Operations on Hochschild Complexes of Hopf-like Algebras

Espen Auseth Nielsen

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Advisor:	Nathalie Wahl University of Copenhagen, Denmark
Assessment committee:	Marcel Bökstedt Aarhus University, Denmark
	Birgit Richter Universität Hamburg, Germany
	Søren Galatius (chair) University of Copenhagen, Denmark

Espen Auseth Nielsen
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 København Ø
Denmark
espen180@gmail.com
espen.sites.ku.dk

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Abstract

This thesis has two main parts. The first part, consisting of two papers, concerns the algebraic structure on Hochschild complexes of commutative Hopf algebras and their weaker cousins, such as commutative quasi-Hopf algebras and commutative Hopfish algebras. For any of the above, we equip the Hochschild complex with a natural Hopf algebra structure up to coherent homotopy.

In the first paper, we study the interplay between the Hochschild complex and the Dold-Kan equivalence between connective chain complexes and simplicial modules over a commutative ring. As an application, we obtain a strictification of the coherent commutative Hopf algebra structure on the Hochschild complex of a commutative Hopf algebra.

In the second paper we study the functoriality of the Hochschild complex with respect to bimodules. This allows us to upgrade the Hochschild complex to a symmetric monoidal functor of quasi-categories from a certain nerve of the (2,1)-category of bimodules between algebras to the quasi-category of chain complexes. This is conditioned on the assumption that the simplicial nerve of the category of non-negative chain complexes over a ring admits a symmetric monoidal structure which on the homotopy category agrees with the derived tensor product. Using the fact that certain families of Hopf-like algebras are special cases of Hopfish algebras, we obtain as an application that the Hochschild complexes of such algebras have a natural Hopf algebra structure up to coherent homotopy.

The second part of the thesis is a work in progress, generalizing the work of Wahl and Westerland on operations on Hochschild complexes to construct operations on topological Hochschild homology. Our main theorem, conditioned on a technical quasi-category-theoretical conjecture, is the construction of an action of moduli spaces of Riemann surfaces on the topological Hochschild homology of $\mathcal{A}_\infty$-Frobenius algebras.

Resumé

Denne afhandling har to hoveddele. Den første, som består af to artikler, handler om den algebraiske struktur besittet af Hochschild-komplekser af kommutative Hopf-algebraer of deres svagere varianter, som for eksempel kommutative kvasi-Hopf-algebraer og kommutative Hopfish-algebraer. For disse typer algebraer udstyrer vi Hochschild-komplexet med en naturlig Hopf-algebrastruktur op til koherent homotopi.

I våres første artikel studerer vi vekselvirkningen mellem Hochschild-komplekset og Dold-Kan-ekvivalensen mellem konnektive kedekomplekser og simplisielle moduler over en kommutativ ring. Som en anvendelse af dette opnår vi en streng modell for den koherente kommutative Hopf-algebrastrukturen på Hochschild-komplekset til en kommutativ Hopf-algebra.

I våres andre artikel studerer vi funktorialiteten til Hochschild-komplekset med hensyn til bimoduler og opgraderer Hochschild-komplekset til en symmetrisk monoidal funktor af kvasikategorier fra en givet nerve af (2,1)-kategorien af bimoduler mellem algebraer til kvasikategorien af kedekomplekse. Dette afhenger af antagelsen om, at den simplicielle nerven til kategorien af kedekomplexe kan gives en symmetrisk monoidal struktur med det deriverte tensorproduktet. Vi bruger at enkelte varianter af Hopf-algebraer er spesielle tilfelde af Hopfish-algebraer til at konstruere koherente Hopf-algebrastrukterer på Hochschild-komplexe af disse.

Den andre del af denne afhandling består af pågående arbeid, der generaliserer resultater af Wahl og Westerland om operationer på Hochschild-komplexe til operationer på topologisk

Hochschild-homologi. Våres hovedresultat, betinget på en kvasikategorisk formodning, konstruerer en virkning af modulirommet af Riemann-flader på THH af $mathcalA_\infty$-Frobenius-algebraer.

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Part 1

Introduction

This thesis consists of the following two papers (A and B) and one work in progress (C):

A. Strictifying Homotopy Coherent Actions on Hochschild Complexes
B. On the Morita Functoriality of the Hochschild Complex
C. Formal Operations on Topological Hochschild Homology (in progress)

1. Overview and Background

We begin by giving an overview of the subject and putting the papers into a bigger picture.

This thesis fits into the program laid out by Wahl-Westerland [**12, 11**]

1.1. The Dold-Kan Equivalence. The main technical tool for Paper A is the Dold-Kan equivalence. Fix a commutative ring k. The Dold-Kan equivalence is an equivalence of categories

$$M : \mathsf{sMod}_k \leftrightarrows \mathsf{Ch}_k^{\geq 0} : \Gamma$$

between simplicial k-modules and connective dg-k-modules. The functor M is called the *normalized Moore complex* and is a symmetric monoidal functor. The inverse functor Γ, however, is not symmetric monoidal, but Richter [**9**] showed that Γ is symmetric monoidal up to coherent homotopy. The failure of the Dold-Kan equivalence to be a symmetric monoidal equivalence is the main technical obstacle in Paper A.

The structure maps which witness the symmetric monoidal structure of M are the shuffle maps

$$\mathsf{sh} : M(A) \otimes M(B) \to M(A \hat{\otimes} B).$$

The shuffle maps satisfy associativity: $\mathsf{sh} \circ (\mathrm{id} \otimes \mathsf{sh}) = \mathsf{sh}(\mathsf{sh} \otimes \mathrm{id})$ as maps

$$M(A) \otimes M(B) \otimes M(C) \to M(A \hat{\otimes} B \hat{\otimes} C),$$

and commutativity, namely the following diagram commutes, where τ is the twist map.

$$\begin{tikzcd}
M(A) \otimes M(B) \arrow[r, "\tau"] \arrow[d, "\mathsf{sh}"'] & M(B) \otimes M(A) \arrow[d, "\mathsf{sh}"] \\
M(A \hat{\otimes} B) \arrow[r, "\tau_*"'] & M(B \hat{\otimes} A)
\end{tikzcd}$$

The shuffle maps are also homotopy equivalences. A homotopy inverse which also satisfies associativity is given by the Alexander-Whitney maps

$$AW : M(A \hat{\otimes} B) \to M(A) \hat{\otimes} M(B)$$

However, the Alexander-Whitney maps do not satisfy commutativity. Instead, they satisfy $\mathbb{E}_\infty$-commutativity, as proven by Richter in [**9**]. In the language of that paper, the functor M is $\mathbb{E}_\infty$-comonoidal.

1.2. Hochschild Homology. Throughout, fix a commutative ring k. Let A be an associative dg-algebra over k. The simplicial chain complex $B^{cyc}(A)$ is given in simplicial degree n by the chain complex $B^{cyc}(A)_n := A^{\otimes n+1}$, and with simplicial face maps

$$d_i(a_0, ..., a_n) = \begin{cases} (a_0, ..., a_i a_{i+1}, ..., a_n) & , 0 \leq i < n \\ (a_n a_0, a_1, ..., a_{n-1}) & , i = n \end{cases}$$

Applying the Dold-Kan equivalence at each differential degree, we get a bicomplex $MB^{cyc}(A)_{*,*}$ with first differential given by the alternating sum of the d_i's, and second differential given by the differential of A. The Hochschild complex $C(A)$ of A is the sum totalization of $MB^{cyc}(A)_{*,*}$. The homology of $C(A)$ is denoted by $HH(A)$ and is called the Hochschild homology of A.

This definition admits an extension from dg-algebras to dg-categories, which we make use of in Paper B.

The Hochschild complex plays diverse roles in algebra and geometry. In algebra, Hochschild homology admits a trace map from algebraic K-theory

$$K_*(A) \to HH_*(A).$$

In geometry, $HH(A)$ gives an algebraic model of differential forms and deRham cohomology, with the deRham differential given by Connes' B-operator.

1.3. Operads and Props. An operad $\mathcal{O}$ in a symmetric monoidal category $\mathcal{C}$ is a collection of objects $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$ equipped with morphisms

$$\mathcal{O}(k) \otimes (\mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k)) \to \mathcal{O}(j_1 + ... + j_k)$$

satisfying certain associativity, unitality and equivariance constraints. An algebra over $\mathcal{O}$ is an object $A \in \operatorname{Ob} \mathcal{C}$ equipped with morphisms

$$\mathcal{O}(k) \otimes A^{\otimes k} \to A$$

satisfying associativity, unitality an equivariance.

Operads are well suited to describing algebraic or coalgebraic structures and admit a convenient homotopy theory, but cannot describe bialgebraic structures - the case when the algebraic and coalgebraic structures interact, for example Hopf algebras.

A prop [**1, 8**] in $\mathcal{C}$ is a symmetric monoidal category P enriched in $\mathcal{C}$ such that the objects $\operatorname{Ob} P$ are in bijection with the natural numbers $\mathbb{N}$, and the symmetric monoidal structure is addition on objects. An algebra over P is an object $A \in \operatorname{Ob} \mathcal{C}$ equipped with morphisms

$$P(n, m) \otimes A^{\otimes n} \to A^{\otimes m}$$

satisfying associativity and unitality. The notion of prop is well suited to describing bialgebraic structures, at the cost of being a less well behaved theory than the theory of operads. In particular, the category of props does not admit the structure of a model category.

Operads are a special case of props, as given an operad $\mathcal{O}$, we can build a prop $P_{\mathcal{O}}$ for which

$$P_{\mathcal{O}}(n, m) = \bigoplus_{n_1 + ... + n_m = n} \mathcal{O}(n_1) \otimes ... \otimes \mathcal{O}(n_m).$$

In particular, $\mathcal{O}$ and $P_{\mathcal{O}}$ have the same algebras.

Example: Let $\mathcal{A}ss$ be the prop generated by the associative operad. In particular, $\mathcal{A}ss(n, 1) = \Sigma_n$, the symmetric group on n letters.

Example: The commutative Hopf prop $\mathcal{CH}opf$ is generated as a symmetric monoidal category by the morphisms

$$\begin{array}{lll} m : 2 \to 1 & \eta : 0 \to 1 & S : 1 \to 1 \\ \Delta : 1 \to 2 & \epsilon : 1 \to 0 & \end{array}$$

such that (m, η) define a commutative algebra, (Δ, ϵ) define an associative coalgebra, Δ and ϵ are algebra homomorphisms and the relations

$$m \circ (1 \otimes S) \circ \Delta = \eta \circ \epsilon = m \circ (S \otimes 1) \circ \Delta$$

hold. In particular, for any symmetric monoidal category C, the category of symmetric monoidal functors $\mathrm{Fun}^{\otimes}(\mathcal{CH}opf, C)$ is equivalent to the category of commutative Hopf algebra objects in C.

1.4. The Morita (2,1)-category. In Paper B we make use of a generalization of categories called (2,1)-categories. A (2,1)-category $\mathcal{C}$ is a category where instead of a set of morphisms between any two objects, we have a groupoid $\mathcal{C}(a, b)$ of morphisms, such that composition of morphisms is associative and unital up to coherent isomorphism. We also make use of notions of symmetric monoidal (2,1)-categories and 2-functor between (2,1)-categories.

Example: Any ordinary category can be viewed as a (2,1)-category with discrete morphisms groupoids. The notions of symmetric monoidal (2,1)-category and 2-functor reduce to the ordinary notions of symmetric monoidal category and functor in this case.

Example: For a commutative ring k, the Morita (2,1)-category has objects given by associative dg-k-algebras, and for each pair A, B of algebras, the groupoid $\mathsf{Mor}_k(A, B)$ is given by the groupoid of (B, A)-bimodules and bimodule isomorphisms. Composition $\mathsf{Mor}_k(B, C) \times \mathsf{Mor}_k(A, B) \to \mathsf{Mor}_k(A, C)$ is given by tensor product $(Q, P) \mapsto Q \otimes_B P$. The Morita (2,1)-category is also symmetric monoidal and admits a symmetric monoidal functor $\mathsf{m} : \mathsf{dgAlg}_k \to \mathsf{Mor}_k$.

A commutative Hopfish algebra over k [**10**] is defined as a symmetric monoidal functor $\mathcal{CH}opf \to \mathsf{Mor}_k$. This notion encompasses commutative Hopf algebras up to Morita equivalence, as well as weaker variants of Hopf algebras such as quasi-Hopf algebras [**2**], in which the co-associativity condition is relaxed.

1.5. Quasi-categories. In Paper B, we make use of the theory of quasi-categories to handle algebraic structures which are defined up to coherent homotopy. Originally defined in [**1**], the theory was developed in [**6, 7**] and is now the standard framework for homotopy theoretical category theory.

The theory is remarkably similar to the theory of ordinary categories, with just about any commonly occurring category theoretical statement generalizing to an analogous statement in quasi-category theory.

A quasi-category is a simplicial set X which admits inner horn fillers. Namely, for each $n \in \mathbb{N}$ and each $0 < i < n$, let Λ^n_i be the boundary of the n-simplex minus the face opposite the i'th vertex. Then given the solid arrows below, the dotted arrow exists such that the diagram commutes.

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}
$$

If X and Y are quasi-categories, then a functor $X \to Y$ is just a morphism of simplicial sets.

Example: Let C be a category. Then the nerve NC, given by $NC_n = \mathrm{Fun}([n], C)$, where $[n]$ is the poset $0 \to ... \to n$, is a quasi-category.

Example: In Paper B, we also consider a nerve construction for (2,1)-categories, called the Duskin nerve N_D [**3**]. If $\mathcal{C}$ is a (2,1)-category, then $N_D(\mathcal{C})$ is a quasi-category.

Example: There is also a nerve construction N_Δ for simplicial categories, i.e. categories enriched in simplicial sets, called the coherent nerve. If $\mathcal{C}$ is a simplicial category such that for any pair of objects $x, y \in \mathrm{Ob}\,\mathcal{C}$, the hom-space $\mathcal{C}(x, y)$ is a Kan complex, then $N_\Delta\mathcal{C}$ is a quasi-category.

1.6. Operations on Hochschild Homology. This thesis fits into the general program laid out by Wahl and Westerland in [**12, 11**] on the algebraic structure of Hochschild complexes. Let $\mathcal{A}ss$ denote the associative prop, i.e. the prop associated to the associative operad in Ch_k, and let P be a prop in Ch_k such that there is a symmetric monoidal functor $\mathcal{A}ss \to P$ inducing a bijection on objects. Then one can define the Hochschild complex of a P-algebra by restricting to its associative algebra structure. The goal of the program laid out in [**12, 11**] is to find, and classify, operations

$$C(A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C(A)^{\otimes n_2} \otimes A^{\otimes m^2}$$

which are natural in algebras over P. Examples of previous work include symmetric Frobenius algebras [**12**], commutative algebras [**5**] and commutative Frobenius algebras [**4**].

In Paper A we address the case when P is given by the Boardman-Vogt tensor product $\mathcal{A}ss \otimes P'$ for some prop P. In particular, this covers the case of commutative Hopf algebras.

In Paper B we consider algebraic structures which are more general than algebras over props, and so it does not fit directly into the framework of Wahl-Westerland. Instead, we make use of Hochschild complexes of dg-categories to define a Hochschild complex functor from the Morita (2,1)-category, constructed as a symmetric monoidal functor of quasi-categories, and use this to examine operations on Hochschild complexes of some variants of commutative Hopf algebras, such as commutative quasi-Hopf algebras.

1.7. Open and Open-Closed Field Theories. In Paper C we are concerned with operations on topological Hochschild homology of open conformal field theories. We describe the idea below.

The open-closed cobordism category $\mathcal{OC}$ is the quasi-category whose objects are one-dimensional manifolds with boundary, and whose space of morphisms $\mathcal{OC}(M, N)$ is the moduli space

$$\mathcal{OC}(M, N) = \coprod_{\substack{[\Gamma] \\ \bar{M} \sqcup N \hookrightarrow \partial\Gamma}} B\mathrm{Diff}(\Gamma; \bar{M} \sqcup N)$$

i.e. the moduli space of Riemannian cobordisms with corners from M to N, and composition is given by gluing cobordisms. It can be constructed as the coherent nerve of a simplicial category. The open cobordism category is the full subcategory of $\mathcal{OC}$ on the objects $\sqcup_n \mathbb{R}$. $\mathcal{O}$ and $\mathcal{OC}$ are both symmetric monoidal quasi-categories under disjoint union. An open (resp. open-closed) conformal field theory is a symmetric monoidal functor $\mathcal{O} \to \mathsf{Sp}$ (resp. $\mathcal{OC} \to \mathsf{Sp}$).

$\mathcal{O}$ is an example of a quasi-categorical prop, and algebras over $\mathcal{O}$ are homotopy coherent associative symmetric Frobenius algebras.

2. Summary of Paper A

The motivating question for Paper A is the following: For a commutative Hopf algebra A, what are the operations on the Hochschild complex $C(A)$ which are natural in homomorphisms of Hopf algebras?

The Dold-Kan equivalence is an $\mathbb{E}_\infty$-monoidal equivalence [**9**], and this implies that $C(A)$ carries a commutative Hopf algebra structure up to coherent homotopy. In this paper we give a strict model for this structure. Its ingredients are the commutative Hopf algebra structure on A and the structure maps of the Dold-Kan equivalence.

3. Summary of Paper B

Paper B generalizes the motivating question of Paper A by replacing the class of Hopf algebras with variants of Hopf algebras, such as quasi-Hopf algebras and Hopfish algebras.

For these types of algebras, we construct a natural Hopf algebra structure on their Hochschild complexes by realizing the Hochschild complex as a symmetric monoidal functor of quasi-categories from the Duskin nerve of the Morita (2,1)-category to the quasi-category of chain complexes. This is conditioned on the assumption that the simplicial nerve of the category of non-negative chain complexes over a ring admits a symmetric monoidal structure which on the homotopy category agrees with the derived tensor product.

4. Summary of Paper C

In this work in progress, we extend program laid out in [**12, 11**] for operations on Hochschild complexes to a quasi-categorical context in order to investigate operations on topological Hochschild homology.

We consider in particular the case of operations on THH of open conformal field theories, obtaining an extension of a theorem of Wahl-Westerland [**12**, Theorem 6.2] conditioned on a technical conjecture concerning homotopy coends in quasi-categories.

5. Perspectives

Throughout the papers A-C there are conjectures indicating possible lines of future research. In this section we give a broad overview.

First of all, the Hochschild complex on any associative algebra admits an operation called Connes' B-operator, which is a natural degree 1 operation

$$B : C_*(A) \to C_{*+1}(A)$$

satisfying $B \circ B = 0$. The Hochschild differential and Connes' B-operator can be used to define the *mixed complex* $CC_*(A)$, which computes cyclic homology. An immediate question is weather the operations we find on Hochschild complexes generalize to operations on the mixed complex.

Question 1: How does the homotopy coherent Hopf algebra structure carried by the Hochschild complex in the various situations in Papers A and B interact with Connes' B-operator?

Paper C itself can be read as a plan for future research, and the following questions are attractive directions that would follow its completion.

Question 2a: Can the $\mathcal{OC}$-module structure on THH of open conformal field theories be generalized to higher dimensional field theories?

Question 2b: Can we use the operations constructed in Paper C to lift the string topology operations constructed in [**12, 11**] to the level of spectra?

Bibliography

[1] Michael Boardman and Rainer Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, 1973.

[2] V. G. Drinfeld, *Quasi-Hopf algebras*, Algebra i Analiz, 1989, Volume 1, Issue 6, 114-–148

[3] John Duskin, *Simplicial matrices and the nerves of weak n-categories I: nerves of bicategories*, Theory and Applications of Categories, Vol. 9, No. 10, 2002, pp. 198-–308.

[4] Angela Klamt, *Natural operations on the Hochschild complex of commutative Frobenius algebras via the complex of looped diagrams*, arXiv:1309.4997, 19 Sep 2013.

[5] Angela Klamt, *The complex of formal operations on the Hochschild chains of commutative algebras*, arXiv:1309.7882, 30 Sep 2013.

[6] Jacob Lurie, *Higher Topos Theory*, Princeton University Press, Princeton and Oxford, 2009

[7] Jacob Lurie, *Higher Algebra*, ver. 16 May 2016.

[8] Saunders MacLane, *Natural Associativity and Commutativity*, Rice Institute Pamphlet - Rice University Studies, 49, no. 4 (1963).

[9] Birgit Richter, *Symmetry Properties of the Dold-Kan Correspondence*, Mathematical Proceedings of the Cambridge Philosophical Society, 134(1) (2003), pp. 95—102.

[10] Xiang Tang, Alan Weinstein, Chenchang Zhu, *Hopfish algebras*, arxiv:math/0510421v2, 2008.

[11] Nathalie Wahl, *Universal Operations in Hochschild Homology*, J. Reine Angew. Math. 720 (2016), 81-127.

[12] Nathalie Wahl and Craig Westerland, *Hochschild homology of structured algebras*, Adv. Math. 288 (2016), 240-307.

Part 2

Paper A

STRICTIFYING HOMOTOPY COHERENT ACTIONS ON HOCHSCHILD COMPLEXES

ESPEN AUSETH NIELSEN

Abstract. If P is a dg-operad acting on a dg-algebra A via algebra homomorphisms, then P acts on the Hochschild complex of A. In the more general case when P is a dg-prop, we show that P still acts on the Hochschild complex, but only up to coherent homotopy. We moreover give a functorial dg-replacement of P that strictifies the action. As an application, we obtain an explicit strictification of the homotopy coherent commutative Hopf algebra structure on the Hochschild complex of a commutative Hopf algebra.

1. Introduction

A *dg-prop* [9, Section 24] is a symmetric monoidal dg-category P whose monoid of objects is isomorphic to $(\mathbb{N}, +)$. If P is a dg-prop and $\mathcal{C}$ a symmetric monoidal dg-category, then a *P-algebra* in $\mathcal{C}$ is a symmetric monoidal functor $P \to \mathcal{C}$. A *morphism* of dg-props is a symmetric monoidal dg-functor which induces an isomorphism of object monoids, and such a morphism is called a *quasi-equivalence* if it induces quasi-isomorphisms on Hom-complexes. Let dgprop be the subcategory of dgCat generated by the dg-props and morphisms of dg-props. Examples of dg-props arise from dg-operads $\mathcal{O}$ (see e.g. [10, Example 60]) by the formula

$$P(n,m) = \bigoplus_{n_1+\ldots+n_m=n} \mathcal{O}(n_1) \otimes \ldots \otimes \mathcal{O}(n_m)$$

and composition defined using the composition product of $\mathcal{O}$. With this definition, an algebra over an operad is precisely an algebra over the dg-prop it generates (see e.g. [6, p.10]. In the following, we will therefore not distinguish between a dg-operad and the dg-prop it generates.

The Hochschild complex is a functor from $\mathcal{A}_\infty$-algebras to chain complexes. If P is a prop equipped with a morphism of props $\mathcal{A}_\infty \to P$, the Hochschild complex restricts to a functor from P-algebras. It is an open problem to compute the operations on the Hochschild complex of algebras over such props P. Partial results have been obtained in many cases, see e.g. [14, 15]. One such case is the following.

If P is a dg-operad (or a dg-prop) the Boardman-Vogt tensor product $\mathcal{A}ss \otimes P$ (see [1, Section II.3]) is the dg-prop characterized by the equivalence

$$\mathrm{Fun}^{\otimes}(\mathcal{A}ss \otimes P, \mathcal{C}) \simeq \mathrm{Fun}^{\otimes}(P, \mathsf{Alg}(\mathcal{C}))$$

for any symmetric monoidal dg-category $\mathcal{C}$. Evaluating the right hand side at $1 \in P$, we obtain a functor from $\mathcal{A}ss \otimes P$-algebras to $\mathcal{A}ss$-algebras. The Hochschild complex of an $\mathcal{A}ss \otimes P$-algebra is by definition the Hochschild complex of the associated $\mathcal{A}ss$-algebra. The Hochschild complex functor is lax monoidal. The structure morphisms may be used to prove that for a dg-operad P and an algebra A over the tensor product $\mathcal{A}ss \otimes P$, the Hochschild

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complex of A admits a P-algebra structure (see [2] and [14, Section 6.9]). On the other hand, this fails if P is a more general dg-prop. This is due to the failure of the Dold-Kan equivalence to be a symmetric monoidal equivalence. It is however true up to coherent homotopy, as the Dold-Kan equivalence is an $\mathbb{E}_\infty$-monoidal equivalence [12, Section 5]. In this paper, we give an explicit functorial strictification of the natural homotopy coherent P-algebra structure on the Hochschild complex of a $(P \otimes \mathcal{A}ss)$-algebra. Formally this is encoded in the following result.

Theorem A. *Let κ be an inaccessible cardinal and let k be a commutative ring with cardinality less than κ. Let Ch_k be the category of chain complexes over k with cardinality less than κ and let* dgprop *be the category of dg-props over k. There is a functor*

$$\widetilde{(-)}\colon \mathsf{dgprop} \to \mathsf{dgprop}$$

equipped with a natural quasi-equivalence $\widetilde{(-)} \to id$ and a natural transformation

$$\widetilde{\alpha}\colon \mathit{Fun}^{\otimes}(\mathcal{A}ss \otimes -, \mathsf{Ch}_k) \to \mathit{Fun}^{\otimes}(\widetilde{(-)}, \mathsf{Ch}_k)$$

of functors $\mathsf{dgprop}^{op} \to \mathsf{Cat}$ such that for a dg-prop P and an $\mathcal{A}ss \otimes P$-algebra

$$\Phi\colon \mathcal{A}ss \otimes P \to \mathsf{Ch}_k$$

the value $\widetilde{\alpha}_P(\Phi)(1)$ is equal to the Hochschild complex of $\Phi(1)$.

We use explicit generators and relations to construct the functor $\widetilde{(-)}$, fattening the input prop with the structure maps of the Dold-Kan equivalence. The functor $\widetilde{(-)}$ also admits the structure of a non-unital monad.

Example. (Example 3.14) If $\Phi\colon \mathcal{CH}opf \to \mathsf{dgAlg}_k$ is a commutative Hopf algebra over any ring k, then

$$\widetilde{\alpha}_{\mathcal{CH}opf}(\Phi)\colon \widetilde{\mathcal{CH}opf} \to \mathsf{Ch}_k$$

gives an explicit strict model for the coherent commutative Hopf algebra structure of the Hochschild complex of $\Phi(1)$.

Given a dg-prop P, one may ask whether $\widetilde{P}$ is cofibrant in a model structure on dg-props. In [7], Fresse constructs a model structure on the category of props over a field of characteristic zero, and a semi-model structure on certain sub-families of props in positive characteristic. However, for example Hopf algebras in positive characteristic cannot be treated in his framework. Additionally, in characteristic zero, our replacement $\widetilde{P}$ will not be cofibrant. For example, our replacement of the commutative prop still has a strictly commutative multiplication.

Further Questions. Theorem A displays $\widetilde{P}$ as a sub-prop of the prop of natural operations on the Hochschild complex. On the other hand, it leaves open the interaction of the P-action with Connes' B-operator. The determination of the total prop of natural operations on $C(A)$ is still an interesting open problem with a view toward operations on cyclic homology.

The structure of the paper is as follows. In Section 2 we define the Dold-Kan structure maps and their action on Hochschild complexes, and establish necessary properties. In Section 3 we define the fattening functor $\widetilde{(-)}$ for dg-props and prove the main theorem.

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2. The cyclic bar construction and the Dold-Kan equivalence

In this section we will build a dg-category $\widetilde{N}^{\Sigma}$ from the structure maps of the Dold-Kan correspondence and establish the action of $\widetilde{N}^{\Sigma}$ on Hochschild complexes of dg-algebras. This dg-category is a key ingredient for the fattening functor we will construct in Section 3.

Convention 2.1. Throughout, we fix a commutative base ring k. All algebras are assumed to be algebras over k. We employ the Kozul sign convention for chain complexes. In particular, our convention for bicomplexes are that the differentials anti-commute. Furthermore, throughout the paper we fix an inaccessible cardinal κ. All abelian groups (in particular, all simplicial modules and chain complexes over k) are assumed to have cardinality less than κ.

We begin by recalling some basic notions from homological algebra.

We will work with the categories sMod_k of simplicial k-modules and Ch_k of non-negatively graded chain complexes over k with k-linear chain maps, where we consider Ch_k as a category enriched in itself. sMod_k is a symmetric monoidal simplicial category with tensor product given by the degreewise tensor product of k-modules. We denote this tensor product by $\hat{\otimes}$. Similarly, Ch_k is a symmetric monoidal category with tensor product denoted by $\otimes$ and given by $(A \otimes B)_* = \oplus_{p+q=*} A_p \otimes B_q$ and differential $d_{A\otimes B}(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b)$. The category of monoids in sMod_k is denoted by sAlg_k, and is a symmetric monoidal category with the levelwise tensor product.

For a simplicial chain complex $A = A_{*,\bullet}$ over k, call $*$ the differential degree and $\bullet$ the simplicial degree. Write $d_A^{a,b}\colon A_{a,b} \to A_{a-1,b}$ for the differential and $d_i^{a,b}\colon A_{a,b} \to A_{a,b-1}$ for the simplicial face maps. Write sCh_k for the category of simplicial chain complexes over k.

The Dold-Kan equivalence

$$N\colon \mathsf{sMod}_k \rightleftarrows \mathsf{Ch}_k : \Gamma$$

gives an equivalence of categories between simplicial k-modules and connective chain complexes over k. The functor $N\colon \mathsf{sMod}_k \to \mathsf{Ch}_k$, is called the *normalized Moore complex* functor, and takes a simplicial k-module $M_\bullet$ to the chain complex NM_* with $NM_p = M_p/sM_{p-1}$, the quotient of M_p by the degenerate simplices, and $d\colon NM_p \to NM_{p-1}$ given by the alternating sum $d = \sum_{i=0}^{p} (-1)^i d_i$. The inverse functor $\Gamma\colon \mathsf{Ch}_k \to \mathsf{sMod}_k$ is called the Dold-Kan construction. We can also apply N degreewise to a simplicial chain complex as follows:

Definition 2.2. (1) For $A \in \mathsf{sCh}_k$, the *bicomplex associated to* $A_{*,\bullet}$ is denoted by $N_\epsilon(A_{*,\bullet})_*$ and is obtained by applying the Moore complex functor levelwise and shifting the differentials by the differential degree of A. Writing this out, we have

$N_\epsilon(A_{a,\bullet})_b = A_{a,b}/sA_{a,b-1}$, the horizontal differential is $d_h = d_A$, and the vertical differential is

$$d_v^{a,b} = (-1)^a \sum_{i=0}^{b} (-1)^i d_i^{a,b}$$

We write

$$N_\delta(A) \colon = \operatorname{Tot}\left(N_\epsilon(A_{*,\bullet})_*\right).$$

(2) Let A and B be simplicial chain complexes over k and denote by $A\hat{\otimes}B$ the simplicial chain complex which in simplicial degree p is given by $A_{*,p} \otimes B_{*,p}$. The differential of $A\hat{\otimes}B$ is given by

$$d^{n,p}_{A\hat{\otimes}B}(a \otimes b) = d_A^{|a|,p} a \otimes b + (-1)^{|a|+p} a \otimes d_B^{|b|,p} b.$$

Definition 2.3. [8, Section 5.3.2]

(1) The *cyclic bar construction* is the functor $B^{cy} \colon \mathsf{dgAlg}_k \to \mathsf{sCh}_k$ given in simplicial degree p by $B^{cy}_p(A) = A^{\otimes p+1}$. The face maps $d_i \colon B^{cy}_p(A) \to B^{cy}_{p-1}(A)$ are given by

$$d_i \colon a_0 \otimes ... \otimes a_p \mapsto \begin{cases} a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_p & , i = 0, ..., p-1 \\ (-1)^{|a_p|(|a_0|+...+|a_{p-1}|)} a_p a_0 \otimes a_1 \otimes ... \otimes a_{p-1} & , i = p \end{cases}$$

and the degeneracies $s_i \colon B^{cy}_p(A) \to B^{cy}_{p+1}(A)$ are given by

$$s_i \colon a_0 \otimes ... \otimes a_p \mapsto a_0 \otimes ... \otimes a_i \otimes 1 \otimes a_{i+1} \otimes ... \otimes a_p$$

making $B^{cy}_\bullet(A)$ into a simplicial chain complex, called the *cyclic bar construction* of A.

(2) For a dg-algebra A, the complex

$$C(A) \colon = N_\delta B^{cy}_*(A)$$

is called the *Hochschild complex* of A.

Note: The following lemma is doubtlessly well known, but the author was unable to find a reference which proves the result, so we give a proof here.

Lemma 2.4. The cyclic bar construction

$$B^{cy} \colon \mathsf{dgAlg}_k \to \mathsf{sCh}_k$$

is symmetric monoidal.

Proof. Let A and B be dg-algebras over k. Define the natural transformation $B^{cy}(A)\hat{\otimes}B^{cy}(B) \to B^{cy}(A \otimes B)$ is given in simplicial degree p by permuting tensor factors:

$$A^{\otimes p+1} \otimes B^{\otimes p+1} \xrightarrow{\sigma} (A \otimes B)^{\otimes p+1}$$

$$a_0 \otimes ... \otimes a_p \otimes b_0 \otimes ... \otimes b_p \mapsto (-1)^{\operatorname{sgn}(a,b,\sigma)} a_0 \otimes b_0 \otimes ... \otimes a_p \otimes b_p$$

where $\operatorname{sgn}(a,b,\sigma) \in \mathbb{Z}/2$ is the sign of σ weighted by the elements a_i, b_j, which can be computed as

$$\operatorname{sgn}(a,b,\sigma) \equiv \sum_{i=0}^{p-1} |b_i| \left(\sum_{j=i+1}^{p} |a_j| \right) \pmod 2$$

To check that this defines a chain map in simplicial degree p, we must verify that there are no sign issues. It is sufficient to consider each summand of the differential separately. For the differential acting on a_k for $1 \leq k \leq p+1$, the sign we get by permuting first (i.e. the sign associated to $d^{A\otimes B} \circ \sigma$) is

$$\mathrm{sgn}(a,b,\sigma) + \left(\sum_{i=0}^{k-1} |a_i| + |b_i|\right)$$

where the second term comes from the placement of a_k after permuting. The sign we get by permuting second is

$$\left(\sum_{i=0}^{k-1} |a_i|\right) + \mathrm{sgn}(a,b,\sigma) + \sum_{j=0}^{k-1} |b_j|$$

where the third term is the correction to $\mathrm{sgn}_{a,b}\sigma$ when the degree of a_k is decreased by one. We see that the two are equal. For the differential acting on b_k, the sign we get by permuting first is

$$\mathrm{sgn}(a,b,\sigma) + \left(\sum_{i=0}^{k-1} |a_i| + |b_i|\right) + |a_k|$$

and the sign we get by permuting second is

$$\left(\sum_{i=0}^{p} |a_i|\right) + \left(\sum_{j=0}^{k-1} |b_j|\right) + \mathrm{sgn}(a,b,\sigma) + \sum_{j=k+1}^{p} |a_j|$$

where the fourth term is the correction to $\mathrm{sgn}_{a,b}\sigma$ when the degree of b_k is decreased by one. Again we see that the two expressions are equal mod 2, hence we have a chain map.

We now verify that σ is a symmetric monoidal transformation. Let τ be the symmetric monoidal twist map of Ch_k, given by $A\otimes B \to B \otimes A$, $a\otimes b \mapsto (-1)^{|a||b|} b\otimes a$. We check that $\sigma \circ \tau_{p+1,p+1} = (\tau^{\otimes p+1}) \circ \sigma$. The left hand side has sign

$$\mathrm{sgn}_L = \left(\sum_{i=0}^{p} |a_i|\right)\left(\sum_{j=0}^{p} |b_j|\right) + \sum_{i=0}^{p-1} |a_i| \left(\sum_{j=i+1}^{p} |b_j|\right) = \sum_{i=0}^{p} |a_i| \left(\sum_{j=0}^{i} |b_j|\right)$$

and the right hand side has sign

$$\mathrm{sgn}_R = \sum_{i=0}^{p-1} |b_i| \left(\sum_{j=i+1}^{p} |a_j|\right) + \left(\sum_{i=0}^{p} |a_i||b_i|\right) = \sum_{i=0}^{p} |b_i| \left(\sum_{j=i}^{p} |a_j|\right)$$

Rearranging the order of summation shows that $\mathrm{sgn}_L = \mathrm{sgn}_R$, and we conclude that B^{cy} is symmetric monoidal as claimed. $\square$

Before discussing the monoidality properties of the cyclic bar construction and the Hochschild complex, we recall some of the monoidality properties of the Dold-Kan equivalence.

Definition 2.5. [4, 5.3] Let A and B be simplicial k-modules. The *shuffle map* (also called the *Eilenberg-Zilber map*):

$$\mathsf{sh}_{A,B} \colon N(A) \otimes N(B) \to N(A \hat{\otimes} B)$$

is defined on elementary tensors $a \otimes b \in A_p \otimes B_q$ as

$$\mathsf{sh}_{A,B}(a\otimes b) = \sum_{\sigma\in\Sigma_{(p,q)}} \mathrm{sgn}(\sigma) s_{\sigma(p+q)} \ldots s_{\sigma(p+1)} a \otimes s_{\sigma(p)} \ldots s_{\sigma(1)} b$$

When there is no risk of confusion, we will omit A and B from the notation and simply write sh for the shuffle map.

Definition 2.6. [5, 2.9] Let A and B be simplicial k-modules. The *Alexander-Whitney map*

$$AW_{A,B}\colon N(A\hat{\otimes}B) \to N(A)\otimes N(B)$$

is defined on elementary tensors $a\otimes b \in A_n \otimes B_n$ as

$$AW_{A,B}\colon (a\otimes b) \mapsto \sum_{i=0}^{n} d_{i+1}...d_{n-1}d_n a \otimes (d_0)^i b$$

As with the shuffle map, we omit A, B from the notation $AW_{A,B}$ when there is no risk of confusion.

Lemma 2.7. [4, Theorem 5.4] Let A, B and C simplicial k-modules. Then

- $\mathsf{sh}_{A\otimes B,C}\circ(\mathsf{sh}_{A,B}\otimes \mathrm{id}) = \mathsf{sh}_{A,B\otimes C}\circ(\mathrm{id}\otimes \mathsf{sh}_{B,C})$, i.e. the shuffle maps are associative.
- For $a \in A_p$ and $b \in B_q$ and τ denoting the twist morphism, we have $\mathsf{sh}_{A,B}(a\otimes b) = (-1)^{pq}\tau_*\mathsf{sh}_{B,A}(b\otimes a)\tau^{-1}$, i.e. the shuffle maps are graded symmetric.

Lemma 2.8. [5, Theorem 2.1] The shuffle and Alexander-Whitney maps are mutually inverse natural homotopy equivalences.

Lemma 2.9. [5, Corollary 2.2] The Alexander-Whitney map is associative, i.e. for A, B and C simplicial k-modules, the morphisms $(\mathrm{id}\otimes AW_{B,C})\circ AW_{A,B\hat{\otimes}C}$ and $(AW_{A,B}\otimes\mathrm{id})\circ AW_{A\hat{\otimes}B,C}$ from $N(A\hat{\otimes}B\hat{\otimes}C)$ to $N(A)\otimes N(B)\otimes N(C)$ are equal.

Proof. Let $a\otimes b\otimes c \in A_n\otimes B_n\otimes C_n$. For brevity, we write $\widetilde{d_i^n} = d_{i+1}...d_{n-1}d_n$. Then the two compositions

$$N(A\hat{\otimes}B\hat{\otimes}C) \to N(A)\otimes N(B)\otimes N(C)$$

are

$$(a\otimes b\otimes c) \xmapsto{AW_{A,B\hat{\otimes}C}} \sum_{p=0}^{n} \widetilde{d_p^n}a \otimes d_0^p b\otimes d_0^p c$$

$$\xmapsto{\mathrm{id}\otimes AW_{B,C}} \sum_{p=0}^{n}\sum_{s=0}^{n-p} \widetilde{d_p^n}a\otimes \widetilde{d_s^{n-p}}d_0^p b\otimes d_0^{p+s}c$$

and

$$(a\otimes b\otimes c) \xmapsto{AW_{A\hat{\otimes}B,C}} \sum_{q=0}^{n} \widetilde{d_q^n}a \otimes \widetilde{d_q^n} b\otimes d_0^q c$$

$$\xmapsto{AW_{B,C}\otimes\mathrm{id}} \sum_{q=0}^{n}\sum_{t=0}^{q} \widetilde{d_t^q}\widetilde{d_q^n}a\otimes d_0^t\widetilde{d_q^n} b\otimes d_0^{q}c.$$

Note that $\widetilde{d_t^q}\widetilde{d_q^n} = \widetilde{d_t^n}$. Using the simplicial identity $d_id_j = d_{j-1}d_i$ when $i<j$, observe that $d_0^t\widetilde{d_q^n} = \widetilde{d_{q-t}^{n-t}}d_0^t$. Writing $(q,t) = (p+s,p)$, we now see that the two expressions are equal. □

To summarize the above theorems, the shuffle and Alexander-Whitney maps are mutually inverse quasi-isomorphisms. In particular, $AW\circ\mathsf{sh} = \mathrm{id}$ and $\mathsf{sh}\circ AW \simeq \mathrm{id}$. The shuffle map is a lax symmetric monoidal transformation witnessing that the normalized Moore complex functor $N\colon \mathsf{sMod}_k \to \mathsf{Ch}_k$, and hence also the Hochschild chains functor $C\colon \mathsf{sAlg}_k \to \mathsf{Ch}_k$ is lax symmetric monoidal. The Alexander-Whitney map is an oplax monoidal transformation witnessing that N, and hence C is oplax monoidal. However, the Alexander-Whitney map is *not* symmetric. Still, it is $\mathbb{E}_\infty$ in the following sense (see Lemma 2.13).

Definition 2.10. ([11, p.552]) A functor $F\colon \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories is $\mathbb{E}_\infty$*-monoidal* if there is an $\mathbb{E}_\infty$ operad $\mathcal{O}$ in $\mathcal{D}$ and maps

$$\mu_n\colon \mathcal{O}(n) \otimes (F(A_1) \otimes ... \otimes F(A_n)) \to F(A_1 \otimes ... \otimes A_n)$$

such that

(1) the action is unital, i.e. if I denotes the monoidal unit of $\mathcal{D}$ and $\eta\colon I \to \mathcal{O}(1)$ is the unit of the operad, then the following diagram commutes:

$$\begin{tikzcd} I \otimes F(A) \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\simeq"'] & \mathcal{O}(1) \otimes F(A) \arrow[d, "\mu_1"] \\ & F(A) \end{tikzcd}$$

(2) The action is equivariant: for each $\sigma \in \Sigma_n$, the action μ_n is compatible with the action of Σ_n on $\mathcal{O}(n)$ and by permuting indices of the A_i. I.e. the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(n) \otimes F(A_1) \otimes ... \otimes F(A_n) \arrow[r, "\mu_n"] \arrow[d, "\sigma \otimes \sigma"] & F(A_1 \otimes ... \otimes A_n) \arrow[d, "F(\sigma)"] \\ \mathcal{O}(n) \otimes F(A_{\sigma^{-1}(1)}) \otimes ... \otimes F(A_{\sigma^{-1}(n)}) \arrow[r, "\mu_n"] & F(A_{\sigma^{-1}(1)} \otimes ... \otimes A_{\sigma^{-1}(n)}) \end{tikzcd}$$

(3) The action is associative, i.e. is compatible with the operad multiplication.

$\mathbb{E}_\infty$-comonoidal functors are similarly defined by using structure maps

$$\nu_n\colon \mathcal{O}(n) \otimes F(A_1 \otimes ... \otimes A_n) \to F(A_1) \otimes ... \otimes F(A_n).$$

We will now define chain complexes which assemble into a dg-operad (and later a symmetric monoidal dg-category) witnessing that AW is an $\mathbb{E}_\infty$-comonoidal transformation.

Definition 2.11. Define the functors

$$N^{\hat{\otimes} n}, N^{\otimes n}\colon \mathsf{sMod}_k^{\times n} \to \mathsf{Ch}_k$$

given by

$$N^{\hat{\otimes} n}(A_1, ..., A_n) = N(A_1 \hat{\otimes} ... \hat{\otimes} A_n)$$
$$N^{\otimes n}(A_1, ..., A_n) = N(A_1) \otimes ... \otimes N(A_n)$$

and let

$$\mathcal{O}(n) := \mathrm{Nat}_{\mathsf{sMod}_k^{\times n}}(N^{\hat{\otimes} n}, N^{\otimes n})$$

Notation 2.12. Since the elements of $\mathcal{O}(n)$, and of the complex $\widetilde{N}^{\Sigma}((n),(n))$ which we define below, are natural transformations, we can in particular view them as 2-morphisms in the 2-category of categories, and so 2-categorical constructions, like horizontal composition, can be applied to them. For a 4-tuple of morphisms $f, f'\colon a \to b$ and $g, g'\colon b \to c$ and a pair of 2-morphisms $\alpha\colon f \to f'$ and $\beta\colon g \to g'$, we write $\beta * \alpha$ for their horizontal composition $\beta * \alpha\colon gf \to g'f'$.

Lemma 2.13. The complexes $\mathcal{O}(n)$ assemble into an $\mathbb{E}_\infty$ operad witnessing that $N^{\otimes n}$ and $N^{\hat{\otimes} n}$ are $\mathbb{E}_\infty$-comonoidal functors and that $AW\colon N^{\otimes 2} \to N^{\hat{\otimes} 2}$ is an $\mathbb{E}_\infty$-comonoidal transformation.

Proof. Let $n_1 + ... + n_i = n$ be natural numbers. The operad structure on $\mathcal{O}$ is given by the maps

$$\mathcal{O}(i) \otimes (\mathcal{O}(n_1) \otimes ... \otimes \mathcal{O}(n_i)) \to \mathcal{O}(n)$$

given by $(\phi, \gamma_1, ..., \gamma_i) \mapsto \phi \circ (\gamma_1 * ... * \gamma_i)$. The Σ_n-action is given by conjugation, i.e. for $\chi \in \Sigma_n$ and $\psi \in \mathcal{O}(n)$ we have $\chi \cdot \psi = \chi \circ \psi \circ \chi^{-1}$. It is known (see [3, Satz 1.6]) that the complex of natural transformations

$$\mathcal{O}(n) = \mathrm{Nat}_{\mathsf{sMod}_k^{\times n}}(N^{\hat{\otimes} n}, N^{\otimes n})$$

is acyclic with zero'th homology k. It follows (see [11, Section 7] and [12, Section 5]) that the functors N is an $\mathbb{E}_\infty$-comonoidal functor and that $AW\colon N^{\otimes 2} \to N^{\hat{\otimes} 2}$ is an $\mathbb{E}_\infty$-comonoidal transformation. □

We will look at the complex

$$\widetilde{N}((n),(n)) := \mathrm{Nat}_{\mathsf{sMod}_k^{\times n}}(N^{\hat{\otimes} n}, N^{\hat{\otimes} n})$$

which is homotopy equivalent to $\mathrm{Nat}_{\mathsf{sMod}_k^{\times n}}(N^{\hat{\otimes} n}, N^{\otimes n})$, seen by post-composing with shuffle and Alexander-Whitney maps, but with the difference that maps in $\widetilde{N}((n),(n))$ may be composed, giving rise to an algebra structure. In the rest of this section, we will construct a dg-category with morphism complexes built from $\mathrm{Nat}_{\mathsf{sMod}_k^{\times n}}(N^{\hat{\otimes} n}, N^{\hat{\otimes} n})$, and the notation is chosen with this in mind.

Definition 2.14. The symmetric group Σ_n acts on $\mathsf{sMod}_k^{\times n}$ by

$$\chi(A_1, ..., A_n) = (A_{\chi^{-1}(1)}, ..., A_{\chi^{-1}(n)}).$$

Let $\widetilde{N}^\Sigma((n),(n))$ be the complex

$$\widetilde{N}^\Sigma((n),(n)) = \bigoplus_{\chi \in \Sigma_n} \mathrm{Nat}_{\mathsf{sMod}^{\times n}}(N^{\hat{\otimes} n}, N^{\hat{\otimes} n} \circ \chi) =: \bigoplus_{\chi \in \Sigma_n} \widetilde{N}^\Sigma_\chi((n),(n)).$$

Lemma 2.15. The chain complex $\widetilde{N}^\Sigma((n),(n))$ admits a Σ_n-graded algebra structure and contracts to $k\Sigma_n$ in degree 0.

Proof. Let $f \in \widetilde{N}^\Sigma_\chi((n),(n))$ and $g \in \widetilde{N}^\Sigma_{\chi'}((n),(n))$. We treat f and g as 2-morphisms in the 2-category of dg-categories as in Remark 2.12. The product of g and f is given by $(g * \mathrm{id}_\chi) \circ f\colon N^{\hat{\otimes} n} \to N^{\hat{\otimes} n} \circ (\chi'\chi)$, which may also be visualized by the pasting diagram

$$\begin{tikzcd}
\mathsf{sMod}_k^{\times n} \arrow[d, "\chi"] \arrow[drr, "N^{\hat{\otimes} n}"] \arrow[dr, phantom, "\Downarrow f"] & & \\
\mathsf{sMod}_k^{\times n} \arrow[rr, "N^{\hat{\otimes} n}"] \arrow[d, "\chi'"] \arrow[dr, phantom, "\Downarrow g"] & & \mathsf{Ch}_k \\
\mathsf{sMod}_k^{\times n} \arrow[urr, "N^{\hat{\otimes} n}"'] & &
\end{tikzcd}$$

This gives the graded algebra structure. As for the contraction, the components $\widetilde{N}^{\Sigma}_{\chi}((n),(n))$ are isomorphic to $\widetilde{N}((n),(n))$ by pre-composition by χ and χ^{-1}. As $\widetilde{N}((n),(n))$ contracts onto $\mathrm{id}_{N^{\hat{\otimes} n}}$, $\widetilde{N}^{\Sigma}_{\chi}((n),(n))$ contracts similarly to χ. □

Definition 2.16. • For $A_1, ..., A_k$ simplicial k-modules, we introduce the shorthand

$$N^{(k_1,...,k_n)}(A_1,...,A_k) = N(A_1\hat{\otimes}...\hat{\otimes}A_{k_1}) \otimes ... \otimes N(A_{k_1+...+k_{n-1}+1}\hat{\otimes}...\hat{\otimes}A_k)$$

where $k = k_1 + ... + k_n$. Let $m_1 + ... + m_l = k$. Writing $\vec{k} = (k_1, ..., k_n)$ and similarly for $\vec{m}$, define the complex

$$\widetilde{N}(\vec{k},\vec{m}) := \mathrm{Nat}_{\mathsf{sMod}_k^{\times k}}(N^{\vec{k}}, N^{\vec{m}})$$

Its symmetrized version $\widetilde{N}^{\Sigma}(\vec{k},\vec{m})$ is defined as before by

$$\widetilde{N}^{\Sigma}(\vec{k},\vec{m}) = \bigoplus_{\chi\in\Sigma_k} \mathrm{Nat}_{\mathsf{sMod}_k^{\times k}}(N^{\vec{k}}, N^{\vec{m}}\circ\chi) =: \bigoplus_{\chi\in\Sigma_k} \widetilde{N}^{\Sigma}_{\chi}(\vec{k},\vec{m})$$

• We will refer to a finite sequence of integers $\vec{k} = (k_1, ..., k_n)$ as a *vector*. The sum of the entries of a vector is called its *length* and denoted by $|\vec{k}| := k_1 + ... + k_n$.
• We write $\widetilde{N}^{\Sigma}$ for the dg-category whose objects are vectors $\vec{k}$, and whose morphism complexes are given by the $\widetilde{N}^{\Sigma}(\vec{k},\vec{m})$ defined above.

Notation 2.17. For any $\vec{k} = (k_1, ..., k_n)$ with $|\vec{k}| = k$, by Lemma 2.7 composing shuffle maps gives rise to a well-defined shuffle map which we denote by $\mathsf{sh}_{\vec{k}} \colon N^{\vec{k}} \to N^{(k)} = N^{\hat{\otimes} k}$. Similarly, the Alexander-Whitney map is associative by Lemma 2.9, so composing AW-maps gives rise to a well-defined map $AW_{\vec{k}} \colon N^{(k)} \to N^{\vec{k}}$. Note that $AW_{\vec{k}} \circ \mathsf{sh}_{\vec{k}} \simeq \mathrm{id}_{N^{\vec{k}}}$ and $\mathsf{sh}_{\vec{k}} \circ AW_{\vec{k}} \simeq \mathrm{id}_{N^{(k)}}$.

Lemma 2.18. For every pair $\vec{k} = (k_1, ..., k_n), \vec{m} = (m_1, ..., m_l)$, the assignment

$$\phi\colon f \mapsto AW_{\vec{m}} \circ f \circ \mathsf{sh}_{\vec{k}}$$

defines a homotopy equivalence $\widetilde{N}^{\Sigma}((n),(n)) \to \widetilde{N}^{\Sigma}(\vec{k},\vec{m})$ with homotopy inverse

$$\psi\colon g \mapsto \mathsf{sh}_{\vec{m}} \circ g \circ AW_{\vec{k}}$$

In particular, $\widetilde{N}^{\Sigma}(\vec{k},\vec{m})$ contracts onto the degree zero subcomplex of elements of the form $AW_{\vec{m}} \circ \chi_* \circ \mathsf{sh}_{\vec{k}}$ for some $\chi \in \Sigma_k$.

Proof. Fix homotopies $\alpha_{\vec{k}} \colon AW_{\vec{k}}\mathsf{sh}_{\vec{k}} \to \mathrm{id}$ and $\beta_{\vec{k}} \colon \mathsf{sh}_{\vec{k}}AW_{\vec{k}} \to \mathrm{id}$. Then we get homotopies

$$\beta_{\vec{m}} * \mathrm{id} * \beta_{\vec{k}} \colon \psi\circ\phi \to \mathrm{id}$$

$$\alpha_{\vec{m}} * \mathrm{id} * \alpha_{\vec{k}} \colon \phi\circ\psi \to \mathrm{id}$$

so that ϕ and ψ are mutually inverse homotopy equivalences. Now the composition

$$k\Sigma_n \hookrightarrow \widetilde{N}^\Sigma((n),(n)) \to \widetilde{N}^\Sigma(\vec{k},\vec{m})$$

takes χ to $AW_{\vec{m}} \circ \chi_* \circ \mathsf{sh}_{\vec{k}}$ and is a homotopy equivalence since $k\Sigma_n \hookrightarrow \widetilde{N}^\Sigma((n),(n))$ is by Lemma 2.15. The inverse

$$\widetilde{N}^\Sigma(\vec{k},\vec{m}) \to \widetilde{N}^\Sigma((n),(n)) \to k\Sigma_n$$

sends $AW_{\vec{m}} \circ \chi_* \circ \mathsf{sh}_{\vec{k}}$ to χ, so $\widetilde{N}^\Sigma(\vec{k},\vec{m})$ contracts as claimed. □

We now turn to establishing the action of $\widetilde{N}^\Sigma$ on Hochschild complexes of dg-algebras. We begin by constructing a way of differentially extending functors between additive categories.

Construction 2.19. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and let $\mathrm{Fun}^{pt}(\mathcal{A},\mathcal{B})$ be the category of pointed functors between them, that is, functors $F\colon \mathcal{A} \to \mathcal{B}$ such that there is an isomorphism $F(0) \simeq 0$. Note that $\mathrm{Fun}^{pt}(\mathcal{A},\mathcal{B})$ is itself an additive category. Denote by m-$\mathsf{Ch}(B)$ the additive category of m-fold chain complexes in $\mathcal{B}$. We will produce an additive functor

$$(-)_\epsilon\colon \mathrm{Fun}^{pt}(\mathcal{A}^{\times n}, m\text{-}\mathsf{Ch}(\mathcal{B})) \to \mathrm{Fun}^{pt}(\mathsf{Ch}(\mathcal{A})^{\times n}, (n+m)\text{-}\mathsf{Ch}(\mathcal{B})).$$

Let $F\colon \mathcal{A}^{\times n} \to m\text{-}\mathsf{Ch}(\mathcal{B})$ be a ponited functor. Then F_ϵ sends an n-tuple of chain complexes $(A^1,...,A^n)$ in $\mathcal{A}$ to the $(n+m)$-fold chain complex in $\mathcal{B}$ given in multidegree $(p_1,...,p_n,q_1,...,q_m)$ by $F(A^1_{p_1},...,A^n_{p_n})_{q_1,...,q_m}$. In the same multidegree, the differentials are given by

$$d_i = \begin{cases} (-1)^{p_1+...+p_{i-1}} d^{A^i} & , 1 \le i \le n \\ (-1)^{p_1+...+p_n+q_1+...+q_{i-n-1}} d_{i-n}^{F(A^1,...,A^n)} & , n+1 \le i \le n+m \end{cases}$$

Since F is a pointed functor, this does indeed define an $(n+m)$-fold chain complex in $\mathcal{B}$. Similarly, F_ϵ sends an n-tuple of morphisms $(f^i\colon A^i \to B^i)_{1\le i\le n}$ to the morphism given on the first n multidegrees $(p_1,...,p_n)$ by the morphism

$$F(f^1_{p_1},...,f^n_{p_n})\colon F(A^1_{p_1},...,A^n_{p_n}) \to F(B^1_{p_1},...,B^n_{p_n})$$

Hence F_ϵ is indeed a functor.

Now let $F,G\colon \mathcal{A}^{\times n} \to m\text{-}\mathsf{Ch}(\mathcal{B})$ be pointed functors and let $\alpha\colon F \to G$ be a natural transformation. Then α_ϵ is the natural transformation given by applying α levelwise, i.e. for an n-tuple of chain complexes $(A^1,...,A^n)$ in $\mathcal{A}$, the morphism

$$(\alpha_\epsilon)_{(A^1,...,A^n)}\colon F_\epsilon(A^1,...,A^n) \to G_\epsilon(A^1,...,A^n)$$

is given in the first n multidegrees $(p_1,...,p_n)$ by the morphism

$$\alpha_{(A^1_{p_1},...,A^n_{p_n})}\colon F(A^1_{p_1},...,A^n_{p_n}) \to G(A^1_{p_1},...,A^n_{p_n}).$$

With this definition it is clear that $(-)_\epsilon$ is an additive functor.

Definition 2.20. Building on Construction 2.19 we define the functor

$$(-)_\delta\colon \mathrm{Fun}^{pt}(\mathcal{A}^{\times n}, m\text{-}\mathsf{Ch}(\mathcal{B})) \to \mathrm{Fun}^{pt}(\mathsf{Ch}(\mathcal{A})^{\times n}, \mathsf{Ch}(\mathcal{B}))$$

as the composition of $(-)_\epsilon$ with the totalization functor $(n+m)\text{-}\mathsf{Ch}(\mathcal{B}) \to \mathsf{Ch}(\mathcal{B})$.

Before considering the monoidality properties of the functor $(-)_\delta$, we need an observation about totalizations of n-fold chain complexes.

Observation 2.21. Let $\mathcal{A}$ be an additive category. The symmetric group on n letters acts on the category of n-fold chain complexes in $\mathcal{A}$ by reordering the differentials. Specifically, if A is an n-fold chain complex in $\mathcal{A}$ and $\chi \in \Sigma_n$, we have

$$(\chi \cdot A)_{p_1,...,p_n} = A_{p_{\chi^{-1}(1)},...,p_{\chi^{-1}(n)}}$$

and the differentials are similarly reordered. Let $\chi_{(p_1,...,p_n)}$ be the image of χ under the blow-up homomorphism $\Sigma_n \to \Sigma_{p_1+...+p_n}$. There is a natural transformation $g_\chi \colon \mathrm{Tot} \to \mathrm{Tot} \circ \chi$ given in multidegree $(p_1, ..., p_n)$ by the sign of $\chi_{(p_1,...,p_n)}$. Composition of these transformations has the same effect as applying the sign associated to the composite permutation, such that $(g_{\chi'} * \mathrm{id}_\chi) \circ g_\chi = g_{\chi'\chi}$ for any pair $\chi, \chi' \in \Sigma_n$.

Lemma 2.22. Let the pointed functor $F \colon \mathcal{A}^{\times n} \to \mathsf{Ch}(\mathcal{B})$ be given by $F(A^1, ..., A^n) = F'(A^1, ..., A^i) \otimes F''(A^{i+1}, ..., A^n)$ for a pair of pointed functors $F' \colon \mathcal{A}^{\times i} \to \mathsf{Ch}(\mathcal{B})$ and $F'' \colon \mathcal{A}^{\times n-i} \to \mathsf{Ch}(\mathcal{B})$. Then there is a natural isomorphism

$$F_\delta(A^1, ..., A^n) \simeq F'_\delta(A^1, ..., A^i) \otimes F''_\delta(A^{i+1}, ..., A^n).$$

Furthermore, this isomorphism is associative.

Proof. The tensor product $F'(A^1, ..., A^i) \otimes F''(A^{i+1}, ..., A^n)$ is the totalization of a bicomplex, so we can lift F_δ to a functor $F_\epsilon \colon \mathsf{sCh}_k^{\times n} \to (n+2)\text{-}\mathsf{Ch}_k$ such that $\mathrm{Tot} \circ F_\epsilon = F_\delta$. As before, the differentials in multidegree $(p_1, ..., p_n, q_1, q_2)$ are given by

$$d_i = \begin{cases} (-1)^{p_1+...+p_{i-1}} d^{A^i} & , 1 \le i \le n \\ (-1)^{p_1+...+p_n} d^{F'(A^1,...,A^i)} & , i = n+1 \\ (-1)^{p_1+...+p_n+q_1} d^{F''(A^{i+1},...,A^n)} & , i = n+2 \end{cases}$$

Now the tensor product $F'_\delta(A^1, ..., A^i) \otimes F''_\delta(A^{i+1}, ..., A^n)$ is obtained by reordering the differentials and totalizing. Specifically, we must pass d_{n+1} past $d_{i+1}, ..., d_n$, which in multidegree $(p_1, ..., p_n, q_1, q_2)$ incurs a sign $(-1)^{q_1(p_{i+1}+...+p_n)}$. The natural isomorphism in the statement of the lemma is thus obtained as the transformation g_{χ_i} of Observation 2.21 where χ_i is the cycle $(i+1, ..., n, n+1) \in \Sigma_{n+2}$.

The above recipe generalizes readily to a version with more than two tensor factors by replacing χ_i with the permutation $\chi_{i_1,...,i_m} \in \Sigma_{n+m}$ given by the composition of cycles

$$\chi_{i_1,...,i_m} = (i_1 + ... + i_m + 1, ..., n+m) \circ ... \circ (i_1 + 1, ..., n+1)$$

To see that this is associative, it is sufficient to look at the case of three factors:

$$F(A_1, ..., A_n) = F^1(A_1, ..., A_i) \otimes F^2(A_{i+1}, ..., A_{i+j}) \otimes F^3(A_{i+j+1}, ..., A_n)$$

Associativity of the natural isomorphism above now follows from the identity

$$(i+j+2, ..., n+1, n+2) \circ (i+1, ..., n, n+1)$$

$$= (i+1, ..., n+1, n+2) \circ (i+j+1, ..., n+1, n+2)$$

in Σ_{n+2}. □

Definition 2.23. Write ζ_n for the natural isomorphism

$$\zeta_n \colon (F^1 \otimes ... \otimes F^n)_\delta \xrightarrow{\sim} F^1_\delta \otimes ... \otimes F^n_\delta$$

of functors $\mathcal{A}^{\times i} \to \mathsf{Ch}(\mathcal{B})$ given by Lemma 2.22.

Notation 2.24. Recall the cyclic bar construction from Definition 2.3 (2). We extend this definiton to the functor $C^{\vec{k}}\colon \mathsf{dgAlg}_k \to \mathsf{Ch}_k$ by

$$A \mapsto C^{\vec{k}}(A) = C(A^{\otimes k_1}) \otimes ... \otimes C(A^{\otimes k_n})$$

i.e. $C^{\vec{k}}(A) = N_\delta B^{cy}(A^{\otimes k_1}) \otimes ... \otimes N_\delta B^{cy}(A^{\otimes k_n})$.

Definition 2.25. Recall that the cyclic bar construction $B^{cy}\colon \mathsf{dgAlg}_k \to \mathsf{sCh}_k$ is a symmetric monoidal functor. We denote the natural structure isomorphism by

$$\theta_n\colon B^{cy}(A_1)\hat{\otimes}...\hat{\otimes}B^{cy}(A_n) \to B^{cy}(A_1 \otimes ... \otimes A_n)$$

The isomorphism is given in simplicial degree $k-1$ (in which we have nk tensor factors) by the permutation $\chi_{n,k} \in \Sigma_{nk}$ sending $i + dk$ to $d + 1 + (i-1)n$ for $0 < i \le k$ and $0 \le d < n$, with a sign like that in the proof of Lemma 2.4.

Proposition 2.26. The dg-category $\widetilde{N}^\Sigma$ acts on Hochschild complexes of dg-algebras. That is, we have natural transformations

$$\widetilde{N}^\Sigma(\vec{k}, \vec{m}) \otimes C^{\vec{k}} \to C^{\vec{m}}$$

of functors $\mathsf{dgAlg}_k \to \mathsf{Ch}_k$ compatible with composition. This action exhibits $C\colon \mathsf{dgAlg}_k \to \mathsf{Ch}_k$ as a symmetric monoidal, $\mathbb{E}_\infty$-comonoidal functor.

Proof. For $\vec{k} = (k_1, ..., k_n)$ denoting

$$N_\delta\theta_{\vec{k}} = (N_\delta(\theta_{k_1}) \otimes ... \otimes N_\delta(\theta_{k_n})) \circ \zeta_n\colon$$
$$N_\delta^{\vec{k}}(\Delta_k B^{cy}(A)) \to N_\delta B^{cy}(A^{\otimes k_1}) \otimes ... \otimes N_\delta B^{cy}(A^{k_n}) = C^{\vec{k}}(A)$$

where $\Delta_k\colon \mathsf{sCh} \to \mathsf{sCh}^{\times k}$ is the diagonal functor. If $f \in \widetilde{N}^\Sigma$, we write $(f_\delta)_{(A_1,...,A_n)}$ for the component of f_δ at the n-tuple $(A_1, ..., A_n)$ of simplicial chain complexes over k. We now have a composite morphism

$$\begin{tikzcd} C^{\vec{k}}(A) \arrow[r, "N_\delta(\theta_{\vec{k}})^{-1}"] & N_\delta^{\vec{k}}(B^{cy}(A), ..., B^{cy}(A)) \arrow[d, "(f_\delta)_{(B^{cy}(A),...,B^{cy}(A))}"] \\ C^{\vec{m}}(A) & N_\delta^{\vec{m}}(B^{cy}(A), ..., B^{cy}(A)) \arrow[l, "N_\delta(\theta_{\vec{m}})"'] \end{tikzcd}$$

We therefore get a morphism

$$i_{\vec{k},\vec{m}}\colon \widetilde{N}^\Sigma(\vec{k}, \vec{m}) \to \mathrm{Nat}(C^{\vec{k}}(A), C^{\vec{m}}(A))$$
$$f \mapsto N_\delta\theta_{\vec{m}} \circ (f_\delta * \mathrm{id}_{\Delta_k \circ B^{cy}}) \circ (N_\delta\theta_{\vec{k}})^{-1}$$

whose adjoint is the morphism in the statement of the proposition. The contractibility of $\widetilde{N}$ (see Lemma 2.18) now implies that $C\colon \mathsf{dgAlg}_k \to \mathsf{Ch}_k$ is $\mathbb{E}_\infty$-monoidal and $\mathbb{E}_\infty$-comonoidal. However, since the shuffle maps are strictly symmetric, it is in fact symmetric monoidal as claimed.

□

Lemma 2.27. The images of $\widetilde{N}^\Sigma_\chi((n),(n))$ and $\widetilde{N}^\Sigma_{\chi'}((n),(n))$ in $\mathrm{End}_{\mathsf{Ch}_k}(C((-)^{\otimes n}))$ are disjoint for $\chi \neq \chi'$.

Proof. Note that for A a commutative algebra we have

$$H_0(C((-)^{\otimes n})) = HH_0((-)^{\otimes n}) = (-)^{\otimes n}$$

so the induced action of $f \in \widetilde{N}^{\Sigma}((n),(n))$ on $H_0(C((-)^{\otimes n}))$ is given by permuting tensor factors. Namely, $\widetilde{N}^{\Sigma}_e((n),(n)) = \widetilde{N}((n),(n))$ acts as the identity since each $f \in \widetilde{N}((n),(n))$ is homotopic to the identity map. Now each $\widetilde{N}^{\Sigma}_{\chi}((n),(n))$ is isomorphic to $\widetilde{N}((n),(n))$, the map given by postcomposition by χ_*, and it follows that each $f \in \widetilde{N}^{\Sigma}_{\chi}((n),(n))$ is homotopic to χ_*. In particular, for $A = k[x]$ we see that χ and χ' act differently on $k[x]^{\otimes n} \simeq k[x_1,...,x_n]$. This proves the claim. □

3. DG-fattening of props

In this section we will build the fattening functor for dg-props and prove Theorem A. The fattening functor will associate to a dg-prop P a certain full subcategory of the free symmetric monoidal dg-category on P and $\widetilde{N}^{\Sigma}$, modulo relations expressing that the Dold-Kan morphisms are natural with respect to the morphisms of P.

Remark 3.1. To spell out what Theorem A means, to each dg-prop P, there is a natural homotopy-coherent P-action on the Hochschild complex of $\mathcal{A}ss \otimes P$-algebras. The homotopies that make up the coherencies are encoded in a replacement dg-prop $\widetilde{P}$ which strictify the homotopy-coherent P-action. This strictification is moreover functorial in the prop.

In order to produce the functor $\widetilde{(-)}$, we first construct an auxiliary functor $Q\colon \mathsf{dgprop} \to \mathsf{dgCat}^{\otimes}$ landing in symmetric monoidal dg-categories. Q is constructed using a natural family of generators and relations and will contain $\widetilde{(-)}$ as a full subfunctor, i.e. there will be a natural transformation $\widetilde{(-)} \to Q$ whose components are inclusions of full subcategories.

Construction 3.2. • Let C be a dg-category. We define a symmetric monoidal dg-category $C^{\otimes}$ given as follows. The objects of $C^{\otimes}$ is the free monoid on the objects of C. Given two words $a = a_1 \otimes ... \otimes a_n$ and $b = b_1 \otimes ... \otimes b_n$ in $\operatorname{Ob} C^{\otimes}$, the morphism complex $a \to b$ in $C^{\otimes}$ is given by

$$\operatorname{Hom}_{C^{\otimes}}(a,b) = \bigoplus_{\sigma\in\Sigma_n} \bigotimes_{i=1}^{n} \operatorname{Hom}_C(a_i, b_{\sigma(i)}).$$

We write $(f_1 \otimes ... \otimes f_n)_\sigma$ for an elementary tensor in the summand of $\sigma \in \Sigma_n$. Note that for the empty product $\emptyset \in \operatorname{Ob} C^{\otimes}$, the tensor product is indexed over the empty set, which by convention means that $\operatorname{Hom}_{C^{\otimes}}(\emptyset,\emptyset) = k$ concentrated in degree 0.

If a and b are words of different lengths, i.e. $a = a_1 \otimes ... \otimes a_n$ and $b = g_1 \otimes ... \otimes b_m$ with $n \neq m$, then $\operatorname{Hom}_{C^{\otimes}}(a,b) = 0$.

The composition

$$\operatorname{Hom}_{C^{\otimes}}(b,c) \otimes \operatorname{Hom}_{C^{\otimes}}(a,b) \to \operatorname{Hom}_{C^{\otimes}}(a,c)$$

is given on elementary tensors by sending $f\colon a \to b$ and $g\colon b \to c$, where $f = (f_1 \otimes ... \otimes f_n)_\sigma$ and $g = (g_1 \otimes ... \otimes g_n)_{\sigma'}$, to

$$g \circ f = (-1)^{\operatorname{sgn}(f,g,\chi)}(g_{\sigma(1)} \circ f_1 \otimes ... \otimes g_{\sigma(n)} \circ f_n)_{\sigma'\sigma}$$

where χ is the permutation of

$$(g_1 \otimes ... \otimes g_n \otimes f_1 \otimes ... \otimes f_n)$$

into

$$(g_{\sigma(1)} \otimes f_1 \otimes g_{\sigma(2)} \otimes f_2 \otimes ... \otimes g_{\sigma(n)} \otimes f_n)$$

and $\mathrm{sgn}(f, g, \chi)$ is the weighted sign of χ.

The symmetric monoidal structure on $C^{\otimes}$ is given on objects by multiplication in the free monoid on $\mathrm{Ob}\, C$ and on morphisms by the inclusion

$$\mathrm{Hom}_{C^{\otimes}}(a, b) \otimes \mathrm{Hom}_{C^{\otimes}}(a', b') \to \mathrm{Hom}_{C^{\otimes}}(a \otimes a', b \otimes b')$$

$$\left(\bigoplus_{\sigma \in \Sigma_n} \bigotimes_{i=1}^{n} \mathrm{Hom}_C(a_i, b_{\sigma(i)})\right) \otimes \left(\bigoplus_{\sigma' \in \Sigma_m} \bigotimes_{j=1}^{m} \mathrm{Hom}_C(a'_j, b'_{\sigma'(j)})\right) \to \left(\bigoplus_{\sigma'' \in \Sigma_{n+m}} \bigotimes_{l=1}^{n+m} \mathrm{Hom}_C(a''_l, b''_{\sigma''(l)})\right)$$

where the summand of $\sigma \in \Sigma_n, \sigma' \in \Sigma_m$ lands in the summand of

$$\sigma'' = \sigma \times \sigma' \in \Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}$$

through the canonical inclusion, and an elementary tensor

$$f_1 \otimes ... \otimes f_n \otimes f'_1 \otimes ... \otimes f'_m$$

is sent to

$$f''_1 \otimes ... \otimes f''_{n+m},$$

where $f''_l \colon a''_l \to b''_{\sigma''(l)}$ is given by

$$\begin{array}{ll} f_l \colon a_l \to b_{\sigma(l)} & \text{, if } 1 \leq l \leq n \\ f'_{l-n} \colon a'_{l-n} \to b'_{\sigma'(l-n)} & \text{, if } n+1 \leq l \leq n+m \end{array}$$

We check that the monoidal product is functorial, i.e. we wish to verify that the following diagram commutes . Consider morphisms

$$a^i \xrightarrow{f^i} b^i \xrightarrow{g^i} c^i \quad , \quad i = 0, 1$$

such that each f^i, g^i is an elementary tensor

$$f^i = (f^i_1 \otimes ... \otimes f^i_n)_{\sigma_i}$$

$$g^i = (g^i_1 \otimes ... \otimes g^i_n)_{\sigma'_i}$$

and compare the operations

$$H_0(g^0, g^1, f^0, f^1) := (g^0 \otimes g^1) \circ (f^0 \otimes f^1)$$

and

$$H_1(g^0, g^1, f^0, f^1) := (g^0 \circ f^0) \otimes (g^1 \circ f^1).$$

From the recipes given for composition and monoidal product, we see that H_0 and H_1 are at least equal up to sign on elementary tensors. The signs are in both cases the weighted sign of the same permutation, so they are equal. It follows that the monoidal product is functorial. Note that the monoidal product is strictly associative and unital, where the unit for the monoidal structure is given by the empty product $\emptyset$.

For $a = a_1 \otimes ... \otimes a_n$ and $b = b_1 \otimes ... \otimes b_m$, the twist map $\tau \colon a \otimes b \to b \otimes a$ is given by the elementary tensor

$$(\mathrm{id}_{a_1} \otimes ... \otimes \mathrm{id}_{a_n} \otimes \mathrm{id}_{b_1} \otimes ... \otimes \mathrm{id}_{b_m})_{\tau_{n,m}}$$

where $\tau_{n,m} \in \Sigma_{n+m}$ is the block permutation permuting the first n letters past the latter m letters.

Observe that for any elementary tensor $f = f_1 \otimes \ldots \otimes f_n \in \mathrm{Hom}_{C^{\otimes}}(a, b)$ and any $\sigma \in \Sigma_n$, conjugation of f by σ produces the elementary tensor

$$f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)} \in \mathrm{Hom}_{C^{\otimes}}(\sigma \cdot a, \sigma \cdot b).$$

In particular, for morphisms $f \colon a \to b$ and $f' \colon a' \to b'$ in $C^{\otimes}$, the conjugation of $f \otimes f'$ by $\tau_{n,m}$ is precisely $f' \otimes f$, such that the twist morphism τ is a symmetry for the monoidal structure.

• Let $\vec{\Sigma}$ be the category whose objects are vectors, and whose morphisms are permutations of the entries of vectors. Note that $\vec{\Sigma}$ is concentrated in degree 0, and can also be defined as $\mathbb{N}^{\otimes}$, the free symmetric monoidal dg-category on the discrete dg-category $\mathbb{N}$. Here by discrete we mean that $\mathrm{Hom}_{\mathbb{N}}(n, m) = k$ in degree 0 if $n = m$, and zero otherwise. Then there are symmetric monoidal functors $\vec{\Sigma} \to \widetilde{N}^{\Sigma}$ and $\vec{\Sigma} \to P^{\otimes}$ which are the identity on objects.

• By viewing the morphism complexes of $P^{\otimes}$ as being concentrated in horizontal degrees, we give $P^{\otimes}$ an enrichment in bicomplexes. Similarly, we view $\widetilde{N}^{\Sigma}$ as being enriched in bicomplexes, concentrated in vertical degrees. We define a category enriched in bicomplexes $Q_0(P)$ as follows.

The objects of $Q_0(P)$ are vectors $\vec{k} = (k_1, \ldots, k_n)$. To give a description of the morphism complexes $Q_0(P)(\vec{k}, \vec{m})$, we define the following auxiliary notation. For $\vec{k}, \vec{m} \in \mathrm{Ob}\ Q_0(P)$ and $j \in \mathbb{N}$, let $\mathcal{C}_{j,l}(\vec{k}, \vec{m})$ be given by

$$\mathcal{C}_j(\vec{k}, \vec{m}) = \begin{cases} P^{\otimes}(\vec{k}, \vec{m}) & \text{, if } j \text{ even,} \\ \widetilde{N}^{\Sigma}(\vec{k}, \vec{m}) & \text{, if } j \text{ odd.} \end{cases}$$

Then $Q_0(P)(\vec{k}, \vec{m})$ is given by the direct sum

$$Q_0(P)(\vec{k}, \vec{m}) = \left(\bigoplus_{\substack{i \in \mathbb{N} \\ \vec{x}_1, \ldots, \vec{x}_i \in \mathrm{Ob}\, Q_0(P) \\ l = 0,1}} \mathcal{C}_{i+l}(\vec{x}_i, \vec{m}) \otimes_{\Sigma} \mathcal{C}_{i-1+l}(\vec{x}_{i-1}, \vec{x}_i) \otimes_{\Sigma} \ldots \otimes_{\Sigma} \mathcal{C}_l(\vec{k}, \vec{x}_1) \right) \Big/ \sim_{red},$$

where the tensor product $\otimes_{\Sigma}$ means taking the colimit of the diagrams

$$\mathcal{C}_{j+l}(\vec{x}_j, \vec{x}_{j+1}) \otimes \vec{\Sigma}(\vec{x}_j, \vec{x}_j) \otimes \mathcal{C}_{j-1+l}(\vec{x}_{j-1}, \vec{x}_j) \rightrightarrows \mathcal{C}_{j+l}(\vec{x}_j, \vec{x}_{j+1}) \otimes \mathcal{C}_{j-1+l}(\vec{x}_{j-1}, \vec{x}_j),$$

where $\vec{\Sigma}$ acts via its inclusions into $P^{\otimes}$ and $\widetilde{N}^{\Sigma}$ described above. The equivalence relation $\sim_{red}$ is generated as an equivalence relation by the following relations:

(1) Permutation morphisms on the ends are absorbed into their neighbour. For example, the word
$$\vec{k}_0 \xrightarrow{\chi} \vec{k}_1 \xrightarrow{\phi} \vec{k}_2$$
where $\chi \in \vec{\Sigma}$ and $\phi \in \widetilde{N}^{\Sigma}$ is equivalent to the word
$$k_0 \xrightarrow{\chi \circ \phi} \vec{k}_2$$
using the inclusion $\vec{\Sigma} \to \widetilde{N}^{\Sigma}$.

(2) If a morphism in the middle of a word is a permutation morphism, then it is absorbed into either its left or right hand neighbour.

(3) If two adjacent morphisms are either both from $P^{\otimes}$ of both from $\widetilde{N}^{\Sigma}$, then they are composed.

In other words a morphism in P is a reduced word composed of morphisms in $P^{\otimes}$ and $\widetilde{N}^{\Sigma}$, where the subcategories $\vec{\Sigma} \subset P^{\otimes}$ and $\vec{\Sigma} \subset \widetilde{N}^{\Sigma}$ are identified. Composition is given by concatenating words and reducing, similarly to the case of free products of algebras. Note that when we consider a general morphism in $Q_0(P)$ which can be represented by an elementary tensor, we can add identity arrows at the ends as needed to ensure that the first arrow is from $P^{\otimes}$ and the last is from $\widetilde{N}^{\Sigma}$. When we work with such morphisms later, we will implicitly choose a representative of this form.

As for the enrichment in bicomplexes, $(f) \in P(n,m)$ has bidegree $(|f|, 0)$ and $\psi \in \widetilde{N}^{\Sigma}(\vec{n}, \vec{m})$ has bidegree $(0, |\psi|)$. The horizontal and vertical differentials act on P and $\widetilde{N}^{\Sigma}$ respectively. To be precise, the horizontal differential of a word

$$\vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \dots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m}$$

where $\phi^i \in P^{\otimes n_i}(\vec{k}_{2i}, \vec{k}_{2i+1})$ and $\gamma^i \in \widetilde{N}^{\Sigma}(\vec{k}_{2i+1}, \vec{k}_{2i+2})$, is the sum of applications of the differential to each ϕ^i, with sign $(-1)^{|\phi^0|+|\gamma^0|+\dots+|\gamma^{i-i}|}$, and the vertical differential is defined similarly.

There is one exceptional object in $Q_0(P)$, the empty vector $\vec{0} = ()$. The morphism space $\mathrm{Hom}_{Q_0}(\vec{0}, \vec{0})$ is declared to be the ground ring k in bidegree $(0,0)$. Note that $\vec{0}$ does not admit any morphism to or from a non-empty vector, as generators in $\widetilde{N}^{\Sigma}$ cannot change the length of a vector and generators in $P^{\otimes}$ cannot change the number of entries.

• We write $*$ for the concatenation of vectors. For a morphism $f\colon \vec{k}_0 \to \vec{k}_1$ in $P^{\otimes}$, and a vector $\vec{m}$, let

$$f \otimes^P \mathrm{id}\colon \vec{k}_0 * \vec{m} \to \vec{k}_1 * \vec{m}$$

be the tensor product of f and $\mathrm{id}_{\vec{m}}$ using the symmetric monoidal structure of $P^{\otimes}$. The morphisms $\mathrm{id} \otimes^P f$ is defined similarly. For a morphism $\psi\colon \vec{k}_0 \to \vec{k}_1$ in $\widetilde{N}^{\Sigma}$, we also define

$$\psi \otimes^N \mathrm{id}\colon \vec{k}_0 * \vec{m} \to \vec{k}_1 * \vec{m}$$

and similarly $\mathrm{id} \otimes^N \psi$ using the symmetric monoidal structure of $\widetilde{N}^{\Sigma}$.

Let $Q_1(P)$ be the quotient of $Q_0(P)$ with respect to the following relations. For each pair of morphisms $f\colon \vec{k}_0 \to \vec{k}_1$ in $P^{\otimes}$ and $\psi\colon \vec{m}_0 \to \vec{m}_1$ in $\widetilde{N}^{\Sigma}$, we impose an *interchange relation*, i.e. that the following diagrams commute.

$$\begin{tikzcd}
\vec{k}_0 * \vec{m}_0 \arrow[r, "\mathrm{id} \otimes^N \psi"] \arrow[d, "f \otimes^P \mathrm{id}"'] & \vec{k}_0 * \vec{m}_1 \arrow[d, "f \otimes^P \mathrm{id}"] \\
\vec{k}_1 * \vec{m}_0 \arrow[r, "\mathrm{id} \otimes^N \psi"'] & \vec{k}_1 * \vec{m}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\vec{m}_0 * \vec{k}_0 \arrow[r, "\psi \otimes^N \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes^P f"'] & \vec{m}_1 * \vec{k}_0 \arrow[d, "\mathrm{id} \otimes^P f"] \\
\vec{m}_0 * \vec{k}_1 \arrow[r, "\psi \otimes^N \mathrm{id}"'] & \vec{m}_1 * \vec{k}_1
\end{tikzcd}$$

To be precise, $Q_1(P)$ is the quotient of $Q_0(P)$ with respect to the ideal

$$I = \Big((f \otimes^P \mathrm{id}) \circ (\mathrm{id} \otimes^N \psi) - (\mathrm{id} \otimes^N \psi) \circ (f \otimes^P \mathrm{id}),$$
$$(f \otimes^P \mathrm{id}) \circ (\mathrm{id} \otimes^N \psi) - (\mathrm{id} \otimes^N \psi) \circ (f \otimes^P \mathrm{id}) \;\Big|\; f \in P^{\otimes}, \psi \in \widetilde{N}^{\Sigma}\Big)$$

Note that this relation is an identification of generators which respects bidegrees of morphisms and is compatible with concatenation of words. It also respects the differentials of morphisms, hence the result is a well-defined category enriched in bicomplexes. We expand this notation to general morphisms. Given a morphism $f\colon \vec{k} \to \vec{m}$ in $Q_1(P)$ represented by a word

$$\vec{k} = \vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \dots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m} = \vec{m}$$

where $\phi^i \in P^{\otimes n_i}(\vec{k}_{2i}, \vec{k}_{2i+1})$ and $\gamma^i \in \widetilde{N}^\Sigma(\vec{k}_{2i+1}, \vec{k}_{2i+2})$, and a vector $\vec{l}$, we write $f \otimes_1 \mathrm{id}$ for the morphism represented by the word

$$(\vec{k}_0 * \vec{l}) \xrightarrow{\phi^0 \otimes^P \mathrm{id}} (\vec{k}_1 * \vec{l}) \xrightarrow{\gamma^0 \otimes^N \mathrm{id}} \dots \xrightarrow{\gamma^{m-1} \otimes^N \mathrm{id}} (\vec{k}_{2m} * \vec{l})$$

We also define $\mathrm{id} \otimes_1 f$ similarly. Note that if $\vec{l} = \vec{0}$ then

$$f \otimes_1 \mathrm{id} = f = \mathrm{id} \otimes f.$$

To see that this construction is well-defined, take morphisms $\phi\colon \vec{k}_0 \to \vec{k}_1$ and $\phi'\colon \vec{k}_1 \to \vec{k}_2$ in $P^\otimes$ and vectors $\vec{m}_0, \vec{m}_1$. Then the following properties of the operation $- \otimes^P \mathrm{id}$ are immediate.

(1) $\mathrm{id} \otimes^P (\phi' \circ \phi) = (\mathrm{id} \otimes^P \phi') \circ (\mathrm{id} \otimes^P \phi)$ and similarly for $(\phi' \circ \phi) \otimes^P \mathrm{id})$,
(2) for the identity morphism $\mathrm{id}_{\vec{k}}\colon \vec{k} \to \vec{k}$ in $P^\otimes$, we have $\mathrm{id} \otimes \mathrm{id}_{\vec{k}} = \mathrm{id}_{\vec{m} * \vec{k}}$ and $\mathrm{id}_{\vec{k}} \otimes \mathrm{id} = \mathrm{id}_{\vec{k} * \vec{m}}$ in $P^\otimes$,
(3) $((\phi \otimes^P \mathrm{id}) \otimes^P \mathrm{id}) = \phi \otimes^P \mathrm{id}\colon \vec{k}_0 * \vec{m}_0 * \vec{m}_1 \to \vec{k}_1 * \vec{m}_0 * \vec{m}_1$,
(4) $(\mathrm{id} \otimes^P (\phi \otimes^P \mathrm{id})) = ((\mathrm{id} \otimes^P \phi) \otimes^P \mathrm{id})\colon \vec{m}_0 * \vec{k}_0 * \vec{m}_1 \to \vec{m}_0 * \vec{k}_1 * \vec{m}_1$,
(5) $d(\mathrm{id} \otimes^P f) = \mathrm{id} \otimes^P df$ and $d(f \otimes^P \mathrm{id}) = (df) \otimes^P \mathrm{id}$.

Similar identities hold for morphisms in $\widetilde{N}^\Sigma$. We regard property (2) as a definition if $\vec{k} = \vec{0}$. Together these properties imply that if f, g are morphisms $\vec{k}_0 \to \vec{k}_1$ in $Q_0(P)$ such that $f \sim g$ according to the interchange relation, then also $(f \otimes_1 \mathrm{id}) \sim (g \otimes_1 \mathrm{id})$ and $(\mathrm{id} \otimes_1 f) \sim (\mathrm{id} \otimes_1 g)$, and that the operations $f \mapsto f \otimes^P \mathrm{id}$ and $f \mapsto \mathrm{id} \otimes^P f$, and the same operations for $\widetilde{N}^\Sigma$ are well-defined morphisms of bicomplexes.

The interchange relation now implies that

$$(\mathrm{id} \otimes_1 f) \circ (g \otimes_1 \mathrm{id}) = (g \otimes_1 \mathrm{id}) \circ (\mathrm{id} \otimes_1 f)$$

for any pair of morphisms $f\colon \vec{k}_0 \to \vec{k}_1$ and $g\colon \vec{m}_0 \to \vec{m}_1$ in $Q_1(P)$.

Finally, observe that for a pair of morphisms

$$\vec{k}_0 \xrightarrow{f_0} \vec{k}_1 \xrightarrow{f_1} \vec{k}_2$$

in $Q_1(P)$ and a vector $\vec{m}$, we have

$$(f_1 \circ f_0) \otimes_1 \mathrm{id} = (f_1 \otimes_1 \mathrm{id}) \circ (f_0 \otimes_1 \mathrm{id})$$

and

$$\mathrm{id} \otimes_1 (f_1 \circ f_0) = (\mathrm{id} \otimes_1 f_1) \circ (\mathrm{id} \otimes_1 f_0),$$

such that $(- \otimes_1 \mathrm{id})$ and $(\mathrm{id} \otimes_1 -)$ are endofunctors on $Q_1(P)$.

• Recall that for a pair of categories $\mathcal{C}$ and $\mathcal{D}$ enriched in bicomplexes, their tensor product $\mathcal{C} \otimes \mathcal{D}$ has objects $\mathrm{Ob}\,(\mathcal{C} \otimes \mathcal{D}) = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$ and morphisms

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((a,b),(a',b')) = \mathrm{Hom}_{\mathcal{C}}(a,a') \otimes \mathrm{Hom}_{\mathcal{D}}(b,b').$$

We define a symmetric monoidal structure on $Q_1(P)$ as follows. The functor

$$\otimes_1 \colon Q_1(P) \otimes Q_1(P) \to Q_1(P)$$

is given on objects by concatenating vectors. Given two morphisms $f \colon \vec{k}_0 \to \vec{k}_1$ and $g \colon \vec{m}_0 \to \vec{m}_1$ in $Q_1(P)$, we define the monoidal product

$$f \otimes_1 g \colon \vec{k}_0 * \vec{m}_0 \to \vec{k}_1 * \vec{m}_1$$

by the formula

$$f \otimes_1 g := (\mathrm{id} \otimes_1 g) \circ (f \otimes_1 \mathrm{id}).$$

Since $(- \otimes_1 \mathrm{id})$ is a well defined operation on morphisms in $Q_1(P)$, this gives rise to a well defined morphism of bicomplexes

$$\mathrm{Hom}_{Q_1(P)}(\vec{k}_0, \vec{k}_1) \otimes \mathrm{Hom}_{Q_1(P)}(\vec{m}_0, \vec{m}_1) \xrightarrow{-\otimes_1-} \mathrm{Hom}_{Q_1(P)}(\vec{k}_0 * \vec{m}_0, \vec{k}_1 * \vec{m}_1).$$

We check that $\otimes_1$ is a functor. Given morphisms

$$\vec{k}_0 \xrightarrow{f_0} \vec{k}_1 \xrightarrow{f_1} \vec{k}_2$$

and

$$\vec{m}_0 \xrightarrow{g_0} \vec{m}_1 \xrightarrow{g_1} \vec{m}_2,$$

we have

$$\begin{aligned}(f_1 \circ f_0) \otimes_1 (g_1 \circ g_0) &= (\mathrm{id} \otimes_1 (g_1 \circ g_0) \circ ((f_1 \circ f_0) \otimes_1 \mathrm{id}) \\ &= (\mathrm{id} \otimes_1 g_1) \circ (\mathrm{id} \otimes_1 g_1) \circ (f_1 \otimes_1 \mathrm{id}) \circ (f_0 \otimes \mathrm{id}).\end{aligned}$$

Now the interchange relation implies that

$$(\mathrm{id} \otimes_1 g_1) \circ (f_1 \otimes_1 \mathrm{id}) = (f_1 \otimes_1 \mathrm{id}) \circ (\mathrm{id} \otimes_1 g_1)$$

as morphisms in $Q_1(P)$, so that

$$\begin{aligned}(f_1 \circ f_0) \otimes_1 (g_1 \circ g_0) &= (\mathrm{id} \otimes_1 g_1) \circ (f_1 \otimes_1 \mathrm{id}) \circ (\mathrm{id} \otimes_1 g_1) \circ (f_0 \otimes \mathrm{id}) \\ &= (f_1 \otimes_1 g_1) \circ (f_0 \otimes_1 g_0)\end{aligned}$$

as claimed.

The unit for the monoidal structure is given by the empty vector $\vec{0} = ()$.

Note also by properties (3) and (4) above that $\otimes_1$ is strictly associative, so $\otimes_1$ defines a strict monoidal structure on $Q_1(P)$.

To give a symmetry for $\otimes_1$, we first describe how the symmetric group Σ_n acts on $Q_1(P)^{\otimes n}$. Given an n-tuple of vectors $(\vec{k}_1, ..., \vec{k}_n)$, the permutation

$$(\sigma, \vec{k}_1, ..., \vec{k}_n) \colon \vec{k}_1 * ... * \vec{k}_n \to \vec{k}_{\sigma(1)} * ... * \vec{k}_{\sigma(n)}$$

is given by acting on the source by the block permutation $\sigma_{|\vec{k}_1|,...,|\vec{k}_n|} \in \Sigma_{|\vec{k}_1|+...+|\vec{k}_n|}$, i.e. if $\sigma(i) = j$, then for all $1 \leq l \leq |\vec{k}_i|$, we have

$$\sigma_{|\vec{k}_1|,...,|\vec{k}_n|}\left(|\vec{k}_1| + ... + |\vec{k}_{i-1}| + l\right) = |\vec{k}_1| + ... + |\vec{k}_{j-1}| + l.$$

If $\phi \colon \vec{k}_0 \to \vec{k}_1$ is a morphism in $P^{\otimes}$, $\vec{m}$ is a vector, and $\tau \in \Sigma_2$ is the twist permutation, then in $P^{\otimes}$ we have

$$(\mathrm{id} \otimes^P \phi) \circ \tau_{|\vec{k}_0|,|\vec{m}|} = \tau_{|\vec{k}_1|,|\vec{m}|} \circ (\phi \otimes^P \mathrm{id})$$

and

$$(\phi \otimes^P \mathrm{id}) \circ \tau_{|\vec{m}|,|\vec{k}_0|} = \tau_{|\vec{m}|,|\vec{k}_1|} \circ (\mathrm{id} \otimes^P \phi)$$

and similarly for $\gamma\colon \vec{k}_0 \to \vec{k}_1$ in $\widetilde{N}^{\Sigma}$. This implies that for any morphism $f\colon \vec{k}_0 \to \vec{k}_1$ in $Q_1(P)$, we have

$$(\mathrm{id} \otimes_1 f) \circ \tau_{|\vec{k}_0|,|\vec{m}|} = \tau_{|\vec{k}_1|,|\vec{m}|} \circ (f \otimes_1 \mathrm{id})$$

and

$$(f \otimes_1 \mathrm{id}) \circ \tau_{|\vec{m}|,|\vec{k}_0|} = \tau_{|\vec{m}|,|\vec{k}_1|} \circ (\mathrm{id} \otimes_1 f).$$

In particular, for $g\colon \vec{m}_0 \to \vec{m}_1$ another morphism in $Q_1(P)$ we have

$$(g \otimes_1 f) \circ \tau_{|\vec{k}_0|,|\vec{m}_0|} = \tau_{|\vec{k}_1|,|\vec{m}_1|} \circ (f \otimes_1 g)$$

such that τ is a symmetry for $\otimes_1$.

Notation 3.3. • We write $(1)^n$ for the vector $(1, ..., 1)$ of length n.

- For a vector $\vec{a} = (a_1, ..., a_n)$, we write $l(\vec{a}) = n$ for its number of elements.

Remark 3.4. The following definition describes a way of functorially arranging the entries of a vector $\vec{k} = (k_1, ..., k_n)$ (in particular, $l(\vec{k}) = n$) according to the entries of a vector $\vec{a}$ of length n, which we use to define the functor Q. Informally one should think of $\mathrm{Par}_{\vec{k}}(\vec{a})$ as given by arranging the entries of $\vec{k}$ according to the entries of $\vec{a}$. Similarly, for a morphism $\gamma\colon \vec{a} \to \vec{b}$, one may think of $\mathrm{Par}_{\vec{k}}(\gamma)$ as the natural transformation $N^{\mathrm{Par}_{\vec{k}}(\vec{a})} \to N^{\mathrm{Par}_{\vec{k}}(\vec{b})}$ of functors $\mathsf{sMod}^{\times k} \to \mathsf{Ch}_k$ whose $(A_1, ..., A_k)$-component equals the $(A_1 \hat{\otimes} ... \hat{\otimes} A_{k_1}\ , \ ...\ , \ A_{k_1+...+k_{n-1}+1} \hat{\otimes} ... \hat{\otimes} A_k)$-component of γ.

Definition 3.5. For each vector $\vec{k} = (k_1, ..., k_n)$, $k = |\vec{k}|$, let $\iota_{\vec{k}}\colon \mathsf{sMod}^{\times k} \to \mathsf{sMod}^{\times n}$ be the functor taking a k-tuple $(A_1, ..., A_k)$ to the n-tuple $(B_1, ..., B_n)$ where

$$B_i = A_{k_1+...+k_{i-1}+1} \hat{\otimes} ... \hat{\otimes} A_{k_1+...+k_i}.$$

Writing N_n for the full subcategory of $\widetilde{N}$ on the objects $\vec{a}$ with $|\vec{a}| = n$, let

$$\mathrm{Par}_{\vec{k}}\colon N_n \to N_k$$

be the functor taking $\vec{a} = (a_1, ..., a_l)$ to

$$\mathrm{Par}_{\vec{k}}(\vec{a}) := (k_1 + ... + k_{a_1}\ ,\ k_{a_1+1} + ... + k_{a_1+a_2}\ ,\ ...\ ,\ k_{a_1+...+a_{l-1}+1} + ... + k_n)$$

i.e. the unique vector such that $N^{\vec{a}} \circ \iota_{\vec{k}} = N^{\mathrm{Par}_{\vec{k}}(\vec{a})}$. In particular we have $|\mathrm{Par}_{\vec{k}}(\vec{a})| = |\vec{k}|$ and $l(\mathrm{Par}_{\vec{k}}(\vec{a})) = l(\vec{a})$. For a morphism $\gamma\colon \vec{a} \to \vec{b}$, $\mathrm{Par}_{\vec{k}}(\gamma)$ is given by $\gamma * \mathrm{id}_{\iota_{\vec{k}}}$. In particular, the following diagram commutes.

$$\begin{tikzcd}
\mathsf{sMod}^{\times k} \arrow[rr, bend left, "\iota_{\vec{k}}"] \arrow[dr, bend left, "N^{\mathrm{Par}_{\vec{k}}(\vec{a})}"] \arrow[dr, bend right=60, "N^{\mathrm{Par}_{\vec{k}}(\vec{b})}"'] \arrow[dr, phantom, shift right=3ex, "\mathrm{Par}_{\vec{k}}(\gamma)\ \Downarrow"] & & \mathsf{sMod}^{\times n} \arrow[dl, bend right, "N^{\vec{a}}"'] \arrow[dl, bend left=60, "N^{\vec{b}}"] \arrow[dl, phantom, shift left=3ex, "\Downarrow\gamma"] \\
& \mathsf{Ch}_k &
\end{tikzcd}$$

In particular, for any vector $\vec{a}$, we have $\mathrm{Par}_{\vec{a}}((1)^{l(\vec{a})}) = \vec{a}$.

Construction 3.6. • Let $\vec{a} = (a_1, ..., a_l)$ be an object of N_n (i.e. $|\vec{a}| = n$) and $f = f_1 \otimes ... \otimes f_n \colon \vec{k} \to \vec{m}$ a morphism in $P^{\otimes}$, where $l(\vec{k}) = l(\vec{m}) = n$. For each $| \leq i \leq l$ we have a morphism

$$(k_{a_1+...a_i+1} + ... + k_{a_1+...a_{i+1}}) \xrightarrow{(f_{a_1+...a_i+1} \otimes ... \otimes f_{a_1+...a_{i+1}})} (m_{a_1+...a_i+1} + ... + m_{a_1+...a_{i+1}})$$

in P, and we write $\mathrm{Par}_f(\vec{a}) \colon \mathrm{Par}_{\vec{k}}(\vec{a}) \to \mathrm{Par}_{\vec{m}}(\vec{a})$ for the tensor product in $P^{\otimes}$ of these morphisms for $1 \leq i \leq l$. Similarly to Definition 3.5, one can think of this as grouping the factors of f according to the entries of $\vec{a}$. Below is an example where $\vec{a} = (1, 3)$. In this example,

$$\mathrm{Par}_f(\vec{a}) = \mathrm{Par}_f(\vec{a})_1 \otimes^P \mathrm{Par}_f(\vec{a})_2$$

with

$$\begin{array}{rl} a_1 = 1 \left\{ \begin{array}{c} f_1 \colon k_1 \to m_1 \end{array} \right\} & \mathrm{Par}_f(\vec{a})_1 = f_1 \colon k_1 \to m_1 \\ a_2 = 3 \left\{ \begin{array}{c} f_2 \colon k_2 \to m_2 \\ f_3 \colon k_3 \to m_3 \\ f_4 \colon k_4 \to m_4 \end{array} \right\} & \mathrm{Par}_f(\vec{a})_2 = \bigotimes_{i=2}^4 f_i \colon \sum_{i=2}^4 k_i \to \sum_{i=2}^4 m_i. \end{array}$$

Let $Q(P)$ be the quotient of $Q_1(P)$ with respect to the ideal generated by the following relations. For each morphism $\gamma \colon \vec{a} \to \vec{b}$ in N_n and each morphism $f \colon \vec{k} \to \vec{m}$ in $P^{\otimes}$ with $l(\vec{k}) = l(\vec{m}) = n$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Par}_{\vec{k}}(\vec{a}) \arrow[r, "\mathrm{Par}_f(\vec{a})"] \arrow[d, "\mathrm{Par}_{\vec{k}}(\gamma)"'] & \mathrm{Par}_{\vec{m}}(\vec{a}) \arrow[d, "\mathrm{Par}_{\vec{m}}(\gamma)"] \\ \mathrm{Par}_{\vec{k}}(\vec{b}) \arrow[r, "\mathrm{Par}_f(\vec{b})"'] & \mathrm{Par}_{\vec{m}}(\vec{b}) \end{tikzcd}$$

We call this the *partition relation*. If two morphisms f, g are equivalent under the partition relation, we write $f \sim_{\mathrm{Par}} g$. The partition relation captures in a more general manner the naturality of shuffle and Alexander-Whitney maps with respect to algebra homomorphisms, see Observation 3.8.

These relations are compatible with concatenations of words and respect bidegrees of morphisms. They are also compatible with horizontal and vertical differentials as the relation only depends on the source and target of morphisms, hence the result is a well-defined category enriched in bicomplexes.

To see that the symmetric monoidal structure is compatible with these relations, observe that

$$\mathrm{Par}_{\vec{k}}(\vec{a}) * \vec{r} = \mathrm{Par}_{\vec{k} * \vec{r}}(\vec{a} * (1)^{l(\vec{r})}),$$

which implies that the following diagrams are equal and commute.

$$\begin{tikzcd}
\mathrm{Par}_{\vec{k}}(\vec{a}) * \vec{r} \arrow[r, "\mathrm{Par}_f(\vec{a}) \otimes^P \mathrm{id}"] \arrow[d, "\mathrm{Par}_{\vec{k}}(\gamma) \otimes^N \mathrm{id}"'] & \mathrm{Par}_{\vec{m}}(\vec{a}) * \vec{r} \arrow[d, "\mathrm{Par}_{\vec{m}}(\gamma) \otimes^N \mathrm{id}"] \\
\mathrm{Par}_{\vec{k}}(\vec{b}) * \vec{r} \arrow[r, "\mathrm{Par}_f(\vec{b}) \otimes^P \mathrm{id}"'] & \mathrm{Par}_{\vec{m}}(\vec{b}) * \vec{r}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Par}_{\vec{k}*\vec{r}}(\vec{a} * (1)^{l(\vec{r})}) \arrow[r, "\mathrm{Par}_{f \otimes^P \mathrm{id}}(\vec{a} * (1)^{l(\vec{r})})"] \arrow[d, "\mathrm{Par}_{\vec{k}*\vec{r}}(\gamma \otimes^N \mathrm{id})"'] & \mathrm{Par}_{\vec{m}*\vec{r}}(\vec{a} * (1)^{l(\vec{r})}) \arrow[d, "\mathrm{Par}_{\vec{m}*\vec{r}}(\gamma \otimes^N \mathrm{id})"] \\
\mathrm{Par}_{\vec{k}*\vec{r}}(\vec{b} * (1)^{l(\vec{r})}) \arrow[r, "\mathrm{Par}_{f \otimes^P \mathrm{id}}(\vec{b} * (1)^{l(\vec{r})})"'] & \mathrm{Par}_{\vec{m}*\vec{r}}(\vec{b} * (1)^{l(\vec{r})})
\end{tikzcd}$$

The same statement also holds for tensoring with id on the left.

In particular this implies that if $f, g\colon \vec{k}_0 \to \vec{k}_1$ are morphisms in $Q_1(P)$ such that $f \sim_{\mathrm{Par}} g$, and $\vec{m}$ is a vector, then

$$(f \otimes_1 \mathrm{id}) \sim_{\mathrm{Par}} (g \otimes_1 \mathrm{id})$$

and

$$(\mathrm{id} \otimes_1 f) \sim_{\mathrm{Par}} (\mathrm{id} \otimes_1 g),$$

which in turn implies that the monoidal product respects the partition relation. It follows that the symmetric monoidal structure on $Q_1(P)$ descends to a symmetric monoidal structure on $Q(P)$.

- Let P, P' be dg-props and let $g\colon P \to P'$ be a morphism of dg-props. The functor g induces a symmetric monoidal functor $g^{\otimes}\colon P^{\otimes} \to P'^{\otimes}$. Define $Q_0(g)\colon Q_0(P) \to Q_0(P')$ to be the induced functor.It is the identity on objects and on generators coming from $\widetilde{N}^{\Sigma}$, and acts by $g^{\otimes}$ on generators coming from $P^{\otimes}$. This functor induces a symmetric monoidal functor $Q(P) \to Q(P')$. To see this, observe that since g is a prop morphism, we have $Q_0(g)(f \otimes_1 \mathrm{id}) = Q_0(g)(f) \otimes_1 \mathrm{id}$ for any morphism in $Q_0(P)$ and $Q(\mathrm{Par}_f(\vec{a})) = \mathrm{Par}_{g^{\otimes}(f)}(\vec{a})$ for any morphism in $P^{\otimes}$. This implies that $Q_0(g)$ descends to a functor $Q(g)\colon Q(P) \to Q(P')$. Symmetric monoidality of this functor can be seen by observing that

$$Q(g)(f_0) \otimes_1 Q(g)(f_1) = (\mathrm{id} \otimes_1 Q(g)(f_1)) \circ (Q(g)(f_0) \otimes_1 \mathrm{id})$$
$$= Q(g)(\mathrm{id} \otimes_1 f_1) \circ Q(g)(f_0 \otimes_1 \mathrm{id}) = Q(g)\left((\mathrm{id} \otimes_1 f_1) \circ (f_0 \otimes_1 \mathrm{id})\right) = Q(g)(f_0 \otimes_1 f_1)$$

and that since $Q(g)$ is the identity on $\widetilde{N}^{\Sigma}$, the image of the twist morphism in $Q(P)$ is the twist morphism in $Q(P')$.

Lemma 3.7. The canonical functors $P^{\otimes} \to Q(P)$ and $\widetilde{N}^{\Sigma} \to Q(P)$ are symmetric monoidal.

Proof. Let $F\colon P^{\otimes} \to Q(P)$ be the canonical functor taking a morphism $f\colon \vec{k} \to \vec{m}$ to the morphism represented by the singleton word f. Note that for $g\colon \vec{m} \to \vec{l}$ another morphism in $P^{\otimes}$, we have

$$F(g) \circ F(f) \sim_{red} F(g \circ f)$$

such that this is indeed a functor. Since the twist morphism in $Q(P)$ lives in $\vec{Sigma}$, the functor F takes the twist morphism of $P^{\otimes}$ to the twist morphism of $Q(P)$. To see that

F is symmetric monoidal, it is therefore sufficient to observe that for a pair of morphisms $f_i\colon \vec{k}_{0,i} \to \vec{m}_i$, $i = 0, 1$ in $P^{\otimes}$, we have

$$(\mathrm{id} \otimes^P f_1) \circ (f_0 \otimes^P \mathrm{id}) \sim_{red} f_0 \otimes^P f_1,$$

such that $F(f_0) \otimes_1 F(f_1) = F(f_0 \otimes^P f_1)$ in $Q(P)$. The case for $\widetilde{N}^{\Sigma}$ is identical. $\square$

Observation 3.8. The shuffle and Alexander-Whitney maps introduced in Notation 2.17 are special cases of the partition construction. In particular, if $\vec{k}$ has $l(\vec{k}) = n$, then

$$\mathsf{sh}_{\vec{k}} = \mathrm{Par}_{\vec{k}}(\mathsf{sh}_n) : \vec{k} \to (|\vec{k}|)$$

and similarly

$$AW_{\vec{k}} = \mathrm{Par}_{\vec{k}}(AW_n) : (|\vec{k}|) \to \vec{k}.$$

The partition relations imply that if $\vec{k}_0$ and $\vec{k}_1$ are vectors with $l(\vec{k}_0) = l(\vec{k}_1) = n$ and $f = (f_1, ..., f_n)\colon \vec{k}_0 \to \vec{k}_1$ is a morphism in $P^{\otimes}$, then the following squares commute.

$$\begin{tikzcd}
\vec{k}_0 \arrow[r, "f"] \arrow[d, "\mathrm{Par}_{\vec{k}_0}(\mathsf{sh}_n) = \mathsf{sh}_{\vec{k}_0}"'] & \vec{k}_1 \arrow[d, "\mathsf{sh}_{\vec{k}_1}"] \\
(|\vec{k}_0|) \arrow[r, "\mathrm{Par}_f((n))"'] & (|\vec{k}_1|)
\end{tikzcd}
\qquad
\begin{tikzcd}
(|\vec{k}_0|) \arrow[r, "\mathrm{Par}_f((n))"] \arrow[d, "AW_{\vec{k}_0}"'] & (|\vec{k}_1|) \arrow[d, "AW_{\vec{k}_1} = \mathrm{Par}_{\vec{k}_1}(AW_n)"] \\
\vec{k}_0 \arrow[r, "f"'] & \vec{k}_1
\end{tikzcd}$$

Observation 3.9. • For any n-tuple $f = (f_1, ..., f_n)$ of morphisms in P, we have $\mathrm{Par}_f((1)^n) = f$.

- For a sequence

$$\vec{k}_0 \xrightarrow{f} \vec{k}_1 \xrightarrow{g} \vec{k}_2$$

of morphisms in $P^{\otimes}$, where the $l(\vec{k}_i) = n$ and $\vec{a} \in N_n$ we have $\mathrm{Par}_g(\vec{a}) \circ \mathrm{Par}_f(\vec{a}) = \mathrm{Par}_{g \circ f}(\vec{a})$.

Definition 3.10. For $\mathcal{C}$ a category enriched in bicomplexes, let $\mathrm{Tot}(\mathcal{C})$ be the dg-category whose morphism complexes are the $\oplus$-totalization of the morphism bicomplexes in $\mathcal{C}$.

Lemma 3.11. There is a natural symmetric monoidal functor $F\colon \mathrm{Tot}(Q(P)) \to P$ defined on objects by taking $\vec{k}$ to $|\vec{k}|$, and on morphisms by taking $f = (f_1, ..., f_n)\colon \vec{k} \to \vec{m}$ in $P^{\otimes}(\vec{k}, \vec{m})$ to $\mathrm{Par}_f((n))\colon |\vec{k}| \to |\vec{m}|$ in P, and $\gamma\colon \vec{k} \to \vec{m}$ in $\widetilde{N}(\vec{k}, \vec{m})_i$ to $\mathrm{id}_{|\vec{k}|}$ if $i = 0$ and 0 otherwise.

Proof. It is clear that the assignment is natural in P if it is well-defined, which we now verify. Given a morphism $\gamma\colon \vec{a} \to \vec{b}$ in N_n and $f = (f_1, ..., f_n)\colon \vec{k} \to \vec{m}$ in $P^{\otimes}$, we must verify that the diagram

$$\begin{tikzcd}
\mathrm{Par}_{\vec{k}}(\vec{a}) \arrow[r, "\mathrm{Par}_f(\vec{a})"] \arrow[d, "\mathrm{Par}_{\vec{k}}(\gamma)"'] & \mathrm{Par}_{\vec{m}}(\vec{a}) \arrow[d, "\mathrm{Par}_{\vec{m}}(\gamma)"] \\
\mathrm{Par}_{\vec{k}}(\vec{b}) \arrow[r, "\mathrm{Par}_f(\vec{b})"'] & \mathrm{Par}_{\vec{m}}(\vec{b})
\end{tikzcd}$$

remains commutative after applying F. It is sufficient to assume that $\gamma \in \widetilde{N}(\vec{a},\vec{b})_0$. But F takes $\mathrm{Par}_{\vec{k}}(\gamma)$ to the identity and we have in general that

$$\mathrm{Par}_{\mathrm{Par}_f(\vec{a})}((l(\vec{a})) = \mathrm{Par}_f((n))$$

such that

$$F(\mathrm{Par}_f(\vec{a})) = \mathrm{Par}_f((n)) = F(\mathrm{Par}_f(\vec{b})),$$

so F is well defined. To see that F preserves the differentials on each morphism complex, note that for a morphism

$$f = \left\{ \vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \dots \xrightarrow{\gamma^m} \vec{k}_{2m} \right\}$$

where $\phi^i \in P^{\otimes}(\vec{k}_{2i},\vec{k}_{2i+1})$ and $\gamma^i \in \widetilde{N}(\vec{k}_{2i+1},\vec{k}_{2i+2})$, the differential is given by

$$\begin{aligned} df = (d_v\gamma^m) \circ \phi^m \circ \dots \circ \gamma^0 \circ \phi^0 + (-1)^{|\gamma^m|}\gamma^m \circ (d_h\phi^m) \circ \dots \circ \gamma^0 \circ \phi^0 + \dots \\ +(-1)^{|\gamma^m|+|\phi^m|+\dots+|\gamma^0|}\gamma^m \circ \phi^m \circ \dots \circ \gamma^0 \circ d_h\phi^0 \end{aligned}$$

In the case that Ff is non-zero (i.e. each $|\gamma^i| = 0$) this differential is identical to the differential in P. ☐

Lemma 3.12. Let P be a dg-prop and let $\vec{k},\vec{m} \in \mathrm{Ob}\, Q(P)$. The map

$$\mathrm{Hom}_{\mathrm{Tot}(Q(P))}(\vec{k},\vec{m}) \to \mathrm{Hom}_P(|\vec{k}|,|\vec{m}|)$$

induced by F is a quasi-isomorphism.

Proof. Denote by $c_v\mathrm{Hom}_P(|\vec{k}|,|\vec{m}|)$ the bicomplex which has $\mathrm{Hom}_P(|\vec{k}|,|\vec{m}|)$ concentrated in vertical degree 0, and consider the map of bicomplexes

$$A\colon c_v\mathrm{Hom}_P(|\vec{k}|,|\vec{m}|) \simeq \widetilde{N}((|\vec{m}|),(|\vec{m}|)) \otimes \mathrm{Hom}_P(|\vec{k}|,|\vec{m}|) \to \mathrm{Hom}_{Q(P)}(\vec{k},\vec{m})$$

where the left map is the homotopy equivalence taking $f \in \mathrm{Hom}_P(|\vec{k}|,|\vec{m}|)$ to $\mathrm{id} \otimes f$, and the right map takes $\gamma \otimes f$ to $AW_{\vec{m}} \circ \gamma \circ f \circ \mathsf{sh}_{\vec{k}}$. We will show that the totalization of A is a quasi-isomorphism and a quasi-inverse to the map induced by F on Hom-complexes.

Let $f\colon \vec{k} \to \vec{m}$ with $|f| = (d,d')$ in $Q(P)$ be a composition of generators of $Q(P)$, i.e. f is represented by an elementary tensor

$$\vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \dots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m}$$

in $Q_0(P)(\vec{k},\vec{m})$, where $\phi^i \in P^{\otimes}(\vec{k}_{2i},\vec{k}_{2i+1})$ and $\gamma^i \in \widetilde{N}(\vec{k}_{2i+1},\vec{k}_{2i+2})_0$. We write

$$n_i := l(\vec{k}_{2i}) = l(\vec{k}_{2i+1}).$$

For such an f, we write $G(f) \in \mathrm{Hom}_P(|\vec{k}_0|,|\vec{k}_{2m}|)$ for the morphism

$$G(f) = \mathrm{Par}_{\phi^{m-1}}((n_{m-1})) \circ \dots \circ \mathrm{Par}_{\phi^0}((n_0)).$$

Note that $G(f)$ does not depend on the representative of f. In particular, well-definedness with respect to $\sim_{\mathrm{Par}}$ can be seen by the identity

$$\mathrm{Par}_{\mathrm{Par}_\phi(\vec{a})}((l(\vec{a})) = \mathrm{Par}_\phi((n)).$$

for any $\phi\colon \vec{k} \to \vec{m}$ in $P^{\otimes}$ with $l(\vec{k}) = n$.

If for any ϕ which is represented by an elementary tensor, the homology class of ϕ is represented by a composition $AW_{\vec{m}} \circ \widetilde{\gamma} \circ (G(\phi)) \circ \mathsf{sh}_{\vec{k}}$, where $\widetilde{\gamma} \in \widetilde{N}((|\vec{m}|),(|\vec{m}|))_{d'}$, then the map A above is a quasi-isomorphism after totalizing. Indeed, assume that $f \in \mathrm{Hom}_{Q(P)}(\vec{k}, \vec{m})$ is a cycle with respect to the vertical differential. It is given by a sum

$$f = f_1 + \ldots + f_n \in \mathrm{Hom}_{Q(P)}(\vec{k}, \vec{m})_{d,n}.$$

where each f_i is represented by an elementary tensor. We may assume that each f_i has the form $AW_{\vec{m}} \circ \gamma_i \circ (G(f_i)) \circ \mathsf{sh}_{\vec{k}}$. Using the contractibility of $\widetilde{N}$, we may in fact assume that the γ_i are identical, such that f represents the same vertical homology class as

$$AW_{\vec{m}} \circ \gamma \circ (G(f)) \circ \mathsf{sh}_{\vec{k}}$$

for some $\gamma \in \widetilde{N}((|\vec{m}|), (|\vec{m}|))$. Now the vertical differential acts only on γ, which must be a cycle, hence a boundary in $\widetilde{N}((|\vec{m}|), (|\vec{m}|))$ unless $d = 0$, hence this cycle represents a trivial homology class if $d > 0$. In the case $d = 0$, $\gamma = \mathrm{id}$ is a cycle which is not a boundary. It follows that on homology,

$$H_*(\mathrm{Hom}_{Q(P)}(\vec{k}, \vec{m}); d_v) \simeq \mathrm{Hom}_P(|\vec{k}|, |\vec{m}|).$$

We get an isomorphism of E_1-pages of the spectral sequence for a double complex:

$$H_*(\widetilde{N}((|\vec{m}|), (|\vec{m}|)) \otimes \mathrm{Hom}_P(|\vec{k}|, |\vec{m}|), d_v) \xrightarrow{\sim} H_*(\mathrm{Hom}_{Q(P)}(\vec{k}, \vec{m}); d_v)$$

hence A is a quasi-isomorphism after totalizing. Now, for any $f \in \mathrm{Hom}_P(k, m)$ we have

$$F \circ \mathrm{Tot}(A)(f) = F(AW_{\vec{m}} \circ (f) \circ \mathsf{sh}_{\vec{k}}) = f$$

such that $F \circ \mathrm{Tot}(A)$ is the identity. By the 2-out-of-3 property for quasi-isomorphisms, F induced quasi-isomorphisms on Hom-complexes.

In the following, for a, b elements of a bicomplex C with $|a| = |b| = (d, d')$, a *vertical homotopy* $h \colon a \simeq b$ means an element h of C with $|h| = (d, d' + 1)$ such that $d_v h = b - a$.

We are left to show that each morphism $f \in \mathrm{Hom}_{Q(P)}(\vec{k}, \vec{m})$ which is a composition of generators admits a homotopy to the desired form. This is done by performing (strong) induction on the vertical degree d'. For $d' = 0$, the morphism f is given *a priori* by a sequence of generators

$$\vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \ldots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m}$$

where $\phi^i \in P^{\otimes}(\vec{k}_{2i}, \vec{k}_{2i+1})$ and $\gamma^i \in \widetilde{N}(\vec{k}_{2i+1}, \vec{k}_{2i+2})_0$. Write $n_i := l(\vec{k}_{2i}) = l(\vec{k}_{2i+1})$. To begin, we may fix for each γ^i a vertical homotopy $c(\gamma^i) \colon \gamma^i \simeq AW_{\vec{k}_{2i+2}} \circ \mathsf{sh}_{\vec{k}_{2i+1}}$. Applying the $c(\gamma^i)$ we obtain a new morphism

$$g = \vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{AW_{\vec{k}_2} \circ \mathsf{sh}_{\vec{k}_1}} \vec{k}_2 \xrightarrow{\phi^1} \ldots \xrightarrow{AW_{\vec{k}_{2m}} \circ \mathsf{sh}_{\vec{k}_{2m-1}}} \vec{k}_{2m}$$

equipped with a vertical homotopy $f \simeq g$. Now repeated application of the relations

$$\begin{tikzcd}
\vec{k}_{2i} \arrow[r, "\phi^i"] \arrow[d, "\mathsf{sh}_{\vec{k}_{2i}}"'] & \vec{k}_{2i+1} \arrow[d, "\mathsf{sh}_{\vec{k}_{2i+1}}"] \\
(|\vec{k}_{2i}|) \arrow[r, "\mathrm{Par}_{\phi^i}((n_i))"'] & (|\vec{k}_{2i+1}|)
\end{tikzcd}
\qquad
\begin{tikzcd}
(|\vec{k}_{2i}|) \arrow[r, "\mathrm{Par}_{\phi^i}((n_i))"] \arrow[d, "AW_{\vec{k}_{2i}}"'] & (|\vec{k}_{2i+1}|) \arrow[d, "AW_{\vec{k}_{2i+1}}"] \\
\vec{k}_{2i} \arrow[r, "\phi^i"'] & \vec{k}_{2i+1}
\end{tikzcd}$$

allows us to rewrite g as the composition

$$g = AW_{\vec{k}_{2m}} \circ \mathrm{Par}_{\phi^{m-1}}((n_{m-1})) \circ (AW_{\vec{k}_{2m-2}} \circ \mathsf{sh}_{\vec{k}_{2m-2}}) \circ \ldots$$
$$\ldots \circ \mathrm{Par}_{\phi^1}((n_1)) \circ (AW_{\vec{k}_2} \circ \mathsf{sh}_{\vec{k}_2}) \circ \mathrm{Par}_{\phi^0}((n_0)) \circ \mathsf{sh}_{\vec{k}_0}$$

Now choose vertical homotopies $\beta_{\vec{k}_{2i}} \colon (AW_{\vec{k}_{2i}} \circ \mathsf{sh}_{\vec{k}_{2i}}) \to \mathrm{id}_{(|\vec{k}_{2i}|)}$, giving us a composition of the desired form. This completes the base case.

For $d' > 0$, we may again write f as a sequence of generators

$$\vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \ldots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m}$$

where $\phi^i \in P^{\otimes}(\vec{k}_{2i}, \vec{k}_{2i+1})$ with $n_i := l(\vec{k}_{2i}) = l(\vec{k}_{2i+1})$ and now $\gamma^i \in \widetilde{N}(\vec{k}_{2i+1}, \vec{k}_{i+1,0})_{d'_i}$. We now consider two cases. Assume first that $d'_i < d'$ for each i. Let j be the least i such that $|\gamma^i| > 0$. By our assumption on the d'_i, $j < m-1$. Write f' for the composition

$$\vec{k}_{2j+2} \xrightarrow{\phi^{j+1}} \vec{k}_{2j+3} \xrightarrow{\gamma^{j+1}} \vec{k}_{j+2,0} \xrightarrow{\phi^{j+2}} \ldots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m}$$

and write f'' for the composition

$$\vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \ldots \xrightarrow{\gamma^j} \vec{k}_{2j+2}.$$

By induction, we may rewrite f' and f'' up to homotopy as

$$f' \simeq AW_{\vec{k}_{2m}} \circ \widetilde{\gamma}' \circ (G(f')) \circ \mathsf{sh}_{\vec{k}_{2j+2}}$$

where $\widetilde{\gamma}' \in \widetilde{N}((m),(m))_{d'_{j+1}+\ldots+d'_{m-1}}$, and

$$f'' \simeq AW_{\vec{k}_{2j+2}} \circ \widetilde{\gamma}'' \circ (G(f'')) \circ \mathsf{sh}_{\vec{k}_0}$$

where $\widetilde{\gamma}'' \in \widetilde{N}((|\vec{k}_{2j+2}|),(|\vec{k}_{2j+2}|))_{d'_0+\ldots+d'_j}$. Hence we get a vertical homotopy

$$f \simeq AW_{\vec{k}_{2m}} \circ \widetilde{\gamma}' \circ (G(f')) \circ \widetilde{\gamma}'' \circ (G(f'')) \circ \mathsf{sh}_{\vec{k}_0}$$

By induction, we may now rewrite

$$(G(f')) \circ \widetilde{\gamma}'' \circ (G(f'')) \simeq \widetilde{\gamma}''' \circ (G(f')) \circ (G(f'')) = \widetilde{\gamma}''' \circ (G(f))$$

for some $\widetilde{\gamma}''' \in \widetilde{N}((|\vec{k}_{2m}|),(|\vec{k}_{2m}|))_{d'_0+\ldots+d'_j}$ to obtain a composition of the desired form. This completes the case when $d'_i < d'$ for all i.

Finally, assume that there is a j such that $d'_j = d'$. If $j = m-1$, then the result follows from the base case and contractibility of $\widetilde{N}((|\vec{m}|),(|\vec{m}|))$. Namely, for a morphism f in $Q(P)$ represented by a sequence of generators

$$\vec{k}_0 \xrightarrow{\phi^0} \vec{k}_1 \xrightarrow{\gamma^0} \vec{k}_2 \xrightarrow{\phi^1} \ldots \xrightarrow{\gamma^{m-1}} \vec{k}_{2m}$$

as above, where $|\gamma^{m-1}| = d'$, we get vertical homotopies

$$f \simeq \gamma^{m-1} \circ AW_{\vec{k}_{2m-1}} \circ \gamma' \circ (G(f)) \circ \mathsf{sh}_{\vec{k}_0}$$
$$\simeq AW_{\vec{k}_{2m}} \circ \gamma \circ (G(f)) \circ \mathsf{sh}_{\vec{k}_0}$$

for some $\gamma' \in \widetilde{N}((|\vec{k}_{2m-1}|),(|\vec{k}_{2m-1}|))_0$ and $\gamma \in \widetilde{N}((|\vec{k}_{2m}|),(|\vec{k}_{2m}|))_{d'}$, where the first homotopy comes from the base case, and the second by contractibility of $\widetilde{N}((|\vec{m}|),(|\vec{m}|))$.

If $j < m-1$, we will provide a vertical homotopy between f and another morphism f' for which $d'_{j+1} = d'$. By the above, this will finish the argument. We apply a homotopy $\gamma^j \simeq AW_{\vec{k}_{2j+2}} \circ \bar{\gamma}^j \circ \mathsf{sh}_{\vec{k}_{2j+1}}$ where $\bar{\gamma}^j \in \widetilde{N}((|\vec{k}_{2j+2}|),(|\vec{k}_{2j+2}|))_{d'}$. By contractibility, we may assume that $\bar{\gamma}_j$ is of the form $\mathrm{Par}_{\vec{k}_{2j+2}}(\gamma'_j)$ for some $\gamma'_j \in \widetilde{N}((n_{j+1}),(n_{j+1}))_{d'}$. To name a concrete such element, one can use the (higher) homotopies witnessing sh and AW as mutual homotopy inverses. Now using the relations

$$\begin{tikzcd}
(|\vec{k}_{2j+2}|) \arrow[r, "\mathrm{Par}_{\phi^{j+1}}((n_j))"] \arrow[d, "\mathrm{Par}_{\vec{k}_{2j+2}}(\gamma'_j)"'] & (|\vec{k}_{2j+3}|) \arrow[d, "\mathrm{Par}_{\vec{k}_{2j+3}}(\gamma'_j)"] \\
(|\vec{k}_{2j+2}|) \arrow[r, "\mathrm{Par}_{\phi^{j+1}}((n_j))"'] & (|\vec{k}_{2j+3}|)
\end{tikzcd}$$

we obtain a composition

$$f \simeq f' = \left\{ \vec{k}'_0 \xrightarrow{\phi'^0} \vec{k}'_1 \xrightarrow{\gamma'^0} \vec{k}'_2 \xrightarrow{\phi'^1} \dots \xrightarrow{\gamma'^{m-1}} \vec{k}'_{2m} \right\}$$

where for $i \neq j+1$ we have $\vec{k}'_{2i} = \vec{k}_{2i}$ and $\vec{k}'_{2i+1} = \vec{k}_{2i+1}$, for $i \neq j+1$ we have $\phi'^i = \phi^i$, and for $i \neq j+1, j$ we have $\gamma^i = \gamma'^i$. Finally, $\vec{k}'_{2j+2} = (|\vec{k}_{2j}|)$, $\vec{k}'_{2j+3} = (|\vec{k}_{2j+3}|)$, $\phi'^{j+1} = \mathrm{Par}_{\phi^{j+1}}((n_{j+1}))$, $\gamma'^j = \mathsf{sh}_{\vec{k}_{2j+1}}$ and $\gamma'^{j+1} = \gamma^{j+1} \circ AW_{\vec{k}_{2j+3}} \circ \mathrm{Par}_{\vec{k}_{2j+3}}(\gamma'_j)$. Collecting the differences between f and f' in a diagram, we have

$$\begin{tikzcd}
\vec{k}_{2j+1} \arrow[rr, "\gamma^j"] \arrow[dr, "\gamma'^j = \mathsf{sh}_{\vec{k}_{2j+1}}"'] \arrow[from=1-1, to=2-4, phantom, "(\sim)"] & & \vec{k}_{2j+2} \arrow[rr, "\phi^{j+1}"] & & \vec{k}_{2j+3} \arrow[rr, "\gamma^{j+1}"] & & \vec{k}_{2j+4} \\
& (|\vec{k}_{2j+1}|) \arrow[rr, "\mathrm{Par}_{\vec{k}_{2j+2}}(\gamma'_j)"] \arrow[dr, "\phi'^{j+1} = \mathrm{Par}_{\phi^{j+1}}((n_{j+1}))"'] & & (|\vec{k}_{2j+1}|) \arrow[ul, "AW_{\vec{k}_{2j+2}}"'] \arrow[dr, "\mathrm{Par}_{\phi^{j+1}}((n_{j+1}))"'] & & & \\
& & (|\vec{k}_{2j+3}|) \arrow[rr, "\mathrm{Par}_{\vec{k}_{2j+2}}(\gamma'_j)"] \arrow[to=1-7, bend right, "\gamma'^{j+1}"'] & & (|\vec{k}_{2j+3}|) \arrow[uu, "AW_{\vec{k}_{2j+3}}"'] & &
\end{tikzcd}$$

where the square marked $(\sim)$ commutes up to vertical homotopy.

We see now that for the composition f', $|\gamma'^{j+1}| = d'$, and this finishes the argument. $\square$

Recall that for a prop P, a P-algebra is a symmetric monoidal functor $\Phi\colon P \to \mathsf{Ch}_k$ and a $\mathcal{A}ss \otimes P$-algebra is the same as a symmetric monoidal functor $P \to \mathsf{dgAlg}_k$.

Lemma 3.13. The functor $\mathrm{Tot}(Q(-))\colon \mathsf{dgprop} \to \mathsf{dgCat}^{\otimes}$ has the property that there is a natural transformation of functors $\mathsf{dgprop}^{op} \to \mathsf{Cat}$

$$\alpha\colon \mathrm{Fun}^{\otimes}(-, \mathsf{dgAlg}_k) \to \mathrm{Fun}^{\otimes}(\mathrm{Tot}(Q(-)), \mathsf{Ch}_k)$$

such that $\alpha_P(\Phi)(1) = C(\Phi(1))$.

Proof. We divide the proof into several steps. First we construct the functors $\alpha_P(\Phi)$. Then we show functoriality in Φ. Finally we will show naturality in P.

Step 1: Constructing $\alpha_P(\Phi)$.

Let P be a dg-prop, and let $\Phi\colon P \to \mathsf{dgAlg}_k$ be a symmetric monoidal functor. Throughout this section of the proof, we write $A = \Phi(1)$ for ease of notation. We will produce a symmetric monoidal functor $\alpha_P(\Phi)\colon \mathrm{Tot}(Q(P)) \to \mathsf{Ch}_k$ that sends $\vec{k}$ to $C^{\vec{k}}(A)$ (recall Notation 2.24). We describe the action of $\alpha_P(\Phi)$ on morphisms in terms of the generators of $\mathrm{Tot}(Q(P))$. If $f\colon n \to m$ is in P, then (f) acts by

$$C^{(n)}(A) \simeq C(\Phi(n)) \xrightarrow{C(f)} C(\Phi(m)) \simeq C^{(m)}(A).$$

This determines the action of morphisms in $P^{\otimes}$. Furthermore, $\widetilde{N}$ acts according to (the proof of) Proposition 2.26. This determines how the generators of $\mathrm{Tot}(Q(P))$ act. Note that the relations $\sim_{red}$ are preserved by this assignment. Furthermore, if $f\colon \vec{k}_0 \to \vec{k}_1$ is in $P^{\otimes}$ and $\psi\colon \vec{m}_0 \to \vec{m}_1$ is in $\widetilde{N}^{\Sigma}$, we have isomorphisms $C^{\vec{k}_0 * \vec{m}_0}(A) = C^{\vec{k}_0}(A) \otimes C^{\vec{m}_0}(A)$, and modulo these isomorphisms, we have

$$\alpha_P(\Phi)(\mathrm{id}_{\vec{m}_0} \otimes^P f) = \mathrm{id}_{C^{\vec{m}_0}(A)} \otimes \alpha_P(\Phi)(f)$$

and

$$\alpha_P(\Phi)(\phi \otimes^N \mathrm{id}_{\vec{k}_0}) = \alpha_P(\Phi)(\psi) \otimes \mathrm{id}_{C^{\vec{k}_0}(A)}$$

such that the interchange relation is preserved. Since

$$\left(\mathrm{id}_{C^{\vec{m}_0}(A)} \otimes \alpha_P(\Phi)(f)\right) \circ \left(\alpha_P(\Phi)(\psi) \otimes \mathrm{id}_{C^{\vec{k}_0}(A)}\right) = (\alpha_P(\Phi)(f) \otimes \alpha_P(\Phi)(\psi))$$

we have

$$\alpha_P(\Phi)(f \otimes_1 g) = \alpha_P(\Phi)(f) \otimes \alpha_P(\Phi)(g)$$

such that $\alpha_P(\Phi)$ defines a symmetric monoidal functor $\mathrm{Tot}(Q_1(P)) \to \mathsf{Ch}_k$.

To see that this assignment descends to a symmetric monoidal functor $\mathrm{Tot}(Q(P)) \to \mathsf{Ch}_k$, we are left to verify that the partition relations are preserved. Let $f = (f_1, ..., f_n)\colon \vec{k} \to \vec{m}$ be an n-tuple of morphisms in P and let $\gamma\colon \vec{a} \to \vec{b}$ be a morphism in N_n. We are will verify that the relations

$$\begin{tikzcd}
\mathrm{Par}_{\vec{k}}(\vec{a}) \arrow[r, "\mathrm{Par}_f(\vec{a})"] \arrow[d, "\mathrm{Par}_{\vec{k}}(\gamma)"'] & \mathrm{Par}_{\vec{m}}(\vec{a}) \arrow[d, "\mathrm{Par}_{\vec{m}}(\gamma)"] \\
\mathrm{Par}_{\vec{k}}(\vec{b}) \arrow[r, "\mathrm{Par}_f(\vec{b})"'] & \mathrm{Par}_{\vec{m}}(\vec{b})
\end{tikzcd}$$

are preserved under the assignment $\alpha_P(\Phi)$. There is an isomorphism (see Definition 2.25 and the proof of Proposition 2.26)

$$\alpha_P(\Phi)(\mathrm{Par}_{\vec{k}}(\vec{a})) = C^{\mathrm{Par}_{\vec{k}}(\vec{a})}(A) \overset{N\theta_{\vec{a}}^{-1}}{\to} N^{\vec{a}}(B^{cy}(A^{\otimes k_1}), ..., B^{cy}(A^{\otimes k_n})).$$

Write $N^{\vec{a}}(B^{cy}(A^{\vec{k}}))$ for the latter. Consider the following diagrams:

$$
\begin{tikzcd}
N^{\vec{a}}(B^{cy}(A^{\vec{k}})) \arrow[r, "N^{\vec{a}}(B^{cy}(f))"] \arrow[d, "\gamma_{B^{cy}(A^{\vec{k}})}"'] & N^{\vec{a}}(B^{cy}(A^{\vec{m}})) \arrow[d, "\gamma_{B^{cy}(A^{\vec{m}})}"] \\
N^{\vec{b}}(B^{cy}(A^{\vec{k}})) \arrow[r, "N^{\vec{b}}(B^{cy}(f))"'] & N^{\vec{b}}(B^{cy}(A^{\vec{m}}))
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{\mathrm{Par}_{\vec{k}}(\vec{a})}(A) \arrow[r, "\alpha_P(\Phi)(\mathrm{Par}_f(\vec{a}))"] \arrow[d, "N\theta_{\vec{a}}^{-1}"'] & C^{\mathrm{Par}_{\vec{m}}(\vec{a})}(A) \arrow[d, "N\theta_{\vec{a}}^{-1}"] \\
N^{\vec{a}}(B^{cy}(A^{\vec{k}})) \arrow[r, "N^{\vec{a}}(B^{cy}(f))"'] & N^{\vec{a}}(B^{cy}(A^{\vec{m}}))
\end{tikzcd}
$$

The left diagram commutes by the definition of $\widetilde{N}$, while the right diagram commutes by the naturality of the symmetric monoidal structure maps of B^{cy}. Finally, observe that

$$\alpha_P(\Phi)(\mathrm{Par}_{\vec{k}}(\gamma)) = N\theta_{\vec{b}} \circ \gamma_{B^{cy}(A^{\vec{k}})} \circ N\theta_{\vec{a}}^{-1}$$

Together these facts imply that the relations in $\mathrm{Tot}(Q(P))$ are preserved by the action. To be precise, we have a commutative diagram

$$
\begin{tikzcd}
C^{\mathrm{Par}_{\vec{k}}(\vec{a})}(A) \arrow[r, "\alpha_P(\Phi)(\mathrm{Par}_f(\vec{a}))"] \arrow[d, "N\theta_{\vec{a}}^{-1}"] \arrow[ddd, bend right=60, "\alpha_P(\Phi)(\mathrm{Par}_{\vec{k}}(\gamma)"'] & C^{\mathrm{Par}_{\vec{m}}(\vec{a})}(A) \arrow[d, "N\theta_{\vec{a}}^{-1}"] \arrow[ddd, bend left=60, "\alpha_P(\Phi)(\mathrm{Par}_{\vec{m}}(\gamma)"] \\
N^{\vec{a}}(B^{cy}(A^{\vec{k}})) \arrow[r, "N^{\vec{a}}(B^{cy}(f))"] \arrow[d, "\gamma_{B^{cy}(A^{\vec{k}})}"] & N^{\vec{a}}(B^{cy}(A^{\vec{m}})) \arrow[d, "\gamma_{B^{cy}(A^{\vec{m}})}"] \\
N^{\vec{b}}(B^{cy}(A^{\vec{k}})) \arrow[r, "N^{\vec{b}}(B^{cy}(f))"] \arrow[d, "N\theta_{\vec{b}}"] & N^{\vec{b}}(B^{cy}(A^{\vec{m}})) \arrow[d, "N\theta_{\vec{b}}"] \\
C^{\mathrm{Par}_{\vec{k}}(\vec{b})}(A) \arrow[r, "\alpha_P(\Phi)(\mathrm{Par}_f(\vec{b}))"] & C^{\mathrm{Par}_{\vec{m}}(\vec{b})}(A)
\end{tikzcd}
$$

Thus $\alpha_P(\Phi)$ is a functor from $\mathrm{Tot}(Q(P))$ as claimed.

Step 2: Showing that α_P is a functor.

We will notationally identify an object in $\mathrm{Fun}^{\otimes}(\mathcal{A}ss \otimes P, \mathsf{Ch}_k)$ with its value at 1. Let $\phi\colon A \to B$ be a morphism in $\mathrm{Fun}^{\otimes}(P, \mathsf{dgAlg}_k)$. We will produce a natural transformation $\alpha_P(A) \to \alpha_P(B)$. The component at $\vec{l} \in Q(P)$ is given by applying $\phi_1\colon A \to B$ componentwise, i.e. $\alpha_P(\phi)_{\vec{l}} = C^{\vec{l}}(\phi_1)\colon C^{\vec{l}}(A) \to C^{\vec{l}}(B)$. It is sufficient to check naturality against the generators of $Q(P)$. If $f\colon n \to m$ is a morphism in P, then the following diagram commutes because it commutes before applying $C(-)$.

$$
\begin{tikzcd}
C(A^{\otimes n}) \arrow[r, "C(\phi_n)"] \arrow[d, "C(f_A)"'] & C(B^{\otimes n}) \arrow[d, "C(f_B)"] \\
C(A^{\otimes m}) \arrow[r, "C(\phi_m)"'] & C(B^{\otimes m})
\end{tikzcd}
$$

Let $\gamma\colon \vec{k} \to \vec{m}$ be a morphism in $\widetilde{N}$ and consider the following diagrams:

$$\begin{tikzcd} C^{\vec{k}}(A) \arrow[r, "N\theta_{\vec{k}}^{-1}"] \arrow[d, "C^{\vec{k}}(\phi)"'] & N^{\vec{k}}(B^{cy}(A)) \arrow[d, "N^{\vec{k}}(B^{cy}(\phi))"] \\ C^{\vec{k}}(B) \arrow[r, "N\theta_{\vec{k}}^{-1}"'] & N^{\vec{k}}(B^{cy}(B)) \end{tikzcd} \qquad \begin{tikzcd} N^{\vec{k}}(B^{cy}(A)) \arrow[r, "N^{\gamma}(B^{cy}(A))"] \arrow[d, "N^{\vec{k}}(B^{cy}(\phi))"'] & N^{\vec{m}}(B^{cy}(A)) \arrow[d, "N^{\vec{m}}(B^{cy}(\phi))"] \\ N^{\vec{k}}(B^{cy}(B)) \arrow[r, "N^{\gamma}(B^{cy}(B))"'] & N^{\vec{m}}(B^{cy}(B)) \end{tikzcd}$$

The left diagram commutes by naturality of the symmetric monoidal structure maps of B^{cy} and the right diagram commutes by the definition of $\widetilde{N}$. Since γ acts by

$$\alpha_P(A)(\gamma) = N\theta_m \circ N^{\gamma}(B^{cy}(A)) \circ N\theta_{\vec{k}}^{-1}$$

the commutativity of these two families of diagrams implies naturality with respect to the morphisms in $\widetilde{N}^{\Sigma}$.

Step 3: Showing that α is natural in P.

Let $i\colon P \to P'$ be a morphism of dg-props. We need to check commutativity of the diagram

$$\begin{tikzcd} \mathrm{Fun}^{\otimes}(P', \mathsf{dgAlg}_k) \arrow[r, "\alpha_{P'}"] \arrow[d, "i^*"'] & \mathrm{Fun}^{\otimes}(\mathrm{Tot}(Q(P')), \mathsf{Ch}_k) \arrow[d, "\mathrm{Tot}(Q(i))^*"] \\ \mathrm{Fun}^{\otimes}(P, \mathsf{dgAlg}_k) \arrow[r, "\alpha_P"'] & \mathrm{Fun}^{\otimes}(\mathrm{Tot}(Q(P)), \mathsf{Ch}_k) \end{tikzcd}$$

Let $\Phi\colon P' \to \mathsf{dgAlg}_k$ be a symmetric monoidal functor. We first show that the functors $\alpha_P(i^*\Phi)$ and $\mathrm{Tot}(Q(i))^*\alpha_{P'}(\Phi)$ are equal. Since i and $\mathrm{Tot}(Q(i))$ are isomorphisms on objects, we have

$$\alpha_P(i^*\Phi)(\vec{k}) = C^{\vec{k}}(i^*\Phi(1)) = C^{\vec{k}}(\Phi(1))$$

and

$$\mathrm{Tot}(Q(i))^*\alpha_{P'}(\Phi)(\vec{k}) = \alpha_{P'}(\Phi)(\vec{k}) = C^{\vec{k}}(\Phi(1))$$

so they are equal on objects. Let $\gamma\colon \vec{k} \to \vec{m}$ be a morphism in $\widetilde{N}^{\Sigma}$. Since $\mathrm{Tot}(Q(i))$ is the identity on $\widetilde{N}^{\Sigma}$, we similarly have

$$\mathrm{Tot}(Q(i))^*\alpha_{P'}(\Phi)(\gamma) = \alpha_{P'}(\Phi)(\gamma) = N\theta_m \circ N^{\gamma}(B^{cy}(\Phi(1))) \circ N\theta_{\vec{k}}^{-1}$$

and

$$\alpha_P(i^*\Phi)(\gamma) = N\theta_m \circ N^{\gamma}(B^{cy}(i^*\Phi(1))) \circ N\theta_{\vec{k}}^{-1} = N\theta_m \circ N^{\gamma}(B^{cy}(\Phi(1))) \circ N\theta_{\vec{k}}^{-1}$$

so the action of $\widetilde{N}^{\Sigma}$ coincides as well. We now compare the action by a morphism $f\colon k \to m$ in P. We have

$$\alpha_P(i^*\Phi)(f)\colon C^{(k)}(\Phi(1)) \simeq C(\Phi(k)) \xrightarrow{C(i(f))} C(\Phi(m)) \simeq C^{(m)}(\Phi(1)).$$

Notice that $\alpha_P(i^*\Phi)(f) = \alpha_{P'}(\Phi)(i(f))$. Now since $\mathrm{Tot}(Q(i))(f) = i(f)$ in $\mathrm{Tot}(Q(P'))$, we have

$$\mathrm{Tot}(Q(i))^*\alpha_{P'}(\Phi)(f) = \alpha_{P'}(\Phi)(i(f))$$

so the two functors coincide on objects.

Before we verify that the functors also agree on morphisms, we recall a basic fact about compositions of natural transformations. If $\mathcal{C}, \mathcal{C}', \mathcal{D}$ are categories, $j\colon \mathcal{C} \to C'$ is a functor and $\alpha\colon F \Rightarrow G\colon \mathcal{C}' \to \mathcal{D}$ is a natural transformation, then the pullback of α along j is given componentwise by $(\alpha * \mathrm{id}_j)_c = \alpha_{j(c)}$.

For a morphism $\psi\colon \Phi \to \Psi$, we have the natural transformations

$$\alpha_P(i^*\psi)\colon \alpha_P(i^*\Phi) \to \alpha_P(i^*\Psi)$$

and

$$\mathrm{Tot}(Q(i))^*\alpha_{P'}(\psi)\colon \mathrm{Tot}(Q(i))^*\alpha_{P'}(\Phi) \to \mathrm{Tot}(Q(i))^*\alpha_{P'}(\Psi)$$

of functors $\mathrm{Tot}(Q(P)) \to \mathsf{Ch}_k$. It is sufficient to check that they coincide on components. Let $\vec{k}$ be an object of $\mathrm{Tot}(Q(P))$. Then since i is the identity on objects, we get

$$\alpha_P(i^*\psi)(\vec{k}) = C^{\vec{k}}(i^*\psi(1)) = C^{\vec{k}}(\psi(1))$$

and

$$\mathrm{Tot}(Q(i))^*\alpha_{P'}(\psi)(\vec{k}) = \alpha_{P'}(\psi)(\vec{k}) = C^{\vec{k}}(\psi(1))$$

so they are equal. $\square$

Proof of Theorem A: Define $\widetilde{(-)}\colon \mathsf{dgprop} \to \mathsf{dgprop}$ to be the functor taking a dg-prop P to the full subcategory of $\mathrm{Tot}(Q(P))$ generated by the objects $\{(1)^n\}_{n\geq 0}$. To see that this defines a functor, recall from Construction 3.6 that for a morphism of dg-props $P \to P'$, the induced symmetric monoidal functor $\mathrm{Tot}(Q(P)) \to \mathrm{Tot}(Q(P'))$ is the identity on object monoids, hence it restricts to a prop morphism $\widetilde{P} \to \widetilde{P}'$.

<u>The natural quasi-equivalence $\widetilde{(-)} \to \mathrm{id}$.</u>

Let $F|_{\widetilde{P}}\colon \widetilde{P} \to P$ be the composition

$$\widetilde{P} \hookrightarrow \mathrm{Tot}(Q(P)) \xrightarrow{F} P$$

It is clear that $F|_{\widetilde{P}}$ induces an isomorphism on object monoids. By Lemma 3.12, $F|_{\widetilde{P}}$ also induces quasi-isomorphisms on Hom-complexes, hence it is a quasi-equivalence. Naturality of F and the inclusion $\widetilde{P} \to \mathrm{Tot}(Q(P))$ imply that $F|_{\widetilde{P}}$ is a natural quasi-equivalence.

<u>The natural transformation $\widetilde{\alpha}$.</u>

To produce the natural transformation $\widetilde{\alpha}$, we use the transformation α from Lemma 3.13. Recall that there is an equivalence of categories

$$\mathrm{Fun}^{\otimes}(\mathcal{A}ss \otimes P, \mathsf{Ch}_k) \simeq \mathrm{Fun}^{\otimes}(P, \mathsf{dgAlg}_k).$$

The natural inclusion $i\colon \widetilde{(-)} \to \mathrm{Tot}(Q(-))$ gives us a natural transformation

$$i^*\colon \mathrm{Fun}^{\otimes}(\mathrm{Tot}(Q(-)), \mathsf{Ch}_k) \to \mathrm{Fun}^{\otimes}(\widetilde{(-)}, \mathsf{Ch}_k)$$

and we define $\widetilde{\alpha}$ to be the composition

$$\widetilde{\alpha} = i^* \circ \alpha\colon \mathrm{Fun}^{\otimes}(\mathcal{A}ss \otimes -, \mathsf{Ch}_k) \to \mathrm{Fun}^{\otimes}(\widetilde{(-)}, \mathsf{Ch}_k).$$

Because $\widetilde{\alpha}$ is a restriction of α, we have that for any prop P, and symmetric monoidal functor $\Phi\colon \mathcal{A}ss \otimes P \to \mathsf{Ch}_k$, there is an equality $\widetilde{\alpha}_P(\Phi)(1) = \alpha_P(\Phi)(1) = C(\Phi(1))$, hence $\widetilde{\alpha}$ has the stated properties.

$\square$

Example 3.14. Consider the example $P = \mathcal{CH}opf$, the prop encoding a commutative Hopf algebra structure. Note that every morphism in $\mathcal{CH}opf$ is an algebra homomorphism, hence we have an equivalence $\mathcal{A}ss \otimes \mathcal{CH}opf \simeq \mathcal{CH}opf$ and Theorem A gives a recipe for the natural coherent commutative Hopf algebra structure on Hochschild chains of commutative Hopf algebras. In particular, $\widetilde{\mathcal{CH}opf}$ is generated in degree 0 by the morphisms

$$() \xrightarrow{\eta} (1)$$

$$(1,1) \xrightarrow{\mathsf{sh}} (2) \xrightarrow{m} (1)$$

$$(1) \xrightarrow{\epsilon} ()$$

$$(1) \xrightarrow{\Delta} (2) \xrightarrow{AW} (1,1)$$

$$(1) \xrightarrow{S} (1)$$

An example of a generator in degree 1 is the bialgebra relation, in which we need the homotopy $\theta \in \widetilde{N}^{\Sigma}((2,2),(2,2))_1$ to interpolate between the upper and lower legs of the diagram. Here $F \in \Sigma_4$ is the transposition $(2,3)$.

$$\begin{tikzcd}
(1,1) \arrow[r, "{(\Delta,\Delta)}"] \arrow[d, "\mathsf{sh}"] & (2,2) \arrow[r, "{F \circ (AW, AW)}"] \arrow[d, "\mathrm{Par}_{(2,2)}(\mathsf{sh})"'] \arrow[dr, phantom, "\theta\,\Swarrow"] & (1,1,1,1) \arrow[d, "{(\mathsf{sh},\mathsf{sh})}"] \\
(2) \arrow[r, "\mathrm{Par}_{(\Delta,\Delta)}((2))"] \arrow[d, "m"] & (4) \arrow[r, "\mathrm{Par}_{(2,2)}(AW) \circ F_*"'] \arrow[d, "\mathrm{Par}_{(m,m)}((2)) \circ F_*"'] & (2,2) \arrow[d, "{(m,m)}"] \\
(1) \arrow[r, "(\Delta)"] & (2) \arrow[r, "AW"] & (1,1)
\end{tikzcd}$$

In a similar way, we need the contraction $\alpha_{(1,1)}\colon AW \circ \mathsf{sh} \simeq \mathrm{id} \in \widetilde{N}^{\Sigma}((1,1),(1,1))$ for the antipode diagrams.

Note that $\widetilde{\mathcal{CH}opf}$ still has a strictly commutative multiplication. If $\mathcal{C}\mathbb{E}_n\mathcal{H}opf$ encodes commutative and $\mathbb{E}_n$-cocommutative Hopf algebras, then $\widetilde{\mathcal{C}\mathbb{E}_n\mathcal{H}opf}$ will also be $\mathbb{E}_n$ cocommutative for $n \leq \infty$, but if $\mathcal{CCH}opf$ is the prop encoding a Hopf algebra structure which is both commutative and cocommutative, then $\widetilde{\mathcal{CCH}opf}$ is strictly commutative but only $\mathbb{E}_\infty$-cocommutative, since AW is not a symmetric monoidal transformation.

References

[1] Michael Boardman, Rainer Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, (1973).

[2] Morten Brun, Zbigniew Fiedorowicz, and Rainer M. Vogt, *On the multiplicative structure of topological Hochschild homology*, Algebr. Geom. Topol., 7:1633—1650, 2007

[3] Albrecht Dold, *Über die Steenrodschen Kohomologieoperationen*, Ann. of Math. 73, 1961, 258–294.

[4] Samuel Eilenberg and Saunders Mac Lane, *On the Groups* $H(\Pi, n)$*, I*, Annals of Mathematics, Vol. 58, No. 1 (Jul., 1953), pp. 55–106.

[5] Samuel Eilenberg and Saunders MacLane, *On the Groups* $H(\Pi, n)$*, II: Methods of Computation*, Annals of Mathematics, Vol. 60, No. 1 (Jul., 1954), pp. 49–139.

[6] D. Fiorenza, *An introduction to the Language of Operads*, `www.mat.uniroma1.it/ fiorenza/ilo.ps.gz`, 2006.

[7] Benoit Fresse, *Props in model categories and homotopy invariance of structures*, arXiv:0812.2738v4, 5 Dec 2008.

[8] Jean-Louis Loday, *Cyclic Homology*, Springer Verlag, 1992.

[9] Saunders MacLane, *Categorical Algebra* Bull. Amer. Math. Soc., Volume 71, Number 1 (1965), 40–106.

[10] Martin Markl, *Operads and PROPs*, arXiv:math/0601129v3, 6 Jan 2006.

[11] Birgit Richter, $\mathbb{E}_\infty$*-structure for* $Q_*(R)$, Math. Ann. 316, 547–564 (2000).

[12] Birgit Richter, *Symmetry Properties of the Dold-Kan Correspondence*, Mathematical Proceedings of the Cambridge Philosophical Society, 134(1), pp. 95-–102, 2003.

[13] Stefan Schwede and Brooke Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol., Volume 3, Number 1 (2003), 287–334.

[14] Nathalie Wahl and Craig Westerland, *Hochschild homology of structured algebras*, Advances in Math. 288 (2016), 240–307.

[15] Nathalie Wahl, *Universal operations in Hochschild homology*, J. Reine Angew. Math. 720 (2016), 81–127 2016.

[16] Harvey Wolff, *V-cat and V-graph*, J. Pure Appl. Algebra 4 (1974), 123-–135.

Part 3

Paper B

ON THE MORITA FUNCTORIALITY OF THE HOCHSCHILD COMPLEX

ESPEN AUSETH NIELSEN

Abstract. We construct the Hochschild complex as a symmetric monoidal functor of quasi-categories from the Morita (2,1)-category to the quasi-category of chain complexes, conditioned on the existence of a certain symmetric monoidal structure on the latter. As an application, the Hochschild complex on a commutative Hopfish algebra, of which a commutative quasi-Hopf algebra is an example, obtains a the structure of a commutative Hopf algebra object in the quasi-category of chain complexes.

Note: The main theorem of this paper depends on the statement that the simplicial nerve of the category of connective chain complexes, whose simplicial structure comes from the Dold-Kan equivalence, can be given the structure of a symmetric monoidal quasi-category which on the homotopy category agrees with the derived tensor product. Although probably true, this statement does not appear in the literature to the knowledge of the author, and should be taken as an additional assumption in the theorem.

1. Introduction

In this paper we will prove that the Hochschild chain functor extends to a "weak symmetric monoidal functor" from the Morita (2,1)-category to chain complexes. To make this precise, we employ the theory of quasi-categories. As an application, we obtain a natural homotopy coherent Hopf algebra structure on the Hochschild complex of several variations of Hopf algebras, such as quasi-Hopf algebras and Hopfish algebras.

The Hochschild complex of an algebra (also called the cyclic bar complex) takes a k-algebra A to the chain complex $C(A)_*$ where $C(A)_n = A^{\otimes n+1}$ and the differential is the alternating sum of multiplying the i'th and $i+1$'st coordinates modulo $n+1$. This definition has a natural extension to dg-algebras and dg-categories which we give in Definition 5.1.

Fixing an inaccessible cardinal κ, let k be a commutative ring, dgAlg_k the category of dg-algebras over k. We write Ch_k for the dg-category of chain complexes over k. We require that the ring k and all k-modules we consider are smaller than κ. It is known by the work of Richter [27] that the Hochschild complex $\mathsf{dgAlg}_k \to \mathsf{Ch}_k$ is a so-called $\mathbb{E}_\infty$-monoidal functor. In other words, the nerve of the Hochschild complex functor,

$$N(\mathsf{dgAlg}_k) \to N_\Delta(\mathsf{Ch}_k),$$

can be given the structure of a symmetric monoidal functor of quasi-categories.

The Morita (2,1)-category Mor_k of k is the (2,1)-category of k-algebras and their bimodules. It is defined in Definition 3.1 and admits a symmetric monoidal 2-functor of (2,1)-categories

$$\mathsf{m}\colon \mathsf{dgAlg}_k \to \mathsf{Mor}_k,$$

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where dgAlg_k is considered as a (2,1)-category with only identity 2-morphisms. Our main theorem is that the symmetric monoidal of quasi-categories $N(\mathsf{dgAlg}_k) \to N_\Delta(\mathsf{Ch}_k)$, factors through m.

Main Theorem: (Conditioned on Assumption 2.32) *Let k be a commutative ring.*

- (Theorem 5.22) *The Hochschild complex gives rise to a functor of quasi-categories $N_D\mathsf{Mor}_k \to N_\Delta\mathsf{Ch}_k$ such that the following diagram of quasi-categories commutes up to homotopy.*

$$\begin{tikzcd} N(\mathsf{dgAlg}_k) \arrow[r, "N_D(\mathsf{m})"] \arrow[dr, "C(-)"'] & N_D(\mathsf{Mor}_k) \arrow[d, "C(-)"] \\ & N_\Delta(\mathsf{Ch}_k) \end{tikzcd}$$

- (Theorem 6.7) *The above diagram upgrades to a homotopy commutative diagram of symmetric monoidal functors of quasi-categories which fits into a homotopy commutative diagram in the category of symmetric monoidal quasi-categories:*

$$\begin{tikzcd} N(\mathsf{dgAlg}_k)^{\otimes} \arrow[r, "N_D(\mathsf{m})"] \arrow[dr, "C(-)"'] & N_D(\mathsf{Mor}_k)^{\otimes} \arrow[d, "C(-)"] \\ & N_\Delta(\mathsf{Ch}_k)^{\otimes} \end{tikzcd}$$

The functor $N_D\mathsf{Mor}_k \to N_\Delta\mathsf{Ch}_k$ will be constructed as a composition:

$$C(-)\colon N_D(\mathsf{Mor}_k) \to N_D(\mathsf{dgCat}_k^{(2,1)}) \cong N_\Delta(\mathsf{dgCat}_{k,\Delta}^{(2,1)}) \to N_\Delta(\mathsf{Ch}_k)$$

where the first map is induced on the level of (2,1)-categories and the last map is induced on the level of simplicial categories. The compatibilities of the symmetric monoidal structures is Theorem 6.7.

As an application of the theorem, we obtain a homotopy coherent Hopf algebra structure on the Hochschild chains on commutative Hopfish algebras in the sense of [30], of which commutative quasi-Hopf algebras are examples, see Section 7, in particular Proposition 7.6.

The paper is structured as follows. In Section 2, we review the theory of (2,1)-categories and quasi-categories. In Section 3, we construct the (2,1)-category of algebras, bimodules and bimodule isomorphisms associated to a ground ring k. In Section 4, we review some material on the Dold-Kan equivalence and its monoidality properties which we need to compare symmetric monoidal structures. In Section 5 we construct the vertical functor in the Main Theorem. In Section 6, we finish the proof of the Main Theorem by showing that the vertical functor is symmetric monoidal comparing the symmetric monoidal structures in the diagram. In Section 7 we explore some consequences for the algebraic structure on Hochschild complexes of Hopf-like algebras.

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2. Some categorical notions

In this section we introduce some elementary notions of the theory of (2,1)-categories and quasi-categories which we need in the coming sections. We will define nerve constructions N_D (Definition 2.10) and N_Δ (Definition 2.12) which take (2,1)-categories, resp. simplicial categories, to their associated quasi-categories. Furthermore, N_D and N_Δ are equivalent for strict 2-categories whose 2-morphisms are invertible (Lemma 2.21). We also review basic theory about symmetric monoidal quasi-categories. When applying the functor N_Δ to a dg-category, we implicitly treat the dg-category as a simplicial category by using the Dold-Kan equivalence. For a more thorough introduction to quasi-category theory, see [15], and for a detailed account, see [22] and [23].

Convention 2.1. We fix an inaccessible cardinal κ. We call a set, category, algebra etc. *small* if it has cardinality less than κ, and *large* otherwise. Unless stated otherwise, all introduced objects are assumed to be small.

2.1. **(2,1)-categories and their nerves.**

Definition 2.2. [1, Definition 1.1] A *(2,1)-category* $\mathcal{C}$ is the data of

- a set of objects $\mathrm{Ob}\,\mathcal{C}$,
- for each pair $a, b \in \mathrm{Ob}\,\mathcal{C}$, a groupoid of morphisms $\mathcal{C}(a,b)$,
- for each object $a \in \mathrm{Ob}\,\mathcal{C}$, an identity object $1_a \in \mathcal{C}(a,a)$,
- for each triple $a,b,c \in \mathrm{Ob}\,\mathcal{C}$, a functor $\circ_{a,b,c} \colon \mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)$,
- for each 4-tuple $a,b,c,d \in \mathrm{Ob}\,\mathcal{C}$, a natural isomorphism called an *associator*

$$\alpha_{a,b,c,d} \colon \circ_{a,b,d}(\circ_{b,c,d} \times \mathrm{id}) \to \circ_{a,c,d}(\mathrm{id} \times \circ_{a,b,c}),$$

- and for each pair $a, b \in \mathrm{Ob}\,\mathcal{C}$, isomorphisms called left and right *unitors*

$$\lambda_{a,b} \colon \circ_{a,b,b}(1_b, -) \to \mathrm{id}_{\mathcal{C}(a,b)}$$
$$\rho_{a,b} \colon \circ_{a,a,b}(-, 1_a) \to \mathrm{id}_{\mathcal{C}(a,b)}$$

such that the associators satisfy MacLane's pentagram relation, and the unitors satisfy the triangle relations.

We call the objects of the morphism groupoids $\mathcal{C}(a,b)$ the morphisms of $\mathcal{C}$, and the morphisms of $\mathcal{C}(a,b)$ are called the 2-morphisms of $\mathcal{C}$.

Notation 2.3. For a (2,1)-category $\mathcal{C}$ and a pair of morphisms $f\colon a \to b$ and $g\colon b \to c$ of $\mathcal{C}$, we may denote the composition $\circ_{a,b,c}(g,f)$ by $g * f$ when the sources and targets are understood.

Remark 2.4. The cited definition for Definition 2.2 allows arbitrary categories $\mathcal{C}(a,b)$ instead of just groupoids. We would then obtain the definition of a *bicategory*. In [1] a (2,1)-category is called a bicategory which is *locally a groupoid*. The theory of bicategories is quite exotic when compared to ordinary category theory, but we will only have need for the simpler notion of (2,1)-categories, which behave much closer to ordinary categories.

Definition 2.5. Let $\mathcal{C}$ be a (2,1)-category. A morphism $f\colon a \to b$ in $\mathcal{C}$ is called an *equivalence* if there exists a morphism $g\colon b \to a$ and 2-morphisms $g \circ f \to 1_a$ and $f \circ g \to 1_b$.

Definition 2.6. [1, Definition 2.1] A (2,1)-category $\mathcal{C}$ for which the associators and unitors are identity morphisms is called a *strict (2,1)-category.*

Examples 2.7. • Any (small) ordinary category can be considered a (2,1)-category whose morphism groupoids are discrete.

• There is a large (2,1)-category Cat whose objects are categories, morphisms are functors and 2-morphisms are natural isomorphisms.

• Similarly to the above, let k be a commutative ring and let $\mathsf{dgCat}_k^{(2,1)}$ be the (2,1)-category whose objects are k-linear dg-categories, morphisms are degree-preserving dg-functors and 2-morphisms are natural isomorphisms.

The above examples are all strict (2,1)-categories. In the next section, we will see an example of a non-strict (2,1)-category, namely the Morita category Mor_k. We now discuss the notion of 2-functor between (2,1)-categories.

Definition 2.8. [1, Definition 4.1] Let $\mathcal{C}$ and $\mathcal{D}$ be (2,1)-categories. A *2-functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the data of

- A function $F\colon \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$,
- for each pair $a, b \in \operatorname{Ob}\mathcal{C}$, a functor $F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$,
- for each $a \in \operatorname{Ob}\mathcal{C}$, a 2-morphism $F_{1_a}\colon 1_{Fa} \to F_{a,a}(1_a)$,
- and for each triple $a, b, c \in \operatorname{Ob}\mathcal{C}$, a 2-morphism, natural in $f\colon a \to b$ and $g\colon b \to c$,

$$F_{a,b,c}(g,f)\colon \circ_{Fa,Fb,Fc}(F_{b,c}g, F_{a,b}f) \to F_{a,c}(\circ_{a,b,c}(g,f)),$$

such that

- for each $a, b \in \mathcal{C}$, and each $f \in \mathcal{C}(a,b)$ the following diagrams commute in $\mathcal{D}(Fa, Fb)$,

$$\begin{tikzcd}
1_{Fb} * F_{a,b}f \arrow[r, "\lambda_{Fa}(F_{a,b}f)"] \arrow[d, "F_{1_b} * \mathrm{id}"'] & F_{a,b}f \\
F_{b,b}(1_b) * F_{a,b}f \arrow[r, "F_{a,b,b}(1_b, f)"'] & F_{a,b}(1_b * f) \arrow[u, "F_{a,b}(\lambda_{a,b}(f))"']
\end{tikzcd}$$

$$\begin{tikzcd}
F_{a,b}f * 1_{Fa} \arrow[r, "\rho_{Fa}(F_{a,b}f)"] \arrow[d, "\mathrm{id} * F_{1_a}"'] & F_{a,b}f \\
F_{a,b}f * F_{a,a}(1_a) \arrow[r, "F_{a,a,b}(f, 1_a)"'] & F_{a,b}(f * 1_a) \arrow[u, "F_{a,b}(\rho_{a,b}(f))"']
\end{tikzcd}$$

  and
- for each 4-tuple $a, b, c, d \in \operatorname{Ob}\mathcal{C}$, and morphisms $f\colon a \to b$, $g\colon b \to c$ and $h\colon c \to d$, the following diagram commutes in $\mathcal{D}(Fa, Fd)$:

$$\begin{tikzcd}
(F_{c,d}h * F_{b,c}g) * F_{a,b}f \arrow[r, "\alpha_{Fa,Fb,Fc,Fd}"] \arrow[d, "F_{b,c,d}(h,g) * \mathrm{id}"'] & F_{c,d}h * (F_{b,c}g * F_{a,b}f) \arrow[d, "\mathrm{id} * F_{a,b,c}(g,f)"] \\
F_{b,d}(h * g) * F_{a,b}f \arrow[d, "F_{a,b,d}(h * g, f)"'] & F_{c,d}h * F_{a,c}(g * f) \arrow[d, "F_{a,c,d}(h, g * f)"] \\
F_{a,d}((h * g) * f) \arrow[r, "F_{a,d}(\alpha_{a,b,c,d}(h,g,f))"'] & F_{a,d}(h * (g * f))
\end{tikzcd}$$

Definition 2.9. [1, Remark 4.2 ("strictly unitary morphisms")] A 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ between (2,1)-categories is called *normal* if for each $a \in \operatorname{Ob}\mathcal{C}$ the morphisms $F_{1_a}\colon 1_{Fa} \to F_{a,a}(1_a)$ are identity morphisms. We denote by $\operatorname{Hom}^N_{(2,1)\mathsf{Cat}}(\mathcal{C},\mathcal{D})$ the set of normal 2-functors between a pair of (2,1)-categories.

We write Δ for the standard simplicial category. It has as objects the finite linear posets $[n] = \{0 \to ... \to n\}$ and as morphisms poset maps between these.

Definition 2.10. [10, Section 6] Let $\mathcal{C}$ be a (2,1)-category. Considering the linear posets $[n]$ as (2,1)-categories with only identity 2-morphisms, the *Duskin nerve* of $\mathcal{C}$ is the simplicial set $N_D(\mathcal{C})$ whose set of n-simplices is given by the set $\operatorname{Hom}^N_{(2,1)\mathsf{Cat}}([n],\mathcal{C})$ of normal 2-functors $[n] \to \mathcal{C}$. The assignment

$$[n] \mapsto \operatorname{Hom}^N_{(2,1)\mathsf{Cat}}([n],\mathcal{C})$$

defines a functor $\Delta^{op} \to \mathsf{Set}$, where the simplicial structure maps are given by precomposition: for any $\phi\colon [m] \to [n]$ in Δ and $F \in \operatorname{Hom}^N_{(2,1)\mathsf{Cat}}([n],\mathcal{C})$, we have $\phi^*(F) = F \circ \phi$.

Spelling out this definition, an n-simplex $F \in N_D(\mathcal{C})_n$ is given by the data of

1. for each $0 \le i \le n$, an object $F(i)$,
2. for each pair $0 \le i \le j \le n$, a morphism $F(f_{i,j})\colon F(i) \to F(j)$ in $\mathcal{C}$, and
3. for each triple $0 \le i \le j \le k \le n$, a 2-morphism $\alpha_{i,j,k}\colon F(f_{j,k}) \circ F(f_{i,j}) \to F(f_{i,k})$,

such that for each 4-tuple $0 \le i \le j \le k \le l \le n$, we have an equality

$$\alpha_{i,k,l} \circ (\mathrm{id}_{f_{k,l}} * \alpha_{i,j,k}) = \alpha_{i,j,l} \circ (\alpha_{j,k,l} * \mathrm{id}_{f_{i,j}}).$$

The face maps $d_i\colon N_D(\mathcal{C})_n \to N_D(\mathcal{C})$ act by forgetting all pieces of data where the number i appears. For example, $(d_i\alpha)_{j,k,l} = \alpha_{j',k',l'}$, where

$$j' = \begin{cases} j & \text{, if } 0 \le j < i \\ j+1 & \text{, if } i \le j \le n \end{cases}$$

and so on. Let $F \in N_D(\mathcal{C})_n$ be as above. The degeneracy maps $s_i\colon N_D(\mathcal{C})_n \to N_D(\mathcal{C})_{n+1}$ take F to the element s_iF whose data is given by shifting the indices similarly to the above, i.e.

$$j' = \begin{cases} j & \text{, if } 0 \le j \le i \\ j-1 & \text{, if } i < j \le n+1 \end{cases}$$

with the implicit assumption that $F(f_{j,k})$ and $\alpha_{j,k,l}$ are identity (2-)morphisms if an index is repeated.

2.2. **Simplicial categories and their nerves.** We recall some quasi-categorical notions. Recall that a simplicial set X is a quasi-category if it admits internal horn fillers. That is, if Λ^n_k denotes the boundary of the representable n-simplex, minus the face opposite the k-th vertex (explicitly, Λ^n_k is the subset of Δ^n generated by those simplices $i\colon [k] \to [n]$ such that the image of i does not contain $\{1, ..., k-1, k+1, ..., n\}$) then for each simplicial map $\Lambda^n_k \to X$, where $0 < k < n$, the dashed arrow exists in the diagram below, such that the below diagram commutes.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

A functor between quasi-categories is just a map of simplicial sets. Quasi-categories were originally introduced in [4].

Definition 2.11. A morphism of simplicial sets $f\colon X \to Y$ is an inner fibration, if it has the right lifting property with respect to inner horn inclusions. That is, if the solid part of the below diagram commutes and $0 < k < n$, then the dashed arrow exists and the whole diagram commutes.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Definition 2.12. [22, Section 1.1.5] Let $n \geq 0$. The simplicial category $\mathfrak{C}[\Delta^n]$ has objects $\mathrm{Ob}\,\mathfrak{C}[\Delta^n] = \mathrm{Ob}\,[n] = \{0, ..., n\}$ and morphisms given as follows. For $0 \leq i \leq j \leq n$, define a poset $P_{i,j} = \{I \subseteq \{i, ..., j\} \mid i, j \in I\}$ ordered by set inclusion, and define

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,j) = \begin{cases} NP_{i,j} & , \ i \leq j \\ \emptyset & , \ i > j \end{cases}$$

and composition given by taking unions of subsets.

The assignment $[n] \mapsto \mathfrak{C}[\Delta^n]$ extends to a cosimplicial object $\Delta \to \mathsf{sCat}$ as follows: if $f\colon [n] \to [m]$ is a morphism in Δ, the induced simplicial functor

$$f_*\colon \mathfrak{C}[\Delta^n] \to \mathfrak{C}[\Delta^m]$$

takes the object i to $f(i)$, and for $i \leq j$ in $[n]$, the map

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^m]}(f(i), f(j))$$

is induced by the functor $P_{i,j} \to P_{f(i),f(j)}$ taking $I \subseteq \{i, ..., j\}$ to $f(I) = \{f(k) | k \in I\}$.

The coherent nerve functor $N_\Delta\colon \mathsf{sCat} \to \mathsf{sSet}$ is given by taking $\mathcal{C} \in \mathsf{sCat}$ to

$$N_\Delta(\mathcal{C})_\bullet = \mathrm{Hom}_{\mathsf{sCat}}(\mathfrak{C}[\Delta^\bullet], \mathcal{C})$$

Example 2.13. Let $\mathcal{M}$ be a simplicial model category. Writing $\mathcal{M}_{cf}$ for the subcategory of fibrant-cofibrant objects, the underlying quasi-category of $\mathcal{M}$ is given by $N_\Delta(\mathcal{M}_{cf})$. In this paper we make use of the category of connective chain complexes over a ring k with the projective model structure, whose simplicial structure is obtained by passing its dg-structure through the Dold-Kan equivalence. See also [23, Proposition 1.3.1.17].

Remark 2.14. If $\mathcal{C}$ is a simplicial category, then another method of producing a quasi-category from $\mathcal{C}$ is to first take its levelwise nerve, producing a bisimplicial set, followed by taking its diagonal simplicial set. This is equivalent to taking the coherent nerve up to weak equivalence. To be precise, [2, Theorem 8.6] states that the levelwise nerve is a right Quillen equivalence from simplicial categories to Segal categories, and [17, Theorem 5.7] states that the diagonal is a left Quillen equivalence from Segal categories to quasi-categories.

Definition 2.15. Let $\mathcal{C}$ be a strict (2,1)-category. The simplicial category $\mathcal{C}_\Delta$ has the same objects as $\mathcal{C}$ and morphism spaces given by the nerve of the morphism category in $\mathcal{C}$: $\mathrm{Hom}_{\mathcal{C}_\Delta}(c,c') := N\mathrm{Hom}_{\mathcal{C}}(c,c')$.

Recollection 2.16. ([13, Section IV.3.2]) The truncated simplex category $\Delta_{\leq n}$ is the full subcategory of Δ on the objects $\{[i]\}_{i\leq n}$. The inclusion $i_n\colon \Delta_{\leq n} \hookrightarrow \Delta$ gives rise to a truncation functor $\mathrm{tr}_n\colon \mathsf{sSet} \to \mathsf{Set}^{\Delta^{op}_{\leq n}}$. This functor admits both left and right adjoints. The left adjoint is writen sk_n and called the n-skeleton. The right adjoint is written cosk_n and called the n-coskeleton. A simplicial set which is in the image of cosk_n is called n-coskeletal. We denote the compositions $\mathrm{cosk}_n \circ \mathrm{tr}_n =: \mathbf{cosk}_n$ and $\mathrm{sk}_n \circ \mathrm{tr}_n =: \mathbf{sk}_n$. These are idempotent endofunctors on sSet, i.e. there are isomorphisms $\mathbf{cosk}_n(\mathbf{cosk}_n X) \cong \mathbf{cosk}_n X$. An n-coskeletal simplicial set X can hence be thought of as a simplicial set for which any map of truncated simplicial sets $\mathrm{tr}_n S \to \mathrm{tr}_n X$ admits a unique extension to a simplicial map $S \to X$.

Since tr_n and cosk_n are both right adjoints, we have in particular isomorphisms

$$\mathrm{tr}_n(X \times Y) \cong \mathrm{tr}_n(X) \times \mathrm{tr}_n(Y)$$

for all $X, Y \in \mathsf{sSet}$ and

$$\mathrm{cosk}_n(X \times Y) \cong \mathrm{cosk}_n(X) \times \mathrm{cosk}_n(Y)$$

for all $X, Y \in \mathsf{Set}^{\Delta^{op}_{\leq n}}$, and so the following definition makes sense.

Definition 2.17.
- Let $\mathcal{C}$ be a simplicially enriched category. $\mathrm{tr}_n\mathcal{C}$ is the $\mathsf{Set}^{\Delta^{op}_{\leq n}}$-enriched category for which $\mathrm{Ob}\,\mathrm{tr}_n\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$ and
$$\mathrm{Hom}_{\mathrm{tr}_n\mathcal{C}}(c,c') = \mathrm{tr}_n\mathrm{Hom}_{\mathcal{C}}(c,c')$$
is the n-truncation of $\mathcal{C}(c,c')$. The composition of morphisms is given by
$$\mathrm{tr}_n(\mathrm{Hom}_{\mathcal{C}}(c',c'')) \times \mathrm{tr}_n(\mathrm{Hom}_{\mathcal{C}}(c,c')) \cong \mathrm{tr}_n(\mathrm{Hom}_{\mathcal{C}}(c',c'') \times \mathrm{Hom}_{\mathcal{C}}(c,c'))$$
$$\to \mathrm{tr}_n\mathrm{Hom}_{\mathcal{C}}(c,c'')$$
where the arrow is induced by the composition in $\mathcal{C}$.
- Let $\mathcal{D}$ be a $\mathsf{Set}^{\Delta^{op}_{\leq n}}$-enriched category. $\mathrm{cosk}_n\mathcal{D}$ is the simplicially enriched category for which $\mathrm{Ob}\,\mathrm{cosk}_n\mathcal{D} = \mathrm{Ob}\,\mathcal{D}$ and
$$\mathrm{Hom}_{\mathrm{cosk}_n\mathcal{D}}(d,d') = \mathrm{cosk}_n\mathrm{Hom}_{\mathcal{D}}(d,d')$$
is the n-coskeleton of $\mathcal{D}(d,d')$. The composition is defined as above.

Lemma 2.18. The functors tr_n and cosk_n form an adjunction between $\mathsf{Set}^{\Delta^{op}_{\leq n}}$-enriched categories and simplicially enriched categories.

Proof. Let $\mathcal{C}$ be a simplicially enriched category and let $\mathcal{D}$ be a category enriched in $\mathsf{Set}^{\Delta^{op}_{\leq n}}$. A functor $F\colon \mathrm{tr}_n\mathcal{C} \to \mathcal{D}$ is given by a function $f\colon \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and for each pair of objects $c, c' \in \mathrm{Ob}\,\mathcal{C}$ a morphism

$$F_{c,c'}\colon \mathrm{tr}_n\mathrm{Hom}_{\mathcal{C}}(c,c') \to \mathrm{Hom}_{\mathcal{D}}(fc, fc')$$

in $\mathsf{Set}^{\Delta^{op}_{\leq n}}$.

The adjunction isomorphism is given levelwise on morphism spaces. □

Recollection 2.19. Recall that nerves of categories are characterized as those quasi-categories which have unique inner horn filling in dimensions $n \geq 2$ ([22, Proposition 1.1.2.2]). In contrast, nerves of (2,1)-categories are characterized as those quasi-categories which have unique inner horn filling in dimensions $n \geq 3$ ([10, Theorem 8.6]). The following lemma is not new, but the author was unable to find a reference in the literature.

Lemma 2.20. Let X be a simplicial set which has unique inner horn filling in dimensions $k \geq n$. Then X is n-coskeletal.

Proof. By the adjunction, X is n-coskeletal if and only if the morphism

$$X_k = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^k, X) \xrightarrow{\mathrm{tr}_n} \mathrm{Hom}_{\mathsf{Set}^{\Delta^{op}_{\leq n}}}(\mathrm{tr}_n\Delta^k, \mathrm{tr}_n X) \cong \mathrm{Hom}_{\mathsf{sSet}}(\mathbf{sk}_n\Delta^k, X)$$

is an isomorphism for every $k > n$.

Consider a morphism $f\colon \mathbf{sk}_n\Delta^k \to X$. We proceed by induction, and prove that f extends uniquely to a morphism $f_{n+1}\colon \mathbf{sk}_{n+1}\Delta^k \to X$. Since $\mathbf{sk}_k\Delta^k \cong \Delta^k$, this will finish the proof. Now, since

$$\mathbf{sk}_{n+1}\Delta^k \cong \left(\coprod_{\mathrm{Hom}_{\mathsf{Set}^{\Delta^{op}_{\leq n}}}(\partial\Delta^{n+1}, \mathbf{sk}_n\Delta^k)} \Delta^{n+1} \right) / \sim$$

where $\sim$ glues the faces appropriately, it is sufficient to prove that any morphism $g\colon \partial\Delta^k \to X$ uniquely extends to a morphism $\bar{g}\colon \Delta^k \to X$ when $k > n$. To show this, let Λ^k_i denote the boundary of the k-simplex, minus the face opposite the i'th vertex. By unique inner horn filling in dimension $k-1$, there is an isomorphism

$$\mathrm{Hom}_{\mathsf{sSet}}(\partial\Delta^k, X) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^k_i, X)$$

for $0 < i < k$. To see this, let $h\colon \Delta^{k-1} \to X$ be the restriction of g to the i'th face of $\partial\Delta^k$. Then h is a filler for the the restriction $h|_{\Lambda^{k-1}_j}\colon \Lambda^{k-1}_j \to X$ for any $0 < j < k-1$, which by assumption is unique, so h can be reconstructed from $h|_{\partial\Delta^{k-1}}$ and so g can be reconstructed from $g|_{\Lambda^k_i}$. Now $g|_{\Lambda^k_i}$ admits a unique filler $\bar{g}\colon \Delta^k \to X$, and by uniqueness of inner horn fillers in dimension $k-1$, we have $\bar{g}|_{\partial\Delta^k} = g$. It follows that the map

$$X_k \xrightarrow{\mathrm{tr}_n} \mathrm{Hom}_{\mathsf{Set}^{\Delta^{op}_{\leq n}}}(\mathrm{tr}_n\Delta^k, \mathrm{tr}_n X)$$

is an isomorphism and that X is n-coskeletal as claimed. □

Lemma 2.21. Let $\mathcal{C}$ be a strict (2,1)-category. Then there is an isomorphism $N_D(\mathcal{C}) \cong N_\Delta(\mathcal{C}_\Delta)$.

Proof. We will construct a bijection between the set of normal 2-functors $[n] \to \mathcal{C}$ and the set of simplicially enriched functors $\mathfrak{C}[\Delta^n] \to \mathcal{C}_\Delta$. There is a forgetful map

$$U\colon \mathrm{Hom}_{\mathsf{sCat}}(\mathfrak{C}[\Delta^n], \mathcal{C}_\Delta) \to \mathrm{Hom}^N_{(2,1)\mathsf{Cat}}([n], \mathcal{C})$$

taking a simplicially enriched functor $f\colon \mathfrak{C}[\Delta^n] \to \mathcal{C}_\Delta$ to the normal 2-functor $Uf\colon [n] \to \mathcal{C}$ given as follows.

- The object i is taken to $Uf(i) = f(i)$.
- A morphism $i \to j$ is taken to $Uf(i \to j) = f(\{i, j\})$.
- Let $\phi_{i,j,k} = f\left(\{i, k\} \subset \{i, j, k\}\right) \in C(f(i), f(k)$. In other words, $\phi_{i,j,k}$ is a morphism in $\mathcal{C}(f(i), f(k))$ with source and target

$$p_{i,j,k}\colon f(\{i,k\}) \to f(\{j,k\}) * f(\{i,j\})$$

  The composition 2-morphisms of Uf are given by $(Uf)_{i,j,k} = \phi^{-1}_{i,j,k}$.
- Let $i, j, k, l \in [n]$. Since the pair of 2-morphisms

$$(Uf)_{i,j,l} \circ (\mathrm{id} * (Uf)_{j,k,l})\colon f(\{k,l\}) * f(\{j,k\}) * f(\{i,j\}) \to f(\{i,l\})$$

  and

$$(Uf)_{i,k,l} \circ ((Uf)_{i,j,k} * \mathrm{id})\colon f(\{k,l\}) * f(\{j,k\}) * f(\{i,j\}) \to f(\{i,l\})$$

  both are in the image of $\{i, l\} \subset \{i, j, k, l\}$, they are equal, such that Uf satisfies the conditions of Definition 2.8.

Conversely, given a normal 2-functor $F\colon [n] \to \mathcal{C}$, we construct a functor of categories enriched in $\mathsf{Set}^{\Delta^{op}_{\leq 2}}$

$$RF\colon \mathrm{tr}_2\mathfrak{C}[\Delta^n] \to \mathrm{tr}_2\mathcal{C}_\Delta$$

as follows.

- the object $i \in \{0, ..., n\}$ is taken to $RF(i) = F(i)$.
- On morphisms, the morphism of simplicial sets

$$\mathrm{tr}_2 NP_{0,n} \to \mathrm{tr}_2 N(\mathcal{C}(F(0), F(n))$$

  is given by sending
  $\{0, n\}$ to $F(0 \to n)$,
  $\{0, i, n\}$ to $F(i \to n) * F(0 \to i)$, and
  $\{0, i, j, n\}$ to $F(j \to n) * F(i \to j) * F(0 \to i)$.
  Inclusions of subsets $\{0, n\} \subset \{0, i, n\}$ is taken to $F^{-1}_{0,i,n}$ etc.

For a monoidal category V and V-enriched cateogries C and D, we write $\mathrm{Fun}_V(C, D)$ for the set of V-enriched functors from C to D.

By Recollection 2.19 and Lemma 2.20, nerves of categories are 2-coskeletal, the forgetful map of sets

$$\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}[\Delta^n], \mathcal{C}_\Delta) \xrightarrow{\mathrm{tr}_2} \mathrm{Fun}_{\mathsf{Set}^{\Delta^{op}_{\leq 2}}}(\mathrm{tr}_2\mathfrak{C}[\Delta^n], \mathrm{tr}_2\mathcal{C}_\Delta)$$

is a bijection by the adjunction in Lemma 2.18. The map U also factors through tr_2. It follows that U and $\mathrm{tr}_2^{-1}(R-)$ are mutually inverse maps of sets. Since all of the maps defined are also natural with respect to the cosimplicial structure maps of $\mathfrak{C}[\Delta^\bullet]$, we are done. □

Corollary 2.22. Let C be an ordinary category and let C_Δ be C considered as a simplicial category with discrete morphism spaces. Then $N(C) \cong N_\Delta(C_\Delta)$

2.3. Symmetric monoidal quasi-categories.

Notation 2.23. Write $\mathcal{F}in_*$ for the category of finite pointed sets and basepoint preserving set maps, and let $\langle n\rangle$ be the pointed set $\{*, 1, ..., n\}$.

We will now briefly recall some aspects of symmetric monoidal categories in order to motivate the definitions below, in particular Definition 2.27. A symmetric monoidal structure on a category C can be defined [28, Section 2] as a 2-functor from the category of finite pointed sets to the (2,1)-category of categories:

$$\mathcal{F}in_* \to \mathsf{Cat}$$
$$S \mapsto C^{\times |S|}$$

where $|S|$ denotes the number of non-basepoint elements of S. Using the Grothendieck construction, this is equivalently a so-called Grothendieck opfibration (in the sense of [14])

$$p\colon C^{\otimes} \to \mathcal{F}in_*$$

where $C^{\otimes}$ is the category defined as follows: an object of $C^{\otimes}$ consists of a pair $((c_1, ..., c_n), \langle n\rangle)$, where the c_i are objects of C. A morphism

$$((c_1, ..., c_n), \langle n\rangle) \to ((c_1', ..., c_m'), \langle m\rangle)$$

in $C^{\otimes}$ is a pair $((f_1, ..., f_m), \alpha)$ where $\alpha\colon \langle n\rangle \to \langle m\rangle$ is a map of pointed sets and $f_j\colon \bigotimes_{i\in\alpha^{-1}(j)} c_i \to c_j'$ are morphisms in C.

The structure of a Grothendieck opfibration $p\colon D \to \mathcal{F}in_*$ can be summarized as a functor such that for each morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$ in $\mathcal{F}in_*$, and each object $c \in \operatorname{Ob} D$ such that $p(c) = \langle n\rangle$, there exists a morphism $f\colon c \to c'$ in D such that $p(f) = \alpha$ and such that the induced functor of slice categories

$$D_{f/} \to D_{c'/} \times_{\mathcal{F}in_{*\,p(c')/}} \mathcal{F}in_{*\,p(f)/}$$

is an equivalence of categories. To spell out what this equivalence means, for every morphism of finite pointed sets $\beta\colon \langle m\rangle \to \langle l\rangle$ and every morphism $h\colon c \to c''$ in $C^{\otimes}$ such that $p(h) = \beta\circ\alpha$, there exists a unique morphism $g\colon c' \to c''$ in $C^{\otimes}$ such that $p(g) = \beta$ and $g \circ f = h$. This situation is depicted in the below diagram.

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[dd, dash] \arrow[drr, bend right, "\forall h"'] & c' \arrow[dr, dashed, "\exists! g"] \arrow[dd, dash] & \\
& & c'' \arrow[dd, dash] \\
\langle n\rangle \arrow[r, "\alpha"] \arrow[drr, bend right, "\gamma"'] & \langle m\rangle \arrow[dr, "\forall\beta"] & \\
& & \langle l\rangle
\end{tikzcd}$$

Such a morphism f is called a *coCartesian morphism.* The existence of coCartesian morphisms encodes the fact that the fibers of a Grothendieck opfibration $p\colon D \to \mathcal{F}in_*$ are functorial in the base category.

The remarkable fact is that given such a Grothendieck opfibration such that

$$p^{-1}(\langle n\rangle) \simeq p^{-1}(\{*,1\})^{\times n},$$

the category $p^{-1}(\langle 1\rangle)$ is a symmetric monoidal category, and so this is an equally good definition of symmetric monoidal categories. It turns out that this definition, using a version of Grothendieck opfibrations adapted to quasi-categories, is a robust generalization of symmetric monoidal categories to the quasi-categorical setting.

The join [22, Definition 1.2.8.1] of two simplicial sets S and T is denoted $S \star T$ and has n-simplices given by

$$(S \star T)_n = S_n \sqcup T_n \sqcup \coprod_{i+j=n-1} S_i \times T_j$$

where the face maps $d_i\colon (S\star T)_n \to (S\star T)_{n-1}$ are defined using the i'th face map on S_n and T_n. To define $d_i\colon S_j \times T_k \to (S\star T)_{n-1}$, let $\sigma \in S_j$ and $\tau \in T_k$. Then $d_i(\sigma,\tau)$ is given by

$$\begin{array}{ll} d_i(\sigma,\tau) = (d_i\sigma,\tau) \in S_{j-1}\times T_k & \text{, if } i \le j, j \neq 0, \\ d_i(\sigma,\tau) = (\sigma, d_{i-j-1}\tau) \in S_j \times T_{k-1} & \text{, if } i > j, k \neq 0, \\ d_0(\sigma,\tau) = \tau \in T_{n-1} & \text{, if } j = 0, \\ d_n(\sigma,\tau) = \sigma \in S_{n-1} & \text{, if } k = 0. \end{array}$$

If X is a simplicial set and $f\colon K \to X$ is a simplicial subset, we denote the *slice* of X under f to be the simplicial set written $X_{f/}$ with n-simplices the subset

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X_{f/}) \subset \mathrm{Hom}_{\mathsf{sSet}}(K\star\Delta^n, X)$$

consisting of those maps $g\colon K\star\Delta^n \to X$ such that $g|_K = f$. The simplicial structure comes from the cosimplicial structure of $\{\Delta^\bullet\}$. In particular, if $\mathcal{C}$ is a quasi-category and $f\colon c_0 \to c_1$ is an edge in $\mathcal{C}$, then the slice $\mathcal{C}_{f/}$ is defined as the slice over a morphism $\Delta^1 \to \mathcal{C}$ hitting f. Similarly, the slice over an object $\mathcal{C}_{/c_0}$ is defined using a morphism $\Delta^0 \to \mathcal{C}$ hitting c_0.

Definition 2.24. Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories.

(1) [22, Definition 2.4.1.1] An edge $f\colon c_0 \to c_1$ of $\mathcal{C}$ is p-coCartesian if the canonical map

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_0/} \times_{\mathcal{D}_{p(c_0)/}} \mathcal{D}_{p(f)/}$$

is an acyclic Kan fibration.

(2) [22, Definition 2.4.2.1] p is called a coCartesian fibration if for each edge $g\colon d_0 \to d_1$ of $\mathcal{D}$ and every object c_0 in $\mathcal{C}$ such that $p(c_0) = d_1$, there exists a p-coCartesian edge $f\colon c_0 \to c_1$ of $\mathcal{C}$ such that $p(f) = g$.

Observation 2.25.

- The structure of a coCartesian fibration implies that the fibers of p depend covariantly on the base. This is encapsulated in the straightening-unstraightening theorem [22, Theorem 3.2.0.1].
- Given a coCartesian fibration $p\colon \mathcal{C} \to \mathcal{D}$, an object $c_0 \in \mathrm{Ob}\,\mathcal{C}$ and an edge $g\colon p(c_0) \to p(c_1)$ in $\mathcal{D}$, the space of coCartesian lifts of g with fixed starting point c_0 is contractible. Namely, in the language of [22, Section 3.1], $p\colon \mathcal{C}^\natural \to \mathcal{D}$ is a fibrant marked simplicial set over $\mathcal{D}$ and $\{0\} \to (\Delta^1)^\sharp$ is a marked anodyne morphism. Let $(\Delta^1)^\sharp \to \mathcal{D}$ be a marked simplicial set over $\mathcal{D}$ picking out g. Using using [22, Remark 3.1.3.4], the restriction map

$$r\colon \mathrm{Hom}^\flat_{\mathcal{D}}((\Delta^1)^\sharp, \mathcal{C}^\natural) \to \mathrm{Hom}^\flat_{\mathcal{D}}(\{0\}, \mathcal{C}^\natural) \simeq C_{p(c_0)}$$

where $C_{p(c_0)}$ is the fiber over $p(c)$ in $\mathcal{C}$, is a trivial fibration. In particular, each fiber of r is contractible. In this map, the source simplicial set is the space of all coCartesian morphisms in $\mathcal{C}$.

- An edge $f\colon c_0 \to c_1$ of $\mathcal{C}$ is equivalently p-coCartesian if the diagram

$$\begin{tikzcd} \mathcal{C}(c_1, c_2) \arrow[r, "f^*"] \arrow[d, "p_*"'] & \mathcal{C}(c_0, c_2) \arrow[d, "p_*"] \\ \mathcal{D}(p(c_1), p(c_2)) \arrow[r, "f^*"'] & \mathcal{D}(p(c_0), p(c_2)) \end{tikzcd}$$

is a homotopy pullback square for all objects c_2 of $\mathcal{C}$ [22, Proposition 2.4.4.3]. See [22, Section 2.4] or [15, Section 4] for more details on coCartesian fibrations.

Definition 2.26. [28, Definition 2.1] Let $p\colon \mathcal{C} \to N\mathcal{F}in_*$ be a coCartesian fibration. We denote the fiber over $\langle n\rangle$ by $C_{\langle n\rangle}$.

For $1 \le i \le n$, let $r_{n,i}$ be the map of pointed sets $r_{n,i}\colon \langle n\rangle \to \langle 1\rangle$ such that $r_{n,i}^{-1}(1) = \{i\}$. Choosing coCartesian lifts of $r_{n,i}$ gives a functor $\rho_{n,i}\colon \mathcal{C}_{\langle n\rangle} \to \mathcal{C}_{\langle 1\rangle}$.

The *Segal map* $\rho_n\colon \mathcal{C}_{\langle n\rangle} \to (\mathcal{C}_{\langle 1\rangle})^{\times n}$ is given by the product $\prod_i \rho_{n,i}$.

Definition 2.27. [23, Definition 2.0.0.7] Let $\mathcal{C}$ be a quasi-category. A symmetric monoidal structure on $\mathcal{C}$ is a coCartesian fibration $p\colon \mathcal{C}^\otimes \to N\mathcal{F}in_*$ such that the fiber $\mathcal{C}_{\langle 1\rangle}$ is equivalent to $\mathcal{C}$ and the Segal maps $\rho_n\colon \mathcal{C}^\otimes_{\langle n\rangle} \to (\mathcal{C}^\otimes_{\langle 1\rangle})^{\times n}$ are equivalences for each n. A functor of quasi-categories over $N\mathcal{F}in_*$

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[r, "f"] \arrow[dr, "p"'] & \mathcal{D}^\otimes \arrow[d, "q"] \\ & N\mathcal{F}in_* \end{tikzcd}$$

is symmetric monoidal if it takes p-coCartesian morphisms to q-coCartesian morphisms.

Observation 2.28. By straightening-unstraightening [22, Theorem 3.2.0.1], a symmetric monoidal category is equivalently a functor of quasi-categories $F\colon N\mathcal{F}in_* \to \mathsf{Cat}_\infty$ satisfying the Segal condition (i.e., the canonical maps $F(\langle n\rangle) \to (F(\langle 1\rangle)^{\times n}$ are equivalences, and a symmetric monoidal functor is a natural transformation.

Using the equivalences $F(\langle n\rangle) \to (F(\langle 1\rangle)^{\times n}$, every object of $\mathcal{C}^\otimes$ corresponds to a tuple $(C_1, ..., C_n, \langle n\rangle)$ [23, Remark 2.1.1.5].

The existence of coCartesian morphisms allow us to define the tensor product of the objects C_i by choosing a coCartesian lift of the map $\alpha\colon \langle n\rangle \to \langle 1\rangle$ where $\alpha^{-1}(*) = \{*\}$:

$$(C_1, ..., C_n, \langle n\rangle) \to (D, \langle 1\rangle)$$

and defining $\bigotimes_{1\le i\le n} C_i := D$. This determines the tensor product up to a contractible space of choices.

The following description of morphisms in $\mathcal{C}^\otimes$ is not new, but the author could not find a detailed treatment in the literature and so we include a proof for completeness.

Lemma 2.29. Let $p\colon \mathcal{C}^{\otimes} \to N\mathcal{F}in_*$ be a symmetric monoidal quasi-category and let $\mathcal{C}$ be the fiber at $\langle 1\rangle$. Let C and D be a pair of objects of $\mathcal{C}^{\otimes}$ corresponding under the Segal condition to tuples $(C_1, ..., C_n, \langle n\rangle)$ and $(D_1, ..., D_m, \langle m\rangle)$ respectively. Let $\alpha\colon \langle n\rangle \to \langle m\rangle$ be a map of pointed sets and write $\mathrm{Hom}_\alpha(C, D) \subseteq \mathrm{Hom}_{\mathcal{C}^{\otimes}}(C, D)$ for the space of morphisms lying over α. Then there is a homotopy equivalence

$$\mathrm{Hom}_\alpha(C, D) \simeq \prod_{1\le j\le m} \mathrm{Hom}_{\mathcal{C}}\left(\otimes_{i\in\alpha^{-1}(j)} C_i, D_j\right).$$

Proof. Choosing for $1 \le j \le m$ coCartesian lifts $r_{m,j,!}\colon D \to D_j$, by [23, Definition 2.1.1.10], these morphisms induce a homotopy equivalence

$$\mathrm{Hom}_\alpha(C, D) \simeq \prod_{1\le j\le m} \mathrm{Hom}_{\alpha_j}(C, D_j)$$

where $\alpha_j = r_{m,j} \circ \alpha$. A coCartesian lift of α_j takes $(C_1, ..., C_n, \langle n\rangle)$ to $\alpha_{j,!}C := \otimes_{i\in\alpha^{-1}(j)} C_i$, and by definition of coCartesian morphisms, we have an equivalence

$$\mathcal{C}^{\otimes}_{\alpha_{j,!}C/} \xrightarrow{\sim} \mathcal{C}^{\otimes}_{f/} \xrightarrow{\sim} \mathcal{C}_{C/} \times_{N\mathcal{F}in_{*,\langle n\rangle/}} N\mathcal{F}in_{*,\alpha_j/}.$$

Considering the maximal Kan-subcomplex containing $\{D_j\} \times \{\mathrm{id}\}$, we get a homotopy equivalence of spaces

$$\mathrm{Hom}_{\mathcal{C}}(\otimes_{i\in\alpha^{-1}(j)} C_i, D_j) \simeq \mathrm{Hom}_{\alpha_j}(C, D_j).$$

□

Notation 2.30. Let $p\colon \mathcal{C}^{\otimes} \to N\mathcal{F}in_*$ be a symmetric monoidal quasi-category. To simplify the notation, we may write an object $(C_1, ..., C_n, \langle n\rangle)$ of $C^{\otimes}$ simply as $(C_1, ..., C_n)$.

Examples 2.31.

- Let C be a symmetric monoidal category. Then applying the nerve to the Grothendieck construction $C^{\otimes} \to \mathcal{F}in_*$ exhibits NC as a symmetric monoidal quasi-category.
- [20, Proposition 4.3.13] Let $\mathcal{M}$ be a simplicial symmetric monoidal model category and let $\mathcal{M}^{\otimes}_{cf}$ denote the full subcategory of $\mathcal{M}^{\otimes}$ on the objects $(C_1, ..., C_n)$ such that each C_i is fibrant-cofibrant. Then $\mathcal{M}^{\otimes}_{cf}$ is a simplicial category, and applying N_Δ to the Grothendieck opfibration $\mathcal{M}^{\otimes}_{cf} \to \mathcal{F}in_*$ gives a symmetric monoidal structure for $N_\Delta\mathcal{M}_{cf}$ where the coCartesian lifts of a morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$ are the morphisms $(C_1, ..., C_n) \to (D_1, ..., D_m)$ in $\mathcal{M}^{\otimes}_{cf}$ such that each $\otimes_{i\in\alpha^{-1}(j)} C_i \to D_j$ is a homotopy equivalence.

Assumption 2.32. The quasi-category $N_\Delta(\mathsf{Ch}_k)$ carries a symmetric monoidal structure given by the derived tensor product of chain complexes.

The results [23, Remark 7.1.2.12 and Theorem 7.1.2.13] prove a similar result for the hammock localization of the subcategory Ch_k^c of cofibrant objects.

There is a notion of symmetric monoidal (2,1)-category, but the definition is technical. For the conventional definition, see [16, Section 1.1]. For our purposes, we use the following proxy definition. The compatibility of this definition with the conventional one is Lemma 2.34.

Definition 2.33. A (2,1)-category is called *symmetric monoidal* if its Duskin nerve $N_D(\mathcal{C})$ is a symmetric monoidal quasi-category. A symmetric monoidal functor of (2,1)-categories $\mathcal{C} \to \mathcal{D}$ is given by a symmetric monoidal functor of quasi-categories $N_D(\mathcal{C}) \to N_D(\mathcal{D})$.

The following lemma concerns the compatibility of our proxy definition of symmetric monoidal (2,1)-categories (Definition 2.33) and the conventional definition given in [16]. In order to keep the length of this paper reasonable, we omit the definitions of (3,1)-category and Segal nerve used in the proof, which can be found in [5].

Lemma 2.34. Let $\mathcal{C}$ be a symmetric monoidal (2,1)-category in the sense of [16, Section 1.1]. Then $N_D(\mathcal{C})$ is a symmetric monoidal quasi-category. If $f\colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor between (2,1)-categories, then $N_D(f)$ extends to a symmetric monoidal functor $N_D(\mathcal{C})^{\otimes} \to N_D(\mathcal{D})^{\otimes}$.

Proof. Denote by BiCat the (3,1)-category of (2,1)-categories. By [16, Theorem 2.3], there is a functorial assignment of a pseudo-functor $f\colon \mathcal{F}in_* \to \mathsf{BiCat}$ such that $f(\langle n\rangle) \simeq \mathcal{C}^{\times n}$. Passing to nerves, we get a functor of quasi-categories $N\mathcal{F}in_* \to \Delta(\mathsf{BiCat}) \to \mathsf{Cat}_\infty$, where $\Delta(\mathsf{BiCat})$ denotes the Segal nerve of the (3,1)-category of (2,1)-categories, see [5]. □

Remark 2.35. As stated, the definition of symmetric monoidal (2,1)-category in [16, Section 1.1] is potentially stronger than that of Definition 2.33. Henceforth a "symmetric monoidal (2,1)-category" will mean in the sense of Definition 2.33, unless stated otherwise.

Observation 2.36. By [3, Theorem 3.3.12], a (2,1)-category $\mathcal{C}$ which is symmetric monoidal in the sense of [16] corresponds to a "fibered (2,1)-category" $p\colon \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ such that for each fiber we have $\mathcal{C}_{\langle n\rangle} \simeq \mathcal{C}^{\times n}$. In that case, like in the case for ordinary categories, there is an isomorphism $N_D(\mathcal{C})^{\otimes} \cong N_D(\mathcal{C}^{\otimes})$.

Lemma 2.37. Let $\mathcal{C}$ be a symmetric monoidal (2,1)-category, and let $p\colon N_D(\mathcal{C})^{\otimes} \to N\mathcal{F}in_*$ be the associated coCartesian fibration. Let $\alpha\colon \langle n\rangle \to \langle m\rangle$ be a map of pointed finite sets. Then the coCartesian lifts of α are those morphisms $f\colon (A_1, ..., A_n) \to (B_1, ..., B_m)$ in $N_D(\mathcal{C})^{\otimes}$ such that each of the morphisms $\otimes_{i\in\alpha^{-1}(j)} A_i \to B_j$ is an equivalence in $\mathcal{C}$ (see Definition 2.5).

Proof. Recall from Observation 2.25 that the space of coCartesian lifts of $\alpha\colon \langle n\rangle \to \langle m\rangle$ starting at $(A_1, ..., A_n)$ is contractible. Since the property of being an equivalence on components is preserved by homotopies, to prove the result it is sufficient to show that morphisms of the stated type are coCartesian lifts of α.

Let f be a morphism in $N_D(\mathcal{C})^{\otimes}$ as in the statement. We have to show that the diagram

$$\begin{tikzcd}
N_D(\mathcal{C})^{\otimes}((B_1, ..., B_m), (C_1, ..., C_l)) \arrow[r, "f^*"] \arrow[d, "p_*"] & N_D(\mathcal{C})^{\otimes}((A_1, ..., A_n), (C_1, ..., C_l)) \arrow[d, "p_*"] \\
N\mathcal{F}in_*(\langle m\rangle, \langle l\rangle) \arrow[r, "\alpha^*"] & N\mathcal{F}in_*(\langle n\rangle, \langle l\rangle)
\end{tikzcd}$$

is a homotopy pullback square for each object $(C_1, ..., C_l)$ in $N_D(\mathcal{C})^{\otimes}$. In other words, for each $\beta \in \mathcal{F}in_*(\langle m\rangle, \langle l\rangle)$, we need to show that

$$N_D(\mathcal{C})^{\otimes}((B_1, ..., B_m), (C_1, ..., C_l)) \times_{N\mathcal{F}in_*(\langle n\rangle,\langle l\rangle)} \{\beta\} \to N_D(\mathcal{C})^{\otimes}((A_1, ..., A_n), (C_1, ..., C_l))$$

is a homotopy equivalence onto the component of $N_D(\mathcal{C})^{\otimes}((A_1, ..., A_n), (C_1, ..., C_l))$ lying over $\beta \circ \alpha$ in $\mathcal{F}in_*(\langle n\rangle, \langle l\rangle)$. This means that for each $k \in \langle l\rangle \setminus *$, the induced map

$$f^*\colon N_D(\mathcal{C})\left(\otimes_{j\in\beta^{-1}(k)} B_j, E_k\right) \to N_D(\mathcal{C})\left(\otimes_{i\in(\beta\circ\alpha)^{-1}(k)} A_i, E_k\right)$$

is a homotopy equivalence, which is equivalent to saying that

$$f^*\colon \mathcal{C}\left(\otimes_{j\in\beta^{-1}(k)}B_j, E_k\right) \to \mathcal{C}\left(\otimes_{i\in(\beta\circ\alpha)^{-1}(k)}A_i, E_k\right)$$

is an equivalence of categories, which holds by assumption. □

Definition 2.38. • [22, Definition 3.1.0.1] Let sSet^+ be the category of marked simlicial sets. The objects of sSet^+ are pairs $(X, \mathcal{E})$, where $sX_0 \subseteq \mathcal{E} \subseteq X_1$ is a subset of X_1 containing the degenerate edges. The morphisms $(X, \mathcal{E}) \to (Y, \mathcal{E}')$ in sSet^+ are morphisms of simplicial sets $f\colon X \to Y$ such that $f(\mathcal{E}) \subseteq \mathcal{E}'$.

• Given a simplicial set X, define the marked simplicial sets

$$X^\flat = (X, sX_0) \quad \text{and} \quad X^\sharp = (X, X_1)$$

with markings given by the degenerate and all edges respectively.

• [22, Notation 3.1.0.2] Let $\mathsf{sSet}^+_{/S}$ be the over-category of sSet^+ with respect to $S^\sharp$.

• [22, Definition 3.1.1.8] Given a morphism of simplicial sets $p\colon X \to S$, let $X^\natural = (X, \mathcal{E}, p) \in \mathsf{sSet}^+_{/S}$ be the marked simplicial set over S where $\mathcal{E}$ is the set of p-coCartesian edges in X.

Definition 2.39. [22, Section 3.1.3] If $X \to S$ and $Y \to S$ are simplicial sets over S, define the mapping space $\mathrm{Hom}^\sharp(X, Y) \subseteq \mathrm{Hom}_{\mathsf{sSet}^+/S}(X, Y)$ to be the subspace of the simplicial mapping space defined by

$$\mathrm{Hom}_{\mathsf{sSet}}(K, \mathrm{Hom}^\sharp(X, Y)) \cong \mathrm{Hom}_{\mathsf{sSet}^+_{/S}}(K^\sharp \times X, Y).$$

Lemma 2.40. [22, Proposition 3.1.3.7] There is a simplicial model structure on $\mathsf{sSet}^+_{/S}$, called the *coCartesian model structure*, for which the equivalences are the Joyal equivalences and cofibrations are monomorphisms. In this model structure, the fibrant objects are the marked simplicial sets $X^\natural$ where $p\colon X \to S$ is a coCartesian fibration.

Lemma 2.41. [22, Lemma 3.1.3.6] Consider the morphism F in $\mathsf{Set}^+_{/S}$

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & S \end{tikzcd}$$

such that F is an inclusion of simplicial sets and let $C \to S$ be a coCartesian fibration. Then the induced map $F^*\colon \mathrm{Hom}^\sharp(Y, C^\natural) \to \mathrm{Hom}^\sharp(X, C^\natural)$ is a Kan fibration.

Definition 2.42. Let $p\colon \mathcal{C}^\otimes \to N\mathcal{F}in_*$ be a symmetric monoidal quasi-category. We define its object monoid $\mathrm{Ob}\,\mathcal{C}^\otimes$ by fixing one coCartesian lift $f_!$ for each pair

$$(\langle n\rangle \xrightarrow{f} \langle m\rangle \in N\mathcal{F}in_*, (C_1, ..., C_n) \in \mathrm{Ob}\,\mathcal{C}^{\times n})$$

such that $\mathrm{id}_! = \mathrm{id}$, and taking the full subcategory of $\mathcal{C}^\otimes$ on these morphisms. We obtain an inclusion $i\colon \mathrm{Ob}\,\mathcal{C}^\otimes \to \mathcal{C}^\otimes$ over $N\mathcal{F}in_*$.

Lemma 2.43. Let $p\colon \mathcal{C}^\otimes \to N\mathcal{F}in_*$ and $q\colon \mathcal{D}^\otimes \to N\mathcal{F}in_*$ be symmetric monoidal quasi-categories and let $i\colon \mathrm{Ob}\,\mathcal{C}^\otimes \to \mathcal{C}^\otimes$ be the inclusion of the monoid of objects. If $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ and $G\colon \mathrm{Ob}\,\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ are symmetric monoidal functors and $\gamma\colon G \to F|_{\mathrm{Ob}\,\mathcal{C}^\otimes}$ is a natural equivalence, then F can be pulled back along γ to produce a new symmetric monoidal functor $F' \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ which equals G on objects and on morphisms sends $f \colon x \to y$ to $F'(f) \sim \gamma_y^{-1} \circ F(f) \circ \gamma_x$.

Proof. Let I be the groupoid with objects $0, 1$ and an isomorphism $0 \to 1$. The Kan complex NI classifies homotopy equivalences in quasi-categories [9, Proposition 2.2]. We apply Lemma 2.41 to the inclusion $i \colon \mathrm{Ob}\,\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$ and the symmetric monoidal quasi-category $\mathcal{D}^{\otimes}$, then the induced map of simplicial sets $\mathrm{Hom}^{\sharp}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \to \mathrm{Hom}^{\sharp}(\mathrm{Ob}\,\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ is a Kan fibration by Lemma 2.41. As such, we can find a lift in the square

$$\begin{tikzcd}
\Delta^0 \arrow[r, "f"] \arrow[d, "i_0"'] & \mathrm{Hom}^{\sharp}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d, "i^*"] \\
NI \arrow[r, "h"'] \arrow[ur, "\widetilde{h}"] & \mathrm{Hom}^{\sharp}(\mathrm{Ob}\,\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})
\end{tikzcd}$$

The image of 1 under $\widetilde{h}$ then has the desired property. It follows that $\widetilde{h}(1)$ also preserves coCartesian edges and so is a symmetric monoidal functor. $\square$

Replacing $N\mathcal{F}in_*$ with the point Δ^0 as the base simplicial set and using the same argument results in the following result:

Lemma 2.44. Let $\mathcal{C}$ and $\mathcal{D}$ be quasi-cateogries and let $i \colon \mathrm{Ob}\,\mathcal{C} \to \mathcal{C}$ be the inclusion of the set of objects. If $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathrm{Ob}\,\mathcal{C} \to \mathcal{D}$ are functors and $\gamma \colon G \to F|_{\mathrm{Ob}\,\mathcal{C}}$ is a natural equivalence, then F can be pulled back along γ to produce a new functor $F' \colon \mathcal{C} \to \mathcal{D}$ which equals G on objects and on morphisms sends $f \colon x \to y$ to $F'(f) \sim \gamma_y^{-1} \circ F(f) \circ \gamma_x$.

3. The Morita category

Throughout, fix a commutative ring k. We denote by dgAlg_k the (large) category of dg-k-algebras and algebra homomorphisms.

In this section we introduce the (large) Morita (2,1)-category, which is a 2-categorical version of [18, Section 2.4] and study its properties and relation to the category of algebras over k. In the following, by Morita functoriality we will mean a functor out of the Morita (2,1)-category.

Definition 3.1. Fix a commutative ground ring k. We define the *Morita category* Mor_k as the following (2,1)-category.

- The objects are dg-k-algebras.
- For a pair of dg-algebras A, B, $\mathsf{Mor}_k(A, B)$ is the groupoid of (B, A)-bimodules and (B, A)-bimodule isomorphisms.
- For a triple $A, B, C \in \mathrm{Ob}\,\mathsf{Mor}_k$, the composition functor
$$\mathsf{Mor}_k(B, C) \times \mathsf{Mor}_k(A, B) \to \mathsf{Mor}_k(A, C)$$
is given by tensor product over B: $(P, Q) \mapsto P \otimes_B Q$.
- The unit of a dg-algebra A in Mor_k is given by A considered as an (A, A)-bimodule.
- The associators and unitors are given by the associators and unitors for the tensor product, that is
$$\alpha_{A,B,C,D}(P, Q, R) \colon P \otimes_C (Q \otimes_B R) \xrightarrow{\sim} (P \otimes_C Q) \otimes_B R$$

$$\lambda_{A,B}(P)\colon B \otimes_B P \xrightarrow{\sim} P$$

Lemma 3.2. The following defines a 2-functor $\mathsf{m}\colon \mathsf{dgAlg}_k \to \mathsf{Mor}_k$.

- m is the identity on objects.
- A homomorphism of dg-algebras $f\colon A \to B$ is sent to the (B, A)-bimodule $B_f = B$, with left structure map the multiplication $m\colon B \otimes B \to B$ and right structure map $m \circ (\mathrm{id} \otimes f)\colon B \otimes A \to B$.
- The 2-morphism $\mathsf{m}_{1_a}\colon 1_{\mathsf{m}(A)} \to \mathsf{m}_{A,A}(\mathrm{id}_A)$ is given by the identity morphism $A \to A$ in the category of (A, A)-bimodules.
- For dg-algebra homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, the 2-morphism
$$\mathsf{m}_{A,B,C}\colon C_g \otimes_B B_f \to C_{g\circ f}$$
is given by the natural isomorphism $c \otimes b \mapsto c \cdot g(b)$, where $\cdot$ is the product in C.

Proof. We need to verify the commutativity of the diagrams in Definition 2.8. The vertical arrows in the unitor diagrams are identity morphisms, so it is sufficient to observe that for an algebra homomorphism $f\colon A \to B$, the structure maps

$$\mathsf{m}_{A,B,B}(\mathrm{id}_B, f)\colon B_{\mathrm{id}_B} \otimes_B B_f \to B_f$$

and

$$\mathsf{m}_{A,A,B}(f, \mathrm{id}_A)\colon B_f \otimes_A A_{\mathrm{id}_A} \to B_f$$

are exactly the unitor maps in Mor_k. For the associator diagram, we need to show that for dg-algebra homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$$

the following diagram commutes:

$$\begin{tikzcd}
(D_h \otimes_C C_g) \otimes_B B_f \arrow[r, "\alpha_{A,B,C,D}"] \arrow[d, "\mathsf{m}_{B,C,D}(h,g) \otimes_B B_f"'] & D_h \otimes_C (C_g \otimes_B B_f) \arrow[d, "D_h \otimes_C \mathsf{m}_{A,B,C}(g,f)"] \\
D_{h\circ g} \otimes_B B_f \arrow[d, "\mathsf{m}_{A,B,D}(h\circ g, f)"'] & D_h \otimes_C C_{g\circ f} \arrow[d, "\mathsf{m}_{A,C,D}(h, g\circ f)"] \\
D_{h\circ g\circ f} \arrow[r, "\mathrm{id}"'] & D_{h\circ g\circ f}
\end{tikzcd}$$

this follows from a calculation on elementary tensors. Namely, for $d \in D_h$, $c \in C_g$ and $b \in B_f$, both sides of the diagram yield the composition

$$d \otimes c \otimes b \mapsto d \cdot h(c) \cdot (g \circ h)(b) = d \cdot h(c \cdot g(b))$$

□

Definition 3.3. The 2-functor $\mathsf{m}\colon \mathsf{dgAlg}_k \to \mathsf{Mor}_k$ constructed in Lemma 3.2 is called the *modulation 2-functor*.

Remark 3.4. Note that modulation is not a strict 2-functor as for homomorphisms of k-algebras $A \xrightarrow{f} B \xrightarrow{g} C$, the (C, A)-bimodule $C_g \otimes_B B_f$ is only isomorphic and not equal to $C_{g\circ f}$.

The reason we restrict the 2-morphisms in Mor_k to be isomorphisms is because Lemma 5.11 does not hold for arbitrary bimodule homomorphisms.

Lemma 3.5. ([29, Example 2.10]) Mor_k is a symmetric monoidal (2,1)-category, in the sense of [16], under tensor product over k.

Lemma 3.6. The modulation functor $\mathsf{m}\colon \mathsf{dgAlg}_k \to \mathsf{Mor}_k$ induces a symmetric monoidal functor of quasi-categories

$$\begin{tikzcd} N(\mathsf{dgAlg}_k)^{\otimes} \arrow[r, "\mathsf{m}^{\otimes}"] \arrow[dr, "p"'] & N_D(\mathsf{Mor}_k)^{\otimes} \arrow[d, "q"] \\ & N\mathcal{F}in_* \end{tikzcd}$$

where $\mathsf{m}^{\otimes}$ is defined as follows: it is given by the identity on objects, and for a morphism $f\colon (A_1, ..., A_n) \to (B_1, ..., B_m)$ lying over a map $\alpha\colon \langle n\rangle \to \langle m\rangle$, the image

$$\mathsf{m}^{\otimes}(f)\colon (A_1, ..., A_n) \to (B_1, ..., B_m)$$

is the morphism given on components by $\mathsf{m}(f)_j = \mathsf{m}(f_j) = (B_j)_{f_j}$.

Proof. We first show that $\mathsf{m}^{\otimes}$ is indeed a functor. We will construct a map $\mathrm{Fun}([2], \mathsf{dgAlg}_k^{\otimes}) \to \mathrm{Hom}^N_{(2,1)\mathsf{Cat}}([2], \mathsf{Mor}_k^{\otimes})$. From the construction it will be clear that the general case follows similarly.

Given a finite set I and algebra homomorphisms $g_i\colon A^i \to B^i$ for $i \in I$, the $(\otimes_{i\in I} B^i, \otimes_{i\in I} A^i)$-bimodules $\otimes_{i\in I} B^i_{g_i}$ and $(\otimes_{i\in I} B^i)_{\otimes_{i\in I} g_i}$ are in fact equal.

An element of $\mathrm{Fun}([2], \mathsf{dgAlg}_k^{\otimes})$ is given by a pair of maps of finite pointed sets

$$\langle n\rangle \xrightarrow{\alpha} \langle m\rangle \xrightarrow{\beta} \langle r\rangle$$

and morphisms

$$(A_1, ..., A_n) \xrightarrow{f} (B_1, ..., B_m) \xrightarrow{g} (C_1, ..., C_r)$$

lying over these. The composition $\mathsf{m}^{\otimes}(g)\mathsf{m}^{\otimes}(f)$, given componentwise by

$$(\mathsf{m}^{\otimes}(g)\mathsf{m}^{\otimes}(f))_k\colon \bigotimes_{j\in\beta^{-1}(k)} \bigotimes_{i\in\alpha^{-1}(j)} A_i \xrightarrow{\bigotimes_{j\in\beta^{-1}(k)} \mathsf{m}(f_j)} \bigotimes_{j\in\beta^{-1}(k)} B_j \xrightarrow{\mathsf{m}(g_k)} C_k$$

as a $(C_k, \otimes_i A_i)$-bimodule, this is given by

$$(C_k)_{g_k} \otimes_{\otimes_{j\in\beta^{-1}(k)} B_j} \left(\otimes_{j\in\beta^{-1}(k)} (B_j)_{f_j}\right) = (C_k)_{g_k} \otimes_{\otimes_{j\in\beta^{-1}(k)} B_j} \left(\otimes_{j\in\beta^{-1}(k)} B_j\right)_{\otimes_{j\in\beta^{-1}(k)} f_j}$$

such that the composition 2-morphism $\mathsf{m}^{\otimes}(g)\mathsf{m}^{\otimes}(f) \to \mathsf{m}^{\otimes}(g\circ f)$ is given componentwise by the canonical isomorphism $\mathsf{m}_{\otimes_i A, \otimes_j B, C}$.

In summary, the normal 2-functor $[2] \to \mathsf{Mor}_k^{\otimes}$ obtained from (f, g) takes $(0 \to 1)$ to $\mathsf{m}^{\otimes}(f)$, $(1 \to 2)$ to $\mathsf{m}^{\otimes}(g)$ and $(0 \to 2)$ to $\mathsf{m}^{\otimes}(g\circ f)$, with structure map $\mathsf{m}^{\otimes}(g)\mathsf{m}^{\otimes}(f) \to \mathsf{m}^{\otimes}(g\circ f)$ given componentwise by $\mathsf{m}_{\otimes_i A, \otimes_j B, C}$. The functoriality of $\mathsf{m}^{\otimes}$ therefore reduces to the functoriality of m, and the general case follows similarly.

Let $\alpha\colon \langle n\rangle \to \langle m\rangle$ be a map of finite pointed sets. By Lemma 2.37, the coCartesian lifts of α in $N(\mathsf{dgAlg}_k)$ are precisely those morphisms $f\colon (A_1, ..., A_n) \to (B_1, ..., B_m)$ such that each component

$$f_j\colon \bigotimes_{i\in\alpha^{-1}(j)} A_i \to B_j$$

is an isomorphism. Such an edge is sent by m to an edge where each component is a Morita equivalence, which is coCartesian by Lemma 2.37). It we conclude that $\mathsf{m}^{\otimes}$ preserves coCartesian morphisms and is a symmetric monoidal functor. □

Definition 3.7. Let dgCat_k be the (large) category of dg-categories over k. We consider it as a dg-category. Given a pair of dg-categories $\mathcal{A}$ and $\mathcal{B}$, the complex of functors $\mathrm{Fun}_k(\mathcal{A},\mathcal{B})$ is given in degree n by Ch_k-enriched functors $\mathcal{A} \to \mathcal{B}$ such that for each pair of objects $a_0, a_1 \in \mathrm{Ob}\,\mathcal{A}$, the chain map

$$\mathcal{A}(a_0, a_1) \to \mathcal{B}(Fa_0, Fa_1)$$

has degree n.

We denote the underlying (strict, large) (2,1)-category of dgCat_k by $\mathsf{dgCat}_k^{(2,1)}$. It is given as follows:

- $\mathsf{dgCat}_k^{(2,1)}$ has the same objects as dgCat_k.
- Given as pair of dg-categories $\mathcal{A}, \mathcal{B}$, the category $\mathsf{dgCat}_k^{(2,1)}(\mathcal{A},\mathcal{B})$ has objects given by $Z_0\mathrm{Fun}_k(\mathcal{A},\mathcal{B})$, and for a pair of such functors $F, G\colon \mathcal{A} \to \mathcal{B}$, the two-morphisms $F \to G$ are given by natural transformations.

Definition 3.8. Define a functor $\mathcal{P}_-\colon \mathsf{Mor}_k \to \mathsf{dgCat}_k^{(2,1)}$ by sending an algebra A to the dg-category $\mathcal{P}_A$ of finitely generated projective left dg-A-modules and dg-A-module homomorphisms. Given a (B, A)-bimodule P, assign the functor $P \otimes_A -\colon \mathcal{P}_A \to \mathcal{P}_B$. To a bimodule isomorphism $\lambda\colon P \to P'$, assign the transformation $\lambda \otimes_A \mathrm{id}\colon P \otimes_A - \to P' \otimes_A -$.

4. The Dold-Kan Equivalence

In this section we briefly recall some material on the Dold-Kan equivalence.

The Dold-Kan equivalence is an equivalence of categories

$$M\colon \mathsf{sMod}_k \rightleftarrows \mathsf{Ch}_k^{\geq 0} : \Gamma$$

between simplicial k-modules and connective dg-k-modules.

Definition 4.1. The functor M takes a simplicial k-module $A_\bullet$ to the chain complex MA_* where $MA_p = A_p/sA_{p-1}$, the quotient of A_p by the degenerate simplices. The differential $d\colon MA_p \to MA_{p-1}$ is given by the alternating sum of face maps $d = \sum_{i=0}^{p}(-1)^i d_i$.

sMod_k and $\mathsf{Ch}_k^{\geq 0}$ are symmetric monoidal categories under the degreewise tensor product $\hat{\otimes}$ over k and the usual tensor product $\otimes$ of complexes, respectively. M is an equivalence of monoidal categories, but not of symmetric monoidal categories, as the inverse functor is only an $\mathbb{E}_\infty$-monoidal (we will not need this notion in the present paper; see [27] for a definition).

Definition 4.2. A (p, q)-*shuffle*, is a permutation σ of $(1, ..., p + q)$ such that

$$\sigma(1) < \sigma(2) < ... < \sigma(p)$$

and

$$\sigma(p + 1) < \sigma(p + 2) < ... < \sigma(p + q).$$

Let $\Sigma_{(p,q)} \subseteq \Sigma_{p+q}$ denote the subset of (p,q)-shuffles.

Definition 4.3. [11, 5.3] Let A_1 and A_2 be simplicial k-modules. The *shuffle map* is the natural transformation $M(A_1) \otimes M(A_2) \to M(A_1 \hat{\otimes} A_2)$ given by

$$\mathsf{sh}(a_1 \otimes a_2) = \sum_{\sigma \in \Sigma_{(p,q)}} \operatorname{sgn}(\sigma) s_{\sigma(p+q)} ... s_{\sigma(p+1)} a_1 \otimes s_{\sigma(p)} ... s_{\sigma(1)} a_2.$$

Definition 4.4. [12, 2.9] Let A and B be simplicial k-modules. The *Alexander-Whitney map*

$$AW_{A,B} \colon N(A \hat{\otimes} B) \to N(A) \otimes N(B)$$

is defined on elementary tensors $a \otimes b \in A_n \otimes B_n$ as

$$AW_{A,B} \colon (a \otimes b) \mapsto \sum_{i=0}^{n} d_{i+1} ... d_{n-1} d_n a \otimes (d_0)^i b$$

As with the shuffle map, we omit A, B from the notation $AW_{A,B}$ when there is no risk of confusion.

The shuffle and Alexander-Whitney maps are associative (see [11, Theorem 5.4] and [12, Corollary 2.2]) and mutual homotopy inverses [12, Theorem 2.1].

Notation 4.5. Write sh_n for the natural transformation

$$\mathsf{sh}_n \colon M(A_1) \otimes ... \otimes M(A_n) \to M(A_1 \hat{\otimes} ... \hat{\otimes} A_n)$$

of functors $\mathsf{sMod}_k^{\times n} \to \mathsf{Ch}_k$.

5. Hochschild chains on dg-categories

We first remind the reader of some standard constructions.

Definition 5.1. [18, Section 5.4] The cyclic bar construction on a dg-category $\mathcal{A}$ is given by the simplicial chain complex

$$C_\Delta(\mathcal{A})_n = \bigoplus_{a_0, \dots, a_n \in \operatorname{Ob} \mathcal{A}} \mathcal{A}(a_n, a_0) \otimes \mathcal{A}(a_0, a_1) \otimes ... \otimes \mathcal{A}(a_{n-1}, a_n)$$

with simplicial face maps

$$d_i(f_0 \otimes ... \otimes f_n) = \begin{cases} f_0 \otimes ... \otimes f_{i+1} f_i \otimes ... \otimes f_n & , i < n \\ f_0 f_n \otimes f_1 \otimes ... \otimes f_{n-1} & , i = n \end{cases} .$$

The Hochschild boundary operator is given by $b = \sum_{i=0}^{n} (-1)^i d_i$ and makes $C_\Delta(\mathcal{A})$ into a double complex $C_\epsilon(\mathcal{A})$. We define the Hochschild complex $C(\mathcal{A})$ of $\mathcal{A}$ to be the $\oplus$-total complex of $C_\epsilon(\mathcal{A})$.

If $F \colon \mathcal{A} \to \mathcal{B}$ is a functor between dg-categories, let $C(F) \colon C(\mathcal{A}) \to C(\mathcal{B})$ be given on the level of the cyclic bar construction by

$$\mathcal{A}(a_n, a_0) \otimes ... \otimes \mathcal{A}(a_{n-1}, a_n) \xrightarrow{F_{a_n,a_0} \otimes ... \otimes F_{a_{n-1},a_n}} \mathcal{B}(Fa_n, Fa_0) \otimes ... \otimes \mathcal{B}(Fa_{n-1}, Fa_n).$$

Definition 5.2. Let $s_{p,q} \colon [p+q] \to [p] \times [q]$ be the embedding given by

$$s(a) = \begin{cases} (a, 0) & , 0 \le a \le p \\ (p, a-p) & , p+1 \le a \le p+q \end{cases}$$

Thinking of $s_{p,q}$ as a sequence of moves either vertically or horizontally, the set of (p,q)-shuffles acts on s by permuting the moves.

Definition 5.3. • Let $\mathcal{A}, \mathcal{B}$ be dg-categories over k. Define the dg-category $\mathcal{A} \otimes \mathcal{B}$ with objects $\operatorname{Ob} \mathcal{A} \times \operatorname{Ob} \mathcal{B}$ and morphisms $(\mathcal{A} \otimes \mathcal{B})((a,b),(a',b')) = \mathcal{A}(a,a') \otimes \mathcal{B}(b,b')$.

• Given a Hochschild chain

$$(a_n \xrightarrow{f_0} a_0 \xrightarrow{f_1} \dots \xrightarrow{f_{p-1}} a_{p-1} \xrightarrow{f_p} a_p)$$

in $\mathcal{A}$, write $f\colon k[p] \to \mathcal{A}$ for the restriction to the latter p morphisms.

The *external shuffle map*

$$\overline{\mathsf{sh}}_{\mathcal{A},\mathcal{B}}\colon C(\mathcal{A}) \otimes C(\mathcal{B}) \to C(\mathcal{A} \otimes \mathcal{B})$$

is defined as

$$(a_n \xrightarrow{f_0} a_0 \xrightarrow{f_1} \dots \xrightarrow{f_{p-1}} a_{p-1} \xrightarrow{f_p} a_p) \otimes (b_n \xrightarrow{f_0} b_0 \xrightarrow{g_1} \dots \xrightarrow{g_{p-1}} b_{q-1} \xrightarrow{g_q} b_q) \mapsto$$

$$\sum_{\sigma \in (p,q)} \operatorname{sgn}(\sigma)(f_0 \otimes g_0, ((f \otimes g)(\sigma \cdot s_{p,q})(0 \to 1)), \dots, ((f \otimes g)(\sigma \cdot s_{p,q})(p+q-1 \to p+q))).$$

• Define the *external Alexander-Whitney map*

$$\overline{AW}_{\mathcal{A},\mathcal{B}}\colon C(\mathcal{A} \otimes \mathcal{B}) \to C(\mathcal{A}) \otimes C(\mathcal{B})$$

taking

$$(a_n, b_n) \overset{f_0 \otimes g_0}{\to} (a_0, b_0) \to \dots \to (a_{n-1}, b_{n-1}) \overset{f_n \otimes g_n}{\to} (a_n, b_n)$$

to

$$\sum_{p=0}^{n} (f_0 f_n \dots f_{p+1} \otimes f_1 \otimes \dots \otimes f_p) \otimes (g_p \dots g_0 \otimes g_{p+1} \otimes \dots \otimes g_n)$$

As in the case for algebras, sh and AW are mutually inverse quasi-isomorphisms.

Lemma 5.4. Let A and B be dgas over k, considered as dg-categories with one object. Then $\overline{\mathsf{sh}}_{A,B} = \mathsf{sh}_{A,B}$ and $\overline{AW}_{A,B} = AW_{A,B}$.

Proof. That the Alexander-Whitney maps agree is immediate from the simplicial structure of the cyclic bar construction.

To see that the shuffle maps agree, fix a (p,q)-shuffle σ. Given a degree p Hochschild chain of $a = a_0 \otimes \widetilde{a}$ of A, where $\widetilde{a} \in A^{\otimes p}$, and a degree q Hochschild chain $b = b_0 \otimes \widetilde{b}$ of B, we have

$$\sigma \cdot (\widetilde{a} \otimes \widetilde{b}) s_{p,q}(r \to r+1) = \Big(\sigma \cdot (\widetilde{a} \otimes 1) s_{p,q}(r \to r+1)\Big) \otimes \Big(\sigma \cdot (1 \otimes \widetilde{b}) s_{p,q}(r \to r+1)\Big)$$

where $\sigma \cdot (\widetilde{a} \otimes 1) s_{p,q}(r \to r+1) = 1$ if an only if $\sigma^{-1}(r) > p$. In ohter words,

$$a_0 \otimes \Big(\sigma \cdot (\widetilde{a} \otimes 1) s_{p,q}(0 \to 1), \dots, \sigma \cdot (\widetilde{a} \otimes 1) s_{p,q}(p+q \to p+q+1)\Big) = s_{\sigma(p+q)} \dots s_{\sigma(p+1)} a,$$

and similarly

$$b_0 \otimes \Big(\sigma \cdot (\widetilde{b} \otimes 1) s_{p,q}(0 \to 1), \dots, \sigma \cdot (\widetilde{b} \otimes 1) s_{p,q}(p+q \to p+q+1)\Big) = s_{\sigma(p)} \dots s_{\sigma(1)} b.$$

□

Lemma 5.5. ([18, Section 5.5]) Let A be a dga over k, and regard A as a category with one object $*$. The inclusion $i_A\colon A \to \mathcal{P}_A$ sending $*$ to A induces a natural homotopy equivalence

$$e_A\colon C(A) \to C(\mathcal{P}_A).$$

Definition 5.6. For each dga A over k, let $f_A\colon C(\mathcal{P}_A) \to C(A)$ be a homotopy inverse to e_A.

Remark 5.7. We do not know if f_A can be chosen to be strictly natural in A, but we expect it to be impossible. For ordinary algebras, the known construction corresponds to realizing each projective A-module as a direct summand of a free module, see the proof of [24, Proposition 2.4.3].

By the above, using a non-symmetric monoidal version of Lemma 2.43, a functor $F\colon \mathcal{P}_A \to \mathcal{P}_B$ induces a morphism $f_B \circ F \circ e_A\colon C(A) \to C(B)$, which is compatible with composition up to coherent homotopy in the sense of Lemma 2.44, and this morphism is a homotopy equivalence if the functor is an equivalence. In particular, such a map is induced by a (B, A)-bimodule P, and is a homotopy equivalence when P witnesses a Morita equivalence.

Remark 5.8. It is well known (see [19, Section 4.1.1]) that conjugation by an invertible element induces the identity morphism on Hochschild homology. The same is true for the modulation functor landing in the Morita category, see [18, Section 2.4].

Lemma 5.9. Let $f\colon A \to B$ be a map of dga's. The following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} C(A) \arrow[r, "e_A"] \arrow[d, "f"'] & C(\mathcal{P}_A) \arrow[d, "B_f"] \\ C(B) \arrow[r, "e_B"'] & C(\mathcal{P}_B) \end{tikzcd}$$

and hence the maps $HH(A) \to HH(B)$ induced by f and the (B, A)-bimodule B_f are equal.

Proof. We examine the two compositions at the level of the cyclic bar construction. The lower part of the diagram takes a Hochschild chain $(a_0, ..., a_n) \in A^{\otimes n+1}$ to

$$(f(a_0), ..., f(a_n)) \in B^{\otimes n}.$$

The upper part takes a Hochschild chain $(a_0, ..., a_n) \in A^{\otimes n+1}$ to the chain

$$(1 \otimes a_0, ..., 1 \otimes a_n) \in (B_f \otimes_A A)^{\otimes n}$$

where the natural isomorphism is the unitor $\rho_{B,A}\colon B_f \otimes_A A \cong B$ taking $(1 \otimes a)$ to $f(a)$. □

Lemma 5.10. Let M and N be a pair of (B, A)-bimodules and let $\phi\colon M \to N$ be a bimodule isomorphism. We obtain a natural isomorphism $M \otimes_A - \to N \otimes_A -$ of functors $\mathcal{P}_A \to \mathcal{P}_B$ given componentwise by $\phi \otimes \mathrm{id}\colon M \otimes_A P \to N \otimes_A P$.

Proof. Given a morphism $\psi\colon P \to Q$ of perfect A-modules, the naturality squares are of the form

$$\begin{tikzcd} M \otimes_A P \arrow[r, "\phi \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \psi"'] & N \otimes_A P \arrow[d, "\mathrm{id} \otimes \psi"] \\ M \otimes_A Q \arrow[r, "\phi \otimes \mathrm{id}"'] & N \otimes_A Q \end{tikzcd}$$

and so they commute. □

Lemma 5.11. Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of dg-categories, $F, G\colon \mathcal{A} \to \mathcal{B}$ a pair of dg-functors and $\phi\colon F \to G$ a natural isomorphism. Then ϕ induces a homotopy $h\colon C(F) \to C(G)$ between the induced morphisms $C(F), C(G)\colon C(\mathcal{A}) \to C(\mathcal{B})$.

Proof. We construct a presimplicial homotopy in the sense of [19, 1.0.8] at the level of the cyclic bar constructions. This will imply the statement. For a Hochschild chain

$$a_n \xrightarrow{f_0} a_0 \xrightarrow{f_1} ... \xrightarrow{f_{n-1}} a_{n-1} \xrightarrow{f_n} a_n$$

in $C_\Delta(\mathcal{A})_n$, let $h_i\colon C_\Delta(\mathcal{A})_n \to C_\Delta(\mathcal{B})_{n+1}$ be given by

$$h_i\colon (f_0, ..., f_n) \mapsto (F(f_0) \circ \phi_{a_n}^{-1}, F(f_1), ..., F(f_i), \phi_{a_i}, G(f_{i+1}), ..., G(f_n))$$

We are left to verify the relations

(1) $d_i h_j = h_{j-1} d_i$ for $i < j$,
(2) $d_i h_j = d_i h_{j-1}$ for $0 < i \leq n$ and $i = j$ or $i = j+1$,
(3) $d_i h_j = h_j d_{i-1}$ for $i > j+1$,
(4) $d_0 h_0 = C(G), d_{n+1} h_n = C(F)$

Note that since ϕ is a natural transformation, the below diagram commutes, and $h_i(f_0, ..., f_n)$ can be viewed as the Hochschild chain starting at $G(a_n)$, traversing the upper part of the diagram for the first i morphisms and then goind down to the lower part.

$$\begin{tikzcd}
F(a_n) \arrow[r, "F(f_0)"] \arrow[d, "\phi_{a_n}"] & ... \arrow[r, "F(f_i)"] & F(a_i) \arrow[r, "F(f_{i+1})"] \arrow[d, "\phi_{a_i}"] & ... \arrow[r, "F(f_n)"] & F(a_n) \arrow[d, "\phi_{a_n}"] \\
G(a_n) \arrow[r, "G(f_0)"] & ... \arrow[r, "G(f_i)"] & G(a_i) \arrow[r, "G(f_{i+1})"] & ... \arrow[r, "G(f_n)"] & G(a_n)
\end{tikzcd}$$

Relations (1) and (3) clearly hold. Relation (2) boils down to the commutativity of the following diagram:

$$\begin{tikzcd}
F(a_{l-1}) \arrow[r, "F(f_l)"] \arrow[d, "\phi_{a_{l-1}}"'] & F(a_l) \arrow[d, "\phi_{a_l}"] \\
G(a_{l-1}) \arrow[r, "G(f_l)"'] & G(a_l)
\end{tikzcd}$$

for $i = l$ or $i = l+1$, which holds by naturality. Relation (4) is checked by hand:

$$d_0 h_0(f_0, ..., f_n) = (\phi_{a_0} \circ F(f_0) \circ \phi_{a_n}^{-1}, G(f_1), ..., G(f_n)) = (G(f_0), ..., G(f_n))$$

by the naturality of ϕ. Similarly,

$$d_{n+1} h_n(f_0, ..., f_n) = (F(f_0) \circ \phi_{a_n}^{-1} \circ \phi_{a_n}, F(f_1), ..., F(f_n)) = (F(f_0), ..., F(f_n)).$$

Then $h = \sum_{i=0}^n (-1)^i h_i$ is a homotopy between $C(F)$ and $C(G)$ by [19, Lemma 1.0.9]. $\square$

Corollary 5.12. If $\mathcal{A}$ and $\mathcal{B}$ are dg-categories over k and $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of categories, then F induces a homotopy equivalence $C(F)\colon C(\mathcal{A}) \to C(\mathcal{B})$.

Definition 5.13. Let Δ^n be the groupoid $0 \leftrightarrows 1 \leftrightarrows \ldots \leftrightarrows n$. and let $k\Delta^n \in \mathsf{dgCat}_k$ be the free k-linear category on Δ^n. In other words, for every $0 \leq i, j \leq n$, $k\Delta^n(i,j) = k$ concentrated in degree 0.

Lemma 5.14. The Hochshcild complex $C(k\Delta^n)$ is acyclic.

Proof. This follows from Corollary 5.12 since $k\Delta^n$ is equivalent to its full subcategory on 0. $\square$

Definition 5.15. Let $S_*\Delta_n$ be the reduced simplicial chain complex of the n-simplex, i.e. it has a generator in degree k for each functor $f\colon [k] \to [n]$ such that f is injective on objects, and the differential $d\colon S_k\Delta_n \to S_{k-1}\Delta_n$ is given by $d = \sum_{i=0}^k \delta_i^*$, where $\delta_i\colon [k-1] \to [k]$ is the injection omitting the i'th object.

Definition 5.16. Let $m, n \geq 0$. There is a map $i_m\colon \mathcal{S}_*(\Delta_n^{\times m}) \to C((k\Delta^n)^{\otimes m})$ given by sending a functor $f\colon [p] \to [n]^{\times m}$ to the Hochschild chain

$$i_m(f) = \big(f(0 \to p)^{-1}, f(0 \to 1), \ldots, f((p-1) \to p)\big)$$

For ease of notation, we simply write $i := i_1$.

Lemma 5.17. Let $\Delta\colon \Delta_n \to \Delta_n^{\times m}$ be the diagonal embedding, and write also $\Delta\colon k\Delta^n \to (k\Delta^n)^{\otimes m}$ for the diagonal functor. Then the diagram

$$\begin{tikzcd}
S_*(\Delta_n) \arrow[r, "i"] \arrow[d, "\Delta_*"'] & C(k\Delta^n) \arrow[d, "\Delta_*"] \\
S_*(\Delta_n^{\times m}) \arrow[r, "i_m"'] & C((k\Delta^n)^{\otimes m})
\end{tikzcd}$$

commutes.

Proof. The diagonal map $\Delta_*\colon C(k\Delta^n) \to C((k\Delta^n)^{\otimes m})$ takes a Hochschild chain $(f_0, \ldots, f_p)$ to the diagonal chain $(\Delta f_0, \ldots, \Delta f_p)$, such that the upper part of the diagram takes a functor $f\colon [p] \to [n]$ to

$$\Delta_* i(f) = \big(\Delta f(0 \to p)^{-1}, \Delta f(0 \to 1), \ldots, \Delta f((p-1) \to p)\big)\,.$$

On the other hand, the diagonal map $\Delta_*\colon S_*(\Delta_n) \to S_*(\Delta_n^{\times m})$ takes a functor $f\colon [p] \to [n]$ to the composition

$$[p] \xrightarrow{f} [n] \xrightarrow{\Delta} [n]^{\times m}.$$

such that

$$i_m \circ \Delta_*(f) = i_m(\Delta \circ f) = \big(\Delta f(0 \to p)^{-1}, \Delta f(0 \to 1), \ldots, \Delta f((p-1) \to p)\big) = \Delta_* i(f).$$

So the diagram commutes. $\square$

Lemma 5.18. The following diagram commutes.

$$\begin{tikzcd}
S_*(\Delta_n)^{\otimes m} \arrow[r, "i^{\otimes m}"] \arrow[d, "\mathsf{sh}"'] & C(k\Delta^n)^{\otimes m} \arrow[d, "\overline{\mathsf{sh}}"] \\
S_*(\Delta_n^{\times m}) \arrow[r, "i_m"'] & C((k\Delta^n)^{\otimes m})
\end{tikzcd}$$

Proof. We give a proof for the case $k = 2$. The general case is similar. We have to prove that the following diagram commutes.

$$\begin{tikzcd}
S_*(\Delta_n) \otimes S_*(\Delta_n) \arrow[r, "i \otimes i"] \arrow[d, "\mathsf{sh}"'] & C(k\Delta^n) \otimes C(k\Delta^n) \arrow[d, "\overline{\mathsf{sh}}"] \\
S_*(\Delta_n \times \Delta_n) \arrow[r, "i_2"'] & C(k\Delta^n \otimes k\Delta^n)
\end{tikzcd}$$

Recall the functors $s_{p,q} \colon [p] \times [q] \to [p+q]$ from Definition 5.2 and the action on $s_{p,q}$ by shuffle maps. Then given a pair of functors $f \colon [p] \to [n]$ and $g \colon [q] \to [n]$, the lower part of the diagram evaluates to

$$i_2 \circ \mathsf{sh}(f,g) = i_2 \left(\sum_{\sigma \in \Sigma_{p,q}} (\operatorname{sgn} \sigma)(f \times g) \circ (\sigma \cdot s_{p,q}) \right)$$

$$= \sum_{\sigma \in \Sigma_{p,q}} (\operatorname{sgn} \sigma) \left(f(0 \to p)^{-1} \otimes g(0 \to q)^{-1}, (f \times g) \circ (\sigma \cdot s_{p,q})(0 \to 1), \ldots \right.$$
$$\left. \ldots, (f \times g) \circ (\sigma \cdot s_{p,q})((p+q-1) \to (p+q)) \right)$$

However, this action of the (p,q)-shuffles is precisely the action on Hochschild chains, so this is equal to

$$\sum_{\sigma \in \Sigma_{p,q}} (\operatorname{sgn} \sigma) \sigma \cdot \left(f(0 \to p)^{-1} \otimes g(0 \to q)^{-1}, f \otimes \mathrm{id}, \mathrm{id} \otimes g \right)$$
$$= \overline{\mathsf{sh}}(i(f), i(g))$$

so the diagram commutes. ☐

Observation 5.19. • By the Dold-Kan equivalence, we can consider Ch_k as a simplicial category with morphism spaces

$$\mathrm{Hom}_{\mathsf{Ch}_k}(A,B)_n := Z_0 \mathrm{Hom}_{\mathsf{Ch}_k}(S_* \Delta_n \otimes A, B)$$

i.e. chain maps $S_*\Delta_n \otimes A \to B$. The composition of two degree n morphisms $S_*\Delta_n \otimes A \xrightarrow{f} B$ and $S_*\Delta_n \otimes B \xrightarrow{g} C$ is given by the composition

$$\mathcal{S}_*\Delta_n \otimes A \xrightarrow{\Delta_* \otimes \mathrm{id}} S_*(\Delta_n \times \Delta_n) \otimes A \xrightarrow{AW \otimes \mathrm{id}} S_*\Delta_n \otimes S_*\Delta_n \otimes A \xrightarrow{\mathrm{id} \otimes f} S_*\Delta_n \otimes B \xrightarrow{g} C.$$

• Since $\mathsf{dgCat}_k^{(2,1)}$ is a strict (2,1)-category, it has an associated simplicial category $\mathsf{dgCat}_{k,\Delta}^{(2,1)}$ (see Definition 2.15). A degree n morphism in $\mathsf{dgCat}_{k,\Delta}^{(2,1)}$ is given by a chain

$$F_0 \xrightarrow{\phi_1} F_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_n} F_n$$

of natural isomorphisms of functors $F_i\colon \mathcal{A}\to\mathcal{B}$ in $Z_0(\mathrm{Fun}(\mathcal{A},\mathcal{B}))$. This can equivalently be expressed as a functor $k\Delta^n\otimes\mathcal{A}\to\mathcal{B}$ in $Z_0(\mathrm{Fun}(k\Delta^n\otimes\mathcal{A},\mathcal{B}))$. In other words, there is a natural isomorphism

$$\mathsf{dgCat}^{(2,1)}_{k,\Delta}(\mathcal{A},\mathcal{B})_n \cong Z_0(\mathrm{Fun}(k\Delta^n\otimes\mathcal{A},\mathcal{B})).$$

Lemma 5.20. Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of dg-categories, $F,G\colon\mathcal{A}\to\mathcal{B}$ a pair of dg-functors and $\phi\colon F\to G$ a natural isomorphism. We get an induced morphism

$$S_*\Delta_1\otimes C(\mathcal{A})\xrightarrow{i_1\otimes\mathrm{id}} C(k\Delta^1)\otimes C(\mathcal{A})\xrightarrow{\overline{\mathsf{sh}}} C(k\Delta^1\otimes\mathcal{A})\xrightarrow{\phi_*} C(\mathcal{B}).$$

This map is equal to the homotopy constructed in the proof of Lemma 5.11 when evaluated at the top non-degenerate simplex $[1]\xrightarrow{\mathrm{id}}[1]$ in $S_*\Delta_1$.

Proof. Write $u\colon 0\to 1$ for the morphism in Δ^1 and let

$$a_n\xrightarrow{f_0} a_0 ... a_{n-1}\xrightarrow{f_n} a_n$$

be a Hochschild chain of $\mathcal{A}$. Then the above composition evaluates to

$$\Big([1]\xrightarrow{\mathrm{id}}[1]\Big)\otimes(f_0,...,f_n)\overset{i\otimes\mathrm{id}}{\mapsto}(u^{-1},u)\otimes(f_0,...,f_n)$$

$$\overset{\overline{\mathsf{sh}}}{\mapsto}\sum_{\sigma\in\Sigma_{1,n}}(\operatorname{sgn}\sigma)\sigma\cdot((f_0\otimes u^{-1}),(a_0\otimes u),(1\otimes f_1),...,(1\otimes f_n))$$

$$=\sum_{i=0}^{n}(-1)^i((f_0\otimes u^{-1}),(f_1\otimes 0),...,(f_i\otimes 0),(a_i\otimes u),(f_{i+1}\otimes 1),...,(f_n\otimes 1))$$

$$\overset{\phi_*}{\mapsto}\sum_{i=0}^{n}(-1)^i(F(f_0)\circ\phi^{-1}_{a_n},F(f_1),...,F(f_i),\phi_{a_i},G(f_{i+1}),...,G(f_n))$$

which agrees with the homotopy h constructed in the proof of Lemma 5.11. ☐

Lemma 5.21. By the Dold-Kan equivalence, consider Ch_k as a simplicial category. Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of dg-categories. There is a simplicial map $N\mathrm{Fun}(\mathcal{A},\mathcal{B})\to\mathrm{Hom}(C(\mathcal{A}),C(\mathcal{B}))$ extending to a functor of quasi-categories $C(-)\colon N_\Delta\mathsf{dgCat}^{(2,1)}_{k,\Delta}\to N_\Delta\mathsf{Ch}_k$.

Proof. We will construct a simplicial functor up to coherent homotopy $\mathsf{dgCat}^{(2,1)}_{k,\Delta}\to\mathsf{Ch}_k$. Then such a functor gives rise to a simplicial map between their simplicial nerves.

Let $F_0,...,F_n\colon\mathcal{A}\to\mathcal{B}$ be a collection of dg-functors and let $F_0\xrightarrow{\phi_1}...\xrightarrow{\phi_n}F_n$ be a chain of natural isomorphisms, and let $\phi\colon k\Delta^n\otimes\mathcal{A}\to\mathcal{B}$ be the associated functor sending $i\otimes a$ to $F_i(a)$ and $u_i\otimes a$ to $\phi_{i,a}\colon F_{i-1}(a)\to F_i(a)$.

We precompose the induced chain map of ϕ with the external shuffle map (see Definition 5.3) to obtain a chain map

$$C(\phi)\colon S_*\Delta_n\otimes C(\mathcal{A})\xrightarrow{i\otimes\mathrm{id}} C(k\Delta^n)\otimes C(\mathcal{A})\xrightarrow{\overline{\mathsf{sh}}} C(k\Delta^n\otimes\mathcal{A})\xrightarrow{\phi_*} C(\mathcal{B}).$$

We now verify that this assignment is compatible with composition. If we in addition have dg-functors $G_0,...,G_n\colon\mathcal{B}\to\mathcal{C}$ and natural isomorphisms $G_0\xrightarrow{\psi_1}...\xrightarrow{\psi_n}G_n$, we get an induced functor $\psi\colon k\Delta^n\otimes\mathcal{B}\to\mathcal{C}$ as above. We can also horizontally compose the ψ_i's and ϕ_i's to obtain a chain

$$G_0F_0\xrightarrow{\psi_1*\phi_1}...\xrightarrow{\psi_n*\phi_n}G_nF_n$$

and an associated functor $\psi*\phi\colon k\Delta^n\otimes\mathcal{A}\to\mathcal{C}$. These chains fit into commutative diagram

$$\begin{tikzcd}
k\Delta^n \otimes \mathcal{A} \arrow[d, "\Delta \otimes \mathrm{id}"'] \arrow[rrd, "\psi * \phi"] & & \\
k\Delta^n \otimes k\Delta^n \otimes \mathcal{A} \arrow[r, "\mathrm{id} \otimes \phi"'] & k\Delta^n \otimes \mathcal{B} \arrow[r, "\psi"'] & \mathcal{C}
\end{tikzcd}$$

where $\Delta\colon \Delta^n \to \Delta^n \otimes \Delta^n$ is the diagonal functor, and $\psi * \phi$ is the composition of ϕ and ψ in $\mathsf{dgCat}^{(2,1)}_{k,\Delta}$.

Similar to the map $i\colon S_*(\Delta_n) \to C(k\Delta^n)$, the Hochschild complex $C(k\Delta^n \otimes k\Delta^n)$ contains a product of two n-simplices there is a map

$$i_2\colon S_*(\Delta_n \times \Delta_n) \to C(k\Delta^n \otimes k\Delta^n)$$

The composition $C(\psi) \circ C(\phi)$ is given by the upper part of the below large diagram, which the lower part is given by $C(\psi * \phi)$. The only squares that need to be checked are the leftmost ones. The lower leftmost square commutes by Lemma 5.17, while the upper leftmost square commutes up to homotopy by Lemma 5.18.

$$\begin{tikzcd}
S_*(\Delta_n) \otimes S_*(\Delta_n) \otimes C(\mathcal{A}) \arrow[r, "1 \otimes i \otimes 1"] & S_*(\Delta_n) \otimes C(k\Delta^n) \otimes C(\mathcal{A}) \arrow[r, "1 \otimes \overline{\mathsf{sh}}"] \arrow[d, "i \otimes 1 \otimes 1"'] & S_*(\Delta_n) \otimes C(k\Delta^n \otimes \mathcal{A}) \arrow[r, "1 \otimes \phi_*"] \arrow[d, "i \otimes 1"'] & S_*(\Delta_n) \otimes C(\mathcal{B}) \arrow[d, "i \otimes 1"'] \\
 & C(k\Delta^n) \otimes C(k\Delta^n) \otimes C(\mathcal{A}) \arrow[r, "1 \otimes \overline{\mathsf{sh}}"] \arrow[d, "\overline{\mathsf{sh}} \otimes 1"'] & C(k\Delta^n) \otimes C(k\Delta^n \otimes \mathcal{A}) \arrow[r, "1 \otimes \phi_*"] \arrow[d, "\overline{\mathsf{sh}}"'] & C(k\Delta^n) \otimes C(\mathcal{B}) \arrow[d, "\overline{\mathsf{sh}}"'] \\
S_*(\Delta_n \times \Delta_n) \otimes C(\mathcal{A}) \arrow[uu, "AW"] \arrow[r, "i_2 \otimes 1"] & C(k\Delta^n \otimes k\Delta^n) \otimes C(\mathcal{A}) \arrow[r, "\overline{\mathsf{sh}}"] & C(k\Delta^n \otimes k\Delta^n \otimes \mathcal{A}) \arrow[r, "(1 \otimes \phi)_*"] & C(k\Delta^n \otimes \mathcal{B}) \arrow[d, "\psi_*"'] \\
S_*(\Delta_n) \otimes C(\mathcal{A}) \arrow[u, "\Delta_* \otimes 1"] \arrow[r, "i \otimes 1"] & C(k\Delta^n) \otimes C(\mathcal{A}) \arrow[u, "\Delta_* \otimes 1"] \arrow[r, "\overline{\mathsf{sh}}"] & C(k\Delta^n \otimes \mathcal{A}) \arrow[u, "(\Delta \otimes 1)_*"] \arrow[r, "(\psi * \phi)_*"] & C(\mathcal{C})
\end{tikzcd}$$

The general case for an m-fold composition in degree n is similar, with the homotopies parameterizing the m-fold composition contained in the complex

$$\mathrm{Nat}_{\mathsf{sSet}^{\times m}}\left(S_*(- \times ... \times -), S_*(- \times ... \times -)\right)$$

which is contractible by [6, Satz 1.6]. In other words, we have obtained a functor up to coherent homotopy

$$\mathsf{dgCat}^{(2,1)}_{k,\Delta} \to \mathsf{Ch}_k.$$

Now, let $\mathcal{A}_0, ..., \mathcal{A}_m$ be dg-categories, and for each $1 \le i \le m$, let

$$F_{0,i} \xrightarrow{\phi_{1,i}} F_{1,i} ... \xrightarrow{\phi_{n,i}} F_{n,i}$$

be a sequence of natural isomorphisms defining a functor $\phi_i\colon k\Delta^n \otimes \mathcal{A}_{i-1} \to \mathcal{A}_i$. By the contractibility of the above complex, there is a morphism of simplicial sets

$$NP_{0,m} \to \mathrm{Hom}_{\mathsf{Ch}_k}(\mathcal{A}_0, \mathcal{A}_m)$$

(recall Definition 2.12) sending the subset $I = \{0, i_1, ..., i_k, m\}$ to the composition

$$C(\phi_n * ... * \phi_{i_k+1}) \circ C(\phi_{i_k} * ... * \phi_{i_{k-1}+1}) \circ ... \circ C(\phi_1 * ... \phi_{i_1})$$

and the inclusion $I \setminus \{i_j\} \subset I$ applies the homotopy corresponding to the square

$$\begin{tikzcd}
S_*(\Delta_n^{\times i_{j+1}-i_j}) \otimes S_*(\Delta_n^{\times i_j - i_{j-1}}) \arrow[r, "i_{i_{j+1}-i_j} \otimes i_{i_j - i_{j-1}}"] & C((k\Delta^n)^{\otimes i_{j+1}-i_j}) \otimes C((k\Delta^n)^{\otimes i_j - i_{j-1}}) \arrow[d, "\overline{\mathsf{sh}}"] \\
S_*(\Delta_n^{\times i_{j+1}-i_{j-1}}) \arrow[u, "AW"] \arrow[r, "i_{i_{j+1}-i_{j-1}}"] & C((k\Delta^n)^{\otimes i_{j+1}-i_{j-1}})
\end{tikzcd}$$

Finally, given a natural isomorphism $H \colon \psi \to \phi_2 * \phi_1$ in $N_\Delta \mathsf{dgCat}_{k,\Delta}^{(2,1)}$, the homotopy $C(H)$ is contained in the lower right corner of the big diagram above, so that it does not interact with the homotopies relating the different composites. Indeed, we may extend the assignment above such that $NP_{0,2}$ takes $\{0,1,2\}$ to $C(\phi_2) \circ C(\phi_1)$ and $\{0,2\}$ to $C(\psi)$, where the inclusion $\{0,2\} \subseteq \{0,1,2\}$ applies the homotopies $C(H)$ and $\mathsf{sh} \circ AW \to \mathrm{id}$ in any order. The general case is identical, so the above assignment extends to a morphism of simplicial sets

$$N_\Delta \mathsf{dgCat}_{k,\Delta}^{(2,1)} \to N_\Delta \mathsf{Ch}_k$$

as claimed.

□

Theorem 5.22. The functor $\mathsf{dgAlg}_k \to \mathsf{Ch}_k$ taking A to $C(A)$ factors through the modulation functor $\mathsf{dgAlg}_k \to \mathsf{Mor}_k$. Furthermore, $C(-)$ takes equivalences of dg-categories to homotopy equivalences of chain complexes. We obtain the below diagram, which commutes up to coherent homotopy.

$$\begin{tikzcd}
N(\mathsf{dgAlg}_k) \arrow[r, "\mathsf{m}"] \arrow[dr, "C(-)"'] & N_D(\mathsf{Mor}_k) \arrow[r, "\mathcal{P}_-"] \arrow[d, "C(-)"] & N_D(\mathsf{dgCat}_k^{(2,1)}) \arrow[dl, "C(-)"] \\
& N_\Delta(\mathsf{Ch}_k) &
\end{tikzcd}$$

Proof. The middle and right vertical functors exist by Lemma 5.21. The right triangle commutes by definition, while the left triangle commutes by Lemma 5.9. □

6. Symmetric monoidal structure

In this section we will prove that the left triangle of Theorem 5.22 can be upgraded to a homotopy-commutative diagram of symmetric monoidal quasi-categories. This will complete the proof of the main theorem.

We begin by discussing shuffle maps of Hochschild complexes of dg-categories of finitely generated projective modules.

Notation 6.1. For an n-tuple of simplicial k-algebras $\vec{A} = (A_1, ..., A_n)$, we write

$$C(\mathcal{P}(\vec{A})) = C(\mathcal{P}_{A_1}) \otimes ... \otimes C(\mathcal{P}_{A_n})$$

We apply the maps e_A, f_A, h_A of Definition 5.6 componentwise to obtain maps

$$e_{\vec{A}} \colon C(\vec{A}) \to C(\mathcal{P}(\vec{A}))$$

$$f_{\vec{A}} \colon C(\mathcal{P}(\vec{A})) \to C(\vec{A})$$
$$h_{\vec{A}} \colon e_{\vec{A}} f_{\vec{A}} \to \mathrm{id}_{C(\mathcal{P}(\vec{A}))}$$

Construction 6.2. Consider an n-tuple of dg-algebras $\vec{A} = (A_1, ..., A_n)$ and write

$$A = A_1 \otimes ... \otimes A_n.$$

We define a map $\mathsf{sh}'_n \colon C(\mathcal{P}(\vec{A})) \to C(\mathcal{P}_A)$ by

$$\mathsf{sh}'_n = e_A \circ \mathsf{sh}_n \circ f_{\vec{A}}.$$

Then the following square commutes:

$$\begin{tikzcd} C(\vec{A}) \arrow[r, "e_{\vec{A}}"] \arrow[d, "\mathsf{sh}_n"'] & C(\mathcal{P}(\vec{A})) \arrow[d, "\mathsf{sh}'_n"] \\ C(A) \arrow[r, "e_A"'] & C(\mathcal{P}_A) \end{tikzcd}$$

A homotopy inverse to sh'_n is given by $AW'_n := e_{\vec{A}} \circ AW_n \circ f_A$.

Since f, the homotopy inverse of e, is not a strict natural transformation, the following alternative choice of shuffle maps will be needed to produce a strictly natural shuffle map for dg-categories.

Definition 6.3. • Let

$$(- \otimes -) \colon \mathcal{P}_A \otimes \mathcal{P}_B \to \mathcal{P}_{A \otimes B}$$

be the natural functor given on objects by $(M, N) \mapsto M \otimes N$ and on morphisms by the canonical map

$$\mathrm{Hom}_A(M, M') \otimes \mathrm{Hom}_B(N, N') \to \mathrm{Hom}_{A \otimes B}(M \otimes N, M' \otimes N')$$

induced by the symmetric monoidal structure of Ch_k.

• For A and B dga's over k, we define the *internal shuffle* map as the composition

$$\mathsf{sh} \colon C(\mathcal{P}_A) \otimes C(\mathcal{P}_B) \xrightarrow{\overline{\mathsf{sh}}} C(\mathcal{P}_A \otimes \mathcal{P}_B) \xrightarrow{(-\otimes-)_*} C(\mathcal{P}_{A \otimes B}).$$

Lemma 6.4. The square

$$\begin{tikzcd} C(A) \otimes C(B) \arrow[r, "e_A \otimes e_B"] \arrow[d, "\mathsf{sh}"'] & C(\mathcal{P}_A) \otimes C(\mathcal{P}_B) \arrow[d, "\mathsf{sh}"] \\ C(A \otimes B) \arrow[r, "e_{A \otimes B}"'] & C(\mathcal{P}_{A \otimes B}) \end{tikzcd}$$

commutes. It follows that sh and sh' are homotopic maps. In particular, sh is a homotopy equivalence with inverse AW'.

Proof. This is a consequence of Lemma 5.4. In particular, the image of $\overline{\mathsf{sh}}$ under $e_A \otimes e_B$ ☐

Remark 6.5. One can in fact observe that $\mathsf{sh}' = \mathsf{sh} \circ (e_A \otimes e_B) \circ (f_A \otimes f_B)$. Indeed, $\mathsf{sh} \circ (e_A \otimes e_B) \circ (f_A \otimes f_B) = (e_{A \otimes B}) \circ \mathsf{sh} \circ (f_A \otimes f_B) = \mathsf{sh}'$, and the homotopy relating sh and sh' is simply the homotopy $h_A \otimes h_B \colon (e_A \otimes e_B) \circ (f_A \otimes f_B) \sim \mathrm{id}$.

Lemma 6.6. Let $\vec{P} = (P_1, ..., P_n)\colon (A_1, ..., A_n) \to (B_1, ..., B_n)$ be an n-tuple of morphisms in Mor_k, i.e. for each i, P_i is a (B_i, A_i)-bimodule. We write $A = A_1 \otimes ... \otimes A_n$, $B = B_1 \otimes ... \otimes B_n$ and $P = P_1 \otimes ... \otimes P_n\colon A \to B$. Denote the induced maps on Hochschild complexes by

$$\vec{P}_*\colon C(\mathcal{P}(\vec{A})) \to C(\mathcal{P}(\vec{B}))$$

and

$$P_*\colon C(\mathcal{P}_A) \to C(\mathcal{P}_B).$$

Then the following diagram commutes up to homotopy.

$$\begin{tikzcd}
C(\mathcal{P}(\vec{A})) \arrow[r, "\vec{P}_*"] \arrow[d, "\mathsf{sh}_n"'] & C(\mathcal{P}(\vec{B})) \arrow[d, "\mathsf{sh}_n"] \\
C(\mathcal{P}_A) \arrow[r, "P_*"'] & C(\mathcal{P}_B)
\end{tikzcd}$$

Proof. We give the computation for the case $n = 2$ and at simplicial level 1. The general case is similar.

We begin by factoring sh_n as $(- \otimes -)_* \circ \overline{\mathsf{sh}}_n$ to expend the diagram as follows

$$\begin{tikzcd}
C(\mathcal{P}(\vec{A})) \arrow[r, "\vec{P}_*"] \arrow[d, "\overline{\mathsf{sh}}"'] & C(\mathcal{P}(\vec{B})) \arrow[d, "\overline{\mathsf{sh}}"] \\
C(\mathcal{P}_{A_1} \otimes \mathcal{P}_{A_2}) \arrow[r, "(P_1 \otimes P_2)_*"] \arrow[d, "(- \otimes -)_*"'] & C(\mathcal{P}_{B_1} \otimes \mathcal{P}_{B_2}) \arrow[d, "(- \otimes -)_*"'] \\
C(\mathcal{P}_A) \arrow[r, "(P_1 \otimes P_2)_*"] & C(\mathcal{P}_B)
\end{tikzcd}$$

Now the upper square commutes by naturality of $\overline{\mathsf{sh}}$, and we will produce a homotopy for the bottom square. The legs of the bottom square are both induced by functors

$$\mathcal{P}_{A_1} \otimes \mathcal{P}_{A_2} \to \mathcal{P}_B.$$

By exhibiting a natural isomorphism between them, we will obtain a homotopy by Lemma 5.11. Now the upper leg of the bottom diagram is induced by the functor

$$(M_1, M_2) \mapsto (P_1 \otimes_{A_1} M_1) \otimes (P_2 \otimes_{A_2} M_2)$$

and the lower leg is induced by the functor

$$(M_1, M_2) \mapsto (P_1 \otimes P_2) \otimes_{A_1 \otimes A_2} (M_1 \otimes M_2).$$

There is a canonical isomorphism

$$(P_1 \otimes_{A_1} M_1) \otimes (P_2 \otimes_{A_2} M_2) \to (P_1 \otimes P_2) \otimes_{A_1 \otimes A_2} (M_1 \otimes M_2).$$

This isomorphism is natural, such that given homomorphisms $f_1\colon M_1 \to N_1$ of left A_1-modules and $f_2\colon M_2 \to N_2$ of left A_2-modules, there is a commutative diagram

$$\begin{tikzcd}
(P_1 \otimes_{A_1} M_1) \otimes (P_2 \otimes_{A_2} M_2) \arrow[r, "(\mathrm{id}\otimes f_1)\otimes(\mathrm{id}\otimes f_2)"] \arrow[d, "\sim"'] & (P_1 \otimes_{A_1} N_1) \otimes (P_2 \otimes_{A_2} N_2) \arrow[d, "\sim"] \\
(P_1 \otimes P_2) \otimes_{A_1\otimes A_2} (M_1 \otimes M_2) \arrow[r, "(\mathrm{id}\otimes \mathrm{id})\otimes(f_1\otimes f_2)"] & (P_1 \otimes P_2) \otimes_{A_1\otimes A_2} (N_1 \otimes N_2)
\end{tikzcd}$$

which finishes the proof. □

Theorem 6.7. The Hochschild complex functor $C(-)\colon N_D\mathsf{Mor}_k \to N_\Delta\mathsf{Ch}_k$ of Theorem 5.22 extends to a symmetric monoidal functor of quasi-categories

$$\begin{tikzcd}
N_D\mathsf{Mor}_k^{\otimes} \arrow[r, "C(-)"] \arrow[dr, "p"'] & N_\Delta\mathsf{Ch}_k^{\otimes} \arrow[d, "q"] \\
& N\mathcal{F}in_*
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
N\mathsf{dgAlg}_k^{\otimes} \arrow[r, "N_D\mathsf{m}"] \arrow[dr, "C(-)"'] & N_D\mathsf{Mor}_k^{\otimes} \arrow[d, "C(-)"] \\
& N_\Delta\mathsf{Ch}_k^{\otimes}
\end{tikzcd}$$

is a homotopy commutative diagram of symmetric monoidal quasi-categories.

Proof. The component over $\langle n\rangle \in N\mathcal{F}in_*$ is given by applying the functor $C(-)\colon N_D\mathsf{Mor}_k \to N_\Delta\mathsf{Ch}_k$ of Theorem 5.22 componentwise, sending an n-tuple $(A_1, ..., A_n)$ in $N_D\mathsf{Mor}_k^{\otimes}$ to $(C(\mathcal{P}_{A_1}), ..., C(\mathcal{P}_{A_n})) \in N_\Delta\mathsf{Ch}_k^{\otimes}$. Using the description of morphisms of Lemma 2.29, a morphism $P\colon (A_1, ..., A_n) \to (B_1, ..., B_m)$ lying over a map $g\colon \langle n\rangle \to \langle m\rangle$ with components

$$(P_j\colon \otimes_{i\in g^{-1}(j)} A_i \to B_j)_{j=1,...,m}$$

is sent to the morphism

$$C(P)\colon (C(\mathcal{P}_{A_1}), ..., C(\mathcal{P}_{A_n})) \to (C(\mathcal{P}_{B_1}), ..., C(\mathcal{P}_{B_m}))$$

whose components are given by

$$C(P)_j\colon \otimes_{i\in g^{-1}(j)} C(\mathcal{P}_{A_i}) \xrightarrow{\mathsf{sh}} C(\mathcal{P}_{\otimes_i A_i}) \xrightarrow{C(P_j)} C(\mathcal{P}_{B_j})$$

We extend this assignment to a simplicial map $C(-)\colon N_D\mathsf{Mor}_k^{\otimes} \to N_\Delta\mathsf{Ch}_k^{\otimes}$. In simplicial degree 2, this looks as follows. The general case is similar. Let $[2] \to \mathsf{Mor}_k^{\otimes}$ be a normal 2-functor, corresponding to a diagram

$$\begin{tikzcd}
(A_1, ..., A_n) \arrow[r, "P"] \arrow[dr, "R"'] & (B_1, ..., B_m) \arrow[d, "Q"] \\
& (C_1, ..., C_l)
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle n \rangle \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & \langle m \rangle \arrow[d, "\beta"] \\
& \langle l \rangle
\end{tikzcd}$$

where the left triangle lies over the right triangle and commutes up to an isomorphism $\phi\colon Q \circ P \to R$. In $N_\Delta \mathsf{Ch}_k^\otimes$, we then have a diagram for each non-basepoint element $k \in \langle l \rangle$, in which the upper triangle commutes by associativity of shuffle maps, the square commutes up to homotopy by Lemma 6.6, and the lower triangle commutes up to homotopy as given by Lemma 5.11.

$$\begin{tikzcd}
\bigotimes_{i\in\alpha^{-1}(j)} \bigotimes_{j\in\beta^{-1}(k)} C(\mathcal{P}_{A_i}) \arrow[r, "\mathsf{sh}"] \arrow[dr, "\mathsf{sh}"'] & \bigotimes_{j\in\beta^{-1}(k)} C(\mathcal{P}_{\otimes_{i\in\alpha^{-1}(j)} A_i}) \arrow[r, "\otimes_{j\in\beta^{-1}(k)}(f_j)_*"] \arrow[d, "\mathsf{sh}"] & \bigotimes_{j\in\beta^{-1}(k)} C(\mathcal{P}_{B_j}) \arrow[d, "\mathsf{sh}"] \\
& C(\mathcal{P}_{\otimes_{i\in\gamma^{-1}(k)} A_i}) \arrow[r, "(\otimes_{j\in\beta^{-1}(k)} f_j)_*"] \arrow[dr, "(h_k)_*"'] & C(\mathcal{P}_{\otimes_{j\in\beta^{-1}(k)} B_j}) \arrow[d, "(g_k)_*"] \\
& & C(\mathcal{P}_{C_k})
\end{tikzcd}$$

In this way, the functoriality of $C(-)\colon N_D \mathsf{Mor}_k^\otimes \to N_\Delta \mathsf{Ch}_k^\otimes$ reduces to the functoriality of $C(-)\colon N_D \mathsf{Mor}_k \to N_\Delta \mathsf{Ch}_k$, which we know by Theorem 5.22.

Now, P is p-coCartesian if and only if each P_j is an equivalence in Mor_k (see Lemma 2.37), i.e. the $(B, j, \otimes_{i\in g^{-1}(j)} A_i)$-bimodule P_j is a Morita equivalence, such that the functor $(P_j \otimes_A -)\colon \mathcal{P}_{\otimes_{i\in g^{-1}(j)} A_i} \to \mathcal{P}_{B_j}$ induced by tensoring with P_j is an equivalence of categories, and finally this implies that the induced morphism $C(f_j)$ is a homotopy equivalence by Lemma 5.11. Since sh is a homotopy equivalence by Lemma 6.4, this implies that each $C(P)_j$ is a homotopy equivalence, hence $C(P)$ is coCartesian (ref. Example 2.31). We conclude that $C(-)$ preserves coCartesian edges, such that $C(-)$ is a symmetric monoidal functor.

Consider now the following diagram of categories over $N\mathcal{F}in_*$

$$\begin{tikzcd}
N\mathsf{dgAlg}_k^\otimes \arrow[r, "N_D \mathsf{m}"] \arrow[dr, "C(-)"'] & N_D \mathsf{Mor}_k^\otimes \arrow[d, "C(-)"] \\
& N_\Delta \mathsf{Ch}_k^\otimes
\end{tikzcd}$$

This diagram commutes over each $\langle n \rangle \in N\mathcal{F}in_*$ by Theorem 5.22. Let $g\colon \langle n \rangle \to \langle m \rangle$ be a map of finite pointed sets and let $f\colon (A_1, ..., A_n) \to (B_1, ..., B_m)$ be a morphism in $N\mathsf{dgAlg}_k^\otimes$ lying over g with components

$$(P_j\colon \otimes_{i\in g^{-1}(j)} A_i \to B_j)_{j=1,...,m}.$$

Now by Lemma 6.4 there is a commutative diagram where the upper and lower horizontal morphisms are the images of the f_j under the lower and upper legs of the diagram respectively.

$$\begin{tikzcd}
\otimes_{i\in g^{-1}(j)} C(A_i) \arrow[r, "\mathsf{sh}"] \arrow[d, "\otimes_{i\in g^{-1}(j)} e_{A_i}"] & C(\otimes_i A_i) \arrow[r, "C(f_j)"] \arrow[d, "e_{(\otimes_{i\in g^{-1}(j)} A_i)}"] & C(B_j) \arrow[d, "e_{B_j}"] \\
\otimes_{i\in g^{-1}(j)} C(\mathcal{P}_{A_i}) \arrow[r, "\mathsf{sh}"] & C(\mathcal{P}_{\otimes_i A_i}) \arrow[r, "C(B_{j,f_j})"] & C(\mathcal{P}_{B_j})
\end{tikzcd}$$

We conclude that e defined as natural equivalence between the upper and lower legs of the triangle, which finishes the proof.

□

7. Hochschild complexes of Hopf-like structures

Definition 7.1. Let $\mathcal{CH}opf$ be the prop of commutative Hopf algebras. A *commutative Hopfish algebra* over k is a symmetric monoidal 2-functor

$$\Phi\colon \mathcal{CH}opf \to \mathsf{Mor}_k$$

To expand on this definition, a commutative Hopfish algebra is the data of

- A k-algebra A,
- An $(A, A\otimes A)$-bimodule M,
- A left A-module η,
- An $(A\otimes A, A)$-bimodule Δ,
- A right A-module ϵ, and
- An (A, A)-bimodule S,

together with structure given by

- An isomorphism of $(A, A\otimes A\otimes A)$-bimodules
$$M \otimes_{A\otimes A} (A\otimes M) \cong M \otimes_{A\otimes A} (M\otimes A)$$
- An isomorphism of $(A\otimes A\otimes A, A)$-bimodules
$$(A\otimes \Delta)\otimes_{A\otimes A} \Delta \cong (\Delta\otimes A)\otimes_{A\otimes A}\Delta$$
- Isomorphisms of (A, A)-bimodules
$$M\otimes_{A\otimes A}(\eta\otimes A) \cong A \cong M\otimes_{A\otimes A}(A\otimes\eta)$$
$$(A\otimes\epsilon)\otimes_{A\otimes A}\Delta \cong A \cong (\epsilon\otimes A)\otimes_{A\otimes A}\Delta$$
$$M\otimes_{A\otimes A}(A\otimes S)\otimes_{A\otimes A}\Delta \cong \eta\otimes\epsilon \cong M\otimes_{A\otimes A}(S\otimes A)\otimes_{A\otimes A}\Delta$$
- An isomorphism of $(A\otimes A, A\otimes A)$-bimodules (where $\tau\colon A\otimes A\to A\otimes A$ is the twist map)
$$(M\otimes M)\otimes_{A^{\otimes 4}}(A\otimes \mathsf{m}(\tau)\otimes A)\otimes_{A^{\otimes 4}}(\Delta\otimes\Delta) \cong \Delta\otimes_A M$$
- An isomorphism of right $A\otimes A$-modules
$$\epsilon\otimes_A M \cong \epsilon\otimes\epsilon$$
- An isomorphism of k-modules
$$k \cong \epsilon\otimes_A \eta$$

such that these isomorphisms are compatible with one another in the sense defined by the 2-functoriality of Φ, to the effect that any pair of compositions built from these isomorphisms with the same source and target are equal.

Remark 7.2. Let Φ be a commutative Hopfish algebra and let $A = \Phi(1)$. In [30], a commutative Hopfish algebra is required to send the multiplication $2 \to 1$ in $\mathcal{CH}opf$ to $\mathsf{m}(A \otimes A \xrightarrow{\text{mult.}} A)$, the image of the algebra multiplication under the modulation 2-functor.

Example 7.3. • If A is a commutative Hopf dg-algebra, then A arises as the image of 1 of a symmetric monoidal functor $\Phi\colon \mathcal{CH}opf \to \mathsf{dgAlg}_k$, so the composition by the modulation

$$\mathcal{CH}opf \xrightarrow{\Phi} \mathsf{dgAlg}_k \xrightarrow{\mathsf{m}} \mathsf{Mor}_k$$

is a commutative Hopfish algebra.

• For an example of a commutative Hopfish algebra which is not Morita equivalent to a Hopf algebra, see [30, Section 6], in particular [30, Example 6.11].

There is a variation on Hopf algebras called quasi-Hopf algebras, where the co-associativity of the coalgebra structure is weakened.

Definition 7.4. • [8, Page 116] A *quasi-bialgebra* is a unital associative algebra (A, m, η) equipped with algebra homomorphisms $\Delta\colon A \to A \otimes A$ and $\epsilon\colon A \to k$, and an invertible element $\phi \in A \otimes A \otimes A$ such that

(1) for all $a \in A$, $(\Delta \otimes \mathrm{id})\Delta(a) = \phi(\mathrm{id} \otimes \Delta)\Delta(a)\phi^{-1}$,
(2) $(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (\mathrm{id} \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi \otimes \mathrm{id})$,
(3) $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta$,
(4) $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = 1 \in k$.

• [8, Page 119] A quasi-bialgebra $(A, m, \eta, \Delta, \epsilon, \phi)$ is a *quasi-Hopf algebra* if there is an anti-homomorphism $S\colon A \to A$ and elements $\alpha, \beta \in A$ such that for all $a \in A$,

$$\sum S(a_1)\alpha a_2 = \epsilon(a)\alpha$$

$$\sum a_1 \beta S(a_2) = \epsilon(a)\beta$$

where we use Sweedler notation $\Delta(a) = \sum a_1 \otimes a_2$.

The following is a special case of a theorem of Tang-Weinstein-Zhu.

Theorem 7.5. ([30, Theorem 4.5] If $(A, m, \eta, \Delta, \epsilon, \phi, S, \alpha, \beta)$ is a commutative quasi-Hopf algebra, then the modulation $(A, \mathsf{m}(m), \mathsf{m}(\eta), \mathsf{m}(\Delta), \mathsf{m}(\epsilon), \mathsf{m}(S))$ is a commutative Hopfish algebra.

We now have some examples of algebraic structures on Hochschild complexes of Hopf-like algebras.

Proposition 7.6. (1) Let A be a commutative Hopfish algebra, then $C(A)$ carries a commutative Hopf algebra structure up to coherent homotopy. In particular this is the case if A is a commutative (quasi-)Hopf algebra.
(2) Let A be a (commutative) quasi-bialgebra, then $C(A)$ carries a coalgebra structure (resp. bialgebra structure) up to coherent homotopy.

Proof. (1) Since A is a commutative Hopfish algebra, there is a symmetric monoidal 2-functor $\Phi\colon \mathcal{CH}opf \to \mathsf{Mod}_k$ such that $A = \Phi(1)$. Then post-composition with the Hochschild chain functor gives a symmetric monoidal functor of quasi-categories

$$C(\Phi)\colon N\mathcal{CH}opf \to N_\Delta \mathsf{Ch}_k$$

such that $C(\Phi)(1) \simeq C(\Phi(1)) \simeq C(A)$. In other words, $C(A)$ is a Hopf algebra object in $N_\Delta \mathsf{Ch}_k$, which is precisely a Hopf algebra up to coherent homotopy.

(2) Let $f, g \colon A \to B$ be algebra homomorphisms. Then $\mathsf{m}(f) = B_f$ and $\mathsf{m}(g) = B_g$ if and only if are isomorphic there exists an invertible element $\psi \in B$ such that $f(a) = \psi g(a) \psi^{-1}$ for all $a \in A$ (see [30, Lemma 2.1]). For a quasi-bialgebra $(A, m, \eta, \Delta, \epsilon, \phi)$, let

$$\gamma \colon (A \otimes A)_{\phi(\mathrm{id}\otimes\Delta)\Delta(a)\phi^{-1}} \to (A \otimes A)_{(\mathrm{id}\otimes\Delta)\Delta(a)}$$

be the canonical isomorphism. Then $(A, \mathsf{m}(\Delta), \mathsf{m}(\epsilon), \gamma)$ define a symmetric monoidal 2-functor

$$\mathcal{A}ss^{op} \to \mathsf{Mor}_k$$

from the opposite category of the prop of associative algebras to the Morita (2,1)-category. Post-composing with the Hochschild complex functor gives a symmetric monoidal functor of quasi-categories

$$\Phi \colon N\mathcal{A}ss^{op} \to N_\Delta \mathsf{Ch}_k$$

for which $\Phi(1) \simeq C(A)$.

If A is a commutative algebra, then we get a coherent bialgebra structure on $C(A)$ be a similar argument. □

Remark 7.7. A special case of Proposition 7.6 is when the Hopfish algebra is the modulation of a commutative Hopf algebra. In this case, the author gave a strictification of the commutative Hopf algebra structure on $C(A)$ in [25].

In some cases we do not need to demand that our algebras are commutative to obtain product structures on Hochschild complexes.

Definition 7.8.
- For any dg-k-module A, let $p_{12}, p_{23}, p_{13} \colon A^{\otimes 2} \to A^{\otimes 3}$ be given by
  - $p_{12}(a \otimes b) = a \otimes b \otimes 1$,
  - $p_{23}(a \otimes b) = 1 \otimes a \otimes b$,
  - $p_{13}(a \otimes b) = a \otimes 1 \otimes b$.

  For any element $R \in A \otimes A$, write $R_{ij} := p_{ij}(R)$. Also, write $\tau \colon A \otimes A \to A \otimes A$ for the twist morphism $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$.
- A bialgebra (or Hopf algebra) A is called *quasi-triangular* if there is an invertible element $R \in A \otimes A$ such that for any $a \in A$,
  - $\tau \circ \Delta(a) = R\Delta(a)R^{-1}$,
  - $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,
  - $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

  A is called *triangular* if in addition $\tau(R) = R^{-1}$.

Proposition 7.9. ([7, Section 10]) If A is a quasitriangular (resp. triangular) Hopf algebra, then the module category $\mathcal{P}_A$ is braided monoidal (resp. symmetric monoidal).

Corollary 7.10. If A is a quasi-triangular (resp. triangular) bialgebra or Hopf algebra, then the Hochschild complex $C(A)$ is an $\mathbb{E}_2$-algebra (resp. an $\mathbb{E}_\infty$-algebra) in Ch_k.

Proof. We prove the quasi-triangular case. The triangular case is similar, replacing all instances of $\mathbb{E}_2$ by $\mathbb{E}_\infty$. Writing Cat_∞ for the quasi-category of quasi-categories, the full subcategory on nerves of ordinary categories is equivalent, as a quasi-category, to the Duskin nerve of the (2,1)-category Cat. By [21, Example 1.2.4], the braided monoidal structure on $\mathcal{P}_A$ can therefore be expressed as a symmetric monoidal 2-functor

$$\mathbb{E}_2 \to N_\Delta \mathsf{dgCat}_{k,\Delta}^{(2,1)}$$

which gives a $\mathbb{E}_2$-algebra structure on $C(\mathcal{P}_A)$. The $\mathbb{E}_2$-structure on $C(A)$ then follows by Lemma 2.43. □

Conjecture 7.11. If A is a bialgebra, then $C(A)$ carries a coalgebra structure. How does the algebraic structure of Corollary 7.10 interact with this coalgebraic structure? In particular:

- If A is a quasi-triangular (resp. triangular) bialgebra, is the Hochschild complex $C(A)$ an $\mathbb{E}_2$-commutative bialgebra (resp. an $\mathbb{E}_\infty$-commutative bialgebra) in Ch_k?
- If A is a quasi-triangular (resp. triangular) Hopf algebra, is the Hochschild complex $C(A)$ an $\mathbb{E}_2$-commutative Hopf algebra (resp. an $\mathbb{E}_\infty$-commutative Hopf algebra) in Ch_k?

References

[1] Jean Bénabou, *Introduction to Bicategories*, Reports of the Midwest Category Seminar, Lecture Notes in Mathematics, vol 47. Springer, Berlin, Heidelberg.

[2] Julia E. Bergner, *Three models for the homotopy theory of homotopy theories*, Topology 46 (2007), 397-436.

[3] Mitchell Buckley, *Fibered 2-categories and bicategories*, Journal of Pure and Applied Algebra, Volume 218, Issue 6, June 2014, Pages 1034–1074

[4] Michael Boardman, Rainer Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, (1973).

[5] Antonio M. Cegarra, Benjamín A. Heredia, *Geometric Realizations of Tricategories*, Algebr. Geom. Topol. 14, 2014, 1997–2064.

[6] Albrecht Dold, *Über die Steenrodschen Kohomologieoperationen*, Ann. of Math. 73, 1961, 258–294.

[7] V. G. Drinfel'd, *Quantum groups*, Proc. ICM 1986, Vol. 1, 2 798—820, AMS 1987.

[8] V. G. Drinfeld, *Quasi-Hopf algebras*, Algebra i Analiz, 1989, Volume 1, Issue 6, 114-–148

[9] Daniel Dugger and David I. Spivak *Mapping Spaces in Quasi-Categories* Algebr. Geom. Topol. 11 (2011) 263-325

[10] John Duskin, *Simplicial matrices and the nerves of weak n-categories I: nerves of bicategories*, Theory and Applications of Categories, Vol. 9, No. 10, 2002, pp. 198-–308.

[11] Samuel Eilenberg and Saunders Mac Lane, *On the Groups $H(\Pi, n)$, I*, Annals of Mathematics, Vol. 58, No. 1 (Jul., 1953), pp. 55–106.

[12] Samuel Eilenberg and Saunders MacLane, *On the Groups $H(\Pi, n)$, II: Methods of Computation*, Annals of Mathematics, Vol. 60, No. 1 (Jul., 1954), pp. 49–139.

[13] Paul Goerss and Rick Jardine, *Simplicial homotopy theory*, number 174 in Progress in Mathematics, 1999, Birkhäuser.

[14] John W. Gray, *Fibred and Cofibred Categories*, pp.21-83 in Eilenberg, Harrison, MacLane, Röhrl (eds.), Proceedings of the Conference on Categorical Algebra - La Jolla 1965 , Springer Heidelberg 1966.

[15] Morirz Groth, *A short course on ∞-categories*, 21 Jan 2015

[16] Nick Gurski and Angélica M. Osorno, *Infinite loop spaces, and coherence for symmetric monoidal bicategories*, Adv.Math. 246, 2013, 1–32.

[17] Andre Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, 277–326, Contemp. Math. 431, Amer. Math. Soc., Providence, RI, 2007.

[18] Bernhard Keller, *Invariance and Localization for Cyclic Homology of DG Algebras*, Journal of Pure and Applied Algebra, Volume 123, Issues 1–3, January 1998, Pages 223–273.

[19] Jean-Louis Loday, *Cyclic Homology*, Springer Verlag, 1992.

[20] Jacob Lurie, *Derived Algebraic Geometry III: Commutative Algebra* arXiv:math/0703204v4, 7 Mar 2007

[21] Jacob Lurie, *Derived Algebraic Geometry VI: E_k Algebras* arXiv:0911.0018, 30 Oct 2009.

[22] Jacob Lurie, *Higher Topos Theory*, Princeton University Press, Princeton and Oxford, 2009

[23] Jacob Lurie, *Higher Algebra*, ver. 16 May 2016.

[24] Randy McCarthy, *The cyclic homology of an exact category*, Journal of Pure and Applied Algebra 93 (1994) 251–296.

[25] Espen Auseth Nielsen, *Strictifying Homotopy Coherent Actions on Hochschild Complexes*, arXiv:1805.12466, 31 May 2018.

[26] Birgit Richter, *E_∞-structure for $Q_*(R)$*, Math. Ann. 316, 547-564 (2000).

[27] Birgit Richter, *Symmetry Properties of the Dold-Kan Correspondence*, Mathematical Proceedings of the Cambridge Philosophical Society, 134(1), pp. 95—102, 2003.

[28] Graeme Segal, *Categories and Cohomology Theories*, Topology Vol. 13, pp. 293–312

[29] Michael A. Shulman, *Constructing symmetric monoidal bicategories* arXiv:1004.0993v1, 7 Apr 2010.

[30] Xiang Tang, Alan Weinstein, Chenchang Zhu, *Hopfish algebras*, arxiv:math/0510421v2, 2008.

[31] Bertrand Toën and Michel Vaquié, *Moduli of Objects in DG-Categories*, Ann. Scient. Éc. Norm. Sup., 4e série, t. 40, 2007, p. 387–444.

Part 4

Paper C

FORMAL OPERATIONS ON TOPOLOGICAL HOCHSCHILD HOMOLOGY (WORK IN PROGRESS)

ESPEN AUSETH NIELSEN

Note: The present paper is a work in progress, and as such contains unfinished arguments, loose ends and probably errors.

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Contents

1. Introduction

In this paper we define formal operations on topological Hochschild homology and more generally on factorization homology of structured ring spectra. We use this to obtain extension results for 2D topological quantum field theories generalizing results of Costello [5] and Wahl-Westerland [21].

This work is an extension of the program of Wahl [20] to the setting of ∞-categories. This also extends the results of Klamt [12], who generalized the framework of formal operations to a model categorical setting and obtained a special case of Theorem 2.28 for chain complexes.

Our main tool is the theory of ∞-categories, which is necessary to define and analyze the formal operations. When performing calculations on TQFTs in Section 3, we will relate the homotopy theory of Hochschild functors to moduli spaces of manifolds, which is our second technical tool.

The open-closed cobordism category $\mathcal{OC}$ is the ∞-category whose objects are one-dimensional manifolds with boundary, and whose space of morphisms $\mathcal{OC}(M,N)$ is the moduli space

$$\mathcal{OC}(M,N) = \coprod_{\substack{[\Gamma] \\ \bar{M}\sqcup N \hookrightarrow \partial\Gamma}} B\mathrm{Diff}(\Gamma; \bar{M} \sqcup N)$$

i.e. the moduli space of Riemannian cobordisms with corners from M to N, and composition is given by gluing cobordisms. The open cobordism category is the full subcategory of $\mathcal{OC}$ on the objects $\sqcup_n \mathbb{R}$. $\mathcal{O}$ and $\mathcal{OC}$ are both symmetric monoidal ∞-categories under disjoint union.

Let Sp be the ∞-category of spectra. An open (resp. open-closed) conformal field theory is a symmetric monoidal functor $\mathcal{O} \to \mathsf{Sp}$ (resp. $\mathcal{OC} \to \mathsf{Sp}$).

The main goal of this paper is to prove the following claim, which is a spectral version of [21, Theorem 6.2].

Main Claim: (Theorem (3.26) Let $\Phi : \mathcal{O} \to \mathsf{Sp}$ be an open conformal field theory. Then there is a natural open-closed conformal field theory $\widetilde{\Phi} : \mathcal{OC} \to \mathsf{Sp}$ such that $\widetilde{\Phi}|_{\mathcal{O}} \simeq \Phi$ and $\widetilde{\Phi}(S^1) \simeq THH(\Phi(1))$.

In order to prove this claim, we introduce an ∞-categorical version of Wahl's notion of formal operations on Hochschild homology [20]. For a symmetric monoidal ∞-category P admitting a symmetric monoidal functor $\mathcal{A}_\infty \to P$, we can consider THH of P-algebras by restricting the P-algebra structure to the $\mathcal{A}_\infty$-algebra structure and applying the cyclic bar construction. This gives rise to a functor

$$THH : \mathrm{Fun}^{\otimes}(P, \mathsf{Sp}) \to \mathsf{Sp}$$

Natural operations on THH of P-algebras are then defined to be morphisms of the form

$$THH(\Phi)^{\otimes n_1} \otimes (\Phi(1))^{\otimes m_1} \to THH(\Phi)^{\otimes n_2} \otimes (\Phi(1))^{\otimes m_2}$$

which are natural in $\Phi \in \mathrm{Fun}^{\otimes}(P, \mathsf{Sp})$. In fact the above procedure does not require the functor $P \to \mathsf{Sp}$ to be symmetric monoidal, and se can still define THH of functors $\Phi : P \to \mathsf{Sp}$ which are not symmetric monoidal. Roughly speaking, operations on this kind of THH are called formal operations.

The paper is structured as follows. In Section 2 we recall some ∞-categorical machinery and define Hochschild functors and their formal operations. We then give a technical result on how to compute formal operations. Finally, we relate the formal operations to natural operations of Hochschild functors. In Section 3, we relate our Hochschild functors to factorization homology and apply the results of Section 2 to the problem of universally extending open conformal field theories to open-closed conformal field theories.

2. Hochschild functors and their formal operations

2.1. **Review of ends and coends.** We follow [10].

Assumption 2.1. Throughout this section, let $\mathcal{C}$ be a presentable closed symmetric monoidal ∞-category in the sense of [13, Definition 5.5.0.1],[14, Definition 2.0.0.7].

Definition 2.2. Given a small ∞-category I, the *twisted arrow category* $\widetilde{\mathcal{O}}_I$ is given by $(\widetilde{\mathcal{O}}_I)_n = I_{2n+1}$ with face and degeneracy maps

$$\widetilde{d}_i x = d_{n-i} d_{n+1+i} x$$
$$\widetilde{s}_i x = s_{n-i} s_{n+1+i} x$$

The opposite category $(\widetilde{\mathcal{O}}_I)^{op}$ is given the special notation $\widetilde{\mathcal{O}}^I$. We define the forgetful functor $\mathcal{H}_I : \widetilde{\mathcal{O}}_I \to I^{op} \times I$ taking $x \in (\widetilde{\mathcal{O}}_I)_n$ to $x|_{[0,n]} \times x|_{[n+1,2n+1]}$, and write $\mathcal{H}^I := (\mathcal{H}_I)^{op}$.

Definition 2.3. Let $T : I \times I^{op} \to \mathcal{C}$ be a functor. The *coend* of T is the colimit

$$\int^I T := \operatorname*{colim}_{\widetilde{\mathcal{O}}^I} T \circ \mathcal{H}^I$$

Dually, let $T' : I^{op} \times I \to \mathcal{C}$ be a functor. The *end* of T' is the limit

$$\int_I T := \lim_{\widetilde{\mathcal{O}}_I} T' \circ \mathcal{H}_I$$

Definition 2.4. Given a pair of functors $F : I \to \mathcal{C}$ and $G : I^{op} \to \mathcal{C}$, we obtain a functor $F \boxtimes G$ by the composition

$$I \times I^{op} \xrightarrow{F \times G} \mathcal{C} \times C \xrightarrow{\otimes} \mathcal{C}$$

and we write

$$F \otimes_I G := \int^I F \boxtimes G$$

Notation 2.5. To eliminate confusion, we distinguish between two Hom-space notations. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Denote the Hom-space functor $\mathrm{Hom}_{\mathcal{D}}(-,-) : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{S}$. We write $\mathrm{Hom}_{\mathcal{D}}(F, G) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$ for the composition $\mathrm{Hom}_D \circ F^{op} \times G$. On the other hand, by $\mathrm{Nat}_{\mathcal{C}}(F, G)$ we mean the space of natural transformations from F to G.

Proposition 2.6. ([10, Proposition 2.3], [7, Proposition 5.1]) Let $\mathcal{D}$ be an ∞-category and let $F, G : I \to \mathcal{D}$ be functors. Then there is a natural equivalence

$$\int_I \mathrm{Hom}(F(-), G(-)) \simeq \mathrm{Nat}_I(F, G)$$

Lemma 2.7. Let $\mathcal{C}$ be a small ∞-category, let $\mathcal{D}$ be a presentable ∞-category and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C}^{op} \to \mathcal{D}$ be functors. Let $S(F, G)$ be the colimit of the simplicial diagram given by

$$[n] \mapsto \operatorname*{colim}_{\alpha \in \mathrm{Map}([n], \mathcal{C})} F(\alpha(0)) \otimes G(\alpha(n))$$

where $\mathrm{Map}([n], \mathcal{C})$ is the maximal sub-∞-groupoid of $\mathrm{Fun}([n], \mathcal{C})$. Then there is a homotopy equivalence

$$F \otimes_{\mathcal{C}} G \simeq S(F, G)$$

Proof. See Appendix A. □

Definition 2.8. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{D}$ be a presentable ∞-category and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C}^{op} \to \mathcal{D}$ be functors. Let $B(F, \mathcal{C}, G)_\bullet$ be the simplicial object given by

$$[n] \mapsto \bigoplus_{c_0, \ldots, c_n \in \pi_0 \mathcal{C}} F(c_0) \otimes \mathcal{C}(c_0, c_1) \otimes \ldots \otimes \mathcal{C}(c_{n-1}, c_n) \otimes G(c_n)$$

Write $B(F, \mathcal{C}, G)$ for the colimit of this simplicial diagram.

Lemma 2.9. Let $\mathcal{C}$ be the nerve of an ordinary category, let $\mathcal{D}$ be a presentable ∞-category and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C}^{op} \to \mathcal{D}$ be functors. Then there is a natural equivalence

$$S(F, G) \xrightarrow{\sim} B(F, \mathcal{C}, G)$$

Proof. Since $\mathcal{C}$ has unique horn fillers above dimension 1, the indexing space $\mathrm{Map}([n], \mathcal{C})$ decomposes as

$$\mathrm{Map}([n], \mathcal{C}) \simeq \coprod_{c_0,\dots,c_n \in \pi_0 \mathcal{C}} C(c_0, c_1) \times \dots \times C(c_{n-1}, c_n)$$

such that $S(F, G)_n$ splits as

$$S(F, G)_n \simeq \bigoplus_{c_0,\dots,c_n \in \pi_0 \mathcal{C}} \operatorname*{colim}_{\substack{f_i \in \mathcal{C}(c_i, c_{i+1}) \\ i=0,\dots,n-1}} F(c_0) \otimes G(c_n).$$

Since the latter colimit is over a constant object, it is equivalent to tensoring by the indexing space $\mathcal{C}(c_0, c_1) \times \dots \times \mathcal{C}(c_{n-1}, c_n)$, such that

$$S(F, G)_n \simeq \bigoplus_{c_0,\dots,c_n \in \pi_0 \mathcal{C}} F(c_0) \otimes \mathcal{C}(c_0, c_1) \otimes \dots \otimes \mathcal{C}(c_{n-1}, c_n) \otimes G(c_n)$$

and the claim follows. $\square$

Lemma 2.10. [13, Proposition 5.1.2.3] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let X be an object of $\mathcal{D}$. Then whenever F admits a (co)limit, there are equivalences, natural in X and F, such that

$$\mathrm{Hom}_{\mathcal{D}}(X, \lim_{\mathcal{C}} F) \simeq \lim_{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(X, F(-))$$

$$\mathrm{Hom}_{\mathcal{D}}(\operatorname*{colim}_{\mathcal{C}} F, X) \simeq \lim_{\mathcal{C}^{op}} \mathrm{Hom}_{\mathcal{D}}(F(-), X)$$

2.2. **The Stable Yoneda Lemma.**

Remark 2.11. Despite the literature on spectral ∞-categories being lacking at the time of writing, there is a rich theory of *stable* ∞-categories, which carry a natural spectral enrichment as a result of their internal structure, analogous to the natural enrichment in abelian groups of ordinary additive categories. In this framework, a spectrally enriched functor corresponds to a functor preserving finite limits. Recall that a functor $\mathcal{C} \to \mathcal{D}$ of stable ∞-categories preserves finite limits if and only if they preserve finite colimits ([14, Proposition 1.1.4.1]). In particular this implies that for $B : \mathcal{E}^{op} \to \mathsf{Sp}$ a functor, the coend functor $(-) \otimes_{\mathcal{E}} B : \mathrm{Fun}(\mathcal{E}, \mathsf{Sp}) \to \mathsf{Sp}$ preserves finite limits.

Recollection 2.12. Let $\mathcal{C}$ be a stable ∞-category. Recall that then $\mathcal{C}^{op}$ is also stable, and the functor

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{op}, \mathsf{Sp}) \xrightarrow{\Omega^\infty} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{op}, \mathcal{S})$$

is an equivalence of ∞-categories ([14, Corollary 1.4.2.23]).

Definition 2.13. ([18, Definition 6.1]) Let $\mathcal{C}$ be a stable ∞-category. A functor $F : \mathcal{C}^{op} \to \mathsf{Sp}$ is called representable if it preserves finite limits and the composite $\mathcal{C}^{op} \xrightarrow{F} \mathsf{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$ is representable. If c is a representing object of $\Omega^\infty F$, we say that c represents F.

Lemma 2.14. (Stable Yoneda Lemma, [18, Proposition 6.3 and Remark 6.4]) Let $\mathcal{C}$ be a stable ∞-category. The Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ lifts through Ω^∞ to a fully faithful embedding $\mathcal{C} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{op}, \mathsf{Sp})$. Let $F, G : \mathcal{C} \to \mathsf{Sp}$ be functors such that F is represented by $c \in \mathcal{C}$ and G preserves finite limits. Then

$$\mathrm{Nat}_{\mathcal{C}^{op}}(F, G) \simeq G(c)$$

Corollary 2.15. Let $\mathcal{C}$ be a stable ∞-category and let $F, G : \mathcal{C} \to \mathsf{Sp}$ be functors preserving finite limits. Viewing $\mathrm{Hom}(F, G)$ and a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Sp}$, we have

$$\int_{\mathcal{O}_\mathcal{C}} \mathrm{Hom}(F, G) \simeq \mathrm{Nat}_\mathcal{C}(F, G) \in \mathsf{Sp}$$

Proof. This follows because the two functors $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathsf{Sp})^{op} \times \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathsf{Sp}) \to \mathsf{Sp}$ both preserve finite limits and are equivalent under Ω^∞. $\square$

Observation 2.16. Let $K(n)$ denote the n'th Morava K-theory at an implicit prime, and let $L_{K(n)}\mathsf{Sp}$ denote the category of spectra localized at $K(n)$ and let $i : L_{K(n)}\mathsf{Sp} \to \mathsf{Sp}$ be the (fully faithful) right adjoint of the localization functor. Representable functors $L_{K(n)}\mathsf{Sp}^{op} \to \mathsf{Sp}$ factor through $L_{K(n)}\mathsf{Sp}$, so in the stable Yoneda lemma for $L_{K(n)}\mathsf{Sp}$, we may replace Sp by $L_{K(n)}\mathsf{Sp}$ as follows.

Lemma 2.17. The Yoneda embedding $j : L_{K(n)}\mathsf{Sp} \to \mathrm{Fun}(L_{K(n)}\mathsf{Sp}^{op}, \mathcal{S})$ lifts through $\Omega^\infty \circ i$ to a fully faithful embedding $L_{K(n)}\mathsf{Sp} \to \mathrm{Fun}^{\mathrm{lex}}(L_{K(n)}\mathsf{Sp}^{op}, L_{K(n)}\mathsf{Sp})$. Let $F, G : L_{K(n)}\mathsf{Sp} \to L_{K(n)}\mathsf{Sp}$ be functors such that F is represented by $c \in \mathcal{C}$ and G preserves finite limits. Then

$$\mathrm{Nat}_{L_{K(n)}\mathsf{Sp}^{op}}(F, G) \simeq G(c)$$

2.3. **Hochschild and coHochschild functors.** In this section we lay out the general framework of Hochschild and coHochschild functors, which will be the object of study for the remainder of the paper. For technical reasons, the treatment of the topological and spectral cases need to be separated. The structure of this section will be such that each statement about topological (co)Hochschild functors will be immediately followed by its spectral analogue. The proofs of the latter will only be given at the points where the argument differs from the topological proof.

Notation 2.18. Let $\mathcal{S}$ be the ∞-category of spaces, i.e. the underlying ∞-category of sSet with the Kan-Quillen model structure.

Notation 2.19. In the following, $\mathcal{C}$ will refer to one of $\mathcal{S}, \mathsf{Sp}$ or $L_{K(n)}\mathsf{Sp}$ (or in general any presentably closed symmetric monoidal ∞-category for which the Yoneda lemma holds). If $\mathcal{C}$ is fixed and $\mathcal{E}$ is an ∞-category, then by the functor $\mathcal{E}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C}$ we mean the representable bifunctor composed with the functor $\mathcal{C} \to \mathcal{S}$ given by

$$\begin{cases} \mathrm{id} & \text{, if } \mathcal{C} = \mathcal{S} \\ \Sigma^\infty(\text{infinite suspension}) & \text{, if } \mathcal{C} = \mathsf{Sp} \\ \Sigma^\infty_{K(n)} & \text{, if } \mathcal{C} = L_{K(n)}\mathsf{Sp} \end{cases}.$$

Definition 2.20. Let $\mathcal{E}$ and $\mathcal{E}'$ be ∞-categories equipped with a functor $i : \mathcal{E} \to \mathcal{E}'$. Let $B : \mathcal{E}^{op} \to \mathcal{C}$ be a functor. The Hochschild functor $C_B(-) : \mathrm{Fun}(\mathcal{E}', \mathcal{C}) \to \mathcal{C}$ sends a functor $\Phi : \mathcal{E}' \to \mathcal{C}$ to

$$C_B(\Phi) := i^*\Phi \otimes_\mathcal{E} B$$

Definition 2.21. Let $B, B' : \mathcal{E}^{op} \to \mathcal{C}$ be functors. The formal operations from B to B' are given by the mapping object

$$\mathrm{Hoch}_{\mathcal{E}'}(B, B') := \mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',\mathcal{C})}(C_B(-), C_{B'}(-))$$

Lemma 2.22. Let $\mathcal{E}$ be a small ∞-category, let $\mathcal{D}$ be a closed symmetric monoidal ∞-category and let $F : \mathcal{E} \to \mathcal{D}$ and $G : \mathcal{E}^{op} \to \mathcal{D}$ be functors such that the coend $F \otimes_{\mathcal{E}} G$ exists in $\mathcal{D}$. Then for every object X in $\mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(F \otimes_{\mathcal{E}} G, X) \simeq \mathrm{Nat}_{\mathcal{E}}(F, \mathrm{Hom}(G, X))$$

where $\mathrm{Hom}(G, X) : \mathcal{E} \to \mathcal{D}$ takes $Y \in \mathcal{E}$ to $\mathrm{Hom}_{\mathcal{D}}(G(Y), X)$.

Proof. We have a sequence of natural equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(F \otimes_{\mathcal{E}} G, X) &\simeq \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_{\mathcal{O}^{\mathcal{E}}}(F \boxtimes G \circ \mathcal{H}^{\mathcal{E}}), X)\\ &\simeq \lim_{\mathcal{O}_{\mathcal{E}}} \mathrm{Hom}_{\mathcal{D}}(F \boxtimes G \circ \mathcal{H}^{\mathcal{E}}, X)\\ &\simeq \lim_{\mathcal{O}_{\mathcal{E}}} \mathrm{Hom}_{\mathcal{D}}(F \boxtimes G, X) \circ \mathcal{H}_{\mathcal{E}}\\ &\simeq \lim_{\mathcal{O}_{\mathcal{E}}} \mathrm{Hom}_{\mathcal{D}}(G, \mathrm{Hom}(F, X)) \circ \mathcal{H}_{\mathcal{E}}\\ &\simeq \mathrm{Nat}_{\mathcal{E}^{op}}(G, \mathrm{Hom}(F, X))\end{aligned}$$

□

Corollary 2.23. Let $\mathcal{E}$ and $\mathcal{E}'$ be a small ∞-categories equipped with a functor $i : \mathcal{E} \to \mathcal{E}'$. Moreover, let $B : \mathcal{E}^{op} \to \mathcal{C}$ and $H : \mathrm{Fun}(\mathcal{E}, \mathcal{S}) \to \mathcal{C}$ be functors. Then we have an equivalence

$$\mathrm{Nat}((-) \otimes_{\mathcal{E}} B, H(-)) \xrightarrow{\sim} \mathrm{Nat}(B, \mathrm{Hom}(-, H(-)))$$

of functors $\mathrm{Fun}(\mathcal{E}, \mathcal{C})^{op} \times \mathrm{Fun}(\mathcal{E}, \mathcal{C}) \to \mathcal{C}$. Here $\mathrm{Hom}(-, H(-)) : \mathcal{E}^{op} \to \mathcal{C}$ takes $e \in \mathcal{E}$ to $\mathrm{Nat}_{\mathrm{Fun}(\mathcal{E},\mathcal{C})}(-(e), H(-))$.

Proof. This follows since each equivalence in the proof of Lemma 2.22 is natural in F, G and X. □

Lemma 2.24. Let $\mathcal{E}$ and $\mathcal{E}'$ be a small ∞-categories equipped with a functor $i : \mathcal{E} \to \mathcal{E}'$ and let $B : \mathcal{E}^{op} \to \mathcal{S}$ be a functor. Let e' be an object of $\mathcal{E}'$. Then there is an equivalence

$$\mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',\mathcal{S})}(-(e'), i^*(-) \otimes_{\mathcal{E}} B) \simeq \mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B$$

Proof. Let $\Phi : \mathcal{E}' \to \mathcal{S}$. Then, by the universal property of mapping objects, we obtain a map

$$\mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B \otimes \Phi(e') \simeq \operatorname*{colim}_{\mathcal{O}^{\mathcal{E}}} \left(\mathcal{E}'(e', i(-)) \boxtimes B \circ \mathcal{H}^{\mathcal{E}}\right) \otimes \Phi(e')$$

$$\simeq \operatorname*{colim}_{\mathcal{O}^{\mathcal{E}}} ((\mathcal{E}'(e', i(-)) \otimes \Phi(e')) \boxtimes B) \circ \mathcal{H}^{\mathcal{E}} \xrightarrow{\bar{F}_{\Phi}} \operatorname*{colim}_{\mathcal{O}^{\mathcal{E}}} (i^*\Phi(-) \boxtimes B) \circ \mathcal{H}^{\mathcal{E}}$$

$$\simeq i^*\Phi \otimes_{\mathcal{E}} B$$

Taking the adjoint of this map, we get a map

$$F_{\Phi} : \mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B \to \mathrm{Hom}(\Phi(e'), i^*\Phi \otimes_{\mathcal{E}} B)$$

Since all the maps involved are natural with respect to Φ, this assembles into a map

$$F : \mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B \to \mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',\mathcal{S})}(-(e'), i^*(-) \otimes_{\mathcal{E}} B)$$

Going the other way, evaluating at $\mathcal{E}'(e', -)$ we get a map

$$\mathrm{eval}_{\mathcal{E}'(e',-)} : \mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',\mathcal{S})}(-(e'), i^*(-) \otimes_{\mathcal{E}} B) \to \mathrm{Hom}_{\mathcal{S}}(\mathcal{E}'(e', e'), \mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B)$$

Denote by Id the unit map $\Delta^0 \xrightarrow{\mathrm{Id}} \mathcal{E}'(e',e')$ picking out the identity of e'. Evaluating at $\mathrm{id}_{e'}$ gives a map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}}(\mathcal{E}'(e',e'),\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B) &\simeq \mathrm{Hom}_{\mathcal{S}}(\mathcal{E}'(e',e'),\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B)\otimes\Delta^0\\ &\xrightarrow{\mathrm{id}\otimes\mathrm{Id}} \mathrm{Hom}_{\mathcal{S}}(\mathcal{E}'(e',e'),\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B)\otimes\mathcal{E}'(e',e') \xrightarrow{ev} \mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B\end{aligned}$$

We call G the composition

$$G = ev\circ(\mathrm{eval}_{\mathcal{E}'(e',-)}\otimes\mathrm{Id}) : \mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',\mathcal{S})}(-(e'), i^*(-)\otimes_{\mathcal{E}} B) \to \mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B$$

We now show that F and G are mutually inverse equivalences. The composition $G\circ F$ is given by the right side of the following commutative diagram:

$$\begin{tikzcd}
(\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B)\otimes\Delta^0 \arrow[r, "F\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{Id}"] & \mathrm{Nat}(-(e'), i^*(-)\otimes_{\mathcal{E}} B)\otimes\Delta^0 \arrow[d, "\mathrm{eval}_{\mathcal{E}'(e',-)}\otimes\mathrm{Id}"] \\
(\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B)\otimes\mathcal{E}'(e',e') \arrow[r, "F_{\mathcal{E}(e',-)}\otimes\mathrm{id}"] \arrow[d, "\bar{F}_{\mathcal{E}'(e',-)}"] & \mathrm{Hom}(\mathcal{E}'(e',e'),\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B)\otimes\mathcal{E}'(e',e') \arrow[dl, "ev"] \\
\mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B &
\end{tikzcd}$$

so it is sufficient to show that the left side is homotopic to the identity. But on components, the left side is equivalent to

$$\begin{aligned}&\operatorname*{colim}_{\mathcal{O}^{\mathcal{E}}}(\mathcal{E}'(e',i(-))\boxtimes B)\circ\mathcal{H}^{\mathcal{E}}\otimes\Delta^0\\
&\xrightarrow{\mathrm{id}\otimes\mathrm{Id}} \operatorname*{colim}_{\mathcal{O}^{\mathcal{E}}}(\mathcal{E}'(e',i(-))\boxtimes B)\circ\mathcal{H}^{\mathcal{E}}\otimes\mathcal{E}'(e',e')\\
&\simeq \operatorname*{colim}_{\mathcal{O}^{\mathcal{E}}}((\mathcal{E}'(e',i(-))\otimes\mathcal{E}'(e',e'))\boxtimes B)\circ\mathcal{H}^{\mathcal{E}}\\
&\xrightarrow{\mathrm{comp}\boxtimes\mathrm{id}} \mathcal{E}'(e',i(-))\boxtimes B)\circ\mathcal{H}^{\mathcal{E}}\end{aligned}$$

where comp is composition in $\mathcal{E}'$, and this is homotopic to the identity because $\mathrm{comp}\circ\mathrm{Id}\simeq\mathrm{id}_{\mathcal{E}'(e',-)}$. To investigate the composition $F\circ G$, consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Nat}(-(e'), i^*(-)\otimes_{\mathcal{E}} B)\otimes\Delta^0\otimes-(e') \arrow[d, "\mathrm{id}\otimes\mathrm{Id}\otimes\mathrm{id}"] \arrow[dr, bend left, "G\otimes\mathrm{id}"] & \\
\mathrm{Nat}(-(e'), i^*(-)\otimes_{\mathcal{E}} B)\otimes\mathcal{E}'(e',e')\otimes-(e') \arrow[r, "ev\circ\mathrm{eval}_{\mathcal{E}'(e',-)}\circ\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{comp}"] & \mathcal{E}'(e',i(-))\otimes_{\mathcal{E}} B\otimes-(e') \arrow[dl, "F_{(-)}\otimes\mathrm{id}"] \arrow[d, "\bar{F}_{(-)}"] \\
\mathrm{Nat}(-(e'), i^*(-)\otimes_{\mathcal{E}} B)\otimes-(e') \arrow[r, "ev"] & i^*(-)\otimes_{\mathcal{E}} B
\end{tikzcd}$$

from which it follows that $(F\otimes\mathrm{id})\circ(G\otimes\mathrm{id})\sim\mathrm{id}$. But by the universal property of Hom-objects, this implies that $F\circ G\sim\mathrm{id}$, so we are done. $\square$

Lemma 2.25. Let $\mathcal{E}$ and $\mathcal{E}'$ be a small ∞-categories equipped with a functor $i : \mathcal{E} \to \mathcal{E}'$ and let $B : \mathcal{E}^{op} \to \mathsf{Sp}$ be a functor. Let e' be an object of $\mathcal{E}'$. Then there is an equivalence

$$\mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',\mathsf{Sp})}(-(e'), i^*(-) \otimes_{\mathcal{E}} B) \simeq \Sigma^\infty \mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B$$

Proof. In the proof of Lemma 2.24, replace every instance of $\mathcal{E}'(-,-)$ with $\Sigma^\infty\mathcal{E}'(-,-)$. The map $\bar{F}_\Phi$ is obtained from the adjoint of the evaluation map on spaces $\mathcal{E}'(-,-) \to \Omega^\infty \mathrm{Hom}_{\mathsf{Sp}}(\Phi(-), \Phi(-))$. The equivalences in the proof are all consequences of universal properties which also hold in Sp, so the proof carries over. ☐

A completely analogous argument also gives the following, replacing Σ^∞ by $\Sigma^\infty_{K(n)} := L_{K(n)} \circ \Sigma^\infty$, the infinite suspension composed with the $K(n)$-localization functor.

Lemma 2.26. Let $\mathcal{E}$ and $\mathcal{E}'$ be a small ∞-categories equipped with a functor $i : \mathcal{E} \to \mathcal{E}'$ and let $B : \mathcal{E}^{op} \to L_{K(n)}\mathsf{Sp}$ be a functor. Let e' be an object of $\mathcal{E}'$. Then there is an equivalence

$$\mathrm{Nat}_{\mathrm{Fun}(\mathcal{E}',L_{K(n)}\mathsf{Sp})}(-(e'), i^*(-) \otimes_{\mathcal{E}} B) \simeq \Sigma^\infty_{K(n)} \mathcal{E}'(e', i(-)) \otimes_{\mathcal{E}} B$$

Theorem 2.27. Let $\mathcal{E}$ and $\mathcal{E}'$ be small ∞-categories equipped with a functor $\mathcal{E} \to \mathcal{E}'$ and let $B, B' : \mathcal{E}^{op} \to \mathcal{S}$ be functors. Then there is a homotopy equivalence

$$\mathrm{Hoch}_{\mathcal{E}'}(B, B') \xrightarrow{\sim} D_B C_{B'}(\mathcal{E}'(-,-))$$

Proof. By Corollary 2.23, putting $H = - \otimes_{\mathcal{E}} B'$, we have

$$\mathrm{Nat}(- \otimes_{\mathcal{E}} B, - \otimes_{\mathcal{E}} B') \simeq \mathrm{Nat}(B, \mathrm{Hom}(-, - \otimes_{\mathcal{E}} B'))$$

By Lemma 2.24, we further have $\mathrm{Nat}(-(e'), - \otimes_{\mathcal{E}} B') \simeq \mathcal{E}'(e', -) \otimes_{\mathcal{E}} B'$. Since this equivalence is natural in e', we moreover get

$$\mathrm{Nat}(B, \mathrm{Hom}(-, - \otimes_{\mathcal{E}} B')) \simeq \mathrm{Nat}(B, \mathcal{E}'(-,-) \otimes_{\mathcal{E}} B') \simeq D_B C_{B'}((\mathcal{E}'(-,-))$$

where the last equivalence is by the definition of the Hochschild and coHochschild functors. ☐

Theorem 2.28.

- Let $\mathcal{E}$ and $\mathcal{E}'$ be small ∞-categories equipped with a functor $\mathcal{E} \to \mathcal{E}'$ and let $B, B' : \mathcal{E}^{op} \to \mathsf{Sp}$ be functors. Then there is a homotopy equivalence

$$\mathrm{Hoch}_{\mathcal{E}'}(B, B') \xrightarrow{\sim} D_B C_{B'}(\Sigma^\infty \mathcal{E}'(-,-))$$

- Let $\mathcal{E}$ and $\mathcal{E}'$ be small ∞-categories equipped with a functor $\mathcal{E} \to \mathcal{E}'$ and let $B, B' : \mathcal{E}^{op} \to L_{K(n)}\mathsf{Sp}$ be functors. Then there is a homotopy equivalence

$$\mathrm{Hoch}_{\mathcal{E}'}(B, B') \xrightarrow{\sim} D_B C_{B'}(\Sigma^\infty_{K(n)} \mathcal{E}'(-,-))$$

2.4. **Formal vs. natural operations.**

Assumption 2.29. We now assume that $\mathcal{E}^\otimes$ and $\mathcal{E}'^\otimes$ are symmetric monoidal ∞-categories and $i : \mathcal{E}^\otimes \to \mathcal{E}'^\otimes$ is a symmetric monoidal functor.

Definition 2.30. Let $\mathrm{Fun}^\otimes(\mathcal{E}'^\otimes, \mathcal{S}^\otimes)$ be the category of symmetric monoidal functors from $\mathcal{E}'^\otimes$ to $\mathcal{S}^\otimes$ and let $B, B' : \mathcal{E}^{op} \to \mathcal{S}$ be functors. The *natural operations* from B to B' is the space

$$\mathrm{Hoch}^\otimes_{\mathcal{E}'}(B, B') := \mathrm{Nat}_{\mathrm{Fun}^\otimes(\mathcal{E}'^\otimes, \mathcal{S}^\otimes)}(C_B(-), C_{B'}(-))$$

Observation 2.31. There is a forgetful functor $\mathrm{Fun}^\otimes(\mathcal{E}'^\otimes, \mathcal{S}^\otimes) \to \mathrm{Fun}(\mathcal{E}', \mathcal{S})$, and this gives rise to a restriction map

$$r : \mathrm{Hoch}_{\mathcal{E}'}(B, B') \to \mathrm{Hoch}^\otimes_{\mathcal{E}'}(B, B')$$

Conjecture 2.32. If $i : \mathcal{E} \to \mathcal{E}'$ is the symmetric monoidal envelope of a morphism of ∞-operads $j : \mathcal{O} \to \mathcal{O}'$, then the restriction map r is an embedding.

3. Operations on Factorization Homology

Definition 3.1. Let M be a framed n-dimensional manifold and let $\mathbb{E}_n$ be the symmetric monoidal envelope of the framed n-disk ∞-operad. Then for $\Phi : \mathbb{E}_n \to \mathcal{S}$ a functor, the chiral homology of Φ over M is given by

$$\int_M \Phi := \Phi \otimes_{\mathbb{E}_n} \mathcal{L}_M$$

where $\mathcal{L}_{S^1}$ is the configuration functor of M, which we will define in the next section.

3.1. **Configuration Functors.**

In this section we define the functors $\mathcal{L}_M : \mathbb{E}_n^{op} \to \mathcal{S}$ parameterizing factorization homology and study some of their properties. We begin by recalling some facts about the ∞-categorical Day convolution due to Glasman [9].

Recollection 3.2. Let C and D be symmetric monoidal ∞-categories such that the tensor product in D preserves colimits in each variable. Then the functor category $\mathrm{Fun}(C, D)$ attains a symmetric monoidal structure such that $\mathbb{E}_\infty$-monoids correspond to lax symmetric monoidal functors $C \to D$. Given an s-tuple of functors (F_s), the tensor product is given by the point-set formula

$$\left(\bigotimes_{i \in \langle s \rangle} F_i\right)(X) \simeq \operatorname*{colim}_{(Y_s) \in C^{\otimes}_{\langle s \rangle / X}} m_*(F_s(Y_s))$$

where m is the active map $\langle s \rangle \to \langle 1 \rangle$ and m_* denotes the coCartesian pushforward.

Definition 3.3. Let Mfld_n^{fr} be the ∞-category of framed n-dimensional manifolds without boundary, and framed embeddings between them.

Observation 3.4. Disjoint union of manifolds gives Mfld_n^{fr} the structure of a symmetric monoidal ∞-category, and $\mathbb{E}_n$ embeds into Mfld_n^{fr} as a symmetric monoidal subcategory by sending 1 to $\mathbb{R}^n$ with the standard framing.

Remark 3.5. The ∞-category Mfld_n^{fr} arises naturally as a topological category, but becomes an ∞-category by passing through suitable Quillen equivalences. We will not need point-set level information about it, so we do not concern ourselves with its model-categorical presentation.

Definition 3.6. Let M be a connected framed n-manifold without boundary. Define the *configuration functor* of M to be the restriction to $\mathbb{E}_n^{op}$ of the representable functor $\mathsf{Mfld}_n^{fr,op} \to \mathcal{S}$ associated to M. In other words, $\mathcal{L}_M(m) = \mathrm{Emb}^{fr}(\sqcup_m \mathbb{R}^n, M)$.

Proposition 3.7. [6] The assignment $\mathsf{Mfld}_n \times \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$ given by $(M, A) \mapsto \mathcal{L}_M \otimes_{\mathbb{E}_n} A$ corresponds to factorization homology. In particular, for a ring spectrum A, $\mathcal{L}_{S^1} \otimes_{\mathcal{A}_\infty} A$ is topological Hochschild homology.

Lemma 3.8. The assignment $M \mapsto \mathcal{L}_M$ gives a symmetric monoidal functor $\mathsf{Mfld}_n^{fr} \to \mathrm{Fun}(\mathbb{E}_n^{op}, \mathcal{C})$.

Proof. Let M_i be framed n-manifolds for $i = 1, ..., s$. Let $\mathcal{P}(n,s)$ be the set of partitions of the set $\{1, ..., n\}$ into s disjoint subsets. Using the point-set formula of Recollection 3.2 in the case $\mathcal{C} = \mathbb{E}_n^{op}$, we get that

$$\operatorname*{colim}_{(n_s)\in C^{\otimes}_{\langle s\rangle/n}} m_*((\mathcal{L}_{M_i}(n_i))_{i\in\langle s\rangle}) \simeq \bigoplus_{(a_1,...,a_s)\in\mathcal{P}(n,s)} \bigotimes_{i\in\langle s\rangle} \mathrm{Emb}^{fr}(\sqcup_{a_i}\mathbb{R}^n, M_i)$$

On the other hand, $\mathcal{L}_{\sqcup_i M_i}(n) \simeq \mathrm{Emb}^{fr}(\sqcup_n\mathbb{R}^n, \sqcup_{i=1}^n M_i)$ splits into components according to which copies of $\mathbb{R}^n$ lands in which M_i, such that the functors $\mathcal{L}_{\sqcup_i M_i}$ and $\otimes_i \mathcal{L}_{M_i}$ are objectwise equivalent. Now since the functoriality in both cases is given by precomposition, the result follows. $\square$

Lemma 3.9. There is a homotopy equivalence

$$(\mathcal{L}_M \otimes \mathcal{L}_N) \otimes_{\mathcal{A}_\infty} A \simeq (\mathcal{L}_M \otimes_{\mathcal{A}_\infty} A) \otimes (\mathcal{L}_N \otimes_{\mathcal{A}_\infty} A).$$

Definition 3.10. Let Mfld_n^{fr}, s be the ∞-category of $n-s$-dimensional manifolds P equipped with a framing of $P \times \mathbb{R}^s$, and morphisms the embeddings $P \to P'$ such that $P \times \mathbb{R}^s \to P' \to \mathbb{R}^s$ becomes a framed embedding.

Lemma 3.11. Let $N = P \times \mathbb{R}^s$ be a framed manifold. Then the maps

$$\mathbb{E}_s(n,1) \to \mathrm{Emb}^{fr}((\sqcup_{i=1}^n P \times \mathbb{R}^s, P \times \mathbb{R}^s)$$

sending $f : n \to 1$ to $\mathrm{id} \times f$ gives $P \times \mathbb{R}^s$ an $\mathbb{E}_s$-algebra structure.

Proof. We construct an operad morphism $\mathbb{E}_s^{\otimes} \to \mathsf{Mfld}_n^{fr\otimes}$ witnessing the algebra structure as follows. The unique object of $\mathbb{E}^{\otimes}_{s,\langle t\rangle}$ is sent to $(N, ..., N)_t \in \mathsf{Mfld}^{fr\otimes}_{n,\langle t\rangle}$. Let $\phi : \langle t\rangle \to \langle r\rangle$ be a map of finite sets and $f = (f_1, ..., f_r) : (1, ..., 1)_t \to (1, ..., 1)_r$ a morphism in $\mathbb{E}_s^{\otimes}$ lying over ϕ, such that $f_i : (1, ..., 1)_{t_i} \to (1)$. In other words, for each $1 \le i \le r$, f_i is an element of $\mathbb{E}_s(t_i)$. We sent f to the morphism $(g_1, ..., g_r) : (N, ..., N)_t \to (N, ..., N)_r$ where $g_i = \mathrm{id} \times f_i$. $\square$

3.2. **Examples of non-trivial formal operations.**

3.2.1. *Product structures.* As an example of non-trivial formal operations, we may observe that the $\mathbb{E}_{n-1}$-product structure on THH of $\mathbb{E}_n$-algebras is formal. Namely, it arises from maps $\mathcal{L}_{(\sqcup_k S^1)\otimes\mathbb{R}^{n-1}} \to \mathcal{L}_{S^1\otimes\mathbb{R}^{n-1}}$ given by realizing the fold maps

$$\nabla_k : \sqcup_k S^1 \to S^1$$

as embeddings of n-dimensional manifolds. More precisely, we have maps

$$\mathbb{E}_{n-1}(k) \to \mathrm{Emb}^{fr}((\sqcup_k S^1) \times \mathbb{R}^{n-1}, S^1 \times \mathbb{R}^{n-1})$$

given by $\gamma \mapsto \nabla_k \times \gamma$. By the same method we obtain formal $\mathbb{E}_{n-d}$-algebra structures on factorization homology of $\mathbb{E}_n$-algebras over a d-dimensional Lie group.

3.2.2. *Adams operations.* For factorization homology of $\mathbb{E}_\infty$-algebras, the p-fold covers $S^1 \to S^1$ can be realized as embeddings

$$S^1 \times \mathbb{R}^\infty \hookrightarrow S^1 \times \mathbb{R}^\infty$$

giving a formal action of the multiplicative monoid $(\mathbb{N}, \cdot, 1)$ on THH of commutative algebras. Note that the coherence of the action requires appending an infinite-dimensional euclidean space. However, the embeddings themselves exist with only two extra dimensions, so that we still have the action of $(\mathbb{N}, \cdot, 1)$ on the homotopy groups of THH of $\mathbb{E}_3$-algebras.

3.3. **Open conformal field theories.**

Definition 3.12. Let M and N be one-dimensional compact manifolds with boundary. We define a cobordism $M \to N$ to be a Riemann surface Γ with corners, whose boundary is a union $\partial\Gamma = \partial_{in}\Gamma \sqcup \partial_{free}\Gamma \sqcup \partial_{out}\Gamma$ and equipped with orientation preserving diffeomorphisms $\overline{M} \to \partial_{in}\Gamma$ and $N \to \partial_{out}\Gamma$. We call $\partial_{in}\Gamma$ the incoming boundary, $\partial_{out}\Gamma$ the outgoing boundary and $\partial_{free}\Gamma$ the free boundary.

Definition 3.13. Let $\mathcal{OC}$ the open-closed ∞-category of Riemann surfaces. Its objects are one-dimensional manifolds with boundary. If M and N are two such manifolds, then the space of morphisms is given by

$$\mathcal{OC}(M,N) = \coprod_{\substack{[\Gamma] \\ \partial_i\Gamma = M, \partial_o\Gamma = N}} B\mathrm{Diff}(\Gamma; \partial_{in}\Gamma, \partial_{out}\Gamma)$$

That is, the moduli space of Riemannian cobordisms with corners relative to the incoming and outgoing boundary. For technical reasons, we will only allow those connected components for which Γ has a non-empty incoming or free boundary.

Definition 3.14. Given an open-closed cobordism $(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma)$, define the embedding space

$$\mathrm{Emb}^{\epsilon}(\Gamma, \mathbb{R}^\infty) \subseteq$$

$$\mathrm{Emb}((\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma), (\mathbb{R}^\infty, \{0\} \times \mathbb{R}^{\infty-1}, \{1\} \times \mathbb{R}^{\infty-1}))$$

as the subspace of embeddings $e : \Gamma \to \mathbb{R}^\infty$ which are orthogonal to the first coordinate in an ϵ-neighbourhood of the boundary. That is,

$$e(\Gamma) \cap \left([0,\epsilon) \times \mathbb{R}^{\infty-1}\right) = [0,\epsilon) \times e(\partial_{in}\Gamma)$$

$$e(\Gamma) \cap \left((1-\epsilon,1] \times \mathbb{R}^{\infty-1}\right) = (1-\epsilon,1] \times e(\partial_{out}\Gamma)$$

We write

$$\mathrm{Emb}^{\partial}(\Gamma) \simeq \operatorname*{colim}_{\epsilon \to 0} \mathrm{Emb}^{\epsilon}(\Gamma, \mathbb{R}^\infty)$$

Note that the diffeomorphism group $\mathrm{Diff}(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma)$ acts freely on $\mathrm{Emb}^{\partial}(\Gamma)$. We write

$$I(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma) := \mathrm{Emb}^{\partial}(\Gamma)/\mathrm{Diff}(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma)$$

Observation 3.15. This space is equivalent to $B\mathrm{Diff}(\Gamma; \partial_{in}\Gamma, \partial_{out}\Gamma)$, hence a model for $\mathcal{OC}$ can be given by

$$\mathcal{OC}(M,N) \simeq \coprod_{\substack{[\Gamma] \\ \partial_i\Gamma = M, \partial_o\Gamma = N}} I(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma)$$

We will use this model heavily throughout this section.

Definition 3.16. Let $\mathcal{O}$ be Segal's ∞-category of Riemann surfaces, i.e. the full subcategory of $\mathcal{OC}$ on disjoint unions of intervals.

Lemma 3.17. There is an inclusion $\mathcal{A}_\infty \to \mathcal{O}$ whose essential image consists of those open cobordisms which are disjoint unions of disks with exactly one outgoing boundary component.

Proof. The inclusion is given by sending a linear ordering on $\{1, ..., n\}$ to a disk with one outgoing boundary component and n incoming boundary components corresponding to the linear order. Since a disk with a single outgoing boundary component is determined up to contractible choice by the ordering of its incoming boundary components, and since such a disk has a contractible diffeomorphism group, the inclusion is an equivalence onto its essential image. □

Definition 3.18. Let $\mathcal{C}$ be a symmetric monoidal ∞-category

- An open topological conformal field theory (OTCFT) is a symmetric monoidal functor $\mathcal{O} \to \mathcal{C}$.
- An open-closed topological conformal field theory (OCTCFT) is a symmetric monoidal functor $\mathcal{OC} \to \mathcal{C}$.

Terminology 3.19. Because of the restrictions we put on the connected components of the morphism spaces of $\mathcal{OC}$, our OCTCFTs may also be called *non-positive boundary* OCTCFTs.

3.4. **Operations on THH of Open Field Theories.** In this section we apply the framework of formal operations to prove Theorem 3.26.

To prove this result, it is sufficient to produce a symmetric monoidal functor

$$\mathcal{OC} \to \mathrm{Hoch}_{\mathcal{O}}^{\otimes}$$

Below we will use geometric methods to obtain a functor $\mathcal{OC} \to \mathrm{Nat}_{\mathcal{O}}$. Composing by the restriction map we obtain the desired functor. In order to see that the resulting action on THH is nontrivial, we compare it on rational homology to the action constructed by Wahl-Westerland in [21].

For $f \in \mathcal{L}_{S^1}(n) = \mathrm{Emb}^{fr}(\sqcup_n \mathbb{R}, S^1)$, let f_i be the restriction of f to the i'th copy of $\mathbb{R}$. We assume that for each such f, the closures of the images of the f_i are disjoint. Note that this does not change the homotopy type of $\mathcal{L}_{S^1}(n)$.

Definition 3.20. (Open-closed cylinders) Let

$$\mathrm{Cyl} : \mathcal{L}_{S^1} \to \mathcal{OC}\left(\left[\begin{smallmatrix}0\\-\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)$$

be the map taking $f \in \mathrm{Emb}^{fr}(\sqcup_n \mathbb{R}, S^1)$ to the cylinder $\mathrm{Cyl}(f) = S^1 \times [0,1]$ with $\partial_{out} = S^1 \times \{1\}$ and $\partial_{in} = \sqcup_{i=1}^n \overline{\mathrm{Im}(f_i)}$, the closures of the images of the f_i. The morphisms in $\mathcal{A}_\infty^{op}$ act compatibly on both objects, such that Cyl is a natural transformation.

Lemma 3.21. The transformation Cyl is a natural equivalence onto the connected component of $S^1 \times [0,1]$.

Proof. Let $\mathrm{Cyl}(n)$ be the subspace of $\mathcal{OC}\left(\left[\begin{smallmatrix}0\\n\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)$ of open-closed cobordisms equivalent to $\mathrm{Cyl}(f)$ for some $f \in \mathcal{L}_{S^1}(n)$. The spaces $\mathcal{L}_{S^1}(n)$ and $\mathrm{Cyl}(n)$ both contain one copy of the circle for each cylic ordering of the n copies of $\mathbb{R}$. A full rotation of the copies of $\mathbb{R}$ in $\mathcal{L}_{S^1}(n)$ corresponds to a Dehn twist in $\mathrm{Cyl}(n)$. □

Definition 3.22. Let n_1, n_2, m_1, m_2 be non-negative integers with $n_1 + m_1 \geq 1$. Define the map

$$R : \mathcal{OC}\left(\left[\begin{smallmatrix}n_1\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) \to \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix}n_1-1\\m_1+(-)\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)\right)$$

which takes Γ to $f \mapsto \Gamma \circ \left(\mathrm{id}_{\left[\begin{smallmatrix}n_1-1\\m_1\end{smallmatrix}\right]} \sqcup \mathrm{Cyl}(f)\right)$.

Lemma 3.23. The maps defined in Definition 3.22 give rise to a map

$$\mathcal{OC}\left(\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right) \to \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}}, \mathcal{OC}\left(\begin{bmatrix} 0 \\ - \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right)\right).$$

Proof. Use the map defined in Definition 3.22 and Corollary 2.23 and Lemma 3.8. □

Lemma 3.24. There is a map

$$F : \mathcal{OC}\left(\begin{bmatrix} 0 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right) \to \mathcal{O}(m_1, -) \otimes \mathcal{L}_{\begin{bmatrix} n_2 \\ m_2 \end{bmatrix}}$$

which is a right inverse to composition.

Proof. See Section 3.4.1. □

The two lemmas imply the following calculation of the formal operations on THH of $\mathcal{O}$-algebras, and via the restriction map we obtain the following calculation.

Lemma 3.25. The maps from Lemmas 3.23 and 3.24 give rise to a map

$$j : \mathcal{OC}\left(\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right) \to \mathrm{Nat}_{\mathcal{O}}\left(\mathcal{L}_{\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}}, \mathcal{L}_{\begin{bmatrix} n_2 \\ m_2 \end{bmatrix}}\right)$$

Since the map j of Lemma 3.25 is defined on connected components, it takes disjoint unions to tensor products:

$$j : \Gamma \sqcup \Gamma' \mapsto j(\Gamma) \otimes j(\Gamma)$$

We may now combine this with the composition structure and deduce the following theorem, generalizing [21, Theorem 6.2].

Theorem 3.26. There is a symmetric monoidal functor of ∞-categories

$$F : \mathcal{OC} \to \mathrm{Hoch}_{\mathcal{O}}$$

The following conjecture concerns the compatibility of our computation in spectra with the work of Wahl-Westerland [21, 20] in chain complexes.

Conjecture 3.27. Given a functor $f : \mathcal{C} \to \mathsf{Sp}$, write $H\mathbb{Q} \otimes f : \mathcal{C} \to \mathsf{Mod}_{H\mathbb{Q}}$ for the composition

$$\mathcal{C} \xrightarrow{f} \mathsf{Sp} \xrightarrow{H\mathbb{Q}\otimes -} \mathsf{Mod}_{H\mathbb{Q}}.$$

The morphism

$$\begin{aligned} H\mathbb{Q} \otimes \mathcal{OC}\left(\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right) &\xrightarrow{H\mathbb{Q}\otimes F} H\mathbb{Q} \otimes \mathrm{Hoch}_{\mathcal{O}}\left(\mathcal{L}_{\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}}, \mathcal{L}_{\begin{bmatrix} n_2 \\ m_2 \end{bmatrix}}\right) \\ &\xrightarrow{(H\mathbb{Q}\otimes -)_*} \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left((H\mathbb{Q} \otimes \mathcal{L}_{\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}}), (H\mathbb{Q} \otimes \mathcal{O}(-,-)) \otimes_{\mathcal{A}_\infty} (H\mathbb{Q} \otimes \mathcal{L}_{\begin{bmatrix} n_2 \\ m_2 \end{bmatrix}})\right), \end{aligned}$$

is homotopic to the map described by Wahl-Westerland [21, 20].

The following conjecture is a generalization of [20, Theorem B].

Conjecture 3.28. The functor of Theorem 3.26 is an equivalence of symmetric monoidal ∞-categories.

The following conjecture is the spectral version of [5, Theorem A].

Conjecture 3.29. The assignment $L : \mathrm{Fun}^{\otimes}(\mathcal{O}, \mathsf{Sp}) \to \mathrm{Fun}^{\otimes}(\mathcal{OC}, \mathsf{Sp})$ induced by the map F in Theorem 3.26 is a left adjoint to the restriction functor

$$\mathrm{Fun}^{\otimes}(\mathcal{OC}, \mathsf{Sp}) \to \mathrm{Fun}^{\otimes}(\mathcal{O}, \mathsf{Sp}).$$

3.4.1. *Decompositions of surfaces.* In this section we prove Lemma 3.24.

Idea 3.30. The idea of the proof is to decorate the open-closed cobordisms with arcs connecting the free boundary to each closed outgoing boundary. Pushing the free boundary along these arcs allow us to decompose the cobordism into an open cobordism composed with a disjoint union of cylinders, providing an inverse to the composition map. The arc complex $\mathcal{A}rc_0(S)$ on a surface S is a contractible space carrying a free action by the mapping class groups of S. We can use this to produce an alternative model for $\mathcal{OC}$ admitting a vertical map as pictured in the below diagram. The meat of the proof will therefore be in providing a diagonal map, as in the diagram, that commutes with the action of the diffeomorphism group on each component.

$$\begin{tikzcd}
\widetilde{\mathcal{OC}}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) \arrow[d, "\sim"'] \arrow[dr, "\sim"] & \\
\mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) & \mathcal{O}(m_1,-)\otimes_{\mathcal{A}_\infty}\mathcal{L}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]} \arrow[l, "\mathrm{comp}"]
\end{tikzcd}$$

We begin by defining $\widetilde{\mathcal{OC}}$. For this we need some preliminary definitions.

Observation 3.31. We may assume that an open-closed cobordism comes equipped with a marked point on each open and each closed boundary component, corresponding to the point 0 on S^1 and $(-1,1)$. Doing so does not change the homotopy type of the moduli space.

Definition 3.32. Let $e : S \hookrightarrow \mathbb{R}^\infty$ be an embedded orientable surface representing a morphism in $\mathcal{OC}\left(\left[\begin{smallmatrix}n_1\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$ and let $\Upsilon = \{x_1, ..., x_{n_1+n_2+m_1+m_2}\}$ be the set of marked points described above. We assign a second set Δ of marked points in S as follows. On each closed boundary component choose a point $y_i \neq x_i$, where x_i is the point "marking" the closed boundary component. On the boundary components of S containing open boundary components, choose a point y_i on each free section of the boundary component. Note that these choices are unique up to homotopy and reordering. We now make the following definitions.

- An *essential arc* α_0 in S is an embedded interval
$$\alpha_0 : (I, \partial I) \to (S, \Delta \cup \partial_{free} S)$$
such that α_0 only intersects $\partial S \cup \Delta$ at its endpoints, and such that α_0 does not separate S into two components one of which is disk that intersects Δ only at the endpoints of α_0.
- An *arc set* is a collection of arcs $\alpha = \{\alpha_0, ..., \alpha_k\}$ such that if $i \neq j$, then α_i and α_j have disjoint interiors and are not isotopic rel V.
- An *arc system* $[\alpha]$ is an isotopy class of arc sets rel Δ.
- An arc system is *filling* if it cuts S into contractible subspaces which do not contain elements of Δ in their interior.

Definition 3.33. Let S be an embedded surface representing a morphism in $\mathcal{OC}\left(\left[\begin{smallmatrix}n_1\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$ where $n_1+n_2+m_1+m_2 \geq 1$ and let $\partial_1 S, ..., \partial_{n_2} S$ be the outgoing closed boundary components of S. We say that α is an *admissible arc set if the following conditions hold.*

- *α either contains or can be expanded to contain arcs which cut S into n_2 components such that for each $1 \leq j \leq n_2$, the j'th component contains the interiors of all arcs in α starting or ending at $\partial_j S$.*

- *α does not contain an arc starting and ending at $\partial_j S$ for and $1 \leq j \leq n_2$.*
- *For all $1 \leq j \leq n_2$, let y_j be the marked point of Δ on $\partial_j S$, and let $\alpha_{j_1}, ..., \alpha_{j_r}$ be the arcs in α which start or end at y_j. The arc set α also contains arcs $\beta_{j_1}, ..., \beta_{j_s}$ such that the subspace $((\cup_l \alpha_l) \cup (\cup_l \beta_l)) \setminus y_j$ is connected.*

Let $\mathfrak{B}(S, \Delta)$ be the topological poset of admissible arc sets, and let $\mathfrak{B}_0(S, \Delta)$ be the subposet of the filling admissible arc sets.

Observation 3.34. Note that the first condition in the definition of admissible arc systems implies that no arc can connect two closed outgoing boundary components, and the filling condition implies that there is at least one arc connecting each outgoing closed boundary component to the free boundary.

Lemma 3.35. ([19, Cor. 2.23, Cor. 3.25]) Let S be an open-closed cobordism whose boundary is not completely outgoing closed, then $\mathfrak{B}_0(S, \Delta)$ is contractible and admits a free action of the diffeomorphism group
$Diff(S; \partial_{in} S, \partial_{out} S)$.

We describe the action of the diffeomorphism group on $\mathfrak{B}_0(S, \Delta)$. Let $Diff(S; \Delta)$ be the subgroup of $Diff(S; \partial_{in} S, \partial_{out} S)$ which preserves the marked points Δ. Note that the inclusion $Diff(S; \Delta) \subseteq Diff(S; \partial_{in} S, \partial_{out} S)$ is a homotopy equivalence. Let $f \in Diff(S; \Delta)$ and let $\alpha = \{\alpha_0, ..., \alpha_k\}$ be an arc system. The action of f on α is given by

$$f \cdot \alpha := \{f \circ \alpha_0, ..., f \circ \alpha_k\}.$$

If $f \in Diff(S; \partial_{in} S, \partial_{out} S)$, there is an essentially unique $f' \in Diff(S; \Delta)$ such that $f \simeq f'$, and we define $f \cdot \alpha := f' \cdot \alpha$.

Observation 3.36. In Definition 3.32, we may remove the condition that the arcs are pairwise non-isotopic by considering arcs with multiplicity. This can be thought of as adding degeneracies to a semi-simplicial set and does not change the homotopy type of the arc complex.

Definition 3.37. Let $\mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$ be the space of pairs (e, α), where $e : \Gamma \to \mathbb{R}^\infty$ is an element of $\mathrm{colim}_{\epsilon \to 0} \mathrm{Emb}^\epsilon(\Gamma, \mathbb{R}^\infty)$ is an embedded surface representing an open-closed cobordism and $\alpha \in \mathfrak{B}_0(e(\Gamma), \Delta)$ a filling admissible arc set on $e(\Gamma)$. We denote by $\mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)|_\Gamma$ the component corresponding to a fixed cobordism type Γ. On this component, the diffeomorphism group of Γ acts by

$$f \cdot (e, \alpha) = (e \circ f, f \cdot \alpha)$$

Observation 3.38. Since $\mathfrak{B}_0(\Gamma, \Delta)$ is contractible, the quotient of $\mathcal{A}rc_0$ by the diffeomorphism group action is another model for $BDiff(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma)$.

Lemma 3.39. Let $\Gamma \in \mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$. There is an equivalence

$$\mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)|_\Gamma \simeq \mathrm{Emb}^\partial(\Gamma) \times \mathfrak{B}_0(\Gamma)$$

which is compatible with the action of the mapping class group of Γ.

Proof. The leftward map is given by taking a pair $([e], \alpha)$ of the equivalence class of a representative embedding $e \in \mathrm{Emb}^\epsilon((\Gamma, \mathbb{R}^\infty)$ and an arc set on Γ to the pair $([e], e(\gamma))$. Note that $e(\gamma)$ depends only on the equivalence class $[e]$ and not on the representative e, hence this map is well-defined. The rightward map is constructed similarly, taking a pair $([e], \alpha')$, of an equivalence class of a representative embedding $e \in \mathrm{Emb}^\epsilon((\Gamma, \mathbb{R}^\infty)$ and an arc set on $e(\Gamma)$, to the pair $([e], e^{-1}(\alpha))$. The arc set $e^{-1}(\alpha)$ on Γ is uniquely determined by the requirement that $e(e^{-1}(\alpha)) = \alpha$. These maps are actually mutually inverse homeomorphisms. $\square$

Definition 3.40. Define the space

$$\widehat{\mathcal{OC}}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) = \sqcup_{[\Gamma]\in\mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)} \mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)|_\Gamma / \mathrm{Diff}(\Gamma, \partial_{in}\Gamma, \partial_{out}\Gamma)$$

Observation 3.41. $\mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$ defines the total space of a fiber bundle over $\mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$ whose fiber over the component corresponding to Γ is $\mathfrak{B}_0(\Gamma)$. Hence the projection $\widehat{\mathcal{OC}} \to \mathcal{OC}$ is fully faithful.

The decoration of surfaces by arc sets is sufficient to define an inverse to the composition morphism. The below construction produces a morphism

$$\mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) \to B\left(\mathcal{O}(m_1, m_2 + -), \mathcal{A}_\infty, \mathcal{L}_{\left[\begin{smallmatrix}n_2\\0\end{smallmatrix}\right]}\right)$$

and by Lemma 2.7 this gives a map to $\mathcal{O}(m_1, m_2 + -) \otimes_{\mathcal{A}_\infty} \mathcal{L}_{\left[\begin{smallmatrix}n_2\\0\end{smallmatrix}\right]}$.

Construction 3.42. For an open-closed cobordism $\Gamma \in \mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$, with marking Δ as in Definition 3.32, let $\alpha \in \mathfrak{B}_0(\Gamma, \Delta)$ and let $e \in \mathrm{Emb}^\partial(\Gamma)$. For $1 \leq i \leq n_2$, let $\partial_i\Gamma$ be the i'th outgoing closed boundary component of Γ, let $y_i \in \Delta$ be the marked point on $\partial_i\Gamma$, and let $J_i = \{\alpha_{i,1}, ..., \alpha_{i,r_i}\}$ be the cyclically ordered finite set of arcs intersecting y_i. By perturbing the arcs $\alpha_{i,j}$ in a sufficiently small neighborhood of their endpoints, we may assume that the arcs as a whole are disjoint, not just their interiors. We may then choose tubular neighborhoods $U_{i,j}$ of the arcs such that the closures in Γ are disjoint: $\overline{U_{i,j}} \cap \overline{U_{i',j'}} = \emptyset$ unless $i = i'$ and $j = j'$. For each (i,j), let $V_{i,j} = U_{i,j} \cap \partial_i\Gamma$. and let $W_{i,j}$ be the components of $\partial_i\Gamma \setminus \left(\bigcup_{j=1}^{r_i} V_{i,j}\right)$, cyclically ordered according to the orientation on $\partial_i\Gamma$ such that $W_{i,j}$ separates $V_{i,j}$ and $V_{i,j+1}$. Define the surface

$$\Gamma_\alpha := \Gamma \setminus \left(\bigcup_{i=1}^{n_2}\bigcup_{j=1}^{r_i} U_{i,j}\right)$$

viewed as an open cobordism in $\mathcal{O}(m_1, m_2 + \sum_{i=1}^{n_2} r_i)$, and define the restricted embedding

$$e_\alpha := e|_{\Gamma_\alpha} \in \mathrm{Emb}^\partial(\Gamma_\alpha).$$

Finally, let $f_\alpha \in \mathrm{Emb}^{fr}(\sqcup_{\Sigma r_i}\mathbb{R}, \sqcup_{n_2} S^1)$ be the the map sending the (i,j)'th copy of $\mathbb{R}$ to $W_{i,j}$, where we have identified the i'th copy of S^1 with $\partial_i\Gamma$. We now produce an assignment

$$F_\Gamma : \mathfrak{B}_0(\Gamma, \Delta) \to \mathcal{O}(m_1, m_2 + -) \otimes_{\mathcal{A}_\infty} \mathcal{L}_{\left[\begin{smallmatrix}n_2\\0\end{smallmatrix}\right]}$$

sending α to $\Gamma_\alpha \otimes f_\alpha$. Let $\alpha' \in \mathfrak{B}_0(\Gamma, \Delta)$ be another filling admissible arc system such that $\alpha' \subseteq \alpha$. Assuming for simplicity of exposition that Γ has a single outgoing boundary, this inclusion overlies an inclusion of cyclically ordered finite sets $\theta : J' \hookrightarrow J$. We associate to the inclusion $\alpha' \subseteq \alpha$ an element of $\mathcal{A}_\infty(|J|, |J'|)$, defined up to a cyclic permutation of tensor factors, as follows. Write $s' : J' \to J'$ for the successor function on J'. For each $j' \in J'$, let $J_{j'} \subseteq J$ be the subset $\{\theta(j'), ..., \theta(s'(j')) - 1\}$ and define $m_{\theta,j'} \in \mathcal{A}_\infty(|J_{j'}|, 1)$ to be the inclusion

$$m_{\theta,j'} : \sqcup_{j\in J_{j'}} W_{i,j} \hookrightarrow W_{i,\theta(j')} \cup V_{i,\theta(j')+1} \cup ... \cup W_{i,\theta(s(j'))-1} \simeq \mathbb{R}$$

We now write $m_{\alpha',\alpha}$ for the tensor product

$$m_{\alpha',\alpha} := \bigotimes_{j' \in J'} m_{\theta,j'} \in \mathcal{A}_\infty(|J|,|J'|)$$

Then we have $\Gamma_{\alpha'} \sim m_{\alpha',\alpha} \circ \Gamma_\alpha$ and $f_\alpha \sim f_{\alpha'} \circ m_{\alpha',\alpha}$. By realizing the coend as a two-sided bar construction, we then send a sequence of p composable inclusions $\alpha_0 \subseteq ... \subseteq \alpha_p$ to the $p+1$-simplex

$$\Gamma_{\alpha_p} \otimes m_{\alpha_{p-1},\alpha_p} \otimes ... \otimes m_{\alpha_0,\alpha_1} \otimes f_{\alpha_0}$$

It is then clear that the assignment F_Γ is compatible with the poset structure on $\mathfrak{B}_0(\Gamma,\Delta)$, i.e. up to coherent homotopy, the image of F_Γ does not depend on α.

We describe how F_Γ behaves with respect to the $Diff(\Gamma;\partial_{in}\Gamma,\partial_{out}\Gamma)$-action on $\mathfrak{B}_0(\Gamma,\Delta)$. For a diffeomorphism $g \in Diff(\Gamma;\partial_{in}\Gamma,\partial_{out}\Gamma)$, we have $F_\Gamma(g\circ\alpha) = \Gamma_{g\cdot\alpha} \otimes f_{g\cdot\alpha} = g(\Gamma_\alpha) \otimes f_\alpha$, where $g(\Gamma_\alpha)$ is the image of Γ_α under g as a subspace of Γ. The identity $f_{g\cdot\alpha} \simeq f_\alpha$ is due to g preserving the outgoing boundary pointwise.

From the above, we conclude the following lemma.

Lemma 3.43. The functors

$$F_\Gamma : \mathfrak{B}_0(\Gamma,\Delta) \to \mathcal{O}(m_1, m_2 + -) \otimes_{\mathcal{A}_\infty} \mathcal{L}_{\left[\begin{smallmatrix} n_2 \\ 0 \end{smallmatrix}\right]}$$

induce a morphism

$$\widetilde{F} : \mathcal{OC}\left(\left[\begin{smallmatrix} 0 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right) \to \mathcal{O}(m_1, m_2 + -) \otimes_{\mathcal{A}_\infty} \mathcal{L}_{\left[\begin{smallmatrix} n_2 \\ 0 \end{smallmatrix}\right]}$$

which admits a homotopy

$$\mathrm{comp} \circ \widetilde{F} \simeq \mathrm{id}.$$

Proof. Let $\mathcal{A}rc_0 := \sqcup_\Gamma \mathfrak{B}_0(\Gamma,\Delta)$ and

$$F := \sqcup_\Gamma F_\Gamma : \mathcal{A}rc_0 \to \mathcal{O}(m_1, m_2 + -) \otimes_{\mathcal{A}_\infty} \mathcal{L}_{\left[\begin{smallmatrix} n_2 \\ 0 \end{smallmatrix}\right]}$$

By the universal property of quotients, this induces the morphism $\widetilde{F}$ of the statement. From Construction 3.42, for each Γ the composition $\mathrm{comp} \circ F_\Gamma$ is homotopic to the projection $\mathfrak{B}_0(\Gamma,\Delta) \to \mathcal{OC}\left(\left[\begin{smallmatrix} 0 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$ onto the connected component of Γ. This implies that $\mathrm{comp} \circ F$ is homotopic to the projection. By the universal property of quotients, we then have that $\mathrm{comp} \circ \widetilde{F} \simeq \mathrm{id}$. ∎

A similar argument to the above, using arcs connecting the free boundary to the open outgoing boundary, can be used to construct a natural transformation

$$\mathcal{O}(m_1, m_2 + -) \to \mathcal{O}(m_1, -) \otimes_{\mathcal{A}_\infty} \mathcal{L}_{\left[\begin{smallmatrix} 0 \\ m_2 \end{smallmatrix}\right]}$$

3.4.2. *CoHochschild Reduction of Open-Closed Cobordisms.* In this section we will prove the following statement.

Lemma 3.44. The map R of Lemma 3.23 is a homotopy equivalence.

Lemma 3.45. Let $F \in \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+(-) \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)\right)$. There is a unique open-closed cobordism Γ associated to F such that for $f \in \mathcal{L}_{S^1}(n)$, the last n incoming open boundary components of $F(n)(f)$ are the sole open boundary components on a boundary component of Γ, and appear on that component in the same cyclic order as they do in f.

Proof. Any $f \in \mathcal{L}_{S^1}(n)$ can be written as a composition $\widetilde{f} \circ m_n$, where $m_n \in \mathcal{A}_\infty(n,1)$ and $\widetilde{f} \in \mathcal{L}_{S^1}(1)$. This implies that the last n open boundary components must lie on the same boundary component. To see that these are the only open boundary components on that boundary component of the surface, observe that F must also preserve the cyclic permutations of the inputs, which cannot happen if there are additional open boundary components. Finally, the cyclic order is preserved, otherwise $\mathcal{A}_\infty$-functoriality cannot be preserved. $\square$

Construction 3.46. We will produce a map

$$S : \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+(-) \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)\right) \to \mathcal{OC}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$$

which we will later show is inverse to the map R described in Lemma 3.23. By Lemma 3.45, any $F \in \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+(-) \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)\right)$, is determined up to contractible choice by the restriction to $F(1) : \mathcal{L}_{S^1}(1) \to \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$. To construct S, it is therefore sufficient to consider the connected component of a single open-closed cobordism $\Gamma \in \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$. Let $\partial_1\Gamma$ be the boundary component of Γ which contains the last open boundary component. Let $C_1\Gamma$ be a collar of $\partial_1\Gamma$. The space of collars is contractible, hence $C_1\Gamma$ may be extended to a compatible choice of collar for each surface in the connected component of Γ. By interpreting $\overline{C_1\Gamma} \subset \Gamma$ as an element of $\mathcal{OC}\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right)$, we can realize Γ as the composition of $\overline{C_1\Gamma} \subset \Gamma$ and a surface $\widetilde{\Gamma}_F = \Gamma \setminus C_1\Gamma \in \mathcal{OC}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$. Since the choice of collar is global and we may assume that diffeomorphisms of Γ restrict to the identity in a neighborhood of $\overline{C_1\Gamma}$, we deduce that the assignment $F \mapsto \widetilde{\Gamma}_F$ is compatible with diffeomorphisms.

In summary, we have produced a decomposition

$$F(n)(f) \simeq \widetilde{\Gamma}_F \circ \left(\mathrm{id}_{\left[\begin{smallmatrix} n_1-1 \\ m_1 \end{smallmatrix}\right]} \sqcup \mathrm{Cyl}(f)\right)$$

where $\widetilde{\Gamma}_F \in \mathcal{OC}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$ depends only on F. This gives a map

$$\begin{gathered} \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+(-) \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)\right) \to \\ \mathcal{OC}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right) \times_{S^1} \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix} 0 \\ (-) \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right)\right) \to \\ \mathcal{OC}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right) \end{gathered}$$

taking F to $\widetilde{\Gamma}_F$

From the construction, it is clear that R adds a sequence of cylinders to Γ, and S simply cuts out the same cylinders, such that $S \circ R \simeq \mathrm{id}$. To see that also $R \circ S \simeq \mathrm{id}$, note again that a natural transformation

$$F \in \mathrm{Nat}_{\mathcal{A}_\infty^{op}}\left(\mathcal{L}_{S^1}, \mathcal{OC}\left(\left[\begin{smallmatrix} n_1-1 \\ m_1+(-) \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)\right)$$

is determined by its value at $1 \in \mathcal{A}_\infty$, and it is easy to see that $F(1) \simeq S \circ R(F)(1)$.

3.4.3. *Gluing and Cutting Surfaces.* In this section we give an idea for how to prove that the map F of 3.43 is a homotopy equivalence.

Lemma 3.47. Let $\Gamma \in \mathcal{OC}\left(\left[\begin{smallmatrix} 0 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$ and let I be a set of arcs connecting the outgoing closed boundary to the free boundary. Assume that there is at least one arc intersecting each outgoing closed boundary component. Then I can be extended to an admissible filling arc set.

Proof. We assume that no pair of arcs are isotopic relative to their endpoints, and that no two arcs end at the same point on the free boundary. Let $\{\partial_i\Gamma\}_{0\leq i\leq n_2}$ be the list of outgoing closed boundary components. For each i, let $\{\alpha_{i,j}\}_{j\in I_i}$ be the cyclically ordered set of arcs intersecting $\partial_i\Gamma$. By the assumption that the $\alpha_{i,j}$ connect the outgoing closed boundary to the free boundary, the second condition of admissibility is automatically satisfied. For each pair (i,j) with $0\leq i\leq n_2$ and $j\in I_i$, let $x_{i,j}$ be the point at which $\alpha_{i,j}$ intersects the free boundary, and let $A_{i,j}$ be the closure of an open subset of Γ containing $\alpha_{i,j}$, such that the $A_{i,j}$ are pairwise disjoint away from the endpoints of the $\alpha_{i,j}$. For each i, let C_i be a closed collar of $\partial_i\Gamma$ such that each of the subspaces $\alpha_{i,j}\cap C_i$ and $A_{i,j}\cap C_i$ are connected for $j\in I_j$ and C_i does not intersect any $\alpha_{i',j'}$ such that $i\neq i'$. Finally, let $B_i = C_i\cup\bigcup_{j\in I_i} A_i$. The points $x_{i,j}$ divide the boundary of B_i into one component for each arc $\alpha_{i,j}$, namely there will be a component of the boundary of B_i connecting $x_{i,j}$ to $x_{i,j+1}$ according to the cyclic order of I_j. Call this component $\beta_{i,j}$. It follows that the set $\{\beta_{i,j}\}_{j\in I_i}$ cut Γ into two pieces, one of which containing the interiors of exactly the arcs intersecting $\partial_i\Gamma$. Furthermore, if y_i is the marked point on $\partial_i\Gamma$, then $\cup_{j\in I_i}(\alpha_{i,j}\cup\beta_{i,j})\setminus y_i$ is connected. Applying this procedure to each outgoing closed boundary component, we see that the first and third conditions of admissibility are satisfied. Thus the arcs $\beta_{i,j}$ and $\alpha_{i,j}$ cut Γ into a set of subspaces, only for each $\alpha_{i,j}$ as well as one piece containing the part of the boundary which is not outgoing closed. By adding more arcs, this subspace can be cut into contractible pieces. Thus we have proved that the $\alpha_{i,j}$ can be extended to a filling admissible arc set, finishing the proof. $\square$

Conjecture 3.48. The composition map

$$\mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) \xleftarrow{\text{comp}} \mathcal{O}(m_1, m_2+-)\otimes_{\mathcal{A}_\infty}\mathcal{L}_{\left[\begin{smallmatrix}n_2\\0\end{smallmatrix}\right]}$$

is a homotopy equivalence.

Proof idea: Let $\widetilde{\mathcal{O}}$ be the full subcategory of $\widetilde{\mathcal{OC}}$ on copies of intervals, and let $\widetilde{\mathcal{L}}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]}$ be the $\mathcal{A}_\infty$-submodule of $\widetilde{\mathcal{OC}}\left(\left[\begin{smallmatrix}0\\-\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)$ taking a natural number n to the arc complex of open-closed cylinders with n incoming open boundary components. Observe that if Γ is an open closed cobordism and α is an admissible filling arc set on Γ, then when cutting along arcs as in Construction 3.42, we can remember the remaining arcs in the arc set to obtain an admissible filling arc set on Γ_α. We write $\widetilde{\alpha}\in\mathcal{A}rc_0(\Gamma_\alpha)$ for this arc set. Let $\mathrm{Cyl}_\Gamma(\alpha)\in\mathcal{A}rc_0(\mathrm{Cyl}(f_\alpha))$ be the arc set on $\mathrm{Cyl}(f_\alpha)$ given by connecting each free boundary component to the outgoing circle along the shortest path. The map F lifts to the map cut in the below diagram, taking an admissible filling arc set α to the element

$$\mathrm{cut}(\alpha) = \widetilde{\alpha}\otimes\mathrm{Cyl}_\Gamma(\alpha).$$

Similarly, the contractibility of the arc complex implies that the composition map also lifts:

$$\begin{tikzcd}
\widetilde{\mathcal{OC}}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) \arrow[r, shift left, "\text{cut}"] \arrow[d] \arrow[dr, "F"] & \widetilde{\mathcal{O}}(m_1,-)\otimes_{\mathcal{A}_\infty}\widetilde{\mathcal{L}}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]} \arrow[l, shift left, "\text{comp}"] \arrow[d] \\
\mathcal{OC}\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right],\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right) & \mathcal{O}(m_1,-)\otimes_{\mathcal{A}_\infty}\mathcal{L}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]} \arrow[l, "\text{comp}"]
\end{tikzcd}$$

Fix an open-closed cobordism type Γ and an element $a \in \mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)|_\Gamma$. Contractibility of the arc complex implies that

$$\left(\widetilde{\mathcal{O}}(m_1, -) \otimes_{\mathcal{A}_\infty} \widetilde{\mathcal{L}}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]}\right)|_\Gamma$$

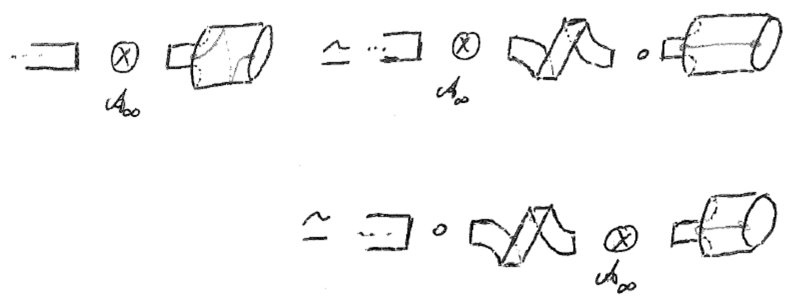
contracts onto $\operatorname{cut}(a)$. Specifically, for any

$$x \in \left(\widetilde{\mathcal{O}}(m_1, -) \otimes_{\mathcal{A}_\infty} \widetilde{\mathcal{L}}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]}\right)|_\Gamma$$

there is a $a' = (f', \alpha') \in \operatorname{Emb}^\partial(\Gamma) \times \mathfrak{B}_0(\Gamma)$ such that $x = \operatorname{cut}(f, \alpha)$. The space of zig-zags connecting a and a' in $\mathcal{A}rc_0\left(\left[\begin{smallmatrix}0\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]\right)|_\Gamma$ is contractible, such that there is a contractible space of homotopies between x and $\operatorname{cut}(a)$. Thus, cut and comp are mutually inverse homotopy equivalences.

It remains to see that they respect the $\operatorname{Diff}(\Gamma, \partial_{in}\Gamma \cup \partial_{out}\Gamma)$-action. The only thing that needs to be checked is the compatibility of the cut map with Dehn twists at the outgoing closed boundary. This generator of the mapping class group can be realized both in $\widetilde{\mathcal{O}}(m_1, -)$ by acting on the image of Γ_α in Γ, and in $\widetilde{\mathcal{L}}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]}$ by acting on the arc system on $\operatorname{Cyl}(f_\alpha)$. We now show that these actions are in fact identified in the tensor product.

We give the details for $n_2 = 1$. The argument in the general case is identical. As the below diagram shows, there is an element $T \in \mathcal{A}_\infty(1,1)$ such that composing by T and acting by the Dehn twist produce identical results up to homotopy, and transporting T across the tensor product one sees that the two Dehn twists are identified.



It now remains to prove that forgetting the arc sets in

$$\left(\widetilde{\mathcal{O}}(m_1, -) \otimes_{\mathcal{A}_\infty} \widetilde{\mathcal{L}}_{\left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]}\right)|_\Gamma$$

corresponds to quotienting by the $\operatorname{Diff}(\Gamma, \partial_{in}\Gamma \cup \partial_{out}\Gamma)$-action.

Appendix A. Coends and Bar Constructions

In this appendix we prove Lemma 2.7. Most of the argument is reproduced verbatim from private communication with Rune Haugseng.

Definition A.1. Given ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the space $\operatorname{Map}(\mathcal{C}, \mathcal{D})$ is the maximal sub-∞-groupoid of the functor ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$.

Definition A.2. Let $\mathcal{C}$ be an ∞-category. The *category of simplices* $\Delta_{/\mathcal{C}}$ is defined as the pullback of ∞-categories

$$\begin{tikzcd}
\Delta_{/\mathcal{C}} \arrow[r] \arrow[d] & \mathsf{Cat}_{\infty/\mathcal{C}} \arrow[d] \\
\Delta \arrow[r] & \mathsf{Cat}_{\infty}
\end{tikzcd}$$

where the lower functor is the embedding $\Delta \hookrightarrow \mathsf{Cat} \overset{N}{\hookrightarrow} \mathsf{Cat}_\infty$. Since $\mathsf{Cat}_{\infty/\mathcal{C}} \to \mathsf{Cat}_\infty$ is a right fibration, so is $\Delta_{/\mathcal{C}} \to \Delta$.

Remark A.3. The functor $\Delta^{op} \to \mathcal{S}$ corresponding to the right fibration $\Delta_{/\mathcal{C}} \to \Delta$ is given by

$$[n] \mapsto \mathrm{Map}([n], \mathcal{C})$$

such that this simplicial space is the ∞-category $\mathcal{C}$ viewed as a complete Segal space.

Lemma A.4. The colimit of the composite functor

$$\Delta_{/\mathcal{C}} \to \Delta \to \mathsf{Cat}_\infty$$

is equivalent to $\mathcal{C}$.

Proof. This follows from the description of Cat_∞ as complete Segal spaces. □

Definition A.5. Let Δ_* be the category with objects given by pairs $([n], i)$, where $[n] \in \Delta$ and $i \in [n]$, and a morphism $([n], i) \to ([m], j)$ is given by a functor $\phi : [n] \to [m]$ and a morphism $\phi(i) \to j$ in $[m]$.

Observation A.6. The projection $\pi : \Delta_* \to \Delta$ is the coCartesian fibration for the inclusion $\Delta \to \mathsf{Cat}_\infty$. The coCartesian morphisms over $\phi : [n] \to [m]$ are the morphisms of the form $([n], i) \to ([m], \phi(i))$.

It follows that the coCartesian fibration for the composite $\Delta_{/\mathcal{C}} \to \Delta \to \mathsf{Cat}_\infty$ is given by

$$\pi_{\mathcal{C}} : \Delta_{/\mathcal{C},*} := \Delta_{/\mathcal{C}} \times_\Delta \Delta_* \to \Delta_{/\mathcal{C}}.$$

Corollary A.7. There is an equivalence of ∞-categories

$$\Delta_{/\mathcal{C},*}[\mathrm{coCart}^{-1}] \simeq \mathcal{C}.$$

Proof. This follows from the description of colimits in Cat_∞ in [13, Section 3.3.4]. □

Definition A.8. • Let $l : \Delta \to \Delta_*$ be the section of π defined on objects by $l([n]) = ([n], n)$, which makes sense because any functor $\phi : [n] \to [m]$ admits a unique morphism $\phi(n) \to m$ in $[m]$.

• Let $\lambda : \Delta_* \to \Delta$ be the functor given on objects by $([n], i) \mapsto [i]$. Given a functor $\phi : [n] \to [m]$ and a morphism $\phi(i) \to j$, then ϕ restricts to a functor $[i] \to [j]$, which we define to be $\lambda(\phi)$.

Observation A.9. • There is a natural isomorphism

$$\mathrm{Hom}_{\Delta_*}(([n], i), l([m])) \simeq \mathrm{Hom}_\Delta([n], [m])$$

such that π is left adjoint to l.

• A morphism $([n], n) \to ([m], i)$ in Δ_* is determined by a morphism $[n] \to \lambda(([m], i))$ in Δ, i.e. we have a natural isomorphism

$$\mathrm{Hom}_{\Delta_*}(l([n], ([m], i)) \simeq \mathrm{Hom}_\Delta([n], \lambda([m], i))$$

such that l is left adjoint to λ.

Definition A.10. Let LV be the set of last vertex morphisms of Δ, i.e. functors $\phi : [n] \to [m]$ such that $\phi(n) = m$.

Lemma A.11.
- λ takes the π-coCartesian morphisms to morphisms in LV.
- l takes morphisms in LV to π-coCartesian morphisms.
- the unit map $[n] \to \lambda l([n])$ is in LV (being in fact the identity map on $[n]$).
- the counit map $l\lambda([n], i) = ([i], i) \to ([n], i)$ in π-coCartesian.

Therefore the adjunction $l \dashv \lambda$ induces an equivalence of ∞-categories

$$\Delta[\mathrm{LV}^{-1}] \simeq \Delta_*[\mathrm{coCart}^{-1}].$$

Lemma A.12. The adjoint triple $\pi \vdash l \vdash \lambda$ induces for all ∞-categories $\mathcal{C}$ an adjoint triple of functors

$$\pi_{\mathcal{C}} \vdash l_{\mathcal{C}} \vdash \lambda_{\mathcal{C}}$$

between $\Delta_{/\mathcal{C},*}$ and $\Delta_{/\mathcal{C}}$. The observations of the previous lemma also hold here, so we get an adjoint equivalence

$$\Delta_{/\mathcal{C}}[\mathrm{LV}^{-1}] \simeq \Delta_{/\mathcal{C},*}[\mathrm{coCart}^{-1}].$$

Corollary A.13. There is a natural equivalence of ∞-categories

$$\Delta_{/\mathcal{C}}[\mathrm{LV}^{-1}] \simeq \mathcal{C}.$$

Proof. This follows from Corollary A.7 and Lemma A.12. □

Definition A.14. Let *textrev* : $\Delta \to \Delta$ be the order-reversing automorphism of Δ, given on objects by $[n] \mapsto [n]$ and on morphisms by sending $\phi : [n] \to [m]$ to $\mathrm{rev}(\phi)(i) = m - \phi(n-i)$.

Lemma A.15. There is a natural pullback square

$$\begin{tikzcd} \Delta_{/\mathcal{C}^{op}} \arrow[r, "\mathrm{rev}_{\mathcal{C}}"] \arrow[d] & \Delta_{\mathcal{C}} \arrow[d] \\ \Delta \arrow[r, "\mathrm{rev}"'] & \Delta \end{tikzcd}$$

Since rev is an equivalence, so is $\mathrm{rev}_{\mathcal{C}}$.

Proof. The pullback of $\Delta_{/\mathcal{C}} \to \Delta$ along rev is the right fibration for the composite

$$\Delta^{op} \xrightarrow{\mathrm{rev}^{op}} \Delta^{op} \xrightarrow{\mathcal{C}} \mathcal{S}$$

and this composite is precisely the complete Segal space corresponding to $\mathcal{C}^{op}$. □

Definition A.16. Let IV be the set of initial-vertex morphisms in Δ, i.e. functors $\phi : [n] \to [m]$ such that $\phi(0) = 0$.

Lemma A.17. The automorphism $\mathrm{rev}_{\mathcal{C}}$ sends morphisms in LV to morphisms in IV, such that there is a natural equivalence of ∞-categories

$$\Delta_{/\mathcal{C}}[IV^{-1}] \simeq \mathcal{C}^{op}.$$

Proof. This follow from Corollary A.7 and Lemma A.15. □

Proposition A.18. The canonical functor $\Delta_{/\mathcal{C}} \to \Delta_{/\mathcal{C}}[\mathrm{LV}^{-1}]$ is coinitial and cofinal.

Proof. By definition of $\Delta_{/\mathcal{C}}[\mathrm{LV}^{-1}]$ there is a pushout square

$$\begin{tikzcd} \mathrm{LV} \arrow[r] \arrow[d] & \mathrm{LV}[\mathrm{LV}^{-1}] \arrow[d] \\ \Delta_{/\mathcal{C}} \arrow[r] & \Delta_{/\mathcal{C}}[\mathrm{LV}^{-1}] \end{tikzcd}$$

By [13, Corollary 4.1.2.6] the map $\mathrm{LV} \to \mathrm{LV}[\mathrm{LV}^{-1}]$ is cofinal, so by [13, Corollary 4.1.2.7] the pushout $\Delta_{/\mathcal{C}} \to \Delta_{/\mathcal{C}}[\mathrm{LV}^{-1}]$ is also cofinal. Applying the same argument on opposite ∞-categories, this map is also coinitial. $\square$

Corollary A.19. The four functors

$$\Delta_{/\mathcal{C}} \to \mathcal{C}, \quad \Delta_{/\mathcal{C}} \to \mathcal{C}^{op}, \quad \Delta_{/\mathcal{C}}^{op} \to \mathcal{C}, \quad \Delta_{/\mathcal{C}}^{op} \to \mathcal{C}^{op}$$

are all coinitial and cofinal.

Corollary A.20. (Bousfield-Kan) Let $\mathcal{D}$ be a cocomplete ∞-category. The colimit of a functor $F : \mathcal{C} \to \mathcal{D}$ is equivalent to the colimit of a simplicial object $\Delta^{op} \to \mathcal{D}$ given by

$$[n] \mapsto \operatorname*{colim}_{\alpha \in \mathrm{Map}([n],\mathcal{C})} F(\alpha(0)).$$

Proof. We can compute the colimit of F after composing with the cofinal map $\Delta_{/\mathcal{C}}^{op} \to \mathcal{C}$, which takes $\alpha : [n] \to \mathcal{C}$ to $\alpha(0)$. This colimit we can inturn compute in two stages, by first taking the left Kan extension along the projection $\Delta_{/\mathcal{C}}^{op} \to \Delta^{op}$, which produces a simplicial object of the given form, and then taking the colimit of this simplicial object. $\square$

Remark A.21. This derivation of the Bousfield-Kan formula is credited to Moritz Groth.

Definition A.22. Given a functor $F : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{D}$, its bar construction $B(F)$ is the colimit of the composite functor

$$\Delta_{/\mathcal{C}}^{op} \to \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{D},$$

where the first functor is induced by the functors $\Delta_{/\mathcal{C}}^{op} \to \mathcal{C}$ and $\Delta_{/\mathcal{C}}^{op} \to \mathcal{C}^{op}$ constructed above.

Lemma A.23. If $\mathcal{D}$ is a cocomplete ∞-category, then the bar construction of a functor $F : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{D}$ can be computed as the colimit of a simplicial object $\Delta^{op} \to \mathcal{D}$ given by

$$[n] \mapsto \operatorname*{colim}_{\alpha \in \mathrm{Map}([n],\mathcal{C})} F(\alpha(0), \alpha(n)).$$

Proof. The colimit over $\Delta_{/\mathcal{C}}^{op}$ can be computed in two steps by first taking the left Kan extension along the projection $\Delta_{/\mathcal{C}}^{op} \to \Delta^{op}$, which gives the desired simplicial object, and then taking the colimit of this simplicial object. $\square$

Definition A.24. Let $\epsilon : \Delta \to \Delta$ be the subdivision endofunctor, given by $[n] \mapsto [n] \star [n]^{op}$.

Lemma A.25. The functor $\epsilon^* : \mathrm{Fun}(\Delta^{op}, \mathcal{S}) \to \mathrm{Fun}(\Delta^{op}, \mathcal{S})$ takes complete Segal spaces to complete Segal spaces.

Definition A.26. The *twisted arrow category* $\mathrm{Tw}(\mathcal{C})$ of an ∞-category $\mathcal{C}$ is the ∞-category escribed by the complete Segal space $\epsilon^*\mathcal{C}$ (with $\mathcal{C}$ viewed as a complete Segal space).

Lemma A.27. There is a natural pullback square

$$\begin{tikzcd} \Delta_{/\mathrm{Tw}(\mathcal{C})} \arrow[r] \arrow[d] & \Delta_{/\mathcal{C}} \arrow[d] \\ \Delta \arrow[r, "\epsilon"'] & \Delta \end{tikzcd}$$

Proof. The pullback of $\Delta_{/\mathcal{C}} \to \Delta$ along ϵ is the right fibration for the composite

$$\Delta \xrightarrow{\epsilon} \Delta \xrightarrow{\mathcal{C}} \mathcal{S},$$

which by definition is $\Delta_{/\mathrm{Tw}(\mathcal{C})}$. □

Lemma A.28. There is a commutative diagram

$$\begin{tikzcd} \Delta^{op}_{/\mathrm{Tw}(\mathcal{C})} \arrow[r] \arrow[d] & \Delta^{op}_{/\mathcal{C}} \arrow[d] \\ \mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C} \times \mathcal{C}^{op} \end{tikzcd}$$

Lemma A.29. $\epsilon^{op} : \Delta^{op} \to \Delta^{op}$ induces homotopy equivalences on colimits valued in spaces.

Proof. This is [2, Lemma 1.1]. In short, the statement is proven for representable simplicial spaces and extending by colimits. □

Lemma A.30. $\epsilon^{op} : \Delta^{op} \to \Delta^{op}$ induces homotopy equivalences on colimits valued in spectra.

Proof. Let X be a simplicial spectrum. Then there is an equivalence

$$X \simeq \operatorname*{colim}_{\alpha:\Sigma^\infty y_{[n]} \to X} \Sigma^\infty y_{[n]} \otimes E_\alpha$$

Here is colimit is taken over the comma-∞-category $\Sigma^\infty y \downarrow X$, which is defined as the pullback

$$\begin{tikzcd} \Sigma^\infty y \downarrow X \arrow[r] \arrow[d] & \mathsf{Sp}^{\Delta^{op}}_{/X} \arrow[d] \\ \Delta \arrow[r, "\Sigma^\infty y"'] & \mathsf{Sp}^{\Delta^{op}} \end{tikzcd}$$

and E_α is a spectrum.

By [2, Lemma 1.1], pulling back along ϵ induces a homeomorphism, in particular a homotopy equivalence, between geometric realizations. In ∞-categorical terms, the realization is precisely the colimit, such that $\epsilon^{op} : \Delta^{op} \to \Delta^{op}$ preserves colimits, hence ϵ^{op} is cofinal, hence ϵ is coinitial. □

Corollary A.31. $\Delta_{/\mathrm{Tw}(\mathcal{C})} \to \Delta_{/\mathcal{C}}$ is coinitial.

Proof. The (the dual of) [13, Proposition 4.1.2.15], the pullback of a coinitial map along a Cartesian fibration is coinitial. ☐

Proposition A.32. Given a functor $F : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{D}$, its coend, given by the colimit of the composite

$$\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{D},$$

is equivalent to the bar construction $B(F)$.

Proof. We have a commutative square

$$\begin{tikzcd}
\Delta^{op}_{/\mathrm{Tw}(\mathcal{C})} \arrow[r] \arrow[d] & \Delta^{op}_{/\mathcal{C}} \arrow[d] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C} \times \mathcal{C}^{op}
\end{tikzcd}$$

where the top horizontal and left vertical maps are cofinal by Corollaries A.19 and A.31. ☐

References

[1] Gregory Arone and Michael Ching, *A classification of Taylor towers of functors of spaces and spectra*, arXiv:1209.5661v3, 25 Sep 2012.

[2] Marcel Bökstedt, W.C. Hsiang and Ib Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Inventiones mathematicae (1993) Volume 111, Issue 3, page 465–540.

[3] Andrew J. Blumberg, David Geppner and Goncalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. 17 (2013) 733-838.

[4] Andrew J. Blumberg, David Geppner and Goncalo Tabuada, *Uniqueness of the multiplicative cyclotomic trace*, arXiv:1103.3923v3, 1 Jul 2015.

[5] Kevin J. Costello, *Topological conformal field theories and Calabi-Yau categories*, arXiv:math/0412149, 7 Dec 2004.

[6] John Francis, *The tangent complex and Hochschild cohomology of E_n-rings*, arXiv:1104.0181v3, 1 Apr 2011.

[7] David Gepner, Rune Haugseng, Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, arXiv:1501.02161v2 9 Jan 2015.

[8] David Gepner, Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Advances in Mathematics 279 (2015), pp. 575-716.

[9] Saul Glasman, *Day convolution for infinity-categories*, arXiv:1308.4940v4, 22 Aug 13.

[10] Saul Glasman, *A Spectrum-level Hodge Filtration on Topological Hochschild Homology*, arXiv:1408.3065v3, 13 Aug 2014.

[11] Thomas G. Googwillie, *Calculus III: Taylor Series*, Geom. Topol. 7 (2003) 645–711.

[12] Angela Klamt, *Natural operations on the Higher Hochschild homology of commutative algebras*, Paper C of PhD thesis, 2013.

[13] Jacob Lurie, *Higher Topos Theory*, ver. April 2017.

[14] Jacob Lurie, *Higher Algebra*, ver. 16 May 2016.

[15] Jacob Lurie, *Derived Algebraic Geometry X: Formal Moduli Problems*, 2011.

[16] Saunders Mac Lane, *Categories for the Working Mathematician*, Second Edition, Springer Verlag, 1997.

[17] Randy McCarthy, *Dual calculus for functors to spectra*, Amer. Math. Soc., Providence, RI, 183–215, 2001.

[18] Thomas Nikolaus, *Stable ∞-Operads and the multiplicative Yoneda lemma*, arXiv:1608.02901, 9 Aug 2016.

[19] Daniela Egas Santander, *Comparing fat graph models of moduli space*, arXiv:1508.03433 14 Aug 2016.

[20] Nathalie Wahl, *Universal Operations in Hochschild Homology*, J. Reine Angew. Math. 720 (2016), 81-127.

[21] Nathalie Wahl and Craig Westerland, *Hochschild homology of structured algebras*, Adv. Math. 288 (2016), 240-307.