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PhD Thesis

Non-commutative Covering Spaces and Their Symmetries

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Abstract

The main goal of this thesis is to propose a notion analogous to covering spaces in classical geometry. This is motivated by the author's long term goal of defining the (étale) fundamental group of a non-commutative space and put forth a good notion of monodromy.

We will present a notion of a non-commutative covering space using Galois theory of Hopf algebroids. We will look at basic properties of classical covering spaces that generalize to the non-commutative framework. Afterwards, we will explore a series of examples. We will start with coverings of a point and central coverings of commutative spaces and see how these are closely tied up. Coupled Hopf algebras will be presented to give a general description of coverings of a point. We will give a complete description of the geometry of the central coverings of commutative spaces using the coverings of a point. A topologized version of Hopf categories will be defined and its corresponding Galois theory. Using this and basic concepts from algebraic geometry and spectral theory, we will give a full description of the general structure of non-central coverings. Examples of coverings of the rational and irrational non-commutative tori will also be studied. Using the non-commutative analogue of the hyperelliptic involution, we will show that unlike the classical case, the non-commutative sphere is a covering of the non-commutative torus. There is a purely non-commutative phenomenon happening to non-commutative coverings, namely, their symmetry is two-sided. We will explain this and relate it to bi-Galois theory. Using the OZ-transform, we will show that non-commutative covering spaces come in pairs. Several categories of covering spaces will be defined and studied. Appealing to Tannaka duality, we will explain how this lead to a notion of an étale fundamental group. Finally, in the last chapter we will discuss possible future projects.

Resumé

Det primære formål med denne afhandling er at introducere et begreb svarende til overdækninger fra klassisk geometry. Dette er motiveret af forfatterens langsigtede mål med at definere (étale) fundamentalgruppen for et ikke-kommutativt rum og fremsætte et begreb om monodromi.

Vi introducerer ikke-kommutative overdækninger ved hjælp af Galoisteori for Hopf algebroids. Vi betragter basale egenskaber for klassiske overdækninger der generaliserer til den ikke-kommutative kontekst. Dernæst udforskes en række eksempler. Vi starter med overdækninger af et punkt og centrale overdækninger a kommutative rum og ser hvordan disse er tæt forbundet. Koblede Hopf algebraer vil blive prsenteret for at give en generel beskrivelse af et punkts overdækninger. En topologiseret version af Hopf kategorier bliver defineret, såvel som dets tilhørende Galoisteori. Ved brug af dette og basale koncepter fra algebraisk geometri og spektralteori, vil vi give en fuldstændig beskrivelse af den generelle struktur for ikke-centrale overdækninger. Eksempler på overdækninger af rationelle og irrationelle ikke-kommutative tori bliver også studeret. Ved brug af den ikke-kommutativ analog til hyperelliptisk involution, vil vi vise at, i modsætning til det klassiske tilfælde, er den ikke-kommutative sfære er en overdækning af den ikke-kommutative torus. Ikke-kommutative overdækninger har den rent ikke-kommutative egenskab at deres symmetri er tosidet. Vi forklarer dette og relaterer det til bi-Galoisteori. Ved brug af OZ-transformationer, viser vi at ikke-kommutative overdækninger kommer i par. Adskillige kategorier af overdkninger vil blive definieret og studeret. Med appel til Tannaka dualitet, forklarer vi hvordan dette fører til et begreb om en étale fundamentalgrupp. Endelig, i det sidste kapitel, diskuterer vi mulige fremtidige projekter.

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Contents

Chapter 1

Introduction

Discovery is the privilege of the child: the child who has no fear of being once again wrong, of looking like an idiot, of not being serious, of not doing things like everyone else.

–Alexander Grothendieck

1.1 Motivation

The main motivation of this PhD project is to put forth a realization of monodromy representation for non-commutative spaces. In its classical sense, monodromy representation refers to a representation of a group G one can write by viewing G as the fundamental group of some topological space X together with a flat vector bundle $E \xrightarrow{p} X$. This is accomplished by viewing $g \in G$ as (based) loops in X and lifting them to paths in E. Formulating monodromy representation for non-commutative spaces requires answering several important preliminary questions: what *is* the fundamental group of a non-commutative space?

Classically, the fundamental group of a topological space X is the homotopy classes of (based) loops in X. The group operation is concatenation (composition followed by reparametrization) of loops. Such a group depends on the choice of the base point. This presents a problem since non-commutative spaces need not have classical points. Even if we ignore the dependence on the based point, the same problem persists since loops and their homotopy involved points of X. One way to go around this difficulty is to use a different but equivalent definition of the fundamental group of a space X. The fundamental group of X (for sufficiently nice X) is also the automorphism group of the universal covering space $\tilde{X}$ of X, see [8]. If we adopt this definition, we will be faced with yet another daunting task: what *is* the universal covering space of a non-commutative space?

The task now boils down into defining what a covering of a non-commutative space is. Then, if things go well, a terminal object in the category of such coverings exists and can be regarded as the universal covering of the given non-commutative space. Alternatively, one can define the universal covering of a non-commutative space, by duality, as the colimit of all of the coverings of that non-commutative space. At any rate, we are faced with the task of defining what a covering space of a non-commutative space should be. This is the path that we are going to take.

The main goal of this thesis to propose a covering theory for non-commutative spaces which we hope, in future works, will lead to a complete realization of the monodromy theorem for non-commutative spaces. A more primitive question remains. Why would one be interested in giving an analogue of the monodromy representation for non-commutative spaces? To answer this, I would like to first quote Cuntz [15]: *The two fundamental "machines" of non-commutative geometry are (bivariant) topological K-theory and cyclic homology... Cyclic theory can be viewed as a far reaching generalization of the classical de Rham cohomology, while bivariant K-theory includes the topological K-theory of Atiyah-Hirzebruch as a special case.* At the present state of non-commutative geometry, invariants of non-commutative spaces are produced with the framework of either (bivariant) K-theory or cyclic homology. One of the goals of the author, as proposed by Ryszard Nest, is to produce a new set of invariants for non-commutative spaces. The problem has been conceptualized during the first quarter of the 2014 but no progress has been made until December of that year. It was during a walk along a Christmas market in Bonn when the author and his supervisor settled with the present formulation.

As far as the limited knowledge of the author goes, there has been no substantial work done towards this endeavor. However, I would like to emphasize that the whole program resembles the path Grothendieck took to define the étale fundamental group for schemes. In other words, we will proceed by partially going against what Grothendieck said (see the quotation in the beginning of this section) by following what Grothendieck did.

1.2 Overview

We have presented in section 1.1 the problems that motivated this thesis. Let us outline our proposed solution to the said challenges. Recall that the reason we are interested in covering spaces is to be able to define the notion of a (étale) fundamental group (or groupoid) for non-commutative spaces. In this generality, this task requires a lot of work. For the mean time, we will only be considering finite Galois covering spaces. Such covering spaces is enough to makes sense of the étale fundamental group for non-commutative spaces. Apart from giving a notion of a non-commutative finite Galois covering spaces, we will also tackle examples and their properties.

In appendix A we discuss the necessary aspects of covering spaces we will be needing in this thesis. Covering spaces are old and well-studied objects. However, instead of referring to existing literature we will explicitly write down aspects of the theory that we will be considering and we will be looking in great detail. In particular, a covering of a topological space X consists of a topological space Y and a continuous surjection $Y \xrightarrow{p} X$ such that the fibers of p are all discrete. With this information, there is a naturally associated discrete group G, the group of deck transformations. We will only deal with the case when the action of G on p is Galois. In the dual picture, X and Y give algebras A and B, respectively. The surjection p turns into an inclusion $A \xrightarrow{i} B$. The group G and its Galois action on Y with invariants X turn into a Hopf algebra H coacting on B with coinvariants A, where the coaction is also Galois. Unlike the classical case, given a general inclusion of algebras $A \subseteq B$ there is no natural way to spit out a Hopf algebra coacting Galois on B whose coinvariants is A. There are even examples of inclusions $A \subseteq B$ and non-isomorphic Hopf algebras H_1 and H_2 with a Galois coaction on B with A as the space of coinvariants. Another important issue at hand, the map p is not just a surjective map. It has discrete fibers. This condition is as important as surjectivity. However, there is no existing notion of discreteness in non-commutative geometry. This means that an extension algebra B of A is not enough to define a covering space. It is not clear what conditions can be imposed to the inclusion $A \subseteq B$ to reflect discreteness. This problem goes away by making the quantum symmetries H of $A \subseteq B$ part of the data and requiring that H is finite-dimensional (in the case when the base ring is a field, otherwise, one can settle for finitely-generated and projective over the base ring). This finiteness condition solves the discreteness problem using the fact that finite subsets of Hausdorff spaces are discrete.

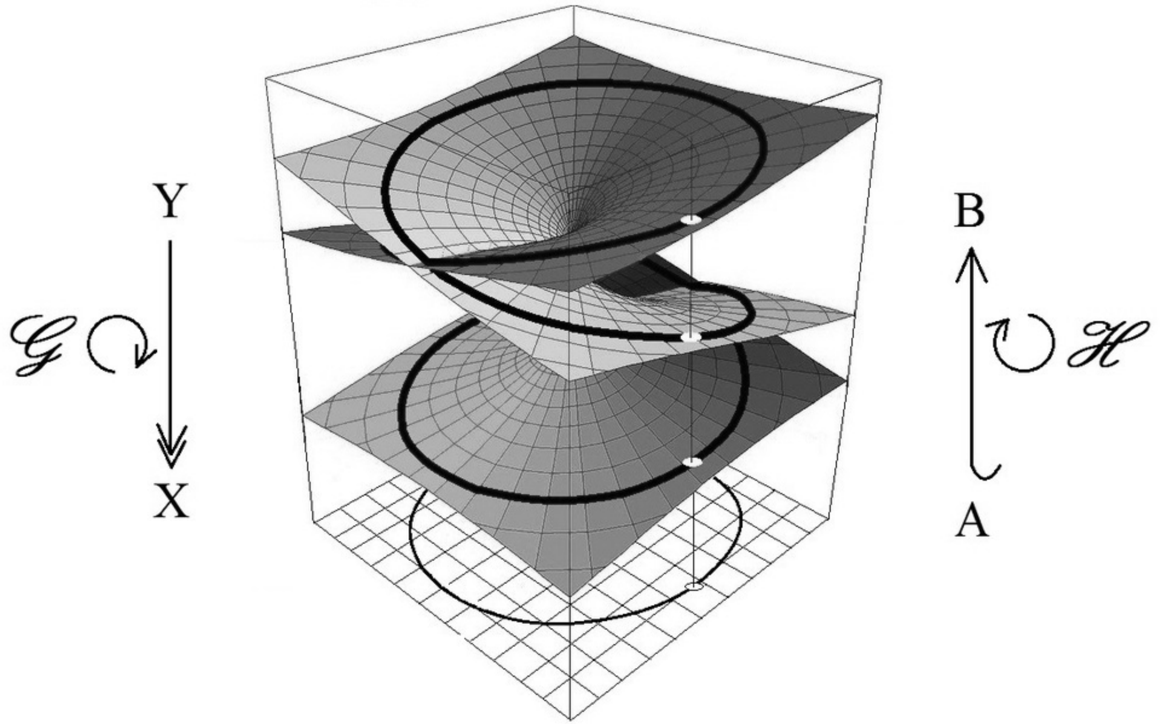


Figure 1.1: Hopf algebroids associated to covering spaces.

In chapter 2, we will discuss Hopf algebroids. Contrary to what we originally conditioned, we will not restrict our formulation to Hopf algebras. Apart from the naturality of the use of Hopf algebroids, it is also at the very core of the tool that we are going to use− Galois theory. With some finiteness condition, non-commutative covering spaces are nothing but Hopf-Galois extensions. We will motivate our choice of a definition in section 2.1 where we will start by constructing a groupoid out of a covering space. Upon convincing ones self that Hopf algebroids are the non-commutative analogue of groupoids, our choice of using groupoids will be justified. The rest of the section of chapter 2 will be devoted for two very crucial task. The first one is to define equivalence of such coverings. Secondly, we will see how composition of covering works in the non-commutative framework.

In chapter 4 we will scrutinize coverings of commutative spaces. We will restrict our attention to central coverings. We will start by looking at coverings of a point and then give a reconstruction theorem for covering spaces whenever the algebras involved are commutative. The nice thing about non-commutative covering spaces is that they already manifest themselves even in the commutative set-up. This is due to the fact that a commutative algebra can be a subalgebra of a non-commutative one. We will give a structure theorem which completely describes the structure of central local non-commutative coverings. In the next chapter, we will deal with the general situation of a non-commutative covering of a commutative space in which the commutative space need not be central in the extension space. We will use topological (coupled) Hopf categories to accomplish this task. Surprisingly, the general picture is captured by a single important example which we will describe in appendix B. In the same appendix, we will introduce topological Hopf categories. These are Hopf categories that carry some continuity structure with it.

We will resume in chapter 6 to give examples and describe coverings of the non-commutative sphere. We will tackle separately the case for the rational and the irrational non-commutative tori. In our formulation of non-commutative covering space, we will be introduce different types of coverings. In particular, we will speak of *local* and *uniform* coverings. In chapter 7 we will discuss the geometry behind such types.

In chapter 8 we will explain a curious situation for non-commutative covering spaces− their quantum symmetries come in pairs. We will relate this two-sidedness of the quantum symmetry with bi-Galois theory. In chapter 9, we will explore other examples of non-commutative covering space. In particular, we will discuss the coverings of the algebra $\mathcal{K}$ of compact operators and the non-commutative sphere. The algebra $\mathcal{K}$ is regarded, in non-commutative geometry, to be infinitesimal neighborhoods. We will show that to some extent, coverings of this algebra is the *same* as the coverings of a point. The other example concerns the non-commutative sphere. In the classical set-up the torus $\mathbb{T}^2$ is a branched covering of the sphere S^2. In the present set-up we will show that it's the other way around, the non-commutative sphere is a covering of the non-commutative torus.

In chapter 10, we will discuss further the categories of coverings of A. We will explore some properties that are unique in the present situation. Finally, in chapter 11 we will enumerate problems time didn't permit to be settled. Also, we will look at natural questions one can ask regarding such covering spaces. In classical geometry, covering spaces enjoy a lot of lifting properties. It is then a natural question to ponder whether same thing happen in the non-commutative set-up. We will formulate this question properly. Another thing the author would have been interested to do is to show how deformation quantization, a very powerful tool to produce interesting examples of non-commutative spaces, gives examples of covering spaces. The last two section of chapter 11 will be speculative and informal in nature. We will give an over-view of what the author expects to happen in the sequel of this project.

1.3 Notations and conventions

We will work with arbitrary base ring k which is assumed to be associative, commutative and unital. In most examples, we will be working with $k = \mathbb{C}$ and we will mention this explicitly. Otherwise, k is arbitrary. All algebras over k are associative and unital. All topological spaces we will be considering are compact and Hausdorff. For simplicity, we will also assume all topological spaces that appears in this manuscript are connected.

We will use Sweedler notation and Einstein summation convention all through out this paper. We will describe such notation when faced by the necessity to do so.

If $\mathscr{C}$ is a category, we will denote by $Hom_{\mathscr{C}}(x, y)$ or $\mathscr{C}_{x,y}$ the hom-set of arrows from x to y depending on which is easier to write. We will also write $x \in \mathscr{C}$ instead of $x \in Ob(\mathscr{C})$ to denote that x is an object of $\mathscr{C}$.

Classical covering spaces will always be finite and Galois unless specifically stated. Since we non-commutative finite, Galois coverings are the main topic of this project, we will simply call them coverings and reserve the term classical coverings for the commutative ones.

We will put $\blacksquare$ to signify the end of a proof while we use $\square$ for definitions and examples. We will use Ⅎ to denote the usual flip of tensor factors, or flip of cartesian product of sets, and the likes. For example, if $A \subset B \times C$ then $\text{Ⅎ}(A) \subseteq C \times B$, where $(c, b) \in \text{Ⅎ}(A)$ if and only if $(b, c) \in A$. This notation will dramatically shorten and compactify equations and diagrams to follow.

For a k-algebra R, we will denote by R^e its universal enveloping algebra, i.e. $R^e = R \otimes R^{op}$.

Chapter 2

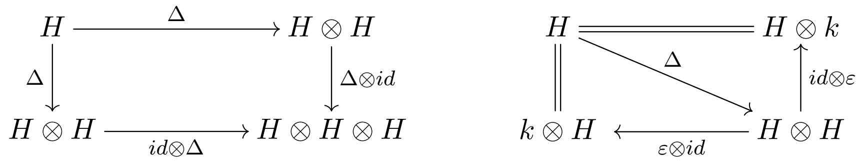
Hopf algebroids

If people do not believe
that mathematics is simple,
it is only because
they do not realize
how complicated life is.

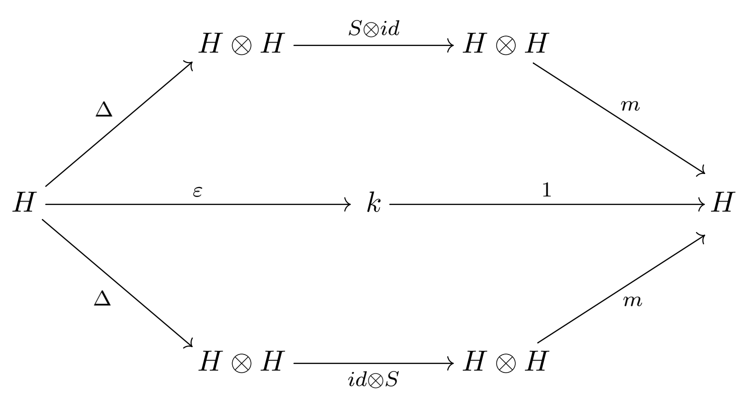
−John von Neumann

2.1 Definitions

It has been a general consensus in noncommutative geometry that the analogue of groups are certain class of Hopf algebras called quantum groups. A *Hopf algebra* H over k is an associative unital algebra $(H, m, 1)$ together with algebra maps $H \xrightarrow{\Delta} H \otimes H$ (*coproduct*), $H \xrightarrow{\varepsilon} k$ (*counit*) and a linear map $H \xrightarrow{S} H$ (*antipode*) making the following diagrams commute.



(2.1)



(2.2)

The diagrams 2.1 express the *coassociativity* of Δ and its *counitality* with respect to ε. With Δ and ε, $End(H)$ becomes a unital ring under convolution

$$f \star g : H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{m} H \tag{2.3}$$

with $H \xrightarrow{\varepsilon} k \xrightarrow{1} H$ as the unit. The diagram 2.2 above expresses the fact that S is the convolution inverse of id. From this, we immediately see that given a bialgebra H (i.e. an algebra H with coproduct and a counit which are algebra maps), there is at most one antipode which makes it into a Hopf algebra. We call a Hopf algebra a *quantum group* if it has a bijective antipode. We will use Sweedler notation convention. Explicitly, for any $h \in H$, instead of writing $\Delta(h) = \sum_{i=1}^{n} (h_1)_i \otimes (h_2)_i$, we will write it as $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

Recently, there has been great interest in Hopf-like structures in which the base ring is not necessarily commutative. Originally, we sought to develop the theory of noncommutative covering spaces using only Hopf algebras but there has been a great need to use a more general structure, one in which the base ring is possibly noncommutative. We will describe one which suits our purpose called a Hopf algebroid. Following the discussion in Böhm [6], we will define what Hopf algebroids are by defining some intermediate structures first. We will also define morphisms between them. Through out the remainder of this section, k will be an associative, commutative unital ring and R and L will be associative unital k-algebras. A Hopf algebroid resembles a Hopf algebra− it will have bialgebra-like structures defined over R and L and an antipode that relates them. Since we are mainly interested in the situation where Hopf algebroids are seen as further generalization of quantum groups, we will assume that Hopf algebroids have bijective antipodes. As it turns out, R and L will be anti-isomorphic k-algebras. However, for notational convenience it will be easier to denote them as such, where we will use R and L to denote right and left structures, respectively. In addition, whenever we have a Hopf-like structure we will use Sweedler notation convention to write down coproduct and coaction images. We will set these notations properly when we get to these coproducts and coactions. For a ring R, we will denote by ${}_R\mathcal{M}$ and by $\mathcal{M}_R$ the categories of left and right R-modules.

Before giving the definition of a Hopf algebroid, let us define first several intermediate structures. An *R-ring* is a monoid object in the category of R-bimodules. Explicitly, an R-ring is a triple (A, μ, η) where $A \otimes_R A \xrightarrow{\mu} A$ and $R \xrightarrow{\eta} A$ are R-bimodule maps satisfying the associativity and unit axioms similar for algebras over commutative rings. A morphism of R-rings is a monoid morphism in category of R-bimodules. It is important to note that the unit $R \xrightarrow{\eta} A$ of the R-ring (A, μ, η) completely determines the R-ring structure of A as the following lemma suggests.

Lemma 1. *There is a bijection between R-rings (A, μ, η) and k-algebras maps $R \xrightarrow{\eta} A$.*

This follows from the universal property of the canonical surjection

$$A \otimes_k A \twoheadrightarrow A \otimes_R A\ .$$

Similar to the case of algebras over commutative rings, we can define modules over R-rings. For an R-ring (A, μ, η), a *right* (resp. *left*) *(A, μ, η)-module* is an algebra for the monad $- \otimes_R A$ (resp. $A \otimes_R -$) on the category $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$) of right (resp. left) modules over R.

We can dualize all the objects we have defined in the previous paragraph. An *R-coring* is a comonoid in the category of R-bimodules, i.e a triple (C, Δ, ϵ) where $C \xrightarrow{\Delta} C \otimes_R C$ and $C \xrightarrow{\epsilon} R$ are R-bimodule maps satisfying the coassociativity and counit axioms dual to those axioms satisfied by the structure maps of an R-ring. A morphism of R-corings is a morphism of comonoids. Given an R-coring (C, Δ, ϵ), similar to coalgebras over commutative rings, we define a *right* (resp. *left*) *(C, Δ, ϵ)-comodule* as a coalgebra for the comonad $- \otimes_R C$ (resp. $C \otimes_R -$) on the category $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$).

Definition 1. A *right* (resp. *left*) *R-bialgebroid B* is an $R \otimes_k R^{op}$-ring (B, μ, η) and an R-coring (B, Δ, ϵ) satisfying:

(a) $\eta : R \otimes R^{op} \longrightarrow B$ determines k-algebra maps $R \xrightarrow{s} B$ and $R^{op} \xrightarrow{t} B$ with commuting images. These maps are compatible to the R-bimodule structure of B as an R-coring thru the following relation:

$$r \cdot b \cdot r' := bs(r')t(r), \quad (\text{resp. } r \cdot b \cdot r' := s(r)t(r')b,) \quad \forall r, r' \in R, b \in B.$$

(b) With the above R-bimodule structure on B one can form $B \otimes_R B$. The coproduct Δ is required to corestrict to a k-algebra map into

$$B \times_R B := \left\{ \sum_i b_i \otimes_R b_i' \,\middle|\, \sum_i s(r) b_i \otimes_R b_i' = \sum_i b_i \otimes_R t(r) b_i', \forall r \in R \right\} \tag{2.4}$$

respectively,

$$B \,{}_R\!\times B := \left\{ \sum_i b_i \otimes_R b_i' \,\middle|\, \sum_i b_i t(r) \otimes_R b_i' = \sum_i b_i \otimes_R b_i' s(r), \forall r \in R \right\}. \tag{2.5}$$

(c) The counit $B \xrightarrow{\epsilon} R$ extends the right (resp. left) regular R-module structure on R to a right (resp. left) (B, s)-module.

A *morphism* of R-bialgebroids is a morphism of $R \otimes R^{op}$-rings and R-corings. □

Remark 1.

(1) Any k-algebra maps $s : R \longrightarrow B$ and $t : R^{op} \longrightarrow B$ with commuting images define an $R \otimes_k R^{op}$-ring structure on B which we will denote by (B, s, t). The maps s and t of condition (a) are called the *source* and *target* maps, respectively.

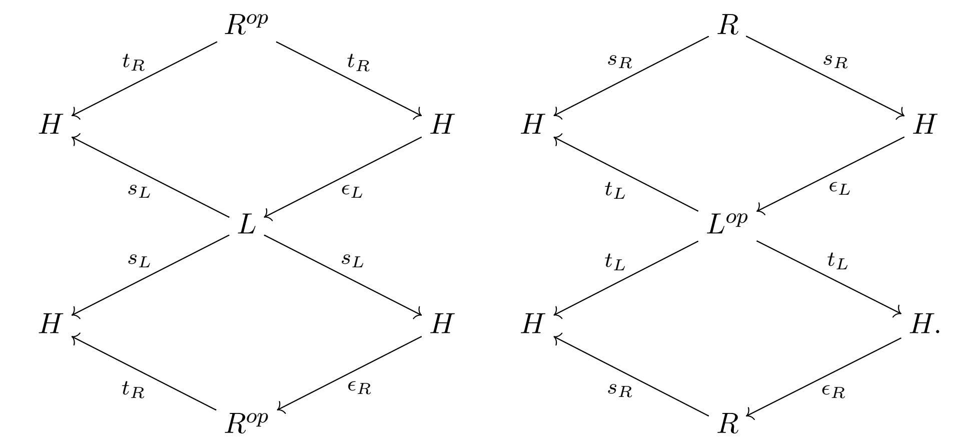
(2) The k-submodule $B \times_R B$ (resp. $B \,{}_R\!\times B$) of $B \otimes_R B$ is a k-algebra with factorwise multiplication. This is called the *Takeuchi product*. The map $R \otimes_k R^{op} \longrightarrow B \times_R B$, $r \otimes_k r' \mapsto t(r') \otimes_R s(r)$ is easily seen to be a k-algebra morphism and hence, $B \times_R B$ is an $R \otimes_k R^{op}$-ring. The corestriction of Δ is an $R \otimes_k R^{op}$-bimodule map. Hence, Δ is an $R \otimes R^{op}$-ring map. The same is true for $B \,{}_R\!\times B$.

(3) The source map s is a k-algebra map and so it defines a unique R-ring structure on B. The right-sided version of condition (c) explicitly means that $r \cdot b := \epsilon(s(r)b)$, $\forall r \in R, b \in B$ defines a right (B, s)-action on R.

We now have the necessary ingredients to define what a Hopf algebroid is.

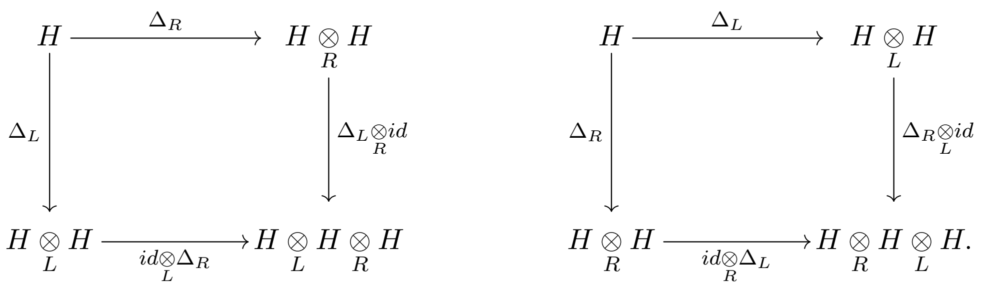
Definition 2. Let k be a commutative, associative, unital ring and let L and R be associative k-algebras. A *Hopf algebroid* $\mathcal{H}$ is a triple $\mathcal{H} = (H_L, H_R, S)$ where H_L and H_R are bialgebroids having the same underlying k-algebra H and S is a bijective k-module map $S : H \longrightarrow H$, called the *antipode*. Specifically, H_L is a left L-bialgebroid with (H, s_L, t_L) and $(H, \Delta_L, \epsilon_L)$ as its underlying $L \otimes_k L^{op}$-ring and L-coring structures, respectively. Similarly, H_R is a right R-bialgebroid with (H, s_R, t_R) and $(H, \Delta_R, \epsilon_R)$ as its underlying $R \otimes_k R^{op}$-ring and R-coring structures, respectively. Let us denote by μ_L (resp. μ_R) the multiplication on (H, s_L) (resp. (H, s_R)). They are subject to the following compatibility conditions.

(a) The source maps s_R, s_L, target maps t_R, t_L and counit maps ϵ_R, ϵ_L fit in commutative diagrams



(2.6)

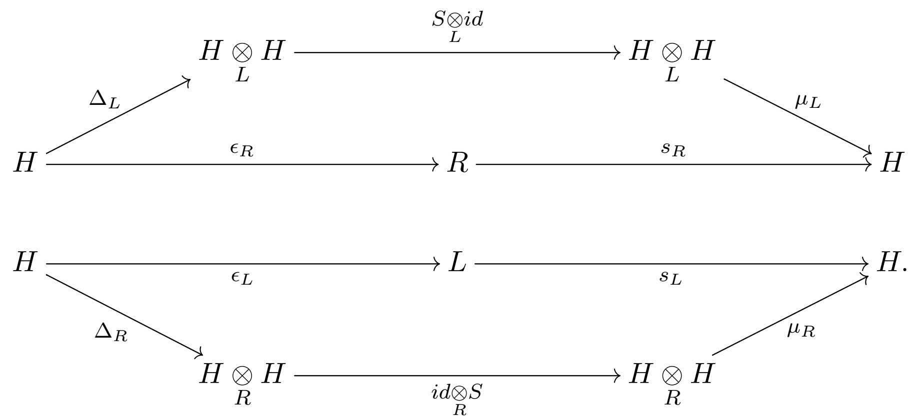
(b) The coproducts Δ_L and Δ_R commute, i.e.



(2.7)

(c) For all $l \in L, r \in R$ and for all $h \in H$ we have $S(t_L(l)ht_R(r)) = s_R(r)S(h)s_L(l)$.

(d) S is the convolution inverse of the identity map i.e., the following diagram commute



(2.8)

□

Remark 2.

(1) Let us note that condition (c) in the definition of a bialgebroid implies that $\epsilon_L \circ s_L : L \longrightarrow L$ is the identity. Similarly, $\epsilon_R \circ s_R : R \longrightarrow R$ is also the identity. Using condition (a) in the definition of a Hopf algebroid, we see that the following compositions define pairs of inverse k-algebra maps.

$$L \xrightarrow{\epsilon_R \circ s_L} R^{op} \xrightarrow{\epsilon_L \circ t_R} L \qquad R \xrightarrow{\epsilon_L \circ s_R} L^{op} \xrightarrow{\epsilon_R \circ t_L} R$$

This is particular implies that R and L are anti-isomorphic k-algebras.

(2) In the constituent bialgebroids H_R and H_L, the counits ϵ_R and ϵ_L extend the regular module structures on the base rings R and L to the R-ring (H, s_R) and to the L-ring (H, s_L), respectively. Equivalently, the counits extend the regular module structures on the base rings R and L to the R^{op}-ring (H, t_R) and to the L^{op}-ring (H, t_L). This particularly implies that the maps $s_L \circ \epsilon_L$, $t_L \circ \epsilon_L$, $s_R \circ \epsilon_R$ and $t_R \circ \epsilon_R$ are idempotents. This means that the images of s_R and t_L coincides in H. Same is true for the images of s_L and t_R.

(3) Part (2) implies that Δ_L, apart from being an L-bimodule map, is also an R-bimodule map. Similarly, Δ_R is an L-bimodule map and so the diagrams in condition (b) make sense.

(4) We can equip H with two (R, L)-bimodule structures one using t_R and t_L and the other using s_R and s_L. Condition (c) relates these two (R, L)-bimodules structures via the antipode S which in turn makes the diagram in condition (d) well-defined.

(5) The convolution structure condition (d) refers to a convolution structure one can define analogous to the one for linear maps from a coalgebra to an algebra. See section 2.4 for this convolution structure. From this, we see that the antipode S of a Hopf algebroid is unique.

(6) Since there are two coproducts involved in a Hopf algebroid, namely Δ_L and Δ_R, we will use different Sweedler notations for their corresponding components. We will write $\Delta_L(h) = h_{[1]} \otimes_L h_{[2]}$ and $\Delta_R(h) = h^{[1]} \otimes_R h^{[2]}$ for $h \in H$.

(7) With a fixed bijective antipode S, the constituent left- and right-bialgebroids of a Hopf algebroid determine each other, see for example proposition 4.3 of Böhm-Szlachányi [7]. In view of this and the fact that L and R are anti-isomorphic, in what follows where we will be mainly interested with Hopf algebroids with bijective antipodes we will simply call $\mathcal{H}$ a Hopf algebroid *over* R instead of explicitly mentioning L.

Hopf algebroids are noncommutative generalization of Hopf algebras and dualization of groupoids. With this, it is natural to expect that there are several natural notions of a morphism between Hopf algebroids. In the next definition, we present algebraic and geometric morphisms of Hopf algebroids. The former sees Hopf algebroids as noncommutative generalization of Hopf algebras while the latter sees Hopf algebroids as dualization of groupoids.

Definition 3. Let (H_L, H_R, S) and (H'_L, H'_R, S') be Hopf algebroids over R. An *algebraic morphism*

$$(H_L, H_R, S) \xrightarrow{(\varphi_L, \varphi_R)} (H'_L, H'_R, S') \tag{2.9}$$

of Hopf algebroids is a pair of a left-bialgebroid morphism φ_L and a right-bialgebroid morphism φ_R for which the following diagrams commute

$$\begin{array}{ccc} H_L & \xrightarrow{S} & H_R \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi_R} \\ H'_L & \xrightarrow[S']{} & H'_R \end{array} \qquad \begin{array}{ccc} H_R & \xrightarrow{S} & H_L \\ \downarrow{\scriptstyle \varphi_R} & & \downarrow{\scriptstyle \varphi_L} \\ H'_R & \xrightarrow[S']{} & H'_L \end{array} \tag{2.10}$$

and composition of such pairs is pairwise.

Let R and R' be k-algebras and (H_L, H_R, S) and $(K_{L'}, K_{R'}, S')$ be Hopf algebroids over R and R', respectively. In view of remark (7) above, denote $L = R^{op}$ and $L' = (R')^{op}$. A *geometric* *morphism* $(H_L, H_R, S) \longrightarrow (K_{L'}, K_{R'}, S')$ of Hopf algebroids is a pair (f, ϕ) of k-algerba maps $R \xrightarrow{f} R'$ and $H \xrightarrow{\phi} K$, where H, K denote the underlying k-algebra structures of the Hopf algebroids under consideration. These two maps satisfy the following compatibility conditions.

(a) f and ϕ intertwines the source, target and counit maps of the left-bialgebroid structures of $\mathcal{H}$ and $\mathcal{K}$, i.e.

$$\begin{array}{ccc} H & \xrightarrow{\epsilon_L^H} & L \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle f^{op}} \\ K & \xrightarrow[\epsilon_L^K]{} & L' \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{t_L^H} & H \\ {\scriptstyle f^{op}}\downarrow & & \downarrow{\scriptstyle \phi} \\ L' & \xrightarrow[t_L^K]{} & K \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{s_L^H} & H \\ {\scriptstyle f^{op}}\downarrow & & \downarrow{\scriptstyle \phi} \\ L' & \xrightarrow[s_L^K]{} & K. \end{array} \tag{2.11}$$

Same goes for the source, target and counit maps of the right-bialgebroid structures.

(b) In view of condition (a), the k-bimodule map $\phi \otimes_k \phi$ defines k-bimodule maps

$$H\ {}_L\!\otimes H \xrightarrow{\phi\ {}_f\!\otimes\ \phi} K\ {}_{L'}\!\otimes K, \qquad H \otimes_R H \xrightarrow{\phi\ \otimes_f \phi} K \otimes_{R'} K.$$

We then require that the following diagrams commute.

$$\begin{array}{ccc} H\ {}_L\!\otimes H & \xrightarrow{\phi\ {}_f\!\otimes\ \phi} & K\ {}_{L'}\!\otimes K \\ {\scriptstyle \mu_L^H}\downarrow & & \downarrow{\scriptstyle \mu_L^K} \\ H & \xrightarrow[\phi]{} & K \end{array} \qquad \begin{array}{ccc} H \otimes_R H & \xrightarrow{\phi\ \otimes_f \phi} & K \otimes_{R'} K \\ {\scriptstyle \mu_R^H}\downarrow & & \downarrow{\scriptstyle \mu_R^K} \\ H & \xrightarrow[\phi]{} & K \end{array} \tag{2.12}$$

(c) Also by of condition (a), the k-bimodule maps $\phi\ {}_f\!\otimes\ \phi$ and $\phi\ \otimes_f \phi$ of condition (b) further define k-bimodule maps

$$H\ {}_L\!\times H \xrightarrow{\phi\ {}_f\!\times\ \phi} K\ {}_{L'}\!\times K, \qquad H \times_R H \xrightarrow{\phi\ \times_f \phi} K \times_{R'} K. \tag{2.13}$$

We then require that the following diagrams commute.

$$\begin{array}{ccc} H & \xrightarrow{\phi} & K \\ {\scriptstyle \Delta_L^H}\downarrow & & \downarrow{\scriptstyle \Delta_L^K} \\ H\ {}_L\!\times H & \xrightarrow[\phi\ {}_f\!\times\ \phi]{} & K\ {}_{L'}\!\times K \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\phi} & K \\ {\scriptstyle \Delta_R^H}\downarrow & & \downarrow{\scriptstyle \Delta_R^K} \\ H \times_R H & \xrightarrow[\phi\ \times_f \phi]{} & K \times_{R'} K \end{array} \tag{2.14}$$

(d) ϕ intertwines the antipodes of $\mathcal{H}$ and $\mathcal{K}$, i.e. $\phi \circ S_H = S_K \circ \phi$.

□

Remark 3.

(1) For a k-algebra R, let us denote by $HALG^{alg}(R)$ the category whose objects are Hopf algebroids over R and arrows are algebraic morphisms. For a fixed k, let us denote by $HALG^{geom}(k)$ the category whose objects are Hopf algebroids over k-algebras and arrows are geometric morphisms. Note that the notion of isomorphism in both categories coincide.

(2) Equip R^e with the Hopf algebroid structure we will define in example 5 of the next section. Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid over R. Then the unit maps η_L, η_R together with the identity map on R define geometric morphisms

$$(id, \eta_L) : R^e \longrightarrow \mathcal{H}, \qquad (id, \eta_R) : R^e \longrightarrow \mathcal{H}.$$

(3) Suppose $R = R^{'}$ (consequently, $L^{'} = L$). Then, a geometric morphism

$$(id, \phi) : (H_L, H_R, S) \longrightarrow (K_L, K_R, S^{'})$$

gives an algebraic morphism $(\phi, \phi) : (H_L, H_R, S) \longrightarrow (K_L, K_R, S^{'})$. This follows immediately from the collapsed of diagrams 2.12, 2.13 and 2.14 into diagrams which describe ϕ as a map of left- and right-bialgebroids.

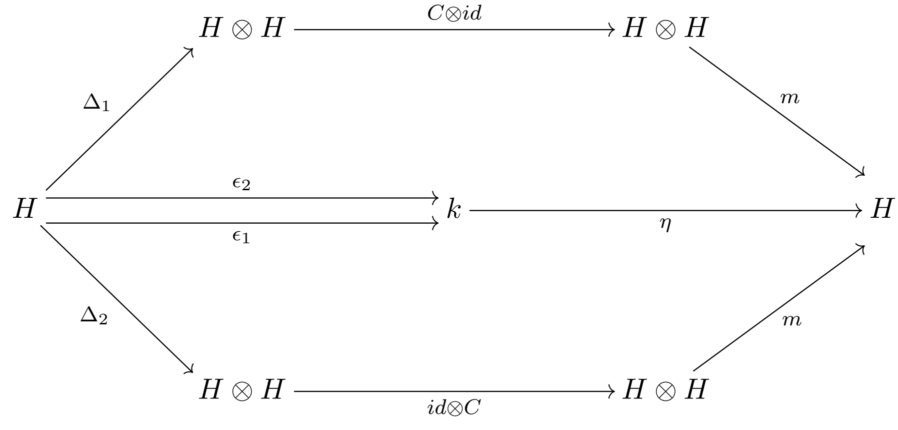
2.2 Examples and properties

In this section, we will enumerate examples of Hopf algebroids that will play a crucial role in the discussions to follow.

Example 1. Hopf algebras. A Hopf algebra H over the commutative unital ring k gives an example of a Hopf algebroid. Here, we take $R = L = k$ as k-algebras, take $s_L = t_L = s_R = t_R = \eta$ to be the source and target maps, set $\epsilon_L = \epsilon_R = \epsilon$ to be the counits, and $\Delta_L = \Delta_R = \Delta$ to be the coproducts. It might be tempting to think that a Hopf algebroid over a k-algebra R is a Hopf algebra as soon as R is commutative. This is not the case as what the following example will show. □

Example 2. Coupled Hopf algebras. Consider two Hopf algebras H_1 and H_2 with the same underlying k-algebra H. Denote by Δ_1 and ϵ_1 the coproduct and counit of H_1, respectively. Likewise, denote by Δ_2 and ϵ_2 the coalgebra structure maps of H_2. Let us denote by m and η the common product and unit maps. Two such Hopf algebras are said to be *coupled* if

(a) there exists a k-module map $C : H_1 \longrightarrow H_2$, called the *coupling map* such that



commutes, and

(b) the coproducts Δ_1 and Δ_2 commute.

We will abuse language by regarding two such Hopf algebras H_1 and H_2 as *a* coupled Hopf algebra and package them as (H_2, H_2, C). Coupled Hopf algebras give rise to Hopf algebroids over k. The left k-bialgebroid is the underlying bialgebra of H_1 while the right k-bialgebroid is the underlying bialgebra of H_2. The coupling map plays the role of the antipode.

Of course, Hopf algebras are examples of coupled Hopf algebras. Let us give a nontrivial class of examples of coupled Hopf algebras. Connes and Moscovici constructed *twisted* antipodes in [14]. Let us show that such a twisted antipode is a coupling map for some coupled Hopf algebras. Let $H = (H, m, 1, \Delta, \epsilon, S)$ be a Hopf algebra. Take $H_1 = H$ as Hopf algebras. Let $\sigma : H \longrightarrow k$ be a character. Define $\Delta_2 : H \longrightarrow H \otimes H$ by $h \mapsto h_{(1)} \otimes \sigma(S(h_{(2)}))h_{(3)}$. Take $\epsilon_2 = \sigma$. Define $S_2 : H \longrightarrow H$ by $h \mapsto \sigma(h_{(1)})S(h_{(2)})\sigma(h_{(3)})$. Coassociativity of Δ_2 follows from coassociativity of Δ. Indeed, for any $h \in H$ we have

$$\begin{aligned}
(id \otimes \Delta_2)\,\Delta_2(h) &= h_{(1)} \otimes \sigma(S(h_{(2)}))\Delta_2(h_{(3)}) \\
&= h_{(1)} \otimes \sigma(S(h_{(2)}))\left(h_{(3)} \otimes \sigma(S(h_{(4)}))h_{(5)}\right) \ \text{(coassociativity of } \Delta) \\
&= h_{(1)} \otimes \sigma(S(h_{(2)}))h_{(3)} \otimes \sigma(S(h_{(4)}))h_{(5)} \\
&= \Delta_2(h_{(1)}) \otimes \sigma(S(h_{(2)}))h_{(3)} \ \text{(coassociativity of } \Delta) \\
&= (\Delta_2 \otimes id)\,\Delta_2(h).
\end{aligned}$$

Δ_2 is counital with respect to σ. Indeed, for any $h \in H$ one has

$$(\sigma \otimes id)\,\Delta_2(h) = \sigma(h_{(1)})\sigma(S(h_{(2)}))h_{(3)} = \sigma(h_{(1)}S(h_{(2)}))h_{(3)} = \epsilon(h_{(1)})h_{(2)}\sigma(1) = h$$

and

$$(id \otimes \sigma)\,\Delta_2(h) = h_{(1)}\sigma(S(h_{(2)}))\sigma(h_{(3)}) = h_{(1)}\sigma(S(h_{(2)})h_{(3)}) = h_{(1)}\epsilon(h_{(2)})\sigma(1) = h.$$

We claim that $H_2 = (H, m, 1, \Delta_2, \epsilon_2, S_2)$ is a Hopf algebra. It is easy to see that σ being a character implies that Δ_2 is multiplicative. The map S_2 is the antipode of the bialgebra H_2. Indeed, for any $h \in H$ we have

$$\begin{aligned}
m\,(S_2 \otimes id)\,\Delta_2(h) = S_2(h_{(1)})\sigma(S(h_{(2)}))h_{(3)} = \sigma(h_{(1)})S(h_{(2)})\sigma(h_{(3)})\sigma(S(h_{(4)}))h_{(5)} \\
= \sigma(h_{(1)})S(h_{(2)})\epsilon(h_{(3)})h_{(4)} = \sigma(h_{(1)})\epsilon(h_{(2)}) \cdot 1 = \sigma(h) \cdot 1
\end{aligned}$$

and

$$\begin{aligned}
m\,(id \otimes S_2)\,\Delta_2(h) = h_{(1)}\sigma(S(h_{(2)}))\sigma(h_{(3)})S(h_{(4)})\sigma(h_{(5)}) = h_{(1)}\sigma(\epsilon(h_{(2)}) \cdot 1)S(h_{(3)})\sigma(h_{(4)}) \\
= h_{(1)}S(h_{(2)})\sigma(h_{(3)}) = \epsilon(h_{(1)})\sigma(h_{(2)}) \cdot 1 = \sigma(\epsilon(h_{(1)})h_{(2)}) \cdot 1 = \sigma(h) \cdot 1.
\end{aligned}$$

Our last claim is that the Hopf algebras H_1 and H_2 are coupled by the twisted antipode $S^\sigma : H \longrightarrow H$ defined by $h \mapsto \sigma(h_{(1)})S(h_{(2)})$. Immediately, the coproducts Δ and Δ_2 commute. For example, Δ and Δ_2 satisfy

$$\begin{aligned}
(\Delta \otimes id)\,\Delta_2(h) = (\Delta \otimes id)\left(h_{(1)} \otimes \sigma(S(h_{(2)}))h_{(3)}\right) = h_{(1)} \otimes h_{(2)} \otimes \sigma(S(h_3))h_{(4)} \\
= (id \otimes \Delta_2)\,(h_{(1)} \otimes h_{(2)}) = (id \otimes \Delta_2)\,\Delta(h)
\end{aligned}$$

for any $h \in H$. Finally, let us verify the lower half of the coupling condition. The proof for the commutativity of the upper half is almost the same. For any $h \in H$ we have

$$m\,(id \otimes S^{\sigma})\,\Delta_2(h) = m\,(id \otimes S^{\sigma})\,(h_{(1)} \otimes \sigma(S(h_{(2)})))h_{(3)} = h_{(1)}\sigma(S(h_{(2)}))\sigma(h_{(3)})S(h_{(4)})$$
$$= h_{(1)}\sigma(S(h_{(2)})h_{(3)})S(h_{(4)}) = h_{(1)}S(\epsilon(h_{(2)})h_{(3)}) = h_{(1)}S(h_{(2)}) = \epsilon(h) \cdot 1.$$

This verifies our claim. □

Proposition 1. *(Eckmann-Hilton argument for coalgebras.) Let Δ_1 and Δ_2 be (possibly non-coassociative) coproducts on a k-module C. Assume Δ_1 has counit ϵ_1 and Δ_2 has counit ϵ_2. Assume also that Δ_1 and Δ_2 commutes, i.e. they make the diagrams 2.7 commute. If one of these coproducts is counital with respect to the counit of the other one, then $\Delta_1 = \Delta_2$. Moreover, they are both coassociative.*

Proof: Write $\Delta_1(h) = h_{(1)} \otimes h_{(2)}$ and $\Delta_2(h) = h^{(1)} \otimes H^{(2)}$, for any $h \in H$. Using commutativity condition 2.7 for the coproducts, we have

$$h^{(1)}_{(1)} \otimes h^{(2)}_{(1)} \otimes h_{(2)} = (\Delta_2 \otimes id)\Delta_1(h) = (id \otimes \Delta_1)\Delta_2(h) = h^{(1)} \otimes h^{(2)}_{(1)} \otimes h^{(2)}_{(2)}$$

which, after applying $id \otimes \epsilon_1 \otimes id$ on both sides (assuming Δ_2 is counital with respect to ϵ_1) yields

$$h_{(1)} \otimes h_{(2)} = h^{(1)}_{(1)}\epsilon_1(h^{(2)}_{(1)}) \otimes h_{(2)} = h^{(1)} \otimes \epsilon_1(h^{(2)}_{(1)})h^{(2)}_{(2)} = h^{(1)} \otimes h^{(2)} = \Delta_2(h).$$

From which we immediately see that $\Delta_2 = \Delta_1$. Coassociativity of Δ_1 (and hence, of Δ_2) follows from either diagrams of 2.7. ■

Corollary 1. *Let (H_1, H_2, C) be a coupled Hopf algebra with coprodcts Δ_1, Δ_2 and respective counits ϵ_1, ϵ_2. If $\epsilon_1 = \epsilon_2$ then the constituent Hopf algebras H_1 and H_2 coincide. In this case, C is the antipode.*

Example 3. Groupoid algebras. Given a small groupoid $\mathcal{G}$ with finitely many objects and a commutative unital ring k, we can construct what is called the groupoid algebra of $\mathcal{G}$ over k, denoted by $k\mathcal{G}$. For such a groupoid $\mathcal{G}$, let us denote by $\mathcal{G}^{(0)}$ its set of objects, $\mathcal{G}^{(1)}$ its set of morphisms, $s, t : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ the source and target maps, $\iota : \mathcal{G}^{(0)} \longrightarrow \mathcal{G}^{(1)}$ the unit map, $\nu : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(1)}$ the inversion map, $\mathcal{G}^{(2)} = \mathcal{G}^{(1)}\,{}_t\times_s\,\mathcal{G}^{(1)}$ the set of composable pairs of morphisms, and $m : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}^{(1)}$ the partial composition. The groupoid algebra $k\mathcal{G}$ is the k-algebra generated by $\mathcal{G}^{(1)}$ subject to the relation

$$ff^{'} = \begin{cases} f \circ f^{'}, & \text{if } f, f^{'} \text{ are composable} \\ \\ 0, & \text{otherwise} \end{cases}$$

for $f, f^{'} \in \mathcal{G}^{(1)}$. The groupoid algebra $k\mathcal{G}$ is a Hopf algebroid as folows. The base algebras R and L are both equal to $k\mathcal{G}^{(0)}$ and the two bialgebroids H_R and H_L are isomorphic as bialgebroids with underlying k-module $k\mathcal{G}^{(1)}$. The partial groupoid composition m dualizes and extends to a multiplication $m : k\mathcal{G}^{(1)} \otimes k\mathcal{G}^{(1)} \longrightarrow k\mathcal{G}^{(1)}$ which then factors through the canonical surjection $k\mathcal{G}^{(1)} \otimes k\mathcal{G}^{(1)} \longrightarrow k\mathcal{G}^{(1)} \otimes_{k\mathcal{G}^{(0)}} k\mathcal{G}^{(1)}$ to give the product $k\mathcal{G}^{(1)} \otimes_{k\mathcal{G}^{(0)}} k\mathcal{G}^{(1)} \longrightarrow k\mathcal{G}^{(1)}$. The source and target maps s, t of the groupoid give the source and target maps $s, t : k\mathcal{G}^{(0)} \longrightarrow k\mathcal{G}^{(1)}$, respectively. The unit map gives the counit map $\epsilon : k\mathcal{G}^{(1)} \longrightarrow k\mathcal{G}^{(0)}$. Finally, the inversion map gives the antipode map $S : k\mathcal{G}^{(1)} \longrightarrow k\mathcal{G}^{(1)}$. Note that the underlying bimodule structures of the right and the left bialgerboid is related by the antipode map.

With this example, we immediately see that if the groupoid is a group, the construction above gives a Hopf algebra over k. This justifies the name Hopf algebroid. □

Example 4. Weak Hopf algebras. Another structure that generalize Hopf algebras, called weak Hopf algebras, also are Hopf algebroids. Explicitly, a weak Hopf algebra H over a commutative unital ring k is a unitary associative algebra together with k-linear maps $\Delta : H \longrightarrow H \otimes H$ (weak coproduct), $\epsilon : H \longrightarrow k$ (weak counit) and $S : H \longrightarrow H$ (weak antipode) satisfying the following axioms:

(i) Δ is multiplicative, coassociative, and weak-unital, i.e.

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = \Delta^{(2)}(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1),$$

(iii) ϵ is counital, and weak-multiplicative, i.e. for any $x, y, z \in H$

$$\epsilon(xy_{(1)})\epsilon(y_{(2)}z) = \epsilon(xyz) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z),$$

(v) for any $h \in H$, $S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$ and

$$h_{(1)}S(h_{(2)}) = \epsilon(1_{(1)}h)1_{(2)}, \qquad S(h_{(1)})h_{(2)} = 1_{(1)}\epsilon(h1_{(2)})$$

Let us sketch a proof why a weak Hopf algebra H is a Hopf algebroid. Consider the maps $p_R : H \longrightarrow H$, $h \mapsto 1_{(1)}\epsilon(h1_{(2)})$ and $p_L : H \longrightarrow H$, $h \mapsto \epsilon(1_{(1)}h)1_{(2)}$. By k-linearity and weak-multiplicativity of ϵ, p_R and p_L are idempotents.

Multiplicativity and coassiociativity of Δ and counitality of ϵ implies that for any $h \in H$,

$$h_{(1)} \otimes p_L(h_{(2)}) = 1_{(1)}h \otimes 1_{(2)} \qquad p_R(h_{(1)}) \otimes h_{(2)} = 1_{(1)} \otimes h1_{(2)}.$$

Now, using these relations and coassiociativity of Δ we get

$$1_{(1)}1_{(1')} \otimes 1_{(2)} \otimes 1_{(2')} = 1_{(1')(1)} \otimes p_L(1_{(1')(2)}) \otimes 1_{(2')} = 1_{(1)} \otimes p_L(1_{(2)}) \otimes 1_{(3)}$$

$$1_{(1)} \otimes 1_{(1')} \otimes 1_{(2)}1_{(2')} = 1_{(1)} \otimes p_L(1_{(2)(1)}) \otimes 1_{(2)(2)} = 1_{(1)(1)} \otimes p_L(1_{(1)(2)}) \otimes 1_{(2)}$$

Thus, the first tensor factor of the left-hand side of the first equation above is in the image of p_R. Similarly, the last tensor factor of the left-hand side of the second equation above is in the image of p_L. Clearly, $p_R(1) = p_L(1) = 1$. Hence, the images of p_R and p_L are unitary subalgebras of H. Denote these subalgebras by R and L, respectively. By the weak-unitality of Δ we see that these subalgebras are commuting subalgebras of H.

Taking the source map s as the inclusion $R \longrightarrow H$ and the target map as $t : R^{op} \longrightarrow H$, $r \mapsto \epsilon(r1_{(1)})1_{(2)}$ equips H with an $R \otimes_k R^{op}$-ring structure. Taking $\epsilon_R = p_R$ and Δ_R as the composition

$$H \xrightarrow{\quad\Delta\quad} H \otimes_k H \longrightarrow\!\!\!\!\rightarrow H \otimes_R H$$

equips H with an R-coring structure $(H, \Delta_R, \epsilon_R)$. The ring and coring structures just constructed gives H a structure of right R-bialgebroid H_R.

Using R^{op} in place of R in the above construction, we get a left R^{op}-bialgebroid $H_{R^{op}}$. Together with the right R-bialgebroid constructed and the existing weak antipode S, we get a Hopf algebroid $(H_{R^{op}}, H_R, S)$. □

Remark 4. Weak Hopf algebras also has a well-understood representation theory. Given a weak Hopf algebra H over a field k, the category ${}_H\mathcal{M}$ of finitely-generated left modules over H is a fusion category. A *fusion category* $\mathcal{C}$ over k is a k-linear rigid semisimple category with finitely-many inequivalent simple objects such that the hom-spaces are finite-dimensional and the endomorphism algebra of the unit object $\mathbb{1}_{\mathcal{C}}$ is k. By Tannaka duality, any fusion category is equivalent to a module category of a weak Hopf algebra. This phenomenon has a nice symmetry. Similar to Hopf algebras, the dual H^* of a finitely generated weak Hopf algebra $H = (H, m, 1, \Delta, \varepsilon, S)$ has a natural weak Hopf algebra structure. Using this idea, one can show that the category $\mathcal{M}^H$ of finitely-generated right comodules over H is a fusion category as well.

Example 5. Group algebras over noncommutative rings. One of the most studied yet mysterious class of a Hopf algebras are group algebras over commutative rings. In this section, we will show a similar construction of a group algebra over a noncommutative base ring and see that such is a Hopf algebroid. This further justifies the banner of Hopf algebroids being a generalization of Hopf algebras over noncommutative rings.

Let A be an associative unital algebra over a commutative ring k. Denote by $A^e = A \otimes A^{op}$ its universal enveloping algebra. Consider a finite group G acting on A via $G \xrightarrow{\alpha} Aut(A)$. This action extends to a kG-module structure on A^e via the usual coproduct on kG. Consider the smash product algebra $A^e\#kG$. The underlying k-module of this algebra is $A^e \otimes kG$. The multiplication is defined as

$$\left(\sum \left(a^1 \otimes a^2\right) \#g\right) \left(\sum \left(b^1 \otimes b^2\right) \#h\right) = \sum \left(a^1 \otimes a^2\right) \alpha_g \left(b^1 \otimes b^2\right) \#gh$$

Note that this construction generalize to the case of a bialgebra H in place of kG where the two appearance of g's in the defining relation for the multiplication is played by the legs of coproduct applied to the appropriate tensor factor. If the action of G is trivial, we get the algebra A^eG which we call the group algebra of G over A^e. Let us show that A^eG is a Hopf algebroid over A. The right A-bialgebroid structure consists of A^eG as the underlying k-module. The right source s_R, target t_R and counit maps ϵ_R are

$$\begin{array}{ccc} A \xrightarrow{s_R} A \otimes A^{op}\#kG & A \xrightarrow{t_R} A \otimes A^{op}\#kG & A \otimes A^{op}\#kG \xrightarrow{\epsilon_R} A. \\ a \longmapsto (a \otimes 1)\#e & a \longmapsto (1 \otimes a)\#e & \left(a \otimes a'\right)\#g \longmapsto aa' \end{array}$$

where e stands for the identity element of G. The right coproduct Δ_R is the following map.

$$\begin{array}{c} A \otimes A^{op}\#kG \xrightarrow{\Delta_R} (A \otimes A^{op}\#kG) \underset{A}{\otimes} (A \otimes A^{op}\#kG) \\ \left(a \otimes a'\right)\#g \longmapsto \left(1 \otimes a'\right)\#g \underset{A}{\otimes} (a \otimes 1)\#g \end{array}$$

The left A-bialgebroid is the opposite co-opposite of the right A-bialgebroid we just constructed. The map

$$\begin{array}{c} A \otimes A^{op}\#kG \xrightarrow{S} A^{op} \otimes A\#kG \\ \left(a \otimes a'\right)\#g \longmapsto \left(a' \otimes a\right)\#g^{-1} \end{array}$$

is the antipode. In particular, taking G to be the trivial group makes A^e a Hopf algebroid over A. Any of the underlying coring structures of A^e is what is commonly known in the literature as the canonical coring associated to A. We call A^e the *canonical Hopf algebroid* over A. □

An interesting feature of a Hopf algebra that its classical counterpart, i.e. groups, do not enjoy is the fact that the structure maps of a Hopf algebra are Hopf algebra maps. Requiring that the maps Δ and ϵ to be algebra maps is equivalent to requiring m and $k \xrightarrow{1} H$ to be coalgebra maps relative to the natural coalgebra structure (H, Δ, ϵ) induce on $H \otimes H$. Note that there is only one coalgebra structure on k. Moreover, equipping k and $H \otimes H$ with these *natural* Hopf algebra structures we see that the maps $m, 1, \Delta$ and ϵ are Hopf algerba maps. Furthermore, the antipode S is a Hopf algebra map $H \xrightarrow{S} H^{op\ cop}$. Let us list some properties of a Hopf algebroid which is similar to these properties.

Lemma 2. *Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid over R. Then:*

(i) The coproducts Δ_L and Δ_R are multiplicative.

(ii) The antipode S is an $R \otimes R^{op}$-ring morphism

$$\Big(H,\ s_R,\ t_R\Big) \longrightarrow \Big(H^{op},\ s_L \circ (\epsilon_L \circ s_R)\,,\ t_L \circ (\epsilon_L \circ s_R)\Big)$$

and an $L \otimes L^{op}$-ring morphism

$$\Big(H,\ s_L,\ t_L\Big) \longrightarrow \Big(H^{op},\ s_R \circ (\epsilon_R \circ s_L)\,,\ t_R \circ (\epsilon_R \circ s_L)\Big).$$

In particular, $S : H \longrightarrow H$ is a k-algebra anti-homomorphism.

(iii) The antipode S is an R-coring morphism

$$\Big(H,\ \Delta_R,\ \epsilon_R\Big) \longrightarrow \Big(H,\ \Delta_L^{cop},\ (\epsilon_R \circ s_L) \circ \epsilon_L\Big)$$

and an L-coring morphism

$$\Big(H,\ \Delta_L,\ \epsilon_L\Big) \longrightarrow \Big(H,\ \Delta_R^{cop},\ (\epsilon_L \circ s_R) \circ \epsilon_R\Big)\ .$$

Here, Δ_L^{cop} (resp. Δ_R^{cop}) is considered as a map $H \longrightarrow H \otimes_{L^{op}} H \cong H \otimes_R H$ (resp. $H \longrightarrow H \otimes_{R^{op}} H \cong H \otimes_L H$) in view of the isomorphisms of remark 2(1).

Proof: (i) Note that the map $R \otimes R^{op} \xrightarrow{\varphi} B \times_R B$, $r \otimes r' \mapsto t(r') \otimes_R s(r)$ is a k-algebra homomorphism. In view of lemma 1, the Takeuchi product $B \times_R B$ is an $R \otimes R^{op}$-ring. We claim that with the $R \otimes R^{op}$-ring structure on B, $\Delta_R : B \longrightarrow B \times_R B$ becomes an $R \otimes R^{op}$-bimodule map.

$$\begin{aligned}
\Delta_R((r \otimes r') \cdot b \cdot (q \otimes q')) &= \Delta_R\Big(t_R(q')s_R(r)bt_R(r')s_R(q)\Big) \\
&= r' \cdot \Big(\Delta_R(t_R(q')s_R(r)b)\Big) \cdot q \\
&= r' \cdot \Big(\Delta_R(t_R(q'))\Delta_R(s_R(r))\Delta_R(b)\Big) \cdot q \\
&= r' \cdot \Big((1 \otimes_R t_R(q'))(s_R(r) \otimes_R 1)(b^{[1]} \otimes_R b^{[2]})\Big) \cdot q \\
&= r' \cdot \Big(s(r)b^{[1]} \otimes_R t(q')b^{[2]}\Big) \cdot q \\
&= (r \otimes r') \cdot \Delta_R(b) \cdot (q \otimes q')
\end{aligned}$$

To complete the proof of (i), let us argue that the following diagram of $R \otimes R^{op}$-bimodules commutes.

$$\begin{array}{ccc} B \underset{R^e}{\otimes} B & \xrightarrow{\mu} & B \\ \downarrow{\scriptstyle \Delta_R \underset{R^e}{\otimes} \Delta_R} & & \downarrow{\scriptstyle \Delta_R} \\ (B \times_R B) \underset{R^e}{\otimes} (B \times_R B) & \xrightarrow[\mu_T]{} & B \times_R B \end{array}$$

$B \times_R B$ is a k-algebra via factorwise multiplication. Since $B \times_R B$ is an $R \otimes R^{op}$-ring, μ descends to a map μ_T given by factorwise multiplication. This shows that the above diagram commutes. With appropriate modifications, one can show Δ_L to be multiplicative as well.
(ii) By condition (c) in definition 2, S is an $R \otimes R^{op}$-bimodule map which intertwines $s_R \otimes t_R$ and $s_L(\epsilon_L \circ s_R) \otimes t_L(\epsilon_L \circ s_R)$, i.e. we have

$$\begin{array}{ccc} & & H \\ & \nearrow{\scriptstyle s_R \otimes t_R} & \\ R \otimes R^{op} & & \downarrow{\scriptstyle S} \\ & \searrow{\scriptstyle s_L(\epsilon_L \circ s_R) \otimes t_L(\epsilon_L \circ s_R)} & \\ & & H^{op} \end{array}$$

In view of lemma 1, S defines an $R \otimes R^{op}$-ring map from H to H^{op}.
(iii) Using condition (c) in definition 1 again, we have a commutative diagram of R-bimodules

$$\begin{array}{ccc} H & \xrightarrow{\Delta_R} & H \underset{R}{\otimes} H \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S \underset{R}{\otimes} S} \\ H & \xrightarrow[\Delta_L^{cop}]{} & H \underset{R}{\otimes} H \end{array}$$

which proves part (iii) of the above lemma. ■

Remark 5.

(1) Note that in part (i) of lemma 2 we did not say explicitly to which product Δ_R and Δ_L are multiplicative. This is because it does not matter, at least not gravely. The proof shows that Δ_R is multiplicative with respect to that R^e-ring structure of H. The same proof shows Δ_R is multiplicative with respect to the R-ring and R^{op}-ring structures of H. Similarly, Δ_L is multiplicative with respect to the L^e-ring, L-ring and L^{op}-ring structures of H.

(2) Parts (ii) and (iii) of the above lemma say that the pairs $(\epsilon_L \circ s_R, S)$ and $(\epsilon_R \circ s_L, S)$ define geometric morphisms

$$(H, s_R, t_R, \Delta_R, \epsilon_R) \xrightarrow{(\epsilon_L \circ s_R,\ S)} (H^{op}, t_R, s_R, \Delta_R^{cop}, \epsilon_R)$$

and

$$(H, s_L, t_L, \Delta_L, \epsilon_L) \xrightarrow{(\epsilon_R \circ s_L,\ S)} (H^{op}, t_L, s_L, \Delta_L^{cop}, \epsilon_L)$$

of bialgebroids, respectively. By a *geometric morphism* of bialgebroids we mean a pair (f, ϕ) satisfying conditions (a) to (c) of definition 3 stated for one bialgebroid. This is the content of proposition 3.2 in [5].

2.3 Representation theory of Hopf algebroids and their descent

In this section, we will look at representations of Hopf algebroids. Towards the end of the section, we will look at the descent theoretic aspect of a special class of modules over Hopf algebroids, the so called relative Hopf modules. Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid with underlying k-module H. H carries both a left L-module sctructure and a left R-module structure via the maps s_L and t_R, respectively. With this, the following definition make sense.

Definition 4. A *right $\mathcal{H}$-comodule* M is a right L-module and a right R-module together with a right H_R-coaction $\rho_R : M \longrightarrow M \otimes_R H$ and a right H_L-coaction $\rho_L : M \longrightarrow M \otimes_L H$ such that ρ_R is an H_L-comodule map and ρ_L is an H_R-comodule map. □

For the coaction ρ_R, let us use the following Sweedler notation:

$$\rho_R(m) = m^{[0]} \underset{R}{\otimes} m^{[1]}$$

and for the coaction ρ_L, let us use the following Sweedler notation:

$$\rho_L(m) = m_{[0]} \underset{L}{\otimes} m_{[1]}.$$

With these notations, the conditions above explicitly means that for all $m \in M$, $l \in L$ and $r \in R$ we have

$$(m \cdot l)^{[0]} \underset{R}{\otimes} (m \cdot l)^{[1]} = \rho_R(m \cdot l) = m^{[0]} \underset{R}{\otimes} t_L(l) m^{[1]}$$

$$(m \cdot r)_{[0]} \underset{L}{\otimes} (m \cdot r)_{[1]} = \rho_L(m \cdot r) = m_{[0]} \underset{L}{\otimes} m_{[1]} s_R(r).$$

We further require that the two coactions satify the following commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{\rho_L} & M \underset{L}{\otimes} H \\ \downarrow{\scriptstyle \rho_R} & & \downarrow{\scriptstyle \rho_R \underset{L}{\otimes} id} \\ M \underset{R}{\otimes} H & \xrightarrow[id \underset{R}{\otimes} \Delta_L]{} & M \underset{R}{\otimes} H \underset{L}{\otimes} H \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_R} & M \underset{R}{\otimes} H \\ \downarrow{\scriptstyle \rho_L} & & \downarrow{\scriptstyle \rho_L \underset{R}{\otimes} id} \\ M \underset{L}{\otimes} H & \xrightarrow[id \underset{L}{\otimes} \Delta_R]{} & M \underset{L}{\otimes} H \underset{R}{\otimes} H \end{array} \tag{2.15}$$

We will denote by $\mathcal{M}^{\mathcal{H}}$ the category of right $\mathcal{H}$-comodules. Symmetrically, we can define left $\mathcal{H}$-comodules and we denote the category of a such by ${}^{\mathcal{H}}\mathcal{M}$.

Comodules over Hopf algebroids are comodules over the constituent bialgebroids. Thus, one can speak of two different coinvariants, one for each bialgebroid. For a given right $\mathcal{H}$-comodule M, they are defined as follows:

$$M^{co\ H_R} = \left\{ m \in M \;\middle|\; \rho_R(m) = m \underset{R}{\otimes} 1 \right\},$$

$$M^{co\ H_L} = \left\{ m \in M \;\middle|\; \rho_L(m) = m \underset{L}{\otimes} 1 \right\}.$$

In the general case, we have $M^{co\ H_R} \subseteq M^{co\ H_L}$. But in our case, where we assume S is bijective more can be said.

Proposition 2. *If the antipode S of the Hopf algebroid $\mathcal{H}$ is bijective then $M^{co\ H_R} = M^{co\ H_L}$.*

PROOF: To see that these coinvariants coincide, consider the following map

$$\Phi_M : M \underset{R}{\otimes} H \longrightarrow M \underset{L}{\otimes} Hm \underset{R}{\otimes} h \mapsto \rho_L(m) \cdot S(h) \tag{2.16}$$

Here, H acts on the right of $M \otimes_L H$ through the second factor. If $m \in M^{co\ H_R}$, then we have

$$\begin{aligned}
\rho_L(m) &= \rho_L(m) \cdot S(h) = \Phi_M(m \underset{R}{\otimes} 1) = \Phi_M(\rho_R(m)) \\
&= \Phi_M(m^{[0]} \underset{R}{\otimes} m^{[1]}) = \rho_L(m^{[0]}) \cdot S(m^{[1]}) \\
&= (m^{[0]}_{[0]} \underset{L}{\otimes} m^{[0]}_{[1]}) \cdot S(m^{[1]}) = m^{[0]}_{[0]} \underset{L}{\otimes} m^{[0]}_{[1]} S(m^{[1]}) \\
&= m_{[0]} \underset{L}{\otimes} m^{[0]}_{[1]} S(m^{[1]}_{[1]}) = m_{[0]} \underset{L}{\otimes} s_L(\epsilon_L(m_{[1]})) \\
&= m_{[0]} s_L(\epsilon_L(m_{[1]})) \underset{L}{\otimes} 1 = m \underset{L}{\otimes} 1
\end{aligned}$$

This shows the inclusion $M^{co\ H_R} \subseteq M^{co\ H_L}$. To show the other inclusion, using the map

$$\Phi_M^{-1} : M \underset{L}{\otimes} H \longrightarrow M \underset{R}{\otimes} H$$

$$m \underset{L}{\otimes} h \mapsto S^{-1}(h) \cdot \rho_R(m),$$

the inverse of Φ_M, one can run the same computation. ■

This will be important in the formulation of Galois theory for Hopf algebroids. In this case, we can simply write $M^{co\ \mathcal{H}}$ for $M^{co\ H_R} = M^{co\ H_L}$ and refer to it as the $\mathcal{H}$-coinvariants of M instead of distinguishing the H_R- from the H_L-coinvariants, unless it is necessary to do so.

Let us now discuss monoid objects in $\mathcal{M}^{\mathcal{H}}$. They are called $\mathcal{H}$-comodule algebras.

Definition 5. A right *$\mathcal{H}$-comodule algebra* is an R-ring (M, μ, η) such that M is a right $\mathcal{H}$-comodule and $\eta : R \longrightarrow M$ and $\mu : M \otimes_R M \longrightarrow M$ are $\mathcal{H}$-comodule maps. □

Using Sweedler notation for coactions, this explicitly means that for any $m, n \in M$ we have

$$(mn)^{[0]} \underset{R}{\otimes} (mn)^{[1]} = \rho_R(mn) = m^{[0]} n^{[0]} \underset{R}{\otimes} m^{[1]} n^{[1]}, \tag{2.17}$$

$$(mn)_{[0]} \underset{L}{\otimes} (mn)_{[1]} = \rho_L(mn) = m_{[0]} n_{[0]} \underset{L}{\otimes} m_{[1]} n_{[1]}, \tag{2.18}$$

$$1_M^{[0]} \underset{R}{\otimes} 1_M^{[1]} = \rho_R(1_M) = 1_M \underset{R}{\otimes} 1_H, \tag{2.19}$$

$$(1_M)_{[0]} \underset{L}{\otimes} (1_M)_{[1]} = \rho_L(1_M) = 1_M \underset{L}{\otimes} 1_H. \tag{2.20}$$

Remark 6. Equations 2.17 and 2.19 imply that the image of ρ_R lands in the following k-submodule of $M \otimes_R H$.

$$M \times_R H := \left\{ \sum_i m_i \underset{R}{\otimes} h_i \,\middle|\, \sum_i r \cdot m_i \underset{R}{\otimes} h_i = \sum_i m_i \underset{R}{\otimes} t_R(r) h_i, \ \forall r \in R \right\}$$

Likewise, equations 2.18 and 2.20 imply that the image of ρ_L lands in

$$M \,{}_L\!\times H := \left\{ \sum_i m_i \underset{L}{\otimes} h_i \,\middle|\, \sum_i l \cdot m_i \underset{L}{\otimes} h_i = \sum_i m_i \underset{L}{\otimes} s_L(l) h_i, \ \forall l \in L \right\}.$$

Notice the resemblance of the above submodule to the Takeuchi product in definition 1.

Definition 6. Let M be a right $\mathcal{H}$-comodule algebra. A *right-right relative* $(M, \mathcal{H})$*-Hopf module* W is a right module of the R-ring M and a right $\mathcal{H}$-comodule such that the module structure, denoted as $(\cdot) : W \otimes_R M \longrightarrow W$, is a right $\mathcal{H}$-comodule map, i.e.

$$(w \cdot m)^{[0]} \underset{R}{\otimes} (w \cdot m)^{[1]} = w^{[0]} \cdot m^{[0]} \underset{R}{\otimes} w^{[1]} m^{[1]}$$

$$(w \cdot m)_{[0]} \underset{L}{\otimes} (w \cdot m)_{[1]} = w_{[0]} \cdot m_{[0]} \underset{L}{\otimes} w_{[1]} m_{[1]}$$

for any $w \in W$ and $m \in M$. We denote by $\mathcal{M}_M^{\mathcal{H}}$ the category of right-right relative $(M, \mathcal{H})$-Hopf modules. One can symmetrically define left-right, left-left and right-left relative Hopf modules, whose categories will be denoted by ${}_M\mathcal{M}^{\mathcal{H}}$, ${}^{\mathcal{H}}_M\mathcal{M}$ and ${}^{\mathcal{H}}\mathcal{M}_M$, respectively. If in the above situation the (left or right) $\mathcal{H}$-comodule algebra M is $\mathcal{H}$, we will drop *relative* and simply call the Hopf modules. $\square$

Proposition 3. *With the previous set-up, where M is a right $\mathcal{H}$-comodule algebra, let us denote by $N = M^{co\ \mathcal{H}_R}$. Then we have the following adjunction*

$$\mathcal{M}_N \underset{(-)^{co\ \mathcal{H}_R}}{\overset{-\otimes_N M}{\rightleftarrows}} \mathcal{M}_M^{\mathcal{H}}$$

The unit and the counit of the adjunction is

$$\begin{array}{ccc} V \longrightarrow (V \underset{N}{\otimes} M)^{co\ \mathcal{H}_R}, & & W^{co\ \mathcal{H}_R} \underset{N}{\otimes} M \longrightarrow W, \\ v \longmapsto v \underset{N}{\otimes} 1 & & w \underset{N}{\otimes} m \longmapsto w \cdot m \end{array}$$

respectively.

The Hopf algebroid $\mathcal{H}$ is itself a right $\mathcal{H}$-comodule algebra whose H_R-coinvariants is the image of t_R, or equivalently the image of $L \xrightarrow{s_L} H$. The associated induction functor $- \otimes_L H : \mathcal{M}_L \longrightarrow \mathcal{M}_H^H$ is an adjoint equivalence.

2.4 Galois theory of Hopf algebroids

Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid with underlying k-module H. A k-algebra extension $A \subseteq B$ is said to be (*right*) H_R-*Galois* if B is a right H_R-comodule algebra with $B^{co\ H_R} = A$ and the map

$$B \underset{A}{\otimes} B \xrightarrow{\mathfrak{gal}_R} B \underset{R}{\otimes} H$$

$$a \underset{A}{\otimes} b \longmapsto ab^{[0]} \underset{R}{\otimes} b^{[1]}$$

is a bijection. The map $\mathfrak{gal}_R$ is called the Galois map associated to the bialgebroid extension $A \subseteq B$. Symmetrically, the extension $A \subseteq B$ is (*right*) H_L-*Galois* if B is a right H_L-comodule algebra with $B^{co\ H_L} = A$ and the map

$$B \underset{A}{\otimes} B \xrightarrow{\mathfrak{gal}_L} B \underset{L}{\otimes} H$$

$$a \underset{A}{\otimes} b \longmapsto a_{[0]}b \underset{L}{\otimes} a_{[1]}$$

is a bijection. We say that a k-algebra extension $A \subseteq B$ is $\mathcal{H}$-*Galois* if it is both H_R-Galois and H_L-Galois. It is not known in general if the bijectivity of $\mathfrak{gal}_R$ and $\mathfrak{gal}_L$ are equivalent. However, if the antipode S is bijective (which is part of our standing assumption) then $\mathfrak{gal}_R$ is bijective if and only if $\mathfrak{gal}_L$. To see this, note that $\mathfrak{gal}_L = \Phi_B \circ \mathfrak{gal}_R$ where Φ_B is the map 2.16 defined in the previous section for $M = B$. Since S is bijective, Φ_B is an isomorphism which gives the desired equivalence of bijectivity of $\mathfrak{gal}_R$ and $\mathfrak{gal}_L$. Thus, the extension $A \subseteq B$ is $\mathcal{H}$-Galois if it is a bialgebroid Galois extension for any of its constituent bialgebroids.

In the case of Galois extension by Hopf algebras, a class of extensions are of particular interest called cleft extensions since they correspond to trivial extensions. Following [6], we will look what cleft extensions are for Hopf algebroids. But before doing so, let us define the category which is the categorification of the convolution algebra one associates to a Hopf algebra.

Definition 7. Let R and L be k-algebras. Let X and Y be k-modules such that X has an R-coring $(X, \Delta_R, \epsilon_R)$ and an L-coring $(X, \Delta_L, \epsilon_L)$ structures and Y has an $L \otimes_k R$-ring structure with multiplications $\mu_R : Y \otimes_R Y \longrightarrow Y$ and $\mu_L : Y \otimes_L Y \longrightarrow Y$. Define the *convolution category* $Conv(X, Y)$ as the category with two objects labelled R and L. For $I, J \in \{R, L\}$, an arrow $I \longrightarrow J$ is a $J - I$ bimodule map $X \longrightarrow Y$. We define the composition $f * g$ as

$$f * g = \mu_J \circ (f \underset{J}{\otimes} g) \circ \Delta_J$$

for arrows $J \xrightarrow{f} I$ and $K \xrightarrow{g} J$ between objects $I, J, K \in \{R, L\}$. □

Now, consider a Hopf algebroid $\mathcal{H} = (H_L, H_R, S)$ and a right $\mathcal{H}$-comodule algebra B. Note that by definition, B only carries an R-ring structure. Since the k-module H already has an R-coring structure coming from H_R and an L-coring structure coming from H_L, if the R-ring structure of B extends to an $L \otimes_k R$-ring structure then we can consider the convolution category $Conv(H, B)$. Since there is no reason for B to carry a compatible L-ring structure, we have to add this to the definition of a cleft extension.

Definition 8. An extension $A \subseteq B$, where $A = B^{co\ \mathcal{H}}$, is *cleft* if

(a) the R-ring structure of B extends to an $L \otimes_k R$-ring structure, and

(b) there is an invertible morphism $R \xrightarrow{c} L$ in $Conv(H, B)$ which is a right $\mathcal{H}$-comodule map.

□

Similar to the case of extensions by Hopf algebras, cleft extensions have Galois-normal basis and crossed product characterizations. Let us state it in the following theorem.

Theorem 1. *Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid with bijective antipode and let B be a right $\mathcal{H}$-comodule algebra with coinvariants A. The following conditions are equivalent:*

(i) $A \subseteq B$ *is a cleft extension.*

(ii) $B \cong A \otimes_L H$ *as left A-modules and right $\mathcal{H}$-comodules (normal basis property) and $A \subseteq B$ is $\mathcal{H}$-Galois.*

(iii) For some invertible A-valued 2-cocycle σ on H_L, we have $B \cong A\#_\sigma H_L$ as left A-modules and as right $\mathcal{H}$-comodule algebras.

Let us expound on the last characterization of cleft extensions. Consider a left L-bialgebroid $\mathcal{B} = (B, s, t, \Delta, \epsilon)$. Let (N, μ, η) be a $\mathcal{B}$-*measured* L-ring, i.e one which is equipped with a k-module map $B \otimes_k N \xrightarrow{(\cdot)} N$ satisfying

(a) $b \cdot 1_N = \eta(\epsilon(b))$,

(b) $(t(l)b) \cdot n = (b \cdot n)\eta(l)$ and $(s(l)b) \cdot n = \eta(l)(b \cdot n)$,

(c) and $b \cdot (nn^{'}) = (b_{(1)} \cdot n)(b_{(2)} \cdot n^{'})$,

for any $b \in B$, $n, n^{'} \in N$ and $l \in L$. Out of these data, we can construct a two-object category $\mathcal{C}(\mathcal{B}, N)$ whose objects are conveniently labelled as I and II. Let us describe the morphism in this category. Consider $B \otimes_k B$ as an L-bimodule by left multiplication of s and t in the first tensor factor. A map $f \in {}_LHom_L(B \otimes_k B, N)$ is said to be of *type* (i, j) if it satisfies condition (i) on the first list and condition (j) on the second list below.

1^{st} List	2^{nd} List
(I) $f(a \underset{k}{\otimes} t(l)b) = f(at(l) \underset{k}{\otimes} b)$	(I) $f(a \underset{k}{\otimes} s(l)b) = f(as(l) \underset{k}{\otimes} b)$
(II) $f(a \underset{k}{\otimes} t(l)b) = f(a_{(1)} \underset{k}{\otimes} b)(a_{(2)} \cdot \eta(l))$	(II) $f(a \underset{k}{\otimes} t(l)b) = (a_{(1)} \cdot \eta(l))f(a_{(2)} \underset{k}{\otimes} b)$

where $a, b \in B$ and $l \in L$. For any $i, j \in \{I, II\}$, a morphism $i \longrightarrow j$ is a map

$$f \in {}_LHom_L(B \otimes_k B, N)$$

of type (i, j). For any $i, j, l \in \{I, II\}$, the composition of $i \xrightarrow{f} j$ and $j \xrightarrow{g} l$ is the following convolution

$$(f * g)\,(a \underset{k}{\otimes} b) = f\left(a_{(1)} \underset{k}{\otimes} a_{(1)}\right) g\left(a_{(2)} \underset{k}{\otimes} b_{(2)}\right).$$

The identity morphism $I \longrightarrow I$ is the map $a \otimes_k b \longmapsto (ab) \cdot 1_N = \eta(\epsilon(ab))$ and the identity morphism $II \longrightarrow II$ is the map $a \otimes_k b \longmapsto a \cdot (b \cdot 1_N)$.

Definition 9. An N-*valued* 2-*cocycle* on $\mathcal{B}$ is a morphism $I \xrightarrow{\sigma} II$ in the category $\mathcal{C}(\mathcal{B}, N)$ satisfying, for any $a, b, c \in B$, the following conditions.

(a) $\sigma(1_B, b) = \eta(\epsilon(b)) = \sigma(b, 1_B)$ *(normality)*,

(b) $(a_{(1)} \cdot \sigma(b_{(1)}, c_{(1)}))\sigma(a_{(2)}, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$ (*cocycle condition*).

If in addition, we have for any $n \in N$ and $a, b \in B$,

(c) $1_B \cdot n = n$ (*unitality*),

(d) $(a_{(1)} \cdot (b_{(1)} \cdot n))\sigma(a_{(2)}, b_{(2)}) = \sigma(a_{(1)}, b_{(1)})(a_{(2)}b_{(2)} \cdot n)$ (*associativity*),

we call the $\mathcal{B}$-measured L-ring N a *σ-twisted $\mathcal{B}$-module*. □

For such a left L-bialgebroid $\mathcal{B}$ and a σ-twisted $\mathcal{B}$-module N, we can construct the *crossed product* $N\#_\sigma\mathcal{B}$ as the k-algebra whose underlying k-module is $N \otimes_L B$ where the left L-module structure on B is the one via multiplication of s. The multiplication in $N\#_\sigma\mathcal{B}$ is defined as

$$(n\#b)(n^{'}\#b^{'}) = n(b_{(1)} \cdot n^{'})\sigma(b_{(2)}, b^{'}_{(1)})\#b_{(3)}b^{'}_{(2)}, \qquad \text{for any } n\#b, n^{'}\#b^{'} \in N\#_\sigma\mathcal{B}.$$

This multiplication is associative by conditions (b) and (d) and unital by conditions (a) and (c).

Going back to the characterization of cleft extensions by crossed products, the 2-cocycle σ is invertible in the sense that it is invertible as a morphism in the category $\mathcal{C}(\mathcal{H}_L, A)$.

Chapter 3

Non-commutative covering spaces

Alice: *"Where should I go?"*

The Cheshire Cat: *"That depends on where you want to end up."*

–Lewis Carroll,
Through the Looking-Glass

In the classical sense, a covering space is a surjective map $Y \twoheadrightarrow X$ with discrete fibers. In formulating the notion of a noncommutative covering space, discreteness plays a serious obstacle. Fortunately, for our purpose we will only be interested with the analogues of finite Galois coverings. We will go around this difficulty by adding the symmetry of the covering as part of the data. In doing so, discreteness will be reflected as a finiteness condition imposed in the associated symmetry. In the classical case, if we have a surjective map $Y \twoheadrightarrow X$ of (locally) compact Hausdorff spaces with finite fibers then it is automatically a covering map. Before embarking into justifying our choice of a definition of a non-commutative covering space, we should emphasize that it is by no means the unique way or the better way. Our choices are guided through by our goal to put up a working definition of an étale fundamental groupoid and étale fundamental group. Classically, these only concerns finite Galois coverings.

3.1 Definitions and properties

The non-commutative analogues of principal bundles are Hopf-Galois extensions by Hopf algebras. A classical covering space $Y \overset{p}{\twoheadrightarrow} X$ gives a principal bundle Y over X with a discrete gauge group G. In the language of covering spaces, this gauge group is called the *group of deck transformation*. This group is determined, up to a choice of a base point $a \in X$, by the surjective map p defining the covering. Naively using the geometry-algebra dictionary, the spaces Y and X will be algebras, the surjective map p will be an injective algebra map going in the opposite direction, and the group G, the symmetry of the system of interest, will be a Hopf-like structure. In the non-commutative framework, we will include the symmetry as part of the data. Using this strategy, after imposing some finiteness conditions, we will be able to capture discreteness. Note that we didn't immediately say that the symmetry in the non-commutative picture is a Hopf algebra or to be more fancy, a quantum group. This is because, with our forthcoming formulation, it is generally not and there is a good reason for that. In the classical set-up, one only gets a *group* by localizing at a point. In non-commutative geometry one should make constructions independent of *points* as much as possible.

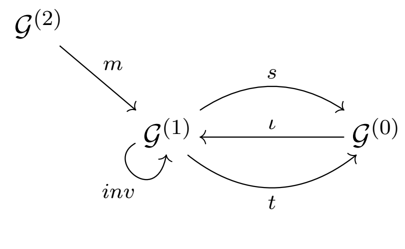
To make our formulation independent of *points*, we will reformulate the notion of a classical covering space using a more global object, that of a *groupoid*. In other words, we will replace the *symmetry group* G by the *symmetry groupoid* $\mathcal{G}$. Finally, as we have emphasized in the introductory paragraph, we are only be interested in defining the étale fundamental group or the étale fundamental groupoid, which means we will reasonable restrict our attention to finite Galois coverings, i.e. coverings in which the action of the group G on $Y \xrightarrow{p} X$ is Galois. Fortunately, Galois actions has a well-established analogue in non-commutative geometry− *Hopf-Galois extensions.*

Let us commence by formulating covering spaces in terms of groupoids. Consider a classical Galois covering space $Y \xrightarrow{p} X$ with finite deck transformation group G. We assume that X has a suitably nice connectivity properties, see for example [27]. Moreover, let us assume Y and X are compact Hausdorff spaces. Denote by A and B the corresponding algebra of continuous functions on X and Y, respectively. The surjection p gives an inclusion $A \subseteq B$.

From a covering $Y \xrightarrow{p} X$ and two points $x, y \in X$, one gets a bijection $p^{-1}(x) \xrightarrow{\gamma^*} p^{-1}(y)$ for each path γ in X from x to y. The bijection γ^* is the correspondence one gets by applying the homotopy lifting property to γ. This gives lifts of γ in Y and by varying the initial point of such lifts among $p^{-1}(x)$, we get a uniquely corresponding point in $p^{-1}(y)$. The bijection we get in this way only depends on the homotopy class of γ. Using this, we can associate two groupoids to any covering $Y \xrightarrow{p} X$ as follows.

Definition 10. Given a covering $Y \xrightarrow{p} X$, the *associated groupoid* $\mathcal{G}$ is the topological groupoid whose set of objects is X and whose arrows from a point x to a point y are bijections γ^* induced from paths γ in X from x to y. The *associated reduced groupoid* of the covering $Y \xrightarrow{p} X$, denoted by $\mathcal{G}_{red}$, is a topological groupoid with the same space of objects as $\mathcal{G}$. The set of arrows from x to y is empty if $x \neq y$. Otherwise, it is the same with the set of arrows in $\mathcal{G}$. □

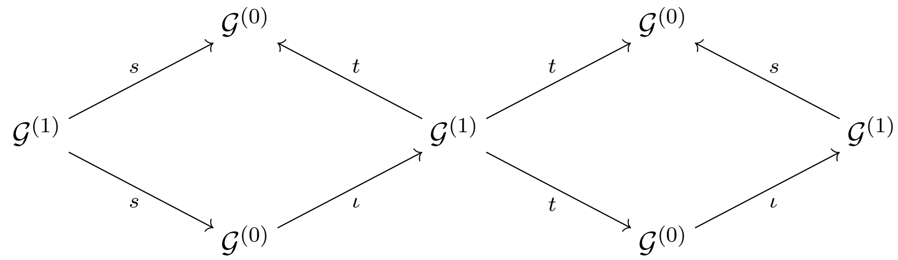
Indeed, the above definition gives a groupoid. Explicitly, $\mathcal{G}^{(1)}$ is the set of induced bijections from homotopy classes of paths in X, the source and target maps $s, t : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$ are the maps giving the base point and the terminal point, respectively, of the path inducing the bijection in $\mathcal{G}^{(1)}$. $\mathcal{G}^{(2)}$ is the fiber product of s and t i.e. the composable morphisms on $\mathcal{G}$, $\iota : \mathcal{G}^{(0)} \longrightarrow \mathcal{G}^{(1)}$ the map sending x to the identity map on $p^{-1}(x)$, and finally $inv : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(1)}$ the map that associates to γ^* the bijection $(\gamma^{-1})^*$. These structure maps



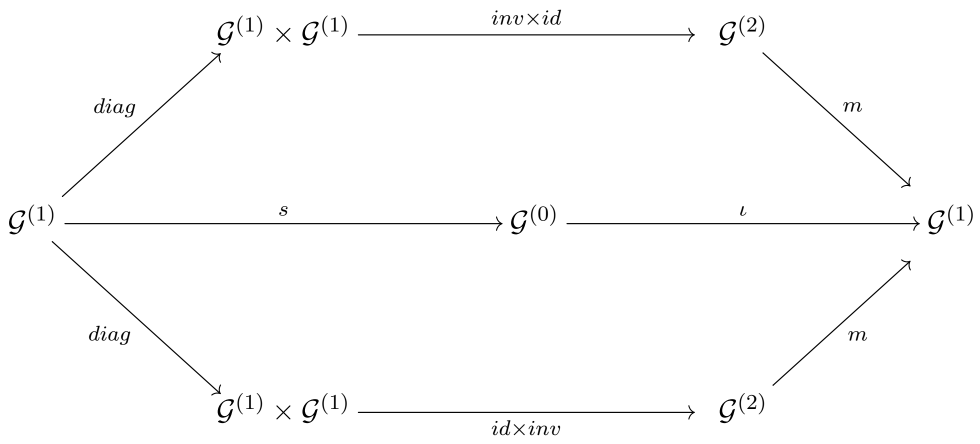
(3.1)

are subject to the following compatibility conditions.

$$\begin{CD} \mathcal{G}^{(3)} @>{m\times id}>> \mathcal{G}^{(2)} \\ @V{id\times m}VV @VV{m}V \\ \mathcal{G}^{(2)} @>>{m}> \mathcal{G}^{(1)} \end{CD} \qquad \begin{CD} \mathcal{G}^{(1)}\times\mathcal{G}^{(1)} @>{s\times id}>> \mathcal{G}^{(0)}\times\mathcal{G}^{(1)} @>{\iota\times id}>> \mathcal{G}^{(2)} \\ @A{diag}AA @. @VV{m}V \\ \mathcal{G}^{(1)} @= {} @= \mathcal{G}^{(1)} \\ @V{diag}VV @. @AA{m}A \\ \mathcal{G}^{(1)}\times\mathcal{G}^{(1)} @>>{id\times s}> \mathcal{G}^{(1)}\times\mathcal{G}^{(0)} @>>{id\times\iota}> \mathcal{G}^{(2)} \end{CD} \tag{3.2}$$



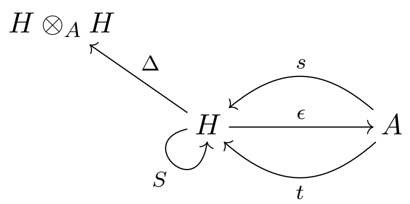
(3.3)



(3.4)

where, for $n \geqslant 2$ and $\mathcal{G}^{(n)}$ denotes the n-fold fiber product of s and t.

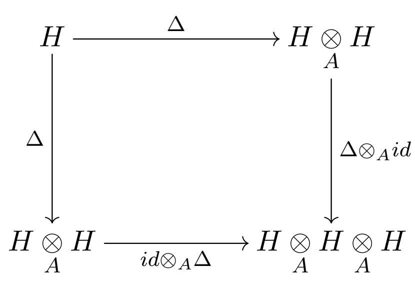
We will explore the properties of this groupoid in relation to the covering it is associated to in section 3.3 The functor $C(-)$ which associates to a topological space X its algebra of continuous complex-valued functions $C(X)$ is a duality (at least for locally compact Hausdorff topological spaces). Applying this functor to the diagram 3.1 gives us the following diagram of A-rings

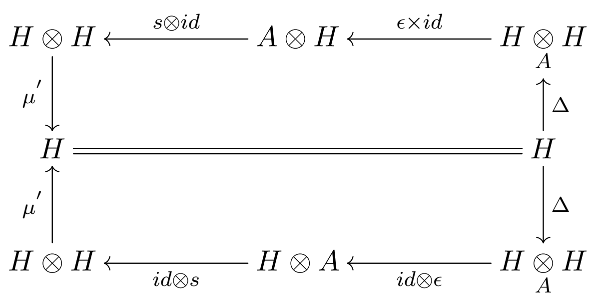


(3.5)

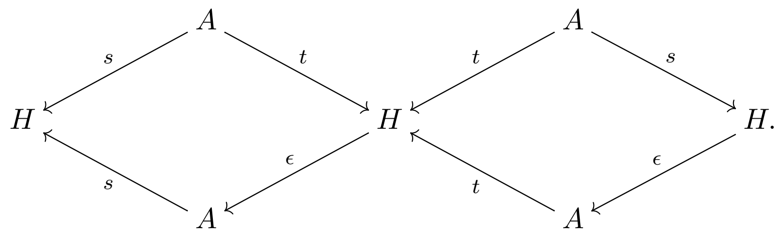
where $H = C(\mathcal{G}^{(1)})$, $\Delta = C(m)$, $\epsilon = C(\iota)$, $S = C(inv)$, and we denote by the same symbol s and t the induced maps of the groupoid's source and target maps.

The diagrams in 3.2 dualize to the following diagrams



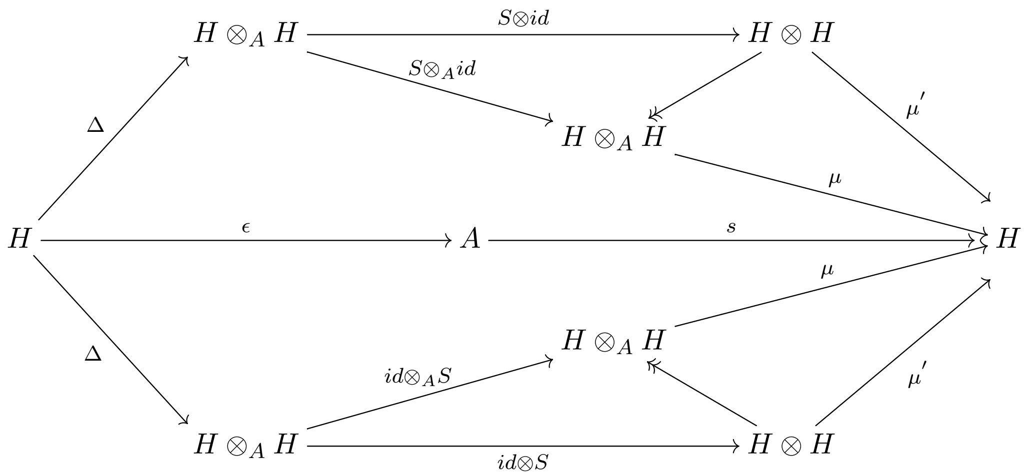


which express the coassociativity of Δ and its counitality with respect to ϵ. Diagram 3.3 dualizes to the following commutative diagram



(3.6)

Note that $C(\mathcal{G}^{(1)}) \otimes C(\mathcal{G}^{(1)}) = H \otimes H$ is in general not the same as $C(\mathcal{G}^{(1)} \times \mathcal{G}^{(1)})$, but only a dense subalgebra. Nonetheless, the image of $S \otimes id$ lands in $H \otimes H$. Thus, diagram 3.4 dualizes to the outer hexagon of the following diagram



(3.7)

The diagonal map *diag* induces a product $H \otimes H \xrightarrow{\mu'} H$. Since $A \xrightarrow{s} H$ is a $\mathbb{C}$-algebra map, by lemma 1 there is a unique A-ring structure on H with product $H \otimes_A H \xrightarrow{\mu} H$. The inner commutative hexagon of 3.7 implies that S is the convolution inverse of id in the convolution category $Conv(H, H)$ defined in section 2.4. All these diagrams tell us that H is a Hopf algebroid with coinciding left- and right-bialgebroid structures and antipode S. Furthermore, S is bijective. Applying the functor $C(-)$ to the associated reduced groupoid instead gives another Hopf algebroid H_{red} over A with coinciding left and right constituent bialgebroids.

Proposition 4. *There is a surjective algebraic morphism* $H \xrightarrow{(\phi,\phi)} H_{red}$ *of Hopf algebroids over* A.

The proposition above follows directly from the functoriality of $C(-)$ and the fact that $\mathcal{G}_{red}$ is a topological subgroupoid of $\mathcal{G}$. Note that in the above proposition, we are suppressing the fact that H and H_{red}, each has two constituent bialgebroids which happen to coincide. The morphisms in the pair (ϕ, ϕ) also coincide.

Going back to the covering $Y \xrightarrow{p} X$ and its associated reduced groupoid $\mathcal{G}_{red}$, there is a groupoid action of $\mathcal{G}_{red}$ on Y defined as follows

$$\mathcal{G}^{(1)}_{red\ s}\times_p Y \xrightarrow{\quad \alpha \quad} Y.$$

$$(\phi, y) \longmapsto \phi(y)$$

Moreover, $Y/\mathcal{G}_{red} \cong X$. Such a groupoid action is called *Galois* if the associated map

$$\mathcal{G}^{(1)}_{red\ s}\times_p Y \xrightarrow{\quad \alpha \quad} Y\ {}_p\times_p Y.$$

$$(\phi, y) \longmapsto (\phi(y), y)$$

Proposition 5. *The covering* $Y \overset{p}{\twoheadrightarrow} X$ *is Galois if and only if the groupoid action of* $\mathcal{G}_{red}$ *on* Y *is Galois.*

To see this, note that the groupoid $\mathcal{G}_{red}$ is isomorphic to the disjoint union of G over X. Since the action of G on $Y \overset{p}{\twoheadrightarrow} X$ is fiberwise, that action is precisely the action of $\mathcal{G}_{red}$ on $Y \overset{p}{\twoheadrightarrow} X$.

Remark 7. The above proposition is a direct consequence of the Galois theory for covering spaces, see for example appendix B.

Dually, the groupoid action of $\mathcal{G}_{red}$ on $Y \overset{p}{\twoheadrightarrow} X$ gives a coaction $B \xrightarrow{\rho} B \otimes_A H_{red}$ of the Hopf algebroid H_{red}. The coinvariants relative to the unit of H_{red} is A. Furthermore, the associated map

$$B \otimes_A B \xrightarrow{\mathfrak{gal}} B \otimes_A H_{red}$$

$$a \otimes_A b \longmapsto (a \otimes_A 1)\rho(b)$$

is a linear bijection in case the covering $Y \xrightarrow{p} X$ is Galois. In other words, $A \subseteq B$ is an H_{red}-Galois extension.

Remark 8. Recall in section 2.3 that a coaction of a Hopf algebroid is a pair of coactions, one for each constituent bialgebroid. In the situation above, the coaction ρ encodes the coinciding coactions of the coinciding constituent bialgebroids of H_{red}.

Let us now look at a reasonable finiteness condition that we can impose to the Hopf-Galois extensions of interest. Consider a faithful finite-dimensional representation π of $\mathcal{G}_{red}$. Explicitly, it is a continuous map $\mathcal{G}_{red} \xrightarrow{\pi} GL(E)$ of groupoids where $E \xrightarrow{q} X$ is a finite-dimensional vector bundle over X and $GL(E)$ is the associated general linear groupoid. As a groupoid, $GL(E)$ has objects points of X and an arrow $x \longrightarrow y$ from x to y is a linear isomorphism $E_x \longrightarrow E_y$. It is clear that the $GL(E)$ acts continuously on E. Construct the topological space $W = W(Y, \pi)$ as the quotient space $\left(Y\,{}_p\!\times_q\! E\right) / \sim$ where $(y, e) \sim (g \cdot y, \pi(g)e)$ for all $y \in Y$, $e \in E$ and $g \in \mathcal{G}^{(1)}_{red}$. Here, $(\cdot)$ refers to the action α of the groupoid $\mathcal{G}_{red}$ on Y. Since the fibers of p are orbits of the $\mathcal{G}$-action on Y, there is a well-defined projection $W \xrightarrow{r} X$, $(y, v) \mapsto p(y)$ making W a finite-dimensional vector bundle over X. The space $W = W(Y, \pi)$ is called the *associated vector bundle* to $Y \xrightarrow{p} X$ along with the representation $\mathcal{G}_{red} \xrightarrow{\pi} GL(E)$.

As before, projection $Y \xrightarrow{p} X$ gives an algebra inclusion $A \subseteq B$ which makes B into an A-module. Also, the global sections $\Gamma(X, W)$ is also a module over A which is finitely generated and projective by the Serre-Swan theorem. Note that by the construction of the associated bundle, $\Gamma(X, W)$ and B are isomorphic as A-modules. Thus, we have proved the following.

Proposition 6. *Given a classical finite Galois covering* $Y \overset{p}{\twoheadrightarrow} X$, $B = C(Y)$ *is a finitely-generated projective module over* $A = C(X)$.

Using the arguments above, we will further restrict our attention to the case when B is a finitely-generated left and right regular A-module. Furthermore, note that H_{red} itself is the space of global sections of a finite-rank vector bundle over X. The vector bundle in question is the vector bundle whose fibers are linear span of elements of G. This means that H_{red} is a finitely-generated projective A-module via any relevant module structure one can extract from our current discussion. With these, we are now ready to give our definition of a non-commutative covering space. We will state it in the most general sense we can afford but upcoming example, we will make reasonable restrictions.

Definition 11. Let A be an algebra over a commutative unital ring k. A *(finite, Galois) non-commutative covering* of A is a pair $(B, \mathcal{H})$ consisting of:

(a) $\mathcal{H}$ is a finitely generated projective Hopf algebroid over $A^{'} \subseteq A$, with bijective antipode S;

(b) $A \subseteq B$ is a right $\mathcal{H}$-Galois extension;

(c) B is a finitely-generated projective left and right regular A-module.

If in addition, B has no non-trivial idempotents as an $A^{'}$-ring, the covering $(B, \mathcal{H})$ is said to be *connected*. If $A^{'} = A$, we will call $(B, \mathcal{H})$ a *local* non-commutative covering of A. Otherwise, it is called *stratified* with stratification datum $A^{'} \subseteq A$. A non-commutative covering $(B, \mathcal{H})$ of A with stratification $k \subseteq A$ is called *uniform* if $\mathcal{H}$ is a Hopf algebra. □

Remark 9.

(1) Since the main concern of present work are non-commutative analogues of (finite, Galois) covering spaces, we will simply refer to a (finite, Galois) non-commutative covering space as a *covering space* and reserve the name classical covering space for classical ones.

(2) It is important to note that, by definition, the Hopf algebroid $\mathcal{H}$ carries several module structure over $A^{'}$. However, bijectivity of S implies that finitely-generated projectivity over $A^{'}$ are all equivalent for the module structures induced by the source and target maps, see proposition 4.5 of [6]. This makes part (a) of definition 11 well-defined.

(3) In a covering $(B, \mathcal{H})$ of A, we call $\mathcal{H}$ the associated *quantum symmetry* or just symmetry for brevity, of the covering. This corresponds to the deck transformation groupoid in the classical set up. Note that for a classical covering space $Y \twoheadrightarrow X$ the deck transformation group is completely determined as $G = Aut_X(Y)$. In the general case, there might be different quantum symmetries $\mathcal{H}_1$ and $\mathcal{H}_2$ making an extension $A \subseteq B$ Hopf-Galois and hence $(B, \mathcal{H}_1)$ and $(B, \mathcal{H}_2)$ are potentially different coverings. See [22] for an example of an extension $A \subset B$ which is Galois for different quantum symmetries. We will abuse language by saying that B *covers* A whenever $(B, \mathcal{H})$ is a covering of A for some Hopf algebroid $\mathcal{H}$.

(4) The motivation we outlined in this section suggests that in a non-commutative covering $(B, \mathcal{H})$ of A, the Hopf algebroid $\mathcal{H}$ is over A. However, as we will see in section 6.2 there are some interesting structures where it is necessary to consider Hopf algebroids over any subalgebra of A. In section 7.1 we will look at the comparison between local and stratified coverings.

Another way to justify these choices is as follows. Ignoring *all the points* of a non-commutative space A is as bad as considering *a point* on it. The option of having an intermediate base $k \subset A^{'} \subset A$ goes around this difficulty.

Lastly about the base, we will assume $A^{'}$ is *contained* in $\mathcal{H}$ and B via the relevant maps. Otherwise, one can always take the image of $A^{'}$ on these spaces and regard that as the space of interest.

(5) With stratification data $A^{'} \subseteq A$, part (b) of definition 11 explicitly means that B is an $A^{'}$-ring and A is an $A^{'}$-subring of B and the maps

$$B \underset{A}{\otimes} B \xrightarrow{\mathfrak{gal}_L} B \underset{(A')^{op}}{\otimes} H \,, \qquad B \underset{A}{\otimes} B \xrightarrow{\mathfrak{gal}_R} B \underset{A'}{\otimes} H$$

are linear bijections.

(6) The analogues of general finite coverings (possibly not Galois) are those extensions whose associated Hopf-Galois map is surjective but not necessarily injective. This is justified by the following observation: the deck transformation group of a covering always act freely. But the covering is normal precisely when, aside from being free, the action is transitive. So to get the analogue of general coverings we simply drop the condition that the action is transitive. But transitivity translates to surjectivity of the associated Galois map. The functor $C(-)$, the one that associates to a space X its algebra of functions $C(X)$, is contravariant. Thus, surjectivity of the associated Galois map is equivalent to the injectivity of the associated Hopf-Galois map. This tells us that general noncommutative coverings are those extensions whose associated Hopf-Galois map is injective.

3.2 Equivalences of coverings

After defining a mathematical object, the next thing to do is to make sense of when two such objects are equivalent. This is precisely what we are going to do in this section. We will focus our attention to local coverings i.e., those coverings $(B, \mathcal{H})$ of A whose quantum symmetry $\mathcal{H}$ is over A. We will briefly discuss how these equivalences work when we are dealing with stratified coverings. The first notion, which we call *naive equivalence* is the direct dualization of equivalence of coverings in the classical sense. The second one, called *Morita equivalence*, is the adaptation of a prominent equivalence in non-commutative geometry (which also goes by the same name) for non-commutative coverings.

Definition 12. Let $(B, \mathcal{H})$ and $(B^{'}, \mathcal{H}^{'})$ be local coverings of a non-commutative space A. We say that $(B^{'}, \mathcal{H}^{'})$ is an *intermediate covering* of $(B, \mathcal{H})$ if there is an intermediate inclusion $A \subseteq B^{'} \subseteq B$ and a monomorphism $\mathcal{H}^{'} \longrightarrow \mathcal{H}$ of Hopf algebroids over A such that the restriction of the coaction of $\mathcal{H}$ on $B^{'} \subseteq B$ corestricts to the coaction of $\mathcal{H}^{'}$ on $B^{'}$, i.e. we have a commutative diagram of A-rings

$$\begin{array}{ccc} B & \xrightarrow{\rho} & B \underset{A}{\otimes} H \\ \uparrow & & \uparrow \\ B^{'} & \xrightarrow[\rho^{'}]{} & B^{'} \underset{A}{\otimes} H^{'} \end{array} \tag{3.8}$$

Two coverings are *naively equivalent* if they are intermediate coverings of each other. □

Remark 10.

(1) In the classical case, $Y \twoheadrightarrow X$ is an intermediate covering of $Z \twoheadrightarrow X$ if there is a (continuous) surjection $Z \twoheadrightarrow Y$ and a group epimorphism $Aut_X(Z) \twoheadrightarrow Aut_X(Y)$. Definition 12 is the naive dualization of this statement, hence the name.

(2) Note that we did not specify which kind of morphism we are using in the above definition. This is because it is not important to do so. Both Hopf algebroids involved in the above definition are Hopf algebroids over A. In view of part (3) of remark 3, algebraic and geometric morphisms coincide in this situation.

Let $(B, \mathcal{H})$ and $(B^{'}, \mathcal{H}^{'})$ be naively equivalent coverings of a noncommutative space A. Immediately, we see that $B \cong B^{'}$ as A-rings. By definition, there are injective maps of Hopf algebroids $\mathcal{H} \xrightarrow{p} \mathcal{H}^{'}$ and $\mathcal{H}^{'} \xrightarrow{q} \mathcal{H}$. Using the map p, $\mathcal{H}$ becomes a right $\mathcal{H}^{'}$-comodule algebra via the coactions ρ_L and ρ_R defined by the compositions

$$\mathcal{H} \xrightarrow{\Delta_L} \mathcal{H} \otimes_A \mathcal{H} \longrightarrow \mathcal{H} \otimes_A \mathcal{H}^{'} \ ,$$

$$\mathcal{H} \xrightarrow{\Delta_R} \mathcal{H} \otimes_A \mathcal{H} \longrightarrow \mathcal{H} \otimes_A \mathcal{H}^{'} \ ,$$

respectively. The coassiociativity of the $\mathcal{H}^{'}$-coaction defined by ρ_L and ρ_R follows from the commutativity of the following diagrams.

$$\begin{array}{ccccc} H & \xrightarrow{\Delta_L} & H \otimes_A H & \xrightarrow{id\otimes p} & H \otimes_A H^{'} \\ \downarrow{\scriptstyle \Delta_R} & & \downarrow{\scriptstyle \Delta_R} & & \downarrow{\scriptstyle \Delta_R^{'}} \\ H \otimes_A H & \xrightarrow{\Delta_L \otimes id} & H \otimes_A H \otimes_A H & \xrightarrow{id\otimes p \otimes p} & H \otimes_A H^{'} \otimes_A H^{'} \\ \downarrow{\scriptstyle id\otimes p} & & \downarrow{\scriptstyle id\otimes id\otimes p} & & \downarrow{\scriptstyle id\otimes id\otimes id} \\ H \otimes_A H^{'} & \xrightarrow[\Delta_L \otimes id]{} & H \otimes_A H \otimes_A H^{'} & \xrightarrow[id\otimes p\otimes id]{} & H \otimes_A H^{'} \otimes_A H^{'} \end{array}$$

$$\begin{array}{ccccc} H & \xrightarrow{\Delta_R} & H \otimes_A H & \xrightarrow{id\otimes p} & H \otimes_A H^{'} \\ \downarrow{\scriptstyle \Delta_L} & & \downarrow{\scriptstyle \Delta_L} & & \downarrow{\scriptstyle \Delta_L^{'}} \\ H \otimes_A H & \xrightarrow{\Delta_R \otimes id} & H \otimes_A H \otimes_A H & \xrightarrow{id\otimes p \otimes p} & H \otimes_A H^{'} \otimes_A H^{'} \\ \downarrow{\scriptstyle id\otimes p} & & \downarrow{\scriptstyle id\otimes id\otimes p} & & \downarrow{\scriptstyle id\otimes id\otimes id} \\ H \otimes_A H^{'} & \xrightarrow[\Delta_R \otimes id]{} & H \otimes_A H \otimes_A H^{'} & \xrightarrow[id\otimes p\otimes id]{} & H \otimes_A H^{'} \otimes_A H^{'} \end{array}$$

Let us determine the coinvariants of $\mathcal{H}$ under this coaction of $H^{'}_R$. An element $a \in H$ is coinvariant if $\rho_R(a) = a \otimes_A 1$. This means that there exist $h \in H$ such that $p(h) = 1$ and $\Delta_R(a) = a \otimes_A h$. Injectivity of p implies that $h = 1$ and hence, $\Delta_R(a) = \Delta_L(a) = a \otimes_A 1$. Thus, the coinvariants of ρ_L and ρ_R coincide with the coinvariants of the regular comodule structure of $\mathcal{H}$ which is A itself.

Meanwhile, using the map q we can equip $\mathcal{H}$ a structure of a right $\mathcal{H}^{'}$-module via

$$H \otimes_A H^{'} \xrightarrow{id\otimes q} H \otimes_A H \xrightarrow{m} H$$

which makes $\mathcal{H}$ a right-right $\mathcal{H}^{'}$-Hopf module. The counit of the adjoint equivalence in 3 with $M = \mathcal{H}^{'}$ provides an isomorphism

$$H^{'} \cong A \otimes_A H^{'} \cong (H)^{co\ \mathcal{H}^{'}_R} \otimes_A H^{'} \xrightarrow{counit} H$$

of right-right $\mathcal{H}^{'}$-Hopf modules. Reversing the roles of $\mathcal{H}$ and $\mathcal{H}^{'}$ in the preceding argument shows that H and $H^{'}$ are also isomorphic as right-right $\mathcal{H}$-Hopf modules. This is enough to conclude that $\mathcal{H}$ and $\mathcal{H}^{'}$ are isomorphic Hopf algebroids. This proves the *only if* part of the following proposition. The *if* part is obvious.

Proposition 7. *Let $(B, \mathcal{H})$ and $(B^{'}, \mathcal{H}^{'})$ be local coverings of a non-commutative space A. Then $(B, \mathcal{H})$ and $(B^{'}, \mathcal{H}^{'})$ are naively equivalent if and only if $B \cong B^{'}$ as A-rings and $\mathcal{H} \cong \mathcal{H}^{'}$ as Hopf algebroids.*

Before stating the second type of equivalence, let us first define what a Hopf bimodule is.

Definition 13. Given Hopf algebroids $\mathcal{H}$ and $\mathcal{H}^{'}$ over R, an $(\mathcal{H}, \mathcal{H}^{'})$-*Hopf bimodule* M is an $(\mathcal{H}, \mathcal{H}^{'})$-bimodule and an $(\mathcal{H}, \mathcal{H}^{'})$-bicomodule such that

(a) the left $\mathcal{H}$-module and the left $\mathcal{H}$-comodule structures on M make it a left-left $(\mathcal{H}, \mathcal{H})$-Hopf module,

(b) the right $\mathcal{H}^{'}$-module and the right $\mathcal{H}^{'}$-comodule structures on M make it a right-right $(\mathcal{H}^{'}, \mathcal{H}^{'})$-Hopf module,

(c) the left $\mathcal{H}$-module and the right $\mathcal{H}^{'}$-comodule structures on M make it a left-right $(\mathcal{H}, \mathcal{H}^{'})$-Hopf module, and

(d) the right $\mathcal{H}^{'}$-module and the left $\mathcal{H}$-comodule structures on M make it a right-left $(\mathcal{H}^{'}, \mathcal{H})$-Hopf module.

□

Note that, by an $(\mathcal{H}, \mathcal{H}^{'})$-bimodule, we mean a left H_L-module and a right $H^{'}_R$-module. In other words, we are only using one ring structure out of the two each of the Hopf algebroids $\mathcal{H}$ and $\mathcal{H}^{'}$ have according to which side they act. On the other hand, both coring structures of each of the Hopf algebroids are used. Notice that for an $(\mathcal{H}, \mathcal{H}^{'})$-Hopf bimodule M, the maps defining the left $\mathcal{H}$-comodule structure are left $\mathcal{H}$-module maps. Similarly, the maps defining the right $\mathcal{H}^{'}$-comodule structure are right $\mathcal{H}^{'}$-module maps.

Definition 14. Two local coverings $(B, \mathcal{H})$ and $(B^{'}, \mathcal{H}^{'})$ of a non-commutative space A are *Morita equivalent* if the following conditions are satisfied:

(a) There exists a $(B, B^{'})$-bimodule $\mathcal{X}$ and a $(B^{'}, B)$-bimodule $\mathcal{Y}$ whose constituent left and right module structures are finitely-generated and projective, and such that

$$\mathcal{X} \underset{B'}{\otimes} \mathcal{Y} \cong B, \qquad \mathcal{Y} \underset{B}{\otimes} \mathcal{X} \cong B^{'}$$

as B-bimodules and $B^{'}$-bimodules, respectively.

(b) There exists an $(\mathcal{H}, \mathcal{H}^{'})$-Hopf bimodule $\mathcal{U}$ and an $(\mathcal{H}^{'}, \mathcal{H})$-Hopf bimodule $\mathcal{V}$ whose constituent left and right module structures are finitely-generated and projective, and such that

$$\mathcal{U} \underset{\mathcal{H}'}{\otimes} \mathcal{V} \cong \mathcal{H}, \qquad \mathcal{V} \underset{\mathcal{H}}{\otimes} \mathcal{U} \cong \mathcal{H}^{'}$$

as $\mathcal{H}$-Hopf bimodules and $\mathcal{H}^{'}$-Hopf bimodules, respectively.

□

In part (b) of the above definition, by the constituent left and right module structures of $\mathcal{U}$ we mean the left H_L-module and the right $H^{'}_R$-module structures of $\mathcal{U}$, since in any case, these are the only relevant module structures in view of the definition of a Hopf module, see definition 6. Thus, the finitely-generated projectivity condition in part (b) refers to the finitely-generated projectivity of the aforementioned modules. A similar clarification works for $\mathcal{V}$.

Remark 11.

(1) Naively equivalent coverings $(B, \mathcal{H})$ and $(B^{'}, \mathcal{H}^{'})$ are Morita equivalent. B and $B^{'}$ provide the bimodules asked in (a) while $\mathcal{H}$ and $\mathcal{H}^{'}$ provide the Hopf bimodules required in (b).

(2) Morita equivalences of coverings coincide with isomorphisms in a suitable category. Denote by $COV_{Morita}(A)$ the category whose objects are local coverings of a non-commutative space A. A morphism $(B, \mathcal{H}) \longrightarrow (B^{'}, \mathcal{H}^{'})$ is a pair $(\mathcal{X}, \mathcal{U})$ of a $(B, B^{'})$-bimodule $\mathcal{X}$ and an $(\mathcal{H}, \mathcal{H}^{'})$-Hopf bimodule $\mathcal{U}$. The composition rule given by

$$\begin{array}{ccc} & (B^{'}, \mathcal{H}^{'}) & \\ {\scriptstyle (\mathcal{X},\mathcal{U})} \nearrow & & \searrow {\scriptstyle (\mathcal{Y},\, \mathcal{V})} \\ (B, \mathcal{H}) & \xrightarrow[\left(\mathcal{X} \underset{B^{'}}{\otimes} \mathcal{Y},\ \mathcal{U} \underset{\mathcal{H}^{'}}{\otimes} \mathcal{V}\right)]{} & (B^{''}, \mathcal{H}^{''}). \end{array}$$

The identity morphism of the object $(B, \mathcal{H})$ is the pair $(B, \mathcal{H})$ itself. It is immediate to see that the isomorphisms in $COV_{Morita}(A)$ are precisely the Morita equivalences of coverings. We will call such invertible arrow $(B, \mathcal{H})$ a *Morita equivalence bimodule.*

(3) Recall that two noncommutative spaces A and $A^{'}$ are *Morita equivalent* if there exist an $(A, A^{'})$-bimodule $\mathcal{P}$ and an $(A^{'}, A)$-bimodule $\mathcal{Q}$ such that

$$\mathcal{P} \underset{A^{'}}{\otimes} \mathcal{Q} \cong A, \qquad \mathcal{Q} \underset{A}{\otimes} \mathcal{P} \cong A^{'}$$

as A-bimodules and $A^{'}$-bimodules, respectively. Notice that Morita equivalence of coverings as defined in 14 puts together Morita equivalence of the extension algebras (part (a)) and the Hopf-adaptation of Morita equivalence for the associated symmetries (part (b)).

(4) In light of remark (3) above, we will say that two Hopf algebroids $\mathcal{H}^{'}$ and $\mathcal{H}^{'}$ over A are *Morita equivalent* if there exists an $(\mathcal{H}, \mathcal{H}^{'})$-Hopf bimodule $\mathcal{U}$ and an $(\mathcal{H}^{'}, \mathcal{H})$-Hopf bimodule $\mathcal{V}$ satisfying condition (b) of definition 14.

The following proposition states that $COV_{Morita}(A)$ is a non-commutative geometric invariants of A.

Proposition 8. *Let A be Morita equivalent to $A^{'}$. Then the categories $COV_{Morita}(A)$ and $COV_{Morita}(A^{'})$ are adjoint equivalent.*

Proof: Consider Morita equivalent noncommutative spaces A and $A^{'}$. Let $(B, \mathcal{H})$ be a local covering of A. We will construct a covering of $A^{'}$ whose Morita equivalence class is uniquely determined by the Morita equivalence class of $(B, \mathcal{H})$.

By assumption, there is an $(A, A^{'})$-bimodule $\mathcal{P}$ and an $(A^{'}, A)$-bimodule $\mathcal{Q}$ such that

$$\mathcal{P} \underset{A'}{\otimes} \mathcal{Q} \cong A, \qquad \mathcal{Q} \underset{A}{\otimes} \mathcal{P} \cong A'.$$

We claim that $\left(B', \mathcal{H}'\right) = (\mathcal{Q} \otimes_A B \otimes_A \mathcal{P}, \mathcal{Q} \otimes_A \mathcal{H} \otimes_A \mathcal{P})$ is a covering of A'. By $\mathcal{Q} \otimes_A \mathcal{H} \otimes_A \mathcal{P}$ we mean the Hopf algebroid with constituent left- and right-bialgebroids $H_L' = \mathcal{Q} \otimes_A H_L \otimes_A \mathcal{P}$ and $H_R' = \mathcal{Q} \otimes_A H_R \otimes_A \mathcal{P}$, respectively.

First, let us show that B' is an A'-ring. The A-bimodule structure maps

$$B \otimes_A B \xrightarrow{\mu} B , \qquad A \xrightarrow{\eta} B$$

of B as an A-ring induce the following A'-bimodule maps

$$B' \otimes_{A'} B' \cong \mathcal{Q} \otimes_A B \otimes_A B \otimes_A \mathcal{P} \xrightarrow{\mathcal{Q} \otimes_A \mu \otimes_A \mathcal{P}} \mathcal{Q} \otimes_A B \otimes_A \mathcal{P} \cong B'$$

$$A' \cong \mathcal{Q} \otimes_A A \otimes_A \mathcal{P} \xrightarrow{\mathcal{Q} \otimes_A \eta \otimes_A \mathcal{P}} \mathcal{Q} \otimes_A B \otimes_A \mathcal{P} \cong B'$$

which satisfy the associativity and the unitality diagrams. These maps make B' into an A'-ring. Note that the above argument is just the application of the functors $\mathcal{Q} \otimes_A -$ and $- \otimes_A \mathcal{P}$ which are both equivalences by the Morita property. Thus, they preserve diagrams. We will make use of this argument in the rest of the proof.

Now, it is easy to see that $\mathcal{H}'$ is a Hopf algebroid over A' since the maps and diagrams that define the Hopf algebroid structure on $\mathcal{H}$ all live in the category of A-bimodules. Applying the functors $\mathcal{Q} \otimes_A -$ and $- \otimes_A \mathcal{P}$ give the structure maps for $\mathcal{H}'$ which satisfy the relevant diagrams. For the same reason, B' carries an $\mathcal{H}'$-comodule structure via

$$B' \cong \mathcal{Q} \otimes_A B \otimes_A \mathcal{P} \underset{\mathcal{Q} \otimes_A \rho_L \otimes_A \mathcal{P}}{\xrightarrow{\mathcal{Q} \otimes_A \rho_R \otimes_A \mathcal{P}}} (\mathcal{Q} \otimes_A B \otimes_A \mathcal{P}) \underset{A'}{\otimes} (\mathcal{Q} \otimes_A \mathcal{H} \otimes_A \mathcal{P}) \cong B' \underset{A'}{\otimes} \mathcal{H}' .$$

The H_R'-coinvariants $\left(B'\right)^{co\ H_R'}$ of this comodule structure is the equalizer of ρ_R' and $- \otimes_{A'} H_R'$, i.e.

$$\left(B'\right)^{co\ H_R'} \longrightarrow B' \underset{\rho_R'}{\overset{\rho_L'}{\rightrightarrows}} B' \underset{A'}{\otimes} \mathcal{H}' .$$

This diagram is the image of the equalizer diagram defining $B^{co\ H_R}$ after applying $\mathcal{Q} \otimes_A -$ and $- \otimes_A \mathcal{P}$. Thus, $\left(B'\right)^{co\ H_R'} \cong \mathcal{Q} \otimes_A B^{co\ H_R} \otimes_A \mathcal{P} \cong \mathcal{Q} \otimes_A A \otimes_A \mathcal{P} \cong A'$.

Finally, finitely-generated projectivity of B' and $\mathcal{H}'$ is equivalent to finitely-generated projectivity of B and $\mathcal{H}$. This proves our claim.

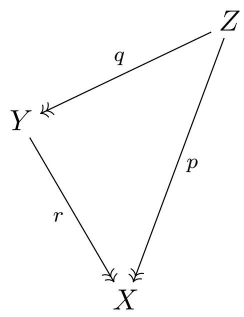
Now, any covering of A' Morita equivalent to $(B', \mathcal{H}')$ is of the form

$$\left(B' \underset{B'}{\otimes} \mathcal{X}, \mathcal{H}' \underset{\mathcal{U}}{\otimes} \mathcal{H}'\right)$$

for some Morita equivalence bimodule $(\mathcal{X}, \mathcal{U})$. Again, by $\mathcal{H}' \otimes_{\mathcal{H}'} \mathcal{U}$ we mean the Hopf algebroid whose consituent bialgebroids are the images of that of $\mathcal{H}'$ under the functor $- \otimes_{\mathcal{H}'} \mathcal{U}$. Invertibility of $(\mathcal{X}, \mathcal{U})$ implies that there exist a Morita equivalence bimodule $(\mathcal{Y}, \mathcal{V})$ such that applying the functor $\mathcal{F} = \mathcal{P} \otimes_{A'} (\mathcal{Y} \otimes -) \otimes_{A'} \mathcal{Q}$ to $B' \otimes_{B'} \mathcal{X}$ and the functor $\mathcal{G} = \mathcal{P} \otimes_{A'} (\mathcal{V} \otimes -) \otimes_{A'} \mathcal{Q}$ to $\mathcal{V} \otimes_{\mathcal{H}'} \mathcal{H}'$ yields a covering of A Morita equivalent to $(B, \mathcal{H})$. This proves the proposition. ■

3.3 Composition of coverings

The following commutative diagram of classical covering spaces



(3.9)

has three different interpretations which are all totally different in the present framework. The first one, by viewing $Y \xrightarrow{r} X$ as an intermediate covering of $Z \xrightarrow{p} X$, one gets the notion of intermediate coverings we defined in section 3.2. The second one, by viewing $Z \xrightarrow{q} Y$ as an arrow from $Z \xrightarrow{p} X$ to $Y \xrightarrow{r} X$ in the category of classical coverings of X, one is lead to the notion of an arrow in the category $COV_{Morita}(A)$ we defined in section 3.2. The third one, which is the main subject of the present section is the analogue of the fact that $Z \xrightarrow{p} X$ is the composition of the coverings $Z \xrightarrow{q} Y$ and $Y \xrightarrow{r} X$.

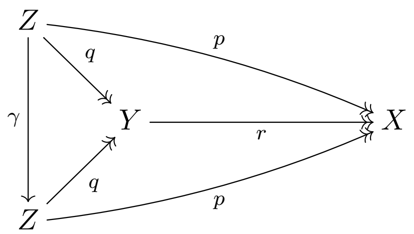
Let $G = Aut(Z \xrightarrow{p} X)$, $H = Aut(Z \xrightarrow{q} Y)$ and $K = Aut(Y \xrightarrow{r} X)$ be the automorphism groups of the indicated classical covering maps in the appropriate over-category. Then, we have the following proposition.

Proposition 9. *If diagram 3.9 commutes then we have an exact sequence*

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0.$$

Conversely, any extension G of K by H gives a commutative diagram as 3.9.

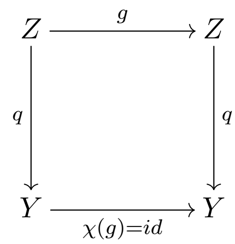
PROOF: Let us outline a proof of this classical fact. Assume 3.9 commutes. Let $\gamma \in H$. Then commutativity of the smaller triangles in the following diagram



implies that $\gamma \in G$. It is immediate to see that this defines an injection $H \longrightarrow G$. Let us define a map $\chi : G \longrightarrow K$ as follows: for $g \in G$, let $\chi(g) : Y \longrightarrow Y$, $y \mapsto qgq^{-1}(y)$. The map $\chi(g)$ is independent of any pre-image of y under q. Also, for any $y \in Y$, we have

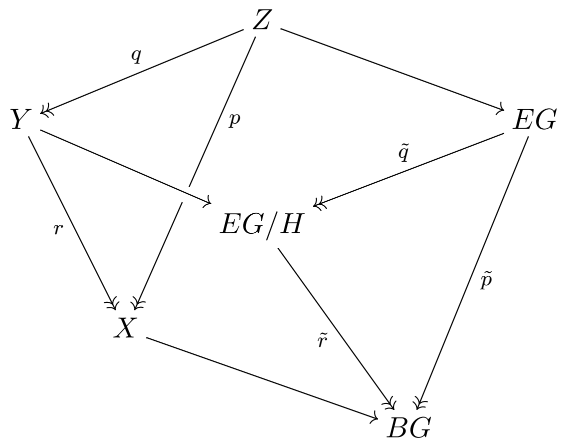
$$r\chi(g)(y) = rqgq^{-1}(y) = pgq^{-1}(y) = pq^{-1}(y) = r(y)$$

which implies that $\chi(g) \in K$. To see that χ is surjective, for any $\gamma \in K$ let γ^* be the pullback of γ along q. Then $\gamma^* \in G$ and $\chi(\gamma^*) = \gamma$. Finally, let us show that $H = ker\ \chi$. Let $g \in G$ such that $\chi(g) = id$. Then we have



which immediately implies that $g \in H$.

For the converse, assume G is an extension of K by H. Consider the classifying space BG of G. By definition, there is a space EG and a surjective map $EG \xrightarrow{\tilde{p}} BG$ which is a G-principal bundle. In other words, $\tilde{p}$ is a classical Galois covering map with G as its deck transformation group. Dividing EG by the restricted action of H gives a diagram



of classical covering spaces with $\tilde{q}$ the canonical surjection and $\tilde{r}$ a covering map with K as its deck transformation group. Pulling-back $\tilde{q}$ and $\tilde{r}$ along the classifying map $X \longrightarrow BG$ gives such a commutative diagram as 3.9. This proves the above proposition. ■

To find the analogue of proposition 9, let us formulate the above proposition in terms of groupoids. Using definition 10, to any classical covering $Y \overset{p}{\twoheadrightarrow} X$ we can associate a topological groupoid $\mathscr{G}_{red}$. We set $\mathscr{G}_{red}^{(0)} = X$, the space of objects. For $x, y \in X$, the hom-set $\mathscr{G}_{red}(x, y)$ is empty unless $x = y$, in which case it is the set of bijections $p^{-1}(x) \longrightarrow p^{-1}(y)$ induced by lifting to Y continuous loops on X based at $x \in X$. The covering $Y \overset{p}{\twoheadrightarrow} X$ is Galois if and only if the associated groupoid action of $\mathscr{G}'$ on $Y \overset{p}{\twoheadrightarrow} X$ is Galois.

A partial converse is true, i.e. one can associate a covering to a sufficiently nice groupoid. But first, we need the following lemma.

Lemma 3. *For a connected groupoid, i.e. one in which there is an arrow between any two objects, the hom-sets are in bijection with each other.*

Let $\mathscr{G}$ be a locally finite, connected Hausdorff groupoid over X. Then

$$Y = \coprod_{x \in X} \mathscr{G}^{(1)}(x, x) \subseteq \mathscr{G}^{(1)},$$

equipped with subspace topology, is a principal G-bundle with $G = \mathscr{G}^{(1)}(x_0, x_0)$ for any fixed $x_0 \in X$. The bundle projection is given by the restriction $p = s|_Y$ of the source map of $\mathscr{G}$ to Y. The hom-sets are finite subsets of a Hausdorff space, which means that they are discrete. This implies that Y, not only is a principal bundle over X, but a covering with deck transformation group G. The next proposition tells us more about the coverings this construction gives.

Proposition 10. *Given a locally finite, connected Hausdorff groupoid $\mathscr{G}$ over X, the map*

$$Y = \coprod_{x \in X} \mathscr{G}^{(1)}(x,x) \xrightarrow{\quad s \quad} X$$

is a finite, Galois covering of X.

PROOF: All that is left to show is that the covering $Y \overset{s}{\twoheadrightarrow} X$ is Galois. This follows from the fact the a connected groupoid acts on itself transitively. To see this, note that a typical fiber of the source map of $\mathscr{G}$ is a group, which acts transitively on itself. One can transport the action of this typical fiber to any other fiber via the conjugation by an arrow between the base points of the fibers involved. This proves that the covering we will get from the construction above is Galois. ■

Corollary 2. *The construction of a groupoid from a covering defined in 10 is not inverse to the construction we used for proposition 10.*

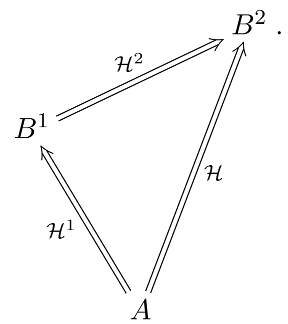
The above corollary follows immediately from the fact that the latter construction spits out a Galois covering while the former works for any covering. Let us now build on stating an analogue of proposition 9. Our first task is to make sense of diagram 3.9 for non-commutative coverings.

Consider locally finite, connected (Hausdorff) groupoids $\mathscr{G}$ and $\mathscr{K}$ over X. Let $\mathscr{G} \xrightarrow{\psi} \mathscr{K}$ be a groupoid homomorphism which is identity on objects and surjective on hom-sets. The construction preceding proposition 10 is clearly functorial. Denote by $Z \xrightarrow{p} X$ and $Y \xrightarrow{q} X$ the associated covering spaces to $\mathscr{G}$ and $\mathscr{K}$, respectively. The groupoid map ψ then induces a map of classical covering spaces $Z \xrightarrow{\psi^*} Y$. It is easy to see that ψ^* is itself a covering. The groupoid $\mathscr{H}$ associated to ψ^* is given as $\mathscr{H}^{(0)} = Y$ and $\mathscr{H}^{(1)}(k_1, k_2) = \psi^{-1}(k_2^{-1}k_1)$. This gives an exact sequence of groupoids

$$\begin{array}{ccccc} \mathscr{H} & \hookrightarrow & \mathscr{G} & \overset{\psi}{\twoheadrightarrow} & \mathscr{K} \\ \\ Y & \twoheadrightarrow & X & \xrightarrow{id} & X \\ \mathscr{H}^{(1)} & \hookrightarrow & \mathscr{G}^{(1)} & \twoheadrightarrow & \mathscr{K}^{(1)}. \end{array}$$

The proposition and the construction above motivate the following definition. Let us diagrammatically write $P \overset{\mathcal{S}}{\Longrightarrow} Q$ when $(Q, \mathcal{S})$ is a local covering of P.

Definition 15. Consider inclusions of k-algebras $A \subseteq B^1 \subseteq B^2$, Hopf algebroids $\mathcal{H}$ and $\mathcal{H}^1$ over A, a Hopf algebroid $\mathcal{H}^2$ over B^1 such that $(B^1, \mathcal{H}^1)$, $(B^2, \mathcal{H}^2)$ and $(B^2, \mathcal{H})$ are (local) non-commutative coverings of A, B^1 and A, respectively. In terms of diagrams, we have



(3.10)

Let us denote by $\mathfrak{gal}$, $\mathfrak{gal}_1$ and $\mathfrak{gal}_2$ the respective Galois maps associated to the coactions $B^2 \xrightarrow{\rho} B^2 \otimes_A \mathcal{H}$, $B^1 \xrightarrow{\rho_1} B^1 \otimes_A \mathcal{H}^1$ and $B^2 \xrightarrow{\rho_2} B^2 \otimes_{B^1} \mathcal{H}^2$. We say that diagram 3.10 *commutes* if the following conditions are satisfied:

(a) There is a geometric morphism $\mathcal{H}^1 \xrightarrow{(id,\phi)} \mathcal{H}$ of Hopf algebroids such that ϕ is injective and the following diagram commutes.

$$\begin{array}{ccc} B^1 \otimes_A B^1 & \xrightarrow{\mathfrak{gal}_1} & B^1 \otimes_A \mathcal{H}^1 \\ \Big\downarrow {\scriptstyle id \underset{A}{\otimes} id} & & \Big\downarrow {\scriptstyle id \underset{A}{\otimes} \phi} \\ B^2 \otimes_A B^2 & \xrightarrow[\mathfrak{gal}]{} & B^2 \otimes_A \mathcal{H} \end{array}$$

(b) There is a geometric morphism $\mathcal{H} \xrightarrow{(f,\psi)} \mathcal{H}^2$ of Hopf algebroids such that f is the inclusion $A \subseteq B^1$, ψ is surjective and the following diagram commutes.

$$\begin{array}{ccc} B^2 \otimes_A B^2 & \xrightarrow{\mathfrak{gal}} & B^2 \otimes_A \mathcal{H} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle id \otimes_f \psi} \\ B^2 \otimes_{B^1} B^2 & \xrightarrow[\mathfrak{gal}_2]{} & B^2 \otimes_{B^1} \mathcal{H}^2 \end{array}$$

□

Remark 12.

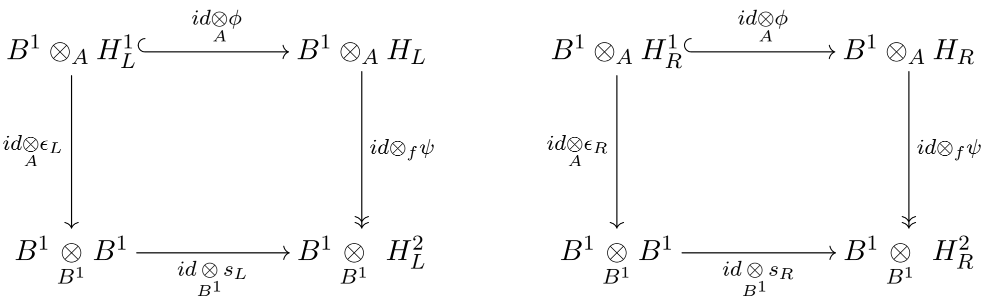
(1) Note that we are suppressing a lot of notations in the above definition. First, when we denote by ρ the coaction of $\mathcal{H}$ on B^2 we mean a pair of maps ρ_L and ρ_R as described in section 2.3. Same goes for ρ^1 and ρ^2. Correspondingly, by $\mathfrak{gal}$ we mean a pair of maps $\mathfrak{gal}_L$ and $\mathfrak{gal}_R$ associated to ρ_L and ρ_R, respectively.

(2) At present writing of this paper, there is no existing Galois correspondence for Hopf-Galois extensions for Hopf algebras let alone for Hopf algebroids. The two conditions listed above are the minimum requirements one needs to have a non-commutative analogue of proposition 9.

(3) The above definition is specifically for local coverings. For general stratified coverings, $\mathcal{H}$ is a Hopf algebroid over $A^{'} \subset A$, $\mathcal{H}_1$ is a Hopf algebroid over $A_1 \subseteq A$ and $\mathcal{H}^2$ is a Hopf algebroid over $A_2 \subseteq B^1$. For the definition of commutativity of diagram 3.10 in this situation, in addition to the existence of ϕ and ψ we also assert the existence of k-algebra morphisms $f_1 : A_1 \longrightarrow A^{'}$ and $f_2 : A_2 \longrightarrow A^{'}$. In the appropriate diagrams, we replace (id, ϕ) by (f_1, ϕ), (f, ψ) by (f_2, ψ), $id \underset{A}{\otimes} \phi$ by $id \underset{f_1}{\otimes} \phi$ and $id \otimes_f \psi$ by $id \underset{f_2}{\otimes} \psi$.

If diagram 3.10 commutes, we will refer to the local covering $A \overset{\mathcal{H}}{\Longrightarrow} B^2$ as the *composition* of $A \overset{\mathcal{H}^1}{\Longrightarrow} B^1$ and $B^1 \overset{\mathcal{H}^2}{\Longrightarrow} B^2$. Note that the commutativity of diagram 3.10 depends on ϕ and ψ. We will call the pair (ϕ, ψ) the *commutativity datum* of diagram 3.10 of local coverings. The *commutativity datum* of stratified coverings is the quadruple (f_1, f_2, ϕ, ψ) as described in (3) of the above remarks. The following proposition states the non-commutative analogue of the first part of proposition 9 for local coverings.

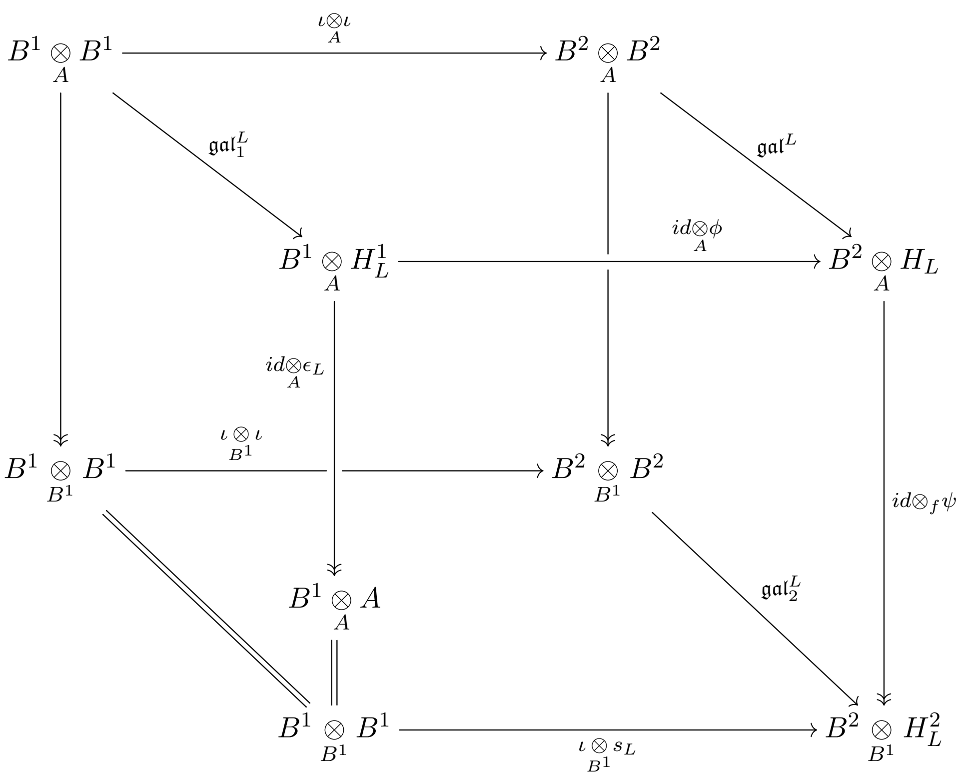
Proposition 11. *(Exact fitting for local coverings.) Let $(B^2, \mathcal{H})$ and $(B^1, \mathcal{H}^1)$ be local coverings of A and let $(B^2, \mathcal{H}^2)$ be a local covering of B^1. Suppose the associated diagram as in 3.10* *commutes with commutativity datum* (ϕ, ψ). *Then, up to extending scalars, the composite map* $\psi \circ \phi$ *factors through the source maps* $B^1 \xrightarrow{s_L,\ s_R} \mathcal{H}^2$, *i.e. following diagram of k-modules commute*



(3.11)

where s_L, s_R *are the source maps of* $\mathcal{H}^2$ *and* ϵ_L, ϵ_R *denote the counit maps of* $\mathcal{H}^1$.

Proof: For $(B^2, \mathcal{H})$ and $(B^1, \mathcal{H}^1)$ local coverings of A and $(B^2, \mathcal{H}^2)$ a local covering of B^1, denote by $\mathfrak{gal}, \mathfrak{gal}_1$ and $\mathfrak{gal}_2$ the associated Galois maps, respectively. We will only prove the left diagram of 3.11. The proof for the right diagram goes the same way. Assuming diagram 3.10 commutes with commutativity datum (ϕ, ψ) we have a commutative diagram

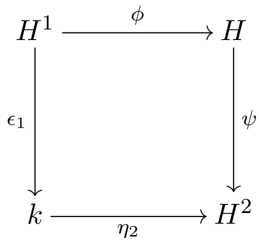


The top and right squares are the commutative diagrams in definition 15. The commutativity of the back square, where the arrows going downwards are the canonical surjections, is obvious. To see the commutativity of the left square, take $b, b' \in B^1$. Then

$$(id \underset{A}{\otimes} \epsilon_L)\mathfrak{gal}^L_1(b \underset{A}{\otimes} b') = (id \underset{A}{\otimes} \epsilon_L)(bb'_{[0]} \underset{A}{\otimes} b'_{[1]}) = bb'_{[0]} \underset{A}{\otimes} \epsilon_L(b'_{[1]})$$
$$= bb'_{[0]}\epsilon_L(b'_{[1]}) \underset{A}{\otimes} 1 = bb' \underset{A}{\otimes} 1 = bb' \underset{B^1}{\otimes} 1 = b \underset{B^1}{\otimes} b'.$$

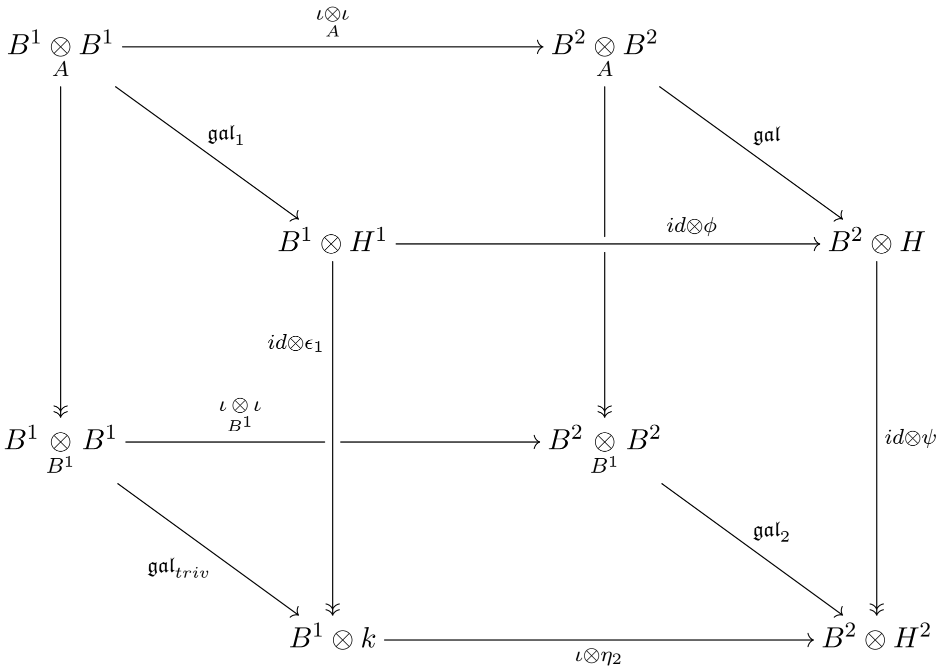
Same computation holds for $\mathfrak{gal}^R_1$ and ϵ_R. The commutativity of the bottom square is due to the fact that the module structure on H^2_L used to form the tensor product $B^2 \underset{B^1}{\otimes} H^2_L$ is the one provided by the source maps. Commutativity of the back, right, left, top and bottom squares imply that the front square commutes. By inspection, the front square reduce to the left square asserted by the proposition. ■

Proposition 12. *(Exact fitting for uniform coverings.) Let* (B^1, H^1) *and* (B^2, H) *be uniform coverings of* A *and* (B^2, H^2) *a uniform covering of* B^1*. Suppose at least one of* B^1 *and* B^2 *is faithfully* k*-flat and suppose the associated diagram as in 3.10 commutes with commutativity datum* (f_1, f_2, ϕ, ψ)*. Then* f_1 *and* f_2 *are both equal to the identity* k*-algebra morphism* $k \longrightarrow k$ *and the composite map* $\psi \circ \phi$ *factors through* k *via the counit* $\epsilon_1 : H^1 \longrightarrow k$ *and the unit* $\eta_2 : k \longrightarrow H^2$*, i.e. the following diagram commutes.*



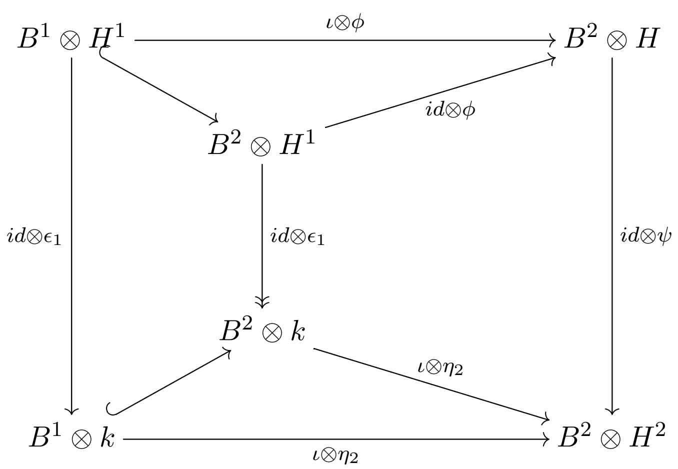
(3.12)

PROOF: Following the proof of proposition 11 we have a cube



(3.13)

with commuting back, right, and top faces. The bottom square commutes by viewing B^1 as a Hopf-Galois extension of B^1 with the trivial coaction of the k-Hopf algebra k. Similar computation as that of the previous proposition implies that the left square commutes as well. Thus, the front square commutes. Finally, the commutative square



and the faithfully k-flatness, say of B^2, implies the desired result. ■

Remark 13. Note that the commutativity of the diagram in proposition 12 is the naive analogue of exactness for a sequence $H^1 \xrightarrow{\phi} H \xrightarrow{\psi} H^2$ of Hopf algebras from the view-point of groups algebras. Note that there is a more categorical wat to see this. The zero object in the category of k-Hopf algebras is k.

Chapter 4

Coverings of commutative spaces: The central case.

If you're in pitch blackness,
all you can do is sit tight
until your eyes get
used to the dark.

–Haruki Murakami,
Norwegian Wood

We mentioned in the introduction, the formulation of a non-commutative covering space should be guided by the following: (1) they should give, as a special case (when the symmetry is a Hopf algebra), non-commutative principal bundles as currently understood (see for example [22]); (2) when the algebras involved are commutative then we should be able to get classical covering spaces i.e., a reconstruction procedure. We will state this reconstruction theorem in this section.

Recall that a covering of a non-commutative space A is in particular, an $\mathcal{H}$-Galois extension $A \subset B$, for some A-ring B and A-Hopf algebroid $\mathcal{H}$. In case A is commutative, there is no reason for B to be. In fact, the striking feature of the notion we put forth is that even for a commutative space, interesting non-commutative covers already exist. We will see examples of a such in chapter 6. Just as there is no reason for B to be commutative, there is also no reason for A to sit centrally in B and $\mathcal{H}$. In the next chapter, we will deal with the more general situation where A need not be central in either B and $\mathcal{H}$. But for the moment, we will look closely to the case when A is central in both B and $\mathcal{H}$. We will do so for the following purposes:

(1) Look at coverings of a point and commutative coverings of commutative spaces. Note that in both cases, A is automatically central.

(2) Plenty of a priori different structures related to A-rings and A-Hopf algebroids collapsed when A is central. For example, the Galois map

$$B \underset{A}{\otimes} B \xrightarrow{\mathfrak{gal}_L} B \underset{A}{\otimes} H$$

is not just a linear bijection but also an A-ring isomorphism. Our goal is to see how these structures collapsed.

(3) We will give a structure theorem for covering spaces which is only valid when A is central. Using such structure theorem, we will be able to look closely to more special classes of coverings.

4.1 Coverings of a point

In this section, we will have a closer look at coverings of a point. In particular, we will see that unlike the classical case, a point has infinitely many connected covers. Nonetheless, as we shall see in section 8.2, there is a corresponding triviality result for such covers. Also, we will characterize the type of Hopf algebroids $\mathcal{H}$ that can arise in a covering $(B, \mathcal{H})$. In non-commutative geometry, a point is represented by the base ring under consideration.

A priori, a covering of a point is a pair $(B, \mathcal{H})$ where $\mathcal{H}$ is a Hopf algebroid over k and $k \subseteq B$ is a right $\mathcal{H}$-Galois extension. In the literature, B is called a *Hopf-Galois* object over k. Let us give some examples of such coverings.

Example 6. Given any finitely generated projective Hopf algebra H over k, we claim that (H, H) is a covering of a point. Here, we use the regular coaction of H on itself. The left and right-bialgebroid structures of H are both isomorphic to the underlying bialgebra of H. All that is left to show is that the Galois map

$$\begin{aligned} H \otimes H &\xrightarrow{\mathfrak{gal}} H \otimes H \\ a \otimes b &\longmapsto ab_{(1)} \otimes b_{(2)} \end{aligned}$$

is bijective. The following map

$$\begin{aligned} H \otimes H &\longrightarrow H \otimes H \\ a \otimes b &\longmapsto aS(b_{(1)}) \otimes b_{(2)} \end{aligned}$$

is its inverse. □

In fact more is true. A bialgebra H is a Hopf algebra if and only if it is an H-Galois extension of the base ring. The above example tells us that any connected k-Hopf algebra is a connected covering of a point. By a connected Hopf algebra we mean connected as an algebra i.e. one in which the only idempotent elements are 0 and 1.

Now, let us look at a more general situation. Let $(B, \mathcal{H})$ be a (finite) covering of k. Explicitly, this means that B is a k-algebra which is finitely generated and projective as a regualr k-module. Also, $\mathcal{H} = (H_L, H_R, S)$ where $H_L = (H, s_L, t_L, \Delta_L, \epsilon_L)$ and $H_R = (H, s_R, t_R, \Delta_R, \epsilon_R)$ are the constituent bialgebroids.

We claim that H_L is a bialgebra. The source and target maps s_L and t_L define a k-algebra map $\eta_L = s_L \otimes t_L : k \longrightarrow H$. The product μ_L on H determined by η_L is associative and unital with respect to η_L. The coproduct $\Delta_L : H \longrightarrow H \otimes H$ is already a k-algebra map since the Takeuchi product $H {}_k\!\times H$ coincide with $H \otimes H$. The coproduct Δ_L is coassociative and counital with respect to ϵ_L. All that is left to show is that $\epsilon_L : H \longrightarrow k$ is a k-algebra map. Part (c) of the definition 1 of a bialgebroid implies that ϵ_L is unital, i.e. $\epsilon_L(1) = 1$. Applying theorem 5.5 of Schauenburg [46] using the identity map $k \longrightarrow k$ and the normalized dual basis of k given by the unit element, we see that H possesses a weak bialgebra structure with coproduct Δ_L and counit ϵ_L. This implies that $\epsilon_L(xy) = \epsilon_L(x1_{[1]})\epsilon_L(1_{[2]}y)$ for any $x, y \in H$. But $1 \otimes 1 = \Delta_L(1) = 1_{[1]} \otimes 1_{[2]}$. Thus, ϵ_L is a unital k-algebra map. This shows that indeed H_L is a bialgebra over k.

Now, H_L admits a Galois extension which is $k \subseteq B$ in this case. By a result of Schauenburg [44], the bialgebra H_L is in fact a Hopf algebra, i.e. there is a k-module map $S_L : H \longrightarrow H$ such that $H_L = (H, \mu_L, \eta_L, \Delta_L, \epsilon_L, S_L)$ is a Hopf algebra over k. Similar argument shows that there is a k-module map $S_R : H \longrightarrow H$ making $H_R = (H, \mu_R, \eta_R, \Delta_R, \epsilon_R, S_R)$ a Hopf algebra over k.

The two Hopf algebras H_L and H_R have the same underlying algebra structure. To see this, note that the multiplication maps of these Hopf algebras factor through $H \otimes H$ via the common underlying complex algebra H of the bialgebroids H_L and H_R. This implies that these maps are the same as the map defining the algebra structure of H.

The antipode S of the Hopf algebroid $\mathcal{H}$ provides a coupling map making $\mathcal{H}_L$ and $\mathcal{H}_R$ coupled Hopf algebras. Thus, we have proved the *only if* part following proposition.

Proposition 13. *$(B, \mathcal{H})$ is a covering of a point if, and only if $\mathcal{H}$ is a coupled Hopf algebra.*

The *if* part follows from the fact that, for a coupled Hopf algebra $\mathcal{H}$ with underlying k-algebra H, the extension $k \subseteq H$ is $\mathcal{H}$-Galois. As we shall see in section 8.2, the constituent Hopf algebras of the coupled Hopf algebra $\mathcal{H}$ has an even tighter relation.

4.2 Commutative coverings of commutative spaces

As we have seen in section 3.1, finite Galois (connected) classical covering $Y \xrightarrow{p} X$ gives a covering $(C(Y), C(\mathcal{G}_{red}))$ in the sense of definition 11 where $\mathcal{G}_{red}$ is the associated reduced groupoid $Y \xrightarrow{p} X$ and $\rho : C(Y) \longrightarrow C(Y) \otimes_{C(X)} C(\mathcal{G}_{red})$ is the induced coaction from the pointwise deck action of $\mathcal{G}_{red}$ on Y. In this section, we will show the converse. That is, commutative coverings give classical covering spaces. Through out this section, we will restrict our attention to local coverings. We will proceed in two ways, one for commutative C^*-algebras and the other one for general commutative unital ring.

Let $(B, \mathcal{H})$ be a covering of A with A and B commutative unital C^*-algebras. Note that as a base algebra for the Hopf algebroid $\mathcal{H}$, only the algebraic structure of A is relevant and not it's analytic structure being a Banach algebra. This might sound like a conflict of data but in fact it's not. In the classical case, the topology of the groupoid $\mathcal{G}_{red}$ associated with a covering is completely determined by the topologies on Y and X. The isotropy groups of $\mathcal{G}_{red}$ are discrete and hence, have no nontrivial contribution to the topology of $\mathcal{G}_{red}$. In the classical case, if a topological group G happens to be the deck transformation group of a covering $Y \xrightarrow{p} X$, one is forced to consider the discrete topology on G as what the definition of a covering begs. Our proposed interpretation of the present formulation is to encapsulate the topology of the covering in B while the symmetry is in $\mathcal{H}$.

Now, A and B being commutative implies that $B \otimes_A B$ carries an algebra structure by tensorwise product. The Galois maps

$$\begin{array}{ccc} B \underset{A}{\otimes} B & \xrightarrow{\mathfrak{gal}_L} & B \underset{A}{\otimes} \mathcal{H}_L \\ a \underset{A}{\otimes} b & \longmapsto & ab_{[1]} \underset{A}{\otimes} b_{[2]} \end{array} \qquad \begin{array}{ccc} B \underset{A}{\otimes} B & \xrightarrow{\mathfrak{gal}_R} & B \underset{A}{\otimes} \mathcal{H}_R \\ a \underset{A}{\otimes} b & \longmapsto & ab^{[1]} \underset{A}{\otimes} b^{[2]} \end{array}$$

then become algebra maps. To see this, given $a \otimes_A b, a' \otimes_A b' \in B \otimes_A B$ we have

$$\begin{aligned} \mathfrak{gal}_L\left(\left(a \underset{A}{\otimes} b\right)\left(a' \underset{A}{\otimes} b'\right)\right) &= \mathfrak{gal}_L\left(aa' \underset{A}{\otimes} bb'\right) = aa'b_{[0]}b'_{[0]} \underset{A}{\otimes} b_{[1]}b'_{[1]} \\ &= \left(ab_{[0]} \underset{A}{\otimes} b_{[1]}\right)\left(a'b'_{[0]} \underset{A}{\otimes} b'_{[1]}\right) \\ &= \mathfrak{gal}_L\left(a \underset{A}{\otimes} b\right)\mathfrak{gal}_L\left(a' \underset{A}{\otimes} b'\right). \end{aligned}$$

The map $\mathfrak{gal}_L$ being a linear bijection implies that it is an algebra isomorphism. Since B and $\mathcal{H}$ are finitely-generated projective modules over a commutative unital C^*-algebra A, the map

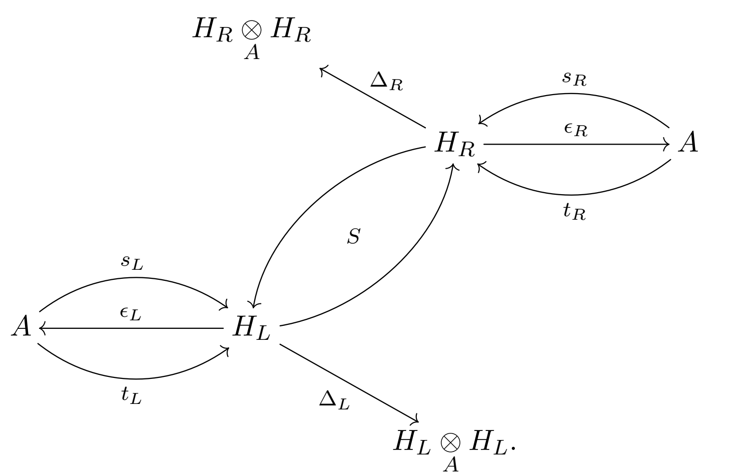
$T : \mathcal{H} \longrightarrow B \otimes_A \mathcal{H}$, $h \mapsto 1 \otimes_A h$ is injective. Thus, the map $\mathfrak{gal}_L \circ T : \mathcal{H} \longrightarrow B \otimes_A B$ is injective and hence, the underlying A-ring structure of $\mathcal{H}_L$ is commutative. Similar argument using $\mathfrak{gal}_R$ shows that $\mathcal{H}_R$ is a commutative A-bialgebroid.

By the Gelfand duality, there are compact Hausdorff spaces $\widehat{A}$ and $\widehat{B}$ such that $A = C(\widehat{A})$ and $B = C(\widehat{B})$. Explicitly, $\widehat{B}$ is the set of unital *-homomorphisms $B \xrightarrow{\varphi} \mathbb{C}$. B being a commutative unital Banach algerba forces $\|\varphi\| = 1$. Thus, $\widehat{B} \subset B^*$ and we can equip $\widehat{B}$ with the subspace topology it inherits from the weak$-*$ topology on B^*. Similarly, we topologize $\widehat{A}$ this way. The inclusion $A \subset B$ induces a projection $\widehat{B} \xrightarrow{p} \widehat{A}, \varphi \mapsto \varphi|_A$. We claim that this projection is a classical covering map.

First, we need the following lemma generalizing the result in algebraic geometry saying that the category of commutative Hopf algebras is dual to the category of affine group schemes.

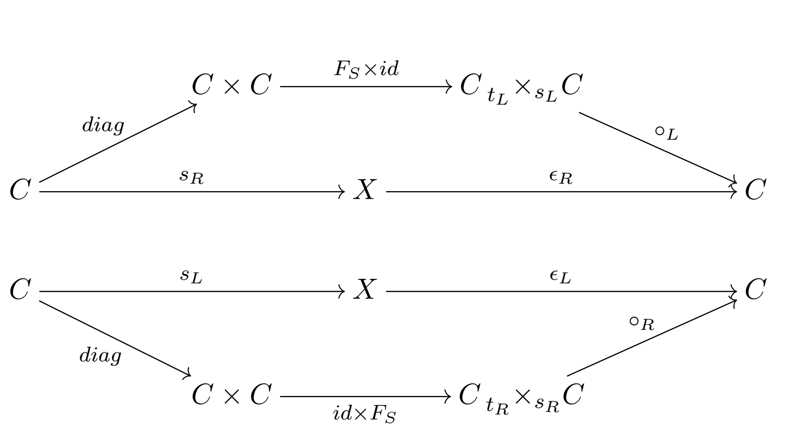
Lemma 4. *Let $\mathcal{H} = (H_L, H_R, S)$ be a commutative Hopf algebroid (i.e. one whose constituent bialgebroids are commutative) over a commutative algebra A with bijective antipode S. Then there is a topological groupoid $\mathcal{G}$ whose algebra of continuous functions is by $\mathcal{H}$.*

Proof: Applying the *Spec* functor in the following diagram of commutative A-algebras describing the Hopf algebroid $\mathcal{H}$



gives topologically enriched small categories $\mathcal{C}_R = Spec(H_R)$ and $\mathcal{C}_L = Spec(H_L)$ over $X = Spec(A)$. To be precise, the underlying space of arrows of these categories come from the commutative A-ring structures of H_L and H_R. The categorical compositions and the units come from the A-coring structures. We abuse notation by writing $\mathcal{C}_R$ (resp. $\mathcal{C}_L$) for the space of arrows of the category $\mathcal{C}_R$ (resp. $\mathcal{C}_L$). Note that $\mathcal{C}_L$ and $\mathcal{C}_R$ have the same underlying space C as this space is precisely $Spec(H)$ where H is the common underlying k-algebra of H_L and H_R.

The antipode S induces a continuous map $C \xrightarrow{F_S} C$. The following diagram of spaces describes the properties of F_S in relation with the rest of the categorical structures of $\mathcal{C}_L$ and $\mathcal{C}_R$.



(4.1)

Here, we denoted by the same notation the maps induced by the source, target and counit maps. As we mentioned above, the counit maps induced the unit maps of the two categories. By part (2) of remark 2, we see that the orientations of elements of C viewed as arrows of $\mathcal{C}_L$ are opposite those orientations when viewed as arrows of $\mathcal{C}_R$. In particular, this means that the two categories have the same units. Using this fact, we can show that more is true. The two categories are groupoids. Let us show that any $\varphi \in \mathcal{C}_R$ is invertible. Using the lower part of diagram 4.1 implies that for any $f \in H$, we have

$$\begin{aligned} f\left(\varphi \circ_R F_S(\varphi)\right) &= f^{[1]}(\varphi) f^{[2]}(F_S(\varphi)) = f^{[1]}(\varphi) S(f^{[2]})(\varphi) \\ &= f^{[1]} S(f^{[2]})(\varphi) = \left(s_L \circ \epsilon_L\right)(f)(\varphi) = \epsilon_L(f)(s_L(\varphi)) = f(id_{s_L(\varphi)}). \end{aligned}$$

Thus, $F_S(\varphi)$ is the inverse of φ in the category $\mathcal{C}_R$. The proof for $\mathcal{C}_L$ being a groupoid goes the same way.

At this point, we have two groupoids $\mathcal{C}_L$ and $\mathcal{C}_R$ whose space of units coincide. Recall that the categorical compositions $\circ_L$ and $\circ_R$ are functorially induced by the coproducts Δ_L and Δ_R, respectively. These coproducts commute by 2.7. Thus, the categorical compositions $\circ_L$ and $\circ_R$ commute as well. By the groupoid version of Eckmann-Hilton argument, the two compositions are the same. This shows that the groupoids are opposite each other. One can pick either of these groupoids to get the groupoid asserted by the lemma. ■

Remark 14.

(1) The proof above provides adjoint equivalence between the category of commutative Hopf algebroids and groupoid schemes. This is formally the same as the adjoint equivalence between commutative Hopf algebras and affine group schemes. The only additional ingredient is Grothendieck's relative point of view for schemes. This may lead one to think that $\mathcal{H}$ being a Hopf algebroid over a commutative algebra R, $\mathcal{H}$ is simply a Hopf algebra over R. This need not be the case, see for example weak Hopf algebras in 2.2. Also, the proof of the lemma 4 involves a construction inverse to the one we had when we constructed Hopf algebroids from groupoids in 3.1.

(2) There is an equivalent way to argue that the groupoids in the proof above are opposite each other. Since the groupoids have the same space of units, the constituent bialgebroids of $\mathcal{H}$ must have the same counits. Using the fact that the coproducts Δ_L and Δ_R commute, one can use the coring version of proposition 1 to show that these coproducts are the same. And hence, the induced compositions are the same as well.

(3) Lemma 4 is the version, in view of Grothendieck's relative principle for schemes, of the fact that commutative Hopf *algebras* over algebraically closed fields are dual group algebras.

Going back to the local covering $(B, \mathcal{H})$ of A, the coaction $B \xrightarrow{\rho} B \otimes_A \mathcal{H}$ defines a groupoid action $\widehat{B}\,{}_p{\times}_s \mathcal{G} \xrightarrow{\alpha} \widehat{B}$ as follows. Using lemma 4 we have an isomorphism $B \otimes_A B = C(\widehat{B}) \otimes_{C(\widehat{A})} C(\mathcal{G}) \cong C(\widehat{B}\,{}_p{\times}_s \mathcal{G})$, and hence we can write $C(\widehat{B}) \xrightarrow{\rho} C(\widehat{B}\,{}_p{\times}_s \mathcal{G})$. Define the action $\widehat{B}\,{}_p{\times}_s \mathcal{G} \xrightarrow{\alpha} \widehat{B}$ as: for any $\varphi \in \widehat{B}$ and $g \in \mathcal{G}$ such that $p(\varphi) = s(g)$, $\varphi \cdot g \in \widehat{B}$ is defined as $(\varphi \cdot g)(b) = \rho(b)(\varphi, g^{-1})$, for any $b \in B$. Using the identification $C(\widehat{B}) \cong B$ we have $(\varphi \cdot g)(b) = \varphi(b_{[0]}) b_{[1]}(g^{-1})$. Let us show that indeed, this defines an action. Let $e \in \mathcal{G}$ be an identity arrow of $\mathcal{G}$. Then for any $\varphi \in \widehat{B}$ and $b \in B$ with $p(\varphi) = s(e)$ we have

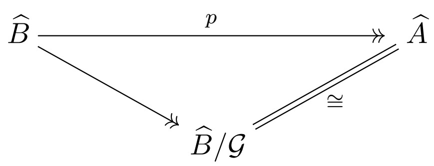
$$(\varphi \cdot e)(b) = \varphi(b_{[0]}) b_{[1]}(e) = \varphi(b_{[0]}\left(b_{[1]}(e)\right)) = \varphi(b_{[0]} \epsilon(b_{[1]})) = \varphi(b)$$

using the definition of the counit ϵ of $\mathcal{H}$ and the counit axiom, respectively. Thus, units $e \in G$ act trivially as desired. For the associativity of the action, let $\varphi \in \widehat{B}$ and $g_1, g_2 \in \mathcal{G}$ with $p(\varphi) = s(g_1) = s(g_2)$. Then for any $b \in B$ we have

$$\begin{aligned}((\varphi \cdot g_1) \cdot g_2)(b) &= (\varphi \cdot g_2)(b_{[0]})b_{[1]}(g_1^{-1}) = \varphi(b_{[0][0]})b_{[0][1]}(g_2^{-1})b_{[1]}(g_1^{-1}) \\ &= \varphi(b_{[0]})b_{[1][0]}(g_2^{-1})b_{[1][1]}(g_1^{-1}) = \varphi(b_{[0]})b_{[1]}(g_2^{-1}g_1^{-1}) \\ &= \varphi(b_{[0]})b_{[1]}((g_1g_2)^{-1}) = (\varphi \cdot (g_1g_2))(b)\end{aligned}$$

using the coassociativity of ρ and the definition of the comultiplication on $\mathcal{H}$, respectively.

Let us show that $\widehat{B}/\mathcal{G} \cong \widehat{A}$. Notice that for $g \in \mathcal{G}$ and $\varphi \in \widehat{B}$ with $p(\varphi) = s(g)$, $\varphi \cdot g$ defines the same function on the set of all $b \in B$ for which $b_{[0]} = b$ and $b_{[1]} = 1$. Thus, such $b \in B$ satisfies $\rho(b) = b \otimes 1$ which implies that $b \in A$. Thus, classes in $\widehat{B}/\mathcal{G}$ defines an element of $\widehat{A}$. Conversely, any element in $\widehat{A}$ is invariant under the induced action of $\mathcal{G}$. Thus, we have a commutative diagram of $\mathcal{G}$-equivariant continuous maps



This in particular shows that $\mathcal{G}$ acts by deck transformations on $\widehat{B} \xrightarrow{p} \widehat{A}$. This means that $\widehat{B} \xrightarrow{p} \widehat{A}$ is a covering space of degree the order of fiber groups of $\mathcal{G}$.

However, $(B, \mathcal{H})$ being a covering space of A in the sense of definition 11 is giving us more. In particular, this tells us that $\widehat{B} \xrightarrow{p} \widehat{A}$ is in fact a Galois covering. This follows immediately from the fact that $B \otimes_A B \xrightarrow{\mathfrak{gal}} B \otimes_A \mathcal{H}$ is bijective. At the level of topological spaces, $\mathfrak{gal}$ induces the corresponding bijective Galois map $\widehat{B}\,{}_p\!\times_s \mathcal{G} \xrightarrow{\mathfrak{gal}'} \widehat{B} \times_{\widehat{A}} \widehat{B}$, showing that fiberwise, $\mathcal{G}$ acts transitively. In addition, $\mathcal{H}$ being finitely-generated projective over A implies that the groupoid $\mathcal{G}$ is locally finite. Thus, we have shown the following theorem.

Theorem 2. *(Reconstruction Theorem.) Let A be a commutative unital C^*-algebra. Let $(B, \mathcal{H})$ be a local covering of A with B a commutative unital C^*-algebra. Then, $\mathcal{H}$ is a commutative Hopf algebroid. Moreover, there is a classical finite Galois covering $Y \xrightarrow{p} X$ with finite deck transformation group G such that $A = C(X)$, $B = C(Y)$ and G is the isotropy group of the groupoid $\mathcal{G}$ where $\mathcal{G}$ is the groupoid determined by $\mathcal{H}$.*

Remark 15. The above theorem is the inverse to the construction we did to motivate definition 11. This is the reconstruction theorem we promised in the beginning of this chapter.

Now let us look at the case of general commutative rings. Let k be a commutative unital ring and A a commutative algebra over k. Let $(B.\mathcal{H})$ be a local covering of A with B a commutative algebra. Then $\mathcal{H}$ is a commutative Hopf algebroid. A general version of lemma 4 can be stated using groupoid schemes as follows.

Lemma 5. *Let $\mathcal{H}$ be a commutative Hopf algebroid over a commutative k-algebra A. Then there is an affine groupoid scheme $\mathcal{G}$ over $X = Spec(A)$ such that $\mathcal{H} = Hom_k(\mathcal{G}, k)$. Moreover, this association gives an adjoint equivalence between the category of commutative Hopf algebroids over A and the category of group schemes over $X = Spec(A)$.*

The above lemma can be proven in a similar way as the proof of lemma 4. Let us go back to the covering $(B, \mathcal{H})$ of A. The inclusion $A \subseteq B$ gives a surjective map $Spec(B) \xrightarrow{p} Spec(A)$. Similar to the case of C^*-algebras, the coaction $B \xrightarrow{\rho} B \otimes_A \mathcal{H}$ gives a groupoid action

$$Spec(B) \times_{Spec(A)} \mathcal{G} \xrightarrow{\alpha} Spec(B).$$

Since the coinvariants of the coaction ρ is A, we have $Spec(B)/\mathcal{G} \cong Spec(A)$. Bijectivity of the Galois map $B \otimes_A B \xrightarrow{\mathfrak{gal}} B \otimes_A \mathcal{H}$ translates to bijectivity of the following map.

$$Spec(B) \times_{Spec(A)} \mathcal{G} \xrightarrow{\mathfrak{gal}'} Spec(B) \times_{Spec(A)} Spec(B)$$

This tells us that theorem 2 is valid in the case of a general commutative unital rings. With this, we have the following theorem.

Theorem 3. *Let A be a commutative unital algebra over k and let $X = Spec(A)$. Then, there is a bijection between commutative local coverings $(B, \mathcal{H})$ of A and finite Galois coverings $Y \xrightarrow{p} X$.*

Here, by a *commutative* covering $(B, \mathcal{H})$ of A we mean a covering where B is commutative.

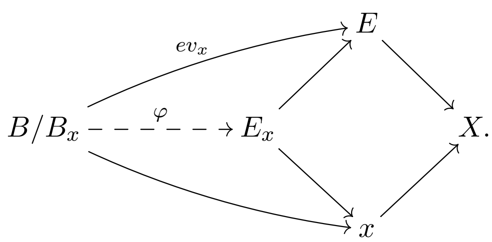
4.3 Non-commutative coverings of commutative spaces

As we mentioned in the beginning of this chapter, we will devote our attention to central coverings. By a *central* covering $(B, \mathcal{H})$ of A we mean a covering in which A is central in both B and $\mathcal{H}$. Note that the coverings we dealt with in the previous two sections are necessarily central coverings. In this section, we will consider central local coverings $(B, \mathcal{H})$ of A where B and $\mathcal{H}$ need not be commutative.

Let A be a commutative unital C^*-algebra. Let $(B, \mathcal{H})$ be a central local covering of A, where B is a unital C^*-algebra. By Gelfand-Naimark duality, $A = C(X)$ where X is a compact Hausdorff space. Specifically, X is the spectrum of A, the space of unitary equivalence classes of irreducible $*$-representations of A. Since A is commutative, X coincides with the primitive spectrum of A, the space of primitive ideals of A with the hull-kernel topology. Since B is a finitely-generated projective module over $C(X)$, the Serre-Swan theorem implies that $B \cong \Gamma(X, E)$ for some finite rank vector bundle $E \xrightarrow{p} X$.

Let $x \in X$ and let $B_x = \{\sigma \in \Gamma(X, E) | \sigma(x) = 0\}$. Then B_x is an ideal of B. To see this, given any $\sigma \in B_x$, write $\sigma = f \cdot \sigma'$ for some $\sigma' \in B$ and $f \in C(X)$ such that $f(x) = 0$. Now, given any $\tau \in B$, we have $(\sigma\tau)(x) = f(x)\left(\sigma'\tau\right)(x) = 0$. Centrality of A in B implies that B/B_x is an $\mathbb{C}$-algebra where we identify $\mathbb{C}$ with A/I_x, $I_x = \{f \in A | f(x) = 0\}$.

The evaluation map $ev_x : B \longrightarrow E$ at x lifts to a map $e : B/B_x \longrightarrow E$. Since E_x is the pullback of $x \longrightarrow X \longleftarrow E$, we have a linear map φ such that the following diagram commutes



In fact, φ is an isomorphism from B/B_x to E_x. To see this, note that any element $e \in E_x$ can be extended to a section $\sigma \in B$ and any other extension is a section having the same value e at x. Thus, they define the same element in B/B_x. This gives us the following proposition.

Proposition 14. *Let $A \subseteq B$ be an algebra extension with $A = C(X)$ central in B and B finitely generated and projective as a regular A-module. Then B is a bundle of complex algebras over X such that the algebra structure of B is pointwise.*

Consider the left bialgebroid structure H_L of $\mathcal{H}$. By assumption, the left and right A-module structure of H_L is finitely-generated and projective. In particular, using the same argument we used for proposition 14 we see that as a left A-module, $H_L \cong \Gamma(X, H^L)$ for some finite rank vector bundle $H^L \xrightarrow{q} X$. Moreover, each fiber has an algebra structure such that the A-ring structure on H_L is isomorphic to the A-ring structure one gets by pointwise multiplication in $\Gamma(X, H^L)$.

By Serre-Swan theorem, the covariant functor $\Gamma(X, -)$ has a left adjoint Σ

$$\left\{\begin{array}{c}\text{finitely-generated}\\ \text{projective module}\\ \text{over } C(X)\end{array}\right\} \underset{\Gamma(X,-)}{\overset{\Sigma}{\rightleftarrows}} \left\{\begin{array}{c}\text{finite rank}\\ \text{vector bundle}\\ \text{over } X\end{array}\right\}.$$

Explicitly, for a $C(X)$-module M the vector bundle $\Sigma(M)$ is constructed as follows. Let $\mathcal{O}_X$ denote the structure sheaf of X and define the presheaf $P(M)$ of $\mathcal{O}_X$-modules by

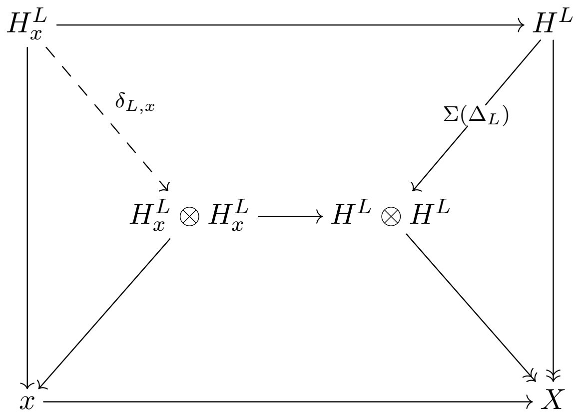
$$P(M)(U) = M \otimes_{C(X)} \mathcal{O}_X(U)$$

and denote by $\Sigma(M)$ its sheafification.

Applying the functor Σ to the coproduct $H_L \xrightarrow{\Delta_L} H_L \underset{A}{\otimes} H_L$ gives a map

$$H^L \xrightarrow{\Sigma(\Delta_L)} H^L \otimes H^L$$

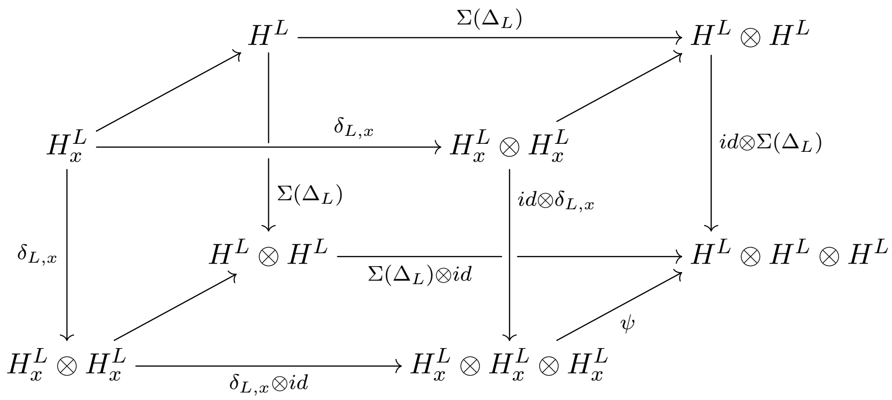
of vector bundles. By definition, the fiber of $H^L \otimes H^L$ at $x \in X$ is $H^L_x \otimes H^L_x$. Thus, there is a linear map $\delta_{L,x}$ making the following diagram commute.



(4.2)

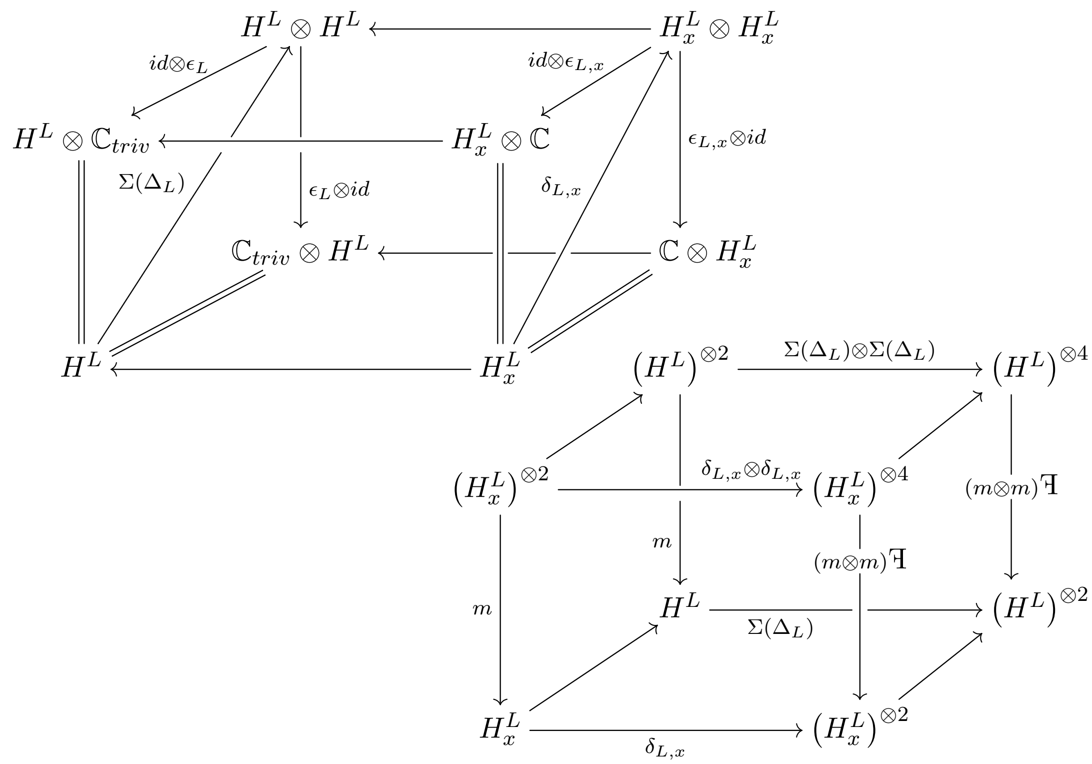
Viewing A itself as a finitely-generated projective module over $C(X)$ and applying the functor Σ on the counit map $H_L \xrightarrow{\epsilon_L} A$ gives a map $H^L \xrightarrow{\Sigma(\epsilon_L)} \mathbb{C}_{triv}$ of vector bundles, where $\mathbb{C}_{triv}$ denotes the trivial line bundle $X \times \mathbb{C}$ over X. Since H^L_x is the pullback of the diagram $x \longrightarrow X \longleftarrow H^L$ we see that we get a linear map $H^L_x \xrightarrow{\epsilon_{L,x}} \mathbb{C}$.

We claim that $\delta_{L,x}$ is coassociative and counital with respect to $\epsilon_{L,x}$. The back face of the following cube commutes by coassociativity of Δ_L and functoriality of Σ



(4.3)

while the lateral faces of diagram 4.3 commute since they are essentially the upper commuting square of diagram 4.2. Commutativity of the five faces and the fact that the map ψ of diagram 4.3 is injective implies that the front face commutes, i.e. $\delta_{L,x}$ is coassociative. Using the same line of reasoning, we can show counitality of $\delta_{L,x}$ with respect to $\epsilon_{L,x}$ using the leftmost diagram in 4.4 below



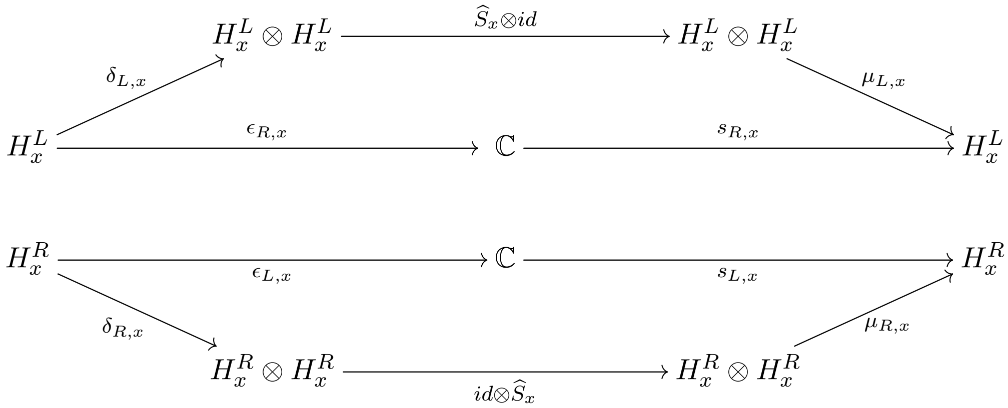
(4.4)

whose front, back, top, bottom and left faces are easily seen to commute implying that the right face is commutative as well. In the leftmost diagram, we denoted by $\mathbb{C} \otimes H^L$ the tensor product of the trivial line bundle $X \times \mathbb{C}$ and the bundle H^L. We also claim that $\delta_{L,x}$ is multiplicative. This follows from the commutativity of the bottom cube in diagram 4.4 above.

Thus, each fiber H^L_x carries a multiplicative coring structure such that the (left) operations on $\mathcal{H}$ are pointwise, i.e $H_L \cong \Gamma(X, H^L)$ is an isomorphism, not just of $C(X)$-modules but also of A-rings and A-corings. In particular, there is a left $\mathbb{C}$-biagebroid structure on H^L_x for every $x \in X$.

Carrying out the same arguments for the right bialgebroid structure H_R of $\mathcal{H}$, we get a finite rank vector bundle $H^R \xrightarrow{r} X$ such that $H_R \cong \Gamma(X, H^R)$ as right A-modules. Each fiber H^R_x of H^R carries an algebra structure such that the A-ring structure of $\mathcal{H}_R$ is isomorphic to the A-ring structure of $\Gamma(X, H^R)$ given by pointwise multiplication. Also, H^R_x carries a multiplicative coring structure such that the coring structure on $\mathcal{H}_R$ is pointwise. Symmetrically, we get a right $\mathbb{C}$-bialgebroid structure on each fiber H^R_x of H^R.

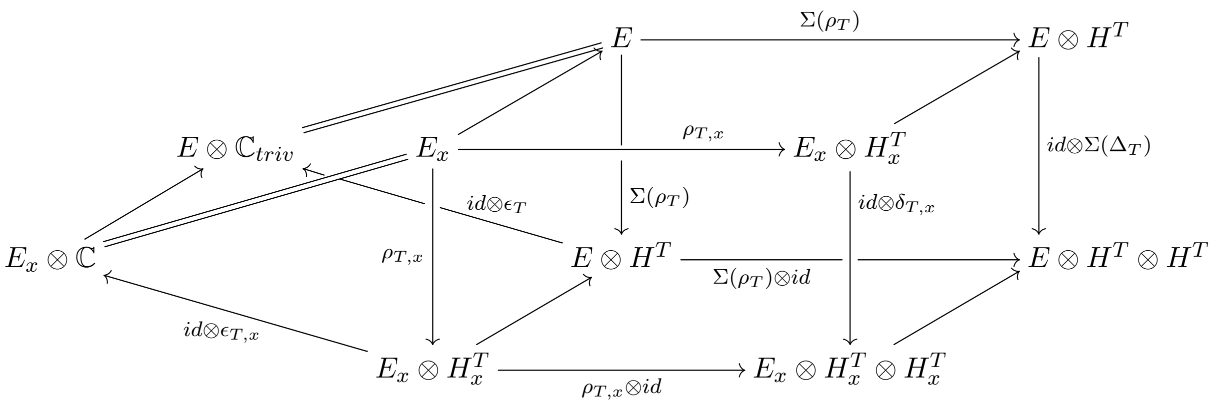
The antipode S defines a $\mathbb{C}$-module map $\Gamma(X, H^L) \xrightarrow{S} \Gamma(X, H^R)$. Part (c) of definition 2 implies that S induces a fiberwise linear map $H^L \xrightarrow{\widehat{S}} H^R$. We then have the following commutative diagram.



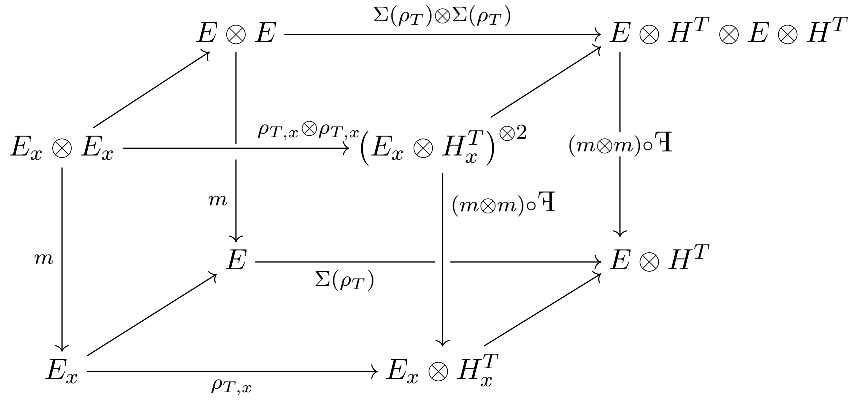
Here, $s_{R,x}$ and $s_{L,x}$ denote the fiber maps at $x \in X$ of the bundle maps $(s_L)_* : \mathbb{C}_{triv} \longrightarrow H^L$ and $(s_R)_* : \mathbb{C}_{triv} \longrightarrow H^R$ induced by $s_L, s_R : A \longrightarrow H$, respectively. Thus, we have the following result.

Theorem 4. *A finitely-generated projective Hopf algebroid $\mathcal{H} = (H_L, H_R, S)$ over $C(X)$, in which the images of $C(X)$ under the source and target maps are central, is a bundle of $\mathbb{C}$-Hopf algebroids $\mathcal{H}_x = \left(H^L_x, H^R_x, \widehat{S}_x\right)$ over X.*

Since $(B, \mathcal{H})$ is a covering of A, B comes with right coactions $B \xrightarrow{\rho_R} B \underset{A}{\otimes} H_R$ and $B \xrightarrow{\rho_L} B \underset{A}{\otimes} H_L$ by $\mathcal{H}$ whose common coinvariant is A. Note that both coactions ρ_R and ρ_L are A-module maps. Thus, applying the functor Σ gives vector bundles maps $E \xrightarrow{\Sigma(\rho_R)} E \otimes H^R$ and $E \xrightarrow{\Sigma(\rho_L)} E \otimes H^L$. Each of these bundle maps induce coactions $\rho_{L,x}$ and $\rho_{R,x}$ of the fiber Hopf algebroids $\mathcal{H}_x$ of $\mathcal{H}$ on the fiber algebras E_x of B by the commutativity of the diagrams below for $T = R, L$.



Meanwhile, the commutativity of the following diagram



shows that $\rho_{T,x}$ is multiplicative for $T = L, R$.

The coinvariants A of the coaction ρ_R is the equalizer of ρ_R and $id \otimes_A 1$. Similarly, A is the equalizer of ρ_L and $id \otimes_A 1$. In other words, we have the following diagrams of A-modules.

$$A \longrightarrow B \underset{id \underset{A}{\otimes} 1}{\overset{\rho_R}{\rightrightarrows}} B \underset{A}{\otimes} H_R$$

$$A \longrightarrow B \underset{id \underset{A}{\otimes} 1}{\overset{\rho_L}{\rightrightarrows}} B \underset{A}{\otimes} H_L$$

Applying the functor Σ to the first diagram gives us the following

$$\mathbb{C}_{triv} \longrightarrow E \underset{id\otimes 1}{\overset{\Sigma(\rho_R)}{\rightrightarrows}} E \otimes H^R$$

which is also an equalizer diagram since Σ is an equivalence. Thus, the coinvariant of the induced fiber coaction $\rho_{R,x}$ is $\mathbb{C}$. Similarly, $\mathbb{C}$ is the coinvariant of the induced fiber coaction $\rho_{L,x}$.

Now, let us show that associated Hopf-Galois map $\mathfrak{gal}_{R,x}$ to $\rho_{R,x}$ is a bijective. The A-module isomorphism $B \otimes_A B \overset{\mathfrak{gal}_R}{\longrightarrow} B \otimes_A \mathcal{H}_R$ induces a bundle isomorphism

$$E \otimes E \xrightarrow{\Sigma(\mathfrak{gal}_R)} E \otimes H^R$$

which on fibers give the isomorphism

$$E_x \otimes E_x \xrightarrow{\mathfrak{gal}_x} E_x \otimes H_x^R.$$

Similarly, the associated Hopf-Galois maps $\mathfrak{gal}_{L,x}$ to the coactions induced on the fibers by ρ_L are all bijective. These give the following result.

Theorem 5. *Let $(B, \mathcal{H})$ be a central local covering of $A = C(X)$. With the notation as above, $(E_x, \mathcal{H}_x)$ is a covering of the point x.*

Remark 16. The above theorem is the non-commutative analogue of the fact that the fibers of a classical covering space are themselves coverings of a point.

Using proposition 13 and the previous theorem, we get the following corollary.

Corollary 3. *Let $(B, \mathcal{H})$ be a central local covering of $C(X)$. Then $\mathcal{H}$ gives two bundles $H^1, H^2 \twoheadrightarrow X$ of coupled Hopf algebras over X.*

The fiber coverings $(E_x, \mathcal{H}_x)$ in theorem 5 need not be isomorphic even within a connected component of X. As a matter of fact, we already have an example for this in the commutative case.

Example 7. Consider the algebras $E_t = \mathbb{C}[x]/(x^n - t)$. The underlying vector space of these algebras are all n-dimensional and they constitute a vector bundle $E \xrightarrow{p} \mathbb{C}$ over the complex plane where $p^{-1}(t) = E_t$. Note that the each fiber carries a natural algebra structure making E an algebra bundle over $\mathbb{C}$ with non-isomorphic fibers. In particular, the fiber algebra E_0 has a nilpotent element while E_1 has none. Furthermore, each fiber algebra is spanned by $\{1, x, ..., x^{n-1}\}$. The group $G = \mathbb{Z}/n\mathbb{Z}$ acts on each fiber algebra E_t via $(m \cdot x) \mapsto \lambda^m x$ extended into an algebra isomorphism where λ is a primitive n^{th} root of 1. This action extends to a Galois action of the group algebra $\mathbb{C}G$ and hence, the function algebra $C(G)$ coacts on $B = \Gamma(\mathbb{C}, E)$. This turns $(B, C(\mathbb{C}) \otimes C(G))$ into a local covering of $C(\mathbb{C})$. □

4.4 Coverings with semisimple fibers

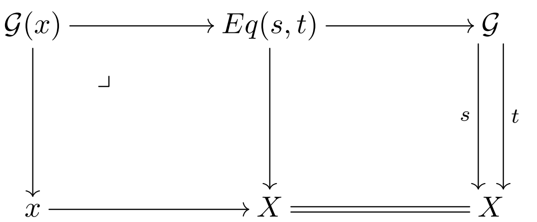
In this section, we will continue to look at the case when $A = C(X)$ is central in the local covering $(B, \mathcal{H})$. In addition, we will assume the fiber coverings of theorem 5 are semisimple. For simplicity, let us also assume that X is connected. This means that any vector bundle E for which $B = \Gamma(X, E)$ and any vector bundle H for which $\mathcal{H} = \Gamma(X, H)$, the underlying complex algebras of the fiber algebras E_x and the fiber Hopf algebroids $(H_x^L, H_x^R, \widehat{S}_x)$ are semisimple complex algebras. By Wedderburn's theorem, E_x is the finite product of matrix algebras i.e.,

$$E_x = M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_j}(\mathbb{C})$$

for some positive integers $n_1, n_2, ..., n_j$. This decomposition determines (and is completely determined by) a set of central orthogonal idempotent $\{e_i \in E_x | i = 1, 2, ..., j\}$ summing up to 1. Explicitly, $M_{n_i}(\mathbb{C}) \cong e_i E_x$ for all $i = 1, ..., j$. Let us call the (unordered) j-tuple $(n_1, n_2, ..., n_j)$ the *Wedderburn shape* of the semisimple algebra E_x. Part of the content of Wedderburn's theorem says that the Wedderburn shape of a semisimple algebra is unique (up to ordering).

Example 8. Let us consider the extreme case when, for all $x \in X$, $E_x \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$ as algebras. In this case, the Hopf algebroid $\mathcal{H}$ is commutative by the bijectivity of the associated Hopf-Galois maps. By assumption, the antipode S is bijective. The coproduct and the counit are unital maps. Thus, lemma 4 implies that there is a groupoid $\mathcal{G}$ such that $\mathcal{H} \cong C(\mathcal{G})$.

Bijectivity of $\mathfrak{gal}_x$ above implies that the underlying $\mathbb{C}$-vector space of H_x is finite dimensional for any $x \in X$. Specifically, each H_x is of dimension n. Now, given $x \in X$ consider the following diagram in the category of topological spaces



where the left square is a pull-back square and in the right square, $Eq(s,t)$ is the equalizer of s and t. Applying the functor $C(-)$ gives the following diagram

$$\begin{array}{ccccc} C(\mathcal{G}(x)) & \longleftarrow & C(Eq(s,t)) & \longleftarrow & \mathcal{H} \\ \uparrow & \ulcorner & \uparrow & & t \uparrow\uparrow s \\ \mathbb{C} & \overset{ev_x}{\twoheadleftarrow} & A & = & A \end{array}$$

where the left square is a push-out diagram. The right square being a coequalizer implies that the large rectangular diagram (using either s or t) is a push-out diagram. The counit of the adjunction $C(-) \dashv Spec$ provides a $\mathbb{C}$-algebra isomorphism $C(\mathcal{G}_x) \cong H_x$. This extends to a bialgebroid isomorphism since the coring structure maps of H_x and $C(\mathcal{G}(x))$ are morphisms of commutative unital $\mathbb{C}$-algebras. Since $\mathcal{G}(x)$ is a group, H_x is then a Hopf algebra. Note that a priori, G_x depends on $x \in X$ but connectivity of X implies that the groups G_x are all isomorphic, denoted accordingly as G. When dualized, the coaction $\rho_x : \mathbb{C}^n \longrightarrow \mathbb{C}^n \otimes H_x$ gives an action $\rho_x^* : \mathbb{C}^n \otimes \mathbb{C}G \longrightarrow \mathbb{C}^n$. Note that $\mathbb{C}^n \otimes \mathbb{C}G \cong \mathbb{C}[Y] \otimes \mathbb{C}G \cong \mathbb{C}[Y \times G]$ where Y is a set consisting of n points and the multiplication in the algebra $\mathbb{C}[Y \times G]$ is pointwise in Y but convolution in G. The map ρ_x^* is completely determines by the map $Y \times G \xrightarrow{\alpha} Y$ which is an action by the virtue of ρ_x being a coaction. The bijectivity of the Hopf-Galois map translate to the bijectivity of the associated map

$$Y \times G \longrightarrow Y \times Y, \qquad (y, g) \mapsto (yg, y)$$

which means that the action α is free and transitive. Thus, $G \leqslant S_n$ is a transitive subgroup with $|G| = n$. $\square$

Let us consider the general case when the fibers of E are non-commutative algebras. In this case, $E_x = M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_j}(\mathbb{C})$ where the Wedderburn shape $(n_1, n_2, \cdots, n_j)$ of E_x a priori depends on $x \in X$. Let us describe how these dependence works.

Consider the center $Z(B)$ of B. Since $B = \Gamma(X, E)$ equipped with pointwise multiplication, we see that $\sigma \in Z(B)$ precisely when $\sigma(x) \in Z(E_x)$ for all $x \in X$. The center $Z(B)$ is a C^*-subalgebra of B. In particular, it is a commutative C^*-algebra and by the Gelfand duality, there is a compact Hausdorff space Y such that $Z(B) = C(Y)$. Note that $A = C(X)$ sits inside $Z(B) = C(Y)$. Thus, there is a continuous surjective map $Y \overset{p}{\twoheadrightarrow} X$. Consider the following stratification of X. Denote by $X^{(n)} = \{x \in X | \#(p^{-1}(x)) = n\}$ where $\#(S)$ denotes the cardinality of the set S. Note that $X^{(n)}$'s are generally not connected. Define $X^{(n,i)}$, $i \in I_n$ to be the connected components of $X^{(n)}$. Note that the $X^{(n,i)}$'s forms a partition of X and that the $X^{(n,i)}$'s are generally not closed in X. We call $\{X^{(n,i)} | n \in \mathbb{N}, i \in I_n\}$ the stratification of X and each $X^{n,i}$ as a stratum. Let us denote by $Y^{(n,i)} = p^{-1}(X^{(n,i)})$. Then $Y^{(n,i)} \overset{p}{\twoheadrightarrow} X^{(n,i)}$ is a covering space in the classical sense.

Surjectivity of p implies that $X^{(0)} = \emptyset$. We claim that $X^{(n,i)} = \emptyset$ as well for $n \geqslant m$ for some sufficiently large m. To see this, note that semisimplicity of E_x implies that $Z(E_x) \subseteq E_x$ is complemented. This implies that the dimension of $Z(E_x)$ is bounded above by the dimension of E_x. By theorem 2, we see that this dimension is bounded by $dim\ H < \infty$. The center $Z(E_x)$ of E_x is linearly generated by the central orthogonal idempotents $\{e_i\}$ giving the Wedderburn factors. These central orthogonal idempotents can be extended continuously to relative sections $\{\sigma_i \in \Gamma(X^{(n,j)}, Z(E)) | x \in X^{(n,j)}, \sigma_i(x) = e_i\}$. Since the rank of an idempotent is locally constant, we see that Wedderburn factors are all the same for all $x \in X^{(n,j)}$. Thus, we see that Wedderburn shape of the fibers E_x of E only depend on the stratum of $x \in X$.

On the other hand, much can be said about the fiber Hopf algebroids. From section 4.1 such a Hopf algebroid is a coupled Hopf algebra. There are only finitely many semisimple complex Hopf algebras of a given fixed dimension. Thus, there are only finitely many coupled Hopf algebras of that same dimension. Since the fiber Hopf algebroids have the same dimension, this implies that there are only finitely many posibilities for their structure. Connectivity of X, discreteness of the collection of such coupled Hopf algebras, and continuity of the structure maps of the bundle Hopf algebroid, imply that the fiber Hopf algebroids must be isomorphic, say to a fixed one $H_0 = (H^L_{x_0}, H^R_{x_0}, \widehat{S}_{x_0})$.

Proposition 15. *For any $x, y \in X$, $H_x \cong H_y$ as coupled Hopf algebras.*

Specializing the notion of an algebraic morphism of Hopf algebroids from section 2.1, tells us that a morphism $(H^L_1, H^R_1, S_1) \xrightarrow{\phi} (H^L_2, H^R_2, S_2)$ of coupled Hopf algebras is a linear map ϕ which defines Hopf algebra maps $H^L_1 \xrightarrow{\phi} H^L_2$ and $H^R_1 \xrightarrow{\phi} H^R_2$ intertwining the coupling maps. The notation makes sense since H^L_1 and H^R_1 have the same underlying algebra. Same goes for (H^L_2, H^R_2, S_2).

Let $G = Aut(H_0)$ and let $\phi \in G$. Finite dimensionality of $H^L_{x_0}$ and $H^R_{x_0}$ implies that they are Frobenius algebras. Thus, they are equipped with nondegenerate pairings $\langle, \rangle_L$ and $\langle, \rangle_R$ making them finite-dimensional Hilbert spaces. The automorphism ϕ in particular defines automorphisms of these two Frobenius algebras, i.e. ϕ preserves the inner products $\langle, \rangle_L$ and $\langle, \rangle_R$. Thus, each $\phi \in G$ is a unitary map with respect to both inner products (actually, since there is a unique Hilbert space up to isomorphism for a particular dimension, the two inner product defines the same Hilbert space structure on $H^L_{x_0}$ and $H^R_{x_0}$). Hence, we have the following proposition.

Proposition 16. *$G \subseteq U(n)$ where $n = dim\ H^L_{x_0}$.*

The two propositions give a continuous map $\alpha : X \longrightarrow G$, $\alpha(x) : H_x \xrightarrow{\simeq} H_{x_0}$. By Radford [41], the group of automorphisms of a semisimple Hopf algebra over a field of characteristic 0 is finite. Hence, the group of automorphisms of a semisimple coupled Hopf algebra over $\mathbb{C}$ is finite as well. This implies that G is a finite subgroup of $U(n)$ and thus, discrete. Hence, α is a Čech 1-cocycle since it is locally constant. Therefore, $H \twoheadrightarrow X$ is an *algebra bundle*, i.e. the local transition maps rather than just being linear maps, are algebra maps. For comparison, this is different to a *bundle of algebras*. The latter is just a vector bundle such that the fibers are algebras and such algebra structures vary continuously. The associated Čech 1-cocycle is just α followed by the inclusion $G \subseteq GL_n(\mathbb{C})$.

Proposition 17. *$G \subseteq GL_n(\mathbb{C})$ is finite and $H \twoheadrightarrow X$ is an algebra bundle.*

As we have argued after example 7, the fibers algebras of a central covering of $C(X)$ need not be isomorphic. Let us discuss a necessary and sufficient condition for the fiber of a bundle of algebras to be all isomorphic. For this purpose, we will specialize in the smooth case. Let X be a connected smooth manifold.

Definition 16. Let $E \twoheadrightarrow X$ be a smooth vector bundle such that the fibers are algebras whose multiplications depend on $x \in X$ continuously. A *differential connection* ∇ on E is a smooth connection such that for any vector field ν on X, we have

$$\nabla_\nu(\sigma_1\sigma_2) = \sigma_1\nabla_\nu(\sigma_2) + \nabla_\nu(\sigma_1)\sigma_2$$

for any sections $\sigma_1, \sigma_2 \in \Gamma(X, E)$. □

Surprisingly, existence of such connections is a sufficient condition for the fiber algebras to be isomorphic. For a necessary condition, one need a stronger assumption than just having isomorphic fiber algebras. We will formalize these statements in the next two propositions.

Proposition 18. *If E has a differential connection ∇ then the fiber algebras of $E \twoheadrightarrow X$ are all isomorphic.*

Proof: Assume E has a differential connection ∇. Let $x, y \in X$ and let $\gamma : I \longrightarrow X$ be a (piecewise) smooth path in X with $\gamma(0) = x$ and $\gamma(1) = y$. Using the connection ∇, we have a parallel transport map

$$\Phi(\gamma)_x^y : E_x \longrightarrow E_y$$

which is a linear isomorphism. Thus, all we have to show is that $\Phi(\gamma)_x^y$ is multiplicative. Given $b_1, b_2 \in E_x$, there are unique smooth sections σ_1 and σ_2 of E along γ such that $\nabla_{\vec{\gamma}}\sigma_1 = \nabla_{\vec{\gamma}}\sigma_2 = 0$ and $\sigma_1(x) = b_1$ and $\sigma_2(x) = b_2$. Here, $\vec{\gamma}$ denotes the smooth tangent vector field of γ. Note that the product $\sigma_1\sigma_2$ is the unique smooth section of $E \longrightarrow\!\!\!\!\rightarrow X$ along γ such that $(\sigma_1\sigma_2)(x) = \sigma_1(x)\sigma_2(x) = b_1 b_2$ and

$$\nabla_{\vec{\gamma}}(\sigma_1\sigma_2) = \sigma_1\nabla_{\vec{\gamma}}(\sigma_2) + \nabla_{\vec{\gamma}}(\sigma_1)\sigma_2 = 0.$$

Thus, by definition of the parallel transport map $\Phi(\gamma)_x^y$ we have

$$\Phi(\gamma)_x^y(b_1 b_2) = (\sigma_1\sigma_2)(y) = \sigma_1(y)\sigma_2(y) = \Phi(\gamma)_x^y(b_1)\Phi(\gamma)_x^y(b_2)$$

which shows that $\Phi(\gamma)_x^y$ is multiplicative. ■

A strong converse of the above proposition, where the isomorphisms among fibers satisfy some coherence conditions, holds. By a coherent collection

$$\mathscr{P} = \{\Phi(\gamma)_x^y : E_x \longrightarrow E_y | \forall x, y \in X, \gamma : I \longrightarrow X \ smooth\}$$

of isomorphisms among fibers of $E \twoheadrightarrow X$, we mean a collection satisfying

(a) $\Phi(\gamma)_x^x = id$,

(b) $\Phi(\gamma)_u^y \circ \Phi(\gamma)_x^u = \Phi(\gamma)_x^y$,

(c) and Φ depends smoothly on γ, y and x.

We then have the following proposition.

Proposition 19. *A coherent collection $\mathscr{P}$ of algebra isomorphisms on fibers of $E \twoheadrightarrow X$ gives a differential connection ∇ on E.*

Proof: Using the collection $\mathscr{P}$ we can immediately write an infinitessimal connection ∇ as follows: for any vector V on X we have

$$\nabla_V(\sigma) = \lim_{t \to 0} \frac{\Phi(\gamma)_{\gamma(t)}^x \sigma(\gamma(t)) - \sigma(x)}{t} = \left.\frac{d}{dt}\right|_{t=0} \Phi(\gamma)_{\gamma(t)}^x \sigma(\gamma(t))$$

for any $\sigma \in B$ and $x = \gamma(0)$. That ∇ is a differential connection follows from the multiplicativity of $\Phi(\gamma)_x^y$ and the Leibniz property of $\left.\frac{d}{dt}\right|_{t=0}$. ■

Example 9. In this example, we will show that the Wedderburn shape of fibers need not be constant even over a connected base space. Let G be a finite group of *central type*, i.e. G fits in an extension

$$1 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

such that Γ has an irreducible representation $\Gamma \xrightarrow{\rho} GL(V)$ of dimension $\sqrt{[\Gamma : Z(\Gamma)]}$.

Now, the group extension above determines a 2-cocycle $\beta : G \times G \longrightarrow Z(\Gamma)$. Then the composition

$$\begin{array}{ccccc} G \times G & \xrightarrow{\quad\beta\quad} & Z(\Gamma) & \xrightarrow{\quad\rho\quad} & GL(V) \\ & & \searrow & & \nearrow \\ & & & \mathbb{C}^{\times} & \end{array}$$

determines a 2-cocycle α such that the associated twisted group algebra $\mathbb{C}^{\alpha}G \cong M_n(\mathbb{C})$, where $n = \sqrt{[\Gamma : Z(\Gamma)]}$. The twisted group algebra $\mathbb{C}^{\alpha}G$ is a Hopf algebra with the same coproduct, counit and unit as that of $\mathbb{C}G$ with product given by $g \cdot g^{'} = \alpha(g, g^{'})gg^{'}$ for any $g, g^{'} \in G$. Such a cocycle can be rescaled to get a family of cocycles α_t for every $t \in \mathbb{C}$ with $\alpha_0 = 1$ and α_t nondegenerate for $t \neq 0$. This means $\mathbb{C}^{\alpha_t}G \cong M_n(\mathbb{C})$ for $t \neq 0$ while $\mathbb{C}G$ may decompose nontrivially as a direct sum of matrix algebras over $\mathbb{C}$. This gives a bundle of Hopf algebras $E = \coprod_{t\in\mathbb{C}} \mathbb{C}^{\alpha_t}G \xrightarrow{p} \mathbb{C}$. The algebra $B = \Gamma(\mathbb{C}, E)$ is then a Hopf-Galois extension of $C(\mathbb{C})$. $\square$

4.5 Coverings with cleft fibers

In this section, we are still interested with the case $A = C(X)$ and $(B, \mathcal{H})$ is a local covering in which A is central. As before, $B \cong \Gamma(X, E)$ and $\mathcal{H} \cong \Gamma(X, F)$ where E and F is an algebra bundle and a Hopf algebroid bundle both over X, respectively. Moreover, for any $x \in X$, (E_x, F_x) is a covering of $\mathbb{C}$. In addition, suppose that $(B, \mathcal{H})$ is a *cleft* covering i.e., $A \subseteq B$ is a cleft extension. Recall from theorem 1 that this implies that $B \cong A \otimes_A H$ as left A-modules and as right $\mathcal{H}$-comodules. These conditions descend to the bundle structures of E and F, i.e. $E_x \cong \mathbb{C} \otimes F_x$ as left $\mathbb{C}$-modules and as right F_x-comodules. Since (E_x, F_x) is a covering of $\mathbb{C}$, again by theorem 1 we see that (E_x, F_x) is a cleft covering of $\mathbb{C}$. In other words, central cleft coverings of commutative spaces have cleft fibers.

Proposition 20. *With the assumption of this section, $A \overset{\mathcal{H}}{\Longrightarrow} B$ is a cleft covering implies that the fiber coverings are also cleft.*

It is then a natural question to ask whether the converse is true, i.e. if the fibers of a central covering of a commutative space are cleft, is the given covering also cleft? This is not necessarily so.

If the fibers are cleft, then for any $x \in X$ there is an invertible complex-valued 2-cocycle $\sigma_x : F_x \otimes F_x \longrightarrow \mathbb{C}$. If we can choose these cocycles such that they assemble into a continuous bundle map $\sigma^{'} : F \otimes F \longrightarrow (X \times \mathbb{C})$ then such bundle map gives an A-bimodule map $\sigma : H_L \otimes H_L \longrightarrow A$. It is then immediate to see that B is the σ-twisted crossed product of A and H_L by defining the product pointwise. By theorem 6.4.12 of [16], this is the case if and only if the bundle E is trivial.

Let us end this chapter by noting that the requirement that A is central in both B and $\mathcal{H}$ is actually redundant. If we only require A to be central in B, then $B \otimes_A B$ is an algebra with factor-wise product and the Galois map will force A to be central in $\mathcal{H}$.

Chapter 5

Coverings of commutative spaces: The non-central case.

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

–David Hilbert

In this chapter, we will consider coverings $(B, \mathcal{H})$ of $A = C(X)$ where A need not be central in either B or $\mathcal{H}$. Obviously, the central case which we discussed in chapter 4 is a special case of what we will do here. As we promised, we will use a different machinery in the present chapter− algebraic geometric and spectral theoretic in nature.

The geometric description of non-central coverings necessitates structures closely related to Hopf categories, but which have not appeared in the literature as far as the author's knowledge. We will define such structures in appendix B. Specifically, they are called *topological Hopf categories* and *coupled Hopf categories*. We will also formulate their respective Galois theory in that appendix.

This chapter has two main result. The first one is theorem 6. It gives a bijective correspondence between finitely-generated projective Hopf algebroids over $C(X)$ and topological coupled Hopf categories of finite type. Using algebraic geometric and spectral theoretic methods, spanning the entirety of the first 3 sections of this chapter, we will prove this result. The second is theorem 7, which states that, not only is there a bijection between Hopf algebroids and topological Hopf categories, their Galois theories also matched in a bijective manner. Following David Hilbert's statement at the top of this page, this chapter has been motivated by an example which we described in section B.2 of appendix B. That example illustrates the essence of theorem 6.

5.1 Local eigenspace decomposition

Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid over A, a commutative unital C^*-algebra. Assume that H_L is finitely-generated and projective as a left- and a right-A-module via the source and target maps. With our standing assumption, H_R has the same properties.

Let us first consider the left bialgebroid H_L. The Gelfand duality implies that $A \cong C(X)$ for some compact Hausdorff space X. For simplicity, assume X is connected. The Serre-Swan theorem applied to the left A-module H_L gives us a finite-rank vector bundle $E \xrightarrow{p} X$ such that $H_L \cong \Gamma(X, E)$ as left modules, where the left $C(X)$-module structure on $\Gamma(X, E)$ is by pointwise multiplication, i.e. $(f \cdot \sigma)(x) = f(x)\sigma(x)$ for all $x \in X, f \in C(X)$ and $\sigma \in \Gamma(X, E)$. By the bimodule nature of H_L, the right A-module structure of $\Gamma(X, E)$ commutes with the left A-module which implies that we have a representation $C(X) \xrightarrow{\rho} End(E)$ of $C(X)$ into the endomorphism bundle of $E \xrightarrow{p} X$. Since $C(X)$ is abelian and ρ is a $*$-morphism, $\rho(C(X))$ lands in a maximal abelian subalgebra $D(n)$ of $End(E)$. Since B is finitely-generated and projective as a right-module over $A = C(X)$, locally, these endomorphisms act freely.

Choose a finite collection of open sets $\{U_i | i = 1, 2, ..., m\}$ that cover X over which E is trivializable. Choose a system of coordinates such that E trivial over each U_i, i.e. $E|_{U_i} \cong U_i \times V$, where V is a finite-dimensional vector space. Choosing a basis $v_1, v_2, ..., v_n \in V$ one has $End(E|_{U_i}) = C_b(U_i, M_n(\mathbb{C}))$ where n is the rank of E. Here, $C_b(U_i, M_n(\mathbb{C}))$ denotes the algebra of bounded $M_n(\mathbb{C})$-valued functions on U_i. Since X is compact, $\Gamma(X, End(E))$ consists of bounded (finite-rank) operator-valued functions on X. Localizing over U_i gives bounded $M_n(\mathbb{C})$-valued functions. Commutativity of $C(X)$ implies that up to unitaries $V_i \in U(n)$, we have

$$C(X) \xrightarrow{\quad \rho \quad} C_b(U_i, Diag(n))$$

where $Diag(n)$ denotes the subalgebra of diagonal matrices on $M_n(\mathbb{C})$ and

$$V_i \cdot C_b(U_i, Diag(n)) \cdot V_i^* = D(n)|_{U_i}.$$

For each $i = 1, 2, ..., m$, choosing a set of central orthogonal idempotents $\{e_j | j = 1, ..., n\}$ gives n projections p^i_j given by the following composition

$$C(X) \xrightarrow{\quad \rho_i \quad} C_b(U_i, Diag(n)) \cong \bigoplus_{k=1}^{n} C_b(U_i) \xrightarrow{\quad proj_j \quad} C_b(U_i)$$

These projections are in particular continuous C^*-morphisms. Hence, they give, for each $i = 1, 2, ..., m$, (possibly non-distinct) n continuous injective maps $U_i \xrightarrow{\varphi^i_j} X$, $j = 1, ..., n$. Geometrically, the situation is depicted figure 5.1.

Let us describe the nature of the set $Z = \bigcup_{i,j} \varphi^i_j(U_i)$ over the intersections $U_\alpha \cap U_\beta$. Over $U_\alpha \cap U_\beta \subseteq U_\alpha$ we get a unitary V_α which gives n central orthogonal idempotents and up to ordering of such idempotents, one gets the sets $\varphi^i_j(U_i)$. The union $\bigcup_j \varphi^i_j(U_i)$ does not depend on the ordering of these idempotents. Thus, over $U_\alpha \cap U_\beta$ one gets unitaries V_α and V_β which simultaneously diagonalize $\rho(C(X))$. Thus, we have

$$\bigcup_j \varphi^\alpha_j(U_\alpha \cap U_\beta) = \bigcup_j \varphi^\beta_j(U_\alpha \cap U_\beta)$$

from which we get that

$$\left(\bigcup_j \varphi^\alpha_j(U_\alpha)\right) \cap \left(\bigcup_j \varphi^\beta_j(U_\beta)\right) = \bigcup_j \varphi^\alpha_j(U_\alpha \cap U_\beta)$$

that is, the sets $\bigcup_j \varphi^i_j(U_i)$ agree on the intersections.

A subset $T \subseteq X \times X$ is called *transverse* if

$$proj_1|_T : X \times X \longrightarrow X, \qquad proj_2|_T : X \times X \longrightarrow X$$

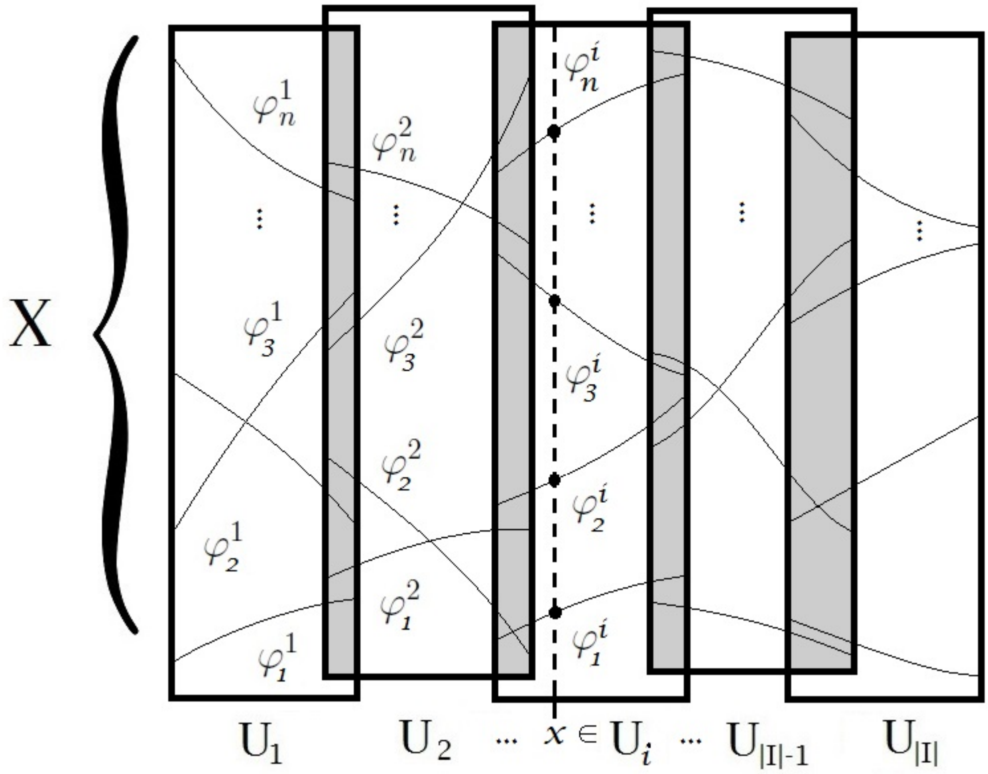


Figure 5.1: Local eigenspace decomposition of E.

are homeomorphisms, where $proj_1$ and $proj_2$ denotes the projection onto the first and second factor, respectively. In particular, T is homeomorphic to X. Using the above argument, we have the following proposition.

Proposition 21. *For every $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ the set $\varphi_j^i(U_i)$ extends to a transverse subset of $X \times X$ completely contained in Z. In particular, Z is the union of n (possibly overlapping) transverse subsets of $X \times X$.*

This means that the curves in figure 5.1 overlap.

Remark 17. Another way to see why the closed subset $Z \subset X \times X$ is the union of transverse subsets of $X \times X$ is by the fact the we can run the construction of the sets $\varphi_j^i(U_i)$ described in the beginning of this section in a symmetric fashion, one for each factor of $X \times X$.

The whole picture 5.1 is a decomposition of $X \times X$ into $X \times U_i$, $i \in I$. The graphs of φ_j^i are labelled accordingly. Note that each $f(x) \in End(E_x)$, $f \in C(X)$ are diagonalizable since they commute with their adjoint $f(x)^* \in C(X)$. And since such operators commute with each other, the collection $\{f(x) \in End(E_x) | f \in C(X)\}$ is simultaneously diagonalizable. Over a point $x \in U_1$, the fiber E_x decomposes into joint eigenspaces of $\{f(x) \in End(E_x) | f \in C(X)\}$. The dimension of these eigenspaces are determined by the number of intersections of the vertical dotted line through $x \in U_1$ with the graphs of φ_j^1. Using this eigenspace decomposition, we have the following proposition which describes geometrically the right $C(X)$-module structure of H_L.

Proposition 22. *Given $\sigma \in \Gamma(X, E)$ and $f \in C(X)$ the section $\sigma \cdot f \in \Gamma(X, E)$ is given as*

$$(\sigma \cdot f)(x) = \sum_{j=1}^{n} f(\varphi_j^i(x)) e_j \cdot \sigma(x). \tag{5.1}$$

where $x \in U_i$ *and* $\sigma(x) = \sum_{j=1}^{n} e_j \cdot \sigma(x)$.

Remark 18.

(1) In case $C(X)$ is central in H_L, the above picture reduce to $\{U_i | i \in I\}$ the trivial cover and $\varphi : X \longrightarrow X$ is the identity, i.e. the graph in the above picture is the diagonal of $X \times X$. The action defined by equation 5.1 then reduces to pointwise multiplication which then coincides with the left $C(X)$-module structure of $H_L \cong \Gamma(X, E)$.

(2) One can understand the right action above as *pointwise-eigenvalue-scaled* action. Compared to the central case, every $f \in C(X)$ acts on a $\sigma \in \Gamma(X, E)$ in a way that $f(x)$ acts diagonally on $\sigma(x)$, i.e. E_x constitutes a single eigenspace for the operator $f(x)$ corresponding to the eigenvalue $f(x) \in \mathbb{C}$. In the noncentral case, the action is still pointwise. However, the operator $f(x)$ no longer has a single eigenspace. The eigenspaces are labelled by the points $\varphi_j^i(x) \in X$ where $x \in U_i$ and the eigenvalues of $f(x)$ are $f\left(\varphi_j^i(x)\right)$, $j = 1, ..., n$.

Proposition 23. *As a* $C(X)$*-bimodule,* $H_L \cong \Gamma(Z, \mathcal{E})$ *where* $\mathcal{E}$ *is a sheaf of complex vector spaces over* $X \times X$ *supported on a closed subset* $Z \subset X \times X$*. The* $C(X)$*-bimodule structure on* $\Gamma(Z, \mathcal{E})$ *is defined as*

$$(f \cdot \sigma \cdot g)(x, y) = f(x)\sigma(x, y)g(y)$$

for $f, g \in C(X)$ *and* $\sigma \in \Gamma(Z, \mathcal{E})$.

The $C(X) \otimes C(X)^{op}$ is dense in $C(X \times X)$ thus we can extend the $C(X) \otimes C(X)^{op}$-module structure of H_L to a $C(X \times X)$-module structure. Consider the annihilator of H_L,

$$Ann(H_L) = \{f \in C(X \times X) | f \cdot \sigma = 0, \text{ for all } \ \sigma \in B_L\}.$$

Then, there is an open set $U \subset X \times X$ such that $Ann(H_L) = C(U)$. Then $Z = (X \times X) - U$, the support of the bimodule H_L.

Proposition 24. *The subset* $Z \subseteq X \times X$ *is completely determined by the* $C(X)$*-bimodule structure of* H_L*. Moreover,* Z *is the support of* $H_L \cong \Gamma(X \times X, \mathcal{E})$.

By proposition 21, Z is the union of transverse subsets of $X \times X$ which is individually are unions of graphs of φ_j^i. Let

$$E_{(x,y)} = \bigoplus_{\varphi_j^i(x) = y} (E_x)_{\varphi_j^i(x)}$$

be the fiber of $\mathcal{E}$ over $(x, y) \in Z$, where $(E_x)_{\varphi_j^i(x)}$ denotes the eigensubspace of E_x over the point $\varphi_j^i(x)$. This defines a sheaf of vector spaces on $X \times X$ supported on Z. A section of $\tau \in \Gamma(X, E)$ defines a section $\hat{\tau} \in \Gamma(Z, \mathcal{E})$ whose value at a point (x, y) is

$$\hat{\tau}(x, y) = \begin{cases} proj_{ij}\tau(x), & \text{if } y = \varphi_j^i(x) \text{ for some } \ i, j \\ \\ 0, & \text{otherwise,} \end{cases}$$

where $proj_{ij}$ denotes the projection $E_x \longrightarrow (E_x)_{\varphi_j^i(x)}$. Conversely, any section $\tau \in \Gamma(Z, \mathcal{E})$ defines a section $\check{\tau} \in \Gamma(X, E)$ by

$$\check{\tau}(x) = \sum_y \tau(x, y).$$

Now, given $h \in C(X) \otimes C(X)$ we have

$$h(x,y) = \sum_k f_k(x) g_k(y)$$

for some $f_k, g_k \in C(X)$. For any $\sigma \in B \cong \Gamma(X,E)$ we have

$$\begin{aligned}
(h \cdot \hat{\sigma})(x,y) &= \sum_k f_k(x)\hat{\sigma}(x,y) g_k(y) \\
&= \sum_k f_k(x) g_k(\varphi_j^i(x)) e_j(\sigma(x)) \\
&= proj_{ij}\left(\sum_k f_k \cdot \sigma \cdot g_k\right)(x,y)
\end{aligned}$$

which shows that $\,^\wedge : \Gamma(X,E) \longrightarrow \Gamma(Z,\mathcal{F})$, $\tau \mapsto \hat{\tau}$ is a bimodule map whose inverse is the map $\,^\vee : \Gamma(Z,\mathcal{F}) \longrightarrow \Gamma(X,E)$, $\tau \mapsto \check{\tau}$.

Using proposition 23, we can relate the vector bundles the Serre-Swan theorem gives when applied to the left and right $C(X)$-module structure of H_L as follows.

Proposition 25. *Let $E_1 \xrightarrow{p_1} X$ and $E_2 \xrightarrow{p_2} X$ be the vector bundles given by the Serre-Swan theorem applied to the finitely-generated projective left and right $C(X)$-module H_L, respectively. Then E_1 and E_2 are the direct-images of the sheaf $\mathcal{E}$ along π_1 and π_2, respectively.*

First, the direct-image of $\mathcal{E}$ along π_1 is easily seen to be a vector bundle and the space of sections $\Gamma(X,(\pi_1)_*\mathcal{E})$ is easily seen to be isomorphic as left $C(X)$-modules to the left $C(X)$-module $\Gamma(Z,\mathcal{E})$. By proposition 23, $\Gamma(Z,\mathcal{E}) \cong \Gamma(X,E_1)$ as left $C(X)$-modules. Thus, by corollary 2.8 of [19] we see that E_1 and $(\pi_1)_*\mathcal{E})$ are isomorphic as vector bundles. Similar argument works for E_2.

Let us say more about the nature of the eigenspaces $E_{(x,y)}$, for $x,y \in X$ in relation to the subset $Z \subset X \times X$.

Proposition 26.

(i) $E_x = \bigoplus_{y \in X} E_{(x,y)}$

(ii) *$dim\left(E_{(x,y)}\right)$ is the number of transverse subsets of $X \times X$ contained in Z passing through (x,y), with multiplicities.*

(iii) $dim\left(\bigoplus_{x\in X} E_{(x,y)}\right) = n$ *for any* $y \in X$.

5.2 The geometry of *C(X)*-ring structures

The previous section describes the geometry of H_L using its bimodule structure over $C(X)$. But H_L has more structure than just being a bimodule. In particular, it is a $C(X)$-ring via the left source map $C(X) \xrightarrow{s_L} H_L$. In this section, we will look at what this additional structure contributes to the geometry of H_L. We will keep the notations of the previous section.

The $C(X)$-ring structure on $H_L \cong \Gamma(X,\mathcal{E})$ via the source map s_L consists of a pair of $C(X)$-bimodule maps

$$\Gamma(Z,\mathcal{E}) \underset{C(X)}{\otimes} \Gamma(Z,\mathcal{E}) \xrightarrow{\mu} \Gamma(Z,\mathcal{E})$$

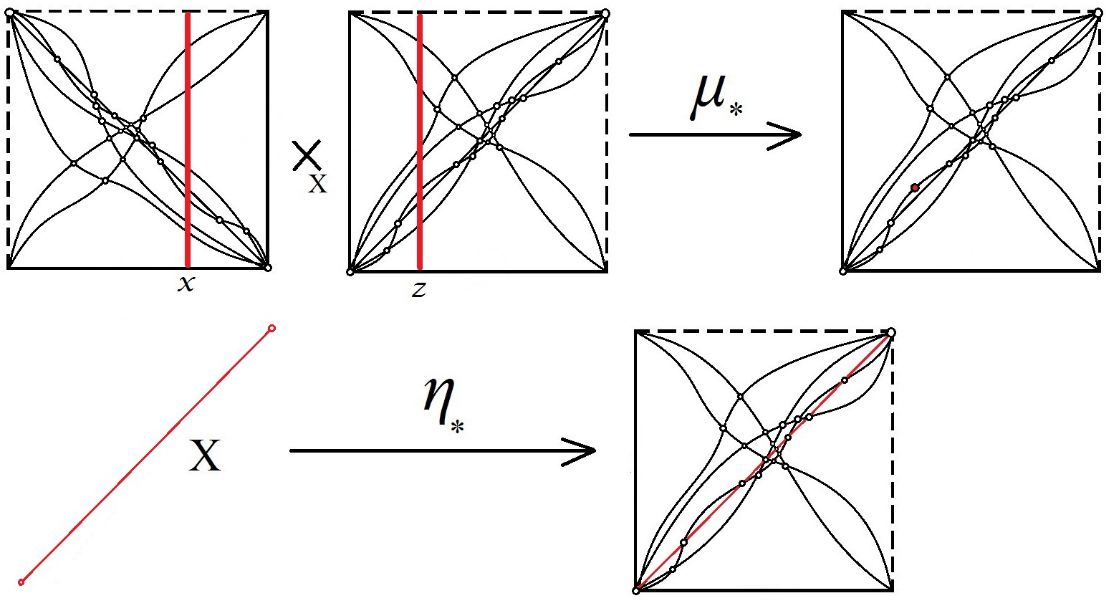


Figure 5.2: The geometry of the product and unit maps.

$$C(X) \xrightarrow{\eta} \Gamma(Z, \mathcal{E})$$

satisfying the associativity and unitality conditions. For brevity we will write $\eta = s_L$.

The unit map η gives an element $1 \in \Gamma(Z, \mathcal{E})$ satisfying $f \cdot 1 = 1 \cdot f$ for all $f \in C(X)$. Since X is Hausdorff, if $x \neq y$ then we can find an $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$. Thus, for $x \neq y$ we have

$$1(x, y) = f(x)1(x, y) = (f \cdot 1)(x, y) = (1 \cdot f)(x, y) = 1(x, y)f(y) = 0.$$

Thus, the source map $A \xrightarrow{s_L} H_L$ is implemented by $C(X) \longrightarrow \Gamma(Z, \mathcal{E})$, $f \mapsto f \cdot 1$. This means that $s_L \circ f(x) = f(x)1(x, x)$ and choosing f such that $f(x) \neq 0$ and $s_L \circ f(x) \neq 0$ we see that $1(x, x) \in E_{(x,x)}$ is a nonzero element. Thus, the diagonal Δ of $X \times X$ is in Z. See figure 5.3.

Note that $\Gamma(Z, \mathcal{E}) \underset{C(X)}{\otimes} \Gamma(Z, \mathcal{E}) \cong \Gamma(Z, \mathcal{E}^{(2)})$ where $\mathcal{E}^{(2)}$ is the sheaf of vector spaces whose fiber at a point $(x, z) \in Z$ is the vector space

$$\bigoplus_{y \in X} \left(E_{(x,y)} \otimes E_{(y,z)}\right)$$

due to the balancing condition $\sigma \cdot f \otimes_{C(X)} \tau = \sigma \otimes_{C(X)} f \cdot \tau$ for $\sigma, \tau \in \Gamma(Z, \mathcal{E})$ and $f \in C(X)$. Notice that all but finitely many summands above are zero. Specifically, only those $y \in X$ for which (x, y) and (y, z) are both in Z contribute nontrivially. Let us denote these $y \in X$ as $y_1, y_2, ..., y_n$.

By proposition 23, $\Gamma(Z, \mathcal{E}) \cong \Gamma(X, E)$ as $C(X)$-bimodules. Since $\Gamma(X, -)$ is a fully faithful functor by corollary 2.8 of [19], we can convert the global ring structures μ and η into something fiber-wise. In particular, the product map μ induces a map

$$E_{(x,y_1)} \otimes E_{(y_1,z)} \oplus ... \oplus E_{(x,y_n)} \otimes E_{(y_n,z)} \xrightarrow{\quad \mu_* \quad} E_{(x,z)} \tag{5.2}$$

illustrated in figure 5.2. By the universal property of direct sums, there are maps

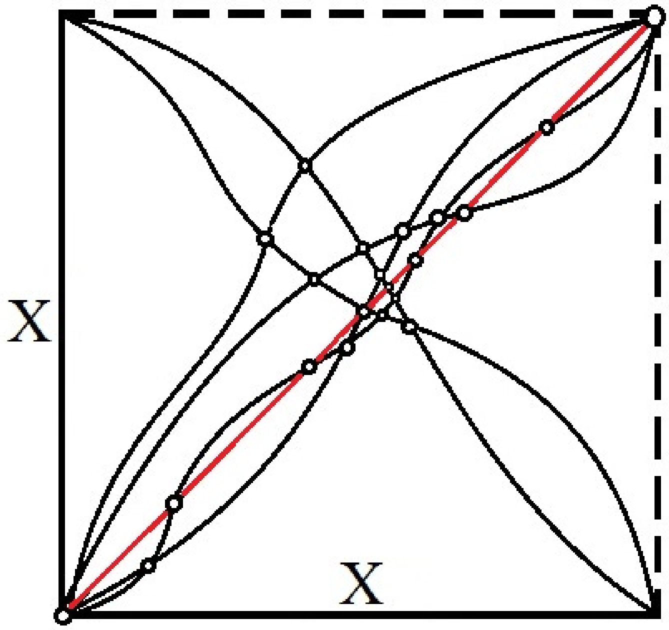


Figure 5.3: Support Z of the bimodule B.

$$E_{(x,y_i)} \otimes E_{(y_i,z)} \xrightarrow{\quad \mu_*^{y_i} \quad} E_{(x,z)}$$

one for each y_i. The collection of these maps satisfy a set of conditions which, though derivable from associativity, is complicated to write down. See (3) of the remark below for these conditions. However, for the maps $E_{(x,x)} \otimes E_{(x,x)} \xrightarrow{\mu_*^x} E_{(x,x)}$ these conditions are precisely the associativity condition. Likewise, the map η induces maps $\eta_*^{x,y} : \mathbb{C} \longrightarrow E_{(x,y)}$ which is nonzero when $x = y$ and zero otherwise. The map μ_*^x together with $\eta_*^x = \eta_*^{x,x}$ makes the vector space $E_{(x,x)}$ a unital algebra, whose dimension depend on the multiplicity of the associated eigenvalue. The following proposition is then immediate from these arguments.

Proposition 27. *Let $A^{'}$ be the $C(X)$-sub-bimodule of $\Gamma(Z, \mathcal{E})$ supported on the diagonal Δ. Then $A^{'}$ is an A-subring of H_L where the multiplication is pointwise. Moreover, $A^{'}$ is the centralizer of A in H_L.*

Remark 19.

(1) Using abuse of notation, let us identify A with its image in H_L. In case A is central in H_L, the fibers of the vector bundle $E \longrightarrow X$ are algebras. These algebras correspond to $E_{(x,x)}$ together with the maps $E_{(x,x)} \otimes E_{(x,x)} \xrightarrow{\mu_*^x} E_{(x,x)}$ and $\mathbb{C} \xrightarrow{\eta_*^x} E_{(x,x)}$ since in the central case, $E_{(x,x)} = E_x$. Thus, $A^{'} = H_L$ in the central case which is not surprising at all knowing that $A^{'}$ is the centralizer of A.

(2) The maps $E_{(x,y_i)} \otimes E_{(y_1,z)} \xrightarrow{\mu_*^i} E_{(x,z)}$ are only restricted by the associativity of μ. Since $\Gamma(Z, \mathcal{E}) \cong \Gamma(X, E)$ and $\Gamma(X, -)$ is known to be a fully faithful functor by corollary 2.8 of [19], we have

$$\begin{array}{ccc}
\bigoplus_{y_i,y_j}\Big(E_{(x,y_i)}\otimes E_{(y_i,y_j)}\otimes E_{(y_j,z)}\Big) & \xrightarrow{\left(\bigoplus_i \mu_*^i\right)\otimes id} & \bigoplus_{y_j}\Big(E_{(x,y_j)}\otimes E_{(y_j,z)}\Big) \\
\Big\downarrow{\scriptstyle id\otimes\left(\bigoplus_j \mu_*^j\right)} & & \Big\downarrow{\scriptstyle \bigoplus_j \mu_*^j} \\
\bigoplus_{y_i}\big(E_{(x,y_i)}\otimes E_{(y_i,z)}\big) & \xrightarrow[\bigoplus_i \mu_*^i]{} & E_{(x,z)}.
\end{array}$$

Universal property of direct sums gives us

$$\begin{array}{ccc}
E_{(x,y_i)}\otimes E_{(y_i,y_j)}\otimes E_{(y_j,z)} & \xrightarrow{\mu_*^i\otimes id} & E_{(x,y_j)}\otimes E_{(y_j,z)} \\
\Big\downarrow{\scriptstyle id\otimes\mu_*^j} & & \Big\downarrow{\scriptstyle \mu_*^j} \\
E_{(x,y_i)}\otimes E_{(y_i,z)} & \xrightarrow[\mu_*^i]{} & E_{(x,z)}.
\end{array}$$

This justifies the argument before proposition 27. We can also use this to say more about the fibers of $\mathcal{E}$ which we state in the next proposition.

Proposition 28. *$E_{(x,y)}$ is a left $E_{(x,x)}-E_{(y,y)}-$bimodule for every $x,y\in X$.*

Remark 20. Using remark 19 (2) above, we can construct a small category $\mathscr{H}_L$ enriched over the category of complex vector spaces. The set of objects of $\mathscr{H}_L$ is X. For every $x,y\in X$, we define

$$Hom(x,y) := \begin{cases} E_{(x,y)}, & \text{if } y=\varphi_j^i(x) \text{ for some } \ i,j \\ \\ \{0\}, & \text{otherwise.}\end{cases}$$

We will call $\mathscr{H}_L$ the *associated category* of the left A-bialgebroid H_L. In the next section, we will see the additional properties of $\mathscr{H}_L$ coming from the A-coring structure of H_L. On a different note, let us give a complete geometric description of the A-ring structure of H_L.

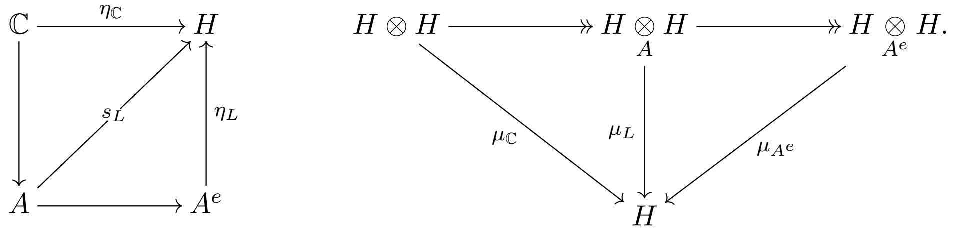
Proposition 29. *Denote by $a *_i b := \mu_*^{y_i}(a,b)$, $a\in E_{(x,y_i)}$ and $b\in E_{(y_i,z)}$. The product of $\sigma,\tau\in\Gamma(Z,\mathcal{E})$ takes the form*

$$(\sigma\tau)(x,z) = \sum_i \sigma(x,y_i) *_i \tau(y_i,z)$$

for all $(x,z)\in Z$.

This follows immediately from equation 5.2. Notice the resemblance of this formula to the one for matrix multiplication. This should remind the reader of an example we discussed in section B.2. One can view a $C(X)$-ring to be a "matrix" of vector spaces whose entries are indexed by $X\times X$ and what sits in entry (x,y) is the vector space $E_{(x,y)}$. As we have defined after proposition 24, the vector space $E_{(x,y)}$ is the zero vector space if $(x,y)\notin Z$. For matrix algebras $M_n(\mathbb{C})$, X would be an n-element set and the vector spaces $E_{(x,y)}$ would all be $\mathbb{C}$. There are a plethora of algebraic structures package into a bialgebroid let alone in a Hopf algebroid. Before we end this section, let us take a detour to describe the relationships among the structures of H: being a $\mathbb{C}$-algebra, the A-ring and the A^e-ring structures being a left-bialgebroid over $A = C(X)$.

For the purpose of this discussion, let us denote by $(\mu_{\mathbb{C}}, \eta_{\mathbb{C}})$ the $\mathbb{C}$-algebra structure of H and recall that (μ_L, s_L) and (μ_{A^e}, η_L) denote the relevant A-ring and A^e-ring structures of H, respectively. Again by lemma 1, for a k-algebra R, R-ring structures are in bijection with k-algebra maps $\eta : k \longrightarrow R$. Thus, the complex algebra structure of H is uniquely determined by the unit map $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow H$. Similarly, the A-ring and the A^e-ring structures are determined by the $\mathbb{C}$-linear maps s_L and η_L. These maps satisfy the following commutativity relations.



In terms of the local eigenspace decomposition, the map $\mu_{\mathbb{C}}$ induces maps

$$E_{(x,w)} \otimes E_{(z,y)} \longrightarrow E_{(x,y)}$$

while, by 5.2, we have maps

$$E_{(x,z)} \otimes E_{(z,y)} \longrightarrow E_{(x,y)} .$$

On the other hand, because the $C(X \times X)$-bimodule structure of H is given as follows,

$$(f \cdot \sigma)(x, y) = f(x, y)\sigma(x, y), (\sigma \cdot f)(x, y) = f(y, x)\sigma(x, y),$$

for any $f \in C(X \times X)$, $\sigma \in H$, and $x, y \in X$, the product μ_{A^e} induces maps

$$E_{(x,z)} \otimes E_{(z,x)} \longrightarrow E_{(x,x)} .$$

Another way of seeing this is by noting that the product μ_L uses the tensor product $\otimes_A$ which kills products $E_{(x,w)} \otimes E_{(z,y)} \longrightarrow E_{(x,y)}$ for which $w \neq z$. Likewise, the tensor product $\otimes_{A^e}$ kills products $E_{(x,z)} \otimes E_{(z,y)} \longrightarrow E_{(x,y)}$ for which $x \neq y$.

5.3 The geometry of *C(X)*-coring structures

In this section, using the techniques and results we have developed in sections 5.1 and 5.2 we will describe what the coring structure of H_L contributes to the geometry of $\mathcal{E}$. We will keep the notations of the previous two sections.

The $C(X)$-bimodule structure of the underlying A-coring structure of H_L is related to the $C(X)$-bimodule structure of the underlying A-ring via

$$(f \cdot \sigma \cdot g)(x, y) = f(x)g(x)\sigma(x, y) \tag{5.3}$$

for $\sigma \in \Gamma(Z, \mathcal{E})$, $f, g \in C(X)$, and $x, y \in X$. The left-hand side of equation 5.3 concerns the bimodule structure one has for the underlying A-coring of H_L while the right-hand side concerns its A-ring structure. This, in particular, implies that if we run the construction we have in section 5.1 for the bimodule structure of the A-coring of H_L, we will get the same sheaf $\mathcal{E}$ supported over the same closed subset Z.

The coproduct $H_L \xrightarrow{\Delta_L} H_L \otimes_A H_L$ uses a different A-bimodule structure from the A-bimodule structure involved in the A-ring structure. Thus, $\otimes_{C(X)}$ means different from the $\otimes_{C(X)}$ we have in the product μ. With this, let us denote by $\boxtimes_A$ this new tensor product. thus, we have

$$\Gamma(Z,\mathcal{E}) \xrightarrow{\quad \Delta_L \quad} \Gamma(Z,\mathcal{E}) \boxtimes_{C(X)} \Gamma(Z,\mathcal{E}) \; . \tag{5.4}$$

However, using the relation 5.3 the codomain of Δ_L can be expressed as

$$\Gamma(Z,\mathcal{E}) \boxtimes_{C(X)} \Gamma(Z,\mathcal{E}) \cong \Gamma(Z,\mathcal{E}^{\langle 2\rangle}),$$

where $\mathcal{E}^{\langle 2\rangle}$ is the sheaf of vector spaces whose fiber at $(x,z) \in X \times X$ is

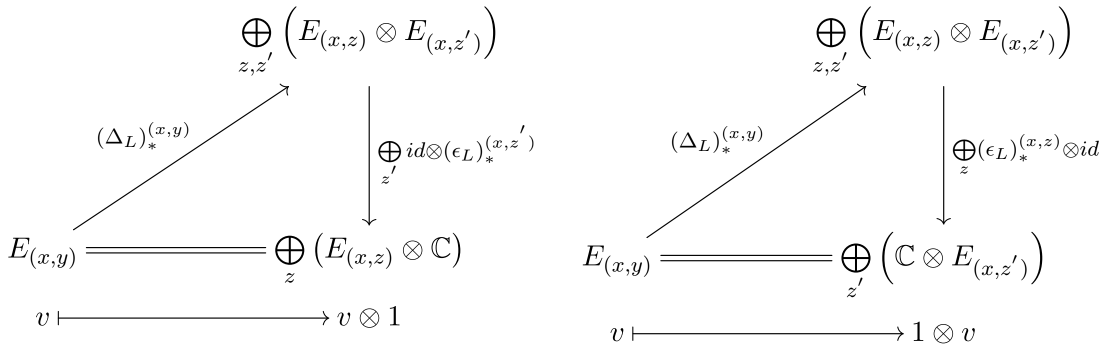
$$\bigoplus_{y',y'' \in X} \left(E_{(x,y')} \otimes E_{(x,y'')} \right) .$$

Using the same argument we used in the previous section, the map Δ_L induces a map $(\Delta_L)_* : \mathcal{E} \longrightarrow \mathcal{E}^{\langle 2\rangle}$ of sheaves over Z. Over point a $(x,y) \in Z$, we have a map

$$E_{(x,y)} \xrightarrow{\quad (\Delta_L)_*^{(x,y)} \quad} \bigoplus_{z',z'' \in X} \left(E_{(x,z')} \otimes E_{(x,z'')} \right) \tag{5.5}$$

Meanwhile, the counit $\epsilon_L : \Gamma(Z,\mathcal{E}) \longrightarrow C(X)$ induces a map $\mathcal{E} \longrightarrow \mathcal{E}^{'}$ of sheaves over Z and $\not\Delta$, respectively. Here, $Z \longrightarrow \not\Delta$ is the map $(x,y) \mapsto (x,x)$ for any $(x,y) \in Z$ and $\mathcal{E}^{'}$ is the subsheaf of $\mathcal{E}$ where the fiber of $\mathcal{E}^{'}$ at (x,y) is $\{0\}$ unless $x = y$, to which the fiber is $\mathbb{C}$ viewed as the one-dimensional subalgebra of $E_{(x,x)}$ spanned by its unit $1(x,x)$. Hence, over a point $(x,y) \in Z$ we have $(\epsilon_L)_*^{(x,y)} : E_{(x,y)} \longrightarrow \mathbb{C}$.

Counitality of Δ_L with respect to ϵ_L implies that for fixed but arbitrary $x, y \in X$ we have



(5.6)

The bottom isomorphisms imply that $(\epsilon_L)_*^{(x,z)}$ and $(\epsilon_L)_*^{(x,z')}$ are nonzero maps for $z = y$ and $z^{'} = y$. Since x and y are arbitrary to start with, we have the following proposition.

Proposition 30. *For any $(x,y) \in Z$, we have $(\epsilon_L)_*^{(x,y)} \neq 0$.*

Another thing we can infer from the diagrams 5.6, using the isomorphisms in the bottom and the fact that y is among the z and $z^{'}$ that appears as indices, is that the image of $(\Delta_L)_*^{(x,y)}$ is contained in

$$\left(E_{(x,y)} \otimes E_{(x,y)} \right) \oplus \bigoplus_{z,z'} \left(ker \left(id \otimes (\epsilon_L)_*^{(x,z')} \right) + ker \left((\epsilon_L)_*^{(x,z)} \otimes id \right) \right)$$

We will show in the next section that more can be said. In fact, the image of $(\Delta_L)_*^{(x,y)}$ is completely contained in $E_{(x,y)} \otimes E_{(x,y)}$.

5.4 Hopf algebroids over *C(X)*

In this section, we will complete our description of the geometry of the Hopf algebroid $\mathcal{H}$ over $C(X)$. In doing so, we will be able to illustrate the main point of this article. That to such a Hopf algebroid, one can associate a highly structured category.

So far, we have considered only the constituent left bialgebroid H_L of $\mathcal{H}$. Running the arguments we have presented in sections 5.1 and 5.2 for H_R, we see that there is a sheaf of vector spaces $\mathcal{E}'$ over $X \times X$ such that $H_R \cong \Gamma(X \times X, \mathcal{E}')$. Let us denote by Z' the support of H_R under the isomorphism $H_R \cong \Gamma(X \times X, \mathcal{E}')$. The following proposition relates these two sheaves.

Proposition 31. *For $H_L \cong \Gamma(X \times X, \mathcal{E})$ and $H_R \cong \Gamma(X \times X, \mathcal{E}')$ as $C(X)$-bimodules as constructed in sections 5.1 and 5.2, where $\mathcal{E}$ and $\mathcal{E}'$ are sheaves of vector spaces supported on $Z, Z' \subseteq X \times X$, we have*

(i) $Z = Z'$.

(ii) $\mathcal{E} \cong \mathcal{E}'$ *as sheaves over* Z.

Proof: Condition (c) of the definition of a Hopf algebroid implies that the antipode S of $\mathcal{H}$ flips the $C(X)$-bimodule structure used for the $C(X)$-ring structure of H_L to that of the $C(X)$-bimodule structure used for the $C(X)$-ring structure of H_R. Likewise, S flips the bimodule structures of the underlying $C(X)$-coring structures of H_L and H_R. In particular, this tells us that S induces a map $S_* : \mathcal{E} \longrightarrow \mathcal{E}'$ which on fibers does $S_*(E_{(x,y)}) = E_{(y,x)}$ for any $(x, y) \in Z$. Symmetrically, we also have a map denoted the same, $S_* : \mathcal{E}' \longrightarrow \mathcal{E}$, which on fibers does $S_*(E_{(y,x)}) = E_{(x,y)}$ for any $(y, x) \in Z'$. This proves proposition 31. ■

Remark 21. Proposition 31 tells us that the closed subset $Z \subseteq X \times X$ must be symmetric, i.e. $Ⅎ(Z) = Z$.

In view of proposition 31, we have $\Delta_R : \Gamma(Z, \mathcal{E}) \longrightarrow \Gamma(Z, \mathcal{E})$. Similar to equation 5.5, Δ_R induces maps

$$E_{(x,y)} \xrightarrow{(\Delta_R)_*^{(x,y)}} \bigoplus_{z',z'' \in X} \big(E_{(z',y)} \otimes E_{(z'',y)}\big) \tag{5.7}$$

for $(x, y) \in Z$. As we promised at the end of section 5.3, $(\Delta_L)_*^{(x,y)}$ maps $E_{(x,y)}$ into $E_{(x,y)} \otimes E_{(x,y)}$, for any $(x, y) \in Z$. same holds for $(\Delta_R)_*^{(x,y)}$. Let us summarize these statements into the following proposition.

Proposition 32. *For every $(x, y) \in Z$,*

(i) $E_{(x,y)}$ is a coalgebra with coproduct $(\Delta_L)_*^{(x,y)}$ and counit $(\epsilon_L)_*^{(x,y)}$, and

(ii) $E_{(x,y)}$ is a coalgebra with coproduct $(\Delta_R)_*^{(x,y)}$ and counit $(\epsilon_R)_*^{(x,y)}$.

Proof: We will only prove part (ii). The proof for part (i) is similar. The second commutation relation of Δ_L and Δ_R in part (b) of the definition 2 gives the following diagram

$$
\begin{array}{ccc}
E_{(x,y)} & \xrightarrow{(\Delta_L)_*^{(x,y)}} & \bigoplus_{z',z''}\left(E_{(x,z')}\otimes E_{(x,z'')}\right) \\
 & & \Big\downarrow {\scriptstyle \bigoplus_{z'}(\Delta_R)_*^{(x,z')}\otimes id} \\
\Big\downarrow {\scriptstyle (\Delta_R)_*^{(x,y)}} & & \bigoplus_{z',z''}\bigoplus_{\alpha',\alpha''}\left(E_{(\alpha',z')}\otimes E_{(\alpha'',z')}\otimes E_{(x,z'')}\right) \\
 & & \Big\| \\
\bigoplus_{\beta',\beta''}\left(E_{(\beta',y)}\otimes E_{(\beta'',y)}\right) & \xrightarrow[\bigoplus_{\beta''} id\otimes(\Delta_L)_*^{(\beta'',y)}]{} & \bigoplus_{\beta',\beta''}\bigoplus_{\gamma',\gamma''}\left(E_{(\beta',y)}\otimes E_{(\beta'',\gamma')}\otimes E_{(\beta'',\gamma'')}\right)
\end{array}
\tag{5.8}
$$

for fixed but arbitrary $(x,y)\in Z$. In the composite

$$\left(\bigoplus_{z'}(\Delta_R)_*^{(x,z')}\otimes id\right)\circ(\Delta_L)_*^{(x,y)},$$

the third leg lands in $\bigoplus_{z''}E_{(x,z'')}$. On the other hand, the third leg of the composite

$$\left(\bigoplus_{\beta''} id\otimes(\Delta_L)_*^{(\beta'',y)}\right)\circ(\Delta_R)_*^{(x,y)}$$

lands in $\bigoplus_{\beta'',\gamma''}E_{(\beta'',\gamma'')}$. This implies that for $\beta''\neq x$, we have $E_{(\beta'',y)}\subseteq ker\ (\Delta_L)_*^{(\beta'',y)}$. From our last statement in section 5.3, $(\Delta_L)_*^{(\beta'',y)}\left(E_{(\beta'',y)}\right)$ is contained in

$$\left(E_{(\beta'',y)}\otimes E_{(\beta'',y)}\right)\oplus\bigoplus_{f',f''}\left(ker\left(id\otimes(\epsilon_L)_*^{(\beta'',f')}\right)+ker\left((\epsilon_L)_*^{(\beta'',f'')}\otimes id\right)\right).$$

Counitality of Δ_L with respect to ϵ_L, implemented locally by diagram 5.6, gives

$$
\begin{array}{ccc}
E_{(\beta'',y)} & \xrightarrow{\cong} & \bigoplus_{f'}\left(id\otimes(\epsilon_L)_*^{(\beta'',f')}\right)(\Delta_L)_*^{(\beta'',y)}\left(E_{(\beta'',y)}\right) = \{0\}. \\
v & \longmapsto & v\otimes 1
\end{array}
$$

By assumption, $E_{(\beta'',y)}$ are nontrivial. This is a contradiction unless the summands corresponding to $\beta''\neq x$ of the direct sum in the lower left corner of diagram 5.8 do not intersect the image of $(\Delta_R)_*^{(x,y)}$.

Using the first commutation relation in part (b) of the definition 2, we have a diagram similar to diagram 5.8. Inspecting that resulting diagram tells us that the image of $(\Delta_R)_*^{(x,y)}$ does not intersect those summands of the direct sum in the lower left corner of diagram 5.8 corresponding to $\beta'\neq x$. This shows that, indeed,

$$(\Delta_R)_*^{(x,y)}:E_{(x,y)}\longrightarrow E_{(x,y)}\otimes E_{(x,y)}.$$

The coassociativity of $(\Delta_R)_*^{(x,y)}$ follows from coassociativity of Δ_R and its counitality with respect to $(\epsilon_R)_*^{(x,y)}$ follows from counitality of Δ_R with respect to ϵ_R. This proves part (ii) of the above proposition. Exchanging the roles of Δ_L and Δ_R with minor modifications proves part (i). ■

Following the arguments in sections 5.1, 5.2 and 5.3 for H_R, we see that we can similarly associate a category $\mathscr{H}_R$ enriched over $\mathcal{V}$. Denoting by $C(\mathcal{V})$ by the category of coalgebras on $\mathcal{V}$, we have the following proposition.

Proposition 33. *The categories $\mathscr{H}_L$ and $\mathscr{H}_R$ are enriched over $C(\mathcal{V})$.*

These categories are strongly related. By proposition 31, we have the following corollary.

Corollary 4. *The $C(\mathcal{V})$-enriched categories $\mathscr{H}_L$ and $\mathscr{H}_R$ have isomorphic underlying $\mathcal{V}$-enriched categories.*

Note that the underlying $\mathcal{V}$-enriched category of $\mathscr{H}_L$ and $\mathscr{H}_R$ only depends on the $C(X)$-ring structures of H_L and H_R, respectively. Another way to prove corollary 4 is to use the fact that H_L and H_R have the isomorphic $C(X)$-ring structures. To see why H_L and H_R have isomorphic $C(X)$-ring structures, note that the source map of H_L is the target map of H_L while the target map of H_L is the source map of H_R. In the general definition of a Hopf algebroid, one can either use the source or the target map to select a particular ring structure to consider, see for example [6]. Using the general fact that for a general k-algebra R, R-rings (A, μ, η) corresponds uniquely to k-algebra maps η, we see that H_L and H_R are isomorphic as $C(X)$-rings.

Remark 22. Another way to see why H_L and H_R are isomorphic as $C(X)$-rings is the fact that general Hopf algebroids $\mathcal{H}$ with bijective antipode over a *commutative* ring K is a coupled K-Hopf algebra.

Unlike the ring structures, the $C(X)$-coring structures of H_L and H_R can vary wildly as illustrated by coupled Hopf algebras. This implies that the $C(\mathcal{V})$-enrichments $\mathscr{H}_L$ and $\mathscr{H}_R$ need not be isomorphic. However, they form a topological coupled Hopf category. The coupling functor is the one induced by the antipode S of the Hopf algebroid $\mathcal{H}$. We formalize this in the following theorem.

Theorem 6. *Given a finitely-generated projective Hopf algebroid $\mathcal{H}$ over $C(X)$ with bijective antipode, one can associate a topological coupled Hopf category $\mathscr{H}$ via the construction we presented in sections 5.1 and 5.2. Conversely, to any topological coupled Hopf category $\mathscr{H}$, the space of sections $\Gamma(X \times X, H)$ of the associated sheaf H of $\mathscr{H}$ is a Hopf algebroid over $C(X)$.*

The proof of the first statement is basically the breadth of chapter 5. For the second statement, one can consider the bimodule structures presented in proposition 23. The rest of the structures are given by the rest of the structure maps of $\mathscr{H}$. The above theorem is a generalization of the example in [1] where they constructed out of a k-linear category with finitely many objects a weak Hopf algebra. The theorem not only recovers an inverse to the construction they presented but it also work for weak Hopf algebra as long as the subalgebra spanned by the left and the right units are commutative. The above theorem is the generalization of the example we discussed in section B.2.

5.5 The central case

Although $C(X)$ is commutative, it may not be central in $\mathcal{H}$. In chapter 5 we discussed the case when $C(X)$ is central in $\mathcal{H}$. Let us revisit this case from the perspective offered by the present chapter. For simplicity, we will blur the disctinction between $C(X)$ at its images under the source maps of $\mathcal{H}$.

Let us consider first the constituent left bialgebroid H_L of $\mathcal{H}$. By proposition 27, H_L is supported along the diagonal $\Delta \subseteq X \times X$. This means that the sheaf $\mathcal{E}$ coincides with the vector bundle $E \longrightarrow X$. We can simply identify the diagonal Δ with X. With this, the multiplication μ_L in H_L via the identification $H_L \cong \Gamma(X, E)$ is pointwise, i.e. the fibers of the vector bundle $E \longrightarrow X$ are (possibly nonisomorphic) unital complex alegrbas $(E_x, (\mu_L)_*^x, (s_L)_*^x)$, where $(\mu_L)_*^x$ and $(s_L)_*^x$ are the maps induced by μ_L and s_L on the fiber E_x.

By proposition 32, the coproduct Δ_L and counit ϵ_L of H_L also descends into a coproduct $(\Delta_L)_*^x$ and a counit $(\epsilon_L)_*^x$ for the fibers E_x, $x \in X$, making them coalgebras. Using condition (b) in the definition of a bialgebroid, we see that $(\Delta_L)_*^x$ is multiplicative for any $x \in X$. Meanwhile, using condition (c) of the definition of a bialgebroid we see that $(\epsilon_L)_*^x$ is multiplicative for any $x \in X$. This gives us the following proposition.

Proposition 34. *If $C(X)$ is central in H_L, then for any $x \in X$,*

$$(E_x, (\mu_L)_*^x, (s_L)_*^x, (\Delta_L)_*^x, (\epsilon_L)_*^x)$$

is a bialgebra. Moreover, the bialgebroid H_L is a bundle of bialgebras via $H_L \cong \Gamma(X, E)$.

Similar statement holds for the constituent right bialgebroid H_R. Since for very $x \in X$ the maps $(s_L)_*^x$ and $(s_R)_*^x$ induced by the source maps s_L and s_R are the same, the multiplications $(\mu_L)_*^x$ and $(\mu_R)_*^x$ coincide. Assuming mild nondegeneracy conditions for $(\Delta_L)_*^x$ and $(\Delta_R)_*^x$, we get the following proposition.

Proposition 35. *Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid over $A = C(X)$ where A is central in both H_L and H_R. Denote by H the underlying complex algebra of $\mathcal{H}$. Suppose that the maps*

$$\begin{array}{ccc} H \underset{A}{\otimes} H & \xrightarrow{\mathfrak{gal}_L} & H \underset{A}{\otimes} H \\ a \underset{A}{\otimes} b & \longmapsto & ab_{[1]} \underset{A}{\otimes} b_{[2]} \end{array} \quad , \qquad \begin{array}{ccc} H \underset{A}{\otimes} H & \xrightarrow{\mathfrak{gal}_R} & H \underset{A}{\otimes} H \\ a \underset{A}{\otimes} b & \longmapsto & ab^{[1]} \underset{A}{\otimes} b^{[2]} \end{array}$$

are bijections. Then

(i) H is a coupled Hopf algebra with constituent Hopf algebras H_L and H_R and coupling map S.

(ii) Each fiber E_x is a Hopf algebra and $H_L \cong \Gamma(X, E)$ as Hopf algebras, where the structure maps of $\Gamma(X, E)$ are all pointwise. Same is true for H_R.

(iii) $\mathcal{H}$ is a bundle of coupled Hopf algebras over X such that the constituent Hopf algebras at a point $x \in X$ are the fiber Hopf algebras of H_L and H_R.

Proof: Centrality of A in both H_L and H_R implies that H_L and H_R are in fact bialgebras over A (not just bialgebroids). The nondegeneracy conditions assumed in the proposition implies that H is a Galois extension for both bialgebras H_L and H_R. By [44], the bialgebras H_L are H_R are in fact Hopf algebras, i.e. the identity maps $H_L \xrightarrow{id} H_L$ and $H_R \xrightarrow{id} H_R$ are invertible in the respective convolution algebras associated to the bialgebras H_L and H_R. The rest of the conditions for $\mathcal{H}$ to be a Hopf algebroid imply that H_L and H_R are coupled Hopf algebras with coupling map S, the antipode of $\mathcal{H}$. This proves part (i).

To prove part (ii), we argue that the maps $\mathfrak{gal}_L$ and $\mathfrak{gal}_R$ are A-bimodule maps. Thus, there descend into fiberwise bijections. Using the same argument we did for part (i), see that the fibers are coupled Hopf algebras. Part (iii) readily follows from the proofs of parts (i) and (ii). ■

5.6 Correspondence of Galois extensions

In this section, we will see that the correspondence between Hopf algebroids and coupled Hopf categories we established in theorem 6 persists to their corresponding Galois theories. To be precise, we will prove the following theorem.

Theorem 7. *Let $\mathcal{H} = (H_L, H_R, S)$ be a Hopf algebroid over $A = C(X)$ for some compact Hausdorff space X. Let $\mathscr{H}$ be the corresponding topological coupled Hopf category of $\mathcal{H}$. Then $\mathcal{H}$-Galois extensions of A corresponds bijectively to $\mathscr{H}$-Galois extensions of $\mathbb{I}^X$.*

Before proving the above theorem, let us comment on what we mean by Galois extension by a (topological) coupled Hopf category $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$. By this, we mean an inclusion of categories $\mathbb{I}^X \subseteq \mathscr{M}$ which is simultaneously $\mathscr{H}_L$-Galois and $\mathscr{H}_R$-Galois in the sense of section B.3. Note that by definition 22, we are not requiring $\mathscr{H}_L$ and $\mathscr{H}_R$ to be Hopf categories (individually, they are only $C(\mathcal{V})$-enriched categories). In particular, they do not necessarily have antipodes. Fortunately, Galois extension in the sense described in section B.3 does not really make use of the antipode.

Proof: Let B be a (left) $\mathcal{H}$-Galois extension of A. In particular, B is an A-ring. Note that the arguments we used in sections 5.1 and 5.2 only use the A-ring structure of the Hopf algebroid $\mathcal{H}$. Using the same arguments, $B \cong \Gamma(X \times X, \mathcal{B})$ where $\mathcal{B}$ is a sheaf of vector spaces over $X \times X$. By the Galois condition, we see that $\mathcal{B}$ has the same support $Z \subseteq X \times X$ as the sheaf $\mathcal{E}$ we get from either H_L or H_R. Similar to remark 20, we get a small category $\mathscr{B}$ over X enriched over $\mathcal{V}$ whose associated sheaf is $\mathcal{B}$.

The (right) H_L-coaction $\rho_L : B \longrightarrow B \otimes_A H$ induces a map $\mathcal{B} \longrightarrow \mathcal{B}\,{}_X\!\times_X \mathcal{E}$ of sheaves of $\mathcal{O}_X$-bimodules over $X \times X$. By definition, B is a right A-module and a right A^{op}-module. Using this, the A-bimodule structure on B is as follows:

$$a \cdot b \cdot a^{'} = b(aa^{'})$$

for any $a, a^{'} \in A$ and $b \in B$. Similar to 5.5, the right H_L-coaction induces, for every $(x, y) \in Z$, linear maps

$$B_{(x,y)} \xrightarrow{(\rho_L)_*^{(x,y)}} \bigoplus_{z^{'},z^{''} \in X} B_{(z^{'},y)} \otimes E_{(z^{''},y)} \tag{5.9}$$

where $B_{(x,y)}$ the fiber of $\mathcal{B}$ at the point (x, y). As before, $E_{(x,y)}$ denotes the fiber of $\mathcal{E}$ over (x, y). Likewise, the right H_R-coaction ρ_R induces linear maps

$$B_{(x,y)} \xrightarrow{(\rho_R)_*^{(x,y)}} \bigoplus_{z^{'},z^{''} \in X} B_{(x,z^{'})} \otimes E_{(x,z^{''})} \tag{5.10}$$

By 2.15, we have

$$\begin{array}{ccc}
B_{(x,y)} & \xrightarrow{\quad(\rho_L)_*^{(x,y)}\quad} & \bigoplus\limits_{z^{'},z^{''}} \Big(B_{(x,z^{'})} \otimes E_{(x,z^{''})}\Big) \\
 & & \Big\downarrow {\scriptstyle \bigoplus_{z^{'}} (\rho_R)_*^{(x,z^{'})} \otimes id} \\
\Big\downarrow {\scriptstyle (\rho_R)_*^{(x,y)}} & & \bigoplus\limits_{z^{'},z^{''}} \bigoplus\limits_{\alpha^{'},\alpha^{''}} \Big(B_{(\alpha^{'},z^{'})} \otimes E_{(\alpha^{''},z^{'})} \otimes E_{(x,z^{''})}\Big) \\
 & & \Big\| \\
\bigoplus\limits_{\beta^{'},\beta^{''}} \Big(B_{(\beta^{'},y)} \otimes E_{(\beta^{''},y)}\Big) & \xrightarrow[\bigoplus_{\beta^{''}} id \otimes (\Delta_L)_*^{(\beta^{''},y)}]{} & \bigoplus\limits_{\beta^{'},\beta^{''}} \bigoplus\limits_{\gamma^{'},\gamma^{''}} \Big(B_{(\beta^{'},y)} \otimes E_{(\beta^{''},\gamma^{'})} \otimes E_{(\beta^{''},\gamma^{''})}\Big)
\end{array} \tag{5.11}$$

Meanwhile, counitality of the left coaction ρ_L implies that

$$\begin{array}{ccc} & & \bigoplus_{z',z''} \left(B_{(x,z')} \otimes E_{(x,z'')} \right) \\ & \overset{(\rho_L)_*^{(x,y)}}{\nearrow} & \Big\downarrow {\scriptstyle \bigoplus_{z''} id \otimes (\epsilon_L)_*^{(z,y)}} \\ B_{(x,y)} & = & \bigoplus_{z'} \left(B_{(x,z')} \otimes \mathbb{C} \right) \\ v & \longmapsto & v \otimes 1 \end{array}$$

from which, using a similar argument we to the proof of proposition 32(1), gives

$$B_{(x,y)} \xrightarrow{(\rho_L)_*^{(x,y)}} B_{(x,y)} \otimes E_{(x,y)} .$$

Similarly, we have

$$B_{(x,y)} \xrightarrow{(\rho_R)_*^{(x,y)}} B_{(x,y)} \otimes E_{(x,y)} .$$

These tell us that $\mathscr{B}$ is a right $\mathscr{H}_L$- and a right $\mathscr{H}_R$-comodule. The composition $\circ$ in $\mathscr{B}$ is induced by the A-product on B. By equations 2.17 to 2.20, this composition $\circ$ is a map of right $\mathscr{H}_L$- and a right $\mathscr{H}_R$-modules. Thus, $\mathscr{B}$ is a right $\mathscr{H}_L$- and a right $\mathscr{H}_R$-comodule-category. It is not hard to see that the right coactions of $\mathscr{H}_L$ and $\mathscr{H}_R$ on $\mathscr{B}$ are both Galois whose subcategories of coinvariants are both the same as I_X. These imply that $\mathscr{B}$ is a Galois extension of I_X by the topological coupled Hopf category $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$.

The inverse of this correspondence is easily seen as the the one that associates to an $(\mathscr{H}_L, \mathscr{H}_R, S)$-Galois extension $I_X \subseteq \mathscr{B}$ the (H_L, H_R, S)-Galois extension $A \subseteq B$. Here, we denote by H_L, H_R, B and A the spaces of global sections of the associated sheaves to $\mathscr{H}_L$, $\mathscr{H}_R$, $\mathscr{B}$ and I_X, respectively. The compatibility conditions in the categorical side precisely correspond to the analogous compatibility conditions in the algebraic side. ■

Theorem 7 together with theorem 6 give the following complete geometric description of possibly non-central local coverings of $A = C(X)$.

Theorem 8. *Let $(B, \mathcal{H})$ be a local covering of $A = C(X)$. Then there is a topological coupled Hopf category $\mathscr{H}$ over X and a right $\mathscr{H}$-comodule-category $\mathscr{B}$ such that the global sections of the associated sheaves are $\mathcal{H}$ and B, respectively. Moreover, $\mathscr{B}$ is an $\mathscr{H}$-Galois extension of I_X.*

Comparing to the central case, the algebraic structures of B and $\mathcal{H}$ are no longer pointwise but *convoluted*. We will see a more explicit example of such structure in section 6.1 when we discuss rational non-commutative tori.

Chapter 6

Coverings of the noncommutative torus

Do. Or do not.
There is no try.

–Yoda,
The Empire
Strikes Back

In chapter 5, we dealt with the general situation of non-central local coverings $(B, \mathcal{H})$ of a commutative space $A = C(X)$. In the first section of this chapter, we will see a particular example of such coverings. In particular, we will discuss the case of the commutative torus. We will also discuss along with it, the case of the rational non-commutative torus. As we shall see in the next section, these two cases are not that far apart.

The first truly non-commutative example we will deal with is that of an irrational non-commutative torus. Using methods of K-theory, we will show that coverings of such tori, which are themselves irrationally non-commutative, are of the same *type*.

6.1 Commutative and rational non-commutative tori

Let $q \in \mathbb{C}$ be a primitive n^{th} root of unity. Let B be the universal C^*-algebra generated by unitaries U and V satisfying $UV = qVU$. Let A be the C^*-subalgebra generated by U and V^n. Then U and V commutes. Thus, A is isomorphic as a C^*-algebra to the algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. As an A-module, B is finitely-generated and projective. In fact, it is free with generators $\left\{1, V, ..., V^{n-1}\right\}$. Thus, by the Serre-Swan theorem $B \cong \Gamma(\mathbb{T}^2, \mathbb{E})$ for some finite-rank vector bundle $\mathbb{E}$ over $\mathbb{T}^2$. However, the multiplication in B is not the pointwise multiplication on $\Gamma(\mathbb{T}^2, \mathbb{E})$ since A is not central in B. Let us describe the product in B as an A-ring. Since B is free over A, we have an isomorphism

$$B \cong \bigoplus_{i=0}^{n-1} A \cdot V^i.$$

Let us index the generating set of B as an A-module by $\mathbb{Z}/n$, the group of integers modulo n. Elements f and g of B are of the form

$$f = \sum_{i \in \mathbb{Z}/n} a_i V^i, \qquad g = \sum_{i \in \mathbb{Z}/n} b_i V^i$$

for some $a_i, b_i \in A, i = 0, ..., n-1$.

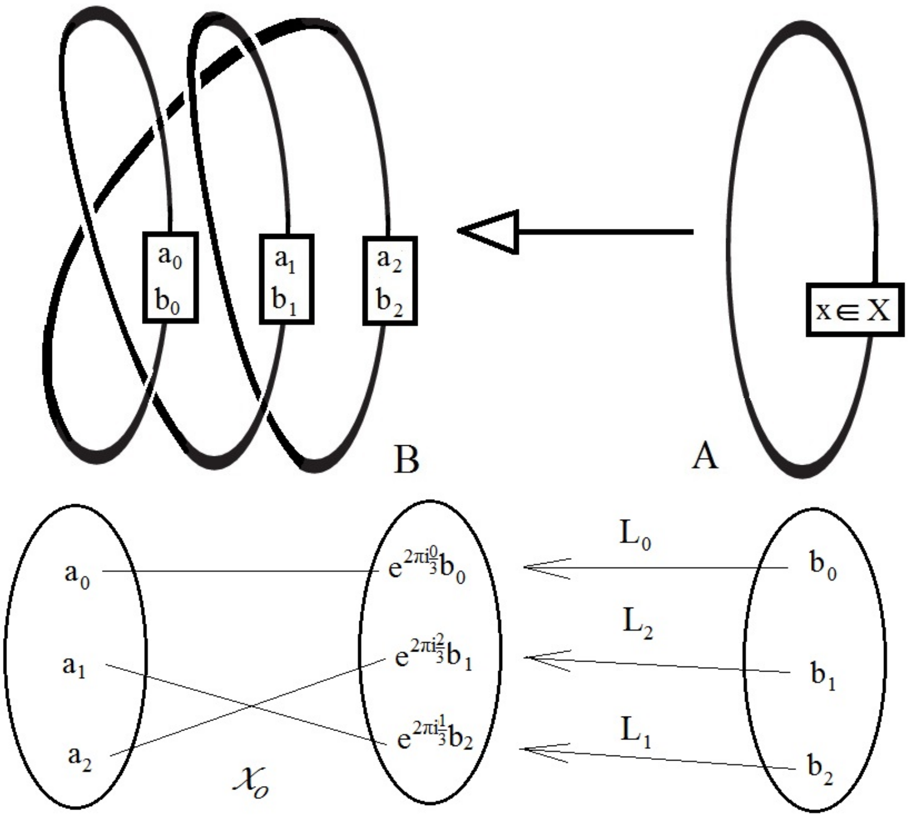


Figure 6.1: Convolution-pointwise product

Then the product of f and g is

$$fg = \sum_{k \in \mathbb{Z}/n} \chi_k (\alpha, \beta) V^k$$

for $\chi_k(\alpha, \beta) \in A,\ k = 0, ..., n-1$ where $\alpha = (a_0, a_1, ..., a_{n-1})$ and $\beta = (b_0, b_1, ..., b_{n-1})$. Let us describe χ_k. Denote by $L : A \longrightarrow A$ the diagonal operator defined on linear generators of A by $L(U^x V^{ny}) = q^{-x} U^x V^{ny}$. Consider the group table of $\mathbb{Z}/n$ considered as a matrix, denoted as Ω. Fix $k \in \mathbb{Z}/n$. From Ω, construct a matrix Ω_k by changing those entries different from $k \in \mathbb{Z}/n$ to 0 and changing the entries with k to L^{i-1}, if that entry is in the i^{th} row. Then

$$\chi_k(\alpha, \beta) = \alpha \Omega_k \beta^T = \begin{pmatrix} a_0, & a_1, & ..., & a_{n-1} \end{pmatrix} \Omega_k \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = \sum_{i=0}^{n-1} a_i L^i (b_{k-i}).$$

for $k = 0, ..., n-1$. As an example, for $n = 3$ we have

$$\Omega_0 = \begin{pmatrix} L^0 & & \\ & & L^1 \\ & L^2 & \end{pmatrix}, \qquad \Omega_1 = \begin{pmatrix} & L^0 & \\ L^1 & & \\ & & L^2 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} & & L^0 \\ & L^1 & \\ L^2 & & \end{pmatrix}$$

and so

$$\chi_0 = a_0 L^0(b_0) + a_1 L^1(b_2) + a_2 L^2(b_1)$$
$$\chi_1 = a_0 L^0(b_1) + a_1 L^1(b_0) + a_2 L^2(b_2)$$
$$\chi_2 = a_0 L^0(b_2) + a_1 L^1(b_1) + a_2 L^2(b_0).$$

The A-ring structure of B is pointwise-convolution as illustrated in figure 2. Denote by $\mathcal{H} = C(G, A)$, where $G = \mathbb{Z}/n$. We claim that $\mathcal{H}$ is a commutative Hopf algebroid. The left- and right-bialgebroid structures of $\mathcal{H}$ are isomorphic, with pointwise product, whose source, target, counit and antipode map is

$$A \xrightarrow{s,t} \mathcal{H}, \qquad \mathcal{H} \xrightarrow{\epsilon} A, \qquad \mathcal{H} \xrightarrow{S} \mathcal{H},$$
$$1 \longmapsto 1 \qquad f \longmapsto f(1) \qquad f \longmapsto Sf, \quad Sf(x) = f(x^{-1})$$

respectively, and whose coproduct is

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \underset{A}{\otimes} \mathcal{H} \cong C(G \times G, A)$$
$$f \longmapsto \Delta f, \qquad \Delta f(x, y) = f(xy).$$

The group G acts on B as follows: $g \cdot U = U$, $g \cdot V = qV$ where $g \in G$ is a generator. This action extends to a module structure over the group algebra $\mathcal{H}^* = AG$, the A-dual of the Hopf algebroid $\mathcal{H}$. The $\mathcal{H}^*$-invariants of B is A. Thus, B carries a coaction of $\mathcal{H}$ whose coinvariants is A. This implies that $(B, \mathcal{H})$ is a local covering of A. □

Remark 23.

(1) The covering $(B, \mathcal{H})$ of A above is an example of a covering where A is a commutative space which is not central in B. However, the images of A under the source and target map is central in $\mathcal{H}$ as it is a commutative Hopf algebroid. This implies that $\mathcal{H}$ is a bundle of Hopf algebroids (actually, of Hopf algebras) but the coaction is not pointwise.

(2) We can generalize the example above as follows. Given integers n and m, let q be a primitive nm^{th} root of unity. Let B be the universal C^*-algebra generated by unitaries U and V satisfying $UV = qVU$) and let A be the C^*-subalgebra generated by commuting unitaries U^n and V^m. Thus, $A \cong C(\mathbb{T}^2)$. Take $\mathcal{H}$ to be the commutative Hopf algebroid $C(G, A)$ over A where $G = \mathbb{Z}/n \times \mathbb{Z}/m$. As a matter of fact, we can construct coverings of $C(\mathbb{T}^2)$ for any finite quotient G of $\mathbb{Z}^2$. We outline this construction in the next section.

Let $\theta = \frac{n}{m} \in \mathbb{Q}$ where n and m are coprime integers. The center of the noncommutative torus $\mathbb{T}^2_\theta$ is the C^*-subalgebra generated by U^m and V^m. The computation above implies that rational noncommutative tori give local coverings of the commutative torus with commutative quantum symmetries. Thus, we get the following proposition.

Proposition 36. *Let $\theta = \frac{n}{m}$ for coprime integers n and m with $m > 0$. Let $\mathbb{T}^2_\theta$ be the non-commutative torus with parameter θ. Then there is a commutative Hopf algebroid $\mathcal{H}$ such that $(\mathbb{T}^2_\theta, \mathcal{H})$ is a covering of $Z(\mathbb{T}^2_\theta) = C(\mathbb{T}^2)$.*

Remark 24. We have an explicit presentation of $\mathbb{T}^2_\theta$ as a bundle over $\mathbb{T}^2$. Consider the following elements of $\mathbb{T}^2_\theta \cong \Gamma(\mathbb{T}^2, M_m(\mathbb{C}))$.

$$U(x,y) = \begin{pmatrix} \exp\left(\frac{2\pi i x}{m}\right) & & & & \\ & \exp\left(\frac{2\pi i(n+x)}{m}\right) & & & \\ & & \exp\left(\frac{2\pi i(2n+x)}{m}\right) & & \\ & & & \ddots & \\ & & & & \exp\left(\frac{2\pi i((m-1)n+x)}{m}\right) \end{pmatrix},$$

$$V(x,y) = \begin{pmatrix} & \exp\left(\frac{2\pi i(n+y)}{m}\right) & & & \\ & & \exp\left(\frac{2\pi i y}{m}\right) & & \\ & & & \ddots & \\ & & & & \exp\left(\frac{2\pi i y}{m}\right) \\ \exp\left(\frac{2\pi i y}{m}\right) & & & & \end{pmatrix}, \qquad x, y \in [0,1].$$

They satisfy the canonical commutation relation relation

$$U(x,y)V(x,y) = e^{2\pi i\theta} V(x,y)U(x,y)$$

for any $x, y \in [0,1]$. Taking m^{th} powers give the toroidal coordinates

$$U(x,y)^m = e^{2\pi i x} I \quad \text{and} \quad V(x,y)^m = e^{2\pi i y} I.$$

6.2 Irrational noncommutative tori

The situation of a rational non-commutative torus is closely related to that of the commutative torus as we saw in the previous section. However, the case for an irrational non-commutative torus is far challenging to describe. If we try to mimic the construction of a local covering in section 6.1, a natural choice for the quantum symmetry is $\mathbb{T}^2_\theta \rtimes G$ but this is in general not a Hopf algebroid over $\mathbb{T}^2_\theta$. The problem is that there are no nice maps $s, t : \mathbb{T}^2_\theta \longrightarrow \mathbb{T}^2_\theta \rtimes G$ with commuting images since $\mathbb{T}^2_\theta$ is centrally simple for θ irrational. In this section, we will construct uniform coverings of $\mathbb{T}^2_\theta$ instead. Before we continue discussing the situation of the irrational non-commutative torus, let us first recall a characterization of finite classical coverings of the commutative torus.

Proposition 37. *Any finite classical covering of the 2-torus $\mathbb{T}^2$ is again a 2-torus. Moreover, the covering map takes the form*

$$\begin{array}{rcl} \mathbb{T}^2 & \xrightarrow{\quad p \quad} & \mathbb{T}^2 \\ (\zeta_1, \zeta_2) & \longmapsto & (\zeta_1^n, \zeta_2^m). \end{array}$$

for some integers n, m.

In the non-commutative framework, there is no obvious reason for a covering of a non-commutative torus to be a non-commutative torus as well. This is easily seen with comparison with the non-commutative point having more than one connected covering space. Let us restrict our attention to coverings of an irrational non-commutative torus $\mathbb{T}^2_\theta$ which are themselves non-commutative tori. We will prove that in that case, the non-commutative tori involved are of the same type.

Definition 17. We say that two irrational numbers θ and η are of the *same type* if $\theta = n + m\eta$ for some integers n, m. $\square$

Let $0 < \theta \in \mathbb{R}$ be an irrational number. Let $\mathbb{T}^2_\theta$ be the universal C^*-algebra generated by unitaries U and V satisfying $UV = e^{2\pi i\theta}VU$. It is well known that $\mathbb{T}^2_\theta$ is centrally simple. The K-theory groups of $\mathbb{T}^2_\theta$ are $K_0(\mathbb{T}^2_\theta) \cong K_1(\mathbb{T}^2_\theta) \cong \mathbb{Z}^2$. More precisely, $K_0(\mathbb{T}^2_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ as an ordered group.

Consider an injective unital C^*-morphism $\mathbb{T}^2_\theta \xrightarrow{j} \mathbb{T}^2_\eta$. There is an induced map $\mathbb{Z} + \theta\mathbb{Z} \xrightarrow{j_*} \mathbb{Z} + \eta\mathbb{Z}$ in K_0, a map of ordered groups. Without loss of generality, we may assume $0 < \theta < 1$. Let $j_*(\theta) = n + m\eta$ for some integers n, m. By unitality of j, we have $j_*(1) = 1$. We claim that θ and η are of the same type. Suppose otherwise. In particular, this implies that $n + m\eta \neq \theta$. Without loss of generality, assume $n + m\eta > \theta$. Then, there is an integer N such that $N\theta < M < N(n + m\eta)$ for some integer M. Thus, $N\theta < M$ and $M < N(n + m\eta)$. This implies that $N\theta < M$ and $j_*(M) < j_*(N\theta)$, which contradicts the fact that j_* is order-preserving. This proves the following proposition.

Proposition 38. *If $(\mathbb{T}^2_\theta, \mathcal{H})$ is a covering of $\mathbb{T}^2_\eta$ then θ and η are of the same type.*

In the classical case it is enough to specify the surjective map defining the covering space. In our framework, one has to specify the symmetry. The next proposition tells us that for non-commutative tori, specifying the inclusion is enough to construct a covering.

Proposition 39. *Given an injective *-homomorphism $\mathbb{T}^2_\theta \xrightarrow{\phi} \mathbb{T}^2_\eta$, up to approximately-unitary equivalence, there is a uniform covering $(\mathbb{T}^2_\eta, H)$ of $\mathbb{T}^2_\theta$ such that the inclusion $\mathbb{T}^2_\theta \subseteq \mathbb{T}^2_\eta$ is given by ϕ.*

Proof: Using theorem 3.2.6 and proposition 3.2.7 in [42], any injective $*$-homomorphism $\mathbb{T}^2_\theta \xrightarrow{\phi} \mathbb{T}^2_\eta$ is approximately unitarily equivalent to an injective $*$-map $\mathbb{T}^2_\theta \xrightarrow{\alpha} \mathbb{T}^2_\eta$ with $K_1\alpha : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$, $(x, y) \mapsto (n_1x + m_1y, n_2x + m_2y)$. In particular, the map

$$\mathbb{T}^2_\theta \xrightarrow{\alpha} \mathbb{T}^2_\eta, \qquad U \mapsto P^{n_1}Q^{m_1}, \qquad V \mapsto P^{n_2}Q^{m_2}$$

does the job. Here, P, Q and U, V are the unitary generators of $\mathbb{T}^2_\eta$ and $\mathbb{T}^2_\theta$, respectively. Let $G = \mathbb{Z}^2 / \langle (n_1, m_1), (n_2, m_2) \rangle$, a group of order $N = n_1m_2 - n_2m_1$. Let $H = C(G)$, the Hopf algebra dual to $\mathbb{C}G$.

Let us show that G acts on $\mathbb{T}^2_\eta$ with invariants $\mathbb{T}^2_\theta$ and hence, H coacts on $\mathbb{T}^2_\eta$ with coinvariants $\mathbb{T}^2_\theta$. Consider a fundamental domain for G. One can for example take the integral region in $\mathbb{Z}^2$ inside the parallelogram with vertices $(0, 0), (n_1, m_1), (n_2, m_2)$ and $(n_1 + n_2, m_1 + m_2)$ including $(0, 0)$. This fundamental region can be identified with the Pontryagin dual $\widehat{G}$ of G. As an $\mathbb{T}^2_\theta$-module, $\mathbb{T}^2_\eta$ is freely generated by elements of the form P^nQ^m where $(n, m) \in \widehat{G}$. Consider the canonical pairing $\langle , \rangle : G \times \widehat{G} \longrightarrow \mathbb{S}^1$. Then G acts on $\mathbb{T}^2_\eta$ by algebra isomorphisms defined for all $(n, m) \in \mathbb{Z}^2$ by

$$(i, j) \cdot P^nQ^m = \langle (i, j), (n, m) \rangle P^nQ^m, \qquad \text{for } (i, j) \in G.$$

Note that an element of $\mathbb{T}^2_\eta$ is invariant with this action precisely when (n, m) is in the integral span of (n_1, m_1) and (n_2, m_2). This shows that the space of invariants is $\mathbb{T}^2_\theta$. This proves our

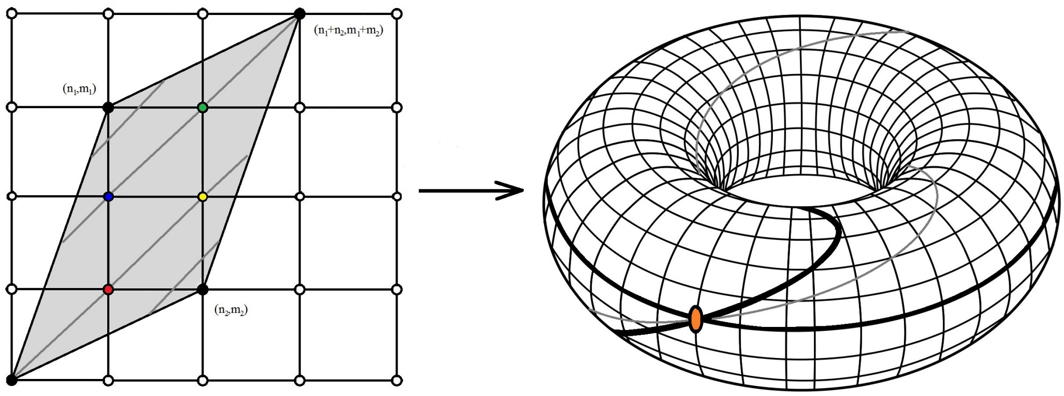


Figure 6.2: Action of G on $\mathbb{T}^2_\eta$

claim. To show that the extension $\mathbb{T}^2_\theta \subseteq \mathbb{T}^2_\eta$ is H-Galois, we have to check that the following linear map

$$\mathbb{T}^2_\eta \underset{\mathbb{T}^2_\theta}{\otimes} \mathbb{T}^2_\eta \longrightarrow \mathbb{T}^2_\eta \otimes \mathbb{C}G$$

is an isomorphism. But this is immediate from the fact that G acts freely and transitively on the $\mathbb{T}^2_\theta$-module generators of $\mathbb{T}^2_\eta$. This gives us a uniform covering $(\mathbb{T}^2_\eta, \mathcal{H})$ of $\mathbb{T}^2_\theta$. ■

Remark 25. Since $UV = e^{2\pi i\theta}VU$, we have $\alpha(U)\alpha(V) = e^{2\pi i\theta}\alpha(V)\alpha(U)$. This implies that $e^{2\pi i(\theta-(n_1m_2-n_2m_1)\eta)} = 1$, and hence $\theta - (n_1m_2 - n_2m_1)\eta \in \mathbb{Z}$. This directly verifies that θ and η are of the same type. At the same time, this gives explicitly the multiplier $N = n_1m_2 - n_2m_1$ witnessing the equivalence of θ and η.

Example 10. Let us construct another covering of $\mathbb{T}^2_\theta$. One which is stratified with stratification $C(S^1) \subseteq \mathbb{T}^2_\theta$. Let $n \in \mathbb{N}$ and let

$$B = \mathbb{T}^2_{\theta/n} = C^*\left\langle U, V | U^*U = UU^* = 1 = V^*V = VV^*, UV = e^{\frac{2\pi i\theta}{n}}VU \right\rangle$$

and let A be the C^*-subalgebra of B generated by U and V^n. Note that $A \cong \mathbb{T}^2_\theta$. Let $A^{'} = C^*\langle U\rangle \subseteq A$. Note that $A^{'} \cong C(S^1)$. Consider the Hopf algebroid $\mathcal{H} = C(G, A^{'})$ where $G = \left\{1, \zeta, \zeta^2, ..., \zeta^{n-1}\right\}$, the group of n^{th} roots of unity. G acts on $\mathbb{T}^2_{\theta/n}$ as follows: $\zeta \cdot U = U$ and $\zeta \cdot V = \zeta V$. This action extends to an action of the Hopf algebra $A^{'}G$ with invariants A. Thus, $\mathcal{H}$ coacts on $\mathbb{T}^2_{\theta/n}$ with coinvariants $\mathbb{T}^2_\theta$. Using similar argument as the previous example, $A \subseteq B$ is an $\mathcal{H}$-Galois extension. This gives us a stratified covering of $\mathcal{T}^2_\theta$ with stratification $A^{'} \cong C(S^1)$. □

Chapter 7

Locality and uniformity of non-commutative covering spaces

It is easier to square a circle
than to get round a mathematician.

–Augustus De Morgan

7.1 Geometric interpretation of stratification of coverings

Let us describe the contrast between local and stratified coverings. We aim to give a geometric intuition behind such stratifications and we will be less precise in doing so. First, note that local coverings can be regarded as a stratified coverings whose stratification is trivial (i.e., stratification by points). However, it will be useful to use local as we shall see soon.

In chapter 6 we have constructed coverings of non-commutative tori with stratifications $A^{'} = A$, $A^{'} = C(S^1)$ and $A^{'} = \mathbb{C}$. Pretending A has points, these stratifications correspond to geometric stratifications illustrated in figure 7.1.

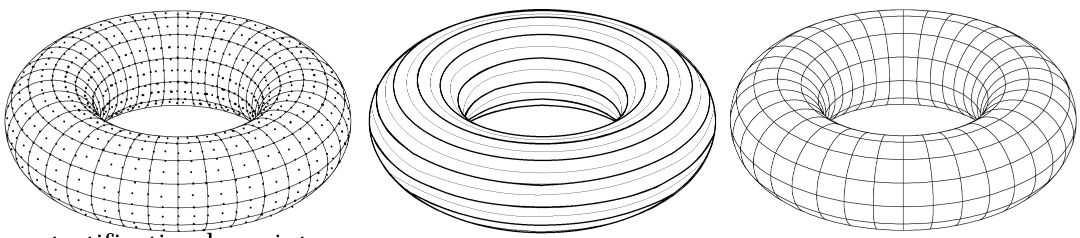


Figure 7.1: Geometric stratifications associated with $A^{'} = A$, $A^{'} = C(S^1)$ and $A^{'} = \mathbb{C}$.

A covering $(B, \mathcal{H})$ of A with stratification $A^{'} \subseteq A$, by definition, has its quantum symmetry defined over $A^{'}$. By the duality between noncommutative spaces and algebras, the inclusion $A^{'} \subseteq A$ induces a surjection $\widehat{A} \twoheadrightarrow \widehat{A^{'}}$. This suggests that the quantum symmetry varies within the leaves of the stratification defined by the fibers of $\widehat{A} \twoheadrightarrow \widehat{A^{'}}$ but the variation is the *same* among the leaves. As a concrete illustration, let us consider coverings of the (commutative) torus $\mathbb{T}^2$ with stratifications $A^{'} = C(\mathbb{T}^2)$, $A^{'} = C(S^1)$ and $A^{'} = \mathbb{C}$. The covering with stratification

$A^{'} = C(\mathbb{T}^2)$ has its quantum symmetry a Hopf algebroid $\mathcal{H}$ defined over the commutative algebra $C(\mathbb{T}^2)$. If $C(\mathbb{T}^2)$ is central in $\mathcal{H}$ then $\mathcal{H}$ is a bundle of complex Hopf algebroids over $\mathbb{T}^2$. These fiber Hopf algebroids need not be isomorphic. This suggest that the quantum symmetry can vary over $A^{'} = C(\mathbb{T}^2)$. For the second case, $A^{'} = C(S^1)$ using the same argument and assumptions imply that $\mathcal{H}$ is a bundle of complex Hopf algebroids over S^1 whose fibers may be nonisomorphic. These fibers Hopf algebroid varries among the fibers of $\mathbb{T}^2 \overset{p}{\twoheadrightarrow} S^1$ which defines the stratification. If $C(S^1)$ is the largest subalgebra of $A = C(\mathbb{T}^2)$ for which $\mathcal{H}$ is defined over then by the Galois condition, $\mathcal{H}$ must be constant along each fibers of p. The third case suggest that we have the same quantum symmetry $\mathcal{H}$ over each point of $\mathbb{T}^2$.

Meanwhile, uniform coverings are a special case of stratified coverings. Aside from $A^{'} = k$ we also require that $\mathcal{H}$ is a Hopf algebra. This in particular requires that the bialgebroid structures to coincide.

Remark 26. The relation between stratifications is not at all clear. It might be tempting to think that a covering $(B, \mathcal{H})$ of A with stratification $A^{'}$ is also a covering with stratification $A^{''}$ where $A^{''} \subseteq A^{'}$ by simply *refining* the stratifications as figure 7.2 suggests.

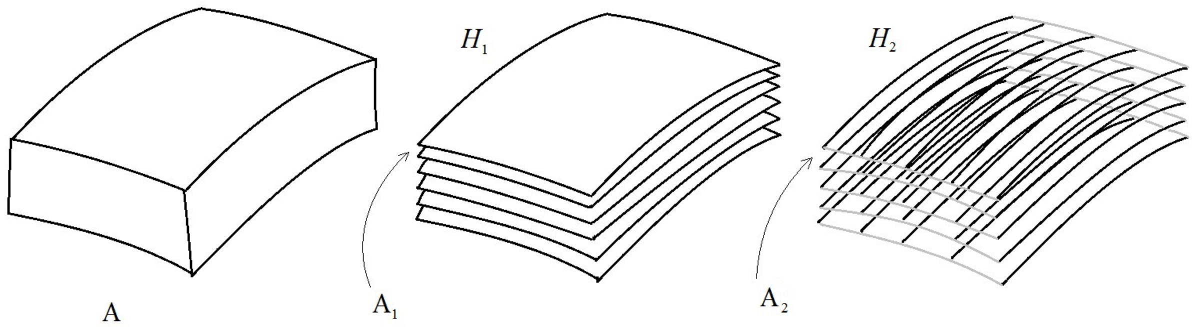


Figure 7.2: Stratifications of the quantum symmetries H_1 and H_2.

This can be done in the commutative case. However, it is not clear in the general case. For one, a Hopf algebroid $\mathcal{H}$ over $A^{'}$ cannot be simply viewed as a Hopf algebroid over $A^{''} \subseteq A^{'}$. The main problem is lifting the coproducts of $\mathcal{H}$ over $A^{'}$ to coproducts over $A^{''}$.

$$\begin{array}{ccc} & H \underset{A''}{\otimes} H & \\ \nearrow & & \searrow \\ H & \xrightarrow{\Delta_L} & H \underset{A'}{\otimes} H \end{array}$$

In the commutative case, one can use decent theoretic methods.

7.2 Uniform coverings

Uniform coverings are, without a doubt, the easiest to handle. For one, the symmetry of the covering is given by Hopf algebras. Though not completely understood, Hopf algebras are far more understood than general Hopf algebroids. In this section, we will look at properties of uniform coverings.

Recall from theorem 5 that a central local covering of $A = C(X)$ is a bundle of coverings of a point. A similar statement for uniform coverings holds.

Proposition 40. *Let (B, H) be a uniform covering of $A = C(X)$ such that A is central in B. Then there is bundle of algebras $E \twoheadrightarrow X$ such that $B \cong \Gamma(X, E)$ as algebras. Moreover, for each $x \in X$, (E_x, H) is a covering of a point.*

The above proposition is hardly a surprise since a uniform covering (B, H) of a commutative space A gives a local covering $(B, A \otimes H)$. Applying theorem 5 to this local covering proves proposition 40. A nice consequence of uniformity is the following proposition.

Proposition 41. *Let (B, H) be a uniform covering of $A = C(X)$ such that A is central in B. The underlying vector bundle of E of proposition 40 is rationally trivial.*

PROOF: Let H_{triv} be the trivial vector bundle over X with typical fiber H. The Galois map $\mathfrak{gal}$ induces the following isomorphism of vector bundles over X.

$$E \otimes_X E \xrightarrow{\cong} E \otimes_X H_{triv}$$

Applying the Chern character map ch on both sides and noting that it is multiplicative, we get $ch(E)^2 = n \cdot ch(E)$. Since $ch(E) \in H^*(X, \mathbb{Q})$ is invertible, we have $ch(E) = n$. Thus, E is rationally trivial. ■

Remark 27. For comparison, let $(B, \mathcal{H})$ be a central local covering of $A = C(X)$, section 4.3 gives bubdles $E \overset{p}{\twoheadrightarrow} X$ and $F \overset{q}{\twoheadrightarrow} X$ associated to B and $\mathcal{H}$, respectively. Using a similar argument as the proof above, we see that $ch(E) = ch(F)$. Thus, E and F are stably equivalent vector bundles.

Theorems 2 and 3 have analogues for uniform covers. We only state the one for 2. The analogue of theorem 3 will be apparent.

Proposition 42. *Let A be a commutative unital C^*-algebra and let (B, H) be a uniform covering of A. Suppose B is commutative. Then H is a commutative Hopf algebra and there is a classical finite Galois covering $Y \overset{p}{\twoheadrightarrow} X$ of compact Hausdorff spaces with $A = C(X)$, $B = C(Y)$ with deck transformation group G such that $H = C(G)$.*

The proof of proposition 42 is similar and much easier than the proof of theorem 2. For one, a commutative Hopf algebra over $\mathbb{C}$ immediately gives a group G for which $H = C(G)$. Recall that we needed lemma 4 for this task.

Chapter 8

Duality

E. Galois (1811-1832) would certainly be surprised to see how often his name is mentioned in the mathematical books and articles of the twentieth century, in topics which are so far from his original work.

–Borceux & Janelidze, [8].

8.1 The OZ-Transform

Apart from the non-commutative point having plenty of non-commutative covers, another interesting phenomenon happens for non-commutative coverings– they come in pairs. The goal of this section is to make this precise and explore its geometric implications.

This requires introducing a handful of intermediary works. Oystaeyen-Zhang [38] illustrated a way to get another Hopf algebra from a Hopf-Galois extension of commutative rings. In [43], upon generalizing the results of Oystaeyen-Zhang to Hopf-Galois extension of non-commutative rings, Schauenburg introduced bi-Galois extensions. The main use of bi-Galois extensions in our present situation is as follows. Given a Hopf algebra H over k; and B, a right H-Galois extension of k, there is a uniquely determined Hopf algebra K for which B is a left K-Galois extension of k. Moreover, the coactions of H and K on B commute making B an K-H-bicomodule algebra. We will call K the *left OZ-transform of* H *relative to the extension* $k \subseteq B$, and denote it as

$$K = OZ^{left}_{[B/k]}(H). \tag{8.1}$$

This construction is involutive, i.e. H is also uniquely determined by the left K-Galois extension $k \subseteq B$. With this, we will denote

$$H = OZ^{right}_{[B/k]}(K), \tag{8.2}$$

and call H the *right OZ-transform of* K *relative to the extension* $k \subseteq B$.

We will look at the case when $A \subseteq B$ a right H-Galois extension where A need not be k. However, we will assume that A is central in B. Denote by ρ the right coaction of H on B defining the Galois extension under consideration.

Proposition 43. *If $A \subseteq B$ is a right H-Galois extension, A central in B and B faithfully flat over A. Then there is a Hopf algebra K over A such that $A \subseteq B$ is a left K-Galois extension. Moreover, the coactions of H and K commute.*

This follows directly from theorem 3.5 of Schauenburg [43]. Changing the base ring k of H into A gives a Hopf algebra $H^{'} = A \otimes H$ over A such that $A \subseteq B$ is a right $H^{'}$-Galois extension. This is due to the following commutative diagram.

$$\begin{array}{ccc} B \otimes_A B & \xrightarrow{\mathfrak{gal}} & B \otimes H \\ \| & & \downarrow \cong \\ B \otimes_A B & \xrightarrow[\mathfrak{gal}^{'}]{} & B \otimes_A (A \otimes H) \end{array}$$

Thus, we get a Hopf algebra $K = OZ^{left}_{[B/A]}(H^{'})$ coacting on the left of B whose subalgebra of coinvariants is A commuting with the right coaction of $H^{'}$. In terms of our language, we have the following proposition.

Proposition 44. *Let (B, H) be a (right) uniform covering of $A = C(X)$ such that A is central in B and B is faithfully flat over A. Then there is a Hopf algebra K over A such that (B, K) is a (left) central local covering of A.*

Notice that theorem 5 applies in this situation giving us a vector bundle $\mathbb{H} \longrightarrow X$ such that the fibers are Hopf algebras. Note that there is no reason for these fiber Hopf algebras to be isomorphic.

8.2 Coupled Hopf algebras and bi-Galois extensions

As we discussed in example 1, in a coupled Hopf algebra, the two constituent Hopf algebras are only related by the coupling map. In this section, we will see that the two Hopf algebras are related by an even stronger relation: one is the cocycle deformation of the other one. In definition 9, a general notion of a cocycle is defined to accommodate cocycle deformations of bialgebroids. Let us discuss explicit what this cocycle deformation means for Hopf algebras.

For the purpose of the the following discussion, assume k is a field. Given an invertible 2-cocycle σ on H, i.e. a convolution invertible element of $Hom(H \otimes H, k)$, we can form another Hopf algebra, denoted by H^{σ}, called the *σ-double deformation* or *σ-double twist* of H. It is the Hopf algebra with the same coalgebra structure as H but the multiplication is given by

$$g \cdot h := \sigma(g_{(1)}, h_{(1)}) g_{(2)} h_{(2)} \sigma^{-1}(g_{(3)}, h_{(3)})$$

for any $g, h \in H$. We can assume σ is normalized, i.e. $\sigma(g, 1) = \sigma(1, g) = 1$. With this, the unit of H^{σ} is the same as the unit of H.

Let us return to the situation that $(B, \mathcal{H})$ is a covering of k and $\mathcal{H}$, as a coupled Hopf algebra, has H_L and H_R as constituent Hopf algebras. Denote by ρ_L and ρ_R the respective coactions of H_L and H_R on B. Since k is a field and H_R is finite-dimensional, theorem 1.9 of [3] implies that any H_R-Galois object is cleft. In particular, B being a (right) H_R-Galois over k, we have $B \cong k\#_{\sigma} H_R$ for some invertible 2-cocycle σ.

Meanwhile, by lemma 4.5.11 of [30] a right H_L-comodule algebra is a left H_L^{cop}-comodule algebra. This implies that B is both a left H_L^{cop}-comodule algebra and a right H_R-comodule algebra. The these two coactions need not commute. As a reminder, the relation they satisfy is given by diagram 2.15. However, if ρ_R equalizes the maps $\rho_L^{cop} \otimes id$ and $(\Finv \otimes id) \circ (id \otimes \Delta_L)$, where ρ_L^{cop} is the co-opposite of the coaction ρ_L, we get a stronger relation between the Hopf algebras H_L and H_R. Before stating this relation, let us first give a name to the condition we just mentioned since we will refer to it again.

Definition 18. Consider a coupled Hopf algebra (H_L, H_R, C) coacting on B via ρ_L and ρ_R. We say that the coaction is *right-twisted* (resp. *left-twisted*) if the following

$$B \xrightarrow{\rho_R} B \otimes H \underset{(Ⅎ\otimes id)\circ(id\otimes\Delta_L)}{\overset{\rho_L^{cop}\otimes id}{\rightrightarrows}} H \otimes B \otimes H$$

commutes, (resp. if the following

$$B \xrightarrow{\rho_L} B \otimes H \underset{(Ⅎ\otimes id)\circ(id\otimes\Delta_R)}{\overset{\rho_R^{cop}\otimes id}{\rightrightarrows}} H \otimes B \otimes H$$

commutes). □

If the Galois coaction of a coupled Hopf algebra $\mathcal{H}$ on B is right-twisted then the coactions ρ_L^{cop} and ρ_R commute. In other words, B is an H_L^{cop}-H_R-bicomodule algebra. Note that B is faithfully k-flat since k is a field. Thus, in the terminology of [47], B is an H_L^{cop}-H_R-bi-Galois object. Using proposition 3.1.6 of [47], $H_L^{cop} = H_R^{\sigma}$, where σ is the same cocycle appearing in the isomorphism $B \cong k\#_{\sigma} H_R$ we mentioned above. The following proposition summarizes what we just argued.

Proposition 45. *Let $(B, \mathcal{H})$ be a covering of a point and let H_L and H_R be the constituent Hopf algebras of $\mathcal{H}$ in view of proposition 13. If the Galois coaction of $\mathcal{H}$ on B is right-twisted, then $H_L^{cop} = H_R^{\sigma}$ as Hopf algebras, for some cocycle σ.*

Remark 28.

(1) In a covering $(B, \mathcal{H})$ of a point, the constituent Hopf algebras of $\mathcal{H}$ play a symmetric role. Thus, one gets a version of proposition 45 by interchanging the indices L and R. However, it is unknown to the best of knowledge of the author how the cocycle one gets from this version relates to that of proposition 45.

(2) The cocycle deformation we described above is dual to Drinfeld twists.

Recall that coupled Hopf algebras H_L and H_R have the same algebra structure. Also, H_L^{cop} and H_L have the same algebra structure. Thus, the cocycle σ one gets in proposition 45 twists the algebra structure of H_R within its isomorphism class. We have the following proposition.

Corollary 5. *Suppose H_L is cocommutative. Then:*

(i) H_R is also cocommutative.

(ii) σ vanishes in the Sweedler cohomology group $H^2(H_R, k)$.

Part (i) is an obvious consequence of proposition 45. Part (ii), making sense in view of part (i), follows from Sweedler's classification of cleft extensions by cocommutative Hopf algebras, see [48].

Let us end this section with a triviality statement for coverings of a point. As we mentioned in section 4.1, unlike the classical case in which a point only has one connected covering– itself, in the present set-up a point has more than one such connected cover. As a specific instance of example 6, Nikshych in [36] constructed a simple Hopf algebra by cocycle-twisting the group algebra of the alternating group $\mathbb{A}_5$. Simple Hopf algebras do not have non-trivial idempotents. However, cleft extensions are the analogues of trivial principal bundles. Using theorem 1.9 of [3] (proved by Kreimer and Cook) we see that if $(B, \mathcal{H})$ is a covering of a point then B is a cleft extension by both H_L and H_R.

8.3 Two-sidedness

Non-commutative coverings spaces possess a two-sided nature in two different but related ways. The first one is the fact that quantum symmetries have two interacting left and right structures (the consituent bialgebroids). One might think that this is because of the use of Hopf algebroids which, as built in their structure, have two such constituent bialgebroids. However, even if we settled with uniform coverings proposition 44 gives another quantum symmetry which illustrates the second way non-commutative coverings shows this two-sided nature. Let us look closer to these two-sidedness nature.

According to lemma 2, the antipode S of a Hopf algebroid $\mathcal{H} = (H_L, H_R, S)$ is an anti-homomorphism $H \xrightarrow{S} H^{op}$ of the underlying k-algebra H of $\mathcal{H}$. As we noted in remark 2 part (7), if S is bijective then H_L determines H_R uniquely, and vice versa. Explicitly, from [5] we have the following proposition.

Proposition 46. *Let H_L be a left L-bialgebroid whose underlying k-algebra is H. If S is an anti-isomorphism of H, then there is at most one right bialgebroid H_R such that (H_L, H_R, S) is a Hopf algebroid over L^{op}.*

The case when there is no such right bialgebroid is when S fails to satisfy conditions (3.5) through (3.8). The above proposition tells us that, in the case of coupled Hopf algebras with bijective coupling map, (H_1, H_2, C) is completely determined by any two among the triple.

In the case of bi-Galois extensions, each Hopf algebra determines the other Hopf algebra uniquely since the OZ-transforms 8.1 and 8.2 are inverses of each other, see for example remark 3.7 of [43]. We should point out that neither bi-Galois extension nor Galois extension of coupled Hopf algebras is much general than the other one. The only thing that relates the two is the condition defined in definition 18. Note that the left regular and the right regular Galois coaction of a coupled Hopf algebra (H_L, H_R, C) already commutes by definition. Thus, the underlying k-algebra H is an H_L-H_R-bi-Galois extension of k. This means that $H_L = H_R^{\sigma}$ for some cocycle σ according to proposition 3.1.6 of [47]. Thus, we have the following proposition.

Proposition 47. *Let $\mathcal{H} = (H_L, H_R, C)$ be a coupled Hopf algebra that admits a right-twisted right Galois coaction. Then H_L is cocommutative.*

To see this, if $\mathcal{H}$ admits a right-twisted right Galois coaction then proposition 45 says that $H_L^{cop} = H_R^{\sigma}$ as Hopf algebras for some cocycle σ. But H itself is an H_L-H_R-bi-Galois extension of k which implies that $H_L = H_R^{\tau}$ for some cocycle τ. Note that cocycle deformation only deforms the algebra structure and not the coalgebra part. Thus, $H_R^{\sigma} = H_R^{\tau}$ as coalgebras which then implies that $H_L^{cop} = H_L$ as coalgebras.

Chapter 9

Other examples of non-commutative covering spaces

It is a curious historical fact that the modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schrödinger and the matrix algebra of Heisenberg. The two apparently dissimilar approaches were proved to be mathematically equivalent.

–Richard Feynman

In chapters 4 to 6 we discussed covering spaces of commutative spaces and non-commutative tori. Let us stretch our collection of examples by considering non-commutative spaces one encounters frequently. Our first example in this chapter concerns matrix algebras. As what section B.2 says, $M_n(\mathbb{C})$ is a covering of $A = \mathbb{C}^n$. But here, we will be interested with coverings of $M_n(\mathbb{C})$ itself. Using similar arguments, we will describe coverings of the algebra $\mathcal{K}(H)$ of compact operators.

9.1 The algebra of compact operators

For this section, let us fix $k = \mathbb{C}$ and let $A = M_n(\mathbb{C})$. Then A is Morita equivalent to $\mathbb{C}$. One can take the $M_n(\mathbb{C})$-$\mathbb{C}$-bimodule $P = \mathbb{C}^n$ and the $\mathbb{C}$-$M_n(\mathbb{C})$-bimodule $Q = \mathbb{C}^n$ as the inverse pairs of bimodules witnessing the equivalence. By proposition 8, the categories $COV_{Morita}(A)$ and $COV_{Morita}(\mathbb{C})$ are adjoint equivalent. The proof of that proposition tells us explicitly how to construct coverings of A from coverings of $\mathbb{C}$, and vice versa. Using this idea, let us characterize local coverings of $M_n(\mathbb{C})$.

Let $(B, \mathcal{H})$ be a covering of $\mathbb{C}$. By proposition 13, $\mathbb{H}$ is a coupled Hopf algebra, i.e. there are two Hopf algebras $H_1 = (H, m, 1, \Delta_1, \epsilon_1, S_1)$ and $H_2 = (H, m, 1, \Delta_2, \epsilon_2, S_2)$ and a coupling map $C : H \longrightarrow H$ satisfying the conditions we enumerated in example 2. Using the proof of 8,

$(B', \mathcal{H}')$ is a covering of A, where $B' = P \otimes_{\mathbb{C}} B \otimes_{\mathbb{C}} Q$ and $\mathcal{H}'$ is the Hopf algebroid over A whose constituent left- and right-bialgebroid structures are

$$\begin{aligned} H_L &= P \otimes_{\mathbb{C}} H \otimes_{\mathbb{C}} Q, & H_R &= P \otimes_{\mathbb{C}} H \otimes_{\mathbb{C}} Q, \\ \Delta_L &= id \otimes_{\mathbb{C}} \Delta_1 \otimes_{\mathbb{C}} id & \Delta_R &= id \otimes_{\mathbb{C}} \Delta_2 \otimes_{\mathbb{C}} id \\ \epsilon_L &= id \otimes_{\mathbb{C}} \epsilon_1 \otimes_{\mathbb{C}} id & \epsilon_R &= id \otimes_{\mathbb{C}} \epsilon_2 \otimes_{\mathbb{C}} id \end{aligned}$$

respectively. The antipode of $\mathcal{H}'$ is $S = id \otimes C \otimes id$. The source and target maps of $\mathcal{H}'$ are the maps determined by $\eta' = id \otimes_{\mathbb{C}} \eta \otimes_{\mathbb{C}} id$ where η is the map $\mathbb{C} \cong \mathbb{C} \otimes \mathbb{C}^{op} \longrightarrow H$ giving the unit $1 \in H$. Similarly, the multiplications on H_L and H_R are the ones induced by the product m of H. In fact, more is true.

Proposition 48. *Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space $\mathbb{H}$. Then $COV_{Morita}(\mathcal{K})$ and $COV_{Morita}(\mathbb{C})$ are equivalent.*

The algebra $\mathcal{K}$ is strongly Morita equivalent to $\mathbb{C}$. The Hilbert space $\mathbb{H}$ viewed as a $\mathbb{C}$-$\mathcal{K}$-Hilbert bimodule and a $\mathcal{K}$-$\mathbb{C}$-Hilbert bimodule gives a pair of inverse equivalence bimodules. By [2], strongly Morita equivalent C^*-algebras are Morita equivalent. The above proposition follows from this and proposition 8.

9.2 Non-commutative sphere

Let us view the 2-torus $\mathbb{T}$ as $S^1 \times S^1$ where S^1 is viewed as a subset of the complex plane. Consider the involution $J : S^1 \times S^1 \longrightarrow S^1 \times S^1$ given by $J(z_1, z_2) = (\overline{z}_1, \overline{z}_2)$. It is not hard to see that the quotient of $\mathbb{T}$ by $\mathbb{Z}_2$ implemented by J is homeomorphic to the 2-sphere. Figure 9.1 illustrates this involution.

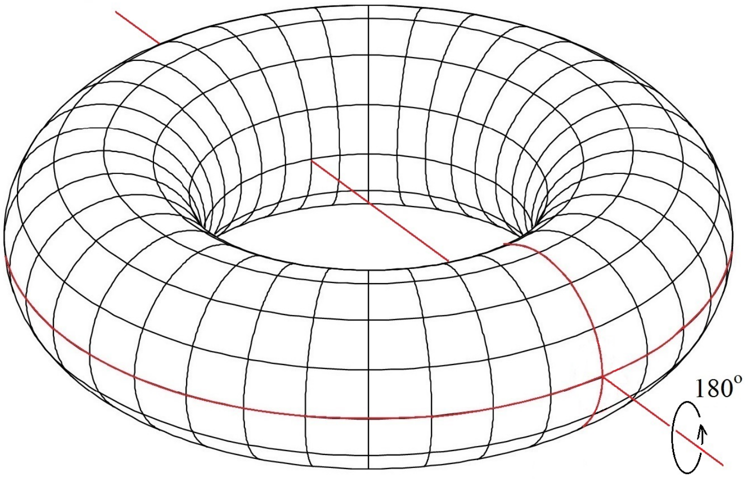


Figure 9.1: The hyperelliptic involution on the torus

Dualizing this construction gives us the non-commutative 2-sphere. For $\theta \in \mathbb{R}$, consider the non-commutative torus $\mathbb{T}^2_\theta$ with unitary toroidal generators U and V, cf. chapter 6. Consider the involution J of $\mathbb{T}^2_\theta$ given by $J(U) = U^*$ and $J(V) = V^*$. In non-commutative geometry, quotients by group actions correspond to crossed products with group actions. Thus, the *non-commutative 2-sphere* S^2_θ is given by $\mathbb{T}^2_\theta \rtimes_J \mathbb{Z}_2$. Equivalently, the non-commutative sphere S^2_θ is the universal C^*-algebra generated by unitaries U, V, and W subject to the relations

$$UV = e^{2\pi i\theta} VU, \qquad WUW = U^*, \qquad WVW = V^*, \quad \text{and} \qquad W^2 = 1$$

Here, W is the unitary of S^2_θ implementing the involution J.

Proposition 49. *Let $H = C(\widehat{\mathbb{Z}_2})$ be the dual group algebra of $\widehat{\mathbb{Z}_2}$ with complex coefficients. Then (S^2_θ, H) is a uniform covering of $\mathbb{T}^2_\theta$.*

Proof: From the identification $S^2_\theta = \mathbb{T}^2_\theta \rtimes_J \mathbb{Z}_2$, we see that S^2_θ carries an action of the dual group $\widehat{\mathbb{Z}_2}$ via a non-degenerate pairing $\langle , \rangle : \mathbb{Z}_2 \times \widehat{\mathbb{Z}_2} \longrightarrow \mathbb{C}$. The invariant subalgebra is $\mathbb{T}^2_\theta$. This action extends to an action of $\mathbb{C}\widehat{\mathbb{Z}_2}$ and hence, a coaction of H with coinvariants $\mathbb{T}^2_\theta$. Since the action of $\mathbb{Z}_2$ is implemented by a unitary involution, the action of $\mathbb{Z}_2$ on S^2_θ is Galois over $\mathbb{T}^2_\theta$. Thus, the coaction of H is Galois as well. This proves the proposition. ■

Remark 29. The classical 2-torus is a branched covering of the classical 2-sphere. Using the hyperelliptic involution described above, this covering has 4 branch points, the 4 intersection points of the red line with the torus in figure 9.1. Meanwhile, there is a reversal of roles in proposition 49, the non-commutative sphere is the covering of the non-commutative torus.

Let θ be rational. Then, from section 6.1, the center of $\mathbb{T}^2_\theta$ is isomorphic to the commutative torus $\mathbb{T}^2_0$. Note that the center $Z(\mathbb{T}^2_\theta)$ is stable under the involution J. Thus, the action of $\mathbb{Z}_2$ restricts to $Z(\mathbb{T}^2_\theta)$. Doing the same construction, we have $S^2_0 = \mathbb{T}^2_0 \rtimes_J \mathbb{Z}_2$ and the following holds.

Proposition 50. *Let $H = C(\widehat{\mathbb{Z}_2})$. Then (S^2_0, H) is a uniform covering of $\mathbb{T}^2_0$ and the diagram*

$$\begin{array}{ccccc}
\mathbb{T}^2_0 & \longrightarrow & S^2_0 & \underset{id\otimes 1}{\overset{\rho^0}{\rightrightarrows}} & S^2_0 \otimes H \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{T}^2_\theta & \longrightarrow & S^2_\theta & \underset{id\otimes 1}{\overset{\rho^\theta}{\rightrightarrows}} & S^2_\theta \otimes H
\end{array}$$

commutes. The right square commutes with appropriate pairing of the top and bottom arrows.

Let us mention some known facts about S^2_θ. One can consult [51] for these properties. The algebra inclusion $\mathbb{T}^2_\theta \longrightarrow S^2_\theta$ makes S^2_θ a free $\mathbb{T}^2_\theta$-module of rank 2, generated by 1 and W. Moreover, there is a unital C^*-embedding $S^2_\theta \xrightarrow{\phi} M_2(\mathbb{T}^2_\theta)$ given as

$$\phi\,(a + bW) = \begin{pmatrix} a & b \\ WbW & WaW \end{pmatrix}, \qquad a, b \in \mathbb{T}^2_\theta.$$

Note that ϕ is well-defined since, by the defining relations of S^2_θ, the subalgebra $\mathbb{T}^2_\theta \subseteq S^2_\theta$ is stable under conjugation by W. Since $\mathbb{T}^2_\theta$ and $M_2(\mathbb{T}^2_\theta)$ are Morita equivalent, they have the same Morita category of (local) covering spaces. The unital C^*-embedding ϕ induces a faithful functor from the category of $M_2(\mathbb{T}^2_\theta)$-bimodules to the category of S^2_θ-bimodules. In particular, any $M_2(\mathbb{T}^2_\theta)$-ring is also an S^2_θ-ring. This gives us the following proposition.

Proposition 51. *There is a faithful functor* $COV_{Morita}(\mathbb{T}^2_\theta) \longrightarrow COV_{Morita}(S^2_\theta)$*.*

Remark 30. Propositions 49 and 51 give a mysterious relationship between the coverings of the non-commutative torus and of the non-commutative sphere. Proposition 51 roughly says that (up to Morita equivalence) local coverings of $\mathbb{T}^2_\theta$ *are* local coverings of S^2_θ. Meanwhile, proposition 49 says S^2_θ covers $\mathbb{T}^2_\theta$. However, this is not enough to conclude that (local) coverings of S^2_θ are (local) coverings of $\mathbb{T}^2_\theta$. There are two main difficulties: the first one is the fact that it is not clear how local coverings and uniform coverings are related; secondly, transitivity of non-commutative coverings is not as simple as transitivity of classical coverings. To be precise, if $Z \xrightarrow{q} Y$ and $Y \xrightarrow{p} X$ are classical coverings then obviously $Z \xrightarrow{pq} X$ is a classical covering as well. As we have seen in section 3.3, composition of non-commutative coverings is much complicated than its classical counterpart.

Chapter 10

The category of non-commutative covering spaces

Mathematicians seem to have
no difficulty in creating
new concepts faster than
the old ones become well-understood.

–Edward Norton Lorenz

It is perhaps misleading that the title of this chapter is '*the* category of non-commutative covering spaces' as we shall see that, unlike the classical case where there is only *one* naturally occurring category of covering spaces, a lot of things happen in the non-commutative framework. A philosophical way to see this is to note that there might be several notions in non-commutative geometry that collapses into the same notion in the classical geometry. For example as we have pointed out in section 3.3, the commutative diagram 3.9 of classical covering spaces has three different interpretations in the present set-up. In fact, this is the root why there are several equally *natural* category of covering spaces. We will introduce them here.

10.1 The naive category of covering spaces

The first category of covering spaces that we will deal with is called the *naive category*, denoted as $COV_{naive}(A)$ for a non-commutative space A. Its objects are local coverings of A. An arrow from $(B^1, \mathcal{H}^1)$ to $(B^2, \mathcal{H}^2)$ is a pair (f, α) where $B^1 \xrightarrow{f} B^2$ is a unital inclusion of algebras and α is an monomorphism $\mathcal{H}^1 \xrightarrow{\alpha} \mathcal{H}^2$ of Hopf algebroids making $(B^1, \mathcal{H}^1)$ an intermediate covering of $(B^2, \mathcal{H}^2)$. Note that unitality of f implies that it is a morphism of A-rings. Proposition 7 tells us what are the equivalent objects in $COV_{naive}(A)$.

Proposition 52. *The category $COV_{naive}(A)$ always has an initial object for any A.*

Proof: The pair (A^e, A^e) is a local covering of A. The pair of maps $(\iota, s_R \otimes t_R)$ where ι is the map that send A to the factor A of A^e and s_R, t_R are the right source and right target maps provide the unique arrow $(A, A) \longrightarrow (B, \mathcal{H})$ for any local covering $(B, \mathcal{H})$ of A. If A is non-commutative then (A^e, A^e) is the initial object of $COV_{naive}(A)$. If A is commutative then (A, A) is a local covering of A which admits a unique arrow $(A, A) \longrightarrow (A^e, A^e)$. This shows that (A, A) is the initial object of $COV_{naive}(A)$. ■

Remark 31. Note that the category $COV_{naive}(A)$ is never pointed. The reason being that the first map in a pair (f, α) defining an arrow in $COV_{naive}(A)$ is required to be an algebra inclusion.

The category $COV_{naive}(k)$, apart from being the most trivial example one can write, plays an important role in the structure on non-commutative coverings of general algebras A. Before we continue on this result let us first state a lemma.

Lemma 6. *Let A be a k-algebra and let $\mathcal{H}^1$ and $\mathcal{H}$ be Hopf algebroids over A and k, respectively. Then the tensor product $H^1 \otimes H$ is a Hopf algebroid over A, where H^1 and H are the underlying k-algebras of $\mathcal{H}^1$ and $\mathcal{H}$.*

PROOF: H^1 carries several bimodule structures in relation to being an $A \otimes A^{op}$-ring and an A-coring. These bimodule structures transfer to $H^1 \otimes H$. For example, an A-bimodule structure of H^1 transfers to $H^1 \otimes H$ via

$$a \cdot \left(h^1 \otimes h\right) \cdot a' = \left(a \cdot h^1 \cdot a'\right) \otimes h, \qquad h^1 \in H^1, h \in H, a, a' \in A.$$

Let us consider the left-bialgebroid structures of $\mathcal{H}^1$ and $\mathcal{H}$. With the transference method described above applied to the underlying A^e-bimodule structure of H^1 being an A^e-ring, we have an isomorphism of A^e-bimodules

$$\left(H^1 \otimes H\right) \otimes_{A^e} \left(H^1 \otimes H\right) \cong \left(H^1 \otimes_{A^e} H^1\right) \otimes (H \otimes H).$$

Composing with $\left(H^1 \otimes_{A^e} H^1\right) \otimes (H \otimes H) \longrightarrow H^1 \otimes H$, the tensor product of the multiplication maps of H^1 and H gives a map

$$\left(H^1 \otimes H\right) \otimes_{A^e} \left(H^1 \otimes H\right) \longrightarrow H^1 \otimes H.$$

Together with the A^e-bimodule map $A^e \xrightarrow{\eta} H^1 \longrightarrow H^1 \otimes H$, $h \longmapsto h \otimes 1$, the product $H^1 \otimes H$ becomes an A^e-ring. Using similar arguments, the A-coring structure of H^1 and the k-coring structure of H makes $H^1 \otimes H$ an A-coring. For the right-bialgebroid structures of H^1 and H we do the same arguments. If S_{H^1} and S_H are the antipodes of H^1 and H then $S = S_{H^1} \otimes S_H$ is the antipode for $H^1 \otimes H$. This proves the lemma. ■

Proposition 53. *Let k be a field. A covering $(B, \mathcal{H}) \in COV_{naive}(k)$ induces an endofunctor of $COV_{naive}(A)$, for any k-algebra A.*

PROOF: Let $(B^1, \mathcal{H}^1) \in COV_{naive}(A)$. We claim that $(B^1 \otimes B, \mathcal{H}^1 \otimes \mathcal{H}) \in COV_{naive}(A)$. Here, $\mathcal{H}^1 \otimes \mathcal{H}$ is the Hopf algebroid structure asserted by lemma 6. First, we need to show that $\mathcal{H}^1 \otimes \mathcal{H}$ coacts on $B^1 \otimes B$ whose coinvariants is A. Note that specifying such a (right) coaction means specifying a (right) coaction of the constituent left- and the right-bialgebroids of $\mathcal{H}^1 \otimes \mathcal{H}$, as discussed in definition 4. We will only construct the one for the left bialgebroid. The one for the right-bialgebroid is constructed similarly. Denoted by ρ_L^1 and ρ_L the right coactions of H_L^1 and H_L, respectively. Then the following composite defines a right coaction of $H_L^1 \otimes H_L$ on $B^1 \otimes B$.

$$B^1 \otimes B \xrightarrow{\rho_L^1 \otimes \rho_L} B^1 \otimes_A H^1 \otimes B \otimes H \xrightarrow{id \otimes \daleth \otimes id} \left(B^1 \otimes B\right) \otimes_A \left(H^1 \otimes H\right).$$

Denote this coaction by α_L. Denote by α_R the coaction one gets by using the right coactions of the right-bialgebroids of $\mathcal{H}^1$ and $\mathcal{H}$. Note that $id \otimes \daleth \otimes id$ makes sense by the transference method we described in the proof of lemma 6. Let us show that $\left(B^1 \otimes B\right)^{co\ (H_L^1 \otimes H_L)} = A$. Let $\sum b^1 \otimes b^2 \in B^1 \otimes B$ such that $\alpha_L \left(\sum b^1 \otimes b^2\right) = \left(\sum b^1 \otimes b^2\right) \otimes_A (1 \otimes 1)$. Then

$$\left(\sum b^1 \otimes b^2\right) \otimes_A (1 \otimes 1) = \alpha_L \left(\sum b^1 \otimes b^2\right) = \sum b^1_{(0)} \otimes b^2_{(0)} \otimes_A b^1_{(1)} \otimes b^2_{(1)}. \qquad (10.1)$$

Since k is a field, we can assume that the $b^2_{(1)}$'s are linearly independent, which then gives $b^2_{(1)} = 1$. Thus $b^2_{(0)} = b^2 \in k$. Thus, up to scalars, we have

$$\sum b^1 \otimes b^2 = \left(\sum b^1\right) \otimes 1 = b \otimes 1$$

where $b = \sum b^1$. Equation 10.1 implies that

$$b \otimes 1 \otimes_A 1 \otimes 1 = \alpha_L(b \otimes 1) = \sum b^1_{(0)} \otimes 1 \otimes_A b^1_{(1)} \otimes 1$$

Assuming the $b^1_{(1)}$'s are linearly independent, we get $b^1_{(1)} = 1$ and hence $b^1_{(0)} = b^1$. This shows that $\rho^1_L(b^1) = b^1 \otimes_A 1$. Thus, $b^1 \in A$. Doing the same for α_R proves that $\left(B^1 \otimes B\right)^{co\ (H_L^1 \otimes H_L)} = A$.

Denote by $- \otimes (B, \mathcal{H})$ the endofunctor induced by $(B, \mathcal{H})$, i.e the endofunctor that sends $(B^1, \mathcal{H}^1)$ to $(B^1 \otimes B, \mathcal{H}^1 \otimes \mathcal{H})$. Then, it is easy to see that $- \otimes (B, \mathcal{H})$ respects arrows in $COV_{naive}(A)$. This finishes the proof of the above proposition. ■

10.2 Morita category of covering spaces

Morita equivalence is a prevalent equivalence in non-commutative geometry. In section 3.2, we defined Morita equivalence of (local) coverings, see definition 14. In doing so, we came up with a category whose isomorphisms are the said Morita equivalences, see remark 11(2). Given a non-commutative space A, recall that the *Morita category* of covering spaces of A, denote by $COV_{Morita}(A)$, is the category whose objects are local coverings of A. An arrow from $(B, \mathcal{H})$ to $(B', \mathcal{H}')$ is a pair $(\mathcal{X}, \mathcal{U})$ of a (B, B')-bimodule $\mathcal{X}$ and an $(\mathcal{H}, \mathcal{H}')$-Hopf bimodule $\mathcal{U}$. Composition is explained in the same remark.

In its original form, Morita equivalence of rings are equivalence of their module categories. The existence of equivalence bimodules is a consequence of Watt's theorem. However, in defining Morita equivalence of coverings we went directly into asserting bimodules and Hopf bimodules exist, witnessing the Morita equivalence of the extension algebras and the Hopf-adapted Morita equivalence of the associated quantum symmetries. Our goal in this chapter is to better understand this equivalence in hopes of understanding the category $COV_{Morita}(A)$ better. In particular, we will look at what this equivalence do for classical covering spaces.

Let $Y \overset{p}{\twoheadrightarrow} X$ and $Z \overset{q}{\twoheadrightarrow} X$ be classical covering spaces. Denote by $\mathcal{Y}$ and $\mathcal{Z}$ the associated reduced groupoids to the coverings p and q, respectively (see definition 10). By section 3.1, $(C(Y), C(\mathcal{Y}))$ and $(C(Z), C(\mathcal{Z}))$ are local coverings of $A = C(X)$. For simplicity, let us denote by $B_Y = C(Y)$, $B_Z = C(Z)$, and $\mathcal{H}_Y$ and $\mathcal{H}_Z$ the Hopf algebroids $C(\mathcal{Y})$ and $C(\mathcal{Z})$, respectively. Suppose that the coverings $(B_Y, \mathcal{H}_Y)$ and $(B_Z, \mathcal{H}_Z)$ are Morita equivalent. In particular, the A-rings B_Y and B_Z are Morita equivalent. Thus, there is a B_Y-B_Z-bimodule P and a B_Z-B_Y-bimodule Q witnessing the categorical equivalence between the module categories of B_Y and B_Z.

The constituent left and right module structures of P and Q are finitely-generated and projective. By Serre-Swan theorem, there is a finite-rank vector bundle $E \twoheadrightarrow Y$ such that $P \cong \Gamma(Y, E)$ as left B_Y-modules. But P is also has a right B_Z-action commuting with the left B_Y-action. Thus, B_Z acts on the right of $\Gamma(Y, E)$ and this action is pointwise. In particular, B_Z acts on the fiber E_y, for each $y \in Y$. Since B_Z is a commutative $*$-algebra, elements of B_Z

act diagonally on E_y. Moreover, for each $y \in Y$, E_y decomposes into joint eigenspaces for these diagonal operators. Let $y \in Y$ be fixed. Then there is an algebra map $B_Z \xrightarrow{\Phi} End(E_y)$. With a choice of a local trivializing neighborhood of $y \in Y$ and an orthonormal basis of the typical fiber over such trivializing neighborhood, Φ factors through $Diag_n(\mathbb{C})$, i.e.

$$\begin{array}{ccc} B_Z & \xrightarrow{\quad\Phi\quad} & M_n(\mathbb{C}) \\ & {\scriptstyle\Phi}\searrow \quad \nearrow & \\ & Diag_n(\mathbb{C}) & \end{array}$$

The algebra $Diag_n(\mathbb{C})$ determines n central orthogonal idempotents. The composition of Φ with the projection onto the ranges of these idempotents give n homomorphisms $B_Z \xrightarrow{\phi_i} \mathbb{C}$. Each such homomorphism gives an element $z_i \in Z$ such that $\phi_i(f) = f(z_i)$, for $f \in B_Z$ and $i = 1, 2, ..., n$. Note that the z_i's depend on y and this dependence is continuous. Similar to our argument from section 5.1, we have a subset $W \subseteq Y \times Z$. Let us denote by $E_{(y,z_i)}$ the eigenspace one gets from the decomposition we described above of the fiber E_y associated to the point $z_i \in Z$, for $i = 1, 2, ..., n$. This gives us a sheaf $\mathcal{E}$ of complex vector spaces over $Y \times Z$ supported on W. As a B_Y-B_Z-bimodule, $P \cong \Gamma(Y \times Z, \mathcal{E})$ where the B_Y-B_Z-bimodule structure on $\Gamma(Y \times Z, \mathcal{E})$ is given as

$$(f \cdot \sigma \cdot g)\,(y, z) = f(y)\sigma(y, z)g(z), \quad \text{for} \;\; y \in Y, z \in Z.$$

Doing the same argument for Q we see that as a B_Z-B_Y-bimodule, $Q \cong \Gamma(Z \times Y, \mathcal{F})$ for some sheaf $\mathcal{F}$ of complex vector spaces supported on a subset $W' \subseteq Z \times Y$. Since $P \otimes_{B_Z} Q \cong B_Y$ as B_Y-bimodules, the fiber product $W \times_Z W'$ must be the diagonal in $Y \times Y$. Similarly, the isomorphism $Q \otimes_{B_Y} P \cong B_Z$ of B_Z-bimodules implies that the fiber product $W' \times_Y W$ must be the diagonal in $Z \times Z$. This can only happen if $\Finv(W) = W'$. In fact, more is true. The sheaf $\mathcal{F}$ is the pull-back of the sheaf $\mathcal{E}$ along the flip map $\Finv : Z \times Y \longrightarrow Y \times Z$. Keeping the notation of this section, we have proven the following proposition.

Proposition 54. *If the commutative A-rings $B_Y = C(Y)$ and $B_Z = C(Z)$ are Morita equivalent, then there is a sheaf $\mathcal{E}$ of complex vector spaces over $Y \times Z$ such that the B_Y-B_Z-bimodule $\Gamma(Y \times Z, \mathcal{E})$ witnesses this equivalence. The inverse bimodule is given by $\Gamma(Z \times Y, \mathcal{E}^T)$ where $\mathcal{E}^T$ denotes the pull-back of $\mathcal{E}$ along the flip map $Z \times Y \longrightarrow Y \times Z$.*

Actually, we can say something stronger. We have isomorphisms of B_Y-bimodules

$$P \otimes_{B_Z} Q \cong \Gamma(Y \times Z, \mathcal{E}) \otimes_{C(Z)} \Gamma(Z \times Y, \mathcal{E}^T) \cong \Gamma(Y \times Y, \mathcal{E} \otimes_Z \mathcal{E}^T)$$

where $\mathcal{E} \otimes_Z \mathcal{E}^T$ is the sheaf of complex vector spaces over $Y \times Y$ whose fiber at a point $(a, b) \in Y \times Y$ is given by

$$\left(\mathcal{E} \otimes_Z \mathcal{E}^T\right)_{(a,b)} = \bigoplus_{z \in Z} E_{(a,z)} \otimes E_{(z,b)}. \tag{10.2}$$

But $P \otimes_{B_Z} Q \cong B_Y$ as B_Y-bimodules. We can consider B_Y as the global section of the sheaf of complex vector spaces over $Y \times Y$ supported along the diagonal and whose fiber at $(y, y) \in Y \times Y$ is the one-dimensional vector space $\mathbb{C}$. With these, we see that $\left(\mathcal{E} \otimes_Z \mathcal{E}^T\right)_{(a,b)} = \mathbb{C}$ for $a = b$. Otherwise, it is the zero vector space. This implies that the direct sum in equation 10.2 contains only one non-zero summand, a tensor product of two one-dimensional complex vector spaces. Thus, we have the following proposition.

Proposition 55. *The sheaf $\mathcal{E}$ of proposition 54 is supported on a tranverse subset $W \subseteq Y \times Z$, i.e. a subset such that $\pi_1|_W : Y \times Z \longrightarrow Y$ and $\pi_2|_W : Y \times Z \longrightarrow Z$ are homeomorphisms. In particular, Y and Z are homeomorphic.*

In the above proposition, π_1 and π_2 denote the projection onto the first and second factor, respectively. From the above proposition, we get the familiar fact that Morita equivalent commutative rings are isomorphic as rings. With the assumptions of this section, we deduce the following.

Corollary 6. *The classical coverings $Y \overset{p}{\twoheadrightarrow} X$ and $Z \overset{q}{\twoheadrightarrow} X$ are equivalent if and only if the coverings $(B_Y, \mathcal{H}_Y)$ and $(B_Z, \mathcal{H}_Z)$ they determine are Morita equivalent.*

Note that in the above discussion, we did not make use of the Hopf algebroids $\mathcal{H}_Y$ and $\mathcal{H}_Z$. This is because the groupoids $\mathcal{Y}$ and $\mathcal{Z}$ are completely determined by the coverings p and q, which in turn, determine the Hopf algebroids in question.

In the above discussion, we assumed that $(B_Y, \mathcal{H}_Y)$ and $(B_Z, \mathcal{H}_Z)$ are Morita equivalent. Using what we have done so far, let us see the case when there is an arrow from $(B_Y, \mathcal{H}_Y)$ to $(B_Z, \mathcal{H}_Z)$ in the category $COV_{Morita}(A)$ that is not necessarily an isomorphism. Take this arrow to be (P, U). By definition,

$$B_Y \underset{B_Y}{\otimes} P \cong B_Z, \qquad \mathcal{H}_Y \underset{\mathcal{H}_Y}{\otimes} U \cong \mathcal{H}_Z$$

as an (A, B_Z)-bimodule and an $(A^e, \mathcal{H}_Z)$-Hopf bimodule, respectively. Just like what we have before, $P \cong \Gamma(Y \times Z, \mathcal{E})$ as B_Y-B_Z-bimodules, for some sheaf $\mathcal{E}$ of complex vector spaces. Likewise, we have $B_Y \cong \Gamma(X \times Y, \mathcal{L})$ as A-B_Y-bimodule, where $\mathcal{L}$ is a sheaf of complex vector spaces. As for B_Z, it is isomorphic to $\Gamma(X \times Z, \mathcal{N})$ as a A-B_Z-bimodule where, similarly, $\mathcal{N}$ is a sheaf of complex vector spaces over $X \times Z$. Then $B_Y \otimes_{B_Y} P \cong \Gamma(X \times Z, \mathcal{L} \otimes_Y \mathcal{E})$ as A-B_Z-bimodules, where $\mathcal{L} \otimes_Y \mathcal{E}$ is the sheaf of complex vector spaces over $X \times Z$, whose fiber at a point $(a, b) \in X \times Z$ is

$$(\mathcal{L} \otimes_Y \mathcal{E})_{(a,b)} = \bigoplus_{y \in Y} L_{(a,y)} \otimes E_{(y,b)}.$$

Here, $L_{(a,y)}$ denotes the fiber of $\mathcal{L}$ at $(a, y) \in X \times Y$. Thus, we have the following proposition.

Proposition 56. *If (P, U) is an arrow from $(B_Y, \mathcal{H}_Y)$ to $(B_Z, \mathcal{H}_Z)$ then there are sheaves $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{N}$ over $X \times Y$, $Y \times Z$ and $X \times Z$, whose global sections are B_Y, P, and B_Z, respectively. Moreover, we have $\mathcal{L} \otimes_Y \mathcal{E} \cong \mathcal{N}$.*

The discussions we did so far only concern the extension algebras and their Morita equivances and Morita arrows. Propositions 54 and 55 can be applied to the underlying A-ring structures of the Hopf algebroids we have in these section, proving that the groupoids $\mathcal{Y}$ and $\mathcal{Z}$ are isomorphic. However, this already follows from the fact that the coverings $Y \overset{p}{\twoheadrightarrow} X$ and $Z \overset{q}{\twoheadrightarrow} X$ are equivalent.

10.3 Towards étale fundamental groupoid of non-commutative spaces

In algebraic geometry, the étale fundamental group $\pi_1^{et}(X)$ of a scheme X is defined as inductive limit

$$\pi_1^{et}(X) := \varprojlim Gal(k(X^{'})/k(X))$$

where $X^{'}$ runs among étale coverings of X such that the field of fractions $k(X^{'})$ of the coordinate ring of $X^{'}$ is a Galois extension of the field of fractions of the coordinate ring of X. Let us call such coverings $X^{'}$ *Galois*. The reason such an inductive limit make sense is because the collection of Galois coverings of X forms a cofiltered category. In particular, pullbacks of étale coverings is again étale.

In our case, pullbacks of non-commutative coverings of A are in general, not finite over A. See for example [39]. The categories $COV_{naive}(A)$ and $COV_{Morita}(A)$ seem to have very few useful properties. In particular, it is not clear whether such categories are close to being a fusion category even for commutative A. With this, we propose the following definitions.

Pseudo-definition. Let A be a k-algebra. Define

(a) the *naive étale fundamental groupoid* of A to be $COV_{naive}(A)$,

(b) the *Morita étale fundamental groupoid* of A to be $COV_{Morita}(A)$, and

(c) the *étale fundamental group* of A to be the category $COV_{uni}(A)$ whose objects are uniform coverings of A. An arrow from (B^1, H^1) to (B^2, H^2) is a pair (f, α) where $B^1 \xrightarrow{f} B^2$ is a monomorphism of A-rings and $H^1 \xrightarrow{\alpha} H^2$ is a monomorphism of Hopf algebras. □

If one wants to associate a Hopf-like structure to A as its étale fundamental group(oid), one should ask how close do the categories $COV_{naive}(A)$ and $COV_{Morita}(A)$, if they are not, from being fusion categories? In this case, one of the author's future project is to study these categories and see how much quantum information they encode for some interesting non-commutative space A. In the next chapter, we will explore several possibilities of future work.

Chapter 11

Future directions

These [cosmos and quantum] are the parts of the intellectual map where we're still groping for the truth– where, in the fashion of ancient cartographers, we must still inscribe 'here be dragons'.

–Martin Rees

In this chapter, we will look at problems the author is planning to work on in relation with this thesis. The first chapter will lay down concrete problems that are supplementary in understanding non-commutative covering spaces. Most of the latter sections will be speculative in nature.

11.1 "Basic" problems that are left unsolved

The main goal of this thesis is to put forth a notion generalizing covering spaces to non-commutative geometry. Along with that goal, we investigated which "basic" properties of classical covering spaces generalize well to the non-commutative set-up and which do not. Results like theorem 5 and proposition 11 are in line with this. However, it is no secret that there are a lot of "basic" questions that are left unanswered. Let us mention them in this section.

In example 2 of section 2.2, we defined and discussed coupled Hopf algebras and illustrated how a construction of Connes and Moscovici [14] lead to an example. Ehud Meir, through a personal discussion during the conference *Topological Quantum Groups and Hopf Algebras* in Warsaw, believes that the only example of such coupled Hopf algebras are the ones given by the Connes-Moscovici construction as discussed in example 2.

Question 1. *Given a coupled Hopf algebra (H_1, H_2, C), is H_2 the ϵ_2-twist of H_1 as described in example 2?*

In the classical situation, finite connected coverings of a torus are themselves torus. Since a (non-commutative) point has plenty of connected covers besides itself, there is no reason to expect that connected (finite) coverings of a non-commutative torus are also non-commutative tori. This begs the following question.

Question 2. *How far is a connected (finite) covering of a non-commutative torus from being a non-commutative torus?*

In algebraic geometry, the étale fundamental group of a curve X is defined to be the limit of the projective system of Galois groups one associates to étale covers $X^{'}$ of X, i.e.

$$\pi_{et}(X) := \varprojlim Gal(k(X^{'})/k(X))$$

where $k(C)$ denotes the function field of C. In the non-commutative framework, one would be interested to have an inductive system of coverings of a non-commutative space A. In particular, one is interested to have a transitive oredering on the collection of coverings of A such that any finite subcollection is bounded above. The transitive ordering for local and uniform coverings is provided by section 3.3. For the upper bound, one asks:

Question 3. *Given local (resp. uniform) coverings* $(B_1, \mathcal{H}_1)$ *and* $(B_2, \mathcal{H}_2)$ *of* A*, can we find a local (resp. uniform) covering* $(B_3, \mathcal{H}_3)$ *of* A *which covers* B_1 *and* B_2*?*

In section 9.2 we discussed the non-commutative sphere. Classically, spheres of dimension greater than 2 have no non-trivial classical covering spaces. In the present set-up, it is reasonable to expect that the non-commutative sphere has non-trivial coverings. This is because, eben highly connected spaces like a point has non-trivial coverings. Thus, it is reasonable to ask the following question.

Question 4. *Let* θ *be an irrational number. Does* S^2_θ *have non-trivial connected coverings?*

Notice that for θ rational, we already have immediate answers. For example, if $\theta = 0$, one can consider a connected non-trivial covering (B, H) of a point. Then $(S^2_0 \otimes B, S^2_0 \otimes H)$ is a connected non-trivial covering of S^2_0.

In appendix B, we defined topological Hopf categories as Hopf categories with associated sheaves of vectors spaces, see definition 21. We used such topological (coupled) Hopf categories to describe the general structure of non-central local coverings of a commutative space. A natural question to ponder is the following.

Question 5. *Is the sheaf* H^{sh} *uniquely determined by* $\mathscr{H}$ *and the topology on* X*?*

Another interesting task to do is to see how the OZ-transform behaves for Hopf algebroids. If the OZ-transform extends to Hopf algebroids then this will complete the picture we painted in chapter 8. Formally, we ask:

Question 6. *Does* $OZ^{left}_{[B/A]}(-)$ *make sense for Hopf algebroids? If so, is it also involutive?*

11.2 Compact quantum groups

Classically, any covering $G^{'}$ of a topological group G is itself, a group. The group structure on $G^{'}$ is unique up to a choice of the identity element among those points in the fiber of the identity e of G. It is then a natural question to ask, whether this is still true in the non-commutative set-up. To be precise, we have the following.

Question 7. *Let* A *be a compact quantum group and let* (B, H) *be a uniform covering of* A*. Is* B *necessarily a compact quantum group?*

The most basic case in this direction is when A is a group C^*-algebra of some locally compact group G. It seems that B need not be a group C^*-algebra itself. However, in case it is, we have the following conjecture.

Conjecture 1. *Let* $A = C^*(G)$*. Let* (B, H) *be a uniform covering of* A *where* $B = C^*(G^{'})$ *for some locally compact group* $G^{'}$*. Then* H *is the complex group algebra of some finite group* Γ*.*

Note that one difficulty in the above conjecture is the fact that an inclusion $C^*(G) \longrightarrow C^*(G^{'})$, if it will say something about the group G and $G^{'}$, it is far from a statement saying $G^{'}$ is a covering of G. Hence, machinery of classical covering spaces is not readily available.

11.3 Spectral triples

Another nice property of classical covering spaces is as follows. If (X, g) is a Riemannian manifold and Y is a smooth manifold covering X via the map $Y \xrightarrow{p} X$, then there is a unique Riemannian metric $g^{'}$ on Y making p a local isometry.

In non-commutative geometry, the analogue of Riemannian manifolds are spectral triples. A *spectral triple* $(A, \mathbb{H}, D)$ consists of a C^*-algebra A with a representation $\pi : A \longrightarrow \mathcal{B}(\mathbb{H})$, where $\mathbb{H}$ is a separable Hilbert space, and an operator D on $\mathbb{H}$ satisfying the following conditions:

(a) D, called the *Dirac operator*, is self-adjoint,

(b) the *resolvent* $(D-\lambda)^{-1}$ is a compact operator for any $\lambda \notin \mathbb{R}$, and

(c) $[D, \pi(a)] \in \mathcal{B}(\mathbb{H})$ for all a in a dense involutive subalgebra of A.

With all this, we have the following question.

Question 8. *Given a spectral triple $(A, \mathbb{H}, D)$ and a uniform covering (B, H) of A, can we equip B with a structure of a spectral triple? If so, how then is H related to the rest of the structures involved?*

Let us call the spectral triple structure on B answering question 8 a *lift* of $(A, \mathbb{H}, D)$. As communicated to me by Olivier Gabriel, the answer to the first question seems to be affirmative but is far from unique. If so, is it possible to classify all such lifts?

11.4 Deformation quantization of covering spaces

...but when these two
sciences [algebra and geometry] have been
united, they have lent each mutual forces,
and have marched together towards perfection.

–Joseph-Louis Lagrange

An on-going project is to produce examples of non-commutative covering spaces via deformation quantization. Let us describe this briefly. Given a symplectic manifold (M, ω) and a finite classical Galois covering $N \xrightarrow{p} M$, there is a unique symplectic structure on N making p a symplectomorphism. This symplectic form is precisely $\omega^{'} = p^*\omega$.

By a *formal deformation quantization* of M, we mean an associative, unital $\mathbb{C}[\![\hbar]\!]$-linear product $\star$ on $C^\infty(M)[\![\hbar]\!]$ of the form

$$f \star g = fg + \sum_{j=1}^{\infty} (i\hbar)^j D_j(f, g)$$

for all $f, g \in C^\infty(M)$, where the D_j, $j \in \mathbb{N}$ are bidifferential operators on $C^\infty(M)$. The product $\star$ is called a *star product.* In Nest-Tsygan [35], formal deformation quantizations and hence, star products have been classified. In particular, formal deformation quantization of (M, ω) is classified by the affine space $\frac{\omega}{i\hbar} + H^2(M)[\![\hbar]\!]$. Now, to give a deformation quantization of M one only needs to specify a class $\theta \in \frac{\omega}{i\hbar} + H^2(M)[\![\hbar]\!]$. Pulling-back via p, we have a class $p^*\theta \in \frac{p^*\omega}{i\hbar} + H^2(N)[\![\hbar]\!]$ which defines a formal deformation quantization of N. With these, we have the following questions.

Question 9. *Let $\star$ be the product on $A = C^\infty(M)[\![\hbar]\!]$ defined by $\theta \in \dfrac{\omega}{i\hbar} + H^2(M)[\![\hbar]\!]$ and let $*$ be the star product $p^*\theta$ defines on $B = C^\infty(N)[\![\hbar]\!]$. Is $(A, \star)$ a subalgebra of $(B, *)$?*

The question above seems like an easy question, though due to lack of time the author did not manage to fully delve into it. In case the answer to the question 9 is yes, we have follow up questions.

Question 10. *Is B a covering of A with quantum symmetry $H = C(G)$ where G is the deck transformation group of p?*

Question 11. *Can we describe all uniform (or local) coverings (B, H) of $A = C^\infty(M)[\![\hbar]\!]$? Can we classify those uniform coverings (B, H) for which B itself is a formal deformation quantization of some manifold N covering M?*

11.5 Covering theory for other models of non-commutative geometry

...non-commutative geometry lacks common foundations: for many interesting constructions of "non-commutative spaces" we cannot even say for sure which of them lead to isomorphic spaces, because they are not objects of an all-embracing category..

–Yuri Manin, 2005.

In this thesis, we followed Connes' idea of non-commutative geometry. Although our formulation of non-commutative coverings covers a much general algebraic situation and not just C^*-algebras, most of our examples are. There are plenty of formulation of non-commutative geometry and each one focuses on an aspect of interest. Connes' formulation of non-commutative spaces is especially powerful when dealing with metric aspects of geometry. However, homotopy theoretic aspects of such non-commutative spaces are not that well-addressed. In part, this thesis tries to partially answer this.

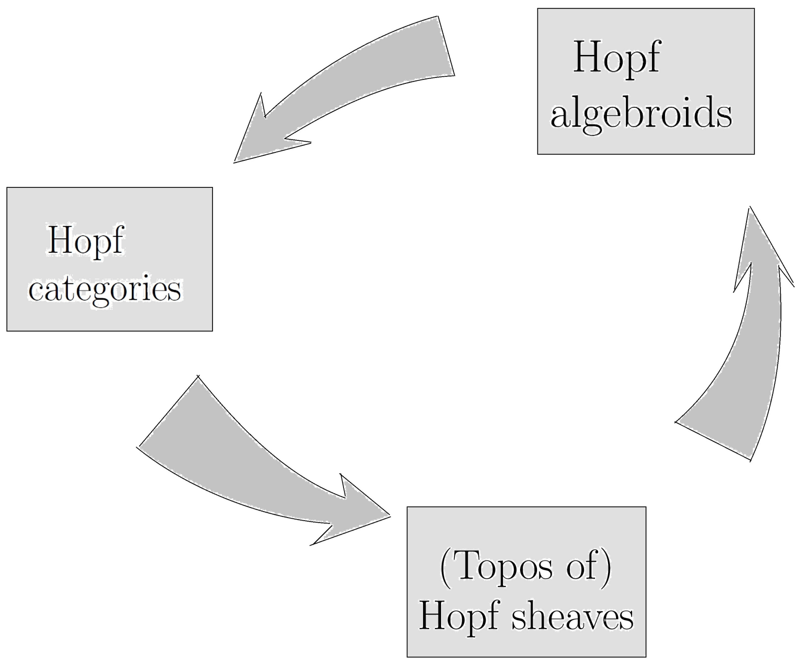
For homotopy theoretic aspects, one can use *differential graded categories* as models of non-commutative spaces. Chapter 5 gives an insight on how our present formulation can be related to categories. In this case, we expect that Hopf categories will play the role Hopf algebroids played in the formulation of non-commutative coverings in this thesis.

Another model that is beginning to take prominence is that of a *topos.* The category of sheaves $Sh(X)$ on a topological space X is a particular example of a topos and to some extent, regarded as the classical case of this model. But as we have seen again in chapter 5 and in appendix B, there is a strong relation between algebras, categories and sheaves. In this case, we expect that the *collection* of Hopf sheaves will play the role of quantum symmetries though we have not yet verified whether such a collection is a topos or not. See definition 21 of a Hopf sheaf.

11.6 Monodromy

A problem worthy of attack
proves its worth by fighting back.

–Piet Hein



The original motivation of this thesis is to put forth a useful notion of monodromy for non-commutative spaces. This was stalled by the fact that there is not existing notion of a fundamental group for non-commutative spaces. In trying to define such fundamental groups (or groupoid), the author was further dragged into defining a more basic structure− that of a covering space. Let us end this chapter by saying why one would be interested to have a notion of monodromy.

$$\left\{ \begin{array}{c} \text{covering} \\ \text{spaces} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{fundamental} \\ \text{group} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{monodromy} \\ \text{representation} \end{array} \right\}$$

Connes' formulation of non-commutative spaces have been successful in avoiding bad quotients, as he called them. The use of operator algebras completely banish the singularities. In having such a notion of monodromy, one can then study directly these singularities, i.e. using only the lattice structure of ideals of a non-commutative space. However, it seems like such an undertaking is far from realization since, as we have seen in section 11.1, there are lots of important questions to be addressed first in the level of covering spaces.

The advantage of using Hopf algebroids instead of just Hopf algebras in the fact that we may be able to encode local quantum symmetries that may vary *point to point*. As Pierre Cartier pointed out in his article [12], the fundamental group $\pi_1(X, a)$ is regarded as the symmetry of the point $a \in X$. Monodromy pushes this idea further by looking at representations of $\pi_1(X, a)$. As it stands, this thesis will be the author's first step towards formulating monodromy representations for non-commutative spaces.

Appendices

The following appendices covers materials that are important in the synergy of the thesis but which does not coherently fit with the narrative. The first one is an elementary recap of classical covering spaces. A more detailed exposition can be found on [27]. In the second appendix, building of the works develop in [1], we will introduce topological Hopf categories and Hopf sheaves.

CONTENT:

Appendix A

Covering Spaces

(What is it like to understand advanced mathematics?)
"..imagine describing what a snowflake
looks like to a blind man."

–Joseph Wang, 2013.

In this appendix, we will recall basic definitions and properties surrounding classical covering spaces. In part, this will serve for the purpose of self-containment. On another part, this will serve as a backbone of what has transpired in the thesis. The first section discusses basic definitions and fact about covering spaces. The second section discusses Galois theory for covering spaces. The last section discusses applications of covering spaces to classical geometry.

A.1 What are covering spaces?

In this section, we will assume topological spaces have nice connectivity properties. For example, properties such as connectedness, local connectedness, path-connectedness, and local simply-connectedness.

Definition 19. Let X be a topological space. An *(unramified) covering* of X is a space Y together with a continuous surjection $Y \overset{p}{\twoheadrightarrow} X$ such that any point $x \in X$ has an open neighborhood U whose preimage under p is a disjoint union of homeomorphic copies of U, i.e. $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$ where each V_α are homeomorphic via p to U. A *ramified covering* of X is a space Y together with a continuous surjection $Y \overset{p}{\twoheadrightarrow} X$ such that outside a nowhere dense set $O \subseteq X$, p is a unramified. The smallest such nowhere dense set O is called the ramification locus of p. □

We will briefly refer to unramified coverings as coverings. We should point out that, as an immediate consequence of definition 19, the fiber $p^{-1}(x)$ of p at any point $x \in X$ is a discrete subset of Y. This might not sound like a big deal but this is one of the main obstacle in defining general non-commutative covering spaces. Covering spaces are among a class of nice maps called fibration. These satisfy homotopy lifting properties. For covering spaces, this takes the following form.

Lemma 7. *(Homotopy lifting property.) Let* $Y \overset{p}{\twoheadrightarrow} X$ *be a covering and let* $\gamma : [0,1] \longrightarrow X$ *be a path in* X *with initial point* x_0*. Given any* $y \in p^{-1}(x_0)$*, there is a unique path* $\tilde{\gamma} : [0,1] \longrightarrow Y$

such that $\tilde{\gamma}(0) = y$ *and* $p \circ \tilde{\gamma} = \gamma$.

Using the above lemma, one can prove that for any $x, y \in X$, the fibers $p^{-1}(x)$ and $p^{-1}(y)$ are in bijection. A particular instance of this idea is when γ is a loop based at, say $x \in X$. In this case, γ induces a auto-bijection of $p^{-1}(x)$.

The collection of all coverings of a given space X forms a category $COV(X)$. A morphism from a covering $Y \xrightarrow{p} X$ to a covering $Z \xrightarrow{q} X$ is a continuous map $Y \xrightarrow{r} Z$ such that $p = q \circ r$. It is obvious that r itself is a covering map. Given a covering $Y \xrightarrow{p} X$, we can associate a group $Aut_X(Y)$ consisting of homeomorphisms $Y \xrightarrow{f} Y$ that commutes with p, i.e.

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y. \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

This group is called the group of *deck transformations* of the covering $Y \xrightarrow{p} X$. It is easy to see that this group acts on fibers of p. We say that $Y \xrightarrow{p} X$ is *Galois* if this group acts transitively on the fibers. Note that $Aut_X(Y)$ always act freely on fibers of p. In some text, free and transitive action actions are also called Galois actions, hence the name.

There is another characterization of covering spaces. The category $COV(X)$ is equivalent to the functor category on the fundamental groupoid of X with values in the category of sets. The latter category is easily seen to be complete and cocomplete. The fundamental groupoid $\Pi(X)$ of X is the topological groupoid with X as its space of objects and arrows from x to y are homotopy classes of paths from x to y.

We say that a (pointed) covering $(Y, b) \xrightarrow{p} (X, a)$ is *intermediate* to the covering $(Z, c) \xrightarrow{q} (X, a)$ if there is a (pointed) map $(Z, c) \xrightarrow{\varphi} (Y, b)$ such that $p \circ \varphi = q$. This induces a partial order on the set of coverings of X and incidentally gives a notion of equivalence. The group of auto-equivalences of $(Y, b) \xrightarrow{p} (X, a)$ is precisely the group of deck transformations. We will be mostly interested in the case of connected covers Y.

A.2 Galois theory for covering spaces

Denote by $\pi_1(X, a)$ the fundamental group of X based at $a \in X$ (we will just write $\pi_1(X)$ if the group is independent of the base point, the case when for example X is path-connected). The covering map p induces a map p_* between fundamental groups

$$\pi_1(Y, b) \xrightarrow{p_*} \pi_1(X, a)$$

by mapping a loop in Y down to a loop in X. By the homotopy lifting property, this map is a monomorphism. The following theorem is called the classification theorem for coverings (cf [27]).

Theorem 9. *(Classification of pointed coverings.) For every subgroup* $G \leqslant \pi_1(X)$ *there is a connected covering* $(Y, b) \xrightarrow{p} (X, a)$ *such that* $p_*(\pi_1(Y, b)) = G$. *Two (pointed) coverings* $(Y, b) \xrightarrow{p} (X, a)$ *and* $(Z, c) \xrightarrow{q} (X, a)$ *are equivalent (as pointed coverings) if* $p_*(\pi_1(Y, b)) = q_*(\pi_1(Z, c))$ *as subgroups of* $\pi_1(X, a)$.

More generally, a covering $(Y, b) \xrightarrow{p} (X, a)$ associated to the subgroup G_Y is intermediate to the covering $(Z, c) \xrightarrow{q} (X, a)$ associated to the subgroup G_Z if and only if $G_Z \subseteq G_Y$. If we drop the assumption regarding base points, coverings are equivalent if their associated subgroups are conjugate in $\pi_1(x, a)$. If G is normal in $\pi_1(X)$ then $Aut_X Y = \pi_1(X)/G$. In this case, $Aut_X Y$

acts transitively on the fibers of $(Y, b) \xrightarrow{p} (X, a)$. In general, $Aut_X Y = Nor(G)/G$ where $Nor(G)$ stands for the normalizer of G in $\pi_1(X)$.

The above discussion will be briefly referred to as the Galois theory for coverings. In analogy with the Galois theory for fields, Galois coverings correspond to Galois extensions, intermediate coverings correspond to intermediate extensions, and deck transformation groups correspond to Galois groups. Note that in classical Galois theory, a Galois extension is an algebraic extension which is both normal and separable. Separable extensions correspond to unramified coverings. Some authors call those coverings in which the deck transformation group act transitively on fibers *normal* instead of Galois. With that terminology, Galois coverings are unramified and normal. Since we are dealing with exclusively with unramified coverings, normal coverings are as good as Galois coverings. For a detailed exposition on this correspondence, one may consult Khovanskii [27].

One way to think of covering spaces is that they are *approximations* of the space being covered, say of X. Let us explain what we meant by an approximation here. Let us denote by $\tilde{X}$ the universal cover, if it exists, of X. The subgroup of $\pi_1(X, a)$ associated to $\tilde{X}$ is the trivial subgroup since there are no non-trivial loops in $\tilde{X}$. But the trivial group is normal in $\pi_1(X, a)$. Thus, $Aut_X(\tilde{X}) = \pi_1(X, a)$. In other words, $\pi_1(X, a)$ acts freely and transitively on fibers of $\tilde{X} \longrightarrow X$. Choose a subgroup $G \subseteq \pi_1(X, a)$. One can then form $\tilde{X}/G$. The universal covering map $\tilde{X} \twoheadrightarrow X$ factors through the canonical projection $\tilde{X} \twoheadrightarrow X/G$. Thus, $\tilde{X}/G$ is a covering of X. The subgroup of $\pi_1(X, a)$ associated to $\tilde{X}/G$ is G. With this, we see that all loops in X are killed when lifted to $\tilde{X}$. For $Y = \tilde{X}/G$, loops γ in X are killed when lifted to Y precisely when the homotopy class $[\gamma]$ is in G. Non-trivial loops in X detect *topological holes*. Thus, climbing up a tower of covering spaces banishes these holes.

A.3 What are they good for?

Covering spaces allow powerful tool of Galois theory accessible for topological spaces. Sometimes, one is interested with local properties of a topological space rather than its global behavior. In this case, one can just look at covering spaces since they are simpler than the original space but locally similar. For example, the Lie algebra $\mathfrak{g}$ of a Lie group G depends only in the neighborhood of the identity. A well known fact about Lie groups tells us that Lie groups with the same identity component have isomorphic Lie algebras. Actually, a stronger statement holds. Two Lie groups with the same universal covering space have the same Lie algebras. It is quite easy to find in the literature that the universal covering space $\tilde{G}$ of a Lie group G is itself a Lie group. The Lie algebra $\mathfrak{g}$ of $\tilde{G}$ is the same as the Lie algebra of G.

Covering spaces have applications to differential geometry. A volume form only exists for an orientable manifold M which allows integration over regions in M. For a non-orientable manifold M, one can proceed as follows. Take an orientable double cover $M' \xrightarrow{p} M$ and consider a volume form ω on M'. To integrate $f \in C^\infty(M)$ over a region $U \subseteq M$, integrate $p^* f\omega$ over the disjoint union $U_1 \cup U_2 = p^{-1}(U)$, then divide by 2. If the preimage of U under p is not disjoint, one can proceed by subdiving U first into subsets with disjoint preimages under p.

The last application we will discuss is the most important for our purpose. Let Σ denote the category whose objects are $\pi_1(X)$-sets and whose morphisms are $\pi_1(X)$-equivariant maps. We will show that $COV(X)$ and Σ are equivalent categories. Given a covering $Y \xrightarrow{p} X$, there is an induced action of $\pi_1(X)$ on $p^{-1}(a)$. This defines a functor from $COV(X)$ to Σ. Now, let S be a $\pi_1(X)$-set. Let $S = \coprod_{\alpha \in I} S_\alpha$ be its decomposition into $\pi_1(X)$-orbits. Given a representative s_α of S_α, we get a bijection between S_α and $\pi_1(X)/stab(s_\alpha)$ by the orbit-stabilizer theorem. Then $stab(s_\alpha)$ acts on $\tilde{X}$ and turns $\tilde{X}/stab(s_\alpha)$ into a covering of X. Thus, we get $Y = \coprod_{\alpha \in I} \tilde{X}/stab(s_\alpha)$ as a covering of X. This defines a functor inverse to the previous one. Note under this equivalence, the connected coverings are precisely the ones corresponding to homogenous $\pi_1(X)$-sets. A natural question to ask is whether $\pi_1(X)$ is completely determined by $COV(X)$. The answer turns out to be affirmative using the following result:

Theorem 10. *The group of natural automorphisms of the forgetful functor from $\pi_1(X)$-Sets to Sets is isomorphic to $\pi_1(X)$.*

By an automorphism α of the forgetful functor $\mathcal{F}$ we mean a family of automorphism $\mathcal{F}(S) \stackrel{\alpha_S}{\to} \mathcal{F}(S)$ such that for any morphism of $\pi_1(X)$-sets $S \stackrel{\sigma}{\to} T$, the following commutes

$$\begin{array}{ccc}
\mathcal{F}(S) & \xrightarrow{\mathcal{F}(\sigma)} & \mathcal{F}(T) \\
\Big\downarrow {\scriptstyle \alpha_S} & & \Big\downarrow {\scriptstyle \alpha_T} \\
\mathcal{F}(S) & \xrightarrow[\mathcal{F}(\sigma)]{} & \mathcal{F}(T).
\end{array}$$

The theorem above is a very important theorem. Since the categories $COV(X)$ and Σ are equivalent, a problem concerning $COV(X)$ is equally difficult in Σ. However, one can *approximate* the answer by considering nice full subcategories of Σ and the automorphism group of the forgetful functors for those subcategories. For example, if one considers the full subcategory of finite dimensional $\pi_1(X)$-representations, one gets the algebraic hull of $\pi_1(X)$. If one considers the full subcategory of finite $\pi_1(X)$-sets, the automorphism of the forgetful functor to sets is the profinite completion of $\pi_1(X)$. In a way, the above theorem serves as our guide in formulating the notion of a fundamental group and fundamental groupoid for a noncomutative space.

Appendix B

Topological Hopf categories

A mathematician is a person who can
find analogies between theorems;
a better mathematician is one who
can see analogies between proofs
and the best mathematician can
notice analogies between theories.
One can imagine that the ultimate
mathematician is one who can
see analogies between analogies.

–Stefan Banach

B.1 Definitions and properties

Batista et al. [1] introduced the notion of a Hopf category over an arbitrary strict braided monoidal category $\mathcal{V}$. In this section, we will introduce its topological version. For this purpose, we specialize $\mathcal{V}$ as the category of complex vector spaces whose braiding is the usual flip of tensor factors. Also, we will assume that the underlying categories of such Hopf categories are small. We will be primarily interested with *finite-type* $\mathcal{V}$-enriched categories, by which we mean the hom-sets are finite-dimensional vector spaces. Before giving the definition of a Hopf category, let us introduce some notation first. For two $\mathcal{V}$-enriched categories $\mathscr{A}$ and $\mathscr{B}$ with the same set of objects X, we define $\mathscr{A} \otimes_X \mathscr{B}$ to the the $\mathcal{V}$-enriched category with X as the set of objects and for $x, y \in X$, the hom-set of arrows from x to y is the vector space

$$(\mathscr{A} \otimes_X \mathscr{B})_{x,y} := \mathscr{A}_{x,y} \otimes \mathscr{B}_{x,y}. \tag{B.1}$$

We call $\mathscr{A} \otimes_X \mathscr{B}$ the *tensor product* of $\mathscr{A}$ and $\mathscr{B}$. With this $\otimes_X$, the category of $\mathcal{V}$-enriched categories over X becomes a strict monoidal category whose monoidal unit, denoted by $\mathbb{I}^X$, is the category over X such that for any $x, y \in X$ we have $\mathbb{I}^X_{x,y} = \mathbb{C}$.

Definition 20. A *Hopf category* $\mathscr{H}$ over X is a $\mathcal{V}$-enriched category satisfying the following conditions.

(a) There are functors

$$\mathscr{H} \xrightarrow{\ \Delta\ } \mathscr{H} \underset{X}{\otimes} \mathscr{H}\ , \qquad \mathscr{H} \xrightarrow{\ \epsilon\ } \mathbb{I}^X$$

called the *coproduct* and *counit*, respectively, such that Δ is *coassociative* and *counital* with respect to ϵ, i.e. the diagram of functors

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\ \Delta\ } & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle id \otimes_X \Delta} \\ \mathscr{H} \underset{X}{\otimes} \mathscr{H} & \xrightarrow[\ \Delta \otimes_X id\ ]{} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \underset{X}{\otimes} \mathscr{H} \end{array} \qquad \begin{array}{ccc} \mathscr{H} & = & \mathscr{H} \underset{X}{\otimes} \mathbb{I}^X \\ \Big\| & \searrow^{\Delta} & \Big\uparrow{\scriptstyle is \otimes_X \epsilon} \\ \mathbb{I}^X \underset{X}{\otimes} \mathscr{H} & \xleftarrow[\ \epsilon \otimes_X id\ ]{} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \end{array}$$

commute.

(b) There is a functor $S : \mathscr{H} \longrightarrow \mathscr{H}^{op}$, called the *antipode*, satisfying

$$\begin{array}{ccccc} & \nearrow^{\Delta} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \xrightarrow{\ S \otimes_X id\ } \mathscr{H}^{op} \underset{X}{\otimes} \mathscr{H} & \searrow^{\circ} & \\ \mathscr{H} & \xrightarrow{\ \epsilon\ } & \mathbb{I}_X & \xrightarrow{\ \eta\ } & \mathscr{H} \\ & \searrow_{\Delta} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \xrightarrow[\ id \otimes_X S\ ]{} \mathscr{H} \underset{X}{\otimes} \mathscr{H}^{op} & \nearrow_{\circ} & \end{array}$$

Here, $\circ$ denotes the bifunctor induced by the categorical composition in $\mathscr{H}$ and η is the functor that send $1 \in \mathbb{I}^X_{x,y}$ to the identity element of $\mathscr{H}_{x,y}$. □

Remark 32. Functoriality of Δ and ϵ means that for any $x, y \in X$, we have linear maps

$$\mathscr{H}_{x,y} \xrightarrow{\ \Delta_{x,y}\ } \mathscr{H}_{x,y} \otimes \mathscr{H}_{x,y} \qquad \mathscr{H}_{x,y} \xrightarrow{\ \epsilon_{x,y}\ } \mathbb{C}$$

where $\Delta_{x,y}$ is coassociative and counital with respect to $\epsilon_{x,y}$ in the usual sense. This implies that $\mathscr{H}_{x,y}$ is a coalgebra. If we denote by $C(\mathcal{V})$ the category of coalgebras on $\mathcal{V}$, another way to package part (a) of definition 20 is to say that $\mathscr{H}$ is enriched over $C(\mathcal{V})$.

For the main results of this paper, we will be mostly interested with the case X is a topological space. In such a case, it makes sense to reflect *continuity* on the functors Δ, ϵ and S along with the categorical structure maps. This calls for the following definition.

Definition 21. Let X be a topological space and $\mathcal{O}_X$ the sheaf of continuous complex-valued functions on X. A *topological Hopf category* $\mathscr{H}$ over X is a Hopf category together with a sheaf H^{sh} over $X \times X$ (with the product topology) of $\mathcal{O}_X$-bimodules satisfying the following conditions.

(a) Denote by $\pi_1, \pi_2 : X \times X \longrightarrow X$ the projection onto the first and second factor, respectively. Over an open set $U \subseteq X \times X$, for any $\sigma \in H^{sh}(U)$, $f \in \mathcal{O}_X(\pi_1 U)$ and $g \in \mathcal{O}_X(\pi_2 U)$ we have

$$(f \cdot \sigma \cdot g)\,(x, y) = f(x)\sigma(x, y)g(y)$$

for any $(x, y) \in U$.

(b) $\mathscr{H}_{x,y}$ is the fiber of H^{sh} at $(x, y) \in X \times X$.

(c) $\circ$, η, Δ, ϵ and S are the induced maps on global sections of the following map of sheaves

$$H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh} \xrightarrow{\circ^{sh}} H^{sh}, \qquad \mathcal{O}_X \xrightarrow{\eta^{sh}} H^{sh},$$

$$H^{sh} \xrightarrow{\Delta^{sh}} H^{sh} \underset{\mathcal{O}_{X\times X}}{\otimes} H^{sh}, \qquad H^{sh} \xrightarrow{\epsilon^{sh}} \mathcal{O}_X,$$

$$H^{sh} \xrightarrow{S^{sh}} \left(H^{sh}\right)^{op}$$

respectively. Here, $\left(H^{sh}\right)^{op}$ is the pullback of the sheaf H^{sh} along the map $X\times X \longrightarrow X\times X$ flipping the factors.

A sheaf H^{sh} of $\mathcal{O}_X$-bimodules over $X \times X$ equipped with maps $\circ$, η, Δ, ϵ, and S described in (c) is called a *Hopf sheaf.* □

Remark 33.

(1) The tensor product $\underset{\mathcal{O}_X}{\otimes}$ used for $\circ^{sh}$, it is the one the identifies

$$\sigma \cdot f \underset{\mathcal{O}_X}{\otimes} \tau = \sigma \underset{\mathcal{O}_X}{\otimes} f \cdot \tau$$

for any $\sigma, \tau \in H^{sh}(X \times X)$ and $f \in \mathcal{O}_X$. The $\mathcal{O}_X$-bimodule structure on the global section of $H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh}$ is given as

$$f \cdot \left(\sigma \underset{\mathcal{O}_X}{\otimes} \tau\right) \cdot g = (f \cdot \sigma) \underset{\mathcal{O}_X}{\otimes} (\tau \cdot g)$$

for any $f, g \in \mathcal{O}_X$ and $\sigma, \tau \in H^{sh}(X \times X)$.

(2) The tensor product $\underset{\mathcal{O}_{X\times X}}{\otimes}$ used in part (c) for Δ^{sh} of definition 21 assures that we have

$$f \cdot \sigma \underset{\mathcal{O}_{X\times X}}{\otimes} \tau \cdot g = \sigma \cdot g \underset{\mathcal{O}_{X\times X}}{\otimes} f \cdot \tau$$

for any $f, g \in \mathcal{O}_X$ and $\sigma, \tau \in H^{sh}(X \times X)$.

(3) The map Δ^{sh} lifts, denoted the same way, to a map $\Delta^{sh} : H^{sh} \longrightarrow H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh}$ where the tensor product $\otimes_{\mathcal{O}_X}$ in this case identifies

$$f \cdot \sigma \underset{\mathcal{O}_X}{\otimes} \tau = \sigma \underset{\mathcal{O}_X}{\otimes} f \cdot \tau$$

for any $\sigma, \tau \in H^{sh}(X \times X)$ and $f \in \mathcal{O}_X$. The antipode property assures us that the direct image of H^{sh} along the lift Δ^{sh} is a subsheaf of $H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh}$ satisfying

$$\sigma \cdot f \underset{\mathcal{O}_X}{\otimes} \tau = \sigma \underset{\mathcal{O}_X}{\otimes} \tau \cdot f$$

for any $\sigma, \tau \in H^{sh}(X \times X)$ and $f \in \mathcal{O}_X$.

(4) By virtue of the previous remark, H^{sh}, $H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh}$, and $H^{sh} \underset{\mathcal{O}_{X\times X}}{\otimes} H^{sh}$ are all sheaves over $X \times X$. The sheaf $\mathcal{O}_X$ can also be viewed as a sheaf over $X \times X$ supported on the diagonal. Thus, the maps enumerated in part (c) of the above definition are maps of sheaves over the same space.

(5) Let us expound more on what we meant by a Hopf sheaf. Consider a sheaf $\mathcal{W}$ of $\mathcal{O}_X$-bimodules over $X \times X$ together with maps $\circ^w$, η^w, Δ^w, ϵ^w and S^w similar to part (c) above. The sheaf $\mathcal{W}$ is a Hopf sheaf if the following conditions are satisfied. The sheaf $\mathcal{W}$ is supported on a closed subset $Z \subset X \times X$ such that the preimages of the projection onto the first and second factors are finite for all $x \in X$. On global sections $\circ^w$ and η^w make $\mathcal{W}(X \times X)$ a unital associative $\mathcal{O}_X$-ring. We require that Δ^w lifts to a map of sheaves

$$\mathcal{W} \xrightarrow{\Delta^w} \mathcal{W} \underset{\mathcal{O}_X}{\otimes} \mathcal{W}$$

where $\mathcal{W} \otimes_{\mathcal{O}_X} \mathcal{W}$ is the sheaf over $X \times X$ defined as $\lambda^* \left((\pi_X)_* \theta_* (\mathcal{W} \otimes \mathcal{W}) \right)$ where the maps λ, θ, and π_X

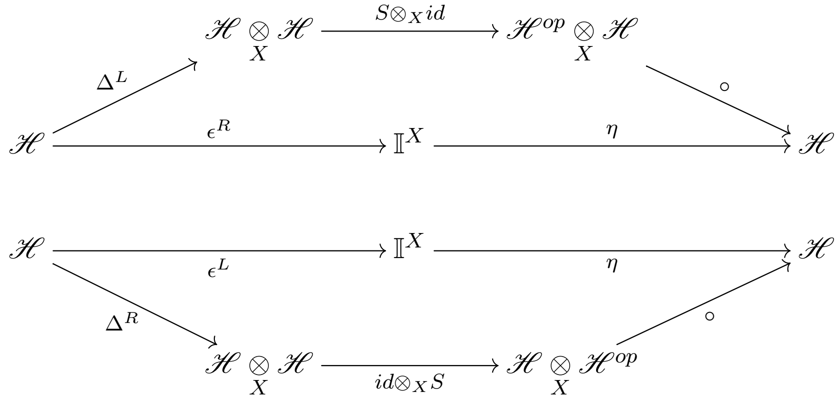
$$Z \times Z \xrightarrow{\theta} Z \times_X Z \xrightarrow{\pi_X} X \xleftarrow{\lambda} Z$$

are the canonical projection, the projection onto the X-factor and the projection onto the first factor, respectively. Together with ϵ^w, we require that the lift of Δ^w makes $\mathcal{W}(X \times X)$ a counital coassociative $\mathcal{O}_X$-coring. Lastly, we require S^w to satisfy a convolution diagram. Note that these conditions are satisfied by H^{sh} and all its accompanying maps listed in part (c) because of condition (b) in definition 21 and the fact that $\mathscr{H}$ is a Hopf category.

The following, which will play an important role in our formulation of the main result, is the categorification of a coupled Hopf algebra.

Definition 22. A *coupled Hopf category* $\mathscr{H}$ is a $\mathcal{V}$-enriched category with two enrichments over $C(\mathcal{V})$, denoted by $\mathscr{H}_L$ and $\mathscr{H}_R$, with coproducts Δ^L, Δ^R and counits ϵ^L, ϵ^R, respectively; there is a functor $S : \mathscr{H} \longrightarrow \mathscr{H}^{op}$, called the *coupling* functor; and all these satisfy the following conditions:

(a) The following diagrams, indicating the *coupling condition*, commute.



(b) The coproducts Δ^L and Δ^R commute, i.e.

$$
\begin{array}{ccc}
\mathscr{H} & \xrightarrow{\Delta^R} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \\
\Big\downarrow{\scriptstyle \Delta^L} & & \Big\downarrow{\scriptstyle \Delta^L \underset{X}{\otimes} id} \\
\mathscr{H} \underset{X}{\otimes} \mathscr{H} & \xrightarrow[id \underset{X}{\otimes} \Delta^R]{} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \underset{X}{\otimes} \mathscr{H}
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{H} & \xrightarrow{\Delta^L} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \\
\Big\downarrow{\scriptstyle \Delta^R} & & \Big\downarrow{\scriptstyle \Delta^R \underset{X}{\otimes} id} \\
\mathscr{H} \underset{X}{\otimes} \mathscr{H} & \xrightarrow[id \underset{X}{\otimes} \Delta^L]{} & \mathscr{H} \underset{X}{\otimes} \mathscr{H} \underset{X}{\otimes} \mathscr{H}
\end{array}
$$

□

Remark 34.

(1) Coupled Hopf categories are almost the categorification of coupled Hopf algebras. While the constituent bialgebras of a coupled Hopf algebra is a Hopf algebras in itself, the constituent categories $\mathscr{H}_L$ and $\mathscr{H}_R$ of a coupled Hopf category $\mathscr{H}$ need not be Hopf categories.

(2) Just like Hopf categories, we can also *topologize* coupled Hopf categories. We can take definition 21: assert the existence of a sheaf H^{sh} over $X \times X$ of $\mathcal{O}_X$-bimodules, take conditions (a) and (b) as they are, and replace condition (c) by

(c') Δ^L, Δ^R, ϵ^L, ϵ^R and S are the induced maps on global sections of the following map of sheaves

$$H^{sh} \xrightarrow{(\Delta^L)^{sh}} H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh}, \qquad H^{sh} \xrightarrow{(\epsilon^L)^{sh}} \mathcal{O}_X,$$

$$H^{sh} \xrightarrow{(\Delta^R)^{sh}} H^{sh} \underset{\mathcal{O}_X}{\otimes} H^{sh}, \qquad H^{sh} \xrightarrow{(\epsilon^R)^{sh}} \mathcal{O}_X,$$

$$H^{sh} \xrightarrow{S^{sh}} \left(H^{sh}\right)^{op}$$

respectively, making the following diagram

$$
\begin{array}{ccccc}
 & H^{sh}(U) \underset{\mathcal{O}_X(U)}{\otimes} H^{sh}(U) & \xrightarrow{S_U \underset{\mathcal{O}_X(U)}{\otimes} id} & H^{sh}(U)^{op} \underset{\mathcal{O}_X(U)}{\otimes} H^{sh}(U) & \\
{\scriptstyle (\Delta^L)^{sh}(U)} \nearrow & & & & \searrow {\scriptstyle \mu_U} \\
H^{sh}(U) & \xrightarrow{(\epsilon^R)^{sh}(U)} & \mathcal{O}_X(U) & \xrightarrow{\eta_U} & H^{sh}(\pi_2^{diag} U)
\end{array}
$$

$$
\begin{array}{ccccc}
H^{sh}(U) & \xrightarrow[(\epsilon^L)^{sh}(U)]{} & \mathcal{O}_X(U) & \xrightarrow[\eta_U]{} & H^{sh}(\pi_1^{diag} U) \\
{\scriptstyle (\Delta^R)^{sh}(U)} \searrow & & & & \nearrow {\scriptstyle \mu_U} \\
 & H^{sh}(U) \underset{\mathcal{O}_X(U)}{\otimes} H^{sh}(U) & \xrightarrow[id \underset{\mathcal{O}_X(U)}{\otimes} S_U]{} & H^{sh}(U) \underset{\mathcal{O}_X(U)}{\otimes} (H^{sh})(U)^{op} &
\end{array}
$$

commute for any $U \subseteq X \times X$. Here, μ_U and η_U denote the maps induced by the composition and unit maps of $\mathscr{C}$. The maps π_1^{diag} and π_2^{diag} denote $X \times X \longrightarrow X \times X$, $(x, y) \mapsto (x, x)$ and $X \times X \longrightarrow X \times X$, $(x, y) \mapsto (y, y)$, respectively.

B.2 A good example of a Hopf category

In this section, we will look at a very important example of a Hopf category. This example will be a representative example of one of the main result of section 5. This is a special case of proposition 7.1 of [1]. Consider a finite set X whose elements are conveniently labelled as $1, 2, ..., n$. Equip X with the discrete topology. Consider the category C whose set of objects is X and define $C_{x,y} = \mathbb{C}$. The category C is obviously a Hopf category. By proposition 7.1 of [1], $\mathcal{H} = \bigoplus_{x,y\in X} C_{x,y}$ is a weak Hopf algebra. Using the arguments in example 4 of section 2.2, $\mathcal{H}$ is a Hopf algebroid over $A = \mathbb{C}^n = \mathcal{O}_X(X)$.

The Hopf algebroid $\mathcal{H}$ has a more familiar form. It is isomorphic, as a Hopf algebroid, the algebra $M_n(\mathbb{C})$ over its diagonal $D_n = Diag_n(\mathbb{C})$. With the D_n-bimodule structure on $M_n(\mathbb{C})$ defined as

$$P \cdot M \cdot Q := MPQ, \qquad P, Q \in D_n, M \in M_n(\mathbb{C}),$$

the coproduct Δ_R and the counit ϵ_R are given as

$$\Delta_R(E_{ij}) = E_{ij} \otimes_{D_n} E_{ij}, \qquad \epsilon_R(M) = \sum_{i=1}^{n} E_{ii}\phi(ME_{ii})$$

where ϕ is the linear functional defined by $\phi(E_{ij}) = 1$ for all $i, j \in X$. With the usual matrix multiplication and unit, Δ_R and ϵ_R constitutes a right D_n-bialgebroid structure on $M_n(\mathbb{C})$. For completeness, let us define the structure maps of the left D_n-bialgebroid structure of $M_n(\mathbb{C})$. Consider the D_n-bimodule structure on $M_n(\mathbb{C})$ defined as

$$P \cdot M \cdot Q := PQM, \qquad P, Q \in D_n, M \in M_n(\mathbb{C}).$$

The coproduct Δ_L and the counit ϵ_L are defined as

$$\Delta_L(E_{ij}) = E_{ij} \otimes_{D_n} E_{ij}, \qquad \epsilon_L(M) = \sum_{i=1}^{n} \phi(E_{ii}M)E_{ii}$$

where ϕ is the same linear functional used to defined ϵ_R. The antipode S of this Hopf algebroid is defined as $S(E_{ij}) = E_{ji}$.

As a weak Hopf algebra, ϕ is the counit of $\mathcal{H}$. The coproducts Δ_L and Δ_R are the extension of the weak coproduct Δ to $M_n(\mathbb{C}) \otimes_{D_n} M_n(\mathbb{C})$ relative to the D_n-bimodule structure used. As we will see in chapter 5, this is not a coincidence. This is in fact a special case of a more general result which we shall prove at the end of that section.

B.3 Galois extensions of Hopf categories

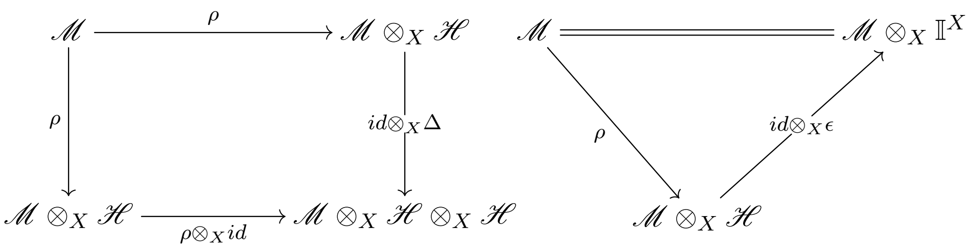
Formulation of Galois theory for Hopf category is straightforward. Recall that in the case of Hopf algebras, only the underlying bialgebra structure is relevant. In the coaction picture, the coalgebra is used to make sense of a coaction while the algebra structure is used to make sense of the Galois map. All these ingredients are already present in the case of a Hopf category. We will discuss the situation for topological Hopf categories. The case for Hopf categories follow almost immediately by dropping any manifestation of topology.

Before giving the definition of the categorical analogue of a comodule algebra, let us first discuss what a topological category is, at least for our purpose. A $\mathcal{V}$-enriched category $\mathscr{M}$ over a space X is a *topological category* if there is a sheaf M^{sh} of $\mathcal{O}_X$-bimodules such that conditions (a), (b) and the relevant part of condition (c) of definition 21 hold.

Definition 23. Let $\mathscr{H}$ be a topological Hopf category with space of objects X, coproduct Δ, counit ϵ and antipode S with associated sheaf H^{sh}.

(1) A topological category $\mathscr{M}$ over X enriched over $\mathcal{V}$, with associated sheaf M^{sh}, is a *right $\mathscr{H}$-comodule* if there is a functor $\rho : \mathscr{M} \longrightarrow \mathscr{M} \otimes_X \mathscr{H}$ such that the following conditions hold.

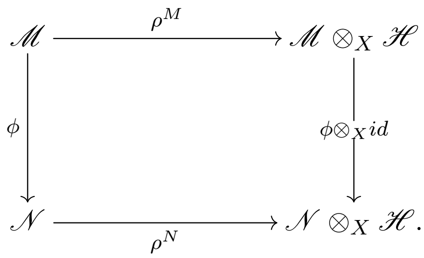
 (a) ρ is coassociative with respect to Δ and counital with respect to ϵ, i.e. the diagrams of functors



 commute, and

 (b) the functor ρ is the map induced by the map of sheaves $M^{sh} \longrightarrow M^{sh} \otimes_{\mathcal{O}_X} H^{sh}$ where the tensor product is the same as the first one we described in remark 33. A *left $\mathscr{H}$-comodule* can be symmetrically defined.

(2) A *morphism* $\mathscr{M} \xrightarrow{\phi} \mathscr{N}$ of right $\mathscr{H}$-comodules is a functor that commutes with the right coactions, i.e. one which makes the following diagram commute



Here, ρ^M and ρ^N are the coactions of $\mathscr{H}$ on $\mathscr{M}$ and $\mathscr{N}$, respectively.

(3) A right $\mathscr{H}$-comodule $\mathscr{M}$ is a *right $\mathscr{H}$-comodule-category* if in addition, the composition map $\mathscr{M} \otimes_X \mathscr{M} \xrightarrow{\circ} \mathscr{M}$ is a map of right $\mathscr{H}$-comodules, where $\mathscr{M} \otimes_X \mathscr{M}$ is equipped with the diagonal coaction.

(4) The *coinvariants* of a right $\mathscr{H}$-comodule-category $\mathscr{M}$ is the subcategory $\mathscr{M}^{co\ \mathscr{H}}$ whose space of objects is X and whose hom-sets are defined as

$$\left(\mathscr{M}^{co\ \mathscr{C}}\right)_{x,y} := \{\alpha \in \mathscr{M}_{x,y} | \rho(\alpha) = \alpha \otimes id_y\}$$

for any $x, y \in X$.

□

Remark 35. A Hopf category is the categorification of a Hopf algebra with categorical composition corresponding to the algebra product. A right $\mathscr{H}$-comodule $\mathscr{M}$ is in particular a category, it already has a composition. This means that we only need to impose requirement (3) in definition 23 to get a categorification of the notion of a comodule-algebra. In the classical set-up, one has to require the existence of a product and assert its compatibility with the comodule structures.

In the set-up of Hopf-Galois theory with respect to Hopf algebras, there is a well-understood notion for extensions of k-algebras $A \subseteq B$ to be H-Galois for a Hopf algebra H even if $A \neq k$. This is because $B \otimes_A B$ makes sense as a k-module. All that is left to do is require $A = B^{co\ H}$ and that the map $B \otimes_A B \longrightarrow B \otimes H$, $a \otimes b \mapsto (a \otimes 1)\rho(b)$ is bijective. On the other hand, in the situation of a Hopf category $\mathscr{H}$ and extensions of comodule-categories $\mathscr{A} \subseteq \mathscr{M}$ with $\mathscr{A} = (\mathscr{M})^{co\ \mathscr{H}}$, we can only make sense of the product $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{M}$ in the case $\mathscr{A}$ is the subcategory of $\mathscr{M}$ whose hom-sets $\mathscr{A}_{x,y}$ are all zero except when $x = y$, in which case $\mathscr{A}_{x,x} = \mathbb{C}$. In this case, we identify $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{M}$ with $\mathscr{M} \otimes_X \mathscr{M}$. Let us call such a category the *trivial linear category* over X, and denote by I_X. There might be a way to consider Galois extensions by Hopf categories in which the subcategory of coinvariants is strictly larger than I_X, but at present it is not clear to the author how to make sense of it. Fortunately, for our purpose of proving theorem 7 it is enough to have I_X as the subcategory of coinvariants.

Definition 24. A right $\mathscr{H}$-comodule-category $\mathscr{M}$ is a $\mathscr{H}$-*Galois extension* of I_X provided

(a) $\mathscr{M}^{co\ \mathscr{H}} = I_X$, and

(b) the functor

$$\begin{aligned} \mathscr{M} \otimes_X \mathscr{M} &\xrightarrow{\mathfrak{gal}} \mathscr{M} \otimes_X \mathscr{H} \\ \alpha \otimes \beta &\longmapsto \big(\alpha \circ \beta_{[0]}\big) \otimes \beta_{[1]} \end{aligned}$$

called the *Galois morphism*, is fully faithful. □

Remark 36.

(1) We are using Sweedler notation for the legs of the coaction

$$\rho : \mathscr{M} \longrightarrow \mathscr{M} \otimes_X \mathscr{H}.$$

In other words, for any $x, y \in X$ and $\alpha \in \mathscr{M}_{x,y}$, we have $\rho(\alpha) = \alpha_{[0]} \otimes \alpha_{[1]}$, where $\alpha_{[0]} \in \mathscr{M}_{x,z}$ and $\alpha_{[1]} \in \mathscr{H}_{z,y}$ for some $z \in X$. This, in particular, tells us that the map $\mathfrak{gal}$ above make sense.

(2) Galois extension by a coupled Hopf category $\mathscr{H} = (\mathscr{H}_L, \mathscr{H}_R, S)$ means simultaneous Galois extensions of the constituent $C(\mathcal{V})$-enriched categories $\mathscr{H}_L$ and $\mathscr{H}_R$.

Appendix C

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