. An example for such a subspace $\mathcal{M}'$ is given by the *objects stably isomorphic to an object $A \in \mathcal{M}$* by which we mean the union of the path components of objects B for which $B \oplus X^{\oplus n}$ is in the component of $A \oplus X^{\oplus m}$ for some $n, m \geq 0$.

Example 2.21. Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra $\mathcal{A}$ and let G be a group acting on $\mathcal{M}$, preserving the grading. If the actions of $\mathcal{A}$ and G on $\mathcal{M}$ commute, then the Borel construction $EG \times_G \mathcal{M}$ inherits a graded E_1-module structure. The choice of a point in EG induces a morphism

$$\begin{array}{ccc} R_\bullet(\mathcal{M}) & \longrightarrow & R_\bullet(EG \times_G \mathcal{M}) \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & EG \times_G \mathcal{M} \end{array}$$

of augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces, which induces weak equivalences on homotopy fibres. An application of Lemma 1.4 implies that the respective canonical resolutions have the same connectivity.

Remark 2.22. Some constructions of this section work in greater generality. The category UO and the augmented UO-space $B_\bullet(\mathcal{M})$ can be defined for any coloured operad. UO then still admits a functor to

$\mathcal{FI}$, but might not be homotopy discrete nor admit a section on $\Delta_{\mathrm{inj}} \subseteq \mathcal{FI}$. The point-set assumption on the action of Σ_k on $O(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m})$ can be avoided by constructing UO using $O(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m})$ instead of $O(k)$, which involves taking care of permutations corresponding to preimages of the quotient map $O(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \to O(k)$.

2.3. **The stable genus.** We extend the notion of the *stable genus* of a manifold, as introduced in [GR18], to our context, providing us with a general way of grading modules over braided monoidal categories and, by Remark 2.4, also of grading E_1-modules over E_2-algebras.

Let $(\mathcal{M}, \oplus)$ be a right-module over a braided monoidal category $(\mathcal{A}, \oplus, b, 0)$. Recall the free braided monoidal category on one object $\mathcal{B} = \coprod_{n\geq 0} B_n$, consisting of the the braid groups B_n. A choice of an object X in $\mathcal{A}$ induces a functor $\mathcal{B} \to \mathcal{A}$ and hence a right-module structure on $\mathcal{M}$ over $\mathcal{B}$. With respect to this module structure, a grading of $\mathcal{M}$ that is compatible with the canonical grading on $\mathcal{B}$ is equivalent to a grading $g_{\mathcal{M}}$ on $\mathcal{M}$ as a category such that $g_{\mathcal{M}}(A \oplus X) = g_{\mathcal{M}}(A) + 1$ holds for all objetcs A in $\mathcal{M}$.

Definition 2.23. Let X be an object of $\mathcal{A}$ and A an object of $\mathcal{M}$.

(i) The *X-genus* of A is defined as

$$g^X(A) = \sup\{k \geq 0 \mid \text{there exists an object } B \text{ in } \mathcal{M} \text{ with } B \oplus X^{\oplus k} \cong A\} \in \bar{\mathbb{N}}.$$

(ii) The *stable X-genus* of A is defined as

$$\bar{g}^X(A) = \sup\{g^X(A \oplus X^{\oplus k}) - k \mid k \geq 0\} \in \bar{\mathbb{N}}.$$

As $\bar{g}^X(A \oplus X) = \bar{g}^X(A) + 1$ holds by definition, the stable X-genus provides a grading of $\mathcal{M}$ when considered as a module over $\mathcal{B}$ via X. This stands in contrast with the (unstable) X-genus, which does in general not define a grading, because the inequality $g^X(A) + 1 \leq g^X(A \oplus X)$ might be strict. For an E_1-module $\mathcal{M}$ over an E_2-algebra $\mathcal{A}$, the choice of a point $X \in \mathcal{A}$ induces an E_1-module structure on $\mathcal{M}$ over the free E_2-algebra on a point $O^{\mathfrak{a}}$ (see Remark 2.19). After taking fundamental groupoids, this results in the module structure of $\Pi(\mathcal{M})$ over $\mathcal{B}$ discussed above, so the stable X-genus provides a grading for $\mathcal{M}$ as an E_1-module over $O^{\mathfrak{a}}$.

Remark 2.24. If the connectivity assumption of Theorem A is satisfied for an E_1-module $\mathcal{M}$, graded with the stable X-genus, then the cancellation result Corollary B implies $g^X(A \oplus X) = g^X(A) + 1$ for objects A of positive stable genus, which, in turn, implies that for such A, the genus and the stable genus coincide.

3. Stability with constant and abelian coefficients

Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilising object X and structure maps θ. We prove Theorem A via a spectral sequence obtained from the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$. All spaces $R_p(\mathcal{M})$ and $|R_\bullet(\mathcal{M})|$ are considered graded by pulling back the grading from $\mathcal{M}$ along the augmentation.

3.1. **The spectral sequence.** Given a local system L on $\mathcal{M}$, the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ (see Section 2.2) gives rise to a tri-graded spectral sequence

$$E^1_{p,q,n} \cong \begin{cases} \mathrm{H}_q(R_p(\mathcal{M})_n; L) & \text{if } p \geq 0 \\ \mathrm{H}_q(\mathcal{M}_n; L) & \text{if } p = -1 \end{cases} \implies \mathrm{H}_{p+q+1}(\mathcal{M}_n, |R_\bullet(\mathcal{M})|_n; L), \tag{5}$$

with differential $d^1\colon E^1_{p,q,n} \to E^1_{p-1,q,n}$, induced by the augmentation for $p = 0$ and the alternating sum $\sum_{i=0}^p (-1)^i (\tilde{d}_i; \mathrm{id})_*$ for $p > 0$, where $\tilde{d}_i$ is any choice of ith face map of $R_\bullet(\mathcal{M})$ (see Sections 1.3 and 1.5). As the differentials do not change the n-grading, it is a sum of spectral sequences, one for each $n \in \bar{\mathbb{N}}$. To identify the E^1-page in terms of the stabilisation $s\colon \mathcal{M} \to \mathcal{M}$, recall from Section 2.1 that the fundamental groupoid $(\Pi(\mathcal{M}), \oplus)$ is a graded module over the graded braided monoidal groupoid $(\Pi(\mathcal{A}), \oplus, b, 0)$.

Lemma 3.1. *We have $E^1_{p,q,n+1} \cong \mathrm{H}_q(\mathcal{M}_{n-p}; (s^{p+1})^* L)$ and $d^1\colon E^1_{p,q,n+1} \to E^1_{p-1,q,n+1}$ identifies with*

$$\textstyle\sum_{i=0}^p (-1)^i (s; \eta_i)_*\colon \mathrm{H}_q\left(\mathcal{M}_{n-p}; (s^{p+1})^* L\right) \to \mathrm{H}_q\left(\mathcal{M}_{n-p+1}; (s^p)^* L\right),$$

where η_i denotes the natural transformation

$$L(- \oplus b_{X^{\oplus i}, X} \oplus X^{\oplus p-i})\colon L(- \oplus X^{\oplus p+1}) \longrightarrow L(- \oplus X^{\oplus p+1}).$$

In particular, d^1 corresponds for $p = 0$ to the stabilisation $(s; \mathrm{id})_*\colon \mathrm{H}_q(\mathcal{M}_n; s^* L) \to \mathrm{H}_q(\mathcal{M}_{n+1}; L)$. Thus, if L is constant, d^1 identifies with $s_*\colon \mathrm{H}_q(\mathcal{M}_{n-p}; L) \to \mathrm{H}_q(\mathcal{M}_{n-p+1}; L)$ for p even and vanishes for p odd.

Proof. Using the choice of face maps $\tilde{d}_i\colon R_p(\mathcal{M})_{n+1} \to R_{p-1}(\mathcal{M})_{n+1}$ of Lemma 2.15, we consider the square

$$\begin{array}{ccc} (\mathcal{M}_{n-p};(s^{p+1})^*L) & \xrightarrow{(\iota_p;\mathrm{id})} & (R_p(\mathcal{M})_{n+1};L) \\ {\scriptstyle (s;\eta_i)}\downarrow & & \downarrow{\scriptstyle (\tilde{d}_i;\mathrm{id})} \\ (\mathcal{M}_{n-p+1};(s^p)^*L) & \xrightarrow{(\iota_{p-1};\mathrm{id})} & (R_{p-1}(\mathcal{M})_{n+1};L), \end{array}$$

where ι_q denotes the canonical equivalence $\mathcal{M}_{n-q} \to R_q(\mathcal{M})_{n+1}$, mapping A to $(A, \mathrm{const}_{s^{q+1}(A)})$. A point $A \in \mathcal{M}_{n-p}$ is mapped by the two compositions in the square to $(s(A), \theta(\mu_i; A, X^{p+1}))$ and $(s(A), \mathrm{const}_{s^{p+1}(A)})$, respectively, which are connected by a preferred homotopy following the path μ_i chosen in Lemma 2.15 to its endpoint. The commutativity of the triangle ensuring that this homotopy extends to one of spaces with local systems (see Section 1.2) is equivalent to $L(\theta(\mu_i; -, X^{p+1}))\eta_i = \mathrm{id}$. But, by the choice of μ_i, the path $\theta(\mu_i; -, X^{p+1})$ corresponds to the braid $- \oplus b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}$, so the required relation holds and the square commutes up to homotopy. Taking vertical mapping cones and homology results in the claimed identification. If L is constant, the η_i coincide for all i, so the terms in the alternating sum cancel out. □

Lemma 3.2. *If the local system L is abelian, then the following compositions are homotopic for all $0 \le i \le p$*

$$(\mathcal{M};(s^{p+2})^*L) \xrightarrow{(s;\mathrm{id})} (\mathcal{M};(s^{p+1})^*L) \xrightarrow{(s;\eta_i)} (\mathcal{M};(s^p)^*L).$$

The proof of Lemma 3.2 uses a self-homotopy of $s^2\colon \mathcal{M} \to \mathcal{M}$ which is crucial for various other arguments. Using the notation of Section 2.2, it is given by

$$\begin{array}{rcl} [0,1] \times \mathcal{M} & \to & \mathcal{M} \\ (t, A) & \mapsto & \theta(\mu(t); A, X^2), \end{array} \tag{6}$$

where μ is a choice of loop of length 1, based at $c_2 \in \mathcal{O}(2)$, such that $[(1_{\mathfrak{m}}, \mu)] \in \pi_0(U\mathcal{O}([1],[1]))$ corresponds to the class $[b^{-1}_{X,X}] \in U\mathcal{B}([1],[1])$ via the isomorphism $\pi_0(U\mathcal{O}) \cong U\mathcal{B}$ fixed in Section 2.2. Since μ is unique up to homotopy, this describes the homotopy of s^2 uniquely up to homotopy of homotopies.

Proof of Lemma 3.2. By the recollection of Section 1.2, the selfhomotopy (6) of s^2 extends to a homotopy of maps of spaces with local systems between the ith and $(i+1)$st composition in question if the triangle

$$\begin{array}{ccc} L(-\oplus X^{\oplus p+2}) & \xrightarrow{L(-\oplus X \oplus b_{X^{\oplus i+1},X} \oplus X^{\oplus p-i-1})} & L(-\oplus X^{\oplus p+2}) \\ & \searrow^{L(-\oplus X \oplus b_{X^{\oplus i},X} \oplus X^{\oplus p-i})} & \downarrow{\scriptstyle L(-\oplus b^{-1}_{X,X} \oplus X^{\oplus p})} \\ & & L(-\oplus X^{\oplus p+2}) \end{array}$$

commutes. The braid relations give $(-\oplus X \oplus b_{X^{\oplus i},X} \oplus X^{\oplus p-i}) = (-\oplus b^{-1}_{X,X} \oplus X^{\oplus p+1})(-\oplus b_{X^{\oplus i+1},X} \oplus X^{\oplus p-i})$, so the claim follows if we show that $[b_{X^{\oplus i+1},X} \oplus X] = [X \oplus b_{X^{\oplus i+1},X}]$ holds in the abelianisation. But the braid relation $(b_{X,X} \oplus X)(X \oplus b_{X,X})(b_{X,X} \oplus X) = (X \oplus b_{X,X})(b_{X,X} \oplus X)(X \oplus b_{X,X})$ abelianises to $[b_{X,X} \oplus X] = [X \oplus b_{X,X}]$ from which the claimed identity follows by induction on i. □

3.2. **The proof of Theorem A.** We prove Theorem A by induction on n, using the spectral sequence (5). As $|R_\bullet(\mathcal{M})|_{n+1} \to \mathcal{M}_{n+1}$ is assumed to be $(\frac{n-1+k}{k})$-connected for a $k \ge 2$ in the constant or a $k \ge 3$ in the abelian coefficients case, the summand of degree $(n+1)$ of the spectral sequence converges to zero in the range $p + q \le \frac{n-1}{k}$. By Lemma 3.1, the differential $d^1\colon E^1_{0,i,n+1} \to E^1_{-1,i,n+1}$ identifies with the stabilisation map $(s;\mathrm{id})_*\colon \mathrm{H}_i(\mathcal{M}_n; s^*L) \to \mathrm{H}_i(\mathcal{M}_{n+1}; L)$. Since there are no differentials targeting $E^k_{-1,0,n+1}$ for $k \ge 1$, the stabilisation has to be surjective for $i = 0$ if $E^\infty_{-1,0,n+1}$ vanishes, which is the case, since we have $-1 \le \frac{n-1}{k}$ for all $n \ge 0$. In particular, this implies the case $n = 0$ for both constant and abelian coefficients, because the isomorphism claims for $n = 0$ are vacuous.

Constant coefficients. Assume the claim for constant coefficients holds in degrees smaller than n. By Lemma 3.1, the differential $d^1\colon E^1_{p,q,n+1} \to E^1_{p-1,q,n+1}$ identifies with $s_*\colon \mathrm{H}_q(\mathcal{M}_{n-p}; L) \to \mathrm{H}_q(\mathcal{M}_{n-p+1}; L)$ for p even and is zero for p odd. From the induction assumption, we draw the conclusion that $E^2_{p,q,n+1}$ vanishes for (p,q) if p is even with $0 < p \le n$ and $q \le \frac{n-p-1}{k}$, and for (p,q) if p is odd with $0 \le p < n$ and $q \le \frac{n-p-3+k}{k}$. So in particular, $E^2_{p,q,n+1}$ vanishes if both $0 < p < n$ and $q \le \frac{n-p-1}{k}$ hold. As

$d^1\colon E^1_{1,i,n+1} \to E^1_{0,i,n+1}$ is zero for all i, injectivity of $s_*\colon \mathrm{H}_i(\mathcal{M}_n;L) \to \mathrm{H}_i(\mathcal{M}_{n+1};L)$ holds if $E^\infty_{0,i,n+1} = 0$ and $E^2_{p,q,n+1} = 0$ hold for $p+q = i+1$ with $q < i$. This is the case for $i \leq \frac{n-1}{k}$, as claimed, which follows from the established vanishing ranges of E^∞ and E^2. Similarly, the map in question is surjective in degree i if $E^\infty_{-1,i,n+1} = 0$ and $E^2_{p,q,n+1} = 0$ hold for $p+q = i$ with $q < i$, which is true for $i \leq \frac{n-1+k}{k}$. □

Abelian coefficients. Assume the statement holds for degrees smaller than n. The differential $d^1\colon E^1_{p,q,n+1} \to E^1_{p-1,q,n+1}$ identifies with $\sum_i (-1)^i (s,\eta_i)_*\colon \mathrm{H}_q(\mathcal{M}_{n-p};(s^{p+1})^*L) \to \mathrm{H}_q(\mathcal{M}_{n-p+1};(s^p)^*L)$. By Lemma 3.2, in the range where $(s,\mathrm{id})_*\colon \mathrm{H}_q(\mathcal{M}_{n-p-1};(s^{p+2})^*L) \to \mathrm{H}_q(\mathcal{M}_{n-p};(s^{p+1})^*L)$ is surjective, d^1 identifies with the stabilisation map $(s,\mathrm{id})_*\colon \mathrm{H}_q(\mathcal{M}_{n-p};(s^{p+1})^*L) \to \mathrm{H}_q(\mathcal{M}_{n-p+1};(s^p)^*L)$ for p even and vanishes for p odd. By induction, this happens for (p,q) with $0 \leq p \leq n-1$ and $q \leq \frac{n-p-1}{k}$, so by using the induction again, we conclude $E^2_{p,q,n+1} = 0$ for (p,q) if p is even satisfying $0 < p \leq n-1$ and $q \leq \frac{n-p+1-k}{k}$, and for (p,q) with p odd satisfying $0 \leq p < n-1$ and $q \leq \frac{n-p-2}{k}$. The rest of the argument proceeds as in the constant case, adapting the ranges and using that $d^1\colon E^1_{1,i,n+1} \to E^1_{0,i,n+1}$ is zero for $i \leq \frac{n-1}{k}$. □

Remark 3.3. If $g_{\mathcal{M}}$ is a grading of $\mathcal{M}$, then so is $g_{\mathcal{M}} + m$ for any fixed number $m \geq 0$. Consequently, if the canonical resolution of $\mathcal{M}$ is graded $(\frac{g_{\mathcal{M}}-m+k}{k})$-connected for an $m \geq 2$, then we can apply Theorem A and C to $\mathcal{M}$, graded by $g_{\mathcal{M}} + (m-2)$, which results in a shift in the stability range. By adapting the ranges in the previous proof appropriately, requiring more specific connectivity assumptions improve the stability ranges in Theorem A as follows.

(i) If the canonical resolution is graded $(\frac{g_{\mathcal{M}}-m+k}{k})$-connected for an $m \geq 3$, the surjectivity range in Theorem A for constant coefficients can be improved from $i \leq \frac{n-m+k}{k}$ to $i \leq \frac{n-m+k+1}{k}$ and the one for abelian coefficients from $i \leq \frac{n-m+2}{k}$ to $i \leq \frac{n-m+3}{k}$.
(ii) If the canonical resolution is graded $(g_{\mathcal{M}}-1)$-connected in degrees ≥ 1, then the isomorphism range in Theorem A for constant coefficients can be improved from $i \leq \frac{n-1}{2}$ to $i \leq \frac{n}{2}$, similar to the proof of [Ran13, Thm 5.1] for symmetric groups.

4. Stability with twisted coefficients

This section serves to introduce a notion of twisted coefficient systems and to prove Theorem C. Many ideas in this section are inspired by [RW17, Sect. 4], which is itself a generalisation of work by Dwyer, van der Kallen, and Ivanov [Dwy80; Iva93; Kal80]. We use similar notation to [RW17] to emphasise analogies and refer to Remarks 4.11 and 4.12 for a comparison of their notion of coefficient systems to ours.

4.1. Coefficient systems of finite degree.

We define coefficients system of finite degree for graded modules over graded braided monoidal categories, such as fundamental groupoids of graded E_1-modules over E_2-algebras, as described in Section 2.1.

Let $(\mathcal{M},\oplus)$ be a graded right-module over a braided monoidal category $(\mathcal{A},\oplus,b,0)$ in the sense of Section 1.1. We fix a *stabilising object* X, i.e. an object of $\mathcal{A}$ of degree 1, and recall the free braided monoidal category $\mathcal{B} = \coprod_{n\geq 0} B_n$ on one object, built from the braid groups B_n. The choice of X induces a functor $\mathcal{B} \to \mathcal{A}$, so in particular homomorphisms $B_n \to \mathrm{Aut}_{\mathcal{A}}(X^{\oplus n})$ and a module-structure on $\mathcal{M}$ over $\mathcal{B}$.

Definition 4.1. A *coefficient system* F for $\mathcal{M}$ is a functor

$$F\colon \mathcal{M} \longrightarrow \mathcal{Ab}$$

to the category of abelian groups, together with a natural transformation

$$\sigma^F\colon F \longrightarrow F(-\oplus X),$$

called the *structure map of* F, such the image of the canonical morphism $B_m \to \mathrm{Aut}_{\mathcal{A}}(X^{\oplus m})$ acts trivially on the image of $(\sigma^F)^m\colon F \to F(-\oplus X^{\oplus m})$ for all $m \geq 0$. A *morphism between coefficient systems* F and G for $\mathcal{M}$ is a natural transformation $F \to G$ that commutes with the structure maps σ^F and σ^G.

Remark 4.2. The category of coefficient systems for $\mathcal{M}$ is abelian, so in particular has (co)kernels. More concretely, it is a category of abelian group-valued functors on a category $\langle\mathcal{M},\mathcal{B}\rangle$ (see Remark 4.12).

Definition 4.3. Define the *suspension* ΣF of a coefficient system F for $\mathcal{M}$ as

$$\Sigma F = F(- \oplus X),$$

together with the structure map $\sigma^{\Sigma F}\colon \Sigma F \longrightarrow \Sigma F(- \oplus X)$, defined as the composition

$$\Sigma F = F(- \oplus X) \xrightarrow{\sigma^F(-\oplus X)} F(- \oplus X^{\oplus 2}) \xrightarrow{F(-\oplus b_{X,X})} F(- \oplus X^{\oplus 2}) = \Sigma F(- \oplus X).$$

The structure map σ^F of F induces a morphism $F \longrightarrow \Sigma F$, called the *suspension map*, whose (co)kernel is the *kernel* $\ker(F)$ respectively *cokernel* $\operatorname{coker}(F)$ of F. We call F *split* if the suspension map is split injective in the category of coefficient systems.

Lemma 4.4. *The suspension ΣF and the suspension map $F \longrightarrow \Sigma F$ are well-defined.*

Proof. The triviality condition for ΣF is implied by the one for F, since $(\sigma^{\Sigma F})^m$ agrees with $F(- \oplus b_{X^{\oplus m},X})\sigma^F(- \oplus X)^m$, which follows by induction on m, using the braid relation $(X \oplus b_{X,X^{\oplus m-1}})(b_{X,X} \oplus X^{\oplus m-1}) = b_{X^{\oplus m},X}$. The fact that the suspension map is a morphism of coefficient system is a consequence of the triviality condition on F, more specifically of $F(- \oplus b_{X,X})(\sigma^F)^2 = (\sigma^F)^2$. □

Remark 4.5. The suspension map gives rise to a natural transformation $\mathrm{id} \longrightarrow \Sigma$ of endofunctors on the category of coefficient systems for $\mathcal{M}$.

For the remainder of the section, we fix a coefficient system F for the module $\mathcal{M}$.

Definition 4.6. We denote by F_n for $n \geq 0$ the restriction of F to the full subcategory $\mathcal{M}_n \subseteq \mathcal{M}$ of objects of degree n and define the *degree* and *split degree* of F at an integer N inductively by saying that F has

(i) (split) degree ≤ -1 at N if $F_n = 0$ for $n \geq N$,
(ii) degree r at N for a $r \geq 0$ if $\ker(F)$ has degree -1 at N and $\operatorname{coker}(F)$ has degree $(r-1)$ at $(N-1)$, and
(iii) split degree r at N for a $r \geq 0$ if F is split and $\operatorname{coker}(F)$ is of split degree $(r-1)$ at $(N-1)$.

Remark 4.7. Note that for all $N \leq 0$, F is of (split) degree r at 0 if and only if it is of (split) degree r at N, and that the property of being of (split) degree r at 0 is independent of the chosen grading. However, being of degree r at N depends on the grading if N is positive. If $g_{\mathcal{M}}$ is a grading for $\mathcal{M}$, then so is $g_{\mathcal{M}} + k$ for any $k \geq 0$ and by induction on r, one proves that for $k \geq 0$, F is of (split) degree r at N with respect to a grading $g_{\mathcal{M}}$ if and only if it is of (split) degree r at $(N-k)$ with respect to the grading $g_{\mathcal{M}} + k$

Lemma 4.8. *The iterated suspension $\Sigma^i F$ for $i \geq 0$ is given by $\Sigma^i F = F(- \oplus X^{\oplus i})$ with structure map*

$$\Sigma^i F = F(- \oplus X^{\oplus i}) \xrightarrow{\sigma^F(-\oplus X^{\oplus i})} F(- \oplus X^{\oplus i} \oplus X) \xrightarrow{F(-\oplus b_{X^{\oplus i},X})} F(- \oplus X \oplus X^{\oplus i}) = \Sigma^i F(- \oplus X).$$

Proof. This follows by induction on i, using the braid relation $(b_{X,X} \oplus X^{\oplus i})(X \oplus b_{X^{\oplus i},X}) = b_{X^{\oplus i} \oplus X,X}$. □

Lemma 4.9. *Let F be a coefficient system for $\mathcal{M}$.*

(i) For all $i \geq 0$, $\Sigma^i(\ker(F))$ and $\Sigma^i(\operatorname{coker}(F))$ are isomorphic to $\ker(\Sigma^i F)$ and $\operatorname{coker}(\Sigma^i F)$, respectively.
(ii) If F is split, then $\Sigma^i F$ is split for all $i \geq 0$.
(iii) If F is of (split) degree r at N, then $\Sigma^i F$ is of (split) degree r at $(N-i)$ for $i \geq 0$.

Proof. Using Lemma 4.8 and $(X \oplus b^{-1}_{X^{\oplus i},X})(b_{X^{\oplus i+1},X}) = (b_{X^{\oplus i+1},X})(b^{-1}_{X^{\oplus i},X} \oplus X)$, the natural transformation

$$\Sigma^{i+1}F(-) = F(- \oplus X^{\oplus i+1}) \xrightarrow{F(-\oplus b^{-1}_{X^{\oplus i},X})} F(- \oplus X^{\oplus i+1}) = \Sigma^{i+1}F(-)$$

can be seen to commute with the structure map of $\Sigma^{i+1}F$, so defines an automorphism $\Phi\colon \Sigma^{i+1}F \longrightarrow \Sigma^{i+1}F$. Lemma 4.8 also implies the relation $\Sigma^i(\sigma^F) = \Phi\sigma^{\Sigma^i F}$ and therefore $\Sigma^i((\mathrm{co})\ker(\sigma^F)) = (\mathrm{co})\ker(\Phi\sigma^{\Sigma^i F})$. Hence, the coefficient systems in comparison are (co)kernels of morphisms that differ by an automorphism. This proves the first claim. Given a splitting $s\colon \Sigma F \longrightarrow F$ for F, the composition $\Sigma^i(s)\Phi$ splits $\Sigma^i F$, which shows the second. Finally, the third follows from the first two by induction on r. □

Remark 4.10. If $\mathcal{M}$ is a groupoid such that all subcategories $\mathcal{M}_n$ are connected, then a coefficient system for $\mathcal{M}$ is equivalently given as a sequence of $\mathrm{Aut}(A \oplus X^{\oplus n-g(A)})$-modules F_n for an element A of minimal degree $g(A)$, together with $(- \oplus X)$-equivariant morphisms $F_n \longrightarrow F_{n+1}$ such that the image of B_m in $\mathrm{Aut}(X^{\oplus m})$ acts via $(A \oplus X^{\oplus n-g(A)} \oplus -)$ trivially on the image of F_n in F_{n+m} for all n and m.

Remark 4.11. A *pre-braided* monoidal category in the sense of [RW17] is a monoidal category $(C, \oplus, b, 0)$ whose unit 0 is initial and whose underlying groupoid $C^{\sim}$ is braided monoidal satisfying a certain condition (see [RW17, Def. 1.5]). In their work, a *coefficient system* for C at a pair of objects (A, X) is an abelian group valued functor F^{RW} defined on the full subcategory $C_{A,X} \subseteq \mathcal{D}$ generated by $A \oplus X^{\oplus n}$ for $n \geq 0$. Considering $C^{\sim}_{A,X}$ as a module over the braided monoidal groupoid $C^{\sim}_{0,X}$, such a functor F^{RW} gives a coefficient system F in our sense by restricting F^{RW} to $C^{\sim}_{A,X}$ and defining the structure map as $\sigma^F(-) := F^{\mathrm{RW}}(- \oplus \iota_X)$, where $\iota_X : 0 \to X$ is the unique morphism. In [RW17], the transformation $- \oplus \iota_X : \mathrm{id}_C \to - \oplus X$ is denoted by σ^X, so we have the suggestive identity $F^{\mathrm{RW}}(\sigma^X) = \sigma^F$. Assigning a coefficient system for C at (A, X) in the sense of [RW17] to one for $C^{\sim}_{A,X}$ in our sense yields a functor between the respective categories of coefficient systems, which can be seen to preserve the *suspension* and *degree* in the sense of [RW17] and in ours, at least up to isomorphism. See Section 7.3 for a general comparison between [RW17] and our work.

Remark 4.12. The category of coefficient systems for $\mathcal{M}$ is isomorphic to the category of abelian group-valued functors on a category $\langle \mathcal{M}, \mathcal{B} \rangle$. To construct this category, recall *Quillen's bracket construction* $\langle \mathcal{E}, \mathcal{F} \rangle$ of a monoidal category $\mathcal{F}$ that acts via $\oplus \colon \mathcal{E} \times \mathcal{F} \to \mathcal{E}$ on a category $\mathcal{E}$ [Gra76, p.219]. It has the same objects as $\mathcal{E}$, and a morphism from C to C' is an equivalence class of pairs (D, f) with $D \in \mathrm{ob}\, \mathcal{F}$ and $f \in \mathcal{E}(C \oplus D, C')$, where (D, f) and (D', f') are equivalent if there is an isomorphism $g \in \mathcal{F}(D, D')$ satisfying $f' = f(C \oplus g)$. Using this construction, we obtain the category $\langle \mathcal{M}, \mathcal{B} \rangle$ encoding coefficient systems by letting the free braided monoidal category on one object $\mathcal{B}$ act on $\mathcal{M}$ via the functor $\mathcal{B} \to \mathcal{A}$ induced by X, followed by the action of $\mathcal{A}$ on $\mathcal{M}$. The multiplication by X on $\mathcal{M}$ induces an endofunctor

$$\Sigma \colon \langle \mathcal{M}, \mathcal{B} \rangle \longrightarrow \langle \mathcal{M}, \mathcal{B} \rangle$$

by mapping a morphism $[D, f] \colon C \to C'$ to $[D, (f \oplus X)(C \oplus b_{X,D})] \colon C \oplus X \to C' \oplus X$. This functor comes together with a natural transformation $\sigma \colon \mathrm{id} \to \Sigma$ given by $[X, \mathrm{id}]$, such that the suspension of a coefficient system F, seen as a functor on $\langle \mathcal{M}, \mathcal{B} \rangle$, is the composition $(F \circ \Sigma)$ and its suspension map is $F(\sigma) \colon F \to (F \circ \Sigma)$. From this point of view and using the notation of the previous remark, the functor from coefficient systems in the sense of [RW17] to ones in ours, described in the previous remark, is given by precomposition with a functor $\langle C^{\sim}_{A,X}, \mathcal{B} \rangle \to C_{A,X}$ that is the identity on objects and maps a morphism $[X^{\oplus k}, f]$ in $\langle C^{\sim}_{A,X}, \mathcal{B} \rangle$ from C to C' to $f(C \oplus \iota_{X^{\oplus k}})$.

4.2. **Twisted stability of E_1-modules over E_2-algebras.** We fix a graded E_1-module $\mathcal{M}$ over an E_2-algebra $\mathcal{A}$ with stabilising object X for the rest of the section. Recall from Section 2.1 that its fundamental groupoid $(\Pi(\mathcal{M}), \oplus)$ is a graded right-module over the graded braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$.

Definition 4.13. A *coefficient system* for $\mathcal{M}$ is a coefficient system for $\Pi(\mathcal{M})$ in the sense of Definition 4.1.

The structure map of a coefficient system F for $\mathcal{M}$ enhance the stabilisation map $s \colon \mathcal{M} \to \mathcal{M}$ to a map

$$(s; \sigma^F) \colon (\mathcal{M}; F) \longrightarrow (\mathcal{M}; F)$$

of spaces with local systems, which stabilises homologically by Theorem C if the canonical resolution is sufficiently connected and F is of finite degree. This remainder of this section is devoted to the proof.

Remark 4.14. In the course of the proof of Theorem C, it will be convenient to have fixed a notion of a *homotopy commutative square of spaces with local systems*, by which we mean a square

$$\begin{array}{ccc} (X; F) & \longrightarrow & (Y; G) \\ \downarrow & & \downarrow \\ (X'; F') & \longrightarrow & (Y'; G'), \end{array}$$

together with a specified homotopy between the two compositions, which might be nontrivial, even if the diagram is strictly commutative. Taking singular chains results in a homotopy commutative square of chain complexes (see Section 1.2), and taking vertical mapping cones of the square induces a morphism

$$\mathrm{H}_* \big((X'; F'), (X; F)\big) \to \mathrm{H}_* \big((Y'; G'), (Y; G)\big), \tag{7}$$

which depends on the homotopy. However, homotopies that are homotopic as homotopies give homotopic morphisms on mapping cones, hence they induce the same morphism (7). Horizontal composition of such squares, including the homotopies, induces the respective composition of (7). Even though (7) depends on the homotopy, the long exact sequences of the mapping cones still fit into a commutative ladder.

We denote by $\mathrm{Rel}_*(F) = \mathrm{H}_*((\mathcal{M};F),(\mathcal{M};F))$ the relative groups with respect to the stabilisation $(s;\sigma^F)$, equipped with the additional grading $\mathrm{Rel}_*(F) = \bigoplus_{n\in\tilde{\mathbb{N}}} \mathrm{H}_*((\mathcal{M}_{n+1};F),(\mathcal{M}_n;F))$. Although the square

$$\begin{array}{ccc} (\mathcal{M};F) & \xrightarrow{(s;\sigma^F)} & (\mathcal{M};F) \\ {\scriptstyle (s;\sigma^F)}\downarrow & & \downarrow{\scriptstyle (s;\sigma^F)} \\ (\mathcal{M};F) & \xrightarrow{(s;\sigma^F)} & (\mathcal{M};F), \end{array} \tag{8}$$

commutes strictly, we consider it as homotopy commutative via the homotopy (6) of Section 3.1, which extends one of spaces with local systems (see Section 1.2), since the triviality condition on coefficient systems gives $F(-\oplus b^{-1}_{X,X})(\sigma^F)^2 = (\sigma^F)^2$. This homotopy commutative square induces a relative stabilisation

$$(s;\sigma^F)^{\sim}_*\colon\ \mathrm{Rel}_*(F) \longrightarrow \mathrm{Rel}_*(F)$$

of degree 1, where the superscript $\sim$ indicates the twist by the homotopy. The homotopy commutative square (8) admits a factorisation into a composition of homotopy commutative squares

$$\begin{array}{ccccc} (\mathcal{M};F) & \xrightarrow{(\mathrm{id};\sigma^F)} & (\mathcal{M};\Sigma F) & \xrightarrow{(s;\mathrm{id})} & (\mathcal{M};F) \\ {\scriptstyle (s;\sigma^F)}\downarrow & & \downarrow{\scriptstyle (s;\sigma^{\Sigma F})} & & \downarrow{\scriptstyle (s;\sigma^F)} \\ (\mathcal{M};F) & \xrightarrow{(\mathrm{id};\sigma^F)} & (\mathcal{M};\Sigma F) & \xrightarrow{(s;\mathrm{id})} & (\mathcal{M};F)), \end{array}$$

in which the square on the left strictly commutes because of the triviality condition, and we equip it with the trivial homotopy. The square on the right is homotopy commutative using the same homotopy as for (8). This induces a factorisation of the relative stabilisation map as

$$\mathrm{Rel}_*(F) \overset{(\mathrm{id};\sigma^F)_*}{\longrightarrow} \mathrm{Rel}_*(\Sigma F) \overset{(s;\mathrm{id})^{\sim}_*}{\longrightarrow} \mathrm{Rel}_*(F), \tag{9}$$

with the first map being of degree 0 and the second of degree 1.

Lemma 4.15. *The composition* $\mathrm{Rel}_*(F) \xrightarrow{(s;\sigma^F)^{\sim}_*} \mathrm{Rel}_*(F) \xrightarrow{(s;\sigma^F)^{\sim}_*} \mathrm{Rel}_*(F)$ *is trivial.*

Proof. The mapping cones defining $\mathrm{Rel}_*(F)$ induce a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_*(\mathcal{M};F) & \longrightarrow & \mathrm{H}_*(\mathcal{M};F) & \longrightarrow & \mathrm{Rel}_*(F) & \xrightarrow{h_3} & \mathrm{H}_{*-1}(\mathcal{M};F) & \xrightarrow{h_1} & \ldots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_2} & & \\ \ldots & \longrightarrow & \mathrm{H}_*(\mathcal{M};F) & \longrightarrow & \mathrm{H}_*(\mathcal{M};F) & \xrightarrow{h_6} & \mathrm{Rel}_*(F) & \xrightarrow{h_5} & \mathrm{H}_{*-1}(\mathcal{M};F) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow{\scriptstyle h_8} & & \downarrow{\scriptstyle h_7} & & \downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_*(\mathcal{M};F) & \xrightarrow{h_{10}} & \mathrm{H}_*(\mathcal{M};F) & \xrightarrow{h_9} & \mathrm{Rel}_*(F) & \longrightarrow & \mathrm{H}_{*-1}(\mathcal{M};F) & \longrightarrow & \ldots \end{array}$$

in which h_7h_4 agrees with the composition in consideration. As h_1 and h_2 both equal $(s;\sigma^F)_*$, we conclude $0 = h_1h_3 = h_2h_3 = h_5h_4$, so the image of h_4 is in the kernel of h_5, which is the image of h_6. Hence it is enough to show $h_7h_6 = 0$. Since $h_8 = h_{10}$ for the same reason as $h_1 = h_2$, we obtain the claim from the identity $h_7h_6 = h_9h_8 = h_9h_{10} = 0$. □

4.3. **The relative spectral sequence.** We prove Theorem C via a relative analogue of the spectral sequence (5) of Section 3.1, which we derive from a map of augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces

$$\begin{array}{ccc} R_\bullet(\mathcal{M}) & \cdots\rightarrow & R_\bullet(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{s} & \mathcal{M} \end{array}$$

covering the stabilisation map s. Indicated by the dotted arrow, this morphism will only be defined up to up to higher coherent homotopy, and we obtain it from replacing the canonical resolution $R_\bullet(\mathcal{M})$ with an equivalent bar construction $B(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M}))$ which admits a strict morphism of the desired form.

To this end, recall from Section 2.2 the homotopy discrete category $U\mathcal{O}$, the isomorphism $\pi_0(U\mathcal{O}) \cong U\mathcal{B}$, and the augmented $U\mathcal{O}$-space $B_\bullet(\mathcal{M})$ whose restriction to the subcategory $\widetilde{\Delta}_{\mathrm{inj}} \subseteq U\mathcal{O}$ is $R_\bullet(\mathcal{M})$, where $\widetilde{\Delta}_{\mathrm{inj}}$ is the union of components hit by the section $\Delta_{\mathrm{inj}} \to U\mathcal{B}$. Define the $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O})$-space $U\mathcal{O}(\bullet,\blacksquare)$ and the $(\Delta^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{B})$-space $U\mathcal{B}(\bullet,\blacksquare)$ by restricting the hom-functors of $U\mathcal{O}$ and $U\mathcal{B}$ appropriately.

Taking components gives a weak equivalence $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O} \to \Delta^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{B}$ of enriched categories and one $U\mathcal{O}(\bullet,\blacksquare) \to U\mathcal{B}(\bullet,\blacksquare)$ of $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O})$-spaces, which fits into a chain of weak equivalences of $\widetilde{\Delta}_{\mathrm{inj}}$-spaces

$$R_\bullet(\mathcal{M}) \xleftarrow{\simeq} B\big(U\mathcal{O}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})\big) \xrightarrow{\simeq} B\big(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})\big), \tag{10}$$

augmented over $\mathcal{M}$, the left arrow being the restriction of the bar resolution of $B_\bullet(\mathcal{M})$ to $\widetilde{\Delta}_{\mathrm{inj}}$ (see Section 1.4). Consider the functor $t\colon U\mathcal{O} \to U\mathcal{O}$ that maps $[p]$ to $[p+1]$ and is on morphisms defined as

$$t\colon \begin{array}{ccc} U\mathcal{O}\big([q],[p]\big) & \longrightarrow & U\mathcal{O}\big(t([q]), t([p])\big) \\ (d,\mu) & \longmapsto & (d, \gamma(c;\mu, 1_{\mathfrak{a}})), \end{array}$$

using the operadic composition γ and the element c with which we defined the iterated operations $c_p \in \mathcal{O}(p)$ in Section 2.2. Accompanying this functor, there is a morphism of augmented $U\mathcal{O}$-spaces

$$\begin{array}{ccc} B_\blacksquare(\mathcal{M}) & \longrightarrow & B_{t(\blacksquare)}(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\quad s \quad} & \mathcal{M}, \end{array} \tag{11}$$

defined by making use of the module structure θ of $\mathcal{M}$ to assign to a p-simplex (A,ζ) in $B_p(\mathcal{M})$ the element $(A, \theta(c;\zeta, X))$ in $B_{p+1}(\mathcal{M})$. Last but not least, we define a morphism of $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O})$-spaces

$$U\mathcal{B}\big(\bullet,\blacksquare\big) \longrightarrow U\mathcal{B}\big(\bullet, t(\blacksquare)\big) \tag{12}$$

by consider the braid groups $\coprod_{n\geq 0} B_n$ as the free braided monoidal category in one object X to define

$$\begin{array}{rcl} U\mathcal{B}\big([q],[p]\big) = B_{p+1}/B_{p-q} & \longrightarrow & B_{p+2}/B_{p-q+1} = U\mathcal{B}\big([q], t([p])\big) \\ {[b]} & \longmapsto & [(b \oplus X)(X^{\oplus p-q} \oplus b^{-1}_{X^{\oplus p+1},X})]. \end{array}$$

Lemma 4.16. *The assignment* (12) *defines indeed a morphism of* $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O})$*-spaces.*

Proof. Recall that $U\mathcal{B}(\bullet,\blacksquare)$ is induced from a $(\Delta^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{B})$-space via the equivalence $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O} \to \Delta^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{B}$. The semi-simplicial direction of $U\mathcal{B}(\bullet,\blacksquare)$ comes from the section $\Delta_{\mathrm{inj}} \to U\mathcal{B}$ of Lemma 2.7, which maps a face map d_i in $\Delta_{\mathrm{inj}}([q-1],[q])$ to the class $[b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus q-i}]$ in $U\mathcal{B}([p-1],[p])$, so (12) is natural in the semi-simplicial direction if the two braids

$$(X^{\oplus p-q} \oplus b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus q-i} \oplus X)(X^{\oplus p-q+1} \oplus b^{-1}_{X^{\oplus q},X}) \quad \text{and} \quad (X^{\oplus p-q} \oplus b^{-1}_{X^{\oplus q+1},X})(X^{\oplus p-q+1} \oplus b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus q-i})$$

define the same class in $U\mathcal{B}([q-1],[p+1]) = B_{p+2}/B_{p-q+2}$. An application of braid relations shows that these two braids agree up to right-multiplication with $(X^{\oplus p-q} \oplus b^{-1}_{X,X} \oplus X^{q})$, so coincide in B_{p+2}/B_{p-q+2}, which shows the claim since the naturality in the $U\mathcal{B}$-direction is immediate. ◻

The functor $t\colon U\mathcal{O} \to U\mathcal{O}$, together with the morphisms (11) and (12), induces a map

$$\begin{array}{ccc} B(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})) & \xrightarrow{t} & B(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})) \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\qquad s \qquad} & \mathcal{M}. \end{array}$$

of augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces. Pulling back a coefficient system F for the graded module $\mathcal{M}$ along the augmentations, this morphism enhances to one of graded $\widetilde{\Delta}_{\mathrm{inj}}$-spaces with local coefficients that covers the stabilization map $(s;\sigma^F)\colon (\mathcal{M};F) \longrightarrow (\mathcal{M};F)$. Identifying $R_\bullet(\mathcal{M})$ with $B(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M}))$ via the zig-zag (10) by abuse of notation, we get a tri-graded spectral sequence of the form

$$E^1_{p,q,n} \cong \begin{cases} \mathrm{H}_q\left((R_p(\mathcal{M})_{n+1};F), (R_p(\mathcal{M})_n;F)\right) & \text{if } p \geq 0 \\ \mathrm{H}_q\left((\mathcal{M}_{n+1};F), (\mathcal{M}_n;F)\right) & \text{if } p = -1 \end{cases} \tag{13}$$
$$\Longrightarrow \mathrm{H}_{p+q+1}\left((\mathcal{M}_{n+1}, |R_\bullet(\mathcal{M})|_{n+1};F), (\mathcal{M}_n, |R_\bullet(\mathcal{M})|_n;F)\right),$$

which is a sum of spectral sequences, one for each $n \in \bar{\mathbb{N}}$ (see Sections 1.3 and 1.5). Using Lemma 4.8 and

$$(b^{-1}_{X,X} \oplus X^{\oplus p})(X \oplus b_{X^{\oplus i},X} \oplus X^{\oplus p-i})(b_{X^{\oplus p+1},X}) = (X \oplus b_{X^{\oplus p},X})(b_{X^{\oplus i},X} \oplus X^{\oplus p-i+1}),$$

one checks that selfhomotopy (6) of s^2 witnesses homotopy commutativity of the square

$$\begin{array}{ccc} (\mathcal{M};\Sigma^{p+1}F) & \xrightarrow{(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus p-i}))} & (\mathcal{M};\Sigma^{p}F) \\ {\scriptstyle (s;\sigma^{\Sigma^{p+1}F})}\downarrow & & \downarrow{\scriptstyle (s;\sigma^{\Sigma^{p}F})} \\ (\mathcal{M};\Sigma^{p+1}F) & \xrightarrow{(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus p-i}))} & (\mathcal{M};\Sigma^{p}F), \end{array}$$

which thus induces a morphism $(s;\eta_i)_*^{\sim}\colon \operatorname{Rel}_*(\Sigma^{p+1}F)\to\operatorname{Rel}_*(\Sigma^pF)$ of degree 1; the superscript $\sim$ indicates the twist by the homotopy (6). This morphism serves us to identify the spectral sequence (13) as follows.

Lemma 4.17. *We have $E^1_{p,q,n+1}\cong\operatorname{Rel}_q(\Sigma^{p+1}F)_{n-p}$, and the E^1-differential identifies with*

$$\textstyle\sum_{i=0}^{p}(-1)^i(s;\eta_i)_*^{\sim}\colon\ \operatorname{Rel}_q(\Sigma^{p+1}F)_{n-p}\to\operatorname{Rel}_q(\Sigma^pF)_{n-p+1}.$$

In particular, the differential $d^1\colon E^1_{0,*,n+1}\to E^1_{-1,*,n+1}$ corresponds to the second map of (9) *in degree n.*

Proof. On p-simplices, the first equivalence of (10) has a preferred homotopy inverse induced by the extra degeneracy given by inserting the identity of $U\mathcal{O}([p],[p])$ (see Section 1.4). Composing it with the second equivalence of (10) yields an equivalence that forms the vertical arrows of a square

$$\begin{array}{ccc} R_p(\mathcal{M}) & \xrightarrow{\quad\tilde t\quad} & R_p(\mathcal{M}) \\ \simeq\downarrow & & \downarrow\simeq \\ B(U\mathcal{B}([p],\blacksquare),U\mathcal{O},B_\blacksquare(\mathcal{M})) & \xrightarrow{\ t\ } & B(U\mathcal{B}([p],\blacksquare),U\mathcal{O},B_\blacksquare(\mathcal{M})), \end{array}$$

where $\tilde t$ is defined by mapping (A,ζ) to $(s(A),s(\zeta)\cdot\theta(\alpha_p;A,X^{p+2}))$. Here $\alpha_p\in\Omega_{c_{p+2}}\mathcal{O}(p+2)$ is any loop that corresponds to $b^{-1}_{X^{\oplus p+1},X}$ under the equivalence $\pi_1(\mathcal{O}(p+2);c_{p+2})\cong B_{p+2}$ (see Section 2.2). This choice of α_p guarantees that the previous square commutes, which is why it is sufficient to show that

$$\begin{array}{ccc} (\mathcal{M}_{n-p};\Sigma^{p+1}F) & \xrightarrow{(s;\sigma^{\Sigma^{p+1}F})} & (\mathcal{M}_{n-p+1};\Sigma^{p+1}F) \\ {\scriptstyle(\iota;\mathrm{id})}\downarrow & & \downarrow{\scriptstyle(\iota;\mathrm{id})} \\ (R_p(\mathcal{M})_{n+1};F) & \xrightarrow{(\tilde t;\sigma^F)} & (R_p(\mathcal{M})_{n+2};F) \end{array}$$

homotopy commutes in order to prove $E^1_{p,q,n+1}\cong\operatorname{Rel}_q(\Sigma^{p+1}F)_{n-p}$, where ι denotes the canonical equivalence mapping A to $(A,\operatorname{const}_{s^{p+1}(A)})$. On the space-level, the two compositions are given by assigning to $A\in\mathcal{M}_{n-p}$ the elements $(s(A),\operatorname{const}_{s^{p+2}(A)})$ and $(s(A),\theta(\alpha_p;A,X^{p+2}))$, respectively. As we have $\sigma^{\Sigma^{p+1}F}(-)=F(-\oplus X^{\oplus p+1})\sigma^F(-\oplus X^{\oplus p+1})$ by Lemma 4.8, the homotopy induced by following α_p to its endpoint is one of maps with local cofficients, which implies the first claim of the lemma. A relative version of the proof of Lemma 3.1 establishes the identification of the differentials and finishes the proof. □

Lemma 4.18. *The following composition is zero for $n\geq 1$,*

$$\operatorname{Rel}_*(F)_{n-1}\xrightarrow{(\mathrm{id};\sigma^F)}\operatorname{Rel}_*(\Sigma F)_{n-1}\xrightarrow{(\mathrm{id};\sigma^{\Sigma F})}\operatorname{Rel}_*(\Sigma^2F)_{n-1}\cong E^1_{1,*,n+1}\xrightarrow{d^1}E^1_{0,*,n+1}\cong\operatorname{Rel}_*(\Sigma F)_n.$$

Proof. Using Lemma 4.17, the composition in question is the difference between the morphisms in degree $(n-1)$ induced by the compositions of the two homotopy commutative squares

$$\begin{array}{ccccc} (\mathcal{M};F) & \xrightarrow{(\mathrm{id};\sigma^{\Sigma F}\sigma^F)} & (\mathcal{M};\Sigma^2F) & \xrightarrow{(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus 1-i}))} & (\mathcal{M};\Sigma F) \\ {\scriptstyle(s;\sigma^F)}\downarrow & & \downarrow{\scriptstyle(s;\sigma^{\Sigma^2F})} & & \downarrow{\scriptstyle(s;\sigma^{\Sigma F})} \\ (\mathcal{M};F) & \xrightarrow{(\mathrm{id};\sigma^{\Sigma F}\sigma^F)} & (\mathcal{M};\Sigma^2F) & \xrightarrow{(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus 1-i}))} & (\mathcal{M};\Sigma F), \end{array}$$

for $i=0$ and $i=1$, where the homotopy of the left square is trivial. The nontrivial homotopy of the right square becomes trivial after composing with the left square by the triviality condition for coefficient systems, so the composition in question is the difference of the morphisms induced by the two strictly commutative outer squares. But, again by the triviality condition, we have $F(-\oplus b_{X,X})\sigma^{\Sigma F}\sigma^F=\sigma^{\Sigma F}\sigma^F$, so the two outer squares coincide and hence the difference of the induced morphisms vanishes. □

4.4. **The proof of Theorem C.** The long exact sequence

$$\ldots \longrightarrow \mathrm{Rel}_{*+1}(F) \longrightarrow \mathrm{H}_*(\mathcal{M};F) \overset{(s;\sigma^F)_*}{\longrightarrow} \mathrm{H}_*(\mathcal{M};F) \longrightarrow \mathrm{Rel}_*(F) \longrightarrow \mathrm{H}_{*-1}(\mathcal{M};F) \longrightarrow \ldots$$

exhibits Theorem C as a consequence of the next result.

Theorem 4.19. *Let F be a coefficient system for $\mathcal{M}$ of degree r at $N \geq 0$. If the canonical resolution of $\mathcal{M}$ is graded $(\frac{g_{\mathcal{M}}-2+k}{k})$-connected in degrees ≥ 1 for a $k \geq 2$, then*
- *(i) the group $\mathrm{Rel}_i(F)_n$ vanishes for $n \geq \max(N+1, k(i+r))$, and*
- *(ii) if F is of degree split r at $N \geq 0$, then $\mathrm{Rel}_i(F)_n$ vanishes for $n \geq \max(N+1, ki+r)$.*

We prove Theorem 4.19 via a double induction on r and $i \geq 0$ by considering the following statement.

($\mathbf{H}_{r,i}$) The vanishing ranges of Theorem 4.19 hold for all F of degree $< r$ at any $N \geq 0$ in all homological degrees i, and for all F of degree r at any $N \geq 0$ in homological degrees $< i$.

The claim ($\mathbf{H}_{r,i}$) holds trivially if $r < 0$ or if $(r,i) = (0,0)$. If ($\mathbf{H}_{r,i}$) holds for a fixed r and all i, then ($\mathbf{H}_{r+1,0}$) follows, since there is no requirement on coefficient systems of degree $(r+1)$. Hence, to prove the theorem, it is sufficient to show that ($\mathbf{H}_{r,i}$) implies ($\mathbf{H}_{r,i+1}$) for $i, r \geq 0$. As the composition

$$\mathrm{Rel}_i(F)_n \overset{(s;\sigma^F)_*^{\sim}}{\longrightarrow} \mathrm{Rel}_i(F)_{n+1} \overset{(s;\sigma^F)_*^{\sim}}{\longrightarrow} \mathrm{Rel}_i(F)_{n+2}$$

is zero by Lemma 4.15, it is enough to show injectivity of both maps in the claimed range. Using the factorisation (9), this is implied by the following lemma.

Lemma 4.20. *Let $r \geq 0$ and $i \geq 0$ satisfying ($\mathbf{H}_{r,i}$), and let F be of degree r at some $N \geq 0$.*
- *(i) The morphism $(\mathrm{id}, \sigma_X)_*\colon \mathrm{Rel}_*(F)_n \longrightarrow \mathrm{Rel}_*(\Sigma F)_n$ is injective for $n \geq \max(N, k(i+r))$ and surjective for $n \geq \max(N, k(i+r-1))$. If F is of split degree r at $N \geq 0$, then the map is split injective for all n and surjective for $n \geq \max(N, ki+r-1)$.*
- *(ii) The morphism $(s,\mathrm{id})_*^{\sim}\colon \mathrm{Rel}_i(\Sigma F)_n \longrightarrow \mathrm{Rel}_i(F)_{n+1}$ is injective in degrees $n \geq \max(N+1, k(i+r))$. If F is of split degree r at $N \geq 0$, then the map is injective for $n \geq \max(N+1, ki+r)$.*

Proof. We begin by proving the first part of the statement. As $\mathrm{Rel}_*(-)$ is functorial in the coefficient system, injectivity of the split case is clear. The remaining claims of the first statement follow from the long exact sequences in $\mathrm{Rel}_*(-)$ induced by the short exact sequences

$$0 \to \ker(F) \to F \to \mathrm{im}(F \to \Sigma F) \to 0 \quad \text{and} \quad 0 \to \mathrm{im}(F \to \Sigma F) \to \Sigma F \to \mathrm{coker}(F) \to 0$$

by applying ($\mathbf{H}_{r,i}$), using that $\ker(F)$ has degree -1 at N and that $\mathrm{coker}(F)$ has (split) degree $(r-1)$ at $(N-1)$. To prove (ii), we use the spectral sequence (13) and Lemma 4.17. Since $|R_\bullet(\mathcal{M})|_m \to \mathcal{M}_m$ is assumed to be $(\frac{m-2+k}{k})$-connected for $m \geq 1$, the groups $\mathrm{H}_*(\mathcal{M}_m, |R_\bullet(\mathcal{M})|_m; F)$ vanish for $* \leq \frac{m-2}{k}$, from which we conclude $E^\infty_{p,q,n+1} = 0$ for $p+q \leq \frac{n}{k}$. We claim that the differential $E^1_{1,i,n+1} \to E^1_{0,i,n+1}$ vanishes for $n \geq \max(N+1, k(i+r))$ in the nonsplit case, and for $n \geq \max(N+1, ki+r)$ in the split one. By Lemma 4.18, this is the case if the maps $\mathrm{Rel}_*(F)_{n-1} \to \mathrm{Rel}_*(\Sigma F)_{n-1} \to \mathrm{Rel}_*(\Sigma^2 F)_{n-1}$ are surjective in that range, which holds by (i). Since the map we want to prove injectivity of identifies with the differential $E^1_{0,i,n+1} \to E^1_{-1,i,n+1}$ by Lemma 4.17, it is therefore enough to show that, in the ranges of the statement, $E^\infty_{0,i,n+1} = 0$ and $E^2_{p,q,n+1} = 0$ holds for (p,q) with $p+q = i+1$ and $q < i$. By the vanishing range of E^∞ noted above, we have $E^\infty_{0,i,n+1} = 0$ in the required range. The claimed vanishing of E^2 follows from the vanishing even on the E^1-page, which is proved by observing that, by ($\mathbf{H}_{r,i}$) and Lemma 4.17, the groups $E^1_{p,q,n+1} \cong \mathrm{Rel}_q(\Sigma^{p+1}F)_{n-p}$ vanish for (p,q) with $q < i$ and $n \geq \max(N-p, k(q+r))$ in the nonsplit, and for (p,q) satisfying $q < i$ and $n \geq \max(N-p, kq+r)$ in the split case, since $\Sigma^{p+1}F$ has (split) degree r at $(N-p-1)$ by Lemma 4.9. □

5. Configuration spaces

The *ordered configuration space* of a manifold W with labels in a Serre fibration $\pi\colon E \to W$ is given by

$$F_n^\pi(W) = \{(e_1,\ldots,e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j \text{ and } \pi(e_i) \in W \backslash \partial W\},$$

and the *unordered configuration space* is the quotient by the canonical action of the symmetric group,

$$C_n^\pi(W) = F_n^\pi(W)/\Sigma_n.$$

To establish an E_1-module structure on the unordered configuration spaces of W, we assume that W has nonempty boundary, fix a collar $(-\infty, 0] \times \partial W \to W$, and attach an infinite cylinder to the boundary,

$$\tilde{W} = W \cup_{\{0\}\times\partial W} [0,\infty) \times \partial W.$$

Collar and cylinder assemble to an embedding $\mathbf{R} \times \partial W \subseteq \tilde{W}$ of which we make frequent use henceforth. We extend the fibration π over $\tilde{W}$ by pulling it back along the retraction $\tilde{W} \to W$ and define the space

$$\tilde{C}_n^\pi(W) = \{(s, e) \in [0,\infty) \times C_n^\pi(\tilde{W}) \mid \pi(e) \subseteq W \cup (-\infty, s) \times \partial W\},$$

which is an equivalent model for $C_n^\pi(W)$, since the inclusion in $\tilde{C}_n^\pi(W)$ as the subspace with $s = 0$ can be seen to be an equivalence by choosing an isotopy of $\tilde{W}$ that pushes $[0,\infty) \times \partial W$ into $(-\infty, 0) \times \partial W$. We furthermore fix an embedded cube $(-1,1)^{d-1} \subseteq \partial W$ of codimension 0, together with a section $l \colon (-1,1)^{d-1} \to E$ of π, which we extend canonically to a section l on $[0,\infty) \times (-1,1)^{d-1} \subseteq \mathbf{R} \times \partial W \subseteq \tilde{W}$.

Lemma 5.1. *Configurations $\coprod_{n\geq 0} C_n(D^d)$ in a disc form a graded E_d-algebra with configurations $\coprod_{n\geq 0} \tilde{C}_n^\pi(W)$ in a d-manifold W with nonempty boundary as an E_1-module over it, graded by the number of points.*

Proof. The operad $\mathcal{D}^\bullet(D^d)$ of little d-discs acts on $\coprod_{n\geq 0} C_n(D^d)$ by

$$\theta\colon \begin{array}{ccc} \mathcal{D}^k(D^d) \times \big(\coprod_{n\geq 0} C_n(D^d)\big)^k & \longrightarrow & \coprod_{n\geq 0} C_n(D^d) \\ \big((\phi_1,\ldots,\phi_k),(\{d_i^1\},\ldots,\{d_i^k\})\big) & \longmapsto & \bigcup_{j=1}^k \phi_j(\{d_i^j\}), \end{array}$$

and this action extends to one of $\mathcal{SC}_d$ (see Definition 2.1) on the pair $(\coprod_{n\geq 0} \tilde{C}_n^\pi(W), \coprod_{n\geq 0} C_n(D^d))$ via

$$\theta\colon \begin{array}{ccc} \mathcal{SC}_d(\mathfrak{m},\mathfrak{a}^k;\mathfrak{m}) \times \coprod_{n\geq 0} \tilde{C}_n^\pi(W) \times \big(\coprod_{n\geq 0} C_n(D^d)\big)^k & \longrightarrow & \coprod_{n\geq 0} \tilde{C}_n^\pi(W) \\ \big((s,\phi_1,\ldots,\phi_k),(s',\{e_i\}),\{d_i^1\},\ldots,\{d_i^k\}\big) & \longmapsto & \big(s'+s, \{e_i\} \cup (\cup_{j=1}^k l(\phi_j(\{d_i^j\}) + s'))\big), \end{array}$$

using the section l and the translation $(-+s')$ by s' in the $[0,\infty)$-coordinate, as illustrated in Figure 4. □

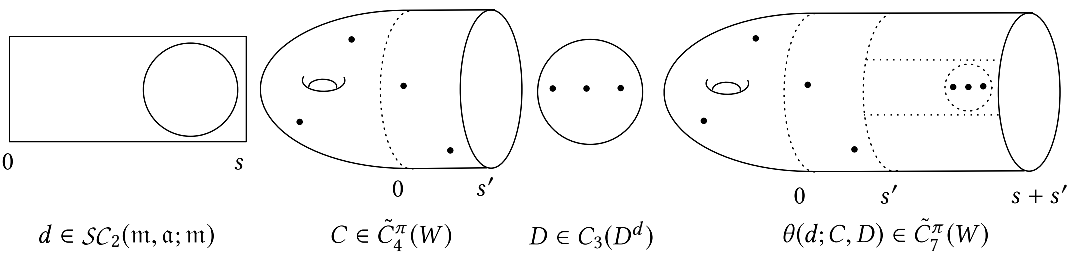


Figure 4. The E_1-module structure on unordered configuration spaces

5.1. **The resolution by arcs.** Let W be a smooth connected manifold of dimension $d \geq 2$ with nonempty boundary and $\pi\colon E \to W$ a Serre fibration with path-connected fibre. By Lemma 5.1, the configuration spaces $\mathcal{M} = \coprod_{n\geq 0} \tilde{C}_n^\pi(W)$ form a graded E_1-module over $\mathcal{A} = \coprod_{n\geq 0} C_n(D^d)$ considered as an E_2-algebra via the canonical morphism $\mathcal{SC}_2 \to \mathcal{SC}_d$ (see Section 2.1). The stabilisation map $s\colon \mathcal{M} \to \mathcal{M}$ with respect to a chosen stabilising object $X \in C_1(D^d)$, restricted to the subspace of elements of degree n, has the form

$$s\colon \tilde{C}_n^\pi(W) \longrightarrow \tilde{C}_{n+1}^\pi(W).$$

Remark 5.2. With respect to the described equivalence $C_n^\pi(W) \simeq \tilde{C}_n^\pi(W)$, the stabilisation map corresponds to the map $C_n^\pi(W) \to C_{n+1}^\pi(W)$ that adds a point "near infinity" [McD75; Seg79].

We prove high-connectivity of the canonical resolution of $\mathcal{M}$ (see Section 2.2) by identifying it with the *resolution by arcs*—an augmented semi-simplicial space of geometric nature, known to be highly connected.

Definition 5.3. The *resolution by arcs* is the augmented semi-simplicial space $R_\bullet^{\bullet-}(\mathcal{M}) \to \mathcal{M}$ with

$$R_p^{\bullet-}(\mathcal{M}) \subseteq \mathcal{M} \times \big(\mathrm{Emb}([-1,0],\tilde{W}) \times \mathrm{Maps}([-1,0],E)\big)^{p+1},$$

consisting of tuples $((s,\{e_i\}),(\varphi_0,\eta_0),\ldots,(\varphi_p,\eta_p))$ such that

(i) the arcs φ_i are pairwise disjoint and connect points in the configuration $\varphi_i(-1) \in \pi(\{e_i\}) \subseteq \tilde{W}$ to points $\varphi(0) \in \{s\} \times (-1,1) \times \{0\}^{d-2} \subseteq [0,\infty) \times \partial W$ in the order $\varphi_0(0) < \ldots < \varphi_p(0)$,
(ii) the interior of the arcs lie in $W \cup [0,s) \times \partial W$ and are disjoint from the configuration $\pi(\{e_i\})$,
(iii) the path of labels η_i satisfies $(\pi \circ \eta_i) = \varphi_i$ and connects the label of $\varphi_i(-1) \in \pi(\{e_i\})$ to $\eta_i(0) = l(\varphi_i(0))$,
(iv) there exists $\varepsilon \in (0,1)$ with $\varphi_i(t) = (s+t, \varphi_i(0), 0, \ldots, 0) \in (-\infty, s] \times (-1,1)^{d-1} \subseteq \tilde{W}$ for $t \in (-\varepsilon, 0]$.

The space $R_p^{\bullet-}(\mathcal{M})$ is topologised using the compact-open topology on $\mathrm{Maps}([-1,0],E)$ and the C^∞-topology on $\mathrm{Emb}([-1,0],\tilde{W})$. The ith face map forgets (φ_i, η_i). The rightmost graphic of Figure 5 depicts an example.

Theorem 5.4. *The resolution by arcs $R_\bullet^{\bullet-}(\mathcal{M}) \to \mathcal{M}$ is graded $(g_{\mathcal{M}} - 1)$-connected.*

Proof. Setting $s = 0$ in the definition of $R_p^{\bullet-}(\mathcal{M})$ yields a sub-semi-simplicial space $\bar{R}_\bullet^{\bullet-}(\mathcal{M}) \subseteq R_\bullet^{\bullet-}(\mathcal{M})$, augmented over $\bar{\mathcal{M}} = \coprod_{n \geq 0} C_n^\pi(W)$. As the inclusion is a weak equivalence by the same argument as for $C_n^\pi(W) \subseteq \tilde{C}_n^\pi(W)$, the augmented semi-simplicial space $R_\bullet^{\bullet-}(\mathcal{M}) \to \mathcal{M}$ is as connected as $\bar{R}_\bullet^{\bullet-}(\mathcal{M}) \to \bar{\mathcal{M}}$ is. The latter is the standard resolution by arcs for configurations of unordered points with labels in W, which is known to have the claimed connectivity (see e.g. the proof of [KM14, Thm A.1]). □

Theorem 5.5. *The canonical resolution and the one by arcs are weakly equivalent as augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces.*

Assuming Theorem 5.5, Theorem 5.4 ensures graded $(g_{\mathcal{M}} - 1)$-connectivity of the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ (see Section 2.2), which in turn implies Theorem D by an application of Theorem A and C.

We prove Theorem 5.5 by constructing a zig-zag of weak equivalences of augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces

$$R_\bullet(\mathcal{M}) \overset{①}{\longleftarrow} B\big(UO(\bullet,\blacksquare), UO, B_\blacksquare(\mathcal{M})\big) \overset{②}{\simeq} B\big(UO^{\bullet-}_{\bullet,\blacksquare}, UO, B_\blacksquare(\mathcal{M})\big) \overset{③}{\longrightarrow} R_\bullet^{\bullet-}(\mathcal{M})^{\mathrm{fib}} \tag{14}$$

between the canonical resolution $R_\bullet(\mathcal{M})$ and the fibrant replacement $R_\bullet^{\bullet-}(\mathcal{M})^{\mathrm{fib}}$ of the resolution by arcs, which is weakly equivalent to the resolution by arcs itself (see Section 1.4). The remainder of this subsection serves to explain the weak equivalences ①–③. We abbreviate the $E_{1,2}$-operad $\mathcal{SC}_2$ by O.

①. Recall from Section 2.2 the category UO and the contravariant UO-space $B_\bullet(\mathcal{M})$ over $\mathcal{M}$ whose restriction to the subcategory $\widetilde{\Delta}_{\mathrm{inj}} \subseteq UO$ is $R_\bullet(\mathcal{M})$. Using the $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times UO)$-space $UO(\bullet,\blacksquare)$ obtained by restricting the hom-functor of UO, the equivalence ① is defined as the restriction of the bar resolution of $B_\bullet(\mathcal{M})$ to $\widetilde{\Delta}_{\mathrm{inj}}$ (see Section 1.4). For the other parts of the zig-zag (14), we define an analogue of the resolution by arcs for the free graded E_1-module $O^{\mathfrak{m}} = \coprod_{n\geq 0} O(\mathfrak{m}, \mathfrak{a}^n; \mathfrak{m})/\Sigma_n$ (see Example 2.18). For simplification, we choose the centre $X = \{0\} \in C_1(D^d)$ as stabilising object and write s_d for the parameter of elements $d = (s_d, \{\phi_i\}) \in O^{\mathfrak{m}}$ and $g(d)$ for their degree, i.e. the cardinality of the set of embeddings $\{\phi_i\}$.

Definition 5.6. Define the augmented semi-simplicial space $R_\bullet^{\bullet-}(O^{\mathfrak{m}}) \to O^{\mathfrak{m}}$ with p-simplices

$$R_p^{\bullet-}(O^{\mathfrak{m}}) \subseteq O^{\mathfrak{m}} \times \mathrm{Emb}\big([-1,0], (0,\infty) \times (-1,1)\big)^{p+1},$$

consisting of tuples $((s, \{\phi_j\}), \varphi_0, \ldots, \varphi_p)$ such that

(i) the arcs φ_i are pairwise disjoint and connect centre points $\varphi_i(-1) \in \{\phi_j(0)\}$ of the discs to $\varphi_i(0) \in \{s\} \times (-1,1)$ in the order $\varphi_0(0) < \ldots < \varphi_p(0)$,
(ii) the interior of the arcs lie in $(0,s) \times (-1,1)$ and are disjoint from the centre points $\{\phi_j(0)\}$, and
(iii) there exists an $\varepsilon \in (0,s)$ such that $\varphi_i(t) = (s+t, \varphi_i(0)) \in (0,s] \times (-1,1)$ holds for all $t \in (-\varepsilon, 0]$.

The third graphic of Figure 5 exemplifies a 0-simplex in $R_\bullet^{\bullet-}(O^{\mathfrak{m}})$.

②. To explain the second equivalence of (14), we note that $O^{\mathfrak{m}}$ becomes a topological monoid by multiplying elements d and e in $O^{\mathfrak{m}}$ by $\gamma(e; d, 1^{g(e)})$. The multiplication map is covered by a simplicial action

$$\Psi\colon \begin{array}{ccc} O^{\mathfrak{m}} \times R_\bullet^{\bullet-}(O^{\mathfrak{m}}) & \longrightarrow & R_\bullet^{\bullet-}(O^{\mathfrak{m}}) \\ \big(d, (e, \varphi_0, \ldots, \varphi_p)\big) & \longmapsto & \big(\gamma(e; d, 1^{g(e)}), \varphi_0 + s_d, \ldots, \varphi_p + s_d\big), \end{array}$$

where $(- + s_d)$ is the translation in the $(0,\infty)$-coordinate. This action leads to a $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times UO)$-space $UO^{\bullet-}_{\bullet,\blacksquare}$, serving us as mediator between the canonical resolution and the one by arcs. On objects $([p],[k])$, it is

$$UO^{\bullet-}_{p,k} = \mathrm{hofib}_{c_{k+1}}\big(R_p^{\bullet-}(O^{\mathfrak{m}}) \to O^{\mathfrak{m}}\big) = \{(e, \varphi_0, \ldots, \varphi_p, \mu) \in R_p^{\bullet-}(O^{\mathfrak{m}}) \times \mathrm{Path}_{c_{k+1}} O^{\mathfrak{m}} \mid \omega(\mu) = e\},$$

where $\omega(-)$ denotes the endpoint of a Moore path and the elements $c_i \in O^{\mathrm{m}}$ are defined as in Section 2.2. The $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$-direction of $UO^{\bullet-}_{\bullet,\blacksquare}$ is induced by the semi-simplicial structure of $R^{\bullet-}_{\bullet}(O^{\mathrm{m}})$ via the functor $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$ (see Section 2.2). The UO-direction is defined by

$$\begin{array}{ccc} UO([k],[l]) \times UO^{\bullet-}_{\bullet,k} & \longrightarrow & UO^{\bullet-}_{\bullet,l} \\ \big((d,\mu),(e,\varphi_0,\ldots,\varphi_p,\zeta)\big) & \longmapsto & \big(\Psi(d,(e,\varphi_0,\ldots,\varphi_p)),\mu\cdot\gamma(\zeta;d,1^{k+1})\big). \end{array}$$

The claimed functoriality of $O^{\bullet-}_{\bullet,\blacksquare}$ follows directly from the associativity of the operadic composition γ. Having introduced the objects involved, the following lemma provides the weak equivalence ②.

Lemma 5.7. *The $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times UO)$-spaces $UO^{\bullet-}_{\bullet,\blacksquare}$ and $UO(\bullet,\blacksquare)$ are weakly equivalent.*

Proof. Choose arcs $\varphi^p = (\varphi^p_0,\ldots,\varphi^p_p) \in \mathrm{Emb}([-1,0],(0,\infty)\times(-1,1))^{p+1}$ such that (c_{p+1},φ^p) forms an element of $R^{\bullet-}_p(O^{\mathrm{m}})$ for which the order of the embeddings $\{\phi_i\}$ in $c_{p+1} = (s_{c_{p+1}},\{\phi_i\})$, induced by the order of the arcs φ^p_i they are connected to, agrees with the order of $\{\phi_i\}$ induced by the $(0,\infty)$-coordinate. Acting on $(c_{p+1},\varphi^p,\mathrm{const}_{c_{p+1}}) \in UO^{\bullet-}_{p,p}$, the $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times UO)$-space $UO^{\bullet-}_{\bullet,\blacksquare}$ induces a morphism of UO-spaces

$$UO([p],\blacksquare) \to UO^{\bullet-}_{p,\blacksquare}, \tag{15}$$

agreeing on $[k] \in \mathrm{ob}(UO)$ with the induced map on diagonal homotopy fibres of the commuting triangle

$$\begin{array}{ccc} O^{\mathrm{m}} & \xrightarrow{\qquad\qquad} & R^{\bullet-}_p(O^{\mathrm{m}}) \\ & {\scriptstyle \gamma(c_{p+1};-,1^{p+1})} \searrow \quad O^{\mathrm{m}} \quad \swarrow & \end{array}$$

at c_{k+1}, where the right diagonal map is the augmentation and the horizontal arrow is given by acting on (c_{p+1},φ^p) via Ψ. There is a map $R^{\bullet-}_p(O^{\mathrm{m}}) \to O^{\mathrm{m}}$ that forgets the arcs and the discs attached to them using which the horizontal map can be seen to be an equivalence by following discs along arcs they are attached to. Hence, (15) is an equivalence of UO-spaces, which in particular shows that $UO^{\bullet-}_{\bullet,\blacksquare}$ is homotopy discrete, as $UO(\bullet,\blacksquare)$ is so by Lemma 2.11. Therefore, to prove the claim, it is sufficient to show that the equivalence (15) is natural in $[p]$ up to homotopy, which would follow from the homotopy commutativity of

$$\begin{array}{ccc} UO([p],[k]) & \longrightarrow & UO^{\bullet-}_{p,k} \\ {\scriptstyle (\tilde{d}_i)_*}\downarrow & & \downarrow{\scriptstyle (d_i)_*} \\ UO([p-1],[k]) & \longrightarrow & UO^{\bullet-}_{p-1,k}, \end{array}$$

using the choice of face maps $\tilde{d}_i = (c,\mu_i) \in \widetilde{\Delta}_{\mathrm{inj}}([p-1],[p])$ provided by Lemma 2.15. The two compositions of the latter diagram map an element (d,ζ) in $UO([p],[k])$ to

$$\Big(\Psi\big(d,(c_{p+1},\varphi^p_0,\ldots,\widehat{\varphi^p_i},\ldots,\varphi^p_p)\big),\zeta\Big) \quad \text{and} \quad \Big(\Psi\big(d,(c_{p+1},\varphi^{p-1}_0+s_c,\ldots,\varphi^{p-1}_{p-1}+s_c)\big),\zeta\cdot\gamma(\mu_i;d,1^{p+1})\Big),$$

respectively, where $\widehat{(-)}$ indicates that the element is omitted. Recalling that, via the isomorphism $\pi_1(O^{\mathrm{m}},c_{p+1}) \cong B_{p+1}$ fixed in Section 2.2, the loop $\mu_i \in \Omega_{c_{p+1}}O^{\mathrm{m}}$ corresponds to the braid $b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}$ in B_{p+1}, we see that our choice of the arcs φ^j_i ensures the existence of a path in $R^{\bullet-}_{p-1}(O^{\mathrm{m}})$ between

$$(c_{p+1},\varphi^p_0,\ldots,\widehat{\varphi^p_i},\ldots,\varphi^p_p) \quad \text{and} \quad (c_{p+1},\varphi^{p-1}_0+s_c,\ldots,\varphi^{p-1}_{p-1}+s_c)$$

that maps via the augmentation $R^{\bullet-}_{p-1}(O^{\mathrm{m}}) \to O^{\mathrm{m}}$ to μ_i, or at least to its homotopy class. Such a path induces a homotopy between the two compositions of the square, which finishes the proof. □

③. For the rightmost equivalence of (14), we use the module structure θ to define the simplicial map

$$\Phi\colon \begin{array}{ccc} R^{\bullet-}_{\bullet}(O^{\mathrm{m}}) \times M & \longrightarrow & R^{\bullet-}_{\bullet}(M) \\ \big((e,\varphi_0,\ldots,\varphi_p),A\big) & \longmapsto & \big(\theta(e,A,1^{g(e)}),(\varphi_0+s_A,l(\varphi_0+s_A)),\ldots,(\varphi_0+s_A,l(\varphi_p+s_A))\big), \end{array} \tag{16}$$

using the embedding $[0,\infty)\times(-1,1)\times\{0\}^{d-2} \subseteq [0,\infty)\times\partial W$, the section $l\colon [0,\infty)\times(-1,1)^{d-1} \to E$, and the translation in the $[0,\infty)$-coordinate, as illustrated in Figure 5. This yields simplicial maps for $k \geq 0$,

$$\begin{array}{ccc} UO^{\bullet-}_{\bullet,k} \times B_k(M) & \longrightarrow & R^{\bullet-}_{\bullet}(M)^{\mathrm{fib}} \\ \big((e,\varphi_0,\ldots,\varphi_p,\mu),(A,\zeta)\big) & \longmapsto & \big(\Phi((e,\varphi_0,\ldots,\varphi_p),A),\zeta\cdot\theta(\mu;A,X^{k+1})\big), \end{array} \tag{17}$$

which induce a morphism $B(UO^{\bullet-}_{\bullet,\blacksquare}, UO, B_\blacksquare(\mathcal{M})) \to R^{\bullet-}_\bullet(\mathcal{M})^{\mathrm{fib}}$, since they equalise the diagram

$$\textstyle\coprod_{f\in UO([k],[l])} UO^{\bullet-}_{\bullet,k} \times B_l(\mathcal{M}) \underset{f_*\times\mathrm{id}}{\overset{\mathrm{id}\times f^*}{\rightrightarrows}} \coprod_{[k]} UO^{\bullet-}_{\bullet,k} \times B_k(\mathcal{M}).$$

This explains the morphism ③, which is a weak equivalence by the following lemma that completes the proof of Theorem 5.5, as the morphisms ①–③ are all compatible with the augmentation to $\mathcal{M}$.

Lemma 5.8. *The morphism ③ is a weak equivalence.*

Proof. On p-simplices, the weak equivalence ① and the morphism ③ fit into a commutative square

$$\begin{array}{ccc} B(UO([p],\blacksquare), UO, B_\blacksquare(\mathcal{M})) & \xrightarrow{①} & R_p(\mathcal{M}) \\ \simeq\downarrow & & \downarrow\simeq \\ B(UO^{\bullet-}_{p,\blacksquare}, UO, B_\blacksquare(\mathcal{M})) & \xrightarrow{③} & R^{\bullet-}_p(\mathcal{M})^{\mathrm{fib}}, \end{array}$$

in which the left morphism is induced by the equivalence (15) which has the form $UO([p],\blacksquare) \to UO^{\bullet-}_{p,\blacksquare}$ and was defined via the action of $UO^{\bullet-}_{\bullet,\blacksquare}$ on a certain element $(c_{p+1}, \varphi^p, \mathrm{const}_{c_{p+1}}) \in UO^{\bullet-}_{p,p}$. The right map is induced by the same element, but using (17), and it is a weak equivalence by an analogous argument as for (15). Consequently, the arrow ③ is a weak equivalence as well and we conclude the assertion. □

Remark 5.9. The right vertical arrow of the previous square can be enhanced to a weak equivalence up to higher coherent homotopy between $R_\bullet(\mathcal{M})$ and $R^{\bullet-}_\bullet(\mathcal{M})^{\mathrm{fib}}$, leading to an alternative proof of Theorem 5.5.

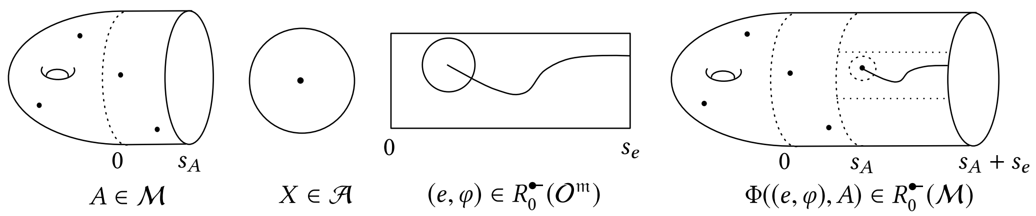


FIGURE 5. The resolution by arcs and the map Φ

5.2. **Coefficient systems for configuration spaces.** Recall from Section 2.1 that the E_1-module structure on $\mathcal{M} = \coprod_{n\geq 0} \tilde{C}^\pi_n(W)$ over $\mathcal{A} = \coprod_{n\geq 0} C_n(D^d)$ induces a right-module structure $\oplus$ on the fundamental groupoid $\Pi(\mathcal{M})$ over the braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$ and hence, after fixing a stabilising object $X \in C_1(D^n)$, also one over the free braided monoidal category $\mathcal{B}$ on one object. Denoting by $A \in C^\pi_0(W)$ the empty configuration, a coefficient system for $\coprod_{n\geq 0} \tilde{C}^\pi_n(W)$ is by Remark 4.10 specified by

(i) a $\pi_1(\tilde{C}^\pi_n(W), A \oplus X^{\oplus n})$-module M_n for each $n \geq 0$, together with
(ii) $(-\oplus X)$-equivariant morphisms $\sigma\colon M_n \to M_{n+1}$ such that B_m acts via $(A \oplus X^{\oplus n} \oplus -)$ trivially on the image of $\sigma^m\colon M_n \to M_{n+m}$.

Equivalently, a coefficient system is an abelian group-valued functor on Quillen's bracket construction

$$C^\pi(W) := \langle \textstyle\coprod_{n\geq 0} \pi_1(\tilde{C}^\pi_n(W)), \mathcal{B}\rangle,$$

compare Remark 4.12. Using the ordering of $A \oplus X^{\oplus n}$ induced by the $[0,\infty)$-coordinate, a loop γ in $\tilde{C}^\pi_n(W)$ induces a permutation in n letters, as well as n ordered loops in E by connecting the paths in E forming γ to a basepoint in E via paths in the image of the section $l\colon [0,\infty) \times (-1,1)^{d-1} \to E$. This induces a morphism

$$\pi_1(\tilde{C}^\pi_n(W)) \longrightarrow \pi_1(E) \wr \Sigma_n \tag{18}$$

to the wreath product, which we use to relate $C^\pi(W)$ to other categories via a commutative diagram

$$
\begin{array}{ccccc}
C^\pi(W) & \longrightarrow & \langle \pi_1(E) \wr \Sigma, \Sigma\rangle & & \\
\downarrow & & \downarrow & & \\
\mathcal{B}^\pi(W) & \longrightarrow & \mathcal{FI}_{\pi_1(E)} & \longrightarrow & \mathcal{FI} \\
\cap\text{|} & & \cap\text{|} & & \cap\text{|} \\
\mathcal{B}^\pi(W)^\sharp & \longrightarrow & \mathcal{FI}^\sharp_{\pi_1(E)} & \longrightarrow & \mathcal{FI}^\sharp
\end{array}
\tag{19}
$$

on which we elaborate in the following.

The category $\langle \pi_1(E) \wr \Sigma, \Sigma\rangle$ results from the action of $\Sigma = \coprod_{n\geq 0} \Sigma_n$ on $\pi_1(E) \wr \Sigma = \coprod_{n\geq 0} \pi_1(E) \wr \Sigma_n$. It receives a functor from $C^\pi(W)$, induced by the morphisms (18). The category $\mathcal{FI}_{\pi_1(E)}$ of finite sets and injective $\pi_1(E)$-maps [Cas16; GL15; Ram17; SS16a] is isomorphic to $\langle \pi_1(E) \wr \Sigma, \pi_1(E) \wr \Sigma\rangle$, so is the target of a functor from $\langle \pi_1(E) \wr \Sigma, \Sigma\rangle$, induced by the inclusion $\Sigma \subseteq \pi_1(E) \wr \Sigma$. By forgetting $\pi_1(E)$, the category $\mathcal{FI}_{\pi_1(E)}$ maps to the category $\mathcal{FI}$ of finite sets and injections, on which functors are studied in the context of representation stability (see e.g. [CEF15; CEFN14]). Both $\mathcal{FI}$ and $\mathcal{FI}_{\pi_1(E)}$ are subcategories of larger categories $\mathcal{FI}^\sharp$ and $\mathcal{FI}^\sharp_{\pi_1(E)}$ of partially defined ($\pi_1(E)$-)-injections [CEF15; SS16a]. The category of *partial braids* $\mathcal{B}^\pi(W)^\sharp$ has the nonnegative integers as its objects and a morphism from n to m is a pair (k, μ) with $k \leq \min(n, m)$ and μ a morphism in $\Pi(\tilde{C}^\pi_k(W))$ from a subset of $A \oplus X^{\oplus n}$ to one of $A \oplus X^{\oplus m}$. For trivial π, the category $\mathcal{B}^\pi(W)^\sharp$ was studied by Palmer [Pal18], who also introduced the subcategory $\mathcal{B}^\pi(W) \subseteq \mathcal{B}^\pi(W)^\sharp$ of *full braids*, consisting of morphisms $(k, \mu)\colon n \to m$ with $k = n$. There is a functor $C^\pi(W) \to \mathcal{B}^\pi(W)$ which is the identity on objects and maps a morphism

$$[\gamma] \in C^\pi(W)(n, m) = \pi_1(\tilde{C}^\pi_m(W), A \oplus X^{\oplus m})/B_{m-n}$$

to the path in $\tilde{C}^\pi_n(W)$ that forms the first n paths in E of γ, i.e. the ones starting at $A \oplus X^{\oplus n} \subseteq A \oplus X^{\oplus m}$. For $W = D^2$ and $\pi = \mathrm{id}_{D^2}$, the category $\mathcal{B}^\pi(W)$ was considered by Schlichtkrull and Solberg [SS16b].

Remark 5.10. If W is of dimension $d \geq 3$, then the morphisms (18) are isomorphisms [Til16, Lem. 4.1], from which it follows that the three left horizontal functors in the diagram (19) are isomorphisms. If E is in addition simply connected, then all functors except for the lower vertical inclusions are isomorphisms.

We call an abelian group valued functor on a category C of the diagram (19) a *coefficient system* on C. There is a notion of being of *(split) degree* r at an integer N for coefficient systems on any of the categories C, defined analogously to Definition 4.6 by using an endofunctor Σ on C together with a natural transformation $\sigma\colon \mathrm{id} \to \Sigma$, similar to $C^\pi(W)$ (see Remark 4.12). Most categories of the diagram are of the form $\langle \mathcal{N}, \mathcal{G}\rangle$ for a braided monoidal groupoid $\mathcal{G}$ acting on a category $\mathcal{N}$ and for such, Σ and σ are defined as in Remark 4.12. For $\mathcal{B}^\pi(W)^\sharp$, the functor Σ maps a morphism (k, μ) to $(k+1, s(\mu))$ using the stabilisation, and σ consists of the constant paths at $A \oplus X^{\oplus n}$. For $\mathcal{B}^\pi(W)$, we obtain Σ and σ by restriction from $\mathcal{B}^\pi(W)^\sharp$. For $\mathcal{FI}^\sharp$ and $\mathcal{FI}^\sharp_{\pi_1(E)}$, the definition is analogous. Note that the morphisms σ of the categories with a $\sharp$-superscript admit left-inverses, which results in all coefficient systems on them being split.

As all functors in the diagram are compatible with Σ and σ, the property of being of (split) degree r at N is preserved by pulling back coefficient systems along them. In conclusion, by pulling back to $C^\pi(W)$, all coefficient systems of finite degree on any of the categories in the diagram induce coefficient systems for which the homology of $\tilde{C}^\pi_n(W)$ stabilises by Theorem D. The degree of coefficient systems on some of the categories has been examined before, providing us with a wealth of examples.

Example 5.11.
(i) In [RW17], the (split) degree of coefficient systems on *prebraided monoidal categories* was introduced. This includes $\langle \pi_1(E) \wr \Sigma, \Sigma\rangle$, $\mathcal{FI}_{\pi_1(E)}$, $\mathcal{FI}^\sharp_{\pi_1(E)}$, $\mathcal{FI}$, and $\mathcal{FI}^\sharp$.
(ii) A *finitely generated* coefficient system F on $\mathcal{FI}_{\pi_1(E)}$ in the sense of [SS16a] is of finite degree, provided that $\pi_1(E)$ is finite (see [SS16a, Prop. 3.4.2]). By [SS16a, Rem. 3.4.3], this implication remains valid if $\pi_1(E)$ is virtually polycyclic (see the introduction for a definition) and even holds for arbitrary $\pi_1(E)$ if F is *presented in finite degree* or if F extends to $\mathcal{FI}^\sharp_{\pi_1(E)}$.
(iii) More quantitatively, coefficient systems on $\mathcal{FI}$ that are *generated in degree* $\leq k$ and *related in degree* $\leq d$, as defined in [CE17, Def. 4.1], are of degree k at $d + \min(k, d)$ by [RW17, Prop. 4.18].

(iv) The degree of a coefficient system on $\mathcal{B}^\pi(W)^\sharp$ has been studied by Palmer [Pal18], who also provides examples of finite degree coefficient systems on $\mathcal{FI}^\sharp$ (see [Pal18, Sect. 4]). Note that the degree and the split degree of coefficient systems on these categories coincide.

(v) For $W = D^2$ and $\pi = \mathrm{id}_{D^2}$, the category $C^\pi(W)$ is isomorphic to the category $U\mathcal{B}$ as recalled in Definition 2.6. The *Burau representation* gives rise to an example of a coefficient system of degree 1 at 0 on $U\mathcal{B}$ [RW17, Ex. 3.14]. On the basis of this example, Soulié [Sou17] has constructed coefficient systems on $U\mathcal{B}$ of arbitrary degree, using the so-called *Long-Moody construction*.

Remark 5.12. Inspired by work of Betley [Bet02], Palmer [Pal18] proved homological stability for $C_n^\pi(W)$ for trivial fibrations π and coefficient systems of finite degree on $\mathcal{B}^\pi(W)^\sharp$. His a surjectivity range agreeing with ours, but his result includes split injectivity in all degrees—a phenomenon special to configuration spaces and not captured by our general approach. In Remark 1.13, Palmer suspects stability for coefficient systems of finite degree on $\mathcal{B}^\pi(W)$. Theorem D confirms this and extends his result to a larger class of coefficient systems and nontrivial labels.

5.3. **Applications.** We complete the proofs of Corollary F and G sketched in the introduction. Unless stated otherwise, W denotes a manifold satisfying the assumptions of Theorem D.

5.3.1. *Configuration spaces of embedded discs.* Recall from the introduction the configuration spaces of (un)ordered k-discs $C_n^k(W)$ and $F_n^k(W)$ of W, the related subgroups $\mathrm{PDiff}^k_{\partial,n}(W) \subseteq \mathrm{Diff}^k_{\partial,n}(W) \subseteq \mathrm{Diff}_\partial(W)$ of diffeomorphisms fixing or permuting n chosen k-discs in W, respectively, and the orientation-preserving variants denoted with a (+)-superscript for $k = d$ and oriented W. The action of $\mathrm{Diff}_\partial(W)$ on $C_n^{\pi_k}(W)$ extends to one on $\coprod_{n\geq 0} \tilde{C}_n^{\pi_k}(W)$ by extending diffeomorphisms of W to $\tilde{W}$ via the identity. This action commutes with the E_d-action of $\coprod_n C_n(D^d)$, so the Borel construction $E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} \mathcal{M}$ inherits a graded E_1-module structure whose canonical resolution is highly-connected by Example 2.21. Consequently, Theorem A and C imply (twisted) stability for $E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} \tilde{C}_n^{\pi_k}(W)$ for $k < d$ and, as the equivalence $C_n^k(W) \to C_n^{\pi_k}(W) \subseteq \tilde{C}_n^{\pi_k}(W)$ (see the introduction for the first map) is equivariant, also for $E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} C_n^k(W)$. The same argument applies to $E\,\mathrm{Diff}^+_\partial(W) \times_{\mathrm{Diff}^+_\partial(W)} C_n^{d,+}(W)$. As announced in the introduction, we identify these homotopy quotients with classifying spaces of certain diffeomorphism groups. This proves Corollary F.

Lemma 5.13. *For $k < d$, the Borel constructions $E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} F_n^k(W)$ and $E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} C_n^k(W)$ are models for the classifying spaces $B\,\mathrm{PDiff}^k_{\partial,n}(W)$ and $B\,\mathrm{Diff}^k_{\partial,n}(W)$, respectively. For $k = d$ and W being oriented, the analogue identifications for the variants with (+)-superscripts hold.*

Proof. It suffices to show that $\mathrm{Diff}_\partial(W)$ acts transitively on $F_n^k(W)$ and $C_n^k(W)$, since the stabilisers of these actions are precisely the subgroups $\mathrm{PDiff}^k_{\partial,n}(W)$ and $\mathrm{Diff}^k_{\partial,n}(W)$, respectively. The required transitivity follows from the fact that the map $\mathrm{Diff}_\partial(W) \to \mathrm{Emb}(\coprod^n D^k, W \backslash \partial W)$, given by acting on n fixed disjoint parametrised k-discs, is by [Pal60] a fibre bundle with path-connected base space $\mathrm{Emb}(\coprod^n D^k, W\backslash\partial W) \simeq F_n^{\pi_k}(W)$. This same argument applies to $\mathrm{PDiff}^{d,+}_{\partial,n}(W)$ and $\mathrm{Diff}^{d,+}_{\partial,n}(W)$ by using orientation preserving diffeomorphisms and embeddings, as the fibre of the bundle π_d^+ of oriented d-frames is path-connected. □

5.3.2. *Representation stability.* We prove Corollary G, using the notation of the introduction.

Lemma 5.14. *Let W and π be as in Theorem D and $\lambda \vdash n$ a partition. The V_λ-multiplicity in $\mathrm{H}^i(F_n^\pi(W);\mathbf{Q})$ is the dimension of $\mathrm{H}_i(C_n^\pi(W);V_\lambda)$, where $\pi_1(C_n^\pi(W))$ acts on V_λ via the morphism $\pi_1(C_n^\pi(W)) \to \Sigma_n$.*

Proof. Delooping the covering space $\Sigma_n \to F_n^\pi(W) \to C_n^\pi(W)$ once results in a fibration sequence with base space $B\Sigma_n$. We consider the induced Serre spectral sequence, twisted by the local system V_λ on $B\Sigma_n$,

$$E^2_{p,q} \cong \mathrm{H}_p\left(B\Sigma_n; \mathrm{H}_q(F_n^\pi(W);V_\lambda)\right) \implies \mathrm{H}_{p+q}\left(C_n^\pi(W);V_\lambda\right).$$

Since the action of $\pi_1(F_n^\pi(W))$ on V_λ is trivial, we conclude

$$\mathrm{H}_p\left(B\Sigma_n; \mathrm{H}_q(F_n^\pi(W);V_\lambda)\right) \cong \mathrm{H}_p\left(B\Sigma_n; \mathrm{H}_q(F_n^\pi(W);\mathbf{Q}) \otimes V_\lambda\right).$$

These groups vanishes for $p \neq 0$ as Σ_n has no rational cohomology in positive degree. Hence, the E_2-page is trivial, except for the 0th column, which is isomorphic to the coinvariants $(\mathrm{H}_q(F_n^\pi(W);\mathbf{Q}) \otimes V_\lambda)_{\Sigma_n}$, which are in turn isomorphic to the invariants $(\mathrm{H}_q(F_n^\pi(W);\mathbf{Q}) \otimes V_\lambda)^{\Sigma_n}$. As a result of this, the spectral sequence collapses and we can identify $\mathrm{H}_q(C_n^\pi(W); V_\lambda)$ with $(\mathrm{H}_q(F_n^\pi(W);\mathbf{Q}) \otimes V_\lambda)^{\Sigma_n}$, whose dimension equals the V_λ-multiplicity in $\mathrm{H}_q(F_n^\pi(W);\mathbf{Q})$, since $V_\mu \otimes V_\lambda$ for a partition $\mu \vdash n$ contains a trivial representation if and only if $\mu = \lambda$ and, in that case, it is 1-dimensional (see [FH91, Ex. 4.51]). This proves the claim, because the V_λ-multiplicity in $\mathrm{H}^i(F_n^\pi(W);\mathbf{Q})$ equals the one in $\mathrm{H}_i(F_n^\pi(W);\mathbf{Q})$ by the universal coefficient theorem. □

Corollary 5.15. *For W and π as in Theorem D, the $V_{\lambda[n]}$-multiplicity in $\mathrm{H}^i(F_n^\pi(W);\mathbf{Q})$ is independent of n for n large relative to i.*

Proof. By [CEF15, Prop. 3.4.1], the Σ_n-representations $V_{\lambda[n]}$ assemble into a finitely generated $\mathcal{FI}$-module $V(\lambda)$ with $V(\lambda)_n \cong V_{\lambda[n]}$, which pulls back along $C^\pi(W) \to \mathcal{FI}$ of (19) to a coefficient system of finite degree for $\coprod_{n\geq 0} \tilde{C}_n(M)$ by Example 5.11 ii). Combining Theorem C with Lemma 5.14 gives the claim. □

Proof of Corollary G. Corollary 5.15 settles the statement for $F_n^\pi(W)$. To derive the claim about $F_n^k(W)$, observe that the equivalence $C_n^k(W) \to C_n^{\pi_k}(W)$ discussed in the introduction is covered by a Σ_n-equivariant equivalence $F_n^k(W) \to F_n^{\pi_k}(W)$, so we have $\mathrm{H}^i(F_n^k(W);\mathbf{Q}) \cong \mathrm{H}^i(F_n^{\pi_k}(W);\mathbf{Q})$ as Σ_n-modules. The remaining part concerning $B\,\mathrm{PDiff}^k_{\partial,n}(W)$ is shown by using the model $B\,\mathrm{PDiff}^k_{\partial,n}(W) \simeq E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} F_n^k(W)$ provided by Lemma 5.13, and adapting the argument of Lemma 5.14 and 5.15 by replacing the covering space $\Sigma_n \to F_n^\pi(W) \to C_n^\pi(W)$ with

$$\Sigma_n \to E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} F_n^k(W) \to E\,\mathrm{Diff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} C_n^k(W).$$

The statements about the variants $F_n^{d,+}(W)$ and $\mathrm{PDiff}_n^{d,+}(W)$ are proved in the same way. □

The following ranges resulted from a discussion with Peter Patzt whom the author would like to thank.

Remark 5.16. To obtain explicit ranges for Corollary G, one can show that the $\mathcal{FI}$-module $V(\lambda)$, used in the proof of Corollary 5.15, is generated in degree $|\lambda|+\lambda_1$ and related in degree $|\lambda|+\lambda_1+1$, so the corresponding coefficient system has degree $|\lambda|+\lambda_1$ at $2|\lambda|+2\lambda_1+1$ by Example 5.11 iii). Consequently, one deduces that the $V_{\lambda[n]}$-multiplicities in the cohomology groups of Corollary G are constant for $i \leq \frac{n}{2} - (|\lambda| + \lambda_1 + 1)$. Note that our range is not uniform, i.e. is dependent on the partition. In contrast, the range for $\mathrm{H}^i(F(W);\mathbf{Q})$ obtained by Church [Chu12] is $i \leq \frac{n}{2}$ if the dimension is $d \geq 3$ and $i \leq \frac{n}{4}$ for $d = 2$, at least for the manifolds W to which his result applies.

6. Moduli spaces of manifolds

Throughout the section, we fix a closed manifold P of dimension $(d-1)$, together with an embedding

$$P \subseteq \mathbf{R}^{d-1} \times \mathbf{R}^\infty$$

which contains the open unit cube $(-1,1)^{d-1} \times \{0\} \subseteq \mathbf{R}^{d-1} \times \mathbf{R}^\infty$ and satisfies $P \subseteq \mathbf{R}^{d-1} \times [0,\infty)^\infty$. We consider compact manifolds W with a specified identification $\partial W = P$ and denote by $\mathrm{Diff}_\partial(W)$ the group of diffeomorphisms fixing a neighbourhood of the boundary, equipped with the C^∞-topology. To construct our preferred model of its classifying space, we choose a collar $c\colon (-\infty,0] \times P \to W$ and denote by $\mathrm{Emb}_\varepsilon(W, (-\infty,0] \times \mathbf{R}^d \times \mathbf{R}^\infty)$ for $\varepsilon > 0$ the space of embeddings e satisfying $(e \circ c)(t,x) = (t,x)$ for $t \in (-\varepsilon, 0]$, using the C^∞-topology.

We define the *moduli space of W-manifolds* $\mathcal{M}(W)$ as the space of submanifolds

$$W' \subseteq (-\infty,0] \times \mathbf{R}^{d-1} \times \mathbf{R}^\infty$$

such that

(i) there is an $\varepsilon > 0$ with $W' \cap (-\varepsilon,0] \times \mathbf{R}^{d-1} \times \mathbf{R}^\infty = (-\varepsilon,0] \times P$ and
(ii) there is a diffeomorphism $\phi\colon W \to W'$ that satisfies $\phi \circ c|_{(-\varepsilon,0]\times P} = \mathrm{inc}_{(-\varepsilon,0]\times P}$,

where inc denotes the inclusion ensured by (i). The space $\mathcal{M}(W)$ is topologised as the quotient of

$$\mathrm{Emb}_\partial(W, (-\infty,0] \times \mathbf{R}^{d-1} \times \mathbf{R}^\infty) = \mathrm{colim}_{\varepsilon\to 0}\, \mathrm{Emb}_\varepsilon(W, (-\infty,0] \times \mathbf{R}^{d-1} \times \mathbf{R}^\infty)$$

by the action of $\mathrm{Diff}_\partial(W)$ via precomposition. The space $\mathrm{Emb}_\partial(W, (-\infty,0] \times \mathbf{R}^{d-1} \times \mathbf{R}^\infty)$ is weakly contractible by Whitney's embedding theorem, and as the action of $\mathrm{Diff}_\partial(W)$ is free and admits slices by [BF81], the moduli space $\mathcal{M}(W)$ provides a model for the classifying space $B\,\mathrm{Diff}_\partial(W)$. In the case of P being the sphere S^{d-1}, we define a weakly equivalent variant $\mathcal{M}^s(W)$ of $\mathcal{M}(W)$, consisting of submanifolds

$$W' \subseteq D^d \times \mathbf{R}^\infty$$

such that

(i) the interior of W' lies in $(D^d \backslash \partial D^d) \times (-\infty, 0]^\infty$,
(ii) there exists an $\varepsilon > 0$ for which $c' : (-\varepsilon, 0] \times S^{d-1} \longrightarrow W'$, mapping (t, x) to $((1+t)x, 0)$, is a collar, and
(iii) there is a diffeomorphism $\phi : W \longrightarrow W'$ satisfying $\phi \circ c|_{(-\varepsilon,0]\times P} = c'|_{(-\varepsilon,0]\times P}$.

We call $\mathcal{M} = \coprod_{[W]} \mathcal{M}(W)$ the *moduli space of manifolds with P-boundary*, the union taken over compact manifolds W with an identification $\partial W = P$, one in each diffeomorphism class relative to P. Analogously, the *moduli space of manifolds with sphere boundary* is $\mathcal{A} = \coprod_{[N]} \mathcal{M}^s(N)$ for N with $\partial N = S^{d-1}$.

Lemma 6.1. *The moduli space $\mathcal{A}$ of manifolds with sphere boundary forms an E_d-algebra with the moduli space $\mathcal{M}$ of manifolds with P-boundary as an E_1-module over it.*

Proof. The operad $\mathcal{D}^\bullet(D^d)$ of little d-discs acts on $\coprod_{[N]} \mathcal{M}^s(N)$ by gluing manifolds along their sphere boundary into a disc, instructed by little d-discs. Formally, define

$$\theta : \begin{array}{ccc} \mathcal{D}^k(D^d) \times (\coprod_{[N]} \mathcal{M}^s(N))^k & \longrightarrow & \coprod_{[N]} \mathcal{M}^s(N) \\ ((\phi_1, \dots, \phi_k), (N_1, \dots, N_k)) & \longmapsto & ((D^d \backslash \cup_{i=1}^k \operatorname{im} \phi_i) \times \{0\}) \cup (\cup_{i=1}^k r_i N_i + b_i), \end{array}$$

where r_i is the radius and b_i the centre of $\phi_i : D^d \longrightarrow D^d$, and $r_i N_i + b_i$ is obtained from N_i by scaling by r_i and translating by b_i, both in the D^d-coordinate. The conditions (i) and (ii) in the definition of $\mathcal{M}^s(N)$ ensure that θ is well-defined. This action extends to an action of $\mathcal{SC}_d$ (see Section 2.1) on $(\coprod_{[W]} \mathcal{M}(W), \coprod_{[N]} \mathcal{M}^s(N))$ via

$$\theta : \mathcal{SC}_d(\mathfrak{m}, \mathfrak{a}^k, \mathfrak{m}) \times \coprod_{[W]} \mathcal{M}(W) \times \big(\coprod_{[N]} \mathcal{M}^s(N)\big)^k \longrightarrow \coprod_{[W]} \mathcal{M}(W),$$

mapping $((s, \phi_1, \dots, \phi_k), M, N_1, \dots, N_k)$ to the submanifold obtained from

$$M \cup \big(([0, s] \times P) \backslash (\cup_{i=1}^k \operatorname{im} \phi_i \times \{0\})\big) \cup \big(\cup_{i=1}^k r_i N_i + b_i\big) \tag{20}$$

by translating in the first coordinate by s to the left, where $r_i N_i + b_i$ is obtained from N_i by scaling by the radius r_i of $\phi_i : D^d \longrightarrow (0, 1) \times (-1, 1)^{d-1}$ and translating by the centre b_i of ϕ_i, both in the $\mathbf{R}^d$-coordinate. Loosely speaking, we attach a cylinder to the boundary of W, glue in the N_i via the little d-discs, and shift everything to the left, as in Figure 6. This yields indeed a smooth submanifold, since the threefold union (20) is one: our conditions on $P \subseteq \mathbf{R}^{d-1} \times [0, \infty)^\infty$ and on the manifolds $N_i \subseteq D^d \times (-\infty, 0]^\infty$ in $\mathcal{M}^s(N)$ ensure that $[0, s] \times P$ and $\cup_{i=1}^k r_i N_i + b_i$ intersect only in $(0, s) \times (-1, 1)^{d-1} \times \{0\}$, which, together with properties (i)–(ii) of $\mathcal{M}^s(N)$, implies that the second union of (20) is a smooth submanifold. This manifold intersects with M only in $\{0\} \times P$, so the whole union (20) forms by property (i) of $\mathcal{M}(W)$ and (ii) of $\mathcal{M}^s(N)$ a smooth submanifold as well. □

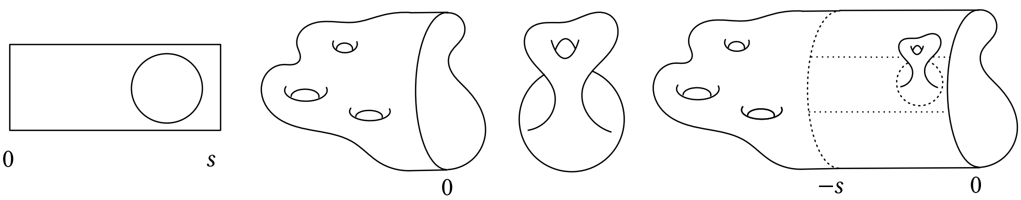


FIGURE 6. The E_1-module structure on the moduli space of manifolds

6.1. **The resolution by embeddings.** By virtue of Lemma 6.1, the moduli space $\mathcal{M}$ of manifolds with P-boundary forms in dimensions $d \geq 2$ an E_1-module over the one of manifolds with sphere boundary $\mathcal{A}$, considered as an E_2-algebra via the embedding $\mathcal{SC}_2 \to \mathcal{SC}_d$ of Section 2.1. For $A \in \mathcal{M}$ and $X \in \mathcal{A}$, the stabilised manifold $A \oplus X$ is a model for the boundary connected sum $A \natural X$ of A and X. But in contrast to the usual construction of the boundary connected sum, the manifold $A \oplus X$ contains A as a canonically embedded submanifold, and the boundary of $A \oplus X$ is canonically identified with the boundary of X (cf. Figure 6). On components, the stabilisation takes the form $s : \mathcal{M}(A) \to \mathcal{M}(A \natural X)$, modeling the map

$$s : B\operatorname{Diff}_\partial(A) \longrightarrow B\operatorname{Diff}_\partial(A \natural X)$$

induced by extending diffeomorphisms by the identity.

As we did for configuration spaces in Section 5.1, we identify the canonical resolution of $\mathcal{M}$ with an augmented semi-simplicial space $R_\bullet^{\multimap}(\mathcal{M})$ of geometric nature, which is a generalisation of one introduced by Galatius and Randal-Williams in [GR18]. To that end, denote by H_X for $X \in \mathcal{A}$ the manifold obtained from X by gluing in $[-1,0] \times D^{d-1}$ along the embedding

$$\begin{array}{ccc} \{-1\} \times D^{d-1} & \longrightarrow & \partial X = S^{d-1} \\ x & \longmapsto & (\sqrt{1-|x|}, x). \end{array}$$

The resulting manifolds is, after smoothing corners, diffeomorphic to X, but contains a canonically embedded strip $[-1,0] \times D^{d-1} \subseteq H_X$. When considering embeddings of H_X into a manifold with boundary, we always implicitly require that $\{0\} \times D^d$ is sent to the boundary and the rest of H_X to the interior.

Definition 6.2. Let W be a d-manifold, equipped with an embedding $e\colon (-\varepsilon, 0] \times \mathbf{R}^{d-1} \to W$ for an $\varepsilon > 0$, satisfying $e^{-1}(\partial W) = \{0\} \times \mathbf{R}^{d-1}$. Define a semi-simplicial space $K_\bullet^X(W)$ with the space of p-simplices given by tuples $((\varphi_0, t_0), \ldots, (\varphi_p, t_p)) \in (\mathrm{Emb}(H_X, W) \times \mathbf{R})^{p+1}$ of embeddings with parameters, such that

(i) the embeddings φ_i are pairwise disjoint,
(ii) there exists an $\delta \in (0, \varepsilon)$ such that $\phi_i(s,p) = e(s, p + t_i e_1)$ holds for $(s,p) \in (-\delta, 0] \times D^{d-1} \subseteq H_X$, where $e_1 \in \mathbf{R}^{d-1}$ is the first basis vector, and
(iii) the parameters are ordered by $t_0 < \ldots < t_p$.

The embedding space is topologised in the C^∞-topology. The ith face map forgets (φ_i, t_i).

For submanifolds $W \in \mathcal{M}$, we use the embedding $e\colon (-\varepsilon, 0] \times \mathbf{R}^{d-1} \to W$ that is obtained from the canonically embedded cube $(-\varepsilon, 0] \times (-1,1)^{d-1} \subseteq (-\varepsilon, 0] \times P \subseteq W$ by use of the diffeomorphism

$$\begin{array}{ccc} \mathbf{R} & \longrightarrow & (-1,1) \\ x & \longmapsto & \frac{2}{\pi}\arctan(x). \end{array} \tag{21}$$

The group $\mathrm{Diff}_\partial(W)$ acts simplicially on $K_\bullet^X(W)$ by precomposition, so the levelwise Borel construction results in an augmented semi-simplicial space

$$\mathrm{Emb}_\partial(W, (-\infty, 0] \times \mathbf{R}^d \times \mathbf{R}^\infty) \times_{\mathrm{Diff}_\partial(W)} K_\bullet^X(W) \longrightarrow \mathcal{M}(W) \tag{22}$$

in terms of which we define the *resolution by embeddings* as the augmented semi-simplicial space

$$R_\bullet^{\multimap}(\mathcal{M}) \to \mathcal{M}$$

obtained by taking coproducts of the semi-simplicial spaces (22) over compact manifolds W with P-boundary, one in each diffeomorphism class relative P. This is the analogue of the resolution by arcs for configuration spaces. A point in $R_\bullet^{\multimap}(\mathcal{M})$ consists of a manifold $W \in \mathcal{M}$ and $(p+1)$ embeddings of H_X into W that form an element of $K_p^X(W)$ (see the rightmost graphic of Figure 7 for an example). Since the augmentation is by construction a levelwise fibre bundle, the resolution by embeddings is fibrant. In particular, its fibre $K_\bullet^X(A)$ at $A \in \mathcal{M}$ is equivalent to the respective homotopy fibre.

Theorem 6.3. *The canonical resolution and the resolution by embeddings are weakly equivalent as augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces. In particular, $K_\bullet^X(A)$ for $A \in \mathcal{M}$ is weakly equivalent to the space of destabilisations $W_\bullet(A)$ of A.*

We closely follow the proof of Theorem 5.5 for configuration spaces to prove Theorem 6.3, adopting the notation of Section 5.1. More specifically, we construct a zig-zag of weak equivalences

$$R_\bullet(\mathcal{M}) \overset{①}{\longleftarrow} B\big(UO(\bullet, \blacksquare), UO, B_\blacksquare(\mathcal{M})\big) \overset{②}{\simeq} B\big(UO^{\multimap}_{\bullet,\blacksquare}, UO, B_\blacksquare(\mathcal{M})\big) \overset{③}{\longrightarrow} R_\bullet^{\multimap}(\mathcal{M})^{\mathrm{fib}} \tag{23}$$

of augmented $\widetilde{\Delta}_{\mathrm{inj}}$-spaces between the canonical resolution and the fibrant replacement of the resolution by embeddings—analogous to the one for configuration spaces, labelled by (14). The first equivalence ① of (14) carries over to (23) verbatim. To construct ②, we replace the semi-simplicial space $R_\bullet^{\bullet\!-}(O^{\mathrm{m}})$ with an equivalent variant $R_\bullet^{\multimap}(O^{\mathrm{m}})$, essentially by including a contractible choice of tubular neighbourhoods of the arcs. To this end, consider for $s > 0$ the simplicial space $K_\bullet^{D^d}((0,s] \times (-1,1)^{d-1})$ for which we use the embedding $e\colon (-s, 0] \times \mathbf{R}^{d-1} \to (0,s] \times (-1,1)^{d-1}$ obtained from (21) and the translation by s. Call a

0-simplex (φ, t) therein a *little d-disc with thickened tether* if the restriction $\varphi|_{D^d}\colon D^d \to (0,s)\times(-1,1)^{d-1}$ is a composition of a scaling and a translation. The embedding $\varphi\colon H_{D^d} \to (0,s]\times(-1,1)^{d-1}$ induces an arc

$$\varphi^{\bullet\!-} := \varphi|_{[-1,0]\times\{0\}}\colon [-1,0] \to (0,s)\times(-1,1)^{d-1},$$

called the *tether* of φ, which connects the little d-disc to the boundary. The embedding φ furthermore induces a trivialisation of the normal bundle of the tether, which we consider as a map $[-1,0]\to V_{d-1}(\mathbf{R}^d)$ to the space of $(d-1)$-frames in $\mathbf{R}^d$. We call a little d-disc with thickened tether (φ, t) *two-dimensional*, if

(i) the little d-disc $\varphi|_{D^d}$ is the image of a little 2-disc in $(0,s)\times(-1,1)$ under $\mathcal{SC}_2 \to \mathcal{SC}_d$ (see Section 2.1),
(ii) the induced tether $\varphi^{\bullet\!-}$ lies in the slice $(0,s)\times(-1,1)\times\{0\}^{d-2}$, and
(iii) the induced trivialisation $[-1,0]\to V_{d-1}(\mathbf{R}^d)$ equals, up to scaling by a smooth function $[-1,0]\to(0,\infty)$, the parallel transport of the frame $(e_2,\ldots,e_d)\in V_{d-1}(\mathbf{R}^d)$ at $\varphi^{\bullet\!-}(0)$ along the tether $\varphi^{\bullet\!-}$, where $e_i\in\mathbf{R}^d$ denotes the ith basis vector.

Definition 6.4. Define the augmented semi-simplicial space $R_\bullet^{\multimap}(\mathcal{O}^{\mathfrak{m}}) \to \mathcal{O}^{\mathfrak{m}}$ with p-simplices

$$R_p^{\multimap}(\mathcal{O}^{\mathfrak{m}}) \subseteq \mathcal{O}^{\mathfrak{m}} \times \big(\mathrm{Emb}(H_{D^d},(0,\infty)\times(-1,1))\times\mathbf{R}\big)^{p+1}$$

consisting of $((s,\{\phi_j\}),(\varphi_0,t_0),\ldots,(\varphi_p,t_p))$ such that $(\varphi_i,t_i)\in K_p^{D^d}((0,s]\times(-1,1)^{d-1})$ and all (φ_i,t_i) are two-dimensional little d-discs with thickened tethers whose induced little 2-disc is one of the ϕ_j. The third graphic of Figure 7 illustrates a 0-simplex of this semi-simplicial space.

As a two-dimensional little 2-disc with thickened tether is, up to a contractible choice of a thickening, determined by the associated little 2-disc and its tether, the $\widetilde{\Delta}_{\mathrm{inj}}$-spaces $R_\bullet^{\multimap}(\mathcal{O}^{\mathfrak{m}})$ and $R_\bullet^{\bullet\!-}(\mathcal{O}^{\mathfrak{m}})$ are weakly equivalent. The $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}\times U\mathcal{O})$-space $U\mathcal{O}_{\bullet,\blacksquare}^{\multimap}$ in (23) is defined in the same way as $U\mathcal{O}_{\bullet,\blacksquare}^{\bullet\!-}$, but using $R_\bullet^{\multimap}(\mathcal{O}^{\mathfrak{m}})$ instead of $R_\bullet^{\bullet\!-}(\mathcal{O}^{\mathfrak{m}})$. Making use of the equivalence between $R_\bullet^{\multimap}(\mathcal{O}^{\mathfrak{m}})$ and $R_\bullet^{\bullet\!-}(\mathcal{O}^{\mathfrak{m}})$, the proof of Lemma 5.7 carries over to the manifold case and shows that $U\mathcal{O}_{\bullet,\blacksquare}^{\multimap}$ and $U\mathcal{O}(\bullet,\blacksquare)$ weakly equivalent $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}\times U\mathcal{O})$-spaces, which establishes the equivalence ②. Finally, we construct the remaining equivalence ③ via an analogue

$$\Phi\colon \mathcal{M}\times R_\bullet^{\multimap}(\mathcal{O}^{\mathfrak{m}}) \longrightarrow R_\bullet^{\multimap}(\mathcal{M}), \tag{24}$$

of the simplicial map (16), mapping $(A,(e,(\varphi_0,t_0),\ldots,(\varphi_p,t_p)))$ to the manifold $\theta(e;A,X^{p+1})$, equipped with the embeddings of H_X obtained from the φ_i by replacing D^d by X (see Figure 7). Using (24) instead of (16) in the definition of (17), we obtain simplicial maps $U\mathcal{O}_{\bullet,k}^{\multimap}\times B_k(\mathcal{M})\to R_\bullet^{\multimap}(\mathcal{M})^{\mathrm{fib}}$, which induce a morphism of the form $B\big(U\mathcal{O}_{\bullet,\blacksquare}^{\multimap}, U\mathcal{O}, B_\blacksquare(\mathcal{M})\big)\to R_\bullet^{\multimap}(\mathcal{M})^{\mathrm{fib}}$, as in the case of configuration spaces. This is the last morphism ③ in the zig-zag (23), and it is a weak equivalence by the argument of the proof of Lemma 5.8, minorly modified using the following lemma, which completes the proof of Theorem 6.3.

Lemma 6.5. *For all* $p\geq 0$ *and elements of the form* $(c_{p+1},(\varphi_0,t_0\ldots,\varphi_p,t_p))\in R_p^{\multimap}(\mathcal{O}^{\mathfrak{m}})$, *the simplicial map* Φ *induces a weak equivalence* $\mathcal{M}\to R_p^{\multimap}(\mathcal{M})$.

Proof. The line of argument of [GR18, Lem. 6.10] for $X = D^{2p}\sharp S^p\times S^p$ generalises verbatim. □

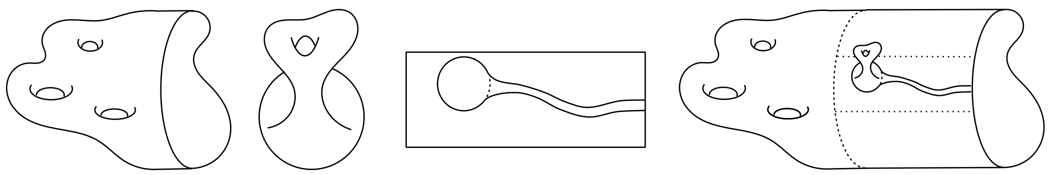


FIGURE 7. The resolution by embeddings and the map Φ

Galatius and Randal-Williams [GR18] proved high-connectivity of $K_\bullet^X(A)$ if A is simply connected and $X\cong D^{2p}\sharp(S^p\times S^p)$ for $p\geq 3$. On the basis of this, Friedrich [Fri17] and Perlmutter [Per16a] proved connectivity results for other choices of A and X. To state their results, recall the stable X-genus $\bar{g}^X$, as introduced in Section 2.3, and denote by $\mathrm{usr}(\mathbf{Z}[G])$ for a group G the *unitary stable rank* [KM02, Def. 6.3] of its group ring $\mathbf{Z}[G]$, considered as a ring with an anti-involution.

Theorem 6.6. *The realisation of $K^X_\bullet(A)$ for a connected manifold $A \in \mathcal{M}$ is*

(i) $\frac{1}{2}(\bar{g}^X(A)-4)$*-connected if $X \cong D^{2p}\sharp(S^p\times S^p)$, $p \geq 3$, and A is simply connected,*
(ii) $\frac{1}{2}(\bar{g}^X(A)-\mathrm{usr}(\mathbf{Z}[\pi_1(A)])-3)$*-connected if $X \cong D^{2p}\sharp(S^p\times S^p)$ and $p\geq 3$, and*
(iii) $\frac{1}{2}(\bar{g}^X(A)-4-m)$*-connected if $X\cong D^{p+q}\sharp(S^p\times S^q)$, $0<p<q<2p-2$, and A is $(q-p+2)$-connected, where m is the smallest number such that there exists an epimorphism of the form $\mathbf{Z}^m \to \pi_q(S^p)$.*

Proof. The first two parts are [Fri17, Thm 4.7; GR18, Cor. 5.10]. Corollary 7.3.1 of [Per16a] proves the third claim for the genus $g^X(B)$ instead of its stable variant $\bar{g}^X(B)$. However, the proof given therein goes through for $\bar{g}^X(B)$ if one replaces the relation between the genus of a manifold B satisfying the assumption in (ii) and the rank of its associated Wall form (see [Per16a, Prop. 6.1]) with the analogous statement relating the stable genus to the stable rank. □

We denote by $\bar{g}^X_A$ for a manifold $A \in \mathcal{M}$ the grading of $\mathcal{M}$ obtained by localising the stable X-genus at objects stably isomorphic to A (see Remark 2.20). Combining Theorem 6.3 with 6.6 implies the following.

Corollary 6.7. *The canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is graded*

(i) $\frac{1}{2}(\bar{g}^X_A-2)$*-connected for $X\cong D^{2p}\sharp(S^p\times S^p)$, $p\geq 3$, and any simply-connected $A\in\mathcal{M}$.*
(ii) $\frac{1}{2}(\bar{g}^X_A-\mathrm{usr}(\mathbf{Z}[\pi_1(A)])-1)$*-connected for $X\cong D^{2p}\sharp(S^p\times S^p)$, $p\geq 3$, and any connected $A\in\mathcal{M}$.*
(iii) $\frac{1}{2}(\bar{g}^X_A-2-m)$*-connected for $X\cong D^{p+q}\sharp(S^p\times S^q)$, $0<p<q<2p-2$, and any $(q-p+2)$-connected $A\in\mathcal{M}$ with m defined as in Theorem 6.6.*

Remark 6.8. In the case $d=2$, one can use [HV17, Prop. 5.1] to show that $K^X_\bullet(A)$ is $\frac{1}{2}(\bar{g}^X(A)-3)$-connected for $X\cong D^2\sharp(S^1\times S^1)$ and $A\in\mathcal{M}$ an orientable surface, which implies stability results for diffeomorphism groups of surfaces. Their homotopy discreteness [EE67; Gra73] ensures their equivalence to their mapping class group for which stability has a longstanding history, going back to a breakthrough result by Harer [Har85], improved in manifold ways since then [Bol12; CM09; Iva93; Ran16; RW17; Wah08].

By Remark 3.3, Theorem A and C apply to $\mathcal{M}$ when graded by $\bar{g}^X_A+2$, by $\bar{g}^X_A+\mathrm{usr}(\mathbf{Z}[\pi_1(A)])+1$, or by $\bar{g}^X_A+m+2$ for X and A as in the respective three cases of Corollary 6.7. On path components, this implies Theorem H, noting that in the relevant ranges, the genus and the stable genus agree (see Remark 2.24).

6.2. **Coefficient systems for moduli spaces of manifolds.** Recall from Section 4.2 that coefficient systems for the moduli space of manifolds with P-boundary $\mathcal{M}$ are defined in terms of the module structure of the fundamental groupoid $(\Pi(\mathcal{M}),\oplus)$ over the braided monoidal category $(\Pi(\mathcal{A}),\oplus,b,0)$, induced by the E_1-module structure of $\mathcal{M}$ over the moduli space of manifolds with sphere boundary $\mathcal{A}$. In the following, we provide an alternative description for the fundamental groupoids $\Pi(\mathcal{M})$ and $\Pi(\mathcal{A})$ that is more suitable to construct coefficient systems on $\mathcal{M}$.

Define the categories $\mathrm{mcg}(\mathcal{M})$ and $\mathrm{mcg}(\mathcal{A})$ having the same objects as $\Pi(\mathcal{M})$ and $\Pi(\mathcal{A})$, respectively, and the set of mapping classes $\pi_0(\mathrm{Diff}_\partial(M,N))$ as morphisms between M and N, where $\mathrm{Diff}_\partial(M,N)$ is the space of diffeomorphisms that preserve a germ of the canonical collars of M and N ensured by condition i) in the definition of $\mathcal{M}(W)$. The composition in $\mathrm{mcg}(\mathcal{M})$ and $\mathrm{mcg}(\mathcal{A})$ is the evident one.

Lemma 6.9. *The category* $\mathrm{mcg}(\mathcal{M})$ *is canonically isomorphic to* $\Pi(\mathcal{M})$*, and* $\mathrm{mcg}(\mathcal{A})$ *to* $\Pi(\mathcal{M})$*.*

Proof. Recall the fibre bundle from the construction of $\mathcal{M}(\mathcal{A})$ in the beginning of the chapter,

$$\mathrm{Diff}_\partial(A)\to\mathrm{Emb}_\partial(A,(-\infty,0]\times\mathbf{R}^d\times\mathbf{R}^\infty)\to\mathcal{M}(A).$$

Lifting a path from A to B in $\mathcal{M}$ to a path in the total space starting at the inclusion $A\subseteq(-\infty,0]\times\mathbf{R}^d\times\mathbf{R}^\infty$ gives a path of embeddings that ends at an embedding with image B and hence provides a diffeomorphism from A to B by restricting to the image. This provides a functor from $\mathrm{mcg}(\mathcal{M})$ to $\Pi(\mathcal{M})$, whose inverse is induced by considering a diffeomorphism as an embedding, choosing a path in the contractible space $\mathrm{Emb}_\partial(A,(-\infty,0]\times\mathbf{R}^d\times\mathbf{R}^\infty)$ from the inclusion $A\subseteq(-\infty,0]\times\mathbf{R}^d\times\mathbf{R}^\infty$ to the embedding obtained from the diffeomorphism, and mapping this path to $\mathcal{M}(A)$. The argument for $\mathrm{mcg}(\mathcal{A})\cong\Pi(\mathcal{A})$ is analogous. □

The module structure of $\Pi(\mathcal{M})$ over $\Pi(\mathcal{A})$ can be transported via the identification of the preceding lemma to one of $\mathrm{mcg}(\mathcal{M})$ over $\mathrm{mcg}(\mathcal{A})$, considered as a braided monoidal category by making use of the isomorphism $\mathrm{mcg}(\mathcal{A}) \cong \Pi(\mathcal{A})$. In concrete terms, the monoidal structure on $\mathrm{mcg}(\mathcal{A})$ is on objects given by the one of $\Pi(\mathcal{A})$ induced by the E_2-multiplication and on morphisms by multiplying $f \in \mathrm{Diff}_\partial(A, B)$ and $g \in \mathrm{Diff}_\partial(A', X')$ as $f \oplus g \in \mathrm{Diff}_\partial(A \oplus A', B \oplus B')$, defined by extending f and g via the identity. The description of the module structure on $\mathrm{mcg}(\mathcal{M})$ is analogous. Coefficient systems for $\mathcal{M}$ are then given by coefficient systems for the module $\mathrm{mcg}(\mathcal{M})$ over $\mathrm{mcg}(\mathcal{A})$ in the sense of Definition 4.1.

To illustrate how this identification can be used to construct coefficient systems on $\mathcal{M}$, we discuss one example in detail. Consider for $i \geq 0$ the functor $\mathrm{H}_i\colon \mathrm{mcg}(\mathcal{M}) \to \mathcal{Ab}$ that assigns a manifold $A \in \mathcal{M}$ its ith singular homology group $\mathrm{H}_i(A)$. The inclusions $A \subseteq A \oplus X$ induce a natural transformation $\sigma^{\mathrm{H}_i}\colon \mathrm{H}_i(-) \to \mathrm{H}_i(- \oplus X)$ that satisifies the triviality condition for coefficient systems (see Definition 4.1). To calculate the degree of H_i, we consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_i(A) & \longrightarrow & \mathrm{H}_i(A\oplus X) & \longrightarrow & \tilde{\mathrm{H}}_i(\mathrm{Cone}(P)\natural X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma^{\mathrm{H}_i}} & & \downarrow{\scriptstyle \sigma^{\Sigma \mathrm{H}_i}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{H}_i(A\oplus X) & \longrightarrow & \mathrm{H}_i(A\oplus X \oplus X) & \longrightarrow & \tilde{\mathrm{H}}_i(\mathrm{Cone}(P)\natural X) & \longrightarrow & 0 \end{array}$$

induced by the long exact sequence of pairs together with the equivalences $\mathrm{H}_i(A \oplus X^{\oplus k}, A \oplus X^{\oplus k-1}) \cong \tilde{\mathrm{H}}_i(\mathrm{Cone}(P)\natural X)$ obtained by collapsing $A \oplus X^{\oplus k-1}$. The leftmost vertical map is induced by the inclusion and the second one by the inclusion followed by $A \oplus b_{X,X}$. Naturality of the diagram in A implies triviality of the kernel of H_i, and also that its cokernel is constant, so of degree 0 if $\tilde{\mathrm{H}}_i(\mathrm{Cone}(P)\natural X) \neq 0$ and of degree -1 else wise. Hence, H_i is of degree 1 at 0 if $\tilde{\mathrm{H}}_i(\mathrm{Cone}(P)\natural X) \neq 0$ and of degree 0 at 0 if $\tilde{\mathrm{H}}_i(\mathrm{Cone}(P)\natural X) = 0$, from which Corollary I is implied by an application of Theorem H.

6.3. **Extensions.**

6.3.1. *Stabilisation by $(2n{-}1)$-connected $(4n{+}1)$-manifolds.* Perlmutter [Per16b] established high-connectivity of the semi-simplicial spaces $K^X_\bullet(A)$ for 2-connected manifolds A of dimension $(4n + 1)$ with $n \geq 2$ and certain specific $(2n - 1)$-connected stably-parallelisable manifolds X with finite $\mathrm{H}_{2n}(X; \mathbf{Z})$ and trivial $\mathrm{H}_{2n}(X, \mathbf{Z}/2\mathbf{Z})$. From this, he derived homological stability with constant coefficients of

$$B\,\mathrm{Diff}_\partial(A) \to B\,\mathrm{Diff}_\partial(A\natural X) \tag{25}$$

for these specific A and X. By using classification results of closed $(2n - 1)$-connected stably parallelisable $(4n + 1)$-manifolds due to Wall [Wal67] and De Sapio [De 70], he furthermore concluded that (25) stabilises in fact for all X with X having the properties described above and not just the specific ones considered before. The methods of this section can be used to extend his homological stability result to abelian coefficients and coefficient systems of finite degree.

6.3.2. *Automorphisms of topological and piecewise linear manifolds.* In [Kup15], Kupers explains how one can adapt the methods of Galatius and Randal-Williams [GR18] to prove high-connectivity of the relevant semi-simplicial spaces of locally flat embeddings to prove homological stability for classifying spaces of homeomorphisms of topological manifolds and PL-automorphisms of piecewise linear manifolds. By extending the ideas of this section, our framework applies also to these examples, resulting in an extension of Kupers' stability results to coefficients systems of finite degree.

7. Homological stability for modules over braided monoidal categories

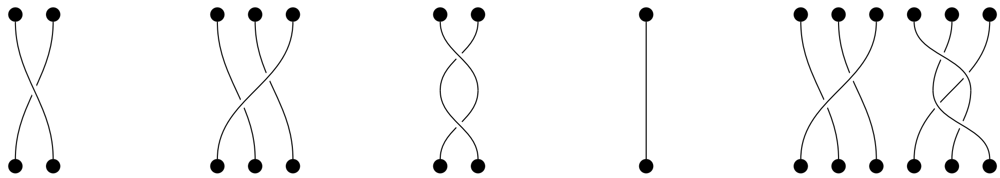
We explain the applicability of our framework to modules over braided monoidal categories, and make a comparison to the theory for braided monoidal groupoids developed by Randal-Williams–Wahl [RW17].

7.1. **E_1-modules over E_2-algebras from modules over braided monoidal categories.** Recall the categorical *operad of coloured braids $\mathcal{CoB}$* (see e.g. [Fre17, Ch. 5]). The category of n-operations is the groupoid $\mathcal{CoB}(n)$ with linear orderings of $\{1, \ldots, n\}$ as objects and braids connecting the spots as prescribed by the orderings as morphisms. The operadic composition is given by "cabling". Algebras over $\mathcal{CoB}$ are exactly strict braided monoidal categories, and the topological operad obtained by taking classifying spaces is E_2 (see e.g. [Fre17, Thm 5.2.12; FSV13, Ch. 8]). Extending this, we construct a two-coloured operad whose algebras are modules over braided monoidal categories and whose classifying space is $E_{1,2}$ (see Section 2.1).

Definition 7.1. Define a categorical operad $\mathcal{C}o\mathcal{BM}$ with colours $\mathfrak{m}$ and $\mathfrak{a}$ whose operations $\mathcal{C}o\mathcal{BM}(\mathfrak{m}^k, \mathfrak{a}^l; \mathfrak{m})$ are empty for $k \neq 1$ and equal $\mathcal{C}o\mathcal{B}(l)$ otherwise. The operations $\mathcal{C}o\mathcal{BM}(\mathfrak{m}^k, \mathfrak{a}^l; \mathfrak{a})$ are empty for $k \neq 0$ and equal $\mathcal{C}o\mathcal{B}(l)$ elsewise. Restricted to the $\mathfrak{a}$-colour, $\mathcal{C}o\mathcal{BM}$ is defined as $\mathcal{C}o\mathcal{B}$. Requiring commutativity of

$$\begin{array}{ccc} \mathcal{C}o\mathcal{BM}(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m}) \times \big(\mathcal{C}o\mathcal{BM}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \times \mathcal{C}o\mathcal{BM}(\mathfrak{a}^{i_1}; \mathfrak{a}) \times \ldots \times \mathcal{C}o\mathcal{BM}(\mathfrak{a}^{i_l}; \mathfrak{a})\big) & \xrightarrow{\gamma_{\mathcal{C}o\mathcal{BM}}} & \mathcal{C}o\mathcal{BM}(\mathfrak{m}, \mathfrak{a}^{k+i}; \mathfrak{m}) \\ \downarrow \tau & & \| \\ \mathcal{C}o\mathcal{B}(k) \times \mathcal{C}o\mathcal{B}(l) \times \mathcal{C}o\mathcal{B}(i_1) \times \ldots \times \mathcal{C}o\mathcal{B}(i_l) \xrightarrow{\mathrm{id} \times \gamma_{\mathcal{C}o\mathcal{B}}} \mathcal{C}o\mathcal{B}(k) \times \mathcal{C}o\mathcal{B}(\sum_j i_j) & \xrightarrow{\oplus} & \mathcal{C}o\mathcal{B}(k + \sum_j i_j) \end{array}$$

defines the remaining composition $\gamma_{\mathcal{C}o\mathcal{BM}}$, where τ interchanges the first two factors, $\gamma_{\mathcal{C}o\mathcal{B}}$ is the composition of $\mathcal{C}o\mathcal{B}$, and $\oplus$ is $\gamma_{\mathcal{C}o\mathcal{B}}(\mathrm{id}_{\{1<2\}}; -, -)$, i.e. puts braids next to each other (see Figure 8 for an example).



$d \in \mathcal{C}o\mathcal{BM}(\mathfrak{m}, \mathfrak{a}^2; \mathfrak{m})$ $e \in \mathcal{C}o\mathcal{BM}(\mathfrak{m}, \mathfrak{a}^3; \mathfrak{m})$ $f \in \mathcal{C}o\mathcal{BM}(\mathfrak{a}^2; \mathfrak{a})$ $g \in \mathcal{C}o\mathcal{BM}(\mathfrak{a}^1; \mathfrak{a})$ $\gamma(d; e, f, g) \in \mathcal{C}o\mathcal{BM}(\mathfrak{m}, \mathfrak{a}^6; \mathfrak{m})$

FIGURE 8. The operadic composition in $\mathcal{C}o\mathcal{BM}$

Recall the notion of a right-module $(\mathcal{M}, \oplus)$ over a monoidal category $(\mathcal{A}, \oplus, 0)$: a category $\mathcal{M}$ with a functor $\oplus\colon \mathcal{M} \times \mathcal{A} \to \mathcal{M}$ that is unital and associative up to coherent isomorphism (see Section 1.1).

Lemma 7.2. *The structure of a (graded) $\mathcal{C}o\mathcal{BM}$-algebra on a pair of categories $(\mathcal{M}, \mathcal{A})$ is equivalent to a strict (graded) braided monoidal structure on $\mathcal{A}$ and a strict (graded) right-module structure on $\mathcal{M}$ over it. Furthermore, the topological operad obtained from $\mathcal{C}o\mathcal{BM}$ by taking levelwise classifying spaces is $E_{1,2}$.*

Proof. The proof of the corresponding result for $\mathcal{C}o\mathcal{B}$ in [Fre17, Ch. 5] carries over mutatis mutandis. □

As a consequence of the previous lemma, the classifying space of a graded module over a braided monoidal category carries the structure of a graded E_1-module over an E_2-algebra.

Remark 7.3. The operad of *parenthesised coloured braids* encodes non-strict braided monoidal categories, and its classifying space operad is E_2 as well [Fre17, Ch. 6]. By considering a parenthesised version of $\mathcal{C}o\mathcal{BM}$, this extends in a similar fashion to non-strict right-modules over non-strict braided monoidal categories, whose classifying spaces hence also give E_1-modules over E_2-algebras.

7.2. **Homological stability for groups and monoids.** Let $(\mathcal{M}, \oplus)$ be a graded right-module over a braided monoidal category $(\mathcal{A}, \oplus, b, 0)$ with a stabilising object X, i.e. an object of $\mathcal{A}$ of degree 1. Taking classifying spaces results by Lemma 7.2 in a graded E_1-module $B\mathcal{M}$ over the E_2-algebra $B\mathcal{A}$ with stabilising object $X \in B\mathcal{A}$, hence provides a suitable input for Theorem A and C. In the following, we introduce a condition on $\mathcal{M}$ that ensures a simplification of the canonical resolution of $B\mathcal{M}$.

Definition 7.4. The module $(\mathcal{M}, \oplus)$ is called *injective* at an object A of $\mathcal{M}$ if the stabilisation

$$(- \oplus X^{\oplus p+1})\colon \ \mathrm{Aut}(B) \to \mathrm{Aut}(B \oplus X^{\oplus p})$$

is injective for all objects B for which $B \oplus X^{\oplus p}$ is isomorphic to A for a $p \geq 0$.

Definition 7.5. Define for an object A of $\mathcal{M}$ a semi-simplicial set $W_\bullet^{\mathrm{RW}}(A)$ with p-simplices given as equivalence classes of pairs (B, f) of an object B of $\mathcal{M}$ and a morphism $f \in \mathcal{M}(B \oplus X^{\oplus p+1}, A)$, where (B, f) and (B', f') are equivalent if there is an isomorphism $g \in \mathcal{M}(B, B')$ satisfying $f' \circ (g \oplus X^{\oplus p+1}) = f$. The ith face of a p-simplex $[B, f]$ is defined as $[B \oplus X, f \circ (B \oplus b^{-1}_{X^{\oplus i}, X} \oplus X^{\oplus p-i})]$.

Recall the spaces of destabilisations $W_\bullet(A)$, i.e. the fibres of the canonical resolution (see Definition 2.14).

Lemma 7.6. *If $\mathcal{M}$ is a groupoid, then the semi-simplicial set of path components $\pi_0(W_\bullet(A))$ for an object A of $\mathcal{M}$ is isomorphic to $W_\bullet^{\mathrm{RW}}(A)$. Moreover, $W_\bullet(A)$ is homotopy discrete if and only if $\mathcal{M}$ is injective at A.*

Proof. The inclusion of the 0-simplices $\operatorname{ob}\mathcal{M} \subseteq B\mathcal{M}$, together with the natural map $\operatorname{mor}\mathcal{M} \to \operatorname{Path}\mathcal{M}$, induces a preferred bijection $W_p^{\mathrm{RW}}(A) \to \pi_0(W_p(A))$ for all $p \geq 0$, since every path in $B\mathcal{M}$ between 0-simplices is homotopic relative to its endpoints to a one simplex, i.e. to a path in the image of $\operatorname{mor}\mathcal{M} \to \operatorname{Path}\mathcal{M}$. By the definition of the respective face maps, these bijections assemble to an isomorphism of simplicial sets, which proves the first claim. The homotopy fibre $W_p(A)$ of the map $B(-\oplus X^{\oplus p+1})\colon B\mathcal{M} \to B\mathcal{M}$ at A is homotopy discrete if and only if the induced morphisms on π_1 based at all objects B with $B \oplus X^{\oplus p+1} \cong A$ for $p \geq 0$ are injective, which is clearly equivalent to $\mathcal{M}$ being locally injective at A. □

Remark 7.7. If $\mathcal{M}$ and $\mathcal{A}$ are groupoids, then $\mathcal{M} \simeq \Pi(B\mathcal{M})$ holds naturally as a module over $\mathcal{A} \simeq \Pi(B\mathcal{A})$, so coefficient systems for $B\mathcal{M}$ (see Definition 4.13) are coefficient systems for $\mathcal{M}$ as in Section 4.1.

Remark 7.8. Since the connectivity of the canonical resolution can be tested on the spaces of destabilisations $W_\bullet(A)$ (see Remark 2.17), Lemma 7.6 and 7.7 imply a version of Theorem A and C that is phrased entirely in terms of (discrete) categories and semi-simplicial sets. This provides a simplified toolkit for proving homological stability for graded modules over braided monoidal categories with a stabilising object X for which the multiplication $(-\oplus X)\colon \operatorname{Aut}(B) \to \operatorname{Aut}(B \oplus X)$ is injective for all objects B of finite degree.

7.3. **Comparison with the work of Randal-Williams and Wahl.** Let $(\mathcal{G}, \oplus, b, 0)$ be a braided monoidal groupoid. In [RW17], it is shown that, for objects A and X in $\mathcal{G}$, the maps

$$B(-\oplus X)\colon B\operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n}) \longrightarrow B\operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n+1}) \tag{26}$$

satisfy homological stability with constant, abelian, and a class of coefficient systems if a certain family of associated semi-simplicial sets $W_n(A,X)_\bullet$ (see [RW17, Def. 2.1]) is sufficiently connected and $\mathcal{G}$ satisfies

(i) injectivity of the stabilisation map $(-\oplus X)\colon \operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n}) \to \operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n+1})$ for all $n \geq 0$,
(ii) *local cancellation at* (A,X), i.e. $Y \oplus X^{\oplus m} \cong A \oplus X^{\oplus n}$ for $Y \in \mathcal{G}$ and $1 \leq m \leq n$ implies $Y \cong A \oplus X^{\oplus m-n}$,
(iii) no zero-divisors, i.e. $U \oplus V \cong 0$ implies $U \cong 0$, and
(iv) the unit 0 has no nontrivial automorphisms.

As indicated by our choice of notation, if we consider $\mathcal{G}$ as a module over itself, the simplicial set $W_n(A,X)_\bullet$ of [RW17] equals $W_\bullet^{\mathrm{RW}}(A \oplus X^{\oplus n})$ as specified in Definition 7.5. To compare [RW17] with our work, define the module $\mathcal{G}_{A,X} = \coprod_{n\geq 0} \operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n})$ over the braided monoidal category $\mathcal{G}_X = \coprod_{n \geq 0}\operatorname{Aut}_{\mathcal{G}}(X^{\oplus n})$, both graded in the evident way. By Theorem A and C, the maps (26) stabilise homologically—without assumptions on $\mathcal{G}$—if the canonical resolution of $B\mathcal{G}_{A,X}$ is sufficiently connected, or equivalently, if the spaces of destabilisations $W_\bullet(A \oplus X^{\oplus n})$ associated to $B\mathcal{G}_{A,X}$ are (see Remark 2.17).

The semi-simplicial sets $W_n(A,X)_\bullet$ of [RW17] are equivalent to the spaces of destabilisations $W_\bullet(A \oplus X^{\oplus n})$ of $B\mathcal{G}_{A,X}$ if conditions (i) and (ii) hold. Indeed, assumption (ii) implies that $W_n(A,X)_\bullet$ agrees with the semi-simplicial set $W_\bullet^{\mathrm{RW}}(A \oplus X^{\oplus n})$ associated to $\mathcal{G}_{A,X}$ and hence also with $\pi_0(W_\bullet(A \oplus X^{\oplus n}))$ by Lemma 7.6. The first condition imposes injectivity of $\mathcal{G}_{A,X}$ at all objects $A \oplus X^{\oplus n}$, which is by Lemma 7.6 equivalent to the homotopy discreteness of the space of destabilisations $W_\bullet(A \oplus X^{\oplus n})$ of $B\mathcal{G}_{A,X}$.

Hence, if one prefers to work in a discrete setting as in [RW17], i.e. using semi-simplicial sets, condition (i) is necessary. Condition (ii) ensures that the semi-simplicial sets of [RW17] agree with our spaces of destabilisations $W_\bullet(A \oplus X^{\oplus n})$, whose high-connectivity always imply stability by Theorem A and C. The last two conditions are redundant, i.e. imposing (i) and (ii) already implies (twisted) homological stability of (26) under the connectivity assumptions of [RW17]. The presence of these additional assumptions in [RW17] is due to their usage of Quillen's construction $\langle \mathcal{G}, \mathcal{G}\rangle$ since the conditions (iii) and (iv) guarantee that the automorphism groups $\operatorname{Aut}_{\langle \mathcal{G},\mathcal{G}\rangle}(A \oplus X^{\oplus n})$ and $\operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n})$ coincide. If (i)–(iii) are satisfied and the $W_n(A,X)$ are highly-conected, then [RW17] implies stability for $\operatorname{Aut}_{\langle \mathcal{G},\mathcal{G}\rangle}(A \oplus X^{\oplus n})$. Hence, in this case, high-connectivity of $W_n(A,X)$ shows stability for both $\operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n})$ and $\operatorname{Aut}_{\langle \mathcal{G},\mathcal{G}\rangle}(A \oplus X^{\oplus n})$. The reason for this is that, although these automorphism groups might differ, their quotients $\operatorname{Aut}(A \oplus X^n)/\operatorname{Aut}(A \oplus X^{\oplus n-p-1}) \cong W_p^{\mathrm{RW}}(A \oplus X^{\oplus n})$, forming the corresponding semi-simplicial sets, agree.

Remark 7.9. The coefficient systems [RW17] deals with are functors of finite degree on the subcategory $C_{A,X} \subseteq \langle \mathcal{G}, \mathcal{G}\rangle$ generated by the objects $A \oplus X^{\oplus n}$. In contrast, Theorem C is applicable to functors of finite degree on $\langle \mathcal{G}_{A,X}, \mathcal{B}\rangle$ (see Remarks 4.12 and 7.7). As the canonical functors $\mathcal{G}_{A,X} \to \mathcal{G}$ and $\mathcal{B} \to \mathcal{G}$ induce $\langle \mathcal{G}_{A,X}, \mathcal{B}\rangle \to C_{A,X}$, every coefficient system of [RW17] gives one in ours (cf. Remarks 4.11 and 4.12).

Remark 7.10. The ranges for coefficient systems of finite degree provided by Theorem C agree with the ones of [RW17] in the situations in which their work is applicable. The ranges for abelian coefficients of Theorem A improve the ones of [RW17] marginally, and so does the surjectivity range for constant coefficients in the case $k > 2$. Note that, by Remark 3.3, these ranges can in some cases be further improved.

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