Centre for Symmetry and Deformation – Department of Mathematical Sciences
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PhD Thesis

# Group Actions on Deformation Quantizations and an Equivariant Algebraic Index Theorem

Niek de Kleijn

This PhD thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen.

Niek de Kleijn
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 København Ø
Denmark

n.kleijn@math.ku.dk
niekdekleijn@gmail.com

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| Advisor: | Ryszard Nest<br>Københavns Universitet, Denmark |
| Assessment committee: | Simone Gutt, Université Libre de Bruxelles, Belgium<br>Dan Petersen (chairman), Københavns Universitet, Denmark<br>Stefan Waldmann, Julius-Maximilians-Universität Würzburg, Germany |


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**Abstract**

Group actions on algebras obtained by formal deformation quantization are the main topic of this thesis. We study these actions in order to obtain an equivariant algebraic index theorem that leads to explicit formulas in terms of equivariant characteristic classes. The Fedosov construction, as realized in a deformed version of Gelfand's formal geometry, is used to obtain the results.

We describe the main points of Gelfand's formal geometry in the deformed case and show how it leads to Fedosov connections and the well-known classification of formal deformation quantization in the direction of a symplectic structure.

A group action on a deformation quantization induces an action on the underlying symplectic manifold. We consider the lifting problem of finding group actions inducing a given action by symplectomorphisms. We reformulate some known sufficient conditions for existence of a lift and show that they are not necessary. Given a particular lift of an action by symplectomorphisms to the deformation quantization, we obtain a classification of all such lifts satisfying a certain technical condition. The classification is in terms of a first non-Abelian group cohomology. We supply tools for computing these sets in terms of a commuting diagram with exact rows and columns. Finally we consider some examples to formulate vanishing and non-vanishing results.

In joint work with A. Gorokhovsky and R. Nest, we prove an equivariant algebraic index theorem. The equivariant algebraic index theorem is a formula expressing the trace on the crossed product algebra of a deformation quantization with a group in terms of a pairing with certain equivariant characteristic classes. The equivariant characteristic classes are viewed as classes in the periodic cyclic cohomology of the crossed product by using the inclusion of Borel equivariant cohomology due to Connes.

**Resumé**

Denne afhandling handler om gruppevirkninger på algebraer, der kommer fra formelle deformationskvantisering. Vi studerer disse virkninger for at opnå en ækvivariant algebraisk index sætning, hvilket giver anledning til eksplicitte formler i ækvivariante karakteristiske klasser. Fedosov konstruktionen, som realiseret i en deformeret udgave af Gelfands formelle geometri, bruges til at udlede resultaterne.

Vi beskriver hovedpointerne i Gelfands formelle geometri i det deformerede tilfælde og viser hvordan det leder til Fedesov connections og velkendte klassifikationen af formelle deformationskvantisering i retning af en symplektiske struktur.

En gruppevirkning på en deformationskvantisering inducerer en virkning på den underliggende symplektiske mangfoldighed. Vi undersøger løfteproblemet om at finde en gruppevirkning som inducerer en given virkning gennem symplektomorfier. Vi omformulerer nogen velkendte tilstrækkelige betingelse for eksistensen af et løft og viser at de er overflødige. Givet et løft af en gruppevirkning gennem symplektomorfier på deformationskvantiseringenen, klassificerer vi alle sådanne løft som opfylder en bestemt teknisk betingelse. Klassifikationen består af en første ikke-Abelsk gruppe cohomologi. Vi giver metoder til at bestemme disse mængder i form af et kommuterende diagram med eksakte række og kolonner. Endeligt gennemgår vi nogle eksempler for at formulere når de forsvinder eller ikke.

I samarbejde med A. Gorokhovsky og R. Nest, viser vi en ækvivariant algebraisk index sætning. Den ækvivariante algebraiske index sætning er en formel, som udtrykker sporet på en krydsprodukt algebra af en deformation quantization og en gruppe i form af en parring med bestemte ækvivariante karakteristiske klasser. De ækvivariante karakteristiske klasser er betragtes som klasser i den periodiske cykliske cohomologi af krydsproduktet ved brug af Connes' inklusion af Borel ækvivarant cohomologi.

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# Contents

CHAPTER 1

# Introduction

*There are no limits. There are plateaus,*
*but you must not stay there,*
*you must go beyond them.*
Bruce Lee

As the title of this thesis suggests, it is mainly concerned with the topics of group actions on deformation quantizations and corresponding versions of the algebraic index theorem. We do not expect the reader to be immediately aware of the meaning of this title. First of all the theory of deformation quantization has many wildly varying aspects [**105**] and we should be more precise about what we mean by "deformation quantization". The algebraic index theorem [**45, 86**] is an adaptation of the Atiyah-Singer index theorem [**2**] from a specific quantization, pseudo-differential operators, to a more general class of deformation quantizations. It deserves an introduction even more, because of the fact that it is less well-known than its celebrated analytic counterpart.

In this introduction we will motivate why the main objects of study, deformation quantizations and algebraic index theorems, are of any interest at all. Simultaneously we will fix some conventions and notations that are used in the rest of this thesis. Finally, we will formulate some objectives that we pursue in the course of the thesis and give a clear picture of the degree to which different parts of the thesis are attributable to the author.

In particular, we shall provide all of the motivation coming from mathematical and theoretical physics in 1.1. We shall provide the definition of formal deformation quantization, some motivation for this definition and some general remarks about the definition in section 1.2. Section 1.3 is devoted to a very brief reminder of the Atiyah-Singer index theorem in order to motivate the algebraic index theorem, which we introduce in section 1.4. The objectives and attributions are handled in section 1.5.

Let us describe the structure of the thesis as a whole. In chapter 2 we set up the framework of formal geometry. We will use a deformed version of this framework in order to derive the main results of this thesis. In chapter 3 we will give a rather complete discussion of the formal Moyal–Weyl algebra. This will allow us to prove a great deal of the statements in the rest of the thesis with relative ease, since the corresponding formal computations are carried out in chapter 3. In chapter 4 we present the deformed analog of the framework of formal geometry and the Fedosov construction [**46**]. In chapter 5 we present the results on existence and, mostly, classification of group actions on deformation quantizations, these are part of the main results of this thesis. In chapter 6 we formulate and prove our equivariant version of the algebraic index theorem. Chapter 7 is devoted to a summary of the main results and a discussion of the further research directions offered by these results. Appendix A contains the definitions of the various chain and cochain complexes used in the main body of the thesis, as well as several general results about these complexes. Finally, appendix B contains a brief summary of the deformation theory of associative algebras.

NOTATION 1.0.1. Let us collect here some notation that is used throughout the thesis. First of all, a manifold will almost always mean a smooth, i.e. $C^\infty$, finite dimensional real manifold and will often be denoted $M$. The algebra of smooth (complex valued) functions will be denoted $C^\infty(M)$ and the differential forms on $M$ will be denoted by $\Omega^\bullet(M)$. We shall denote the manifold with boundary $[0,1]$ by $I$ and shall often denote the natural coordinate on $I$ by $t$. Given a vector bundle $E \to M$, we shall denote the space of smooth sections by $\Gamma(E)$. We shall always denote the tangent bundle of $M$ by $TM$ and the cotangent bundle by $T^*M$. We shall denote tensor products by $\otimes$ and add a subscript to indicate a specific ring if this is needed. We shall denote exterior products, i.e. the anti-symmetrization of $\otimes$, by $\wedge$. We shall denote the pull-back along a map $\varphi$ by $\varphi^*$ and the push-forward by $\varphi_*$. In the case of smooth maps $\varphi\colon M \to M'$, we shall denote the differential by $d\varphi\colon TM \to TM'$ or $T\varphi$. Let us mention in particular that we will always write the action of a map $f\colon X \to Y$ on an element $x \in X$ as $f(x)$ or $fx$, even when this map is induced by some action of an algebraic object. For example, if $G$ is a group acting on the set $X$ from the right, we have by definition

$$g(h(x)) = g(hx) = hgx = hg(x),$$

for all $g, h \in G$ and $x \in X$. Beyond this, we use various well-established conventions that should speak for themselves, like $\oplus$ for direct sum and $\mathbb{N}$ for the natural numbers.

## 1.1. Classical and Quantum Mechanics

Although it is not always at the surface, the theory of deformation quantization is about (unifying) the mathematical formalisms behind the theory of mechanics. We mean this in the following way. Before the beginning of the 20th century (Western calendar), the mathematical formalism behind mechanics consisted essentially of time-dependent three dimensional Euclidean geometry. A unified, and extremely powerful, approach was achieved in the form of the Lagrangian and Hamiltonian formulations of classical mechanics. Einstein's theories of special and general relativity (generalizing Galileo's theory of relativity) required adjustments to these formalisms. It speaks volumes to the power of pure mathematical thought that the needed adjustments had, in a sense, already been made in the purely mathematical treatment of non-Euclidean geometry. Namely, instead of considering the time-dependent *Euclidean* geometry, one considers the full *(pseudo)-Riemannian* geometry of space-time. At the same time, however, another form of mechanics began to play a fundamental role in our understanding of nature: quantum mechanics.

The mathematical formalism behind the (non-relativistic) mechanics of the very small, quantum mechanics, is, at first sight, completely different from the formalism of classical mechanics. In particular it does not seem to come with a clean geometrical interpretation like the rest of the theory of mechanics. At closer inspection, one finds that the kernels of the formalism of classical mechanics are still very much a part of the formalism of quantum mechanics. This leads to the consideration of deformation quantization. It is a way of solidifying the fact that, by generalizing our notion of geometry, as in the case of general relativity, we can clearly frame the theory of quantum mechanics in a geometric picture. Especially when we consider the more recent development of *non-commutative* geometry, we find that, in the formalism of classical mechanics, commutative geometry was used to approximate non-commutative geometry, just as Euclidean geometry was once used to approximate (pseudo)-Riemannian geometry.

In this section we shall present a very brief overview of the mathematical formalisms behind (non-relativistic) classical and quantum mechanics. We do this partly to motivate the following and partly to fix some conventions and notations. This thesis is not a survey of the philosophy behind and development of deformation quantization, however. This section is included rather to allow the reader, should they be so inclined, to consider the contents of the thesis in their broader context within mathematical physics.

**1.1.1. Classical Mechanics.** In this section we will discuss the Hamiltonian formulation of classical mechanics. In particular we will show that the Hamiltonian formulation leads us, in a natural way, to consider symplectic and Poisson geometry. As mentioned above, the mathematical formalism describing classical mechanics can be formulated using either the Lagrangian or the Hamiltonian formulation. In many cases the two will in fact be equivalent. The theory of quantum mechanics is usually phrased in terms of the Hamiltonian formulation rather than the Lagrangian formulation. We should note that this is only true for the non-relativistic quantum mechanics. When one considers for instance quantum field theories, they are usually phrased in terms of the Lagrangian formulation. The theory of deformation quantization boils down to a deformation theory, see appendix B, of the algebra of functions on Poisson manifolds and so definitely considers the Hamiltonian formulation of classical mechanics. Since this thesis is not an introduction to the mathematical formalism behind classical mechanics, we shall only present (a small part) of the Hamiltonian formulation. We refer to the first chapter of the excellent book [**107**] for a full treatment. In fact we follow the presentation of [**107**] for both the classical and quantum mechanical formalisms.

Mathematically, classical mechanics is concerned with finding paths in the configuration space $C$ from $x_0$ at time $t_0$ to $x_1$ at time $t_1$. The configuration space is a smooth manifold in which each point represents the possible configuration of a number of "point particles". The *phase* of a classical mechanical system is a point $s$ in the cotangent bundle $T^*C$ of the configuration space. In the Hamiltonian formulation, the Newtown-Laplace determinancy principle, one of the main postulates of classical mechanics, states that the phase of a (closed) classical mechanical system at a certain time uniquely determines the phase and thus the configuration at all future and all past times.

This means that the Hamiltonian formulation needs to provide a way to select paths $\gamma\colon I \to T^*C$ that correspond to physical situations. This is done by employing the principle of least action, also called Hamilton's principle. This principle postulates that there exists a functional $S$, the so-called action functional, on the space of paths in $T^*C$ (with fixed end-points in $C$) such that the critical points correspond to physical trajectories in the phase space $T^*C$. The Hamiltonian formulation goes on to assert that the action functional $S$ has the form

$$S(\gamma) = \int_I \gamma^*\theta + (\gamma^*H)dt \tag{1.1.1}$$

where $\theta$ denotes Liouville's canonical 1-form, $t$ is the standard coordinate on the interval $I$ and the function $H \in C^\infty(T^*C)$ is called the *Hamiltonian* of the physical system. Liouville's canonical 1-form is defined by $\theta_p = p \circ d\pi$ for all $p \in T^*C$, where $\pi\colon T^*C \to C$ denotes the projection. Thus, up to computation, the Hamiltonian formulation reduces the study of classical mechanics to the careful selection of the Hamiltonian $H$.

There are reasons to postulate an action like the one in equation (1.1.1), of course. Essentially, the reasoning proceeds by interpreting the value of $H$ at a certain phase $p$ as the "energy" of that phase $p$. The expected properties of energy then lead us to consider symplectic and Poisson geometry immediately [**115**] and the action functional above provides the way of picking out physical trajectories in the phase space if the energy is distributed according to $H$.

Starting from the action (1.1.1), the principle of least action yields certain differential equations. Consider a coordinate system $(x^1, \dots, x^n)$ on $C$ and the corresponding coordinates on $T^*C$ given by $(\xi^1, x^1, \dots, \xi^n, x^n)$, where $\xi^i := dx^i$. In these terms the principle of least action leads to Hamilton's equations

$$\frac{d\xi^i(\gamma(t))}{dt} = -\frac{\partial H}{\partial x^i}(\gamma(t)) \quad \text{and} \quad \frac{dx^i(\gamma(t))}{dt} = \frac{\partial H}{\partial \xi^i}(\gamma(t)).$$

Definition 1.1.1. A symplectic manifold $(M, \omega)$ is a smooth manifold $M$ equipped with a closed non-degenerate two-form $\omega \in \Omega^2(M)$.

By non-degenerate we mean that, for any $m \in M$ and $0 \neq v \in T_mM$, there exists $w \in T_mM$ such that $\omega_m(v, w) \neq 0$. Note that non-degeneracy of $\omega$ implies that the maps

$$I_\omega \colon T_mM \longrightarrow T^*_mM,$$

given by $I_\omega(v)(w) = \omega(v, w)$ for all $v, w \in T_mM$, are isomorphisms for all $m \in M$. Thus we find the map

$$I_\omega^{-1} \colon \Omega^1(M) \longrightarrow \mathcal{X}(M),$$

where $\mathcal{X}(M)$ denotes the space of smooth vector fields on $M$. The manifold $T^*C$ is the quintessential example of a symplectic manifold, the two-form $d\theta$ defines the symplectic two-form. This is easily seen from the formula

$$d\theta = \sum_{i=1}^{n} d\xi^i \wedge dx^i$$

in coordinates as in the last paragraph. Now we note that Hamilton's equations correspond to the requirement that $\gamma$ is an integral curve for the *Hamiltonian vector field*

$$X_H := I_\omega^{-1}(dH).$$

Thus the Hamiltonian formulation shows that the formalism of classical mechanics is naturally given by symplectic geometry. One might say that the discussion above only shows that the formalism of classical mechanics is given by very specific symplectic manifolds, namely cotangent bundles. However, when considering specific classical mechanical systems one very often applies various schemes of reductions and simplifications that constrict the phase space to be a (possibly lower dimensional) general symplectic manifold. The mathematical formalism outlined above goes under the name *Hamiltonian dynamics*. It can be formulated equally well on a Poisson manifold, a symplectic manifold is a special case of a Poisson manifold. In terms of viewing $H$ as the energy distribution on phase space, the requirement that the manifold be symplectic instead of just Poisson means that we ask that, for any two phases $p$ and $p'$, there is always a sequence of energies $H_i$ and times $t_i$ which changes the phase from $p$ to $p'$ [**115**].

Definition 1.1.2. A *Poisson bracket* on the $k$-algebra $A$ is a Lie bracket

$$\{\cdot, \cdot\} \colon A \otimes A \to A$$

such that $\{f, -\}$ and $\{-, f\}$, considered as $k$-linear endomorphisms, are derivations for all $f \in A$. A manifold equipped with a Poisson bracket on the algebra of smooth functions is called a Poisson manifold.

The bracket given by $\{f, g\} = \omega(X_f, X_g)$ gives a symplectic manifold $(M, \omega)$ the structure of a Poisson manifold. Here we use the notation $X_f$ and $X_g$ for the Hamiltonian vector fields corresponding to $f$ and $g$ respectively. Note that for a general Poisson manifold (not necessarily symplectic) we can still define Hamiltonian vector fields corresponding to functions by $X_f = \{f, -\}$. Note that, since $\omega(-, X_g) = -dg$ defines the Hamiltonian vector field in the symplectic case and $X_f(g) = dg(X_f)$ in general, the two definitions agree in the symplectic case.

The observational principle [**90**] tells us that, if two physical systems return the same value for every observable quantity, then the systems are the same. In terms of geometry, the observational principle implies that, although the paths in phase space allow for evaluation of all observables, it is the observables that matter. The *classical algebra of observables* is given by the algebra of smooth real-valued functions on the phase space. This means that observable quantities are determined completely by the phase of the system. Values of observables become implicitly dependent on time when we consider the time-evolution of a system along physical trajectories. The Hamiltonian formulation allows us to write down differential equations for any observable $f$ directly. Namely, since the paths that correspond to physical trajectories are integral curves of the Hamiltonian vector field $X_H$, we find the differential equation

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}, \tag{1.1.2}$$

for $f$ as a function of time. In fact it follows from the one-parameter group of diffeomorphisms $\psi_H(t)$ given by the flow corresponding to the vector field $X_H$ (if it exists). Thus we find the time-evolution operators $\psi_H(t)^*\colon C^\infty(T^*C) \to C^\infty(T^*C)$, which can formally be expressed as "$\psi_H(t) = e^{-t\{H,-\}}$".

Using the above, we can only consider deterministic systems, not statistical ones. This means that, in order to apply the formalism, we need to have exact knowledge of the phase of a system at some time. The formalism must therefore be adapted to allow for more elaborate notions of state. Namely, a state is given by a map $\mu\colon C^\infty(T^*C) \to \mathcal{P}(\mathbb{R})$ to the probability measures on $\mathbb{R}$. Then for any Borel subset $E \subset \mathbb{R}$ the number $\mu(f)(E)$ is the probability of a measurement of observable $f$ returning a value in $E$. So for the completely deterministic systems we simply consider the state that sends $f$ to the indicator function of $f(p)$ where $p$ is the phase of the system.

The discussion above is very far from complete. It should serve, however, to indicate that the main ingredients going into determining the time-evolution of a classical mechanical system are a symplectic/Poisson manifold $M$ and a Hamiltonian $H \in C^\infty(M)$.

**1.1.2. Quantum Mechanics.** As mentioned, the mathematical formalism behind the theory of quantum mechanics is very different from the Hamiltonian formulation of classical mechanics at first sight. In this section we will show how the mathematical formalism behind quantum mechanics can be interpreted as a kind of deformed Poisson geometry. In particular we will show that the Poisson algebra $C^\infty(T^*C)$ of classical observables can, possibly, be seen as a first order approximation to a quantum algebra of observables, see also appendix B and section 1.2. There is a need for a quantum theory in order to describe the mechanics of physical systems that are so "small" that any measurement will cause significant disturbance of the system. When we considered the mathematical formalism of classical mechanics, we started describing the first, least abstract, version. As mentioned, the formalism can eventually be given in terms of a general symplectic manifold, in other words the manifold is not necessarily the cotangent bundle of configuration space. Similarly, we will present first the least abstract version of the formalism behind quantum mechanics and subsequently generalize it.

Mathematically, quantum mechanics is concerned with finding the complex valued wave-function $\psi(x,t)$ dependent on position and time. The square of the norm of the wave-function at time $t$ represents the probability distribution for the configuration of the system. In other words, for all $t \in \mathbb{R}$ we have $\psi(x,t) \in L^2(C)$ and the probability that the system is in a configuration in the subset $A \subset C$ at time $t$ is

$$\int_A \|\psi(x,t)\|^2 \, dx.$$

Now the Newton-Laplace determinancy principle makes way for a new determinancy principle. Namely, instead of the phase of the system determining the future and past configurations of the system, the state, provided by the wave-function, at the time $t$ provides the future and past states of the system. In other words, we should be able to formulate a time-evolution equation for the wave-function. Note however that we do find that all wave-functions must satisfy that the integral of their norm squared over all of $C$ equals 1.

Instead of doing this, however, let us first consider the mathematical formalism of quantum mechanics in the abstract. Instead of the phase space of classical mechanics we have the *space of states* given by a separable complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The algebra of observables $A$ is given by the algebra of linear operators on $\mathcal{H}$. The spectrum of the observables will represent the possible outcomes of a measurement of that observable. The likelihood of a certain range of the spectrum to be measured is determined by the state of the system, which we define below. Since, in nature, observables are required to produce real values, the physical observables $\mathcal{O}$ are given by the *self-adjoint* linear operators on $\mathcal{H}$. Note that the self-adjoint bounded operators $A_0 = \mathcal{O} \cap \mathbb{B}(\mathcal{H})$ form a real vector space. The states of the quantum system are given by the positive operators $S$ of trace 1, i.e. $\operatorname{Tr} S = 1$. Note that the Hilbert space provides the special *pure states* $P_\psi$ given by orthogonal projection onto the span of $\psi \in \mathcal{H}$, here $\|\psi\| = 1$. The state $S$ assigns to every observable $A$ the probability measure $S_A$ on $\mathbb{R}$ by the Born-von Neumann formula

$$S_A(E) = \operatorname{Tr} P_A(E)M,$$

where $E \subset \mathbb{R}$ is a Borel subset and $P_A$ is the *projector valued measure* on $\mathbb{R}$ associated to the self-adjoint operator $A$ through von Neumann's spectral theorem [**100**].

An aspect of classical mechanics that is only implicitly present in section 1.1.1 is that there are essentially two ways to consider the time-evolution of a system. Either one considers an evolution of the state $\mu$ of the system, for instance in the deterministic systems this corresponds to the phase-flow directly, or one considers the evolution operators on the algebra of observables. These two points of view are sometimes referred to as Liouville's or Hamilton's picture of classical mechanics [**107**] respectively. In the classical case the distinction is not as often pointed out as in the case of quantum mechanics. The corresponding pictures in the case of quantum mechanics are called the Schrödinger or Heisenberg pictures of quantum mechanics. Thus, the time-evolution of the quantum mechanical system in the Heisenberg picture is given by a strongly continuous one-parameter group of unitaries $U(t)$ through the formula

$$A(t) = U(t)A(0)U(t)^*,$$

for any $A \in A_0$. In the Schrödinger picture the states $S$ evolve similarly according to

$$S(t) = U(t)MU(t)^*.$$

The analog of the Hamiltonian comes in the form of the ansatz $U(t) = e^{\frac{2\pi i t H}{h}}$ for some *constant* self-adjoint operator $H \in \mathcal{O}$, called the (quantum) Hamiltonian and where $h$ denotes Planck's constant [**63**]. This leads to the differential equations

$$\frac{dA}{dt} = \frac{2\pi}{ih}[A, H] + \frac{\partial A}{\partial t} \tag{1.1.3}$$

called *Heisenberg's equations of motion* in the Heisenberg picture. Here we have denoted the commutator of operators by square brackets.

The problem now becomes to find out what combination of Hilbert space and Hamiltonian to associate with a given physical quantum mechanical system. Suppose we want to consider a "free particle" constrained only to lie on a 1-dimensional space. In this case we are immediately led to consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ as above. Note that, in essence, the problem of finding a Hilbert space is solved by finding out the dimension, but, as we will see, a good choice of Hilbert space simplifies the Hamiltonian. The *correspondence* principle, which can be formulated mathematically, states that, if, in units determined by the characteristic dimensions of the system, $h \approx 0$, then the quantum system reduces to a classical system. In this case it means simply that, since the classical Hamiltonian $H_c$ for a free particle on a line is given by the kinetic energy and so $H_c \propto v^2$ where $v$ denotes the velocity, we consider the Hamiltonian $H \propto -h^2\partial_x^2$, since in this case the expectation values of measurements are exactly the classically predicted values.

The last paragraph leads to the notion of *quantization*. The idea is that one can construct physically relevant quantum systems from classical systems. This is motivated by the similarities between the formalisms of quantum and classical mechanics. In particular we note that $A_0$ is a Lie algebra for the bracket $\frac{1}{i}[-,-]$. So, we see that the equations of motion (1.1.2) and (1.1.3) are exactly the same when we consider them as equations for evolution of $A \in \mathfrak{g}$, given a triple $(\mathfrak{g}, L, X)$ of a real vector space $\mathfrak{g}$ with a Lie bracket $L$ and a specific element $X \in \mathfrak{g}$. Namely in the quantum case $\mathfrak{g} = A_0$,

$L = \frac{2\pi}{ih}[\cdot,\cdot]$ and $X = H$, while in the classical case $\mathfrak{g} = C^\infty(M)$, $L = \{\cdot,\cdot\}$ and $X = H_c$. Furthermore, if we consider the previous example, we arrive at the expression $H \propto \hat{p}^2$ for $\hat{p} = ih\partial_x$. So, denoting the operator of multiplication by the identity function by $\hat{x}$, which the correspondence principle implies, we find that

$$[\hat{x},\hat{p}] = ih = ih\{x,\xi\}$$

where we denote the canonical coordinates on $\mathbb{R}^2 = T^*\mathbb{R}$ by $(x,\xi)$. This motivates the mathematically rigorous definition of the correspondence principle.

Rigorously then, quantization of the classical mechanical system given by the Poisson manifold $(M,\{\cdot,\cdot\})$ with the Hamiltonian $H_c$ is given by injective *quantization maps*

$$Q_\hbar \colon C^\infty(M) \to \mathcal{O}$$

from the classical observables to the quantum observables, i.e. to self-adjoint operators on a Hilbert space $\mathcal{H}$, satisfying the following properties. First of all the maps $Q_\hbar$ depend on the variable $\hbar > 0$ (although the range may be any subset of $\mathbb{R}$ that has 0 as a limit point). Secondly, the restriction of $Q_\hbar$ to the subspace $C_0^\infty(M)$ of bounded functions is a linear map to $A_0$. The quantization maps satisfy the equations

$$\lim_{\hbar\to 0} \frac{1}{2} Q_\hbar^{-1}\left(Q_\hbar(f)Q_\hbar(g) + Q_\hbar(g)Q_\hbar(f)\right) = fg \tag{1.1.4}$$

and

$$\lim_{\hbar\to 0} Q_\hbar^{-1}\left(\frac{1}{i\hbar}[Q_\hbar(f),Q_\hbar(g)]\right) = \{f,g\} \tag{1.1.5}$$

for all $f,g \in C_0^\infty(M)$. This last equation (1.1.5) is the mathematical version of the correspondence principle.

One way of obtaining quantization maps is by *deforming* the commutative product on $C^\infty(M)$ along the parameter $\hbar$. This can be seen since equation (1.1.4) means that, up to higher than 0th order in $\hbar$, the product on $A_0$ behaves like the product on $C_0^\infty(M)$. The equation (1.1.5) puts a constraint on the 1st order behaviour in $\hbar$. In higher orders the constraints are coming from the fact that we map into a $C^*$-algebra. The theory of deformation quantization is exactly about this method of producing quantum mechanical systems from classical mechanical systems. The example of $L^2(\mathbb{R})$ comes from the Moyal deformation of $\mathbb{R}^{2d} = T^*\mathbb{R}^d$ with the standard symplectic structure $\omega_{st} = d\theta$. The deformation is provided by the formula (3.0.2), where $(q,p) = (x,\xi)$. This product is obtained by considering the quantization maps given by the Wigner-Weyl transform [**117**].

One can consider the problem of deforming a Poisson manifold to a quantum mechanical system as a combination of two problems. Firstly, there is the purely algebraic problem of finding an associative product on $C^\infty(M)$, which deforms the pointwise product in the direction of the Poisson structure in the sense of equations (1.1.4) and (1.1.5). Secondly, there is the analytic problem of constructing a Hilbert space such that the deformed product is actually realized as the product of operators on that Hilbert space. Formal deformation quantization concerns itself only with the first problem. This means that it is about studying the formal deformations of $C^\infty(M)$ as explained in appendix B and section 1.2. There is a very large body of work on this topic, see [**105, 42, 65**] for nice surveys of and notes on the field, or [**114**] for an introductory textbook.

In this thesis we will concern ourselves only with *formal* deformation quantizations. Let us point out, however, that, as can also be seen in chapters 2 and 4, a solution of the full problem follows from a formal solution, if one has an adequate way to check which formal solutions correspond to full solutions. In the chapters 2 and 4 this way is provided in the form of the Grothendieck and Fedosov connections respectively. We should mention, as does Fedosov in [**48**], that the algebraic index theorem, see sections 1.5 and 6.1, provides a way to show that a formal solution cannot be obtained from full solutions.

Remark 1.1.3. The discussion of the formalism behind quantum mechanics above is, of course, very far from complete. We should mention in particular that we have not said anything about the domains of operators. It has, sadly, become rather common practice in physics courses to ignore a discussion of domains of operators on Hilbert spaces. We will do the same, however, since the (functional) analytic aspects of quantum mechanics will not play any role in the thesis. Let us refer again to [**107**] for an in-depth treatment of the mathematical formalism behind quantum mechanics.

## 1.2. Formal Deformation Quantization

In this section we will introduce the main object of study of this thesis. Namely, the algebras obtained by the process of formal deformation quantization of the algebra of smooth functions on a manifold. We will see how this always induces the structure of a Poisson manifold. So, we note that from this point of view the appearance and importance of the Poisson structure in the Hamiltonian formulation of classical mechanics are explained by the fact that it yields a first order approximation to the corresponding quantum mechanical algebra of observables.

We saw in the previous section 1.1 how the Hamiltonian formulations of classical and quantum mechanics lead naturally to the consideration of deformation quantization. We should mention that the notion was introduced in the influential paper [**4**] and has taken on a vibrant life afterwards. As mentioned, the first step one may take in considering the problem of deformation quantization is to consider the problem of formal deformation quantization. In this thesis we will concern ourselves only with this formal deformation quantization. We will therefore frequently drop the word “formal” and sometimes we will even drop the word quantization for reasons that should be evident from appendix B. Let us point out that in some cases, for instance in the case of index theorems [**88**], it is in some sense enough to consider the formal case.

The notion of formal deformation quantization follows from the notion of (deformation) quantization in section 1.1.2 by considering the expansion of $Q_\hbar^{-1}(Q_\hbar(f)Q_\hbar(g))$ in powers of $\hbar$. In this way the quantization will provide an associative product on $C^\infty(M)[\![\hbar]\!]$, formal power series with coefficients in $C^\infty(M)$. The equation (1.1.4) implies that this product is a deformation of the algebra $C^\infty(M)$ in the sense of appendix B. We do consider one more constraint on the kind of deformations we consider. The constraint arises from the notion of *locality* in physics. It is expressed by the fact that, given a deformation quantization of the classical mechanical system $(M, \{\cdot,\cdot\})$, restriction to an open submanifold $U \subset M$ should give us a deformation quantization of the restricted classical mechanical system $(U, \{\cdot,\cdot\})$ as well. In mathematical terms we could say we want to deform the *sheaf* of algebras $C^\infty_M$, it is given by $C^\infty_M(U) = C^\infty(U)$ for all open subsets $U \subset M$. These considerations lead to the following definition.

Definition 1.2.1. A formal deformation quantization of the manifold $M$ is given by a sequence of linear maps

$$B_k \colon C^\infty(M) \otimes C^\infty(M) \longrightarrow C^\infty(M)$$

for all $k \in \mathbb{N}$ satisfying the following conditions. Firstly the $\mathbb{C}[\![\hbar]\!]$-linear product $\star$ on the vector space $C^\infty(M)[\![\hbar]\!] := C^\infty(M) \otimes \mathbb{C}[\![\hbar]\!]$, given by the formula

$$f \star g = fg + \sum_{k>0}^{\infty} (i\hbar)^k B_k(f,g) \tag{1.2.1}$$

for all $f, g \in C^\infty(M)$, is associative. Here $\mathbb{C}[\![\hbar]\!]$ denotes the ring of formal power series in the *formal* variable $\hbar$ with coefficients in $\mathbb{C}$. Secondly the maps $g \mapsto B_k(f,g)$ and $g \mapsto B_k(g,f)$ are differential operators for all $f \in C^\infty(M)$.

NOTATION 1.2.2. Note that when we consider a deformation quantization of $M$ we are considering the *complex valued* smooth functions on $M$, since we want to find the relation

$$[-,-] = i\hbar\{-,-\} + O(\hbar^2).$$

In the rest of this thesis $C^\infty(M)$ shall denote the complex valued smooth functions, similarly we will denote by $\Omega^\bullet(M)$ the complex valued differential forms on $M$. An exception is the chapter 2 where we consider undeformed formal geometry and can therefore always consider real valued functions. We should note however that the definition still makes sense for real functions, if we replace $i\hbar$ by $\hbar$ in the formulas. In other words the difference is only a rescaling of the formal variable. In fact most considerations in this thesis would still makes sense for real valued functions. The convention comes from the fact that we would like the real momentum (vertical coordinate in $T^*C$) to correspond to the self-adjoint operator $i\hbar\partial_x$.

NOTATION 1.2.3. Note that a deformation quantization defines the algebra $(C^\infty(M)[\![\hbar]\!], \star)$. In fact, this is the main point and therefore we shall often refer to this algebra as a deformation quantization, or sometimes a deformation quantization algebra, of $M$. Bilinear operators that satisfy the second property that the $B_k$ above should satisfy are called *bidifferential* operators. Note that the multiplication $f \otimes g \mapsto fg$ is also a bidifferential operator. Setting $B_0(f,g) = fg$ we find that the $\star$-product gets the form

$$f \star g = \sum_{k\geq 0} (i\hbar)^k B_k(f,g).$$

Note that the requirement that $\star$ defines an associative product can be interpreted as a series of equalities involving the bidifferential operators $B_k$. Namely, we find that

$$\sum_{k+l=n} B_k(f, B_l(g,h)) = \sum_{k+l=n} B_k(B_l(f,g),h)$$

for all $f,g,h \in C^\infty(M)$ and all $n \in \mathbb{Z}_{\geq 0}$. For $n=0$ this just reflects the associativity of the pointwise product on $C^\infty(M)$. The first non-trivial condition is therefore given by the equation

$$B_1(f,gh) + fB_1(g,h) = B_1(fg,h) + B_1(f,g)h$$

for all $f,g,h \in C^\infty(M)$. From this equation it follows that $2B_1^-$ defines a Poisson structure, here $B_1^-$ is the anti-symmetric part of $B_1$, i.e. $B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f))$. Of course it was expected, since the commutator defines a Poisson bracket and

$$[f,g] = 2i\hbar B_1^-(f,g) + O(\hbar^2).$$

Thus, a deformation quantization induces the structure of a Poisson manifold on $M$. When $\{B_k\}$ defines a deformation quantization we will say it is in the direction of the Poisson structure $2B_1^-$.

DEFINITION 1.2.4. A *gauge equivalence* is a sequence $T_k$ of differential operators such that $T_0 = \mathrm{Id}$ and

$$\sum_{k\geq 0} (i\hbar)^k T_k \colon (C^\infty(M)[\![\hbar]\!], \star) \longrightarrow (C^\infty(M)[\![\hbar]\!], \star')$$

is an algebra isomorphism for some deformation quantizations $\star$ and $\star'$.

Note that it is simply the definition of gauge equivalence in definition B.1.3 with the added requirement that the gauge equivalence is given by *differential* operators. In fact it is precisely the same definition, since the requirement that the linear operators $T_k$ yield an algebra isomorphism between deformation quantizations implies that they are differential operators [**77, 30**]. When we consider deformation quantization in the framework of deformation theory of associative algebras as in appendix B, we see that the bidifferential operators define a Maurer-Cartan element in $\mathfrak{g}(C^\infty(M))$, see definitions B.4.4 and B.3.6 specifically. Note that the formal deformation $\{B_k\}$ induces the $\mathbb{C}[\hbar]/\langle\hbar^2\rangle$-deformation given by $f \otimes g \mapsto fg + \hbar B_1(f,g)$, in the language of appendix B. The Maurer-Cartan equation in definition B.3.6 over the ring $R = \mathbb{C}[\hbar]/\langle\hbar^2\rangle$ is simply the cocycle condition for a Hochshild

2-cochain. Let us be a bit more precise here and mention that we are actually considering an adaptation of the Hochschild cochain complex presented in section B.4, see remark B.4.6. Namely, instead of considering all multi-linear maps $C^\infty(M)^{\otimes p} \to C^\infty(M)$, we shall consider only those that are given by multi-differential operators. Now it is well-known that the inclusion of the poly-vectorfields $\Gamma(\bigwedge^\bullet TM)$, equipped with the trivial differential, is a quasi-isomorphism [**112, 25, 71**]. In particular, we see that it means that $B_1 = B_1^- + dT$ for some differential operator $T$.

Corollary 1.2.5. *Every deformation quantization is gauge equivalent to one where* $2B_1$ *is the induced Poisson bracket.*

Proof.
This is a corollary of the discussion above since the map $t = \mathrm{Id} - i\hbar T$ provides the gauge equivalence. Note that the inverse is given by $\sum_{i=0}^\infty (i\hbar T)^i$. It follows since $\|-i\hbar T\| = \frac{1}{2}$ for the norm induced by the $\hbar$ filtration on the space of $\mathbb{C}[\![\hbar]\!]$-endomorphisms of $C^\infty(M)[\![\hbar]\!]$, see definition 5.3.15. So we find the gauge equivalent $\star$-product

$$f \star_t g := t^{-1}(t(f) \star t(g)) = fg + i\hbar B_1^-(f,g) + O(\hbar^2).$$

□

Remark 1.2.6. In a similar way to the proof above one can obtain normalizations of a deformation quantization [**31**], see also [**55**]. A normalized deformation quantization satisfies the additional conditions

$$1 \star f = f \quad \text{and} \quad f \star 1 = f$$

for all $f \in C^\infty(M)$. From now on we will always assume that deformation quantizations are normalized.

By corollary 1.2.5, we see that the space of infinitesimal deformations of $C^\infty(M)$ (up to gauge equivalence), i.e. $\mathbb{C}[\hbar]/\langle\hbar^2\rangle$-deformations, is isomorphic to the space of Poisson structures on $M$ (up to gauge equivalence). More loosely speaking, the 1st order neighborhood of $C^\infty(M)$ in the Moduli space of algebra structures is the space of Poisson structures on $M$. One is led to ask whether the space of formal deformations of $C^\infty(M)$ (up to gauge equivalence) is isomorphic to the space of formal Poisson structures (up to gauge equivalence). We note that the space $\mathfrak{h} := \Gamma(\bigwedge^\bullet TM)$ of poly-vector fields can be equipped with the Gerstenhaber structure given by the wedge product, Schouten-Nijenhuis bracket and trivial differential [**53, 102, 92**]. Thus we see that corollary 1.2.5 comes from the fact that the quasi-isomorphism (of cochain complexes) $\mathfrak{h} \hookrightarrow \mathfrak{g}(C^\infty(M))$ induces an isomorphism $\mathrm{Def}_R(\mathfrak{h}, 0) \to \mathrm{Def}_R(\mathfrak{g}(C^\infty(M)), B_0)$ for $R = \mathbb{C}[\hbar]/\langle\hbar^2\rangle$. Formal Poisson structures are exactly the Maurer-Cartan elements of $\mathfrak{h}$ with respect to the local complete ring $\mathbb{C}[\![\hbar]\!]$. Thus we see that the formal deformations are indeed parametrized by formal Poisson structures if the *dgl algebras* $\mathfrak{h}$ and $\mathfrak{g}(C^\infty(M))$ are quasi-isomorphic, i.e. there exists a chain map that is also a Lie algebra homomorphism and such that the induced map on cohomology is an isomorphism. This requirement turns out to be too strong. However, the same reasoning shows that the formal deformations are parametrized by formal Poisson structures if the $L_\infty$-algebras $\mathfrak{h}$ and $\mathfrak{g}(C^\infty(M))$ are quasi-isomorphic [**79**]. This is exactly what was proved by Kontsevich in his celebrated paper [**71**], see also [**70, 108, 109**].

Although we would be remiss if we did not mention Kontsevich's classification result above, in this thesis we will stick to symplectic manifolds. We will call deformation quantizations in the direction of a Poisson structure induced by a symplectic structure: symplectic deformation quantizations. In this case the classification takes on an even better form, since the formal Poisson structures where the first order term is fixed to be non-degenerate are parametrized by the second de Rham cohomology of the symplectic manifold $M$, see definition A.2.18. This result has many incarnations, in this form it was shown in [**50, 74**]. We will present a proof, based on [**89**], in chapter 4. Beyond that, there is also a (completely different) proof by Deligne in [**30**]. All of these lead to the definition of a *characteristic class* $[\theta] \in \frac{\omega}{i\hbar} + H^2(M)[\![\hbar]\!]$, see definition 4.3.2, of the deformation quantization. This class also goes under the names *Weyl curvature* or *Deligne's class*, [**48, 65**].

The version of the classification result for symplectic deformation quantizations that we will present in chapter 4 is based on a slightly earlier result than the full classification of deformation quantizations. We mean Fedosov's construction of symplectic deformation quantizations [**46**], which leads to the classification results above by the results of [**87**] and [**6**]. This construction, which we say a lot more about in chapters 3 and 4, is closely related, by design, to the algebraic index theorem [**45**]. Since a large part of this thesis revolves around the algebraic index theorem, we should introduce it as well. Before we do that however, we should give a short introduction of the motivation for the algebraic index theorem, namely the *Atiyah-Singer index theorem* 1.3.4.

## 1.3. The Atiyah-Singer Index Theorem

The Atiyah-Singer index theorem is widely seen as one of the most important theorems in modern mathematics, this is exemplified by the great number of adaptations and generalizations, [**104, 3, 110, 56, 28**] to name a few. Another reason is that the theorem itself generalizes some very consequential theorems like the Gauss-Bonnet, see chapter 9 of [**75**] and Riemann-Roch, see chapter 6 of [**83**], theorems. The Atiyah-Singer index theorem is closely related to deformation quantization. This is shown in particular by the algebraic index theorem [**45, 86**], which we shall discuss below, and was the basis of a lot of Fedosov's work on the subject [**48**]. The relation is also simply seen by observing that the example of quantization given in section 1.1.2 yielded (pseudo)-differential operators.

In this section we shall give a very brief overview of the main points of the Atiyah-Singer index theorem that will come up in the rest of this thesis. For a more in depth discussion of the theorem we refer the reader to the notes [**72, 103**] and [**91**]. We will start by defining the notion of index of elliptic differential operators and its relation to K-theory. We will then state a version of the index theorem in K-theoretic terms. From this we will derive the statement in terms of cohomology classes. We want to phrase the theorem in terms of cohomology, since the algebraic index theorem follows from a formula in periodic cyclic cohomology and the corresponding homology is a non-commutative replacement for de Rham cohomology of a manifold [**26**].

The Atiyah-Singer index theorem provides an expression for the index of an elliptic pseudo-differential operator in terms of purely topological data. So let us first mention what exactly the index is.

Definition 1.3.1. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and $F$ is a linear operator between them. We call $F$ a *Fredholm operator* if both the kernel and cokernel of $F$ are finite dimensional. The index of $F$, denoted Ind $F$, is defined by the formula

$$\text{Ind } F = \text{Dim Ker} F - \text{Dim Coker} F.$$

If we restrict our attention to the bounded operators $\mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ we see, by Atkinson's theorem [**66**], that an operator $F \in \mathbb{B}(\mathcal{H})$ is Fredholm if and only if its class in the Calkin algebra $\mathbb{Q}(\mathcal{H}) = \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ is invertible. Here we have denoted the compact operators on $\mathcal{H}$ by $\mathbb{K}(\mathcal{H})$. This allows us to consider the Fredholm index as a construction in K-theory immediately by considering the exact sequence

$$0 \to \mathbb{K}(\mathcal{H}) \longrightarrow \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{Q}(\mathcal{H}) \to 0.$$

This sequence induces a six term exact sequence in (topological) K-theory. We note that Fredholm operators represent classes in $\mathrm{K}_1(\mathbb{Q}(\mathcal{H}))$ and are therefore mapped to $\mathrm{K}_0(\mathbb{K}(\mathcal{H})) \simeq \mathbb{Z}$ under the boundary map. The image of the class of a Fredholm operator under this boundary map is exactly the index. This shows, by homotopy invariance of K-theory, that the index is invariant under homotopy of Fredholm operators. We also mention the construction above since similar constructions of the index of elliptic pseudo-differential operators exist. We will not present these construction in any detail here however, see the last section of [**20**].

The Atiyah-Singer index theorem considers certain pseudo-differential operators. A pseudo-differential operator of order $m \in \mathbb{Z}$ on $\mathbb{R}^n$ is an operator $A$ on functions $f$ on $\mathbb{R}^n$ given by the formula

$$Af(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi) d\xi$$

where $\hat{f}$ denotes the Fourier transform of $f$ and the function $a(x,\xi)$ on $\mathbb{R}^{2n}$ is called the *symbol* of $A$. The symbols of pseudo-differential operators are required to lie in so-called symbol classes $S^m \subset C^\infty(\mathbb{R}^{2n})$ satisfying certain decay conditions

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|},$$

where we use the notation of section 2.1. We note that $S^n \subset S^m$ whenever $n < m$. The principal symbol of the pseudo-differential operator $A$ of order $m$ is defined as the class of the symbol of $A$ in the quotient $S^m/S^{m-1}$. We can generalize this definition to consider pseudo-differential operators

$$A\colon \Gamma(E) \longrightarrow \Gamma(F)$$

between vector bundles $E$ and $F$ over a smooth manifold $M$. This is done by simply requiring the operator to be given by a matrix of symbols of the form above on any neighborhood of $M$ that trivializes both $E$ and $F$. A pseudo-differential operator is called elliptic if the principal symbol is invertible away from the (dimension $n$) submanifold defined by $x = 0$. We note that the principal symbols patch together to define a function on the cotangent bundle $T^*M$ of $M$. More precisely, we obtain the bundle homomorphism

$$\sigma_m(A)\colon \pi^*E \longrightarrow \pi^*F,$$

between the pull-backs of the vector bundles along $\pi\colon T^*M \to M$, from the principal symbol of the pseudo-differential operator $A$ of degree $m$. We also call this bundle homomorphism the principal symbol. So we see that, if $M$ is compact, every elliptic pseudo-differential operator $A$ defines the class $[\sigma_m(A)] \in \mathrm{K}^0(T^*M)$. In fact every class can be obtained in this way [**72**].

LEMMA 1.3.2. *Elliptic pseudo-differential operators are Fredholm.*

Here we consider the pseudo-differential operators as maps $L^2(M,E) \to L^2(M,F)$ for some choice of Hermitian structure on $E$ and $F$. A proof of the lemma can be found for instance in [**48**]. Thus we find that the index Ind $A$ of the elliptic pseudo-differential operator $A$ is well-defined. On the other hand we can construct a certain map $\mathrm{K}^0(T^*X) \to \mathbb{Z}$ as follows. Given an inclusion $\iota\colon X \hookrightarrow Y$ of a compact manifold into another manifold we can construct the wrong-way (or shriek) map

$$\iota_!\colon \mathrm{K}^0(T^*X) \longrightarrow \mathrm{K}^0(T^*Y),$$

see [**72**].

DEFINITION 1.3.3. The topological index map $\mathrm{Ind}_t$ is defined as the composite

$$\mathrm{K}^0(T^*X) \xrightarrow{\iota_!} \mathrm{K}^0(T^*\mathbb{R}^N) \xrightarrow{i_!^{-1}} \mathbb{Z},$$

where $\iota$ denotes the inclusion $X \to \mathbb{R}^N$ for some sufficiently large $N$ and $i$ denotes the inclusion of the origin in $\mathbb{R}^N$.

It turns out that the topological index does not depend on the choice of $\iota$. The Atiyah-Singer index theorem states that

$$\mathrm{Ind}\, A = \mathrm{Ind}\,_t([\sigma_m(A)])$$

for all pseudo-differential operators of degree $m$. Note that we can also view the analytic index Ind $A$ as a map from $\mathrm{K}^0(T^*X)$, for compact manifolds at least, by the fact that all classes in $\mathrm{K}^0(T^*X)$ can be realized as principal symbols of pseudo-differential operators [**72**]. So the index theorem essentially states that two maps from $\mathrm{K}^0(T^*X)$ to the integers are equal.

One of the applications of the index theorem is, of course, to be able to compute the index of certain special operators, which can be very hard in general. Although the statement of the index theorem in terms of K-theory is illuminating on a conceptual level, the statement of the index theorem in terms of characteristic classes, see theorem 1.3.4, is more suited to computation. The way to translate the above K-theoretical statement to a statement about characteristic classes is by using the Chern character map [**68**]

$$\mathrm{ch}\colon \mathrm{K}^0(M) \longrightarrow \mathrm{H}_c^{ev}(M),$$

where $\mathrm{H}_c^{ev}(M)$ denotes the parts of even degree of the de Rham cohomology for compactly supported forms, and the Thom isomorphisms [**48, 12**]

$$\mathrm{K}^0(T^*X) \longrightarrow K^0(X) \quad \text{and} \quad \mathrm{H}^\bullet_{cv}(T^*X) \longrightarrow \mathrm{H}^{\bullet-n}_{\mathrm{dR}}(X),$$

where $\mathrm{H}^\bullet_{cv}(T^*X)$ denotes the de Rham cohomology for forms that have compact supports in the vertical direction. This leads to the following formulation of the Atiyah-Singer index theorem.

THEOREM 1.3.4 (Atiyah-Singer Index Theorem). *Suppose $A$ is a pseudo-differential operator with principal symbol $\sigma(A)$ on the compact manifold $X$, then we have*

$$\mathit{Ind}\, A = \int_X \mathit{ch}\,(\sigma(A))\, \mathit{Td}(X)$$

*where $\mathit{Td}(X)$ denotes the Todd class of the complexified tangent bundle of $X$.*

The appearance of $\mathrm{Td}(X)$ is a consequence of the fact that the square

$$\begin{array}{ccc} \mathrm{K}^0(T^*X) & \longrightarrow & \mathrm{H}^{ev}_{cv}(T^*X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{K}^0(X) & \longrightarrow & \mathrm{H}^{ev}_c(X) \end{array}$$

commutes only up to multiplication by the Todd class [**48**]. Note that one consequence of the Atiyah-Singer index theorem is that the integral on the right hand side in theorem 1.3.4 is forced to be integer valued.

As mentioned, the connection between the Atiyah-Singer index theorem and deformation quantization can be seen explicitly in the example in section 1.1.2. Namely, the assignment of a pseudo-differential to its symbol is a quantization of the phase space $T^*C$. In this sense, a deformation quantization can be seen as a generalized symbol calculus for the pseudo-differential operators [**48**]. It turns out that there is indeed an index theorem associated to any formal deformation quantization of a manifold [**45, 33, 86**]. We will discuss a proof of (the symplectic case of) this algebraic version of the index theorem in section 6.1. In the next section we will give an introduction to this theorem.

## 1.4. The Algebraic Index Theorem

The Atiyah-Singer index theorem 1.3.4 has an algebraic analog in the algebraic index theorem 6.1.22. The main observation behind the analogy is that the algebra of pseudo-differential operators is a deformation quantization of the cotangent bundle by considering the map that associates a pseudo-differential operator to its symbol [**48**]. In other words, just as a formal deformation quantization is a quantization up to certain analytic considerations, so is the algebraic index theorem a version of the Atiyah-Singer index theorem up to certain analytic considerations. In this section we shall give a short introduction to some of the main ideas behind the algebraic index theorems. In section 6.1 we shall present a proof of the algebraic index theorem, inspired by the proof in [**14, 15**], which generalizes to an equivariant version.

The idea for an algebraic index theorem was first formulated by Fedosov in [**44**]. It was proved by Fedosov in [**47**] and, a more general version was independently proved by Nest-Tsygan in [**86**]. A similar result was obtained, again independently, in [**27**]. In essence, it is a theorem expressing an explicit formula for computation of the trace of certain K-theory classes. In [**86**] it is stressed that this means we are considering an instance of the pairing of K-theory and periodic cyclic cohomology, by use of the Chern-Connes character 6.1.17. Comparing this with a certain proof of the index theorem, which we shall sketch below, shows that the algebraic index theorem is really an algebraic analog of the Atiyah-Singer index theorem. Moreover, in the article [**88**] it is shown that, for the deformation quantization given by the symbol calculus on the cotangent bundle, one recovers the theorem 1.3.4 from 6.1.22.

Let us present here a sketch of the proof of the Atiyah-Singer index theorem 1.3.4 for $\mathbb{R}^n$ which appeared in [**40**]. This proof motivates the algebraic index theorem, a sketch also appeared in [**86**] as a reason for considering the algebraic index theorem an algebraic analog of the Atiyah-Singer index theorem. Suppose $D$ is an elliptic differential operator on $\mathbb{R}^n$, we will assume that $D = 1$ outside some compact $K \subset \mathbb{R}^n$. We can naturally associate two projectors to $D$. First consider the projection $p_D \colon L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ onto the graph of $D$, we consider it as a pseudo-differential operator in the trivial $\mathbb{C}^2$ bundle on $\mathbb{R}^n$. Secondly, denoting by $P_D$ the projection onto the kernel of $D$ and by $Q_D$ the projection onto the cokernel of $D$, we can consider the projector $q_D = P_D \oplus (1 - Q_D)$ (where the $\oplus$ denotes block sum of matrices). Note that the index of $D$ equals $\mathrm{Tr}(q_D - (0 \oplus 1))$. It turns out that one can construct a homotopy of projections $e_D(t)$ such that $e_D(0) = q_D$ and $e_D(1) = p_D$. So, one would hope to compute the index of $D$ using the projection $p_D$ instead of $q_D$. The problem is that $e_D - (0 \oplus 1)$ is not trace class. So we try to replace the trace by a suitable "higher trace", i.e. a cyclic cocycle. Thus one considers the cocycle $\Theta$ given by

$$\Theta(a_0, \ldots, a_{2n}) = \frac{1}{n!} \sum_{\sigma \in S_{2n}} \epsilon(\sigma) \,\mathrm{Tr}\left(a_0[\chi_{\sigma(1)}, a_1][\chi_{\sigma(2)}, a_2] \ldots [\chi_{\sigma(2n)}, a_{2n}]\right),$$

for all integral operators $a_0, \ldots, a_{2n}$ and where $S_{2n}$ denotes the symmetric group in $2n$ letters, $\epsilon(\sigma)$ denotes the sign of $\sigma$ and $\chi_{2i+2} = x_i$ while $\chi_{2i+1} = \partial_{x_i}$ for all $i = 1, \ldots, n$. One proceeds to prove that $\Theta$ is cohomologous to Tr. This implies that

$$\mathrm{Tr}(q_D - (0 \oplus 1)) = \Theta(q_D, \ldots, q_D).$$

At this point there are two slight complications. First, one needs to show that the application of $\Theta$ to $2n$ copies of $p_D$ still makes sense. Secondly, one needs to show that the computation showing that the evaluation of a cyclic cocycle along a continuous path of idempotents is constant still holds for the path $e_D(t)$. Let us denote by $D_h$ the differential operator obtained from $D$ by scaling all the $\partial_{x_i}$ by $h \in \mathbb{R}$. Then we find that $\mathrm{Ind}\, D = \mathrm{Ind}\, D_h$ for all $h \in \mathbb{R}$. On the other hand we find that

$$\lim_{h \to 0} \Theta(p_{D_h}, p_{D_h}, \ldots, p_{D_h}) = \frac{1}{n!} \int_{\mathbb{R}^{2n}} \sigma(D) d\sigma(D) \ldots d\sigma(D).$$

From this one deduces the Atiyah-Singer index theorem.

When we consider the algebraic index theorem we note that the analytic subtleties disappear. Thus, the main thing we take away from the sketch above is that the algebraic input entering the index theorem is simply the equality of certain cyclic cocycles. So suppose $\mathbb{A}_\hbar(M)$ is a deformation quantization of the symplectic manifold $M$. Fedosov shows in [**48**] that there is a unique trace, in the sense of definition 6.1.19, on $\mathbb{A}_\hbar(M)$. This trace will be the first periodic cyclic cocycle. In [**86**] the authors construct a "Poincaré duality" map

$$\overline{CC}_{\bullet-1}(A) \longrightarrow CC^{-\bullet}_{per}(A)$$

from the reduced cyclic homology complex of an algebra $A$, see [**78**], to the periodic cyclic cohomology complex of $A$, see definition A.2.2, given a trace Tr on $A$. Another way to see it, especially in the case where there is a unique trace, is as an action of the reduced cyclic complex on the periodic cyclic complex. The cohomologous cocycle is now obtained as the image of the generator of a reduced cyclic homology complex under this Poincaré duality map. The following theorem 6.1.22 is then shown by explicit computation of this second cocycle.

THEOREM 1.4.1 (Algebraic Index Theorem). *Suppose $e, f \in M_N(\mathbb{A}_\hbar(M))$, for some $N > 0$, are idempotents such that $e - f \in M_N(\mathbb{A}_{\hbar c}(M))$. Then we have*

$$Tr_N(e-f) = \int_M \mathcal{E}(ch\,(\sigma(e)) - ch\,(\sigma(f)))\left(\hat{A}(T_{\mathbb{C}}M)e^{\theta}\right),$$

*where $\mathcal{E}\colon CC^{per}_\bullet(C^\infty(M)) \to \Omega^\bullet(M)$ denotes the Connes-Hochschild-Kostant-Rosenberg map* [**25, 95**]*, $Tr_N$ denotes the composition of the unique normalized trace on $\mathbb{A}_\hbar(M)$ (see proposition 6.1.20) with the matrix trace, $\sigma$ denotes the map given by setting $\hbar = 0$ and finally $ch$ denotes the Chern-Connes map defined in definition 6.1.17.*

In this thesis we will consider another proof of the algebraic index theorem, however. This proof, see [**14, 15**] and [**89**], uses the ideas of Gelfand's formal geometry, see chapter 2, more seriously. We will develop the framework of formal geometry in chapter 2 and the deformed version in section 4.1. Using this it can be shown that the algebraic index theorem can already be proved in the formal neighborhood, leading to the universal algebraic index theorem 6.1.12. The algebraic index theorem then follows straightforwardly from the application of the Gelfand-Fuks maps constructed through the framework of formal geometry, see definitions 2.3.9, 4.1.7 and 6.2.9.

## 1.5. Objectives and Attribution

In sections 1.1, 1.2, 1.3 and 1.4 we have introduced the concept of deformation quantization and index theorems. These are the main objects of study in this thesis. Let us use this section to elaborate on what the objectives of (the research behind) this thesis are. As expected, we have tried to add to the already vast pool of knowledge around the topics mentioned above. Thus, we will also give an impression of the work (that we are aware of) that has been carried out in the direction of these objectives.

The objectives can be divided into three main projects, namely:

(1) derivation of an algebraic index theorem for crossed products of a deformation quantization with a group;
(2) classification of the actions of (discrete) groups on deformation quantizations;
(3) concretely realizing the Fedosov construction in Gelfand's framework of formal geometry.

These three objectives are heavily related to each other, of course. The main objective is objective (1), one might even say it is the only objective and the other two are simply prerequisites. The relation between objectives (1) and (2) comes from the simple fact that, in order to understand the crossed product of a group with a deformation quantization algebra, one should first understand the action of the group on the algebra to some extent. The classification of such actions is a consequence that is of independent interest. The relation between objectives (1) and (3) comes from the proofs of the algebraic index theorem given in [**86**] and [**14**], in both proofs certain essential ideas from Gelfand's formal geometry are used, although it may be hard to realize. On the other hand it can be hard to conceptualize the ideas behind the Fedosov construction [**46**]. One way to do this is by considering it in the framework of formal geometry and, while many people must certainly be aware of this [**21, 8, 86**], a straightforward treatment of the relation between the Fedosov construction and formal geometry seems to be missing.

In this section we shall say a little bit about the three objectives above and how they relate to the established body of research. We shall conclude this section with an account of precisely which parts of the thesis can be attributed to the author and which parts of the thesis are partly or wholly the work of others.

**1.5.1. Objective (3).** We shall start our discussion of the objectives of this thesis in reverse order. We do this since, as was mentioned, the ordering above is by reverse dependence. Of the three objectives, it is perhaps hardest to determine the degree of success in the case of objective (3). This is because objective (3) is essentially about supplying an adequate conceptualization of the Fedosov construction. Thus the degree of success depends greatly on the reader. The objective is therefore, mainly, to present the Fedosov construction in a way that connects well with the readers whose frame of mind agrees with Gelfands framework of formal geometry. As mentioned, however, we will also explicitly use the realization of the Fedosov construction in the framework of formal geometry in the pursuit of the other two objectives.

The framework of formal geometry is a name that we have given to a set of ideas proposed by Gelfand at the 1970 ICM in Nice [**51, 52**] A rigorous formulation and treatment of these ideas is hard to find, as mentioned for instance in [**8**], even though the content seems to be well-known among a large group of mathematicians [**86, 8, 21, 71**]. In [**8**] the authors present a (brief) treatment of the theory, since they use the theory in that paper, they mention that they were introduced to the ideas at B. Feigin's Moscow seminar. We have similarly been introduced to the topic through private discussions with the author's advisor R. Nest and (as of yet) unpublished notes written by R. Nest. Therefore, we have taken the opportunity to relay a written account of the main aspects of the framework of formal geometry in chapter 2.

The framework of formal geometry consists of two main parts. First, one considers the geometry of the formal neighborhood of a point in some higher dimensional space (also called the formal polydisc [**8**]). Secondly, one finds a way of globalizing the results of the first step, i.e. "gluing together the formal neighborhoods". We shall perform the first step in the deformed case in chapter 3 by studying the formal Moyal–Weyl algebra. This algebra appears in all the sources considering the Fedosov construction [**114, 89, 46**] in the symplectic setting, although there are essentially two ways to construct it. Our discussion in chapter 3 is mainly based on the treatment of the Moyal–Weyl algebras in [**14**] and [**89**].

Objective (3) is carried out in chapter 4. There we consider the second part of the framework of formal geometry in the deformed case. We note that it is expected that the framework generalizes, partly because of the ideas of non-commutative geometry. Namely, one expects the deformation quantization to behave algebraically like a manifold, except that it is not commutative. In chapter 4 we first show how the process of globalization applied in the undeformed case can be translated to the deformed case and yields Fedosov connections [**114, 89**], this is done in section 4.1. Secondly, in section 4.2, we present the Fedosov construction as carried by B. Fedosov in [**48**]. We conclude by considering the characteristic class of the deformation quantization in section 4.3.

We should note that, as mentioned, the ideas of formal geometry can be found, in varying degrees, in many works on deformation quantization, notably [**71, 32, 8, 86, 46**]. We would like to mention the paper [**21**] specifically. In this paper the authors consider an analog of Kontsevich's construction of $\star$-products on Poisson manifolds by means of a Fedosov-like construction using the framework of formal geometry.

**1.5.2. Objective (2).** Objective (2) arose from the need to understand the notion of group actions on deformation quantizations well. The topic of group actions on deformation quantizations is historically well-studied [**65, 7**]. Especially in the case of Lie groups, where one tries to generalize the notion of momentum map [**41**] and perform symplectic reductions [**81, 41**]. We shall only seriously consider actions by discrete groups, although some results do apply to the Lie group setting as well.

When one considers the action of a group on a deformation quantization algebra it is split essentially into a deformed part and an undeformed part. The action will induce an action on the underlying Poisson manifold. So one would like to know how much of the action is encoded by this induced action. By classification we mean classification of the different group actions on the deformation quantization that induce the same action on the manifold. Thus, we reduce the problem to considering actions by Poisson maps and existence of a lift of such actions to the deformation quantization. In chapter 5 we consider first the question of existence of lifts and proceed to find a classification of the actions inducing a fixed (liftable) action on the underlying (symplectic) manifold. We phrase the question of classification of group actions completely in the setting of the Fedosov construction and the framework of formal geometry. Although we should take care to be clear about the generality of the obtained results.

We should note that a lot of work had already been carried out towards objective (2). Notably, one can consider the (somehow reversed) question of invariance of $\star$-products. In other words, one can try to parametrize those $\star$-products that are invariant under the action on the underlying manifold. In the article [**7**] a complete parametrization of such $\star$-products up to equivariant equivalence is provided. In the, more recent, work [**98**] these results were extended to include a notion of quantum momentum map. Another result of interest is the construction of an equivariant formality map in [**32**], which works for arbitrary Poisson manifolds.

The main difference between the treatment of group actions in this thesis and the notion of invariant $\star$-products is that we allow the action to differ in higher orders of $\hbar$ from the action on the manifold. We do this to create an independence of the treatment from the choice of Fedosov connection realizing the deformation quantization. This has the benefit that in our treatment of the equivariant index theorem, in section 6.2, we will not have to impose any restrictions on the action or the deformation quantization.

**1.5.3. Objective (1).** The main objective of this thesis is to explain and prove an equivariant version of the algebraic index theorem 6.2.23. By equivariant we mean that we consider the action of a discrete (countable) group $\Gamma$ on a deformation quantization $\mathbb{A}_\hbar(M)$ and subsequently let the role of the deformation quantization in the algebraic index theorem 6.1.22 be played by the crossed product $\mathbb{A}_\hbar(M) \rtimes \Gamma$, see definition A.3.1. This corresponds roughly to replacing the underlying manifold by the Borel construction $M \times_\Gamma E\Gamma$. We call this algebraic index theorem for crossed products an *equivariant* algebraic index theorem since the periodic cyclic cohomology of the crossed product is identified as the equivariant version [**113, 62, 67**] when one compares with K-theory (or rather KK-theory).

The crossed product is a well-known non-commutative analog of the quotient [**26**]. Suppose a group $\Gamma$ acts on a manifold $M$ by diffeomorphisms and suppose the quotient $M/\Gamma$ remains a smooth manifold, eg. the group acts freely and properly. In that case the crossed product $C_0(M) \rtimes \Gamma$ is Morita equivalent to the quotient $C_0(M/\Gamma) = C_0(M)^\Gamma$ [**99**]. In general, however, we can not be assured that $M/\Gamma$ is a smooth manifold and the functions $C_0(M)^\Gamma$ do not accurately describe the (non-commutative) geometry of the quotient. It turns out that the crossed-product still does. Note also that we can sometimes realize the crossed product as the convolution algebra on the action groupoid $M \rtimes \Gamma$.

Since the algebraic index theorem is essentially a statement about periodic cyclic cocycles, the hypothesis naturally arises that there exists a formula, similar to the one in theorem 6.1.22, for the crossed product algebra $\mathbb{A}_\hbar(M) \rtimes \Gamma$. A consequence of the Fedosov construction of deformation quantizations [**48, 89**] is that there exists a quasi-isomorphism from the deformation quantization $\mathbb{A}_\hbar(M)$ to the complex of differential forms with values in the bundle with fibers given by the formal Moyal–Weyl algebra equipped with the Fedosov connection, see proposition 4.1.9. This can be combined with the well-known fact that the Borel equivariant cohomology $\mathrm{H}^\bullet_\Gamma(M)$ includes in the periodic cyclic cohomology of the crossed product $C^\infty(M) \rtimes \Gamma$ [**23, 26**]. Thus we try to define a map from the crossed product $\mathbb{A}_\hbar(M) \rtimes \Gamma$ to a model for differential forms on the Borel construction with values in the formal Moyal–Weyl algebra equipped with an equivariant version of the Fedosov connection. This process is executed in section 6.2.

There have been many generalizations and adaptations since the conception of the algebraic index theorems. Notably there is a version that can be applied to Poisson manifolds [**33**]. There are versions tailored to symplectic Lie algebroids [**89**], complex manifolds [**14**] and gerbes [**15**]. Recently there were adaptations to more general Lie algebroids [**9**]. The adaptation to the case of symplectic Lie groupoids and orbifolds carried out in the papers [**97**] and [**96**] are of particular interest to us. Such Lie groupoids can arise from actions on deformation quantizations like the ones that we consider.

**1.5.4. Attribution.** As is to be expected, the material in this thesis is not exclusively attributed to the author. In this section we will try to outline, chapter by chapter, to what degree the material is completely original research and to what degree it should be attributed to other sources.

None of the material in the introduction can be attributed in any way to the author. The material in section 1.1 is mostly based on (the first two chapters of) the book [**107**] by L. Takhtajan. The material in section 1.2 is mostly common knowledge in the field of deformation quantization. Particularly good surveys of this "common knowledge" can be found in [**65**] and [**42**]. The same goes for the material in section 1.3, we refer the interested readers to the excellent article [**72**]. A particularly relevant reference, that we have used extensively, is the book [**48**] by B. Fedosov. The material in section 1.4 is mostly based on the articles [**86**], [**14**] and the book [**48**].

As mentioned earlier, the material in chapter 2 is common knowledge for a large group of mathematicians, although finding a comprehensive source is hard. The presentation in this thesis is inspired nearly exclusively by private discussions with and unpublished notes of the author's advisor R. Nest. Most of the proofs were reproduced independently, however.

The material in chapter 3 is well-known as well. Section 3.1 is based in particular on the articles [**89**] and [**14**]. The results of section 3.2 were rederived independently, although some of the results already appear in the sources mentioned above. The results of section 3.3 are based on the article [**14**]. The proofs of section 3.1 and 3.2 were rederived independently, while the proofs in section 3.3 are directly based on the proofs of [**14**].

The material in chapter 4 is the deformed version of the material in chapter 2. The material in section 4.1 is based on the article [**86**] and to a small degree on [**89**]. The material in section 4.2, including proofs, is simply a rephrasing of sections 5.1 and 5.2 of [**48**] and is included for the convenience of the reader. The material in section 4.3 is to some extent a comparison of section 4.1 and 4.2, as such that material is based on [**86**], [**89**] and [**48**].

The material in chapter 5 is based on the preprint [**69**], written by the author. The material in sections 5.1 and 5.2 was essentially known, but was rephrased in a way that fits well into the general structure of this thesis. The results of section 5.2 were rederived independently. The results of section 5.3 were obtained completely independently and are, as far as the author is aware, original. Of course we mean to exclude supporting lemmas, like lemmas 5.3.23 and 5.3.16, from this statement.

Section 6.1 is an adaptation of the proofs of the universal algebraic index theorem and algebraic index theorem in [**14**] and [**15**]. Section 6.2 on the other hand is based on work carried out in collaboration with A. Gorokhovsky and R. Nest. A preprint should appear in the near future, possibly adapted from the exposition given in this thesis. Accordingly the results of this section are original and the proofs were obtained in collaboration with A. Gorokhovsky and R. Nest.

The material contained in appendix A is, of course, well-known and not at all original. Section A.1 is based for the most part on the appendix of the book [**79**], although the proofs were rederived independently. Section A.2 consists of various standard definitions found for instance in [**116**] or [**78**] for section A.2.1; in [**17**] and [**58**] for sections A.2.2 and A.3.2; in [**12**] for section A.2.3 and finally [**116**] for section A.2.4. The well-known proposition A.2.12 was relayed to the author by D. Sprehn in a private discussion and the proof of proposition A.3.4 is based on a similar proof in [**12**] or [**1**]. The content of section A.3 was relayed to the author through private communication with A. Gorokhovsky and is based further on the articles [**1**] and [**93**].

The material contained in appendix B is also well-known and not at all original. This material is based mostly on the lecture notes [**35**] and some general knowledge. The constructions and proofs, especially in the differential graded case, were rederived independently, however.

CHAPTER 2

# The Framework of Formal Geometry

In this chapter we will set up the framework of Gelfand-style formal geometry [**51, 52, 8, 21**]. This is a framework in which one first studies the geometry of the formal neighborhood of a point in some higher dimensional space and studies a manifold by its associated cover by formal neighborhoods. By "formal neighborhood" we mean the smallest neighborhood of the point that is large enough to exhibit the value of every derivative of a function at that point. Thus the algebra of functions on the formal neighborhood will be isomorphic to an algebra of formal power series, hence the name.

We will start by building up the framework in this sense. It can be generalized however, see [**5**], to consider foliated manifolds. This is done by considering the formal neighborhood of a point in a foliated higher dimensional space, see section 2.4. We will not present this generalization in detail, however. In the next sections we will present a generalization of the framework of formal geometry to the context of symplectic formal deformation quantization and show how this leads to a natural conceptualization of the Fedosov construction and the associated classification of symplectic formal deformation quantizations.

We choose to present the undeformed case for the trivial foliation first, since it is most easily understood, while the ideas and proofs carry over mutatis mutandi. The main result and the main application of the framework in this thesis will be the construction of the *Gelfand-Fuks* maps

$$GF_M\colon C^\bullet_{Lie}(\mathbb{W}_n,\mathfrak{h}_n;\mathbb{L}) \longrightarrow \Omega^\bullet(M;\mathbb{L})$$

from the relative Lie algebra cohomology of a certain (fixed) pair of Lie algebras $\mathfrak{h}_n \subset \mathbb{W}_n$ with values in a module $\mathbb{L}$ to the de Rham cohomology of the $n$-dimensional smooth manifold $M$ with values in a vector bundle, with fibers given by $\mathbb{L}$, associated to the frames bundle of $M$.

Remark 2.0.1. It is helpful to note the similarity to the case of homogeneous spaces. In this setting the Lie algebra $\mathbb{W}_n$ corresponds to the Lie algebra of the Lie group $G$, $\mathfrak{h}_n$ corresponds to the Lie algebra of the subgroup $H \subset G$ and $M$ would be the homogeneous space $G/H$. In fact we will see that the Gelfand-Fuks map arises from a principal bundle $\widetilde{M} \to M$ that only fails to be a Lie group if there exists no "global translation" on $M$. More explicitly the principal bundle $\widetilde{M}$ fails to be a Lie group if certain lowest degree vectors in $\mathbb{W}_n$ cannot be integrated globally.

Remark 2.0.2. When considering (differential) geometric properties of a manifold $M$, it is often useful to split up these properties in terms of local data and global data. Čech cohomology, subordinate to a good cover, offers a tidy way to encode this, for an example see the proof of 5.3.22. A "good cover" is usually defined as a cover by open subsets that are diffeomorphic to $\mathbb{R}^n$/contractible and such that all non-empty (iterated) intersections of these subsets are also diffeomorphic to $\mathbb{R}^n$/contractible. Of course, for certain problems, different covers may be considered "good". It is often useful to consider the framework of formal geometry as a way to make sense of the "cover by formal neighborhoods".

The chapter is set up as follows. In section 2.1 we will define and analyze the $k$th order neighborhoods of a point in Euclidean space. This will naturally lead to the definition of the jet bundles of coordinate charts and eventually the manifold of *non-linear frames* in the section 2.2. In section 2.3 we will show that this manifold carries a natural Maurer-Cartan type connection and we will show how this allows us to define the Gelfand-Fuks maps. Finally, we shall provide some rudimentary examples of the application of the framework in section 2.4. The main example is of course the application in the deformed case, which will be carried out in the subsequent chapters.

## 2.1. $k$th Order Neighborhoods

The notion of a $k$th order neighborhood of a point is well-known in algebraic geometry. For instance, the dual numbers over some field $L$ form a 1st order or infinitesimal neighborhood of a point in the corresponding affine line. In differential geometry the concept seems to be a little less well-known. As we will see, it can still be quite useful however. In the spirit of non-commutative and algebraic geometry we define these $k$th order neighborhoods in terms of their algebras of functions. Thus, we will study first the algebra of $k$-jets at a point of a manifold. This is an algebra which encodes the "up to order $k$" behaviour of the manifold at a certain point. In particular, we will show that the $k$-jets at a point are classified, up to algebra isomorphism, by dimension of the manifold. We will also determine the automorphisms and derivations of these algebras in explicit terms.

DEFINITION 2.1.1. We define the algebra of $k$-jets at $m \in M$ as

$$\mathcal{A}_{m,k} := C^\infty(M) \Big/ (\operatorname{Ker} ev_m)^{k+1}$$

where $ev_m$ denotes the evaluation at $m \in M$. We shall also denote

$$\mathbb{A}_k := C^\infty(\mathbb{R}^n) \Big/ (\operatorname{Ker} ev_0)^{k+1} = \mathcal{A}_{0,k}.$$

REMARK 2.1.2. Note that we abuse notation by denoting $\mathbb{A}_k$ instead of $\mathbb{A}_{n,k}$. In the following section we fix the dimension $n$. Many objects should therefore be considered as implicitly carrying a subscript $n$, which may appear if the need for such clarity arises.

NOTATION 2.1.3. We will denote the quotient maps by

$$J^k_m \colon C^\infty(M) \longrightarrow \mathcal{A}_{m,k}.$$

NOTATION 2.1.4. We will often use the symbol $\alpha$ to denote multi-indices, i.e. $\alpha \in (\mathbb{Z}_{\geq 0})^n$. When we do this, we denote

$$|\alpha| := \sum_{i=1}^n \alpha_i, \quad x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}, \quad \alpha! := \prod_{i=1}^n \alpha_i! \quad \text{and} \quad \partial_x^\alpha := \prod_{i=1}^n \partial_{x_i}^{\alpha_i},$$

where $\partial_{x_i}$ denotes the operator of partial derivative in the direction $x_i$.

The notion of jets of functions is intimately tied up with the notion of differential operators. In fact, we shall see that the "bundle of $\infty$-jets" (a limit of a bundle with fibers given by the $k$-jets defined above) is naturally identified as the $C^\infty(M)$-dual of differential operators on $M$. So, let us give a particularly useful definition of differential operators. It is left to the reader to consolidate this definition with the definition the reader is familiar with.

NOTATION 2.1.5. Denote by $\mathcal{X}_M$ the sheaf of Lie algebras on $M$ given by the smooth vector fields with the Lie bracket given by Lie derivative, i.e. $[X,Y] = \mathcal{L}_X Y$. Note that $\mathcal{X}_M$ is a module over the sheaf of smooth functions $C^\infty_M$. By abuse of notation, we will often write $a \in \mathcal{F}$ to mean a section $a$ in the sheaf $\mathcal{F}$. By this we simply mean $a \in \mathcal{F}(U)$ for some/any open set $U$. As above, we shall denote in particular $C^\infty_n := C^\infty_{\mathbb{R}^n}$ and $\mathcal{X}_n := \mathcal{X}_{\mathbb{R}^n}$.

DEFINITION 2.1.6. Let $\mathrm{Op}_M$ denote the sheaf of differential operators on $M$. It is given by the following construction. Let $\mathcal{T}(\mathcal{X}_M)$ denote the sheaf which associates the tensor algebra generated by $C^\infty_M(U)$ and $\mathcal{X}_M(U)$ to an open $U$, i.e. it is the free $\mathbb{R}$-algebra on those generators. Then $\mathrm{Op}_M$ is given by taking the quotient of $\mathcal{T}(\mathcal{X}_M)$ by the ideal $\mathcal{I}_M$ generated by the elements

$$X \otimes Y - Y \otimes X - [X,Y], \quad X \otimes fY - fX \otimes Y - X(f)Y \quad \text{and} \quad f \otimes X - fX$$

for all $X, Y \in \mathcal{X}_M$ and $f \in C^\infty_M$ (the last relation is also included for $X \in C^\infty_M$). We consider the sections of $\mathcal{T}(\mathcal{X}_M)$ as a graded algebra by assigning $C^\infty_M$ the degree 0 and $\mathcal{X}_M$ the degree 1. We denote the sections of degree $k$ by $\mathcal{T}(\mathcal{X}_M)_k$. This grading induces a filtration on $\mathrm{Op}_M$. We will denote the elements of $\mathrm{Op}_M$ of degree lower than $p \in \mathbb{Z}$ by $F_p\mathrm{Op}_M$, i.e. we have

$$0 = F_{-1}\mathrm{Op}_M \subset F_0\mathrm{Op}_M \subset \ldots \subset F_p\mathrm{Op}_M \subset F_{p+1}\mathrm{Op}_M \subset \ldots,$$

where $F_p\mathrm{Op}_M$ is the image of $\bigoplus_{k=0}^{p} \mathcal{T}(\mathcal{X}_M)_k \subset \mathcal{T}(\mathcal{X}_M)$ under the quotient map. Note that $\mathrm{Op}_M$ is a $C^\infty_M$ module for the module structure induced from $\mathcal{T}(\mathcal{X}_M)$. As above, we shall denote in particular $\mathrm{Op}_n := \mathrm{Op}_{\mathbb{R}^n}$.

REMARK 2.1.7. Note that every $D \in \mathrm{Op}_M$ defines a linear operator on $C^\infty_M$ by

$$(X_1 \otimes \ldots \otimes X_p)f = X_1 \left(X_2 \left(\ldots \left(X_p(f)\right)\right)\right),$$

where $X_i \in \mathcal{X}_M$ acts on functions by the usual [**111**] identification with derivations and $X_i \in C^\infty_M$ acts by multiplication.

LEMMA 2.1.8. *Suppose $f \in C^\infty_n$, then $f \in (\mathit{Ker}\, ev_0)^{k+1}$ iff $Df(0) = 0$ for all differential operators $D \in F_k\mathit{Op}_n$.*

PROOF.
Suppose $f \in (\mathrm{Ker}\, ev_0)^{k+1}$, then

$$f = \sum_{i_1,\ldots,i_{k+1}} f_{i_1} f_{i_2} \ldots f_{i_{k+1}}$$

for some $f_{i_j} \in \mathrm{Ker}\, ev_0$. Now suppose $D \in F_k\mathrm{Op}_n$, then, by repeated application of the product rule, we see that $Df$ will be a sum of terms of the form

$$(D_1 f_{i_1})(D_2 f_{i_2}) \ldots (D_{k+1} f_{i_{k+1}}),$$

for some differential operators $D_j$ such that $D_1 D_2 \ldots D_{k+1} \in F_k\mathrm{Op}_n$. Thus we see that there is at least one $0 \le j \le k+1$ such that $D_j \in F_0\mathrm{Op}_n (= C^\infty_n)$. Thus $Df(0) = 0$ since $D_j f_{i_j}(0) = D_j(0) f_{i_j}(0) = 0$, since $f_{i_j} \in \mathrm{Ker}\, ev_0$.

Conversely, suppose $Df(0) = 0$ for all $D \in F_k\mathrm{Op}_n$. Keeping this in mind, Taylor's theorem [**111**] says that we have

$$f(x) = \sum_{|\alpha|=k} R_\alpha(x) x^\alpha$$

for all $x \in \mathbb{R}^n$. Here the $R_\alpha$ are smooth functions such that $R_\alpha(0) = 0$. Thus we see that indeed $f \in (\mathrm{Ker}\, ev_0)^{k+1}$. □

COROLLARY 2.1.9. *The algebras $\mathcal{A}_{m,k}$ are classified, up to isomorphism, by the dimension of $M$.*

PROOF. This follows since all $n$-dimensional manifolds are locally diffeomorphic to $\mathbb{R}^n$ by definition and differential operators are local [**111**]. □

PROPOSITION 2.1.10. *The algebra $\mathbb{A}_k$ is isomorphic to $\tilde{\mathbb{A}}_k := \mathbb{R}[x_1, \ldots, x_n] \big/ \langle x_1, \ldots, x_n \rangle^{k+1}$ for all $k \geq 0$.*

PROOF.
Consider the inclusion

$$i \colon \mathbb{R}[x_1, \ldots, x_n] \hookrightarrow C^\infty(\mathbb{R}^n)$$

given by the standard coordinates on $\mathbb{R}^n$. Let $\varphi_k := J^k_0 \circ i$, then, by lemma 2.1.8, we see that $\mathrm{Ker}\, \varphi_k = \langle x_1, \ldots, x_n \rangle^{k+1}$. So there is an induced map

$$\overline{\varphi}_k \colon \tilde{\mathbb{A}}_k \longrightarrow \mathbb{A}_k.$$

On the other hand we can consider the map

$$e_k \colon C^\infty(\mathbb{R}^n) \longrightarrow \tilde{\mathbb{A}}_k$$

given by

$$f \mapsto \left[ \sum_{|\alpha| \le k} \frac{1}{\alpha!} \left(\partial_x^\alpha f\right)(0) x^\alpha \right].$$

Again by 2.1.8, we find that $\operatorname{Ker} e_k = (\operatorname{Ker} ev_0)^{k+1}$. So we have the induced map $\overline{e}_k$. Note that $\overline{e}_k = \overline{\varphi}_k^{-1}$ follows from the definitions. □

From now on we will implicitly equate $\tilde{\mathbb{A}}_k$ and $\mathbb{A}_k$ (using the isomorphism from proposition 2.1.10).

Proposition 2.1.11. *The $k$-th jet group $\overline{G}_k := Aut(\mathbb{A}_k)$ is isomorphic to $\mathbb{R}^{N_k} \rtimes GL(n, \mathbb{R})$ as a Lie group, for some $N_k \in \mathbb{N}$ and some Lie group structure on $\mathbb{R}^{N_k}$.*

Proof.
Note that giving an algebra endomorphism (a unital one) of $\mathbb{A}_k$ is equivalent to giving the images $\varphi_1, \ldots, \varphi_n$ of the generators $x_1, \ldots, x_n$. So suppose $\varphi$ is the endomorphism given by

$$\varphi_p = \sum_{|\alpha| \le k} a_{p,\alpha} x^\alpha,$$

with $a_{p,\alpha} \in \mathbb{R}$ for all $1 \le p \le n$ and $\alpha \in (\mathbb{Z}_{\ge 0})^n$.

If the $\varphi_p$ give a well-defined algebra endomorphism $\varphi$, we have

$$0 = \varphi(x_p^{k+1}) = \varphi_p^{k+1} = a_{p,(0,\ldots,0)}^{k+1} + S,$$

where $S \in \langle x_1, \ldots, x_n \rangle$. This implies that $a_{p,(0,\ldots,0)} = 0$. Suppose, on the other hand, that the $a_{p,(0,\ldots,0)}$ vanish for all $1 \le p \le n$ and denote by $\overline{\varphi}$ the map

$$\overline{\varphi} \colon \mathbb{R}[x_1, \ldots, x_n] \longrightarrow \mathbb{A}_k$$

given by extending $\overline{\varphi}(x_p) = \varphi_p$ as (unital) algebra homomorphism. One verifies that

$$\langle x_1, \ldots, x_k \rangle^{k+1} \subset \operatorname{Ker} \overline{\varphi}$$

and thus $\overline{\varphi}$ induces an algebra endomorphism of $\mathbb{A}_k$. This endomorphism is exactly given by $\varphi$. So we see that the $\varphi_p$ provide a well-defined algebra endomorphism $\varphi$ if and only if $a_{p,(0,\ldots,0)} = 0$ for all $p$.

So consider the algebra endomorphism $\varphi$ given by

$$\varphi_p = \sum_{0 < |\alpha| \le k} a_{p,\alpha} x^\alpha.$$

Then the condition for $\varphi$ to be (invertible) an automorphism is exactly that the matrix $\overline{\Phi}$ with entries $\Phi_{ij}$ given by the coefficient of $x_i$ in $\varphi_j$ is invertible. This is apparent since constructing the inverse is an unobstructed process once one knows that $\overline{\Phi}$ has an inverse. Note that the corresponding map $\overline{G_k} \longrightarrow \mathrm{GL}(n, \mathbb{R})$, which maps $\varphi$ to $\overline{\Phi}$, is a group homomorphism. Moreover, given an invertible matrix $(a_{ij})_{i,j=1}^n \in \mathrm{GL}(n, \mathbb{R})$, the polynomials $\varphi_j = \sum_{j=1}^n a_{ij} x_i$ define an automorphism $A$ of $\mathbb{A}_k$ and $A \in \overline{G}_k$ maps to $(a_{ij})_{i,j=1}^n \in \mathrm{GL}(n, \mathbb{R})$. Note also that the map $\mathrm{GL}(n, \mathbb{R}) \to \overline{G}_k$, given by $(a_{ij})_{i,j=1}^n \mapsto A$, is a group homomorphism for all $k$. So we find that the exact sequences

$$0 \to K \longrightarrow \overline{G}_k \longrightarrow \mathrm{GL}(n, \mathbb{R}) \to 0$$

split for all $k$. Since there are no constraints on the coefficients $a_{p,\alpha}$ for $|\alpha| > 1$, we find that the kernel $K$ can be parametrized by $\mathbb{R}^{N_k}$. The induced group multiplication on $\mathbb{R}^{N_k}$ can be expressed in terms of polynomials and therefore it supplies $\mathbb{R}^{N_k}$ with the structure of a Lie group (in fact it supplies $\mathbb{R}^{N_k}$ with the structure of a smooth algebraic group). □

REMARK 2.1.12. Note that the proof of proposition 2.1.11 makes explicit the implied Lie group structure on $\mathbb{R}^{N_k}$. In fact it also makes clear that

$$\frac{N_k}{n} = \sum_{i=2}^{k} \binom{n+i-1}{n-1}.$$

This is because the $a_{p,\alpha}$ with $|\alpha| > 1$ form a coordinate system. For each $1 \leq p \leq n$ there are as many $a_{p,\alpha}$ as there are elements $\alpha \in (\mathbb{Z}_{\geq 0})^n$ such that $1 < |\alpha| < k+1$. This number is in turn equal to the sum over $1 < i < k+1$ of the number of weak compositions $c_n(i)$ of $i$ into exactly $n$ parts. The formula for $N_k$ now follows from the fact that $c_n(i) = \binom{n+i-1}{n-1}$.

DEFINITION 2.1.13. The maps

$$J^k \colon \mathrm{Diff}(\mathbb{R}^n, 0)^{op} \longrightarrow \overline{G}_k,$$

where $\mathrm{Diff}(\mathbb{R}^n, 0)$ denotes the group of those diffeomorphisms $\Phi$ of $\mathbb{R}^n$ such that $\Phi(0) = 0$, are defined by the requirement that the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, "\gamma^*"] \arrow[d] & C^\infty(\mathbb{R}^n) \arrow[d] \\ \mathbb{A}_k \arrow[r, "J^k\gamma"'] & \mathbb{A}_k \end{tikzcd}$$

commutes for any diffeomorphism $\gamma \in \mathrm{Diff}(\mathbb{R}^n, 0)$, i.e. $J^k\gamma \circ J^k_0 = J^k_0 \circ \gamma^*$.

PROPOSITION 2.1.14. *The Lie algebras $\overline{\mathbb{W}}_k := Der(\mathbb{A}_k)$ of derivations are given by*

$$\overline{\mathbb{W}}_k = \left\{ \sum_{i=1}^{n} \hat{P}_i \partial_{x_i} \mid \hat{P}_i \in Ker\, ev_0 \subset \mathbb{A}_k \right\}$$

*where we view $\hat{P}_i \partial_{x_i}$ as the composition of the operator $\partial_{x_i}$ given by*

$$\sum_\alpha a_\alpha x^\alpha \mapsto \sum_\alpha a_\alpha \alpha_i x^{\tilde{\alpha}},$$

*where $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n)$ and the second sum runs over only those $\alpha \in (\mathbb{Z}_{\geq 0})^n$ such that $\alpha_i > 0$ and $|\alpha| < k+1$, and the operator of multiplication by $\hat{P}_i$.*

PROOF.
As was the case for endomorphisms, giving the derivation $D$ is equivalent to giving the images $D_p$ of the generators $x_p$. Again the polynomials

$$D_p = \sum_{|\alpha| \leq k} a_{p,\alpha} x^\alpha$$

yield a well-defined derivation if and only if $a_{p,(0,\ldots,0)} = 0$ for all $p$. This is since, on the one hand,

$$0 = D(x_p^{k+1}) = (k+1)x_p^k D_p = (k+1)x_p^k a_{p,(0,\ldots,0)} + x_p^k S$$

where $S \in \langle x_1, \ldots, x_n \rangle$, which implies $a_{p,(0,\ldots,0)} = 0$. On the other hand, suppose we have that $a_{p,(0,\ldots,0)} = 0$, then consider the map $\overline{D} \colon \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{A}_k$ given by composing the derivation of $\mathbb{R}[x_1, \ldots, x_n]$ determined by the polynomials $D_p$ with the projection to $\mathbb{A}_k$. Then we see that $\langle x_1, \ldots, x_n \rangle^{k+1} \subset \mathrm{Ker}\, \overline{D}$ and thus $\overline{D}$ induces an operator $\mathbb{A}_k \to \mathbb{A}_k$. This operator is exactly $D$ and so we see that the polynomials $D_p$ yield a well-defined derivation. Now note that clearly $D = \sum_{p=1}^n D_p \partial_{x_p}$ and the condition that $a_{p,(0,\ldots,0)} = 0$ simply means that $D_p \in \mathrm{Ker}\, ev_0$. $\square$

## 2.2. Jet Bundles and the Formal Neighborhood

In this section we will define the bundles of $k$-jets associated to an $n$-dimensional manifold $M$. These will be principal bundles in which each point is given by the $k$-jet of a coordinate system centered at a point $m \in M$. We will also give a description of the manifold structure of the bundles of $k$-jets. One then naturally considers the bundle of non-linear frames (the $\infty$-jet bundle). This will also lead to the definition of the algebra of $\infty$-jets at a point, i.e. the formal neighborhood of a point. In this section we will also describe certain automorphisms and derivations of the formal neighborhood, i.e. formal coordinate changes fixing a point and formal vector fields.

Definition 2.2.1. Let $j^k(M)$ denote the $k$th jet manifold of $M$ defined by

$$j^k(M) := \left\{ \varphi_{m,k} \colon \mathcal{A}_{m,k} \xrightarrow{\sim} \mathbb{A}_k \right\}.$$

We equip $j^k(M)$ with the smooth structure obtained from noting that it is a $\overline{G}_k$-principal bundle over $M$. Let us describe the smooth structure more explicitly. First consider the maps

$$P_k \colon j^k(M) \longrightarrow M$$

given by $\varphi_{m,k} \mapsto m$. We will obtain the smooth structure on $j^k(M)$ by providing the bijections

$$P_k^{-1}(U) \longrightarrow U \times \overline{G}_k$$

for coordinate charts $U \subset M$ and showing that they are smoothly compatible for the smooth structure on $\overline{G}_k$ given in proposition 2.1.11.

Notation 2.2.2. For each $y \in \mathbb{R}^n$ we will denote by $T_y$ the diffeomorphism of $\mathbb{R}^n$ given by translation by $y$, i.e. $T_y(x) = x + y$.

Suppose

$$\psi \colon \mathbb{R}^n \xrightarrow{\sim} U \subset M$$

is a coordinate chart. Let

$$\psi_k \colon \mathbb{R}^n \times \overline{G}_k \longrightarrow P_k^{-1}(U) \tag{2.2.1}$$

be given by

$$\psi_k(x, \chi) = \chi \circ (\psi \circ T_x)^* \colon \mathcal{A}_{\psi(x),k} \longrightarrow \mathbb{A}_k.$$

Note that this is well-defined since, if $f \in \operatorname{Ker} ev_{\psi(x)}$, then $f \circ \psi \circ T_x \in \operatorname{Ker} ev_0$. The image is in $j^k(M)$, since $T_x$ is a diffeomorphism while $\psi$ is a local diffeomorphism. Also, since $\psi$ is a local diffeomorphism, we can consider the inverse given by

$$\varphi_{m,k} \mapsto \left(\psi^{-1}(m), \varphi_{m,k} \circ (T_{-\psi^{-1}(m)} \circ \psi^{-1})^*\right) \tag{2.2.2}$$

where one checks that the map is well-defined, as before. By showing that the maps $\psi_k$ are smoothly compatible we will have defined a smooth structure on the jet manifolds $j^k(M)$. So suppose we have two coordinate charts $\psi_U \colon \mathbb{R}^n \to U \subset M$ and $\psi_V \colon \mathbb{R}^n \to V \subset M$ on $M$. We note that the images of $\psi_{U,k}$ and $\psi_{V,k}$ are disjoint if and only if $U \cap V = \emptyset$. So let us assume that $U \cap V \neq \emptyset$. We find that

$$\psi_{V,k}^{-1} \circ \psi_{U,k} \colon \psi_U^{-1}(U \cap V) \times \overline{G}_k \longrightarrow \psi_V^{-1}(U \cap V) \times \overline{G}_k$$

is given by

$$(x, \chi) \mapsto \left(\psi_V^{-1} \circ \psi_U(x), \chi \circ (\psi_U \circ T_x)^* \circ (T_{-\psi_V^{-1}\psi_U(x)} \circ \psi_V^{-1})^*\right).$$

Now it is checked using proposition 2.1.11 that smoothness of $\psi_V^{-1} \circ \psi_U$ implies smoothness of $\psi_{V,k}^{-1} \circ \psi_{U,k}$.

Remark 2.2.3. Note that $\overline{G}_1 \simeq \mathrm{GL}(n, \mathbb{R})$ by proposition 2.1.11, since $N_1 = 0$. One verifies that, as expected, $j^1(M) \to M$ is isomorphic to the general linear frame bundle of $M$. In fact, it is easily verified that $j^k(M)$ is a $\overline{G}_k$-principal bundle over $M$ for all $k \geq 0$ and a $\overline{G}_k/\mathrm{GL}(n, \mathbb{R})$-principal bundle over $j^1(M)$ for $k \geq 1$. Note also that by proposition 2.1.11 the Lie groups $\overline{G}_k/\mathrm{GL}(n,\mathbb{R}) \simeq \mathbb{R}^{N_k}$ are contractible and so for $k \geq 1$ we find that

$$j^k(M) \simeq j^1(M) \times \mathbb{R}^{N_k}$$

as bundles over $j^1(M)$.

Definition 2.2.4. We define the *bundle of non-linear frames in* $M$, denoted $\widetilde{M}$, by

$$\widetilde{M} := j^\infty(M) := \varprojlim j^k(M)$$

where the inverse system is given by the maps $j^k(M) \to j^{k-1}(M)$, mapping $\varphi_{m,k}$ to $\varphi_{m,k-1}$, where $\varphi_{m,k-1}$ is the unique map that makes the diagram

$$\begin{tikzcd}
\mathcal{A}_{m,k} \arrow[r, "\varphi_{m,k}"] \arrow[d] & \mathbb{A}_k \arrow[d] \\
\mathcal{A}_{m,k-1} \arrow[r, "\varphi_{m,k-1}"'] & \mathbb{A}_{k-1}
\end{tikzcd}$$

commute. Here the vertical arrows are given by the induced quotient maps. The uniqueness, and the fact that this yields an automorphism, derive from the fact that any isomorphism $\mathcal{A}_{m,k} \to \mathbb{A}_k$ must map the unique maximal ideal $\mathrm{Ker}\, ev_m$ to the unique maximal ideal $\mathrm{Ker}\, ev_0$.

Notation 2.2.5. We denote the *algebra of* $\infty$*-jets at* $m \in M$, defined as $\varprojlim \mathcal{A}_{m,k}$, by $\hat{\mathcal{A}}_m$. We shall also denote the limit of the maps $J^k_m$ over $k$ by

$$J^\infty_m \colon C^\infty(M) \longrightarrow \hat{\mathcal{A}}_m.$$

We shall denote in particular

$$\hat{\mathbb{A}} := \varprojlim \mathbb{A}_k = \mathbb{R}[\![x_1, \ldots, x_n]\!].$$

Note that $\hat{\mathcal{A}}_m$ is the algebra of functions on the formal neighborhood of $m \in M$.

Remark 2.2.6. The algebras $\hat{\mathcal{A}}_m$ are equipped with the $\mathrm{Ker}\, ev_m$-adic topology. This topology arises naturally from their definition as a limit. Note that any continuous isomorphism

$$\hat{\mathcal{A}}_m \xrightarrow{\sim} \hat{\mathbb{A}} := \varprojlim \mathbb{A}_k = \mathbb{R}[\![x_1, \ldots, x_n]\!]$$

is given by a compatible sequence of isomorphisms $\mathcal{A}_{m,k} \to \mathbb{A}_k$ and thus we find the equivalent description

$$\widetilde{M} = \left\{ \varphi_m \colon \hat{\mathcal{A}}_m \xrightarrow{\sim} \hat{\mathbb{A}} \right\}.$$

Remark 2.2.7. Since $\widetilde{M}$ is given by the sequence $\left(j^k(M)\right)_{k\geq 0}$ of smooth manifolds it has the structure of a pro-finite dimensional manifold. In the following we will consider several differential geometric objects associated to $\widetilde{M}$. These will always be defined as the appropriate limits of the corresponding differential geometric objects on the $j^k(M)$, for example $C^\infty\left(\widetilde{M}\right) = \varinjlim C^\infty(j^k(M))$. In particular we have

$$T_{\varphi_m}\widetilde{M} := \varprojlim T_{\varphi_{m,k}} M.$$

Note that, by definition of the limit topology, giving a path in $\widetilde{M}$ is equivalent to giving compatible paths in all the $j^k(M)$ and so

$$T_{\varphi_m}\widetilde{M} \simeq \left\{ \gamma \colon ]-1,1[ \longrightarrow \widetilde{M} \mid \gamma(0) = \varphi_m \right\} / \sim. \tag{2.2.3}$$

Here the equivalence relation $\sim$ is given by

$$\gamma \sim \gamma' \quad \text{iff} \quad \partial_s f \circ \gamma|_{s=0} = \partial_s f \circ \gamma'|_{s=0} \quad \text{for all } f \in C^\infty\left(\widetilde{M}\right).$$

Note that $f \in C^\infty\left(\widetilde{M}\right)$ is an equivalence class of functions $j^k(M) \to \mathbb{R}$ for the relation that $f_k \sim f_l$ if $p_{k,l}^* f_l = f_k$, where $p_{k,l} \colon j^k(M) \to j^l(M)$ for $k \geq l$ are the maps in the inverse system. Note that, since the system is sequential and terminates at $j^0(M) = M$, we can always find the unique representative $f_k \colon j^k(M) \to \mathbb{R}$ for $f \in C^\infty\left(\widetilde{M}\right)$ such that there is no $l < k$ with $p_{k,l}^* f_l = f_k$. Let us also denote the maps $\widetilde{M} \to j^k(M)$ by $\pi_k$ (they are the $p_{\infty,k}$ if you will). Now we see that the defining equation for $\sim$ simply says that $\gamma \sim \gamma'$ if for all $f \in C^\infty\left(\widetilde{M}\right)$ we have that

$$\partial_s(f_k \circ \pi_k \circ \gamma)|_{s=0} = \partial_s(f_k \circ \pi_k \circ \gamma')|_{s=0}.$$

Other geometric objects we will encounter are for instance $T\widetilde{M}$ and $\Omega^\bullet\left(\widetilde{M}\right)$ the definitions of which should now be reasonably clear. A nice characterization of a smooth map of pro-finite dimensional manifolds $\widetilde{M} \to \widetilde{N}$ is that it is a map which induces an algebra homomorphism $C^\infty\left(\widetilde{N}\right) \to C^\infty\left(\widetilde{M}\right)$.

Definition 2.2.8. We define the pro-finite dimensional Lie group $\widehat{G}$ of *$\infty$-jets of diffeomorphisms of $\mathbb{R}^n$ fixing $0 \in \mathbb{R}^n$* by

$$\widehat{G} := \varprojlim \overline{G}_k.$$

Remark 2.2.9. Note that, since the maps $\overline{G}_k \to \overline{G}_l$ for $k \geq l$ in the limit above are Lie group homomorphisms, we find a group structure on $\widehat{G}$ and in fact the multiplication and inverse are smooth maps in the sense of pro-finite dimensional manifolds. Note that by the same reasoning as remark 2.2.6 we find that $\widehat{G}$ is the group $\mathrm{Aut}^0\hat{\mathbb{A}}$ of continuous algebra automorphisms of $\hat{\mathbb{A}}$.

For any manifold $M$ the group $\widehat{G}$ acts from the left on $\widetilde{M}$ by post-composition. One verifies that this action is smooth, free, preserves the fibers of the map

$$P_\infty \colon \widetilde{M} \longrightarrow M$$

which sends $\varphi_m \colon \hat{\mathcal{A}}_m \to \hat{\mathbb{A}}$ to $m \in M$ and acts transitively on these fibers. Note also that since the $P_k \colon j^k(M) \to M$ are all fiber bundles the same is true for $P_\infty$. In other words $P_\infty$ is a $\widehat{G}$-principal bundle.

Remark 2.2.10. Note that the action of $\widehat{G}$ on $\widetilde{M}$ and in fact the actions of the $\overline{G}_k$ on the $j^k(M)$ are from the left. It is more usual in the literature to require principal bundles to carry right actions. Note however that in the usual construction of the general linear frame bundle $j^1(M)$ the action of $\mathrm{GL}(n, \mathbb{R})$ is directly on the vector space. In our case, we consider the induced action, *by pull-back*, on the 1-jets. This explains why we also obtain actions on the left instead of actions on the right in our definition of the principal bundles $j^k(M)$ and $\widetilde{M}$.

Notation 2.2.11. As before, we denote the limit of the system of maps $J^k \colon Diff(\mathbb{R}^n, 0)^{op} \to \overline{G}_k$ by

$$J^\infty \colon Diff(\mathbb{R}^n, 0)^{op} \longrightarrow \widehat{G}.$$

Note that, given a local diffeomorphism $\Phi \colon (\mathbb{R}^n, 0) \to (M, m)$, we get similarly a continuous isomorphism

$$J_m^\infty \Phi \colon \hat{\mathcal{A}}_m \longrightarrow \hat{\mathbb{A}}.$$

So let us denote also

$$J_m^\infty \colon Diff((\mathbb{R}^n, 0), (M, m)) \longrightarrow Iso^0(\hat{\mathcal{A}}_m, \hat{\mathbb{A}}),$$

where $Iso^0$ denotes the set of continuous isomorphisms.

As for $\widetilde{M}$ above, we have that $T_{\mathrm{Id}}\widehat{G} = \varprojlim T_{\mathrm{Id}}\overline{G}_k$, which is a limit of the Lie algebras $\overline{\mathbb{W}}_k$ corresponding to the Lie groups $\overline{G}_k$. Let us determine in rather clear terms what this Lie algebra is.

DEFINITION 2.2.12. We define $\mathbb{W}$ to be the Lie algebra of continuous derivations of $\hat{\mathbb{A}}$

$$\mathbb{W} := \mathrm{Der}^0(\hat{\mathbb{A}}).$$

The degree filtration

$$\hat{\mathbb{A}} = F_0\hat{\mathbb{A}} \supset F_1\hat{\mathbb{A}} \supset \ldots \supset F_k\hat{\mathbb{A}} \supset F_{k+1}\hat{\mathbb{A}} \supset \ldots$$

where $F_k\hat{\mathbb{A}} := \langle x_1, \ldots, x_n\rangle^k$, induces the filtration

$$\mathbb{W} = F_{-1}\mathbb{W} \supset F_0\mathbb{W} \supset F_1\mathbb{W} \supset \ldots \supset F_p\mathbb{W} \supset F_{p+1}\mathbb{W} \supset \ldots$$

where

$$F_p\mathbb{W} := \left\{D \in \mathrm{Der}^0(\hat{\mathbb{A}}) \mid D(F_k\hat{\mathbb{A}}) \subset F_{k+p}\hat{\mathbb{A}} \ \ \forall k \geq 0\right\}.$$

REMARK 2.2.13. Note that the filtration on $\hat{\mathbb{A}}$ also induces the filtration

$$\widehat{G} = G_0 \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_k \rhd \ldots$$

given by

$$G_k := \left\{\varphi \in \widehat{G} \mid \varphi = \mathrm{Id} \mod (\mathrm{Ker}\, ev_0)^{k+1}\right\}.$$

Note that all the embeddings are normal and $\overline{G}_k \simeq G_0/G_k$.

It is actually quite easy to be more explicit about the Lie algebra $\mathbb{W}$ and provide a presentation of it. Note that, since $\mathbb{W}$ consists of continuous derivations, giving $D \in \mathbb{W}$ is equivalent to giving the formal power series $D(x_i)$ for all $i$. In this case there are no conditions at all, so we find that

$$\mathbb{W} \simeq \left\{\sum_{i=1}^n P_i\partial_{x_i} \mid P_i \in \hat{\mathbb{A}}\right\}$$

where $\partial_{x_i}$ denotes the continuous extension of partial derivative from polynomials to formal power series. We will therefore also call $\mathbb{W}$ the Lie algebra of *formal vector fields*. From this description it is clear that the filtration $F_\bullet\mathbb{W}$ actually derives from a grading given by the linear isomorphism $\mathbb{W} \simeq \hat{\mathbb{A}}^n$ mapping $\partial_{x_i}$ to 1 in the $i$th copy of $\hat{\mathbb{A}}$. Namely by shifting this grading by 1 (since $\partial_{x_i}$ should be of degree $-1$). We will denote the space of degree $p$ elements by $\mathbb{W}_p$. So we find that

$$\mathbb{W} = \prod_{p\geq -1} \mathbb{W}_p.$$

NOTATION 2.2.14. We will denote the *formal vector fields vanishing at* 0 by

$$\overline{\mathbb{W}} := \prod_{k\geq 0} \mathbb{W}_k.$$

We will denote the induced filtration by $F_\bullet\overline{\mathbb{W}}$ and the induced grading by $\overline{\mathbb{W}}_p$.

Consider the map

$$\mathfrak{gl}(n,\mathbb{R}) \longrightarrow \overline{\mathbb{W}}$$

given by $(a_{ij})_{i,j=1}^n \mapsto \sum_{i,j=1}^n a_{ij}x_i\partial_j$. Note that it provides the identification $\mathfrak{gl}(n,\mathbb{R}) \simeq \overline{\mathbb{W}}_0$. Thus we find that

$$\overline{\mathbb{W}} = (F_1\overline{\mathbb{W}}) \rtimes \mathfrak{gl}(n,\mathbb{R}).$$

PROPOSITION 2.2.15. *We have*

$$\overline{\mathbb{W}}_k \simeq (F_1\overline{\mathbb{W}}/F_k\overline{\mathbb{W}}) \rtimes \mathfrak{gl}(n,\mathbb{R})$$

*where* $\overline{\mathbb{W}}_k$ *denotes the Lie algebra of* $\overline{G}_k$.

PROOF.
It follows from the explicit description of $\mathbb{W}$, the proposition 2.1.11 and the remark 2.2.13. $\square$

Note that, since $F_1\overline{\mathbb{W}} = \varprojlim F_1\overline{\mathbb{W}}/F_k\overline{\mathbb{W}}$, we have deduced that

$$T_{\mathrm{Id}}\widehat{G} = \varprojlim \overline{\mathbb{W}}_k \simeq F_1\overline{\mathbb{W}} \rtimes \mathfrak{gl}(n, R) \simeq \overline{\mathbb{W}}.$$

REMARK 2.2.16. Note that any continuous linear endomorphism of $\hat{\mathbb{A}}$ is obtained as a limit of a compatible system of linear endomorphisms $\mathbb{A}_k \to \mathbb{A}_k$. Similarly, as noted above, any continuous algebra automorphism of $\hat{\mathbb{A}}$ is obtained as a limit of a compatible system of linear automorphisms $\mathbb{A}_k \to \mathbb{A}_k$. However, the limits of compatible systems of derivations $\mathbb{A}_k \to \mathbb{A}_k$ will only recover $\overline{\mathbb{W}} \subset \mathbb{W}$ by proposition 2.1.14. So we see that the derivations $\partial_{x_i}$ are limits of compatible systems of linear endomorphisms which are not derivations.

## 2.3. Gelfand-Fuks Map

In this section we will construct the Gelfand-Fuks map mentioned in the introduction to this chapter. This will also make the analogy with Čech cohomology and the idea of the "cover by formal neighborhoods" clear. To do this we will first construct a certain connection one-form on the bundle of non-linear frames of a manifold. The pull-back of this one-form will yield both the "Čech complex" and the Gelfand-Fuks map.

We will need the following well-known theorem [**85**].

THEOREM 2.3.1 (Borel). *The map*

$$J_m^\infty : C^\infty(M) \longrightarrow \hat{\mathcal{A}}_m$$

*is surjective.*

COROLLARY 2.3.2. *Any continuous isomorphism*

$$\varphi_m : \hat{\mathcal{A}}_m \longrightarrow \hat{\mathbb{A}}$$

*is induced by a local diffeomorphism* $\Phi\colon (\mathbb{R}^n, 0) \to (M, m)$*, i.e.* $\varphi_m = J_m^\infty \Phi$.

PROOF.
It follows from theorem 2.3.1, since, if $f_1, \dots, f_n \in C^\infty(M)$ are such that $J_m^\infty f_i = \varphi_m^{-1}(x_i)$, then they form a local coordinate system by the inverse function theorem.
$\square$

We obtain the following well-known theorem [**5, 71, 21**] providing an action of $\mathbb{W}$ on $\widetilde{M}$.

THEOREM 2.3.3. *There is a natural isomorphism*

$$\omega_M(\varphi_m) : T_{\varphi_m}\widetilde{M} \longrightarrow \mathbb{W}$$

*for all* $\varphi_m \in \widetilde{M}$*. The induced map* $\omega_M$ *defines a one-form in* $\Omega^1\left(\widetilde{M}\right) \otimes \mathbb{W}$ *satisfying*

$$d\omega_M + \frac{1}{2}[\omega_M, \omega_M] = 0 \tag{2.3.1}$$

*where* $d$ *denotes the exterior derivative.*

PROOF.
Given $X \in T_{\varphi_m}\widetilde{M}$ we determine $\omega_M(\varphi_m)(X)$ by providing its action on $\hat{\mathbb{A}}$. Recall (2.2.3) in remark 2.2.7 and let $X = \frac{d}{dt}\gamma(t)|_{t=0}$ for some $\gamma\colon ]-1, 1[ \to \widetilde{M}$ such that $\gamma(0) = \varphi_m$. By corollary 2.3.2 we can pick a coordinate system $\psi_U : (\mathbb{R}^n, 0) \xrightarrow{\sim} (U, m) \subset (M, m)$ such that $J_m^\infty \psi_U = \varphi_m$. Then the system of diffeomorphisms $\psi_{U,k}$, given by (2.2.1) for $\psi = \psi_U$, yields the diffeomorphism

$$\psi_{U,\infty} : \mathbb{R}^n \times \widehat{G} \xrightarrow{\sim} P_\infty^{-1}(U).$$

Note that we may assume without loss of generality that the image of $\gamma$ is contained in $P_\infty^{-1}(U)$. Let us denote $m_{\gamma,U}(t) = pr_1\left(\psi_{U,\infty}^{-1}(\gamma(t))\right)$ and $\varphi_{\gamma,U}(t) = pr_2\left(\psi_{U,\infty}^{-1}(\gamma(t))\right)$. Then we set

$$\omega_M(\varphi_m)(X)(\hat{f}) = -\frac{d}{dt}\left(\varphi_{\gamma,U}(t) \circ J_0^\infty \circ T^*_{m_{\gamma,U}(t)}(f)\right)\Big|_{t=0}$$

for any $\hat{f} \in \hat{\mathbb{A}}$, here $f \in C^\infty(\mathbb{R}^n)$ is such that $J_0^\infty f = \hat{f}$. To make sense of the differentiation note that by proposition 2.1.10 we have

$$\varphi_{\gamma,U}(t) \circ J_0^\infty \circ T^*_{m_{\gamma,U}(t)}(f) = \sum_\alpha \frac{1}{\alpha!}\left(\partial_x^\alpha f\right)(m_{\gamma,U}(t))\varphi_{\gamma,U}(t)(x^\alpha), \tag{2.3.2}$$

where we recall that any automorphism of $\hat{\mathbb{A}}$ is given by the images of the $x_i$'s. Now the differentiation in $t$ simply refers to differentiation in the coefficients which are smooth in $t$ since $\gamma$ is smooth in $t$. Note that, if $\omega_M$ is well-defined, it is automatically natural. One needs to check that:

(1) $\omega_M(\varphi_m)(X)$ is a derivation for any $X \in T_{\varphi_m}\widetilde{M}$,
(2) $\omega_M(\varphi_m)$ is a linear isomorphism from $T_{\varphi_m}\widetilde{M}$ to $\mathbb{W}$,
(3) $\omega_M(\varphi_m)(X)(\hat{f})$ does not depend on the choice of $f$,
(4) $\omega_M(\varphi_m)(X)$ does not depend on the choice of $\gamma$,
(5) $\omega_M(\varphi_m)(X)$ does not depend on the choice of $\psi_U$,
(6) $\omega_M(\varphi_m)$ depends smoothly on $\varphi_m$ and
(7) $\omega_M$ satisfies the Maurer-Cartan equation (2.3.1).

1) This follows directly from the fact that $\varphi_{\gamma,U}(t) \circ J_0^\infty \circ T^*_{m_{\gamma,U}(t)}$ is a composition of algebra homomorphisms for all $t$, while $\varphi_{\gamma,U}(0) = \mathrm{Id}$ and $T^*_{m_{\gamma,U}(0)} = \mathrm{Id}$.

2) Let us unravel the description of $\omega_M(\varphi_m)$ above. The description can be given in two steps. First we identify $T_{\varphi_m}\widetilde{M}$ with $T_{(0,\mathrm{Id})}\left(\mathbb{R}^n \times \widehat{G}\right)$ by means of the local diffeomorphism $\psi_{U,\infty}$. Secondly, we identify $T_{(0,\mathrm{Id})}\left(\mathbb{R}^n \times \widehat{G}\right)$ with $\mathbb{W}$ by noting that

$$T_{(0,\mathrm{Id})}\left(\mathbb{R}^n \times \widehat{G}\right) = T_0\mathbb{R}^n \oplus T_{\mathrm{Id}}\widehat{G} \simeq \mathbb{W}_{-1} \oplus \overline{\mathbb{W}} = \mathbb{W}.$$

Note that both these identifications are by means of linear isomorphisms.

3) Note that, by linearity, it is enough to show that

$$\frac{d}{dt}\left(\varphi_{\gamma,U}(t) \circ J_0^\infty \circ T^*_{m_{\gamma,U}(t)}(e)\right)\Big|_{t=0} = 0 \tag{2.3.3}$$

for any $e$ such that $J_0^\infty e = 0$. We have $\frac{d}{dt}(\partial_x^\alpha e)(m_{\gamma,U}(t))|_{t=0} = V(\partial_x^\alpha e)$ where $V$ is the derivation at 0 given by $\frac{d}{dt}m_{\gamma,U}(t)|_{t=0} \in T_0\mathbb{R}^n$ for any $\alpha \in \mathbb{Z}^n_{\geq 0}$. Thus, since $\partial_x^\beta e(0) = 0$ for all $\beta \in \mathbb{Z}^n_{\geq 0}$, we find that $\frac{d}{dt}(\partial_x^\alpha e)(m_{\gamma,U}(t))|_{t=0} = 0$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$. So, we see that (2.3.3) follows from (2.3.2) and the product rule.

4) Recall the unraveling of the definition of $\omega_M$ at 2) above. This clearly also implies that $\omega_M(\varphi_m)(X)$ only depends on the value of $\gamma$ at 0 and the first derivative of $\gamma$ at 0. In other words it only depends on $X$ and not the choice of $\gamma$.

5) Suppose that $\psi_V : (\mathbb{R}^n, 0) \xrightarrow{\sim} (V, m) \subset (M, m)$ is another coordinate neighborhood such that $J_m^\infty \psi_V = \varphi_m$. Note that, since we can shorten the path $\gamma$ arbitrarily without loss of generality, we can assume that $V = U$ without loss of generality. Note that the formula (2.2.2) means that

$$-\omega_M(\varphi_m)(X)(\hat{f}) = \frac{d}{dt}\left(\gamma(t) \circ (\psi_U^{-1})^* \circ T^*_{-m_{\gamma,U}(t)} \circ J_0^\infty \circ T^*_{m_{\gamma,U}(t)} f\right)\Big|_{t=0},$$

which yields explicitly

$$-\sum_{\alpha} \frac{1}{\alpha!} \frac{d}{dt} \left( \left( \partial_x^{\alpha} f \right) (m_{\gamma,U}(t)) \gamma(t) \circ (\psi_U^{-1})^* \left( x - m_{\gamma,U}(t) \right) \right)^{\alpha} \Big|_{t=0}.$$

Then we see from this, the chain rule and the hypothesis that $J_m^\infty \psi_V = J_m^\infty \psi_U$ that $\omega_M(\varphi_m)(X)$ does not depend on the choice of $\psi_U$.

6) Note first that smoothness of $\omega_M$ for any manifold $M$ follows from smoothness of $\omega_{\widetilde{\mathbb{R}^n}}$ by naturality. In other words, we may show the smoothness locally. Also, we may identify $\widetilde{\mathbb{R}^n}$ with $\mathbb{R}^n \times \widehat{G}$ by using the map $\mathrm{Id}_\infty$. As explained at 2) $\omega_M(\varphi_m)$ is essentially given by the differential of the map $\psi_{U,\infty}^{-1}$. Going through the identifications mentioned above, this means $\mathrm{Id}_\infty^* \omega_{\widetilde{\mathbb{R}^n}}(x,\varphi)$ is given by the differential of the map

$$(y,\psi) \mapsto \left( \Phi^{-1}(y-x), \psi \circ T^*_{(y-x)} \circ (\Phi^{-1})^* \circ T^*_{-\Phi^{-1}(y-x)} \right)$$

where $J_0^\infty \Phi = \varphi$. Note that this differential will depend smoothly on $(x,\varphi)$.

7) Using the well-known formula for the exterior derivative we can rewrite the Maurer-Cartan equation as

$$\mathcal{L}_X(\omega_M(Y)) - \mathcal{L}_Y(\omega_M(X)) - \omega_M(\mathcal{L}_X(Y)) + [\omega_M(X), \omega_M(Y)]$$

where $\mathcal{L}_X$ denotes the Lie derivative along $X$ and the bracket is the bracket of $\mathbb{W}$. This equation can be checked explicitly by using the expression of Lie derivative in terms of local flows and the definition of $\omega_M$.

□

Remark 2.3.4. The one-form defined in the theorem 2.3.3 is sometimes called the Kazdan connection on $M$. Note that, given any $\mathbb{W}$-module $V$, we find the cochain complex $\left(\Omega^\bullet\left(\widetilde{M}\right) \otimes V, d + \omega_M \wedge\right)$ by the Maurer-Cartan equation for $\omega_M$. It is easily verified that this also yields the map

$$(C^\bullet_{Lie}(\mathbb{W};V), \partial_{Lie}) \longrightarrow \left(\Omega^\bullet\left(\widetilde{M};V\right), d + \omega_M \wedge\right)$$

from the Gelfand-Fuks cohomology complex of $\mathbb{W}$ with values in $V$, see remark A.2.33, given by

$$(X_1 \wedge \ldots \wedge X_p \mapsto \chi(X_1,\ldots,X_p)) \mapsto ((\varphi_m, Y_1 \wedge \ldots \wedge Y_p) \mapsto \chi(\omega_M(\varphi_m)(Y_1), \ldots, \omega_M(\varphi_m)(Y_p)).$$

The Gelfand-Fuks map mentioned earlier is supposed to land in the differential forms on $M$ however. We will need some extra definitions.

Definition 2.3.5. Suppose $\mathbb{L}$ is a $\mathrm{GL}(n,\mathbb{R})$-module. Then we denote the vector bundle associated to the general linear frames bundle $j^1(M) \to M$ by $\mathbb{L}_M := j^1(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{L} \to M$. We define the differential forms on $M$ with values in $\mathbb{L}_M$, denoted $\Omega^\bullet(M;\mathbb{L})$, by

$$\Omega^\bullet(M;\mathbb{L}) := \left\{ \eta \in \left(\Omega^\bullet\left(j^1(M)\right) \otimes \mathbb{L}\right)^{\mathrm{GL}(n,\mathbb{R})} \mid \iota_X \eta = 0 \ \forall X \in \mathfrak{gl}(n,\mathbb{R}) \right\}.$$

Here the superscript $\mathrm{GL}(n,\mathbb{R})$ refers to taking the invariants with respect to the action given by $\alpha \otimes v \mapsto g^*\alpha \otimes g^{-1}v$ for $g \in \mathrm{GL}(n,\mathbb{R})$, the $\iota_X$ denotes contraction with the vector field $X$, $\mathfrak{gl}(n,\mathbb{R})$ denotes the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ and associated to the vector $X \in \mathfrak{gl}(n,\mathbb{R})$ we consider the vector field $T_{\mathrm{Id}} R_x(X)$ also denoted by $X$, here $R_x \colon \mathrm{GL}(n,\mathbb{R}) \to j^1(M)$ is given by $R_x(g) = gx$.

Remark 2.3.6. The requirements that elements in $\Omega^\bullet\left(j^1(M)\right) \otimes \mathbb{L}$ need to satisfy in order to be differential forms on $M$ with values in $\mathbb{L}$ ensure that they actually only depend on data supported on the manifold $M$ in the following way. Suppose $X_1,\ldots,X_p$ are vector fields on $M$ and suppose $\tilde{X}_1,\ldots,\tilde{X}_p$ and $\tilde{X}'_1,\ldots,\tilde{X}'_p$ are two sets of lifts along the projection map $P_1 \colon j^1(M) \to M$. Then we find that, for all $\eta \in \Omega^p(M;\mathbb{L})$, we have that $\eta(\tilde{X}_1,\ldots,\tilde{X}_p) = \eta\left(\tilde{X}'_1,\ldots,\tilde{X}'_p\right)$, since the differences $\tilde{X}_i - \tilde{X}'_i$ lie in the kernel of $TP_1$ and $\eta$ vanishes on this kernel. Suppose $m \in M$ and $f_m$ and $f'_m$ are two lifts of $m$ along $P_1$. then there is a unique $g \in \mathrm{GL}(n,\mathbb{R})$ such that $f_m = gf'_m$ and, since $\eta$ is invariant, we find that $\eta(X_1,\dots,X_p)(f'_m) = g(\eta(X_1,\dots,X_p)(f_m))$. One can prove that the definition above agrees with the definition $\Omega^\bullet(M;\mathbb{L}) = \Gamma\left(\left(\bigwedge^\bullet T^*M\right)\otimes \mathbb{L}_M\right)$, the definition 2.3.5 will be more convenient for our purposes however.

LEMMA 2.3.7. *There exists a $GL(n,\mathbb{R})$-equivariant section*

$$F\colon j^1(M) \longrightarrow \widetilde{M}$$

*of the obvious map $\pi_1\colon \widetilde{M}\to j^1(M)$.*

PROOF.
The map $\pi_1\colon \widetilde{M}\to j^1(M)$ is a principal fiber bundle with the structure group $G_1$. By proposition 2.2.15 we find that $G_1 = \exp F_1\overline{\mathbb{W}}$ and so by pro-nilpotence of $F_1\overline{\mathbb{W}}$ we find that $G_1$ is diffeomorphic to a pro-finite dimensional vector space and thus contractible. This means that $\widetilde{M}$ trivializes over $j^1(M)$. The $\mathrm{GL}(n,\mathbb{R})$-equivariance now follows from the decomposition $\widehat{G}\simeq G_1\rtimes \mathrm{GL}(n,\mathbb{R})$ given by proposition 2.1.11. □

Note that the section $F$ is not at all unique. Let us simply fix such a $\mathrm{GL}(n,\mathbb{R})$-equivariant section $F$. Now we can pull-back the one-form $\omega_M$ by $F$ to obtain

$$O_F := F^*\omega_M \in \Omega^1(j^1(M))\otimes\mathbb{W}.$$

Note that $O_F$ also satisfies the Maurer-Cartan equation (2.3.1) simply because $\omega_M$ does.

DEFINITION 2.3.8. We define a $(\mathbb{W},\mathrm{GL}(n,\mathbb{R}))$-module $\mathbb{L}$ as a $\mathbb{W}$-module such that the induced action of $\mathfrak{gl}(n,\mathbb{R})$ integrates to an action of $\mathrm{GL}(n,\mathbb{R})$.

It is straightforward to check that, by definition of $\omega_M$ and equivariance of $F$, we have that, if $\eta\in\Omega^p(M;\mathbb{L})$ for some $(\mathbb{W},\mathrm{GL}(n,\mathbb{R}))$-module $\mathbb{L}$, then $d\eta + O_F\wedge\eta$ is a differential form on $M$ (of degree $p+1$) with values in $\mathbb{L}_M$. Of course we mean the form given by

$$X_1\wedge\ldots\wedge X_{p+1}\mapsto \frac{1}{p!}\sum_{\tau\in S_{p+1}}\epsilon(\tau)O_F(X_{\tau(1)})\left(\eta(X_{\tau(2)},\ldots,X_{\tau(p+1)})\right)$$

by $O_F\wedge\eta$, here $S_{p+1}$ denotes the symmetric group in $p+1$ letters and $\epsilon(\tau)$ denotes the sign of $\tau$.

PROPOSITION 2.3.9. *Suppose $\mathbb{L}$ is a $(\mathbb{W}, GL(n,\mathbb{R}))$-module, then the map*

$$GF_M\colon \left(C^\bullet_{Lie}\left(\mathbb{W},\mathfrak{gl}(n,\mathbb{R});\mathbb{L}\right),\partial_{Lie}\right)\longrightarrow\left(\Omega^\bullet(M;\mathbb{L}),\nabla_F\right),$$

*where $\nabla_F = d + O_F\wedge$, given by*

$$GF_M(\chi)(X_1,\ldots,X_p)(f_m) = \chi(O_F(X_1)(f_m),\ldots,O_F(X_p)(f_m)),$$

*for $\chi\in C^p_{Lie}\left(\mathbb{W},\mathfrak{gl}(n,\mathbb{R});\mathbb{L}\right)$, $f_m\in j^1(M)$ and $X_1,\ldots,X_p$ vector fields on $j^1(M)$, is a well-defined map of complexes.*

PROOF.
$\nabla_F^2 = 0$ by the Maurer-Cartan equation. The rest of the proof follows by straightforward computation. □

REMARK 2.3.10. The examples in the next section will show in what sense the above complex of differential forms on $M$ with values in $\mathbb{L}_M$ can be thought of as the Čech complex subordinate to a cover by formal neighborhoods. Let us make some elementary observations about proposition 2.3.9 above. Note that the power of the proposition is that, although the map $GF_M$ and the complex of differential forms with values in $\mathbb{L}_M$ depend on $M$, the Lie algebra complex does not. It should be shown then that the map $GF_M$ is not trivial in general. One can think of the Lie algebra cohomology as the cohomology of the formal neighborhood of points in the manifold, the Gelfand-Fuks map is then a way of globalizing cohomology classes along the manifold.

REMARK 2.3.11. The connection $\nabla_F$ is sometimes called the *Grothendieck connection* [**8, 65, 89**]. When we consider the quintessential $(\mathbb{W}, \mathrm{GL}(n, \mathbb{R}))$-module $\hat{\mathbb{A}}$, as we will do in example 2.4.1 below, we find that the corresponding vector bundle $\hat{\mathbb{A}}_M$ is isomorphic (by use of the section $F$) to the bundle underlying the sheaf of $\infty$-jets $J_n^\infty$, see definition 3.1.1 and proposition 3.1.15. Under this isomorphism the connection $\nabla_F$ takes the more usual form of the Grothendieck connection $\nabla_G$ given by

$$(\nabla_G l)(X)(D) = X(l(D)) - l(XD)$$

for all $l \in J_n^\infty$, $D \in \mathrm{Op}_n$ and $X \in \mathcal{X}_n$.

## 2.4. Examples

Let us demonstrate that the framework was set up properly. We will do this by first showing that the complexes of differential forms with values in $(\mathbb{W}, \mathrm{GL}(n, \mathbb{R}))$-modules are actually quasi-isomorphic to well-known interesting objects. Secondly, we will show that the Gelfand-Fuks map is not trivial in general, by showing that certain Pontrjagin classes lie in the image. After this we will also say some words about the possible generalization of the framework to consider non-trivially foliated manifolds. The main example of a generalization, the fact that the Gelfand-Fuks map is not trivial and the applicability of the framework will however be the deformed case concerning deformation quantization. This will be developed in the subsequent sections.

EXAMPLE 2.4.1. Let us consider the quintessential $(\mathbb{W}, \mathrm{GL}(n, \mathbb{R}))$-module $\hat{\mathbb{A}}$. As mentioned earlier, we want to view $\left(\Omega^\bullet(M; \hat{\mathbb{A}}), \nabla_F\right)$ as a Čech cohomology complex subordinate to the cover by formal neighborhoods. The formal neighborhood of the point $m \in M$ is defined as the "manifold" with algebra of functions given by $\hat{\mathbb{A}}$. Thus the complex associated to the module $\hat{\mathbb{A}}$ has the smooth functions on $M$ as underlying sheaf. The Čech complex corresponding to the sheaf $C_M^\infty$ and a good cover is quasi-isomorphic to $C^\infty(M)$ with trivial differential. So the following result should not be surprising

PROPOSITION 2.4.2. *The map*

$$J_F^\infty \colon \left(C^\infty(M), 0\right) \longrightarrow \left(\Omega^\bullet\left(M; \hat{\mathbb{A}}\right), \nabla_F\right)$$

*given by $f \mapsto (p \mapsto F(p) J^\infty_{\pi(p)} f)$ for all $f \in C^\infty(M)$ and $p \in j^1(M)$ is a quasi-isomorphism of differential graded associative algebras. In other words $C^\infty(M) \simeq Ker \nabla_F$ as algebras and $\left(\Omega^\bullet\left(M; \hat{\mathbb{A}}\right), \nabla_F\right)$ is acyclic.*

PROOF.
First of all we may reduce this to a local computation since the sheaf (on $M$) of differential graded algebras

$$U \mapsto \left(\Omega^\bullet\left(U; \hat{\mathbb{A}}\right), \nabla_F\right)$$

admits a partition of unity. So let us consider the case where $M = \mathbb{R}^n$. Note that since the tangent bundle of $\mathbb{R}^n$ is trivial we find that

$$\Omega^\bullet(\mathbb{R}^n; \hat{\mathbb{A}}) \simeq \Omega^\bullet(\mathbb{R}^n) \otimes \hat{\mathbb{A}}.$$

So 0-forms are simply functions $\hat{f}(x, \hat{x}) = \sum_\alpha \hat{f}_\alpha(x) \hat{x}^\alpha$ in the variable $x$ on $\mathbb{R}^n$ and the formal variable $\hat{x}$, i.e. we will use hats to differentiate between actual and formal variables. In this case we can be very explicit about a choice of section $F$. Simply take the map

$$j^1(\mathbb{R}^n) \xrightarrow{Id_1^{-1}} \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R}) \xrightarrow{\iota} \mathbb{R}^n \times \widehat{G} \xrightarrow{Id_\infty} \widetilde{\mathbb{R}^n} \tag{2.4.1}$$

where $\iota$ denotes the inclusion. Thus we find, by definition of the form $\omega_{\mathbb{R}^n}$ and $F$, that

$$O_F = -\sum_{i=1}^{n} dx_i \otimes \partial_{\hat{x}_i}$$

where we write $\partial_{\hat{x}_i}$ for the derivations on $\hat{\mathbb{A}}$ given by $\partial_{\hat{x}_i}\hat{x}_j = \delta_{ij}$. Then we see that for a 0-form $\hat{f}$ to be in the kernel of $\nabla_F$ we must have that $\partial_{x_i}\hat{f} = \partial_{\hat{x}_i}\hat{f}$ for all $i$. In other words $\hat{f}(x,\hat{x}) = f(x+\hat{x})$ for some smooth function $f \in C^\infty(\mathbb{R}^n)$. This establishes the isomorphism given in the proposition.

It is left to show that the complex $\left(\Omega^\bullet\left(\mathbb{R}^n;\hat{\mathbb{A}}\right), \nabla_F\right)$ has no cohomology in higher degrees. To see this we note that we may view $\Omega^\bullet\left(\mathbb{R}^n;\hat{\mathbb{A}}\right)$ as the sections of the (trivial) bundle with fiber given by $\bigwedge^\bullet \mathbb{R}_n \otimes \widehat{\mathcal{S}}(\mathbb{R}_n)$ where $\widehat{\mathcal{S}}(\mathbb{R}_n)$ denotes the completion of the symmetric algebra on the dual $\mathbb{R}_n$ of $\mathbb{R}^n$ in the $\mathbb{R}_n$-adic topology. The formula for $\nabla_F$ above then makes clear that it simply acts as the dual of the Koszul differential on the symmetric algebra $\mathcal{S}(\mathbb{R}^n)$.

To summarize, the proposition follows from the calculation of cohomology of the double complex $\left(\check{C}^\bullet(\mathcal{U},\mathcal{F}^\bullet),\check{\partial},\nabla_F\right)$, where $\check{C}^\bullet$ refers to the Čech complex (for some good cover $\mathcal{U}$) and $\mathcal{F}^\bullet$ denotes the sheaf of graded vector spaces given by $\Omega^\bullet\left(-;\hat{\mathbb{A}}\right)$. This sheaf may be identified with the sections of a vector bundle with fibers given by the Koszul resolution associated to the general linear frames bundle (as mentioned in the previous paragraph). So, by existence of a partition of unity, the spectral sequence associated to the above double complex collapses on the first page to the Čech complex of the sheaf of smooth functions. By existence of a partition of unity this complex collapses to $C^\infty(M)$ on the second page. The explicit isomorphism mentioned in the proposition is obtained by paying attention to the explicit local definition of $\nabla_F$.

□

Example 2.4.3. The other obvious example of a $(\mathbb{W},\mathrm{GL}(n,\mathbb{R}))$-module is the trivial module $\mathbb{R}$. Since the action is trivial we have that $\nabla_F = d_{\mathrm{dR}}$ and so we simply find that $(\Omega^\bullet(M;\mathbb{R}),\nabla_F)$ is the usual de Rham complex of the manifold $M$. This means that by considering the trivial module $\mathbb{R}$ we can actually try to construct formal analogs of characteristic classes in the Lie algebra cohomology of $\mathbb{W}$ relative to $\mathfrak{gl}(n,\mathbb{R})$ with values in $\mathbb{R}$. For example, we can construct the following classes. Consider the $\mathrm{GL}(n,\mathbb{R})$-equivariant projection

$$\mathfrak{p}\colon \mathbb{W} \longrightarrow \mathfrak{gl}(n,\mathbb{R})$$

given by sending $\sum_{\alpha,j} P_{\alpha,j}x^\alpha\partial_{x_j}$ to the matrix with entries $P_{ij}$. Then we define the *curvature defined by* $\mathfrak{p}$ to be the map

$$R_{\mathfrak{p}}\colon \bigwedge^2 \mathbb{W} \longrightarrow \mathfrak{gl}(n,\mathbb{R})$$

given by $R_{\mathfrak{p}}(X,Y) = [\mathfrak{p}(X),\mathfrak{p}(Y)] - \mathfrak{p}([X,Y])$. Note that the map $R_{\mathfrak{p}}$ measures the failure of $\mathfrak{p}$ to be a Lie algebra homomorphism. Denote by $\mathcal{S}^p(\mathfrak{gl}_n^*)^{\mathrm{GL}(n,\mathbb{R})}$ the space of invariant symmetric multilinear functions on $\mathfrak{gl}(n,\mathbb{R})$ as usual in the Chern-Weil theory of characteristic classes [**38**]. Then we can define a map

$$CW_p\colon \mathcal{S}^p(\mathfrak{gl}_n^*)^{\mathrm{GL}(n,\mathbb{R})} \longrightarrow C^{2p}_{Lie}(\mathbb{W},\mathfrak{gl}(n,\mathbb{R});\mathbb{R})$$

by

$$\chi \mapsto (Y_1\wedge Z_1\wedge Y_2\wedge Z_2\wedge\ldots\wedge Y_p\wedge Z_p \mapsto \chi\left(R_{\mathfrak{p}}(Y_1,Z_1),\ldots,R_{\mathfrak{p}}(Y_p,Z_p)\right)).$$

Recall that a symmetric $p$-linear function $\chi$ is determined by the values $\chi(v,\ldots,v)$ of the corresponding symmetric polynomial in $p$ variables [**38**]. We define the degree $k$ symmetric invariant linear functions $P_{\frac{k}{2}}$ by the following formula

$$Det\left(\lambda I_n - \frac{A}{2\pi}\right) = \sum_{k=0}^{n} P_{\frac{k}{2}}(A,\ldots,A)\lambda^{n-k}$$

for all $A \in \mathfrak{gl}(n,\mathbb{R})$. Now, since $\mathfrak{p} \circ O_F$ defines a $\mathfrak{gl}(n,\mathbb{R})$-valued connection form on $M$, we see by the Chern-Weil theory of characteristic classes, that the classes of $GF(CW_k(P_{\frac{k}{2}}))$ in $H^{2k}_{\mathrm{dR}}(M)$ are exactly the Pontrjagin classes of the tangent bundle of $M$. This shows in particular that the Gelfand-Fuks map is not trivial in general.

REMARK 2.4.4. Note that example 2.4.3 also shows that we can view the framework of formal geometry as a way to obtain a generalized Chern-Weil theory of characteristic classes.

The theory developed above can be generalized in at least two notable ways. The case concerning deformation quantization will be discussed at length in the rest of this thesis, but we would be remiss if we didn't also mention the case of a foliated manifold, see [**5**]. One can use the above theory to define characteristic (secondary) classes of a foliation. The recipe to obtain generalizations is in principle unchanged. The only thing that we need to do is to construct the corresponding foliated formal neighborhood, manifold of non-linear frames, Lie group of jets of local automorphisms and the corresponding Lie algebra.

So suppose we have a codimension $k$ foliation $\mathcal{F}$ of an $n+k$ dimensional manifold $M$. Then the manifold is locally $\mathbb{R}^n \times \mathbb{R}^k$ and we remember that the $\mathbb{R}^n$ are leafwise directions. Thus the formal neighborhood of 0 is given by $\hat{\mathbb{A}}_{n+k}$ plus a splitting, i.e.

$$C^\infty\widehat{(\mathbb{R}^n \times \mathbb{R}^k)}_0 \simeq \hat{\mathbb{A}}_n \otimes \hat{\mathbb{A}}_k \simeq \mathbb{R}[\![x_1,\dots,x_n,\lambda_1,\dots,\lambda_k]\!]$$

where the $\lambda_i$ correspond to transversal directions and the $x_i$ correspond to leafwise directions. The manifold $\widetilde{M}_{\mathcal{F}}$ of non-linear frames subordinate to the foliation is given by those non-linear frames that respect the foliation, i.e.

$$\widetilde{M}_{\mathcal{F}} := \left\{\varphi_m \colon \hat{\mathcal{A}}_m \xrightarrow{\sim} \hat{\mathbb{A}}_n \otimes \hat{\mathbb{A}}_k \mid \varphi_m\left(\hat{\mathcal{A}}_m^{\mathcal{F}}\right) \subset \mathbb{R} \otimes \hat{\mathbb{A}}_k \subset \hat{\mathbb{A}}_n \otimes \hat{\mathbb{A}}_k\right\}$$

where $\hat{\mathcal{A}}_m^{\mathcal{F}}$ is given by the subalgebra of $\infty$-jets of functions that are constant along leaves. The group $\widehat{G}$ is replaced by $\widehat{G}^{\mathcal{F}}$ in a similar fashion:

$$\widehat{G}^{\mathcal{F}} := \left\{g \in \widehat{G} \mid g(\lambda_i) \in \mathbb{R} \otimes \hat{\mathbb{A}}_k \subset \hat{\mathbb{A}}_n \otimes \hat{\mathbb{A}}_k\right\}.$$

Finally the Lie algebra $\mathbb{W}$ is replaced by the Lie algebra $\mathbb{W}^{\mathcal{F}}$, given as

$$\mathbb{W}^{\mathcal{F}} := \mathbb{W}_k \ltimes \mathbb{W}_n[\![\lambda_1,\dots,\lambda_k]\!]$$

with the obvious action of $\mathbb{W}_k$ on $\mathbb{W}_n[\![\lambda_1,\dots,\lambda_k]\!]$. Again one can construct a Kazdan connection and a section from the "$\mathcal{F}$-adapted" $\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{R})$-principal bundle. Since $\mathbb{W}_k$ is a quotient of $\mathbb{W}^{\mathcal{F}}$ we get a map in Lie algebra cohomology which yields the secondary characteristic classes of the foliation mentioned above. Note that the previous sections in this chapter simply correspond to the trivial foliation, which yields the usual characteristic classes.

CHAPTER 3

# The Formal Moyal–Weyl Algebra

In order to apply the framework of formal geometry developed above to deformation quantization we will need to go through the process of generalization exemplified by the case of a foliation at the end of the last section. In particular, we will need to find a deformation quantization of a formal neighborhood, its automorphisms and the Lie algebra of formal vector fields. In this chapter we shall describe these in quite a lot of detail. We do this since most of the results in this thesis will be obtained by careful globalization of the corresponding results for the formal neighborhood. It is also in this section that we see the great benefit of considering symplectic deformation quantizations. They are in some sense "flat" for two reasons:

- symplectic manifolds are always locally symplectomorphic to Euclidean space with the standard symplectic structure, by Darboux's theorem [**19**], and
- up to gauge equivalence the symplectic deformation quantizations of Euclidean space are classified by their dimension [**65**].

These two results mean that we can consider a constant local model of symplectic deformation quantizations and this is very useful when applying the framework of formal geometry.

We will consider the Moyal product (3.0.2), introduced by Groenewold in [**64**], as our constant local model for symplectic deformation quantization of the $2d$-dimensional Euclidean space $(\mathbb{R}^{2d}, \omega_{st})$, equipped with the standard symplectic structure. It is given by

$$(f \star g)(q,p) = \exp\left(\frac{i\hbar}{2}\sum_{i=1}^{d} \partial_{\xi_i}\partial_{y_i} - \partial_{\eta_i}\partial_{x_i}\right) f(x,\xi)g(y,\eta)\bigg|_{\substack{x=y=q,\\ \xi=\eta=p}}, \tag{3.0.2}$$

where we have denoted by $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$, and similar for $y, \eta$ and $q, p$, the standard (Darboux) coordinates on $\mathbb{R}^{2d}$. We will call the corresponding algebra of $\infty$-jets at 0: *the formal Moyal–Weyl algebra*. It is the starting point of Fedosov's approach to deformation quantization of symplectic manifolds [**46**].

We will use this chapter to give three explicit constructions of the formal Moyal–Weyl algebra, analyze its continuous derivations and automorphisms and present computations of its Hochschild and cyclic homology. While the first objectives are necessary to even start applying the framework of formal geometry, the last objective, the computation of homology, is done specifically in order to provide a proof of the algebraic index theorem 6.1.22 later on.

## 3.1. Constructions

We will give three constructions of the formal Moyal–Weyl algebra and show that they are equivalent. Each construction will have a specific advantage over the others. We start with the geometric construction of the formal Moyal–Weyl algebra, which follows the concept sketched above most closely. In fact this geometric construction is given by supplying the $\infty$-jets of functions on $\mathbb{R}^{2d}$ with the Moyal product. The second construction shall be more algebraic and has the advantage that the definition of the product resembles the definition of the Moyal product (3.0.2) more closely. In this second construction the relation with the symmetric algebra/algebra of formal power series will be clear. Closer inspection will reveal that it is simply a reformulation of the first construction. The final construction is most simple and therefore lends itself most easily to explicit computations. It is most simple mainly because it is closest to naive quantization in the sense that one simply extrapolates the implications of $[v, w] = i\hbar\{v, w\}$. This last construction also resembles the usual construction of the classical Weyl algebras most clearly.

**3.1.1. The Geometric Construction.** As mentioned above, this first construction will proceed by equipping the $\infty$-jets of smooth functions at the origin in $\mathbb{R}^{2d}$ with the Moyal product. In the second construction we will do this rather more ad hoc and use the definition $\hat{\mathbb{A}}$ of $\infty$-jets of functions at 0, given in notation 2.2.5. The following will be made easier by also defining the sheaf (of algebras) of $\infty$-jets in a slightly more subtle geometric way, however. Our presentation of the first construction is based on the article [**89**].

Since the Moyal product is a product on the formal power series in $\hbar$ with coefficients in *complex* valued smooth functions, all our constructions will be over the ground field $\mathbb{C}$ from now on. Note that most definitions considered below are equally valid over $\mathbb{R}$ however. For the following definitions one should recall the definition 2.1.6 of the sheaf of differential operators.

Definition 3.1.1. We define the sheaf of $\infty$-jets $J^\infty_{2d}$ by

$$J^\infty_{2d}(U) := \mathrm{Hom}_{C^\infty(U)}(\mathrm{Op}_{2d}(U), C^\infty(U))$$

on an open set $U \subset \mathbb{R}^{2d}$. Let

$$J^\infty \colon C^\infty_{2d} \longrightarrow J^\infty_{2d}$$

be the map given by

$$J^\infty(f)(D) = D(f)$$

for all $D \in \mathrm{Op}_{2d}(U)$ and $f \in C^\infty_{2d}(U)$ for $U$ as above. We denote by the left inverse of $J^\infty$, given by evaluation at $1 \in \mathrm{Op}_{2d}$, by $ev_1$.

The sheaf of $\infty$-jets $J^\infty_{2d}$ comes equipped with an algebra structure which is given by the convolution product corresponding to the usual pointwise multiplication of functions and a coalgebra structure on $\mathrm{Op}_{2d}$. In the following we will describe this coalgebra structure. To avoid conflict of notation we will denote the product (given by the tensor product) in $\mathrm{Op}_{2d}$ and $\mathcal{T}(\mathcal{X}_{2d})$ by simple concatenation of elements.

Definition 3.1.2. Let

$$\Delta_0 \colon \mathrm{Op}_{2d} \longrightarrow \mathrm{Op}_{2d} \otimes_{C^\infty_{2d}} \mathrm{Op}_{2d}$$

be the map descending from the map on $\mathcal{T}(\mathcal{X}_{2d})$ given by $C^\infty_{2d}$-linear extension of

$$\Delta_0(X) = X \otimes 1 + 1 \otimes X \quad \text{and} \quad \Delta_0(1) = 1 \otimes 1$$

for all $X \in \mathcal{X}_{2d}$ and the rule that

$$\Delta_0(D_1 D_2) = \Delta_0(D_1)\Delta_0(D_2)$$

for all $D_1, D_2 \in \mathcal{T}(\mathcal{X}_{2d})$, i.e. $\Delta_0$ should be a map of $C^\infty_{2d}$-algebras.

Example 3.1.3. Note that this means that, for instance,

$$\Delta_0(XY) = (XY) \otimes 1 + X \otimes Y + Y \otimes X + 1 \otimes (XY),$$

for $X, Y \in \mathcal{X}_{2d}$

It can be seen that $\Delta_0$ is well-defined since it preserves the ideal $\mathcal{I}_{2d}$ defined in definition 2.1.6, i.e.

$$\Delta_0(\mathcal{I}_{2d}) \subset \mathcal{I}_{2d} \otimes_{C^\infty_{2d}} \mathcal{T}(\mathcal{X}_{2d}) + \mathcal{T}(\mathcal{X}_{2d}) \otimes_{C^\infty_{2d}} \mathcal{I}_{2d}.$$

REMARK 3.1.4. The coproduct $\Delta_0$ actually comes from the fact that $\mathrm{Op}_{2d}$ is the universal enveloping algebra of the (quintessential) Lie-Rinehart pair $(C^\infty_{2d}, \mathcal{X}_{2d})$. Such algebras are naturally Rinehart bialgebras [**84**].

PROPOSITION 3.1.5. *The map $\Delta_0$ supplies $Op_{2d}$ with the structure of a coassociative cocommutative $C^\infty_{2d}$-coalgebra. Moreover it is counital with counit $J^\infty(1)$.*

PROOF.
It is easily seen, from explicit computation, that

$$(\Delta_0 \otimes \mathrm{Id})\,(\Delta_0\,(X)) = (\mathrm{Id} \otimes \Delta_0)\,(\Delta_0\,(X))$$

for $X \in \mathcal{X}_{2d}$ and $X = 1$. The coassociativity follows since

$$(\Delta_0 \otimes \mathrm{Id})(D_1 D_2) = (\Delta_0 \otimes \mathrm{Id})(D_1)(\Delta_0 \otimes \mathrm{Id})(D_2)$$

for all $D_1, D_2 \in \mathrm{Op}_{2d} \otimes_{C^\infty_{2d}} \mathrm{Op}_{2d}$ and similarly for $(\mathrm{Id} \otimes \Delta_0)$. Denote by $\tau$ the flip endomorphism of $\mathrm{Op}_{2d} \otimes_{C^\infty_{2d}} \mathrm{Op}_{2d}$ given by $D_1 \otimes D_2 \mapsto D_2 \otimes D_1$. Again it is easy to see that

$$\tau(\Delta_0(X)) = \Delta_0(X) \quad \text{and} \quad \tau(\Delta_0(1)) = \Delta_0(1)$$

for all $X \in \mathcal{X}_{2d}$. So the cocommutativity follows since we have

$$\tau(D_1 D_2) = \tau(D_1)\tau(D_2)$$

for all $D_1, D_2 \in \mathrm{Op}_{2d} \otimes_{C^\infty} \mathrm{Op}_{2d}$. Finally, suppose we have $D \in F_p\mathrm{Op}_{2d}$, then $D = \sum_{k=0}^{p} D_k$ with $D_k = \sum_{i_1,\ldots,i_k} X_{i_1} \ldots X_{i_k}$, with $X_j \in \mathcal{X}_{2d}$ for all indices and where the sum is finite. Thus we find that

$$\Delta_0(D) = \sum_{k=0}^{p} \Delta_0(D_k)$$

and

$$\Delta_0(D_k) = 1 \otimes D_k + D_k \otimes 1 + R_k,$$

for $k > 0$, where the rest terms $R_k$ have $X_j$'s on each leg, while

$$\Delta_0(D_0) = D_0 \otimes 1 = 1 \otimes D_0$$

(since $D_0 \in C^\infty_{2d}$). On the other hand, if $X_1, \ldots, X_p \in \mathcal{X}_{2d}$, then we have

$$J^\infty(1)(X_p \ldots X_1) = X_p \ldots X_1(1) = 0.$$

So we get

$$(\mathrm{Id} \otimes J^\infty(1))(\Delta_0(D)) = D \otimes 1 \quad \text{and} \quad (J^\infty(1) \otimes \mathrm{Id})(\Delta_0(D)) = 1 \otimes D$$

for all $D \in \mathrm{Op}_{2d}$, which means exactly that $J^\infty(1)$ is a counit for $\Delta_0$.

□

COROLLARY 3.1.6. *The map*

$$J^\infty_{2d} \otimes_{C^\infty_{2d}} J^\infty_{2d} \longrightarrow J^\infty_{2d}$$

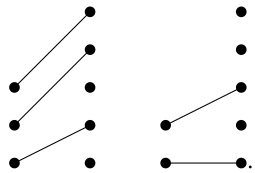
*given by convolving $\Delta_0$ and the pointwise multiplication $\mu$ of smooth functions, i.e.*

$$l_1 l_2 = \mu \circ l_1 \otimes l_2 \circ \Delta_0,$$

*gives $J^\infty_{2d}$ the structure of an associative commutative $C^\infty_{2d}$-algebra. Again we have that $J^\infty(1)$ is a unit for this algebra structure. Moreover, the map $J^\infty$ is a unital $C^\infty_{2d}$-algebra homomorphism.*

PROOF.
The associativity, commutativity and unitality of the product follow immediately from the corresponding properties of $\Delta_0$ and $\mu$. Let us prove the last statement. Let $[m]$ denote the ordered set $\{1, \ldots, m\}$ for all $m \in \mathbb{N}$. For $m < n$ let $\Delta_+(m, n)$ denote the set of strictly increasing maps from $[m]$ to $[n]$ and let $\Delta_+(0, n) = \{e_n\}$ and $\Delta_+(n, n) = \{f_n\}$. The rest of this proof should be significantly easier to

follow if one keeps in mind that the maps in $\Delta_+(m,n)$ can be uniquely described in terms of diagrams of the form (examples for $\Delta_+(3,5)$ and $\Delta_+(2,5)$):



For $0 < m < n$ and $\varphi \in \Delta_+(m,n)$, we denote by $\overline{\varphi} \in \Delta_+(n-m,n)$ the complementary map, i.e. the unique strictly increasing map such that $\mathrm{Im}\,\varphi \cap \mathrm{Im}\,\overline{\varphi} = \emptyset$ (note that the diagrams above correspond to complementary maps), by $\varphi^+$ the induced map in $\Delta_+(m+1,n+1)$ given by $\varphi^+(k) = \varphi(k)$ for all $k \le m$ and $\varphi^+(m+1) = n+1$ and by $\varphi^i \in \Delta(m,n+1)$ the composition of $\varphi$ and the inclusion $i_n \colon [n] \to [n+1]$, given by $i_n(k) = k$ for all $k \le n$. We also set $\overline{f_n} := e_n$ and $\overline{e_n} := f_n$, $f_n^+ := f_{n+1}$, $e_n^i := e_{n+1}$, $f_n^i := i_n$ and finally $e_n^+ \in \Delta_+(1,n+1)$ is given by $e_n^+(1) = n+1$. Given $X_1,\dots,X_n \in \mathcal{X}_{2d}$ and $\varphi \in \Delta_+(m,n)$ we set

$$X_\varphi := \begin{cases} X_{\varphi(m)} X_{\varphi(m-1)} \dots X_{\varphi(1)} & \text{if } m < n \\ X_n \dots X_1 & \text{if } \varphi = f_n \\ 1 & \text{if } \varphi = e_n. \end{cases}$$

Note that we have $X_{n+1} X_\varphi = X_{\varphi^+}$, $X_\varphi = X_{\varphi^i}$ (whenever this makes sense), $\overline{\varphi^+} = \overline{\varphi}^i$ and $\overline{\overline{\varphi}} = \varphi$. Note also that for $0 \le m < n$ we have that $\Delta_+(m,n)^+ \cup \Delta_+(m+1,n)^i = \Delta(m+1,n+1)$ while $\Delta_+(m,n)^+ \cap \Delta_+(m+1,n)^i = \emptyset$ and $\Delta_+(n,n)^+ = \Delta_+(n+1,n+1)$.

Using the above notation and identities it is easily shown by induction that

$$\Delta_0(X_n \dots X_1) = \Delta_0(X_n) \dots \Delta_0(X_1) = \sum_{m=0}^{n} \sum_{\varphi \in \Delta_+(m,n)} X_\varphi \otimes X_{\overline{\varphi}}. \tag{3.1.1}$$

On the other hand we obtain similarly that

$$X_n \dots X_1(fg) = \sum_{m=0}^{n} \sum_{\varphi \in \Delta_+(m,n)} X_\varphi(f) X_{\overline{\varphi}}(g) \tag{3.1.2}$$

using the product rule. The above means that

$$J^\infty(fg) = J^\infty(f) J^\infty(g),$$

by definition. □

It turns out that the sheaf of commutative $C^\infty_{2d}$-algebras $J^\infty_{2d}$ is in fact given by the sections of a pro-finite dimensional commutative $\mathbb{C}$-algebra bundle (associated to $\widetilde{M}$ with fiber $\hat{\mathbb{A}}$ in the real case). The second construction presented below will be a direct consequence of this fact as well as many other implications in the following. The *formal* Moyal–Weyl algebra will be given as a deformation of this commutative algebra structure on the fiber. It is possible to describe the algebra structure on the sections of the corresponding bundle of non-commutative algebras analogously to the description of the commutative structure above. Let us first give a description of this deformation of the algebra structure on $\infty$-jets.

From now on deformation will enter the picture. Let us denote a formal parameter by $\hbar$ and the sheaf given by $\mathcal{F}[\![\hbar]\!](U) = \mathcal{F}(U)[\![\hbar]\!]$ by $\mathcal{F}[\![\hbar]\!]$, e.g. $C^\infty_{2d}[\![\hbar]\!]$ will denote the sheaf of formal power series in $\hbar$ with coefficients in the smooth functions.

Consider the standard symplectic structure $\omega$ on $\mathbb{R}^{2d}$, i.e., if $\{x^1,\dots,x^d,\xi^1,\dots,\xi^d\}$ are the standard coordinates on $\mathbb{R}^{2d}$, we have $\omega=\sum_{i=1}^d d\xi^i\wedge dx^i$. The symplectic structure defines an isomorphism of the tangent and cotangent bundle given by

$$T_x\mathbb{R}^{2d}\ni v\mapsto \omega_x(v,-)\in T_x^*\mathbb{R}^{2d}\qquad \text{for all } x\in\mathbb{R}^{2d} \tag{3.1.3}$$

and we denote by $\bar\omega$ the image of $\omega\in\Gamma(\bigwedge^2 T^*\mathbb{R}^{2d})$ under the induced isomorphism

$$\Gamma\left(\textstyle\bigwedge^2 T^*\mathbb{R}^{2d}\right)\simeq\Gamma\left(\textstyle\bigwedge^2 T\mathbb{R}^{2d}\right),$$

i.e. $\bar\omega=\sum_{i=1}^d\partial_{\xi^i}\wedge\partial_{x^i}$. Consider the map

$$Alt\colon \textstyle\bigwedge^n\mathcal{X}_{2d}\longrightarrow\mathcal{T}(\mathcal{X}_{2d})$$

given by

$$Alt(X_1\wedge\ldots\wedge X_n)=\frac{1}{n!}\sum_{\tau\in S_n}\epsilon(\tau)X_{\tau(1)}\ldots X_{\tau(n)} \tag{3.1.4}$$

where we denote the symmetric group in $n$ letters by $S_n$ and by the sign of $\tau$ $\epsilon(\tau)$. Then we can also consider $\bar\omega\in\mathrm{Op}_{2d}\otimes_{C^\infty_{2d}}\mathrm{Op}_{2d}$, i.e. we denote the class of $Alt(\bar\omega)$ by $\bar\omega$ as well. The Moyal deformation (3.0.2) of $\mathbb{R}^{2d}$ is given by

$$f\star g=\mu\circ e^{i\hbar\bar\omega}(f\otimes g),$$

for all $f,g\in C^\infty_{2d}[\![\hbar]\!]$. Here we have denoted the $\mathbb{C}[\![\hbar]\!]$-linear extension of the usual product of $C^\infty_{2d}$ by $\mu$. This leads us to the following definition.

Definition 3.1.7. We define the product $\star$ on $J^\infty_{2d}$ by

$$l_1\star l_2=\mu\circ l_1\otimes l_2\circ r_{e^{i\hbar\bar\omega}}\circ\Delta_0,$$

where we have denoted the operation of right multiplication by $B\in\mathrm{Op}_{2d}\otimes_{C^\infty_{2d}}\mathrm{Op}_{2d}$ by $r_B$.

Proposition 3.1.8. *The pair $(J^\infty_{2d}[\![\hbar]\!],\star)$ is a sheaf of associative $\mathbb{C}[\![\hbar]\!]$-algebras and the $\mathbb{C}[\![\hbar]\!]$-linear extension of $J^\infty$ defines a $\mathbb{C}[\![\hbar]\!]$-algebra homomorphism*

$$J^\infty\colon (C^\infty_{2d}[\![\hbar]\!],\star)\longrightarrow(J^\infty_{2d}[\![\hbar]\!],\star).$$

Proof.
Let us denote $E(\bar\omega):=e^{i\hbar\bar\omega}$. Note that the associativity of $\star$ follows from the equation

$$(\Delta_0\otimes\mathrm{Id})(E(\bar\omega))(E(\bar\omega)\otimes 1)=(\mathrm{Id}\otimes\Delta_0)(E(\bar\omega))(1\otimes E(\bar\omega)). \tag{3.1.5}$$

Let $D\in\mathrm{Op}_{2d}\otimes\mathrm{Op}_{2d}$ be given by $D=\sum_{j=1}^k D_{j1}\otimes D_{j2}$, we introduce the following (standard) notation

$$D_{12}=D\otimes 1,\qquad D_{13}=\sum_{j=1}^k D_{j1}\otimes 1\otimes D_{j2}\quad\text{and}\quad D_{23}=1\otimes D.$$

Then, if we expand equation (3.1.5) for $E(\bar\omega)$ and equate the coefficients of different powers of $\hbar$, we get the equivalent series of equations

$$\sum_{l=0}^k\frac{1}{l!(k-l)!}(\bar\omega_{13}+\bar\omega_{23})^l\bar\omega_{12}^{(k-l)}=\sum_{l=0}^k\frac{1}{l!(k-l)!}(\bar\omega_{13}+\bar\omega_{12})^l\bar\omega_{23}^{(k-l)}$$

for all $k\geq 0$. Now note that $\bar\omega_{ij}\bar\omega_{pq}=\bar\omega_{pq}\bar\omega_{ij}$ for all possible $i,j,p$ and $q$ and so we can use Newton's binomial formula to obtain the equivalent series of equations

$$\sum_{l=0}^k\sum_{r=0}^l\frac{1}{(k-l)!(l-r)!r!}\bar\omega_{13}^{(l-r)}\bar\omega_{23}^r\bar\omega_{12}^{(k-l)}=\sum_{l=0}^k\sum_{r=0}^l\frac{1}{(k-l)!(l-r)!r!}\bar\omega_{13}^{(l-r)}\bar\omega_{12}^r\bar\omega_{23}^{(k-l)},$$

which is easily seen to be satisfied. This shows that $\star$ is associative on $J^\infty_{2d}[\![\hbar]\!]$.
Note that (3.1.1) and (3.1.2) mean that

$$D(\mu(f \otimes g)) = \mu(\Delta_0(D)(f \otimes g)) \tag{3.1.6}$$

for $f, g \in C^\infty_{2d}$ and $D \in \mathrm{Op}_{2d}$. This still holds when one considers the $\mathbb{C}[\![\hbar]\!]$-linear extensions on $C^\infty_{2d}[\![\hbar]\!]$. Thus we have

$$J^\infty(f \star g)(D) = D\mu(E(\bar{\omega})f \otimes g) = \mu((\Delta_0(D)E(\bar{\omega}))f \otimes g) = J^\infty(f) \star J^\infty(g)(D)$$

for all $f, g \in C^\infty_{2d}[\![\hbar]\!]$ and $D \in \mathrm{Op}_{2d}$. □

Remark 3.1.9. Note that (3.1.6) together with the proof of associativity of $\star$ on $J^\infty_{2d}[\![\hbar]\!]$ also gives the proof of associativity of $\star$ on $C^\infty_{2d}[\![\hbar]\!]$.

Remark 3.1.10. It is worth mentioning that everything we have been considering here can be done in much greater generality. The exact same constructions as above yield the $C^\infty_M$-algebra of $\infty$-jets $J^\infty_M$. One also obtains the unital and injective algebra homomorphism

$$J^\infty \colon C^\infty_M \longrightarrow J^\infty_M.$$

More importantly, given a deformation quantization $\star$ of $C^\infty_M[\![\hbar]\!]$, one also obtains a deformation of the product on $J^\infty_M$. The structure is obtained in much the same manner as above for the Moyal product. Namely, we note that a deformation quantization is given by a sequence of bidifferential operators $B_k$ in the sense that

$$f \star g = \sum_{k=0}^{\infty} (i\hbar)^k B_k(f, g).$$

Thus we can consider the element

$$B = \sum_{k=0}^{\infty} (i\hbar)^k B_k \in (\mathrm{Op}_M \otimes_{C^\infty_M} \mathrm{Op}_M)[\![\hbar]\!]$$

and define the product

$$l_1 \star l_2 = \mu \circ l_1 \otimes l_2 \circ r_B \circ \Delta_0,$$

for all $l_1, l_2 \in J^\infty_M[\![\hbar]\!]$. Then, by equation (3.1.6), we find that associativity of $\star$ on $C^\infty_M[\![\hbar]\!]$ implies associativity of $\star$ on $J^\infty_M[\![\hbar]\!]$.

Remark 3.1.11. We should note that, when we consider the differential operators as the universal enveloping Rinehart bialgebra of the Lie-Rinehart pair $(C^\infty_{2d}, \mathcal{X}_{2d})$, we find that $E(\bar{\omega})$ from proposition 3.1.8 defines a so-called (formal) Drinfeld twist [**43, 36**]. A Drinfeld twist $B \in H \otimes H$ in a Hopf algebra $(H, \Delta, \epsilon)$ is an invertible element that satisfies the equations

$$(\Delta \otimes \mathrm{Id})(B)(B \otimes 1) = (\mathrm{Id} \otimes \Delta)(B)(1 \otimes B) \quad \text{and} \quad (\epsilon \otimes 1)(B) = 1 = (1 \otimes \epsilon)(B).$$

Note, however, that the equations make sense for any unital and counital bialgebra. A formal Drinfeld twist is a Drinfeld twist in formal power series that is a deformation of the trivial Drinfeld twist, i.e. instead of invertibility one asks for the stronger condition that

$$B = 1 \otimes 1 + O(\hbar).$$

Note that remark 3.1.10 says that all normalized formal deformation quantizations are obtained from formal Drinfeld twists in differential operators.

To finish our construction of the formal Moyal–Weyl algebra, we should consider the induced structure on $\infty$-jets of functions at a point. To do this we will use a different yet equivalent definition of $\infty$-jets of functions at a point.

DEFINITION 3.1.12. Denote by $\widehat{C^\infty_{2d}}_x$ the *∞-jets of functions at* $x \in \mathbb{R}^{2d}$. It is given by the classes of $J^\infty_{2d}(\mathbb{R}^{2d})$ for the equivalence relation $\sim$ given by

$$l_1 \sim l_2 \quad \text{if} \quad ev_x \circ l_1 = ev_x \circ l_2$$

where $ev_x \colon C^\infty(\mathbb{R}^{2d}) \to \mathbb{C}$ denotes the evaluation of a function at the point $x$.

REMARK 3.1.13. Note that the ∞-jets of functions at $x \in \mathbb{R}^{2d}$ are naturally a $\mathbb{C}$-vector space. This is even more clear when one consider that $ev_x$ is a $\mathbb{C}$-algebra homomorphism. This means thatMeaning that the induced map on $J^\infty_{2d}(\mathbb{R}^{2d})$, given by post composition, is also an algebra homomorphism (to the dual of differential operators) for the convolution product. Then it is easy to see that

$$\widehat{C^\infty_{2d}}_x = J^\infty_{2d}(\mathbb{R}^{2d}) \big/ \mathrm{Ker}\, ev_x$$

yielding a natural $\mathbb{C}$-algebra structure.

REMARK 3.1.14. Note that the ∞-jets of functions at $x \in \mathbb{R}^{2d}$ are a complexification of the algebras $\hat{\mathcal{A}}_x$. This can be seen by examining the definitions 2.1.6 and 2.1.1 and the lemma 2.1.8.

PROPOSITION 3.1.15. *The sheaf of commutative algebras $J^\infty_{2d}$ is naturally isomorphic to the sheaf of sections of the vector bundle with the fiber $\widehat{C^\infty_{2d}}_x$ over $x \in \mathbb{R}^{2d}$.*

PROOF.
We should show that $J^\infty_{2d}$ is given by a vector bundle. To do this consider the standard coordinates $\{x_1, \ldots, x_{2d}\}$ of $\mathbb{R}^{2d}$. They give rise to the ordered bases $\{\partial_{x_1}|_q, \ldots, \partial_{x_{2d}}|_q\}$ at each $q \in \mathbb{R}^{2d}$. Given a differential operator $D \in F_p\mathrm{Op}_{2d}$, we put it in a normal form by using the relations defining $\mathrm{Op}_{2d}$ to make the degree $p$ part of $D$ have the form

$$\sum_{i_1 \le i_2 \le \ldots \le i_p} f_{i_1 \ldots i_p} \partial_{x_{i_1}} \ldots \partial_{x_{i_p}},$$

where $f_{i_1 \ldots i_p} \in C^\infty_{2d}$ for all indices. This fixes the degree $p$ part of $D$ and we proceed by induction. It is part of the content of the Poincaré-Birkhoff-Witt theorem that this normal form is well-defined. Thus we see that $\mathrm{Op}_{2d}$ is generated over $C^\infty_{2d}$ by the elements

$$e_\alpha = \prod_{i=1}^{2d} \frac{\partial^{\alpha_i}_{x_i}}{\alpha_i!} \quad \text{where} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\ge 0})^{2d}.$$

Denote by $l_\alpha$ the dual elements in $J^\infty_{2d}$, i.e. such that

$$l_\alpha(e_\beta) = \delta_{\alpha\beta}$$

where we denote by $\delta_{\alpha\beta} = \delta_{\alpha_1\beta_1} \ldots \delta_{\alpha_n\beta_n}$ the product of Kronecker deltas. Note that the formula (3.1.1) shows that we have

$$l_\alpha l_\beta(e_\gamma) = \mu(l_\alpha \otimes l_\beta(\Delta_0(e_\gamma))) = \delta_{(\alpha+\beta)\gamma}$$

(this is why there is the $\frac{1}{\alpha_i!}$ in the definition above) and thus $l_\alpha l_\beta = l_{\alpha+\beta}$. We denote by $C$ the algebra generated by 1 and the $l_\alpha$ with the relations that state 1 is the unit and $l_\alpha l_\beta = l_{\alpha+\beta}$. Then $C$ carries the filtration given by the grading induced by

$$|l_\alpha| = \sum_{i=1}^{2d} \alpha_i.$$

We denote by $\widehat{C}$ the completion of $C$ in the topology induced by this grading. Clearly we have

$$J^\infty_{2d} \simeq C^\infty_{2d} \otimes_{\mathbb{C}} \widehat{C}$$

showing that the $\infty$-jets are indeed given by sections of a vector bundle. Thus we can write every $\infty$-jet $l$ in the form $l = \sum_\alpha f_\alpha l_\alpha$, meaning $l(e_\alpha) = f_\alpha$. This yields the isomorphisms $\widehat{C^\infty_{2d}}_x \to \widehat{C}$ for all $x \in \mathbb{R}^{2d}$ given by

$$l \mapsto l(e_\alpha)(x) l_\alpha.$$

$\square$

REMARK 3.1.16. Note that the bundle constructed in the previous proposition is exactly the bundle with fiber $\hat{\mathbb{A}}_{2d} \otimes \mathbb{C}$ associated to the bundle $\widetilde{\mathbb{R}^{2d}}$ or, by using the equivariant section $F\colon j^1(\mathbb{R}^{2d}) \to \widetilde{\mathbb{R}^{2d}}$, the bundle $j^1(\mathbb{R}^{2d})$.

The fact that the product $\star$ of 3.1.8 on $J^\infty_{2d}[\![\hbar]\!]$ is defined in terms of differential operators means that it will restrict to a product on $\widehat{C^\infty_{2d}}_0$.

REMARK 3.1.17. Note that the map

$$J^\infty_0 : C^\infty(\mathbb{R}^{2d}) \longrightarrow \widehat{C^\infty_{2d}}_0$$

given by

$$J^\infty_0(f) = [J^\infty(f)]$$

coincides under the natural identification $\widehat{C^\infty_{2d}}_0 \simeq \hat{\mathbb{A}} \otimes \mathbb{C}$ with the previous definition of $J^\infty_0$. We will denote the $\mathbb{C}[\![\hbar]\!]$-linear extension of $J^\infty_0$ by $J^\infty_0$ also.

DEFINITION 3.1.18. We define the product $\star$ on $\widehat{C^\infty_{2d}}_0[\![\hbar]\!]$ as

$$[l_1] \star [l_2] = J^\infty_0(f_1 \star f_2)$$

where $f_i \in C^\infty(\mathbb{R}^{2d})$ such that $J^\infty_0(f_i) = [l_i]$ for $i = 1, 2$.

Suppose that

$$\Delta_0(D) = \sum_{k=1}^{p} D_{1,k} \otimes D_{2,k}, \quad E(\bar\omega) = \sum_{l=0}^{\infty} \sum_{j=1}^{p_l} (i\hbar)^l E_{1,l,j} \otimes E_{2,l,j} \quad \text{and} \quad J^\infty_0(f) = J^\infty_0(g)$$

for $f, g \in C^\infty_{2d}(\mathbb{R}^{2d})$ and $D \in \mathrm{Op}_{2d}$. Then we find that

$$D(f \star h)(0) = \mu(\Delta_0(D)E(\bar\omega)(f \otimes h)))(0) = \sum_{k=1}^{p} \sum_{l=0}^{\infty} \sum_{j=1}^{p_l} (i\hbar)^l D_{1,k} E_{1,l,j}(f)(0) D_{2,k} E_{2,l,j}(h)(0) =$$

$$\sum_{k=1}^{p} \sum_{l=0}^{\infty} \sum_{j=1}^{p_l} (i\hbar)^l D_{1,k} E_{1,l,j}(g)(0) D_{2,k} E_{2,l,j}(h)(0) = \mu(\Delta_0(D)E(\bar\omega)(g \otimes h)))(0) = D(g \star h)(0).$$

Note that, since $J^\infty_0(f) = J^\infty_0(g)$ if and only if $D(f)(0) = D(g)(0)$ for all $D \in \mathrm{Op}_{2d}$, this means that $\star$ above is a well-defined associative product.

REMARK 3.1.19. Suppose $l_1, l_2 \in J^\infty_{2d}$ and $f_1, f_2, f_{12} \in C^\infty_{2d}$ such that $J^\infty_0(f_1) = [l_1]$, $J^\infty_0(f_2) = [l_2]$ and $J^\infty_0(f_{12}) = [l_1 \star l_2]$ then

$$D(f_{12})(0) = (l_1 \star l_2)(D)(0) = \mu((l_1 \otimes l_2)(\Delta_0(D)E(\bar\omega)))(0)$$

while

$$D(f_1 \star f_2)(0) = \mu(\Delta_0(D)E(\bar\omega)(f_1 \otimes f_2))(0)$$

for all $D \in \mathrm{Op}_{2d}$. Writing these expressions out as above we find that

$$[l_1 \star l_2] = [l_1] \star [l_2]$$

as expected.

Definition 3.1.20. We define the *$2d$-dimensional formal Moyal–Weyl algebra* $\hat{\mathbb{A}}_{2d}^{\hbar}$ as the space $\widehat{C^{\infty}_{2d\,0}}[\![\hbar]\!]$ equipped with the product $\star$.

Note that the "$2d$-dimensional" refers to the dimension of $\mathbb{R}^{2d}$, the algebra $\hat{\mathbb{A}}_{2d}^{\hbar}$ is infinite dimensional both over $\mathbb{C}$ and over $\mathbb{C}[\![\hbar]\!]$.

Remark 3.1.21. Note that, by Borel's theorem 2.3.1, the map

$$J_0^{\infty} : C^{\infty}_{2d}(\mathbb{R}^{2d})[\![\hbar]\!] \longrightarrow \hat{\mathbb{A}}_{2d}^{\hbar},$$

given by mapping $f \in C^{\infty}_{2d}(\mathbb{R}^{2d})[\![\hbar]\!]$ to the class of $J^{\infty}(f)$ in $\hat{\mathbb{A}}_{2d}^{\hbar}$, is surjective. This introduces a filtration

$$\hat{\mathbb{A}}_{2d}^{\hbar} = F_0\hat{\mathbb{A}}_{2d}^{\hbar} \supset F_1\hat{\mathbb{A}}_{2d}^{\hbar} \supset \ldots \supset F_p\hat{\mathbb{A}}_{2d}^{\hbar} \supset F_{p+1}\hat{\mathbb{A}}_{2d}^{\hbar} \supset \ldots$$

given by $l \in F_p\hat{\mathbb{A}}_{2d}^{\hbar}$ if there is $f \in C^{\infty}_{2d}(\mathbb{R}^{2d})[\![\hbar]\!]$ such that $J_0^{\infty}(f) = l$ and the Taylor expansion of $f$ at $0 \in \mathbb{R}^{2d}$ in standard coordinates does not contain terms of degree lower then $p$, where we count the degree of a term as 2 times the power of $\hbar$ plus the order of the monomial in standard coordinates. For example $J_0^{\infty}(\hbar^2 x_1^3 x_2 + \hbar x_1 x_2^2 + x_1^5) \in F_p\hat{\mathbb{A}}_{2d}^{\hbar}$ for $p \leq 5$. We supply $\hat{\mathbb{A}}_{2d}^{\hbar}$ with the topology induced from this filtration. Note that $\hat{\mathbb{A}}_{2d}^{\hbar}$ is complete in this topology. This topology and filtration will be made more clear in the second and third constructions.

Remark 3.1.22. Suppose $M$ is a symplectic smooth manifold and $\star$ is a star product on $C^{\infty}_{M}[\![\hbar]\!]$. Then we can consider the corresponding deformation of $J^{\infty}_{M}$ as explained in remark 3.1.10. We can also define the $\infty$-jets of functions at $m \in M$. Let us denote the corresponding deformation by $\hat{\mathcal{A}}_{m,\hbar}$, e.g. $\hat{\mathcal{A}}_{0,\hbar} = \hat{\mathbb{A}}_{2d}^{\hbar}$ for $\star$ the Moyal product (3.0.2). The Moyal deformation is the unique symplectic deformation quantization of $\mathbb{R}^{2d}$ up to gauge equivalence. From this (and the fact that all symplectic manifolds are locally symplectomorphic to $\mathbb{R}^{2d}$ with the standard symplectic structure) it follows that $(C^{\infty}(U)[\![\hbar]\!], \star)$ is isomorphic to the Moyal deformation of $\mathbb{R}^{2d}$ (where $\mathrm{Dim}\, M = 2d$) for any Darboux coordinate chart $U \subset M$ (in fact the isomorphism is induced in lowest order by a symplectomorphism $U \simeq \mathbb{R}^{2d}$). Since the deformation of the algebra of $\infty$-jets at a point $m \in U$ clearly only depends on local behaviour, this implies that $\hat{\mathcal{A}}_{m,\hbar} \simeq \hat{\mathbb{A}}_{2d}^{\hbar}$. In fact, the isomorphism will even be continuous for the corresponding topologies induced by the filtrations. Note, however, that the isomorphisms are <u>not</u> canonical.

**3.1.2. A Deformed Symmetric Algebra.** The second construction of the formal Moyal–Weyl algebra is motivated by the fact that the sheaf of commutative algebras $J^{\infty}_{2d}$ is in fact given as the sections of a bundle of algebras, see proposition 3.1.15. Thus we may also define the product directly on the fiber of this bundle at $0 \in \mathbb{R}^{2d}$. Another way to think of this is to note that $\infty$-jets are simply given by Taylor expansions and the formula for the Moyal product can be extended from polynomials to formal power series.The following presentation of the second construction is based on the article [**14**].

Let $(V, \omega)$ be a complex symplectic vector space of dimension $2d$. Let $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ denote the linear dual of $V$. Then we can consider the Moyal product restricted to the space of polynomials on $V$.

Definition 3.1.23. Suppose $W$ is a vector space over the field $L$, then we denote the tensor algebra of $W$ (i.e. the free algebra over $L$ generated by $W$) by $\mathcal{T}(W)$. We define the *symmetric algebra* on $W$, denoted $\mathcal{S}^{\bullet}(W)$ as the quotient of $\mathcal{T}(W)$ by the ideal generated by elements of the form

$$v \otimes w - w \otimes v$$

with $v, w \in W$. The $\bullet$ refers to the grading, which is induced from the grading of $\mathcal{T}(W)$, i.e. $\mathcal{S}^n(W)$ consists of words of length $n$ in $W$ considered up to order.

Definition 3.1.24. We extend $\omega$ to a bilinear form on $\mathcal{S}^n(V)$ by $\mathbb{C}$-linear extension of

$$\omega(v_1 \dots v_n, w_1 \dots w_n) := \sum_{\sigma,\tau \in S_n} \frac{1}{(n!)^2} \prod_{i=1}^n \omega(v_{\sigma(i)}, w_{\tau(i)}) = \sum_{\sigma \in S_n} \frac{1}{n!} \prod_{i=1}^n \omega(v_{\sigma(i)}, w_i)$$

and $\omega(1,1) = 1$ for $\mathcal{S}^0(V)$. Here $S_n$ denotes the symmetric group on $n$ letters. We also extend $\omega$ to a map

$$(\mathcal{S}^\bullet(V^*) \otimes \mathcal{S}^n(V))^{\otimes 2} \longrightarrow \mathcal{S}^\bullet(V^*)$$

by $\mathbb{C}$-linear extension of

$$\omega(f \otimes \alpha, g \otimes \beta) = fg\omega(\alpha, \beta).$$

Definition 3.1.25. For $n \in \mathbb{Z}_{\geq 0}$, we define the maps

$$d_n \colon \mathcal{S}^\bullet(V^*) \longrightarrow \mathcal{S}^\bullet(V^*) \otimes \mathcal{S}^n(V)$$

as follows. Note first that $\mathcal{S}^\bullet(V^*)$ can be identified with the space of polynomial functions $V \to \mathbb{C}$, see [**38**]. Now consider the map

$$d \colon \mathcal{S}^\bullet(V^*) \otimes \mathcal{S}^\bullet(V) \longrightarrow \mathcal{S}^\bullet(V^*) \otimes \mathcal{S}^\bullet(V)$$

given by

$$d = (1 \otimes \mu_S) \circ (1 \otimes I_\omega^{-1} \otimes 1) \circ (d_{\mathrm{dR}} \otimes 1)$$

where $d_{\mathrm{dR}}$ denotes the restriction of the exterior derivative to polynomial functions, $I_\omega$ denotes the isomorphism $V \simeq V^*$ induced by $\omega$ and $\mu_S$ denotes the symmetric product restricted to $V \otimes \mathcal{S}^\bullet(V)$. Note that we have implicitly identified the tangent space of the vector space $V$ with $V$. Now consider the inclusion

$$\iota \colon \mathcal{S}^\bullet(V^*) \hookrightarrow \mathcal{S}^\bullet(V^*) \otimes \mathcal{S}^\bullet(V)$$

given by $\iota(f) = f \otimes 1$ for all $f \in \mathcal{S}^\bullet(V^*)$. Then we define

$$d_n = d^n \circ \iota.$$

Definition 3.1.26. Let $\star$ be the product on $\mathcal{S}^\bullet(V^*)[\hbar]$ ($\hbar$ is a formal variable) given by

$$f \star g := \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n \omega(d_n f, d_n g). \tag{3.1.7}$$

Proposition 3.1.27. *The product $\star$ is a well-defined associative product.*

Proof.
Note that the formula for $\star$ above is definitely well-defined, since for all polynomials $f$ there is $k \in \mathbb{N}$ such that $d_{k+1} f = 0$ (namely the order of the polynomial). Thus the sum is finite. The associativity follows from the fact that by choosing a symplectic basis $\{x_1, \dots, x_d, \xi_1, \dots, \xi_d\}$ for $V$ it becomes clear that $\star$ is simply the restriction of the associative product (3.0.2) (in fact we could have defined it that way too). □

As mentioned we want to extend the above formula from polynomials to all Taylor expansions of functions at 0. Note that, by Borel's theorem 2.3.1, this means we want to extend the formula to formal power series.

Definition 3.1.28. Let

$$ev_0 \colon \mathcal{S}^\bullet(V^*) \longrightarrow \mathbb{C}$$

denote the map which evaluates a polynomial at $0 \in V$. Denote

$$\mathfrak{m} = \operatorname{Ker} ev_0.$$

We define the (commutative) algebra of formal power series on $V$ by

$$\widehat{\mathcal{S}^\bullet(V^*)} := \varprojlim_n \frac{\mathcal{S}^\bullet(V^*)}{\mathfrak{m}^n}$$

where the limit runs over the quotient maps

$$\frac{\mathcal{S}^\bullet(V^*)}{\mathfrak{m}^n} \longrightarrow \frac{\mathcal{S}^\bullet(V^*)}{\mathfrak{m}^k}$$

if $k \leq n$.

PROPOSITION 3.1.29. *The formula* (3.1.7) *for* $\star$ *extends to define an associative* $\mathbb{C}[\![\hbar]\!]$*-algebra, denoted* $\mathbb{W}^\hbar(V)$*, with underlying* $\mathbb{C}[\![\hbar]\!]$*-module given by* $\widehat{\mathcal{S}^\bullet(V^*)}[\![\hbar]\!]$.

This proposition will be a corollary of the following proposition, which will also justify that we call $\mathbb{W}^\hbar(V)$ the formal Moyal–Weyl algebra associated to $(V, \omega)$.

PROPOSITION 3.1.30. *There is an isomorphism* $\varphi_V \colon \hat{\mathbb{A}}^\hbar_{2d} \longrightarrow \widehat{\mathcal{S}^\bullet(V^*)}[\![\hbar]\!]$ *of* $\mathbb{C}[\![\hbar]\!]$*-modules satisfying that*

- $\varphi_V$ *is an algebra isomorphism for the commutative algebra structures and*
- *the pushforward of* $\star$ *on* $\hat{\mathbb{A}}^\hbar_{2d}$ *to* $\widehat{\mathcal{S}^\bullet(V^*)}[\![\hbar]\!]$ *agrees with the extension of* (3.1.7).

PROOF.
Note that the commutative algebra structure on the $\mathbb{C}[\![\hbar]\!]$-module underlying $\hat{\mathbb{A}}^\hbar_{2d}$ is given in the remark 3.1.13. The underlying $\mathbb{C}[\![\hbar]\!]$-module is of course unchanged. Let $\{v_1, \ldots, v_{2d}\}$ be a symplectic basis of $V$ with the dual basis $\{\overline{v_1}, \ldots, \overline{v_{2d}}\}$. Now set

$$c_\alpha := \prod_{i=1}^{n} \overline{v_i}^{\alpha_i}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^{2d}$ similar to the definition of the $l_\alpha$'s in proposition 3.1.15. Note that this provides the isomorphism with $\widehat{C}$ and thus $\widehat{C^\infty_{2d\,x}}$ (see the proof of proposition 3.1.15). Thus by considering the $\mathbb{C}[\![\hbar]\!]$-linear extension of this isomorphism we find the isomorphism $\varphi_V$ providing the first item.

It is left to check the second item. Note that it is sufficient to check that

$$c_\alpha \star c_\beta = \varphi_V([l_\alpha \star l_\beta]),$$

since $\star$ is clearly $\mathbb{C}[\![\hbar]\!]$-bilinear and continuous and we have $[l_\alpha] \star [l_\beta] = [l_\alpha \star l_\beta]$ from remark 3.1.19. Consider also the smooth functions (monomials) $f_\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. Then we have that $J^\infty(f_\alpha) = l_\alpha$. We denote the inclusion

$$\iota_V \colon \mathcal{S}^\bullet(V^*)[\![\hbar]\!] \hookrightarrow C^\infty(\mathbb{R}^{2d})[\![\hbar]\!],$$

given by the identification of $V$ and $\mathbb{R}^{2d}$ by way of the symplectic basis $\{v_i\}_{i=1}^{2d}$ and the standard (Darboux) coordinates $\{x_i\}_{i=1}^{2d}$, by $\iota_V$. Then it is clear that

$$\iota_V(c_\alpha \star c_\beta) = f_\alpha \star f_\beta$$

and we have by 3.1.8 that

$$J^\infty(f_\alpha \star f_\beta) = l_\alpha \star l_\beta.$$

On the other hand we note that $\varphi_V \circ J_0^\infty \circ \iota_V$ is simply the inclusion $\mathcal{S}^\bullet(V^*)[\![\hbar]\!] \hookrightarrow \widehat{\mathcal{S}^\bullet(V^*)}[\![\hbar]\!]$. So we find that

$$c_\alpha \star c_\beta = \varphi_V(J_0^\infty(\iota_V(c_\alpha \star c_\beta))) = \varphi_V([l_\alpha \star l_\beta]),$$

which completes the proof.

□

REMARK 3.1.31. Note that the proposition means that $\hat{\mathbb{A}}^\hbar_{2d} \simeq \mathbb{W}^\hbar(V)$ as $\mathbb{C}[\![\hbar]\!]$-algebras for all $2d$-dimensional symplectic vector spaces $V$. The isomorphism is not canonical however and depends on the choice of a symplectic basis of $V$ in order to identify it with $\mathbb{R}^{2d}$ through the standard (Darboux) coordinates.

REMARK 3.1.32. Note that we could have started the first construction with an arbitrary $2d$-dimensional symplectic vector space $V$, considered with the canonical smooth structure. This would have yielded the corresponding bundle of $\infty$-jets $J_V^\infty$, the map $J_v^\infty\colon C_V^\infty \to J_V^\infty$ and so on. Had we done this the corresponding map $\varphi_V$ in proposition 3.1.30 would of course have been canonical.

DEFINITION 3.1.33. Note that $\mathbb{W}^\hbar(V)$ allows for two filtrations

- $\mathbb{W}^\hbar(V) = F_0\mathbb{W}^\hbar(V) \supset F_1\mathbb{W}^\hbar(V) \supset \ldots \supset F_p\mathbb{W}^\hbar(V) \supset F_{p+1}\mathbb{W}^\hbar(V) \supset \ldots$ given by
$$F_p\mathbb{W}^\hbar(V) = \sum_{i+2j\geq p} \mathfrak{m}^i\hbar^j$$
- $\mathbb{W}^\hbar(V) = F_0^\hbar(\mathbb{W}^\hbar) \supset F_1^\hbar\mathbb{W}^\hbar(V) \supset \ldots \supset F_p^\hbar\mathbb{W}^\hbar(V) \supset F_{p+1}^\hbar\mathbb{W}^\hbar(V) \supset \ldots$ given by
$$F_p^\hbar\mathbb{W}^\hbar(V) = \hbar^p\mathbb{W}^\hbar(V).$$

We supply $\mathbb{W}^\hbar(V)$ with the topology induced from the first filtration and note that $\mathbb{W}^\hbar(V)$ is complete for this topology.

Note that
$$F_p^\hbar\mathbb{W}^\hbar(V) \Big/ F_{p+1}^\hbar\mathbb{W}^\hbar(V) \simeq \widehat{\mathcal{S}^\bullet(V^*)}$$
for all $p \geq 0$ while
$$F_p\mathbb{W}^\hbar(V) \Big/ F_{p+1}\mathbb{W}^\hbar(V) = \hbar^{\frac{p}{2}}\mathbb{C} \oplus \hbar^{\frac{p-2}{2}}\mathfrak{m}^2 \oplus \ldots \oplus \hbar\mathfrak{m}^{p-2} \oplus \mathfrak{m}^p$$
if $p \geq 0$ is even and
$$F_p\mathbb{W}^\hbar(V) \Big/ F_{p+1}\mathbb{W}^\hbar(V) = \hbar^{\frac{p-1}{2}}\mathfrak{m} \oplus \hbar^{\frac{p-3}{2}}\mathfrak{m}^3 \oplus \ldots \oplus \hbar\mathfrak{m}^{p-2} \oplus \mathfrak{m}^p$$
if $p \geq 0$ is odd. Note that this implies that the $F_p\mathbb{W}^\hbar(V)$ filtration comes from a grading. We will denote the homogeneous elements of degree $p \in \mathbb{Z}$ by $\mathbb{W}^\hbar(V)_p$. This grading is given by assigning elements of $V^*$ the degree 1 and $\hbar$ the degree 2. This will be more clear in the following construction.

REMARK 3.1.34. Note that the symplectic groups $\mathrm{Sp}(V)$ and $\mathrm{Sp}(2d,\mathbb{C})$ act on both $\mathbb{W}^\hbar(V)$ (through the action on $V$) and $\hat{\mathbb{A}}_{2d}^\hbar$ (through the action of $\mathrm{Sp}(2d,\mathbb{R})$ on $\mathbb{R}^{2d}$ induced up to the complexified tangent space) respectively. These actions respect the filtrations $F_\bullet\mathbb{W}^\hbar(V)$ and $F_\bullet\hat{\mathbb{A}}_{2d}^\hbar$, so the actions are by continuous automorphisms. The continuity is obvious for the case $\mathbb{W}^\hbar(V)$ and for $\hat{\mathbb{A}}_{2d}^\hbar$ it is deduced from the fact that the action of $\mathrm{Sp}(2d,\mathbb{R})$ on $\mathbb{R}^{2d}$ is by *linear* transformations. By picking symplectic bases in the definition of $\varphi_V$ we ensure that this map is $\mathrm{Sp}(2d,\mathbb{C})$-equivariant (the basis also yields the isomorphism $\mathrm{Sp}(V) \simeq \mathrm{Sp}(2d,\mathbb{C})$).

REMARK 3.1.35. Note that $\varphi_V$ corresponding to any choice of symplectic basis will respect the filtrations $F_\bullet\hat{\mathbb{A}}_{2d}^\hbar$ and $F_\bullet\mathbb{W}^\hbar(V)$. So these $\varphi_V$ are in fact continuous $\mathrm{Sp}(2d,\mathbb{C})$-equivariant $\mathbb{C}[\![\hbar]\!]$-algebra isomorphisms (as are the $\varphi_V^{-1}$'s).

**3.1.3. The Weyl Construction.** Lastly let us present a construction of the formal Moyal–Weyl algebra that differs from the above two mainly in the fact that the product is not defined using a Moyal-type formula. It is simply obtained as a quotient of a tensor algebra, as is the case with the classical Weyl algebra. This last algebra is also where the motivation comes from as it is the algebra generated by $\{x_1,\ldots,x_d,\partial_{x_1},\ldots,\partial_{x_d}\}$ with the relations given by interpreting them as differential operators, which is well-known to form a quantization of the algebra of polynomials on a $2d$-dimensional vector space. The following presentation is based mostly on the article [**89**].

Let $(V,\omega)$ be a $2d$-dimensional symplectic vector space as above and let $(V^*,\bar{\omega})$ be the dual symplectic vector space, i.e. $\bar{\omega}(\alpha,\beta) = \omega(v_\alpha,v_\beta)$ for all $\alpha,\beta \in V^*$. Here $v_\alpha \in V$ is defined as the unique element satisfying the equations $\omega(v_\alpha,w) = \alpha(w)$ for all $w \in V$.

DEFINITION 3.1.36. Consider the tensor algebra $\mathcal{T}(V^*)$ and let $\widehat{\mathcal{T}(V^*)}$ be the completion in the $V^*$-adic topology. We define the formal Moyal–Weyl algebra $\mathbb{W}_\hbar(V)$ as the quotient of $\widehat{\mathcal{T}(V^*)}[\![\hbar]\!]$ by the ideal $\mathcal{I}_{\bar{\omega}}$ generated by the elements

$$\alpha \otimes \beta - \beta \otimes \alpha - i\hbar\bar{\omega}(\alpha, \beta)$$

for $\alpha, \beta \in V^*$. We consider $\widehat{\mathcal{T}(V^*)}[\![\hbar]\!]$ as a graded algebra with the degree given by setting $|\alpha| = 1$ for all $\alpha \in V^*$ and $|\hbar| = 2$. The grading descends to a grading on $\mathbb{W}_\hbar(V)$, since the generators of $\mathcal{I}_{\bar{\omega}}$ are homogeneous. We denote by $\mathbb{W}_\hbar(V)_p$ the homogeneous elements of degree $p \in \mathbb{Z}$ and by

$$\mathbb{W}_\hbar(V) = F_0\mathbb{W}_\hbar(V) \supset F_1\mathbb{W}_\hbar(V) \supset \ldots \supset F_p\mathbb{W}_\hbar(V) \supset F_{p+1}\mathbb{W}_\hbar(V) \supset \ldots$$

the corresponding filtration.

REMARK 3.1.37. We consider $\mathbb{W}_\hbar(V)$ as a topological algebra for the topology induced by the grading. Note that $\mathbb{W}_\hbar(V)$ is complete for this topology.

Note that the definition of $\widehat{\mathcal{T}(V^*)}$ is analogous to the definition of $\widehat{\mathcal{S}^\bullet(V^*)}$. We should justify the fact that we also call $\mathbb{W}_\hbar(V)$ the formal Moyal–Weyl algebra of $V$. To this end we note that the $\mathbb{C}[\![\hbar]\!]$-modules underlying both $\mathbb{W}_\hbar(V)$ and $\mathbb{W}^\hbar(V)$ are quotients of the same algebra $\widehat{\mathcal{T}(V^*)}[\![\hbar]\!]$. Thus we have the diagram

$$\begin{tikzcd} & \widehat{\mathcal{T}(V^*)}[\![\hbar]\!] \arrow[dl, "P_S"'] \arrow[dr, "P_W"] & \\ \widehat{\mathcal{S}^\bullet(V^*)}[\![\hbar]\!] & & \mathbb{W}_\hbar(V) \end{tikzcd} \tag{3.1.8}$$

DEFINITION 3.1.38. Let

$$S_P : \widehat{\mathcal{S}^\bullet(V^*)}[\![\hbar]\!] \longrightarrow \widehat{\mathcal{T}(V^*)}[\![\hbar]\!]$$

be the map given by $\mathbb{C}[\![\hbar]\!]$-linear extension of the well-known section

$$\alpha_1 \ldots \alpha_n \mapsto \frac{1}{n!} \sum_{\tau \in S_n} \alpha_{\tau(1)} \otimes \ldots \otimes \alpha_{\tau(n)}$$

of $P_S$.

DEFINITION 3.1.39. Let

$$Q_V = P_W \circ S_P \colon \mathbb{W}^\hbar(V) \longrightarrow \mathbb{W}_\hbar(V).$$

PROPOSITION 3.1.40. *The map $Q_V$ is an isomorphism of $\mathbb{C}[\![\hbar]\!]$-algebras.*

PROOF.
Consider the gradings on $\mathbb{W}^\hbar(V)$ and $\mathbb{W}_\hbar(V)$ and note that $Q_V$ is a $\mathbb{C}[\![\hbar]\!]$-linear and degree preserving map, simply because $P_W$ and $S_P$ have these properties. Furthermore, note that, by picking an ordered basis $\{\alpha_1, \ldots, \alpha_{2d}\}$ of $V^*$, we see that

$$\mathrm{Dim}\ \mathbb{W}^\hbar(V)_p = \mathrm{Dim}\ P_S\left(\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \hbar^k (V^*)^{\otimes p-2k}\right) = \mathrm{Dim}\ P_W\left(\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \hbar^k (V^*)^{\otimes p-2k}\right) = \mathrm{Dim}\ \mathbb{W}_\hbar(V)_p.$$

So in order to show that $Q_V$ is an isomorphism of $\mathbb{C}[\![\hbar]\!]$-modules we only need to show that it is an injective $\mathbb{C}$-linear map in each total degree. But then, by $\mathbb{C}[\![\hbar]\!]$-linearity, it is enough to show that

$$Q_V(\alpha_{i_1} \ldots \alpha_{i_k}) = 0$$

implies that $\alpha_{i_1} \dots \alpha_{i_k} = 0$ for all $i_1 \leq \dots \leq i_k$. Let us denote the product in $\mathbb{W}_\hbar(V)$ by $\star$. Note that

$$Q_V(\alpha_{i_1} \dots \alpha_{i_k}) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha_{i_{\sigma(1)}} \star \dots \star \alpha_{i_{\sigma(k)}} = \alpha_{i_1} \star \dots \star \alpha_{i_k} + \hbar O$$

where $O$ is a rest term where every term has a tensor degree lower then $k$. In other words in the last expression we have written the elements in terms of the basis determined by the ordering mentioned above. This shows that $Q_V(\alpha_{i_1}\alpha_{i_2} \dots \alpha_{i_k}) = 0$ implies that $\alpha_{i_1} \star \alpha_{i_2} \star \dots \star \alpha_{i_k} = 0$. Note that, by writing the elements of $\mathbb{W}_\hbar(V)$ in normal form, by considering the ordered basis given above, it is evident that $\mathbb{W}_\hbar(V)$ does not contain zero divisors. So we have that $\alpha_{i_1} \star \alpha_{i_2} \star \dots \star \alpha_{i_k} = 0$ implies that $\alpha_{i_1}\alpha_{i_2} \dots \alpha_{i_k} = 0$ and so $Q_V$ is a $\mathbb{C}[\![\hbar]\!]$-linear degree preserving isomorphism.

It is left to show that $Q_V$ is also a $\mathbb{C}[\![\hbar]\!]$-algebra isomorphism. Note that, since $Q_V$ preserves the grading, it is continuous and thus it is enough to show that

$$Q_V((\alpha_1 \dots \alpha_p) \star (\beta_1 \dots \beta_q)) = Q_V(\alpha_1 \dots \alpha_p) \star Q_V(\beta_1 \dots \beta_q)$$

for $\alpha_i, \beta_i \in V^*$. Suppose for convenience that $p \leq q$ (the other case is completely analogous) and note that we have

$$(\alpha_1 \dots \alpha_p) \star (\beta_1 \dots \beta_q) = \sum_{n=0}^{p} \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n \omega(d_n(\alpha_1 \dots \alpha_p), d_n(\beta_1 \dots \beta_q))$$

since $d_n(\alpha_1 \dots \alpha_p) = 0$ for all $n > p$. Furthermore

$$\omega(d_n(\alpha_1 \dots \alpha_p), d_n(\beta_1 \dots \beta_q)) = (n!)^2 \sum_{I,J} \alpha_{I^c}\beta_{J^c}\omega(v_{\alpha_I}, v_{\beta_J}).$$

Here we sum over all cardinality $n$ subsets $I$ and $J$ of $[p]$ and $[q]$ respectively. We have denoted the complements of $I$ and $J$ by $I^c$ and $J^c$ respectively and we set $v_{\alpha_I} := v_{\alpha_{i_1}} \dots v_{\alpha_{i_n}}$ when $I = \{i_1, \dots, i_n\}$ and similarly for $v_{\beta_J}$, $\alpha_{I^c}$ and $\beta_{J^c}$. In the expression above we have

$$\omega(v_{\alpha_I}, v_{\beta_J}) = \sum_{\sigma \in S_n} \frac{1}{n!} \prod_{k=1}^{n} \bar{\omega}(\alpha_{i_{\sigma(k)}}, \beta_{j_k}).$$

So bringing all of it together and applying $Q_V$ we find that $Q_V((\alpha_1 \dots \alpha_p) \star (\beta_1 \dots \beta_q))$ equals

$$\sum_{n=0}^{p} \sum_{I,J} \sum_{\substack{\sigma \in S_n \\ \tau \in S_{p+q-2n}}} \left(\frac{i\hbar}{2}\right)^n \frac{\gamma^{I,J}_{a_{\tau(1)}} \star \dots \star \gamma^{I,J}_{a_{\tau(p+q-2n)}}}{(p+q-2n)!} \prod_{k=1}^{n} \bar{\omega}(\alpha_{i_{\sigma(k)}}, \beta_{j_k}), \tag{3.1.9}$$

where we have relabeled $I^c \cup J^c = \{a_1, \dots, a_{p+q-2n}\}$ (where $a_1, \dots, a_{p-n} \in I^c$) and $\gamma^{I,J}_{a_r} = \alpha_{a_r}$ for all $r \leq p - n$ and $\gamma^{I,J}_{a_r} = \beta_{a_r}$ for all $r > p - n$. On the other hand

$$Q_V(\alpha_1 \dots \alpha_p) \star Q_V(\beta_1 \dots \beta_q) = \frac{1}{p!q!} \sum_{\sigma \in S_p} \sum_{\tau \in S_q} \alpha_{\sigma(1)} \star \dots \star \alpha_{\sigma(p)} \star \beta_{\tau(1)} \star \dots \star \beta_{\tau(q)}$$

To demonstrate that (3.1.9) agrees with this expression let us start with the term with the lowest power of $\hbar$. Note that the $n = 0$ term in (3.1.9) is

$$\frac{1}{(p+q)!} \sum_{\tau \in S_{p+q}} \gamma_{\tau(1)} \star \dots \star \gamma_{\tau(p+q)}$$

where $\gamma_r = \alpha_r$ if $r \leq p$ and $\gamma_r = \beta_{r-p}$ if $r > p$, so, since the number of $(p,q)$-shuffles in $S_{p+q}$ is $\frac{(p+q)!}{p!q!}$, we find that after reordering we get exactly $Q_V(\alpha_1 \dots \alpha_p) \star Q_V(\beta_1 \dots \beta_q)$. This is because the cosets $S_{p+q}/S_p \times S_q$ can be labeled exactly by the $(p,q)$-shuffles, here we consider the embedding $S_p \times S_q \hookrightarrow S_{p+q}$ corresponding to separate permutation of the first $p$ and last $q$ letters. The terms for $n > 0$ will be killed by the reordering as follows. The reordering will produce terms multiplied by a factor of $i\hbar\bar{\omega}(\beta_r, \alpha_s)$ when we reorder $\ldots\beta_r\ldots\alpha_s\ldots$ to $\ldots\alpha_s\beta_r\ldots$, note that we only need to reorder such that $\alpha$'s end up to the left of $\beta$'s. It will produce every permutation of the factor multiplying $i\hbar\bar{\omega}(\beta_r, \alpha_s)$ exactly $\frac{(p+q)!}{2!(p+q-2)!}$ times, since this is the number of $(2, p+q-2)$-shuffles. This is because the permutations that place $\beta_r$ to the left of $\alpha_s$ can be decomposed as the permutation $((p+r), s, 1, 2, \ldots)$ followed by a permutation of the last $p+q-2$ letters and then followed by a $(2, p+q-2)$-shuffle. This accounts for all of the $n = 1$ terms in (3.1.9). More generally, for $0 \leq m \leq p$, the reordering will produce terms multiplied by a factor of

$$(i\hbar)^m \prod_{l=1}^{m} \bar{\omega}(\beta_{r_l}, \alpha_{s_l}).$$

As for the case $n = 1$, it will produce every permutation of the factor multiplying it exactly $\frac{(p+q)!}{2^m(p+q-2m)!}$ times. This is because we can decompose all the permutations that put $\beta_{r_l}$ to the left of $\alpha_{s_l}$ for all $1 \leq l \leq m$ as the permutation $((r_1+p), s_1, (r_2+p), s_2, \ldots, (r_m+p), s_m, 1, 2, \ldots)$ followed by a $(2, \ldots, 2)$-shuffle ($m$ times 2) of the first $2m$ letters followed by a permutation of the last $(p+q-2m)$ letters followed by a $(2m, p+q-2m)$-shuffle and there are $\frac{(2m)!}{(2!)^m}$ shuffles of the type $(2, \ldots, 2)$ and $\frac{(p+q)!}{(2m)!(p+q-2m)!}$ shuffles of the type $(2m, p+q-2m)$. This accounts for all the $n = m$ terms in (3.1.9). So we see that $Q_V(\alpha_1 \ldots \alpha_p) \star Q_V(\beta_1 \ldots \beta_q)$ agrees with (3.1.9), which means $Q_V$ is a $\mathbb{C}[\![\hbar]\!]$-algebra isomorphism.

□

REMARK 3.1.41. Note that the action of $\mathrm{Sp}(V)$ on $V^*$ extends to an action on $\mathbb{W}_\hbar(V)$ since $\mathrm{Sp}(V)$ preserves $\mathcal{I}_{\bar{\omega}}$. This action is by continuous automorphisms since it preserves the grading. Clearly the isomorphism $Q_V$ is also $\mathrm{Sp}(V)$-equivariant. Note also that, unlike $\varphi_V$ and thus $Q_V \circ \varphi_V$, the isomorphism $Q_V$ is canonical and thus we are even more justified in calling both $\mathbb{W}_\hbar(V)$ and $\mathbb{W}^\hbar(V)$ by the same name.

REMARK 3.1.42. Note finally that the map $Q_V$ respects the filtration $F_\bullet\mathbb{W}^\hbar(V)$ and the filtration induced by the grading on $\mathbb{W}_\hbar(V)$. Thus $Q_V$ and its inverse are in fact continuous $\mathrm{Sp}(V)$-equivariant $\mathbb{C}[\![\hbar]\!]$-algebra isomorphisms.

## 3.2. Derivations and Automorphisms

In this section we will give a rather complete description of the $\mathbb{C}[\![\hbar]\!]$-linear continuous derivations and automorphisms of $\mathbb{W}_\hbar(V)$ for any symplectic vector space $(V, \omega)$. This description is very useful when implementing the framework of formal geometry and the Fedosov construction. The following is based on [**14**] and [**89**].

**3.2.1. Derivations.** We will denote the Lie algebra $\mathrm{Der}^0(\mathbb{W}_\hbar(V))$ of continuous $\mathbb{C}[\![\hbar]\!]$-linear derivations of $\mathbb{W}_\hbar(V)$ by $\mathfrak{g}^\hbar = \mathfrak{g}^\hbar_V$.

PROPOSITION 3.2.1. *There is an exact sequence of Lie algebras*

$$0 \to \frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!] \longrightarrow \frac{1}{i\hbar}\mathbb{W}_\hbar(V) \xrightarrow{P_{\mathfrak{g}^\hbar}} \mathfrak{g}^\hbar_V \to 0. \tag{3.2.1}$$

*Here we consider $\frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!]$ as Abelian Lie algebra and on $\frac{1}{i\hbar}\mathbb{W}_\hbar(V)$ we consider the bracket*

$$[\frac{1}{i\hbar}f, \frac{1}{i\hbar}g] = \left(\frac{1}{i\hbar}\right)^2 [f, g],$$

*where on the right hand side $[f, g]$ denotes the commutator. The first map is given by the inclusion while*

$$P_{\mathfrak{g}^\hbar}(\frac{1}{i\hbar}f)(g) = \frac{1}{i\hbar}[f, g].$$

PROOF.
Note first that the Lie algebra structures on $\mathbb{W}_\hbar(V)$ and $P_{\mathfrak{g}^\hbar}$ are well-defined, since the commutator always produces a factor of $i\hbar$. It is easily verified that $P_{\mathfrak{g}^\hbar}$ is a map of Lie algebras by using the Jacobi identity for the commutator bracket.

Let us fix a symplectic basis $\{x_1, \dots, x_d, y_1, \dots, y_d\}$ of $V^*$, i.e. s.t.

$$\bar{\omega}(x_i, x_j) = \bar{\omega}(y_i, y_j) = 0 \quad \text{and} \quad \bar{\omega}(y_i, x_j) = \delta_{ij} = -\bar{\omega}(x_j, y_i),$$

where $\delta_{ij}$ denotes the Kronecker delta. Note that we can write every element of $\mathbb{W}_\hbar(V)$ as a formal power series in $\hbar$ and the $x_i$ and $y_i$ in a unique way, by requiring that all monomials appear in the ordering

$$\hbar^k x_1^{i_1} \dots x_d^{i_d} y_1^{j_1} \dots y_d^{j_d},$$

i.e. first the $x$'s and then the $y$'s. Here we have denoted the product in $\mathbb{W}_\hbar(V)$ by concatenation. In fact, this gives us an identification of $\mathbb{W}_\hbar(V)$ with the space of formal power series $\mathbb{C}[\![\hbar, x_1, \dots, x_d, y_1, \dots, y_d]\!]$. Note that

$$[x_k, (i\hbar)^k x_1^{i_1} \dots x_d^{i_d} y_1^{j_1} \dots y_d^{j_d}] = -(i\hbar)^{k+1} j_k x_1^{i_1} \dots x_d^{i_d} y_1^{j_1} \dots y_k^{j_k - 1} \dots y_d^{j_d}$$

and similarly

$$[y_k, (i\hbar)^k x_1^{i_1} \dots x_d^{i_d} y_1^{j_1} \dots y_d^{j_d}] = (i\hbar)^{k+1} i_k x_1^{i_1} \dots x_k^{i_k - 1} \dots x_d^{i_d} y_1^{j_1} \dots y_d^{j_d}.$$

This invites the notation

$$\frac{1}{i\hbar}[x_k, f] =: -\partial_{y_k} f \quad \text{and} \quad \frac{1}{i\hbar}[y_k, f] =: \partial_{x_k} f. \tag{3.2.2}$$

In fact, under the identification of $\mathbb{W}_\hbar(V)$ and formal power series, mentioned above, the operators $\frac{1}{i\hbar}[x_k, -]$ and $\frac{1}{i\hbar}[y_k, -]$ are intertwined with the operators $-\partial_{y_k}$ and $\partial_{x_k}$ respectively. Here the definition of partial differentiation is simply the $\mathbb{C}[\![\hbar]\!]$-linear continuous extension of the definition on polynomials.

Suppose $f, g \in \mathbb{W}_\hbar(V)$ satisfy

$$\frac{1}{i\hbar}[f, h] = \frac{1}{i\hbar}[g, h]$$

for all $h \in \mathbb{W}_\hbar(V)$. Then, using the identification above, we find that

$$\partial_{x_k} f = \partial_{x_k} g \quad \text{and} \quad \partial_{y_k} f = \partial_{y_k} g$$

for all indices $1 \leq k \leq d$. This implies that $f - g \in \frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!]$, which shows exactness at $\mathbb{W}_\hbar(V)$.

The proposition is proved by showing that, for any continuous derivation $D \in \mathfrak{g}^\hbar$, we can find an element $F \in \mathbb{W}_\hbar(V)$ such that

$$D(g) = \frac{1}{i\hbar}[F, g]$$

for all $g \in \mathbb{W}_\hbar(V)$. It is enough to verify the equations above on the $x_k's$ and $y_k's$ since they generate $\mathbb{W}_\hbar(V)$ over $\mathbb{C}[\![\hbar]\!]$ (as a topological algebra) and $D$ is a $\mathbb{C}[\![\hbar]\!]$-linear continuous derivation. Under the identification (3.2.2), this means we want to show that the system of differential equations

$$\partial_{x_k} F = -D(y_k) \quad \text{and} \quad \partial_{y_k} F = D(x_k)$$

can be solved for $F$ for any derivation $D \in \mathfrak{g}^\hbar$. Let us denote

$$\hat{\Omega}_V^r := \mathbb{C}[\![\hbar, x_1, \dots, x_d, y_1, \dots, y_d]\!] \otimes_{\mathbb{C}} \textstyle\bigwedge^r V^*,$$

where $0 \leq r \leq 2d$ and for the anti-symmetric powers we denote the basis of $V^*$ by $dx_k$ and $dy_k$. Also define

$$\hat{d}_{\mathrm{dR}} \colon \hat{\Omega}_V^k \longrightarrow \hat{\Omega}_V^{k+1}$$

as the continuous $\mathbb{C}[\![\hbar]\!]$-linear extension of the exterior derivative on polynomials. Explicitly

$$\hat{d}_{\mathrm{dR}}(f \otimes \eta) = \sum_{k=1}^{d} (\partial_{x_k} f) \otimes dx_k \wedge \eta + (\partial_{y_k} f) \otimes dy_k \wedge \eta,$$

for $f$ a formal power series and $\eta \in \bigwedge^{\bullet} V^*$. Finally, for $D \in \mathfrak{g}^{\hbar}$, define

$$\hat{D} := \sum_{i=1}^{d} D(x_i) \otimes dy_i - D(y_i) \otimes dx_i \in \hat{\Omega}^1_V.$$

With this rephrasing, the exactness at $\mathfrak{g}^{\hbar}$ translates to showing that, given $D \in \mathfrak{g}^{\hbar}$, one can always find $F \in \hat{\Omega}^0_V$ such that

$$\hat{d}_{\mathrm{dR}} F = \hat{D}.$$

Now, by the formal Poincaré lemma [**76**], this is equivalent to showing that

$$\hat{d}_{\mathrm{dR}} \hat{D} = 0$$

for all $D \in \mathfrak{g}^{\hbar}$. Note that, for any $D \in \mathfrak{g}^{\hbar}$ and any $a, b \in \mathbb{W}_{\hbar}(V)$, we have

$$D([a,b]) = [D(a), b] + [a, D(b)],$$

since $D$ is a derivation. This implies that

$$\frac{1}{i\hbar}[D(x_i), x_j] = \frac{1}{i\hbar}[D(x_j), x_i]$$
$$\frac{1}{i\hbar}[y_j, D(y_i)] = \frac{1}{i\hbar}[y_i, D(y_j)]$$

and

$$\frac{1}{i\hbar}[y_j, D(x_i)] = \frac{1}{i\hbar}[x_i, D(y_j)]$$

since $D(\mathbb{C}[\![\hbar]\!]) = 0$ for all $D \in \mathfrak{g}^{\hbar}$. Using the identification (3.2.2), the above equations are

$$\partial_{y_j} D(x_i) = \partial_{y_i} D(x_j), \qquad \partial_{x_j} D(y_i) = \partial_{x_i} D(y_j) \qquad \text{and} \qquad \partial_{x_j} D(x_i) = -\partial_{y_i} D(y_j),$$

which implies $\hat{d}_{\mathrm{dR}} \hat{D} = 0$ for all $D \in \mathfrak{g}^{\hbar}$. □

REMARK 3.2.2. The short exact sequence above gives rise to a class $\hat{\theta}$ in the continuous Lie algebra cohomology (Gelfand-Fuks cohomology) group $\mathrm{H}^2_{Lie}(\mathfrak{g}^{\hbar}, \mathfrak{a})$, where we have denoted the Abelian Lie algebra $\frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!]$ by $\mathfrak{a}$. The class is represented by the cocycle given by

$$X \wedge Y \mapsto s([X,Y]) - [s(X), s(Y)]$$

where $s \colon \mathfrak{g}^{\hbar} \to \frac{1}{i\hbar}\mathbb{W}_{\hbar}(V)$ is a (continuous) linear section. For instance one may choose $s$ by considering an ordered symplectic basis for $V^*$ and considering the corresponding splitting (as a vector space)

$$\frac{1}{i\hbar}\mathbb{W}_{\hbar}(V) = \mathfrak{g}^{\hbar} \oplus \mathfrak{a}.$$

It is an easy standard check that $\hat{\theta}$ is well-defined. We will call $\hat{\theta}$ the *formal Weyl curvature*. It is the formal analog of the characteristic class of a deformation quantization (also called Weyl curvature or Deligne's characteristic class [**48, 65**]).

DEFINITION 3.2.3. We denote the filtration on $\mathfrak{g}^{\hbar}$ induced by the filtration $F_{\bullet}\mathbb{W}_{\hbar}(V)$ by

$$\mathfrak{g}^{\hbar} = F_{-1}\mathfrak{g}^{\hbar} \supset F_0\mathfrak{g}^{\hbar} \supset \ldots \supset F_p\mathfrak{g}^{\hbar} \supset F_{p+1}\mathfrak{g}^{\hbar} \supset \ldots,$$

i.e.

$$F_p\mathfrak{g}^{\hbar} = \left\{ D \in \mathfrak{g}^{\hbar} \mid D(F_k\mathbb{W}_{\hbar}(V)) \subset F_{k+p}\mathbb{W}_{\hbar}(V) \quad \text{for all } k \right\}$$

Note that this makes $\mathfrak{g}^\hbar$ a filtered Lie algebra, i.e. $[F_p\mathfrak{g}^\hbar, F_q\mathfrak{g}^\hbar] \subset F_{p+q}\mathfrak{g}^\hbar$, and the $\mathfrak{g}^\hbar$-module structure of $\mathbb{W}_\hbar(V)$ respects the filtration, i.e. $F_p\mathfrak{g}^\hbar F_q\mathbb{W}_\hbar(V) \subset F_{p+q}\mathbb{W}_\hbar(V)$. Note that the filtration on $\mathfrak{g}^\hbar$ actually comes from a grading, i.e., denoting $\mathfrak{g}^\hbar_p = P_{\mathfrak{g}^\hbar}(\frac{1}{i\hbar}\mathbb{W}_\hbar(V)_{p+2})$, we have that

$$\mathfrak{g}^\hbar = \prod_{k=-1}^{\infty} \mathfrak{g}^\hbar_k.$$

PROPOSITION 3.2.4. *There are canonical isomorphisms $V \simeq \mathfrak{g}^\hbar_{-1}$ and $\mathfrak{sp}(V) \simeq \mathfrak{g}^\hbar_0$, where we have denoted the Lie algebra of $Sp(V)$ by $\mathfrak{sp}(V)$. Under these isomorphisms the action of $\mathfrak{g}^\hbar_0$ on $\mathfrak{g}^\hbar_{-1}$ coincides with the natural action of $\mathfrak{sp}(V)$ on $V$ and we have*

$$\mathfrak{g}^\hbar_{-1} \oplus \mathfrak{g}^\hbar_0 \simeq V \rtimes \mathfrak{sp}(V)$$

*as Lie algebras.*

PROOF.
Note that, since $\mathfrak{g}^\hbar_p = 0$ for $p < -1$, we find that the induced bracket on $\mathfrak{g}^\hbar_{-1}$ vanishes, so that it forms an Abelian Lie algebra. Recall that the kernel of $P_{\mathfrak{g}^\hbar}$ only contains elements of even degree and that $\frac{1}{i\hbar}\mathbb{W}_\hbar(V)_1 = \frac{1}{i\hbar}V^*$. This yields the sequence of isomorphisms

$$V \longrightarrow \frac{1}{i\hbar}V^* \longrightarrow \frac{1}{i\hbar}\mathbb{W}_\hbar(V)_1 \longrightarrow \mathfrak{g}^\hbar_{-1}.$$

The first map is given by $\frac{1}{i\hbar}I_\omega$, where we have denoted the isomorphism $V \simeq V^*$ induced by $\omega$ by $I_\omega$, i.e. $I_\omega(v)(w) = \omega(v, w)$ for all $v, w \in V$. Note that the above isomorphism is canonical for the symplectic vector space $(V, \omega)$.

For the degree 0 case, we start by noting that we have the usual inclusion $\mathfrak{sp}(V) \hookrightarrow V \otimes V^*$. Now we can apply the restriction of $I_\omega^{-1} \otimes \mathrm{Id}$ to $\mathfrak{sp}(V)$ to obtain $\mathfrak{sp}(V) \hookrightarrow V^* \otimes V^*$. We note that the condition

$$\omega(v, \varphi(w)) + \omega(\varphi(v), w) = 0,$$

defining $\mathfrak{sp}(V)$, implies that the image of $\mathfrak{sp}(V)$ in $V^* \otimes V^*$ consists of symmetric tensors. We proceed by applying the restriction of the map $\frac{1}{i\hbar}P_W$, where $P_W$ is as in (3.1.8), to the image of $\mathfrak{sp}(V)$ in $V^* \otimes V^*$. Finally, we follow this map by $P_{\mathfrak{g}^\hbar}$ to obtain a map $\mathfrak{sp}(V) \to \mathfrak{g}^\hbar$. Note that the image clearly lies in $\mathfrak{g}^\hbar_0$ and thus we find the needed map

$$\mathfrak{sp}(V) \longrightarrow \mathfrak{g}^\hbar_0.$$

Note that this map is constructed canonically given the symplectic vector space $(V, \omega)$. Note that $\mathfrak{sp}(V) \to V^* \otimes V^*$ is an inclusion, the image is given by symmetric tensors and $Q_V$ from proposition 3.1.40 factors through $P_W$. So we find that $\mathfrak{sp}(V) \to \frac{1}{i\hbar}\mathbb{W}_\hbar$ is injective. Since the image does not intersect $\frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!]$ we find that $\mathfrak{sp}(V) \to \mathfrak{g}^\hbar_0$ is injective. Thus, since

$$\mathrm{Dim}\ \mathfrak{sp}(V) = 2d^2 + d = \mathrm{Dim}\ \mathfrak{g}^\hbar_0,$$

we find that $\mathfrak{sp}(V) \simeq \mathfrak{g}^\hbar_0$ as vector spaces. Since $[F_0\mathfrak{g}^\hbar, F_0\mathfrak{g}^\hbar] \subset F_0\mathfrak{g}^\hbar$ and $[F_1\mathfrak{g}^\hbar, F_1\mathfrak{g}^\hbar] \subset F_2\mathfrak{g}^\hbar$, we find that the bracket restricts to $\mathfrak{g}^\hbar_0$ and it is left to show that the isomorphism above respects the brackets.

To show this, Consider a symplectic basis $\{\hat{x}_1, \dots, \hat{x}_d, \hat{y}_1, \dots, \hat{y}_d\}$ for $V$ with the corresponding dual basis $\{x_1, \dots, x_d, y_1, \dots, y_d\}$. This induces the basis

$$\hat{S}_{ij} = \hat{y}_i \otimes y_j - \hat{x}_j \otimes x_i, \quad \hat{Y}_{ij} = \hat{y}_i \otimes x_j + \hat{y}_j \otimes x_i \quad \text{and} \quad \hat{V}_{ij} = \hat{x}_i \otimes y_j + \hat{x}_j \otimes y_i$$

of $\mathfrak{sp}(V)$ (identified with the image in $V \otimes V^*$). It gets mapped to the basis

$$S_{ij} = \frac{1}{i\hbar}[x_i y_j, -], \quad Y_{ij} = \frac{1}{i\hbar}[x_i x_j, -] \quad \text{and} \quad V_{ij} = \frac{-1}{i\hbar}[y_i y_j, -] \tag{3.2.3}$$

of $\mathfrak{g}^\hbar_0$. Using these bases it is explicitly verified that the isomorphism $\mathfrak{sp}(V) \simeq \mathfrak{g}^\hbar_0$ respects the brackets.

Note that, since $[\mathfrak{g}_0^\hbar, \mathfrak{g}_{-1}^\hbar] \subset \mathfrak{g}_{-1}^\hbar$, we indeed find an action of $\mathfrak{g}_0^\hbar$ on $\mathfrak{g}_{-1}^\hbar$. Consider the bases given above and note that

$$\hat{y}_i \mapsto \frac{1}{i\hbar}[x_i, -] \quad \text{and} \quad \hat{x}_i \mapsto \frac{-1}{i\hbar}[y_i, -]$$

gives the corresponding basis for $\mathfrak{g}_{-1}^\hbar$. Again it is verified using these bases that the action of $\mathfrak{g}_0^\hbar$ on $\mathfrak{g}_{-1}^\hbar$ coincides with the action of $\mathfrak{sp}(V)$ on $V$ under the above identifications. Note that, since $[\mathfrak{g}_0^\hbar, \mathfrak{g}_0^\hbar] \subset \mathfrak{g}_0^\hbar$, $[\mathfrak{g}_{-1}^\hbar, \mathfrak{g}_0^\hbar] \subset \mathfrak{g}_{-1}^\hbar$ and $[\mathfrak{g}_{-1}^\hbar, \mathfrak{g}_{-1}^\hbar] \subset \mathfrak{g}_{-1}^\hbar$, we find that the bracket restricts to a bracket on $\mathfrak{g}_0^\hbar \oplus \mathfrak{g}_{-1}^\hbar =: \mathfrak{g}_{0,-1}^\hbar$. Using the identifications above we find the exact sequence

$$0 \to V \longrightarrow \mathfrak{g}_{0,-1}^\hbar \longrightarrow \mathfrak{sp}(V) \to 0$$

of Lie algebras. The inclusion $\mathfrak{g}_0^\hbar \to \mathfrak{g}_{0,-1}^\hbar$ provides a section of the sequence above. This shows that indeed

$$\mathfrak{g}_{0,-1}^\hbar \simeq V \rtimes \mathfrak{sp}(V).$$

□

The following proposition will deal with the remaining part of $\mathfrak{g}^\hbar = \mathfrak{g}_{0,-1}^\hbar \oplus F_1\mathfrak{g}^\hbar$ and conclude our discussion of the Lie algebra of continuous derivations of the formal Moyal–Weyl algebra.

Proposition 3.2.5. *The Lie algebra $F_1\mathfrak{g}^\hbar$ is pro-nilpotent.*

Proof.
Note that, since $[F_1\mathfrak{g}^\hbar, F_p\mathfrak{g}^\hbar] \subset F_{p+1}\mathfrak{g}^\hbar$, the bracket descends to define the family of Lie algebras

$$F_1^p\mathfrak{g}^\hbar := F_1\mathfrak{g}^\hbar \Big/ F_p\mathfrak{g}^\hbar.$$

Note also that, given $X_1, \ldots, X_p \in F_1\mathfrak{g}^\hbar$, we have $[X_1, [X_2, [\ldots, [X_{p-1}, X_p]]\ldots] \in F_p\mathfrak{g}^\hbar$ and thus $F_1^p\mathfrak{g}^\hbar$ is nilpotent for all $p$. It is easily verified that

$$F_1\mathfrak{g}^\hbar \simeq \varprojlim_p F_1^p\mathfrak{g}^\hbar,$$

where the limit runs over the projection maps $F_1^i\mathfrak{g}^\hbar \to F_1^j\mathfrak{g}^\hbar$ if $j \leq i$. □

Corollary 3.2.6. *The subalgebra $F_0\mathfrak{g}^\hbar \subset \mathfrak{g}^\hbar$ decomposes as*

$$F_0\mathfrak{g}^\hbar \simeq F_1\mathfrak{g}^\hbar \rtimes \mathfrak{sp}(V).$$

Proof.
Note that $F_1\mathfrak{g}^\hbar \subset F_0\mathfrak{g}^\hbar$ is an ideal. The identification $\mathfrak{g}_0^\hbar \simeq \mathfrak{sp}(V)$ yields the exact sequence

$$0 \to F_1\mathfrak{g}^\hbar \longrightarrow F_0\mathfrak{g}^\hbar \longrightarrow \mathfrak{sp}(V) \to 0$$

and so the section $\mathfrak{sp}(V) \to F_0\mathfrak{g}^\hbar$, given by the above identification and inclusion of $\mathfrak{g}_0^\hbar$, yields the result. □

Remark 3.2.7. Consider the action of $\mathfrak{g}_0^\hbar \simeq \mathfrak{sp}(V)$ on the Lie algebras $\mathfrak{a}$, $\mathfrak{g}^\hbar$, and $\frac{1}{i\hbar}\mathbb{W}_\hbar(V)$. Proposition 3.2.4 and corollary 3.2.6 show that the exact sequence (3.2.1) is $\mathfrak{g}_0^\hbar$-equivariant. Note also that the sequence (3.2.1) has the subsequence

$$0 \to \mathbb{C} \longrightarrow \frac{1}{i\hbar}\mathbb{W}_\hbar(V)_2 \longrightarrow \mathfrak{g}_0^\hbar \to 0.$$

Suppose $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ is a symplectic basis for $V^*$. Then define the map

$$\sigma\colon \mathfrak{g}_0^\hbar \longrightarrow \frac{1}{i\hbar}\mathbb{W}_\hbar(V)_2$$

by

$$\sigma(S_{ij}) = \frac{1}{i\hbar}x_iy_j + \delta_{ij}, \quad \sigma(Y_{ij}) = \frac{1}{i\hbar}x_ix_j \quad \text{and} \quad \sigma(V_{ij}) = \frac{-1}{i\hbar}y_iy_j$$

where $\delta_{ij}$ denotes the Kronecker delta and the $S_{ij}$, $V_{ij}$ and $Y_{ij}$ are as in (3.2.3). Note that $\sigma$ defines a section and so we have

$$\frac{1}{i\hbar}\mathbb{W}_\hbar(V)_2 \simeq \mathbb{C} \oplus \mathfrak{sp}(V)$$

as Lie algebras. We can extend the section $\sigma$ to a continuous $\mathbb{C}[\![\hbar]\!]$-linear map

$$\tilde{\sigma} \colon \mathfrak{g}^\hbar \longrightarrow \frac{1}{i\hbar}\mathbb{W}_\hbar(V)$$

by following the procedure laid out in remark 3.2.2. Note that the section $s$ proposed in remark 3.2.2 and $\tilde{\sigma}$ only differ by their action on the $S_{ij}$. Now the cochain

$$X \wedge Y \mapsto \tilde{\sigma}([X,Y]) - [\tilde{\sigma}(X), \tilde{\sigma}(Y)]$$

lifts to $\mathfrak{g}^\hbar/\mathfrak{g}^\hbar_0 \wedge \mathfrak{g}^\hbar/\mathfrak{g}^\hbar_0$. This shows that in fact

$$\hat{\theta} \in \mathrm{H}^2_{Lie}(\mathfrak{g}^\hbar, \mathfrak{sp}(V); \mathfrak{a}),$$

i.e. the formal Weyl curvature is a relative continuous Lie algebra cohomology class.

**3.2.2. Automorphisms.** We will denote the group $\mathrm{Aut}^0(\mathbb{W}_\hbar(V))$ of continuous $\mathbb{C}[\![\hbar]\!]$-linear automorphisms of $\mathbb{W}_\hbar(V)$ (with continuous inverses) by $\widehat{G}^\hbar_V = \widehat{G}^\hbar$. First of all, we will show that $\widehat{G}^\hbar_V$ can be integrated from $F_0\mathfrak{g}^\hbar$, by showing the decomposition of $\widehat{G}^\hbar_V$ analogous to the decomposition in the corollary 3.2.6.

Proposition 3.2.8. *The group $\widehat{G}^\hbar$ decomposes (canonically) as*

$$\widehat{G}^\hbar \simeq \exp F_1\mathfrak{g}^\hbar \rtimes Sp(V).$$

Proof.
Suppose $\varphi \in \widehat{G}^\hbar_V$, then by continuity it will preserve the filtration $F_p\mathbb{W}_\hbar(V)$, i.e.

$$\varphi(F_p\mathbb{W}_\hbar(V)) = F_p\mathbb{W}_\hbar(V).$$

Let $\varphi_n \colon \mathbb{W}_\hbar(V)_n \to \mathbb{W}_\hbar(V)_n$ be given by

$$\varphi_n(w) = \varphi(w) \mod F_{n+1}\mathbb{W}_\hbar(V)$$

for all $n > 0$. Note that $\varphi_n$ is a well-defined $\mathbb{C}$-linear map, it is invertible with inverse given by $\varphi_n^{-1} = (\varphi^{-1})_n$ and it satisfies $\varphi_n(\hbar w) = \hbar\varphi_n(w)$. Note that, since

$$\mathbb{W}_\hbar(V)_n = \mathbb{W}_\hbar(V)_1\mathbb{W}_\hbar(V)_{n-1}$$

for all $n > 0$ and

$$\varphi_{n+m}(vw) = \varphi_n(v)\varphi_m(w)$$

for all $v \in \mathbb{W}_\hbar(V)_n$ and $w \in \mathbb{W}_\hbar(V)_m$, we find that $\varphi_n$ is determined by $\varphi_1$.

Consider the map $p \colon \widehat{G}^\hbar \to \mathrm{Sp}(V)$ given by $\varphi \mapsto \varphi_1$. Indeed, if $v, w \in V^*$, we have

$$i\hbar\bar{\omega}(\varphi_1(v), \varphi_1(w)) = [\varphi_1(v), \varphi_1(w)] = \varphi_1([v,w]) = i\hbar\bar{\omega}(v,w),$$

which shows that $\varphi_1 \in \mathrm{Sp}(V)$. Suppose on the other hand that $A \in \mathrm{Sp}(V)$, then the natural action on $V^*$ extends to an algebra automorphism of $\mathbb{W}_\hbar(V)$ (also denoted $A$) and we clearly have $A_1 = A$ as elements of $\mathrm{Sp}(V)$. Note that this actually defines a section of $p$. Thus, to show the proposition, we only need to show that

$$\mathrm{Ker}\, p = \exp F_1\mathfrak{g}^\hbar.$$

Suppose $\varphi \in \mathrm{Ker}\, p$, then $\varphi_1 = \mathrm{Id}$ and thus $\varphi_n = \mathrm{Id}$ for all $n > 0$. This means that

$$(\varphi - \mathrm{Id})(F_p\mathbb{W}_\hbar(V)) \subset F_{p+1}\mathbb{W}_\hbar$$

for all $p$. Let us denote

$$D_\varphi := \operatorname{Log}\varphi := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(\varphi - \mathrm{Id})^k.$$

Note that $D_\varphi$ is a well-defined continuous endomorphisms of $\mathbb{W}_\hbar(V)$, since $(\varphi - \mathrm{Id})$ raises degrees. A routine computation shows that $D_\varphi$ is always a derivation. So, since $D_\varphi$ raises degrees, we find that $D_\varphi \in F_1\mathfrak{g}^\hbar$.

On the other hand, suppose $D \in F_1\mathfrak{g}^\hbar$. Since $D$ raises degrees, the map $\varphi_D \in \mathrm{End}(\mathbb{W}_\hbar(V))$ given by

$$\varphi_D := \exp D := \sum_{k=0}^{\infty} \frac{D^k}{k!}$$

is well-defined. Another routine computation shows that $\varphi_D(ab) = \varphi_D(a)\varphi_D(b)$ and $\varphi_D$ is invertible with inverse $\varphi_{-D} = (\varphi_D)^{-1}$. Thus we have a map $F_1\mathfrak{g}^\hbar \to \widehat{G}^\hbar$. Note that, by definition of $\varphi_D$ and the fact that $D \in F_1\mathfrak{g}^\hbar$ raises degrees, we find that $(\varphi_D)_1 = \mathrm{Id}$. In other words $\varphi_D \in \operatorname{Ker} p$ for all $D \in F_1\mathfrak{g}^\hbar$. Now note that, by direct computation,

$$\operatorname{Log}\,\exp D = D \qquad \text{and} \qquad \exp \operatorname{Log}\varphi = \varphi$$

for all $D \in F_1\mathfrak{g}^\hbar$ and $\varphi \in \operatorname{Ker} p$. This shows finally that $\operatorname{Ker} p = \exp F_1\mathfrak{g}^\hbar$. So we have the sequence

$$0 \to \exp F_1\mathfrak{g}^\hbar \longrightarrow \widehat{G}^\hbar \longrightarrow \mathrm{Sp}(V) \to 0$$

which splits, as noted above. This shows that $\widehat{G}^\hbar \simeq \exp F_1\mathfrak{g}^\hbar \rtimes \mathrm{Sp}(V)$. □

REMARK 3.2.9. Note that, since $F_1\mathfrak{g}^\hbar = P_{\mathfrak{g}^\hbar}(\frac{1}{i\hbar}\mathbb{W}_\hbar(V)_3)$ by proposition 3.2.1, we can go a bit further. Namely, given $D \in F_1\mathfrak{g}^\hbar$, there is $S_D \in F_3\mathbb{W}_\hbar(V)$ such that $\frac{1}{i\hbar}[S_D, -] = D$. For $S \in F_3\mathbb{W}_\hbar(V)$ let us set

$$\mathrm{Ad}\,\exp\left(\frac{S}{i\hbar}\right)(a) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k\binom{n}{k}}{(i\hbar)^n n!} S^{n-k} a S^k$$

for all $a \in \mathbb{W}_\hbar(V)$. Note that this is a well-defined automorphism, since it is easily deduced that

$$\exp D(a) = \mathrm{Ad}\,\exp\left(\frac{S_D}{i\hbar}\right)(a).$$

Thus we see that

$$\exp F_1\mathfrak{g}^\hbar = \mathrm{Ad}\,\exp \frac{1}{i\hbar}\mathbb{W}_\hbar(V)_3 \subset \widehat{G}^\hbar$$

and so

$$\widehat{G}^\hbar \simeq \mathrm{Ad}\,\exp\left(\frac{\mathbb{W}_\hbar(V)_3}{i\hbar}\right) \rtimes \mathrm{Sp}(V).$$

DEFINITION 3.2.10. The filtration $F_p\mathfrak{g}^\hbar$ of $F_1\mathfrak{g}^\hbar$ induces the filtration

$$\widehat{G}^\hbar = G_0^\hbar \rhd G_1^\hbar \rhd \ldots \rhd G_p^\hbar \rhd G_{p+1}^\hbar \rhd \ldots$$

given by $G_p^\hbar = \exp F_p\mathfrak{g}^\hbar$ for $p > 0$. The remark 3.2.9 shows that the $G_i^\hbar \hookrightarrow G_{i-1}^\hbar$ are normal inclusions.

PROPOSITION 3.2.11. *The group $\widehat{G}^\hbar$ has the structure of a pro-finite dimensional Lie group.*

PROOF.
As shown above, the map

$$\exp\colon F_1\mathfrak{g}^\hbar \longrightarrow G_1^\hbar$$

is a bijection. Thus we topologize $G_1^\hbar$ accordingly. This provides $\widehat{G}^\hbar$ with a topology, i.e. disregarding the group structure we have $\widehat{G}^\hbar \simeq F_1\mathfrak{g}^\hbar \times \mathrm{Sp}(V)$. This will also provide the quotients

$$\overline{G}_i^\hbar := G_0^\hbar \Big/ G_i^\hbar$$

with the quotient topology. Note that $\overline{G}_1^\hbar \simeq \mathrm{Sp}(V)$ as Lie groups and

$$\overline{G}_i^\hbar \simeq \mathbb{C}^{n_i} \times \mathrm{Sp}(V)$$

as topological spaces. Here

$$n_i = \sum_{k=1}^{i-1} \binom{2d+k-1}{k} S_k^i,$$

where

$$S_k^i := \#\{0 < q < i | q = k \bmod 2 \quad \text{and} \quad k \leq i\}.$$

Note that each of the $\overline{G}_i^\hbar$ for $i > 0$ comes with the quotient map $\overline{G}_i^\hbar \to \mathrm{Sp}(V)$. This is a continuous group homomorphism and allows for the section $\mathrm{Sp}(V) \to G_0^\hbar \to \overline{G}_i^\hbar$ where the first map is as in proposition 3.2.8. Recall the nilpotent Lie algebras $F_1^p\mathfrak{g}^\hbar$ of proposition 3.2.5. Since they are nilpotent, the Campbell-Baker-Hausdorff formula provides them with the structure of a Lie group which we will denote $G_1^{\hbar p}$. It follows that we have the exact sequences of topological groups

$$0 \to G_1^{\hbar p} \longrightarrow \overline{G}_p^\hbar \longrightarrow \mathrm{Sp}(V) \to 0$$

and thus we find that

$$\overline{G}_i^\hbar \simeq G_1^{\hbar i} \rtimes \mathrm{Sp}(V)$$

defines the $2d^2 + d + n_i$-dimensional Lie group structure on the quotients $\overline{G}_i^\hbar$.

Consider the sequence

$$\ldots \longrightarrow \overline{G}_{p+1}^\hbar \longrightarrow \overline{G}_p^\hbar \longrightarrow \ldots \longrightarrow \overline{G}_2^\hbar \longrightarrow \mathrm{Sp}(V) \longrightarrow \{\mathrm{Id}\}$$

of Lie group homomorphisms. Clearly we have

$$\widehat{G}_V^\hbar = G_0^\hbar = \varprojlim_p \overline{G}_p^\hbar$$

giving $\widehat{G}_V^\hbar$ the structure of a pro-finite dimensional Lie group.

□

## 3.3. Hochschild and Cyclic Homology

Finally, let us consider some invariants of the formal Moyal–Weyl algebra, namely its Hochschild and cyclic homologies. They will not play a role in the Fedosov construction or the classification of group actions on symplectic deformation quantizations. However, since the algebraic index theorem 6.1.22 is essentially a product formula in cyclic cohomology, they will play a key role in the (equivariant) index theorem. The following is based on the article [**14**]. In the following we will usually denote the completed tensor product over $\mathbb{C}[\![\hbar]\!]$ by $\otimes$.

**3.3.1. Hochschild Homology.** Let us start by considering (and computing) the Hochschild homology of $\mathbb{W}_\hbar(V)$. The computation will be greatly simplified by considering a smaller complex with which to compute the Hochschild homology of $\mathbb{W}_\hbar(V)$ than the one introduced in the appendix A.2.1. This smaller complex also clearly shows the link between cyclic homology of the formal Moyal–Weyl algebra ($\infty$-jets) and (formal) de Rham cohomology.

Definition 3.3.1.

- For $p \in \mathbb{Z}_{\geq 0}$, let

$$K_\hbar^p(V) := \mathbb{W}_\hbar(V) \otimes \bigwedge^p V^*,$$

where the tensor product is over $\mathbb{C}$, and define

$$d_K\colon K_\hbar^p(V) \longrightarrow K_\hbar^{p-1}(V)$$

by $\mathbb{C}$-linear extension of

$$w \otimes v_1 \wedge \ldots \wedge v_p \mapsto \sum_{i=1}^{p} (-1)^i [v_i, w] \otimes v_1 \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_p$$

where the bracket denotes the commutator in $\mathbb{W}_\hbar(V)$ ($V^* \hookrightarrow \mathbb{W}_\hbar(V)$ is given by the inclusion in the tensor algebra) and the hat denotes omission. Note that $d_K^2 = 0$ yielding the complex $(K_\hbar^\bullet, d_K)$, which we call the *Koszul* complex of $\mathbb{W}_\hbar(V)$.

- Let us denote the commutative algebra $\widehat{\mathcal{S}^\bullet(V)}$ by $\hat{\mathcal{O}}_V$. Recall the remark 3.1.32 and note that it implies that $\hat{\mathcal{O}}_V$ has a $C^\infty(V)$ module structure through the map

$$\varphi_V \circ J^\infty_{v,0} \colon C^\infty(V) \longrightarrow \hat{\mathcal{O}}_V$$

(see remarks 3.1.21 and 3.1.32) and the commutative algebra structure on $\hat{\mathcal{O}}_V$. Denote by $\Omega^\bullet(V)$ the $C^\infty(V)$ module of differential forms on $V$ and denote by $d_{\mathrm{dR}}$ the usual (de Rham) exterior derivative. We denote

$$\hat{\Omega}_V^\bullet := \hat{\mathcal{O}}_V \underset{C^\infty(V)}{\otimes} \Omega^\bullet(V).$$

Let

$$\tilde{d}_{\mathrm{dR}} \colon \hat{\mathcal{O}}_V \longrightarrow \hat{\mathcal{O}}_V \underset{C^\infty(V)}{\otimes} \Omega^1(V)$$

be defined by

$$\tilde{d}_{\mathrm{dR}}(\varphi_V(J^\infty_{v,0}(f)))(X) = \varphi_V(J^\infty_{v,0}(J^\infty_v(f)(X)))$$

for all smooth vector fields $X \in Vect(V)$. Let

$$\epsilon \colon \Omega^1(V) \underset{C^\infty(V)}{\otimes} \Omega^\bullet(V) \longrightarrow \Omega^{\bullet+1}(V)$$

be given by

$$\epsilon(\eta \otimes \omega) = \eta \wedge \omega.$$

Finally, we define

$$\hat{d} \colon \hat{\Omega}_V^p \longrightarrow \hat{\Omega}_V^{p+1}$$

as

$$\hat{d} = (1 \otimes \epsilon) \circ (\tilde{d}_{\mathrm{dR}} \otimes 1) + 1 \otimes d_{\mathrm{dR}}.$$

Note that $\hat{d}$ is well-defined (although $\tilde{d}_{\mathrm{dR}} \otimes 1$ and $1 \otimes d_{\mathrm{dR}}$ are not). It is easily verified that $\hat{d}^2 = 0$ and we call $(\hat{\Omega}_V^\bullet, \hat{d})$ the *formal de Rham* complex of $V$.

Note the similarities between the formal de Rham complex and the objects with the same notation in the proof of proposition 3.2.1.

PROPOSITION 3.3.2. *There is an isomorphism of chain complexes of* $\mathbb{C}[\![\hbar]\!]$*-modules*

$$(K_\hbar^\bullet(V), d_K) \longrightarrow (\hat{\Omega}_V^{-\bullet}[2d][\![\hbar]\!], i\hbar\hat{d}).$$

PROOF.
Recall the map

$$Q_V \colon \mathbb{W}^\hbar(V) \longrightarrow \mathbb{W}_\hbar(V)$$

from definition 3.1.39 and note that it yields, in particular, an isomorphism of the underlying $\mathbb{C}[\![\hbar]\!]$-modules

$$q_V \colon \mathbb{W}_\hbar(V) \longrightarrow \hat{\mathcal{O}}_V[\![\hbar]\!].$$

Note also that, for the vector space $V$, we have the canonical identifications $T_vV \simeq V$ for all $v \in V$. These yield the isomorphism

$$c_V \colon C^\infty(V) \otimes_{\mathbb{C}} \bigwedge\nolimits^\bullet V^* \longrightarrow \Omega^q(V).$$

Putting these two together, we obtain the isomorphism of $\mathbb{C}[\![\hbar]\!]$-modules

$$\mathbb{W}_\hbar \underset{\mathbb{C}}{\otimes} \bigwedge^{\bullet} V^* \xrightarrow{\sim} \hat{\mathcal{O}}_V[\![\hbar]\!] \underset{C^\infty(V)}{\otimes} C^\infty(V) \underset{\mathbb{C}}{\otimes} \bigwedge^{\bullet} V^* \xrightarrow{\sim} \hat{\mathcal{O}}_V[\![\hbar]\!] \underset{C^\infty(V)}{\otimes} \Omega^\bullet(V).$$

Now consider the isomorphism

$$\bigwedge^{\bullet} V^* \longrightarrow \bigwedge^{2d-\bullet} V^*$$

given by

$$v_1 \wedge \ldots \wedge v_q \mapsto \iota_{\overline{v}_1} \ldots \iota_{\overline{v}_q} \omega^{\wedge d}$$

where $\iota_v$ denotes the interior product with $v$ and $\overline{v}_i$ is characterized by $\omega(\overline{v}_i, w) = v_i(w)$ for all $w \in V$. Combining this with the isomorphism above provides the isomorphism

$$K^\bullet_\hbar(V) = \mathbb{W}_\hbar \underset{\mathbb{C}}{\otimes} \bigwedge^{\bullet} V^* \xrightarrow{\sim} \hat{\mathcal{O}}_V[\![\hbar]\!] \underset{C^\infty(V)}{\otimes} \Omega^{2d-\bullet}(V) \xrightarrow{\sim} \hat{\Omega}^{2d-\bullet}_V[\![\hbar]\!]$$

which we denote by $K_V$. The formula (3.1.7) yields that, under the identification $K^\bullet_\hbar(V) \simeq \hat{\Omega}^\bullet_V$, we have

$$d_K(f \otimes v_1 \wedge \ldots \wedge v_q) = \sum_{k=1}^{q} i\hbar(-1)^{k-1}(\iota_{\overline{v}_k} \hat{d} f) \otimes v_1 \wedge \ldots \wedge \widehat{v_k} \wedge \ldots \wedge v_q.$$

Thus we have

$$K_V \circ d_K(f \otimes v_1 \wedge \ldots \wedge v_q) = i\hbar \sum_{k=1}^{q} (-1)^{k-1}(\iota_{\overline{v}_k} \hat{d} f)\iota_{\overline{v}_1} \ldots \widehat{\iota_{\overline{v}_k}} \ldots \iota_{\overline{v}_q}\omega^{\wedge d}.$$

On the other hand

$$i\hbar\hat{d} \circ K_V(f \otimes v_1 \wedge \ldots \wedge v_q) = i\hbar\hat{d}(f\iota_{\overline{v}_1} \ldots \iota_{\overline{v}_q}\omega^{\wedge d}) = i\hbar(\hat{d}f)\iota_{\overline{v}_1} \ldots \iota_{\overline{v}_q}\omega^{\wedge d},$$

where the last equality holds since $\omega$ and the $v_k$ are constant. Suppose $0 \neq f \otimes v_1 \wedge \ldots \wedge v_q$ and note that this implies that the $v_k$ are linearly independent. Thus we can extend them to a basis $\{v_1, \ldots, v_{2d}\}$ of $V^*$. Denote the dual basis of $V^*$ by $\{\overline{v}^*_k\}_{k=1}^{2d}$. Finally, the facts that $\hat{d}f = \sum_{k=1}^{2d}(\iota_{\overline{v}_k}\hat{d}f)\overline{v}^*_k$ and $\overline{v}^*_k \wedge \iota_{\overline{v}_k}\eta = \eta$, for any differential form $\eta$, imply that

$$K_V \circ d_K = i\hbar\hat{d} \circ K_V.$$

□

PROPOSITION 3.3.3. *The natural inclusion*

$$i_V \colon (K^\bullet_\hbar(V), d_K) \longrightarrow \big(C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)), b\big)$$

*is a quasi-isomorphism.*

PROOF.
First of all, by the natural inclusion we mean the map

$$w \otimes v_1 \wedge \ldots \wedge v_q \mapsto \sum_{\tau \in S_q} \frac{\epsilon(\tau)}{q!} w \otimes v_{\tau(1)} \otimes \ldots \otimes v_{\tau(q)}$$

for all $w \in \mathbb{W}_\hbar(V)$ and $v_k \in V^*$ where $\epsilon(\tau)$ denotes the sign of the permutation $\tau$. It is easily checked that this identifies $(K^\bullet_\hbar(V), d_K)$ with a subcomplex of $(C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)), b)$. As noted in the remark A.2.3, the Hochschild complex represents the left derived tensor product of $\mathbb{W}_\hbar(V)$ with itself over $\mathbb{W}_\hbar(V)^e$. In fact it is given by tensoring the normalized bar complex

$$\ldots \longrightarrow \mathbb{W}_\hbar(V)^e \otimes \overline{\mathbb{W}_\hbar(V)}^{\otimes 2} \longrightarrow \mathbb{W}_\hbar(V)^e \otimes \overline{\mathbb{W}_\hbar(V)} \longrightarrow \mathbb{W}_\hbar(V)^e$$

with $\mathbb{W}_\hbar(V)$ over $\mathbb{W}_\hbar(V)^e$. Note that in this last complex the tensor product is over $\mathbb{C}[\![\hbar]\!]$, completed and the maps are continuous. Similarly, we have the sequence of free $\mathbb{W}_\hbar(V)$ bimodules

$$\ldots \longrightarrow \mathbb{W}_\hbar(V)^e \otimes \bigwedge^2 V^* \longrightarrow \mathbb{W}_\hbar(V)^e \otimes V^* \longrightarrow \mathbb{W}_\hbar(V)^e$$

where the maps are given by restriction of the maps in the bar complex (again viewing this as a subcomplex in the complex above using the equivalent of $\iota_V$). Note that the tensor products are over $\mathbb{C}$ in this complex and that tensoring with $\mathbb{W}_\hbar(V)$ over $\mathbb{W}_\hbar(V)^e$ yields the Koszul complex of $\mathbb{W}_\hbar(V)$. If we can show this second complex is acyclic, this means both define a resolution of $\mathbb{W}_\hbar(V)$ and this proves the statement of the proposition.

In order to determine acyclicity, let us consider the filtration $F^\hbar_\bullet\mathbb{W}_\hbar(V)$ (i.e. the filtration given by powers of $\hbar$ analogous to $F^\hbar_\bullet\mathbb{W}^\hbar(V)$). Note that the maps in the sequence above respect this filtration and all the graded quotients are isomorphic to the (formal) symmetric algebra of $V^*$. This means that, if we can show that the complex

$$\ldots \longrightarrow \widehat{\mathcal{S}^\bullet(V^*)}^e \otimes \bigwedge^2 V^* \longrightarrow \widehat{\mathcal{S}^\bullet(V^*)}^e \otimes V^* \longrightarrow \widehat{\mathcal{S}^\bullet(V^*)}^e$$

is acyclic, the complex concerning $\mathbb{W}_\hbar(V)$ is also acyclic.

The complex concerning $\widehat{\mathcal{S}^\bullet(V^*)}$ is graded by adding the grading of $\widehat{\mathcal{S}^\bullet(V^*)}$ (the $\bullet$) and of the exterior powers, i.e.

$$|f \otimes v_1 \wedge \ldots \wedge v_q| = |f| + q$$

for homogeneous $f \in \widehat{\mathcal{S}^\bullet(V^*)}^e$. Considering the corresponding filtration and spectral sequence, we find that the complex above is acyclic if and only if

$$\ldots \longrightarrow \mathcal{S}^\bullet(V^*)^e \otimes \bigwedge^2 V^* \longrightarrow \mathcal{S}^\bullet(V^*)^e \otimes V^* \longrightarrow \mathcal{S}^\bullet(V^*)^e$$

is. This can be seen more directly as follows. Let us denote by $D$ and $\hat{D}$ the differentials in the complexes considering $\mathcal{S}^\bullet(V^*)$ and $\widehat{\mathcal{S}^\bullet(V^*)}^e$ respectively. Then if $\hat{D}(\sum_{n=0}^\infty \tau_n) = 0$, where we denote by $\tau_n \in \widehat{\mathcal{S}^\bullet(V^*)}^e \otimes \bigwedge^q V^*$ the homogeneous degree $n$ components, we also have $D(\tau_n) = 0$ separately. Thus if the complex concerning $D$ is acyclic we can find homogeneous elements $\sigma_n$ such that $D(\sigma_n) = \tau_n$, but then $\hat{D}(\sum_{n=0}^\infty \sigma_n) = \sum_{n=0}^\infty \tau_n$ since $\hat{D}$ is continuous. The complex concerning $D$ is in fact acyclic, since the Koszul dual of $\mathcal{S}^\bullet(V^*)$ is $\bigwedge^\bullet V$ [**79**] (also explaining the name of the Koszul complex). This shows the acyclicity of the above complex, since it is the Koszul resolution of a Koszul algebra. $\square$

Corollary 3.3.4. *We have*

$$HH_\bullet(\mathbb{W}_\hbar(V)) \simeq \begin{cases} 0 & \text{if } \bullet > 2d \\ \mathbb{C}[\![\hbar]\!] & \text{if } \bullet = 2d \\ \hat{d}\hat{\Omega}_V^{2d-(\bullet+1)} & \text{if } 0 \leq \bullet < 2d. \end{cases}$$

The corollary follows for $\bullet > 2d$ since the Koszul complex is finite. For $\bullet = 2d$ it follows from proposition 3.3.2 and since the formal de Rham 0-cocycles are given by the ($\infty$-jets of) constants $\mathbb{C}$, while $\hat{\Omega}_V^{-1} = 0$. For $0 < \bullet \leq 2d$ the proposition follows from the isomorphism in proposition 3.3.2 and the formal Poincaré lemma [**76**]. Note that in this last case we still find the $2d - \bullet$-coboundaries because of the factor $i\hbar$ multiplying $\hat{d}$. The results will take a much nicer (and more useful) form if we get rid of these last cocycles by formally inverting $\hbar$. In other words, we can extend scalars from the ring $\mathbb{C}[\![\hbar]\!]$ to the field $\mathbb{C}[\hbar^{-1}, \hbar]\!]$ of Laurent series in $\hbar$. Note that this means we will be considering the Hochschild (and later cyclic) homology of $\mathbb{W}_\hbar(V)[\hbar^{-1}]$ as a $\mathbb{C}[\hbar^{-1}, \hbar]\!]$-module, i.e. we should replace the relevant tensor products by tensor products over this field.

Proposition 3.3.5. *There is an isomorphism*

$$(\hat{\Omega}_V^\bullet[\hbar^{-1}, \hbar]\!], i\hbar\hat{d}) \longrightarrow (\hat{\Omega}_V^\bullet[\hbar^{-1}, \hbar]\!], \hat{d})$$

Proof.
Consider the map

$$\hat{I}\colon \hat{\Omega}_V^\bullet[\hbar^{-1}, \hbar]\!] \longrightarrow \hat{\Omega}_V^\bullet[\hbar^{-1}, \hbar]\!]$$

given by multiplication by $(i\hbar)^{-p}$ on $\hat{\Omega}^p_V[\hbar^{-1},\hbar]\!]$. This is clearly an isomorphism that intertwines $i\hbar\hat{d}$ and $\hat{d}$. $\square$

Corollary 3.3.6. *The Hochschild homology $HH_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}])$ vanishes unless $\bullet = 2d$ and we have $HH_{2d}(\mathbb{W}_\hbar(V)[\hbar^{-1}]) = \mathbb{C}[\hbar^{-1},\hbar]\!]$.*

This corollary follows easily from the above previous propositions and the formal Poincaré lemma.

Remark 3.3.7. Note that we also find the explicit generator $1 \otimes \frac{\omega^d}{d!}$ of $\mathrm{HH}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}])$ over $\mathbb{C}[\hbar^{-1},\hbar]\!]$. Note that both $\widehat{G}^\hbar_V$ and $\mathfrak{g}^\hbar_V$ act on $C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}])$. However $1\otimes\frac{\omega^d}{d!}$ is not a fixed point for these actions. It is a fixed point for the action of the subgroup $\mathrm{Sp}(V) \hookrightarrow \widehat{G}^\hbar_V$ and Lie subalgebra $\mathfrak{sp}(V) \hookrightarrow \mathfrak{g}^\hbar_V$, however.

Remark 3.3.8. Note that the preceding discussion supplies an explicit quasi-isomorphism

$$\left(\hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]\!][2d],\hat{d}\right) \longrightarrow \left(C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}]), b\right),$$

given by the isomorphism $\left(\hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]\!][2d],\hat{d}\right) \simeq (K^\bullet_\hbar(V)[\hbar^{-1}], d_K)$ defined above and the inclusion of the latter complex into $C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}])$. It does not supply a quasi-inverse automatically. A quasi-inverse is given by fixing a complementary vector space $W$ such that

$$C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}]) \simeq i_V\left(K^\bullet_\hbar(V)[\hbar^{-1}]\right) \oplus W$$

or even fixing a space $W$ complementary to the space spanned by the generator mentioned above. Of course the quasi-isomorphism then factors through the projection onto the complement of $W$.

### 3.3.2. Cyclic Homology.

Finally let us compute the cyclic homology of $\mathbb{W}_\hbar(V)[\hbar^{-1}]$.

Proposition 3.3.9. *We have*

$$HC^{per}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}]) \simeq \begin{cases} 0 & \text{if } \bullet = 1 \bmod 2 \\ \mathbb{C}[\hbar^{-1},\hbar]\!] & \text{if } \bullet = 0 \bmod 2. \end{cases}$$

$$HC^{-}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}]) \simeq \begin{cases} 0 & \text{if } \bullet = 1 \bmod 2 \\ \mathbb{C}[\hbar^{-1},\hbar]\!] & \text{if } \bullet = 2p \text{ where } p \le d \\ 0 & \text{if } \bullet = 2p \text{ where } p > d. \end{cases}$$

*and*

$$HC_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}]) \simeq \begin{cases} 0 & \text{if } \bullet = 1 \bmod 2 \\ 0 & \text{if } \bullet = 2p \text{ where } p < d \\ \mathbb{C}[\hbar^{-1},\hbar]\!] & \text{if } \bullet = 2p\ \ p \ge d. \end{cases}$$

Proof.
We can consider the (naive) spectral sequence associated to the double complex used to define periodic cyclic homology $(C^{Hoch}_\bullet(\mathbb{W}_\hbar(V)[\hbar^{-1}]), b, B)$. In other words, suppose

$$A_0 = (a_0, ua_2, \dots, u^p a_{2p}, \dots) \in CC^{per}_0(\mathbb{W}_\hbar(V)[\hbar^{-1}])$$

is a periodic cyclic 0-cycle. This means we have

$$Ba_{2p-2} + ba_{2p} = 0$$

for all $p \ge 0$, where we set $a_{-2} = 0$. In particular $ba_0 = 0$ and the computation of Hochschild homology implies there exists $a_1 \in C^{Hoch}_1(\mathbb{W}_\hbar(V)[\hbar^{-1}])$ such that $ba_1 = a_0$ (unless $d = 0$). Then we have

$$b(a_2 - Ba_1) = Ba_0 + ba_2 = 0$$

implying existence of $a_3 \in C_3^{Hoch}(\mathbb{W}_\hbar(V)[\hbar^{-1}])$ such that $ba_3 + Ba_1 = a_2$ (unless $d = 1$). Continuing in this way we obtain $(a_1, a_3, \ldots, a_{2d-1})$ such that

$$ba_{2p+1} + Ba_{2p-1} = a_{2p}.$$

Thus we find that $b(a_{2d} - Ba_{2d-1}) = 0$. Since the Hochschild homology does not vanish in this degree, we cannot assume existence of $a_{2d+1} \in C_{2d+1}^{Hoch}(\mathbb{W}_\hbar(V)[\hbar^{-1}])$ such that

$$ba_{2d+1} + Ba_{2d-1} = a_{2d}.$$

Assume for a moment that we can find such $a_{2d+1}$, then we are back in the position we were before and can continue to obtain

$$(a_1, ua_3, \ldots, u^p a_{2p+1}, \ldots) \overset{b+uB}{\mapsto} (a_0, ua_2, \ldots, u^p a_{2p}, \ldots).$$

This shows that, if $C_0$ is another 0-cycle, then the class of $C_0$ equals the class of $A_0$ if and only if the class of $a_{2d} - Ba_{2d-1}$ equals the class of $c_{2d} - Bc_{2d-1}$ in Hochschild homology. If we would have started with a 1-cycle $A_1$, there would have been no obstruction at all to finding a 2-chain $A_2$ such that $(b + uB)A_2 = A_1$, since the Hochschild homology is generated in even degree. Note that, since the periodic cyclic homology is 2-periodic, we are done. The computation of the negative cyclic and cyclic homology is done in a similar way.

□

REMARK 3.3.10. Note that the representative of the generator of Hochschild homology $1 \otimes \frac{\omega^d}{d!}$ also provides the generators for the cyclic homologies (after multiplication by the suitable powers of $u$). In other words, we find that

$$\mathrm{HC}_\bullet^{per}\left(\mathbb{W}(V)\left[\hbar^{-1}\right]\right) = k^{per} \otimes_{\mathbb{C}} \mathbb{C}\langle\alpha\rangle[-2d]$$
$$\mathrm{HC}_\bullet^{-}\left(\mathbb{W}_\hbar(V)\left[\hbar^{-1}\right]\right) = k^{-} \otimes_{\mathbb{C}} \mathbb{C}\langle\alpha\rangle[-2d]$$

and

$$\mathrm{HC}_\bullet\left(\mathbb{W}_\hbar(V)\left[\hbar^{-1}\right]\right) = k \otimes_{\mathbb{C}} \mathbb{C}\langle\alpha\rangle[-2d],$$

where $\alpha$ represents $[1 \otimes \frac{\omega^d}{d!}]$, i.e. it is a degree $2d$ element, while

$$k^{per} := \mathbb{C}[\hbar^{-1}, \hbar]\![u^{-1}, u]\!]$$
$$k^{-} := \mathbb{C}[\hbar^{-1}, \hbar]\!][\![u]\!]$$

and

$$k := \mathbb{C}[\hbar^{-1}, \hbar]\!][u^{-1}].$$

We recall that $|u| = -2$ and we are considering cyclic homology of $\mathbb{C}[\hbar^{-1}, \hbar]\!]$ modules.

REMARK 3.3.11. We will not compute the cyclic homologies of $\mathbb{W}_\hbar(V)$ without localising at $\hbar$, mainly because the result is messy and not very illuminating. It can be computed in much the same way as above, however one will have to use the Hochschild homology of $\mathbb{W}_\hbar(V)$ instead. Note that, after localising at $\hbar$, both the Hochschild and cyclic homologies are directly analogous to the Hochschild and cyclic homologies of the classical (non-formal) Weyl algebras [**78**].

CHAPTER 4

# Formal Deformed Geometry and the Fedosov Construction

In this chapter we will present the framework of formal geometry in the deformed setting and the well-known Fedosov construction of deformation quantization algebras [**46**]. To do this we will show that the framework of formal geometry, developed in chapter 2, has a counterpart when starting with a symplectic deformation quantization $\mathbb{A}_\hbar(M)$ instead of the smooth functions on a manifold. This will allow us to define the characteristic class of the deformation quantization as the image of the formal Weyl curvature, mentioned in the remarks 3.2.2 and 3.2.7, under a certain Gelfand-Fuks map. After this, we will present the Fedosov construction to show every possible class can be reached by a deformation quantization. Putting them together, we will have presented a proof of the well-known result that the characteristic class is a complete invariant of symplectic deformation quantizations up to gauge equivalence.

We will see that, in a sense, the framework of formal deformed geometry applies the Fedosov construction in reverse. Starting with a deformation quantization and producing a Fedosov connection and characteristic class. The Fedosov construction, on the other hand, starts with a class, constructs a connection from it and produces the deformation quantization from the connection. Both viewpoints will be relevant in the following chapters.

## 4.1. Formal Deformed Geometry

In this section we will finish the adaptation of the framework of formal geometry to deformation quantization. Note that the first three parts of our "dictionary" have been obtained in the previous section. We replace $\hat{\mathbb{A}}$ by $\hat{\mathbb{A}}^\hbar$, $\widehat{G}$ by $\widehat{G}^\hbar$ and $\mathbb{W}$ by $\mathfrak{g}^\hbar$. Note that these replacements have many properties analogous to the undeformed counterparts. For instance they come with gradings/filtrations, corresponding topologies and splittings. In this section we will give the rest of the counterparts, i.e. the manifold of non-linear frames $\widetilde{M}_\hbar$ and the Kazdan connection $\omega_M^\hbar$. We will also show how this leads to Fedosov connections.

In order to define the framework of formal geometry, we will need to fix a deformation quantization. So, for the rest of this section $(M, \omega)$ is a $2d$-dimensional symplectic manifold and $\mathbb{A}_\hbar(M)$ is a fixed deformation quantization of this manifold. Many constructions will depend on the specific deformation quantization we choose, however we will mostly surpress any notation referring to it directly.

Definition 4.1.1. We will denote the *principal symbol map*, given by setting $\hbar = 0$, by

$$\sigma\colon \mathbb{A}_\hbar(M) \longrightarrow C^\infty(M).$$

Note that $\sigma$ is an algebra homomorphism. Given an algebra homomorphism $\varphi_\hbar\colon \mathbb{A}_\hbar(M) \to \mathbb{A}_\hbar(X)$, where $\mathbb{A}_\hbar(M)$ and $\mathbb{A}_\hbar(X)$ are both deformation quantizations, we say it *extends* or *lifts* the smooth map $\varphi\colon X \to M$, if the following diagram commutes:

$$\begin{tikzcd} \mathbb{A}_\hbar(M) \arrow[r, "\varphi_\hbar"] \arrow[d, "\sigma"'] & \mathbb{A}_\hbar(X) \arrow[d, "\sigma"] \\ C^\infty(M) \arrow[r, "\varphi^*"'] & C^\infty(X) \end{tikzcd}.$$

An $\mathbb{A}_\hbar(M)$-adapted chart consists of a coordinate chart $\varphi\colon \mathbb{R}^{2d} \to M$ together with an isomorphism $\varphi_\hbar$ from $\mathbb{A}_\hbar(U)$ to the Moyal deformation of $\mathbb{R}^{2d}$ which lifts $\varphi$.

Definition 4.1.2. We define the manifold of non-linear deformed frames $\widetilde{M}_\hbar$ as

$$\widetilde{M}_\hbar := \left\{\varphi_m\colon \hat{\mathcal{A}}_{m,\hbar} \xrightarrow{\sim} \hat{\mathbb{A}}^\hbar_{2d}\right\},$$

where we mean <u>continuous</u> algebra isomorphism and we recall the definition of $\hat{\mathcal{A}}_{m,\hbar}$ in remark 3.1.22.

The pro-finite dimensional manifold structure on $\widetilde{M}_\hbar$ is given in essentially the same way as in the undeformed case. This is possible because the translations

$$T_{(l,i)}\colon (\mathbb{R}^{2d}, 0, \omega) \to (\mathbb{R}^{2d}, (l,i), \omega)$$

are symplectic transformations, for any $(l,i) \in \mathbb{R}^{2d}$, and moreover they define algebra automorphisms of the Moyal deformation by translation invariance of the product (3.0.2). So we construct charts for $\widetilde{M}_\hbar$, as in the undeformed case, by starting with an $\mathbb{A}_\hbar(M)$-adapted chart $U$ on $M$ and then covering $\widetilde{M}_\hbar$ by charts of the form $\mathbb{R}^{2d} \times \widehat{G}^\hbar$ as in (2.2.1) and (2.2.2). Note that we have already provided the manifold structure on $\widehat{G}^\hbar$ in proposition 3.2.11.

Proposition 4.1.3. *There is a natural homomorphism*

$$\omega^\hbar_M(\varphi_m)\colon T_{\varphi_m}\widetilde{M}_\hbar \longrightarrow \mathfrak{g}^\hbar,$$

*for all $\varphi_m \in \widetilde{M}^\hbar$, such that $\omega^\hbar_M$ defines a one-form in $\Omega^1\left(\widetilde{M}_\hbar\right) \otimes \mathfrak{g}^\hbar$ satisfying*

$$d\omega^\hbar_M + \frac{1}{2}[\omega^\hbar_M, \omega^\hbar_M] = 0, \tag{4.1.1}$$

*where $d$ denotes the exterior derivative.*

Proof.
Note that the proposition is simply a duplicate of theorem 2.3.3 where we replaced every object by their deformed counterpart. The construction of $\omega^\hbar_M$ and the proof of the proposition are in fact identical so we will refrain from writing it here. □

In order to proceed with our considerations of the framework of formal geometry we would like to know that $\widetilde{M}_\hbar$ is a principal bundle both over the manifold $M$ and over the frames bundle of $M$. Instead of considering the general linear frames bundle, we will have to reduce to the symplectic frames bundle, since the isomorphisms $\varphi_m\colon \hat{\mathcal{A}}_{m,\hbar} \to \hat{\mathbb{A}}^\hbar_{2d}$ always induce symplectomorphisms of the complexified cotangent spaces. Note also that we find the complexified cotangent bundle instead of the real cotangent bundle. Thus, we will need to make the following adaptations to the undeformed case.

First of all, we adapt the definition 2.3.5 by using the bundle of symplectic frames $\mathrm{Sp}_M \to M$ instead of $j^1(M)$ and the symplectic group and Lie algebra instead of the general linear one. Suppose $\mathbb{L}$ is a $\mathrm{GL}(2d,\mathbb{R})$-module, then, since $\mathrm{Sp}(2d,\mathbb{R}) \subset \mathrm{GL}(2d,\mathbb{R})$, it is also an $\mathrm{Sp}(2d,\mathbb{R})$-module. We note that the two definitions of differential forms on $M$ with values in $\mathbb{L}_M$ coincide. This is to be expected due to the remark 2.3.6. To deal with the issue of complexification we note that, if $(V,\omega)$ is a complex symplectic vector space such that $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega = \omega_{\mathbb{R}} \otimes 1$ for some real symplectic vector space $(V_{\mathbb{R}}, \omega_{\mathbb{R}})$, then we have the inclusions $V_{\mathbb{R}} \hookrightarrow \mathbb{W}_\hbar(V)$ and $\mathrm{Sp}(V_{\mathbb{R}}) \hookrightarrow \mathrm{Sp}(V)$. This leads to the following adaptations.

Definition 4.1.4. Suppose $(V,\omega) = (V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \omega_{\mathbb{R}} \otimes 1)$ is a complexified real symplectic vector space. We define the reduced group $\widehat{G}^\hbar_{r,V} = \widehat{G}^\hbar_r$ as the subgroup $\widehat{G}^\hbar_r \subset \widehat{G}^\hbar$ given by

$$\widehat{G}^\hbar_r = \exp F_1\mathfrak{g}^\hbar \rtimes \mathrm{Sp}(V_{\mathbb{R}}) \hookrightarrow \exp F_1\mathfrak{g}^\hbar \rtimes \mathrm{Sp}(V) = \widehat{G}^\hbar,$$

where we use the canonical decomposition of the lemma 3.2.8 and the inclusion $\mathrm{Sp}(V_{\mathbb{R}}) \subset \mathrm{Sp}(V)$ given by the fact that $V$ is a complexification.

Definition 4.1.5. We define the reduced manifold of non-linear deformed frames $\widetilde{M_{\hbar,r}} \subset \widetilde{M_\hbar}$ to be the submanifold given by those isomorphisms $\varphi_m$ such that the induced isomorphism

$$(\varphi_m)_1 : T^*_m M \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow T^*_0\mathbb{R}^{2d} \otimes_{\mathbb{R}} \mathbb{C}$$

is induced by a local symplectomorphism $(\mathbb{R}^{2d}, 0) \to (M, m)$.

Now note that, if we denote the symplectic linear frame bundle (as associated to the cotangent bundle, i.e. with a left action of $\mathrm{Sp}(2d,\mathbb{R})$) by $\mathrm{Sp}_M$, then we have the map $\pi_{1,r} \colon \widetilde{M_{\hbar,r}} \to \mathrm{Sp}_M$, which defines a $\widehat{G}^\hbar_r$-principal bundle. We have the following lemma analogous to lemma 2.3.7.

Lemma 4.1.6. *There exists a* $Sp(2d,\mathbb{R})$*-equivariant section*

$$F_r \colon Sp_M \longrightarrow \widetilde{M^\hbar_r}$$

*of the map* $\pi_{1,r} \colon \widetilde{M^\hbar_r} \to Sp_M$.

Proof.
The proof of lemma 2.3.7 still holds by proposition 3.2.8 and definition 4.1.4. □

Let us fix such a $\mathrm{Sp}(2d,\mathbb{R})$-equivariant section $F_r$, denote the inclusion $\widetilde{M_{\hbar,r}} \hookrightarrow \widetilde{M_\hbar}$ by $\iota_r$ and denote the composition $\iota_r \circ F_r$ by $F$. Again we can pull-back the one-form $\omega^\hbar_M$ by $F$ to obtain

$$A^\hbar_F := F^*\omega^\hbar_M \in \Omega^1(\mathrm{Sp}_M) \otimes \mathfrak{g}^\hbar.$$

Note that $A^\hbar_F$ also satisfies the Maurer-Cartan equation (4.1.1) simply because $\omega^\hbar_M$ does.

We define the notion of a $(\mathfrak{g}^\hbar, \mathrm{Sp}(2d,\mathbb{R}))$-module $\mathbb{L}$ analogously to definition 2.3.8, note that we consider real Lie algebras. Again we find that $d_{\mathrm{dR}} + A^\hbar_F\wedge$ defines a square-zero operator on $\Omega^\bullet(M;\mathbb{L})$ for any $(\mathfrak{g}^\hbar, \mathrm{Sp}(2d,\mathbb{R}))$-module $\mathbb{L}$.

Proposition 4.1.7. *Suppose* $\mathbb{L}$ *is a* $(\mathfrak{g}^\hbar, Sp(2d,\mathbb{R}))$*-module, then the map*

$$GF^\hbar_M \colon \left(C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \mathbb{L}\right), \partial_{Lie}\right) \longrightarrow \left(\Omega^\bullet(M;\mathbb{L}), \nabla^\hbar_F\right),$$

*where* $\nabla^\hbar_F = d_{dR} + A^\hbar_F\wedge$*, given by*

$$GF^\hbar_M(\chi)(X_1,\ldots,X_p)(f_m) = \chi(A^\hbar_F(X_1)(f_m),\ldots,A^\hbar_F(X_p)(f_m)),$$

*for* $\chi \in C^p_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \mathbb{L}\right)$*,* $f_m \in Sp_M$ *and* $X_1,\ldots,X_p$ *vector fields on* $Sp_M$*, is a well-defined map of complexes.*

Proof.
Again the proof is identical to the proof of proposition 2.3.9. □

Notation 4.1.8. We shall denote the most usual $\mathbb{W}_\hbar(V)$, for $(V,\omega) = (\mathbb{R}^{2d} \otimes_{\mathbb{R}} \mathbb{C}, \omega_{st} \otimes 1)$, simply by $\mathbb{W}_\hbar$.

We will see that the differential $\nabla_F^\hbar$ is a Fedosov connection in the sense of [**89**]. From the point of view of the Fedosov construction we would assume that the kernel of $\nabla_F^\hbar$ on $\Omega^\bullet(M;\mathbb{W}_\hbar)$ is isomorphic as an algebra to $\mathbb{A}_\hbar(M)$. From the formal geometric point of view we would expect the same due to proposition 2.4.2 and the fact that the deformed bundle of jets is easily identified with the bundle associated to $\mathrm{Sp}_M$ (or $\widetilde{M_\hbar}$) with fiber $\mathbb{W}_\hbar$.

PROPOSITION 4.1.9. *The map*

$$J^\infty_{F,\hbar}\colon\ (\mathbb{A}_\hbar(M),0) \longrightarrow \left(\Omega^\bullet\left(M;\mathbb{W}_\hbar\right),\nabla_F^\hbar\right)$$

*given by $f\mapsto (p\mapsto F(p)J^\infty_{\pi(p)}f)$ for all $f\in\mathbb{A}_\hbar(M)$ and $p\in Sp_M$ is a quasi-isomorphism of differential graded associative algebras. In other words $\mathbb{A}_\hbar(M)\simeq Ker\nabla_F^\hbar$ as algebras and $\left(\Omega^\bullet\left(M;\mathbb{W}_\hbar\right),\nabla_F^\hbar\right)$ is acyclic.*

PROOF.
Again the proof is essentially identical to the proof of proposition 2.4.2 when one considers the spectral sequences associated to the $\hbar$ filtrations on $\mathbb{W}_\hbar$ and $\mathbb{A}_\hbar(M)$. □

REMARK 4.1.10. Suppose for a moment that $M=\mathbb{R}^{2d}$ with the standard symplectic structure and let $\mathbb{A}_\hbar(\mathbb{R}^{2d})$ denote the Moyal deformation. Then both $\mathrm{Sp}_{\mathbb{R}^{2d}}$ and $\widetilde{\mathbb{R}^{2d}}$ are trivial bundles over $\mathbb{R}^{2d}$. The trivialization is given by the standard (Darboux) coordinates $x_1,\dots,x_d,\xi_1,\dots,\xi_d$. Then note that, by similar reasoning as in the proof of proposition 2.4.2, we find that

$$-\sum_{i=1}^d dx_i\otimes\partial_{\hat{x}_i}+d\xi_i\otimes\partial_{\hat{\xi}_i}=A_F\in\Omega^1(M)\otimes\mathfrak{g}^\hbar.$$

Or in other words, we have $A_F(\partial_{x_i})=-\partial_{\hat{x}_i}$ and $A_F(\partial_{\xi_i})=-\partial_{\hat{\xi}_i}$, which can also be expressed as

$$A_F(X)=\frac{1}{i\hbar}[\omega(X,-),-].$$

Now suppose, denoting the coordinates $(x_1,\dots,\xi_d)$ simply by $x$, we have $\hat{f}(x,\hat{x})\in\Omega^0(M;\mathbb{W}_\hbar)$, then we see that, in these terms, $\nabla_F^\hbar\hat{f}=0$ simply means that $\hat{f}(x,\hat{x})=f(x+\hat{x})$ for some $f\in C^\infty(\mathbb{R}^{2d})[\![\hbar]\!]$. Note that a general symplectic manifold is locally symplectomorphic to $\mathbb{R}^{2d}$ with the standard symplectic structure and deformation quantization is local. This means that we can always find a cover by coordinate neighborhoods $U$ such that the deformation quantization is given by sections of the form $g(x+\hat{x})\in\Omega^0(U,\mathbb{W}_\hbar)$ for some $g\in C^\infty(U)[\![\hbar]\!]$.

Let us conclude this section on the framework of formal deformed geometry here, even though we have not yet given any application of the Gelfand-Fuks maps constructed in proposition 4.1.7. These applications will follow in the last section of this chapter, where we consider the characteristic class of the deformation quantization, and more intensively in the chapter on the algebraic index theorem. For now we should like to make the relation between the above framework of formal deformed geometry and the Fedosov construction of deformation quantization clear. To do this, we will first need to present the Fedosov construction itself, of course.

## 4.2. The Fedosov Construction

The Fedosov construction of deformation quantization will be used in this thesis to present the classification of symplectic deformation quantizations and also to give the classification of group actions on symplectic deformation quantizations in the following chapter. Thus it is a good idea to recall the construction here. There are very many excellent sources on this topic however, [**114, 48, 89, 65**] to name a few. The Fedosov construction refers to the process of defining a deformation quantization, given a two-form, by constructing, in an iterative manner, a certain flat connection in the formal Weyl algebras bundle [**48**]. One then shows that the flat sections for this connection are isomorphic to a deformation quantization of the manifold $M$. The following, including proofs, is from the book [**48**] by B. Fedosov.

DEFINITION 4.2.1. The Weyl algebras bundle $\mathcal{W} \to M$ is the bundle of algebras associated to $\mathrm{Sp}_M$ with the fiber $\mathbb{W}_\hbar$.

We shall sometimes denote the product on $\mathcal{W}$ by $\star$. It shall be convenient to work in the setting adopted in [**114**] and [**48**]. This means that instead of the construction $\mathbb{W}_\hbar$ we shall use the (naturally isomorphic) construction $\mathbb{W}^\hbar$. Note that, in this setting, we have three (natural) gradings on the underlying $\mathbb{C}$-module.

- Firstly, we have the degree induced by the weight in the tensor algebra, i.e. the element $\hbar^p v_1 \otimes \ldots \otimes v_l$ is of tensor degree $l$. We shall denote the homogeneous elements of tensor degree $l$ by $\mathbb{W}^\hbar_{\otimes(l)}$.
- Secondly, we have the degree induced by the powers of $\hbar$, i.e. $\hbar^p v_1 \otimes \ldots \otimes v_l$ is of degree $p$. We shall denote the homogeneous elements of $\hbar$-degree $p$ by $\mathbb{W}^\hbar_{d\hbar(p)}$.
- Finally, we have the total degree which combines the tensor degree and twice the $\hbar$-degree, i.e. $\hbar^p v_1 \otimes \ldots \otimes v_l$ is of total degree $2p + l$. We shall denote the homogeneous elements of total degree $k$ by $\mathbb{W}^\hbar_{t(k)}$.

Note that, with the undeformed product, $\mathbb{W}^\hbar$ is a graded algebra for each choice of grading. For the deformed product, $\mathbb{W}^\hbar$ is only a graded algebra when one considers the total degree.

DEFINITION 4.2.2. We define the one-form $A_{-1} \in \Omega^1\left(M; \mathbb{W}^\hbar_{t(1)}\right)$ as the composition

$$\Gamma(TM) \longrightarrow \Gamma(T^*M) \longrightarrow \Gamma(\mathcal{W}).$$

Here the first map is the isomorphism given by the symplectic structure and the second map is given by the fact that $T^*M$ is the bundle associated to $\mathrm{Sp}_M$ with fiber $\mathbb{R}_{2d}$ (the dual of $\mathbb{R}^{2d}$) and $\mathbb{R}_{2d}$ includes, $\mathrm{Sp}(2d, \mathbb{R})$-equivariantly, in $\mathbb{W}^\hbar$. We define $D_{-1}$ as the one form with values in $\mathfrak{g}^\hbar$ given by $D_{-1} = P_{\mathfrak{g}^\hbar}(\frac{1}{i\hbar} A_{-1})$.

The following lemma summarizes some facts about $A_{-1}$ and $D_{-1}$. We will use the Einstein summation convention in this section.

LEMMA 4.2.3.

(1)
$$D_{-1} \colon \Omega^k\left(M; \mathbb{W}^\hbar_{t(l)}\right) \longrightarrow \Omega^{k+1}\left(M; \mathbb{W}^\hbar_{t(l-1)}\right)$$

(2) $D_{-1}(A_{-1}) = \omega \in \Omega^2\left(M; \mathbb{W}^\hbar_{t(0)}\right)$

(3) $D_{-1}^2 = 0$

(4) $D_{-1}$ *is an anti-derivation w.r.t. the degree of differential forms.*

(5) $A_{-1}|_U = \omega_{ij}\xi^j \otimes dx^i$

(6)
$$D_{-1}|_U(\xi^{i_1} \ldots \xi^{i_k} \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_l}) = -\sum_{p=1}^{k} \xi^{i_1} \ldots \hat{\xi}^{i_p} \ldots \xi^{i_k} \otimes dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l},$$

*where $(U, x^1, \ldots, x^{2d})$ is a coordinate chart on $M$, $\omega_{ij}$ is the symplectic form tensor in these coordinates and $\xi^i = dx^i$ as a section of the bundle with fiber $\mathbb{W}^\hbar$ over $U$ and the hat signifies omission in 6.*

PROOF.
Numbers 1, 3, 4 and 5 follow directly from the definitions, number 6 follows from an explicit computation and number 2 follows from number 6. □

Proposition 4.2.4. *Suppose $U$, $x^i$ and $\xi^i$ are as in lemma 4.2.3. The map*

$$W_{+1}\colon \Omega^\bullet\left(M;\mathbb{W}^\hbar\right) \longrightarrow \Omega^\bullet\left(M;\mathbb{W}^\hbar\right)$$

*given by*

$$W_{+1}|_U\left(\left(\xi^{i_1}\dots\xi^{i_k}\right)\otimes\left(dx^{j_1}\wedge\dots\wedge dx^{j_l}\right)\right) =$$

$$= \frac{1}{k+l}\sum_{p=1}^{l}(-1)^p\left(\xi^{j_p}\xi^{i_1}\dots\xi^{i_k}\right)\otimes\left(dx^{j_1}\wedge\dots\wedge\widehat{dx^{j_p}}\wedge\dots\wedge dx^{j_l}\right),$$

*where the hat signifies omission, is globally well-defined and satisfies the properties*

1. $(W_{+1})^2 = 0$.
2. $W_{+1}:\Omega^l\left(M,\mathbb{W}^\hbar_{t(k)}\right)\to\Omega^{l-1}\left(M,\mathbb{W}^\hbar_{t(k+1)}\right)$
3. $(D_{-1}\circ W_{+1}+W_{+1}\circ D_{-1})(a) = a$ *for all* $a\in\Omega^k\left(M,\mathbb{W}^\hbar_{\otimes(l)}\right)$ *such that* $k+l>0$.

Proof.
The fact that $W_{+1}$ is globally well-defined follows from explicit computation of behaviour under coordinate transformations. The other properties can all be checked explicitly from the local formula for $W_{+1}$ and the local formula for $D_{-1}$ given in lemma 4.2.3. □

Definition 4.2.5. We define the *symbol* map [**48**]

$$\sigma_\hbar\colon \Omega^\bullet\left(M;\mathbb{W}^\hbar\right) \longrightarrow \Omega^\bullet\left(M;\mathbb{W}^\hbar\right)$$

by

$$\sigma_\hbar = \mathrm{Id} - D_{-1}\circ W_{+1} + W_{+1}\circ D_{-1}.$$

Note that $\sigma_\hbar = \mathrm{Id}$ on $\Omega^0\left(M;\mathbb{W}^\hbar_{\otimes(0)}\right)$ while $\sigma_\hbar = 0$ on $\Omega^k\left(M,\mathbb{W}^\hbar_{\otimes(l)}\right)$ if $k+l>0$. Note also that $\sigma_\hbar$ factors as

$$\Omega^\bullet\left(M;\mathbb{W}^\hbar\right)\longrightarrow C^\infty(M)[\![\hbar]\!]\hookrightarrow\Gamma\left(\mathcal{W}\right),$$

where the inclusion comes from the inclusion of scalars in the tensor algebra.

The space of connections in a given bundle is isomorphic to the space of one-forms with values in the linear endomorphisms of that bundle as affine spaces. This means that, relative to a given connection, every connection is uniquely determined by such a one-form. The Fedosov construction proceeds by solving certain equations involving the one-forms used in defining a Fedosov connection relative to some symplectic connection. Let us first introduce some notation and consider some facts about such symplectic connections. Given any linear connection, we can, by the usual trick used for instance in [**97**], construct a torsion-free symplectic connection. Since a linear connection always exists [**48**], let us fix a choice of torsion-free symplectic connection $\nabla_0$. Recall that a symplectic connection is a linear connection such that $\nabla_0\omega = 0$. Note that any symplectic connection defines a connection in $\mathcal{W}$, since we can extend the action on the generators $(T^*M)$ of the tensor algebra to a derivation and we have

$$\nabla_0([v,w]) - \nabla_0(i\hbar\bar\omega(v,w)) = (\nabla_0 v)\otimes w + v\otimes(\nabla_0 w) - (\nabla_0 w)\otimes v - w\otimes(\nabla_0 v) - \nabla_0(i\hbar\bar\omega(v,w)) =$$

$$[\nabla_0 v, w] + [v,\nabla_0 w] - \nabla_0(i\hbar\bar\omega(v,w)) = i\hbar\bar\omega(\nabla_0 v,w) + i\hbar\bar\omega(v,\nabla_0 w) - \nabla_0(i\hbar\bar\omega(v,w)) = 0.$$

Definition 4.2.6. We define the two-form $R_0\in\Omega^2\left(M;\mathbb{W}^\hbar_{t(2)}\right)$ by the local expression

$$\frac{1}{4}\omega_{im}R^m_{jkl}(\xi^i\star\xi^j)\otimes(dx^k\wedge dx^l),$$

where again we use the notation of lemma 4.2.3 and $R^m_{jkl}$ is given by the usual curvature tensor corresponding to $\nabla_0$.

Lemma 4.2.7. *The symplectic connection $\nabla_0$ satisfies the following properties:*

1. $\nabla_0 A_{-1} = 0$,

(2) $D_{-1} \circ \nabla_0 + \nabla_0 \circ D_{-1} = 0$ *and*
(3) $\nabla_0^2 a = \frac{1}{i\hbar}[R_0, a]$ *for all* $a \in \Omega^\bullet(M; \mathbb{W}^\hbar)$.

PROOF.
The first property follows from the torsion-freeness of $\nabla_0$, the second one follows from the definition of $D_{-1}$ and the third property follows from the definition of the lifting of the *linear* symplectic connection $\nabla_0$ to $\mathcal{W}$. □

NOTATION 4.2.8. Given $A \in \Omega^1\left(M; \mathbb{W}^\hbar\right)$, we denote

$$\nabla_A := \nabla_0 + \frac{1}{i\hbar}[A, -].$$

We also denote $A_{\mathbb{N}} := A - A_{-1}$.

By definition, a connection in a bundle of algebras is given by a connection in the underlying vector bundle that acts by derivations. By proposition 3.2.1, this means that any connection in $\mathcal{W}$ is given by $\nabla_A$ for some $A$ and some $\nabla_0$.

DEFINITION 4.2.9. A *Fedosov connection* is defined to be a connection $\nabla_A$, for $A \in \Omega^1\left(M; \mathbb{W}^\hbar\right)$, such that $A_{\mathbb{N}} \in \Omega^1\left(M; \mathbb{W}^\hbar_{t(\geq 3)}\right)$ and $\nabla_A^2 = 0$.

DEFINITION 4.2.10. For $A \in \Omega^1\left(M; \mathbb{W}^\hbar\right)$, we define the curvature $\Omega_A$ of $\nabla_A$ by

$$\Omega_A := R_0 + \nabla_0(A) + \frac{1}{i\hbar} A \star A.$$

REMARK 4.2.11. Note that we have $\nabla_A^2 a = \frac{1}{i\hbar}[\Omega_A, a]$, by definition of $\Omega_A$, and $\nabla_A \Omega_A = 0$ by a simple computation.

We have the following corollary of lemmas 4.2.3 and 4.2.7 and definition 4.2.9.

COROLLARY 4.2.12. *Suppose* $\nabla_A$ *is a Fedosov connection, then*

$$\Omega_A = R_0 + \omega + D_{-1}(A_{\mathbb{N}}) + \nabla_0(A_{\mathbb{N}}) + \frac{1}{i\hbar} A_{\mathbb{N}} \star A_{\mathbb{N}}$$

*and* $\Omega_A$ *is central.*

DEFINITION 4.2.13. Given a Fedosov connection $\nabla_A$ we define

$$\theta_A := R_0 + D_{-1}(A_{\mathbb{N}}) + \nabla_0(A_{\mathbb{N}}) + \frac{1}{i\hbar} A_{\mathbb{N}} \star A_{\mathbb{N}}.$$

Note that $\theta_A \in \Omega^2(M)[\![\hbar]\!]$ is a closed two-form by remark 4.2.11.

THEOREM 4.2.14. *(Fedosov)*
*For any closed two-form* $\theta \in \Omega^2(M)[\![\hbar]\!]$ *there is a Fedosov connection* $\nabla_A$ *such that*

$$\theta_A = i\hbar\theta.$$

PROOF.
Note that the theorem requires us to show that there always exists a solution $A_{\mathbb{N}}$ to the equation for $\theta_A$ above. However, let us first show that there exists a one-form $B \in \Omega^1\left(M; \mathbb{W}^\hbar_{t(\geq 3)}\right)$ such that

$$B = -W_{+1}(R_0 - i\hbar\theta) - W_{+1}(\nabla_0 B + \frac{1}{i\hbar} B \star B).$$

To show this, let $A_0 = -W_{+1}(R_0 - i\hbar\theta)$ and consider the recursively defined sequence

$$A_n = A_0 - W_{+1}(\nabla_0 A_{n-1} + \frac{1}{i\hbar} A_{n-1} \star A_{n-1}),$$

for $n \in \mathbb{N}$. We will show that $(A_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a Cauchy sequence in the complete metric topology given by the total degree filtration. Note that this means it is enough to show that

$$A_n - A_{n+1} \in \Omega^1\left(M; \mathbb{W}^{\hbar}_{t(\geq n+4)}\right)$$

for all $n \in \mathbb{Z}_{\geq 0}$.

We will show this by 2 inductions. First, note that $A_0 = -W_{+1}(R_0 - i\hbar\theta)$ is of degree 3 since both $R_0$ and $i\hbar\theta$ are in degree 2 and $W_{+1}$ raises degree by $+1$. Now, suppose $A_k$ is of degree greater than 3 for all $k$ below some $n \in \mathbb{N}$, then $A_n = A_0 - W_{+1}(\nabla_0 A_{n-1} + \frac{1}{i\hbar}A_{n-1} \star A_{n-1})$ is also of degree greater then 3 since all the summands are. In particular $A_n + A_{n-1}$ is of degree greater than 3 for all $n \in \mathbb{N}$.

Secondly, note that

$$A_0 - A_1 = W_{+1}(\nabla_0 A_0 + \frac{1}{i\hbar}A_0 \star A_0).$$

On the other hand $\nabla_0 A_0 \in \Omega^1(M; \mathbb{W}^{\hbar}_{t(3)})$ since $\nabla_0$ preserves degree and obviously $\frac{1}{i\hbar}A_0 \star A_0$ is in degree 4. So, $A_0 - A_1 \in \Omega^1(M; \mathbb{W}^{\hbar}_{t(\geq 4)})$ (since $W_{+1}$ increases degree) and the base case of our induction is established. Suppose $A_k - A_{k+1}$ is of degree greater then $k+4$ for all $k < n \in \mathbb{Z}_{\geq 0}$. Now

$$A_n - A_{n+1} = W_{+1}(\nabla_0(A_n - A_{n-1}) + \frac{1}{2i\hbar}[A_{n-1} + A_n, A_{n-1} - A_n]),$$

which must be in degree greater than $n+4$, since $A_{n-1} - A_n$ is in degree greater than $n+3$ and $A_{n-1} + A_n$ is in degree greater than 3.

So, the sequence $(A_n)_{n\in\mathbb{Z}_{\geq 0}}$ is Cauchy and we denote $\mathcal{A} := \lim_{n\to\infty} A_n$. Now note that, since it increases the degree of two-forms, $W_{+1}$ is continuous on two-forms. Similarly, because it preserves the degree of one-forms, $\nabla_0$ is continuous on one-forms. Furthermore, the map

$$\frac{\star}{i\hbar} : \Omega^\bullet\left(M; \mathbb{W}^{\hbar}_{t(\geq 2)}\right) \longrightarrow \Omega^\bullet\left(M; \mathbb{W}^{\hbar}_{t(\geq 2)}\right)$$

given by $B \mapsto \frac{1}{i\hbar}B \star B$, also increases degree and is therefore continuous. Now note that, since $A_n$ is of degree greater than 3 for all $n$, so is $\mathcal{A}$ and so we have

$$A_0 - W_{+1}(\nabla_0 \mathcal{A} + \frac{1}{i\hbar}\mathcal{A} \star \mathcal{A}) = \lim_{n\to\infty} A_0 - W_{+1}(\nabla_0 A_n + \frac{1}{i\hbar}A_n \star A_n) = \lim_{n\to\infty} A_{n+1} = \mathcal{A}.$$

Note that $\mathcal{A}$ is completely determined by $A_0$ and therefore by $\theta$ and $R_0$. In fact, by simply writing out the degrees of the equation for $\mathcal{A}$, one sees that $\mathcal{A}$ is the unique solution that is of degree greater than 3. Lastly note that, by construction, $W_{+1}\mathcal{A} = 0$, since $(W_{+1})^2 = 0$.

Denote $A_{\mathbb{N}} := \mathcal{A}$ and $A := A_{-1} + A_{\mathbb{N}}$, we have

$$W_{+1}(\Omega_A - \omega - i\hbar\theta) = W_{+1}(R_0 - i\hbar\theta + \nabla_0 A_{\mathbb{N}} + \frac{1}{i\hbar}A_{\mathbb{N}} \star A_{\mathbb{N}}) + W_{+1}D_{-1}A_{\mathbb{N}} = -A_{\mathbb{N}} + A_{\mathbb{N}} = 0.$$

Thus we find that, since $\nabla_A \Omega_A = 0$ and $\nabla_A(\omega + i\hbar\theta) = d(\omega + i\hbar\theta) = 0$, we have

$$\Omega_A - \omega - i\hbar\theta = W_{+1}D_{-1}(\Omega_A - \omega - i\hbar\theta) = W_{+1}(D_{-1} - \nabla_A)(\Omega_A - \omega - i\hbar\theta).$$

Thus we see that we can compute $\Omega_A - \omega - i\hbar\theta$ recursively, since $W_{+1}(D_{-1} - \nabla_A) = -W_{+1}\nabla_{A_{\mathbb{N}}}$ raises the degree. Now note that the degree 0 and 1 parts of $\Omega_A - \omega - i\hbar\theta$ vanish since $A_{\mathbb{N}}$ is of degree greater than 3, but this shows that $\Omega_A - \omega - i\hbar\theta = 0$. Thus we see that in fact $\Omega_A = \omega + i\hbar\theta$ and thus $\nabla_A$ is a Fedosov connection and $\theta_A = i\hbar\theta$. □

The proof implies the following normalization.

COROLLARY 4.2.15. *Given a symplectic torsion-free connection $\nabla_0$ and a closed two-form $\theta \in \Omega^2(M)[\![\hbar]\!]$, there is a unique $A \in \Omega^1\left(M; \mathbb{W}^{\hbar}\right)$ such that*

- $\nabla_A$ *is a Fedosov connection,*
- *the degree of* $A_{\mathbb{N}}$ *is greater than* 3,
- $W_{+1}A_{\mathbb{N}} = 0$ *and*
- $\theta_A = i\hbar\theta$.

PROOF.
The fact that $\theta_A = i\hbar\theta$ in fact implies $-D_{-1}A_{\mathbb{N}} = R_0 + \nabla_0 A_{\mathbb{N}} + \frac{1}{i\hbar}A_{\mathbb{N}} \star A_{\mathbb{N}}$. This last equation and the condition $W_{+1}A_{\mathbb{N}} = 0$ imply that $A_{\mathbb{N}} = -W_{+1}(R_0 - i\hbar\theta) - W_{+1}(\nabla_0 A_{\mathbb{N}} + \frac{1}{i\hbar}A_{\mathbb{N}} \star A_{\mathbb{N}})$, which has a unique solution of degree greater than 3. □

The final part of the Fedosov construction consists in showing that the kernel of the constructed Fedosov connection $\nabla_A$, given by the closed two form $\theta$ and the torsion-free symplectic connection $\nabla_0$, is a deformation quantization. Let us fix a closed two-form $\theta$ and let us preemptively denote $\mathbb{A}_\hbar(M) := \operatorname{Ker}\nabla_A$, then we have the following proposition.

PROPOSITION 4.2.16. *The map* $\sigma_\hbar$ *is a linear isomorphism onto the image when restricted to* $\mathbb{A}_\hbar(M)$ *and the induced product on* $C^\infty(M)[\![\hbar]\!]$ *is a deformation quantization of* $(M,\omega)$.

PROOF.
Let $f \in C^\infty(M)[\![\hbar]\!]$, we shall also denote the image of $f$ in $\Gamma(\mathcal{W})$, see definition 4.2.5, by $f$. Consider the equation

$$a = f - W_{+1}(\nabla_A - D_{-1})a.$$

Note that, since the operator $W_{+1}(\nabla_A - D_{-1})$ raises degree, we find that, if the equation has a solution $a$, then this solution is completely determined by $f$.

As before, consider the sequence defined recursively by

$$a_n = f - W_{+1}(\nabla_A - D_{-1})a_{n-1}$$

and $a_0 = f$, i.e. $a_n = \sum_{i=0}^{n}(-W_{+1}(\nabla_A - D_{-1}))^i f$. So we see that for $k > l$ we have

$$a_k - a_l = \sum_{i=1}^{k-l}(-W_{+1}(\nabla_A - D_{-1}))^{l+i} f.$$

Thus the degree of $a_k - a_l$ is at least greater than $l$ and the sequence is Cauchy. Denote by $a_f$ the (unique) solution to the equation above. Note that

$$\begin{gathered}\sigma_\hbar(a_f) = \sigma_\hbar(f) - \sigma_\hbar(W_{+1}(\nabla_A - D_{-1})a_f) = \\ f - (W_{+1}D_{-1} + D_{-1}W_{+1} - \mathrm{Id})W_{+1}(D_{-1} - \nabla_A)a_f = \\ f + W_{+1}(D_{-1} - \nabla_A)a_f - W_{+1}(D_{-1} - \nabla_A)a_f = f.\end{gathered}$$

Note also that

$$-W_{+1}\nabla_A a_f = -W_{+1}(\nabla_A - D_{-1})a_f - W_{+1}D_{-1}a_f = a_f - f - a_f + \sigma_\hbar(a_f) = 0.$$

Then we see that $\nabla_A a_f$ solves the equation

$$b = -W_{+1}(\nabla_A - D_{-1})b$$

for $b$, but note that, as twice before, if this equation has a solution then it is unique, so we may conclude that $\nabla_A a_f = 0$.

Thus we have obtained the map $\tau\colon C^\infty(M)[\![\hbar]\!] \to \mathbb{A}_\hbar(M)$ given by $\tau(f) = a_f$. Note that $\tau$ is linear by definition. Moreover, we have shown that $\sigma_\hbar(\tau(f)) = f$. So, note that, if $a \in \mathbb{A}_\hbar(M)$, then

$$-W_{+1}(\nabla_A - D_{-1})a = W_{+1}D_{-1}a = a - \sigma_\hbar(a),$$

which shows that $\tau(\sigma_\hbar(a)) = a$ and thus $\tau = \sigma_\hbar^{-1}$.

The fact that the product on $\mathbb{A}_\hbar(M)$ induces a $\star$-product on $C^\infty(M)[\![\hbar]\!]$ follows by construction. □

This concludes our recollection of the Fedosov construction. In the following section we will show, using the framework of formal geometry, that the construction actually only depends on the cohomology class of the two form $\theta$. Moreover, we will show that the class of $\theta$ provides a complete invariant of symplectic deformation quantizations up to gauge equivalence.

## 4.3. The Characteristic Class of $\mathbb{A}_\hbar(M)$

In this section we will show the relation between the Fedosov construction, presented in the previous section 4.2 and the framework of formal deformed geometry presented in section 4.1. To do this, we should first show that the connection $\nabla_F^\hbar$ defined in proposition 4.1.7 is in fact a Fedosov connection as in definition 4.2.9. After we have established this, we will proceed to define the characteristic class associated to a deformation quantization. We will show that it is a gauge equivalence invariant and we will use the Fedosov construction to show that it is a *complete* invariant.

Note that the definition of a Fedosov connection can be put into words in the following way. A Fedosov connection is a flat connection on the Weyl algebras bundle which has values in the continuous derivations of $\mathbb{W}_\hbar$. This allows us to split up the connection in terms of the grading of $\mathfrak{g}^\hbar$ and a Fedosov connection should have a specific degree $-1$ part, while the degree 0 part should be given by a symplectic connection.

Proposition 4.3.1. *The operator $\nabla_F^\hbar$ defined in proposition 4.1.7 is a Fedosov connection.*

Proof.
We consider the operator $\nabla_F^\hbar$ of proposition 4.1.7 for the $(\mathfrak{g}^\hbar, \mathrm{Sp}(2d,\mathbb{R}))$-module $\mathbb{W}_\hbar$, i.e. as an operator in $\mathcal{W}$. We want to show that it is a Fedosov connection in the sense of the last paragraph. Since it is, by construction, a flat connection on the Weyl algebras bundle with values in the continuous derivations of $\mathbb{W}_\hbar$, we actually only have to show that it is given by $D_{-1}$ in lowest degree and by a symplectic connection in degree 0. To show this, note first that, by proposition 3.2.1, we can give $\nabla_F^\hbar$ relative to a symplectic connection by providing one-forms with values in $\frac{1}{i\hbar}\mathcal{W}$, i.e.

$$\nabla_F^\hbar - \nabla_0 = P_{\mathfrak{g}^\hbar}\left(\frac{1}{i\hbar}B_{-1}\right) + P_{\mathfrak{g}^\hbar}\left(\frac{1}{i\hbar}B_0\right) + P_{\mathfrak{g}^\hbar}\left(\frac{1}{i\hbar}B_1\right) + \dots.$$

The one-form corresponding to $D_{-1}$ in this way is simply $X \mapsto \frac{1}{i\hbar}\omega(X,-) \in \Gamma(\mathcal{W})$ for all vector fields $X$. Now, note that, by remark 4.1.10, we find that locally

$$\nabla_F^\hbar|_{\mathbb{R}^{2d}} = d_{\mathrm{dR}} - \sum_{i=1}^d dx_i \otimes \partial_{\hat{x}_i} + d\xi_i \otimes \partial_{\hat{\xi}_i}$$

in standard (Darboux) coordinates. More explicitly, we use the section given by

$$\mathrm{Sp}_{\mathbb{R}^{2d}} \longrightarrow \mathbb{R}^{2d} \times \mathrm{Sp}(2d,\mathbb{R}) \longrightarrow \mathbb{R}^{2d} \times \widehat{G}_r^\hbar \longrightarrow \widetilde{\mathbb{R}^{2d}}_{r,\hbar}$$

analogous to (2.4.1). Which, by the identification (3.2.2), means that $\nabla_F^\hbar|_{\mathbb{R}^{2d}} = d_{\mathrm{dR}} + D_{-1}$, where we note that on $\mathbb{R}^{2d}$ the exterior derivative is a symplectic linear connection. Now we see that, by naturality of $\omega_M^\hbar$; the fact that $\left(\frac{1}{i\hbar}\mathbb{W}_\hbar\right)_{-2} = \frac{\mathbb{C}}{i\hbar}$ and the fact that $\left(\frac{1}{i\hbar}\mathbb{W}_\hbar\right)_{-1} = \frac{1}{i\hbar}(\mathbb{R}^{2d} \otimes_{\mathbb{R}} \mathbb{C})^*$, the degree $-1$ part of $\nabla_F^\hbar$ must be given by $D_{-1}$. The fact that the degree 0 part of $\nabla_F^\hbar$ is given by a symplectic connection follows from the fact that it is given by a section from $\mathrm{Sp}_M$ to the reduced non-linear frames $\widetilde{M}_{r,\hbar}$.

□

In some sense, the framework of formal deformed geometry supplies a reverse Fedosov construction. Rather than producing a deformation quantization from a Fedosov connection, it produces a Fedosov connection, given a deformation quantization. Moreover, the produced Fedosov connection recovers the deformation quantization we started with, in the sense of the Fedosov construction, by 4.1.9. We will now show how this reverse Fedosov construction also recovers the two-form $\theta$ appearing in the Fedosov construction.

Definition 4.3.2. The characteristic class, sometimes called Weyl curvature, of the deformation $\mathbb{A}_\hbar(M)$ is defined to be the class of $\theta := GF^\hbar_M(\hat{\theta})$ where $GF^\hbar_M$ is the Gelfand-Fuks map defined in proposition 4.1.7 and the cocycle $\hat{\theta} \in C^2_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}\right)$ is the formal Weyl curvature defined in remarks 3.2.2 and 3.2.7, where we construct the map $\tilde{\sigma}$ using the standard symplectic basis of $\mathbb{R}^{2d}$.

Remark 4.3.3. Note that in the remarks 3.2.2 and 3.2.7 the class $\hat{\theta}$ was actually constructed relative to $\mathfrak{sp}(\mathbb{R}^{2d}\otimes_\mathbb{R}\mathbb{C})$. However, the inclusion of $\mathrm{Sp}(2d,\mathbb{R})$in $\mathrm{Sp}(\mathbb{R}^{2d}\otimes_\mathbb{R}\mathbb{C})$ allows us to define the class relative to $\mathfrak{sp}(2d,\mathbb{R})$ as well.

Note that, similar to example 2.4.3, the fact that the action of $\mathfrak{g}^\hbar$ on $\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}$ is trivial means that the complex $\left(\Omega^\bullet\left(M;\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}\right),\nabla^\hbar_F\right)$ has cohomology $\frac{\mathrm{H}^\bullet(M)[\![\hbar]\!]}{i\hbar}$.

Proposition 4.3.4. *The class* $[\theta] \in \frac{H^2(M)[\![\hbar]\!]}{i\hbar}$ *is a gauge equivalence invariant.*

Proof.
Suppose $\mathbb{A}_\hbar(M)'$ is another deformation quantization of $M$ and $\varphi\colon \mathbb{A}_\hbar(M)\to\mathbb{A}_\hbar(M)'$ is a gauge equivalence. Note that, in the terminology of 4.1.1, $\varphi$ is a $\mathbb{C}[\![\hbar]\!]$-linear algebra isomorphism that lifts the identity on $M$. Since $\varphi$ is defined in terms of differential operators and lifts the identity, we see that it also induces continuous isomorphisms

$$\varphi\colon \hat{\mathcal{A}}_m \longrightarrow \hat{\mathcal{A}}'_m,$$

of the deformed algebras of $\infty$-jets at each $m\in M$ This allows us to define the diffeomorphism $\widetilde{M}'_\hbar \to \widetilde{M}_\hbar$ given by $\Phi(\varphi_m) = \varphi_m\circ\varphi$ (note that it preserves the reduced submanifolds). Since $\varphi$ lifts the identity, this defines the $\mathrm{Sp}(2d,\mathbb{R})$-equivariant map $\mathrm{Sp}_M \to \widetilde{M}'_\hbar$ given by $\Phi^{-1}\circ F$. Now, by naturality of the Kazdan connection, we find that

$$(\Phi^{-1}\circ F)^*\omega'^\hbar_M = F^*((\Phi^{-1})^*\omega'^\hbar_M) = F^*\omega^\hbar_M.$$

Thus it is enough to show that the class $[\theta]$ is independent of the choice of section $F_r$.

By contractibility of $G^\hbar_1$ (as a pro-finite dimensional manifold) the map

$$\pi^*_{1,r}\colon \left(\Omega^\bullet(\mathrm{Sp}_M)\otimes\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}, d_{\mathrm{dR}}\right) \longrightarrow \left(\Omega^\bullet(\widetilde{M}_{r,\hbar})\otimes\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}, d_{\mathrm{dR}}\right)$$

is a quasi-isomorphism. Since $\pi_{1,r}$ is $\mathrm{Sp}(2d,\mathbb{R})$-equivariant, we find that $\widetilde{M}_{r,\hbar}/\mathrm{Sp}(2d,\mathbb{R})$ is a fiber bundle with contractible fibers over $M$ and so the induced map

$$\pi^*_{1,r}\colon \left(\Omega^\bullet\left(M;\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}\right), d_{\mathrm{dR}}\right) \longrightarrow \left(\Omega^\bullet(\widetilde{M}_{r,\hbar})^{\mathrm{Sp}(2d)}\otimes\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}, d_{\mathrm{dR}}\right)$$

is also a quasi-isomorphism. Now suppose $F'_r$ is another $\mathrm{Sp}(2d,\mathbb{R})$-equivariant section. Then we find that both $F_r$ and $F'_r$ induce maps

$$F^*_r, F'^*_r\colon \left(\Omega^\bullet\left(\widetilde{M}_{r,\hbar}\right)^{\mathrm{Sp}(2d)}\otimes\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}, d_{\mathrm{dR}}\right) \longrightarrow \left(\Omega^\bullet\left(M;\frac{\mathbb{C}[\![\hbar]\!]}{i\hbar}\right), d_{\mathrm{dR}}\right).$$

Since $\pi_{1,r}\circ F_r = \pi_{1,r}\circ F'_r = \mathrm{Id}$, we see that they must induce the same map in cohomology. Now note that the maps $GF^\hbar_M$ and $GF'^\hbar_M$ corresponding to $F_r$ and $F'_r$ are simply the compositions, of $F^*_r$ and

$F_r'^*$ respectively, with the map analogous to the one in remark 2.3.4 (using $\iota_r^*\omega_M^\hbar$ instead of $\omega_M$). So we see that they induce the same map in cohomology and thus $[\theta]$ does not depend on the choice of $F_r$.

□

REMARK 4.3.5. For degree reasons, only $D_{-1}$ contributes to the $\frac{\mathrm{H}^2(M)}{i\hbar}$ component of $[\theta]$. Using the explicit description of $\tilde{\sigma}$ in remark 3.2.7 and the definition of $D_{-1}$, one verifies, by local computation, that this component must equal $\frac{\omega}{i\hbar}$. Thus we find the restriction

$$[\theta] \in \frac{\omega}{i\hbar} + \mathrm{H}^2(M)[\![\hbar]\!]$$

on the characteristic class or Weyl curvature of a symplectic deformation quantization.

To conclude this chapter we apply the result of the Fedosov construction to show that the characteristic class is indeed a *complete* invariant of symplectic deformation quantizations. Suppose $[\theta_{\geq 1}] \in \mathrm{H}^2(M)[\![\hbar]\!]$, then we can construct a deformation quantization by picking a representative $\theta_{\geq 1}$ and applying the Fedosov construction. This means we construct $\nabla_A$, along the lines of theorem 4.2.14 and corollary 4.2.15, and identify $\mathbb{A}_\hbar(M)$ with $\operatorname{Ker}\nabla_A$ using $\sigma_\hbar$. Once we have such a deformation quantization $\mathbb{A}_\hbar(M)$, we can construct the corresponding bundles $\widetilde{M}_\hbar$ and $\widetilde{M}_{r,\hbar}$ of non-linear frames and, choosing a section $F_r$, we obtain the Fedosov connection $\nabla_F^\hbar$. By proposition 4.1.9 the kernel of $\nabla_F^\hbar$ is isomorphic to the deformation quantization constructed through the Fedosov construction. This means, by proposition 4.3.4, that the characteristic class of the deformation quantization defined in 4.3.2 coincides with the class $\frac{\omega}{i\hbar} + [\theta_{\geq 1}]$. So, we have the following result.

COROLLARY 4.3.6. *The affine space* $\frac{\omega}{i\hbar} + H^2(M)[\![\hbar]\!]$ *provides a parametrization of the gauge equivalence classes of symplectic deformation quantizations of* $(M, \omega)$.

CHAPTER 5

# Group Actions on Deformation Quantizations

The interaction of deformation quantization with groups of symplectic symmetries is interesting for a variety of reasons, both for Lie groups of continuous symmetries and for discrete symmetries. In the first case one may study the problem of symplectic reduction of deformation quantizations as is done for instance in [**18**]. In general, group actions by automorphisms on a deformation quantization represent symmetries of a quantum mechanical system these may be exploited in computations or provide further insight in the algebra itself. In this chapter we shall study group actions on deformation quantizations in some generality.

Two main topics will be of interest. First of all, it is easily deduced that "quantum symmetries", i.e. a group action on a deformation quantization, induce corresponding "classical symmetries", i.e. a group action on the underlying symplectic manifold. Thus it is natural to ask whether all classical symmetries come from quantum symmetries. This question can be phrased in terms of a lifting problem, which we will do in section 5.2. The second topic is the problem of classification. Since, a priori, different quantum symmetries may induce the same classical symmetry, it is also natural to ask whether one can parametrize the quantum symmetries inducing the same classical symmetries in some meaningful way. We devote section 5.3 to this second topic. In particular, we define a certain parametrizing object, see theorem 5.3.12, and show in sections 5.3.2 and 5.3.3 that this object can be determined in familiar terms. We will approach these two topics through the framework of deformed formal geometry and the Fedosov construction. In section 5.1 we set up a framework of considering group actions in terms of the Fedosov construction.

REMARK 5.0.7. Some work towards both the classification and existence results of group actions on formal deformation quantizations of symplectic manifolds was already carried out. First of all, Fedosov comments on the problem in both the paper [**46**] and the book [**48**]. Furthermore, when one restricts to so called invariant star products (a definition will follow), a classification up to equivariant equivalence was found by Bertelson-Bieliavsky-Gutt [**7**] and recently this classification was extended to include a notion of quantum moment maps by Reichert-Waldmann [**98**].

NOTATION 5.0.8. We keep using all the conventions of the previous chapter, i.e. $(M, \omega)$ is a symplectic manifold of dimension $2d$; We will denote by $\mathbb{A}_\hbar(M)$ or even $\mathbb{A}_\hbar$ a fixed deformation quantization of $M$; $\mathbb{W}_\hbar = \mathbb{W}_\hbar(\mathbb{R}^{2d} \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathcal{W}$ denotes the formal Weyl algebras bundle, i.e. the bundle associated to $\mathrm{Sp}_M$ with fiber $\mathbb{W}_\hbar$.

## 5.1. Group Actions Through Fedosov

The Fedosov construction of deformation quantization and the added framework of formal geometry provide a rigid environment for considering group actions by automorphisms on deformation quantizations. One of the advantages noted in Fedosov's original paper is that there is a simple way to lift any symplectomorphism of the manifold to an automorphism of the deformation quantization. It pays to keep in mind, however, that these lifts do not obviously respect compositions and are in general not unique. Since group actions on the symplectic manifold can be interpreted as symmetries of the classical mechanical system, the question of whether one can lift a group action naturally arises and Fedosov comments on this in his paper [**46**].

In this section we will lay some groundwork towards the existence and classification results in the following two sections. We will set up a language of group actions on deformation quantizations in terms of actions by symplectomorphisms on the underlying manifolds. More explicitly, we will define what we understand by the extension of an action by symplectomorphisms to an action on the deformation quantization and give a motivation for this definition.

Before we consider the case of group actions on deformation quantizations, let us consider the action of a single automorphism. If $\varphi$ is a $\mathbb{C}[\![\hbar]\!]$-linear algebra automorphism of $\mathbb{A}_\hbar(M)$, the assignment $f \mapsto \varphi(f) \mod \hbar$ defines an algebra automorphism of $C^\infty(M)$. Thus there is a map

$$\mathrm{Aut}(\mathbb{A}_\hbar(M)) \longrightarrow \mathrm{Symp}(M,\omega),$$

given by noting that any automorphism of $C^\infty(M)$ is given (uniquely) by the pull-back under a diffeomorphism (see, for instance, chapter 7 of [**90**]). The fact that the induced diffeomorphism is a symplectomorphism can be seen by applying it to a commutator. Let us recall for clarity the part of definition 4.1.1 that will be most relevant in this section.

Definition 5.1.1. Suppose $\varphi \in \mathrm{Symp}(M,\omega)$, we say $\alpha_\varphi \in \mathrm{Aut}(\mathbb{A}_\hbar(M))$ extends $\varphi$ if

$$\alpha_\varphi(f) = \varphi^* f + \hbar\Phi_f,$$

with $\Phi_f \in \mathbb{A}_\hbar$, for all $f \in C^\infty(M)[\![\hbar]\!]$. Here $\varphi^*$ denotes the $\mathbb{C}[\![\hbar]\!]$-linear extension of the pull-back $\varphi^* f = f \circ \varphi$.

Before we consider extending group actions, we should know which symplectomorphisms allow for lifts. It turns out that the Fedosov construction provides a clear answer to this question [**46, 89**]. So let us fix a Fedosov connection $\nabla_A$, given as

$$\nabla_A = D_{-1} + \nabla_0 + \frac{1}{i\hbar}[A_{\mathbb{N}}, -]$$

as in section 4.2. It will turn out that automorphisms of the deformation quantization can be split up into a global part and a fiberwise part. The following definition will be helpful.

Definition 5.1.2. Note that $\mathrm{Sp}(2d,\mathbb{R})$ acts on $G_1^\hbar$ by conjugation. This yields a sheaf of groups $\mathcal{G}_1^\hbar$ given by the sections of $\mathrm{Sp}_M \times_{\mathrm{Sp}(2d)} G_1^\hbar$. Here $\mathrm{Sp}_M \times_{\mathrm{Sp}(2d)} G_1^\hbar$ denotes the quotient of $\mathrm{Sp}_M \times G_1^\hbar$ by the relation $(pg,\gamma) = (p, g\gamma g^{-1})$ for any $g \in \mathrm{Sp}(2d,\mathbb{R})$.

Proposition 5.1.3. *The map*

$$\mathit{Aut}(\mathbb{A}_\hbar(M)) \longrightarrow \mathit{Symp}(M,\omega)_{[\theta]}$$

*is a surjection. Here $[\theta] \in H^2(M)[\![\hbar]\!]$ denotes the characteristic class of $\mathbb{A}_\hbar(M)$ and $\mathit{Symp}(M,\omega)_{[\theta]}$ denotes the stabilizer for the obvious (right) action on $H^2(M)[\![\hbar]\!]$.*

Proof.
Note that the lemma asserts that, for a symplectomorphism $\varphi$ to extend to the deformation, it is both a necessary and sufficient condition that it preserves the characteristic class of the deformation. So let us fix a symplectomorphism $\varphi$.

We will begin by showing that, if there exists an automorphism $\alpha_\varphi$ lifting $\varphi$, then $\varphi$ preserves the characteristic class. Let $\mathbb{A}_\hbar(M)^\varphi$ denote the deformation quantization given by twisting the product $\star$ of $\mathbb{A}_\hbar(M)$ by $\varphi$, i.e.

$$f \star_\varphi g = \varphi^*(\varphi_*(f) \star \varphi_*(g)),$$

where $\varphi^*$ denotes the pull-back by $\varphi$ and $\varphi_*$ denotes the pull-back by $\varphi^{-1}$. We note that $\varphi_*$ is an algebra isomorphism from $\mathbb{A}_\hbar(M)$ to $\mathbb{A}_\hbar(M)^\varphi$. Now suppose there exists an extension $\alpha_\varphi$ of $\varphi$. Then we can consider the composition $G_\varphi := \varphi_* \circ \alpha_\varphi$. By a partition of unity argument (using the $\star$-product), we find that $\alpha_\varphi$ restricts to a map $\mathbb{A}_\hbar(U) \to \mathbb{A}_\hbar(\varphi^{-1}(U))$ for every open $U \subset M$. This shows that $G_\varphi$ must in fact be a gauge equivalence and thus the characteristic class of $\mathbb{A}_\hbar(M)^\varphi$ must be the same as the class of $\mathbb{A}_\hbar(M)$ by 4.3.4. On the other hand, it is not hard to see that $\star_\varphi$ is induced by the Fedosov connection $(\varphi^{-1})^*\nabla_A$ and thus we find that preserving $[\theta]$ is a necessary condition.

Suppose now that $\varphi$ preserves the characteristic class of $\mathbb{A}_\hbar(M)$. Note that the pull-back $\varphi^*$ is a well-defined algebra automorphism of $\Omega^\bullet(M;\mathbb{W}_\hbar)$. Clearly, it is an isomorphism

$$\varphi^*\colon\ \operatorname{Ker}\nabla_A \xrightarrow{\sim} \operatorname{Ker}\varphi^*\nabla_A.$$

We will show sufficiency, i.e. we will show that there exists a lift $\alpha_\varphi$ of $\varphi$, by constructing a map in the opposite direction given by a section $U_\varphi$ of $\mathcal{G}_1^\hbar$. This means that the composite map $U_\varphi\circ\varphi^*$ extends $\varphi$. The construction of $U_\varphi$ can also be found in [**89**] or section 5.5 of [**48**]. The construction of $U_\varphi$ is by an iterative procedure. Note that, since $\varphi$ preserves the characteristic class, we find that $\varphi^*\nabla_A$ defines the same class. Moreover, since adding any central form to $\varphi^*\nabla_A$ does not change its kernel, we may as well assume the curvatures (in the sense of 4.2.10), of the connections $\nabla_A$ and $\varphi^*\nabla_A$, are equal.

We have fixed a lift of $\nabla_A$ to $\frac{1}{i\hbar}\mathbb{W}_\hbar$ relative to $\nabla_0$ above, namely $A=\frac{1}{i\hbar}(A_{-1}+A_{\mathbb{N}})$. This provides the one-forms

$$r_k := A_k-\varphi^*A_k \ \text{ for }\ k>0 \qquad\text{and}\qquad r_0=\tilde\sigma\left(\nabla_0-\varphi^*\nabla_0\right).$$

Note that $r_k$ has values in the elements of total degree $k$ in $\frac{1}{i\hbar}\mathbb{W}_\hbar$. Note also that, by lemma 4.2.3, the form $A_{-1}$ is a diffeomorphism invariant. Now, since the curvatures of $\nabla_A$ and $\varphi^*\nabla_A$ are equal, we find for degree reasons that

$$\frac{1}{i\hbar}[A_{-1},r_0]=0.$$

By acyclicity of the complex with differential $D_{-1}$, see the proof of proposition 2.4.2 and the remark 4.1.10, we may choose $\delta_1\in\Omega^1\left(M,\left(\frac{1}{i\hbar}\mathbb{W}_\hbar\right)_1\right)$ such that

$$r_0=\frac{1}{i\hbar}[A_{-1},\delta_1].$$

Then, by remark 3.2.9, we can consider the automorphism $\operatorname{Ad}\,\exp(\delta_1)$, note that it is a section of $\mathcal{G}_1^\hbar$. We find that

$$\nabla_A-\operatorname{Ad}\,\exp(\delta_1)\circ\varphi^*\nabla_A\circ\operatorname{Ad}\,\exp(-\delta_1)=0 \bmod \mathfrak{g}^\hbar_{>0}.$$

In other words if we replace $\varphi^*\nabla_A$ by $\operatorname{Ad}\,\exp(\delta_1)\circ\varphi^*\nabla_A\circ\operatorname{Ad}\,\exp(-\delta_1)$ we find that $r_0=0$. This means for degree reasons that

$$\frac{1}{i\hbar}[A_{-1},r_1]=0,$$

so we know there exists $\delta_2$ such that

$$r_1=\frac{1}{i\hbar}[A_{-1},\delta_2].$$

Continuing like this we can iteratively define $\delta_i$ for all $i\in\mathbb{N}$. Since the degrees of the $\delta_i$ necessarily increase we find the well-defined section

$$U_\varphi := \operatorname{Ad}\ \ldots\exp\delta_i\exp\delta_{i-1}\ldots\exp\delta_2\exp\delta_1$$

of $\mathcal{G}_1^\hbar$ such that

$$U_\varphi\circ\varphi^*\nabla_A\circ U_\varphi^{-1}=\nabla_A. \tag{5.1.1}$$

Note that this implies that

$$U_\varphi\colon\ \operatorname{Ker}\varphi^*\nabla_A\longrightarrow\operatorname{Ker}\nabla_A$$

is a well-defined isomorphism.

□

Remark 5.1.4. Note that the proof of sufficiency in proposition 5.1.3 provides a (highly non-unique) choice of section of the map from algebra automorphisms to symplectomorphisms preserving the characteristic class $[\theta]$. However, it is not guaranteed that such a section is a group homomorphism. So let us consider, instead of a single symplectomorphism, a group $\Gamma$ of symplectomorphisms. In other words, suppose we have a group $\Gamma$ acting (from the left) by symplectomorphisms on $M$, i.e. a group homomorphism $\Gamma \to \mathrm{Symp}(M,\omega)$. Note that we have, by proposition 5.1.3, a map

$$\mathrm{Hom}_{\mathrm{Gr}}\left(\Gamma^{op}, \mathrm{Aut}\left(\mathbb{A}_\hbar(M)\right)\right) \longrightarrow \mathrm{Hom}_{\mathrm{Gr}}\left(\Gamma, \mathrm{Symp}(M,\omega)_{[\theta]}\right), \tag{5.1.2}$$

although we cannot be assured of surjectivity any longer.

Remark 5.1.5. Note that, by proposition 3.2.8, we find that $G_1^\hbar = \exp F_1\mathfrak{g}^\hbar$, while, by proposition 3.2.1, we have that $F_1\mathfrak{g}^\hbar = P_{\mathfrak{g}^\hbar}\left(\frac{1}{i\hbar}\left(\mathbb{W}_\hbar\right)_{\geq 3}\right)$. So we find that sections of $\mathcal{G}_1^\hbar$ can always locally be given as (conjugation by) exponentials of sections of $\frac{1}{i\hbar}\mathcal{W}_{\geq 3}$.

Proposition 5.1.6. *A lift $\alpha_\varphi$ of a symplectomorphism $\varphi$ is of the form*

$$\alpha_\varphi = c_\varphi \circ \varphi^*$$

*where $c_\varphi$ is a section of $\mathcal{G}_1^\hbar$.*

Proof.
The automorphism $\alpha_\varphi$ of $\mathbb{A}_\hbar(M)$ induces isomorphisms

$$\hat{\alpha}_{\varphi,m}\colon \hat{\mathcal{A}}_{\varphi(m),\hbar} \longrightarrow \hat{\mathcal{A}}_{m,\hbar}.$$

Thus we find the diffeomorphism $\zeta_\varphi$ of $\widetilde{M}_\hbar$ given by $\zeta_\varphi(\varphi_m) = \varphi_m \circ \hat{\alpha}_{\varphi,m}$. Note that $\zeta_\varphi$ is $\widehat{G}^\hbar$-equivariant, since the action of $\widehat{G}^\hbar$ is by pre-composition. We also see that $\alpha_\varphi$ preserves $\widetilde{M}_{r,\hbar}$ since it lifts a symplectomorphism. Note that we may consider the Weyl algebras bundle as the bundle with fiber $\mathbb{W}_\hbar$ associated to the principal bundle $\widetilde{M}_{r,\hbar}$, by choosing the $\mathrm{Sp}(2d,\mathbb{R})$-equivariant section $F_r\colon \mathrm{Sp}_M \to \widetilde{M}_{r,\hbar}$. This means $\zeta_\varphi$ induces an automorphism of the sections of the formal Weyl algebras bundle, we shall denote this automorphism by

$$\tilde{\alpha}_\varphi := \left(\pi_{1,r} \circ \zeta_\varphi \circ F_r\right)^*.$$

Suppose $U \subset M$ is an open subset such that we have trivializations

$$t_U\colon \mathcal{W}|_U \longrightarrow U \times \mathbb{W}_\hbar \quad \text{and} \quad t_{\varphi(U)}\colon \mathcal{W}|_{\varphi(U)} \longrightarrow \varphi(U) \times \mathbb{W}_\hbar.$$

Then we see that $t_{\varphi(U)} \circ \tilde{\alpha}_\varphi \circ t_U^{-1} = (\varphi, \nu_\varphi)$ where $\nu_\varphi\colon U \to G_1^\hbar$ is a smooth map.

□

In order to apply the Fedosov construction in the context of group actions we need to ask for a certain compatability between the group action and a Fedosov connection. Combining this with the previous proposition motivates the following definition.

Definition 5.1.7. An extension of the left action of $\Gamma$ on $M$ by symplectomorphisms is defined to be a right action $\alpha\colon \Gamma^{op} \to \mathrm{Aut}\left(\mathbb{A}_\hbar\left(M\right)\right)$ such that

$$\alpha_\gamma = c_\gamma \circ \gamma^*, \tag{5.1.3}$$

for $\gamma \in \Gamma$ and $c_\gamma$ is a section of $\mathcal{G}_1^\hbar$, such that

$$c_\gamma \circ \gamma^* \nabla_A \circ c_\gamma^{-1} = \nabla_A$$

for some Fedosov connection $\nabla_A$ such that $\mathrm{Ker}\,\nabla_A \simeq \mathbb{A}_\hbar(M)$. Here we have also denoted the symplectomorphism by $\gamma$.

REMARK 5.1.8. In [**7, 98**] and other sources the authors consider only those extensions where the $c_\gamma$ are trivial. These are of course the most natural extensions (if they exist). The reason to consider not only the most natural extensions of the action to the deformation quantization is that two equivalent (Fedosov) star products that allow such a natural action are not necessarily equivariantly equivalent. This means that, when one transports the natural group action from one to the other (using the equivalence) it will not be of this most natural form. However, there is in general no clear reason to prefer one (Fedosov) star product over the other, when one starts from a given characteristic class. Thus, when one considers all extensions, as defined in definition 5.1.7, one becomes free to consider an arbitrary Fedosov connection with curvature in the characteristic class. Another, more obvious, reason is that there may be cases where extensions as in definition 5.1.7 exist while the extension with trivial $c_\gamma$ does not.

REMARK 5.1.9. Note that the condition of compatibility with the Fedosov connection will be superfluous in many cases, see [**46**]. In particular given a symplectomorphism that preserves not only the class $[\theta]$, but also a representative $\theta$ of that class we can always lift it in a way compatible with a Fedosov connection.

## 5.2. Existence of Extensions of Group Actions

In this section we investigate the problem of lifting a group action $\Gamma \to \mathrm{Symp}\,(M,\omega)_{[\theta]}$ to a group action $\Gamma^{op} \to \mathrm{Aut}\,(\mathbb{A}_\hbar\,(M))$. The question of existence of lifts is exactly the question of surjectivity of the map (5.1.2). We will rephrase the question in a rather compact form. We will also give some sufficient conditions for the existence of a lift and show that they are not necessary in general.

So, let us fix the group action $\Gamma \to \mathrm{Symp}\,(M,\omega)_{[\theta]}$. As mentioned above, we can, as in proposition 5.1.3, always find a map $\alpha\colon \Gamma^{op} \to \mathrm{Aut}\,(\mathbb{A}_\hbar\,(M))$ such that $\alpha_\gamma \mapsto \gamma \in \mathrm{Symp}\,(M,\omega)_{[\theta]}$ by the map mentioned in the previous section. Note also that the proof of proposition 5.1.3 shows that $\alpha_\gamma := c_\gamma \circ \gamma^*$, where $c_\gamma \circ \gamma^* \nabla_A \circ c_\gamma^{-1} = \nabla_A$, for some section $c_\gamma$ of $\mathcal{G}_1^\hbar$ and a given Fedosov connection $\nabla_A$. The following observation is also present in Fedosov's paper [**46**]. We have

$$\alpha_\gamma \circ \alpha_\mu = c_\gamma \gamma\,(c_\mu) \circ (\mu\gamma)^*\,,$$

where the action of $\Gamma$ on sections of $\mathcal{G}_1^\hbar$ is by conjugation, i.e. $\gamma(c) = \gamma^* \circ c \circ \gamma_*$. On the other hand $\alpha_{\mu\gamma} = c_{\mu\gamma} \circ (\mu\gamma)^*$. So we find that the $c_\gamma$ should satisfy a cocycle condition in order for $\alpha$ to be a group homomorphism. Indeed, $\alpha$ defines a group action exactly when

$$c_\gamma \gamma\,(c_\mu)\, c_{\mu\gamma}^{-1} = \mathrm{Id} \tag{5.2.1}$$

for all $\gamma, \mu \in \Gamma$.

COROLLARY 5.2.1. *The action $\Gamma \to Symp\,(M,\omega)_{[\theta]}$ lifts to an action on the deformation quantization in the sense of definition 5.1.7 iff there exist sections $c_\gamma$ of $\mathcal{G}_1^\hbar$, such that $c_\gamma \circ \gamma^* \nabla_A \circ c_\gamma^{-1} = \nabla_A$ for all $\gamma \in \Gamma$, that form a cocycle.*

REMARK 5.2.2. Although it is in general not easy to check the cocycle condition (5.2.1), more can be said about the question of existence in more general terms. As noted above in corollary 4.2.15 and in [**48**], one can construct a Fedosov connection $\nabla_A$ with characteristic class $[\theta]$ *uniquely*, given a representative $\theta \in \Omega^2\,(M, \mathbb{C}[\![\hbar]\!])$ and a symplectic torsion-free connection. Moreover, given an invariant linear connection, one can construct an invariant torsion-free symplectic connection in a canonical way, this is done (again) by the trick also applied in [**97**].

PROPOSITION 5.2.3. *Suppose there exists an invariant linear connection $\nabla_{00}$ on $M$, i.e. such that $\gamma^* \nabla_{00} = \nabla_{00}$ for all $\gamma \in \Gamma$. Then the action extends to any deformation quantization which has characteristic class in the image of the map*

$$\frac{\omega}{i\hbar} + H^2_\Gamma\,(M)\,[\![\hbar]\!] \longrightarrow \frac{\omega}{i\hbar} + H^2_{dR}\,(M)^\Gamma\,[\![\hbar]\!], \tag{5.2.2}$$

*where we denote by the cohomology of the complex* $\left(\Omega^\bullet(M)^\Gamma[\![\hbar]\!], d\right)$ *by* $H^\bullet_\Gamma(M)[\![\hbar]\!]$ *and we denote the invariants of* $H^\bullet_{dR}(M)$ *by* $H^\bullet_{dR}(M)^\Gamma$.

Proof.
The proposition follows immediately from the remark above and is also noted in [**48**]. For the situation as described in the lemma it is possible to construct a Fedosov connection $\nabla_A$ such that $\gamma^*\nabla_A = \nabla_A$ for all $\gamma \in \Gamma$. Then $\gamma^*$ defines an automorphism of $\operatorname{Ker}\nabla_A$ and thus $\alpha_\gamma = \gamma^*$ extends the action. □

Remark 5.2.4. Note that the class $\frac{\omega}{i\hbar} + [\omega]$ (and similar classes) will always be in the image of the map stated above. Note also that proposition 5.2.3 is comparable to the results in [**7**], there the authors consider more restrictive extensions (i.e. $\alpha_\gamma = \gamma^*$) and show that any class in the left hand cohomology induces an *invariant* star product up to *equivariant* equivalence. Recently, these results where extended to include quantum moment maps in [**98**].

Corollary 5.2.5. *Suppose* $\Gamma$ *is a compact Lie group, then any (smooth) action of* $\Gamma$ *by symplectomorphisms can be extended to any deformation quantization by* $\alpha_\gamma = \gamma^*$.

Proof.
By averaging an arbitrary linear connection one obtains an invariant linear connection and by averaging an arbitrary representative of a characteristic class one obtains a representative of the pre-image in $\frac{\omega}{i\hbar} + \mathrm{H}^2_\Gamma(M)[\![\hbar]\!]$. □

Remark 5.2.6. Note that the proof of corollary 5.2.5 actually also applies to the case where $\Gamma$ is not itself a compact Lie group, but the action of $\Gamma$ factors through the (smooth) action of a compact Lie group.

Remark 5.2.7. It is possible to phrase the first condition in proposition 5.2.3 in terms of a cohomological obstruction in the following way. Consider the right-action of $\Gamma$ on $\Omega^1(M, \operatorname{End}(TM))$ by pull-back. Given an affine connection $\nabla_{00}$, we define

$$D\colon \Gamma \longrightarrow \Omega^1(M, \operatorname{End}(TM))$$

$$\gamma \longmapsto \gamma^*\nabla_{00} - \nabla_{00}$$

Note that this defines a group cocycle in the group cohomology complex of $\Gamma^{op}$ with values in the vector space $\Omega^1(M, \operatorname{End}(TM))$. Clearly, if $\nabla_{00}$ is invariant we find that $D = 0$. It is easy to check that the class of $D$ in $\mathrm{H}^1\left(\Gamma^{op}, \Omega^1(M, \operatorname{End}(TM))\right)$ does not depend on $\nabla_{00}$ and in fact $[D] = 0$ if and only if there exists an invariant affine connection on $M$. The class defined above can be identified as a certain notion of the Atiyah class [**73**].

Proposition 5.2.3 gives a sufficient condition for the existence of a lift of a group action. It includes many group actions on manifolds, so a natural question is whether it is possible to extend actions that do not satisfy the hypotheses of the proposition. In other words, one might wonder if there exist actions that allow extension, but do not satisfy the hypotheses of proposition 5.2.3, i.e. one wonders whether the conditions of proposition 5.2.3 are necessary.

Example 5.2.8. Consider the cotangent bundle $\left(T^*S^2, d\eta\right)$ of the 2-sphere with the canonical symplectic structure $d\eta$. Let $\Gamma$ be the group of orientation preserving diffeomorphisms of $S^2$ that fix the equator $Eq\colon S^1 \hookrightarrow S^2$ pointwise. It is an elementary fact of symplectic geometry, found for instance in [**19**], that the group of diffeomorphisms of a manifold lifts to symplectomorphisms of the corresponding cotangent bundle. Note that

$$\mathrm{H}^2_{\mathrm{dR}}\left(T^*S^2\right)[\![\hbar]\!] = \mathbb{C}[\![\hbar]\!]\pi^*[\omega],$$

where $\pi$ denotes the projection $T^*S^2 \to S^2$ and $\omega$ denotes the standard symplectic structure of $S^2$. Since $\Gamma$ consists of orientation preserving diffeomorphisms we find that the class $\pi^*[\omega] \in \mathrm{H}^2\left(T^*S^2\right)^\Gamma$

is invariant. On the other hand, if $\beta$ is any 1-form on $T^*S^2$ such that $\pi^*\omega + d\beta \in \Omega^2\left(T^*S^2\right)^\Gamma$, then we find

$$\omega + d\left(z^*\beta\right) = z^* f_\sharp^*\left(\pi^*\omega + d\beta\right) = f^* z^*\left(\pi^*\omega + d\beta\right) = f^*\left(\omega + dz^*\beta\right)$$

for all $f \in \Gamma$, here we have denoted the zero-section of $T^*S^2 \to S^2$ by $z$ and the symplectomorphism of $T^*S^2$ induced by $f$ by $f_\sharp$. However, note that any non-zero two-form on $S^2$ will have support on some open $U$ which is disjoint from $Eq(S^1)$ and subject to a multitude of "local" diffeomorphisms in $\Gamma$. So since $\omega + dz^*\theta$ is invariant under all such diffeomorphisms we find that it vanishes identically, but this implies that $[\omega] = 0$. Thus we are led to a contradiction, which shows that $[\omega]$ is not in the image of the map (5.2.2), in fact this map is 0. Thus this action does not satisfy the criteria of proposition 5.2.3 and there does not exist any invariant Fedosov connection with non-trivial characteristic class for this action. On the other hand, we note that, by considering the clutching construction of vector bundles on $S^2$, we can lift the action of $\Gamma$ to an action on the line bundle corresponding to the inclusion $S^1 \hookrightarrow \mathbb{C}^\times$ (which has Chern class $[\omega]$). This means we can extend the action of $\Gamma$ to differential operators on smooth sections of this line bundle. Thus by the results in [**11**] we find that the action of $\Gamma$ does lift to the deformation quantization with characteristic class $\hbar[\pi^*\omega]$.

By example 5.2.8, we see that although the conditions of proposition 5.2.3 are sufficient to conclude existence of extended group actions (as defined in definition 5.1.7), they are not necessary.

## 5.3. Classification of Extended Group Actions

In this section we will turn to the question of classification of extended group actions. We will show that extended group actions are classified, up to a technical condition, by the first cohomology of the group $\Gamma$ with values in a certain non-Abelian group $\overline{\mathcal{G}_\nabla}$. We will finish the classification, by first providing the essential tools for computing this first cohomology and subsequently considering some examples. The classification will be carried out relative to a given extended group action. So for this section we will fix a Fedosov connection $\nabla$ (dropping the subscript $A$) and an extended action $\alpha\colon \Gamma^{op} \to \mathrm{Aut}\left(\mathbb{A}_\hbar\left(M\right)\right)$.

**5.3.1. Abstract Classification.** In this section we will give a classification of the lifts of group actions as in definition 5.1.7 in abstract terms, up to a certain technical condition. We will provide methods of computation of these abstract objects in the next section. To do this, we will first, following Fedosov [**46, 48**], introduce an extension of the Weyl algebras bundle such that a subgroup of the group of invertible sections surjects locally onto the sections of $\mathcal{G}_1^\hbar$ in a natural way. This will allow us to understand the bundle $\mathcal{G}_1^\hbar$ better, in order to provide tools for computation in the following section and also define a subgroup of the total sections providing the abstract classification.

Definition 5.3.1. Consider the algebra $\mathbb{W}_{(\hbar)} := \mathbb{W}_\hbar \otimes \mathbb{C}[\hbar^{-1}, \hbar\rrbracket$ where the tensor product is over $\mathbb{C}\llbracket\hbar\rrbracket$, i.e. $\mathbb{W}_{(\hbar)} = \mathbb{W}_\hbar[\hbar^{-1}]$. Note that $\mathbb{W}_{(\hbar)}$ carries a grading induced by the grading of $\mathbb{W}_\hbar$, i.e. $|\hbar^{-1}| = -2$. We define the algebra $\mathbb{W}_\hbar^+ \subset \mathbb{W}_{(\hbar)}$ as the subalgebra $F_0\mathbb{W}_{(\hbar)}$ of elements with degree greater than 0. In other words we allow power series with negative powers of $\hbar$ as long as the total degree is still greater than 0. Similarly, we denote the bundle associated to $\mathrm{Sp}_M$ with fibers given by $\mathbb{W}_\hbar^+$ by $\mathcal{W}^+$ and the bundle associated to $\mathrm{Sp}_M$ with fibers given by $\mathbb{W}_{(\hbar)}$ by $\mathcal{W}^f$.

Remark 5.3.2. Note that the Fedosov connection $\nabla$ is well-defined on the bundle $\mathcal{W}^f$. We will denote the center of $\mathcal{W}^+$ and $\mathcal{W}$ by $\mathcal{Z}$. Note that $\mathcal{Z} \simeq C_M^\infty\llbracket\hbar\rrbracket$ by the inclusion $\mathbb{C}\llbracket\hbar\rrbracket \hookrightarrow \mathbb{W}_\hbar^+$.

Definition 5.3.3. We define the sheaf of *fiberwise transformations* by assigning the sections of $\mathcal{W}^+(U)$ which are given by exponentials of elements of $\left(\frac{1}{i\hbar}\mathcal{W}(U)\right)_{\geq 1}$ to the open subset $U$. We will denote this sheaf by $\mathcal{T}_\mathcal{F}$. Note that $\mathcal{T}_\mathcal{F}$ i a sheaf of groups by the Campbell-Baker-Hausdorff formula.

Remark 5.3.4. Note that there is map

$$\mathrm{Ad} : \mathcal{T}_\mathcal{F} \longrightarrow \mathcal{G}_1^\hbar$$

given by assigning to the section $E$ of $\mathcal{T}_{\mathcal{F}}$ the automorphism of conjugation by $E$. Note that, by the proposition 3.2.8 and remark 3.2.9, we find that Ad is locally surjective.

Suppose that, for $E_\varphi, E'_\varphi \in \mathcal{T}_{\mathcal{F}}$, $U_\varphi := \mathrm{Ad}\ E_\varphi$ and $U'_\varphi := \mathrm{Ad}\ E'_\varphi$ both satisfy (5.1.1). Then we find that, denoting $E = E'_\varphi E_\varphi^{-1}$,

$$\mathrm{Ad}\ E \circ \nabla \circ \mathrm{Ad}\ E^{-1} = \nabla. \tag{5.3.1}$$

Conversely, if $U_\varphi$ satisfies (5.1.1) and $E$ satisfies (5.3.1), then clearly $U'_\varphi := \mathrm{Ad}\ E \circ U_\varphi$ also satisfies (5.1.1).

Remark 5.3.5. The discussion above suggests the following technical condition on the kind of actions we allow. Namely, we will only consider those actions that are of the form

$$\gamma \mapsto \mathrm{Ad}\ E_\gamma \circ \alpha_\gamma,$$

with $E_\gamma \in \mathcal{T}_{\mathcal{F}}$ for all $\gamma \in \Gamma$.

Remark 5.3.6. It serves now to compare the technical conditions in definition 5.1.7 and remark 5.3.5. First of all we note that they are in fact compatible. Secondly we compare them to the notion of isomorphism (and automorphism) of quantum algebras as defined in [**48**] section 5.5. We note that the condition in definition 5.1.7 is in fact weaker than the condition of a connection preserving isomorphism in [**48**]. On the other hand, the condition in remark 5.3.5 is, in the terms of [**48**], exactly the condition that the action is given, relative to $\alpha_\gamma$, by a fiberwise automorphism preserving the connection $\nabla$. Thus, if we use the definition of automorphism in section 5.5 of [**48**], the only technical condition is preservation of the Fedosov connection $\nabla$. The reason, also offered in [**48**], to consider actions through these kinds of actions seriously is that they correspond to the time evolution operators through Heisenberg's equations of motion, see equation (1.1.3).

Definition 5.3.7. We define *Fedosov's fiberwise* $\nabla$ *preserving isomorphisms* by

$$\mathcal{G}_\nabla := \left\{ E \in \mathcal{T}_{\mathcal{F}} \mid \nabla\left(E^{-1}\right) E \in \Omega^1\left(M; \mathbb{C}[\![\hbar]\!]\right) \right\}.$$

We should mention that, although it is not presented quite in this form, a lot of the following (excluding the classification) is implicit in the work of Fedosov [**48, 46**]. We should justify the nomenclature of $\mathcal{G}_\nabla$.

Lemma 5.3.8. *The section* $E \in \mathcal{T}_{\mathcal{F}}$ *satisfies* (5.3.1) *if and only if* $E \in \mathcal{G}_\nabla$.

Proof. Suppose $E \in \mathcal{T}_{\mathcal{F}}$ satisfies (5.3.1), then for all $\sigma \in \mathcal{W}$ we find

$$\nabla\sigma = E^{-1}\left(\nabla\left(E\sigma E^{-1}\right)\right) E = \nabla\sigma + [E^{-1}\nabla E, \sigma],$$

which implies $E \in \mathcal{G}_\nabla$, since $E^{-1}\nabla E = -\nabla(E^{-1})E$. The equation above also shows the converse statement. □

Notation 5.3.9. We denote the group of automorphisms of $\mathcal{W}$ that are given by sections of $\mathcal{T}_{\mathcal{F}}$ and induce automorphisms of $\mathrm{Ker}\,\nabla$ by $\mathrm{Loc}\,(\mathcal{W} \mid \nabla)$.

Lemma 5.3.10.

- **(i):** $\mathcal{G}_\nabla$ *is a subgroup of the invertibles* $\left(\mathcal{W}^+\right)^\times$ *of* $\mathcal{W}^+$.
- **(ii):** $\mathcal{Z}^\times \triangleleft \mathcal{G}_\nabla$ *and* $(Ker\nabla)^\times \triangleleft \mathcal{G}_\nabla$.
- **(iii):** $\mathcal{G}_\nabla / \mathcal{Z}^\times \simeq Loc(\mathcal{W} \mid \nabla)$.
- **(iv):** $\mathcal{G}_\nabla$ *forms a sheaf of groups on* $M$.
- **(v):** *For any open* $V \subset M$ *such that* $H^1_{dR}(V) = 0$ *we have*

$$\mathcal{G}_\nabla|_V = \mathcal{Z}^\times|_V \cdot \mathbb{A}_\hbar(V)^\times \tag{5.3.2}$$

- **(vi):** $\alpha_\gamma\left(\mathcal{G}_\nabla\right) \subset \mathcal{G}_\nabla$ *for all* $\gamma \in \Gamma$.

PROOF.
"(i)" Suppose $E, B \in \mathcal{G}_\nabla$, we should show that $E^{-1}$ and $EB$ are also in $\mathcal{G}_\nabla$. We have $E^{-1} \in \mathcal{G}_\nabla$, since

$$E^{-1}\nabla E = \mathrm{Ad}\, E\left(E^{-1}\nabla E\right) = (\nabla E)\, E^{-1} = -E\nabla E^{-1}.$$

Similarly, we have $EB \in \mathcal{G}_\nabla$, since

$$B^{-1}E^{-1}\nabla EB = \mathrm{Ad}\, B^{-1}\left(E^{-1}\nabla E\right) + B^{-1}\nabla B = E^{-1}\nabla E + B^{-1}\nabla B.$$

"(ii)" Since $\mathcal{Z}^\times$ is central we see that $\nabla z = dz$ for all $z \in \mathcal{Z}^\times$, this shows that $\mathcal{Z}^\times$ is a subgroup. It is a normal subgroup because it is central. Of course $\nabla k = 0$ for all $k \in (\mathrm{Ker}\,\nabla)^\times$ showing that it is a subgroup. It is normal since

$$\nabla\left(EkE^{-1}\right) = \nabla(E)\, kE^{-1} + Ek\nabla E^{-1} = \mathrm{Ad}\, E\left(\left[E^{-1}\nabla(E), k\right]\right) = 0 \tag{5.3.3}$$

for all $k \in (\mathrm{Ker}\,\nabla)^\times$ and $E \in \mathcal{G}_\nabla$.
"(iii)" Consider the group homomorphism

$$\mathrm{Ad} : \mathcal{G}_\nabla \longrightarrow \mathrm{Loc}\,(\mathcal{W} \mid \nabla)$$

given by restricting the group homomorphism $\mathrm{Ad} : \mathcal{T}_\mathcal{F} \to \mathcal{G}_1^\hbar$. Note that it is well-defined, since 5.3.3 also holds for $k \in \mathrm{Ker}\,\nabla$. Suppose $\alpha = \mathrm{Ad}\, B \in \mathrm{Loc}\,(\mathcal{W} \mid \nabla)$, then we find that

$$0 = \mathrm{Ad}\, B^{-1} \circ \nabla \circ \mathrm{Ad}\, B\,(k) = \left[B^{-1}\nabla(B), k\right]$$

for all $k \in \mathrm{Ker}\,\nabla$. So $\nabla(B)\, B^{-1}$ is in the centralizer of $\mathrm{Ker}\,\nabla$. Given $k_x \in \mathcal{W}_x$ at some $x \in M$ we can (by parallel transport) always find a section $k \in \mathrm{Ker}\,\nabla$ such that $k(x) = k_x$. This shows that the centralizer of $\mathrm{Ker}\,\nabla$ is simply the center and so $B \in \mathcal{G}_\nabla$. Thus, Ad is surjective and, since $\mathrm{Ker}\,\mathrm{Ad} = \mathcal{Z}^\times$ (another proof of the first part of (ii)), we find that (iii) holds.
"(iv)" This should be clear from the definition of $\mathcal{G}_\nabla$.
"(v)" Suppose $V \subset M$ such that $\mathrm{H}^1_{\mathrm{dR}}(V) = 0$. Then suppose $E \in \mathcal{G}_\nabla|_V$ and denote the implied central one-form by $\beta = E^{-1}\nabla E$. Note that $\nabla E = \beta E$. We have

$$d\beta = \nabla\beta = \nabla\left(\nabla(E)\, E^{-1}\right) = \beta \wedge \beta = 0.$$

So, since $\mathrm{H}^1_{dR}(V) = 0$, we find that $\beta = d\alpha$ for some $\alpha \in C^\infty(V)[\![\hbar]\!]$. Then

$$\nabla\left(e^{-\alpha}E\right) = -d\alpha e^{-\alpha}E + e^{-\alpha}\beta E = 0,$$

so we find that $E = e^{\alpha}e^{-\alpha}E$, i.e. $E$ is a product of a central section $e^\alpha$ and a flat section $e^{-\alpha}E$. Finally we note that, if $a_f \in \mathcal{W}^+$ is flat, then it cannot contain any negative powers of $\hbar$, since it is uniquely determined by its image under $\mathbb{W}^+_\hbar \to \mathbb{C}[\![\hbar]\!]$, by the proof of proposition 4.2.16. So we see that $\mathrm{Ker}\,\nabla \simeq \mathbb{A}_\hbar$ still holds when we consider $\nabla$ as acting on $\mathcal{W}^+$.
"(vi)" Suppose $\gamma \in \Gamma$ and $E \in \mathcal{G}_\nabla$, then $\mathrm{Ad}\,\alpha_\gamma(E) = \alpha_\gamma \circ \mathrm{Ad}\, E \circ \alpha_\gamma^{-1} \in \mathrm{Loc}\,(\mathcal{W} \mid \nabla)$. So, by (iii) and the fact that the $\alpha_\gamma$ define automorphisms of $\mathrm{Ker}\,\nabla$ and of $\mathcal{Z}$, we have $\alpha_\gamma(E) \in \mathcal{G}_\nabla$. □

NOTATION 5.3.11. From now on we will be considering group cohomology. Since we are considering right actions, i.e. $\Gamma^{op}$, we should be writing the group cohomology with values in $B$ as $\mathrm{H}^\bullet(\Gamma^{op}; B)$. For notational convenience and since it will not play any role, we will drop the superscript *op*.

THEOREM 5.3.12. *The group cohomology pointed set* $H^1\left(\Gamma; \overline{\mathcal{G}_\nabla}\right)$ *classifies the actions of* $\Gamma$ *on* $\mathbb{A}_\hbar(M)$ *that extend a given action on* $M$*, in the sense of 5.1.7 and satisfying the condition in remark 5.3.5, up to conjugation by a fixed element of* $Loc\,(\mathcal{W} \mid \nabla)$. *Here we denote* $\overline{\mathcal{G}_\nabla} := \mathcal{G}_\nabla / \mathcal{Z}^\times$.

PROOF.
Note first of all that the action $\alpha$ also descends to $\overline{\mathcal{G}_\nabla}$, since it restricts to an automorphism of the center (via the identification $\mathcal{Z} \simeq C^\infty(M)[\![\hbar]\!]$ this is simply the action induced by the action on the manifold). Suppose $\tilde{S} \colon \Gamma \to \overline{\mathcal{G}_\nabla}$ is a cocycle for the action $\alpha$. Pick any lift $S$ of $\tilde{S}$ to $\mathcal{G}_\nabla$. Then we have

$$S_\gamma \alpha_\gamma(S_\mu)\, S_{\mu\gamma}^{-1} \in \mathcal{Z}^\times \quad \forall\, \gamma, \mu \in \Gamma.$$

We see that

$$\beta\colon \Gamma^{op} \longrightarrow \operatorname{Aut}\left(\mathbb{A}_\hbar\left(M\right)\right),$$

defined by $\beta_\gamma = \operatorname{Ad} S_\gamma \circ \alpha_\gamma$, is a group homomorphism. Note that $\beta$ is well-defined by point (iii) of 5.3.10 and it is a group homomorphism since

$$\beta_\gamma \circ \beta_\mu = \alpha_\gamma \circ \operatorname{Ad} \alpha_\gamma^{-1}\left(S_\gamma\right) S_\mu \circ \alpha_\mu = \alpha_\gamma \circ \operatorname{Ad} \alpha_\gamma^{-1}\left(S_{\mu\gamma}\right) \circ \alpha_\mu = \operatorname{Ad} S_{\mu\gamma} \circ \alpha_\gamma \circ \alpha_\mu = \beta_{\mu\gamma}.$$

Note that $\beta$ does not depend on the particular lift of $\tilde{S}$.

Conversely, suppose $\beta_\gamma = \alpha_\gamma \circ \operatorname{Ad} E_\gamma$ where $E\colon \Gamma \to \mathcal{G}_\nabla$ and $\beta$ defines an action. Note that $\beta$ only depends on the induced map $\tilde{E}$ into $\overline{\mathcal{G}_\nabla}$. Since $\beta_{\mu\gamma} = \beta_\gamma \circ \beta_\mu$, we find immediately that

$$\operatorname{Ad} \alpha_\gamma\left(E_\gamma\right) \alpha_{\mu\gamma}\left(E_\mu\right) = \operatorname{Ad} \alpha_{\mu\gamma}\left(E_{\mu\gamma}\right),$$

which implies that $\gamma \mapsto \alpha_\gamma\left(\tilde{E}_\gamma\right)$ defines a cocycle. Note that the action corresponding to it by the construction above is $\beta$.

Finally, let us pass to cohomology. So, suppose $\tilde{S}$ and $\tilde{S}'$ are two cohomologous $\Gamma^{op}$-cocycles in $\overline{\mathcal{G}_\nabla}$. Denote by $\beta$ and $\beta'$ the corresponding actions. Then there is an element $\tilde{C} \in \overline{\mathcal{G}_\nabla}$ such that

$$\tilde{C}\tilde{S}_\gamma\alpha_\gamma\left(\tilde{C}^{-1}\right) = \tilde{S}'{}_\gamma.$$

Then, picking any lifts $C, S$ and $S'$ of $\tilde{C}$, $\tilde{S}$ and $\tilde{S}'$ to $\mathcal{G}_\nabla$, we find the equations

$$\operatorname{Ad} CS_\gamma\alpha_\gamma\left(C^{-1}\right) = \operatorname{Ad} S'_\gamma.$$

So,

$$\beta'_\gamma = \operatorname{Ad} S'_\gamma \circ \alpha_\gamma = \operatorname{Ad} CS_\gamma\alpha_\gamma\left(C^{-1}\right) \circ \alpha_\gamma = \operatorname{Ad} C \circ \beta_\gamma \circ \operatorname{Ad} C^{-1}$$

for all $\gamma \in \Gamma$. Conversely, if $\beta, \beta'$ are two actions with corresponding cocycles $\tilde{S}$ and $\tilde{S}'$ and lifts $S$, $S'$ and $C \in \mathcal{G}_\nabla$ satisfies $\operatorname{Ad} C \circ \beta_\gamma \circ \operatorname{Ad} C^{-1} = \beta'_\gamma$ for all $\gamma \in \Gamma$, then we have $\operatorname{Ad} CS_\gamma\alpha_\gamma\left(C^{-1}\right) = \operatorname{Ad} S'_\gamma$ and thus $\tilde{S}$ and $\tilde{S}'$ are cohomologous. □

**5.3.2. Computational Tools.** In this section we will provide some tools that should aid in the concrete computation of the pointed set $\mathrm{H}^1\left(\Gamma; \overline{\mathcal{G}_\nabla}\right)$ of equivalence classes of extended group actions. The main burden of proof will be in showing that there exists a surjective map $\mathbb{D}$ from $\mathcal{G}_\nabla$ to $Z^1(M)[\![\hbar]\!]$ (formal power series of closed one forms). This will provide the diagram with exact columns and exact rows

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{C}[\![\hbar]\!]^\times \arrow[r] \arrow[d] & \mathcal{Z}^\times \arrow[r] \arrow[d] & \mathcal{Z}^\times \big/ \mathbb{C}[\![\hbar]\!]^\times \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{A}_\hbar{}^\times \arrow[r] \arrow[d] & \mathcal{G}_\nabla \arrow[r] \arrow[d] & Z^1(M)[\![\hbar]\!] \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathbb{A}_\hbar{}^\times \big/ \mathbb{C}[\![\hbar]\!]^\times \arrow[r] \arrow[d] & \overline{\mathcal{G}_\nabla} \arrow[r] \arrow[d] & T^1_\hbar(M) \arrow[r] \arrow[d] & 0 \\
 & 1 & 1 & 0 &
\end{tikzcd} \qquad (5.3.4)$$

where $T^1_\hbar(M) := \mathrm{H}^1(M;\mathbb{C}) \big/ \mathrm{H}^1(M;\mathbb{Z}) \oplus \hbar\mathrm{H}^1(M)[\![\hbar]\!]$. This matches well with the appearance of $\hbar T^1_\hbar(M)$ as a parametrization of equivalence classes of certain formal representations and equivalence classes of certain formal connections in [**11**]. Note that, although we will be working with non-Abelian cohomology, we still obtain (truncated) exact sequences from short exact sequences of coefficient groups (see section 2.7 of[**58**]). In the following section we will show how one can exploit this diagram to compute $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$.

NOTATION 5.3.13. From now on we will identify $\operatorname{Ker}\nabla$ with the deformation quantization $\mathbb{A}_\hbar$. We will often implicitly identify $\mathcal{Z} \simeq C^\infty(M)[\![\hbar]\!]$. For elements of graded algebras (or invertibles of graded algebras) a subscript will always refer to the degree in this section.

LEMMA 5.3.14. *We have the following short exact sequence of sheaves of groups*

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{A}_\hbar{}^\times & \hookrightarrow & \mathcal{G}_\nabla & \xrightarrow{\mathbb{D}} & Z^1[\![\hbar]\!] & \to & 0 \\ & & & & g & \longmapsto & g^{-1}\nabla g & & . \end{array} \tag{5.3.5}$$

PROOF.
Note first that $\mathbb{D}$ is well-defined since $d\left(g^{-1}\nabla g\right) = \nabla\left(g^{-1}\nabla g\right) = -\left(g^{-1}\nabla g\right)^2 = 0$. The proof follows from the decomposition (5.3.2) and the fact that de Rham cohomology vanishes locally. □

We will show that the above sequence induces an exact sequence of groups on global sections. Note that this means we should prove surjectivity of $\mathbb{D}$. To do this we will use Čech cohomology and some facts about rings of formal power series. So let us fix a good cover $\mathcal{U} = \{U_i\}_{i\in J}$ and recall that for smooth manifolds this choice will not play a role, see section A.2.3. Recall also that $\mathbb{A}_\hbar(M)$ and $C^\infty(M)[\![\hbar]\!]$ are complete with respect to the $\hbar$-adic topology.

DEFINITION 5.3.15. The $\hbar$-adic topology is given by the norm $\|f\| = 2^{-k}$, where $k$ is the smallest non-negative integer such that $f_k \neq 0$, it satisfies the (in)equalities $\|f+g\| \leq \max\{\|f\|,\|g\|\}$ and $\|fg\| = \|f\|\,\|g\|$.

We have the following fact (see chapter 3 of [**60**]).

LEMMA 5.3.16. *Suppose $R$ is a commutative unital ring that contains a copy of the rationals, then $(\hbar R[\![\hbar]\!],+) \simeq (1+\hbar R[\![\hbar]\!],\cdot)$ by the maps*

$$exp(f) = \sum_{n=0}^{\infty}\frac{f^n}{n!} \quad and \quad Log(1+f) = \sum_{n=1}^{\infty}\frac{(-f)^n}{n}.$$

LEMMA 5.3.17. *Suppose $f \in \left(C^\infty(M)[\![\hbar]\!]\right)^\times \simeq \mathcal{Z}^\times$ and $f_0 = 1$, then $\frac{df}{f}$ is exact.*

PROOF.
It is easily verified that $d$ satisfies the product rule on formal power series. This shows that we have $d\exp g = (\exp g)\,dg$ for all $g \in \hbar C^\infty(M)[\![\hbar]\!]$. But then let $g = \operatorname{Log}(f)$ and we see that $\frac{df}{f} = dg$. □

PROPOSITION 5.3.18. *Let us denote the subgroup of exact forms in $Z^1(M)[\![\hbar]\!]$ by $d\mathcal{Z}$ and the restriction of $\mathbb{D}$ to $\mathcal{Z}^\times$ by $D$. Then we have $d\mathcal{Z} \subset Im\ D$ and $Im\ D \big/ d\mathcal{Z} \simeq H^1(M;\mathbb{Z})$.*

PROOF.
Suppose $dg \in d\mathcal{Z}$ such that $(dg)_0 = 0$. Then we may assume that $g \in \hbar C^\infty(M)[\![\hbar]\!]$. We want to show that there exists an $f \in \mathcal{Z}^\times$ such that $Df = dg$. Clearly $f = \exp(g)$ will do the job. This shows that every exact form $dg$ with $(dg)_0 = 0$ is in $\operatorname{Im} D$. Now suppose $dg$ is a general element of $d\mathcal{Z}$, then $\exists f \in \mathcal{Z}^\times$ such that $Df = \hbar\,(dg)_1 + \hbar^2\,(dg)_2 + \dots$, but then $dg = De^{g_0} + Df = D\left(e^{g_0}f\right)$. This shows the first claim of the proposition.

To show the second claim consider the map

$$C_\mathcal{U}\colon \operatorname{Im} D \to \check{\mathrm{H}}^1(\mathcal{U};\mathbb{C})[\![\hbar]\!]$$

where $C_{\mathcal{U}}(Df)$ is represented by $\eta(i,j) = g(i) - g(j)$ if $Df|_{U_i} = dg(i)$. We leave the routine check that this map is a well-defined group homomorphism to the reader. Note that $\operatorname{Ker} C_{\mathcal{U}} = d\mathcal{Z}$ and in fact the map $C_{\mathcal{U}}$ is simply given by the map which implements the isomorphism of de Rham en Čech cohomology, see theorems A.2.24 and A.2.22. It is left to show that $\operatorname{Im} C_{\mathcal{U}} \simeq \mathrm{H}^1(M;\mathbb{Z})$. Note that every element in $\mathcal{Z}^\times$ can be written as a product of a nowhere vanishing function and a function as in lemma 5.3.17. So we find that $C_{\mathcal{U}}(Df) = C_{\mathcal{U}}(Df_0) = [\eta]$, where $\eta(i,j) = g(i) - g(j)$ with $e^{g(i)} = f_0|_{U_i}$. So we see that $[\eta] \in \check{\mathrm{H}}^1(\mathcal{U};\mathbb{Z}) \hookrightarrow \check{\mathrm{H}}^1(\mathcal{U};\mathbb{C})[\![\hbar]\!]$. Where the inclusion comes from the exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i\cdot} \mathbb{C} \xrightarrow{e^{\cdot}} \mathbb{C}^\times \to 1, \tag{5.3.6}$$

since it is also a short exact sequence of groups. So $\operatorname{Im} C_{\mathcal{U}} \subset \check{\mathrm{H}}^1(\mathcal{U},\mathbb{Z})$. On the other hand consider the exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i\cdot} C^\infty(M) \xrightarrow{e^{\cdot}} C^\infty(M)^\times \to 1. \tag{5.3.7}$$

Now we see that the first connecting map $\partial$ in the corresponding long exact sequence in Čech cohomology is surjective, since $\check{\mathrm{H}}^1(\mathcal{U};C^\infty(M)) = 0$, and $\partial(f_0) = [\lambda]$ where $\lambda$ is given by $\lambda(i,j) = g(i) - g(j)$ such that $e^{g(i)} = f_0|_{U_i}$, but then $Df_0|_{U_i} = dg(i)$ so $\check{\mathrm{H}}^1(\mathcal{U};\mathbb{Z}) \subset \operatorname{Im} C_{\mathcal{U}}$. To get the result, simply note that $\check{\mathrm{H}}^1(\mathcal{U};\mathbb{Z}) \simeq \mathrm{H}^1(M;\mathbb{Z})$ by theorem A.2.22 and proposition A.2.23. □

REMARK 5.3.19. Note that the arguments in the proof of proposition 5.3.18 above are simply the standard considerations when one notices the fact that $D$ agrees locally with the differential of the logarithm and one notices the fact that $C_{\mathcal{U}} \circ D$ factors through the non-vanishing functions (by noting that we have the isomorphism $\mathcal{Z}^\times \simeq C^\infty(M)^\times \times (1 + \hbar C^\infty(M)[\![\hbar]\!])$).

LEMMA 5.3.20. *There are maps*

$$P_1 \colon \check{\mathrm{H}}^1(\mathcal{U};\mathcal{Z}^\times) \xrightarrow{\sim} \check{\mathrm{H}}^1\left(\mathcal{U};C^\infty(M)^\times\right),$$

$$P_2 \colon \check{\mathrm{H}}^1(\mathcal{U};\mathbb{A}_\hbar{}^\times) \longrightarrow \check{\mathrm{H}}^1\left(\mathcal{U};C^\infty(M)^\times\right)$$

*and*

$$P_3 \colon \check{\mathrm{H}}^1(\mathcal{U};1+\hbar\mathbb{C}[\![\hbar]\!]) \xrightarrow{\sim} \hbar\check{\mathrm{H}}^1(\mathcal{U};\mathbb{C})[\![\hbar]\!],$$

*where $P_2$ has trivial kernel (note that as its domain is not necessarily a group this does not imply that the map is injective) and $P_1$ and $P_2$ are induced by the map $f_0 + \hbar f_1 + \ldots \mapsto f_0$.*

PROOF.
The proof for $P_1$ is analogous to the proof for $P_2$ if not simply easier. So we will explicitly show the proof only for $P_2$. Consider the decreasing filtration given by $(\mathbb{A}_\hbar{}^\times)_n = 1 + \hbar^n\mathbb{A}_\hbar$ for $n > 0$ and $(\mathbb{A}_\hbar{}^\times)_0 = \mathbb{A}_\hbar{}^\times$. Then we have that $(\mathbb{A}_\hbar{}^\times)_{n+1} \lhd (\mathbb{A}_\hbar{}^\times)_n$ and $(\mathbb{A}_\hbar{}^\times)_n \star (\mathbb{A}_\hbar{}^\times)_m \subset (\mathbb{A}_\hbar{}^\times)_{\min(n,m)}$. So we have the short exact sequence of (sheaves of) groups

$$1 \to \left(\mathbb{A}_\hbar{}^\times\right)_1 \longrightarrow \mathbb{A}_\hbar{}^\times \longrightarrow C^\infty(M)^\times \to 1. \tag{5.3.8}$$

Note that, if $f, g \in (\mathbb{A}_\hbar{}^\times)_n$, then $f \star g = 1 + \hbar^n(f_n + g_n) \mod (\mathbb{A}_\hbar{}^\times)_{n+1}$ so we also have the short exact sequences of (sheaves of) groups

$$1 \to \left(\mathbb{A}_\hbar{}^\times\right)_{n+1} \longrightarrow \left(\mathbb{A}_\hbar{}^\times\right)_n \longrightarrow C^\infty(M) \to 0, \tag{5.3.9}$$

for all $n \in \mathbb{N}$. The map $P_2$ is induced in the long exact sequence corresponding to 5.3.8. In order to show that it has trivial kernel, we should show that $\check{\mathrm{H}}^1(\mathcal{U};(\mathbb{A}_\hbar{}^\times)_1)$ vanishes. Note first of all that, since $\check{\mathrm{H}}^1(\mathcal{U};C^\infty(M)) = 0$, we find surjections $\check{\mathrm{H}}^1(\mathcal{U};(\mathbb{A}_\hbar{}^\times)_{n+1}) \twoheadrightarrow \check{\mathrm{H}}^1(\mathcal{U};(\mathbb{A}_\hbar{}^\times)_n)$ (which in fact have trivial kernel). Suppose $S\colon J^2 \to (\mathbb{A}_\hbar{}^\times)_1$ is a cocycle. Then, by the surjection above, $\exists a_1 \colon J \to (\mathbb{A}_\hbar{}^\times)_1$ such that the cochain $a_1 \cdot S \colon J^2 \to (\mathbb{A}_\hbar{}^\times)_2$ given by

$$a_1 \cdot S(i,j) = a_1(i) \star S(i,j) \star a_1(j)^{-1}$$

is a cocycle. Iterating this process yields a sequence $a_k\colon J \to (\mathbb{A}_\hbar{}^\times)_k$ such that, if we denote

$$b_k := \mathop{\bigstar}_{j=1}^{k} a_j := a_k \star a_{k-1} \star \ldots \star a_1,$$

then $b_k \cdot S : J^2 \to (\mathbb{A}_\hbar{}^\times)_{k+1}$ is a cocycle (cohomologous to $S$). Let us denote $S_k := b_k \cdot S$. Then we might consider its values in $\mathbb{A}_\hbar$. Suppose $k \geq l \in \mathbb{N}$, we have

$$\begin{aligned}\|S_k(i,j) - S_l(i,j)\| &= \left\|(b_k(i) - b_l(i)) \star S(i,j) \star b_k(j)^{-1} + b_l(i) \star S(i,j) \star (b_k(j)^{-1} - b_l(j)^{-1})\right\| \\ &\leq \max\left\{\|b_k(i) - b_l(i)\|, \left\|b_k(j)^{-1} - b_l(j)^{-1}\right\|\right\} = \\ = \max&\left\{\left\|\left(-1 + \mathop{\bigstar}_{\alpha=l+1}^{k} a_\alpha(i)\right) \star b_l(i)\right\|, \left\|b_l(j)^{-1}\left(-1 + \left(\mathop{\bigstar}_{\alpha=l+1}^{k} a_\alpha(j)\right)^{-1}\right)\right\|\right\} \leq 2^{-1-l}.\end{aligned}$$

This means the sequence $S_k(i,j)$ is Cauchy in $\mathbb{A}_\hbar$ (which is complete) and therefore has a limit $S_\infty(i,j) \in \mathbb{A}_\hbar$ for all $(i,j) \in J^2$. Now note that since all the $S_k$ have values in $(\mathbb{A}_\hbar{}^\times)_1$ so does $S_\infty$. Similarly we can show that the sequence $b_k$ has a limit $b\colon J \to (\mathbb{A}_\hbar{}^\times)_1$. Moreover, by the same computation, we see that $S_\infty = b \cdot S$. In particular, $S$ is cohomologous to $S_\infty$ and, since $\varprojlim \left(\mathbb{A}_\hbar{}^\times\right)_n = 1$, we see that $[S] = [1] \in \check{\mathrm{H}}^1(\mathcal{U}; (\mathbb{A}_\hbar{}^\times)_1)$. Since we started with an arbitrary cocycle this shows that the last cohomology pointed set is in fact trivial. This shows the claim about $P_2$ and an easier version of the same argument shows that $P_1$ is injective (since this is a group homomorphism). Since $C^\infty(M)[\![\hbar]\!]^\times \simeq C^\infty(M)^\times \times (1 + \hbar C^\infty(M)[\![\hbar]\!])$, in the obvious way, we see that $P_1$ is also surjective.

The map $P_3$ is simply induced by exp from lemma 5.3.16.

□

Remark 5.3.21. To show lemma 5.3.20 we have used a method to pass from cohomology of $\mathbb{A}_\hbar{}^\times$ to cohomology of $C^\infty(M)^\times$ by showing that $\check{\mathrm{H}}^1\left(\mathcal{U}; \left(\mathbb{A}_\hbar{}^\times\right)_1\right) = \{1\}$, using the fact that, by existence of a smooth partition of unity, $\check{\mathrm{H}}^1(\mathcal{U}; C^\infty(M)) = 0$. This method works equally well for group cohomology when $\mathrm{H}^1(\Gamma; C^\infty(M)) = 0$ and we will use it in the next section. When we do this we will simply refer to the proof of lemma 5.3.20, instead of basically repeating the proof. Note that, in general, the method provides an indication of how the cohomology of $\mathbb{A}_\hbar{}^\times$ and $C^\infty(M)[\![\hbar]\!]^\times$ are both given by infinitely many copies of the cohomology of $C^\infty(M)$ and the cohomology of $C^\infty(M)^\times$.

Proposition 5.3.22. *$\mathbb{D}$ is surjective on total sections.*

Proof.
In order to show the proposition, we will show that

$$\frac{\mathrm{Im}\,\mathbb{D} \big/ \mathrm{Im}\,D}{\check{\mathrm{H}}^1(\mathcal{U}; 1 + \hbar\mathbb{C}[\![\hbar]\!])} \simeq \frac{\mathrm{H}^1(M;\mathbb{C})}{\mathrm{H}^1(M;\mathbb{Z})}.$$

Then, a triple application of the five lemma will show that $\mathrm{Im}\,\mathbb{D} = Z^1(M)[\![\hbar]\!]$. The inclusion of $\check{\mathrm{H}}^1(\mathcal{U}; 1 + \hbar\mathbb{C}[\![\hbar]\!])$ in $\frac{\mathrm{Im}\,\mathbb{D}}{\mathrm{Im}\,D}$ should be evident from the proof.

Note that, if $g \in \mathcal{G}_\nabla$ and $U \subset M$ is a coordinate neighborhood (or any other neighborhood such that $\mathrm{H}^1_{\mathrm{dR}}(U) = 0$), then $g|_U = f \cdot k$ with $f \in \mathcal{Z}^\times|_U$ and $k \in \mathrm{Ker}\,\nabla^\times|_U \simeq \mathbb{A}_\hbar{}^\times|_U$ by point (v) of lemma 5.3.10. Now let $H\colon \mathrm{Im}\,\mathbb{D} \to \check{\mathrm{H}}^1(\mathcal{U}; \mathbb{C}[\![\hbar]\!]^\times)$ be the map given by $H(\mathbb{D}g) = [\eta]$ where $\eta(i,j) = \frac{f(i)}{f(j)}$ with the $f(i)$ given by the decompositions $g|_{U_i} = f(i) \cdot k(i)$. Again we leave it to the reader to check that this map is well-defined.

Now suppose $\mathbb{D}g \in \mathrm{Ker}\,H$ then $g|_{U_i} = f(i) \cdot k(i)$ and $\frac{f(i)}{f(j)} = \frac{c(j)}{c(i)}$ with $c(i) \in \mathbb{C}[\![\hbar]\!]^\times$ for all $i$. This means that $g|_{U_i} = f(i)\,c(i) \cdot \frac{k(i)}{c(i)}$ and $\frac{f(i)c(i)}{f(j)c(j)} = 1$, so $\exists fc \in \mathcal{Z}^\times$ such that $fc|_{U_i} = f(i)\,c(i)$. On the other hand we have that $k(i) \cdot k(j)^{-1} = \frac{c(i)}{c(j)}$, so $\exists \frac{k}{c} \in \mathbb{A}_\hbar{}^\times$ such that $\frac{k}{c}|_{U_i} = \frac{k(i)}{c(i)}$. This shows that $g = fc \cdot \frac{k}{c}$ and thus $\operatorname{Ker} H = \operatorname{Im} D$. So we conclude that

$$\frac{\operatorname{Im} \mathbb{D}}{\operatorname{Im} D} = \frac{\operatorname{Im} \mathbb{D}}{\operatorname{Ker} H} \simeq \operatorname{Im} H \subset \check{\mathrm{H}}^1\left(\mathcal{U}; \mathbb{C}[\![\hbar]\!]^\times\right).$$

Now suppose $[\lambda] \in \check{\mathrm{H}}^1(\mathcal{U}; \mathbb{C}[\![\hbar]\!]^\times)$ such that $\exists f \in \check{\mathrm{C}}^0(\mathcal{U}; \mathcal{Z}^\times)$ and $k \in \check{\mathrm{C}}^0\left(\mathcal{U}; \mathbb{A}_\hbar{}^\times\right)$ such that $\frac{f(i)}{f(j)} = \lambda(i,j) = k(j)\, k(i)^{-1}$ for all $(i,j) \in J^2$. Then $\frac{f(i)}{f(j)} k(i)\, k(j)^{-1} = \frac{\lambda(i,j)}{\lambda(i,j)} = 1$ for all $(i,j) \in J^2$. So $\exists g \in \mathcal{G}_\nabla$ such that $g|_{U_i} = f(i) \cdot k(i)$ and $H(\mathbb{D}g) = [\lambda]$. Conversely, if $[\lambda] = H(\mathbb{D}g)$ for some $g \in \mathcal{G}_\nabla$, then obviously such 0-cochains exist. So we find that $\operatorname{Im} H = \operatorname{Ker} I \cap \operatorname{Ker} Y$, for $I$ and $Y$ the maps induced by the inclusions

$$\mathbb{A}_\hbar{}^\times \overset{Y}{\hookleftarrow} \mathbb{C}[\![\hbar]\!]^\times \overset{I}{\hookrightarrow} \mathcal{Z}^\times.$$

Now, by the lemma 5.3.20, we see that $\operatorname{Ker} I = \operatorname{Ker} P_1 \circ I$ and $\operatorname{Ker} Y = \operatorname{Ker} P_2 \circ Y$ and in fact these maps agree, i.e. $R := P_1 \circ I = P_2 \circ Y$. They agree since they are all simply the map induced by

$$\begin{array}{ccc} \mathbb{C}[\![\hbar]\!]^\times & \to & C^\infty(M)^\times \\ c & \mapsto & c_0 \end{array}.$$

This map factors like $R = \partial \circ P$, where $P$ is induced by the projection $\mathbb{C}[\![\hbar]\!]^\times \twoheadrightarrow \mathbb{C}^\times$ and $\partial$ is induced by the inclusion of $\mathbb{C}^\times$ in $C^\infty(M)^\times$. Now, since $\mathbb{C}[\![\hbar]\!]^\times \simeq \mathbb{C}^\times \times (1 + \hbar\mathbb{C}[\![\hbar]\!])$ in the obvious way, we see that $\check{\mathrm{H}}^1(\mathcal{U}; 1 + \hbar\mathbb{C}[\![\hbar]\!]) = \operatorname{Ker} P$. So, we find that

$$\frac{\operatorname{Im} H}{\operatorname{Ker} P} = \frac{\operatorname{Ker} R}{\operatorname{Ker} P} \simeq \operatorname{Ker} \partial.$$

If we identify $\check{\mathrm{H}}^1\left(\mathcal{U}; C^\infty(M)^\times\right) \simeq \check{\mathrm{H}}^2(\mathcal{U}; \mathbb{Z})$ using the exponential sequence (5.3.7), then we see that $\partial$ is simply the connecting map in the exponential sequence (5.3.6) and this shows that $\operatorname{Ker} \partial \simeq \frac{\check{\mathrm{H}}^1(\mathcal{U};\mathbb{C})}{\check{\mathrm{H}}^1(\mathcal{U};\mathbb{Z})}$, which proves the claim. $\square$

In light of various exact sequences that appear in the following section, we will be able to use the following proposition.

Proposition 5.3.23. *Let*

$$0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

*be a* $\Gamma$*-equivariant* central *extension, then we also have the exact sequence*

$$0 \to A^\Gamma \longrightarrow E^\Gamma \longrightarrow G^\Gamma \longrightarrow H^1(\Gamma; A) \longrightarrow H^1(\Gamma; E) \longrightarrow H^1(\Gamma; G) \longrightarrow H^2(\Gamma; A).$$

Proof.
As mentioned we only need to extend the sequence to include $\mathrm{H}^2(\Gamma; A)$ (see section 2.7 of [**58**]). We simply do the usual (Abelian) construction and show that it still works because the extension is central. So suppose $[\eta] \in \mathrm{H}^1(\Gamma; G)$ and let $E_\gamma \mapsto \eta_\gamma$ for all $\gamma \in \Gamma$. Then $(\delta E)_{\gamma,\mu} \in \operatorname{Ker} \pi$ for all $\gamma, \mu \in \Gamma$, so $a := \delta E \colon \Gamma^2 \to A$. By writing out and noting that the computation takes place in the Abelian group $A$ we find

$$(\delta a)_{\gamma,\mu,\chi} = (\delta E)^{-1}_{\gamma\mu,\chi} \gamma((\delta E)_{\mu,\chi})(\delta E)_{\gamma,\mu\chi}(\delta E)^{-1}_{\gamma,\mu} = E_{\gamma\mu\chi} E^{-1}_{\gamma\mu}(\delta E)^{-1}_{\gamma,\mu}\gamma(E_\mu) E_\gamma E^{-1}_{\gamma\mu\chi} = 1$$

So that $[a] \in \mathrm{H}^2(\Gamma; A)$ and we will need to show that this map $[\eta] \mapsto [a]$ is well-defined. First suppose that $\tilde{E}_\gamma$ is another lift of $\eta_\gamma$ for each $\gamma \in \Gamma$. Then note that $\tilde{E}_\gamma E^{-1}_\gamma$ and $\tilde{E}^{-1}_\gamma E_\gamma$ are in $A$ and thus central for all $\gamma \in \Gamma$, so we have

$$(\delta\tilde{E})_{\gamma,\mu}(\delta E)^{-1}_{\gamma,\mu} = (\delta\tilde{E}E)_{\gamma,\mu} \quad \forall \gamma, \mu \in \Gamma$$

where $(\tilde{E}E)_\gamma = \tilde{E}_\gamma E_\gamma$. So both lifts will define cohomologous cocycles in $\mathrm{H}^2(\Gamma; A)$. Now suppose $E'_\gamma$ lifts $\gamma(x)\eta_\gamma x^{-1}$ for all $\gamma \in \Gamma$ and some $x \in G$. Then, if $\pi(X) = x$, we note that the element $(X \cdot E')_\gamma := \gamma(X)^{-1} E_\gamma X$ lifts $\eta_\gamma$ for all $\gamma \in \Gamma$. Then note that

$$(\delta(X \cdot E'))_{\gamma,\mu} = \mathrm{Ad}\,(\gamma\mu)(X^{-1})((\delta E)_{\gamma,\mu}) = (\delta E)_{\gamma,\mu} \quad \forall \gamma, \mu \in \Gamma,$$

which shows, when combined with the previous remark (concerning $\tilde{E}$), that the implied map in cohomology $\mathrm{H}^1(\Gamma; G) \to \mathrm{H}^2(\Gamma; A)$ is well-defined. Lastly we should show that the sequence is exact at $\mathrm{H}^1(\Gamma; G)$. So suppose $[\eta] \in \mathrm{H}^1(\Gamma; G)$ maps to 0 under the map described above. Then there is a lift $E_\gamma$ for every $\eta_\gamma$ for all $\gamma \in \Gamma$ such that $(\delta E)_{\gamma,\mu} = (\delta\alpha)_{\gamma,\mu}$ for some $\alpha \colon \Gamma \to A$ and all $\gamma, \mu \in \Gamma$. Then consider $E\alpha^{-1}$ given by $(E\alpha^{-1})_\gamma = E_\gamma \alpha_\gamma^{-1}$ and note that

$$(\delta E\alpha^{-1})_{\gamma,\mu} = (\delta E)_{\gamma,\mu}(\delta\alpha)^{-1}_{\gamma,\mu} = 1 \quad \forall \gamma, \mu \in \Gamma^2,$$

but also $E\alpha^{-1} \mapsto \eta$ under $\pi$. This shows that $[\eta]$ is in the image of $\mathrm{H}^1(\Gamma; E) \to \mathrm{H}^1(\Gamma; G)$. □

**5.3.3. Computations.** Let us show in this section how the tools developed in the previous section can be applied to computation. In particular we will show that, under sufficiently restrictive assumptions on the cohomology of the group $\Gamma$ and the first cohomology of the manifold $M$, we find that there is a unique extension of the action to any deformation quantization. We will also show that in general the extensions are not unique even when the first cohomology of the manifold vanishes and the group acts faithfully. Most of the computations here are done by applying group cohomology to the diagram (5.3.4).

Proposition 5.3.24. *Suppose $H^1_{dR}(M) = 0$ and $|\Gamma| < \infty$, then we have $H^1(\Gamma; \overline{\mathcal{G}_\nabla}) = \{1\}$.*

Proof.
By vanishing of $\mathrm{H}^1_{\mathrm{dR}}(M)$ and lemma 5.3.10 we have the exact sequence

$$1 \to \mathbb{C}[\![\hbar]\!]^\times \longrightarrow \mathbb{A}_\hbar{}^\times \longrightarrow \overline{\mathcal{G}_\nabla} \to 1, \tag{5.3.10}$$

since $\mathcal{G}_\nabla = \mathcal{Z}^\times \cdot \mathbb{A}_\hbar{}^\times$ and $\mathcal{Z}^\times \cap \mathbb{A}_\hbar{}^\times = \mathbb{C}[\![\hbar]\!]^\times$.

The proof proceeds in two steps. First we show that the connecting map

$$\mathrm{H}^1\left(\Gamma; \overline{\mathcal{G}_\nabla}\right) \to \mathrm{H}^2\left(\Gamma; \mathbb{C}[\![\hbar]\!]^\times\right)$$

is trivial. Secondly we show that the map

$$\mathrm{H}^1\left(\Gamma; \mathbb{C}[\![\hbar]\!]^\times\right) \to \mathrm{H}^1\left(\Gamma; \mathbb{A}_\hbar{}^\times\right)$$

is surjective. Given these two facts, the proposition is implied by proposition 5.3.23. In the following we shall denote the differential of the cochain $c$ by $\delta c$.

Suppose $\eta \colon \Gamma \to \overline{\mathcal{G}_\nabla}$ is a cocycle. We will first show that the image of the class of $\eta$ under the connecting map can be represented by a cocycle $c$ with values in $\mathbb{C}^\times$. So suppose $E \colon \Gamma \to \mathbb{A}_\hbar{}^\times$ lifts $\eta$. By proposition 5.3.23 we find that $\delta E$ is a 2-cocycle in $\mathbb{C}[\![\hbar]\!]^\times$ for the trivial action. Then, by the fact that $\mathrm{H}^2(\Gamma; \mathbb{C}) = 0$ and an analogous argument to the one used in the proof of lemma 5.3.20, we find $\lambda \colon \Gamma \to \mathbb{C}[\![\hbar]\!]^\times$ such that $(\delta\lambda)(\delta E)$ has values in $\mathbb{C}^\times$. Now note that, if $\tilde{E}$ denotes the cochain given by $\tilde{E}_\gamma = \lambda_\gamma E_\gamma$ for all $\gamma \in \Gamma$, then $\tilde{E}$ also lifts $\eta$ and $\delta\tilde{E} = (\delta\lambda)(\delta E)$ (since $\mathbb{C}[\![\hbar]\!]^\times$ is central in $\mathbb{A}_\hbar{}^\times$). But then, if we write $\tilde{E} = \sum \hbar^k E_k$, we find that

$$(\delta\tilde{E})_{\gamma,\mu} = (\delta E_0)_{\gamma,\mu} + \hbar S_{\gamma,\mu},$$

for all $\gamma, \mu \in \Gamma$, here we consider $\delta E_0$ as the boundary of the cochain $E_0 \colon \Gamma \to C^\infty(M)^\times$. Now the fact that $\delta\tilde{E}_{\gamma,\mu} \in \mathbb{C}^\times$ implies that $S = 0$. So we find that $\delta\tilde{E} = \delta E_0$ as cochains with values in $\mathbb{C}^\times$. So, we have found the cocycle $c := \delta E_0$ representing the image of the class of $\eta$ under the connecting map. Note that, since $\mathrm{H}^1(M;\mathbb{Z}) = 0$, the inclusion of $\mathbb{C}$ in $C^\infty(M)$ and the exponential sequences (5.3.6) and (5.3.7) induce the commuting diagram with exact rows

$$\begin{tikzcd}
\mathrm{H}^2(\Gamma;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^2(\Gamma;\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{H}^2(\Gamma;\mathbb{C}^\times) \arrow[r] \arrow[d] & \mathrm{H}^3(\Gamma;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^3(\Gamma;\mathbb{C}) \arrow[d] \\
\mathrm{H}^2(\Gamma;\mathbb{Z}) \arrow[r] & \mathrm{H}^2(\Gamma;C^\infty) \arrow[r] & \mathrm{H}^2(\Gamma;(C^\infty)^\times) \arrow[r] & \mathrm{H}^3(\Gamma;\mathbb{Z}) \arrow[r] & \mathrm{H}^3(\Gamma;C^\infty)
\end{tikzcd},$$

where we have abbreviated $C^\infty := C^\infty(M)$. So, we have $\mathrm{H}^2(\Gamma;\mathbb{C}^\times) \simeq \mathrm{H}^2(\Gamma;C^\infty(M)^\times)$ by the five lemma, but then $[c]$ is trivial, since it is trivial in $\mathrm{H}^2(\Gamma;C^\infty(M)^\times)$ by construction. So the map $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla}) \to \mathrm{H}^2(\Gamma;\mathbb{C}[\![\hbar]\!]^\times)$ is the zero map.

Now suppose $\eta\colon \Gamma \to \mathbb{A}_\hbar{}^\times$ is a cocycle. Then, writing $\eta = \sum \hbar^k \eta_k$, we find that $\eta_0\colon \Gamma \to C^\infty(M)^\times$ is also a cocycle. As above, we find the commuting diagram with exact rows

$$\begin{tikzcd}
\mathrm{H}^1(\Gamma;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^1(\Gamma;\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{H}^1(\Gamma;\mathbb{C}^\times) \arrow[r] \arrow[d] & \mathrm{H}^2(\Gamma;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^2(\Gamma;\mathbb{C}) \arrow[d] \\
\mathrm{H}^1(\Gamma;\mathbb{Z}) \arrow[r] & \mathrm{H}^1(\Gamma;C^\infty) \arrow[r] & \mathrm{H}^1(\Gamma;(C^\infty)^\times) \arrow[r] & \mathrm{H}^2(\Gamma;\mathbb{Z}) \arrow[r] & \mathrm{H}^3(\Gamma;C^\infty)
\end{tikzcd}.$$

So, again by the five lemma, we find $f \in C^\infty(M)^\times$ such that $(\delta f)\eta_0$ has values in $\mathbb{C}^\times$. Denote by $\tilde{\eta}$ the cocycle given by $\tilde{\eta}_\gamma = \gamma^*(f)\eta_\gamma f^{-1}$, then

$$\tilde{\eta}((\delta f)\eta_0)^{-1} = 1 + \hbar S$$

for some $S\colon \Gamma \to \mathbb{A}_\hbar$. So, using the exact sequences

$$1 \to 1 + \hbar^{k+1}\mathbb{A}_\hbar \longrightarrow 1 + \hbar^k \mathbb{A}_\hbar \longrightarrow C^\infty(M) \to 0,$$

the fact that $\mathrm{H}^1(\Gamma;C^\infty(M)) = 0$ and the same reasoning as in lemma 5.3.20, we find that there exists $P \in 1 + \hbar\mathbb{A}_\hbar$ such that $\alpha_\gamma(P)(1 + \hbar S_\gamma)P^{-1} = 1$ for all $\gamma \in \Gamma$. Thus we find that

$$[\eta] = [\tilde{\eta}] = [(\delta f)\eta_0] \in \mathrm{H}^1(\Gamma, \mathbb{A}_\hbar{}^\times)$$

which means that the map $\mathrm{H}^1(\Gamma, \mathbb{C}[\![\hbar]\!]^\times) \to \mathrm{H}^1(\Gamma;\mathbb{A}_\hbar{}^\times)$ is surjective. So, by the sequence implied by proposition 5.3.23 and (5.3.10), we find that $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla}) = \{1\}$. ☐

Corollary 5.3.25. *For the conditions in proposition 5.3.24 there is, up to conjugation by a fixed automorphism, a* unique *extension of the group action to* any *deformation quantization by proposition 5.3.24 and corollary 5.2.5.*

Note that proposition 5.3.24 in particular contains the cases of the actions of $\mathbb{Z}/n\mathbb{Z}, D_{2n}, A_4, S_4$ and $A_5$ (finite subgroups of $SO(3)$) by symplectomorphisms on $S^2$.

Remark 5.3.26. Note that the actual properties of the group that were used in the proof of proposition 5.3.24 are

- $\mathrm{H}^i(\Gamma,\mathbb{C}) = \mathrm{H}^i(\Gamma, C^\infty(M)) = 0$ for $i = 1, 2$ (for the trivial action on $\mathbb{C}$)
- $\mathrm{H}^3(\Gamma,\mathbb{C}) \hookrightarrow \mathrm{H}^3(\Gamma, C^\infty(M))$ (induced by the inclusion),

all of which are satisfied by finite groups.

Remark 5.3.27. Let us consider for a moment the group $\Gamma$ acting trivially on the manifold $M$. Then of course we can lift this action to the trivial action on any deformation quantization and the classification comes down to finding representations of $\Gamma$ as self gauge equivalences of the deformation. In this case we find that

$$\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla}) \simeq \mathrm{Hom}(\Gamma^{op},\overline{\mathcal{G}_\nabla}) \big/ \overline{\mathcal{G}_\nabla}$$

where the quotient is taken with respect to the inclusion of $\overline{\mathcal{G}_\nabla}$ in $\mathrm{Aut}(\overline{\mathcal{G}_\nabla})$ as inner automorphisms. It is to be expected that this yields non-equivalent extensions of the trivial action of $\Gamma$. However, they would be rather pathological examples. Let us show that there may be non-equivalent extensions of the group action even when this action $\Gamma \to \mathrm{Symp}(M)$ is injective.

EXAMPLE 5.3.28. Consider the action of $\mathbb{Z}$ on $S^2$ by rotation through an irrational angle $\theta$ around the vertical axis. Note that, since $\mathrm{H}^2_{\mathrm{dR}}(S^2)[\![\hbar]\!] = \mathbb{C}[\![\hbar]\!][\alpha]$ where $\alpha$ denotes the standard symplectic structure on $S^2$, we find that any action by symplectomorphisms that preserves an affine connection extends to any deformation quantization by proposition 5.2.3. In particular, by uniqueness of the Levi-Civita connection, any action by symplectic isometries lifts to any deformation quantization. Note that we have $\mathrm{H}^1(\mathbb{Z};G) \simeq G/\sim$, where $g \sim h$ if $1(b)gb^{-1} = h$, whenever $\mathbb{Z}$ acts on any group $G$. Since we have $\mathrm{H}^2(\mathbb{Z};\mathbb{C}[\![\hbar]\!]^\times) = 0$ and (5.3.10), we find the sequence

$$1 \to \mathbb{C}[\![\hbar]\!]^\times \longrightarrow \left(\mathbb{A}_\hbar{}^\times\right)^{\mathbb{Z}} \longrightarrow \overline{\mathcal{G}_\nabla}^{\mathbb{Z}} \longrightarrow \mathbb{C}[\![\hbar]\!]^\times \longrightarrow \mathbb{A}_\hbar{}^\times/\sim \longrightarrow \overline{\mathcal{G}_\nabla}/\sim \to 0. \tag{5.3.11}$$

Suppose $c, c' \in \mathbb{C}[\![\hbar]\!]^\times$ such that their images in $\mathbb{A}_\hbar{}^\times/\sim$ coincide, i.e. there exists $g \in \mathbb{A}_\hbar{}^\times$ such that $cr_\theta^*(g) = c'g$, where we denote the rotation inducing the action of $\mathbb{Z}$ by $r_\theta$ . Then note that, since the action has a fixed point at the north pole, we can evaluate $g$ there to find that $c = c'$. So we find that the map $\mathbb{C}[\![\hbar]\!]^\times \to \mathbb{A}_\hbar{}^\times/\sim$ is injective. Now suppose $f \in \mathbb{A}_\hbar{}^\times$ such that there is $c \in \mathbb{C}[\![\hbar]\!]^\times$ and $c \sim f$. Then, if we write $c = \sum \hbar^k c_k$ and $f = \sum \hbar^k f_k$, we find that there exists some $g_0 \in C^\infty(S^2)^\times$ such that $c_0 r_\theta^*(g_0) = f_0 g_0$ (for the undeformed product). Now note that this implies that $f_0$ must take the value $c_0$ at both the north and south pole. So, we find that the map $\mathbb{C}[\![\hbar]\!]^\times \to \mathbb{A}_\hbar{}^\times/\sim$ is definitely not surjective, since there exist many non-vanishing functions on $S^2$ that do not have the same value at the north and south pole. Then exactness of (5.3.11) shows that $\mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla}) \simeq \overline{\mathcal{G}_\nabla}/\sim \neq \{1\}$ and so there exist multiple extensions of this action to any deformation quantization.

COROLLARY 5.3.29. *Suppose* $\Gamma = \mathbb{Z}$ *acts on* $(M,\omega)$ *with* $H^1_{dR}(M) = 0$ *and there is more than* 1 *fixed point, then* $H^1(\Gamma;\overline{\mathcal{G}_\nabla}) \neq \{1\}$.

Finally let us show that the diagram (5.3.4) allows us to make conclusions about $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$ even in the case that $\mathrm{H}^1_{\mathrm{dR}}(M) \neq 0$.

EXAMPLE 5.3.30. Consider the action of $\mathbb{Z}$ on $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ by irrational rotation of one of the coordinates. Then we find that the induced action on $T^1_\hbar(\mathbb{T}^2)$ is trivial and so

$$\mathrm{H}^1(\mathbb{Z};T^1_\hbar(\mathbb{T}^2)) = T^1_\hbar(\mathbb{T}^2)$$

by the discussion in the beginning of example 5.3.28. Suppose $0 \neq l \in T^1_\hbar(\mathbb{T}^2)$, then we can lift this to $[0] \neq [z] \in \mathrm{H}^1(\mathbb{Z};Z^1(M)[\![\hbar]\!]) \simeq Z^1(M)[\![\hbar]\!]/\sim$, since $\mathrm{H}^2(\mathbb{Z};\mathcal{Z}^\times/\mathbb{C}[\![\hbar]\!]^\times) = 0$. By proposition 5.3.22 we can find $g \in \mathcal{G}_\nabla$ such that $\mathbb{D}g = z$. In particular $g$ represents a non-trivial class in $\mathrm{H}^1(\mathbb{Z};\mathcal{G}_\nabla)$. Its image $\overline{g}$ in $\mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla})$ must, by commutativity of (5.3.4), be mapped the the non-trivial class $l$ and therefore it cannot be trivial. So we find that $\mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla}) \neq \{1\}$. In fact this shows that

$$T^1_\hbar(\mathbb{T}^2) \hookrightarrow \mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla}).$$

Note also that, since $\mathrm{H}^2(\mathbb{Z};\mathcal{Z}^\times) = 0$, we find that the map $\mathrm{H}^1(\mathbb{Z};\mathcal{G}_\nabla) \to \mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla})$ is surjective and, since $\mathrm{H}^2(\mathbb{Z};\mathbb{C}[\![\hbar]\!]^\times) = 0$, so is the map $\mathrm{H}^1(\mathbb{Z};\mathbb{A}_\hbar{}^\times) \to \mathrm{H}^1(\mathbb{Z};\mathbb{A}_\hbar{}^\times/\mathbb{C}[\![\hbar]\!]^\times)$. Thus we find that $\mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla})$ is essentially given by $T^1_\hbar(\mathbb{T}^2)$ and $\mathrm{H}^1(\mathbb{Z};\mathbb{A}_\hbar{}^\times)$ the last of which is given essentially by $\mathrm{H}^1(\mathbb{Z};C^\infty(\mathbb{T}^2))$ and $\mathrm{H}^1(\mathbb{Z};C^\infty(\mathbb{T}^2)^\times)$.

Many more examples can of course be considered, for instance when $\Gamma$ is finite, but $\mathrm{H}^1_{\mathrm{dR}}(M) \neq 0$. We will leave these examples to the reader.

CHAPTER 6

# Algebraic Index Theorems

In this chapter we will consider the algebraic analog of the Atiyah-Singer index theorem 1.3.4, namely the algebraic index theorem 6.1.22. The algebraic index theorem is in essence a certain product formula for periodic cyclic cohomology classes. By the well-known pairing of K-theory and cyclic cohomology, given by the Chern-Connes map [**78**], we obtain the theorem 6.1.22. The algebraic index theorem was first proved in [**86**] as stated in 6.1.22, although some results had appeared earlier [**48**].

Although the method of proof of the algebraic index theorem in [**86**] is slightly different from the proof we will present below, the framework of formal geometry was already present. In the next section we will present a proof of the algebraic index theorem 6.1.22 similar to the proof presented in [**14**]. In the section following it we will adapt this proof to prove the *equivariant* algebraic index theorem 6.2.23. The equivariant algebraic index theorem is the analog of the algebraic index theorem when we consider a group action on the deformation quantization. The main application is in the case that the action of the group is not "nice" enough to allow for a symplectic quotient manifold.

## 6.1. The Algebraic Index Theorem

In this section we will present a proof of the algebraic index theorem 6.1.22 similar to the proof presented in [**14**]. The proof proceeds in the spirit of formal geometry, by first proving the theorem for the formal neighborhood of a point in $2d$-dimensional Euclidean space and subsequently globalizing by use of an appropriate Gelfand-Fuks map. Thus we will first prove a certain product formula in Lie algebra cohomology, which we will call the universal algebraic index formula 6.1.12, and then we will derive the algebraic index theorem 6.1.22 from this one.

**6.1.1. Universal Algebraic Index Theorem.** In this section we will present a proof, given in [**14**], of the universal algebraic index theorem 6.1.12. Let us first motivate why one expects this kind of theorem. First of all we note that, since the trace Tr appearing in the algebraic index theorem 6.1.22 is defined in terms of integrals, the algebraic index theorem is actually an equation of densities, rather than their integrals. In other words the algebraic index theorem equates two maps

$$\mathrm{HC}^{per}_{\bullet}(\mathbb{A}_\hbar(M)) \longrightarrow \mathrm{H}^{\bullet}_{\mathrm{dR}}(M)[\![\hbar]\!].$$

If this holds for any symplectic manifold $(M,\omega)$ and any deformation quantization of it, then it should certainly hold for the formal neighborhood of $0 \in \mathbb{R}^{2d}$ and the unique deformation quantization of it. The global aspect is given in this case by requiring that the classes we equate are equivariant with respect to the action of the formal vector fields in $\mathfrak{g}^\hbar$. This leads to the universal algebraic index theorem 6.1.12.

NOTATION 6.1.1. From now on there will be instances where algebras initially defined over $\mathbb{C}[\![\hbar]\!]$ will be localized at $\hbar$ and thus defined over $\mathbb{C}[\hbar^{-1},\hbar]\!]$. To avoid cumbersome notation we will turn subscripts $\hbar$ into $(\hbar)$ to signify that we mean the localized version of the algebra. So, we denote for instance $\mathbb{W}_{(\hbar)} := \mathbb{W}_\hbar[\hbar^{-1}]$ and $\mathbb{A}_{(\hbar)} := \mathbb{A}_\hbar[\hbar^{-1}]$.

Note that, in section 3.3.1, we proved the following proposition.

PROPOSITION 6.1.2. *There exists a quasi-isomorphism*

$$\tilde{\mu}_\hbar \colon \left(C^{Hoch}_\bullet\left(\mathbb{W}_{(\hbar)}\right), b\right) \longrightarrow \left(\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][2d], \hat{d}\right).$$

We will now fix a choice of such a quasi-isomorphism $\tilde{\mu}_\hbar$. Note that all choices of quasi-isomorphism are homotopic. Note that in section 3.3.2 we also show the periodic cyclic counterpart of proposition 6.1.2.

PROPOSITION 6.1.3. *There exists a quasi-isomorphism*

$$\mu_\hbar \colon \left(CC^{per}_\bullet\left(\mathbb{W}_{(\hbar)}\right), b + uB\right) \longrightarrow \left(\hat{\Omega}^{-\bullet}[u^{-1}, u]][\hbar^{-1}, \hbar]][2d], \hat{d}\right).$$

*where $u$ is a formal variable of degree $-2$ (i.e. on the right hand side we consider the periodicized formal de Rham complex).*

As before, we will fix such a quasi-isomorphism extending the previously fixed $\tilde{\mu}_\hbar$ in the Hochschild case. For the periodic cyclic homology of the undeformed formal neighborhood we have another canonically defined quasi-isomorphism. In the following we will use $\hat{\mathbb{A}}$ to refer to the complexification of $\hat{\mathbb{A}}_{2d}$.

PROPOSITION 6.1.4. *The map*

$$E \colon CC^{per}_\bullet\left(\hat{\mathbb{A}}\right) \longrightarrow \left(\hat{\Omega}^\bullet[u^{-1}, u]], u\hat{d}\right)$$

*given by $f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 \hat{d} f_1 \wedge \hat{d} f_2 \wedge \ldots \wedge \hat{d} f_n$ is a quasi-isomorphism.*

PROOF.
A direct check shows that $E \circ b = 0$, while $E \circ B = \hat{d}$, so $E$ is a map of complexes. Note that we can view the periodic cyclic complex as the totalization of a double complex in the usual sense, see appendix A.2.1, and we can see the formal de Rham complex as the totalization of the double complex $\left(\hat{\Omega}^\bullet[u^{-1}, u]], 0, \hat{d}\right)$. Comparing the associated spectral sequences, we see that, by the well-known [**78, 79**] fact that

$$\mathrm{HH}_\bullet\left(\hat{\mathbb{A}}\right) \simeq \hat{\Omega}^\bullet,$$

$E$ is an isomorphism on the first pages. This implies that $E$ is a quasi-isomorphism. □

Note that proposition 6.1.4 is the formal analog of the Connes-Hochschild-Kostant-Rosenberg theorem [**25, 95**].

PROPOSITION 6.1.5. *The map*

$$J \colon \left(\hat{\Omega}^\bullet[u^{-1}, u]], u\hat{d}\right) \longrightarrow \left(\hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], \hat{d}\right),$$

*given by $f_0 \hat{d} f_1 \wedge \ldots \wedge \hat{d} f_n \mapsto u^{d-n} f_0 \hat{d} f_1 \wedge \ldots \wedge \hat{d} f_n$, is an isomorphism of complexes.*

PROOF.
A simple check shows that $J \circ u\hat{d} = \hat{d} \circ J$. Now note that the map $Q$, given by

$$f_0 \hat{d} f_1 \wedge \ldots \wedge \hat{d} f_n \mapsto u^{n-d} f_0 \hat{d} f_1 \wedge \ldots \wedge \hat{d} f_n,$$

is seen, by a similar check, to satisfy $Q \circ \hat{d} = u\hat{d} \circ Q$ and clearly $Q = J^{-1}$. □

DEFINITION 6.1.6. We define the *formal topological trace density*

$$\hat{\tau}_t \colon (CC^{per}_\bullet(\mathbb{W}_\hbar), b + uB) \longrightarrow \left(\hat{\Omega}^{-\bullet}[u^{-1}, u]][\hbar^{-1}, \hbar]][2d], \hat{d}\right)$$

as the composite $J \circ E \circ \iota \circ \hat{\sigma}$. Here $\hat{\sigma}$ denotes the map from $CC^{per}_\bullet(\mathbb{W}_\hbar)$ to $CC^{per}_\bullet\left(\hat{\mathbb{A}}[[\hbar]]\right)$ induced by the map that sets $\hbar = 0$; by $\iota$ we mean the map induced in periodic cyclic homology by the extension of scalars from $\mathbb{C}[[\hbar]]$ to $\mathbb{C}[\hbar^{-1}, \hbar]]$ and $E$ and $J$ are $\mathbb{C}[\hbar^{-1}, \hbar]]$-linear analogs of the maps in propositions 6.1.4 and 6.1.5.

NOTATION 6.1.7. We denote by

$$\mathbb{L}^\bullet := \operatorname{Hom}^{-\bullet}\left(CC^{per}_\bullet(\mathbb{W}_\hbar), \hat{\Omega}^{-\bullet}[u^{-1},u]][\hbar^{-1},\hbar]][2d]\right)$$

the hom internal to the category of chain complexes. We will denote the differential on $\mathbb{L}^\bullet$ by $\partial_{\mathbb{L}}$. Similarly, we denote

$$\widetilde{\mathbb{L}}^\bullet := \operatorname{Hom}^{-\bullet}\left(CC^{per}_\bullet\left(\mathbb{W}_{(\hbar)}\right), \hat{\Omega}^{-\bullet}[u^{-1},u]][\hbar^{-1},\hbar]][2d]\right).$$

Suppose $D$ is a derivation of $\mathbb{W}_\hbar$, then we define the derivation $D\colon \hat{\mathbb{A}} \to \hat{\mathbb{A}}$ by the formula $D(\hat{f}) = \hat{\sigma}D(f)$, where, on the right hand side, we consider $f \in \mathbb{W}_\hbar$ in the obvious way. Note that this means that $\mathfrak{g}^\hbar$ acts by derivations on $\hat{\mathbb{A}}$ and thus also on $\hat{\Omega}^{-\bullet}[u^{-1},u]][\hbar^{-1},\hbar]][2d]$. Note that the corresponding action of $\mathfrak{sp}(2d,\mathbb{R})$ integrates to an action of $\mathrm{Sp}(2d,\mathbb{R})$, which makes $\hat{\Omega}^{-\bullet}[u^{-1},u]][\hbar^{-1},\hbar]][2d]$ a $(\mathfrak{g}^\hbar, \mathrm{Sp}(2d,\mathbb{R}))$-module. Note that the actions of $\mathrm{Sp}(2d,\mathbb{R})$ and $\mathfrak{g}^\hbar$ on $\mathbb{W}_\hbar$ extend to $CC^{per}_\bullet(\mathbb{W}_\hbar)$ making this an $(\mathfrak{g}^\hbar, \mathrm{Sp}(2d,\mathbb{R}))$-module also. In particular, we can consider the complex, i.e. the total complex associated to a double complex,

$$\left(C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \mathbb{L}^\bullet\right), \partial_{Lie}, \partial_{\mathbb{L}}\right).$$

Note that, since the formal topological trace density is $\mathfrak{g}^\hbar$-equivariant, we find that

$$[\hat{\tau}_t] \in \mathrm{H}^0\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \mathbb{L}^\bullet\right).$$

On the other hand, as mentioned in the remark 3.3.7, the quasi-isomorphism $\mu_\hbar$ is $\mathfrak{sp}(2d,\mathbb{R})$-equivariant, but not $\mathfrak{g}^\hbar$-equivariant.

LEMMA 6.1.8. *There are cochains*

$$\mu_{\hbar,p} \in C^p_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \widetilde{\mathbb{L}}^{-p}\right),$$

*for all $p > 0$, such that*

$$\mu_\hbar^{eq} := \mu_\hbar + \sum_{p>0} \mu_{\hbar,p}$$

*is a cocycle.*

PROOF.
The proof follows from the fact that the cohomology of $\widetilde{\mathbb{L}}^\bullet$ is one-dimensional, the spectral sequence associated to the double complex

$$\left(C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \widetilde{\mathbb{L}}^\bullet\right), \partial_{Lie}, \partial_{\mathbb{L}}\right)$$

and an argument very similar to the proof of proposition 3.3.9. □

DEFINITION 6.1.9. We define the *formal algebraic trace density*

$$\hat{\tau}_a \in \left(C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \mathbb{L}^\bullet\right), \partial_{Lie}, \partial_{\mathbb{L}}\right)$$

as the composite $\mu_\hbar^{eq} \circ \iota$, where $\iota$ is defined in definition 6.1.6. We denote the analogous cochain starting from Hochschild homology by $\hat{\bar{\tau}}_a$.

Now we have the two classes

$$[\hat{\tau}_t], [\hat{\tau}_a] \in \mathrm{H}^0\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d,\mathbb{R}); \mathbb{L}^\bullet\right).$$

The universal algebraic index theorem states that these two classes are related by a certain product formula, i.e. $[\hat{\tau}_a]$ is given by the product of $[\hat{\tau}_t]$ with a certain class. Another way to say it is that the diagram

$$\begin{tikzcd} CC^{per}_\bullet(\mathbb{W}_\hbar) \arrow[r, "\hat{\sigma}"] \arrow[d, "\iota"'] & CC^{per}_\bullet\left(\hat{\mathbb{A}}\right) \arrow[r, "E"] & \hat{\Omega}^\bullet[u^{-1}, u]\!] \arrow[d, "J"] \\ CC^{per}_\bullet\left(\mathbb{W}_{(\hbar)}\right) \arrow[rr, "\mu_\hbar"'] & & \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]\!][u^{-1}, u]\!][2d] \end{tikzcd},$$

is commutative up to a factor in the derived category of $(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}))$-modules, see [**14**]. Let us construct the needed factor that will make the diagram above commutative.

Recall the Chern-Weil construction of classes in example 2.4.3. Note that this construction can be applied verbatim to the deformed situation. In other words, consider the $\mathrm{Sp}(2d, \mathbb{R})$-equivariant projection

$$\mathfrak{p}\colon \mathfrak{g}^\hbar \longrightarrow \mathfrak{sp}(2d, \mathbb{R})$$

given by the identification of $\mathfrak{g}^\hbar_0$ and $\mathfrak{sp}(2d, \mathbb{C})$ in proposition 3.2.4 and the inclusion of $\mathfrak{sp}(2d, \mathbb{R})$ in $\mathfrak{sp}(2d, \mathbb{C})$. Just as in example 2.4.3, we have the corresponding curvature map

$$R_\mathfrak{p}\colon \mathfrak{g}^\hbar \wedge \mathfrak{g}^\hbar \longrightarrow \mathfrak{sp}(2d, \mathbb{R}).$$

Thus we also find the Chern-Weil homomorphisms

$$CW_p\colon \mathcal{S}^p(\mathfrak{sp}^*_{2d})^{\mathrm{Sp}(2d,\mathbb{R})} \longrightarrow C^{2p}_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R})\right)$$

defined exactly as in example 2.4.3.

Definition 6.1.10. We define the formal A-hat class $\hat{A}_f$ as the image of the symmetric invariant polynomial

$$\sqrt{\det\left(\frac{\mathrm{ad}\ \frac{X}{2}}{\exp\left(\mathrm{ad}\ \frac{X}{2}\right) - \exp\left(-\mathrm{ad}\ \frac{X}{2}\right)}\right)}$$

under the Chern-Weil homomorphism defined above.

Note that we can also view $\hat{A}_f$ as a class in $C^{2\bullet}_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathfrak{a}\right)$, using the notation of remark 3.2.2. Recall also the Weyl curvature $\hat{\theta} \in C^2_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathfrak{a}\right)$ of the remarks 3.2.2 and 3.2.7.

Definition 6.1.11. Suppose $[A] \in \mathrm{H}^{ev}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathbb{C}[\hbar^{-1}, \hbar]\!]\right)$ is an even cohomology class and $\mathcal{L}$ is any $\mathfrak{g}^{(\hbar)}[u^{-1}, u]\!]$-module, then we denote by $A\cdot$ the operator

$$A\cdot\colon C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathcal{L}\right) \longrightarrow C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathcal{L}\right)$$

given by

$$A \cdot c = \sum_{k \geq 0} u^d A_{2d} \wedge c$$

for any $c \in C^\bullet_{Lie}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathcal{L}\right)$, where $A = \sum_{k\geq 0} A_{2k}$ for $[A_{2k}]$ the component of $A$ in the cohomology group $\mathrm{H}^{2k}\left(\mathfrak{g}^\hbar, \mathfrak{sp}(2d, \mathbb{R}); \mathbb{C}[\hbar^{-1}, \hbar]\!]\right)$.

Note that, for any $A$ as in definition 6.1.11 above, the operator $A\cdot$ is of degree 0.

Theorem 6.1.12 (Universal Algebraic Index Theorem). *The cocycle*

$$\hat{\tau}_a - \left(\hat{A}_f e^{\hat{\theta}}\right) \cdot \hat{\tau}_t$$

*is cohomologous to* 0.

The proof of theorem 6.1.12 can be found in [**14**] and implicitly in [**86**]. Since the proof uses some rather involved constructions in cyclic homology we will omit it here. A sketch of the proof is as follows. First of all one notes that, since both the periodic cyclic complex of $\mathbb{W}_{(\hbar)}$ and the formal de Rham complex are one-dimensional in cohomology, the proof only requires computation of $\mu_\hbar(1)$. The proof uses the *reduced* cyclic complex $\overline{CC}_\bullet\left(\mathbb{W}_{(\hbar)}\right)$ of the algebra $\mathbb{W}_{(\hbar)}$, see [**78, 14**] for a definition. In particular it uses the fact that one can define a certain action

$$\overline{CC}_{1-\bullet}\left(\mathbb{W}_{(\hbar)}[\eta]\right)[u^{-1},u]\!]\otimes CC^{per}_\bullet\left(\mathbb{W}_{(\hbar)}\right)\longrightarrow CC^{per}_{-\bullet}\left(\mathbb{W}_{(\hbar)}\right),$$

where $\eta$ is a degree $-1$ variable such that $\eta^2{=}0$, of the reduced cyclic complex on the periodic cyclic complex. In fact, this can be done for any unital algebra [**14**]. The proof proceeds by constructing a certain fundamental cocycle $U$ in the reduced cyclic complex and showing that $\mu_\hbar(U\bullet 1)=1$, where we denote the action of the reduced complex on the periodic complex by $\bullet$. The theorem then follows from an explicit computation of $U$ in terms $\hat{A}_f$ and $\hat{\theta}$. This computation uses the explicit construction, due to Brodzki [**16**], of the connecting morphism in the exact triangle defining the reduced cyclic homology complex.

**6.1.2. Globalization of the Universal Formula.** The algebraic index theorem basically follows from the universal theorem 6.1.12 by applying the Gelfand-Fuks map $GF^\hbar_M$, defined in 4.1.7, obtained through the framework of formal deformed geometry, see section 4.1. Namely, we note that the cocycles $\hat{\tau}_a$ and $\hat{\tau}_t$ provide maps of complexes

$$C^\bullet_{Lie}\left(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});CC^{per}_\bullet(\mathbb{W}_\hbar)\right)\longrightarrow C^\bullet_{Lie}\left(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});\hat{\Omega}^{-\bullet}[u^{-1},u]\!][\hbar^{-1},\hbar]\!][2d]\right).$$

This means that, by applying the Gelfand-Fuks map, we obtain the maps of complexes

$$GF^\hbar_M(\hat{\tau}_a),GF^\hbar_M(\hat{\tau}_t)\colon\Omega^\bullet\left(M;CC^{per}_\bullet(\mathbb{W}_\hbar)\right)\longrightarrow\Omega^\bullet\left(M;\hat{\Omega}^{-\bullet}[u^{-1},u]\!][\hbar^{-1},\hbar]\!][2d]\right).$$

The algebraic index theorem 6.1.22 then follows from the observation that the left hand complex is quasi-isomorphic to $CC^{per}_\bullet\left(\mathbb{A}_\hbar(M)\right)$, by proposition 4.1.9, and the right hand complex is quasi-isomorphic to the usual de Rham complex. The only thing that remains to be done is find out what the images of $\hat{\tau}_a$, $\hat{A}_f$, $\hat{\theta}$ and $\hat{\tau}_t$ under the Gelfand-Fuks map are.

PROPOSITION 6.1.13. *We have $GF^\hbar_M\left(\hat{\theta}\right)=\theta$, the characteristic class, or the Weyl curvature, of $\mathbb{A}_\hbar(M)$ and $GF^\hbar_M\left(\hat{A}_f\right)=\hat{A}\left(T_{\mathbb{C}}M\right)$, the $\hat{A}$-genus of the complexified tangent bundle.*

PROOF.
The proposition follows by the definition of the characteristic class 4.3.2 and the definition of the $\hat{A}$-genus [**48**]. □

LEMMA 6.1.14. *There exists a quasi-isomorphism*

$$T_0\colon\left(\Omega^\bullet(M;\hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]\!][u^{-1},u]\!][2d]),\nabla^\hbar_F+\hat{d}\right)\longrightarrow\left(\Omega^\bullet(M)[\hbar^{-1},\hbar]\!][u^{-1},u]\!][2d],d_{dR}\right).$$

PROOF.
On the left hand side we are referring to the totalization of a (completely) bounded double complex with the differential $\nabla^\hbar_F$ acting vertically and the differential $\hat{d}$ horizontally. Let us also view the right hand side as the totalization of a completely bounded double complex with the differential $d_{\mathrm{dR}}$ acting vertically and the differential 0 acting horizontally. Consider the natural inclusion of the right hand complex into the left hand complex. If we associate to both double complexes the spectral sequence arising from their filtration in terms of the degree of $\Omega^\bullet$, we see that this inclusion induces an isomorphism on the first pages. Moreover the spectral sequences collapse on the second pages. Thus the natural inclusion is a quasi-isomorphism. Now we let $T_0$ be a quasi-inverse to the natural inclusion. □

REMARK 6.1.15. Let us consider
$$GF^{\hbar}_{M}(\hat{\tau}_t) \in Hom^0\left(CC^{\bullet}_{per}(\mathcal{W}_{\hbar}), \hat{\Omega}^{-\bullet}_{M}[\hbar^{-1},\hbar]][u^{-1},u]][2d]\right),$$
where $\hat{\Omega}^{-\bullet}_{M}[\hbar^{-1},\hbar]][u^{-1},u]][2d]$ is the bundle with fibers $\hat{\Omega}^{-\bullet}[u^{-1},u]][\hbar^{-1},\hbar]][2d]$ associated to $\mathrm{Sp}_M$. It is given by
$$GF^{\hbar}_{M}(\hat{\tau}_t)(w_0\otimes\ldots\otimes w_n) = u^{d-n}\sigma(w_0)\hat{d}\sigma(w_1)\wedge\ldots\wedge\hat{d}\sigma(w_n),$$
where $\sigma$ sets $\hbar=0$ as expected. Note that, if $f\in C^{\infty}(M)$, then locally the $\infty$-jet $\hat{f}$ of $f$ in $\mathcal{W}_{\hbar}$ satisfies $d\hat{f}=\hat{d}\hat{f}$, by the remark 4.1.10. This shows that $T_0\circ GF^{\hbar}_{M}(\hat{\tau}_t)\circ J^{\infty}_{F,\hbar}$, where $J^{\infty}_{F,\hbar}$ denotes the map that the map in proposition 4.1.9 induces in cyclic homology and $T_0$ is as in lemma 6.1.14, is exactly the Connes-Hochschild-Kostant-Rosenberg map defined in [**25, 95**] (composed with $\sigma$). We will denote this Connes-Hochschild-Kostant-Rosenberg map by $\mathcal{E}$.

Determining the image of $\hat{\tau}_a$ under the Gelfand-Fuks map is slightly more involved. To do this, let us return briefly to the class $\hat{\hat{\tau}}_a$ defined in terms of the Hochschild homology instead of the cyclic homology. Note that $(\nabla^{\hbar}_{F}+b^{*}+\hat{d})GF^{\hbar}_{M}(\hat{\hat{\tau}}_a)=0$ since $\hat{\hat{\tau}}_a$ is a cocycle. Let us denote by $\mathbb{L}^{\bullet}_{H}$ the counterpart of the complex $\widetilde{\mathbb{L}}^{\bullet}$, see notation 6.1.7, for the Hochschild complex instead of the periodic cyclic complex. Note that, since the cohomology of $\mathbb{L}^{\bullet}_{H}$ is one-dimensional, we know that the extension of $\hat{\hat{\tau}}_a$ to a $\mathfrak{g}^{\hbar}$-equivariant class is unique up to homotopy. These two observations together with remark 4.1.10 show the following corollary.

COROLLARY 6.1.16. *In local Darboux coordinates* $(x_1,\ldots,x_d,\xi_1,\ldots,\xi_d)$ *on the chart* $U\subset M$*, we have that* $GF^{\hbar}_{M}\left(\hat{\hat{\tau}}_a\right)(1)$ *is given by*
$$\frac{\omega(x+\hat{x})^d}{d!(i\hbar)^d}=\frac{d(\xi_1+\hat{\xi}_1)\wedge d(x_1+\hat{x}_1)\wedge\ldots\wedge d(x_d+\hat{x}_d)}{d!(i\hbar)^d} \tag{6.1.1}$$
*up to an exact form in* $\Omega^{\bullet}\left(U,\hat{\Omega}^{-\bullet}[\hbar^{-1},\hbar]][u^{-1},u]][2d]\right)$.

Although the algebraic index theorem is essentially a theorem about certain periodic cyclic cohomology classes, it can be phrased in terms of K-theory by considering the well-known pairing induced by the Chern-Connes map [**78**]. Let us recall the definition of this map here and refer to [**78**] for more information on the pairing.

DEFINITION 6.1.17. Suppose $A$ is an algebra, then we define the Chern-Connes map
$$\mathrm{ch}\colon K_0(A)\longrightarrow \mathrm{HC}^{per}_{0}(A)$$
as follows. Suppose $[e]\in K_0(A)$ is represented by the idempotent $e\in M_N(A)$ then
$$\mathrm{ch}\,([e])=\left[\mathrm{tr}\left(e+\sum_{i\geq 1}(-u)^i\frac{(2i)!}{i!}\left(e-\frac{1}{2}\right)\otimes e^{\otimes 2i}\right)\right],$$
where tr denotes the *generalized* trace [**78**]
$$\mathrm{tr}(a\otimes b\otimes\ldots\otimes c)=\sum a_{i_0i_1}\otimes b_{i_1i_2}\otimes\ldots\otimes c_{i_ni_0}$$
for $a,b,c\in M_N(A)$.

In theorem 6.1.22 below we will consider, instead of this map on cohomology, the underlying assignment which assigns to the idempotent $e$ the periodic cyclic chain underlying the definition of ch $([e])$.

NOTATION 6.1.18. We shall denote the compactly supported sections by a subscript $c$, for any bundle over $M$. In particular $\mathbb{A}_{\hbar c}(M)$ denotes the ideal of compactly supported functions in $\mathbb{A}_{\hbar}(M)$ and $\Omega^{\bullet}_{c}(M)$ denotes the compactly supported differential forms on $M$.

The following is a special definition of trace on $\mathbb{A}_\hbar(M)$ due to Fedosov [**48**]. It uses the Fedosov construction explicitly, so let us fix a Fedosov connection $\nabla$ and identify $\operatorname{Ker}\nabla$ with $\mathbb{A}_\hbar(M)$.

DEFINITION 6.1.19. A trace on the algebra $\mathbb{A}_\hbar(M)$ is defined to be a $\mathbb{C}[\![\hbar]\!]$-linear functional

$$\operatorname{Tr}\colon \mathbb{A}_{\hbar c}(M) \longrightarrow \mathbb{C}[\hbar^{-1},\hbar]\!]$$

such that

$$\operatorname{Tr}\operatorname{Ad} E(f) = \operatorname{Tr} f$$

for all $f \in \mathbb{A}_{\hbar c}(M)$ and all $E \in \mathcal{G}_\nabla$.

Note that the above definition implies the familiar trace identity $\operatorname{Tr} fg = \operatorname{Tr} gf$.

PROPOSITION 6.1.20. *There is a unique normalized trace Tr on any deformation quantization.*

The proof of this proposition may be found in section 5.6 of [**48**]. A sketch of the proof is as follows. First, the trace is defined locally as

$$\operatorname{Tr} f = (i\hbar^{-d}) \int_{\mathbb{R}^{2d}} f \frac{\omega_{st}^d}{d!},$$

this fixes the normalization. It is shown that this formula defines a trace as in the definition 6.1.19. Then the formula is globalized by considering a (Darboux) cover and a partition of unity subordinate to it. Finally it is shown that the globalized formula defines a trace as in definition 6.1.19 and does not depend on the particular cover, trivialization or partition of unity.

REMARK 6.1.21. It is also shown in [**48**] that the trace Tr can be given in terms of a density. Although we shall not explicitly use this fact, we note that the density is implicit in the proof of the theorem 6.1.22 below. Also note that, since we shall show that the trace Tr is actually given by globalization of the formal class $\hat{\hat{\tau}}_a$, existence of a trace density is expected.

The following theorem will now follow easily from the considerations above.

THEOREM 6.1.22 (Algebraic Index Theorem). *Suppose $e, f \in M_N(\mathbb{A}_\hbar(M))$, for some $N > 0$, are idempotents such that $e - f \in M_N(\mathbb{A}_{\hbar c}(M))$. Then we have*

$$Tr_N(e-f) = \int_M \mathcal{E}(ch\,(\sigma(e)) - ch\,(\sigma(f)))\left(\hat{A}(T_{\mathbb{C}}M)e^{\theta}\right),$$

*where $\mathcal{E}\colon CC^{per}_\bullet(C^\infty(M)) \to \Omega^\bullet(M)$ denotes the Connes-Hochschild-Kostant-Rosenberg map* [**25, 95**]*, $Tr_N$ denotes the composition of the unique normalized trace on $\mathbb{A}_\hbar(M)$ (see proposition 6.1.20) with the matrix trace, $\sigma$ denotes the map given by setting $\hbar = 0$ and finally $ch$ denotes the Chern-Connes map defined in definition 6.1.17.*

PROOF.
The theorem follows by applying the compositions

$$CC^{per}_\bullet(\mathbb{A}_{\hbar c}(M)) \xrightarrow{J^\infty_{F,\hbar}} \Omega^\bullet_c(M; CC^{per}_\bullet(\mathbb{W}_\hbar)) \xrightarrow{V} \Omega^\bullet_c(M; \Omega^{-\bullet}[u^{-1},u]\!][\hbar^{-1},\hbar][2d]\!]) \longrightarrow$$

$$\xrightarrow{T_0} \Omega^\bullet_c(M)[u^{-1},u]\!][\hbar^{-1},\hbar][2d] \xrightarrow{\int_M} \mathbb{C}[u^{-1},u]\!][\hbar^{-1},\hbar][2d].$$

Here $J^\infty_{F,\hbar}$ is as in proposition 4.1.9, $T_0$ is as in proposition 6.1.14, $\int_M$ denotes integration over the manifold $M$ and $V$ is either $GF^\hbar_M(\hat{\tau}_a)$ (for the left hand side) or $GF^\hbar_M\left(\left(\hat{A}_f e^{\hat{\theta}}\cdot\right)\hat{\tau}_t\right)$ (for the right hand side). By proposition 6.1.12 application of both these compositions must agree. Note that the expression on the right hand side follows from proposition 6.1.13 and remark 6.1.15.

We need to show that the left hand side is indeed the trace. This follows by returning to the Hochschild complex and $\hat{\hat{\tau}}_a$, since, if we consider the composition above defined with respect to the Hochschild instead of the cyclic complex, then we see that we obtain a cocycle in the complex

$\mathrm{Hom}_{\mathbb{C}[\![\hbar]\!]}\left(C_\bullet^{Hoch}\left(\mathbb{A}_{\hbar c}(M)\right),\mathbb{C}[\hbar^{-1},\hbar]\!]\right)$. By construction of the Gelfand-Fuks map, this cocycle is also invariant under automorphisms lifting the identity, i.e. it is a trace as in definition 6.1.19. This means we only need to check that it is normalized (or rather non-zero, since in that case we can always adjust the map $\hat{\tau}_a$ to take care of the normalization). Note that we can check this normalization locally and thus it follows from equation (6.1.1).

□

## 6.2. An Equivariant Algebraic Index Theorem

In this section we shall formulate and prove the main theorem 6.2.23 of this thesis. It is a generalization of theorem 6.1.22 to include equivariance under a group action. We mean this in the following way. Suppose given an action of the group $\Gamma$ on the deformation quantization $\mathbb{A}_\hbar(M)$. Then in particular we find an action of $\Gamma$ by symplectomorphisms of $M$ (see section 5.1). Now suppose this action is such that there exists a well-defined symplectic quotient manifold $M/\Gamma$, for instance the action is free and proper, and we find the deformation quantization $\mathbb{A}_\hbar(M)^\Gamma$ of the quotient manifold. Then we could apply the theorem 6.1.22 above to obtain an equivariant algebraic index theorem. This would be a very special case, however. In the case that there is no symplectic quotient manifold we can still obtain an equivariant index theorem by considering the replacement $\mathbb{A}_\hbar(M)\rtimes\Gamma$ of $\mathbb{A}_\hbar(M)^\Gamma$. Here $\mathbb{A}_\hbar(M)\rtimes\Gamma$ denotes the *crossed product*, see definition A.3.1. The crossed product is the usual non-commutative manifold replacing the quotient manifold [**26**]. Thus the generalization of theorem 6.1.22 to the equivariant setting is a similar product formula in periodic cyclic cohomology of the crossed product $\mathbb{A}_\hbar(M)\rtimes\Gamma$. Another reason to consider the crossed product as a replacement for the quotient is that the periodic cyclic homology of the crossed product is the equivariant periodic cyclic cohomology of the deformation quantization in the sense of [**113**].

We will adapt the proof of the algebraic index theorem 6.1.22 given above to prove the equivariant algebraic index theorem 6.2.23 given below. The main difficulty is that since the group acts by global transformations it will not act on the formal neighborhood. So, in order to use the universal formula 6.1.12, we will have to define an equivariant version of the Gelfand-Fuks maps given in proposition 4.1.7. Once we have obtained the equivariant Gelfand-Fuks maps we will have to define the pairing of cyclic homology of the crossed product and the Lie algebra cohomology appearing in the universal algebraic index theorem 6.1.12. Finally, we shall need to evaluate the equivariant versions of the characteristic classes appearing in the formula 6.1.22.

**6.2.1. Equivariant Gelfand-Fuks maps.** As mentioned above we will need to define an equivariant version of the Gelfand-Fuks maps. We will do this by considering the usual homotopical replacement for the quotient, namely the Borel construction $M\times_\Gamma E\Gamma$. However, since we work with differential forms, we will need a smooth version of the Borel construction. Recall that the usual construction to obtain $M\times_\Gamma E\Gamma$ is by taking the geometric realization of the simplicial manifold $\Gamma^\bullet\times M$. Thus $M\times_\Gamma E\Gamma$ is a quotient of $\Delta^\bullet\times\Gamma^\bullet\times M$. So, we will replace the complex of differential forms on $M$ by a subcomplex of the complex of differential forms on $\Delta^\bullet\times\Gamma^\bullet\times M$ consisting of forms that are constant along the equivalence relations defining $M\times_\Gamma E\Gamma$. This means that, in order for the Fedosov connection to preserve the subcomplex, we will need to construct sections $F_\bullet$, analogous to the one used to define $\nabla_F^\hbar$ appearing in proposition 4.1.7, that also respect these equivalence relations. Since the Gelfand-Fuks maps are defined in terms of the section $F$, this will also supply the definition of the equivariant Gelfand-Fuks maps.

Let us fix a symplectic deformation quantization $\mathbb{A}_\hbar(M)$ of the symplectic manifold $(M,\omega)$ and a countable discrete group $\Gamma$ acting on it by algebra automorphisms for the rest of this section. Note that this means that $\Gamma$ acts on the underlying manifold $M$, the symplectic frames bundles $\mathrm{Sp}_M$ and the (reduced) manifold of deformed non-linear frames $(\widetilde{M}_{r,\hbar})$ $\widetilde{M}_\hbar$, see the proof of proposition 5.1.6.

Note that these actions commute with the actions of the structure groups. In order to define the equivariant Gelfand-Fuks map, we will first show existence of certain systems of sections

$$F_k \colon \Delta^k \times \Gamma^k \times M \longrightarrow \Delta^k \times \Gamma^k \times \widetilde{M}_{r,\hbar}$$

satisfying appropriate boundary conditions.

Notation 6.2.1. We denote by

$$\Delta^k := \left\{ (t_0, \dots, t_k) \geq 0 \mid \sum_{i=0}^{k} t_i = 1 \right\} \subset \mathbb{R}^{k+1}$$

the standard $k$-simplex, viewed as a manifold with corners. We denote

$$P_\Gamma^k := \Delta^k \times \Gamma^k \times \mathrm{Sp}_M$$

and similarly

$$M_\Gamma^k := \Delta^k \times \Gamma^k \times M,$$
$$\widetilde{M}_{r,\Gamma}^k := \Delta^k \times \Gamma^k \times \widetilde{M}_{r,\hbar}$$

and

$$\widetilde{M}_\Gamma^k := \Delta^k \times \Gamma^k \times \widetilde{M}_\hbar.$$

Note that $P_\Gamma^k \to M_\Gamma^k$ is a principal $\mathrm{Sp}(2d,\mathbb{R})$-bundle, namely the pull-back via the obvious projection to $\mathrm{Sp}_M$. Similarly $\widetilde{M}_{r,\Gamma}^k$ is the pull-back of $\widetilde{M}_{r,\hbar}$. We define $\Omega^\bullet(M_\Gamma^k;\mathbb{L})$ for a $(\mathfrak{g},\mathrm{Sp}(2d,\mathbb{R}))$-module $\mathbb{L}$ as we did for $M$ in definition 2.3.5 above, replacing $j^1(M)$ by $P_\Gamma^k$ and considering the trivial action of the symplectic group on $\Delta^k \times \Gamma^k$.

Notation 6.2.2. In the rest of this section we will denote the diffeomorphisms of $M$, $\mathrm{Sp}_M$, $\widetilde{M}_\hbar$ and $\widetilde{M}_{r,\hbar}$ defined by an element $\gamma \in \Gamma$ through the action of $\Gamma$ on $\mathbb{A}_\hbar(M)$ simply by $\gamma$.

In order to accurately define the subcomplex mentioned above and the boundary conditions that the sections $F_k$ should satisfy, we will introduce the maps used to define the simplicial structure on the simplicial manifold $\Gamma^\bullet \times M$ and the corresponding geometric realization.

Definition 6.2.3. For all $k \geq 0$ and $0 \leq i \leq k$ we define

$$\epsilon_i^k \colon \Delta^{k-1} \hookrightarrow \Delta^k, \quad D_i^k \colon \Gamma^k \times \mathrm{Sp}_M \longrightarrow \Gamma^{k-1} \times \mathrm{Sp}_M \quad \text{and} \quad C_i^k \colon \Gamma^k \times \widetilde{M}_{r,\hbar} \xrightarrow{\sim} \Gamma^k \times \widetilde{M}_{r,\hbar}$$

by

$$\epsilon_i^k(t_0,\dots,t_{k-1}) = \begin{cases} (0, t_0, \dots, t_{k-1}) & \text{if } i = 0 \\ (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1}) & \text{if } 0 < i \leq k, \end{cases}$$

$$D_i^k(\gamma_1,\dots,\gamma_k,p) = \begin{cases} (\gamma_2, \dots, \gamma_k, \gamma_1^{-1}(p)) & \text{if } i = 0 \\ (\gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_k, p) & \text{if } 0 < i < k \\ (\gamma_1, \dots, \gamma_{k-1}, p) & \text{if } i = k \end{cases}$$

and finally

$$C_i^k(\gamma_1,\dots,\gamma_k,\varphi_m) = \begin{cases} (\gamma_1, \dots, \gamma_k, \gamma_1(\varphi_m)) & \text{if } i = 0 \\ (\gamma_1, \dots, \gamma_k, \varphi_m) & \text{if } 0 < i \leq k. \end{cases}$$

Lemma 6.2.4. *There exist sections*

$$\begin{tikzcd} P_\Gamma^k \arrow[r, "F_k"] \arrow[dr, "="'] & \widetilde{M}_{r,\Gamma}^k \arrow[d, "Id_{\Delta^k \times \Gamma^k} \times \pi_{r,1}"] \\ & P_\Gamma^k \end{tikzcd}$$

*such that*

$$F_k \circ (\epsilon_i^k \times Id_{\Gamma^k \times Sp_M}) =$$
$$\left(Id_{\Delta^k} \times C_i^k\right) \circ (pr_1 \times pr_2 \times pr_5) \circ (Id_{\Delta^k \times \Gamma^k} \times F_{k-1}) \circ \left(\left(\epsilon_i^k \times Id_{\Gamma^k}\right) \times \left(Id_{\Delta^{k-1}} \times \ D_i^k\right)\right),$$

*i.e. each $F_k$ satisfies boundary conditions specified by $F_{k-1}$ on each face of $\Delta^k$.*

PROOF.
In lemma 4.1.6 we gave the equivariant section

$$F_0 := F_r \colon P_\Gamma^0 = \mathrm{Sp}_M \longrightarrow \widetilde{M}_{r,\hbar},$$

note that $F_0$ does not need to satisfy any boundary conditions (as there is no boundary of $P_\Gamma^0$). Now suppose we have found $F_l$ satisfying the boundary conditions for all $l < k$. Note that giving $F_k$ is equivalent to giving

$$\begin{tikzcd} P_\Gamma^k \arrow[rr, "f_k"] \arrow[dr, "pr_3"'] & & \widetilde{M}_{r,\hbar} \arrow[dl, "\pi_{r,1}"] \\ & \mathrm{Sp}_M & \end{tikzcd}$$

satisfying the corresponding boundary conditions, since $F_k$ has to be the identity on the component $\Delta^k \times \Gamma^k$. The section $F_r$ from lemma 4.1.6 also yields the trivialization of the principal bundle $\pi_{r,1}$, i.e. $\widetilde{M}_{r,\hbar} \simeq \mathrm{Sp}_M \times G_1^\hbar$. Using this, we see that giving $f_k$ is equivalent to giving a map $s_k \colon P_\Gamma^k \to G_1$ which is fixed on the boundary $\partial P_\Gamma^k$ by the boundary conditions and $F_{k-1}$. Now note that the exponential map gives us the isomorphism $G_1 \simeq F_1\mathfrak{g}^\hbar$. So, since this vector space is contractible, we can always find $F_k$ satisfying the boundary conditions. □

REMARK 6.2.5. Note that, since $\widetilde{M}_{r,\hbar} \to \mathrm{Sp}_M$ is the pull-back of a $\widehat{G}_r^\hbar$ bundle over $M$ and the action of $\Gamma$ is by symplectomorphisms, we can extend the sections $\mathrm{Sp}(2d,\mathbb{R})$-equivariantly. From now on we will assume that we have chosen $\mathrm{Sp}(2d,\mathbb{R})$-equivariant sections $F_k$ for all $k \geq 0$ satisfying the conditions in lemma 6.2.4.

Now, as before in section 4.1, we can use the sections $F_k$ to pull back the canonical connection form from $\widetilde{M}_\Gamma^k$ (which was itself pulled back from $\widetilde{M}_\hbar$), to define a $\mathfrak{g}^\hbar$-valued differential form $A_{F_k}$ on $P_\Gamma^k$ for each $k$.

NOTATION 6.2.6. Suppose $\mathbb{L}$ is a $\left(\mathfrak{g}^\hbar, \widehat{G}_r^\hbar\right)$-module. Then we denote the bundle over $M$ associated to $\widetilde{M}_{r,\hbar}$ by $\mathbb{L}_0$, i.e. the bundle with total space $\widetilde{M}_{r,\hbar} \times_{\widehat{G}_r^\hbar} \mathbb{L}$. We will denote the pull-back to the $M_\Gamma^k$ by the same symbol. Note that, denoting $\pi \colon \mathrm{Sp}_M \to M$, the pullback $\pi^*\mathbb{L}_0$ is exactly the bundle associated to $\widetilde{M}_{r,\hbar} \to \mathrm{Sp}_M$ with the $\left(\mathfrak{g}^\hbar, G_1^\hbar\right)$-module $\mathbb{L}$, given by $G_1^\hbar \hookrightarrow \widehat{G}_r^\hbar$, as fiber, i.e the pull-back has total space $\widetilde{M}_{r,\hbar} \times_{G_1^\hbar} \mathbb{L}$. Thus we will denote $\pi^*\mathbb{L}_0 = \mathbb{L}_1$. Again we will use the same notation for the pull-backs over the $P_\Gamma^k$.

REMARK 6.2.7. Note that, since $\Gamma$ acts on the $\widehat{G}_r^\hbar$-bundle $\widetilde{M}_{r,\hbar} \to M$, we find that $\Gamma$ also acts on $\mathbb{L}_0$ and, since this action lifts to a $\mathrm{Sp}(2d,\mathbb{R})$-equivariant action on $\widetilde{M}_{r,\hbar} \to \mathrm{Sp}_M$, we find a corresponding action on the space $\Omega^\bullet(M;\mathbb{L})$. Let us be a bit more precise about this action. Note first that the section $F_0$ (from lemma 6.2.4) yields a trivialization of $\widetilde{M}_{r,\hbar} \to \mathrm{Sp}_M$. This means that it also yields a trivialization (denoted by the same symbol)

$$F_0 \colon \mathrm{Sp}_M \times \mathbb{L} \xrightarrow{\sim} \widetilde{M}_{r,\hbar} \times_{G_1^\hbar} \mathbb{L},$$

explicitly given by $(p,\ell) \mapsto [F_0(p),\ell]$ with inverse $[\varphi_m,\ell] \mapsto \left(\pi_{r,1}(\varphi_m), \left(\varphi_m \circ F_0(\pi_{r,1}(\varphi_m))^{-1}\right)(\ell)\right)$. In these terms the action is given by

$$\gamma(\eta \otimes \ell) = (\gamma^*\eta) \otimes (F_0^{-1})^*\gamma^* F_0^* \ell$$

where $(F_0^{-1})^*\gamma^*F_0^*\ell$ is the section given by

$$p \mapsto (\gamma(p), F_0(p)\gamma F_0(\gamma(p))^{-1}\ell).$$

We consider the corresponding action of $\Gamma$ on the spaces $\Omega^\bullet\left(M_\Gamma^k;\mathbb{L}\right)$, where we use $F_k$ instead of $F_0$ (or in fact on $\Omega^\bullet(N\times M,\mathbb{L})$ for any $N$).

DEFINITION 6.2.8. Note that the differential forms $A_{F_k}$ define flat connections $\nabla_{F_k}$ on $\Omega^\bullet\left(M_\Gamma^k;\mathbb{L}\right)$ for all $k$, as in proposition 4.1.7, and so we can consider the product complex

$$\left(\prod_{k\geq 0}\Omega^\bullet\left(M_\Gamma^k;\mathbb{L}\right),\tilde{\nabla}_F\right),$$

where

$$\tilde{\nabla}_F := \prod_{k\geq 0}\nabla_{F_k}.$$

PROPOSITION 6.2.9. *Suppose* $\mathbb{L}$ *is a* $(\mathfrak{g}^\hbar, Sp(2d,\mathbb{R}))$*-module, then the map*

$$GF_M^\Gamma\colon \left(C^\bullet_{Lie}\left(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});\mathbb{L}\right),\partial_{Lie}\right)\longrightarrow\left(\prod_{k\geq 0}\Omega^\bullet\left(M_\Gamma^k;\mathbb{L}\right),\tilde{\nabla}_F\right),$$

*given by*

$$GF_M^\Gamma(\chi)_k = \chi\circ A_{F_k}^{\otimes p}, \tag{6.2.1}$$

*for* $\chi\in C^p_{Lie}\left(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});\mathbb{L}\right)$ *and where the subscript* $k$ *refers to taking the* $k$*-th coordinate in the product, is a well-defined map of complexes.*

Note that the definition of $GF_M^\Gamma$ is the same as in proposition 4.1.7 only we now use the compatible system of sections $\{F_k\}_{\geq 0}$.

PROOF.
This proof is exactly the same as in the non-equivariant setting, see propositions 2.3.9 and 4.1.7, carried out coordinate-wise in the product.

□

The *equivariance* of the equivariant Gelfand-Fuks map is demonstrated by the following lemma.

LEMMA 6.2.10. *For all* $\chi\in C^p_{Lie}(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});\mathbb{L})$*, we have that the form*

$$GF_M^\Gamma(\chi)\in\prod_k\Omega^p(M_\Gamma^k;\mathbb{L})$$

*satisfies*

$$(\epsilon_i^k\times Id_{\Gamma^k\times Sp_M})^*GF_M^\Gamma(\chi)_k = (Id_{\Delta^{k-1}}\times D_i^k)^*GF_M^\Gamma(\chi)_{k-1}$$

*if* $0<i\leq k\in\mathbb{N}$ *and*

$$(\epsilon_0^k\times Id_{\Gamma^k\times Sp_M})^*GF_M^\Gamma(\chi)_k|_{\Delta^{k-1}\times\{\gamma\}\times\Gamma^{k-1}\times Sp_M} = \gamma((Id_{\Delta^{k-1}}\times D_0^k)^*(GF_M^\Gamma(\chi)_{k-1}))$$

*where* $\gamma\in\Gamma$ *on the right hand side signifies the action on* $\Omega^p(\Delta^{k-1}\times\Gamma^k\times M;\mathbb{L})$*.*

PROOF.
The boundary conditions put on the sections $F_k$ in lemma 6.2.4 are meant exactly to ensure this property of the Gelfand-Fuks map $GF_M^\Gamma$. The lemma follows straightforwardly from these boundary conditions. □

REMARK 6.2.11. The phrasing of the preceding lemma may obscure its nature a bit. Note that the left hand sides of the equations in the lemma are simply the restrictions of the forms $GF_M^\Gamma(\chi)_k$ to the various boundary components of $P_\Gamma^k$. The lemma states that the forms in the image of the equivariant Gelfand-Fuks map are constant on the equivalence classes of the relation $\sim$ on $M_\Gamma := \coprod_k \Delta^k \times \Gamma^k \times M$ such that $M_\Gamma/\sim =: E\Gamma \times_\Gamma M$ is the Borel construction (through geometric realization of the corresponding simplicial manifold).

**6.2.2. Pairing with $\mathbf{HC}_\bullet^{per}(\mathbb{A}_{\hbar c}(M) \rtimes \Gamma)$.** In order to derive the equivariant version of the algebraic index theorem 6.2.23, we need to pair the periodic cyclic complex of $\mathbb{A}_\hbar(M) \rtimes \Gamma$ with the Lie algebra complex with values in

$$\mathbb{L}^\bullet := Hom^{-\bullet}(CC_\bullet^{per}(\mathbb{W}_\hbar), \hat{\Omega}^{-\bullet}[u^{-1}, u]][\hbar^{-1}, \hbar]][2d]),$$

as in section 6.1.1, in order to interpret the universal formula 6.1.12 in the periodic cyclic cohomology of $\mathbb{A}_\hbar(M) \rtimes \Gamma$. We fix the notation $\mathbb{L}^\bullet$ to mean the above for the rest of this chapter. We will define a trace on the crossed product that is "supported at the identity of $\Gamma$" in a sense that will be clear in section 6.2.3, see definition 6.2.19. In this section it will mean that, instead of pairing the full periodic cyclic complex with the Lie algebra complex, we will only pair the so-called homogeneous chains, see appendix A.3.1. By theorem A.3.6, this means we can pair with the periodic cyclic chains by constructing a map from the Lie algebra complex to

$$C^\bullet(\Gamma, \Omega^\bullet(M; \mathbb{L}^\bullet)),$$

the group cohomology complex with values in differential forms with values in $\mathbb{L}^\bullet$. We will do this (implicitly) in this section.

NOTATION 6.2.12. We denote the composition of the projection to the homogeneous summand with the quasi-isomorphism of theorem A.3.6 by

$$D\colon CC_\bullet^{per}(\mathbb{A}_\hbar(M) \rtimes \Gamma) \longrightarrow C_\bullet(\Gamma, CC_\bullet^{per}(\mathbb{A}_\hbar(M))).$$

DEFINITION 6.2.13. We define the $p$-forms $\varphi \in \prod_{k\geq 0} \Omega^\bullet\left(M_\Gamma^k; \mathbb{L}^\bullet\right)$ of Sullivan-de Rham type as those $\varphi$ that satisfy the conditions of lemma 6.2.10 and such that $\varphi_k$ is a $p$-form for all $k$, but are not necessarily in the image of $GF_M^\Gamma$. We denote the space of $p$-forms of Sullivan-de Rham type by

$$\Omega^p\left(M \times_\Gamma E\Gamma; \mathbb{L}^\bullet\right)$$

and denote by

$$\Omega^\bullet\left(M \times_\Gamma E\Gamma, \mathbb{L}^\bullet\right) := \bigoplus_{p\geq 0} \Omega^p\left(M \times_\Gamma E\Gamma, \mathbb{L}^\bullet\right)$$

the forms of Sullivan-de Rham type.

Since the definition uses lemma 6.2.10, we find the following corollary.

COROLLARY 6.2.14. *The image of the Gelfand-Fuks map $GF_M^\Gamma$ is contained in the forms of Sullivan-de Rham type.*

LEMMA 6.2.15. *There are quasi-isomorphisms*

$$T_k\colon \left(\Omega^\bullet(M_\Gamma^k; \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d]), \nabla_{F_k}^\hbar + \hat{d}\right) \longrightarrow \left(\Omega^\bullet(M_\Gamma^k)[\hbar^{-1}, \hbar]][u^{-1}, u]][2d], d_{dR}\right)$$

*for all $k \geq 0$.*

PROOF.
Note that this is essentially the same lemma as lemma 6.1.14. ☐

DEFINITION 6.2.16. We define the pairing

$$\langle\cdot,\cdot\rangle\colon \Omega^\bullet(M\times_\Gamma E\Gamma;\mathbb{L}^\bullet)\times CC^{per}_{-\bullet}(\mathbb{A}_\hbar(M)\rtimes\Gamma)\longrightarrow\Omega^\bullet(M)[2d]$$

as follows. Given $a\in CC^{per}_k(\mathbb{A}_\hbar(M)\rtimes\Gamma)$ we note that $D(a)\in C_\bullet(\Gamma,CC^{per}_\bullet(\mathbb{A}_\hbar(M)))$ is given by the components $D(a)_p\in CC^{per}_{k-p}(\mathbb{A}_\hbar(M))\otimes(\mathbb{C}\Gamma)^{\otimes p}$ for all $p\in\mathbb{Z}_{\geq 0}$. Then let us denote

$$D(a)_p=\sum_{i=0}^{q}D(a)_{p,i,\mathbb{A}_\hbar}\otimes D(a)_{p,i,\Gamma}$$

with $D(a)_{p,i,\Gamma}=\gamma_{1,i}\otimes\ldots\otimes\gamma_{p,i}$ for some $\gamma_{j,i}\in\Gamma$ for all $j$ and $i$, i.e. both legs of the tensor product are given by the above. Suppose $\varphi\in\Omega^k(M\times_\Gamma E\Gamma;\mathbb{L}^\bullet)$ then

$$\langle\varphi,a\rangle:=\sum_{p\geq 0}\sum_{i=0}^{q}\int_{\Delta^p\times\{(\gamma_{i,1},\ldots,\gamma_{i,p})\}}T_p\varphi_p(J^\infty_{F_k,\hbar}(D(a)_{p,i,\mathbb{A}_\hbar})),$$

where $J^\infty_{F_k,\hbar}$ is the map given by taking the $\infty$-jets of elements of $\mathbb{A}_\hbar(M)$ relative to the Fedosov connection $\nabla_{F_k}$, i.e. it is given by the analog of the map in proposition 4.1.9 for $M^k_\Gamma$ instead of $M$.

We note that the pairing $\langle\cdot,\cdot\rangle$ is well-defined since the integral of $\varphi\in\Omega^k(M\times_\Gamma E\Gamma;\mathbb{L}^\bullet)$ over any simplex $\Delta^p$ for $p>k$ will vanish.

DEFINITION 6.2.17. Define

$$C\colon C^\bullet_{Lie}(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});\mathbb{L}^\bullet)\longrightarrow CC^\bullet_{per}(\mathbb{A}_{\hbar c}(M)\rtimes\Gamma)$$

by

$$C_\varphi(a)=\int_M\langle GF^\Gamma_M(\varphi),a\rangle$$

for all $\varphi\in C^\bullet_{Lie}(\mathfrak{g},\mathfrak{sp}(2d,\mathbb{R});\mathbb{L}^\bullet)$ and $a\in CC^{per}_\bullet(\mathbb{A}_{\hbar c}(M)\rtimes\Gamma)$.

PROPOSITION 6.2.18. *The map*

$$C\colon \left(C^\bullet_{Lie}(\mathfrak{g}^\hbar,\mathfrak{sp}(2d,\mathbb{R});\mathbb{L}^\bullet),\partial_{Lie}+\partial_{\mathbb{L}}\right)\longrightarrow\left(CC^\bullet_{per}(\mathbb{A}_{\hbar c}(M)\rtimes\Gamma),(b+uB)^*\right)$$

*is a map of complexes.*

PROOF.
Suppose $\varphi\in C^r_{Lie}(\mathfrak{g},\mathfrak{sp}(2d,\mathbb{R});\mathbb{L}^s)$ and $a\in CC^{per}_t(\mathbb{A}_{\hbar c}(M)\rtimes\Gamma)$. We will show that

$$C_{(\partial_{Lie}+(-1)^r\partial_{\mathbb{L}})\varphi)}(a)=C_\varphi((b+uB)a).$$

By proposition 6.2.9, we see that

$$\int_M\left\langle GF^\Gamma_M((\partial_{Lie}+(-1)^r\partial_{\mathbb{L}})\chi),a\right\rangle=\int_M\left\langle\left(\tilde\nabla+(-1)^r\partial_{\mathbb{L}}\right)GF^\Gamma_M(\chi),a\right\rangle.$$

Furthermore

$$\nabla_{F_k}GF_\Gamma(\varphi)_k(J^\infty_{F_k,\hbar}D(a)_{i,k,\mathbb{A}_\hbar})=\left(\nabla_{F_k}GF_\Gamma(\varphi)_k\right)(J^\infty_{F_k,\hbar}D(a)_{i,k,\mathbb{A}_\hbar})$$

for all $i$ and $k$ since $\nabla_{F_k}J^\infty_{F_k,\hbar}=0$ by definition of $J^\infty_{F_k,\hbar}$. Now note that

$$\int_M\int_{\Delta^k\times\{(\mu_1,\ldots,\mu_k)\}}T_k\left(\left(\nabla_{F_k}+(-1)^{r+s}\hat d\right)GF^\Gamma_M(\varphi)_k\right)(b)=\int_M\int_{\Delta^k\times\{(\mu_1,\ldots,\mu_k)\}}\left(d_{\Delta^k}T_kGF^\Gamma_M(\varphi)_k\right)(b)$$

for any $k\geq 0$, $\mu_i\in\Gamma$, $\varphi$ and $b$, since $\int_M d_{\mathrm{dR}}=0$ by Stokes' theorem. Note that we have used the decomposition

$$\nabla_{F_k}=d_{\Delta^k}+d_{\mathrm{Sp}_M}+A_{F_k},$$

where we have denoted the exterior derivative on $\Delta^k$ by $d_{\Delta^k}$, for all $k \geq 0$. Combining this with the previous statements we find that $C_{(\partial_{Lie}+(-1)^r\partial_{\mathbb{L}})\varphi)}(a)$ equals

$$\sum_{k\geq 0}\sum_{i=0}^{q}\int_M \left(\int_{\Delta^k\times\{(\gamma_{1,i},\ldots,\gamma_{k,i})\}} (d_{\Delta^k}T_k GF_M^\Gamma(\varphi)_k)(J^\infty_{F_k,\hbar}D(a)_{i,k,\mathbb{A}_\hbar})+(-1)^r T_k GF_M^\Gamma(\varphi)_k\big((b+uB)J^\infty_{F_k,\hbar}D(a)_{i,k,\mathbb{A}_\hbar}\big)\right)$$

where we use the notation of definition 6.2.16. Now one checks that, since $GF_M^\Gamma(\varphi)$ is of Sullivan-de Rham type, by corollary 6.2.14 and Stokes' theorem, the term involving $d_{\Delta^k}$ equals a term where the group boundary operator acts on $D(a) \in C_\bullet(\Gamma, CC_\bullet^{per}(\mathbb{A}_{\hbar c}(M)))$. Combining this with the term involving the $b + uB$ operator yields exactly $C_\varphi((b + uB)a)$.

□

**6.2.3. Evaluation of the Equivariant Classes.** In the previous sections we constructed the map

$$C\colon \mathrm{H}^0(\mathfrak{g}, \mathfrak{sp}(2d); \mathbb{L}^\bullet) \longrightarrow \mathrm{HC}^0_{per}(\mathbb{A}_{\hbar c}(M) \rtimes \Gamma).$$

The last step in proving the main result of this chapter is to evaluate the classes appearing in the universal algebraic index formula 6.1.12. This will lead to a theorem similar to the algebraic index theorem 6.1.22. This means that we should expect to consider an ordinary trace on the crossed product as well as a "higher trace" (the image of the formal topological trace density). In the non-equivariant setting there was a unique candidate by the result of Fedosov 6.1.20. As mentioned, we will consider the natural counterpart of this trace "supported at the identity of Γ".

Definition 6.2.19. We define the trace supported at the identity $\mathrm{Tr}_e\colon \mathbb{A}_{\hbar 0}(M) \rtimes \Gamma \to \mathbb{C}[\hbar^{-1}, \hbar]\!]$ by the formula

$$\mathrm{Tr}_e(f\delta_\gamma) = \delta_{e,\gamma}\,\mathrm{Tr}(f),$$

where the $\delta_{e,\gamma}$ on the right hand side is the Kronecker delta.

Now let us consider the equivariant version of the trace density $GF_M^\Gamma(\hat{\tau}_a)$. By the same reasoning as in section 6.1.2, we see that we get a trace on $\mathbb{A}_{\hbar c}(M) \rtimes \Gamma$. By following the various (quasi)-isomorphisms from $C_\bullet^{Hoch}((\mathbb{A}_\hbar(M) \rtimes \Gamma)_e^\natural)$ to $C_\bullet(\Gamma, C_\bullet^{Hoch}(\mathbb{A}_\hbar(M)))$ constructed in the appendix A.3.1 to obtain theorem A.3.6, we see that this trace comes from the map

$$\tau\colon C_0(\Gamma, C_0(\mathbb{A}_{\hbar c}(M))) \longrightarrow \mathbb{C}[\hbar^{-1}, \hbar]\!]$$

such that $\tau \circ (b + \delta_\Gamma) = 0$. This condition means exactly that the map

$$\tilde{\tau}\colon \mathbb{A}_{\hbar c}(M) \longrightarrow \mathbb{C}[\hbar^{-1}, \hbar]\!]$$

given by $\tilde{\tau}(f) = \tau(f \otimes e)$ is a trace. By construction, we see that $\tilde{\tau}$ is also invariant under fiberwise automorphisms and therefore a trace according to definition 6.1.19. Thus we see that $\tilde{\tau} = \mathrm{Tr}$ and so the trace induced by the equivariant Gelfand-Fuks map $GF_M^\Gamma$ is exactly the map

$$f\delta_\gamma \mapsto \delta_{e,\gamma}\,\mathrm{Tr}(f)$$

defined in definition 6.2.19 above.

In the previous sections we have used some complexes to represent Borel equivariant cohomology [**37**], which are not completely standard. The following proposition shows that they really do represent Borel equivariant cohomology.

Proposition 6.2.20. *We have*

$$H^\bullet(M \times_\Gamma E\Gamma) \simeq H^\bullet_\Gamma(M; \mathbb{C})$$

*where on the left hand side we mean the cohomology of* $\Omega^\bullet(M \times_\Gamma E\Gamma; \mathbb{C})$ *and on the right hand side we mean the cohomology of the Borel construction* $M \times_\Gamma E\Gamma$.

This proposition holds basically by construction (see for instance [**37**]).

DEFINITION 6.2.21. The equivariant Weyl curvature $\theta_\Gamma$ is defined as the image of $\hat{\theta}$ under $GF_M^\Gamma$ followed by $\mathbb{C}[\![\hbar]\!]$-linear extension of the map in 6.2.20. Similarly, the equivariant $\hat{A}$-genus of $M$, denoted $\hat{A}(M)_\Gamma$, is defined as the image of $\hat{A}_f$ under the equivariant Gelfand-Fuks map $GF_M^\Gamma$ followed by $\mathbb{C}[\![\hbar]\!]$-linear extension of the isomorphism in 6.2.20.

NOTATION 6.2.22. In the following, we denote the well-known inclusion of the Borel equivariant cohomology into the periodic cyclic cohomology of the crossed product, as defined (for instance) in [**26**] section 3.2.$\delta$, by

$$\Phi\colon \mathrm{H}^\bullet_\Gamma(M;\mathbb{C}) \hookrightarrow \mathrm{HC}^\bullet_{per}(C^\infty(M) \rtimes \Gamma).$$

Finally we obtain the following theorem as a corollary of the previous three sections.

THEOREM 6.2.23 (Equivariant Algebraic Index Theorem).
*Suppose* $p, q \in M_N(\mathbb{A}_\hbar(M) \rtimes \Gamma)$, *for some* $N > 0$, *are idempotents such that* $p - q \in M_N(\mathbb{A}_{\hbar c}(M) \rtimes \Gamma)$. *Then we have*

$$Tr_{N,e}(p-q) = \left\langle \Phi\left(\hat{A}(M)_\Gamma e^{\theta_\Gamma}\right), ch\left(\sigma(p)\right) - ch\left(\sigma(q)\right)\right\rangle,$$

*where* $Tr_{N,e}$ *denotes the composition of the trace defined in definition 6.2.19 with the matrix trace,* $\sigma$ *denotes the map given by setting* $\hbar = 0$ *and finally* $ch$ *denotes the Chern-Connes map defined in definition 6.1.17.*

Given the results of the previous three sections this theorem follows from the formal algebraic index theorem 6.1.12 and a straightforward, but long, calculation that we omit.

CHAPTER 7

# Conclusions and Prospects

The theorem 6.2.23 concluding the previous chapter 6 is objective (1) as stated in section 1.5. Objectives (2) and (3) were "handled" in the chapters 5 and 4 respectively. We should comment on the degree to which the objectives have been met and provide a clear overview of the main results of this thesis. In this chapter we will provide such an overview of the main results. We will also comment on the possibilities of further research offered by these results. We will do these things in a style similar to section 1.5. So, we shall comment on the degree to which the objective has been met and the possibilities for further research this offers counting down from (3) to (1).

## 7.1. Objective (3)

As mentioned in section 1.5, it will be hardest to determine the degree of success for objective (3). Let us start by giving a quick overview of the results obtained in the pursuit of this objective.

In chapter 4 we have presented a deformed version of the framework of formal geometry presented in chapter 2. The objective was to show that Fedosov's construction can be interpreted naturally in this framework. We have also presented this construction in section 4.2. This led us to proposition 4.3.1, it shows that the connections obtained as the deformed analogs of the Grothendieck connections, see proposition 2.3.9 and remark 2.3.11 are Fedosov connections. In section 3.1 we showed that the formal Weyl algebras bundle appearing in the Fedosov construction [**46**] is isomorphic to the deformed bundle of infinity jets, see proposition 3.1.8 and remark 3.1.10. This shows how one can view $(\Omega^\bullet(M;\mathbb{W}_\hbar),\nabla)$, for a Fedosov connection $\nabla$, as a replacement for the Čech cohomology complex on $M$ with values in the (sheaf given by the) deformation quantization, when we consider the cover by formal neighborhoods. In definition 4.3.2 we define the characteristic class of a deformation quantization in terms of the framework of deformed formal geometry and the Gelfand-Fuks maps 4.1.7. By remark 4.3.5, the class lands in the affine space $\frac{\omega}{i\hbar}+\mathrm{H}^2_{\mathrm{dR}}(M)[\![\hbar]\!]$, as expected. This shows that the Fedosov construction is obtained by solving the equations that arise from assuming a certain class is obtained as the characteristic class of a deformation quantization in the sense of 4.3.2.

Now let us consider the further research that may be carried out in the direction of objective (3). First of all, there are many different generalizations of the Fedosov construction and one could investigate whether each of them allows for a corresponding framework of formal geometry. Most specifically one may consider generalizing to the case of general Poisson manifolds, instead of symplectic manifolds. The problem would be that there is, in general, no constant local model for a Poisson manifold. Even if there is such a local model, it may allow for several inequivalent deformation quantizations. Nonetheless, this generalization is done to some extent in the article [**21**]. In that article the authors relate certain $\star$-products on Poisson manifolds to connections in jet bundles and they mention the underlying formal geometry. They do not consider a framework of deformed formal geometry, however. So, one further direction of research would be to try and combine the results of this thesis and [**21**] to obtain an interpretation of that result as we did for the symplectic case here. Another option in the same vein is to consider the case of dg manifolds [**106**]. In this article the authors consider a Fedosov-type construction for dg manifolds, thus one may consider whether there is a corresponding framework of formal deformed dg geometry. Let us also mention the recent article

[**43**], the authors consider deformations of the algebra $A$ that correspond to a Drinfeld twist in the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra acting on $A$. They use a Fedosov construction in order to obtain such Drinfeld twists. If we consider a Lie-Rinehart pair instead, we can associate the corresponding convolution algebra as a replacement for the algebra of infinite jets, see remark 3.1.4. Drinfeld twists will also yield deformations of this convolution algebra, see 3.1.8 and the remark 3.1.11. Thus we may develop a corresponding notion of formal geometry using this, more general, notion of infinity jets.

Finally, one may use a part of the results summarized above to place the Fedosov construction more squarely in the framework of deformation theory of associative algebras of appendix B. Namely, the remark that the formal Weyl algebras bundle appearing in [**48**] is isomorphic to the deformed bundle of infinity jets. In particular, this bundle is independent of the class of the deformation quantization up to isomorphism. In fact, one can already observe in the case of the Fedosov construction that, up to isomorphism, all deformation quantizations are obtained as subalgebras of the same algebra. When we consider the framework of formal geometry in the context of deformation theory of associative algebras, we note the following. We have the quasi-isomorphism of differential graded algebras

$$J^{\infty}\colon\ (C^{\infty}(M),0) \longrightarrow \left(J^{\infty}_{M}(M)\otimes_{C^{\infty}(M)}\Omega^{\bullet}(M),\nabla_G\right), \tag{7.1.1}$$

where $\nabla_G$ denotes the Grothendieck connection, see proposition 2.3.9 and remark 2.3.11. As mentioned in remark B.4.7, this implies that we obtain a bijection between the equivalence classes of the deformations as $A_\infty$-algebras. On the left hand side of (7.1.1) this just means deformation quantizations, since the differential is trivial. We note that the deformation quantizations of $M$ up to gauge equivalence can be divided into classes parametrized by the induced Poisson bracket, see corollary 1.2.5. Let us denote the classes that induce the Poisson bracket induced by the symplectic structure by $D_\omega(M)$. Let us consider now the deformations of the dg algebra on the right hand side of 7.1.1, in the sense of definition B.1.1. If they lie in the image of $J^\infty$, they should be given by $(\mu_{\geq 1},0)$, i.e. the differential remains undeformed. If they even lie in the image of $D_\omega(M)$ under $J^\infty$, then we should be able to normalize the deformed product to obtain the product on the formal Weyl algebras bundle. This will mean that the differential has to change and thus we obtain the Fedosov connection. A prospect of research is to carry out the sketch above in detail and in particular deal with the fact (that we swept under the rug) that the results of [**34**] show that only the deformations as $A_\infty$-algebras are identified by a quasi-isomorphism.

## 7.2. Objective (2)

The treatment of objective (2) is left entirely to chapter 5. In section 5.1 we recall some well-known facts about group actions on deformation quantizations. We arrive at the definition 5.1.7 of an extension of the action by symplectomorphisms, motivated by the result on lifts of symplectomorphisms in propositions 5.1.3 and 5.1.6.

In section 5.2 we consider the question of existence of lifts of a given group action. The main reason is that we will carry out the classification project relative to a given lift. We arrive at the well-known condition that the group action $\Gamma \to \mathrm{Symp}(M,\omega)$ should be contained in the stabilizer subgroup of the characteristic class. Given an action that satisfies this condition, we can associate (non-uniquely) a cochain $c$ of $\Gamma$ with values in the sections of the bundle $\mathcal{G}_1^{\hbar}$ that satisfies the equations

$$c_\gamma \gamma^* \nabla c_\gamma^{-1} = \nabla,$$

for some Fedosov connection $\nabla$ (such that $\operatorname{Ker}\nabla$ is the deformation quantization). At that point the existence of a lift comes down to the cocycle condition for $c$, see corollary 5.2.1. We proceed to consider more restrictive conditions, leading to the known result, see [**7**], rephrased in proposition 5.2.3. To conclude our treatment of the question of existence, we show that there exist examples of group actions that do not satisfy the conditions of proposition 5.2.3, but can lift nonetheless.

Finally we come to the question of classification relative to a given lift of $\Gamma \to \mathrm{Symp}(M,\omega)_{[\theta]}$ to the deformation quantization. In section 5.3.1 we construct a certain abstract (pointed) set $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$ that parametrizes the equivalence classes of lifts of the action satisfying a certain technical condition. In fact, it is given as the first group cohomology with values in the non-Abelian group $\overline{\mathcal{G}_\nabla}(M)$. This solves the problem in the abstract, but it does not immediately allow for easy conclusions about the number of equivalence classes of lifts. To obtain such results we should supply a method of computation of the sets $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$. In particular we would like to have a way to relate it to some more familiar combination of group cohomology and the cohomology of the manifold.

The main obstacle at this point is the fact that we are dealing with non-Abelian cohomology and thus we cannot employ the usual tools of homological algebra. Nonetheless, we obtain truncated long exact sequences from short exact sequences of coefficient groups. Using these, we first prove the existence of a certain commuting diagram consisting of interlocked exact sequences (5.3.4), in section 5.3.2. We obtain these by considering the non-Abelian Čech cohomology, see the end of section A.2.3 in the appendix. We then use this square to obtain vanishing results for $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$, namely proposition 5.3.24, and non-vanishing results, see examples 5.3.28 and 5.3.30. Especially the last example shows how one may apply the diagram (5.3.4) in order to compute $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$.

Let us also consider some prospects for further research concerning group actions on deformation quantizations and classification. First of all, we may consider the restriction imposed by the definition 5.1.7. Namely, we only consider those lifts of the action on the symplectic manifold that preserve a given Fedosov connection. Further research may point out how severe this restriction is in practice. A point of interest is whether there exist actions that do not preserve any Fedosov connection. One way to proceed is to consider the approach used to prove the equivariant version of the algebraic index theorem in section 6.2. In this case we do not require the section $F$ defining the Fedosov connection to be equivariant with respect to the action of the group, instead we construct a system of sections $P_\Gamma^k \to \widetilde{M}_{\Gamma,r}^k$ satisfying certain boundary conditions.

Secondly, we can consider the equivariant characteristic class $\theta_\Gamma$ of the action of $\Gamma$ on $\mathbb{A}_\hbar(M)$ constructed in section 6.2, see definition 6.2.21. The question naturally arises to what degree this class actually classifies the action. In particular when one considers this class in the light of section 5.3. For instance, in the case of example 5.3.30, we have that $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$ contains a copy of $T_\hbar^1(\mathbb{T}^2)$, on the other hand the equivariant class $\theta_\Gamma$ takes values in $\mathrm{H}^2(\mathbb{T}^2\times_{\mathbb{Z}} E\mathbb{Z})[\![\hbar]\!] \simeq \mathbb{C}^3[\![\hbar]\!]$, this follows from the fact that $\mathrm{H}^\bullet(\mathbb{T}^2\times_{\mathbb{Z}} E\mathbb{Z}) \simeq \mathrm{H}^\bullet(\mathbb{T}^2)\otimes \mathrm{H}^\bullet(B\mathbb{Z})$ in this case. The Borel equivariant cohomology comes with the map $\mathrm{H}^2(\mathbb{T}^2\times_{\mathbb{Z}} E\mathbb{Z}) \to \mathrm{H}^2(\mathbb{T}^2)$ induced by the quotient map $\mathbb{T}^2\times E\mathbb{Z} \to \mathbb{T}^2\times_{\mathbb{Z}} E\mathbb{Z}$, which should map $\theta_\Gamma \mapsto \theta$. The kernel $\mathbb{C}^2$ is obtained as $K := \mathrm{H}^1(\mathbb{T}^2)\otimes \mathrm{H}^1(\mathbb{Z};\mathbb{C})$. At this point we are tempted to conclude that $\mathrm{H}^1(\mathbb{Z};\overline{\mathcal{G}_\nabla}) = e^K$. A more conservative statement is that there should be a more rigorous scheme with which to identify $\mathrm{H}^1(\Gamma;\overline{\mathcal{G}_\nabla})$ with some combination of cohomology of the underlying manifold and group cohomology, of course proposition 5.3.24 also supports this.

## 7.3. Objective (1)

Finally we consider the results obtained in the pursuit of the main objective of this thesis: the equivariant algebraic index theorem 6.2.23. In section 6.1 we consider a proof, which appeared in [**14**] and [**15**], of the algebraic index theorem that proceeds by first proving the universal or formal algebraic index theorem 6.1.12 and subsequently applying a Gelfand-Fuks map, see proposition 4.1.7.

The first problem with incorporating a group action is that the group acts by global transformations. So there is no hope of obtaining an equivariant formal algebraic index theorem. This implies that we should try to obtain an equivariant Gelfand-Fuks map. This amounts to finding an equivariant version of the section $F_r$ in lemma 4.1.6. As mentioned, we do not want to assume that an equivariant section $F_r\colon \mathrm{Sp}_M \to \widetilde{M}_{r,\hbar}$ exists a priori. This amounts to fixing special Fedosov connections and actions of the type of definition 5.1.7. Thus we are led to consider the simplicial manifold $\Gamma^\bullet \times M$ whose geometric realization is the Borel construction. We show, in lemma 6.2.4, that one can find equivariant sections $F_r$ in the form of a system of sections for this simplicial manifold satisfying certain boundary conditions. These yield the equivariance of the Gelfand-Fuks maps, given in proposition 6.2.9, in the sense of lemma 6.2.10. The rest of the program is carried out by following the usual steps. We choose to pair the Lie algebra complex with the periodic cyclic chain complex directly to simplify the computations. Now we can use the well-known results [**93, 57, 49**], recalled in section A.3.1 of the appendix, to obtain the equivariant algebraic index theorem 6.2.23.

There is a lot of room for further investigation of the formula in theorem 6.2.23. First of all we should carry out certain interesting examples, like the irrational rotation of one coordinate on the torus, or the irrational rotation of the sphere proposed in examples 5.3.30 and 5.3.28. Another large class of examples is given by the actions on manifolds that do allow for a symplectic quotient and where the invariants $\mathbb{A}_\hbar(M)^\Gamma$ provide a deformation quantization of the quotient. In this case, we expect the formula to reduce to the usual algebraic index theorem 6.1.22. In particular, it is of interest to compute the classes $\hat{A}(M)_\Gamma$ and $\theta_\Gamma$, defined in definition 6.2.21, in these cases.

Secondly, we should compare the results of theorem 6.2.23 with similar results. In particular we should compare with the (formal analog) of the results in [**101**]. A comparison that would be of particular interest is with the results obtained in the papers [**96, 97**], in particular [**97**]. Although the actions are restricted to a smaller class, the formulas in these articles are more explicit and it may lead to a more explicit formulation of theorem 6.2.23. As mentioned above, we should also compare the class $\theta_\Gamma$ from definition 6.2.21 to the results of section 5.3, i.e. the preprint [**69**].

Thirdly, we may consider generalizing or adapting the proof of theorem 6.2.23 in section 6.2 to obtain similar results for certain adaptations of the algebraic index theorem 6.1.22. For instance we may consider the adaptations put forth in section 1.5.3. Namely, we may consider the case of symplectic Lie algebroids [**89**], gerbes [**15**] or general Lie algebroids [**9**] equipped with an action. The case of a complex manifold, see [**89, 14**], should follow particularly straightforwardly from the treatment in section 6.2. Finally, one may consider the case of more general Poisson manifolds, as in [**33**], although this may be more challenging due to the fact that there is no easy classification of the local models. Another option is to consider an action twisted by a group cocycle instead of an untwisted action. One can still form the corresponding crossed product and a straightforward adaptation of the proof in section 6.2 should provide an analog of theorem 6.2.23 for this case. Finally, one may consider the case of a Lie group acting smoothly rather than a discrete group. In this case the model for the Borel equivariant cohomology, put forth in section 6.2, would have to be adapted accordingly.

Another project following naturally from this one is to "integrate" in the sense of obtaining a corresponding analytic index theorem, like in the article [**88**]. In particular, given a (regular) foliation of a smooth manifold, we could consider operators on the foliation algebra, see section 2.8 of [**26**], which can sometimes be given in terms of a crossed product along an action twisted by a cocycle, that are (pseudo)-differential along a transversal. The formal analog would be given exactly by the crossed product twisted by a cocycle of the deformation quantization corresponding to the symbol calculus in the cotangent bundle of the transversal. Thus, the adaptation of theorem 6.2.23 proposed in the previous paragraph would yield an algebraic version. This would allow us to obtain explicit formulas for instances of the transversal index theorem [**29**].

# Appendices

APPENDIX A

# Simplicial and Cyclic Structures and (Co)Homology Theories

In this appendix we will give definitions of the various (co)homology theories used in the main body of the thesis. To name them precisely we use

- Hochschild (co)homology of associative algebras,
- (periodic) cyclic (co)homology of associative algebras,
- group (co)homology,
- de Rham cohomology,
- Čech cohomology (subordinate to a cover),
- singular cohomology and
- cohomology of (topological infinite-dimensional) Lie algebras.

Note that, of these, the singular cohomology is used in a completely rudimentary way. It turns out that, by considering the notion of simplicial and cyclic structure, we can obtain many of these theories by simply considering the right simplicial and cyclic modules. So, we will start this appendix by giving a recollection of simplicial and cyclic structure. We will then provide the definitions of the various complexes above as well as some remarks that will be of use in the main body of the thesis. This appendix is mostly used to fix conventions with regard to the various complexes used to compute the above homologies, however. Section A.3.1 is an exception. In that section we provide a proof of a well-known result [**93, 57, 49**] that is used in section 6.2.

## A.1. Simplicial and Cyclic Structures

The notion of simplicial structures unifies a lot of (co)homology theories under a single header. In fact, if the (co)homology theory is defined by use of a (positive) chain complex, the Dold-Kan correspondence [**116**] implies that there is an underlying simplicial module. Since most of the (co)homology theories mentioned above are in fact defined by use of (positive) chain complexes, it will be useful to present parts of the theory of simplicial modules here, in order to fix conventions.

Simplicial modules are right modules over a certain category $\triangle$. It turns out that one can expand this to a larger category $\triangle C$, called the cyclic category, in a natural way. Modules over this bigger category are called cyclic modules and they also give rise to chain complexes. The cyclic homology theories we use in the main body of this thesis are obtained from certain cyclic modules associated to associative algebras [**24**]. In this section we shall first give a description of these categories and provide a specific presentation of them. This presentation is in essence a convention. So, since the main aim of this appendix is to fix conventions, the presentation is the main result of this section.

**A.1.1. The Simplex and Cyclic Categories.** The following section is based on appendix B.5 of [**79**]. The simplex category $\triangle$ and the cyclic category $\triangle C$ fit into the larger picture of categories of finite sets, which is neatly expressed in terms of the commutative diagram

$$\begin{tikzcd}
 & \mathbb{F} \arrow[rr, hook] & & \mathbf{Fin} \\
\triangle \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \triangle\mathbb{S} \arrow[u] & \mathbb{S} \arrow[l, hook'] \arrow[r] \arrow[ul, hook] & \mathbf{Bij} \arrow[u] \\
 & \triangle C \arrow[u, hook] & C \arrow[l, hook'] \arrow[u, hook] \arrow[ur, hook]
\end{tikzcd}.$$

The arrows in this diagram represent functors. Excluding $\triangle\mathbb{S} \to \mathbb{F}$, all arrows represent inclusions. In the diagram we have denoted the category of finite sets by **Fin**; the category of finite sets and bijections by **Bij**; the skeleton of **Fin** given by only considering the objects $[n] := \{0, 1, \ldots, n\}$ for all $n \in \mathbb{Z}_{\geq 0}$ by $\mathbb{F}$ and the skeleton of **Bij** with objects $[n] := \{0, 1, \ldots, n\}$ for all $n \in \mathbb{Z}_{\geq 0}$ by $\mathbb{S}$. We shall be a bit more precise about the definitions of the other categories, since they define the simplicial and cyclic structures. Note that

$$\mathrm{Hom}_{\mathbb{S}}([n], [m]) = \begin{cases} S_{n+1} & \text{if } n = m \\ \emptyset & \text{otherwise,} \end{cases}$$

where we denote by $S_p$ the symmetric group in $p$-letters.

Definition A.1.1. We define the *simplex* category $\triangle$ as the category with the totally ordered sets $\langle n \rangle := \{0 < 1 < \ldots < n\}$ for all $n \in \mathbb{Z}_{\geq 0}$ as objects and the weakly increasing maps $f\colon \langle n \rangle \to \langle m \rangle$, i.e. such that $i < j$ implies $f(i) \leq f(j)$, as morphisms.

We define the category $C$ as the subcategory of $\mathbb{S}$ that has all objects $[n]$ for $n \in \mathbb{Z}_{\geq 0}$, but only those morphisms corresponding to the cyclic subgroups $C_{n+1} \subset \mathbb{S}_{n+1}$.

The cyclic category $\triangle C$ will be generated as the so-called bi-crossed product of the matched pair $(\triangle, C)$. In order to explain what this means, let us construct the ambient category of non-commutative sets.

Definition A.1.2. We define the category of *non-commutative sets* $\triangle\mathbb{S}$ as the category with objects $[n]$ for all $n \in \mathbb{Z}_{\geq 0}$ and with morphisms $f\colon [n] \to [m]$ given by maps of sets equipped with a total order on the fibers $f^{-1}(\{i\})$ for every element $i \in [m]$.

We should note the definition of the composition of two morphisms $f\colon [n] \to [m]$ and $g\colon [k] \to [n]$ in $\triangle\mathbb{S}$. As a map of sets $f \circ g$ is simply the composition of the maps $f$ and $g$, so all we need to specify is how we supply the fibers with a total order. Suppose $i \in [m]$, then, if $a, b \in g^{-1}(f^{-1}(\{i\}))$, we set $a < b$, if $g(a) < g(b)$ in the ordering on $f^{-1}(\{i\})$ determined by $f$, or, if $g(a) = g(b)$ and $a < b$, in the ordering on $g^{-1}(g(\{a\})$ determined by $g$.

Note that there is a canonical surjective functor $\triangle\mathbb{S} \to \mathbb{F}$ given by simply forgetting the total order on the fibers. Note also that both $\mathbb{S}$ and $\triangle$ embed into $\triangle\mathbb{S}$ in the obvious way, however neither embedding is full.

Proposition A.1.3. *Any morphism* $f\colon [n] \to [m]$ *in* $\triangle\mathbb{S}$ *can be uniquely decomposed as*

$$f = \varphi \circ \sigma$$

*where* $\varphi\colon [n] \to [m]$ *is in* $\triangle$ *and* $\sigma\colon [n] \to [n]$ *is in* $\mathbb{S}$. *Here we have identified the morphisms in* $\triangle$ *and* $\mathbb{S}$ *with their images in* $\triangle\mathbb{S}$.

Proof.
Suppose we are given a total order on $[n]$, let us denote it $(n)$, e.g. $(5) = \{3 < 2 < 4 < 0 < 1 < 5\}$. We obtain an element $\sigma_{(n)} \in \mathbb{S}_{n+1}$ defined by the requirement that

$$\sigma_{(n)}\colon (n) \longrightarrow \langle n \rangle$$

is order preserving. Here we have identified $\mathbb{S}_{n+1} = \mathrm{Aut}_{\mathbb{S}}([n])$ by shifting 0 to 1, 1 to 2 and so on until we shift $n$ to $n+1$ and then considering the canonical action of $\mathbb{S}_{n+1}$ on $\{1,\ldots,n\}$. In the example given above we would have $\sigma_{(5)} = (5)(421)(30)$ in "cycle" notation. Note that this defines a bijection

$$\sigma\colon \{\text{total orderings of} \quad [n]\} \longrightarrow \mathbb{S}_{n+1}.$$

Suppose $f\colon [n] \to [m] \in \triangle\mathbb{S}$, then we find an induced total order $(n)_f$ given by setting $i < j$ if $f(i) < f(j)$ in $\langle m\rangle$ or if $f(i) = f(j)$ and $i < j$ in $f^{-1}(f(i))$ for all $i, j \in [n]$. Note that $(n)_{\sigma^{-1}_{(n)}} = (n)$. Note also that

$$\langle n\rangle \xrightarrow{\sigma_{(n)_f}} (n)_f \xrightarrow{f} \langle m\rangle$$

is weakly increasing, by definition of $\sigma_{(n)_f}$ and $(n)_f$. So, we find the decomposition

$$f = \varphi \circ \sigma,$$

with $\varphi \in \triangle$ and $\sigma \in \mathbb{S}$ for all $f \in \triangle\mathbb{S}$, by $\varphi = f \circ \sigma_{(n)_f}$ and $\sigma = \sigma^{-1}_{(n)_f}$. This proves the existence of the decomposition.

Now suppose $\psi, \varphi \in \triangle$ and $\sigma, \tau \in \mathbb{S}$ such that

$$\psi \circ \tau = \varphi \circ \sigma$$

in $\triangle\mathbb{S}$. Then $\varphi\circ\sigma\circ\tau^{-1}$ is weakly increasing and so $\varphi(\sigma\tau^{-1}(i)) < \varphi(\sigma\tau^{-1}(j))$ implies that both $i < j$ and $\sigma\tau^{-1}(i) < \sigma\tau^{-1}(j)$ while $\varphi(\sigma\tau^{-1}(i)) = \varphi(\sigma\tau^{-1}(j))$ implies that $i < j$ if and only if $\sigma\tau^{-1}(i) < \sigma\tau^{-1}(j)$ by definition of the ordering on the fibers of compositions and the fact that $\varphi$ is weakly increasing. So we find that $i \leq j$ if and only if $\sigma\tau^{-1}(i) \leq \sigma\tau^{-1}(j)$. Since $\sigma\tau^{-1}$ is a permutation, we find that $\sigma\tau^{-1} = \mathrm{Id}_{[n]}$, i.e. $\sigma = \tau$. Finally $\psi = \varphi \circ \sigma\tau^{-1} = \varphi$ shows uniqueness of the decomposition. □

Definition A.1.4. For $\sigma \in \mathbb{S}_{m+1}$ and $n \in \mathbb{Z}_{\geq 0}$ let

$$\sigma_*\colon \mathrm{Hom}_{\triangle}([n],[m]) \longrightarrow \mathrm{Hom}_{\triangle}([n],[m])$$

be the map that sends $\varphi \in \mathrm{Hom}([n],[m])$ to the unique $\sigma_*(\varphi) \in \mathrm{Hom}([n],[m])$ in the decomposition of $\sigma \circ \varphi$ given by proposition A.1.3.

For every $\varphi \in \mathrm{Hom}_{\triangle}([n],[m])$ let

$$\varphi^*\colon \mathbb{S}_{m+1} \longrightarrow \mathbb{S}_{n+1}$$

be the map that sends $\sigma \in \mathbb{S}_{m+1}$ to the unique $\varphi^*(\sigma) \in \mathbb{S}_{n+1}$ in the decomposition of $\sigma \circ \varphi$ given by proposition A.1.3.

A convenient summary of this definition is given by the defining identity

$$\sigma \circ \varphi = \sigma_*(\varphi) \circ \varphi^*(\sigma).$$

The cyclic category $\triangle C$ is obtained by considering not all morphisms in $\triangle\mathbb{S}$, but only those morphisms that decompose in terms of a cyclic permutation and a weakly increasing map. In order for this to be well-defined, we need the following proposition.

Proposition A.1.5. *We have $\varphi^*(C_{m+1}) \subset C_{n+1}$ for any $\varphi \in Hom_{\triangle}([n],[m])$.*

Proof.
Note that

$$\varphi^*(\sigma) = \sigma^{-1}_{(n)_{\sigma\circ\varphi}}$$

in the notation of the proof of proposition A.1.3. Now, if $t \in C_{m+1}$, we see that, since $\varphi$ is weakly increasing, $t_{(n)_{t\circ\varphi}}$ is in $C_{n+1}$, but then its inverse is also in $C_{n+1}$. □

Definition A.1.6. The cyclic category $\triangle C$ is the subcategory of $\triangle\mathbb{S}$ with all objects and those morphisms $f$ that decompose according to proposition A.1.3 as $f = \varphi \circ c$ with $c$ in a cyclic subgroup.

Although it is nice to have a good definition of the simplicial and cyclic categories, it is usually more useful to consider a certain specific presentation of them by generators and relations. It is easily deduced and well-known [**82**] that the simplex category $\triangle$ allows for the presentation given by generators $\delta_i^n \in \mathrm{Hom}_\triangle([n-1],[n])$ and $\sigma_i^n \in \mathrm{Hom}_\triangle([n+1],[n])$, for $0 \leq i \leq n$, and relations

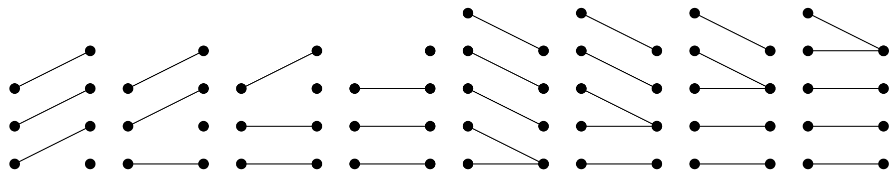
$$\delta_j^n \delta_i^{n-1} = \delta_i^n \delta_{j-1}^{n-1} \quad \text{if} \quad i < j$$

$$\sigma_j^n \sigma_i^{n+1} = \sigma_i^n \sigma_{j+1}^{n+1} \quad \text{if} \quad i \leq j$$

and

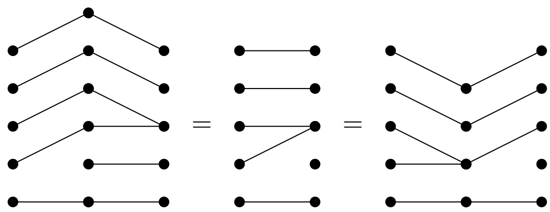
$$\sigma_j^n \delta_i^{n+1} = \begin{cases} \delta_i^n \sigma_{j-1}^{n-1} & \text{if } i < j \\ \mathrm{Id}_{[n]} & \text{if } i = j, j+1 \\ \delta_{i-1}^n \sigma_j^{n-1} & \text{if } i > j+1. \end{cases}$$

Here the $\delta_i^n$ correspond to the maps given by $\delta_i^n(j) = j$ if $j < i$ and $\delta_i^n(j) = j+1$ if $j \geq i$ and the $\sigma_i^n$ correspond to the maps given by $\sigma_i^n(j) = j$ if $j \leq i$ and $\sigma_i^n(j) = j-1$ if $j > i$. It is sometimes useful to consider the maps in $\triangle$ in terms of "dots and bars" diagrams as used in the proof of corollary 3.1.6. In these terms the generators for $n = 3$ are



$\delta_0^3$ $\delta_1^3$ $\delta_2^3$ $\delta_3^3$ $\sigma_0^3$ $\sigma_1^3$ $\sigma_2^3$ $\sigma_3^3$ .

The relation $\sigma_2^4 \delta_1^5 = \delta_1^4 \sigma_1^3$ looks like



in these terms.

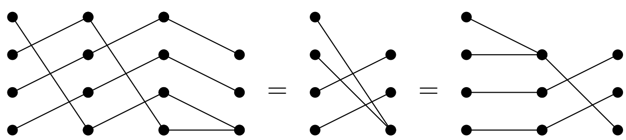
From the presentation of the simplex category $\triangle$ given above, we obtain the presentation of the cyclic category $\triangle C$, by adding the generator $t_n$ of the cyclic group of order $n+1$ for every $n \in \mathbb{Z}_{\geq 0}$. In other words we add the generator $t_n \in \mathrm{Hom}_{\triangle C}([n],[n])$ for all $n \geq 0$ and we add the relations

$$t_n \delta_i^n = \delta_{i+1}^n t_{n-1} \quad \text{if} \quad i < n,$$

$$t_n \sigma_i^n = \sigma_{i+1}^n t_{n+1} \quad \text{if} \quad i < n,$$

$$t_n \sigma_n^n = \sigma_0^n t_{n+1}^2, \quad t_n \delta_n^n = \delta_0^n \quad \text{and} \quad t_n^{n+1} = \mathrm{Id}_{[n]}.$$

Here $t_n$ is the map given by $t_n(j) = j+1$ if $j < n$ and $t_n(n) = 0$. Pictorially we have



$\sigma_0^2 t_3^2$ $t_2 \sigma_2^2$ .

REMARK A.1.7. Note that we could also have provided the generator $t_n^{-1}$ instead of $t_n$. This would have led to slightly different relations and conventions about the $B$ operator, see equation (A.1.1). The reader should be careful to realize which presentation is used when considering such expressions as (A.1.1) for the cyclic operators, which is (partly) why we have included this appendix. For instance in [**93**] the generator $t_n^{-1}$ is used instead of $t_n$. Note that the equation (A.1.1) for $B$ as a formal sum of morphisms in $\triangle C$ is what matters in the end, the only thing that changes when one uses a different presentation is the formula.

**A.1.2. Simplicial and Cyclic Homologies.** As mentioned, modules over the simplex and cyclic categories give rise to chain complexes. Let us fix our conventions and notations of these chain complexes here.

DEFINITION A.1.8. For a commutative ring $R$ we define simplicial/cyclic $R$-modules as contravariant functors from the category $\triangle/\triangle C$ to the category of $R$-modules. Alternatively we can consider the $\mathbb{Z}$-graded $R$-algebras $R[\triangle]$ or $R[\triangle C]$ generated over $R$ by the generators and relations of section A.1.1. In these terms a simplicial/cyclic $R$-module is given by a $\mathbb{Z}$-graded right module over $R[\triangle]/R[\triangle C]$.

DEFINITION A.1.9. A morphism of simplicial/cyclic $R$-modules $\varphi\colon M^\natural \to N^\natural$ is given by a natural transformation of functors or equivalently by a (degree 0) morphism of $R[\triangle]/R[\triangle C]$-modules.

REMARK A.1.10. Note that the definition A.1.8 means explicitly that a simplicial or cyclic $R$-module $M^\natural$ is given by a sequence $M^\natural([n])$ of $R$-modules for each $n \in \mathbb{Z}_{\geq} 0$ and $R$-linear operators

$$f^\natural\colon M^\natural([m]) \longrightarrow M^\natural([n]),$$

for each $f \in \mathrm{Hom}([n],[m])$, such that for $f \in \mathrm{Hom}([n],[m])$ and $g \in \mathrm{Hom}([m],[k])$ we have

$$f^\natural \circ g^\natural = (g \circ f)^\natural.$$

In order to provide such a structure we actually only need to provide the $R$-modules $M([n])$ and the operators given by the generators from section A.1.1 above such that they satisfy the relations given in section A.1.1. When we do this we will always drop the superscript $\natural$ on the operators induced by the generators of section A.1.1. Note that, since we are considering *right* modules, i.e. *contravariant* functors, the operators associated to the generators of section A.1.1 should satisfy the relations of section A.1.1 in reverse order.

DEFINITION A.1.11. Given two cyclic $R$-modules $M^\natural$ and $N^\natural$, we define the product $M\natural N$ as the cyclic $R$-module given by $M\natural N([n]) = M^\natural([n]) \otimes_R N^\natural([n])$ with the diagonal cyclic structure, i.e.

$$f^\natural := f^\natural_M \otimes f^\natural_N\colon M\natural N([m]) \longrightarrow M\natural N([n])$$

for all $f \in \mathrm{Hom}([m],[n])$.

Given a cyclic $R$-module $M^\natural$, we can consider four different complexes associated to the simplicial/cyclic structure. To define them we shall first define two operators: $b$ and $B$.

The first is induced through the Dold-Kan correspondence and uses only the simplicial structure. It is given by

$$b_n = \sum_{i=0}^{n} (-1)^i \delta_i^n\colon M^\natural([n]) \longrightarrow M^\natural([n-1]).$$

By using the simplicial identities above, it is easily verified that $b_{n-1}b_n = 0$. To define the "Hochschild" complex it is enough to have just the operators $b_n$.

To define the three cyclic complexes we shall use the operator

$$B_n = (t_{n+1}^{-1} + (-1)^n) \circ \sigma_n^n \circ \left( \sum_{i=0}^{n} (-1)^{in} t_n^i \right)\colon M^\natural([n]) \longrightarrow M^\natural([n+1]). \tag{A.1.1}$$

Note that $B_{n+1}B_n = 0$, since

$$\sum_{i=0}^{n+1}(-1)^{i(n+1)}t_{n+1}^i \circ (t_{n+1}^{-1} + (-1)^n) = \sum_{i=0}^{n+1}(-1)^{i(n+1)}(t_{n+1}^{i-1} + (-1)^n t_{n+1}^i) = 0. \tag{A.1.2}$$

Vanishing of the expression (A.1.2) follows since the sum telescopes except for the first term $t_{n+1}^{-1}$ and the last term $-(-1)^{n(n+1)}t_{n+1}^{n+1}$, which also cancel each other. Note also that

$$b_{n+1}B_n + B_{n-1}b_n = 0, \tag{A.1.3}$$

this can be seen by writing out both operators as sums of operators in the normal form $\delta_k^n\sigma_l^{n-1}t_n^i$.

From now on we will drop the subscripts of the $b$ and $B$ operators. The cyclic module $M^\natural$ gives rise to a graded module $\{M_n^\natural\}_{n\in\mathbb{Z}_{\geq 0}}$ by $M_n^\natural = M^\natural([n])$. Then we see that the operator $b$ turns $M^\natural$ into a chain complex.

Definition A.1.12. The *Hochschild complex* $(C_\bullet^{Hoch}(M^\natural), b)$ of the cyclic module $M^\natural$ is defined as $C_n^{Hoch}(M^\natural) := M^\natural([n])$ equipped with the boundary operator $b$ (of degree $-1$). The corresponding homology shall be denoted $\mathrm{HH}_\bullet(M^\natural)$.

Note that we have not used the full cyclic structure of $M^\natural$ to construct the Hochschild complex. In fact one can form the Hochschild complex $\left(C_\bullet^{Hoch}\left(M^\triangle\right), b\right)$ of any simplicial $R$-module $M^\triangle$ in exactly the same way.

Note that, by equalities (A.1.3), (A.1.2) and the fact that $b^2 = 0$, we find that $(b+B)^2 = 0$. This implies that we could consider a certain double complex with columns given by the Hochschild complex. Note, however, that, if $b$ is of degree $-1$ on the Hochschild complex, the operator $B$ is naturally of degree $+1$. We can consider a new grading for which the operator $b+B$ is homogeneous of degree $-1$. In order to make this grading clear, it will be useful to introduce the formal variable $u$ of degree $-2$. This leads us to several choices of double complexes.

Definition A.1.13. We define the *cyclic* complex by

$$\left(CC_\bullet(M^\natural), \delta^\natural\right) := \left(C_\bullet^{Hoch}(M^\natural)[u^{-1}, u\rrbracket \middle/ C_\bullet^{Hoch}(M^\natural)\llbracket u\rrbracket, b + uB\right),$$

the *negative cyclic* complex by

$$\left(CC_\bullet^-(M^\natural), \delta_-^\natural\right) := \left(C_\bullet^{Hoch}(M^\natural)\llbracket u\rrbracket, b + uB\right)$$

and finally the *periodic cyclic* complex by

$$\left(CC_\bullet^{per}(M^\natural), \delta_{per}^\natural\right) := \left(C_\bullet^{Hoch}(M^\natural)[u^{-1}, u\rrbracket, b + uB\right).$$

Here $u$ denotes a formal variable of degree $-2$. The corresponding homologies will be denoted $\mathrm{HC}_\bullet(M^\natural)$, $\mathrm{HC}_\bullet^-(M^\natural)$ and $\mathrm{HC}_\bullet^{per}(M^\natural)$ respectively. The cyclic cochain complexes, denoted $CC_{per}^\bullet(M^\natural)$, $CC_-^\bullet(M^\natural)$ and $CC^\bullet(M^\natural)$, are defined as the $R$-duals of the chain complexes.

Notation A.1.14. We shall often omit the superscripts $\natural$ when there can be no confusion as to what the cyclic structures are.

Remark A.1.15. Note that every "flavour" of cyclic homology comes equipped with spectral sequences induced from the fact that they are realized as totalizations of a double complex. The double complex corresponding to cyclic homology is bounded (second octant) and therefore the spectral sequence which starts by taking homology on columns converges to $\mathrm{HC}_\bullet$. The negative (or periodic) cyclic double complex is unbounded, but concentrated in the (second,) third, fourth and fifth octant. This means that the spectral sequence starting with taking homology in the columns converges again to $\mathrm{HC}_\bullet^-$ (or $\mathrm{HC}_\bullet^{per}$). Note, however, that in this case the negative (or periodic) cyclic homology is given by the product totalization.

The remark A.1.15 provides the proof of the following well-known proposition.

PROPOSITION A.1.16. *Suppose $M^\natural$ and $N^\natural$ are two cyclic $R$-modules and $\varphi\colon N^\natural \longrightarrow M^\natural$ is a map of cyclic modules that induces an isomorphism on Hochschild homologies. Then $\varphi$ induces an isomorphism on cyclic, negative cyclic and periodic cyclic homologies as well.*

PROOF.
The proof follows since $\varphi$ induces isomorphisms on the first pages of the relevant spectral sequences, which converge. □

REMARK A.1.17. When considering the Hochschild or cyclic complexes of associative algebras we will usually use more convenient complexes than the ones defined above, see section A.2.1. In fact they come from the general process of normalization. By normalization we mean that there is a certain quotient of the Hochschild complex of any cyclic $R$-module $M$ that is quasi-isomorphic [**61**]. This quotient is obtained by modding out the "degeneracies", i.e. the submodules given by the union of the images of the $\sigma_i^n$'s.

## A.2. Definitions of (Co)Homology Theories

In this section we will finally provide the definitions of the homology theories used in this thesis. We will also provide some remarks about properties that are used in the main body of the thesis. The general scheme for defining these homology theories is by defining the relevant simplicial or cyclic module and then considering the corresponding "Hochschild" or cyclic homologies. Let us fix the commutative ring $k$ for the rest of this chapter.

**A.2.1. (Co)Homology of Associative Algebras.** We shall start by giving the definitions of the Hochschild and various cyclic complexes associated to associative $k$-algebras. To do this, we shall first define the cyclic $k$-module associated to a unital associative $k$-algebra.

DEFINITION A.2.1. Given a unital associative $k$-algebra $A$, we shall define the cyclic $k$-module $A^\natural$ as follows. We let $A^\natural([n]) := A^{\otimes n+1}$ and the operators corresponding to the generators of section A.1.1 are given by

$$\begin{aligned} \delta_i^n(a_0\otimes\ldots\otimes a_n) &= a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n && \text{if}\quad 0\le i<n\\ \delta_n^n(a_0\otimes\ldots\otimes a_n) &= a_na_0\otimes a_1\otimes\ldots\otimes a_{n-1}\\ \sigma_i^n(a_0\otimes\ldots\otimes a_n) &= a_0\otimes\ldots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\ldots\otimes a_n && \text{for all}\quad 0\le i\le n\\ t_n(a_0\otimes\ldots\otimes a_n) &= a_1\otimes\ldots\otimes a_n\otimes a_0. \end{aligned}$$

So, going through the definitions of the Hochschild and cyclic complexes given in section A.1.2, we find that

$$b(a_0\otimes\ldots\otimes a_n) = (-1)^n a_na_0\otimes a_1\otimes\ldots\otimes a_{n-1} + \sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n, \qquad \text{(A.2.1)}$$

while $B(a_0\otimes\ldots\otimes a_n)$ is given by the expression

$$\sum_{i=0}^{n}(-1)^{in}\left(1\otimes a_i\otimes\ldots\otimes a_n\otimes a_0\otimes\ldots\otimes a_{i-1} + (-1)^n a_i\otimes\ldots\otimes a_n\otimes a_0\otimes\ldots\otimes a_{i-1}\otimes 1\right),$$

for all $a_0,\ldots,a_n\in A$. Now note that modding out the degeneracies, as noted in the remark A.1.17, we obtain instead the map

$$B\colon A\otimes\overline{A}^{\otimes n} \longrightarrow A\otimes\overline{A}^{\otimes n+1}$$

given by

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{in} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}, \tag{A.2.2}$$

for $a_0 \in A$ and $a_1, \ldots, a_n \in \overline{A} := A/k$. So we arrive at the following definitions.

Definition A.2.2. We define the Hochschild and cyclic complexes of a unital associative algebra $A$ as follows. For $p \in \mathbb{Z}_{\geq 0}$ set

$$C_p^{Hoch}(A) := A \otimes \overline{A}^{\otimes p},$$

where we have denoted the module quotient

$$\overline{A} := A/k$$

by $\overline{A}$ and the tensor products are over $k$. We let

$$b\colon C_p^{Hoch}(A) \longrightarrow C_{p-1}^{Hoch}(A)$$

be the Hochschild boundary given by $k$-linear extension of formula (A.2.1) for all $p \in \mathbb{Z}_{\geq 0}$. Furthermore we set

$$CC_\bullet^{per}(A) := C_*(A)[u^{-1}, u\rrbracket,$$
$$CC_\bullet^{-}(A) := C_*(A)\llbracket u \rrbracket$$

and

$$CC_\bullet(A) := C_*(A)[u^{-1}, u\rrbracket \Big/ uC_*(A)\llbracket u \rrbracket,$$

where the $\bullet$ degree is given by the $*$ degree minus 2 times the power of $u$, i.e. $u$ is a formal variable of degree $-2$. Finally we let

$$uB\colon CC_p^{per}(A) \longrightarrow CC_{p+1}^{per}(A)$$

be the operator given by composition of the $k$-linear extension of formula (A.2.2) with the operation of multiplication by $u$, and similar for $CC_p^-(A)$ and $CC_p(A)$.

The Hochschild chain complex of the unital associative algebra $A$ is given by $(C_\bullet^{Hoch}(A), b)$, the periodic cyclic chain complex is given by $(CC_\bullet^{per}(A), b + uB)$, the negative cyclic chain complex is given by $(CC_\bullet^-(A), b + uB)$ and, finally, the cyclic chain complex is given by $(CC_\bullet(A), b + uB)$. We denote the corresponding homologies by $\mathrm{HH}_\bullet(A)$, $\mathrm{HC}_\bullet^{per}(A)$, $\mathrm{HC}_\bullet^-(A)$ and $\mathrm{HC}_\bullet(A)$ respectively.

Remark A.2.3. Suppose $A$ is a *flat* $k$-algebra, then it is well-known [**116**] that the Hochschild complex represents $A \otimes_{A^e}^{\mathbb{L}} A$ in the derived category of $k$-modules. Here

$$A^e = A \otimes A^{op}$$

denotes the enveloping algebra of $A$ and so the Hochschild complex represents the left derived tensor product of $A$ with itself over the enveloping algebra. Thus we find that

$$\mathrm{HH}_\bullet(A) \simeq \mathrm{Tor}_\bullet^{A^e}(A, A).$$

This follows from the fact that the Hochschild complex is obtained by tensoring the usual bar complex [**116**] with $A$ and the bar complex is an acyclic projective resolution if $A$ is flat over $k$.

Definition A.2.4. Suppose $A$ is a non-unital $k$-algebra. Then we define the unitalization $A^+$ of $A$ as the minimal unital algebra containing $A$, explicitly it is given by

$$A^+ = A \oplus k,$$

as a $k$-module, with the product

$$(a + \lambda)(b + \mu) = ab + \lambda b + \mu a + \lambda\mu.$$

Definition A.2.5. Suppose $A$ is a non-unital $k$-algebra. Then we define the Hochschild complex of $A$ as
$$C_\bullet^{Hoch}(A) = \operatorname{Ker}\left(C_\bullet^{Hoch}(A^+) \xrightarrow{p} C_\bullet^{Hoch}(k)\right)$$
where the map $p$ is the one induced by the obvious projection $p\colon A \to k$. The definition of the various cyclic complexes is similar.

Definition A.2.6. The Hochschild cochain complex of $A$ is defined by
$$C^\bullet_{Hoch}(A) = \operatorname{Hom}_k(C_\bullet^{Hoch}(A), k)$$
with the differential given by the transpose $b^*$ of $b$. The definition of the various cyclic complexes is similar.

Remark A.2.7. The definitions above all need to be adjusted slightly to be used in the main body of the thesis, since we are working mostly with infinite dimensional topological algebras. The definitions are adjusted by requiring tensor products to be completed (this is unambiguous for the algebras we consider) and for the cochain complexes we consider the continuous dual.

Remark A.2.8. Our definition of "the" Hochschild complex is actually the definition of the Hochschild complex with coefficients in $A$. One could consider similar complexes with values in other $A$-bimodules $M$, by considering $M \otimes \overline{A}^{\otimes \bullet}$ instead of $A \otimes \overline{A}^{\otimes \bullet}$. Since we will only use the Hochschild complex with coefficients in $A$, we shall not give the definition of these other Hochschild complexes here. For the Hochschild cochain complex this remark is more important, since the definition we give is not the Hochschild complex with coefficients in $A$, but rather the Hochschild complex with coefficients in the dual of $A$.

**A.2.2. (Abelian) Group Homology.** As for the case of associative algebras, we can define the group (co)homology by means of a cyclic module. Note that this means that we have to choose a ring over which we consider modules. This means that we will really be considering group (co)homology with values in Abelian groups only. The group (co)homology with values in non-Abelian groups does play a role in the main body of the thesis. We will define group (co)homology with values in a non-Abelian group in section A.3.2.

Definition A.2.9. Given a group $G$, we define the cyclic $k$-module $G^{k\natural}$ as the cyclic $k$-module given by the modules $G^{k\natural}([n]) := (kG)^{\otimes n+1}$ and the operators
$$\begin{aligned}
\delta_i^n(g_0 \otimes \ldots \otimes g_n) &= g_0 \otimes \ldots \otimes \hat{g}_i \otimes \ldots \otimes g_n && \text{for all} \quad 0 \leq i \leq n \\
\sigma_i^n(g_0 \otimes \ldots \otimes g_n) &= g_0 \otimes \ldots \otimes g_i \otimes g_i \otimes g_{i+1} \otimes \ldots \otimes g_n && \text{for all} \quad 0 \leq i \leq n \\
t_n(g_0 \otimes \ldots \otimes g_n) &= g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes g_0.
\end{aligned}$$

Note that $G$ acts on $G^{k\natural}$ from the right by $g \cdot (g_0 \otimes \ldots \otimes g_n) = g^{-1}g_0 \otimes \ldots \otimes g^{-1}g_n$.

Definition A.2.10. Suppose $(M_\bullet, \partial)$ is a right $kG$-chain complex. Then we define the group homology complex of $G$ with values in $M$ as
$$\left(C_\bullet(G; M), \delta_{(G,M)}\right) := \operatorname{Tot}^\Pi M_\bullet \otimes_{kG} C_\bullet^{Hoch}\left(G^{k\natural}\right),$$
where we consider the tensor product of $kG$-chain complexes with the obvious structure of left $kG$-chain complex on $C_\bullet^{Hoch}(G^{k\natural})$. Note that this means that
$$C_n(G; M) = \prod_{p+q=n} M_p \otimes_{kG} C_q^{Hoch}\left(G^{k\natural}\right)$$
and
$$\delta_{(G,M)} = \partial \otimes \operatorname{Id} + \operatorname{Id} \otimes b,$$
where we use the Koszul sign convention, see notation B.2.1. We denote group homology with values in $F_\bullet$ by $\operatorname{H}_\bullet(G; F)$.

REMARK A.2.11. We will often abbreviate the notation $\delta_{(G,M)}$ to $\delta_G$ or even simply $\delta$, if there can be no cause for confusion. We should point out that we have the option to choose either product or sum totalizations in this definition. In the case that $M_\bullet$ is bounded below this does not make any difference of course. We choose to consider the product totalization here because we will need it when considering the periodic cyclic (unbounded) complex of the crossed product algebra in section A.3.1 below. Although the group cohomology complex can be defined in much the same terms as above, we shall not do it here since we will never (explicitly) use the group cohomology complex in this thesis.

PROPOSITION A.2.12. *Suppose $M$ is a right $kG$-module, then $M \otimes kG$ with the diagonal right action is a free $kG$-module.*

PROOF.
Let us denote the $k$-module underlying $M$ by $F(M)$ and thus by $F(M)\otimes kG$ the free (right) $kG$-module induced by the $k$-module underlying $M$. Consider the map

$$M \otimes kG \longrightarrow F(M) \otimes kG$$

given by $m \otimes g \mapsto mg^{-1} \otimes g$. It is obviously a map of $kG$-modules and allows for an inverse, namely $m \otimes g \mapsto mg \otimes g$. □

PROPOSITION A.2.13. *Suppose $F$ is a free right $kG$-module (we view it as a chain complex concentrated in degree 0 with trivial differential), then there exists a contracting homotopy*

$$H_F \colon C_\bullet(G; F) \longrightarrow C_{\bullet+1}(G; F).$$

*Suppose $(F_\bullet, \partial)$ is a quasi-free right $kG$-chain complex (i.e. $F_n$ is a free $kG$-module for all $n$), then the homotopies $H_{F_n}$ give rise to a quasi-isomorphism*

$$Q_F \colon\ ((F_\bullet)_G, \partial) \xrightarrow{\sim} (C_\bullet(G; F), \delta_{(G,F)}),$$

*where the subscript $G$ denotes taking coinvariants (modding out the $G$-action).*

PROOF.
Note that $F \simeq M \otimes kG$, since it is a free module. So we find that

$$C_p(G; F) = (M \otimes kG) \otimes_{kG} (kG)^{\otimes p+1} \simeq M \otimes (kG)^{\otimes p+1}$$

by the map $m\otimes g\otimes g_0 \otimes \ldots \otimes g_p \otimes m\otimes g \mapsto m \otimes gg_0 \otimes \ldots \otimes gg_p$. Using this normalization, we consider the map $H_F$ given by

$$m \otimes g_0 \otimes \ldots \otimes g_p \mapsto m \otimes e \otimes g_0 \otimes \ldots \otimes g_p$$

and note that indeed

$$\delta_G^{p+1} H_F + H_F \delta_G^p = \mathrm{Id}$$

for all $p > 0$.

Now for the second statement, we find that $F_n \simeq M_n \otimes kG$ for each $n$, since it is quasi-free. For each $n$ we have the homotopy $H_{F_n}$ given by the formula above on $C_\bullet(G; F_n)$. Then we consider the map

$$Q_F \colon (F_p)_G \longrightarrow C_p(G; F)$$

given by

$$Q_F([f]) = f - \delta_G^1 Hf + \sum_{q=1}^{\infty} (-H\partial)^q f - \partial(-H\partial)^{q-1} Hf - \delta_G^{q+1}(-H\partial)^q Hf,$$

where we have dropped the subscript from $H$ and we denote by $[f]$ the class of $f$ in the coinvariants $F_G$. One may check by straightforward computation that $Q_F$ is a well-defined morphism of complexes. Now we note that the double complex defining $C_\bullet(G; F)$ is concentrated in the upper half plane and therefore comes with a spectral sequence with first page given by $\mathrm{H}_p(G; F_q)$, which converges to $\mathrm{H}_{p+q}(G; F)$. Note that, since $F_\bullet$ is quasi-free, we find that $\mathrm{H}_p(G; F_q) = 0$ unless $p = 0$ and $\mathrm{H}_0(G; F_q) = (F_q)_G$. Thus, since $Q_F$ induces on isomorphism on the first page, we find that $Q_F$ is a quasi-isomorphism. □

**A.2.3. The de Rham Theorem.** Given a smooth manifold $X$, we can define three chain complexes immediately and, given any vector bundle (or sheaf), we can define them with coefficients. The three chain complexes are the singular chain complex, the de Rham complex and the Čech complex. We shall use this section to give a definition of the singular chain complex and the de Rham complex without coefficients and a more elaborate definition of the Čech complex. We do this since we shall not really use singular cohomology at all, we give a definition of de Rham cohomology with coefficients in the main body of the thesis (see definition 2.3.5) and we make use of the definition of the Čech complex rather extensively in section 5.3.

Recall that we denote the standard (geometric) $n$-simplex by $\Delta^n$, see notation 6.2.1. Denote the *nth singular set of* $X$ by $\mathrm{Sing}_n(X)$, this is the set of continuous maps

$$\sigma\colon \Delta^n \longrightarrow X.$$

Definition A.2.14. We define the cyclic module $X^{k\natural}$ as the cyclic module given by the modules $X^{k\natural}([n]) = k\mathrm{Sing}_n(X)$ endowed with the operators

$$\begin{aligned} \delta_i^n(\sigma) &= (\epsilon_i^n)^*\sigma && \text{for all} \quad 0 \le i \le n \\ \sigma_i^n(\sigma) &= (a_i^n)^*\sigma && \text{for all} \quad 0 \le i \le n \\ t_n(\sigma) &= (c_n)^*\sigma. \end{aligned}$$

Here $\epsilon_i^n$ is as in definition 6.2.3, $a_i^n\colon \Delta^{n+1} \to \Delta^n$ is given by

$$(t_0, \ldots, t_{n+1}) \mapsto (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1})$$

and finally $c_n\colon \Delta^n \to \Delta^n$ is given by

$$(t_0, \ldots, t_n) \mapsto (t_n, t_0, t_1, \ldots, t_{n-1}).$$

We define the *singular cochain complex of $X$ with values in $k$* by

$$(S^\bullet(X;k), \delta) := \left(C^k_{Hoch}(X^{k\natural}), b^*\right).$$

We denote the singular cohomology of $X$ with values in $k$ by $\mathrm{H}^\bullet(X;k)$.

Now let us also define the de Rham and Čech complexes. Although it is possible to give the definitions in terms of a simplicial structure, it is in these cases not more convenient to do so.

Definition A.2.15. We define the differential $p$-forms on $X$ as

$$\Omega^p(X) := \Gamma(\textstyle\bigwedge^p T^*X),$$

where $\bigwedge^p T^*X$ denotes the vector bundle associated to the frames bundle of $X$ with fiber given by the alternating $p$-linear functions on $\mathbb{R}^{\mathrm{Dim}X}$. Differential forms on $X$ are defined as the direct sum

$$\Omega^\bullet(X) = \bigoplus_{p=0}^{\mathrm{Dim}X} \Omega^p(X).$$

The differential forms carry a graded commutative product involved in the definition of the exterior derivative, i.e. the differential in the de Rham complex.

Definition A.2.16. Suppose $\varphi$ is an alternating $p$-linear function on $\mathbb{R}^n$ and $\psi$ is an alternating $q$-linear function on $\mathbb{R}^n$ then we define the *wedge* product $\varphi \wedge \psi$ as the alternating $p+q$-linear function given by

$$\varphi \wedge \psi(v_1, \ldots, v_{p+q}) := \frac{1}{p!q!} \sum_{\tau \in S_{p+q}} \epsilon(\tau)\varphi(v_{\tau(1)}, \ldots, v_{\tau(p)})\psi(v_{\tau(p+1)}, \ldots, v_{\tau(p+q)}),$$

where $S_{p+q}$ denotes the symmetric group in $p+q$ letters and $\epsilon(\tau)$ denotes the sign of $\tau$.

Note that we can extend the wedge product bilinearly to a graded commutative product on the differential forms on $X$.

Definition A.2.17. The exterior derivative

$$d_{\mathrm{dR}}\colon \Omega^\bullet(X) \longrightarrow \Omega^\bullet(X)$$

is defined to be the unique linear operator satisfying

- $d_{\mathrm{dR}}(\alpha \wedge \eta) = (d_{\mathrm{dR}}\alpha) \wedge \eta + (-1)^p \alpha \wedge d_{\mathrm{dR}}\eta$,
- $d_{\mathrm{dR}}^2 = 0$ and
- $d_{\mathrm{dR}} f(Y) = Y(f)$

for all $\alpha \in \Omega^p(X)$, $\eta \in \Omega^\bullet(X)$, vector fields $Y$ on $X$ and where we consider the usual action of vector fields on functions.

It is well-known [**111**] and straightforward to deduce that $d_{\mathrm{dR}}$ is well-defined.

Definition A.2.18. We define the de Rham complex of $X$ as

$$(\Omega^\bullet(X), d_{\mathrm{dR}})$$

and we shall denote the de Rham cohomology of $X$ by $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$.

Finally let us give a definition of the Čech complex subordinate to a cover $\mathcal{U}$. Suppose that $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of $X$ and $\mathcal{F}$ is a sheaf of $k$-modules on $X$. We denote by $I^p \subset I^{\times p}$ the set of $(i_1, \ldots, i_p)$ such that $U_{i_1} \cap \ldots \cap U_{i_p} \neq \emptyset$.

Definition A.2.19. We define the $k$-module of Čech $p$-cochains subordinate to $\mathcal{U}$ with values in $\mathcal{F}$ as

$$\check{C}^p(\mathcal{U}; \mathcal{F}) := \left\{\sigma := \{\sigma(\iota)\}_{\iota \in I^p} \mid \sigma(i_1, \ldots, i_p) \in \mathcal{F}(U_{i_1} \cap \ldots \cap U_{i_p})\right\}.$$

We define $\delta^p_j \colon \check{C}^p(\mathcal{U}; \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}; \mathcal{F})$ by

$$(\delta^p_j \sigma)(i_0, \ldots, i_p) = \sigma(i_0, \ldots, \hat{i}_j, \ldots, i_p),$$

for all $0 \leq j \leq p$, where the hat denotes omission and, on the right hand side, we consider the restriction to the intersection $U_{i_0} \cap \ldots \cap U_{i_p}$. The Čech cochain complex subordinate to $\mathcal{U}$ with values in $\mathcal{F}$ is now given by

$$\left(\check{C}^\bullet(\mathcal{U}; \mathcal{F}), \delta\right)$$

where, on $p$-cochains, $\delta = \sum_{j=0}^p (-1)^j \delta^p_j$. We shall denote the Čech cohomology subordinate to $\mathcal{U}$ with values in $\mathcal{F}$ by $\check{\mathrm{H}}^\bullet(\mathcal{U}; \mathcal{F})$.

Note that for all sheaves $\mathcal{F}$ and all covers $\mathcal{U}$ we have $\check{\mathrm{H}}^0(\mathcal{U}; \mathcal{F}) = \mathcal{F}(X)$. Although in the main body of the thesis it is sufficient to consider Čech cohomology subordinate to a cover, we include the following definition for completeness.

Definition A.2.20. The Čech cohomology with values in $\mathcal{F}$ is defined as the direct limit

$$\check{\mathrm{H}}^\bullet(X; \mathcal{F}) := \varinjlim \check{\mathrm{H}}^\bullet(\mathcal{U}; \mathcal{F}).$$

The maps in the directed system are induced by refinements of covers. Explicitly, refinements are given by a cover $\mathcal{V} = \{V_i\}_{i\in J}$ and a map $f\colon J \to I$ such that $V_j \subset U_{f(j)}$.

Remark A.2.21. A good cover $\mathcal{U}$ of $X$ is a cover such that all intersections $U_{i_1} \cap \ldots \cap U_{i_p}$ for all $(i_1, \ldots, i_p) \in I^p$ are diffeomorphic to $\mathbb{R}^{\mathrm{Dim} X}$. It is a well-known fact that all smooth manifolds allow for a good cover [**12**]. In fact, one can show that every cover has a good refinement [**12**]. This means that in order to compute Čech cohomology on a manifold $X$ one actually only needs to consider good covers. For two opens $V \subset U \subset X$ we will denote the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ by $\rho^U_V$. A partition of unity subordinate to a locally finite cover $\mathcal{V} = \{V_i\}_{i\in J}$ of $X$ is given by maps $\varphi_i\colon \mathcal{F}(V_i) \to \mathcal{F}(X)$ for all $i \in J$ such that

$$\sum_{i\in J} \varphi_i \circ \rho^X_{V_i} = \mathrm{Id}_{\mathcal{F}(X)}.$$

A sheaf is called *fine* if every open subset $U \subset X$ allows for a partition of unity subordinate to any locally finite open cover of $U$. It is also well-known [**13**] that the Čech cohomology of a fine sheaf subordinate to a locally finite cover is acyclic, this is shown by constructing a specific homotopy.

Let us also record, without proof, the very useful result called Leray's theorem.

Theorem A.2.22 (Leray). *If $k > 0$ and $\mathcal{U} = \{U_i\}_{i\in I}$ is a cover of $X$ such that*

$$\check{\mathrm{H}}^k(U_{i_1} \cap \ldots \cap U_{i_p}; \mathcal{F}) = 0$$

*for all $p > 0$ and all $(i_1, \ldots, i_p) \in I^p$, then the natural map*

$$\check{\mathrm{H}}^k(\mathcal{U}; \mathcal{F}) \longrightarrow \check{\mathrm{H}}^k(X; \mathcal{F})$$

*is an isomorphism.*

The proof goes by constructing a certain resolution of $\mathcal{F}$ and applying a spectral sequence argument, see [**13**]. From now on we denote the constant sheaf $U \mapsto k$ by $k$.

Proposition A.2.23. *There are isomorphisms*

$$\check{\mathrm{H}}^\bullet(X; k) \longrightarrow H^\bullet(X; k).$$

A sketch of the proof of the proposition can be found in [**13**]. Thus we arrive at the following version of de Rham's theorem

Theorem A.2.24 (de Rham). *We have isomorphisms*

$$H^\bullet_{dR}(X) \simeq \check{\mathrm{H}}^\bullet(X; \mathbb{R}) \simeq H^\bullet(X; \mathbb{R}).$$

A proof can be found in [**12**]. It follows from the previous proposition A.2.23 and a spectral sequence argument using the double complex given by considering the Čech complex with values in the sheaf $\Omega^\bullet$.

Finally, let us consider non-Abelian Čech cohomology. There is a deep theory of non-Abelian cohomology theories, see [**59, 39**]. In this thesis we will only consider two and only in the most naive sense. We will present the first, non-Abelian Čech cohomology, here and the second, non-Abelian group cohomology, in section A.3.2. The term non-Abelian refers to the coefficients, while these are usually Abelian groups ($\mathbb{Z}$-modules), in non-Abelian cohomology one considers coefficients in non-Abelian groups. So let $\mathcal{G}$ be a sheaf of not necessarily Abelian groups on $X$. Note that, if we would follow the definition A.2.19, we would no longer have $\delta^2 = 0$. Thus we cannot define the cohomology as a quotient $\mathrm{Ker}\,\delta/\mathrm{Im}\,\delta$ in the usual way. One could try to obtain a definition with $\delta^2 = 0$ by reordering the $\delta^p_i$ in the definition of $\delta$, but it turns out that no good reordering would be possible. Instead, we manually define the 0th and 1st cohomologies as follows.

Definition A.2.25. We define

$$\check{\mathrm{H}}^0(\mathcal{U}; \mathcal{G}) := \left\{\sigma \in \check{C}^0(\mathcal{U}; \mathcal{G}) \mid \sigma(i)\sigma(j)^{-1} = e \in \mathcal{G}(U_i \cap U_j) \;\; \forall\, (i,j) \in I^2\right\}.$$

Here $\check{C}^0(\mathcal{U}; \mathcal{G})$ is as in A.2.19 and $e$ denotes the neutral element.

Note that this is actually the usual definition. The set $\check{\mathrm{H}}^0(\mathcal{U}; \mathcal{G})$ inherits a group structure from $\mathcal{G}$ as is the case for Abelian Čech cohomology. In essence, no problems arise from the failure of $\mathcal{G}$ to be Abelian for the 0th cohomology.

Definition A.2.26. We define

$$\check{Z}^1(\mathcal{U};\mathcal{G}) := \left\{\sigma \in \check{C}^1(\mathcal{U};\mathcal{G}) \mid \sigma(j,k)\sigma(i,k)^{-1}\sigma(i,j) = e \in \mathcal{G}(U_i \cap U_j \cap U_k) \ \ \forall\ (i,j,k) \in I^3\right\}.$$

Here $\check{C}^1(\mathcal{U};\mathcal{G})$ is as in definition A.2.19 and $e$ denotes the neutral element. We define

$$\check{\mathrm{H}}^1(\mathcal{U};\mathcal{G}) := \check{Z}^1(\mathcal{U};\mathcal{G})/\sim,$$

where $\sim$ denote the equivalence relation given by $\sigma \sim \tau$ if there exists $a \in \check{C}^0(\mathcal{U};\mathcal{G})$ such that

$$a(i)\sigma(i,j)a(j)^{-1} = \tau(i,j)$$

for all $(i,j) \in I^2$.

Note that this definition does differ (formally) from the previous one, since we need to take the order into account when we consider the equivalence relation $\sim$. In the Abelian case we recover the definition of $\check{\mathrm{H}}^1$, given in definition A.2.19 above. Note that, while $\check{\mathrm{H}}^1(\mathcal{U};\mathcal{G})$ is always a well-defined pointed set (the point is $e(i,j) = e$ for all $(i,j) \in I^2$), it is not a group in general.

**A.2.4. Lie algebra Cohomology.** Finally let us consider the Lie algebra cohomology. If $\mathfrak{g}$ is a finite dimensional Lie algebra, then we can consider its universal enveloping algebra $U(\mathfrak{g})$. It is an associative algebra containing the vector space $\mathfrak{g}$ such that the bracket on $\mathfrak{g}$ coincides with the commutator bracket. Then we can simply consider the Hochschild cohomology of $U(\mathfrak{g})$ with values in some module as the definition of Lie algebra cohomology. Let us not give this definition, since we did not actually define the Hochschild cohomology with values in a general module and we do not use this complex for Lie algebra cohomology in the main body of the thesis. Instead, we use a different complex which is defined as follows.

Definition A.2.27. Suppose $\mathfrak{g}$ is a Lie algebra over the field $L$ and $M$ is a $\mathfrak{g}$-module. Then, for $p \geq 0$, we define

$$C^p_{Lie}(\mathfrak{g};M) := \mathrm{Hom}_L\left(\textstyle\bigwedge^p \mathfrak{g}, M\right)$$

and we define $\partial_{Lie} \colon C^p_{Lie}(\mathfrak{g};M) \to C^{p+1}_{Lie}(\mathfrak{g};M)$ by

$$\begin{aligned}(\partial_{Lie}\chi)(X_0,\ldots,X_p) &:= \sum_{i=0}^{p}(-1)^i X_i\chi(X_0,\ldots,\hat{X}_i,\ldots,X_p)\\ &+ \sum_{0\leq i<j\leq p}(-1)^{i+j}\chi([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p)\end{aligned}$$

for all $\chi \in C^p_{Lie}(\mathfrak{g};M)$, $X_0,\ldots,X_p \in \mathfrak{g}$ and where the hat signifies omission. By direct computation we see that $\partial^2_{Lie} = 0$. The Lie algebra complex with values in $M$ is defined as

$$(C^\bullet_{Lie}(\mathfrak{g};M), \partial_{Lie})$$

and we denote the cohomology by $\mathrm{H}^\bullet_{Lie}(\mathfrak{g};M)$.

Remark A.2.28. Note that, if $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, then the Lie algebra complex with values in $\mathbb{R}$ is very similar to the de Rham complex of $G$. Writing out an explicit definition of $d_{\mathrm{dR}}$ will yield exactly the formula for $\partial_{Lie}$ given in definition A.2.27 above. In fact, proposition A.2.29 below will solidify this similarity. Note that the wedge product defined in definition A.2.16 above also defines a product on the Lie algebra complex.

Proposition A.2.29. *Suppose $\mathfrak{g}$ is the Lie algebra of the compact connected Lie group $G$, then*

$$H^\bullet_{Lie}(\mathfrak{g};\mathbb{R}) \simeq H^\bullet_{dR}(G).$$

A proof may be found in, for instance, [**22**]. It is easy to see that the Lie algebra cohomology will be the same as the cohomology of the subcomplex of $G$-invariant differential forms, the harder part of the proof is to show that the de Rham complex contracts onto the complex of $G$-invariant forms. Proposition A.2.29 leads to the more general proposition A.2.31. It describes the de Rham cohomology of a homogeneous space $G/H$, for $G$ a compact connected Lie group and $H$ a connected closed subgroup, in terms of the Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ of $G$ *relative* to the Lie algebra $\mathfrak{h}$ of $H$.

Definition A.2.30. Suppose $\mathfrak{h} \subset \mathfrak{g}$ is an inclusion of Lie algebras over the field $L$ and $M$ is a $\mathfrak{g}$-module, then, for $p \geq 0$, we define

$$C^p_{Lie}(\mathfrak{g}, \mathfrak{h}; M) := \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^p (\mathfrak{g}/\mathfrak{h}), M\right).$$

We define the Lie algebra complex with values in $M$ relative to $\mathfrak{h}$ as

$$(C^\bullet_{Lie}(\mathfrak{g}, \mathfrak{h}; M), \partial_{Lie}),$$

where $\partial_{Lie}$ is as in definition A.2.27. We denote the relative Lie algebra cohomology with values in $M$ by $\mathrm{H}^\bullet_{Lie}(\mathfrak{g}, \mathfrak{h}; M)$.

Proposition A.2.31. *Suppose $H \subset G$ is a connected closed subgroup of a compact connected Lie group $G$, then we have*

$$H^\bullet_{dR}(M) \simeq H^\bullet_{Lie}(\mathfrak{g}, \mathfrak{h}; \mathbb{R}),$$

*where $\mathfrak{h} \subset \mathfrak{g}$ are the Lie algebras of $H \subset G$ and $M = G/H$.*

Note that if $H$ is the trivial subgroup we recover proposition A.2.29.

Remark A.2.32. Proposition A.2.31 motivates the definition of the Gelfand-Fuks maps 2.3.9 (and also definition 4.1.7). Namely, just as $G$ is an $H$-principal bundle over $M$ in proposition A.2.31, the manifold of non-linear frames $\widetilde{M}$ is a $\mathrm{GL}(n, \mathbb{R})$ principal bundle over $\widetilde{M}/\mathrm{GL}(n, \mathbb{R})$, where $n = \mathrm{Dim}\, M$ and this last quotient is homotopic (at least as far as the de Rham theory is concerned) to $M$. We do see, however, that, since $\widetilde{M}$ is not a Lie group, the Gelfand-Fuks maps are not quasi-isomorphisms.

Remark A.2.33. As was the case for associative algebras (see remark A.2.7), the definitions A.2.27 and A.2.30 above are not quite the Gelfand-Fuks cohomology mentioned in the main body of the thesis. This is (again) since we consider mostly infinite dimensional topological Lie algebras. So, the Gelfand-Fuks cohomology is actually given by the above definitions, but where we consider completed tensor products and continuous homomorphisms.

## A.3. Replacements for Certain Complexes

In the previous section A.2 we gave definitions of complexes underlying the (co)homology theories used in the main body of this thesis. In this section we shall provide replacements for a few of these complexes. Namely, we shall provide a replacement for the cyclic complexes associated to the crossed product algebra in section A.3.1 and we shall provide a replacement for the group homology complex in section A.3.2. We shall also give the definition of non-Abelian group cohomology in this last section, since it is defined by analogy with the replacement given in that section. We use this definition of non-Abelian group cohomology extensively in section 5.3.

**A.3.1. Cyclic Homology of Crossed Products.** In section 6.2 we consider the equivariant cyclic cohomology of $\mathbb{A}_\hbar(M)$ with respect to the group $\Gamma$. In order to prove theorem 6.2.23 we need the pairing given in definition 6.2.16. This pairing is defined by observing that one can express the homogeneous summand of the periodic cyclic complex of the crossed product algebra in terms of group homology with values in the periodic cyclic complex of the underlying algebra. In fact, this is part of a more general theory of periodic cyclic (co)homology of crossed products, see [**93, 49, 57**]. We will now present the various definitions and propositions from this theory that we use in the main body of the thesis. For this section we fix the unital associative $k$-algebra $A$, for some field $k$. We also fix a group $G$ acting on $A$ from the left.

DEFINITION A.3.1. We define the *crossed product* $A \rtimes G$ of $A$ and $G$ as the algebra given, as a vector space, by

$$A \rtimes G := A \otimes_k kG$$

equipped with the product given by $k$-linear extension of

$$(a\delta_g)(b\delta_h) = ag(b)\delta_{gh},$$

where we have introduced the notation $a\delta_g := a \otimes g$.

It is easily verified that $A \rtimes G$ is a well-defined associative algebra. We should note that, in the case $A = C^\infty(M)$, the algebra $A \rtimes G$ is the convolution algebra of the action Lie groupoid $M \rtimes G$. Again for the case that $A$ is the algebra of functions on a space $X$, the crossed product is, in many cases, a better behaved non-commutative analog of the functions on the quotient $X/G$ [**26**].

Now let us consider the cyclic module $(A \rtimes G)^\natural$ associated to the crossed product algebra in definition A.2.1. Note that the cyclic structure of $(A \rtimes G)^\natural$ splits over the conjugacy classes of $G$. Namely, given a tensor $a_0\delta_{g_0} \otimes a_1\delta_{g_1} \otimes \ldots \otimes a_n\delta_{g_n}$, the conjugacy class of the product $g_0 \cdot \ldots \cdot g_n$ is invariant under $\delta_i^n$, $\sigma_i^n$ and $t_n$ for all $i$ and $n$.

DEFINITION A.3.2. Denote by $\langle G \rangle$ the set of conjugacy classes of $G$. Then we define the part of $(A \rtimes G)^\natural$ supported at $x \in \langle G \rangle$, denoted $(A \rtimes G)^\natural_x$, as the span of all tensors $a_0\delta_{g_0} \otimes \ldots \otimes a_n\delta_{g_n}$ such that $g_0 \cdot \ldots \cdot g_n \in x$. We shall call $(A \rtimes G)^\natural_e$, where $e$ is the (conjugacy class of) the neutral element, the homogeneous summand (as in [**93**]).

Note that we have the splitting

$$(A \rtimes G)^\natural = \bigoplus_{x \in \langle G \rangle} (A \rtimes G)^\natural_x$$

as a cyclic module.

What follows is a proof of the well-known fact [**93, 49, 57**] that one can express the cyclic homology of the homogeneous summand of the crossed product in terms of group homology of the cyclic homology of the underlying algebra. We present an adaptation of the proof in [**93**].

NOTATION A.3.3. We shall use the specialized notation $A\natural G := A^\natural \natural G^{k\natural}$.

Note that $A\natural G$ carries a right $G$-action given by the diagonal action (the left action on $A$ is converted to a right action by inversion, i.e. $G \simeq G^{op}$). Thus the quotient, also called module of coinvariants,

$$(A\natural G)_G = A\natural G \Big/ \langle a - g(a) \rangle$$

is another cyclic $k$-module.

PROPOSITION A.3.4. *The homogeneous summand of $(A \rtimes G)^\natural$ is isomorphic to the coinvariants of $A\natural G$ as a cyclic module.*

$$(A \rtimes G)^\natural_e \xrightarrow{\sim} (A\natural G)_G.$$

PROOF.
Consider the map given by

$$a_0\delta_{g_0} \otimes \ldots \otimes a_n\delta_{g_n} \mapsto (g_0^{-1}(a_0) \otimes a_1 \otimes g_1(a_2) \otimes \ldots \otimes g_1 \ldots g_{n-1}(a_n))\natural(e \otimes g_1 \otimes g_1g_2 \otimes \ldots \otimes g_1 \cdot \ldots \cdot g_n),$$

it is easily checked to commute with the cyclic structure and allows the inverse given by

$$(a_0 \otimes \ldots \otimes a_n)\natural(g_0 \otimes \ldots \otimes g_n) \mapsto g_n^{-1}(a_0)\delta_{g_n^{-1}g_0} \otimes g_0^{-1}(a_1)\delta_{g_0^{-1}g_1} \otimes \ldots \otimes g_{n-1}^{-1}(a_n)\delta_{g_{n-1}^{-1}g_n}.$$

This last tensor can also be expressed as $\delta_{g_n^{-1}}a_0\delta_{g_0} \otimes \delta_{g_0^{-1}}a_1\delta_{g_1} \otimes \ldots \otimes \delta_{g_{n-1}^{-1}}a_n\delta_{g_n}$. □

As a $kG$-module, we see that $A\natural G([n]) = A^\natural([n]) \otimes G^{k\natural}([n]) = B([n]) \otimes kG$ with the diagonal action, where $B([n]) = A^{\otimes n+1} \otimes kG^{\otimes n}$. So, by proposition A.2.12, we find that the Hochschild and various cyclic chain complexes corresponding to $A\natural G$ are quasi-free. Thus, we can construct the quasi-isomorphisms from proposition A.2.13 for each chain complex associated to the cyclic module $A\natural G$. So, we find four quasi-isomorphisms which we shall denote $Q^{Hoch}$, $Q$, $Q^-$ and $Q^{per}$ corresponding to the Hochschild, cyclic, negative cyclic and periodic cyclic complexes respectively.

Proposition A.3.5. *The map*

$$A\natural G \longrightarrow A^\natural,$$

*given by*

$$(a_0 \otimes \ldots \otimes a_n)\natural(g_0 \otimes \ldots \otimes g_n) \mapsto a_0 \otimes \ldots \otimes a_n,$$

*induces a quasi-isomorphism on all associated complexes.*

Proof.
Note that, by proposition A.1.16, it is sufficient to prove the statement for the Hochschild complexes. Let us denote by $F(G)$ the standard free resolution of $G$ [**17**], note that

$$F(G) = (C^{Hoch}_\bullet(G^{k\natural}), b).$$

The map given above is obtained by first applying the Alexander-Whitney map, see chapter 8 of [**80**],

$$C^{Hoch}_n(A^\natural) \otimes C^{Hoch}_n(G^{k\natural}) \longrightarrow \bigoplus_{p+q=n} C^{Hoch}_p(A^\natural) \otimes C^{Hoch}_q(G^{k\natural}),$$

which yields a quasi-isomorphism

$$C^{Hoch}_\bullet(A\natural G) \xrightarrow{\sim} C^{Hoch}_\bullet(A^\natural) \otimes C^{Hoch}_\bullet(G^{k\natural}),$$

where we consider the tensor product of chain complexes on the right-hand side. Then one simply takes the cap product with the generator in $H^*(F(G)^*) \simeq k$, which is also a quasi-isomorphism. So we find that the map is a quasi-isomorphism for the Hochschild complexes. □

Note that the map given in proposition A.3.5 is also $G$-equivariant and therefore it induces a map

$$C_\bullet(G, A\natural G) \longrightarrow C_\bullet(G, A^\natural),$$

which is a quasi-isomorphism when we consider the group homology complex with values in the various complexes associated to $A^\natural$.

Theorem A.3.6. *The composite maps from the Hochschild and various cyclic complexes associated to $(A \rtimes G)^\natural_e$ to the group homology with values in the various Hochschild and cyclic complexes associated to $A^\natural$ implied by propositions A.3.4 and A.3.5 are quasi-isomorphisms, i.e. there are quasi-isomorphisms*

$$\left(C^{Hoch}_\bullet\left((A \rtimes G)^\natural_e\right), b\right) \xrightarrow{\sim} C_\bullet(G; C^{Hoch}_\bullet(A))$$

$$\left(CC_\bullet\left((A \rtimes G)^\natural_e\right), \delta^\natural\right) \xrightarrow{\sim} C_\bullet(G; CC_\bullet(A))$$

$$\left(CC^-_\bullet\left((A \rtimes G)^\natural_e\right), \delta^\natural_-\right) \xrightarrow{\sim} C_\bullet(G; CC^-_\bullet(A))$$

*and*

$$\left(CC^{per}_\bullet\left((A \rtimes G)^\natural_e\right), \delta^\natural_{per}\right) \xrightarrow{\sim} C_\bullet(G; CC^{per}_\bullet(A)).$$

Remark A.3.7. Note that, since the cyclic and Hochschild complexes are bounded below, the product totalization in our definition of group homology agrees with the (usual) direct sum totalizations. In the periodic cyclic and negative cyclic cases they do not agree in general.

**A.3.2. Group Homology and Non-Abelian Cohomology.** In this section we will first introduce a more convenient complex with which to compute group homology, see definition A.2.10. Secondly we will define group cohomology in the non-Abelian case. This non-Abelian group cohomology is used extensively in section 5.3. In section A.2.2 definition A.2.10 we defined a complex representing group homology. In most applications it is more useful to consider a certain *isomorphic* complex however. In fact in the main body of the thesis we do use this isomorphic complex, see section 6.2. One obtains the new complex from the old complex by explicitly modding out the group action across the tensor product. For this section we consider again the group $G$ and the field $k$. The following is based on [**17**].

Definition A.3.8. Suppose $(M_\bullet, \partial)$ is a right $kG$-chain complex, then we set

$$\tilde{C}_n(G;M) := \prod_{p+q=n} M_q \otimes (kG)^{\otimes p}.$$

We define the operators $\delta_i^p \colon M_\bullet \otimes (kG)^{\otimes p} \to M_\bullet \otimes (kG)^{\otimes p-1}$ by

$$\delta_0^p(m \otimes g_1 \otimes \ldots \otimes g_p) := g_1(m) \otimes g_2 \otimes \ldots \otimes g_p$$

$$\delta_i^p(m \otimes g_1 \otimes \ldots \otimes g_p) := m \otimes g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_p$$

for all $0 < i < p$ and finally

$$\delta_p^p(m \otimes g_1 \otimes \ldots \otimes g_p) := m \otimes g_1 \otimes \ldots \otimes g_{p-1}.$$

We define $\left(\tilde{C}_\bullet(G;M), \tilde{\delta}_{(G,M)}\right)$ to be the chain complex given by

$$\tilde{\delta}_{(G,M)} = \partial \otimes \mathrm{Id} + \mathrm{Id} \otimes \delta_G$$

where $\delta_G^p = \sum_{i=0}^p (-1)^i \delta_i^p$ and we use the Koszul sign convention again, see notation B.2.1.

Proposition A.3.9. *There is an isomorphism of chain complexes*

$$C_\bullet(G;M) \longrightarrow \tilde{C}_\bullet(G;M).$$

Proof.
Consider the map

$$C_n(G;M) \longrightarrow \tilde{C}_n(G;M),$$

given by

$$m \otimes g_0 \otimes \ldots \otimes g_p \mapsto g_0(m) \otimes g_0^{-1} g_1 \otimes g_1^{-1} g_2 \otimes \ldots \otimes g_{p-1}^{-1} g_p.$$

Note that it commutes with the differentials and allows for the inverse given by

$$m \otimes g_1 \otimes \ldots \otimes g_p \mapsto m \otimes e \otimes g_1 \otimes g_1 g_2 \otimes \ldots \otimes g_1 \cdot \ldots \cdot g_p.$$

□

We will usually use this chain complex when dealing with group homology and thus we will drop the tilde in the main body of this thesis.

In section 5.3 we consider group cohomology with values in a certain non-Abelian group. As was the case for non-Abelian Čech cohomology, see section A.2.3, and for the same reasons, we shall only define the 0th and 1st non-Abelian group cohomologies. As mentioned there is a broad and deep theory of non-Abelian cohomology (especially in the case of group cohomology), however, since we don't have need for the full theory, we shall stick to the most naive considerations. For this section we shall fix the action of a group $G$ on a (non-Abelian) group $H$.

Definition A.3.10. We define the 0th group cohomology of $G$ with values in $H$ as

$$\mathrm{H}^0(G;H) := \left\{ h \in H \mid hg(h^{-1}) = e \in H \ \ \forall\, g \in G \right\},$$

where $e$ denotes the neutral element.

Note that this simply means that $\mathrm{H}^0(G;H) = H^G$, the invariants of $H$. Thus, as was the case for Čech cohomology, the fact that $H$ is not Abelian has no effect on the 0th cohomology. In particular $\mathrm{H}^0(G;H)$ is a subgroup of $H$.

Definition A.3.11. We define

$$Z^1(G;H) := \left\{\varphi\colon G \to H \mid \varphi(g_1)g_1(\varphi(g_2))\varphi(g_1g_2)^{-1} = e \in H \ \ \forall\, g_1, g_2 \in G\right\},$$

where $e$ denotes the neutral element. We define

$$\mathrm{H}^1(G;H) = Z^1(G;H)/{\sim}$$

where $\sim$ denotes the equivalence relation given by $\varphi \sim \psi$ if there exists $h \in H$ such that

$$h\varphi(g)g(h^{-1}) = \psi(g)$$

for all $g \in G$.

Again we find that this time the definition is different from the Abelian case. In particular we can no longer be assured that $\mathrm{H}^1(G;H)$ is a group. It is still a pointed set, however. The point is given by $e\colon G \to H$ given by $e(g) = e$ for all $g \in G$. One verifies, as they do in [**58**], that this means we still get truncated exact sequences (of pointed sets)

$$1 \to H_1^G \longrightarrow H_2^G \longrightarrow H_3^G \longrightarrow \mathrm{H}^1(G;H_1) \longrightarrow \mathrm{H}^1(G;H_2) \longrightarrow \mathrm{H}^1(G;H_3)$$

from $G$-equivariant exact sequences of coefficient groups

$$1 \to H_1 \longrightarrow H_2 \longrightarrow H_3 \to 1$$

in the usual way. We show in 5.3.23 that, in case $H_1 \to H_2$ is a central inclusion, we can extend the induced truncated exact sequence to include $\mathrm{H}^2(G;H_1)$, where this last *group* is defined using the complex in definition A.3.8.

APPENDIX B

# Deformation Theory of Associative Algebras

The theory of formal deformation quantization is an example of the deformation theory of associative algebras. Thus it will be helpful to develop some of this theory here. In this appendix we will recall some parts of the deformation theory of associative algebras. The field of deformation theory of algebras (of any type) is far richer than what we will present here, however. For instance, we will only consider $R$-deformations of associative $L$-algebras, where $L$ is a field of characteristic 0 and $R = L[\hbar]/\langle \hbar^k \rangle$ is a $k$-truncation or $R = L[\![\hbar]\!]$ is the ring of formal power series. More loosely speaking, we will only consider deformations into the $k$-th or $\infty$-jet (formal) neighborhood of (differential graded) associative algebras over a field of characteristic 0, see section 2. For a more in-depth discussion of the deformation theory of algebras see the quintessential papers [**54**] or, for a more succinct introduction, [**35**]. In this appendix we will consider the deformation theory of (differential graded) associative algebras in a completely abstract setting. In the section 1.2 we explain the relation to formal deformation quantization. The material in this appendix is based mostly on [**35**] and [**79**].

## B.1. Deformation of DGA Algebras

In this section we will give the most straightforward definition of a deformation of a differential graded associative algebra (sometimes shortened to dga or dg algebra). As mentioned, we will only give the definition of deformations over the local Artinian or local complete rings of $k$-truncations $L[\hbar]/\langle \hbar^k \rangle$ or formal power series $L[\![\hbar]\!]$, which will be denoted by $R$.

Definition B.1.1. An $R$-deformation of the dg $L$-algebra $(A, \mu_0, d_0)$ is given by a dg $R$-algebra $(B, \mu, d)$ equipped with an isomorphism

$$B \simeq A \otimes_L R$$

of $R$-modules and such that there exist $R$-module maps

$$\mu_{\geq 1} \colon B \otimes B \longrightarrow B$$

and

$$d_{\geq 1} \colon B \longrightarrow B,$$

of degree 0 and +1 respectively, such that, under the identification $B \simeq A \otimes R$, we have

$$\mu = \mu_0 + \hbar \mu_{\geq 1} \quad \text{ and } \quad d = d_0 + \hbar d_{\geq 1}.$$

Remark B.1.2. Suppose $(B, \mu, d)$ is an $R$-deformation of $(A, \mu_0, d_0)$, then $\mu$ and $d$ are $R$-module maps and so they are determined by their action on $A \otimes 1 \hookrightarrow A \otimes R$. This means that we can expand

$$\mu = \mu_0 + \hbar \mu_1 + \hbar^2 \mu_2 + \dots \quad \text{ and } \quad d = d_0 + \hbar d_1 + \hbar^2 d_2 + \dots$$

where

$$\mu_i \colon A \otimes A \longrightarrow A \quad \text{ and } \quad d_i \colon A \longrightarrow A$$

are $L$-linear maps of degree 0 and +1 respectively.

The remark B.1.2 means that we could alternatively view an $R$-deformation of the dg algebra $(A, \mu_0, d_0)$ as the sequences of linear maps $\{\mu_i\}$ and $\{d_i\}$ (where $0 < i < k$ or $i \in \mathbb{N}$) together with a list of equations which require $\mu$ to be associative, $d^2 = 0$ and $d$ to be a graded derivation. The first few (relatively short) equations would read

$$
\begin{aligned}
a\mu_1(b,c) + \mu_1(a,bc) &= \mu_1(ab,c) + \mu_1(a,b)c \\
a\mu_2(b,c) + \mu_1(a,\mu_1(b,c)) + \mu_2(a,bc) &= \mu_2(ab,c) + \mu_1(\mu_1(a,b),c) + \mu_2(a,b)c \\
d_1(ab) + d_0\mu_1(a,b) &= (d_1a)b + \mu_1(d_0a,b) + (-1)^{|a|}\left(ad_1b + \mu_1(a,d_0b)\right) \\
d_0d_1a + d_1d_0a &= 0 \\
d_0d_2a + d_1^2a + d_2d_0a &= 0
\end{aligned}
\tag{B.1.1}
$$

for all homogeneous $a \in A$ of degree $|a|$ and all $b, c \in A$. Here we have found it convenient to denote $\mu_0$ by concatenation of elements, i.e. $\mu_0(a,b) =: ab$. In general

$$
\begin{aligned}
\sum_{l=0}^{p} \mu_l(a,\mu_{p-l}(b,c)) &= \sum_{l=0}^{p} \mu_l(\mu_{p-l}(a,b),c) \\
\sum_{l=0}^{p} d_l\mu_{p-l}(a,b) &= \sum_{l=0}^{p} \mu_{p-l}(d_la,b) + (-1)^{|a|}\mu_{p-l}(a,d_lb) \\
\sum_{l=0}^{p} d_ld_{p-l}a &= 0
\end{aligned}
\tag{B.1.2}
$$

for $a$, $b$ and $c$ as above and all $p$ in the range dictated by the choice of $R$.

Definition B.1.3. Suppose $(B,\mu,d)$ and $(B',\mu',d')$ are $R$-deformations of $(A,\mu_0,d_0)$. We will say $B$ and $B'$ are *gauge equivalent* if there exists an $R$-linear algebra isomorphism $\varphi\colon B \longrightarrow B'$ such that the induced map

$$\varphi_0\colon A \simeq B/(\hbar B) \longrightarrow B'/(\hbar B') \simeq A$$

coincides with the identity. We will call such $\varphi$ *gauge equivalences*. Note that gauge equivalence is an equivalence relation on the set of $R$-deformations of a dg algebra $A$.

In order to parametrize and handle $R$-deformations of a given dg algebra, it will be useful, first of all, to write equations like (B.1.2) without reference to any elements of $A$. Usually one considers deformation problems (like this one) by considering a suitable differential graded Lie algebra [**79**] (sometimes abbreviated to dgl algebra). Writing the equations without reference to elements will also bring us one step closer to determining such a differential graded Lie algebra. In this case, a convenient description of this (differential) graded Lie algebra is given in terms of coderivations of a certain coalgebra.

## B.2. Coderivation

In this section we shall show that one can describe the dga algebra structures on a given graded vector space $V$ in terms of coderivations that they induce on the *tensor coalgebra* associated to $V$. This will eventually lead us to the dgl algebra that governs the deformation problem we consider.

Notation B.2.1. Vertical bars will denote the degree of homogeneous elements (or maps). Let us also establish that we will apply the *Koszul sign convention* from now on. Thus, given homogeneous linear maps $\varphi\colon V \to V'$ and $\psi\colon W \to W'$ of graded vector spaces, we shall denote the graded tensor product, given by

$$\varphi \otimes \psi(v \otimes w) = (-1)^{|\psi|\cdot|v|}\varphi(v) \otimes \psi(w)$$

for homogeneous $v \in V$ and $w \in W$, by $\varphi \otimes \psi$. Given a graded vector space $V$, we shall denote the $k$-shift of $V$, given by $V[k]^p = V^{p+k}$, by $V[k]$. We shall denote the degree $+1$ suspension isomorphism by $\uparrow\colon V \to V[-1]$ and the degree $-1$ desuspension isomorphism by $\downarrow\colon V \to V[1]$. Note that $\downarrow\uparrow = \mathrm{Id}_V = \uparrow\downarrow$.

In terms of the notation B.2.1, the equations (B.1.2) are equivalent to

$$\begin{aligned}\sum_{l=0}^{p}\mu_l\circ(\mu_{p-l}\otimes\mathrm{Id}) &= \sum_{l=0}^{p}\mu_l\circ(\mathrm{Id}\otimes\mu_{p-l})\\ \sum_{l=0}^{p}d_l\circ\mu_{p-l} &= \sum_{l=0}^{p}\mu_{p-l}\circ(d_l\otimes\mathrm{Id}+\mathrm{Id}\otimes d_l)\\ \sum_{l=0}^{p}d_l\circ d_{p-l} &= 0.\end{aligned}\tag{B.2.1}$$

Recall that the equations above are given by writing out the equations

$$\mu\circ(\mu\otimes\mathrm{Id})=\mu\circ(\mathrm{Id}\otimes\mu),\qquad d\circ\mu=\mu\circ(d\otimes\mathrm{Id}+\mathrm{Id}\otimes d)\quad\text{and}\quad d\circ d=0$$

in terms of the $\hbar$ grading.

We will describe below how one can associate a coderivation on the tensor coalgebra to every linear map from tensor powers of a (graded) vector space to this vector space. Doing this to the map $m=\mu+d$ will show that the equations above coincide exactly with the vanishing of the graded commutator of this map with itself.

DEFINITION B.2.2. Given a graded vector space $V$, we denote the *tensor coalgebra* by $(TC(V),\Delta)$. Here

$$TC(V)=\bigoplus_{k\geq 1}V^{\otimes k}$$

with the induced grading as a graded vector space, while the coproduct $\Delta$ is given by

$$\Delta(v_1\otimes\ldots\otimes v_k)=\sum_{i=1}^{k-1}(v_1\otimes\ldots\otimes v_i)\otimes(v_{i+1}\otimes\ldots\otimes v_k)\in TC(V)\otimes TC(V)$$

for all $v_i\in V$. We will say $v\in TC(V)$ is of *weight* $n$, if $v\in V^{\otimes n}$, to avoid confusion with the degree induced from the grading of $V$.

Note that $\Delta(v_1\otimes\ldots\otimes v_k)$ can be described as a sum over all ways to partition $\{1,\ldots,k\}$ into two non-empty sets $I_1$ and $I_2$ such that $a\in I_1$ and $b\in I_2$ implies $a<b$. Then it is easily seen that both $(\Delta\otimes\mathrm{Id})\Delta(v_1\otimes\ldots\otimes v_k)$ and $(\mathrm{Id}\otimes\Delta)\Delta(v_1\otimes\ldots\otimes v_k)$ can be described as a sum over all ways to partition $\{1,\ldots,k\}$ into three non-empty sets $I_1$, $I_2$ and $I_3$ such that $a\in I_\alpha$, $b\in I_\beta$ and $\alpha<\beta$ implies $a<b$. So, $\Delta$ is indeed coassociative, justifying the name tensor *coalgebra*. The grading on $V$ induces a grading on $TC(V)$, which makes it a graded coalgebra, since $\Delta$ is of degree 0.

NOTATION B.2.3. For a coproduct $\Delta$ we denote $\Delta^{(1)}:=\Delta$ and we denote

$$\Delta^{(n)}=(\Delta\otimes\mathrm{Id}^{\otimes n-1})\Delta^{(n-1)}$$

recursively for $n\in\mathbb{N}$. Let us also set $\Delta^{(0)}=\mathrm{Id}$.

REMARK B.2.4. The tensor coalgebra $TC(V)$ has the universal property of being a cofree (locally) conilpotent graded coalgebra cogenerated by $V$. Note in particular the *conilpotent* in this description (it is often omitted, which can lead to some confusion). A coalgebra $C$ is said to be (locally) conilpotent if for all $c\in C$ there exists $n\in\mathbb{N}$ such that $\Delta^{(n)}c=0$. So, if $C$ is a coassociative (locally) conilpotent coalgebra and $\varphi\colon C\to V$ is a linear map, then there exists a unique coalgebra map $\tilde\varphi\colon C\to TC(V)$ such that $p\circ\tilde\varphi=\varphi$. Here $p\colon TC(V)\to V$ denotes the projection onto the lowest weight.

Definition B.2.5. A degree $k$ coderivation $\tau$ on the coassociative graded coalgebra $(C, \Delta)$ is a degree $k$ map $\tau\colon C \longrightarrow C$ such that

$$\Delta \circ \tau = (\tau \otimes \mathrm{Id} + \mathrm{Id} \otimes \tau)\, \Delta.$$

We shall denote the space of degree $k$ coderivations of $C$ by $\mathrm{Coder}^k(C)$. If $m \in \mathrm{Coder}^1(C)$ is a coderivation and a differential, i.e. $m \circ m = 0$, we call the triple $(C, \Delta, m)$ a *differential graded coassociative coalgebra* (also abbreviated to dg or dga coalgebra).

Remark B.2.6. Note that, if $\tau$ is a coderivation on the coalgebra $(C, \Delta)$, then it can be verified, by induction on $n \in \mathbb{N}$, that

$$\Delta^{(n)} \circ \tau = \sum_{i=0}^{n} \left(\mathrm{Id}^{\otimes i} \otimes \tau \otimes \mathrm{Id}^{\otimes n-i}\right) \circ \Delta^{(n)}.$$

Proposition B.2.7. *Suppose $V$ is a graded vector space and $p_V : TC(V) \longrightarrow V$ is the projection onto the lowest weight. The map*

$$p_*\colon \mathit{Coder}^k(TC(V)) \longrightarrow \mathit{Hom}^k(TC(V), V),$$

*given by $\tau \mapsto p \circ \tau$, is a linear isomorphism. Here $\mathit{Hom}^k(TC(V), V)$ simply denotes the space of degree $k$ linear maps $TC(V) \to V$.*

Proof.
Note first that the space of degree $k$ coderivations is a linear subspace of the vector space $\mathrm{Hom}^k(TC(V), TC(V))$ and that the map $p_*$ is linear and well-defined, since $p$ is of degree 0.

Let us first prove that $p_*$ is injective by showing that one may express any coderivation in terms of its image under $p_*$. Note first that

$$\Delta^{(n)}\colon V^{\otimes n+1} \longrightarrow V^{\otimes n+1}$$

for all $n > 0$, where on the left hand side $V^{\otimes n+1} \hookrightarrow TC(V)$, while we have $V^{\otimes n+1} \hookrightarrow TC(V)^{\otimes n+1}$ instead on the right hand side. Nonetheless we see that, as a linear map, this restriction of $\Delta^{(n)}$ coincides with the identity. This is evident when one notes that $\Delta^{(n)}(v_0 \otimes \ldots \otimes v_m)$ can be written as a sum over all the ways to partition $\{0, \ldots, m\}$ into non-empty subsets $I_1, \ldots, I_{n+1}$ such that $a \in I_\alpha$, $b \in I_\beta$ and $\alpha < \beta$ implies $a < b$. Of course there is only one such way to partition $\{0, \ldots, n\}$ (and no such ways if $m < n$). Let us denote the component of a vector in the tensor power $V^{\otimes n+1}$ by a superscript $(n)$. Since the components of $\tau$ that end in $V \hookrightarrow TC(V)$ are given by $p_*(\tau)$, we need only consider $n > 0$. Then we find, by the remark B.2.6, that we have

$$\tau(v_1 \otimes \ldots \otimes v_m)^{(n)} = \left(\sum_{i=0}^{n} \left(\mathrm{Id}^{\otimes i} \otimes \tau \otimes \mathrm{Id}^{\otimes n-i}\right) \circ \Delta^{(n)}(v_1 \otimes \ldots \otimes v_m)\right)^{(n)} \tag{B.2.2}$$

for all $\tau \in \mathrm{Coder}^k(TC(V))$, since

$$\left(\Delta^{(n)}\tau(v_1 \otimes \ldots \otimes v_m)\right)^{(n)} = \Delta^{(n)}\tau(v_1 \otimes \ldots \otimes v_m)^{(n)} = \tau(v_1 \otimes \ldots \otimes v_m)^{(n)}.$$

In particular we have $\tau(v_1 \otimes \ldots \otimes v_m)^{(n)} = 0$ if $m \leq n$. Now suppose $m > n$ and write

$$\Delta^{(n)}(v_1 \otimes \ldots \otimes v_m) =: \sum v_{(0)} \otimes \ldots \otimes v_{(n)},$$

then the equation (B.2.2) says that

$$\tau(v_1 \otimes \ldots \otimes v_m)^{(n)} = \left(\sum_{i=0}^{n} \sum v_{(0)} \otimes \ldots \otimes \tau(v_{(i)})^{(0)} \otimes \ldots \otimes v_{(n)}\right)^{(n)}$$

(we only see $\tau(v_{(i)})^{(0)}$ because the left hand side lies in $V^{\otimes n+1}$). Of course $\tau(v)^{(0)} = p_*(\tau)(v)$ for all $v \in TC(V)$ and so we see that $p_*$ is injective, since we have shown that $\tau$ can be given in terms of $p_*(\tau)$.

Finally, let us prove that $p_*$ is surjective by constructing a right inverse. Suppose we have the homomorphism $\varphi \in \mathrm{Hom}^k(TC(V), V)$, then define $\tau_\varphi \colon TC(V) \to TC(V)$ by

$$\tau_\varphi(v_1 \otimes \ldots \otimes v_m)^{(n)} = \left( \sum_{i=0}^{n} \left( \mathrm{Id}^{\otimes i} \otimes \varphi \otimes \mathrm{Id}^{\otimes n-i} \right) \circ \Delta^{(n)}(v_1 \otimes \ldots \otimes v_m) \right)^{(n)} .$$

It is a straightforward, though long, check that we have $\tau_\varphi \in \mathrm{Coder}^k(TC(V))$ and it is deduced easily that $p_*(\tau_\varphi) = \varphi$.

□

REMARK B.2.8. In the proof above we frequently move between $V^{\otimes n}$ as a subspace of $TC(V)^{\otimes n}$ and as a subspace of $TC(V)$ without noting it every time. Nonetheless this should not cause any confusion for the attentive reader.

PROPOSITION B.2.9. *Suppose $C$ is a graded coalgebra. Then the graded commutator*

$$[\tau, \theta] = \tau \circ \theta - (-1)^{kl} \theta \circ \tau$$

*is a coderivation of degree $k + l$ for all $\tau \in \mathit{Coder}^k(C)$ and $\theta \in \mathit{Coder}^l(C)$.*

PROOF.
Clearly $[\tau, \theta]$ is a degree $k + l$ linear map. Verifying that

$$\Delta \circ [\tau, \theta] = ([\tau, \theta] \otimes \mathrm{Id} + \mathrm{Id} \otimes [\tau, \theta]) \circ \Delta$$

is a completely straightforward exercise left to the reader. □

Note that the graded commutator bracket supplies the graded vector space

$$\mathrm{Coder}(TC(V)) := \bigoplus_k \mathrm{Coder}^k(TC(V))$$

with the structure of a graded Lie algebra. It turns out that the equations defining the structure of a dg algebra on $V$ coincide (up to a shift) with the equations that specify the vanishing of the bracket of a certain coderivation with itself.

PROPOSITION B.2.10. *Suppose $V$ is a graded vector space equipped with linear maps*

$$\mu \colon V^{\otimes 2} \longrightarrow V \quad \text{and} \quad d \colon V \longrightarrow V$$

*of degree $0$ and $+1$ respectively. Then $(V, \mu, d)$ is a dg algebra if and only if $[m, m] = 0$ where the coderivation $m \in \mathit{Coder}^1(TC(V[1]))$ is given by*

$$p_*(m) = \downarrow \circ \mu \circ (\downarrow \otimes \downarrow)^{-1} + \downarrow \circ d \circ \uparrow .$$

PROOF.
The "if" statement follows from the straightforward, but long, computation of evaluating

$$0 = \frac{1}{2} p_*([m, m]) = p_*(m) \circ m$$

on $V^{\otimes n}$ for $0 < n < 4$. We leave this computation to the reader.

The "only if" statement follows since $p_*(m)$ vanishes on $V^{\otimes n}$ for all $n > 2$, while, by the fact that $m = \tau_{p_*(m)}$ in the notation of the proof of proposition B.2.7, we see that $m$ lowers the weight by a maximum of 1. This means that the computation proving the "if" statement is the only computation that needs to be done. So, if $(V, \mu, d)$ is a dg algebra, we find that $[m, m]$ is the unique coderivation such that $p_*([m, m]) = 0$ and therefore $[m, m] = 0$.

□

Thus we see that the space of dg algebra structures on a graded vector space $V$ is exactly the space of degree +1 coderivations $m$ of $TC(V[1])$ such that $p_*(m)(V[1]^{\otimes n}) = 0$ for all $n > 2$ and $[m, m] = 0$.

REMARK B.2.11. Strengthening the last condition to $p_*(m)(V[1]^{\otimes n}) = 0$ for all $n > 1$ recovers the notion of differential graded vector space (cochain complex). While dropping the last condition that $p_*(m)(V[1]^{\otimes n}) = 0$ for all $n > 2$ recovers the notion of $A_\infty$ or strong homotopy associative algebra in the sense that the space of degree +1 coderivations of $TC(V[1])$ such that $[m, m] = 0$ is exactly the space of $A_\infty$-structures on $V$.

Proposition B.2.10 shows that, given a dg algebra $(A, \mu_0, d_0)$, the deformations $(A \otimes_L R, \mu, d)$ will be given by degree +1 coderivations as specified. If we denote by $m_0$ the coderivation corresponding to $(A \otimes_L R, \mu_0, d_0)$ ($R$-linear extension of $\mu_0$, $d_0$), we see that $(A \otimes_L R, \mu, d)$ corresponds to the coderivation $m_0 + \hbar m_{\geq 1}$ for a coderivation $m_{\geq 1}$ of degree +1 such that $p_*(m_{\geq 1})(A[1]^{\otimes n}) = 0$ for all $n > 2$ and

$$0 = [m_0 + \hbar m_{\geq 1}, m_0 + \hbar m_{\geq 1}] = \hbar \left(2d(\mu_{\geq 1}) + \hbar[\mu_{\geq 1}, \mu_{\geq 1}]\right). \tag{B.2.3}$$

Here we have denoted $d(a) := [m_0, a]$ and we recall that $[m_0, m_0] = 0$, since $(A, \mu_0, d_0)$ is a differential graded associative algebra. Note that equation (B.2.3) is an incarnation of the Maurer-Cartan equation "$d\omega + \frac{1}{2}[\omega, \omega] = 0$", see (2.3.1) and (4.1.1). We now have all the necessary elements to specify the differential graded Lie algebra previously alluded to and how it controls the deformation problem.

## B.3. Deformation Lie Algebra

In this section we will explain what we mean when we say that the dgl algebra $\mathfrak{g}$ controls $R$-deformations of $(A, \mu, d)$.

Note that our choices of $R$ are local rings for the maximal ideal $\mathfrak{m} = \langle \hbar \rangle$. In fact, most of the theory developed here is straightforwardly generalized to the setting of local Artinian rings and local complete rings $R$. For simplicity we will stick to the $k$-truncations and formal power series however.

NOTATION B.3.1. Given a differential graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot], d)$, we denote the differential graded Lie algebra given by

$$\mathfrak{g}_\mathfrak{m} := \mathfrak{g} \otimes_L \mathfrak{m}, \qquad [X \otimes a, Y \otimes b] = [X, Y] \otimes ab \qquad \text{and} \qquad d(X \otimes a) = (dX) \otimes a,$$

with the grading given by $(\mathfrak{g}_\mathfrak{m})^n = \mathfrak{g}^n \otimes_L \mathfrak{m}$, by $(\mathfrak{g}_\mathfrak{m}, [\cdot, \cdot], d)$.

Note that, since $\mathfrak{m}$ is either nilpotent or pro-nilpotent (a limit of nilpotent algebras), we find that $\mathfrak{g}_\mathfrak{m}$ is also nilpotent or pro-nilpotent respectively. This means that infinite sums such that there are only finitely many terms of each bracket length always converge in $\mathfrak{g}_\mathfrak{m}$.

DEFINITION B.3.2. Given a differential graded Lie algebra $\mathfrak{g}$ we denote the *Gauge group* associated to $\mathfrak{g}_\mathfrak{m}$ by $G(\mathfrak{g}_\mathfrak{m})$. It is given as a set by $(\mathfrak{g}_\mathfrak{m})^0$. The group multiplication is given by the Campbell-Baker-Hausdorff formula

$$X \cdot Y = \mathrm{Log}\,(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + [Y, [Y, X]]) + \ldots$$

where $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ and $\mathrm{Log}\, X = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}(X - 1)^{n+1}$.

Note that $X \cdot 0 = X = 0 \cdot X$ and $X \cdot (-X) = 0 = (-X) \cdot X$ and so the neutral element of $\exp(\mathfrak{g}_\mathfrak{m})$ is given by 0, while the inverse of $X \in \exp(\mathfrak{g}_\mathfrak{m})$ is given by $-X$.

REMARK B.3.3. In the definition B.3.2 we are being a bit sloppy. We mean that $X \cdot Y$ is given by writing out the formal power series $\mathrm{Log}\,(\exp(x)\exp(y))$ of non-commuting formal variables $x$ and $y$ in terms of (iterated) commutators and subsequently replacing all instances of $x$ by $X$ and all instances of $y$ by $Y$. The resulting sum will be well-defined by the comment on nilpotency above.

DEFINITION B.3.4. Given a differential graded Lie algebra $\mathfrak{g}$ and $m \in \mathfrak{g}^1$, we define the *gauge action* with respect to $m$

$$\alpha_m \colon G(\mathfrak{g}_\mathfrak{m}) \longrightarrow \mathrm{Aut}_L\left((\mathfrak{g}_\mathfrak{m})^1\right)$$

by

$$\alpha_m(X)(Y) = \exp(X)(m+Y)\exp(-X) - m.$$

REMARK B.3.5. Note that we have been sloppy in the same way as before. We mean that $\alpha_m(X)(Y)$ is given by writing out the formal power series $\exp(x)(z+y)\exp(-x) - z$ of non-commuting formal variables $x$, $y$ and $z$ in terms of iterated commutators and then substituting all instances of $x$ by $X$, $y$ by $Y$ and $z$ by $m$. The brackets with $m$ should be interpreted in terms of the obvious $R$-linear action of $\mathfrak{g}$ on $(\mathfrak{g}_\mathfrak{m})^0$.

DEFINITION B.3.6. Suppose $(\mathfrak{g}, [\cdot,\cdot], d)$ is a differential graded Lie algebra. Then we define the set of *Maurer-Cartan elements* $MC(\mathfrak{g})$ by

$$MC(\mathfrak{g}) := \left\{ X \in \mathfrak{g}^1 \mid dX + \frac{1}{2}[X,X] = 0 \right\}.$$

REMARK B.3.7. Suppose $(\mathfrak{g}, [\cdot,\cdot])$ is a graded Lie algebra and $m \in \mathfrak{g}^1$ such that $[m,m] = 0$, then it is a trivial exercise to verify that

$$d_m \colon \mathfrak{g} \longrightarrow \mathfrak{g},$$

given by $d_m(X) = [m,X]$, gives $\mathfrak{g}$ the structure of a differential graded Lie algebra.

PROPOSITION B.3.8. *Suppose $\mathfrak{g}$ is a graded Lie algebra with $m \in \mathfrak{g}^1$ such that $[m,m] = 0$, then the gauge action with respect to $m$ preserves the set of Maurer-Cartan elements of $(\mathfrak{g}, [\cdot,\cdot], d_m)$.*

PROOF.
The proposition follows by direct computation. □

DEFINITION B.3.9. We define the set of $R$-deformations up to gauge equivalence controlled by the dgl algebra $(\mathfrak{g}, [\cdot,\cdot], d_m)$ as above, denoted $\mathrm{Def}_R(\mathfrak{g}, m)$, as the quotient

$$\mathrm{Def}_R(\mathfrak{g}, m) := MC(\mathfrak{g}_\mathfrak{m}) \big/ G(\mathfrak{g}_\mathfrak{m})$$

by the gauge action w.r.t. $m$.

When we say that the dgl algebra $\mathfrak{g}$ controls $R$-deformations of $(A, \mu, d)$ we mean that the set of gauge equivalence classes of $R$-deformations is isomorphic to $\mathrm{Def}_R(\mathfrak{g}, m)$ (in a natural way).

REMARK B.3.10. It should be evident that we have not even begun to present the full story here. In general one would find, instead of the sets $\mathrm{Def}_R(\mathfrak{g}, m)$, a functor

$$\mathrm{Def}(\mathfrak{g}) \colon \mathcal{C} \longrightarrow \mathbf{Grpd_2},$$

where $\mathbf{Grpd_2}$ denotes the category of 2-groupoids and $\mathcal{C}$ denotes the category of local Artinian or local complete (commutative) $L$-algebras. Also, the differential in $\mathfrak{g}$ need not be of the form $d_m$ for some $m \in \mathfrak{g}^1$. We will not need the full generality of the theory here however.

## B.4. Deformation Cohomology

Now we are in a position to provide the dgl algebra $\mathfrak{g}(A)$ that controls the $R$-deformations of the dga algebra $(A, \mu_0, d_0)$. We should of course be guided by the equation (B.2.3). Let us introduce first some useful notation.

NOTATION B.4.1. For $i, j \in \mathbb{Z}_{\geq 0}$ and $i > 0$, we denote the space of $L$-linear maps from $A^{\otimes i}$ to $A$ of degree $j$ by

$$C^{ij}(A) := \mathrm{Hom}_L^j(A^{\otimes i}, A).$$

For $n \geq 1$ we also denote

$$C^n(A) := \bigoplus_{i+j=n} C^{ij}(A) \quad \text{and} \quad C^\bullet(A) := \bigoplus_{n\geq 1} C^n(A).$$

DEFINITION B.4.2. For $i, j \in \mathbb{Z}_{\geq 0}$ $(i > 0)$ let $T_{ij} \colon C^{ij}(A) \to C^{i(i+j-1)}(A[1])$ denote the map given by

$$T_{ij}(f) = (-1)^{\frac{i(i-1)}{2}} \downarrow \circ f \circ \uparrow^{\otimes i}.$$

We shall denote the induced map

$$T = \bigoplus_{i,j} T_{ij} \colon C^\bullet(A) \longrightarrow C^\bullet(A[1]).$$

We also denote the map defined in the proof of proposition B.2.7 by

$$\tau \colon C^\bullet(A[1]) \longrightarrow \bigoplus_{k\geq 0} \mathrm{Coder}^k(TC(A[1])).$$

REMARK B.4.3. As shown in proposition B.2.7, the map $\tau$ is a linear isomorphism. Clearly the map $T$ is also a linear isomorphism. If we consider $C^\bullet(A)$ graded by the total degree, i.e. $C^n(A)$ are the elements of degree $n$, and $C^\bullet(A[1])$ graded by the degree of maps, i.e. $\mathrm{Hom}^k(A[1]^{\otimes l}, A)$ are elements of degree $k$, then $T$ is a degree $-1$ map. With this same grading on $C^\bullet(A[1])$ we see that $\tau$ is of degree 0.

DEFINITION B.4.4. Let $(\mathfrak{g}(A), [\cdot,\cdot]_G, d)$ be defined as the dgl algebra given by

$$\mathfrak{g}(A)^n := C^\bullet(A)[1]^n$$

$$\tau \circ T([X, Y]_G) = [\tau \circ T(X), \tau \circ T(Y)]$$

where the brackets on the right hand side are simply the commutator brackets and

$$dX = [\mu_0 + d_0, X]_G.$$

THEOREM B.4.5. *The differential graded Lie algebra $\mathfrak{g}(A)$ is well-defined and controls the deformations of $(A, \mu_0, d_0)$.*

PROOF.
The fact that $[\cdot,\cdot]_G$ is a well-defined graded Lie bracket follows from the fact that the commutator bracket on coderivations is a well-defined Lie bracket and that the composition

$$\tau \circ T \colon \mathfrak{g}(A)^n \longrightarrow \mathrm{Coder}^n(TC(A[1]))$$

is a degree 0 linear isomorphism. This last fact is checked by simply going through the definitions. Thus, by remark B.3.7, we find that $\mathfrak{g}(A)$ is a well-defined dgl algebra if $[\mu_0 + d_0, \mu_0 + d_0]_G = 0$. This last identity follows from proposition B.2.10.

It remains to verify that

$$\mathrm{Def}_R(\mathfrak{g}(A), m) \simeq \{R - \text{deformations of} \quad (A, \mu_0, d_0)\} \big/ \text{gauge equivalence},$$

where $m = \mu_0 + d_0$. Note that, again by proposition B.2.10, we find the map

$$t \colon MC(\mathfrak{g}(A)_\mathfrak{m}) \longrightarrow \{R - \text{deformations of} \quad (A, \mu_0, d_0)\}$$

given by

$$t(c) = (A \otimes_L R, \mu_0 + c, d_0 + c).$$

Note also that, by definition, every $R-$deformation of $(A, \mu_0, d_0)$ is of the form

$$(A \otimes_L R, \mu_0 + c, d_0 + c)$$

up to gauge equivalence. By proposition B.2.10, it follows that if $(B, \mu, d) \simeq (A \otimes_L R, \mu_0 + c, d_0 + c)$ then $c \in MC(\mathfrak{g}(A)_\mathfrak{m})$. Thus it is only left to check that the gauge group $G(\mathfrak{g}(A)_\mathfrak{m})$ acts by gauge equivalences and that every gauge equivalence between deformations (with $B = A \otimes R$) yields an element of the gauge group $G(\mathfrak{g}(A)_\mathfrak{m})$. Note that

$$G(\mathfrak{g}(A)_\mathfrak{m}) \subset \mathfrak{g}(A)^0_\mathfrak{m} = C^1(A) \otimes \mathfrak{m} = \bigoplus_{i+j=1} \mathrm{Hom}^j_L(A^{\otimes i}, A) \otimes \mathfrak{m} = \mathrm{Hom}^0_L(A, A) \otimes \mathfrak{m},$$

since $i > 0$ and $j \geq 0$. So, for $X \in G(\mathfrak{g}(A)_\mathfrak{m})$, the linear map

$$\exp X := \sum_{k=0}^{\infty} \frac{X^{\circ k}}{k!} \colon A \otimes R \longrightarrow A \otimes R$$

is well-defined. Clearly, the linear map $\exp(-X)$ is also well-defined and an inverse to $\exp X$. Now note that, if $c \in MC(\mathfrak{g}(A)_\mathfrak{m})$, then

$$\exp(X) \colon t(c) \longrightarrow t(\alpha_m(X)(c)),$$

where $m := \mu_0 + d_0$, follows from the definition B.3.4 of $\alpha_m$. Note that, by definition, $\exp(X)$ is a gauge equivalence. Thus we find that $G(\mathfrak{g}(A)_\mathfrak{m})$ acts on the set of $R$-deformations by gauge equivalences. On the other hand suppose

$$\varphi \colon (A \otimes_L R, \mu, d) \longrightarrow (A \otimes_L R, \mu', d')$$

is a gauge equivalence of $R$-deformations. Then $\varphi = \mathrm{Id} + \hbar\varphi_{\geq 1}$ and the map

$$\mathrm{Log}\,\varphi = \sum_{k=0}^{\infty} \frac{(-1)^{n+1}}{n+1} (\hbar\varphi_{\geq 1})^{n+1} \colon A \otimes_L R \longrightarrow A \otimes_L R$$

is a well-defined linear map. Note that $\mathrm{Log}\,\varphi \in \mathrm{Hom}^0_L(A, A) \otimes \mathfrak{m}$ and

$$\alpha_m(\mathrm{Log}\,\varphi)(\mu + d - m) = \mu' + d' - m.$$

Thus, every gauge equivalence is implemented by an element of the gauge group.

□

Remark B.4.6. Note that $(C^\bullet(A), \uparrow \circ d_m \circ \downarrow)$ is a subcomplex of the Hochschild cohomology complex of $(A, \mu, d)$ with values in the $A$-bimodule $(A, d)$. The bracket $[\cdot, \cdot]_G$ is usually called the Gerstenhaber bracket and it is part of the Gerstenhaber structure on the Hochschild cohomology of $A$ with values in $A$.

Remark B.4.7. Note that, given an isomorphism $A \to B$ of dg algebras, we obtain an induced isomorphism $\mathrm{Def}_R(\mathfrak{g}(A), m_A) \to \mathrm{Def}_R(\mathfrak{g}(B), m_B)$. When we consider differential graded algebras, we are more often interested in quasi-isomorphism than isomorphism however. In order to make a similar statement about quasi-isomorphisms, we would need to be able to construct a quasi-inverse. This is not always possible. However, if we consider the differential graded algebras as strong homotopy associative algebras ($A_\infty$-algebra) instead, we can always find such inverse $\infty$-isomorphism [**79**]. It is shown in the paper [**34**] that a quasi-isomorphism of dg algebras induces a corresponding isomorphism of the deformation functors. Note, however, that one is required to consider deformations of the dg algebras as $A_\infty$-algebras in this case. This amounts technically to allowing $j \in \mathbb{Z}$ in definition B.4.2.

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