REAL TOPOLOGICAL CYCLIC HOMOLOGY

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PhD Thesis
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PhD thesis in mathematics

PhD thesis submitted to the PhD School of Science, Faculty of Science, University of Copenhagen, Denmark in November 2016.

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Abstract

The main topics of this thesis are real topological Hochschild homology and real topological cyclic homology. The thesis consists of an introduction followed by two papers.

If a ring or a ring spectrum is equipped with an anti-involution, then it induces additional structure on the associated topological Hochschild homology spectrum. The group $O(2) = \mathbb{T} \rtimes G$ acts on the spectrum, where $\mathbb{T}$ is the multiplicative group of complex number of modulus 1 and G denotes the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of order 2. We refer to this $O(2)$-spectrum as the real topological Hochschild homology. This generalization leads to a G-equivariant version of topological cyclic homology, which we call real topological cyclic homology.

The first paper of this thesis computes the G-equivariant homotopy type of the real topological cyclic homology at a prime p of spherical group rings with anti-involution induced by taking inverses in the group.

The second paper of this thesis investigates the derived G-geometric fixed points of the real topological Hochschild homology of an ordinary ring with an anti-involution. The main theorem of the second paper computes the component group of the derived G-geometric fixed points.

Resumé

Hovedemnerne i denne afhandling er reel topologisk Hochschild homology og reel topologisk cyklisk homologi. Afhandlingen består af en introduktion efterfulgt af to artikler.

Hvis en ring eller et ring spektrum er udstyret med en anti-involution, så inducerer det ekstra struktur på det associerede topologiske Hochschild homologi spektrum. Gruppen $O(2) = \mathbb{T} \rtimes G$ virker på spektret, hvor $\mathbb{T}$ er den multiplikative gruppe af komplekse tal med modulus 1 og G betegner gruppen $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ af orden 2. Vi kalder dette $O(2)$-spektrum for reel topologisk Hochschild homology. Denne generalisering giver anledning til en G-ækvivariant version af topologisk cyklisk homologi, som vi kalder reel topologisk cyklisk homologi.

Den første artikel i denne afhandling bestemmer den G-ækvivariante homotopi type af reel topologisk cyklisk homologi ved et primtal p af sfæriske grupperinge med anti-involution induceret af at tage inverser i gruppen.

Den anden artikel i denne afhandling undersøger de afledte G-geometriske fixpunkter af reel topologisk Hochschild homologi af en ring med anti-involution. Hovedresultatet i den anden artikel udregner gruppen af komponenter af de afledte G-geometriske fixpunkter.

Acknowledgments

I would like to express my sincere gratitude to my supervisor Lars Hesselholt for his inspiring guidance and support through three years of research and studies. I'm grateful to Lars for not only teaching me mathematics but also showing me how to be a professional researcher. My special thanks are due to Ib Madsen for originally encouraging my interest in algebraic K-theory and topological cyclic homology.

I would like to thank my fellow PhD students and postdocs at the department for many insightful discussions, seminars and workshops through the years. In particular, I would like to thank Irakli Patchkoria and Kristian Moi for sharing their mathematical knowledge and for always answering my questions. I thank Martin Speirs, Ryo Horiuchi, Espen Auseth Nielsen, Manuel Krannich, and Massimiliano Ungheretti for our many fruitful discussions about the content of this thesis. Finally, I would like to thank my office mates Isabelle, Matthias, Tomasz, Niek, Martin, Manuel, Joshua, and Ryo for making my daily life at the department so enjoyable.

Support from the DNRF Niels Bohr Professorship of Lars Hesselholt is gratefully acknowledged.

Amalie Høgenhaven
Copenhagen, November 2016

Contents

Part I

Thesis overview

Introduction

The main topics of this thesis are real topological Hochschild homology and real topological cyclic homology. The thesis consists of an introduction followed by two papers.

Paper A. *Real topological cyclic homology of spherical group rings.*

Paper B. *On the geometric fixed points of real topological Hochschild homology.*

To put them into context, we give an overview of the historical development of topological Hochschild homology, topological cyclic homology and their real analogues. We describe the real variants in more detail and indicate how the results of the papers fit into the general picture. We conclude the introduction by commenting on future directions naturally extending the results of the two papers.

Historical development

Algebraic K-theory encodes important invariants for a wide range of areas in mathematics, spanning from geometric topology to number theory and it has been a vibrant research area in modern mathematics. Algebraic K-theory is a functor which associates, to a ring R, a spectrum $K(R)$, whose homotopy groups $\pi_n(K(R)) = K_n(R)$ are the higher algebraic K-groups of R introduced in the seminal work of Quillen in [18]. Waldhausen extended the theory to more general input such as ring spectra and exact categories with weak equivalences.

The higher K-groups are related to many important questions in number theory, algebraic geometry and geometric topology and are notoriously difficult to calculate. As an example, the algebraic K-theory of the integers is still not completely understood. The missing information is connected to the Vandiver conjecture in number theory, which states that a prime number p does not divide the class number h_K of the maximal real subfield $K = \mathbb{Q}(\zeta_p)^+$ of the pth cyclotomic field. The conjecture is known to be equivalent to $K_{4i}(\mathbb{Z}) = 0$ for all i.

Another important example is the algebraic K-theory of spherical group rings, which has connections to the geometry of manifolds. Let M be a compact connected topological manifold admitting a smooth structure, and let $P(M)$ be the space of pseudo-isotopies of M, that is, homeomorphisms $h : M \times [0,1] \to M \times [0,1]$ which restrict to the identity on $\partial M \times [0,1] \cup M \times \{0\}$. There is a stabilization map

$P(M) \to P(M \times [0,1])$, given by crossing with the identity, and the stable pseudo-isotopy space $\mathscr{P}(M)$ is the homotopy colimit of the stabilization maps. Igusa showed in [14], building on work by Hatcher in [6], that the inclusion $P(M) \to \mathscr{P}(M)$ is k-connected if k is less than both $(\dim(M)-7)/2$ and $(\dim(M)-4)/3$. The stable pseudo-isotopy space can be expressed in terms of Waldhausen's algebraic K-theory of spaces. If we let Γ be the Kan loop group of M (i.e. a simplicial group such that M is weakly equivalent to the classifying space $B\Gamma$), then by work of Waldhausen [20],[21] and Waldhausen-Rognes-Jahren [22], there is a cofibration sequence of spectra

$$K(\mathbb{S}) \wedge B\Gamma_+ \to K(\mathbb{S}[\Gamma]) \to \Sigma^2\mathscr{P}(M) \to \Sigma\big(K(\mathbb{S}) \wedge B\Gamma_+\big).$$

Over the last two decades, the computational study of algebraic K-theory has been revolutionized by the development of so called trace methods. In analogy with the Chern character from topological K-theory to rational cohomology, algebraic K-theory admits maps to various objects of more homological or homotopical nature. The first example of such an object is Hochschild homology, which is a homology theory for unital, associative rings introduced by Hochschild in the 1940's. Dennis constructed a trace map

$$\mathrm{tr} : K(R) \to \mathrm{HH}(R).$$

from the algebraic K-theory of rings to Hochschild homology in [4]. As a consequence of Connes' theory of cyclic sets, the right hand space is equipped with an action by the circle group $\mathbb{T}$, which allowed the construction of negative cyclic homology $\mathrm{HC}^-(R)$ from Hochschild homology. Jones-Goodwillie factored the trace map

$$\mathrm{tr} : K(R) \to \mathrm{HC}^-(R) \to \mathrm{HH}(R)$$

and Goodwillie showed that it can sometimes be used to compute $K(R)$ rationally.

The seminal idea of Waldhausen was to change the ground ring from the integers to the sphere spectrum and this idea, in the context of Hochschild homology, was carried out by Bökstedt in the late eighties. He defined topological Hochschild homology in [3] based on earlier work by Breen in [2]. At this time, the modern symmetric monoidal categories of spectra had not been invented yet and Bökstedt developed a coherence machinery that enabled a definition of topological Hochschild homology realizing Waldhausen's vision. The theory associates a $\mathbb{T}$-spectrum $\mathrm{THH}(R)$ to a ring R, where $\mathbb{T} \subset \mathbb{C}$ denotes the circle group, and there is a linearization map $\mathrm{THH}(R) \to \mathrm{HH}(R)$, which should be thought of as being induced by the base change from the sphere spectrum to the integers. Furthermore, the Dennis trace map factors as follows

$$\mathrm{tr} : K(R) \to \mathrm{THH}(R) \to \mathrm{HH}(R).$$

Bökstedt-Hsiang-Madsen carried on to construct topological cyclic homology $\mathrm{TC}(R)$, which uses the $\mathbb{T}$-action on topological Hochschild homology in an intricate way, and introduced the cyclotomic trace factoring the topological trace

$$\mathrm{tr} : K(R) \to \mathrm{TC}(R) \to \mathrm{THH}(R).$$

The cyclotomic trace was constructed in the course of resolving the K-theoretic Novikov conjecture for groups satisfying a mild finiteness assumption in [1]. Dundas [5] and McCarthy [16] proceeded to show the following central theorem.

Theorem. *For a surjective homomorphism of rings $f : R \to S$ with nilpotent kernel, the square*

$$\begin{array}{ccc} K(R) & \xrightarrow{\text{tr}} & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ K(S) & \xrightarrow{\text{tr}} & \mathrm{TC}(S) \end{array}$$

becomes homotopy cartesian after completion at any prime number p.

Hesselholt and Madsen have used topological cyclic homology to make extensive calculations in algebraic K-theory including a computational resolution of the Quillen-Lichtenbaum conjecture for local number fields in [10].

Topological Hochschild homology has applications outside algebraic K-theory as well, exemplified by Hesselholt in [8], where topological Hochschild homology is used to give cohomological interpretations of the zeta function for smooth and proper schemes over a finite field.

A duality structure on the input of the algebraic K-theory spectrum induces additional structure on the K-theory. Hesselholt and Madsen defined real algebraic K-theory in [11], which associates a G-spectrum $KR(\mathcal{C}, D, \eta)$ to an exact category $\mathcal{C}$ with duality (D, η) and weak equivalences, where G is the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of order 2. The duality structure consists of an exact functor $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and a natural weak equivalence $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow D \circ D^{\mathrm{op}}$ such that the composition

$$D \xRightarrow{\eta \circ D} D \circ D^{\mathrm{op}} \circ D \xRightarrow{D \circ \eta^{\mathrm{op}}} D$$

is equal to the identity. A good example to keep in mind is the following. Let R be a ring with an anti-involution α, i.e. a ring-isomorphism $\alpha : R^{\mathrm{op}} \to R$ which squares to the identity, and let $\mathcal{P}(R)$ denote the category of finitely generated projective right R-modules with weak equivalences being the isomorphisms. We get a duality structure on $\mathcal{P}(R)$ by setting $D(P) = \mathrm{Hom}_R(P, R)$, with module structure given by $f \cdot r(p) = \alpha(r)f(p)$ and natural weak equivalence $\eta : P \to D(D(P))$ given by the isomorphism $p \mapsto (f \mapsto f(p))$. We let $KR(R, \alpha)$ be the real K-theory of $\mathcal{P}(R)$ with this duality structure. The underlying homotopy type of the real algebraic K-theory spectrum agrees with the usual Waldhausen K-theory of $\mathcal{C}$, and the G-fixed point spectrum agrees with Grothendieck-Witt-theory, or Hermitian K-theory as studied by Karoubi [15], Schlichting [19], Hornbostel [12], [13], and others.

Hesselholt and Madsen also constructed real topological Hochschild homology in [11] as a generalization of topological Hochschild homology to the G-equivariant setting of real algebraic K-theory. The theory takes a ring spectrum A with anti-involution D, and associates to it an $O(2)$-equivariant spectrum $\mathrm{THR}(A, D)$ using a dihedral variant of Bökstedt's model, and there is G-equivariant trace map

$$\mathrm{tr} : KR(A,D) \to \mathrm{THR}(A,D).$$

The generalization to real topological Hochschild homology leads to a G-equivariant version of topological cyclic homology, denoted by $\mathrm{TCR}(A,D)$, and the trace factors through $\mathrm{TCR}(A,D)$. In particular, this makes it possible to use trace methods as a computational tool in Hermitian K-theory.

We return to the example of spherical group rings, in which case the equivariant structure of real algebraic K-theory has geometric meaning. For a compact connected topological manifold M allowing a smooth structure, there is a geometric involution on the space of pseudo-isotopies giving by "turning a pseudo-isotopy upside down," see [7], which in turn induces an involution on the stable pseudo-isotopy space $\mathscr{P}(M)$. By work of Weiss and Williams [23], there is map

$$\widetilde{\mathrm{Homeo}}(M)/\,\mathrm{Homeo}(M) \to \mathscr{P}(M)_{hG}$$

which is at least as connected as the stabilization map $P(M) \to \mathscr{P}(M)$. Here the left hand space is the quotient of the space of block homeomorphisms by the space of homeomorphisms, and $\mathscr{P}(M)_{hG}$ are the homotopy orbits with respect to the involution. The generalization to real algebraic K-theory expresses the equivariant stable pseudo-isotopy space in terms of the real algebraic K-theory of spherical group rings with anti-involution, and an equivariant understanding of the real topological cyclic homology gives, via trace methods, information about the real algebraic K-theory. Paper A in this thesis takes a step in this direction by determining the G-homotopy type of $\mathrm{TCR}(\mathbb{S}[\Gamma])$, where the spherical group ring is equipped with the basic anti-involution induced by taking inverses in the group. In order to obtain geometric applications as indicated, it will be necessary to consider more general anti-involutions.

Real Hochschild homology

This section displays the extra structure on the Hochschild homology spectrum of a ring arising from an anti-involution. If R is a unital associative ring, then the Hochschild homology space $\mathrm{HH}(R)$ is the geometric realization of the simplicial set underlying the simplicial abelian group $\mathrm{HH}(R)_\bullet$, with k-simplices

$$\mathrm{HH}(R)_k = R^{\otimes k+1}$$

and with face and degeneracy maps given by

$$d_i(r_0 \otimes \cdots \otimes r_k) = \begin{cases} r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_k & \text{if } 0 \le i < k, \\ r_k r_0 \otimes r_1 \otimes \cdots \otimes r_{k-1} & \text{if } i = k, \end{cases}$$

$$s_i(r_0 \otimes \cdots \otimes r_k) = (r_0 \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_k), \quad \text{for } 0 \le i \le k.$$

The space $\mathrm{HH}(R)$ is the zeroth space of an Eilenberg-MacLane spectrum, which we also denote $\mathrm{HH}(R)$. The Hochschild homology groups of R are the homotopy groups of this spectrum

$$\mathrm{HH}_i(R) := \pi_i(\mathrm{HH}(R)).$$

The simplicial abelian group $\mathrm{HH}(R)_\bullet$ is a cyclic abelian group in the sense of Connes, meaning that the cyclic group of order $k+1$ acts on the k-simplices. The action is generated by a cyclic structure map $t_k : \mathrm{HH}(R)_k \to \mathrm{HH}(R)_k$ given by

$$t_k(r_0 \otimes \cdots \otimes r_k) = r_k \otimes r_0 \otimes \cdots \otimes r_{k-1}$$

which satisfies $t_k^{k+1} = \mathrm{id}$ and is suitably compatible with the simplicial structure maps. Connes discovered that this structure gives rise to a continuous $\mathbb{T}$-action on the spectrum

$$\mathbb{T}_+ \wedge \mathrm{HH}(R) \to \mathrm{HH}(R).$$

If R is equipped with an anti-involution α, then we can give even more structure to $\mathrm{HH}(R)_\bullet$. The dihedral group of order $2(k+1)$ acts on the k-simplices. The action is generated by the maps $t_k : \mathrm{HH}(R)_k \to \mathrm{HH}(R)_k$ and $w_k : \mathrm{HH}(R)_k \to \mathrm{HH}(R)_k$, given by

$$w_k(r_0 \otimes \cdots \otimes r_n) = \alpha(r_0) \otimes \alpha(r_n) \cdots \otimes \alpha(r_1),$$

which are suitably compatible with the simplicial structure maps. The maps satisfy the relations $t_k^{k+1} = \mathrm{id}$, $w_k^2 = \mathrm{id}$ and $t_k w_k = t_k^{-1} w_k$. This time, the action of the dihedral group of order $2(k+1)$ on the k-simplices gives rise to a continuous $O(2)$-action on the spectrum

$$O(2)_+ \wedge \mathrm{HH}(R) \to \mathrm{HH}(R).$$

We will denote the $O(2)$-spectrum by $\mathrm{HR}(R, \alpha)$ and refer to it as the real Hochschild homology of (R, α).

Real topological Hochschild homology

The topological Hochschild homology of a ring spectrum A is a (genuine) $\mathbb{T}$-spectrum and has a special extra structure. $\mathrm{THH}(A)$ is a cyclotomic spectrum, which means that its geometric fixed points mimic the behavior of the fixed points of free loop spaces, which we now explain. We let $\mathcal{L}X = \mathrm{Map}(\mathbb{T}, X)$ be the free loop space of a space X. The group $\mathbb{T}$ acts on the free loop space by multiplication in $\mathbb{T}$ and the map that takes a loop to the r-fold concatenation with itself,

$$p_r : \mathcal{L}X \to \rho_r^*(\mathcal{L}X)^{C_r}, \quad p_r(\gamma) = \gamma \star \cdots \star \gamma,$$

is an $\mathbb{T}$-equivariant homeomorphism, where $\rho_r : \mathbb{T} \to \mathbb{T}/C_r$ is the root isomorphism given by $\rho_r(z) = z^{\frac{1}{r}} C_r$.

We let $(\mathrm{THH}(A)^c)^{gC_r}$ denote the derived C_r-geometric fixed points. In analogy with the example of the free loop space, the cyclotomic structure of $\mathrm{THH}(A)$ is additional data in the form of compatible $\mathbb{T}$-equivariant maps

$$T_r : \rho_r^* \left((\mathrm{THH}(A)^c)^{gC_r} \right) \to \mathrm{THH}(A),$$

which induce weak equivalences on H-fixed points for all finite subgroups $H \leq \mathbb{T}$. One should note that recently a more flexible version of the notion of cyclotomic spectra was introduced by Nikolaus and Scholze; see [17].

The cyclotomic structure is crucial to the construction of topological cyclic homology and in order to obtain a G-equivariant version of topological cyclic homology, we must ensure that the added G-action on $\mathrm{THR}(A,D)$ is compatible with the cyclotomic structure. This is done in Paper A, Section 3.3. In order to state the compatibility, we introduce the following notation. The group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{T}$ and we let $O(2)$ denote the semi-direct product $\mathbb{T} \rtimes G$. We let $\rho_r : O(2) \to O(2)/C_r$ be the root isomorphism given by $\rho_r(z) = z^{\frac{1}{r}} C_r$ if $z \in \mathbb{T}$ and $\rho_r(x) = x$ if $x \in G$. Then we show that the underlying $\mathbb{T}$-equivariant maps are indeed $O(2)$-equivariant maps

$$T_r : \rho_r^* \left((\mathrm{THR}(A,D)^c)^{gC_r} \right) \to \mathrm{THR}(A,D),$$

and they induce weak equivalences on H-fixed points for all finite subgroups $H \leq O(2)$.

While the C_r-geometric fixed points resemble the topological Hochschild homology spectrum itself, the G-geometric fixed points behave differently. Considering the analogous situation for the free loop space, this is to be expected. Indeed, if X is a G-space, then the free loop space $\mathcal{L}X = \mathrm{Map}(\mathbb{T}, X)$ is an $O(2)$-space, with $\mathbb{T}$-action as described earlier and with G acting on $\mathbb{T}$ by complex conjugation and on the loop space by conjugation. If we let $\omega \in G$ denote complex conjugation, then there is a homeomorphism

$$\mathrm{Map}((I, \partial I), (X, X^G)) \to (\mathcal{L}X)^G; \quad \gamma \mapsto (\omega \cdot \gamma) \star \gamma.$$

The main theorem of the second paper computes the component group of the derived G-geometric fixed points of $\mathrm{THR}(R,\alpha)$ when R is an ordinary ring with anti-involution α. To state the theorem, which can be found in Paper B, Theorem 4.1, we let $(\mathrm{THR}(R,\alpha)^c)^{gG}$ denote the derived G-geometric fixed points of $\mathrm{THH}(R,\alpha)$ and we let $N : R \to R^\alpha$ be the norm map; $N(r) = r + \alpha(r)$.

Theorem. *Let R be a ring with an anti-involution α. There is an isomorphism of abelian groups*

$$\pi_0\big((\mathrm{THR}(R,\alpha)^c)^{gG}\big) \cong (R^\alpha/N(R) \otimes_{\mathbb{Z}} R^\alpha/N(R))/I$$

where I denotes the subgroup generated by the elements $\alpha(s)rs \otimes t - r \otimes st\alpha(s)$ for all $s \in R$ and $r, t \in R^\alpha$.

The theorem implies that the component group of the derived G-geometric fixed points vanish if 2 is invertible in the ring. If $x \in R^\alpha$, then $N(\frac{1}{2}x) = x$, and therefore the norm map surjects onto the fixed points of the anti-involution.

Real topological cyclic homology

The cyclotomic structure on the topological Hochschild homology of a ring spectrum A gives rise to restriction maps

$$R_n : \mathrm{THH}(A)^{C_{p^n}} \to \mathrm{THR}(A)^{C_{p^{n-1}}},$$

and classically, the topological cyclic homology at the prime p is built from the fixed point spectra $\mathrm{THH}(A)^{C_{p^n}}$ using the restriction maps and inclusions of fixed points. If D is an anti-involution on A, then the cyclotomic structure gives rise to G-equivariant maps

$$R_n : \mathrm{THR}(A, D)^{C_{p^n}} \to \mathrm{THR}(A, D)^{C_{p^{n-1}}},$$

and we can define the real topological cyclic homology G-spectrum at a prime p, $\mathrm{TCR}(A, D; p)$ by mimicking the definition by Bökstedt-Hsiang-Madsen.

The calculation of the topological cyclic homology of spherical group rings at a prime p conducted in Paper A, in particular, includes a calculation of the real topological cyclic homology of the sphere spectrum with the identity serving as anti-involution. In order to state the calculation, we let $\mathbb{P}^\infty(\mathbb{C})$ denote the infinite complex projective space with G acting by complex conjugation and we let $\Sigma^{1,1}$ denote suspension with respect to the sign representation of G. The following result can be found in Paper A, Corollary 5.4

Theorem. *After p-completion, there is an isomorphism in the G-stable homotopy category*

$$\mathrm{TCR}(\mathbb{S}, \mathrm{id}; p) \sim \Sigma^{1,1}\mathbb{P}^\infty_{-1}(\mathbb{C}) \vee \mathbb{S},$$

where $\Sigma^{1,1}\mathbb{P}^\infty_{-1}(\mathbb{C})$ denotes the homotopy fiber of the $\mathbb{T}$-transfer $\Sigma^\infty_G \Sigma^{1,1}\mathbb{P}^\infty(\mathbb{C}) \to \mathbb{S}$.

Future directions

The introduction of real algebraic K-theory, real topological Hochschild homology, real topological cyclic homology and the equivariant trace have opened many exciting new directions. We comment on future directions related to the results in the two papers.

Paper A determines the G-equivariant homotopy type of spherical group rings and, as mentioned, the long term goal of this program is to determine the canonical involution on the stable pseudo-isotopy space of a closed connected topological manifold. As a first step in this direction, it is necessary to investigate more general duality structures on spherical group rings.

The perspectives of the results in Paper B are of a computational nature. In order to make the equivariant trace an efficient calculational tool, we must understand the dihedral fixed points $\pi_* \mathrm{THR}(R, \alpha)^{D_r}$. Indeed, many classical calculations using trace methods rely on a good understanding of the cyclic fixed points. For example, the component ring of the C_{p^n}-fixed points, $\pi_0 \mathrm{THH}(R)^{C_{p^n}}$, is completely understood when R is commutative and p is a prime. Hesselholt and Madsen prove in [9], that there is a canonical ring isomorphism identifying $\pi_0 \mathrm{THH}(R)^{C_{p^n}}$ with the p-typical Witt vectors of length $n+1$, and one should be able to get a similar algebraic expression for the component ring of the dihedral fixed points $\pi_0 \mathrm{THR}(R, \alpha)^{D_{p^n}}$ when R is a commutative ring with anti-involution α. The computation of the components of the derived G-geometric fixed points should lead to an understanding of the components of the G-fixed points. From here, we can investigate the components of the dihedral fixed points and in turn $\pi_* \mathrm{THR}(R, \alpha)^{D_{p^n}}$.

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Part II

Papers

REAL TOPOLOGICAL CYCLIC HOMOLOGY OF SPHERICAL GROUP RINGS

AMALIE HØGENHAVEN

Abstract. We compute the G-equivariant homotopy type of the real topological cyclic homology of spherical group rings with anti-involution induced by taking inverses in the group, where G denotes the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The real topological Hochschild homology of a spherical group ring $\mathbb{S}[\Gamma]$, with anti-involution as described, is an $O(2)$-cyclotomic spectrum and we construct a map commuting with the cyclotomic structures from the $O(2)$-equivariant suspension spectrum of the dihedral bar construction on Γ to the real topological Hochschild homology of $\mathbb{S}[\Gamma]$, which induce isomorphisms on C_{p^n}- and D_{p^n}-homotopy groups for all $n \in \mathbb{N}_0$ and all primes p. Here C_{p^n} is the cyclic group of order p^n and D_{p^n} is the dihedral group of order $2p^n$. Finally, we compute the G-equivariant homotopy type of the real topological cyclic homology of $\mathbb{S}[\Gamma]$ at a prime p.

Contents

Date: January 1, 2017.
Assistance from DNRF Niels Bohr Professorship of Lars Hesselholt is gratefully acknowledged.

Introduction

This paper determines the G-equivariant homotopy type of the real topological cyclic homology of spherical group rings with anti-involution induced by taking inverses in the group, where G denotes the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of order 2. Bökstedt-Hsiang-Madsen calculated the topological cyclic homology of spherical group rings in [2, Section 5] and this is a generalization of the classical results. The long term goal of this program is to determine the canonical involution on the stable pseudo-isotopy space of a compact connected topological manifold. The equivariant stable pseudo-isotopy space of a manifold and the real topological cyclic homology of spherical group rings are connected via real algebraic K-theory, as explained below.

Recently, Hesselholt and Madsen defined real algebraic K-theory in [10], which associates a G-spectrum to a ring with involution. Real topological Hochschild homology was also constructed in [10] as a generalization of topological Hochschild homology to the G-equivariant setting. Real topological Hochschild homology associates an $O(2)$-equivariant orthogonal spectrum $\mathrm{THR}(A, D)$ to a ring spectrum A with anti-involution D using a dihedral variant of Bökstedt's model introduced in [3]. The generalization leads to a G-equivariant version of topological cyclic homology, which we denote $\mathrm{TCR}(A, D)$ and refer to as real topological cyclic homology. Hesselholt and Madsen constructed a G-equivariant trace map from the real algebraic K-theory to the real topological Hocschild homology, which factors through real topological cyclic homology.

The real algebraic K-theory of spherical group rings has close connections to the geometry of manifolds. Let M be a compact connected topological manifold admitting a smooth structure. A topological pseudo-isotopy of M is a homeomorphism $h : M \times [0,1] \to M \times [0,1]$ which is the identity on $\partial M \times [0,1] \cup M \times \{0\}$. We let $P(M)$ be the space of such homeomorphisms and we note that there is a stabilization map $P(M) \to P(M \times [0,1])$ given by crossing with the identity. The stable pseudo-isotopy space $\mathscr{P}(M)$ is defined as the homotopy colimit of the stabilization maps. Igusa showed in [12], building on work by Hatcher in [7], that the inclusion $P(M) \to \mathscr{P}(M)$ is k-connected if k is less than both $(\dim(M) - 7)/2$ and $(\dim(M) - 4)/3$. There is a geometric involution on $P(M)$ giving by "turning a pseudo-istopy upside down," see [8], which in turn induces an involution on $\mathscr{P}(M)$. By work of Weiss and Williams [24], there is map

$$\widetilde{\mathrm{Homeo}}(M)/\,\mathrm{Homeo}(M) \to \mathscr{P}(M)_{hG}$$

which is at least as connected as the stabilization map by Igusa, where the left hand space is the quotient of the space of block homeomorphisms of M by is the space of homeomorphisms of M, and $\mathscr{P}(M)_{hG}$ are the homotopy orbits with respect to the involution. Thus information about the involution on the stable pseudo-isotopy yields information about the self-homeomorphisms of the manifold. The stable psuedo-isotopy space can be expressed in terms of Waldhausen's algebraic K-theory of spaces. If we let Γ be the Kan loop group of M, (i.e a simplicial group such that M is weakly equivalent to the classifying space $B\Gamma$) then by work of Waldhuasen [21],[22] and Waldhausen-Rognes-Jahren [23], there is a cofibration sequence of spectra

$$K(\mathbb{S}) \wedge \mathrm{B}\,\Gamma_+ \to K(\mathbb{S}[\Gamma]) \to \Sigma^2\mathcal{P}(M) \to \Sigma\big(K(\mathbb{S}) \wedge \mathrm{B}\,\Gamma_+\big).$$

The long term goal of this project is to express the equivariant stable pseudo-isotopy space in terms of real algebraic K-theory of spherical group rings with anti-involution. An equivariant understanding of the real topological cyclic homology will via trace methods give information about the real algebraic K-theory and therefor yield information about the equivariant stable pseudo-isotopy space. In this paper we consider the basic anti-involution on the spherical group rings induced by taking inverses in the group. In order to obtain geometric applications as indicated, it will be necessary to consider more general anti-involutions.

We proceed to explain the content of this paper. In Section 1 we review the definition of the orthogonal $O(2)$-spectrum $\mathrm{THR}(A,D)$ and we establish an equivariant version of Bökstedt's approximation lemma.

In Section 2 we observe that the cyclotomic structure of the classical topological Hochschild homology spectrum is compatible with the G-action. The compatibility is crucial, if we want a G-equivariant version of topological cyclic homology. For this purpose we introduce the notation of an $O(2)$-cyclotomic spectrum, see Definition 2.6 for details. We let $\mathbb{T}$ denote the multiplicative group of complex numbers of modulus 1. The group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{T}$ and $O(2)$ is the semi-direct product $O(2) = \mathbb{T} \rtimes G$. Let

$$\rho_r : O(2) \to O(2)/C_r$$

be the root isomorphism given by $\rho_r(z) = z^{\frac{1}{r}}C_r$ if $z \in \mathbb{T}$ and $\rho_r(x) = x$ if $x \in G$ and let $\mathrm{THR}(A,D)^{gC_r}$ denote the C_r-geometric fixed points. The $O(2)$-cyclotomic structure is a collection of compatible $O(2)$-equivariant maps

$$T_r : \rho_r^*\left(\mathrm{THR}(A,D)^{gC_r}\right) \to \mathrm{THR}(A,D)$$

which induce weak equivalences on H-fixed points for all finite subgroups $H \leq O(2)$, when pre-composed with the canonical map from the derived C_r-geometric fixed points.

In Section 3 we observe that the $O(2)$-cyclotomic structure on $\mathrm{THR}(A,D)$ gives rise to G-equivariant restriction maps

$$R_n : \mathrm{THR}(A,D)^{C_{p^n}} \to \mathrm{THR}(A,D)^{C_{p^{n-1}}},$$

and we define the real topological cyclic homology at a prime p as an orthogonal G-spectrum $\mathrm{TCR}(A,D;p)$ by mimicking the classical definition by Bökstedt-Hsiang-Madsen in [2]. We define a G-spectrum $\mathrm{TRR}(A,D;p)$ as the homotopy limit over the R_n maps

$$\mathrm{TRR}(A,D;p) := \operatorname*{holim}_{n,R_n} \mathrm{THR}(A,D)^{C_{p^n}}.$$

The Frobenius maps, which are inclusion of fixed points, induce a self-map of the G-spectrum $\mathrm{TRR}(A,D;p)$, which we denote φ, and the real topological cyclic homology of (A,D) at p, $\mathrm{TCR}(A,D;p)$, is the homotopy equalizer of the maps

$$\mathrm{TRR}(A,D;p) \underset{\mathrm{id}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TRR}(A,D;p).$$

Section 4 computes the real topological Hochschild homology and the real topological cyclic homology of a spherical group ring $\mathbb{S}[\Gamma]$ with anti-involution $\mathrm{id}[\Gamma]$ induced by taking inverses in the group. In particular, we obtain a calculation of the real topological cyclic homology of the sphere spectrum with the identity serving as anti-involution.

In the rest of the introduction, we state our computational results for spherical group rings. The proofs of Theorems A, B and C below can be found in Section 4.

We let $B^{\mathrm{di}}\Gamma$ denote the geometric realization of the dihedral bar construction on Γ. The space $B^{\mathrm{di}}\Gamma$ is weakly equivalent to the free loop-space $\mathrm{Map}(\mathbb{T}, B\Gamma)$ by [13, Theorem 7.3.11]. It follows from [14, Section 2.1] and [20, Theorem 4.0.5] that the weak equivalence induces isomorphism on $\pi_*^H(-)$ for all finite subgroups $H \leq O(2)$, if we let $O(2)$ act on the free loop space as follows: The group $O(2)$ acts on $\mathbb{T}$ by multiplication and complex conjugation. Taking inverses in the group induces a G-action on $B\Gamma$ and we view $B\Gamma$ as an $O(2)$-space with trivial $\mathbb{T}$-action. Finally, $O(2)$ acts on the free loop space by the conjugation action. There are $O(2)$-equivariant homeomorphisms

$$p_r : B^{\mathrm{di}}\Gamma \xrightarrow{\cong} \rho_r^*\left(B^{\mathrm{di}}\Gamma^{C_r}\right),$$

which under the identification with the free loop space correspond to the maps, which take a loop to the r-fold concatenation with itself. These maps give the $O(2)$-equivariant suspension spectrum of $B^{\mathrm{di}}\Gamma$ an $O(2)$-cyclotomic structure.

Theorem A. *Let Γ be a topological group. There is a map of $O(2)$-orthogonal spectra*

$$i : \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+ \to \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]),$$

commuting with the cyclotomic structures, which induces isomorphisms on $\pi_*^{C_{p^n}}(-)$ and $\pi_*^{D_{p^n}}(-)$ for all $n \geq 0$ and all primes p.

The G-equivariant restriction maps

$$R_n : \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])^{C_{p^n}} \to \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])^{C_{p^{n-1}}}$$

admit canonical sections, which provide a canonical isomorphism in the G-stable homotopy category

$$\mathrm{TRR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p) \sim \prod_{j=0}^{\infty} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma),$$

where $\mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma)$ is the G-equivariant Borel group homology spectrum of the subgroup C_{p^j} acting on the $O(2)$-spectrum $\Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma$. We have a π_*-isomorphisms of G-spectra $c : \mathrm{H}.(1; B^{\mathrm{di}}\Gamma) \xrightarrow{c} \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+$ given by collapsing a certain classifying space in the construction of the homology spectrum. We let Δ_p denote the G-equivariant composition

$$B^{\mathrm{di}}\Gamma \xrightarrow{p_p} B^{\mathrm{di}}\Gamma^{C_p} \hookrightarrow B^{\mathrm{di}}\Gamma,$$

and we let $\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p=\mathrm{id}\text{”}}$ denote the homotopy equalizer of $\Sigma^\infty {\Delta_p}_+$ and the identity map. We have a canonical inclusion $\iota : \Omega(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+) \to \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p=\mathrm{id}\text{”}}$. The inclusion and the projection

$$\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ \xrightarrow{\mathrm{incl}} \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ \times \prod_{j=1}^{\infty} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) \xrightarrow{\mathrm{proj}} \prod_{j=1}^{\infty} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma)$$

induce the maps I and P in the theorem below, where the homotopy limit is constructed with respect to inclusions.

Theorem B. *The triangle*

$$\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p=\mathrm{id}\text{”}} \xrightarrow{I} \mathrm{TCR}(\mathbb{S}[\Gamma],\mathrm{id}[\Gamma];p) \xrightarrow{P} \operatorname*{holim}_{j\geq 1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma)$$
$$\xrightarrow{-\Sigma(\iota)\circ\varepsilon^{-1}\circ c\circ \mathrm{incl}\circ \mathrm{pr}_1} \Sigma\left(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p=\mathrm{id}\text{”}}\right)$$

is distinguished in the G-stable homotopy category, where $\varepsilon : \Sigma\Omega Y \to Y$ is the counit of the loop-suspension adjunction.

After p-completion, a non-equivariant identification of the homotopy limit in the triangle appears in [2], and the result generalizes immediately to the equivariant setting. We let $O(2)$ act on $\mathbb{C}$ by multiplication and complex conjugation, and on $\mathbb{C}^n$ by the diagonal action. We let

$$S(\mathbb{C}^\infty) = \bigcup_{n=0}^{\infty} S(\mathbb{C}^{n+1}),$$

where $S(\mathbb{C}^{n+1})$ denotes the unit sphere in $\mathbb{C}^{n+1}$. Then there is an isomorphism in the G-stable category after p-completion

$$\Sigma^{1,1} S(\mathbb{C}^\infty)_+ \wedge_{\mathbb{T}} \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+ \to \operatorname*{holim}_{j\geq 1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma),$$

where $\Sigma^{1,1}$ denotes suspension with respect to the sign representation of G. More specifically, a map of orthogonal G-spectra is an isomorphism in the G-stable category after p-completion, if it is an isomorphism in the stable category after p-completion on underlying spectra and derived G-fixed point spectra.

The theorem above leads to a calculation of the topological cyclic homology of the sphere spectrum with the identity as anti-involution. We let $\mathbb{P}^\infty(\mathbb{C}) := S(\mathbb{C}^\infty)/\mathbb{T}$ denote the infinite complex projective space with G acting by complex conjugation and we let $\Sigma^{1,1}$ denote suspension with respect to the sign representation of G.

Theorem C. *After p-completion, there is an isomorphism in the G-stable homotopy category*

$$\mathrm{TCR}(\mathbb{S},\mathrm{id};p) \sim \Sigma^{1,1}\mathbb{P}^\infty_{-1}(\mathbb{C}) \vee \mathbb{S},$$

where $\Sigma^{1,1}\mathbb{P}^\infty_{-1}(\mathbb{C})$ denotes the homotopy fiber of the $\mathbb{T}$-transfer $\Sigma^\infty_G\Sigma^{1,1}\mathbb{P}^\infty(\mathbb{C}) \to \mathbb{S}$.

Acknowledgments. I would like to express my sincere gratitude to my supervisor Lars Hesselholt for his inspiring guidance and support throughout this project. My thanks are due to Ib Madsen for originally encouraging my interest in topological cyclic homology and for teaching me about the classical results underlying this paper. I would also like to thank Irakli Patchkoria, Kristian Moi, and Cary Malkiewich for several helpful and inspiring conversations that have helped me write this paper.

Throughout this paper, $\mathbb{T}$ denotes the multiplicative group of complex numbers of modulus 1 and G is the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \omega\}$ of order 2. The group G acts on $\mathbb{T} \subset \mathbb{C}$ and $O(2)$ is the semi-direct product $O(2) = \mathbb{T} \rtimes G$. Let C_r denote the cyclic subgroup of order r, generated by the rth root of unity $t_r := e^{\frac{2\pi i}{r}}$. Let D_r denote the dihedral subgroup of order $2r$ generated by t_r and ω. The collection $\{C_r, D_r\}_{r \geq 0}$ represents all conjugacy classes of finite subgroups of $O(2)$.

By a space we will always mean a compactly generated weak Hausdorff space and any construction is always carried out in this category.

1. Real topological Hochschild homology

A symmetric ring spectrum X is a sequence of based spaces $X_0, X_1, \dots$ with a left based action of the symmetric group Σ_n on X_n and $\Sigma_n \times \Sigma_m$-equivariant maps $\lambda_{n,m} : X_n \wedge S^m \to X_{n+m}$. Let A be a symmetric ring spectrum with multiplication maps $\mu_{n,m} : A_n \wedge A_m \to A_{n+m}$ and unit maps $1_n : S^n \to A_n$. An anti-involution on A is a self-map of the underlying symmetric spectrum $D : A \to A$, such that

$$D^2 = \mathrm{id}, \quad D_n \circ 1_n = 1_n,$$

and the following diagram commutes:

$$\begin{array}{ccc}
A_m \wedge A_n & \xrightarrow{D_m \wedge D_n} & A_m \wedge A_n \\
 & & \big\downarrow \gamma \\
 & & A_n \wedge A_m \\
\big\downarrow \mu_{m,n} & & \big\downarrow \mu_{n,m} \\
 & & A_{n+m} \\
 & & \big\downarrow \chi_{n,m} \\
A_{m+n} & \xrightarrow{D_{m+n}} & A_{m+n}.
\end{array}$$

Here γ is the twist map and $\chi_{n,m} \in \Sigma_{n+m}$ is the shuffle permutation

$$\chi_{n,m}(i) = \begin{cases} i+m & \text{if } 1 \leq i \leq n \\ i-n & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

If A is commutative, then the identity defines an anti-involution on A.

If Γ is a topological group, then the spherical group ring $\mathbb{S}[\Gamma]$ is the symmetric suspension spectrum of Γ_+. If $m : \Gamma \times \Gamma \to \Gamma$ denotes the multiplication in Γ and $1 \in \Gamma$ is the unit, then the spherical group ring becomes a symmetric ring spectrum with multiplication maps defined to be the compositions

$$\Gamma_+ \wedge S^n \wedge \Gamma_+ \wedge S^m \xrightarrow{\mathrm{id} \wedge \gamma \wedge \mathrm{id}} (\Gamma \times \Gamma)_+ \wedge S^n \wedge S^m \xrightarrow{m_+ \wedge \mu_{n,m}} \Gamma_+ \wedge S^{n+m},$$

and unit maps $S^n \to \Gamma_+ \wedge S^n$ given by $z \mapsto 1 \wedge z$. Taking inverses in the group induce an anti-involution $\mathrm{id}[\Gamma]$ on the spherical group ring, where $\mathrm{id}[\Gamma]_n : \Gamma_+ \wedge S^n \to \Gamma_+ \wedge S^n$ is given by

$$\mathrm{id}[\Gamma]_n(g \wedge x) = g^{-1} \wedge x.$$

The real topological Hochschild homology space $\mathrm{THR}(A, D)$ of a symmetric ring spectrum A with anti-involution D was defined in [10, Sect. 10] as the geometric realization of a dihedral space. Before reviewing the definition, we recall the notion of dihedral sets, and their realization.

Definition 1.1. A dihedral object in a category $\mathcal{C}$ is a simplicial object

$$X[-] : \triangle^{\mathrm{op}} \to \mathcal{C}$$

together with dihedral structure maps $t_k, w_k : X[k] \to X[k]$ such that $t_k^{k+1} = \mathrm{id}$, $w_k^2 = \mathrm{id}$, and $t_k w_k = t_k^{-1}\omega_k$. The dihedral structure maps are required to satisfy the following relation involving the simplicial structure maps:

$$\begin{aligned} d_l w_k &= w_{k-1} d_{k-l}, \quad s_l w_k = w_{k+1} s_{k-l} \quad \text{for } 0 \leq l \leq k, \\ d_l t_k &= t_{k-1} d_{l-1}, \quad s_l t_k = t_{k+1} s_{l-1} \quad \text{for } 0 < l \leq k, \\ d_0 t_k &= d_k, \quad s_0 t_k = t_{k+1}^2 s_k. \end{aligned}$$

We let Sets denote the category of sets and set maps and we let $X[-]$ be a dihedral object in Sets. We use Drinfeld's realization of dihedral sets which is naturally homeomorphic as an $O(2)$-space to the ordinary geometric realization of the underlying simplicial set with $O(2)$-action arising from the dihedral structure. We briefly recall the method; see [18] and [5] for details.

Let $\mathcal{F}$ denote the category with objects all finite subsets of the circle and morphisms set inclusions. A dihedral set $X[-]$ extends uniquely, up to unique isomorphism, to a functor $X[-] : \mathcal{F} \to \mathsf{Sets}$. Given an inclusion $F \subset F' \subset \mathbb{T}$ the degeneracy maps give rise to a map $s_F^{F'} : X[F] \to X[F']$. As a set

$$|X[-]| := \operatorname*{colim}_{F \in \mathcal{F}} X[F].$$

Given an inclusion $F \subset F' \subset \mathbb{T}$ the face maps give rise to a map $d_F^{F'} : X[F'] \to X[F]$. These maps are used to define a topology on the colimit. Finally, the dihedral structure maps give rise to a continuous action of the homeomorphism group $\mathrm{Homeo}(\mathbb{T})$ on the colimit as follows. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ induces a functor $\mathcal{F} \to \mathcal{F}$ given on objects by $F \mapsto h(F)$. The dihedral structure maps give rise to a natural transformations $\phi_h : X[-] \Rightarrow X[-] \circ h$. The action of h on the colimit is given as the composition:

$$\operatorname*{colim}_{F \in \mathcal{F}} X[F] \xrightarrow{\phi_h} \operatorname*{colim}_{F \in \mathcal{F}} X[h(F)] \xrightarrow{\mathrm{ind}_h} \operatorname*{colim}_{F \in \mathcal{F}} X[F].$$

In particular, the subgroup $O(2) < \mathrm{Homeo}(\mathbb{T})$ acts on the realization. Note that the category $\mathcal{F}$ is filtered, and therefore the realization, as a functor valued in Sets, commutes with finite limits. This is also true as functors valued in Top, the category of compactly generated weak Hausdorff spaces; e.g. [18, Corollary 1.2]. The functor $X[-] : \mathcal{F} \to \mathsf{Sets}$ arising from a dihedral set is a special case of an $O(2)$-diagram, as defined below, and we recall how such diagrams behave.

Definition 1.2. Let H be a group and let J be a small category with a left H-action. An H-diagram indexed by J is a functor $X : J \to \mathsf{Sets}$ together with a collection of natural transformations

$$\alpha = \{h \in H \mid \alpha_h : X \Rightarrow X \circ h\}.$$

such that $\alpha_e = \mathrm{id}$ and $(\alpha_{h'})_h \circ \alpha_h = \alpha_{h'h}$. Here $(\alpha_{h'})_h$ is the natural transformation obtained by restricting $\alpha_{h'}$ along the functor $h : J \to J$.

The group H acts on the colimit of the diagram by letting $h \in H$ act as the composition

$$\operatorname*{colim}_J X \xrightarrow{\alpha_h} \operatorname*{colim}_J X \circ h \xrightarrow{\mathrm{ind}_h} \operatorname*{colim}_J X.$$

A natural transformation of H-diagrams indexed by J commuting with the group action induces an H-equivariant map of colimits. Let $\phi : K \to J$ be a functor between small categories with H-actions, such that $\phi \circ h = h \circ \phi$. If $(X : J \to \mathsf{Sets}, \alpha)$ is an H-diagram indexed by J, then $(X \circ \phi : K \to \mathsf{Sets}, \alpha_\phi)$ is an H-diagram indexed by K and the canonical map

$$\operatorname*{colim}_K X \circ \phi \xrightarrow{\mathrm{ind}_\phi} \operatorname*{colim}_J X$$

is H-equivariant. Finally, if J is filtered and $N \leq H$ is a normal subgroup acting trivially on J then $(X^N : J \to \mathsf{Sets}, \alpha^N)$ is an H/N-diagram indexed by J and the canonical inclusion $X(j)^N \hookrightarrow X(j)$ induces an H/N-equivariant bijection

$$\operatorname*{colim}_J (X^N) \xrightarrow{\cong} \big(\operatorname*{colim}_J X\big)^N.$$

In the following, we would like to apply Drinfeld's realization to the case of a dihedral pointed space $Y[-]$, but a priori Drinfeld's description of the topology on the realization only applies to simplical sets. We solve this problem as follows. We have a bijection of underlying sets from the ordinary geometric realization to the colimit as specified in [18] and [5]:

$$\Big(\bigvee_{n=0}^{\infty} Y[n] \times \Delta^n / \sim\Big) \;\to\; \operatorname*{colim}_{F \in \mathcal{F}} Y[F].$$

The bijection is $O(2)$-equivariant and natural with respect to functors of dihedral pointed spaces. We give the right hand colimit the topology which makes the above bijection a homeomorphism. Of course, if $Y[-]$ is discrete, then the topology in the description of Drinfeld's realization makes the bijection into a homeomorphism.

When we describe dihedral spaces in this work, we will give the space $X[F]$ at every object $F \in \mathcal{F}$, for each inclusion $F \subset F' \subset \mathbb{T}$, the maps $d_F^{F'} : X[F] \to X[F']$ and $s_F^{F'} : X[F'] \to X[F]$, and the transformations $X[F] \to X[h(F)]$ for $h \in \mathrm{Homeo}(\mathbb{T})$.

We are now ready to construct to real topological homology of a ring spectrum A with anti-involution D. First let I be the category with objects all non-negative integers. The morphisms from i to j are all injective set maps

$$\{1, \dots, i\} \to \{1, \dots, j\}.$$

The category I has a strict monoidal product $+ : I \times I \to I$ given on objects by addition and on morphisms by concatenation. We note that the initial object $0 \in \mathrm{Ob}(I)$ is the identity for the monoidal product. Given an object i let $\omega_i : i \to i$ denote the involution given by

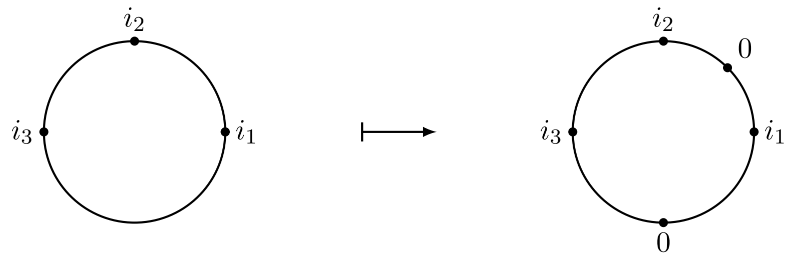
$$\omega_i(s) = i - s + 1,$$

and we define the conjugate of a morphism $\alpha : i \to j$ by $\alpha^\omega := \omega_j \circ \alpha \circ \omega_i^{-1}$.

Let $F \subset F'$ be finite subsets of the circle. We define $s_F^{F'} : I^F \to I^{F'}$ on objects by $(i_z)_{z \in F} \mapsto (i_{z'})_{z' \in F'}$ where

$$i_{z'} = \begin{cases} i_{z'} & \text{if } z' \in F \\ 0 & \text{if } z' \notin F. \end{cases}$$

Hence the functor repeats the initial object $0 \in I$ as pictured in the example:



This also defines the functor on morphisms, since 0 is the initial object in I.

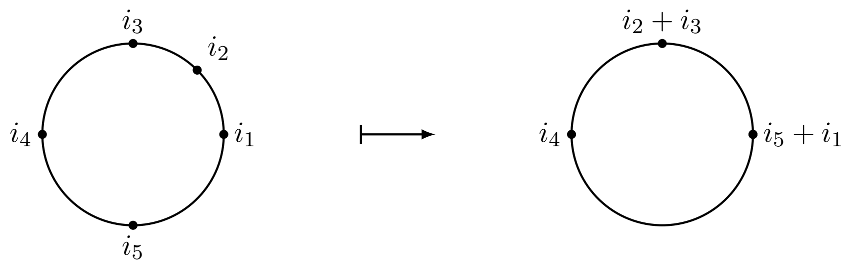
In order to define $d_F^{F'} : I^{F'} \to I^F$ we introduce to following notation. Given $z = e^{2\pi i s} \in F$, let $\overline{s} = \max_t \{s \leq t < 1 + s \mid e^{2\pi i t} \in F\}$. Then we set

$$\mathrm{Arc}_z := \{e^{2\pi i t} \mid \overline{s} < t \leq 1 + s\} \subset \mathbb{T},$$

hence Arc_z is the circle arc starting from the first element clockwise of z and ending at z. We define $d_F^{F'} : I^{F'} \to I^F$ on objects by $(i_{z'})_{z' \in F'} \mapsto (i_z)_{z \in F}$ where

$$i_z = \sum_{z' \in \mathrm{Arc}_z} i_{z'},$$

and similarly on morphisms. Hence the functor adds together the objects counter clockwise around the circle and concatenate morphisms, as pictured in the example:



Given a pointed left $O(2)$-space, we define a dihedral space $\mathrm{THR}(A, D; X)[-]$. Let $F \subset \mathbb{T}$ be finite and define a functor $G_X^F : I^F \to \mathsf{Top}_*$ on objects by

$$G_X^F\left((i_z)_{z \in F}\right) = \mathrm{Map}\Big(\bigwedge_{z \in F} S^{i_z}, \bigwedge_{z \in F} A_{i_z} \wedge X\Big).$$

We define G_X^F on morphisms in the case $|F| = 1$, the general case is similar. Let $\alpha : i \to j$ be a morphism in I. We write α as a composite $\alpha = \sigma \circ \iota$, where

$$\iota : \{1, \ldots, i\} \to \{1, \ldots, j\}$$

is the standard inclusion and $\sigma \in \Sigma_j$. The map $G_X^F(\alpha)$ is the composite

$$\mathrm{Map}(S^i, A_i \wedge X) \xrightarrow{S^{j-i}\wedge(-)} \mathrm{Map}(S^{j-i}\wedge S^i, S^{j-i}\wedge A_i \wedge X) \xrightarrow{(\lambda_{j-i,i}\wedge \mathrm{id})\circ(-)} \mathrm{Map}(S^j, A_j \wedge X) \xrightarrow{\mathrm{Map}(\sigma^{-1},\sigma\wedge\mathrm{id})} \mathrm{Map}(S^j, A_j \wedge X),$$

which is independent of choice of σ, since $\lambda_{j,i}$ is $\Sigma_j \times \Sigma_i$ - equivariant. We set

$$\mathrm{THR}(A, D; X)[F] := \operatorname*{hocolim}_{I^F} G_X^F.$$

Let $F \subset F'$. We define the map $\mathrm{THR}(A, D; X)[F] \to \mathrm{THR}(A, D; X)[F']$ by first defining a natural transformation of functors $(s')_F^{F'} : G_X^F \Rightarrow G_X^{F'} \circ s_F^{F'}$. At an object $(i_z)_{z\in F} \in I^F$ the natural transformation

$$\mathrm{Map}\Big(\bigwedge_{z\in F} S^{i_z}, \bigwedge_{z\in F} A_{i_z} \wedge X\Big) \to \mathrm{Map}\Big(\bigwedge_{z'\in F'} S^{i_{z'}}, \bigwedge_{z'\in F'} A_{i_{z'}} \wedge X\Big)$$

uses the identity map $1_0 : S^0 \to A_0$. We let $s_F^{F'}$ be the composition

$$\operatorname*{hocolim}_{I^F} G_X^F \xrightarrow{(s')_F^{F'}} \operatorname*{hocolim}_{I^F} G_X^{F'} \circ s_F^{F'} \xrightarrow{\mathrm{ind}_{s_F^{F'}}} \operatorname*{hocolim}_{I^{F'}} G_X^{F'}.$$

We define the map $\mathrm{THR}(A, D; X)[F'] \to \mathrm{THR}(A, D; X)[F]$ by first defining a natural transformation of functors $(d')_F^{F'} : G_X^{F'} \Rightarrow G_X^F \circ d_F^{F'}$. At an object $(i_{z'})_{z'\in F'} \in I^{F'}$ the natural transformation

$$\mathrm{Map}\Big(\bigwedge_{z'\in F'} S^{i_{z'}}, \bigwedge_{z'\in F'} A_{i_{z'}} \wedge X\Big) \to \mathrm{Map}\Big(\bigwedge_{z\in F} S^{i_z}, \bigwedge_{z\in F} A_{i_z} \wedge X\Big)$$

uses the multiplication maps in A. We let $d_F^{F'}$ be the composition

$$\operatorname*{hocolim}_{I^{F'}} G_X^{F'} \xrightarrow{(d')_F^{F'}} \operatorname*{hocolim}_{I^{F'}} G_X^F \circ d_F^{F'} \xrightarrow{\mathrm{ind}_{d_F^{F'}}} \operatorname*{hocolim}_{I^F} G_X^F.$$

Finally, we describe the action of $\mathrm{Homeo}(\mathbb{T})$. We restrict our attention to the subgroup $O(2)$ and describe the transformations induced by $t \in \mathbb{T}$ and complex conjugation $\omega \in G$. We define functors $t_F : I^F \to I^{t(F)}$ and $w_F : I^F \to I^{\omega(F)}$ on objects and morphisms by:

$$\begin{aligned} t_F :\ & (i_z)_{z\in F} \mapsto (i_{t^{-1}(y)})_{y\in t(F)}, && (\alpha_z)_{z\in F} \mapsto (\alpha_{t^{-1}(y)})_{y\in t(F)}, \\ w_F :\ & (i_z)_{z\in F} \mapsto (i_{\omega^{-1}(y)})_{y\in \omega(F)}, && (\alpha_z)_{z\in F} \mapsto (\alpha^{\omega}_{\omega^{-1}(y)})_{y\in \omega(F)}. \end{aligned}$$

Let $\omega_i \in \Sigma_i$ be the permutaion given by $\omega_i(s) = i - s + 1$. We define the natural transformations

$$t'_F : G_X^F \Rightarrow G_X^{t(F)} \circ t_F, \quad w'_F : G_X^F \Rightarrow G_X^{\omega(F)} \circ w_F,$$

to be the natural transformations which at $(i_z)_{z\in F} \in \mathrm{Ob}(I^F)$ are the unique maps making the diagrams

$$\begin{array}{ccc}
\bigwedge_{z\in F} S^{i_z} & \xrightarrow{\quad g \quad} & \bigwedge_{z\in F} A_{i_z} \wedge X \\
\uparrow & & \downarrow \\
\bigwedge_{y\in t(F)} S^{i_{t^{-1}(y)}} & \xrightarrow{\quad t'_F(g) \quad} & \bigwedge_{y\in t(F)} A_{i_{t^{-1}(y)}} \wedge X
\end{array}$$

and

$$\begin{array}{ccc}
\bigwedge_{z\in F} S^{i_z} & \xrightarrow{\quad g \quad} & \bigwedge_{z\in F} A_{i_z} \wedge X \\
{\scriptstyle \bigwedge_{z\in F}\omega_{i_z}}\uparrow & & \downarrow{\scriptstyle \bigwedge_{z\in F} D_{i_z}\circ\omega_{i_z}\wedge\mathrm{id}} \\
\bigwedge_{z\in F} S^{i_z} & & \bigwedge_{z\in F} A_{i_z} \wedge X \\
\uparrow & & \downarrow \\
\bigwedge_{y\in \omega(F)} S^{i_{\omega^{-1}(y)}} & \xrightarrow{\quad w'_F(g) \quad} & \bigwedge_{y\in \omega(F)} A_{i_{\omega^{-1}(y)}} \wedge X
\end{array}$$

commute. The unlabelled vertical maps are appropriate permutations of the smash factors. More precisely, given families of pointed spaces $\{X_k\}_{k\in K}$ and $\{X_j\}_{j\in J}$, then a set bijection $\phi : K \to J$ together with a collection of pointed maps $\phi_k : X_k \to X_{\phi(k)}$ induces a map of indexed smash products

$$\bigwedge_{k\in K} X_k \to \bigwedge_{j\in J} X_j, \quad (x_k)_{k\in K} \mapsto (\phi_{\phi^{-1}(j)}(x_{\phi^{-1}(j)}))_{j\in J}.$$

In the case at hand, the space maps are all identity maps and the set bijections $t(F) \to F$, $F \to t(F)$, $\omega(F) \to F$ and $F \to \omega(F)$ are given by t^{-1}, t, ω^{-1} and ω, respectively. The natural transformations are given at $F \in \mathcal{F}$ as the compositions

$$t_F : \operatorname*{hocolim}_{I^F} G^F_X \xrightarrow{t'_F} \operatorname*{hocolim}_{I^F} G^{t(F)}_X \circ t_F \xrightarrow{\mathrm{ind}_{t_F}} \operatorname*{hocolim}_{I^{t(F)}} G^{t(F)}_X,$$
$$w_F : \operatorname*{hocolim}_{I^F} G^F_X \xrightarrow{w'_F} \operatorname*{hocolim}_{I^F} G^{\omega(F)}_X \circ w_F \xrightarrow{\mathrm{ind}_{w_F}} \operatorname*{hocolim}_{I^{\omega(F)}} G^{\omega(F)}_X.$$

We have now defined a dihedral space and we let $\mathrm{THR}(A, D; X)$ be the realization

$$\mathrm{THR}(A, D; X) := \operatorname*{colim}_{F\in\mathcal{F}} \mathrm{THR}(A, D; X)[F]$$

with the topology given by identifying the colimit with the ordinary geometric realization. The space $\mathrm{THR}(A, D; X)$ is in fact an $O(2) \times O(2)$-space, where the action by the first factor comes from the dihedral structure and the action by the second factor comes from the $O(2)$-action on X. We are interested in $\mathrm{THR}(A, D)$ with the diagonal $O(2)$-action.

1.1. **Fixed points.** Let $\triangle : O(2) \to O(2) \times O(2)$ be the diagonal map. We wish to study the $O(2)/C_r$-space $(\triangle^* \operatorname{THR}(A, D; X))^{C_r}$. The image $\triangle(C_r)$ is not normal in $O(2) \times O(2)$, but it is normal in $\triangle(O(2))$, hence we consider $\operatorname{THR}(A, D; X)^{\triangle(C_r)}$ as an $\triangle(O(2))/\triangle(C_r)$-space.

Let $C_r\mathcal{F}$ denote the full subcategory of $\mathcal{F}$ with objects $C_r \cdot F$ for $F \in \operatorname{Ob}(\mathcal{F})$. This subcategory is both cofinal and stable under the group action, and therefore the inclusion $i : C_r\mathcal{F} \to \mathcal{F}$ induces a bijection of $\triangle(O(2))/\triangle(C_r)$-sets:

$$\Big(\operatorname*{colim}_{\mathcal{F}} \operatorname{THR}(A, D; X)[F]\Big)^{\triangle(C_r)} \xleftarrow{\operatorname{ind}_i} \Big(\operatorname*{colim}_{C_r\mathcal{F}} \operatorname{THR}(A, D; X)[C_r \cdot F]\Big)^{\triangle(C_r)}.$$

This is in fact a homeomorphism, since the corresponding map on classical geometric realizations is the homeomorphism from the realization of the r-subdivided simplicial space to the realization of the simplicial space itself. Since the C_r-action on the indexing category is trivial, the inclusion

$$\operatorname{THR}(A, D; X)[C_r \cdot F]^{\triangle(C_r)} \hookrightarrow \operatorname{THR}(A, D; X)[C_r \cdot F]$$

induces an $\triangle(O(2))/\triangle(C_r)$-bijection

$$\Big(\operatorname*{colim}_{C_r\mathcal{F}} \operatorname{THR}(A, D; X)[C_r \cdot F]\Big)^{\triangle(C_r)} \xleftarrow{\sim} \operatorname*{colim}_{C_r\mathcal{F}} \operatorname{THR}(A, D; X)[C_r \cdot F]^{\triangle(C_r)}.$$

This is likewise a homeomorphism, since it corresponds to commuting the classical geometric realization and C_r-fixed points of a C_r-simplicial space, which commutes by [16, Corollary 11.6].

Note that

$$\operatorname{THR}(A, D; X)[C_r \cdot F] = \operatorname*{hocolim}_{I^{C_r \cdot F}} G_X^{C_r \cdot F} = \Big|[k] \mapsto \bigvee_{\underline{i}_0 \to \cdots \to \underline{i}_k} G_X^{C_r \cdot F}(\underline{i}_0)\Big|.$$

The $\triangle(C_r)$-action on the homotopy colimit is simplicial and generated by a simplicial map which we now describe. Let $t_r := e^{\frac{2\pi i}{r}} \in \mathbb{T}$ and recall that we constructed a functor $(t_r)_{C_r \cdot F} : I^{C_r \cdot F} \to I^{C_r \cdot F}$ and a natural transformation

$$t'_{C_r \cdot F} : G_X^{C_r \cdot F} \Rightarrow G_X^{C_r \cdot F} \circ t_{C_r \cdot F}.$$

The t_r-action on X gives rise to a natural transformation

$$X_{t_r} : G_X^{C_r \cdot F} \circ (t_r)_{C_r \cdot F} \Rightarrow G_X^{C_r \cdot F} \circ (t_r)_{C_r \cdot F}.$$

The $\triangle(C_r)$-action is generated by the simplicial map, which takes the summand indexed by $\underline{i}_0 \to \cdots \to \underline{i}_k$ to the one indexed by $(t_r)_{C_r \cdot F}(\underline{i}_0) \to \cdots \to (t_r)_{C_r \cdot F}(\underline{i}_k)$ via

$$G_X^{C_r \cdot F}(\underline{i}_0) \xrightarrow{(t'_r)_{C_r \cdot F}} G_X^{C_r \cdot F} \circ (t_r)_{C_r \cdot F}(\underline{i}_0) \xrightarrow{X_{t_r}} G_X^{C_r \cdot F} \circ (t_r)_{C_r \cdot F}(\underline{i}_0). \tag{1}$$

If a k-simplex is fixed, then it must belong to a wedge summand whose index consists of objects and morphisms in $I^{C_r \cdot F}$ which are fixed by $(t_r)_{C_r \cdot F}$. This is exactly the image of the diagonal functor

$$\triangle_r : I^{C_r \cdot F / C_r} \to I^{C_r \cdot F}, \quad (i_{\overline{z}})_{\overline{z} \in C_r \cdot F / C_r} \mapsto (i_{\overline{z}})_{z \in C_r \cdot F},$$

which is defined similarly on morphism and where $\overline{z}$ denotes the orbit of $z \in C_r \cdot F$. The natural transformation (1) restricts to a natural transformation from $G_X^{C_r \cdot F} \circ \triangle_r$ to itself, hence C_r acts on $G_X^{C_r \cdot F} \circ \triangle_r$ through natural transformations. Since geometric realization commutes with finite limits by [16, Corollary 11.6], we obtain the following lemma:

Lemma 1.3. *The canonical map induces a homeomorphism of non-equivariant spaces:*

$$\operatorname*{hocolim}_{I^{C_r\cdot F/C_r}} \left(G_X^{C_r\cdot F}\circ\triangle_r\right)^{C_r} \xrightarrow{\cong} \left(\operatorname*{hocolim}_{I^{C_r\cdot F}} G_X^{C_r\cdot F}\right)^{C_r}.$$

Furthermore, the maps assemble into an isomorphism of $\triangle(O(2))/\triangle(C_r)$-diagrams $C_r\mathcal{F}\to\mathsf{Top}_*$.

Next, we consider the non-equivariant fixed point space $\mathrm{THR}(A,D;X)^{\triangle(D_r)}$. Let $D_r\mathcal{F}$ denote the full subcategory with objects $D_r\cdot F$ for $F\in\mathrm{Ob}(\mathcal{F})$. For convenience we restrict further to the cofinal subcategory $D_r\mathcal{F}_*$ consisting of objects of $D_r\mathcal{F}$ containing 1 and $t_{2r}:=e^{\frac{2\pi i}{2r}}$. There is a canonical homeomorphism constructed as above

$$\left(\operatorname*{colim}_{\mathcal{F}}\mathrm{THR}(A,D;X)[F]\right)^{\triangle(D_r)}\xleftarrow{\sim}\operatorname*{colim}_{D_r\mathcal{F}_*}\mathrm{THR}(A,D;X)[D_r\cdot F]^{\triangle(D_r)}.$$

Note that

$$\mathrm{THR}(A,D;X)[D_r\cdot F]=\operatorname*{hocolim}_{I^{D_r\cdot F}}G_X^{D_r\cdot F}=\Big|[k]\mapsto\bigvee_{\underline{i}_0\to\cdots\to\underline{i}_k}G_X^{D_r\cdot F}(\underline{i}_0)\Big|.$$

The $\triangle(D_r)$-action on the homotopy colimit is simplicial and generated by two simplicial maps, which we now describe. The first map takes the summand indexed by $\underline{i}_0\to\cdots\to\underline{i}_k$ to the one indexed by $w_{D_r\cdot F}(\underline{i}_0)\to\cdots\to w_{D_r\cdot F}(\underline{i}_k)$ via

$$G_X^{D_r\cdot F}(\underline{i}_0)\xrightarrow{w'_{D_r\cdot F}}G_X^{D_r\cdot F}\circ w_{D_r\cdot F}(\underline{i}_0)\xrightarrow{X_\omega}G_X^{D_r\cdot F}\circ w_{D_r\cdot F}(\underline{i}_0). \tag{2}$$

The second map takes the summand indexed by $\underline{i}_0\to\cdots\to\underline{i}_k$ to the one indexed by $(t_r)_{D_r\cdot F}(\underline{i}_0)\to\cdots\to(t_r)_{D_r\cdot F}(\underline{i}_k)$ via

$$G_X^{D_r\cdot F}(\underline{i}_0)\xrightarrow{(t'_r)_{D_r\cdot F}}G_X^{D_r\cdot F}\circ(t_r)_{D_r\cdot F}(\underline{i}_0)\xrightarrow{X_{t_r}}G_X^{D_r\cdot F}\circ(t_r)_{D_r\cdot F}(\underline{i}_0). \tag{3}$$

If a k-simplex is fixed, then it must belong to a wedge summand whose index consists of objects and morphisms in $I^{D_r\cdot F}$ which are fixed by the D_r-action. In order to describe such objects and morphisms, we note that a fundamental domain for the D_r-action on $\mathbb{T}$ is given by

$$F_{D_r}=\{e^{2\pi it}\mid 0\le t\le 1/2r\}\subset\mathbb{T}.$$

Let z be in the interior of the circle arc F_{D_r}. Then $\overline{z}:=D_r\cdot z=C_r\cdot z\amalg C_r\cdot\omega(z)$ while the orbits of the endpoints 1 and t_{2r} are of the form $\overline{1}=C_r\cdot 1=C_r\cdot\omega(1)$ and $\overline{t_{2r}}=C_r\cdot t_{2r}=C_r\cdot\omega(t_{2r})$.

The objects of $I^{D_r\cdot F}$ are permuted by the generators, while the morphisms are permuted by t_r and both permuted and conjugated by ω. Let I^G denote the subcategory of I with the same objects and all morphisms α which satisfies $\alpha^\omega=\alpha$. Define a functor

$$\triangle_r^e:(I^G)^{\{\overline{1},\overline{t_{2r}}\}}\times I^{D_rF/D_r\setminus\{\overline{1},\overline{t_{2r}}\}}\to I^{D_r\cdot F}$$

on objects by

$$\triangle_r^e((i_{\overline{z}})_{\overline{z}\in D_r\cdot F/D_r})=(i_{\overline{z}})_{z\in D_r\cdot F}$$

and on morphisms by

$$\triangle_r^e((\alpha_{\overline{z}})_{\overline{z}\in D_r\cdot F/D_r}) = (\alpha_z)_{z\in D_r\cdot F},$$

where

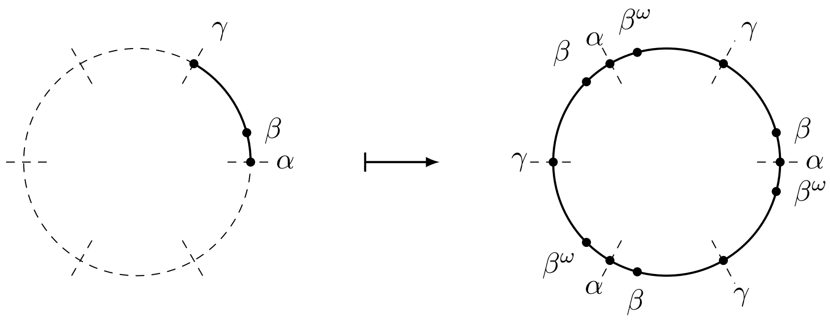
$$\alpha_z = \begin{cases} \alpha_{\overline{y}} & \text{if } z \in C_r \cdot y \\ (\alpha_{\overline{y}})^\omega & \text{if } z \in C_r \cdot \omega(y), \end{cases}$$

and $y \in F_{D_r}$. Then a fixed k-simplex must have an index which consist of elements and morphisms in the image of $\triangle_r^e$.

To clarify the "diagonal" functor $\triangle_r^e$, we draw an example of the functor

$$\triangle_3^e : (I^G)^{\{\overline{1},\overline{t_6}\}} \times I^{D_3F/D_3\setminus\{\overline{1},\overline{t_6}\}} \to I^{D_3\cdot F}$$

on a tuple of morphisms. In the picture below $D_3 \cdot F$ is the subset consisting of all the points on the right hand circle, and the points of the left hand circle are orbit representatives.



The natural transformations (2) and (3) restrict to transformations from the functor $G_X^{D_r\cdot F} \circ \triangle_r^e$ to itself, hence D_r acts on $G_X^{D_r\cdot F} \circ \triangle_r^e$ trough natural transformation. To ease notation we set $I^{D_rF/D_r^-} := I^{D_rF/D_r\setminus\{\overline{1},\overline{t_{2r}}\}}$. Likewise, we omit the over-lines on the orbits 1 and t_{2r}. We obtain the following lemma:

Lemma 1.4. *The canonical map induces a homeomorphism:*

$$\underset{(I^G)^{\{1,t_{2r}\}}\times I^{D_rF/D_r^-}}{\operatorname{hocolim}} (G_X^{D_r\cdot F} \circ \triangle_r^e)^{D_r} \xrightarrow{\cong} \left(\underset{I^{D_r\cdot F}}{\operatorname{hocolim}}\, G_X^{D_r\cdot F}\right)^{D_r}.$$

Furthermore, the maps assemble into an isomorphism of functors $D_r\mathcal{F}_* \to \mathsf{Top}_*$.

Remark 1.5. We describe the D_r-action on

$$G_X^{D_r\cdot F} \circ \triangle_r^e(i) = \operatorname{Map}\Big(\bigwedge_{z\in D_r\cdot F} S^{i_{\overline{z}}}, \bigwedge_{z\in D_r\cdot F} A_{i_{\overline{z}}} \wedge X\Big)$$

at an object $i \in \operatorname{Ob}\big((I^G)^{\{1,t_{2r}\}} \times I^{D_r\cdot F/D_r^-}\big)$ arising from the natural transformations (2) and (3). The spaces

$$\bigwedge_{z\in D_r\cdot F} S^{i_{\overline{z}}} \quad \text{and} \quad \bigwedge_{z\in D_r\cdot F} A_{i_{\overline{z}}} \wedge X$$

are themselves D_r-spaces. On the left hand side, the generator t_r permutes the smash factors and the generator ω permutes the smash factors, then acts by $w_i \in \Sigma_i$ factorwise. On the right hand side the generator t_r permutes the smash factors and acts on X, and the generator ω permutes the smash factors, then acts by $w_i \in \Sigma_i$ and the anti-involution D factor-wise and acts on X. The D_r-action on the mapping space is by conjugation.

1.2. **Equivariant Approximation Lemma.** In this section we consider the functor $G^F := G^F_{S^0}$ for simplicity. The approximation lemma proven below also holds if S^0 is replaced by S^V for any finite dimensional real orthogonal $O(2)$-representation V. Bökstedt's Approximation Lemma states that the canonical inclusion

$$G^F(i) \hookrightarrow \operatorname*{hocolim}_{I^F} G^F$$

can be made as connected as desired by choosing i coordinate-wise big enough under some connectivity conditions on A. Dotto proves in [4] a G-equivariant version of the approximation lemma under some restricted connectivity conditions on (A, D). We use the same connectivity conditions and extend the result to a D_r-equivariant version. For an integer n, we let $\lceil \frac{n}{2} \rceil$ denote the ceiling of $\frac{n}{2}$. Throughout this rest of this paper we make the following connectivity assumptions on (A, D), which hold for $(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])$:

Assumptions 1.6. Let (A, D) be a symmetric ring spectrum with anti-involution. We assume that A_n is $(n-1)$-connected and that the fixed points space $(A_n)^{D_n \circ \omega_n}$ is $\left(\lceil \frac{n}{2} \rceil - 1\right)$-connected. Here $\omega_n \in \Sigma_n$ is the permutation that reverses the order of the elements and $D_n : A_n \to A_n$ is the anti-involution in level n. Furthermore, we assume that there exists a constant $\epsilon \geq 0$, such that the structure map $\lambda_{n,m} : A_n \wedge S^m \to A_{n+m}$ is $(2n + m - \epsilon)$-connected as a map on non-equivariant spaces and the restriction of the structure map $\lambda_{n,m} : A_n^{D_n \circ \omega_n} \wedge (S^m)^{\omega_m} \to (A_{n+m})^{D_{n+m} \circ (\omega_n \times \omega_m)}$ is $(n + \lceil \frac{m}{2} \rceil - \epsilon)$-connected.

Before we state the Equivariant Approximation Lemma, we introduce some notation. Let $f : Z \to Y$ be a map of pointed left H-spaces, where H denotes a compact Lie group. We call f n-connected respectively a weak-H-equivalence if $f^K : Z^K \to Y^K$ is n-connected respectively a weak equivalence for all closed subgroups $K \leq H$. For a natural number $N \in \mathbb{N}$ and $i \in \mathrm{Ob}(I^F)$ we say that $i \geq N$ if $i_z \geq N$ for all $z \in F$. We define a partial order on the set $\mathrm{Ob}(I^F)$ by declaring $(i)_{z \in F} \leq (j)_{z \in F}$ if $i_z \leq j_z$ for all $z \in F$. Given a partially ordered set J we say that a map $\lambda : J \to \mathbb{N}$ tends to infinity on J if for all $N \in \mathbb{N}$ there is a $j_N \in J$ such that $\lambda(j) \geq N$ for all $j \geq j_N$.

Proposition 1.7 (Equivariant Approximation Lemma)**.** *Let (A, D) be a symmetric ring spectrum with anti-involution satisfying Assumptions 1.6. Let F be a finite subset of the circle. For part* (ii) *assume further that $1, t_{2r} \in D_r \cdot F$. For part* (iii) *assume further that $1, t_{4r} \in D_{2r} \cdot F$. Then the following holds:*

i *Given $n \geq 0$, there exists $N \geq 0$ such that the C_r-equivariant inclusion*

$$G^{C_r \cdot F} \circ \triangle_r(i) \hookrightarrow \operatorname*{hocolim}_{I^{C_r \cdot F}} G^{C_r \cdot F}$$

is n-connected for all $i \in \mathrm{Ob}(I^{C_r \cdot F / C_r})$ such that $i \geq N$.

ii *Let r be odd. Given $n \geq 0$, there exists $N \geq 0$ such that the D_r-equivariant inclusion*

$$G^{D_r \cdot F} \circ \triangle_r^e(i) \hookrightarrow \operatorname*{hocolim}_{I^{D_r \cdot F}} G^{D_r \cdot F}$$

is n-connected for all $i \in \mathrm{Ob}\left((I^G)^{\{1, t_{2r}\}} \times I^{D_r \cdot F / D_r^-}\right)$ such that $i \geq N$.

iii Let r be even. Given $n \geq 0$, there exists $N \geq 0$ such that the D_{2r}-equivariant inclusion

$$G^{D_{2r}\cdot F} \circ \triangle^e_{2r}(i) \hookrightarrow \operatorname*{hocolim}_{I^{D_{2r}\cdot F}} G^{D_{2r}\cdot F}$$

is n-connected for all $i \in \mathrm{Ob}\left((I^G)^{\{1,t_{4r}\}} \times I^{D_{2r}\cdot F/D^-_{2r}}\right)$ such that $i \geq N$, when considered as a map of D_r-spaces.

The reason that we distinguish between r even and r odd is because the conjugacy classes of subgroups of D_r behave differently in the two cases. Recall that if s divides r, then D_s denotes the dihedral subgroup of order $2s$ generated by the rotation $t_s \in \mathbb{T}$ and complex conjugation $\omega \in G$. If r is odd, then the collection $\{C_s, D_s\}_{s|r}$ represents all conjugacy classes of subgroups in D_r. If r is even, then D_r contains 2 conjugacy classes of dihedral subgroups of order $2s$, represented by

$$D_s = \langle t_s, \omega\rangle\,, \quad D'_s = \langle t_s, t_r\omega\rangle\,.$$

Hence the collection $\{C_s, D_s, D'_s\}_{s|r}$ represents all conjugacy classes of subgroups in D_r, when r is even. The subgroups D_s and D'_s become conjugate when considered as subgroups of the bigger group D_{2r}.

The result depends on the categories I^F and I^G being good indexing categories and on the connectivity of the functor G^F. Part (i) is proven in [2]. Part (ii) is proven in the case $r = 1$ in [4, Prop. 4.3.2]. We prove part (ii) and (iii) below.

Definition 1.8. A good indexing category is a triple $(J, \overline{J}, \mu)$ where J is a small category, $\overline{J} \subset J$ is a full subcategory and $\mu = \{\mu_j : J \to J\}_{j \in \mathrm{Ob}(\overline{J})}$ is a family of functors. The data is required to satisfy that for every $j \in \mathrm{Ob}(\overline{J})$, there exists a natural transformation $U : \mathrm{id} \Rightarrow \mu_j$ such that $\mu_j \circ U = U \circ \mu_j$.

Let I_{ev} denote the full subcategory of I with objects the even non-negative integers. For $j \in \mathrm{Ob}(I_{\mathrm{ev}})$ let $\mu_j : I \to I$ denote the functor

$$\mu_j(i) = \frac{j}{2} + i + \frac{j}{2}, \qquad \mu_j(\alpha) = \mathrm{id}_{\frac{j}{2}} + \alpha + \mathrm{id}_{\frac{j}{2}}.$$

There is a natural transformation $U : \mathrm{id} \Rightarrow \mu_j$ defined by the middle inclusion. The triple $(I, I_{\mathrm{ev}}, \mu)$ is a good indexing category and the structure restricts to I^G. In [4, Lemma 4.3.8] the following lemma is proved:

Lemma 1.9. *Let $(J, \overline{J}, \mu)$ be a good indexing category with initial object $0 \in \mathrm{Ob}(J)$ such that $\mu_j(0) = j$ for all $j \in \mathrm{Ob}(\overline{J})$. Let $X : J \to \mathsf{Top}_*$ be a functor. If for $j \in \mathrm{Ob}(\overline{J})$*

$$X(j) = X(\mu_j(0)) \to X(\mu_j(i)),$$

induced by $0 \to i$ is n-connected for all $i \in \mathrm{Ob}(J)$, then the canonical map

$$X(j) \to \operatorname*{hocolim}_{J} X$$

is n-connected.

Proof of part (ii). We let $j \in \mathrm{Ob}\left((I^G)^{\{1,t_{2r}\}} \times I^{D_r F/D^-_r}\right)$ and consider the inclusion

$$\iota_j : G^{D_r\cdot F} \circ \triangle^e_r(j) \hookrightarrow \operatorname*{hocolim}_{I^{D_r\cdot F}} G^{D_r\cdot F}.$$

We must show that the connectivity of the induced map on H-fixed points tends to infinity with j for $H \in \{C_s, D_s\}_{s|r}$. It suffices by part (i) to prove that the connectivity on D_r-fixed points tends to infinity with j. Indeed, if $s \cdot t = r$, then $D_r \cdot F = D_s \cdot F'$ and $D_r \cdot F = C_s \cdot F''$ for finite subsets $F', F'' \subset \mathbb{T}$, and

$$G^{D_r \cdot F} \circ \triangle_r^e(j) = G^{D_s \cdot F'} \circ \triangle_s(j'), \quad G^{D_r \cdot F} \circ \triangle_r^e(j) = G^{C_s \cdot F''} \circ \triangle_s(j'').$$

for some $j' \in \mathrm{Ob}\left((I^G)^{\{1,t_{2s}\}} \times I^{D_s \cdot F'/D_s^-}\right)$ and $j'' \in \mathrm{Ob}(I^{C_s \cdot F''/C_s})$ both of which tend to infinity with j.

By Lemma 1.4 the restriction of ι_j to D_r-fixed points is equal to the inclusion

$$\left(G^{D_r \cdot F} \circ \triangle_r^e(j)\right)^{D_r} \hookrightarrow \operatorname*{hocolim}_{(I^G)^{\{1,t_{2r}\}} \times I^{D_r F/D_r^-}} \left(G^{D_r \cdot F} \circ \triangle_r^e\right)^{D_r}.$$

Assume first that j is coordinate-wise even. We let $i \in \mathrm{Ob}\left((I^G)^{\{1,t_{2r}\}} \times I^{D_r \cdot F/D_r^-}\right)$ and define the map Λ to be the composite

$$\bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}} \wedge \bigwedge_{z \in D_r \cdot F} S^{i_{\overline{z}}} \xrightarrow{\cong} \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}} \wedge S^{i_{\overline{z}}} \xrightarrow{\bigwedge_{z \in D_r \cdot F} \lambda_{j_{\overline{z}}, i_{\overline{z}}}} \bigwedge_{z \in D_r \cdot F} A_{i_{\overline{z}} + j_{\overline{z}}}.$$

If we give the domain the diagonal D_r-action and we let D_r act on the target as described in Remark 1.5 but using the permutations $\omega_{j_{\overline{z}}} \times \omega_{i_{\overline{z}}}$ instead of $\omega_{j_{\overline{z}} + i_{\overline{z}}}$, then Λ is D_r-equivariant. The D_r-map $G^{D_r \cdot F} \circ \triangle_r^e(\mu_j(0)) \to G^{D_r \cdot F} \circ \triangle_r^e(\mu_j(i))$ induced by the morphisms $0 \to i$ is equal to the composite:

$$\begin{array}{c}
\mathrm{Map}\Big(\bigwedge_{z \in D_r \cdot F} S^{j_{\overline{z}}}, \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}} \Big) \\
\Big\downarrow {\scriptstyle (-) \wedge (\wedge_{z \in D_r \cdot F} S^{i_{\overline{z}}})} \\
\mathrm{Map}\Big(\bigwedge_{z \in D_r \cdot F} S^{j_{\overline{z}}} \wedge \bigwedge_{z \in D_r \cdot F} S^{i_{\overline{z}}}, \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}} \wedge \bigwedge_{z \in D_r \cdot F} S^{i_{\overline{z}}} \Big) \\
\Big\downarrow {\scriptstyle \Lambda^*} \\
\mathrm{Map}\Big(\bigwedge_{z \in D_r \cdot F} S^{j_{\overline{z}} + i_{\overline{z}}}, \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}} + i_{\overline{z}}} \Big) \\
\Big\downarrow {\scriptstyle \mathrm{Map}(\wedge \alpha_{\overline{z}}^{-1}, \wedge \alpha_{\overline{z}})} \\
\mathrm{Map}\Big(\bigwedge_{z \in D_r \cdot F} S^{j_{\overline{z}} + i_{\overline{z}}}, \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}} + i_{\overline{z}}} \Big),
\end{array}$$

where $\alpha_{\overline{z}} = id_{j_{\overline{z}}/2} \times \xi_{j_{\overline{z}}/2, i_{\overline{z}}} \in \Sigma_{j_{\overline{z}} + i_{\overline{z}}}$. The first map is induced by the adjunction unit,

$$\eta : \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}} \to \mathrm{Map}\Big(\bigwedge_{z \in D_r \cdot F} S^{i_{\overline{z}}}, \bigwedge_{z \in D_r \cdot F} S^{i_{\overline{z}}} \wedge \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}} \Big),$$

followed by the adjunction homeomorphism and a twist homeomorphism. We use the Equivariant Suspension Theorem A.3 to estimate its connectivity. If $s \mid r$ and we set $A := \bigwedge_{z \in D_r \cdot F} A_{j_{\overline{z}}}$, then we have homeomorphisms induced by the appropriate diagonal maps and connectivity estimates following from Assumptions 1.6:

$$A^{C_s} \cong \Big(A_{j_1} \wedge A_{j_{t_{2r}}} \wedge \Big(\bigwedge_{\overline{z}\in D_rF/D_r^-} A_{j_{\overline{z}}}\Big)^{\wedge 2}\Big)^{\wedge r/s},$$

$$\operatorname{conn}(A^{C_s}) \geq \frac{r}{s}\Big(j_1 + j_{t_{2r}} + \sum_{\overline{z}\in D_rF/D_r^-} 2j_{\overline{z}}\Big) - 1,$$

$$A^{D_s} \cong A_{j_1}^{D\circ\omega} \wedge A_{j_{t_{2r}}}^{D\circ\omega} \wedge \big(A_{j_1} \wedge A_{j_{t_{2r}}}\big)^{\wedge \frac{r-s}{2s}} \wedge \Big(\bigwedge_{\overline{z}\in D_rF/D_r^-} A_{j_{\overline{z}}}\Big)^{\wedge \frac{r}{s}},$$

$$\operatorname{conn}(A^{D_s}) \geq \frac{r}{2s}\Big(j_1 + j_{t_{2r}} + \sum_{\overline{z}\in D_rF/D_r^-} 2j_{\overline{z}}\Big) - 1.$$

By the Equivariant Suspension Theorem A.3 and Lemma A.1 wee see that

$$\operatorname{conn}(\eta^{C_s}) \geq \frac{2r}{s}\Big(j_1 + j_{t_{2r}} + \sum_{\overline{z}\in D_rF/D_r^-} 2j_{\overline{z}}\Big) - 1,$$

$$\operatorname{conn}(\eta^{D_s}) \geq \frac{r}{s}\Big(j_1 + j_{t_{2r}} + \sum_{\overline{z}\in D_rF/D_r^-} 2j_{\overline{z}}\Big) - 1,$$

$$\operatorname{conn}((\eta^*)^{D_r}) \geq \frac{j_1}{2} + \frac{j_{t_{2r}}}{2} + \sum_{\overline{z}\in D_rF/D_r^-} j_{\overline{z}}.$$

The connectivity of Λ on fixed points follows from Assumptions 1.6:

$$\operatorname{conn}(\Lambda^{C_s}) \geq \min(j) + \frac{r}{s}\Big(j_1 + i_1 + j_{t_{2r}} + i_{t_{2r}} + \sum_{\overline{z}\in D_rF/D_r^-} 2(j_{\overline{z}} + i_{\overline{z}})\Big) - 1 - \epsilon,$$

$$\operatorname{conn}(\Lambda^{D_s}) \geq \min{}'(j) + \frac{j_1}{2} + \Big\lceil\frac{i_1}{2}\Big\rceil + \frac{j_{t_{2r}}}{2} + \Big\lceil\frac{i_{t_{2r}}}{2}\Big\rceil + \frac{r-s}{2s}\,(j_1 + i_1 + j_{t_{2r}} + i_{t_{2r}}) + \frac{r}{s}\Big(\sum_{\overline{z}\in D_rF/D_r^-} j_{\overline{z}} + i_{\overline{z}}\Big) - 1 - \epsilon,$$

where $\min(j) = \min\{j_{\overline{z}} \mid \overline{z} \in D_rF/D_r\}$ and $\min'(j) = \min\{\frac{j_1}{2}, \frac{j_{t_{2r}}}{2}, j_{\overline{z}} \mid \overline{z} \in D_rF/D_r\}$. By Lemma A.1 $\operatorname{conn}((\Lambda^*)^{D_r}) \geq \min'(j) - 1 - \epsilon$. Hence by Lemma 1.9 there exists $\overline{N} \in \mathbb{N}$ such that the map

$$\big(G^{D_r\cdot F} \circ \triangle_r^e(j)\big)^{D_r} \hookrightarrow \operatorname*{hocolim}_{(I^G)^{\{1,t_{2r}\}}\times I^{D_r\cdot F/D_r^-}} \big(G^{D_r\cdot F} \circ \triangle_r^e\big)^{D_r}$$

is n-connected if j is coordinate-wise even and bigger than $\overline{N}$.

Let $N := \overline{N} + 1$. Finally let $i \in \operatorname{Ob}\big((I^G)^{\{1,t_{2r}\}} \times I^{D_r\cdot F/D_r^-}\big)$ and assume that $i \geq N$. If i is coordinate-wise even, then we have already seen that the inclusion into the homotopy colimit is n-connected. If i is not coordinate-wise even then write $i = i' + j'$, where i' is the coordinate-wise even element given by subtracting 1 from all the odd coordinates of i. The unique map $0 \to j'$ induces the horizontal map in the homotopy commutative diagram below:

$$\begin{array}{ccc} \big(G^{D_rF}\circ\triangle^e_r(i')\big)^{D_r} & \longrightarrow & \big(G^{D_rF}\circ\triangle^e_r(i)\big)^{D_r} \\ & \searrow \qquad \swarrow & \\ & \operatorname*{hocolim}_{(I^G)^{\{1,t_{2r}\}}\times I^{D_rF/D_r^-}} \big(G^{D_rF}\circ\triangle^e_r\big)^{D_r}. & \end{array}$$

Since $i' \geq \overline{N}$, the horizontal map and the left hand diagonal map are n-connected as proven above, hence so is the right hand diagonal map as desired. ☐

Proof of part (iii). We let $j \in \mathrm{Ob}\big((I^G)^{\{1,t_{4r}\}} \times I^{D_{2r}F/D_{2r}^-}\big)$ and consider the D_{2r}-equivariant inclusion

$$G^{D_{2r}\cdot F} \circ \triangle^e_{2r}(j) \hookrightarrow \operatorname*{hocolim}_{I^{D_{2r}\cdot F}} G^{D_{2r}\cdot F}.$$

We want to show that the connectivity on H-fixed points tends to infinity with j for $H \in \{C_s, D_s, D'_s\}_{s|r}$. The subgroups D_s and D'_s are conjugate inside D_{2r}, hence we can reduce to checking connectivity on the H-fixed points for $H \in \{C_s, D_s\}_{s|r}$ and complete the proof as above. ☐

Before we conclude this section, we state the following useful lemma, which can be found in [4, Lemma 4.3.7].

Lemma 1.10. *Let $(J, \overline{J}, \mu)$ be a good indexing category, let $X, Y : J \to \mathsf{Top}_*$ be functors, and let $\Phi : X \Rightarrow Y$ be a natural transformation. Suppose that for all $j \in \mathrm{Ob}(J)$ the map $\Phi_j : X(j) \to Y(j)$ is $\lambda(j)$-connected for a map $\lambda : \mathrm{Ob}(J) \to \mathbb{N}$ that tends to infinity on $\mathrm{Ob}(J)$. In this situation, the induced map on homotopy colimits*

$$\Phi : \operatorname*{hocolim}_J X \to \operatorname*{hocolim}_J Y$$

is a weak equivalence.

2. The cyclotomic structure of $\mathrm{THR}(A, D)$

The space $\mathrm{THR}(A, D; S^0)$ is the 0th space of a fibrant orthogonal $O(2)$-spectrum in the model structure based on the family of finite subgroups of $O(2)$, see Proposition 2.5, and furthermore the spectrum is cyclotomic. Before we establish these results, we briefly recall the category of equivariant orthogonal spectra and the fixed points functors.

We let H denote a compact Lie group. By an H-representation, we will mean a finite dimensional real inner product space on which H acts by linear isometries. We will work in the category of orthogonal H-spectra, defined as diagram H-spaces as in [15, Chapter II.4]. Let $(H\mathsf{Top}_*, \wedge, S^0)$ denote the symmetric monoidal category of based H-spaces and continuous based H-equivariant maps. The collection of all H-spaces together with all based maps gives rise to a category enriched in $(H\mathsf{Top}_*, \wedge, S^0)$, which we denote $\mathscr{T}_H$.

We fix a complete H-universe $\mathcal{U}$. If V and W are H-representations in $\mathcal{U}$, then $L(V, W)$ denotes the H-space of linear isometries from V to W with H-action by conjugation. The H-bundle $E(V, W)$ is the sub-bundle of the product H-bundle $L(V, W) \times W \to W$ consisting of those pairs (α, w) such that w is in the orthogonal complement $W-\alpha(V)$. Let $\mathscr{J}_H(V,W)$ be the Thom H-space of $E(V,W)$. We define composition

$$\circ : \mathscr{J}_H(V',V'') \times \mathscr{J}_H(V,V') \to \mathscr{J}_H(V,V'')$$

by $(\beta,y)\circ(\alpha,x) = (\beta\circ\alpha, \beta(x)+y)$. The point $(\mathrm{id}_V, 0) \in \mathscr{J}_G^U(V,V)$ is the identity morphism. Let $\mathscr{J}_H^{\mathcal{U}}$ be the category enriched over $(H\mathsf{Top}_*, \wedge, S^0)$ with objects all finite dimensional H-representation $V \subset \mathcal{U}$ and morphisms the Thom H-spaces $\mathscr{J}_H(V,W)$.

Definition 2.1. An orthogonal H-spectrum indexed on $\mathcal{U}$ is an enriched functor

$$X : \mathscr{J}_H^{\mathcal{U}} \to \mathscr{T}_H.$$

A morphism of orthogonal H-spectra indexed on $\mathcal{U}$ is an enriched natural transformation. Let $H\mathsf{Sp}_{\mathcal{U}}^{\mathrm{O}}$ denote the category of orthogonal H-spectra indexed on $\mathcal{U}$.

Let X be an orthogonal H-spectrum. For a closed subgroup $K \leq H$ and a non-negative integer q the homotopy groups of X are given as follows:

$$\pi_q^K(X) = \operatorname*{colim}_{V\subset U} \pi_q^K(\Omega^V X(V)), \quad \pi_{-q}^K(X) = \operatorname*{colim}_{\mathbb{R}^q \subset V \subset U} \pi_0^K(\Omega^{V-\mathbb{R}^q} X(V)).$$

The morphisms of H-spectra inducing isomorphism on all homotopy groups are referred to as π_*-isomorphisms. These are the weak equivalences in the stable model structure on orthogonal H-spectra indexed on $\mathcal{U}$ given in [15, Chapter III, 4.1,4.2]. Throughout this paper, we let $j_f : X \to X_f$ denote a fibrant replacement functor in the stable model structure, such that j_f is an acyclic cofibration, and we let $j^c : X^c \to X$ denote a cofibrant replacement functor in the stable model structure such that j^c is an acyclic fibration.

Let $\mathcal{S}$ be a family of subgroups of H, that is $\mathcal{S}$ is a collection of subgroups closed under taking subgroups and conjugates. We say that a morphism of orthogonal H-spectra indexed on $\mathcal{U}$ is an $\mathcal{S}$-equivalence if it induces isomorphisms on $\pi_q^K(-)$ for all subgroups $K \in \mathcal{S}$ and all $q \in \mathbb{Z}$. The $\mathcal{S}$-equivalences constitutes the weak equivalences in the $\mathcal{S}$-model structure, see [15, Chapter IV.6].

2.1. **Pointset fixed point functors.** Let $K \leq H$ be a closed subgroup. If $\mathcal{U}$ is a complete H-universe, then $\mathcal{U}^K$ is a complete $N(K)/K$-universe. There are two fixed point functors which take an orthogonal H-spectrum indexed on U and produce an orthogonal $N(K)/K$-spectrum indexed on U^K. We recall the fixed point functors in case of a normal subgroup N, see [15, Chapter V.4] for details.

We define a category $\mathscr{J}_{H,N}^U$ with the same objects as $\mathscr{J}_H^{\mathcal{U}}$. The morphism spaces are the H/N-spaces of N-fixed points

$$\mathscr{J}_{H,N}^{\mathcal{U}}(V,W) := \mathscr{J}_H^{\mathcal{U}}(V,W)^N.$$

The composition and identity restrict appropriately making $\mathscr{J}_{H,N}^U$ into a category enriched over $(H/N\mathsf{Top}_*, \wedge, S^0)$. Taking N-fixed points levelwise takes an orthogonal H-spectrum X and produces a functor enriched over $(H/N\mathsf{Top}_*, \wedge, S^0)$:

$$\mathrm{Fix}^N(X) : \mathscr{J}_{H,N}^{\mathcal{U}} \to \mathscr{T}_{H/N}.$$

We have two enriched functors comparing the categories $\mathscr{J}_{H,N}^{\mathcal{U}}$ and $\mathscr{J}_{H/N}^{\mathcal{U}^N}$:

$$\mathscr{J}_{H/N}^{\mathcal{U}^N} \xrightarrow{\nu} \mathscr{J}_{H,N}^{\mathcal{U}} \xrightarrow{\phi} \mathscr{J}_{H/N}^{\mathcal{U}^N}.$$

The functor ν takes an H/N-representation V to the H-representation q^*V, where $q : H \to H/N$ is the quotient homomorphism. The functor ϕ sends an H-representation V to the H/N-representation V^N, and it sends a morphism (α, x) to the fixpoints morphism (α^N, x). The functors ϕ and ν induces forgetful functors

$$U_\phi : \mathrm{Fun}_{H/N\mathsf{Top}_*}(\mathscr{J}^{\mathcal{U}^N}_{H/N}, \mathscr{T}_{H/N}) \to \mathrm{Fun}_{H/N\mathsf{Top}_*}(\mathscr{J}^{\mathcal{U}}_{H,N}, \mathscr{T}_{H/N}),$$
$$U_\nu : \mathrm{Fun}_{H/N\mathsf{Top}_*}(\mathscr{J}^{\mathcal{U}}_{H,N}, \mathscr{T}_{H/N}) \to \mathrm{Fun}_{H/N\mathsf{Top}_*}(\mathscr{J}^{\mathcal{U}^N}_{H/N}, \mathscr{T}_{H/N}),$$

where $\mathrm{Fun}_{H/N\mathsf{Top}_*}(-,-)$ is the category of functors enriched in $(H/N\mathsf{Top}_*, \wedge, S^0)$ and enriched natural tranformations. Note that $\phi \circ \nu = \mathrm{id}$, hence $U_\nu \circ U_\phi = \mathrm{id}$. Enriched left Kan extension along ϕ gives an enriched functor

$$P_\phi : \mathrm{Fun}_{H/N\mathsf{Top}_*}(\mathscr{J}^{\mathcal{U}}_{H,N}, \mathscr{T}_{H/N}) \to \mathrm{Fun}_{H/N\mathsf{Top}_*}(\mathscr{J}^{\mathcal{U}^N}_{H/N}, \mathscr{T}_{H/N}),$$

which is left adjoint to U_ϕ.

Definition 2.2. The fixed point functor is the composite functor

$$(-)^N := U_\nu \circ \mathrm{Fix}^N : H\mathsf{Sp}^{\mathrm{O}}_{\mathcal{U}} \to H/N\mathsf{Sp}^{\mathrm{O}}_{\mathcal{U}^N}.$$

We note that $X^N(V) = X(q^*V)^N$ for a H/N-representation V. The fixed point functor preserves fibrations, acyclic fibrations and π_*-isomorphisms between fibrant objects, see [15, Chapter V, Prop. 3.4].

Definition 2.3. The geometric fixed point functor is the composite

$$(-)^{gN} := P_\phi \circ \mathrm{Fix}^N : H\mathsf{Sp}^{\mathrm{O}}_{\mathcal{U}} \to H/N\mathsf{Sp}^{\mathrm{O}}_{\mathcal{U}^N}.$$

The geometric fixed point functor preserves cofibrations, acyclic cofibrations, and π_*-isomorphisms between cofibrant objects, see [15, Chapter V, Prop. 4.5].

Let $\overline{\gamma} : \mathrm{id} \to U_\phi P_\phi$ denote the unit of the adjunction (P_ϕ, U_ϕ). We have a natural transformation of fixed point functors $\gamma : X^N \to X^{gN}$ given as

$$U_\nu(\overline{\gamma}) : U_\nu(\mathrm{Fix}^N(X)) \to U_\nu U_\phi P_\phi(\mathrm{Fix}^N(X)) = P_\phi(\mathrm{Fix}^N(X)).$$

2.2. **The orthogonal spectrum** $\mathrm{THR}(A, D)$**.** We fix a complete $O(2)$-universe

$$\mathcal{U} = \Big(\bigoplus_{\alpha \geq 0} \bigoplus_{n \geq 0} \mathbb{C}(n)\Big) \bigoplus \Big(\bigoplus_{\alpha \geq 0} \bigoplus_{n \geq 0} \mathbb{C}(n) \oplus \mathbb{C}(-n)\Big).$$

Here $\mathbb{C}(n) := \mathbb{C}$ with $\mathbb{T}$-action given by $z \cdot x = z^n x$ for $z \in \mathbb{T}$ and $x \in \mathbb{C}$. On the left hand side $\mathbb{C}(n)$ is the $O(2)$-representation with ω acting by complex conjugation. On the right hand side $\mathbb{C}(n) \oplus \mathbb{C}(-n)$ is the $O(2)$-representation with ω acting by $\omega(x, y) = (y, x)$. We see that

$$\rho_r^* \mathbb{C}(n)^{C_r} = \begin{cases} \mathbb{C}(\frac{n}{r}) & \text{if } r \mid n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\rho_r^* \left(\mathbb{C}(n) \oplus \mathbb{C}(-n)\right)^{C_r} = \begin{cases} \mathbb{C}(\frac{n}{r}) \oplus \mathbb{C}(-\frac{n}{r}) & \text{if } r \mid n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\rho_r^* \mathcal{U}^{C_r} = \Big(\bigoplus_{\alpha \geq 0} \bigoplus_{n \geq 0, r \mid n} \mathbb{C}\left(\frac{n}{r}\right)\Big) \bigoplus \Big(\bigoplus_{\alpha \geq 0} \bigoplus_{n \geq 0, r \mid n} \mathbb{C}\left(\frac{n}{r}\right) \oplus \mathbb{C}\left(-\frac{n}{r}\right)\Big).$$

We let $f_r : \mathcal{U} \to \rho_r^* \mathcal{U}^{C_r}$ be the $O(2)$-equivariant homeomorphism given by mapping the summand $\mathbb{C}(n)$ indexed by (α, n) in $\mathcal{U}$ to the summand $\mathbb{C}(n)$ indexed by (α, rn) in $\rho_r^* \mathcal{U}^{C_r}$ and by mapping the summand $\mathbb{C}(n) \oplus \mathbb{C}(-n)$ indexed by (α, n) in $\mathcal{U}$ to the summand $\mathbb{C}(n) \oplus \mathbb{C}(-n)$ indexed by (α, rn) in $\rho_r^* \mathcal{U}^{C_r}$.

Let X and Y be pointed $O(2)$-spaces. We define a natural transformation of functors $G_X^F \wedge Y \Rightarrow G_{X\wedge Y}^F$ from I^F to Top_* by $(f, y) \mapsto \overline{f}$, where $\overline{f}(t) = f(t) \wedge y$. We compose the induced map on colimits with the canonical homeomorphism commuting the functor that smashes with a fixed pointed $O(2)$-space and the homotopy colimit functor to obtain maps

$$\left(\operatorname*{hocolim}_{I^F} G_X^F\right) \wedge Y \xrightarrow{\cong} \operatorname*{hocolim}_{I^F}(G_X^F \wedge Y) \to \operatorname*{hocolim}_{I^F} G_{X\wedge Y}^F.$$

Since these maps commute with the dihedral structure maps and the $O(2)$-action on X and Y, we obtain an $O(2)$-equivariant map

$$\sigma_{X,Y} : \triangle^*(\mathrm{THR}(A, D; X)) \wedge Y \to \triangle^* \mathrm{THR}(A, D; X \wedge Y),$$

where $\triangle : O(2) \to O(2) \times O(2)$ denotes the diagonal map and $O(2)$ acts diagonally on the domain.

Definition 2.4. Let $V \subset \mathcal{U}$ be an $O(2)$-representation. Let

$$\mathrm{THR}(A, D)(V) = \triangle^* \mathrm{THR}(A, D; S^V).$$

The group $O(V)$ acts on $\mathrm{THR}(A, D)(V)$ through the action on the sphere S^V. The family of $O(V) \rtimes O(2)$-spaces $\mathrm{THR}(A, D)(V)$ together with the structure maps

$$\sigma_{V,W} := \sigma_{S^V, S^W} : \mathrm{THR}(A, D)(V) \wedge S^W \to \mathrm{THR}(A, D)(V \oplus W),$$

defines an orthogonal $O(2)$-spectrum indexed on $\mathcal{U}$, which is denoted $\mathrm{THR}(A, D)$.

It follows from [15, Chapter II, Theorem 4.3] that the data above defines an $O(2)$-spectrum in sense if Definition 2.1 given earlier. The following result extends the classical result for the $\mathbb{T}$-spectrum $\mathrm{THH}(A)$; see [9, Prop. 1.4]. Let $\mathscr{F}$ denote the family of finite subgroups of $O(2)$.

Proposition 2.5. *Let $H < O(2)$ be a finite subgroup. For all finite dimensional $O(2)$-representations $V \subset W$ the adjoint of the structure map*

$$\widetilde{\sigma}_{V,W-V} : \mathrm{THR}(A, D)(V) \to \Omega^{W-V} \mathrm{THR}(A, D)(W)$$

induces a weak equivalence on H-fixed points. In other words, $\mathrm{THR}(A, D)$ *is fibrant in the $\mathscr{F}$-model structure.*

Proof. It suffices to prove the statement for $H \in \{C_r, D_r\}_{r\geq 0}$. The case $H = C_r$ is done in [9, Prop. 1.4]. We assume for convenience that $V = 0$, the general case is analogous.

Assume $H = D_r$ with r odd. Let $i \in \mathrm{Ob}\left((I^G)^{\{1,t_{2r}\}} \times I^{D_r \cdot F/D_r^-}\right)$. We introduce the notation

$$G^{D_r F/D_r} \circ \triangle_r^e(i) \bullet S^W := \mathrm{Map}\left(\Big(\bigwedge_{z\in D_r\cdot F} S^{i_{\overline{z}}}\Big) \wedge S^W, \Big(\bigwedge_{z\in D_r\cdot F} A_{i_{\overline{z}}}\Big) \wedge S^W\right).$$

The domain and target are given the diagonal $O(2)$-action and the action on the mapping space is by conjugation. The map $\widetilde{\sigma}$ becomes the top row in the commutative diagram below, when we restrict to the cofinal subcategory $D_r\mathcal{F}_*$.

$$
\begin{array}{ccc}
\left(\operatorname*{colim}_{D_r\mathcal{F}_*} \operatorname*{hocolim}_{I^{D_rF}} G^{D_r\cdot F}\right)^{D_r} & \xrightarrow{\widetilde{\sigma}} & \left(\Omega^W\left(\operatorname*{colim}_{D_r\mathcal{F}_*} \operatorname*{hocolim}_{I^{D_rF}} G^{D_r\cdot F}_{S^W}\right)\right)^{D_r} \\
 & & \sim\Big\uparrow \gamma^{D_r} \\
 & & \left(\operatorname*{colim}_{D_r\mathcal{F}_*} \Omega^W \operatorname*{hocolim}_{I^{D_rF}} G^{D_r\cdot F}_{S^W}\right)^{D_r} \\
\cong\Big\uparrow \triangle & & \sim\Big\uparrow \mathrm{can} \\
 & & \left(\operatorname*{colim}_{D_r\mathcal{F}_*} \operatorname*{hocolim}_{I^{D_rF}} \Omega^W G^{D_r\cdot F}_{S^W}\right)^{D_r} \\
 & & \cong\Big\uparrow \triangle \\
\operatorname*{colim}_{D_r\mathcal{F}_*} \operatorname*{hocolim}_{\overline{I}} \left(G^{D_r\cdot F}\circ \triangle^e_r\right)^{D_r} & \underset{\sim}{\xrightarrow{(\eta^*)^{D_r}}} & \operatorname*{colim}_{D_r\mathcal{F}_*} \operatorname*{hocolim}_{\overline{I}} \left(G^{D_rF/D_r}\circ \triangle^e_r(i)\bullet S^W\right)^{D_r}
\end{array}
$$

where $\overline{I} := (I^G)^{\{1,t_{2r}\}} \times I^{D_r\cdot F/D_r^-}$. The maps labelled $\triangle$ are homeomorphisms by Lemma 1.4. The bottom map is induced from the adjunction unit

$$\eta : \bigwedge_{z\in D_{2r}\cdot F} A_{i_{\overline{z}}} \to \operatorname{Map}\left(S^W, S^W \wedge \bigwedge_{z\in D_{2r}\cdot F} A_{i_{\overline{z}}}\right)$$

followed by a homeomorphism. The connectivity of $(\eta^*)^{D_r}$ can be estimated using the equivariant suspension Theorem A.3 and Lemma A.1:

$$\operatorname{conn}((\eta^*)^{D_r}) \geq \left\lceil \frac{i_1}{2}\right\rceil + \left\lceil \frac{i_{t_{2r}}}{2}\right\rceil + \sum_{\overline{z}\in D_rF/D_r^-} i_{\overline{z}}.$$

Since the connectivity tends to infinity with i, the induced map on homotopy colimits is a weak equivalence by Lemma 1.10. A similar argument shows that can is a weak equivalence. Finally γ^{D_r} is a weak equivalence by [9, Lemma 1.4].

The case $H = D_r$ with r even, can be done analogously by restricting to the cofinal subcategory $D_{2r}\mathcal{F}_*$, compare Lemma 1.7 part (iii). □

2.3. **The cyclotomic structure.** The $\mathbb{T}$-spectrum underlying $\mathrm{THR}(A, D)$ is cyclotomic; see [9, Def. 1.2, Prop. 1.5]. We prove that the cyclotomic structure is compatible with the G-action. For this purpose, we introduce the notation of an $O(2)$-cyclotomic spectrum. Let

$$\rho_r : O(2) \to O(2)/C_r$$

be the root isomorphism given by $\rho_r(z) = z^{\frac{1}{r}}C_r$ if $z \in \mathbb{T}$ and $\rho_r(x) = x$ if $x \in G$. The isomorphism ρ_r induces isomorphisms of enriched categories

$$\mathscr{I}_{\rho_r^*} : \mathscr{I}^{\mathcal{U}^{C_r}}_{O(2)/C_r} \xrightarrow{\cong} \mathscr{I}^{\rho_r^* U^{C_r}}_{O(2)}, \quad \mathscr{T}_{\rho_r^*} : \mathscr{T}_{O(2)/C_r} \xrightarrow{\cong} \mathscr{T}_{O(2)}.$$

The isomorphism of universes $f_r : \mathcal{U} \xrightarrow{\cong} \rho_r^*\mathcal{U}^{C_r}$ defined earlier induces an isomorphism of enriched categories

$$f_r : \mathscr{I}^{\mathcal{U}}_{O(2)} \xrightarrow{\cong} \mathscr{I}^{\rho_r^*\mathcal{U}^{C_r}}_{O(2)}.$$

If Y is a $O(2)/C_r$-spectrum indexed on $\mathcal{U}^{C_r}$, then we let $\rho_r^* Y$ be the $O(2)$-spectrum indexed on $\mathcal{U}$ given by the composition

$$\mathscr{I}_{O(2)}^{\mathcal{U}} \xrightarrow{f_r} \mathscr{I}_{O(2)}^{\rho_r^* U^{C_r}} \xrightarrow{\mathscr{I}_{\rho_r^*}^{-1}} \mathscr{I}_{O(2)/C_r}^{\mathcal{U}^{C_r}} \xrightarrow{Y} \mathscr{T}_{O(2)/C_r} \xrightarrow{\mathscr{T}_{\rho_r^*}} \mathscr{T}_{O(2)}.$$

Definition 2.6. An $O(2)$-cyclotomic spectrum is an orthogonal $O(2)$-spectrum X together with maps of orthogonal $O(2)$-spectra for all $r \geq 0$,

$$T_r : \rho_r^*(X^{gC_r}) \to X,$$

such that the composite from the derived geometric fixed point functor

$$\rho_r^*(X^c)^{gC_r} \to \rho_r^*(X^{gC_r}) \xrightarrow{T_r} X$$

is an $\mathscr{F}$-equivalence, where the first map is induced by the cofibrant replacement $j^c : X^c \to X$. Furthermore we require that for all $s, r \geq 1$ the following diagram commutes:

$$\begin{array}{ccccc}
\rho_{rs}^*(X^{gC_{rs}}) & \longrightarrow & \rho_s^*((\rho_r^*(X^{gC_r}))^{gC_s}) & \xrightarrow{\rho_s^*(T_r^{gC_s})} & \rho_s^*(X^{gC_s}) \\
\downarrow & & & & \downarrow T_r \\
\rho_r^*((\rho_s^*(X^{gC_s}))^{gC_r}) & \xrightarrow[\rho_r^*(T_s^{gC_r})]{} & \rho_r^*(X^{gC_r}) & \xrightarrow[T_s]{} & X.
\end{array}$$

The geometric fixed point functor X^{gC_r} is defined as a left Kan extension via

$$\phi : \mathscr{I}_{O(2);C_r}^{U} \to \mathscr{I}_{O(2)/C_r}^{U^{C_r}}, \quad \phi(V) = V^{C_r},$$

of the functor $\mathrm{Fix}^{C_r} X$. To construct cyclotomic structure maps it therefore suffices to construct $O(2)$-equivariant maps

$$\tilde{T}_r : \rho_r^*(X(V)^{C_r}) \to X(\rho_r^*(V^{C_r})),$$

for each representation V satisfying certain compatibility conditions to ensure commutativity of the diagram above; compare [9, Lemma 1.2].

Example 2.7. The $O(2)$-equivariant sphere spectrum $\mathbb{S}$ is an $O(2)$-cyclotomic spectrum with structure maps arising from the homeomorphisms: $\rho_r^*((S^V)^{C_r}) \to S^{\rho_r^*(V^{C_r})}$.

Both in the next example and in the construction of the cyclotomic structure map for $\mathrm{THR}(A, D)$, we will need to pull back the group action on a diagram along a group homomorphism. Let $g : K \to H$ be a group homomorphism. Let $(X : J \to \mathsf{Sets}, \alpha)$ be a H-diagram indexed by J. Let $g^* J$ denote the category J with K-action defined by $k := g(k) : J \to J$. We have natural transformations

$$g^* \alpha_k := \alpha_{g(k)} : X \Rightarrow X \circ g(k).$$

This gives a K-diagram indexed by $g^* J$ and there is a unique isomorphism of K-sets:

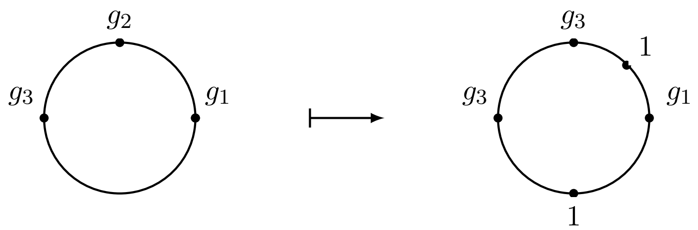
$$\operatorname*{colim}_{g^* J} g^* X \cong g^*(\operatorname*{colim}_{J} X).$$

Example 2.8. Let Γ be a topological group. The geometric realization of the dihedral bar construction on Γ is an $O(2)$-space which we denote $B^{\mathrm{di}}\Gamma$. As explained in the introduction, there is an $\mathscr{F}$-equivalence from $B^{\mathrm{di}}\Gamma$ to the free loop space on the classifying space, $\mathrm{Map}(\mathbb{T}, B\Gamma)$, if we give the free loop space an $O(2)$-action as follows: The group $O(2)$ acts on $\mathbb{T}$ by multiplication and complex cojugation. Taking inverses in the group induces a G-action on $B\Gamma$, and we view $B\Gamma$ as an $O(2)$-space with trivial $\mathbb{T}$-action. We let $O(2)$ act on the free loop space by the conjugation action.

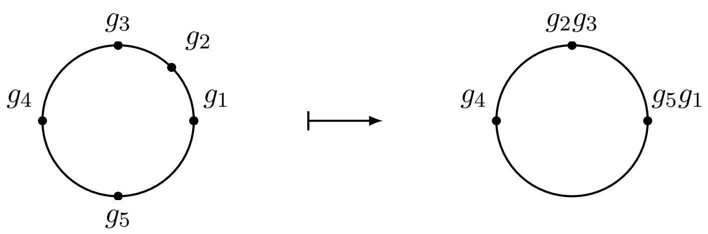
There are $\mathbb{T}$-equivariant homeomorphisms

$$p_r : B^{\mathrm{di}}\Gamma \xrightarrow{\cong} \rho_r^*\big(B^{\mathrm{di}}\Gamma^{C_r}\big).$$

are constructed in [14, Sect. 2.1.7]. We run trough the construction of p_r to ensure that it is $O(2)$-equivariant. Since we already know that p_r is continuous, we will not keep track of continuity. First we recall the dihedral bar constrcution on Γ. For $F \in \mathcal{F}$, let $B^{\mathrm{di}}[F] = \Gamma^F$. Let $F \subset F'$ be finite subsets of the circle. We define $s_F^{F'} : \Gamma^F \to \Gamma^{F'}$ by repeating the identity element $1 \in \Gamma$ as pictured in the example:



We define $d_F^{F'} : \Gamma^{F'} \to \Gamma^F$ by pushing the group elements clockwise around the circle using the multiplication in Γ, as pictured in the example:



We describe the natural transformations $t_F : \Gamma^F \to \Gamma^{t(F)}$ and $\omega_F : \Gamma^F \to \Gamma^{\omega(F)}$, where $t \in \mathbb{T}$ and ω is complex conjugation. They both permute a tuple of group elements accordingly and ω, in addition, takes each label to its inverse:

$$t_F\left((g_z)_{z\in F}\right) = (g_{t^{-1}(y)})_{y\in t(F)}, \quad \omega_F\left((g_z)_{z\in F}\right) = ((g_{\omega^{-1}(y)})^{-1})_{y\in\omega(F)}.$$

The geometric realization $B^{\mathrm{di}}\Gamma := \mathrm{colim}_{F\in\mathcal{F}}\,\Gamma^F$ is an $O(2)$-space.

We proceed to construct the map p_r. Let $\overline{\mathcal{F}}$ denote the category of finite subsets of $\mathbb{T}/C_r$ and set inclusions. We have $O(2)/C_r$-equivariant bijections:

$$\begin{aligned}\left(\operatorname*{colim}_{F\in\mathcal{F}}\Gamma^F\right)^{C_r} &\xleftarrow{\cong}\left(\operatorname*{colim}_{C\cdot F\in C_r\mathcal{F}}\Gamma^{C_r\cdot F}\right)^{C_r}\\ &\xleftarrow{\cong}\operatorname*{colim}_{C_r\cdot F\in C_r\mathcal{F}}\left(\Gamma^{C_r\cdot F}\right)^{C_r}\\ &\xleftarrow{\cong}\operatorname*{colim}_{C_r\cdot F\in C_r\mathcal{F}}\Gamma^{C_r\cdot F/C_r}\\ &\xrightarrow{\cong}\operatorname*{colim}_{\overline{F}\in\overline{\mathcal{F}}}\Gamma^{\overline{F}}.\end{aligned}$$

The first map is induced by the inclusion of categories $C_r\mathcal{F}\hookrightarrow\mathcal{F}$, the second map is induced by the inclusion $(\Gamma^{C_r\cdot F})^{C_r}\hookrightarrow\Gamma^{C_r\cdot F}$, the third map is induced by the diagonal isomorphism $\Gamma^{C_r\cdot F/C_r}\to\left(\Gamma^{C_r\cdot F}\right)^{C_r}$, and, finally the last map is induced by the $O(2)/C_r$-equivariant isomorphism of categories $C_r\mathcal{F}\to\overline{\mathcal{F}}$ given on objects by $C_r\cdot F\mapsto C_r\cdot F/C_r$. If we pull back the diagram $\overline{F}\mapsto\Gamma^{\overline{F}}$ along ρ_r, then the isomorphism of categories $\mathcal{F}\to\overline{\mathcal{F}}$ given by $F\mapsto\rho_r(F)$ induces an $O(2)$-bijection:

$$\rho_r^*\left(\operatorname*{colim}_{\overline{F}\in\overline{\mathcal{F}}}\Gamma^{\overline{F}}\right)\xleftarrow{\cong}\operatorname*{colim}_{F\in\mathcal{F}}\Gamma^{\rho_r(F)}.$$

Finally the isomorphism of dihedral sets $\Gamma^F\to\Gamma^{\rho_r(F)}$ induces an $O(2)$-bijection

$$\operatorname*{colim}_{F\in\mathcal{F}}\Gamma^{\rho_r(F)}\xleftarrow{\cong}\operatorname*{colim}_{F\in\mathcal{F}}\Gamma^F.$$

We combine the maps above to obtain p_r, which is indeed $O(2)$-equivariant. We give the suspension spectrum $\Sigma^\infty_{O(2)}B^{\mathrm{di}}\Gamma_+$ the structure of an $O(2)$-cyclotomic spectrum by letting $\tilde{T}_r$ be the map:

$$S^{\rho_r^*(V^{C_r})}\wedge\rho_r^*(B^{\mathrm{di}}\Gamma)_+^{C_r}\xrightarrow{\mathrm{id}\wedge p_r^{-1}}S^{\rho_r^*(V^{C_r})}\wedge B^{\mathrm{di}}\Gamma_+.$$

Let V be a finite dimensional $O(2)$-representation and let $r\geq 1$. We run through the construction of the map

$$\tilde{T}_r:\rho_r^*\big(\operatorname{THR}(A,D)(V)^{C_r}\big)\to\operatorname{THR}(A,D)\big(\rho_r^*(V^{C_r})\big).$$

defined in [9, Sect. 1.5], to check that it is $O(2)$-equivariant. Since it is already known that the map is continuous, we will not keep track of continuity. Consider the $O(2)/C_r\times O(2)/C_r$-diagram, where the second copy of $O(2)/C_r$ acts trivially on the category:

$$C_r\mathcal{F}\to\mathsf{Top}_*,\quad C_r\cdot F\mapsto\operatorname{THR}(A,D;S^{V^{C_r}})[C_r\cdot F/C_r].$$

We pull the diagram back along

$$D:\triangle(O(2))/\triangle(C_r)\to O(2)/C_r\times O(2)/C_r,\quad D((a,a)\triangle(C_r))=(aC_r,aC_r),$$

and define a natural transformation of $\triangle(O(2))/\triangle(C_r)$-diagrams by

$$\begin{aligned}
\mathrm{THR}(A,D;S^V)[C_r\cdot F]^{\triangle(C_r)} &\xleftarrow{\cong} \operatorname*{hocolim}_{I^{C_r\cdot F/C_r}} \Big(G^{C_rF}_{S^V}\circ \triangle_r\Big)^{\triangle(C_r)} \\
&= \operatorname*{hocolim}_{I^{C_r\cdot F/C_r}} \Big(\mathrm{Map}\Big(\big(\bigwedge_{\overline{z}\in C_r\cdot F/C_r} S^{i_{\overline{z}}}\big)^{\wedge r}, \big(\bigwedge_{\overline{z}\in C_r\cdot F/C_r} A_{i_{\overline{z}}}\big)^{\wedge r}\wedge S^V\Big)\Big)^{\triangle(C_r)} \\
&\xrightarrow{\mathrm{res}} \operatorname*{hocolim}_{I^{C_r\cdot F/C_r}} \mathrm{Map}\Big(\bigwedge_{\overline{z}\in C_r\cdot F/C_r} S^{i_{\overline{z}}}, \bigwedge_{\overline{z}\in C_r\cdot F/C_r} A_{i_{\overline{z}}}\wedge S^{V^{C_r}}\Big) \\
&= \mathrm{THR}(A,D;S^{V^{C_r}})[C_r\cdot F/C_r].
\end{aligned}$$

The map res is induced by the natural transformation obtained by restricting a map to the fixed point space: $\mathrm{Map}(X,Y)^{C_r}\to \mathrm{Map}(X^{C_r},Y^{C_r})$. The natural transformation above induces a map of colimits which we also denote res.

If $d: O(2)/C_r\to \triangle(O(2))/\triangle(C_r)$ denotes the isomorphism $aC_r\mapsto (a,a)\triangle(C_r)$, then there is a commutative diagram of homomorphisms:

$$\begin{array}{ccccc}
O(2) & \xrightarrow{\rho_r} & O(2)/C_r & \xrightarrow{d} & \triangle(O(2))/\triangle(C_r) \\
\big\downarrow \triangle & & & & \big\downarrow D \\
O(2)\times O(2) & & \xrightarrow{\rho_r\times\rho_r} & & O(2)/C_r\times O(2)/C_r.
\end{array}$$

We have a string of $O(2)/C_r$-equivariant set maps:

$$\begin{aligned}
\rho_r^*\Big(\triangle^*\operatorname*{colim}_{F\in\mathcal{F}}\mathrm{THR}(A,D;S^V)[F]\Big)^{C_r} &= (d\circ\rho_r)^*\Big(\operatorname*{colim}_{F\in\mathcal{F}}\mathrm{THR}(A,D;S^V)[F]\Big)^{\triangle(C_r)} \\
&\xleftarrow{\cong} (d\circ\rho_r)^*\Big(\operatorname*{colim}_{C_r\cdot F\in C_r\mathcal{F}}\mathrm{THR}(A,D;S^V)[C_r\cdot F]^{\triangle(C_r)}\Big) \\
&\xrightarrow{\mathrm{res}} (d\circ\rho_r)^*\Big(D^*\Big(\operatorname*{colim}_{C_r\cdot F\in C_r\mathcal{F}}\mathrm{THR}(A,D;S^{V^{C_r}})[C_r\cdot F/C_r]\Big)\Big) \\
&= (\triangle^*)\Big(\rho_r^*\times\rho_r^*\Big(\operatorname*{colim}_{C_r\cdot F\in C_r\mathcal{F}}\mathrm{THR}(A,D;S^{V^{C_r}})[C_r\cdot F/C_r]\Big)\Big) \\
&\xleftarrow{\cong} (\triangle^*)\Big(\rho_r^*\times\rho_r^*\Big(\operatorname*{colim}_{\overline{F}\in\overline{\mathcal{F}}}\mathrm{THR}(A,D;S^{V^{C_r}})[\overline{F}]\Big)\Big) \\
&\xleftarrow{\cong} \triangle^*\operatorname*{colim}_{F\in\mathcal{F}}\mathrm{THR}(A,D;S^{\rho_r^*(V^{C_r})})[\rho_r(F)] \\
&\xleftarrow{\cong} \triangle^*\operatorname*{colim}_{F\in\mathcal{F}}\mathrm{THR}(A,D;S^{\rho_r^*(V^{C_r})})[F].
\end{aligned}$$

The maps labelled $\cong$ are bijections. The map in the second line is induced by the inclusion of categories $C_r\mathcal{F}\hookrightarrow\mathcal{F}$. The map in the fifth line is induced by the isomorphism of $O(2)/C_r$-categories $C_r\mathcal{F}\to\overline{\mathcal{F}}$ given by $C_r\cdot F\mapsto C_r\cdot F/C_r$. The map in the sixth line is induced by the functor $\mathcal{F}\to\overline{\mathcal{F}}$ given by $F\mapsto\rho_r(F)$ and finally the last map is induced by the obvious isomorphism of dihedral spaces. The composition is the cyclotomic structure map $\tilde{T}_r$, which is indeed $O(2)$-equivariant.

Theorem 2.9. *The composite*

$$\rho_r^*(\mathrm{THR}(A,D)^c)^{gC_r} \to \rho_r^* \mathrm{THR}(A,D)^{gC_r} \xrightarrow{T_r} \mathrm{THR}(A,D)$$

is an $\mathscr{F}$-equivalence.

Proof. By [15, Lemma 4.10] it suffices to show that the map of $\mathscr{I}^U_{O(2);C_r}$-spaces

$$\tilde{T}_r : \mathrm{Fix}^{C_r} \mathrm{THR}(A,D) \to (\rho_r^{-1})^*(\mathrm{THR}(A,D) \circ \phi),$$

induces an isomorphism on $\pi_q^H(-)$ for all finite subgroups $H \le O(2)/C_r$, where these are the homotopy groups of $\mathscr{I}^U_{O(2);C_r}$-spaces, see [15, Def. 4.8]. We consider the case $q = 0$, the general case is similar. More specifically, we must show that the connectivity of the induced $O(2)/C_r$-map on H-fixed points

$$\left(\Omega^{V^{C_r}} \mathrm{THR}(A,D)(V)^{C_r}\right)^H \to \left(\Omega^{V^{C_r}} (\rho_r^{-1})^* \mathrm{THR}(A,D)(\rho_r^* V^{C_r})\right)^H$$

tends to infinity with V for all finite subgroups $H \le O(2)/C_r$.

The only non-homeomorphism in the definition of $\tilde{T}_r$, is the restriction map induced by the natural transformation induced by restricting a map to the fixed point space: $\mathrm{Map}(X,Y)^{C_r} \to \mathrm{Map}(X^{C_r}, Y^{C_r})$. We let $H = D_{rs}/C_r$, the case $H = C_{rs}/C_r$ is analogous. We assume that $2 \nmid rs$. The case $2 \mid rs$, can be done analogously by restricting to the cofinal subcategory $D_{2sr}\mathcal{F}_*$, compare Lemma 1.7 part (iii). We restrict to the cofinal subcategory $D_{rs}\mathcal{F}_* \subset C_r\mathcal{F}$. The map in question is then

$$\begin{gathered}\left(\Omega^{V^{C_r}} \mathrm{colim}_{D_{sr}\mathcal{F}_*} \left(\mathrm{THR}(A,D;S^V)[D_{sr}F]\right)^{C_r}\right)^H \\ \downarrow \mathrm{res} \\ \left(\Omega^{V^{C_r}} \mathrm{colim}_{D_{sr}\mathcal{F}_*} D^* \mathrm{THR}(A,D;S^{V^{C_r}})[D_{sr}F/C_r]\right)^H,\end{gathered}$$

where $H = \triangle(D_{rs})/\triangle(C_r)$, $\Omega^{V^{C_r}}$ is viewed as a $\triangle(O(2))/\triangle(C_r)$-space, $D_{rs}\mathcal{F}_*$ as a category with a $\triangle(O(2))/\triangle(C_r)$-action and D is the homomorphism

$$\triangle(O(2))/\triangle(C_r) \to O(2)/C_r \times O(2)/C_r, \quad D(a,a)\triangle(C_r) = (aC_r, aC_r).$$

It follows from [9, Lemma 1.4] that we can move $\Omega^{V^{C_r}}$ past the colimit up to weak equivalence. We can then take fixed points before taking the colimit. Thus we are reduced to showing that the connectivity of the map

$$\left(\Omega^{V^{C_r}} \left(\mathrm{THR}(A,D;S^V)[D_{sr}F]\right)^{C_r}\right)^H \xrightarrow{\mathrm{res}} \left(\Omega^{V^{C_r}} D^* \mathrm{THR}(A,D;S^{V^{C_r}})[D_{sr}F/C_r]\right)^H$$

tends to infinity with V. We can move $\Omega^{V^{C_r}}$ past the homotopy colimit, up to weak equivalence, and by Lemma 1.4 we can take fixed points before taking homotopy colimits. Let $i \in \mathrm{Ob}\left((I^G)^{\{1,t_{2rs}\}} \times I^{D_{sr}F/D_{sr}^-}\right)$. By Lemma 1.10, we are reduced to showing that the connectivity of the map

$$\begin{gathered}\left(\Omega^{V^{C_r}} \mathrm{Map}(\textstyle\bigwedge_{z \in D_{sr}F} S^{i_{\overline{z}}}, \bigwedge_{z \in D_{sr}F} A_{i_{\overline{z}}} \wedge S^V)^{C_r}\right)^H \\ \downarrow \mathrm{res} \\ \left(\Omega^{V^{C_r}} \mathrm{Map}((\textstyle\bigwedge_{D_{sr}F} S^{i_{\overline{z}}})^{C_r}, (\bigwedge_{D_{sr}F} A_{i_{\overline{z}}} \wedge S^V)^{C_r})\right)^H.\end{gathered}$$

tends to infinity with V and i. We can rewrite the map in question as

$$\begin{array}{c}\mathrm{Map}_{D_{rs}}\Big(S^{V^{C_r}} \wedge \bigwedge_{z\in D_{sr}F} S^{i_{\overline{z}}}, \bigwedge_{z\in D_{sr}F} A_{i_{\overline{z}}} \wedge S^V\Big) \\ \big\downarrow \mathrm{res} \\ \mathrm{Map}_{D_{rs}}\Big(S^{V^{C_r}} \wedge (\bigwedge_{D_{sr}F} S^{i_{\overline{z}}})^{C_r}, \bigwedge_{z\in D_{sr}F} A_{i_{\overline{z}}} \wedge S^V\Big).\end{array}$$

This is a fibration with fiber

$$\mathrm{Map}_{D_{rs}}\Big(S^{V^{C_r}} \wedge (\bigwedge_{z\in D_{sr}F} S^{i_{\overline{z}}}/(\bigwedge_{z\in D_{sr}F} S^{i_{\overline{z}}})^{C_r}), \bigwedge_{z\in D_{sr}F} A_{i_{\overline{z}}} \wedge S^V\Big).$$

Let $S^W = \bigwedge_{z\in D_{sr}F} S^{i_{\overline{z}}}$. By Lemma A.2 the connectivity of the fiber is greater than or equal to

$$\min_{K\leq D_{rs}}\Big(\mathrm{conn}\Big((\bigwedge_{z\in D_{sr}F} A_{i_{\overline{z}}} \wedge S^V)^K\Big) - \dim\Big((S^{V^{C_r}} \wedge S^W/S^{W^{C_r}})^K\Big)\Big).$$

Let t divide rs. It follows from the connectivity assumptions 1.6 on (A, D) that

$$\mathrm{conn}(\bigwedge_{z\in D_{sr}F} A_{i_z} \wedge S^V)^{C_t} \geq \frac{rs}{t}\Big(i_1 + i_{t2rs} + \sum_{\overline{z}\in D_{sr}F/D_{sr}^-} 2i_{\overline{z}}\Big) + \dim(V^{C_t}) - 1,$$

$$\mathrm{conn}(\bigwedge_{z\in D_{sr}F} A_{i_z} \wedge S^V)^{D_t} \geq \frac{rs}{t}\Big(\Big\lceil\frac{i_1}{2}\Big\rceil + \Big\lceil\frac{i_{t2rs}}{2}\Big\rceil + \sum_{\overline{z}\in D_{sr}F/D_{sr}^-} i_{\overline{z}}\Big) + \dim(V^{D_t}) - 1.$$

If $C_r \leq C_t$ then the dimension of both $(S^W/S^{W^{C_r}})^{C_t}$ and $(S^W/S^{W^{C_r}})^{D_t}$ is 0, hence the connectivity of the fiber tends to infinity with i. Otherwise, since C_r is normal in D_{rs}, we have a splitting $W = W^{C_r} \oplus W'$, and by Lemma A.4:

$$\dim\Big(S^{V^{C_r}} \wedge (S^W/S^{W^{C_r}})\Big)^{C_t} = \frac{rs}{t}\Big(i_1 + i_{t2rs} + \sum_{\overline{z}\in D_{sr}F/D_{sr}^-} 2i_{\overline{z}}\Big) + \dim(V^{C_rC_t}),$$

$$\dim\Big(S^{V^{C_r}} \wedge (S^W/S^{W^{C_r}})\Big)^{D_t} = \frac{rs}{t}\Big(\Big\lceil\frac{i_1}{2}\Big\rceil + \Big\lceil\frac{i_{t2rs}}{2}\Big\rceil + \sum_{\overline{z}\in D_{sr}F/D_{sr}^-} i_{\overline{z}}\Big) + \dim(V^{C_rD_t}).$$

Thus the connectivity of the fiber is greater than

$$\min_{C_r\nleq K\leq D_{rs}}(\dim(V^K) - \dim(V^{C_rK})) - 1.$$

We can find a D_{rs}-representation V' which is fixed by K but not by the bigger group C_rK. Adding copies of V' to V, we can make the connectivity as big as we want. $\square$

3. Real topological cyclic homology

The $O(2)$-cyclotomic structure on $\mathrm{THR}(A, D)$ allows us to define G-equivariant restriction maps $R : \mathrm{THR}(A, D)^{C_{p^n}} \to \mathrm{THR}(A, D)^{C_{p^{n-1}}}$, and we can define the real topological cyclic homology at a prime p as a G-spectrum $\mathrm{TCR}(A, D; p)$ by mimicking the classical definition. Before we do so, we review some constructions from equivariant stable homotopy theory.

3.1. **The equivariant stable homotopy category.** Let H be a compact Lie group. We work in the H-stable homotopy category which is defined to be the homotopy category of the model category of orthogonal H-spectra on a complete universe with the stable model structure, e.g. [15, Chapter III, 4.1,4.2].

Let X be an orthogonal H-spectrum and let V be an H-representation. The suspension ΣX is defined by $(\Sigma X)(V) = S^1 \wedge X(V)$. The group $O(V) \rtimes H$ acts through the action on $X(V)$, and the structure maps are the suspensions of the structure maps in X. The loop spectrum ΩX is defined by $(\Omega X)(V) = \mathrm{Map}(S^1, X(V))$. The group $O(V) \rtimes H$ acts through the action on $X(V)$. The structure maps are given as the composite

$$\mathrm{Map}(S^1, X(V)) \wedge S^W \to \mathrm{Map}(S^1, X(V) \wedge S^W) \xrightarrow{\mathrm{Map}(\mathrm{id}, \lambda_{V,W})} \mathrm{Map}(S^1, X(V)).$$

The functors are adjoint and both preserve π_*-isomorphisms. We let $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{id}$ denote the counit of the adjunction and $\eta : \mathrm{id} \Rightarrow \Omega\Sigma$ denote the unit of the adjunction. Both ε and η are natural isomorphisms on the homotopy category.

Let $\psi : A \to B$ be a map of pointed H-spaces. We define the mapping cone by

$$C_\psi = B \cup_\psi ([0,1] \wedge A),$$

where $1 \in [0,1]$ is the basepoint for the interval and H acts trivially on the interval. Let $i : B \to C_\psi$ be the inclusion. Collapsing the image of the inclusion to the basepoint defines a map $\delta : C_\psi \to S^1 \wedge A = \Sigma A$. We define the mapping cone C_f of a map of orthogonal H-spectra $f : X \to Y$ by applying this construction lewelwise. The inclusions and the collapse maps assemble into morphisms of orthogonal H-spectra and we obtain a sequence of orthogonal H-spectra

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\delta} \Sigma X. \tag{4}$$

We call a diagram of the form $X \to Y \to Z \to \Sigma X$ in the H-stable homotopy category a triangle. The collection of all triangles isomorphic to triangles of the form (4) gives the H-stable homotopy category the structure of a triangulated category, see [19, Theorem A.12]. More precisely, Schwede takes the distinguished triangles to be all triangles isomorphic to triangles of the form

$$V \xrightarrow{j} W \to W/V \xrightarrow{\partial} \Sigma W,$$

where j is a cofibration and V and W are cofibrant objects and ∂ fits in the homotopy commutative diagram

$$\begin{array}{ccccccc}
V & \xrightarrow{j} & W & \longrightarrow & W/V & \xrightarrow{\partial} & \Sigma V \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \sim}\uparrow{\scriptstyle c} & & \uparrow{\scriptstyle \mathrm{id}} \\
V & \xrightarrow{j} & W & \xrightarrow{i} & C_j & \xrightarrow{\delta} & \Sigma V,
\end{array}$$

where c collapses the cone of V to the base point. In order to see that this choice makes the triangles of the form (4) distinguished, we first note that the map c is a weak equivalence by the gluing lemma, and therefore the lower row is a distinguished triangle. Let $f : X \to Y$ be an arbitrary map. We cofibrantly replace X and Y

and get a map $X^c \xrightarrow{f^c} Y^c$, which we factor as a cofibration $\tilde{f}^c$ followed by a weak equivalence in the following diagram

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & C_f & \xrightarrow{\delta} & \Sigma X \\
{\sim}\Big\uparrow j^c & & {\sim}\Big\uparrow j^c & & {\sim}\Big\uparrow & & {\sim}\Big\uparrow \Sigma(j^c) \\
X^c & \xrightarrow{f^c} & Y^c & \xrightarrow{i} & C_{f^c} & \xrightarrow{\delta} & \Sigma X^c \\
\Big\uparrow \mathrm{id} & & {\sim}\Big\uparrow & & {\sim}\Big\uparrow & & \Big\uparrow \mathrm{id} \\
X^c & \xrightarrow{\tilde{f}^c} & Y' & \xrightarrow{i} & C_{\tilde{f}^c} & \xrightarrow{\delta} & \Sigma X^c.
\end{array}
$$

It follows from the long exact sequence induced by the cofiber sequence that the induced map on cones are π_*-isomorphisms. We note that Y' is cofibrant, hence the bottom triangle is distinguished, and therefore the top triangle is distinguished.

In the rest of this section we establish some identification of distinguished triangles that we will need later.

Let $\psi : A \to B$ be a map of pointed H-spaces. We define the homotopy fiber of ψ as the pointed H-space

$$H_\psi = \{(\gamma, a) \in B^{[0,1]} \times A \mid \gamma(0) = \psi(x),\ \gamma(1) = *\},$$

where $(\gamma_*, *)$ serves as the basepoint and H acts trivially on the interval. Here γ_* refers to the constant path at the basepoint. Let $p : H_\psi \to A$ be the projection $p(\gamma, a) = a$ and let $j : \Omega B \to H_\psi$ be the inclusion $j(\alpha) = (\alpha, *)$. We define the homotopy fiber of a map of orthogonal H-spectra $f : X \to Y$ by applying this construction lewelwise. The projections and inclusions assemble into morphisms of orthogonal H-spectra $p : H_f \to X$ and $j : \Omega Y \to H_f$. See [11, Theorem 7.1.11] for the following lemma.

Lemma 3.1. *The triangle*

$$H_f \xrightarrow{p} X \xrightarrow{f} Y \xrightarrow{\Sigma(j)\circ\varepsilon^{-1}} \Sigma H_f,$$

is distinguished, where $\varepsilon : \Sigma\Omega Y \to Y$ *is the counit of the loop-suspension adjunction.*

Given maps of pointed H-spaces $f, g : A \to B$ we define the homotopy equalizer as the pointed H-space

$$\mathrm{HE}(f, g) = \{(\gamma, a) \in B^{[0,1]} \times A \mid \gamma(0) = f(a),\ \gamma(1) = g(a)\},$$

where $(\gamma_*, *)$ is the basepoint and H acts trivially on the interval. Let $p : \mathrm{HE}(f, g) \to A$ denote the projection $p(\gamma, a) = a$ and $\iota : \Omega B \to \mathrm{HE}(f, g)$ the inclusion $\iota(\alpha) = (\alpha, *)$. We define the homotopy equalizer of maps of orthogonal H-spectra $f, g : X \to Y$ by applying this construction lewelwise, and the projections and inclusions assemble into morphisms of orthogonal H-spectra.

Lemma 3.2. *The triangle*

$$\mathrm{HE}(f, g) \xrightarrow{p} X \xrightarrow{f-g} Y \xrightarrow{\Sigma(\iota)\circ\varepsilon^{-1}} \Sigma\,\mathrm{HE}(f, g),$$

is distinguished, where $\varepsilon : \Sigma\Omega Y \to Y$ *is the counit of the loop-suspension adjunction.*

Before we prove the lemma, we introduce some notation. For $\alpha \in \Omega Y$, let $\overline{\alpha} \in \Omega Y$ denote the inverse loop, i.e. $\overline{\alpha}(t) = \alpha(1-t)$. Given two loops $\alpha, \beta \in \Omega Y$ the concatenated loop $\beta \star \alpha \in \Omega Y$ is given by

$$\beta \star \alpha(t) = \begin{cases} \alpha(2t) & \text{if } t \leq 1/2, \\ \beta(2(t-\frac{1}{2})) & \text{if } t > 1/2. \end{cases}$$

Given a path γ in Y and $s \in [0,1]$, let $\gamma_{\leq s}$ and $\gamma_{\geq s}$ denote the paths in Y given by

$$\gamma_{\leq s}(t) = \gamma(t \cdot s), \quad \gamma_{\geq s}(t) = \gamma(t \cdot (1-s) + s), \quad t \in [0,1].$$

We have a canonical path in ΩY from the loop $\overline{\gamma} \star \gamma$ to the constant loop $*$ given by

$$t \mapsto \overline{\gamma_{\geq t}} \star \gamma_{\leq (1-t)}.$$

Proof. We define the map $F - G : \Omega\Sigma X \to \Omega\Sigma Y$ to be the composition

$$\Omega\Sigma X \xrightarrow{\triangle} \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{\overline{(\Sigma g \circ -)} \star (\Sigma f \circ -)} \Omega\Sigma Y,$$

where $\triangle$ is the diagonal map and consider the commutative diagram:

$$\begin{array}{ccccccc} \Omega Y & \xrightarrow{\iota} & \mathrm{HE}(f,g) & \xrightarrow{p} & X & \xrightarrow{f-g} & Y \\ \sim \big\downarrow \Omega(\eta) & & & & \sim \big\downarrow \eta & & \sim \big\downarrow \eta \\ \Omega(\Omega\Sigma Y) & \xrightarrow{j} & H_{F-G} & \xrightarrow{P} & \Omega\Sigma X & \xrightarrow{F-G} & \Omega\Sigma Y. \end{array}$$

The composition

$$[0,1] \xrightarrow{\gamma} Y \xrightarrow{\eta} \Omega\Sigma Y \xrightarrow{\overline{\eta(g(x))} \star (-)} \Omega\Sigma Y$$

is a path from $\overline{\eta(g(x))} \star \eta(f(x))$ to $\overline{\eta(g(x))} \star \eta(g(x))$. We let $\Psi_{\gamma,x}$ be the concatenation of this path with the canonical path from $\eta(g(x)) \star \overline{\eta(g(x))}$ to the basepoint. We define a map $\mathrm{HE}(f,g) \to H_{F-G}$ by $(\gamma, x) \mapsto (\Psi_{\gamma,x}, \eta(x))$. The constructed map is a π_*-isomorphism and completes the commutative diagram above. □

Consider the following diagram of orthogonal H-spectra

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

We define the homotopy pull-back spectrum $\mathrm{HP}(f,g)$ lewelwise:

$$\mathrm{HP}(f,g)(V) = \{(x, \gamma, y) \in X(V) \times Z(V)^{[0,1]} \times Y(V) \mid \gamma(0) = f(x), \gamma(1) = g(y)\},$$

with H acting trivially on the interval. The projections $p_X : \mathrm{HP}(f,g)(V) \to X(V)$, $p_Y : \mathrm{HP}(f,g)(V) \to Y(V)$ assemble into maps of orthogonal H-spectra. The following diagram is homotopy commutative:

$$\begin{array}{ccccccc} \Omega Y & \longrightarrow & H_{p_Y} & \xrightarrow{p} & \mathrm{HP}(f,g) & \xrightarrow{p_Y} & Y \\ \Omega g \big\downarrow & & s \big\downarrow \sim & & p_X \big\downarrow & & g \big\downarrow \\ \Omega Z & \xrightarrow[j]{} & H_f & \longrightarrow & X & \xrightarrow[f]{} & Z. \end{array}$$

The map s is defined as follows. A point in $H_{p_Y}(V)$ is a triple $(x, \gamma, y) \in \mathrm{HP}(f,g)(V)$ along with a path $\beta : I \to Y$ such that $\beta(0) = y$ and $\beta(1) = *$. Then

$$s(\beta, (x, \gamma, y)) = ((g \circ \beta) \star \gamma, x).$$

The following lemma now follows from Lemma 3.1 and the diagram above:

Lemma 3.3. *The triangle*

$$H_f \xrightarrow{p \circ s^{-1}} \mathrm{HP}(f,g) \xrightarrow{p_Y} Y \xrightarrow{\Sigma(j) \circ \varepsilon^{-1} \circ g} \Sigma H_f,$$

is distinguished.

We end this section by considering the following setup, that will occur in the calculation of the topological cyclic homology of spherical groups rings in Section 4. Assume we have a sequence of H-spectra X_i for $i \geq 0$, maps $f_i : X_i \to X_{i-1}$ and a map $g : X_0 \to X_0$. We recall that the unit of the loop-suspension adjunction

$$\eta : X_0 \xrightarrow{\sim} \Omega\Sigma(X_0)$$

is a π_*-isomorphism and consider the following diagram:

$$\begin{array}{ccccc}
X_0 & \xrightarrow{\ \mathrm{incl} \circ \eta\ } & \Omega\Sigma(X_0) \times \prod_{i=1}^{\infty} X_i & \xrightarrow{\ \mathrm{proj}\ } & \prod_{i=1}^{\infty} X_i \\
g \Big\downarrow \Big\downarrow \mathrm{id} & & ((\eta \circ f_1 \circ \mathrm{pr}_1) \star \Sigma(g), f_2, f_3, \dots) \Big\downarrow \Big\downarrow \mathrm{id} & & q \circ \prod_{i \geq 1} f_i \Big\downarrow \Big\downarrow \mathrm{id} \\
X_0 & \xrightarrow[\ \mathrm{incl} \circ \eta\ ]{} & \Omega\Sigma(X_0) \times \prod_{i=1}^{\infty} X_i & \xrightarrow[\ \mathrm{proj}\ ]{} & \prod_{i=1}^{\infty} X_i,
\end{array}$$

where

$$(\eta \circ f_1 \circ \mathrm{pr}_1) \star \Sigma(g) : \Omega\Sigma(X_0) \times \prod_{i=1}^{\infty} X_i \to \Omega\Sigma(X_0)$$

is the map

$$(\alpha, x) \mapsto \eta \circ f_1 \circ \mathrm{pr}_1(x) \star \Sigma(g) \circ \alpha$$

The diagram gives rise to a distinguished triangle connecting the vertical homotopy equalizers as we now explain. Consider the following diagram where the maps are defined below:

$$\Omega\Sigma X_0 \xrightarrow{\Omega\Sigma(g) - \mathrm{id}} \Omega\Sigma X_0 \xleftarrow{-\eta \circ f_1 \circ \mathrm{pr}_1} \mathrm{HE}\left(q \circ \textstyle\prod_{i \geq 1} f_i, \mathrm{id}\right).$$

The left hand map takes a loop $\alpha \in \Omega\Sigma X_0$ to the loop $\Sigma g \circ \alpha \star \overline{\alpha} \in \Omega\Sigma X_0$. The right hand map takes a pair

$$(\gamma, x) \in \mathrm{HE}\left(q \circ \prod_{i \geq 1} f_i, \mathrm{id}\right) \subseteq \prod_{i=1}^{\infty} X_i^{[0,1]} \times \prod_{i=1}^{\infty} X_i$$

to the loop $\overline{\eta \circ f_1 \circ \mathrm{pr}_1(x)}$. A point in the homotopy pull-back of the diagram is a loop $\alpha \in \Omega\Sigma X_0$, a point $x \in \prod_{i=1}^{\infty} X_i$ and paths

$$\gamma : q \circ \prod_{i \geq 1} f_i(x) \sim x, \quad \Sigma(g) \circ \alpha \star \overline{\alpha} \sim \overline{\eta \circ f_1 \circ \mathrm{pr}_1(x)}.$$

The notation $x \sim y$ means that the path takes the value x at 0 and the value y at 1.

Next consider the diagram:

$$\Omega\Sigma X_0 \times \prod_{i=1}^{\infty} X_i \underset{\mathrm{id}}{\overset{(\eta\circ f_1\circ \mathrm{pr}_1)\star\Sigma(g)\times(f_2,f_3,\dots)}{\rightrightarrows}} \Omega\Sigma X_0 \times \prod_{i=1}^{\infty} X_i.$$

The top map takes a pair $(\alpha, x) \in \Omega\Sigma X_0 \times \prod_{i=1}^{\infty} X_i$ to the pair

$$\Big(\eta \circ f_1 \circ \mathrm{pr}_1(x) \star \Sigma(g) \circ \alpha, q \circ \prod_{i\geq 1} f_i(x)\Big).$$

A point in the homotopy equalizer is therefor a loop $\alpha \in \Omega\Sigma X_0$, a point $x \in \prod_{i=1}^{\infty} X_i$ and paths

$$\eta \circ f_1 \circ \mathrm{pr}_1(x) \star \Sigma(g) \circ \alpha \sim \alpha, \quad q \circ \prod_{i\geq 1} f_i(x) \sim x.$$

Thus there is a homotopy equivalence from the homotopy pullback to the homotopy equalizer. It follows from Lemma 3.3 that there is a distinguished triangle of the form

$$\mathrm{HE}(\Omega\Sigma g, \mathrm{id}) \to \mathrm{HE}((\eta \circ f_1 \star \Sigma g, f_2, f_3, \dots), \mathrm{id}) \to$$
$$\mathrm{HE}(q \circ \prod f_i, \mathrm{id}) \xrightarrow{-\Sigma(\iota)\circ\varepsilon^{-1}\circ\eta\circ f_1\circ\mathrm{pr}_1} \Sigma\,\mathrm{HE}(\Omega\Sigma g, \mathrm{id}),$$

where the first map is induced by the inclusion and the second map is induced by the projection and

$$\iota : \Omega\big(\Omega\Sigma(X_0)\big) \to \mathrm{HE}(\Omega\Sigma(g), \mathrm{id})$$

was defined above when we described the construction of homotopy equalizers. Note that we have a commutative square

$$\begin{array}{ccc} \Omega\Sigma(X_0) & \xleftarrow[\sim]{\ \eta\ } & X_0 \\ {\scriptstyle \Omega\Sigma(g)}\downdownarrows{\scriptstyle \mathrm{id}} & & {\scriptstyle g}\downdownarrows{\scriptstyle \mathrm{id}} \\ \Omega\Sigma(X_0) & \xleftarrow[\sim]{\ \eta\ } & X_0. \end{array}$$

Thus η induces an π_*-isomorphism of homotopy equalizers. The triangle above simplifies to the following triangle under this identification:

$$\mathrm{HE}(g, \mathrm{id}) \xrightarrow{I} \mathrm{HE}((\eta \circ f_1 \star \Sigma g, f_2, f_3, \dots), \mathrm{id}) \xrightarrow{P}$$
$$\mathrm{HE}(q \circ \prod_{i\geq 1} f_i, \mathrm{id}) \xrightarrow{-\Sigma(\iota)\circ\varepsilon^{-1}\circ f_1\circ\mathrm{pr}_1} \Sigma\,\mathrm{HE}(g, \mathrm{id}),$$

where I is induced by

$$\mathrm{incl} \circ \eta : X_0 \to \Omega\Sigma(X_0) \times \prod_{i=1}^{\infty} X_i,$$

P is induced by the projection

$$\mathrm{proj} : \Omega\Sigma(X_0) \times \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{\infty} X_i,$$

and $\iota : \Omega X_0 \to \mathrm{HE}(g, \mathrm{id})$. Finally we note that $\mathrm{HE}(q \circ \prod_{i \geq 1} f_i, \mathrm{id})$ is a model for the homotopy limit $\mathrm{holim}_{i\geq 1} X_i$ and we obtain the following theorem:

Theorem 3.4. *The triangle*

$$\mathrm{HE}(g, \mathrm{id}) \xrightarrow{I} \mathrm{HE}((\eta \circ f_1 \star \Sigma g, f_2, f_3, \dots), \mathrm{id}) \xrightarrow{P} \operatorname*{holim}_{i\geq 1, f_i} X_i \xrightarrow{-\Sigma(\iota)\circ\varepsilon^{-1}\circ f_1 \circ \mathrm{pr}_1} \Sigma\,\mathrm{HE}(g, \mathrm{id})$$

is distinguished.

3.2. **Real topological cyclic homology.** We fix a prime p and we let $\mathscr{F}_p$ denote the family of $O(2)$-subgroups generated by the subgroups D_{p^n} and C_{p^n} for all $n \geq 0$. Let X be a $\mathscr{F}_p$-fibrant $O(2)$-cyclotomic spectrum. We let R denote the $O(2)$-equivariant composition

$$R : \rho_p^* X^{C_p} \xrightarrow{\rho_p^*(\gamma)} \rho_p^* X^{gC_p} \xrightarrow{T_p} X.$$

Let $i : G \hookrightarrow O(2)$ denote the inclusion. We will let $X^{C_{p^n}}$ denote the underlying G-spectrum indexed on $i^*\mathcal{U}$. We have an isomorphism of orthogonal G-spectra

$$(\rho_p^* X^{C_p})^{C_{p^{n-1}}} \cong X^{C_{p^n}},$$

and we define a map of orthogonal G-spectra

$$R_n : X^{C_{p^n}} \cong (\rho_p^* X^{C_p})^{C_{p^{n-1}}} \xrightarrow{R^{C_{p^{n-1}}}} X^{C_{p^{n-1}}}.$$

Remark 3.5. We let X be an $O(2)$-cyclotomic spectrum and let $j_f : X \xrightarrow{\sim} X_f$ be a fibrant replacement and consider the following diagram:

$$\begin{array}{ccccc}
\rho_p^*(X^c)^{gC_p} & \longrightarrow & \rho_p^* X^{gC_p} & \xrightarrow{\quad T_p \quad} & X \\
{\scriptstyle \rho_p^*((j_f)^c)^{gC_p}} \big\downarrow \wr & & {\scriptstyle \rho_p^*(j_f)^{gC_p}} \big\downarrow \wr & & {\scriptstyle j_f} \big\downarrow \wr \\
\rho_p^*((X_f)^c)^{gC_p} & \longrightarrow & \rho_p^*(X_f)^{gC_p} & \overset{\hat{T}_p}{\dashrightarrow} & X_f.
\end{array}$$

Even though X_f is not cyclotomic in the sense of Definition 2.6, we do have maps $\hat{T}_p : \rho_p^*(X_f)^{gC_p} \to X_f$ in the $O(2)$-stable homotopy category. We let R denote the composition

$$R : \rho_p^*(X_f)^{C_p} \xrightarrow{\rho_p^*(\gamma)} \rho_p^*(X_f)^{gC_p} \xrightarrow{\hat{T}_p} X$$

We get maps in the $O(2)$-stable homotopy category

$$R_n : X_f^{C_{p^n}} \to X_f^{C_{p^{n-1}}}$$

by mimicking the above construction.

In order to define the real topological cyclic homology at a prime p, we let

$$\mathrm{TRR}^n(A, D; p) = \mathrm{THR}(A, D)^{C_{p^n}},$$

and define the G-spectrum $\mathrm{TRR}(A, D; p)$ to be the homotopy limit over the R_n maps,

$$\mathrm{TRR}(A, D; p) := \operatorname*{holim}_{n, R_n} \mathrm{TRR}^n(A, D; p).$$

The inclusions of fixed points

$$F_n : X^{C_{p^n}} \to X^{C_{p^{n-1}}},$$

which we refer to as the Frobenius maps, are G-equivariant and induce a self map of $\mathrm{TRR}(A, D; p)$. In order to describe this map, we let N_0 denote the category

$$\cdots \to n \to (n-1) \to \cdots \to 2 \to 1 \to 0,$$

and we let $\mathrm{TRR}^{(-)}(A, D; p) : N_0 \to \mathsf{Top}_*$ denote the functor which sends $n \to n-1$ to $\mathrm{THR}(A, D)^{C_{p^n}} \xrightarrow{R_n} \mathrm{THR}(A, D)^{C_{p^{n-1}}}$. Let $\tau : N_0 \to N_0$ denote the translation functor $\tau(n) = n+1$. The Frobenius maps F_n assemble into a natural transformation

$$F : \mathrm{TRR}^{(-)}(A, D; p) \circ \tau \Rightarrow \mathrm{TRR}^{(-)}(A, D; p),$$

and we let φ denote the composite

$$\operatorname*{holim}_{N_0} \mathrm{TRR}^n(A, D; p) \xrightarrow{\mathrm{ind}_\tau} \operatorname*{holim}_{N_0} \mathrm{TRR}^{n+1}(A, D; p) \circ \tau \xrightarrow{F} \operatorname*{holim}_{N_0} \mathrm{TRR}^n(A, D; p).$$

Definition 3.6. The real topological cyclic homology at p, $\mathrm{TCR}(A, D; p)$, is the homotopy equalizer, $\mathrm{HE}(\varphi, \mathrm{id})$, of the diagram

$$\mathrm{TRR}(A, D; p) \underset{\mathrm{id}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TRR}(A, D; p).$$

We conclude this section by describing the homotopy fiber of the restriction maps. Let $\mathcal{R}$ denote the family of subgroups of $O(2)$ consisting of the trivial subgroup and all order 2 subgroups generated by a reflection of the plane in a line trough the origin;

$$\mathcal{R} = \{1, \langle t\omega \rangle \mid t \in \mathbb{T}, \omega \in G\}.$$

Write $E\mathcal{R}$ for the classifying space of this family, thus $E\mathcal{R}$ is an $O(2)$-CW-complex such that

$$E\mathcal{R}^H \cong \begin{cases} * & \text{if } H \in \mathcal{R} \\ \emptyset & \text{if } H \notin \mathcal{R}. \end{cases}$$

If we let $O(2)$ act on $\mathbb{C}$ by multiplication and complex conjugation, and on $\mathbb{C}^n$ by the diagonal action, then the $O(2)$-space

$$S(\mathbb{C}^\infty) = \bigcup_{n=0}^{\infty} S(\mathbb{C}^{n+1}),$$

where $S(\mathbb{C}^{n+1})$ denotes the unit sphere in $\mathbb{C}^{n+1}$, is a model for $E\mathcal{R}$. The $O(2)$-equivariant homeomorphism $S(\mathbb{C}) \star S(\mathbb{C}) \star \cdots \cong S(\mathbb{C}^\infty)$ shows that the G-fixed points are contractible. We let $\tilde{E\mathcal{R}}$ denote the mapping cone of the map $E\mathcal{R}_+ \to S^0$ which collapses $E\mathcal{R}$ to the non-basepoint. We have a cofibration sequence of based $O(2)$-spaces,

$$E\mathcal{R}_+ \to S^0 \to \tilde{E\mathcal{R}} \to \Sigma E\mathcal{R}_+.$$

and we obtain a distinguished triangle of orthogonal G-spectra by smashing the sequence with an $O(2)$-spectrum X and taking derived C_{p^n}-fixpoints.

$$(5) \qquad (E\mathcal{R}_+ \wedge X)_f^{C_{p^n}} \to X_f^{C_{p^n}} \xrightarrow{\lambda^{C_{p^n}}} (\tilde{E\mathcal{R}} \wedge X)_f^{C_{p^n}} \to \Sigma(E\mathcal{R}_+ \wedge X)_f^{C_{p^n}}.$$

We denote the left hand spectrum by $\mathrm{H.}(C_{p^n}; X)$ and refer to it as the G-equivariant homology spectrum of the subgroup C_{p^n} acting on the $O(2)$-spectrum X. The third term in the triangle identifies with the G-equivariant derived C_{p^n}-geometric fixed points; see [15, Prop. 4.17], giving the well-known isotropy separation sequence.

Lemma 3.7. *Let X be a $O(2)$-cyclotomic spectrum. The triangle*

$$\mathrm{H}.(C_{p^n}; X) \to X_f^{C_{p^n}} \xrightarrow{R_n} X_f^{C_{p^{n-1}}} \to \Sigma\,\mathrm{H}.(C_{p^n}; X),$$

is distinguished in the G-stable homotopy category.

Proof. The result follows immediately using the distinguished triangle (5), the identification of $(\tilde{E\mathcal{R}} \wedge X)_f^{C_{p^n}}$ with the G-equivariant derived C_{p^n}-geometric fixed points and the π_*-isomorphism of G-spectra $(X^c)^{gC_p} \to X^{gC_p} \xrightarrow{T_p} X$. □

Remark 3.8. Let $f : X \to Y$ be a map of $O(2)$-spectra. If both X and Y are cyclotomic and f commutes with the cyclotomic structure maps, then f commutes with R. If f restricts to a π_*-isomorphism of G-spectra, then by induction using the distinguished triangle above, f is a $\mathscr{F}_p$-equivalence.

4. Spherical group rings

In this section we determine the real topological Hochschild homology of the spherical group ring $\mathbb{S}[\Gamma]$ of a topological group Γ with anti-involution $\mathrm{id}[\Gamma]$ induced by taking inverses in the group. We determine the G-homotopy type of $\mathrm{TCR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p)$ where p is a prime. This is a generalization of results by Bökstedt-Hsiang-Madsen in [2, Section 5]; see also [14, Section 4.4].

Theorem 4.1. *Let Γ be a topological group. There is a map of $O(2)$-orthogonal spectra*

$$i : \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+ \to \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]),$$

commuting with the cyclotomic structures, which induces isomorphisms on $\pi_*^{C_{p^n}}(-)$ and $\pi_*^{D_{p^n}}(-)$ for all $n \geq 0$ and all primes p.

Proof. Let V be an $O(2)$-representation and let $F \subset \mathbb{T}$ be a finite subset. We define the map $i_V[F]$ to be the composition

$$\begin{aligned}
\bigwedge_{z\in F} \Gamma_+ \wedge S^V &\cong \mathrm{Map}\Big(\bigwedge_{z\in F} S^0, \big(\bigwedge_{z\in F} S^0\big) \wedge \big(\bigwedge_{z\in F} \Gamma_+\big) \wedge S^V \Big) \\
&\to \operatorname*{hocolim}_{I^F}\ \mathrm{Map}\Big(\bigwedge_{z\in F} S^{i_z}, \big(\bigwedge_{z\in F} S^{i_z}\big) \wedge \big(\bigwedge_{z\in F} \Gamma_+\big) \wedge S^V \Big) \\
&\cong \operatorname*{hocolim}_{I^F}\ \mathrm{Map}\Big(\bigwedge_{z\in F} S^{i_z}, \big(\bigwedge_{z\in F} (S^{i_z} \wedge \Gamma_+)\big) \wedge S^V \Big),
\end{aligned}$$

where the last map is induced by the natural transformation given by permuting the smash factors of the target. These maps commute with the dihedral structure and the $O(2)$-action on V, thus we obtain an $O(2)$-equivariant map on realizations:

$$i_V : B^{\mathrm{di}}\Gamma_+ \wedge S^V \to \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])(V).$$

The map i commutes with the cyclotomic structure, hence by Remark 3.8, it suffices to check that i restricts to a π_*-isomorphism of G-spectra. It follows from [17, Chapter XVI, Thm. 6.4], that it suffices to show that i induces an isomorphism on $\pi_*((-^c)^{gC_p})$ and $\pi_*^e(-)$, hence we must show that the connectivity of the induced map

$$(i_V)^H : (B^{\mathrm{di}}\Gamma_+ \wedge S^V)^H \to (\mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])(V))^H,$$

is $(\dim(V^H)+\epsilon(V))$-connected, where $H \in \{e, G\}$ and $\epsilon(V)$ tends to infinity with V. Let $i \in \mathrm{Ob}\left((I^G)^{\{1,-1\}} \times I^{GF/F^-}\right)$. After we restrict to the cofinal subcategory $G\mathcal{F}_*$, then the map i_V in simplicial level $G \cdot F$ is equal to the composite:

$$\begin{array}{c} \bigwedge_{z \in G\cdot F} \Gamma_+ \wedge S^V \\ \big\downarrow \eta \\ \mathrm{Map}\left(\bigwedge_{z\in G\cdot F} S^{i_{\overline{z}}}, \bigwedge_{z\in G\cdot F} S^{i_{\overline{z}}} \wedge S^V \wedge \bigwedge_{z\in G\cdot F} \Gamma_+\right) \\ \big\downarrow \\ \operatorname*{hocolim}_{I^{G\cdot F}} \mathrm{Map}\left(\bigwedge_{z\in G\cdot F} S^{i_{\overline{z}}}, \bigwedge_{z\in G\cdot F} S^{i_{\overline{z}}} \wedge S^V \wedge \bigwedge_{z\in G\cdot F} \Gamma_+\right). \end{array}$$

The top map is the adjunction unit, which by the Equivariant Suspension Theorem A.3 is at least $2 \cdot \dim(V) - 1$ connected as a map of non-equivariant spaces and $2\cdot\dim(V^G)-1$ connected on G-fixed points. By Lemma 1.4 we can make the lower map as connected as desired as a map of G-spaces by choosing $i \in \mathrm{Ob}\left((I^G)^{\{1,-1\}} \times I^{GF/F^-}\right)$ big enough. Thus the composite has the desired connectivity. □

The calculation of $\mathrm{TCR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p)$ relies on the fact that the restriction map

$$R_n : \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])^{C_{p^n}} \to \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])^{C_{p^{n-1}}}$$

splits. In [6, Lemma 6.2.5.1], a splitting of R_n is constructed and it is straight forward to check that the splitting is indeed G-equivariant. We denote this section S_n. We note that the theorem above makes the splitting apparent. Indeed, recall that we have defined $O(2)$-equivariant homeomorphisms $p_r : B^{\mathrm{di}}\Gamma \to \rho_r^* B^{\mathrm{di}}\Gamma^{C_r}$. The cyclotomic structure map at a representation V and natural number r on $\Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+$ arises from the adjoint homeomorphism

$$S^{\rho_r^*(V^{C_r})} \wedge \rho_r^*(B^{\mathrm{di}}\Gamma)_+^{C_r} \xrightarrow{\mathrm{id}\wedge p_r^{-1}} S^{\rho_r^*(V^{C_r})} \wedge B^{\mathrm{di}}\Gamma_+,$$

which is split by $\mathrm{id} \wedge p_r$. This implies, that the map

$$R_n : \left(\Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+\right)_f^{C_{p^n}} \to \left(\Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+\right)_f^{C_{p^{n-1}}}$$

splits.

We let Δ_r denote the G-equivariant composition

$$B^{\mathrm{di}}\Gamma \xrightarrow{p_r} B^{\mathrm{di}}\Gamma^{C_r} \hookrightarrow B^{\mathrm{di}}\Gamma.$$

There is a commutative diagram in the G-stable homotopy category where i is the $\mathscr{F}_p$-equivalence of Theorem 4.1; see [14, Corollary 4.4.11]

$$\begin{array}{ccc} \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]) & \overset{\sim}{\underset{i}{\longleftarrow}} & \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+ \\ {\scriptstyle F_1 \circ S_1}\big\downarrow & & \big\downarrow {\scriptstyle \Sigma^\infty \Delta_{p_+}} \\ \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]) & \overset{\sim}{\underset{i}{\longleftarrow}} & \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+. \end{array}$$

We set $T := \mathrm{THR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma])$ to ease notation and we recall the notation for the G-equivariant homology spectrum

$$\mathrm{H}.(C_{p^n}; B^{\mathrm{di}}\Gamma) = (E\mathcal{R}_+ \wedge \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+)_f^{C_{p^n}}, \quad n \geq 1.$$

If $n = 0$, then we let

$$\mathrm{H}.(1; B^{\mathrm{di}}\Gamma) := \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+.$$

Let $c : E\mathcal{R}_+ \wedge \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+ \to \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+$ denote the map that collapses $E\mathcal{R}$ to a point. We let c_n denote the composition

$$\mathrm{H}.(C_{p^n}; B^{\mathrm{di}}\Gamma) \xrightarrow{(c_f)^{C_{p^n}}} \big(\Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+\big)_f^{C_{p^n}} \to T^{C_{p^n}},$$

where the last map is induced by the $\mathscr{F}_p$-equivalence i of Theorem 4.1 and fibrant replacement. By Lemma 3.7 we have distinguished triangles of G-spectra

$$\mathrm{H}.(C_{p^n}; B^{\mathrm{di}}\Gamma) \xrightarrow{c_n} T^{C_{p^n}} \xrightarrow{R_n} T^{C_{p^{n-1}}} \to \Sigma\, \mathrm{H}.(C_{p^n}; B^{\mathrm{di}}\Gamma),$$

which split and provide an isomorphism in the G-stable homotopy category

$$S_n \circ \cdots \circ S_1 \circ i \vee \cdots \vee S_n \circ c_{n-1} \vee c_n : \bigvee_{j=0}^{n} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) \xrightarrow{\sim} T^{C_{p^n}}.$$

There is a commutative diagram, where the projection maps collapse the nth summand to the basepoint:

$$\begin{array}{ccccc}
T^{C_{p^n}} & \xleftarrow{\sim} & \bigvee_{j=0}^{n} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) & \xrightarrow{\sim} & \prod_{j=0}^{n} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) \\
\downarrow R_n & & \downarrow \mathrm{proj} & & \downarrow \mathrm{proj} \\
T^{C_{p^{n-1}}} & \xleftarrow{\sim} & \bigvee_{j=0}^{n-1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) & \xrightarrow{\sim} & \prod_{j=0}^{n-1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma).
\end{array}$$

Combined with the canonical map from the limit to the homotopy limit we obtain a canonical isomorphism in the G-stable homotopy category

$$\mathrm{TRR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p) \sim \prod_{j=0}^{\infty} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma).$$

We proceed to identify the Frobenius maps F_n. We have inclusions

$$\mathrm{incl} : (E\mathcal{R}_+ \wedge \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+)_f^{C_{p^n}} \to (E\mathcal{R}_+ \wedge \Sigma^\infty_{O(2)} B^{\mathrm{di}}\Gamma_+)_f^{C_{p^{n-1}}}.$$

We abuse notation slightly and let c denote the composite

$$(E\mathcal{R}_+ \wedge \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+)_f \xrightarrow{j_f^{-1}} E\mathcal{R}_+ \wedge \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ \xrightarrow{c} \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+,$$

where j_f is fibrant replacement. There is a commutative diagram:

$$\begin{array}{ccc} T^{C_{p^n}} & \xleftarrow{\sim} & \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ \vee \bigvee_{j=1}^n \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) \\ \Big\downarrow F_n & & \Big\downarrow \Sigma^\infty \Delta_{p_+} \vee c \circ \mathrm{incl} \vee \mathrm{incl} \vee \cdots \vee \mathrm{incl} \\ T^{C_{p^{n-1}}} & \xleftarrow{\sim} & \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ \vee \bigvee_{j=1}^{n-1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma). \end{array}$$

We can now identify the self-map φ of $\mathrm{TRR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p)$ induced by the Frobenius maps F_n. We have the following commutative diagram, where $\mathrm{TCR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p)$ is the homotopy equalizer of the two middle maps:

$$\begin{array}{ccccc} \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ & \xrightarrow{\mathrm{incl}\circ\eta} & \Omega\Sigma\big(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+\big) \times \prod_{j=1}^\infty \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) & \xrightarrow{\mathrm{proj}} & \prod_{j=1}^\infty \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) \\ \Sigma^\infty\Delta_{p_+} \Big\downarrow\Big\downarrow \mathrm{id} & & X \Big\downarrow\Big\downarrow \mathrm{id} & & q \circ \prod_{i\geq 1}\mathrm{incl} \Big\downarrow\Big\downarrow \mathrm{id} \\ \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+ & \xrightarrow[\mathrm{incl}\circ\eta]{} & \Omega\Sigma\big(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+\big) \times \prod_{j=1}^\infty \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) & \xrightarrow[\mathrm{proj}]{} & \prod_{j=1}^\infty \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma), \end{array}$$

where $X = ((\eta \circ c \circ \mathrm{incl} \circ \mathrm{pr}_1) \star \Sigma(\Sigma^\infty \Delta_{p_+}), \mathrm{incl}, \ldots, \mathrm{incl})$ and the first map in the tuple takes a pair

$$(\alpha, x) \in \Omega\Sigma\big(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+\big) \times \prod_{j=1}^\infty \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma)$$

to the loop

$$\big(\eta \circ c \circ \mathrm{incl} \circ \mathrm{pr}_1(x)\big) \star \big(\Sigma(\Sigma^\infty \Delta_{p_+}) \circ \alpha\big) \in \Omega\Sigma\big(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+\big).$$

We let $\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p = \mathrm{id}\text{”}}$ denote the homotopy equalizer of $\Sigma^\infty\Delta_{p_+}$ and the identity map,

$$\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p = \mathrm{id}\text{”}} := \mathrm{HE}(\Sigma^\infty\Delta_{p_+}, \mathrm{id}).$$

Recall that we constructed a map $\iota : \Omega(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+) \to \Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p = \mathrm{id}\text{”}}$. The map $\mathrm{incl} \circ \eta$ and the projection induce the maps I and P in the following theorem, which follows directly from Theorem 3.4.

Theorem 4.2. *The triangle*

$$\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p = \mathrm{id}\text{”}} \xrightarrow{I} \mathrm{TCR}(\mathbb{S}[\Gamma], \mathrm{id}[\Gamma]; p) \xrightarrow{P} \operatorname*{holim}_{j\geq 1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma)$$
$$\xrightarrow{-\Sigma(\iota)\circ\varepsilon^{-1}\circ c\circ\mathrm{incl}\circ\mathrm{pr}_1} \Sigma\left(\Sigma^\infty_G B^{\mathrm{di}}\Gamma_+^{\text{“}\Delta_p = \mathrm{id}\text{”}}\right)$$

is distinguished in the G-stable homotopy category, where $\varepsilon : \Sigma\Omega Y \to Y$ is the counit of the loop-suspension adjunction.

Remark 4.3. The inclusion of index categories induces a canonical π_*-isomorphism

$$\operatorname*{holim}_{j\geq 0} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma) \xrightarrow{\mathrm{can}} \operatorname*{holim}_{j\geq 1} \mathrm{H}.(C_{p^j}; B^{\mathrm{di}}\Gamma)$$

and the composite $\mathrm{incl} \circ \mathrm{pr}_1$ in the triangle can be replaced by $\mathrm{pr}_0 \circ \mathrm{can}^{-1}$.

When Γ is trivial, the distinguished triangle of the above theorem simplifies to

$$\Omega\mathbb{S} \vee \mathbb{S} \longrightarrow \mathrm{TCR}(\mathbb{S}, \mathrm{id}; p) \longrightarrow \operatorname*{holim}_{i\geq 0} \mathrm{H}.(C_{p^i}; \mathbb{S}) \xrightarrow{-(\varepsilon^{-1} \circ \mathrm{pr}_0)} \Sigma\Omega\mathbb{S} \vee \Sigma\mathbb{S}.$$

If we rotate the distinguished triangle arising from the homotopy fiber of the projection $\mathrm{pr}_0 : \operatorname{holim}_{i\geq 0} \mathrm{H}.(C_{p^i}; \mathbb{S}) \to \mathbb{S}$ given in Lemma 3.1 and add the distinguished triangle $\mathbb{S} \xrightarrow{\mathrm{id}} \mathbb{S} \to * \to \Sigma\mathbb{S}$, then we obtain the distinguished triangle:

$$\Omega\mathbb{S} \vee \mathbb{S} \longrightarrow H_{\mathrm{pr}_0} \vee \mathbb{S} \longrightarrow \operatorname*{holim}_{i\geq 0} \mathrm{H}.(C_{p^i}; \mathbb{S}) \xrightarrow{-(\varepsilon^{-1} \circ \mathrm{pr}_0)} \Sigma\Omega\mathbb{S} \vee \Sigma\mathbb{S}.$$

It follows from the axioms of a triangulated category that there is a non-canonical isomorphism in the G-stable homotopy category $\mathrm{TCR}(\mathbb{S}, \mathrm{id}; p) \sim H_{pr_0} \vee \mathbb{S}$.

A non-equivariant identification after p-completion of the homotopy limit appearing in the triangle in Theorem 4.2 appears in [2] and in more generality in [14, Lemma 4.4.9]. The result generalizes immediately to the equivariant setting. Let X be an $O(2)$-spectrum and let $M(\mathbb{Q}_p/\mathbb{Z}_p, -1)$ be the non-equivariant Moore spectrum. We can view $M(\mathbb{Q}_p/\mathbb{Z}_p, -1)$ as an orthogonal G-spectrum by giving it the trivial G-action, see [15, Chapter V, Sect. 1]. We then define the p-completion of X to be the function spectrum $X_p\hat{} = [M(\mathbb{Q}_p/\mathbb{Z}_p, -1), X]$. A map of orthogonal G-spectra is an isomorphism in the G-stable category after p-completion, if it is an isomorphism in the stable category after p-completion on underlying spectra and fixed point spectra.

The $O(2)$-spectrum $E\mathcal{R}_+ \wedge X$ is C_{p^n}-free. It follows from the generalized Adams isomorphism [17, Chapter XVI, Thm. 5.4] that there are isomorphisms in the G-stable homotopy category

$$E\mathcal{R}_+ \wedge_{C_{p^n}} X \xrightarrow{\sim} (E\mathcal{R}_+ \wedge X)_f^{C_{p^n}}$$

and under this isomorphism the inclusions of fixed points on the right hand side correspond to the G-equivariant transfers on the left hand side

$$\begin{array}{ccc} E\mathcal{R}_+ \wedge_{C_{p^n}} X & \xrightarrow{\sim} & (E\mathcal{R}_+ \wedge X)_f^{C_{p^n}} \\ {\scriptstyle \mathrm{trf}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{incl}} \\ E\mathcal{R}_+ \wedge_{C_{p^{n-1}}} X & \xrightarrow{\sim} & (E\mathcal{R}_+ \wedge X)_f^{C_{p^{n-1}}}. \end{array}$$

We obtain an isomorphism in the G-stable homotopy category

$$\operatorname*{holim}_{\mathrm{trf}} E\mathcal{R}_+ \wedge_{C_{p^i}} X \xrightarrow{\sim} \operatorname*{holim}_{i\geq 1} \mathrm{H}.(C_{p^i}; X),$$

and we can identify the left hand homotopy limit after p-completion:

Theorem 4.4. *Let X be an orthogonal $O(2)$-spectrum. The $\mathbb{T}$-transfer*

$$\mathrm{trf}_{\mathbb{T}} : \Sigma^{1,1} E\mathcal{R}_+ \wedge_{\mathbb{T}} X \to \operatorname*{holim}_{\mathrm{trf}} E\mathcal{R}_+ \wedge_{C_{p^i}} X,$$

induces an isomorphism in the G-stable homotopy category after p-completion, where $\Sigma^{1,1}$ denotes suspension with the sign representation of G.

Proof. Recall that

$$S(\mathbb{C}^\infty) = \bigcup_{n=0}^{\infty} S(\mathbb{C}^{n+1})$$

is a model for $E\mathcal{R}$. We filter $S(\mathbb{C}^\infty)$ by the $O(2)$-subspaces $S(\mathbb{C}^k)$ and obtain the diagram

$$\begin{array}{ccccc}
\Sigma^{1,1} S(\mathbb{C}^k)_+ \wedge_{\mathbb{T}} X & \longrightarrow & \Sigma^{1,1} S(\mathbb{C}^{k+1})_+ \wedge_{\mathbb{T}} X & \longrightarrow & \Sigma^{1,1} S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \wedge_{\mathbb{T}} X \\
\downarrow{\scriptstyle \mathrm{trf}_{\mathbb{T}}} & & \downarrow{\scriptstyle \mathrm{trf}_{\mathbb{T}}} & & \downarrow \\
S(\mathbb{C}^k)_+ \wedge_{C_{p^n}} X & \longrightarrow & S(\mathbb{C}^{k+1})_+ \wedge_{C_{p^n}} X & \longrightarrow & S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \wedge_{C_{p^n}} X.
\end{array}$$

By induction it suffices to show that the right hand vertical map induces isomorphism after p-completion on the homotopy limit of the transfers

$$\mathrm{trf} : S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \wedge_{C_{p^n}} X \to S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \wedge_{C_{p^{n-1}}} X.$$

In order to identify the right vertical map, we first note the following general fact. Let Y be a pointed $O(2)$-space and let $i : G \to O(2)$ be the inclusion. If we let $i(Y)$ denote the $\mathbb{T}$-trivial $O(2)$-space whose underlying G-space is i^*Y, then there is an $O(2)$-equivariant homeomorphism

$$S(\mathbb{C})_+ \wedge Y \xrightarrow{\cong} S(\mathbb{C})_+ \wedge i(Y), \quad (z, y) \mapsto (z, z^{-1}y).$$

The proof of Lemma A.4 provides a homeomorphism of $O(2)$-spaces

$$S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \cong S(\mathbb{C})_+ \wedge \Sigma S(\mathbb{C}^k)$$

and it follows that the right vertical map in the diagram identifies with

$$\Sigma^{1,1} S(\mathbb{C})/\mathbb{T}_+ \wedge i(\Sigma S(\mathbb{C}^k) \wedge X) \xrightarrow{\tau_\infty \wedge \mathrm{id}} S(\mathbb{C})/C_{p^n_+} \wedge i(\Sigma S(\mathbb{C}^k) \wedge X),$$

where $\tau_\infty : \Sigma^{1,1} S(\mathbb{C})/\mathbb{T}_+ \to S(\mathbb{C})/C_{p^n+}$ is the G-equivariant transfer. Likewise the transfer map

$$\mathrm{trf} : S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \wedge_{C_{p^n}} X \to S(\mathbb{C}^{k+1})/S(\mathbb{C}^k) \wedge_{C_{p^{n-1}}} X,$$

identifies with

$$S(\mathbb{C})/C_{p^n_+} \wedge i(\Sigma S(\mathbb{C}^k) \wedge X) \xrightarrow{\tau_n \wedge \mathrm{id}} S(\mathbb{C})/C_{p^{n-1}_+} \wedge i(\Sigma S(\mathbb{C}^k) \wedge X),$$

where $\tau_n : S(\mathbb{C})/C_{p^n+} \to S(\mathbb{C})/C_{p^{n-1}+}$ is the G-equivariant transfer.

As an $O(2)$-space $S(\mathbb{C})_+ = S^0 \vee S(\mathbb{C})$, with $O(2)$ acting trivially on S^0. If we identify $S(\mathbb{C})/C_{p^n}$ with $S(\mathbb{C})$ via the root isomorphism ρ_{p^n}, then $\tau_n = \mathrm{id} \vee p$.

The result now follows exactly as in [14, Lemma 4.4.9] by first arguing that there is a distinguished triangle of the form

$$\Sigma^{1,1} i(\Sigma S^{2k-1} \wedge X) \to \operatorname*{holim}_{\tau_n} S(\mathbb{C})/C_{p^n +} \wedge i(\Sigma S^{2k-1} \wedge X) \to \operatorname*{holim}_{p} i(\Sigma S^{2k-1} \wedge X) \to \Sigma.$$

The mod p homotopy groups of the homotopy limit vanish. Hence the left hand map induces an isomorphism after p-completion as desired. □

If we let $\mathbb{P}^\infty(\mathbb{C}) := S(\mathbb{C}^\infty)/\mathbb{T}$ be the infinite complex projective space with G acting by complex conjugation, then it follows from Theorem 4.4, that we can identify the projection $\mathrm{pr}_0 : \mathrm{holim}_{i \geq 0} \mathrm{H}.(C_{p^i}; \mathbb{S}) \to \mathbb{S}$ with the G-equivariant $\mathbb{T}$-transfer $\Sigma^\infty_G \Sigma^{1,1} \mathbb{P}^\infty(\mathbb{C}) \to \mathbb{S}$ after p-completion. If we let $\Sigma^{1,1}\mathbb{P}^\infty_{-1}(\mathbb{C})$ denote the G-equivariant homotopy fiber of the $\mathbb{T}$-transfer above, then we obtain the following corollary generalizing the classical calculation:

Corollary 4.5. *After p-completion, there is an isomorphism in the G-stable homotopy category*

$$\mathrm{TCR}(\mathbb{S}, \mathrm{id}; p) \sim \Sigma^{1,1}\mathbb{P}^\infty_{-1}(\mathbb{C}) \vee \mathbb{S}.$$

Appendix A. Equivariant homotopy theory

This appendix recalls some results from equivariant homotopy theory, which we need in the paper. Let H be a finite group and let $f : A \to B$ be a map of pointed H-spaces. We call the map f n-connected respectively a weak-H-equivalence if $f^K : A^K \to B^K$ is n-connected respectively a weak equivalence for all subgroups $K \leq H$. Let $\mathrm{Map}_H(A, B)$ denote the space of pointed H-equivariant maps.

The following lemma can be found in [1, Prop. 2.7].

Lemma A.1. *Let $f : B \to C$ be a map of pointed H-spaces and let A be a pointed H-CW-complex. The induced map*

$$f_* : \mathrm{Map}_H(A, B) \to \mathrm{Map}_H(A, C)$$

is n-connected with $n \geq \min_{K \leq H}\{\mathrm{conn}(f^K) - \dim(A^K)\}$, where K runs through all subgroups of H.

Let $i : A' \to A$ be an H-cofibration. For any pointed H-space B, the induced map

$$i^* : \mathrm{Map}_H(A, B) \to \mathrm{Map}_H(A', B)$$

is a fibration with fiber $\mathrm{Map}_H(A/A', B)$. The above Lemma estimates the connectivity of mapping spaces such as the fiber by considering the map $f : B \to *$. The estimate amounts to the following lemma:

Lemma A.2. *Let A be a pointed H-CW-complex and let B be a pointed H-space. Then*

$$\mathrm{conn}(\mathrm{Map}_H(A, B)) \geq \min_{K \leq H}(\mathrm{conn}(B^K) - \dim(A^K)),$$

where K runs through all subgroups of H.

Throughout this paper we will make use of the Equivariant Suspension Theorem, a proof can be found in [1, Theorem 3.3].

Theorem A.3 (Equivariant Freudenthal Suspension Theorem)**.** *Let V be a finite dimensional orthogonal H-representation and let A be a based H-space. The adjunction unit*

$$\eta : A \to \Omega^V \Sigma^V A$$

is n-connected, where

$$n \geq \min\{2 \cdot \mathrm{conn}(A^H) + 1, \mathrm{conn}(A^K) \mid K \leq H \text{ with } \dim\ V^K > \dim\ V^H\}.$$

Finally, we will need the following lemma.

Lemma A.4. *Let V and W be finite dimensional orthogonal H-representations. There is a canonical H-equivariant homeomorphism*

$$S^{V \oplus W}/S^W \cong \Sigma S^W \wedge S(V)_+.$$

Proof. First note that we have canonical H-equivariant homeomorphisms

$$S(V \oplus W) \cong S(V) \star S(W) \cong S(V) \times D(W) \cup_{S(V)\times S(W)} D(V) \times S(W),$$

which gives a canonical H-equivariant homeomorphism

$$S(V \oplus W)/S(W) \cong S^W \wedge S(V)_+.$$

We have a commutative diagram

$$\begin{array}{ccccc} S(W)_+ & \xrightarrow{c} & S^0 & \longrightarrow & S^W \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow \\ S(V\oplus W)_+ & \xrightarrow{c} & S^0 & \longrightarrow & S^{V\oplus W} \\ \downarrow & & \downarrow & & \downarrow \\ S(V\oplus W)/S(W) & \longrightarrow & * & \longrightarrow & S^{V\oplus W}/S^W, \end{array}$$

where c collapses the unit-sphere to the non-basepoint, and we can identify

$$S^{V\oplus W}/S^W \cong \Sigma S^W \wedge S(V)_+.$$

. □

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ON THE GEOMETRIC FIXED POINTS OF REAL TOPOLOGICAL HOCHSCHILD HOMOLOGY

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Abstract. We compute the component group of the derived G-geometric fixed points of the real topological Hochschild homology of a ring with anti-involution, where G denotes the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of order 2.

Introduction

Recently, Hesselholt and Madsen defined real topological Hochschild homology in [11] using a dihedral variant of Bökstedt's model in [4]. The real topological Hochschild homology functor takes a ring R with an anti-involution α, that is a ring isomorphism $\alpha : R^{\mathrm{op}} \to R$ such that $\alpha^2 = \mathrm{id}$, and associates an $O(2)$-equivariant orthogonal spectrum $\mathrm{THR}(R, \alpha)$.

Real topological Hochschild homology was introduced to fit in the framework of real algebraic K-theory, which was also defined by Hesselholt and Madsen in [11]. Real algebraic K-theory associates a G-spectrum $KR(R, \alpha)$ to a ring R with an anti-involution α, where G is the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The underlying non-equivariant spectrum is weakly equivalent to the usual K-theory spectrum of R, and the G-fixed point spectrum is weakly equivalent to the Hermitian K-theory spectrum of (R, α), as defined by Karoubi in [14], when 2 is invertible in the ring. Furthermore, there is a G-equivariant trace map

$$\mathrm{tr} : KR(R, \alpha) \to \mathrm{THR}(R, \alpha).$$

The classical trace maps are often highly non-trivial and several calculations in algebraic K-theory have been carried out using the trace, or more precisely the refinement of the trace to topological cyclic homology, see [2]. Classical calculations using trace methods often rely on a good understanding of $\pi_* \mathrm{THH}(R)^{C_r}$. In order to make the equivariant trace an efficient computational tool, we must understand the dihedral fixed points $\pi_* \mathrm{THR}(R, \alpha)^{D_r}$ and, in particular, the components $\pi_0 \mathrm{THR}(R, \alpha)^{D_r}$. As a first step in this direction, we calculate the group of components of the derived G-geometric fixed points.

The orthogonal spectrum $\mathrm{THR}(R, \alpha)$ is cyclotomic, which means that its derived C_r-geometric fixed points mimic the behavior of the C_r-fixed points of a free loop space $\mathcal{L}X$ of a G-space X, see [10, Prop. 1.5] and [13, Sect. 3.3] for details. In particular, it implies that the spectrum of C_r-geometric fixed points of $\mathrm{THR}(R, \alpha)$ resembles $\mathrm{THR}(R, \alpha)$ itself. The derived G-geometric fixed point, however, behave differently. In the analogy with the free loop space, the derived G-geometric fixed

Date: January 1, 2017.
Assistance from DNRF Niels Bohr Professorship of Lars Hesselholt is gratefully acknowledged.

points of $\mathrm{THR}(R,\alpha)$ corresponds to the G-fixed points of $\mathcal{L}X$, and we briefly investigate how the latter behaves.

We let $\mathcal{L}X = \mathrm{Map}(\mathbb{T}, X)$ be the free loop space of a G-space X. The group $O(2)$ acts on $\mathbb{T}$ by multiplication and complex conjugation and we view X as an $O(2)$-space with trivial $\mathbb{T}$-action. The free loop space becomes an $O(2)$-space by the conjugation action. Let $\mathbb{T} \subset \mathbb{C}$ denote the circle group. The group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{T}$ and we let $O(2)$ denote the semi-direct product $O(2) = \mathbb{T} \rtimes G$. If r is a natural number, then we let

$$\rho_r : O(2) \to O(2)/C_r$$

denote the root isomorphism given by $\rho_r(z) = z^{\frac{1}{r}} C_r$ if $z \in \mathbb{T}$ and $\rho_r(x) = x$ if $x \in G$. The map that takes a loop to the r-fold concatenation with itself,

$$p_r : \mathcal{L}X \to \rho_r^*(\mathcal{L}X)^{C_r}, \quad p_r(\gamma) = \gamma \star \cdots \star \gamma,$$

is an $O(2)$-equivariant homeomorphism, but the G-fixed space of the free loop space looks very different from the loop space itself. Indeed, if $\omega \in G$ is complex conjugation then we have a homeomorphism

$$\mathrm{Map}((I, \partial I), (X, X^G)) \to (\mathcal{L}X)^G, \quad \gamma \mapsto (\omega \cdot \gamma) \star \gamma.$$

The content of this paper is organized as follows. In Section 1 and 2 we review the construction of the orthogonal $O(2)$-spectrum $\mathrm{THR}(R,\alpha)$, and observe that, if R is a commutative ring, then $\mathrm{THR}(R,\alpha)$ has the homotopy type of a commutative $O(2)$-ring spectrum.

In section 3 we prove the main theorem of this paper. In order to state the theorem, we let R be a ring with an anti-involution α, which is a ring isomorphism $\alpha : R^{\mathrm{op}} \to R$ such that $\alpha^2 = \mathrm{id}$, and we let $N : R \to R^\alpha$ denote the norm map

$$N(r) = r + \alpha(r).$$

Theorem A. *Let R be a ring with an anti-involution α. There is an isomorphism of abelian groups*

$$\pi_0\big((\mathrm{THR}(R,\alpha)^c)^{gG}\big) \cong (R^\alpha/N(R) \otimes_{\mathbb{Z}} R^\alpha/N(R))/I,$$

where I denotes the subgroup generated by the elements $\alpha(s)rs \otimes t - r \otimes st\alpha(s)$ for all $s \in R$ and $r,t \in R^\alpha$.

The identification of the component group can be rewritten as

$$\pi_0\big((\mathrm{THR}(R,\alpha)^c)^{gG}\big) \cong R^\alpha/N(R) \otimes_R R^\alpha/N(R).$$

where we view the group $R^\alpha/N(R)$ as a right resp. left R-module via the actions

$$x \cdot r = \alpha(r)xr \quad \text{and} \quad r \cdot x = rx\alpha(r).$$

We note that the G-geometric fixed points vanishes if 2 is invertible in R, since the norm map surjects onto the fixed points of the anti-involution in this case: If $x \in R^\alpha$, then $N(\frac{1}{2}x) = x$.

We end this introduction by stating some immediate consequences of Theorem A.

If R is a commutative ring, then the components of the C_r-fixed points and the components of the D_r-fixed points have ring structures. The component ring of the C_{p^n}-fixed points, $\pi_0 \,\mathrm{THH}(R)^{C_{p^n}}$, is completely understood when p is a prime. Hesselholt and Madsen prove in [10] that there is a canonical ring isomorphism identifying

$\pi_0\,\mathrm{THH}(R)^{C_{p^n}}$ with the p-typical Witt vectors of length $n+1$. The classical construction of the Witt vectors can be understood as a special case of a construction which can be defined relative to any given profinite group, as done by Dress and Siebeneicher in [6]. Furthermore, the p-typical Witt vectors of length $n+1$ are exactly the Witt vectors constructed relative to the group C_{p^n}. In other words,

$$\pi_0\,\mathrm{THH}(R)^{C_{p^n}} \cong W_{C_{p^n}}(R).$$

If R is commutative, then the identity defines an anti-involution on R and it is tempting to guess that the ring $\pi_0\,\mathrm{THR}(R,\mathrm{id})^{D_{p^n}}$ can be identified with the Witt vectors $W_{D_{p^n}}(R)$. However, Theorem A tells us that this is not the case; see Remark 3.4.

When R is a commutative ring, the components of the derived G-geometric fixed points of $\mathrm{THR}(R,\alpha)$ have a ring structure. If R is a commutative ring with the identity serving as anti-involution, then Theorem A implies the functor

$$R \mapsto \pi_0\big((\mathrm{THR}(R,\mathrm{id})^c)^{gG}\big).$$

considered as a functor from the category of commutative rings to the category of sets, is not representable, since the functor does not preserve finite products, see Remark 3.2. This rules out the possibility that $\big((\mathrm{THR}(R,\mathrm{id})^c)^{gG}\big)$ is a ring of Witt vectors as defined by Borger in [3].

Acknowledgments. The author wishes to thank Lars Hesselholt for his guidance and for many valuable discussions. The author would also like to thank Martin Speirs, Dustin Clausen, Irakli Patchkoria and Kristian Moi for many useful conversations concerning the content of this paper.

Throughout this paper, $\mathbb{T} \subset \mathbb{C}$ denotes the circle group and G is the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \omega\}$. The group G acts on $\mathbb{T} \subset \mathbb{C}$ and $O(2)$ is the semi-direct product $O(2) = \mathbb{T} \rtimes G$. We let C_r denote the cyclic subgroup of order r and let D_r denote the dihedral subgroup $C_r \rtimes G$ of order $2r$.

By a space we will always mean a compactly generated weak Hausdorff space and all constructions are always carried out in this category.

1. Real topological Hochschild homology

A symmetric ring spectrum X is a sequence of based spaces $X_0, X_1, \dots$ with a left based action of the symmetric group Σ_n on X_n and $\Sigma_n \times \Sigma_m$-equivariant maps $\lambda_{n,m} : X_n \wedge S^m \to X_{n+m}$. Let A be a symmetric ring spectrum with multiplication maps $\mu_{n,m} : A_n \wedge A_m \to A_{n+m}$ and unit maps $1_n : S^n \to A_n$. An anti-involution on A is a self-map of the underlying symmetric spectrum $D : A \to A$, such that

$$D^2 = \mathrm{id}, \quad D_n \circ 1_n = 1_n,$$

and the following diagram commutes:

$$\begin{array}{ccc}
A_m \wedge A_n & \xrightarrow{D_m \wedge D_n} & A_m \wedge A_n \\
\Big\downarrow \mu_{m,n} & & \Big\downarrow \gamma \\
 & & A_n \wedge A_m \\
 & & \Big\downarrow \mu_{n,m} \\
 & & A_{n+m} \\
 & & \Big\downarrow \chi_{n,m} \\
A_{m+n} & \xrightarrow{D_{m+n}} & A_{m+n}.
\end{array}$$

Here γ is the twist map and $\chi_{n,m} \in \Sigma_{n+m}$ is the shuffle permutation

$$\chi_{n,m}(i) = \begin{cases} i+m & \text{if } 1 \le i \le n \\ i-n & \text{if } n+1 \le i \le n+m. \end{cases}$$

Let R be a unital, associative ring. Then R determines a symmetric ring spectrum HR, called the Eilenberg MacLane spectrum of R, which can be constructed as follows. Let $S^1[-] := \Delta^1[-]/\partial\Delta^1[-]$ denote the pointed simplicial circle and let $S^n[-]$ denote the pointed simplicial n-sphere defined as the n-fold smash product $S^1[-] \wedge \cdots \wedge S^1[-]$. The nth space of the spectrum HR is the realization of the reduced R-linearization of the simplicial n-sphere:

$$HR_n := R(S^n) = |[k] \mapsto R[S^n[k]]/R[*]|.$$

Here $R[S^n[k]]$ is the free R-module generated by the k-simplices $S^n[k]$ and $R[*]$ is the sub-R-module generated by the basepoint $* \in S^n[k]$. The symmetric group Σ_n acts by permutation of the smash factors of $S^n[-]$ and there are natural multiplication and unit maps

$$\mu_{m,n} : HR_m \wedge HR_n \to HR_{m+n}, \quad 1_n : S^n \to HR_n,$$

which are $\Sigma_m \times \Sigma_n$-equivariant and Σ_n-equivariant.

An anti-involution α on R is a ring isomorphism $\alpha : R^{\mathrm{op}} \to R$ such that $\alpha^2 = \mathrm{id}$. If α is an anti-involution on R, then we also let α denote the induced anti-involution on the symmetric ring spectrum HR, which in spectral level n is the geometric realization of the map of simplicial R-modules given by $r \cdot x \mapsto \alpha(r) \cdot x$ for $r \in R$ and $x \in S^n[k]$.

Given a symmetric ring spectrum with anti-involution (A, D), the real topological Hochschild homology space $\mathrm{THR}(A, D)$ was defined in [11] as the geometric realization of a dihedral space, and we briefly recall the notion of a dihedral object. The real topological Hochschild homology of a ring with anti-involution (R, α) is the real topological Hochschild homology of (HR, α), which we simply denote $\mathrm{THR}(R, \alpha)$.

Definition 1.1. A dihedral object in a category $\mathcal{C}$ is a simplicial object

$$X[-] : \triangle^{\mathrm{op}} \to \mathcal{C}$$

together with dihedral structure maps $t_k, w_k : X[k] \to X[k]$ such that $t_k^{k+1} = \mathrm{id}$, $w_k^2 = \mathrm{id}$, and $t_k w_k = t_k^{-1} \omega_k$. The dihedral structure maps are required to satisfy the following relations involving the simplicial structure maps:

$$\begin{aligned} d_l w_k &= w_{k-1} d_{k-l}, \quad s_l w_k = w_{k+1} s_{k-l} \quad \text{if } 0 \le l \le k, \\ d_l t_k &= t_{k-1} d_{l-1}, \quad s_l t_k = t_{k+1} s_{l-1} \quad \text{if } 0 < l \le k, \\ d_0 t_k &= d_k, \quad s_0 t_k = t_{k+1}^2 s_k. \end{aligned}$$

A simplicial object together with structure maps $t_k : X[k] \to X[k]$ satisfying the above relations is called a cyclic object and a simplicial object together with structure maps $w_k : X[k] \to X[k]$ satisfying the above relations is called a real object. The geometric realization of the simplicial space underlying a dihedral (resp. cyclic, resp. real) space carries an action by $O(2)$ (resp. $\mathbb{T}$, resp. G): See [8] for more details.

Let I be the category with objects all non-negative integers. The morphisms from i to j are all injective set maps

$$\{1, \ldots, i\} \to \{1, \ldots, j\}.$$

The category I has a strict monoidal product $+ : I \times I \to I$ given on objects by addition and on morphisms by concatenation. We note that the initial object $0 \in \mathrm{Ob}(I)$ serves as the identity for the monoidal product. For $i \in \mathrm{Ob}(I)$ we let $\omega_i : i \to i$ denote the morphism that reverses the order of the elements:

$$\omega_i(s) = i - s + 1.$$

Given a morphism $\theta : i \to j$ we define the conjugate morphism θ^ω by $\theta^\omega := \omega_j \circ \theta \circ \omega_i^{-1}$. We define a dihedral category $I[-]$ by letting $I[k] = I^{k+1}$ and defining cyclic structure maps $d_i : I[k] \to I[k-1]$, $s_i : I[k] \to I[k+1]$, and $t_k : I[k] \to I[k]$ on objects by

$$\begin{aligned} d_i(i_0, \cdots, i_k) &= (i_0, \cdots, i_i + i_{i+1}, \cdots, i_k), \quad & 0 \le i < k, \\ d_i(i_0, \cdots, i_k) &= (i_k + i_0, \cdots, i_{k-1}), & i = k, \\ s_i(i_0, \cdots, i_k) &= (i_0, \cdots, i_i, 0, i_{i+1}, \cdots, i_k), & 0 \le i \le k, \\ t_k(i_0, \cdots, i_k) &= (i_k, i_0, \cdots, i_{k-1}), \end{aligned}$$

and similarly on morphisms. The structure maps $w_k : I[k] \to I[k]$ is defined on a tuple of objects by

$$w_k(i_0, \ldots, i_k) = (i_0, i_k, i_{k-1}, \ldots, i_1)$$

and on a tuple of morphisms by

$$w_k(\theta_0, \ldots, \theta_k) = (\theta_0^\omega, \theta_k^\omega \ldots, \theta_1^\omega).$$

Let X be a pointed space with a pointed left $O(2)$-action and let (A, D) be a symmetric ring spectrum with an anti-involution. Let $G(A)_X^k : I^{k+1} \to \mathsf{Top}_*$ denote the functor given on objects by

$$G(A)_X^k(i_0, \ldots, i_k) = \operatorname{Map}\Big(S^{i_0} \wedge \cdots \wedge S^{i_k}, A_{i_0} \wedge \cdots \wedge A_{i_k} \wedge X\Big).$$

We will almost always omit the A and simply write G_X^{k+1} if there is no confusion about which spectrum A is used in the construction of the functor. The functor is defined on morphisms using the structure maps of the spectrum; see [10] or [7, Sect. 4.2.2]. We define a dihedral space by setting

$$\operatorname{THR}(A, D; X)[k] := \operatorname*{hocolim}_{I^{k+1}} G_X^k$$

with simplicial structure maps as described in [10] or [7, Sect. 4.2.2]. Let $\omega_i \in \Sigma_i$ be the permutation given by $\omega_i(s) = i - s + 1$. We let

$$t'_k : G_X^k \Rightarrow G_X^k \circ t_k, \quad w'_k : G_X^k \Rightarrow G_X^k \circ w_k$$

be the natural transformations which at $(i_0, \ldots, i_k) \in \operatorname{Ob}(I^k)$ are defined by the following commutative diagrams

$$\begin{array}{ccc}
S^{i_0} \wedge \cdots \wedge S^{i_k} & \xrightarrow{\quad f \quad} & A_{i_0} \wedge \cdots \wedge A_{i_k} \wedge X \\
\tau^{-1} \big\uparrow & & \big\downarrow \tau \wedge X \\
S^{i_k} \wedge S^{i_0} \wedge \cdots \wedge S^{i_{k-1}} & \xrightarrow{\quad t'_k(f) \quad} & A_{i_k} \wedge A_{i_0} \wedge \cdots \wedge A_{i_{k-1}} \wedge X
\end{array}$$

and

$$\begin{array}{ccc}
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} & \xrightarrow{\quad f \quad} & A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge X \\
\upsilon \big\uparrow & & \big\downarrow \omega_{i_0} \wedge \cdots \wedge \omega_{i_k} \wedge \mathrm{id} \\
S^{i_0} \wedge S^{i_k} \wedge \cdots \wedge S^{i_1} & & A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge X \\
\mathrm{id} \big\uparrow & & \big\downarrow D_{i_0} \wedge \cdots \wedge D_{i_k} \wedge \mathrm{id} \\
S^{i_0} \wedge S^{i_k} \wedge \cdots \wedge S^{i_1} & & A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge X \\
\omega_0 \wedge \cdots \wedge \omega_k \big\uparrow & & \big\downarrow \upsilon \\
S^{i_0} \wedge S^{i_k} \wedge \cdots \wedge S^{i_1} & \xrightarrow{\quad w'_k(f) \quad} & A_{i_0} \wedge A_{i_k} \wedge \cdots \wedge A_{i_1} \wedge X
\end{array}$$

where τ cyclically permutes the smash factors to the right, and υ fixes the first smash factor and reverses the order of the rest. The structure maps are given as the compositions of the maps induced by the natural transformations and the canonical maps:

$$t_k : \operatorname*{hocolim}_{I^{k+1}} G_X^k \xrightarrow{t_k'} \operatorname*{hocolim}_{I^{k+1}} G_X^k \circ t_k \xrightarrow{\mathrm{ind}_{t_k}} \operatorname*{hocolim}_{I^{k+1}} G_X^k,$$
$$w_k : \operatorname*{hocolim}_{I^{k+1}} G_X^k \xrightarrow{w_k'} \operatorname*{hocolim}_{I^{k+1}} G_X^k \circ w_k \xrightarrow{\mathrm{ind}_{w_k}} \operatorname*{hocolim}_{I^{k+1}} G_X^k.$$

We have defined a dihedral space and we let $\mathrm{THR}(A, D; X)$ denote the realization

$$\mathrm{THR}(A, D; X) := \Big|[k] \mapsto \mathrm{THR}(A, D; X)[k]\Big|.$$

The space $\mathrm{THR}(A, D; X)$ is in fact an $O(2)\times O(2)$-space, where the action by the first factor comes from the dihedral structure and the action by the second factor comes from the $O(2)$-action on X. We are interested in the space $\mathrm{THR}(A, D; X)$ with the diagonal $O(2)$-action.

Remark 1.2. Let $\triangle : O(2) \to O(2) \times O(2)$ denote the diagonal map. The tool available for investigating the fixed point space $(\triangle^* \mathrm{THR}(A, D; X))^G$ is Segal's real subdivision constructed in [17, Appendix A1]. We briefly recall the real subdivision functor sd^e and refer to Segal's paper for details.

Let $X[-]$ be a dihedral space. There is a (non-simplicial) homeomorphism

$$D^e : |\mathrm{sd}^e X[-]| \to |X[-]|,$$

where $\mathrm{sd}^e X[-]$ is the simplicial space with k-simplices $\mathrm{sd}^e X[k] = X[2k + 1]$ and simplicial structure maps, for $0 \le i \le k$, given by

$$(d_i)^e : \mathrm{sd}^e X[k] \to \mathrm{sd}^e X[k-1], \quad (d_i)^e = d_i \circ d_{2k+1-i},$$
$$(s_i)^e : \mathrm{sd}^e X[k] \to \mathrm{sd}^e X[k+1], \quad (s_i)^e = s_i \circ s_{2k+1-i}.$$

The simplicial set $\mathrm{sd}^e X[-]$ has a simplicial G-action which in simplicial level k is generated by w_{2k+1}. Thus the realization inherits a G-action. The advantage of real subdivision is that the homeomorphism D^e is G-equivariant. In particular, it induces a homeomorphism

$$|\mathrm{sd}^e X[-]^G| \to |X[-]|^G.$$

Note that

$$\mathrm{sd}^e\, \mathrm{THR}(A, D; X)[k] = \operatorname*{hocolim}_{I^{2k+2}} G_X^{2k+1} = \Big|[n] \mapsto \bigvee_{\underline{i}_0 \to \cdots \to \underline{i}_n} G_X^{2k+1}(\underline{i}_0)\Big|,$$

where $\underline{i} \in \mathrm{Ob}(I^{2k+1})$. The G-action on X gives rise to a natural transformation

$$X_\omega : G_X^{2k+1} \circ w_{2k+1} \Rightarrow G_X^{2k+1} \circ w_{2k+1}.$$

The diagonal G-action is generated by the simplicial operator which takes the summand indexed by $\underline{i}_0 \to \cdots \to \underline{i}_n$ to the one indexed by $w_{2k+1}(\underline{i}_0) \to \cdots \to w_{2k+1}(\underline{i}_n)$ via

$$G_X^{2k+1}(\underline{i}_0) \xrightarrow{w_{2k+1'}} G_X^{2k+1} \circ w_{2k+1}(\underline{i}_0) \xrightarrow{X_\omega} G_X^{2k+1} \circ w_{2k+1}(\underline{i}_0). \tag{1}$$

In particular if a k-simplex is fixed, then it must belong to a summand whose index is fixed under the functor w_{2k+1}. Such an index consists of objects of the form

$$(i_0, i_1, \dots, i_k, i_{k+1}, i_k, \dots, i_1) \tag{2}$$

and morphisms of the form

$$(\theta_0, \theta_1, \dots, \theta_k, \theta_{k+1}, \theta_k^\omega, \dots, \theta_1^\omega) \tag{3}$$

where $\theta_0 = \theta_0^\omega$ and $\theta_{k+1} = \theta_{k+1}^\omega$. Let I^G denote the subcategory of I with the same objects and all morphisms θ which satisfies $\theta^\omega = \theta$. Let

$$\triangle^e : I^G \times I^k \times I^G \to I^{2k+2}$$

denote the "diagonal" functor which maps a tuple $(i_0, i_1, \dots, i_k, i_{k+1})$ to the tuple (2) and a tuple of morphisms $(\theta_0, \theta_1, \dots, \theta_k, \theta_{k+1})$ to the tuple (3). The natural transformation (1) restricts to a natural transformation from $G_X^{2k+1} \circ \triangle^e$ to itself, hence G acts on $G_X^{2k+1} \circ \triangle^e$ through natural transformations. Since geometric realization commutes with taking fixed points of the finite group G by [16, Cor. 11.6], we obtain the following lemma.

Lemma 1.3. *The canonical map induces a homeomorphism*

$$\operatorname*{hocolim}_{I^G \times I^k \times I^G} (G_X^{2k+1} \circ \triangle^e)^G \xrightarrow{\cong} \Big(\operatorname*{hocolim}_{I^{2k+2}} G_X^{2k+1} \Big)^G.$$

The G-action on $G_X^{2k+1} \circ \triangle^e$ at $(i, n_1, \dots, n_k, j) \in \mathrm{Ob}\left(I^G \times I^k \times I^G\right)$ can be described as follows. The image of the functor is the mapping space:

$$G_X^{2k+1} \circ \triangle^e(i, n_1, \dots, n_k, j) = \mathrm{Map}\left(\overline{S}, \overline{A} \wedge X\right),$$

where

$$\overline{S} = S^i \wedge S^{n_1} \wedge \cdots \wedge S^{n_k} \wedge S^j \wedge S^{n_k} \wedge \cdots \wedge S^{n_1},$$
$$\overline{A} = A_i \wedge A_{n_1} \wedge \cdots \wedge A_{n_k} \wedge A_j \wedge A_{n_k} \wedge \cdots \wedge A_{n_1}.$$

The spaces above are G-spaces: The non-trivial element $\omega \in G$ fixes the first smash factor and reverses the order of the remaining factors, then acts by the permutation $w_i \in \Sigma_i$ factor-wise. On the space $\overline{A}$, ω further acts by the anti-involution D_i factor-wise. The space $\overline{A} \wedge X$ is given the diagonal action and finally G acts on the mapping space by the conjugation action.

We will need a G-equivariant version of Bökstedt's Approximation Lemma as proven by Dotto; see [5, 4.3.2]. We call a map of G-spaces $f : Z \to Y$ n-connected if $f^K : Z^K \to Y^K$ is n-connected for $K \in \{e, G\}$.

Proposition 1.4 (Equivariant Approximation Lemma)**.** *Let (R, α) be a ring with an anti-involution α and let V be a finite dimensional real G-representation. Given $n \geq 0$, there exists $N \geq 0$ such that the G-equivariant inclusion*

$$G(R)_{S^V}^{2k+1} \circ \triangle^e(i) \hookrightarrow \operatorname*{hocolim}_{I^{2k+2}} G(R)_{S^V}^{2k+1}$$

is n-connected for all $i \in \mathrm{Ob}(I^G \times I^k \times I^G)$ coordinate-wise bigger than N.

2. The real topological Hochschild homology spectrum

The space $\mathrm{THR}(A, D; S^0)$ is the 0th space of a fibrant orthogonal $O(2)$-spectrum in the model structure based on the family of finite subgroups of $O(2)$, see Proposition 2.2. Furthermore, if A is a commutative symmetric ring spectrum, then $\mathrm{THR}(A, D)$ has the homotopy type of an $O(2)$-ring spectrum.

In the classical setup one needs certain connectivity assumptions on the spectrum A to ensure that $\mathrm{THH}(A)$ has the correct homotopy type. We likewise need some connectivity assumption on (A, D). For an integer n we let $\lceil \frac{n}{2} \rceil$ denote the ceiling of $\frac{n}{2}$. Throughout this section we make the following assumptions on (A, D):

Assumptions 2.1. Let (A, D) be a symmetric ring spectrum with anti-involution. We assume that A_n is $(n-1)$-connected as a non-equivariant space and that $(A_n)^{D_n \circ \omega_n}$ is $\big(\lceil \frac{n}{2} \rceil - 1\big)$-connected. Furthermore we assume that there exists a constant $\epsilon \geq 0$ such that the structure map $\lambda_{n,m} : A_n \wedge S^m \to A_{n+m}$ is $(2n + m - \epsilon)$-connected as a map of non-equivariant spaces and such that the restriction of the structure map $\lambda_{n,m} : A_n^{D_n \circ \omega_n} \wedge (S^m)^{\omega_m} \to (A_{n+m})^{D_{n+m} \circ (\omega_n \times \omega_m)}$ is $(n + \lceil \frac{m}{2} \rceil - \epsilon)$-connected.

By an $O(2)$-representation, we will mean a finite dimensional real inner product space on which $O(2)$ acts by linear isometries. We fix a complete $O(2)$-universe $\mathcal{U}$ and work in the category of orthogonal $O(2)$-spectra indexed on $\mathcal{U}$ as defined in [15, Chapter II.4]. Let $V \subset \mathcal{U}$ be a finite $O(2)$-representation. Let

$$\mathrm{THR}(A, D)(V) = \triangle^* \, \mathrm{THR}(A, D; S^V),$$

where $\triangle : O(2) \to O(2) \times O(2)$ is the diagonal map. The orthogonal group $O(V)$ acts on $\mathrm{THR}(A, D)(V)$ through the sphere S^V. It is straightforward to construct spectral structure maps

$$\sigma_{V,W} : \mathrm{THR}(A, D)(V) \wedge S^W \to \mathrm{THR}(A, D)(V \oplus W),$$

see [9] or [13]. The family of $O(V) \rtimes O(2)$-spaces $\mathrm{THR}(A, D)(V)$ together with the maps $\sigma_{V,W}$ defines an orthogonal $O(2)$-spectrum indexed on $\mathcal{U}$, which is denoted $\mathrm{THR}(A, D)$. The following result is proven in [13, Prop. 3.6].

Proposition 2.2. *If V and W are finite $O(2)$-representations, then the adjoint of the structure map*

$$\widetilde{\sigma}_{V,W} : \mathrm{THR}(A, D)(V) \to \mathrm{Map}(S^W, \mathrm{THR}(A, D)(V \oplus W))$$

induces a weak equivalence on H-fixed points for any finite subgroup $H \leq O(2)$.

When A is a commutative symmetric ring spectrum, $\mathrm{THH}(A)$ is a $\mathbb{T}$-ring spectrum, though we must change foundations and work in the category of symmetric orthogonal $\mathbb{T}$-spectra to display this structure; see [12]. The multiplicative and unital structure maps as described in [9, Appendix] are compatible with the added G-action. Thus when A is commutative, $\mathrm{THR}(A, D)$ is a symmetric orthogonal $O(2)$-ring spectrum. We briefly recall the construction and refer to [9, Appendix] for details. Let (n) denote the finite ordered set $\{1, \ldots, n\}$ and let $I^{(n)}$ denote the product category. There is a functor

$$\sqcup_n : I^{(n)} \to I$$

given by addition of objects and concatenation of morphisms according to the order of (n). Let $G^{k,(n)}_X$ denote the composite $G^k_X \circ (\sqcup_n)^{k+1}$. There is a dihedral space $\mathrm{THR}^{(n)}(A,D;X)[-]$ with k-simplices the homotopy colimit

$$\mathrm{THR}^{(n)}(A,D;X)[k] := \operatorname*{hocolim}_{\left(I^{(n)}\right)^{k+1}} G^{k,(n)}_X,$$

and with cyclic structure maps constructed as for $\mathrm{THR}(A,D;X)[-]$ with minor adjustments. We define

$$w^{(n)}_k : \left(I^{(n)}\right)^{k+1} \to \left(I^{(n)}\right)^{k+1}$$

on objects by

$$w^{(n)}_k\big((i_{01},\dots,i_{0n}),\dots,(i_{k1},\dots,i_{kn})\big) = \big((i_{01},\dots,i_{0n}),(i_{k1},\dots,i_{kn}),\dots,(i_{11},\dots,i_{1n})\big)$$

and on morphisms by

$$w^{(n)}_k\big((\alpha_{01},\dots,\alpha_{0n}),\dots,(\alpha_{k1},\dots,\alpha_{kn})\big) = \big((\alpha^{\omega}_{01},\dots,\alpha^{\omega}_{0n}),\dots,(\alpha^{\omega}_{11},\dots,\alpha^{\omega}_{1n})\big)$$

Furthermore we define the natural transformation

$$\left(w^{(n)}_k\right)' : G^{k,(n)}_X \Rightarrow G^{k,(n)}_X \circ w^{(n)}_k$$

at an object $\big((i_{01},\dots,i_{0n}),\dots,(i_{k1},\dots,i_{kn})\big) \in \left(I^{(n)}\right)^{k+1}$ by replacing the permutations $\omega_{i_{j1}+\cdots+i_{jn}}$ by the permutations $\omega_{i_{j1}} \times \cdots \times \omega_{i_{jn}}$ in the defining diagram for the natural transformation w'_k. The dihedral structure map is the composition

$$w^{(n)}_k : \operatorname*{hocolim}_{\left(I^{(n)}\right)^{k+1}} G^{k,(n)}_X \xrightarrow{\left(w^{(n)}_k\right)'} \operatorname*{hocolim}_{\left(I^{(n)}\right)^{k+1}} G^{k,(n)}_X \circ w^{(n)}_k \xrightarrow{\mathrm{ind}_{w^{(n)}_k}} \operatorname*{hocolim}_{\left(I^{(n)}\right)^{k+1}} G^{k,(n)}_X.$$

Let $\mathrm{THR}^{(n)}(A;D;X)$ denote the geometric realization of $\mathrm{THR}(A,D;X)[-]$.

An order preserving inclusion $\iota : (m) \hookrightarrow (n)$ induces a functor

$$\iota : \left(I^{(m)}\right)^{k+1} \to \left(I^{(n)}\right)^{k+1}$$

by inserting the initial element $0 \in \mathrm{Ob}(I)$ into the added coordinates, which in turn induces a map

$$\iota : \mathrm{THR}^{(m)}(A;D;X) \to \mathrm{THR}^{(n)}(A;D;X).$$

When $m \geq 1$ it follows from the most general version of the Equivariant Approximation Lemma as stated in [13, Prop. 2.7] that the map ι induces isomorphisms on $\pi^H_*(-)$ for all finite subgroups $H \leq O(2)$.

We define the symmetric orthogonal spectrum $\mathrm{THR}(A,D)$ as follows. Let n be a non-negative integer, and let V be a finite $O(2)$-representation. The (n,V)th space is defined to be

$$\mathrm{THR}(A,D)(n,V) = \triangle^* \mathrm{THR}^{(n)}(A,D;S^n \wedge S^V)$$

where $\triangle : \Sigma_n \times O(2) \to \Sigma_n \times O(2) \times \Sigma_n \times O(2)$ is the diagonal map. The action by the first $O(2)$-factor arises from the dihedral structure and the action by the second $O(2)$-factor is induced from the $O(2)$-action on V. The action by the first Σ_n-factor is induced from permutation action on $I^{(n)}$ and the action by the second Σ_n-factor is induced from the Σ_n-action on S^n given by permuting the sphere coordinates. The spectrum structure maps and the unit maps are described in [9, Appendix] and one can verify that they are G-equivariant.

To define multiplicative structure maps we first recall that the canonical map

$$\operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_X^{k,(n)} \wedge \operatorname*{hocolim}_{(I^{(m)})^{k+1}} G_Y^{k,(m)} \xrightarrow{\cong} \operatorname*{hocolim}_{(I^{(n+m)})^{k+1}} G_X^{k,(n)} \wedge G_Y^{k,(m)}$$

is a homeomorphism, when the spaces are given the compactly generated weak Hausdorff topology. Next we note that there are natural transformations

$$\mu'_{n,X,m,Y} : G_X^{k,(n)} \wedge G_Y^{k,(m)} \Rightarrow G_{X\wedge Y}^{k,(n+m)}$$

given by smashing together the maps $f \in G_X^{k,(n)}$ and $g \in G_Y^{k,(m)}$ and composing with the multiplication maps in A. The composition of the canonical map and the map induced by the natural transformation μ' commutes with the dihedral structure maps, the $\Sigma_n \times \Sigma_m$-action, and the $O(2)$-action from X and Y. Given $n, m \geq 0$ and finite $O(2)$-representations V and W the multiplicative structure map is the geometric realization

$$\mu_{n,V,m,W} : \mathrm{THR}(A,D)(n,V) \wedge \mathrm{THR}(A,D)(m,W) \to \mathrm{THR}(A,D)(n+m, V\oplus W).$$

3. The components of the G-geometric fixed points

This section is devoted to the proof of Theorem A in the introduction. We start by introducing some notation. We define the G-spheres $S^{1,0} = S^{\mathbb{R}}$ and $S^{1,1} = S^{i\mathbb{R}}$ to be the pointed G-spaces given by the one point compactifications of the 1-dimensional trivial representation and sign representation, respectively. More generally, we set

$$S^{p,q} = \left(S^{1,0}\right)^{\wedge(p-q)} \wedge \left(S^{1,1}\right)^{\wedge(q)}$$

for integers $p \geq q \geq 0$.

Let EG be the free contractible G-CW-complex

$$EG := \bigcup_{n=0}^{\infty} S\big(\bigoplus_{j=1}^{n} i\mathbb{R}\big),$$

where $S(\oplus_{j=1}^n i\mathbb{R})$ denotes the unit sphere in $\oplus_{j=1}^n i\mathbb{R}$. We denote by $\widetilde{E}G$ the reduced mapping cone of the based G-map $EG_+ \to S^0$ which collapses EG to the non-basepoint, hence

$$\widetilde{E}G = \operatorname*{colim}_{k\to\infty} S^{k,k}.$$

If X is an orthogonal G-spectrum, then the derived G-fixed points of $\widetilde{E}G \wedge X$ is a model for the derived G-geometric fixed point of X; see [15, Prop. 4.17]. Consider the inclusion $S^{n,n} \to S^{n+1,n+1}$. There are canonical homeomorphisms of G-spaces

$$S^{n+1,n+1}/S^{n,n} \xrightarrow{\cong} \Sigma S^{n,n} \wedge G_+ \xrightarrow{\cong} S^{n+1} \wedge G_+,$$

where the first map is described in [13, Lemma A.4] and the second map untwists the G-action, that is the map is given by $(x,g) \mapsto (g^{-1}x, g)$. Thus there are cofiber sequences of based G-CW-complexes for $n \geq 0$:

$$S^{n,n} \to S^{n+1,n+1} \to G_+ \wedge S^{n+1}.$$

We smash the cofiber sequence with the orthogonal G-spectrum X and obtain a long exact sequence of G-stable homotopy groups, which contains the segment

$$\cdots \to \pi_{-n}(X) \to \pi_0^G(S^{n,n} \wedge X) \to \pi_0^G(S^{n+1,n+1} \wedge X) \to \pi_{-n-1}(X) \to \cdots$$

If X is connective, then the inclusion $S^{n,n} \to S^{n+1,n+1}$ induces an isomorphism

$$\pi_0^G(S^{n,n} \wedge X) \xrightarrow{\cong} \pi_0^G(S^{n+1,n+1} \wedge X)$$

for $n \geq 1$ and, in particular, the inclusion $S^{1,1} \to \widetilde{E}G$ induces an isomorphism

$$\pi_0^G(S^{1,1} \wedge X) \xrightarrow{\cong} \pi_0^G(\widetilde{E}G \wedge X).$$

We are now ready to prove Theorem A from the introduction.

Proof of Theorem A. Let (R, α) denote a ring R with an anti-involution α. By the discussion above, the components of the derived G-geometric fixed points can be calculated as the homotopy group $\pi_0^G(S^{1,1} \wedge \mathrm{THR}(R, \alpha))$. The actions of smashing an orthogonal G-spectrum with $S^{1,1}$ and shifting the spectrum by $i\mathbb{R}$ yield canonically π_*-isomorphic G-spectra. It follows from Lemma 2.2 that we have an isomorphism of abelian groups:

$$\pi_0^G(S^{1,1} \wedge \mathrm{THR}(R, \alpha)) \cong \pi_0\big(\mathrm{THR}(R, \alpha)(i\mathbb{R})^G\big).$$

Segal's real subdivision described in Remark 1.2 provides a homeomorphism

$$D^e : |\big(\mathrm{sd}^e\, \mathrm{THR}(R, \alpha; S^{1,1})[-]\big)^G| \xrightarrow{\cong} \mathrm{THR}(R, \alpha)(i\mathbb{R})^G.$$

By [16, Lemma 11.11], we can calculate the group of components of the left hand side as the quotient of $\pi_0\big(\mathrm{sd}^e\, \mathrm{THR}(R, \alpha; S^{1,1})[0]^G\big)$ by the equivalence relation generated by $d_0^e(x) \sim d_1^e(x)$ for all $x \in \pi_0\big(\mathrm{sd}^e\, \mathrm{THR}(A; S^{1,1})[1]^G\big)$. It follows from Lemma 1.3 that the diagram

$$\mathrm{sd}^e\, \mathrm{THR}(R, \alpha; S^{1,1})[1]^G \underset{d_1^e}{\overset{d_0^e}{\rightrightarrows}} \mathrm{sd}^e\, \mathrm{THR}(R, \alpha; S^{1,1})[0]^G$$

is homeomorphic to the left hand part of the homotopy commutative diagram

$$\begin{array}{ccc}
\operatorname*{hocolim}_{I^G \times I \times I^G} (G_X^3 \circ \triangle^e)^G & \leftarrow & \mathrm{Map}(S^i \wedge S^n \wedge S^j \wedge S^n, HR_i \wedge HR_n \wedge HR_j \wedge HR_n \wedge S^{1,1})^G \\
d_1 \circ d_2 \downarrow\downarrow d_0 \circ d_3 & & \mathrm{incl}_i \circ d_1' \circ d_2' \downarrow\downarrow \mathrm{incl}_j \circ d_0' \circ d_3' \\
\operatorname*{hocolim}_{I^G \times I^G} (G_X^1 \circ \triangle^e)^G & \longleftarrow & \mathrm{Map}(S^{n+i+n} \wedge S^{n+j+n}, HR_{n+i+n} \wedge HR_{n+j+n} \wedge S^{1,1})^G.
\end{array}$$

Here incl_j is the image of the functor $G^1_{S^{1,1}} \circ \triangle^e$ at the morphism $(\mathrm{id}_{n+i+n}, \mathrm{incl}_{\mathrm{mid}})$ in $I^G \times I^G$ where $\mathrm{incl}_{\mathrm{mid}}$ is middle inclusion $j \to n+j+n$. The map incl_j is defined analogously. The horizontal maps can be made as connected as desired by choosing i, j and n big enough by the Equivariant Approximation Lemma 1.4. We fix a choice of i, j and n such that the horizontal maps are 0-connected and for simplicity we choose i and j to be even.

If X and Y are G-CW-complexes, then the inclusion of fixed points $g : X^G \hookrightarrow X$ induces a fibration

$$g_* : \mathrm{Map}_G(X, Y) \to \mathrm{Map}_G(X^G, Y) = \mathrm{Map}(X^G, Y^G)$$

with fiber $\mathrm{Map}_G(X/X^G, Y)$. It follows from [1, Prop. 2.7] that the connectivity of the fiber can be estimated as follows:

$$\mathrm{conn}(\mathrm{Map}_G(X/X^G, Y)) \geq \min_{K \in \{e,G\}} (\mathrm{conn}(Y^K) - \dim((X/X^G)^K)).$$

In the case at hand, the inclusions of G-fixed points

$$\big(S^i \wedge S^n \wedge S^j \wedge S^n\big)^G \hookrightarrow S^i \wedge S^n \wedge S^j \wedge S^n$$

and

$$\big(S^{n+i+n} \wedge S^{n+j+n}\big)^G \hookrightarrow S^{n+i+n} \wedge S^{n+j+n}$$

induce fibrations with 0-connected fibers, so the right hand part of the diagram evaluated at π_0 is isomorphic to the diagram

$$\begin{array}{c} \pi_0\Big(\mathrm{Map}(S^{\frac{i}{2}} \wedge S^n \wedge S^{\frac{j}{2}}, HR_i^{H\alpha\circ\omega} \wedge HR_n \wedge HR_j^{H\alpha\circ\omega})\Big) \\ \tilde{d}_1 \Big\downarrow\Big\downarrow \tilde{d}_0 \\ \pi_0\Big(\mathrm{Map}(S^{n+\frac{i}{2}+n} \wedge S^{n+\frac{j}{2}+n}, HR_{n+i+n}^{H\alpha\circ\omega} \wedge HR_{n+j+n}^{H\alpha\circ\omega})\Big), \end{array}$$

where $\tilde{d}_0 := \pi_0^*(\mathrm{incl}_j \circ d_0' \circ d_3')$ and $\tilde{d}_1 := \pi_0^*(\mathrm{incl}_i \circ d_1' \circ d_2')$. We have omitted the index on $H\alpha$ and ω. The space HR_n is $(n-1)$-connected and Dotto proves that the space $HR_{2n}^{\alpha\circ\omega}$ is $(n-1)$-connected; see [5, Lemma 6.3.2]. It follows from the Hurewicz isomorphism and the Künneth formula that the diagram above is isomorphic to the diagram

$$\begin{array}{c} \pi_{\frac{i}{2}}(HR_i^{H\alpha\circ\omega}) \otimes \pi_n(HR_n) \otimes \pi_{\frac{j}{2}}(HR_j^{H\alpha\circ\omega}) \\ \tilde{d}_1 \Big\downarrow\Big\downarrow \tilde{d}_0 \\ \pi_{n+\frac{i}{2}}(HR_{n+i+n}^{H\alpha\circ\omega}) \otimes \pi_{n+\frac{j}{2}}(HR_{n+j+n}^{H\alpha\circ\omega}). \end{array}$$

The homotopy groups $\pi_n(HR_{2n}^{\alpha\omega})$ are independent of n when $n \geq 1$ and we therefore calculate $\pi_1(HR_2^{\alpha\omega})$. The space HR_2 is the geometric realization of the simplicial set

$$R[S^1[-] \wedge S^1[-]]/R[*].$$

The action by $H\alpha\circ\omega$ is induced by a simplicial action where α acts on the R-label and ω acts by twisting the smash factors $S^1[-] \wedge S^1[-]$. Since taking fixed points of a finite group commutes with geometric realization, $HR_2^{H\alpha\circ\omega}$ is the geometric realization of the simplicial set

$$\big(R[S^1[-] \wedge S^1[-]]\big)^{\alpha\circ\omega}/R[*].$$

This is a simplicial abelian group and we may therefore calculate $\pi_1(HR_2^{H\alpha\circ\omega})$ as the first homology group of the associated chain complex. Recall that

$$\Delta^1[k] = \mathrm{Hom}_\Delta([k],[1]) = \{x_0, x_1, \ldots, x_{k+1}\}$$

where $\#x_i^{-1}(0) = i$ and with the face maps given by

$$d_s(x_i) = \begin{cases} x_i & \text{if } i \leq s \\ x_{i-1} & \text{if } i > s. \end{cases}$$

The sphere $S^1[-]$ is defined to be the quotient $\Delta^1[-]/\partial\Delta^1[-]$. We have representatives of the simplices in $S^2[-] = S^1[-] \wedge S^1[-]$ as follows:

$$S^2[0] = \{x_0 \wedge x_0\}, \quad S^2[1] = \{x_0 \wedge x_0,\ x_1 \wedge x_1\},$$
$$S^2[2] = \{x_0 \wedge x_0,\ x_1 \wedge x_1,\ x_2 \wedge x_2,\ x_1 \wedge x_2,\ x_2 \wedge x_1\}.$$

The associated chain complex of $\big(R[S^1[-] \wedge S^1[-]]\big)^{\alpha\circ\omega}/R[*]$ begins with the sequence

$$\cdots \to (R \cdot (x_1 \wedge x_1) \oplus R \cdot (x_1 \wedge x_2) \oplus R \cdot (x_2 \wedge x_1))^{\alpha\circ\omega} \xrightarrow{d} R^\alpha \cdot (x_1 \wedge x_1) \to 0.$$

Since $d(x_1 \wedge x_1) = 0$, the first homology group is the cokernel of the map

$$(R \cdot (x_1 \wedge x_2) \oplus R \cdot (x_2 \wedge x_1))^{\alpha\circ\omega} \xrightarrow{d} R^\alpha \cdot (x_1 \wedge x_1),$$

where $d(x_1 \wedge x_2) = d(x_2 \wedge x_1) = -(x_1 \wedge x_1)$. The map

$$R \to (R \cdot (x_1 \wedge x_2) \oplus R \cdot (x_2 \wedge x_1))^{\alpha\circ\omega}$$

which sends r to the tuple $(-r, -\alpha(r))$ is an isomorphism, and d corresponds to the norm map $N : R \to R^\alpha$ given by

$$N(r) = r + \alpha(r)$$

under this isomorphism. We conclude that $\pi_1(HR_2^{H\alpha\circ\omega}) \cong R^\alpha/N(R)$. The diagram from before can now be identified with

$$\begin{array}{c} R^\alpha/N(R) \otimes R \otimes R^\alpha/N(R) \\ \tilde{d}_1 \Big\downarrow \Big\downarrow \tilde{d}_0 \\ R^\alpha/N(R) \otimes R^\alpha/N(R) \end{array}$$

where $\tilde{d}_0(r \otimes s \otimes t) = \alpha(s)rs \otimes t$ and $\tilde{d}_1(r \otimes s \otimes t) = r \otimes st\alpha(s)$. It follows that

$$\pi_0\big((\mathrm{THR}(R,\alpha)^c)^{gG}\big) \cong (R^\alpha/N(R) \otimes_{\mathbb{Z}} R^\alpha/N(R))/I,$$

where I denotes the subgroup generated by the elements $\alpha(s)rs \otimes t - r \otimes st\alpha(s)$ for all $s \in R$ and $r, t \in R^\alpha$. This completes the proof of Theorem A. □

Remark 3.1. When R is a commutative ring, then $\mathrm{THR}(R,\alpha)$ has the homotopy type of an $O(2)$-ring spectrum, hence the components of the G-geometric fixed points have a ring structure. We note that in this case R^α is a subring of R and $N(R)$ is an ideal in R^α. Furthermore the subgroup I generated by the elements $\alpha(s)rs \otimes t - r \otimes st\alpha(s)$ for all $s \in R$ and $r, t \in R^\alpha$ is an ideal. It is generated as an ideal by the elements $\alpha(r) \cdot r \otimes 1 - 1 \otimes r \cdot \alpha(r)$ for all $r \in R$. These observations give

$$(R^\alpha/N(R) \otimes_{\mathbb{Z}} R^\alpha/N(R))/I$$

a natural ring structure.

Remark 3.2. Let us consider the case where R is a commutative ring with the identity serving as anti-involution. The functor

$$R \mapsto \pi_0\big((\mathrm{THR}(R,\mathrm{id})^c)^{gG}\big),$$

considered as a functor from the category of commutative rings to the category of sets, is not representable. This rules out the possibility that the ring $\pi_0\big((\mathrm{THR}(R,\mathrm{id})^c)^{gG}\big)$ is a ring of Witt vectors as defined by Borger in [3]. For example, the functor does not preserve finite products. Indeed, in this case we have an isomorphisms of abelian groups

$$\pi_0\big((\mathrm{THR}(R,\mathrm{id})^c)^{gG}\big) \cong (R/2R \otimes R/2R)/I$$

where I is the ideal in the ring $R/2R \otimes R/2R$ generated as an ideal as follows

$$I = \big(x^2 \otimes 1 - 1 \otimes x^2 \mid x \in R/2R\big)\,.$$

We consider the product $\mathbb{F}_2[x] \times \mathbb{F}_2[y]$. We have a commutative diagram where the top right corner is the functor applied to the product ring and the lower right corner is the product of the functor applied to each factor. The horizontal maps are surjective quotient maps and the vertical maps are induced by the projections.

$$\begin{array}{ccc}
\big(\mathbb{F}_2[x] \times \mathbb{F}_2[y]\big) \otimes \big(\mathbb{F}_2[x] \times \mathbb{F}_2[y]\big) & \twoheadrightarrow & \Big(\big(\mathbb{F}_2[x] \times \mathbb{F}_2[y]\big) \otimes \big(\mathbb{F}_2[x] \times \mathbb{F}_2[y]\big)\Big)/I \\
\downarrow & & \downarrow \\
\big(\mathbb{F}_2[x] \otimes \mathbb{F}_2[x]\big) \times \big(\mathbb{F}_2[y] \otimes \mathbb{F}_2[y]\big) & \twoheadrightarrow & \big(\mathbb{F}_2[x] \otimes \mathbb{F}_2[x]\big)/I \times \big(\mathbb{F}_2[y] \otimes \mathbb{F}_2[y]\big)/I
\end{array}$$

The claim is that the right vertical map is not a bijection. The left vertical map takes the element $(x,0) \otimes (0,y)$ to zero. The ideal

$$I \subset \big(\mathbb{F}_2[x] \times \mathbb{F}_2[y]\big) \otimes \big(\mathbb{F}_2[x] \times \mathbb{F}_2[y]\big)$$

is generated by the elements $\big(p(x)^2, q(y)^2\big) \otimes 1 - 1 \otimes \big(p(x)^2, q(y)^2\big)$, where $p(x)$ and $q(y)$ are polynomials. Hence $(x,0) \otimes (0,y) \notin I$ and the right vertical map is not injective.

In some cases, the calculation of the components of the G-geometric fixed points immediately leads to a calculation of the components of the G-fixed points as a ring. We have a cofibration sequence of G-spaces

$$G_+ \to S^0 \to S^{1,1}.$$

We smash the cofibration sequence with the spectrum $\mathrm{THR}(R,\alpha)$ and obtain a long exact sequence of G-stable homotopy groups which begins with the sequence

$$\cdots \to \pi_0(\mathrm{THR}(R,\alpha)) \xrightarrow{V_e^G} \pi_0^G(\mathrm{THR}(R,\alpha)) \to \pi_0^G\big(S^{1,1} \wedge \mathrm{THR}(R,\alpha)\big) \to 0, \tag{4}$$

where V_e^G denotes the transfer map; see [10, Lemma 2.2] for an identification of the induced map in the long exact sequence and the transfer map. We let

$$F_e^G : \pi_0^G(\mathrm{THR}(R,\alpha)) \to \pi_0(\mathrm{THR}(R,\alpha))$$

denote the restriction map, which is a ring map. By the double coset formula

$$F_e^G \circ V_e^G = N.$$

Example 3.3. If R is a commutative ring with $\frac{1}{2} \in R$ and the identity serving as anti-involution, then it follows from the formula $F_e^G \circ V_e^G = 2 \cdot \mathrm{id}$, that V_e^G is injective. Since the components of the G-geometric fixed points vanish, it follows from the exact sequence (4) that

$$F_e^G : \pi_0^G(\mathrm{THR}(R,\mathrm{id})) \to \pi_0(\mathrm{THR}(R,\mathrm{id})) = R \cdot 1$$

is a ring isomorphism.

Remark 3.4. Hesselholt and Madsen prove in [10] that there is a canonical ring isomorphism identifying $\pi_0 \operatorname{THH}(R)^{C_{p^n}}$ with the p-typical Witt vectors of length $n+1$, when R is a commutative ring and p is a prime. Dress and Siebeneicher introduced a Witt vector construction in [6], which is carried out relative to any given profinite group, and the p-typical Witt vectors of length $n+1$ are the Witt vectors constructed relative to C_{p^n}, i.e. $\pi_0 \operatorname{THH}(R)^{C_{p^n}} \cong W_{C_{p^n}}(R)$. If we let D_{p^n} denote the dihedral group of order $2p^n$, then it is tempting to expect that the ring $\pi_0 \operatorname{THR}(R, \mathrm{id})^{D_{p^n}}$ can be identified with the Witt vectors $W_{D_{p^n}}(R)$ when R is commutative but the example above shows that this is not the case. If $p > 2$, then $W_G(\mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p \times \mathbb{F}_p$, but $\pi_0(\operatorname{THR}(\mathbb{F}_p, \mathrm{id}))^G$ is isomorphic to $\mathbb{F}_p$ by the example above.

Example 3.5. We consider the example $(\mathbb{Z}, \mathrm{id})$. Since $F_e^G \circ V_e^G = 2 \cdot \mathrm{id}$ and there is no 2-torsion in $\mathbb{Z}$, the transfer map is injective and the long exact sequence (4) gives rise to a short exact sequence

$$0 \to \pi_0(\operatorname{THR}(\mathbb{Z}, \mathrm{id})) \xrightarrow{V_e^G} \pi_0^G(\operatorname{THR}(\mathbb{Z}, \mathrm{id})) \to \mathbb{Z}/2 \to 0,$$

where $\pi_0(\operatorname{THR}(\mathbb{Z})) = \mathbb{Z} \cdot 1$. Since $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, there are two possibilities for what this short exact sequence can look like, when considered as short exact sequence of abelian groups. The first possibility is

$$0 \to \mathbb{Z} \xrightarrow{V} \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

with $V(x) = (x, 0)$, hence $F(x, 0) = 2x$. Since $\mathbb{Z}$ is torsion free, $F(0, y) = 0$, hence $F(x, y) = 2x$. But then the pre-image of the unit is empty, which is a contradiction, since F is a ring map. The short exact sequence must therefore be of the form

$$0 \to \mathbb{Z} \xrightarrow{V} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

with $V = 2 \cdot \mathrm{id}$ and $F = \mathrm{id}$. It follows that

$$F_e^G : \pi_0^G(\operatorname{THR}(\mathbb{Z}, \mathrm{id})) \to \pi_0(\operatorname{THR}(\mathbb{Z}, \mathrm{id})) = \mathbb{Z} \cdot 1$$

is a ring isomorphism.

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