

Rational Homological Stability for Automorphisms of Manifolds

PhD thesis by

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This PhD thesis has been submitted to the PhD School of the Faculty of Science, University of Copenhagen.

Submitted: September 2015

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This PhD thesis has been submitted to the PhD School of The Faculty of Science,
University of Copenhagen, in September 2015.

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ISBN 978-87-7078-956-1

Abstract

In this thesis we prove rational homological stability for the classifying spaces of the homotopy automorphisms and block diffeomorphisms of iterated connected sums of products of spheres of a certain connectivity.

The results in particular apply to the manifolds

$$N_g^{p,q} = (\#_g(S^p \times S^q)) \setminus \text{int}(D^{p+q}), \text{ where } 3 \leq p < q < 2p - 1.$$

We show that the homology groups

$$H_*(\text{Baut}_\partial(N_g^{p,q}); \mathbb{Q}) \text{ and } H_*(\widetilde{\text{BDiff}}_\partial(N_g^{p,q}); \mathbb{Q})$$

are independent of g for $* < g/2 - 1$. To prove the homological stability for the homotopy automorphisms we show that the groups $\pi_1(\text{Baut}_\partial(N_g^{p,q}))$ satisfy homological stability with coefficients in the homology of the universal covering, which is studied using rational homology theory. The result for the block diffeomorphisms is deduced from the homological stability for the homotopy automorphisms upon using Surgery theory. The main theorems of this thesis extend the homological stability results in [BM15] where the automorphism spaces of $N_g^{p,p}$ are studied.

Resumé

I denne afhandling beviser vi rationel homologisk stabilitet for klassificerende rum af homotopi-automorfier og blok-diffeomorfier af itererede summer af produkter af sfærer af bestemt konnektivitet.

Resultaterne gælder specielt for mangfoldighederne

$$N_g^{p,q} = (\#_g(S^p \times S^q)) \setminus \text{int}(D^{p+q}), \text{ where } 3 \leq p < q < 2p - 1.$$

Vi viser at homologigrupperne

$$H_*(\text{Baut}_\partial(N_g^{p,q}); \mathbb{Q}) \text{ and } H_*(\widetilde{\text{BDiff}}_\partial(N_g^{p,q}); \mathbb{Q})$$

er uafhængige af g for $* < g/2 - 1$. For at bevise homologisk stabilitet for homotopi-automorfierne viser vi, at grupperne opfylder homologisk stabilitet med koefficienter i homologien af det universelle overlejringsrum, som studeres ved hjælp af rationel homologi-teori. Resultatet for blok-diffeomorfierne udledes fra den homologiske stabilitet for homotopi-automorfierne ved hjælp af kirurgi-teori. Hovedresultaterne i denne afhandling udvider resultaterne om homologisk stabilitet i [BM15], hvor automorfirummen af $N_g^{p,p}$ studeres.

Acknowledgments

First of all I would like to thank my advisor Ib Madsen for suggesting the research problem underlying this thesis to me and guiding me during the last three years.

Among the many people, that I had the chance to discuss with or that taught me something new, I would like to thank in particular Diarmuid Crowley, Dieter Degrijsse, Johannes Ebert, Søren Galatius, Nathan Perlmutter, Oscar Randal-Williams and Nathalie Wahl. Especially Alexander Berglund answered many of my questions about rational homotopy theory. Moreover I would like to thank Irakli Patchkoria and Kristian J. Moi for reading earlier versions of this thesis and providing many helpful suggestions. The three years wouldn't have been as much fun without my office mates and friends at the department - thanks especially for keeping me company during late nights and weekends of work in the last couple of months.

Lastly I would like to thank my mother who always supported me.

The author was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

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1 Introduction

This thesis is concerned with the problem of understanding the rational homological structure of automorphism spaces of manifolds, similar in spirit to the highly connected even dimensional case considered by Berglund and Madsen in [BM13, BM15]. The main theorems of this thesis are homological stability results for the homotopy automorphisms and the block diffeomorphisms of connected sums of products of spheres.

Consider a closed oriented n -manifold M and denote by $B\text{Aut}(M)$ the classifying space of some automorphism group or monoid $\text{Aut}(M)$. A central topic of current and past research is to understand the cohomology ring $H^*(B\text{Aut}(M))$. One example of an automorphism group $\text{Aut}(M)$ that is studied in the literature is $\text{Diff}(M)$ the group of self-diffeomorphisms with the Whitney C^∞ -topology. The cohomology ring $H^*(B\text{Diff}(M))$ is the ring of characteristic classes of smooth manifold bundles with fiber M . Another example is $\text{aut}(M)$, the topological monoid of homotopy self-equivalences. In this case the cohomology ring $H^*(B\text{aut}(M))$ is the ring of characteristic classes of fibrations with fiber M .

In general it is extremely difficult to understand the cohomology ring $H^*(B\text{Aut}(M))$ for a single manifold. One approach is to study a family of manifolds and restrict to information in the so called stable range. For this we consider a compact oriented n -manifold N with boundary $\partial N \cong S^{n-1}$ an $(n-1)$ -sphere. Let

$$V^{p,q} = S^p \times S^q \setminus \text{int}(D_1^n \sqcup D_2^n),$$

be a product of a p -sphere and a q -sphere, such that $p+q=n$, with the interior of two disjoint embedded disks removed. We obtain a new compact oriented manifold with boundary S^{n-1}

$$N \cup_{\partial_1} V^{p,q},$$

by gluing N and $V^{p,q}$ along the boundary of N and $\partial_1 V^{p,q} = \partial D_1^n \subset V^{p,q}$. Let $\text{Aut}_\partial(N)$ be the subspace of some automorphism space of N , fixing the boundary pointwise. We can define a map

$$\begin{aligned} \text{Aut}_\partial(N) &\rightarrow \text{Aut}_\partial(N \cup_{\partial_1} V^{p,q}) \\ f &\mapsto f \cup_{\partial_1} \text{id}_{V^{p,q}}, \end{aligned}$$

by extending an automorphism of N fixing the boundary by the identity on $V^{p,q}$. Since the map respects the composition of maps, it gives us a map of the classifying spaces

$$\sigma : B\text{Aut}_\partial(N) \rightarrow B\text{Aut}_\partial(N \cup_{\partial_1} V^{p,q})$$

We will refer to the induced map in homology

$$\sigma_* : H_*(BAut_{\partial}(N)) \rightarrow H_*(BAut_{\partial}(N \cup_{\partial_1} V^{p,q}))$$

as the stabilization map. Of course one could also define the stabilization map when $V^{p,q}$ is replaced by another cobordism of S^{n-1} with S^{n-1} or even cobordisms of other submanifolds.

Note that $N \cup_{\partial_1} V^{p,q} \cong N \# S^p \times S^q$, where $\#$ denotes the connected sum. We can repeat the process above and get a family of manifolds

$$N_g = N \# (\#_g(S^p \times S^q)),$$

where $\#_g$ denotes the g -fold connected sum. We also get a family of stabilization maps

$$\sigma_* : H_*(BAut_{\partial}(N_g)) \rightarrow H_*(BAut_{\partial}(N_{g+1})).$$

We say that the automorphism spaces of the family N_g satisfy homological stability, if σ_* is an isomorphism for $* < c(N_g)$, where $c(N_g)$ is some increasing function depending on g and possibly N (for example if N already has some $S^p \times S^q$ summands). The function $c(N_g)$ bounds the so called the stable range. Note that homological stability also implies cohomological stability, i.e. σ^* is an isomorphism in the same range by the universal coefficient theorem.

The second step in the process (that we don't treat in this thesis) is to identify the so called stable cohomology

$$H^*(BAut_{\partial}(N_{\infty})) = H^*(\operatorname{hocolim}_{g \rightarrow \infty} BAut_{\partial}(N_g)).$$

The induced map

$$H^*(BAut_{\partial}(N_{\infty})) \rightarrow H^*(BAut_{\partial}(N_g))$$

is an isomorphism in the stable range.

Consider the manifolds

$$N_g^{p,q} = (\#_g S^p \times S^q) \setminus \operatorname{int} D^{p+q}.$$

The homological stability for the mapping class group $\pi_0(\operatorname{Diff}_{\partial}(N_g^{1,1}))$ is a classical result by Harer [Har85]. Combined with the contractability of the components of $B \operatorname{Diff}_{\partial}(N_g^{1,1})$, $g \geq 2$ [EE69] this shows homological stability for $B \operatorname{Diff}_{\partial}(N_g^{1,1})$. Madsen and Weiss identified the stable cohomology with the cohomology of the infinite loop-space of the now called Madsen-Tillman spectrum $\operatorname{MTSO}(2)$ [MW07].

The above was extended by Galatius and Randal-Williams to homological stability for $B\text{Diff}_\partial(M_g^{2d})$, $d \geq 3$ [GRW12]. Moreover they showed in [GRW14] that the map

$$B\text{Diff}_\partial(M_\infty^{2d}) \rightarrow \Omega_\bullet^\infty \text{MTSO}(2d)$$

is an integral homology equivalence. In fact Galatius and Randal-Williams' results are more general, in the sense that they show stability for many more families of even dimensional manifolds with tangential structures and also identify their stable cohomology.

Perlmutter showed homological stability for the diffeomorphism groups of $N \# N_g^{p,q}$, for $p < q < 2p - 2$ and $(q - p + 2)$ -connected N in [Per14a]. In [Per14b] he shows stability with respect to stabilization with all $(d - 1)$ -connected $(2d + 1)$ -manifolds for d even. In [BP15] Botvinnik and Perlmutter identify the stable cohomology of the classifying space $B\text{Diff}_{D^{2n}}(\natural_g(D^{n+1} \times S^n))$ of the group of diffeomorphisms of the g -fold boundary connected sum that restrict to the identity near a disk, embedded in the boundary.

The map from $B\text{Diff}_\partial$ of an n -manifold to the infinite loop-space of the spectrum $\text{MTSO}(n)$ is also defined in odd-dimensions, but Ebert showed in [Ebe13], using index theory, that the induced map in cohomology is not injective. Hence the obvious generalization of the Madsen-Weiss theorem to odd dimensional manifolds fails.

Homological stability for mapping-class groups of certain 3-manifolds is shown by Hatcher and Wahl in [HW10]. The traditional method to prove homological stability results is originally due to Quillen and has been applied in many cases. For an axiomatization and more stability results for automorphism groups check [RWW15].

Berglund and Madsen's approach in [BM15] to show rational homological stability for $B\text{aut}_\partial(N_g^{d,d})$, $d \geq 3$, uses a stability result by Charney [Cha87] for automorphism groups of hyperbolic modules with twisted coefficients. Denote by G_g the image of

$$H_d : \pi_0(\text{aut}_\partial(N_g^{d,d})) \rightarrow \text{Aut}(H_d(N_g^{d,d})).$$

Berglund and Madsen use that G_g is an automorphism group of a certain hyperbolic module and that H_d has a finite kernel. They identify the $\pi_1(B\text{aut}_\partial(N_g^{d,d}))$ -module $H_*(B\text{aut}_\partial(N_g^{d,d})\langle 1 \rangle; \mathbb{Q})$ in terms of the Chevalley-Eilenberg homology of a certain derivation Lie algebra $H_*^{CE}(\text{Der}_\omega(\mathbb{L}_g))$ using rational homotopy theory. The $\pi_1(B\text{aut}_\partial(N_g^{d,d}))$ -action on $H_*^{CE}(\text{Der}_\omega(\mathbb{L}_g))$ is through G_g . They show that $H_*^{CE}(\text{Der}_\omega(\mathbb{L}_g))$ is a coefficient system for G_g that satisfies homological stability. The homological stability result for $B\text{aut}_\partial(N_g^{d,d})$, then follows by an application of the covering spectral sequence:

$$E_{p,q}^2(N_g^{d,d}) = H_p(\pi_1(B\text{aut}_\partial(N_g^{d,d})); H_q(B\text{aut}_\partial(N_g^{d,d})\langle 1 \rangle; \mathbb{Q})) \Rightarrow H_{p+q}(B\text{aut}_\partial(N_g^{d,d}); \mathbb{Q})$$

upon using that $E_{p,q}^2(N_g^{d,d}) \cong H_p(G_g; H_q^{CE}(\text{Der}_\omega(\mathbb{L}_g)))$.

Surgery theory allows them to understand the fiber of the map

$$\widetilde{B\text{Diff}}_\partial(N_g^{d,d}) \rightarrow B\text{aut}_\partial(N_g^{d,d}),$$

where $\widetilde{\text{Diff}}$ denotes the block diffeomorphisms. Using this they extend the homological stability result to $B\widetilde{\text{Diff}}_{\partial}(N_g^{d,d})$.

Moreover they show, using the results in [GRW14] that the covering cohomology spectral sequences for $B\text{aut}_{\partial}(N_g^{d,d})$ and $B\widetilde{\text{Diff}}_{\partial}(N_g^{d,d})$ collapse in the stable range. Borel's work on the cohomology of arithmetic groups now allows them to conclude that

$$H^*(B\text{aut}_{\partial}(N_{\infty}^{d,d}); \mathbb{Q}) \cong H^*(G_{\infty}; \mathbb{Q}) \otimes H_{CE}^*(\text{Der}_{\omega}(\mathbb{L}_{\infty}))^{G_{\infty}},$$

where the first term was calculated by Borel. The second term was calculated by Kontsevich [CV03]. Finally they use Kontsevich's graph homology to identify the stable cohomology ring $H^*(B\widetilde{\text{Diff}}_{\partial}(N_{\infty}^{d,d}); \mathbb{Q})$ with the cohomology of certain discrete groups, generalizing Kontsevich's theorem on $H_{CE}^*(\text{Der}_{\omega}(\mathbb{L}_{\infty}))^{G_{\infty}}$.

Outline of this thesis

Consider a finite indexing set I and for $i \in I$ natural numbers $3 \leq p_i \leq q_i < 2p_i - 1$, such that $p_i + q_i = n$. We define an oriented n -manifold

$$N_I = (\#_{i \in I}(S^{p_i} \times S^{q_i})) \setminus \text{int}(D^n).$$

Note that gluing a $V^{p,q}$ along one boundary component to N_I , where $3 \leq p \leq q \leq 2p - 1$ and $p + q = n$, gives us another manifold

$$N_{I'} \cong N_I \cup_{\partial_1} V^{p,q},$$

where the indexing set is given by $I' = I \cup \{i'\}$, $p_{i'} = p$ and $q_{i'} = q$. Denote by

$$g_p = \begin{cases} \text{rank}(H_p(N_I))/2 & \text{if } p = n/2 \\ \text{rank}(H_p(N_I)) & \text{otherwise.} \end{cases}$$

We think of g_p as a generalized genus - it measures how many summands of $S^p \times S^q$ the manifold N_I has. The main results of this thesis are:

Theorem A and B. *The stabilization maps with respect to $V^{p,q}$, $3 \leq p \leq q < 2p - 1$*

$$H_i(B\text{aut}_{\partial}(N_I); \mathbb{Q}) \rightarrow H_i(B\text{aut}_{\partial}(N_{I'}); \mathbb{Q})$$

and

$$H_i(B\widetilde{\text{Diff}}_{\partial}(N_I); \mathbb{Q}) \rightarrow H_i(B\widetilde{\text{Diff}}_{\partial}(N_{I'}); \mathbb{Q})$$

are isomorphisms for $g_p > 2i + 2$ when $2p \neq n$ and $g_p > 2i + 4$ if $2p = n$ and epimorphisms for $g_p \geq 2i + 2$ respectively $g_p \geq 2i + 4$.

The proof of Theorem A, the homological stability for the homotopy automorphisms, can be found in Chapter 6. The proof of Theorem B, the homological stability for the block diffeomorphisms, is in Chapter 7.

In Section 2.5 we review the definition of hyperbolic modules with form parameters in the sense of Bak and extend the definition to graded hyperbolic modules. The graded hyperbolic modules model the reduced homology $\tilde{H}_*(N_I)$ together with the intersection pairing and in the middle dimension a certain quadratic refinement of it. In Chapter 5 we establish that the homotopy mapping class group surjects onto the automorphisms of the graded quadratic module $\tilde{H}_*(N_I)$. The automorphisms are isomorphic to

$$G_{g_{n/2}} \times \prod_{i=1}^{\lceil n/2 \rceil - 1} Gl_{g_i}(\mathbb{Z}),$$

where the $G_{g_{n/2}}$ only appears for n even and is given by an automorphism group of a hyperbolic module.

In Section 2.6 we review van der Kallen's and Charney's twisted homological stability results for general linear groups and automorphisms of hyperbolic modules with form parameters. We combine the latter two to a stability result for graded hyperbolic modules with twisted coefficient systems induced by polynomial functors (reviewed in Section 2.3). In Section 2.1 we quickly review Quillen's approach to rational homotopy theory and describe some results on the rational homotopy theory of mapping spaces in Section 2.2. We use this to express the rational homotopy groups of the universal covering of $Baut_{\partial}(N_I)$ as $\pi_1(Baut_{\partial}(N_I))$ -modules in terms of a certain derivation Lie algebras in Chapter 5. The rational homotopy theory needed in this thesis is a straightforward generalization of [BM15]. One of the key ingredients is a theorem by Tanré relating the universal cover of the classifying space of the pointed homotopy automorphisms to a certain derivation Lie algebra.

In Chapter 3 we show that the Chevalley-Eilenberg chains of these derivation Lie algebras can be identified with the values of a Schur multifunctor (defined in Section 2.4).

In Chapter 6 we use the latter to show Theorem A, where we need Section 2.7 for technical reasons.

The review of Surgery theory in Section 2.8 is needed for the proof of Theorem B in Chapter 7.

The results in this thesis extend the stability results in [BM15] in the following ways:

- In the case that $n = 2d$ we show rational homological stability of the homotopy automorphisms and block diffeomorphisms for families $N_I \# N_g^{d,d}$.
- We show rational homological stability of the homotopy automorphisms and block diffeomorphisms with respect to stabilization with other products of spheres $S^p \times S^q$.

- In particular we show rational homological stability of the homotopy automorphisms and block diffeomorphisms for odd-dimensional manifolds N_I including the family

$$N_g^{p,p+1} = (\#_g(S^p \times S^{p+1})) \setminus \text{int}(D^{2p+1}), p \geq 3.$$

2 Preliminaries

2.1 Differential graded Lie algebras and rational homotopy theory

In this section we review Quillen's approach to rational homotopy theory. For more details consult Quillen's original paper [Qui69] or for example [Tan83].

We assume all graded objects to be \mathbb{Z} -graded unless otherwise stated and will omit the \mathbb{Z} from the notation. Moreover we assume all vector spaces to be over \mathbb{Q} . Additional structure on a graded vector space $V = V_*$ will be denoted as $(V, -)$, but abbreviated to V , when clear from the context.

Let V, W be graded vector spaces and let $\text{Hom}_n(V, W)$ denote the set of linear maps from V_* to W_{*+n} , this makes all linear maps from V to W into a graded vector space $\text{Hom}(V, W) = \bigoplus_n \text{Hom}_n(V, W)$.

A *graded Lie algebra* $(L, [-, -])$ is a graded vector space L_* together with a bilinear degree 0 map $[-, -]$, such that

1. $[x, y] = (-1)^{|x||y|+1}[y, x]$
2. $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$

We call $[-, -]$ the Lie bracket. A morphism of graded Lie algebras is a degree 0 homomorphism of the underlying vector spaces, respecting the Lie brackets. We will denote the category of graded Lie algebras by $gLie$.

Let V be a graded vector space. We define the reduced tensor algebra on V to be:

$$T(V) = \bigcup_{n>0} V^{\otimes n},$$

where $\otimes n$ denotes the n -fold tensor product. For $x_i, i = 1, \dots, n$ with $|x_i| = d_i$ we set $|\bigotimes_i x_i| = \sum_i d_i$. We can define a Lie bracket on $T(V)$ by $[x, y] = x \otimes y - (-1)^{|x||y|} y \otimes x$, which makes $T(V)$ into a graded Lie algebra.

The *free graded Lie algebra* $\mathbb{L}(V)$ generated by V is the smallest sub Lie algebra of $T(V)$ containing V . Another way to express $\mathbb{L}(V)$ is in terms of generators

$$[[\dots[[e_{i_1}, e_{i_2}], e_{i_3}]\dots]e_{i_n}],$$

where the $\{e_i\}$ form a basis for V and relations generated by the conditions (1) and (2) for the Lie bracket. When we want to emphasize the elements generating a free graded

Lie algebra, we will denote it by $\mathbb{L}[e_1, \dots, e_n] = \mathbb{L}(\mathbb{Q}\langle e_1, \dots, e_n \rangle)$, where $\mathbb{Q}\langle e_1, \dots, e_n \rangle$ denotes the graded vectorspace generated by basis elements e_1, \dots, e_n .

For a connected topological group G with unit e , the *Samelson product*

$$\begin{aligned} \pi_k(G, e) \otimes \pi_l(G, e) &\rightarrow \pi_{k+l}(G, e) \\ [f] \otimes [g] &\mapsto \langle f, g \rangle \end{aligned}$$

is defined as follows: Consider representatives $f : S^k \rightarrow G$ and $g : S^l \rightarrow G$, the pointwise commutator $[f, g](x, y) = f(x)g(y)f(x)^{-1}g(y)^{-1}$ defines a map $S^k \times S^l \rightarrow G$ that is trivial on $S^k \vee S^l \subset S^k \times S^l$. Hence we get a map

$$\langle f, g \rangle : S^{k+l} \simeq S^k \times S^l / S^k \vee S^l \rightarrow G.$$

The rational homotopy groups $\pi_*(G, e) \otimes \mathbb{Q}$ together with the Samelson bracket form a graded Lie algebra. Note that we can extend the definition of the Samelson product to group-like topological monoids, since we can replace them by homotopy equivalent topological groups. For a simply connected topological space (X, x_0) , the rational homotopy groups of the loop space, based at the constant loop, together with the Samelson product $(\pi_*(\Omega X, c_{x_0}) \otimes \mathbb{Q}, \langle -, - \rangle)$ is called the *homotopy Lie algebra* of X .

Another product on homotopy groups is the Whitehead product

$$\begin{aligned} \pi_{k+1}(X, x_0) \otimes \pi_{l+1}(X, x_0) &\rightarrow \pi_{k+l+1}(X, x_0). \\ [f] \otimes [g] &\mapsto [f, g] \end{aligned}$$

It is defined as follows: Consider representatives $f : S^{k+1} \rightarrow X$ and $g : S^{l+1} \rightarrow X$. The product $S^{k+1} \times S^{l+1}$ can be described as a complex with one $(k+l+2)$ -cell attached to $S^{k+1} \vee S^{l+1}$. The *Whitehead product* is defined as the composition

$$[f, g] : S^{k+l+1} \rightarrow S^{k+1} \vee S^{l+1} \xrightarrow{f \vee g} X,$$

where the first map is the attaching map of the $(k+l+2)$ -cell. Denote by ∂ the boundary map $\partial : \pi_{*+1}(X, x_0) \xrightarrow{\cong} \pi_*(\Omega X, c_{x_0})$ in the homotopy exact sequence of the path space fibration

$$\Omega X \rightarrow PX \rightarrow X.$$

The Whitehead product and the Samelson product relate in the following way:

$$\partial[f, g] = (-1)^k \langle \partial[f], \partial[g] \rangle, \text{ where } [f] \in \pi_{k+1}(X, x_0) \text{ and } [g] \in \pi_{l+1}(X, x_0).$$

For graded Lie algebras $(L, [-, -])$ and $(L', [-, -])$ we define the n -derivations as the subset $\text{Der}_n(L, L') \subset \text{Hom}_n(L_*, L'_*)$, such that for $\theta \in \text{Der}_n(L, L')$:

$$\theta([x, y]) = [\theta(x), y] + (-1)^{n|x|}[x, \theta(y)].$$

A *differential graded Lie algebra* (L, ∂) is a graded Lie algebra $(L, [-, -])$ together with a (-1) -derivation ∂ , such that $\partial \circ \partial = 0$. A morphism of differential graded Lie algebras is a morphism of graded vector spaces of degree 0 commuting with the Lie bracket and the differential. We will denote the category of differential graded Lie algebras by $dgLie$. A *free differential graded Lie algebra*, is a differential graded Lie algebra such that the underlying graded Lie algebra is free.

Quillen defines in a functor

$$\lambda : Top_1 \rightarrow dgLie_1,$$

where Top_1 is the category of simply connected pointed topological spaces and $dgLie_1$ the category of 1-reduced differential graded Lie algebras, i.e. differential graded Lie algebras concentrated in degrees ≥ 1 . Quillen's functor is defined as a composite of a chain of functors. On objects it is given by

$$\lambda(X) = N\mathcal{P}\hat{\mathbb{Q}}[G(E_2 \text{Sing } X)],$$

where $E_2 \text{Sing}$ denotes the Eilenberg subcomplex of the singular simplicial set, consisting of simplices whose 1-skeleton is at the basepoint. G denotes Kan's loop group functor (it satisfies $|G(E_2 \text{Sing } X)| \simeq \Omega X$). $\hat{\mathbb{Q}}[-]$ is the complete simplicial Hopf algebra, obtained by completing the simplicial group ring $\mathbb{Q}[-]$ at the augmentation ideal. \mathcal{P} are the primitive elements, which form a simplicial differential graded Lie algebra. Finally the normalized chain functor N gives us a differential graded Lie algebra. One of the key properties of Quillen's functor is that there is a natural isomorphism of graded Lie algebras

$$H_*(\lambda(X)) \cong (\pi_*(\Omega X, c_{x_0}), \langle -, - \rangle).$$

Quillen proves that $\lambda(-)$ induces an equivalence of homotopy categories

$$\text{Ho}_{\mathbb{Q}}(Top_1) \rightarrow \text{Ho}(dgLie_1),$$

where the weak equivalences in Top_1 are rational homotopy equivalences and in $dgLie_1$ quasi-isomorphisms.

Remark 2.1. The category of differential graded Lie algebras has in fact the structure of a model category. The weak equivalences are quasi-isomorphisms, which we will denote by $\simeq_{q.i.}$, and the fibrations are maps that are surjections in degrees ≥ 2 . The cofibrations can be described as follows: A map $i : (K, \partial_K) \rightarrow (L, \partial_L)$ of differential graded Lie algebras is a cofibration if there exist a free differential graded Lie algebra $(\mathbb{L}(V), \partial)$, a map $\rho : (L, \partial_L) \rightarrow (\mathbb{L}(V), \partial)$ of differential graded Lie algebras and a map $f : K * \mathbb{L}(V) \rightarrow L$ of graded Lie algebras(!) such that the following diagram commutes:

$$\begin{array}{ccccc} K & \longrightarrow & K * \mathbb{L}(V) & \longrightarrow & \mathbb{L}(V) \\ & \searrow & \downarrow f & \nearrow \rho & \\ & & L & & \end{array} .$$

The symbol $*$ denotes the coproduct in $dgLie$. It is also called the free product. On free differential graded Lie algebras it can be described as

$$\mathbb{L}(V) * \mathbb{L}(W) = \mathbb{L}(V \oplus W)$$

together with the unique differential extending the differentials of $\mathbb{L}(V)$ and $\mathbb{L}(W)$.

A *Quillen model of a simply connected topological space* X is a free differential graded Lie algebra (L_X, ∂) with a quasi-isomorphism $L_X \rightarrow \lambda(X)$. A Quillen model is called minimal, if $\partial(L_X) \subset [L_X, L_X]$. We call a space coformal, if $\lambda(X) \simeq_{q.i.} H(\lambda(X))$. A *Quillen model of a map* $f : X \rightarrow Y$, is a map L_f of differential graded Lie algebras between Quillen models of the spaces such that

$$\begin{array}{ccc} L_X & \xrightarrow{L_f} & L_Y \\ \downarrow & & \downarrow \\ \lambda(X) & \xrightarrow{\lambda(f)} & \lambda(Y) \end{array}$$

commutes up to chain homotopy equivalence. A *minimal Quillen model of a map* is a Quillen model of a map such that the models of the spaces are minimal.

The *Chevalley-Eilenberg complex* of a differential graded Lie algebra (L, ∂) is the chain complex $C_*^{CE}(L) = \Lambda_* sL$ with differential $\delta = \delta_0 + \delta_1$, where s denotes the suspension (i.e. $sL_* = L_{*+1}$) and Λ_* the free graded commutative algebra. In the simplest cases the differentials are given by

$$\begin{aligned} \delta_0(sx) &= -s\partial x \\ \delta_1(sx_1 \wedge sx_2) &= (-1)^{|x_1|} s[x_1, x_2], \end{aligned}$$

where $x, x_1, x_2 \in L$. For the general definition see e.g. [Tan83]. The *Chevalley-Eilenberg homology* is the homology of this chain complex. Quillen shows that there is a natural isomorphism

$$H_*^{CE}(\lambda(X)) \cong H_*(X; \mathbb{Q}).$$

Denote by $C_p^{CE}(L)^q$ the elements of the Chevalley-Eilenberg complex in degree p and with word length q . Let $H_*^{CE}(L)^q$ be the homology of the complex $(C_*^{CE}(L)^q, \delta_1)$. We can identify the E^2 -page of the spectral sequence coming from the filtration by word length with

$$H_p^{CE}(H_*(L))^q \Rightarrow H_{p+q}^{CE}(L)$$

In the special case that $L = \lambda(X)$ for some simply connected connected space X we get the so called Quillen spectral sequence

$$E_{p,q}^2 = H_p^{CE}((\pi_*(X) \otimes \mathbb{Q}))^q \Rightarrow H_{p+q}(X; \mathbb{Q}).$$

It is clear by construction that it is natural with respect to pointed maps, but this requirement can be loosened:

Theorem 2.2 ([BM15, Proposition 2.1.]). *Let X be a simply connected space. The Quillen spectral sequence*

$$E_{p,q}^2 = H_p^{CE}((\pi_*(X) \otimes \mathbb{Q}))^q \Rightarrow H_{p+q}(X; \mathbb{Q})$$

is natural with respect to unbased maps of simply connected spaces.

Moreover we observe that the Quillen spectral sequence collapses at the E_2 -page, when a space is coformal. Thus we get for a coformal space X a natural isomorphism:

$$H_k(X; \mathbb{Q}) \cong \bigoplus_{p+q=k} H_p^{CE}((\pi_*(X) \otimes \mathbb{Q}))^q.$$

2.2 Rational homotopy theory of mapping spaces

Let X, Y be pointed topological spaces. Denote by $\text{map}_*(X, Y)$ the space of base-point preserving maps with the compact open topology. Denote by

$$\text{aut}_*(X) \subset \text{map}_*(X, X)$$

the subspace of pointed homotopy self-equivalences. Note that the composition makes $\text{aut}_*(X)$ into a grouplike topological monoid. For a subspace $A \subset X$ such that A contains the basepoint of X denote by $\text{aut}_A(X) \subset \text{aut}_*(X)$ the topological sub-monoid of homotopy self-equivalences of X that restrict to the identity on A . In this section we review some results concerning the rational homotopy theory of the mapping spaces above.

Let

$$f : (L, d_L) \rightarrow (K, d_K)$$

be a map of differential graded Lie algebras. We say that a degree n linear map $\theta \in \text{Hom}_n(L, K)$, is an f -derivations of degree n , if

$$\theta[x, y] = [\theta(x), f(y)] + (-1)^{n|x|} [f(x), \theta(y)], \text{ for all } x, y \in L.$$

The f -derivations form a differential graded vector space $\text{Der}_f(L, K)$, with differential given by

$$D(\theta) = d_K \circ \theta - (-1)^{|\theta|} \theta \circ d_L.$$

The derivations of a differential graded Lie algebra (L, d_L) is the special case

$$\text{Der}(L) = \text{Der}_{\text{id}_L}(L, L).$$

We define a bracket on $\text{Der}(L)$, by

$$[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|} \eta \circ \theta,$$

which makes $(\text{Der}(L), D)$ into a differential graded Lie algebra. Let $L' \subset L$ be a sub-differential graded Lie algebra. The derivations relative to L'

$$\text{Der}(L, \text{rel. } L')$$

is the sub-differential graded Lie algebra of $\text{Der}(L)$ of derivations that annihilate the elements of L' . The positive truncation of a differential graded vector space (V, d_V) is the differential graded vector space V^+ given by

$$V_i^+ = \begin{cases} V_i & \text{for } i \geq 2 \\ \ker(d_V : L_1 \rightarrow L_0) & \text{for } i = 1 \\ 0 & \text{for } i \leq 0 \end{cases}$$

with its obvious differential. Note that we can also consider the positive truncation of a differential graded Lie algebra, which is naturally a differential graded Lie algebra.

Theorem 2.3 ([LS07] and [BM15, Theorem 3.6.]). *Let $f : X \rightarrow Y$ be a map of simply connected CW-complexes with X finite and $\phi_f : \mathbb{L}_X \rightarrow \mathbb{L}_Y$ a Quillen model. There are natural isomorphisms of sets*

$$\pi_k(\text{map}_*(X, Y), f) \otimes \mathbb{Q} \cong H_k(\text{Der}_{\phi_f}(\mathbb{L}_X, \mathbb{L}_Y)), \text{ for } k \geq 1,$$

which are vectorspace isomorphisms for $k > 1$. In the case $X = Y$ and $f = \text{id}_X$, there are isomorphisms of vectorspaces

$$\pi_k(\text{aut}_*(X), \text{id}_X) \otimes \mathbb{Q} \cong H_k(\text{Der}(\mathbb{L}_X)), \text{ for } k \geq 1,$$

and the Samelson product corresponds to the Lie bracket.

The topological monoid of homotopy automorphisms is in general not connected and its classifying space not simply connected. Thus in general we can not get Quillen models of them. One way around this is to restrict to the universal covering.

Theorem 2.4 ([Tan83]). *Let \mathbb{L}_X be a minimal Quillen model of a simply connected space of finite \mathbb{Q} -type. Then*

$$(\text{Der}^+(\mathbb{L}_X), D)$$

is a Quillen model for the universal cover $\text{Baut}_*(X)\langle 1 \rangle$.

Berglund and Madsen observe that Tanré's Theorem extends to relative automorphism spaces.

Theorem 2.5 ([BM15, Theorem 3.4]). *Let $i : A \subset X$ be a cofibration of simply connected spaces of finite \mathbb{Q} -type and let $\mathbb{L}_i : \mathbb{L}_A \subset \mathbb{L}_X$ be a cofibration and a Quillen model for i , then*

$$(\text{Der}(\mathbb{L}_X, \text{rel. } \mathbb{L}_A), D)$$

is a Quillen model for $\text{Baut}_A(X)\langle 1 \rangle$.

2.3 Polynomial functors

In this section we give the definition of polynomial functors in the sense of [EML54], slightly modified as in [Dwy80, Section 3]. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a (not necessarily additive) functor between abelian categories. For $k \geq 1$, the k -th cross-effect functor

$$T^k : \mathcal{A}^k \rightarrow \mathcal{B},$$

is uniquely defined up to isomorphism given T^l for $l < k$ by the properties:

1. $T^k(A_1, \dots, A_k) = 0$ if $A_i = 0$ for some i .
2. There is a natural isomorphism $T(A_1 \oplus \dots \oplus A_k) \cong T(0) \oplus \bigoplus_{\{i_1, \dots, i_r\}} T^r(A_{i_1}, \dots, A_{i_r})$,
where the sum runs over all non-empty subsets $\{i_1, \dots, i_r\} \subset \{1, \dots, k\}$.

Definition 2.6 (Polynomial functors). A functor T is polynomial of degree $\leq k$, if T^l is the constant zero functor for $l > k$.

On objects the first cross-effect functor is given by the kernel of the natural map

$$T^1(A_1) = \text{Kernel}(T(A) \rightarrow T(0)).$$

An immediate consequence is that a functor is of degree ≤ 0 if and only if it is constant. The higher cross-effects can be defined using deviations. The k -fold deviation of a k -tuple of maps

$$(f_1, \dots, f_k) : A \rightarrow B$$

in \mathcal{A} is the map

$$T(f_1 \top \dots \top f_k) : T(A) \rightarrow T(B),$$

given by

$$T(f_1 \top \dots \top f_k) = T(0) + \sum_{\{i_1, \dots, i_r\}} (-1)^{k-r} T(f_{i_1} + \dots + f_{i_r}),$$

where the sum runs over all non-empty subsets $\{i_1, \dots, i_r\} \subset \{1, \dots, k\}$ and 0 denotes the canonical map $A \rightarrow 0 \rightarrow B$. Setting $A = A_1 \oplus \dots \oplus A_k$ and denoting by $\pi_i : A \rightarrow A_i$ the projections and by $\iota_i : A_i \rightarrow A$ the inclusions, the k -th cross-effect functor is given on objects by

$$T^k(A_1, \dots, A_k) = \text{Image}(T((\iota_1 \circ \pi_1) \top \dots \top (\iota_k \circ \pi_k))).$$

Proposition 2.7 (See e.g. [Dwy80]).

1. An additive functor is of degree ≤ 1 .
2. The composition of functors of degree $\leq k$ and $\leq l$ is a functor of degree $\leq kl$.
3. If T is a functor of degree $\leq k$, then $T^2(A, -)$ is of degree $\leq k - 1$ for any fixed A .
4. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ and $R : \mathcal{C} \rightarrow \mathcal{B}$ be of degree $\leq k$ and $\leq l$, respectively. The level-wise sum

$$T \oplus R : \mathcal{A} \times \mathcal{C} \rightarrow \mathcal{B}$$

is polynomial of degree $\leq \max\{k, l\}$

2.4 Schur functors

Schur functors give examples of polynomial functors. Note that the following definitions also make sense for general commutative rings, but we are going to restrict our presentation to the category of graded rational vector spaces $Vect_*(\mathbb{Q})$. Schur functors are treated for example in [LV12]. We couldn't find any literature on Schur multifunctors and hence state the facts we need here.

Let $\mathcal{M} = \{\mathcal{M}(n)\}_{n \geq 0} \in Vect_*(\mathbb{Q})$ be a sequence of $\mathbb{Q}[\Sigma_n]$ -modules. We will refer to them as Σ_n -modules but implicitly use the $\mathbb{Q}[\Sigma_n]$ -modules structure, in particular \otimes_{Σ_n} refers to the tensor product over $\mathbb{Q}[\Sigma_n]$. The Schur functor given by \mathcal{M} is defined to be the endofunctor of $Vect_*(\mathbb{Q})$ induced by

$$\mathcal{M}(V) = \bigoplus_k \mathcal{M}(k) \otimes_{\Sigma_k} V^{\otimes k} \text{ for all } V \in Vect_*(\mathbb{Q}),$$

where $V^{\otimes k}$ is the left Σ_k -module with action of $\sigma \in \Sigma_k$ given by

$$\sigma(v_1 \otimes \dots \otimes v_k) = \pm v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(k)}$$

(with sign according to the Koszul sign convention). Note that $\mathcal{M}(0)$ is just a constant summand. A Schur functor \mathcal{M} with $\mathcal{M}(l)$ trivial for $l > k$ is a polynomial functor of degree $\leq k$.

Let $\eta = (n_1, \dots, n_l)$, $n_1, \dots, n_l \geq 0$ be a multi-index. Throughout this thesis we will assume all multi-indices to have non-negative entries. We want to use the following

conventions:

$$\begin{aligned}
 |\eta| &= \sum_{i=1}^l n_i \\
 l(\eta) &= l \\
 \mu + \eta &= (m_1 + n_1, \dots, m_l + n_l), \text{ for } \mu = (m_1, \dots, m_l) \\
 (V_i)^{\otimes \eta} &= V_1^{\otimes n_1} \otimes \dots \otimes V_l^{\otimes n_l}, \text{ where } (V_i) \in (\text{Vect}_*(\mathbb{Q}))^l \\
 \Sigma_\eta &= \Sigma_{n_1} \times \dots \times \Sigma_{n_l}
 \end{aligned}$$

Consider a sequence of $\mathbb{Q}[\Sigma_\eta]$ -modules $\mathcal{N} = \{\mathcal{N}(\eta)\}_{l(\eta)=l} \in \text{Vect}_*(\mathbb{Q})$. As before, we will from now on refer to them as Σ_η -modules. We define the Schur multifunctor given by \mathcal{N} on objects by

$$\mathcal{N}(V_1, \dots, V_l) = \bigoplus_{l(\eta)=l} \mathcal{N}(\eta) \otimes_{\Sigma_\eta} (V_i)^{\otimes \eta} \text{ for all } (V_i) \in (\text{Vect}_*(\mathbb{Q}))^l.$$

Similarly a Schur multifunctor \mathcal{N} is polynomial of degree $\leq k$ if $\mathcal{N}(\eta)$ is trivial for $|\eta| > k$.

Example Consider Schur functors $\mathcal{N}_i : \text{Vect}_*(\mathbb{Q}) \rightarrow \text{Vect}_*(\mathbb{Q})$, $i = 1, \dots, l$ The tensor product

$$\bigotimes \mathcal{N}_i : \text{Vect}_*(\mathbb{Q})^l \rightarrow \text{Vect}_*(\mathbb{Q})$$

is a Schur multifunctor with $(\bigotimes \mathcal{N}_i)(\eta) = \bigotimes \mathcal{N}_i(n_i)$.

We define the tensor product of $\mathcal{M} = \{\mathcal{M}(\mu)\}$ and $\mathcal{N} = \{\mathcal{N}(\eta)\}$, where $l(\mu) = l(\eta)$ as

$$\mathcal{M} \otimes \mathcal{N}(\nu) = \bigoplus_{\mu' + \eta' = \nu} \text{Ind}_{\Sigma_{\mu'} \times \Sigma_{\eta'}}^{\Sigma_\nu} \mathcal{M}(\mu') \otimes \mathcal{N}(\eta').$$

The Schur functor defined by this tensor product is indeed (up to natural isomorphism) the tensor product of the two functors, as we see by the isomorphisms for $\mu' + \eta' = \nu$ with $l(\mu') = l(\eta') = l(\nu)$

$$\begin{aligned}
 & \left(\text{Ind}_{\Sigma_{\mu'} \times \Sigma_{\eta'}}^{\Sigma_\nu} \mathcal{M}(\mu') \otimes \mathcal{N}(\eta') \right) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu} \\
 &= \left(\mathcal{M}(\mu') \otimes \mathcal{N}(\eta') \otimes_{\Sigma_{\mu'} \times \Sigma_{\eta'}} \mathbb{Q}[\Sigma_\nu] \right) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu} \\
 &\cong \left(\mathcal{M}(\mu') \otimes \mathcal{N}(\eta') \right) \otimes_{\Sigma_{\mu'} \times \Sigma_{\eta'}} \left((V_i)^{\otimes \mu'} \otimes (V_i)^{\otimes \eta'} \right) \\
 &\cong \left(\mathcal{M}(\mu') \otimes_{\Sigma_{\mu'}} (V_i)^{\otimes \mu'} \right) \otimes \left(\mathcal{N}(\eta') \otimes_{\Sigma_{\eta'}} (V_i)^{\otimes \eta'} \right).
 \end{aligned}$$

The tensor powers of a $\mathcal{N} = \{\mathcal{N}(\eta)\}$ are (up to natural isomorphism) explicitly described by

$$\mathcal{N}^{\otimes r}(\nu) = \bigoplus_{\Sigma_{\eta_1} \times \dots \times \Sigma_{\eta_r}}^{\Sigma_\nu} \mathcal{N}(\eta_1) \otimes \dots \otimes \mathcal{N}(\eta_r), \quad (2.1)$$

where the sum runs over all r -tuples (η_1, \dots, η_r) , where $l(\eta_i) = l$, such that $\sum_{i=1}^r \eta_i = \nu$. Now consider a Schur functor $\mathcal{M} = \{\mathcal{M}(m)\}$ and a Schur multifunctor $\mathcal{N} = \{\mathcal{N}(\eta)\}$. The composition is a Schur multifunctor isomorphic to the Schur multifunctor given by

$$(\mathcal{M} \circ \mathcal{N})(\nu) = \bigoplus_r \mathcal{M}(r) \otimes_{\Sigma_r} \bigoplus_{\Sigma_{\eta_1} \times \dots \times \Sigma_{\eta_r}}^{\Sigma_\nu} \mathcal{N}(\eta_1) \otimes \dots \otimes \mathcal{N}(\eta_r), \quad (2.2)$$

where the second sum runs over all r -tuples (η_1, \dots, η_r) , where $l(\eta_i) = l$, such that $\sum_{i=1}^r \eta_i = \nu$. The action of Σ_r is by permuting the tuples (η_1, \dots, η_r) by the inverse. Indeed as we check using (2.1):

$$\begin{aligned} (\mathcal{M} \circ \mathcal{N})(V_i) &= \bigoplus_r \mathcal{M}(r) \otimes_{\Sigma_r} \mathcal{N}(V_i)^{\otimes r} \cong \bigoplus_r \mathcal{M}(r) \otimes_{\Sigma_r} \mathcal{N}^{\otimes r}(V_i) \\ &\cong \bigoplus_{r, \nu} \mathcal{M}(r) \otimes_{\Sigma_r} (\mathcal{N}^{\otimes r}(\nu) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu}) \\ &\cong \bigoplus_{r, \nu} \mathcal{M}(r) \otimes_{\Sigma_r} \left(\bigoplus_{\Sigma_{\eta_1} \times \dots \times \Sigma_{\eta_r}}^{\Sigma_\nu} \mathcal{N}(\eta_1) \otimes \dots \otimes \mathcal{N}(\eta_r) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu} \right) \\ &\cong \bigoplus_\nu \bigoplus_r \mathcal{M}(r) \otimes_{\Sigma_r} \left(\bigoplus_{\Sigma_{\eta_1} \times \dots \times \Sigma_{\eta_r}}^{\Sigma_\nu} \mathcal{N}(\eta_1) \otimes \dots \otimes \mathcal{N}(\eta_r) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu} \right). \end{aligned}$$

Remark 2.8. We will later use Schur (multi)functors with domain the category of rational vector spaces - just consider them as graded rational vector spaces concentrated in degree 0.

2.5 Automorphisms of hyperbolic modules over the integers

In this section we introduce hyperbolic modules in the sense of [Bak81]. Fix a $\lambda \in \{+1, -1\}$. Let $\Lambda \subset \mathbb{Z}$, be an additive subgroup, called the form parameter, such that

$$\{z - \lambda z \mid z \in \mathbb{Z}\} \subset \Lambda \subset \{z \in \mathbb{Z} \mid z = -\lambda z\}. \quad (2.3)$$

A Λ -quadratic module is a pair (M, μ) , where M is a \mathbb{Z} -module and μ is a bilinear form, i.e. a homomorphism

$$\mu : M \otimes M \rightarrow \mathbb{Z}.$$

To a Λ -quadratic module (M, μ) we can associate a Λ -quadratic form

$$q_\mu : M \rightarrow \mathbb{Z}/\Lambda, q_\mu(x) = [\mu(x, x)]$$

and a λ -symmetric bilinear form

$$\langle -, - \rangle_\mu : M \otimes M \rightarrow \mathbb{Z},$$

defined by $\langle x, y \rangle_\mu = \mu(x, y) + \lambda\mu(y, x)$. (λ -symmetric bilinear forms like this are called even.) We call a finitely generated free Λ -quadratic module (M, μ) non-degenerate, if the map

$$M \rightarrow M^*, x \mapsto \langle x, - \rangle_\mu$$

is an isomorphism. Denote by $Q^\lambda(\mathbb{Z}, \Lambda)$ the category non-degenerate Λ -quadratic modules and morphisms linear maps preserving the associated λ -symmetric bilinear form and the associated Λ -quadratic form.

Given a finitely generated free \mathbb{Z} -module M , we can define a non-degenerate Λ -quadratic module $H(M) = (M \oplus M^*, \mu_M)$, where $\mu_M((x, f), (y, g)) = f(y)$. We call $H(M)$ the hyperbolic module on M . A Λ -quadratic module is called hyperbolic, if it is isomorphic to $H(N)$ for some finitely generated \mathbb{Z} -module N .

Let $\{e_i\}$ be the standard basis for \mathbb{Z}^g and $\{f_i\}$ the dual basis of $(\mathbb{Z}^g)^*$. The bilinear form of

$$H(\mathbb{Z}^g) = \mathbb{Z}\langle e_1, \dots, e_n \rangle \oplus \mathbb{Z}\langle f_1, \dots, f_n \rangle$$

is determined by

$$\mu_{\mathbb{Z}^g}(e_i, f_j) = f_j(e_i) = \delta_{i,j} \text{ and } \mu_{\mathbb{Z}^g}(e_i, e_j) = \mu_{\mathbb{Z}^g}(f_i, e_j) = \mu_{\mathbb{Z}^g}(f_i, f_j) = 0.$$

Identify $\mathbb{Z}\langle e_1, \dots, e_n \rangle \oplus \mathbb{Z}\langle f_1, \dots, f_n \rangle \cong \mathbb{Z}\langle a_1, \dots, a_{2g} \rangle$ by setting $a_i = e_i$ and $a_{i+g} = f_i$ for $i = 1, \dots, g$. The associated λ -symmetric bilinear form is now given by the matrix

$$(\langle a_i, a_j \rangle_{\mu_{\mathbb{Z}^g}})_{i,j} = \begin{pmatrix} 0 & \lambda 1 \\ 1 & 0 \end{pmatrix},$$

where 1 denotes the $g \times g$ identity matrix.

We consider the automorphisms of the Λ -quadratic module $H(\mathbb{Z}^g)$ as a subgroup of $Gl_{2g}(\mathbb{Z})$. The subgroups can be described as follows:

Proposition 2.9 ([Bak81, Corollary 3.2.]). *The automorphism group of $H(\mathbb{Z}^g)$ in $Q^\lambda(\mathbb{Z}, \Lambda)$ is isomorphic to the subgroup of $Gl_{2g}(\mathbb{Z})$ consisting of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that*

$$D^T A + \lambda B^T C = 1$$

$$D^T B + \lambda B^T D = 0$$

$$A^T C + \lambda C^T A = 0$$

$$C^T A \text{ and } D^T B \text{ have diagonal entries in } \Lambda.$$

Note that if $\lambda = 1$ we necessarily have $\Lambda = 0$. When $\lambda = -1$, the condition (2.3) implies that $2\mathbb{Z} \subset \Lambda \subset \mathbb{Z}$, thus we have the two cases $\Lambda = \mathbb{Z}$ and $\Lambda = 2\mathbb{Z}$. Thus we can list the automorphisms of hyperbolic modules:

1. When $\lambda = 1$ and $\Lambda = 0$, then $\text{Aut}(H(\mathbb{Z}^g)) = O_{g,g}(\mathbb{Z})$ in $Q^1(\mathbb{Z}, 0)$.
2. When $\lambda = -1$ and $\Lambda = \mathbb{Z}$, then $\text{Aut}(H(\mathbb{Z}^g)) = Sp_{2g}(\mathbb{Z})$ in $Q^{-1}(\mathbb{Z}, \mathbb{Z})$.
3. When $\lambda = -1$ and $\Lambda = 2\mathbb{Z}$, then $\text{Aut}(H(\mathbb{Z}^g))$ in $Q^{-1}(\mathbb{Z}, 2\mathbb{Z})$ is the subgroup of $Sp_{2g}(\mathbb{Z})$ described as:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \mid C^T A \text{ and } D^T B \text{ have even entries at the diagonal} \right\}.$$

Let N be a $(d-1)$ -connected $2d$ -manifold. Wall [Wal62] has show that the automorphisms of the homology realized by diffeomorphisms are the automorphisms of a Λ -quadratic module with underlying \mathbb{Z} -module $H_d(N)$. Later we will show a similar statement for connected sums of products of spheres. For this we need a slight variation of Λ -quadratic modules.

Let $n \in \mathbb{N}$ and for $n = 2d$ let $\Lambda \subset \mathbb{Z}$, be an additive subgroup, such that

$$\{z - (-1)^d z \mid z \in \mathbb{Z}\} \subset \Lambda \subset \{z \in \mathbb{Z} \mid z = -(-1)^d z\}.$$

A graded Λ -quadratic module is a pair (M_*, μ) , where M_* is a graded \mathbb{Z} -modules and μ bilinear n -pairing, i.e. a degree 0 homomorphism

$$\mu : M_* \otimes M_* \rightarrow \mathbb{Z}[n],$$

where $\mathbb{Z}[n]$ denotes the graded \mathbb{Z} -module with a \mathbb{Z} in degree n . We can associate to (M_*, μ) a graded symmetric bilinear n -pairing

$$\langle -, - \rangle_\mu : M_* \otimes M_* \rightarrow \mathbb{Z}[n],$$

defined by

$$\langle x, y \rangle_\mu = \mu(x, y) + (-1)^{|x||y|} \mu(y, x).$$

By graded symmetric we mean that

$$\langle x, y \rangle_\mu = (-1)^{|x||y|} \langle y, x \rangle_\mu.$$

When $n = 2d$, we associate a Λ -quadratic form

$$q_\mu : M_d \rightarrow \mathbb{Z}/\Lambda, q_\mu(x) = [\mu(x, x)].$$

We call a finitely generated free graded Λ -quadratic module (M_*, μ) non-degenerate, if the map

$$M_* \rightarrow \text{Hom}(M_*, \mathbb{Z}[n]), x \mapsto \langle x, - \rangle_\mu$$

is an isomorphism.

For n even, we define $Q_*^n(\mathbb{Z}, \Lambda)$ to be the category whose objects are non-degenerate graded Λ -quadratic modules (M_*, μ) , where M_* is a finitely generated free graded \mathbb{Z} -module and the morphisms respect q_μ and $\langle -, - \rangle_\mu$.

For n odd, we define $Q_*^n(\mathbb{Z}, \Lambda)$ to have objects non-degenerate graded Λ -quadratic modules (M_*, μ) , where M_* is a finitely generated free graded \mathbb{Z} -module and morphisms respecting $\langle -, - \rangle_\mu$. Note that in the case that n is odd the Λ in the notation is unnecessary, but we keep it in the notion for convenience.

Let $Q_+^n(\mathbb{Z}, \Lambda)$ be the full subcategory with objects concentrated in positive degrees and hence necessarily concentrated in degrees $1, \dots, n-1$. Now let $|I|$ be a finite indexing set and for $i \in I$ let $p_i \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$. Denote by

$$\mathbb{Z}_I = (\mathbb{Z}^{g_1}[1] \oplus \dots \oplus \mathbb{Z}^{g_{\lfloor n/2 \rfloor}}[\lfloor n/2 \rfloor]),$$

where $g_k = \#\{i \in I | p_i = k\}$. We define a graded Λ -quadratic module

$$H_I = \mathbb{Z}_I \oplus \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_I, \mathbb{Z}[n]).$$

Denote by $\{a_i\}$ the standard basis for

$$\mathbb{Z}^{g_1}[1] \oplus \dots \oplus \mathbb{Z}^{g_{\lfloor n/2 \rfloor}}[\lfloor n/2 \rfloor]$$

and by $\{b_i\}$ the dual basis of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_I, \mathbb{Z}[n])$. The pairing $\mu_{H_I} = \mu_I$ is given by

$$\mu_I(a_i, b_j) = b_j(a_i) = \delta_{i,j} \text{ and } \mu_I(a_i, a_j) = \mu_I(b_i, b_j) = \mu_I(b_j, a_i) = 0.$$

This definition is motivated by the following: Let

$$N_I = (\#_{i \in I}(S^{p_i} \times S^{q_i})) \setminus \text{int}(D^n),$$

where $q_i = n - p_i$. The canonical inclusions

$$\alpha_i : S^{p_i} \hookrightarrow N_I \text{ and } \beta_i : S^{q_i} \hookrightarrow N_I$$

give us a basis $\{a_i\} \cup \{b_i\}$ for $\tilde{H}_*(N_I)$ via the Hurewicz homomorphism. We will later show that the intersection pairing is the associated graded symmetric form of the graded hyperbolic module $H_I \cong \tilde{H}_*(N_I)$.

Denote by $\Gamma_I = \text{Aut}(H_I)$ in $Q_+^n(\mathbb{Z}, \Lambda)$. We get the following cases:

1. When n is odd we get

$$\Gamma_I \cong \prod_{k=1}^{\lfloor n/2 \rfloor} Gl_{g_k}(\mathbb{Z}).$$

2. When $n = 2d$ and d is even we necessarily have $\Lambda = 0$ and

$$\Gamma_I \cong O_{g_d, g_d}(\mathbb{Z}) \times \prod_{k=1}^{n/2-1} Gl_{g_k}(\mathbb{Z}).$$

3. Similarly for $n = 2d$ with d odd the only cases are $\Lambda = \mathbb{Z}, 2\mathbb{Z}$ and we just get a products of general linear groups and $Sp_{2g_d}(\mathbb{Z})$ respectively the subgroup described in the list of automorphism groups above under point 3.

2.6 Van der Kallen's and Charney's homological stability results

In this section we recall van der Kallen's homological stability for general linear groups and Charney's homological stability for automorphisms of hyperbolic quadratic modules. We will combine them to homological stability for the Γ_I defined above with certain coefficient systems induced by polynomial functors.

Remark 2.10. Charney's results hold for Dedekind domains with involutions and van der Kallen's for associative rings with finite stable range, but we restrict our presentation to \mathbb{Z} (with trivial involution).

We begin by reviewing the notion of coefficient systems as discussed in [Dwy80]. A coefficient system for $\{Gl_g(\mathbb{Z})\}_{g \geq 1}$ is a sequence of $Gl_g(\mathbb{Z})$ -modules $\{\rho_g\}_{g \geq 1}$ together with $Gl_g(\mathbb{Z})$ -maps $F_g : \rho_g \rightarrow I^*(\rho_{g+1})$, where I^* denotes the restriction via the upper inclusion

$$I : Gl_g(\mathbb{Z}) \rightarrow Gl_{g+1}(\mathbb{Z}), A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We will denote the system by ρ and call the maps F_g structure maps. A map of coefficient systems ρ and ρ' is a collection of $Gl_g(\mathbb{Z})$ -maps $\{\tau_g\}_{g \geq 1}$ commuting with the structure maps. The level-wise kernels and cokernels are again coefficient systems with the obvious structure maps. Denote by

$$J : Gl_g(\mathbb{Z}) \rightarrow Gl_{g+1}(\mathbb{Z}), A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

the lower inclusion map. For a coefficient system ρ we define the shifted system $\Sigma\rho$ by $\Sigma\rho_g = J^*(\rho_{g+1})$ with structure maps $\Sigma F_g = J^*F_{g+1} : J^*(\rho_{g+1}) \rightarrow I^*J^*(\rho_{g+2})$. Denote

by $s_g \in Gl_g(\mathbb{Z})$ the element permuting the last two standard basis elements. We call a coefficient system central, if s_{g+2} acts trivially on the image of $F_{g+1}F_g : \rho_g \rightarrow \rho_{g+2}$. Denote by $e_{g-1,g} \in Gl_g(\mathbb{Z})$ the element sending all but the g -th standard basis element to itself and the g -th e_g to $e_{g-1} + e_g$. We call a central coefficient system strongly central, if $e_{g+1,g+2}$ acts trivially on the image of $F_{g+1}F_g : \rho_g \rightarrow \rho_{g+2}$.

Let $c_g \in Gl_g(\mathbb{Z})$ ($g > 1$) be the element sending the i -th standard basis element to the $(i + 1)$ -st and the g -th to the first.

Denote by $\mu(c_g)$ the multiplication from the left by c_g . Then the following holds:

Lemma 2.11 ([Dwy80, Lemma 2.1.]). *Let ρ be a central coefficient system. Then we have a map of coefficient systems $\tau : \rho \rightarrow \Sigma\rho$, defined by*

$$\tau_g : \rho_g \xrightarrow{F_g} I^*(\rho_{g+1}) \xrightarrow{\mu(c_{g+2})} J^*(\rho_{g+1}) = \Sigma\rho_g.$$

We say that a central coefficient system ρ splits, if $\Sigma\rho$ is isomorphic to $\rho \oplus \text{coker}(\tau)$ via τ . We then denote $\text{coker}(\tau)$ by $\Delta\rho$. We now define the notion of degree of a strongly central coefficient system ρ inductively. We say it has degree $k = 0$, if it is constant, i.e. the F_g are isomorphism for all g . For $k > 0$ we say that it has degree $\leq k$, if $\Sigma\rho$ splits and $\Delta\rho$ is a strongly central coefficient system of degree $k - 1$.

Theorem 2.12 ([vdK80, p. 291]). *Let ρ be a strongly central coefficient system of degree $\leq k$, then*

$$H_i(Gl_g(\mathbb{Z}), \rho_g) \rightarrow H_i(Gl_{g+1}(\mathbb{Z}), \rho_{g+1})$$

is an isomorphism for $g > 2i + k + 2$ and an epimorphism for $g \geq 2i + k + 2$.

Denote by λ_g the standard representation of $Gl_g(\mathbb{Z})$ on \mathbb{Z}^g and by $\bar{\lambda}_g$ the action by the inverse transposed on \mathbb{Z}^g . Let \mathcal{A} be an abelian category. Given a functor

$$T : \text{Mod}(\mathbb{Z}) \times \text{Mod}(\mathbb{Z}) \rightarrow \mathcal{A},$$

we can define a coefficient system $\{T(\lambda_g, \bar{\lambda}_g)\}_{g \geq 1}$ with structure maps induced by the standard inclusions and actions induced by λ_g and $\bar{\lambda}_g$.

Lemma 2.13 ([vdK80, 5.5.] and [Dwy80, Lemma 3.1.]). *If*

$$T : \text{Mod}(\mathbb{Z}) \times \text{Mod}(\mathbb{Z}) \rightarrow \mathcal{A}$$

is a polynomial functor of degree $\leq k$, then $\{T(\lambda_g, \bar{\lambda}_g)\}_{g \geq 1}$ is a strongly central coefficient system of degree $\leq k$.

Denote by G_g the automorphisms of $H(\mathbb{Z}^g)$ in $Q^\lambda(\mathbb{Z}, \Lambda)$. Denote by e_1, \dots, e_g the standard basis for \mathbb{Z}^g and by f_1, \dots, f_g the dual basis of $(\mathbb{Z}^g)^*$. We can see G_g as a subgroup of

$Gl_{2g}(\mathbb{Z})$, by considering the elements of G_g as $2g \times 2g$ -matrices acting on $H(\mathbb{Z}^g) \cong \mathbb{Z}^{2g}$. We define the upper inclusion

$$I : G_g \rightarrow G_{g+1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and similarly the lower inclusion $J : G_g \rightarrow G_{g+1}$. The definition of a coefficient system is very similar to the one for $Gl_g(\mathbb{Z})$ and we only briefly summarize it. A coefficient system for $\{G_g\}_{g \geq 1}$ is a sequence of $G_g(\mathbb{Z})$ -modules $\{\rho_g\}_{g \geq 1}$ together with G_g -maps $F_g : \rho_g \rightarrow I^*(\rho_{g+1})$. We will denote a coefficient system again by ρ and let maps of coefficient systems be defined analogous to above. The shifted coefficient system $\Sigma\rho$ is the restriction via the lower inclusion as above. A coefficient system is called central if $s_{g+2} \circ s_{2g+4}$ acts trivially on the image of $F_{g+1}F_g : \rho_g \rightarrow \rho_{g+2}$. For a central coefficient system we define the map of coefficient systems $\tau : \rho \rightarrow \Sigma\rho$, by

$$\tau_g : \rho_g \xrightarrow{F_g} I^*(\rho_{g+1}) \xrightarrow{\mu(c_{g+2} \circ c_{2g+4})} J^*(\rho_{g+1}) = \Sigma\rho_g.$$

We call a central coefficient system ρ split, if τ is injective and $\Sigma\rho \cong \tau(\rho) \oplus \text{coker}(\tau)$. For a central coefficient system we define the degree inductively: We say it has degree $k = 0$, if it is constant. For $k \geq 0$ we say that it has degree $\leq k$, if $\Sigma\rho$ splits and $\text{coker}(\tau)$ is a central coefficient system of degree $\leq k - 1$.

Theorem 2.14 ([Cha87, Theorem 4.3.]). *Let ρ be a central coefficient system of degree $\leq k$, then*

$$H_i(G_g, \rho_g) \rightarrow H_i(G_{g+1}, \rho_{g+1})$$

is an isomorphism for $g > 2i + k + 4$ and an epimorphism for $g \geq 2i + k + 4$.

Again we can get a central coefficient system of degree $\leq k$ by considering the standard G_g -action $\lambda_{g,g}$ on $H(\mathbb{Z}^g) \cong \mathbb{Z}^{2g}$, induced by the inclusion $G_g \subset Gl_{2g}(\mathbb{Z})$. Let \mathcal{A} be an abelian category. Given a polynomial functor

$$T : \text{Mod}(\mathbb{Z}) \rightarrow \mathcal{A},$$

of degree $\leq k$ then $\{T(\lambda_{g,g})\}_{g \geq 1}$ is a central coefficient system of degree $\leq k$ for $\{G_g\}_{g \geq 1}$.

Now we are going to combine van der Kallen's and Charney's homological stability results, to get homological stability for $\Gamma_I = \text{Aut}(H_I)$ in $Q_+^n(\mathbb{Z}, \Lambda)$. Recall that I was a

finite indexing set for natural numbers $p_i \in \{1, \dots, \lfloor n/2 \rfloor\}$ and g_k denoted the number of $p_i = k$. Let

$$r_k = \begin{cases} g_k & \text{if } 0 < k < n/2 \\ 2g_k & \text{if } k = n/2 \\ g_{n-k} & \text{if } n/2 < k < n. \end{cases}$$

Denote by λ_I the standard representation of Γ_I on $(\mathbb{Z}^{r_1}, \dots, \mathbb{Z}^{r_{n-1}}) \in \text{Mod}(\mathbb{Z})^{n-1}$ induced by the obvious inclusion $\Gamma_I \subset \prod_{k=1}^{n-1} \text{Gl}_{r_k}(\mathbb{Z})$. A functor

$$T : \text{Mod}(\mathbb{Z})^{n-1} \rightarrow \mathcal{A}$$

induces a Γ_I -module

$$T(\mathbb{Z}^{r_1}, \dots, \mathbb{Z}^{r_{n-1}}).$$

We will denote this Γ_I -module by $T(\lambda_I)$. For a fixed $p \in \mathbb{N}$, such that $0 < p \leq \lfloor n/2 \rfloor$, denote by $\Gamma_{I'}$ the automorphisms of $H_{I'}$, where $I' = I \cup \{i'\}$ with $p_{i'} = p$. We define the stabilization map

$$\sigma_{p,n-p} : H_i(\Gamma_I, T(\lambda_I)) \rightarrow H_i(\Gamma_{I'}, T(\lambda_{I'}))$$

to be the map induced by the obvious upper inclusion $I_{p,n-p} : \Gamma_I \rightarrow \Gamma_{I'}$ and $T(I_{p,n-p})$.

Proposition 2.15. *Let \mathcal{A} be an abelian category, $T : \text{Mod}(\mathbb{Z})^{n-1} \rightarrow \mathcal{A}$ be a polynomial functor of degree $\leq k$. The stabilization map*

$$\sigma_{p,n-p} : H_i(\Gamma_I, T(\lambda_I)) \rightarrow H_i(\Gamma_{I'}, T(\lambda_{I'}))$$

induces an isomorphism for $g_p > 2i + k + 2$ when $2p \neq n$ and $g_p > 2i + k + 4$ if $2p = n$ and an epimorphism for $g_p \geq 2i + k + 2$ respectively $g_p \geq 2i + k + 4$.

Proof. Note that

$$\Gamma_I \subset \prod_{k=1}^{\lfloor n/2 \rfloor} \text{Gl}_{r_k}(\mathbb{Z}) \subset \prod_{k=1}^{n-1} \text{Gl}_{r_k}(\mathbb{Z}),$$

because of the compatibility with the pairing. Denote by Γ_{g_p} the subgroup that sits in $\text{Gl}_{r_p}(\mathbb{Z})$. Let $\Gamma = \text{Aut}(H_{\bar{I}})$, where $\bar{I} = I \setminus \{i \in I | p_i = p\}$. Note that $\Gamma_I = \Gamma \times \Gamma_{g_p}$ and $\Gamma_{I'} = \Gamma \times \Gamma_{g_p+1}$, where Γ_{g_p+1} is defined analogous to Γ_{g_p} . Consider the functor

$$\mathfrak{J}_{p,n-p} : \begin{cases} \text{Mod}(\mathbb{Z}) \rightarrow \text{Mod}(\mathbb{Z})^{n-1} & \text{if } p = n/2 \\ \text{Mod}(\mathbb{Z}) \times \text{Mod}(\mathbb{Z}) \rightarrow \text{Mod}(\mathbb{Z})^{n-1} & \text{otherwise,} \end{cases}$$

defined by sending a module M to $(\mathbb{Z}^{r_1}, \dots, M, \dots, \mathbb{Z}^{r_{n-1}})$, where the M sits at the $(n/2)$ -th position and a pair (M, N) to $(\mathbb{Z}^{r_1}, \dots, M, \dots, N, \dots, \mathbb{Z}^{r_{n-1}})$, where the M sits at the p -th position the N sits at the $(n-p)$ -th position. This functor is clearly additive and

hence of degree ≤ 1 . This implies that the composition $T \circ \mathfrak{I}_{p,n-p}$ is of degree $\leq k$. Note that for $p \neq n/2$ the Γ_{g_p} -module obtained by restricting $T(\lambda_I)$ is naturally isomorphic to $T \circ \mathfrak{I}_{p,n-p}(\lambda_{g_p}, \bar{\lambda}_{g_p})$. For $p = n/2$ the Γ_{g_p} -module obtained by restricting $T(\lambda_I)$ is naturally isomorphic to $T \circ \mathfrak{I}_{p,n-p}(\lambda_{g_p, g_p})$. Hence we get a (strongly) central coefficient system of degree $\leq k$ for

$$\Gamma_{g_p} = \begin{cases} G_{g_p} & \text{if } p = n/2 \\ Gl_{g_p}(\mathbb{Z}) & \text{otherwise.} \end{cases}$$

This implies using Charney's and van der Kallen's stability results that the stabilization maps

$$H_i(\Gamma_{g_p}, T \circ \mathfrak{I}_{n/2, n/2}(\lambda_{g_p, g_p})) \rightarrow H_i(\Gamma_{g_p+1}, T \circ \mathfrak{I}_{n/2, n/2}(\lambda_{g_p+1, g_p+1}))$$

and

$$H_i(\Gamma_{g_p}, T \circ \mathfrak{I}_{p, n-p}(\lambda_{g_p}, \bar{\lambda}_{g_p})) \rightarrow H_i(\Gamma_{g_p+1}, T \circ \mathfrak{I}_{p, n-p}(\lambda_{g_p+1}, \bar{\lambda}_{g_p+1}))$$

are isomorphisms and epimorphisms in the ranges in the statement of the proposition. Observing that the $T \circ \mathfrak{I}_{n/2, n/2}(\lambda_{g_p, g_p})$ respectively $T \circ \mathfrak{I}_{p, n-p}(\lambda_{g_p}, \bar{\lambda}_{g_p})$ are precisely the restrictions of the Γ_I -representation to the subgroup Γ_{g_p} . The results follows by comparing spectral sequences. \square

2.7 Rationally perfect groups

A group G is called rationally perfect, if $H^1(G; V) = 0$ for any finite dimensional rational G -representation V . We will need that the automorphism groups of graded hyperbolic modules are rationally perfect.

Lemma 2.16. *The groups Γ_I are rationally perfect.*

Proof. We begin by observing that being rationally perfect is stable under group extensions, i.e. if in a group extension

$$0 \rightarrow K \rightarrow G \rightarrow C \rightarrow 0,$$

K and C are rationally perfect, then so is G . This follows from the Lyndon spectral sequence, since $H^1(C; H^0(K; V))$ and $H^0(C; H^1(K; V))$ are trivial for any finite dimensional rational G -representation V . In particular products of rationally perfect groups are rationally perfect. Moreover we observe that finite groups are rationally perfect.

It follows from Borel's work on the cohomology of arithmetic groups that the automorphism groups G_g of the hyperbolic modules $H(\mathbb{Z}^g)$ in $Q^\lambda(\mathbb{Z}, \Lambda)$ are rationally perfect for $g \geq 2$ (see e.g. [BM15]).

In [BMS67] it is shown that $Sl_g(\mathbb{Z})$ is rationally perfect for $g \geq 3$. Since $Sl_g(\mathbb{Z})$ is an

index two normal subgroup of $Gl_g(\mathbb{Z})$, this now also implies that $Gl_g(\mathbb{Z})$ is rationally perfect for $g \geq 3$.

Recall that the Γ_I are products of G_g -s and $Gl_g(\mathbb{Z})$ -s. Since $Gl_1(\mathbb{Z})$ is finite and hence rationally perfect, to finish the proof we have to show that G_1 and $Gl_2(\mathbb{Z})$ are rationally perfect.

The group $Sl_2(\mathbb{Z})$ is an extension

$$0 \rightarrow C_2 \rightarrow Sl_2(\mathbb{Z}) \rightarrow C_2 * C_3 \rightarrow 0$$

and $C_2 * C_3$ can be seen to be rationally perfect using a Mayer-Vietoris argument. Hence $Sl_2(\mathbb{Z})$ and also $Gl_2(\mathbb{Z})$ are rationally perfect.

The group G_1 in $Q^{-1}(\mathbb{Z}, \mathbb{Z})$ is $Sp_2(\mathbb{Z})$, which is isomorphic to $Sl_2(\mathbb{Z})$. For $\Lambda = 2\mathbb{Z}$ it follows that G_1 is rationally perfect because it is a finite index subgroup of $Sp_2(\mathbb{Z})$.

Recall that for $\lambda = 1$, we necessarily have $\Lambda = 0$ and $G_1 \cong O_{1,1}(\mathbb{Z}) \cong C_2 \times C_2$ and hence we are done. \square

We will later need the following consequences:

Proposition 2.17 (see e.g. [BM15, Proposition B.5 and Lemma B.1]). *Let G be a rationally perfect group and let C_* be a chain complex of $\mathbb{Q}[G]$ -modules that is degree-wise finite dimensional over \mathbb{Q} . Then there is a chain homotopy equivalence*

$$p_C : C_* \rightarrow H_*(C)$$

of $\mathbb{Q}[G]$ -chain complexes such that $p_C(z) = [z]$ if z is a cycle. Moreover for a chain map $f : C_* \rightarrow D_*$ between $\mathbb{Q}[G]$ -chain complexes as above the following diagram

$$\begin{array}{ccc} C_* & \xrightarrow{f} & D_* \\ p_C \downarrow & & \downarrow p_D \\ H_*(C) & \xrightarrow{H_*(f)} & H_*(D) \end{array}$$

commutes up to chain homotopy of $\mathbb{Q}[G]$ -chain complexes.

2.8 Surgery theory and automorphism spaces of manifolds

Surgery theory is a method to enumerate the number of different compact manifolds of a given dimension $n \geq 5$ within a given homotopy type. We are going to restrict our presentation to homotopy types given by simply connected compact n -manifolds. For a detailed description see e.g. [Bro72] or [Wal99].

Let X be a simply connected compact oriented smooth n -manifold, $n \geq 6$, with boundary ∂X . The structure set $\mathcal{S}^{G/O}(X, \partial X)$ consists of equivalence classes of oriented n -manifolds with boundary $(Y, \partial Y)$ together with orientation preserving homotopy equivalences

$$f : (Y, \partial Y) \rightarrow (X, \partial X),$$

such that $f|_{\partial Y} : \partial Y \rightarrow \partial X$ are diffeomorphisms. Two elements

$$f_i : (Y_i, \partial Y_i) \rightarrow (X, \partial X), \quad i = 1, 2$$

are equivalent, if there exists a diffeomorphism

$$F : (Y_1, \partial Y_1) \rightarrow (Y_2, \partial Y_2),$$

such that

$$\begin{array}{ccc} (Y_1, \partial Y_1) & \xrightarrow{F} & (Y_2, \partial Y_2) \\ & \searrow f_1 & \swarrow f_2 \\ & (X, \partial X) & \end{array}$$

commutes up to homotopy relative to the boundaries. The set of normal invariants $\mathcal{N}^{G/O}(X, \partial X)$ consists of equivalence classes of degree one normal maps, i.e. orientation preserving maps from manifolds $(Y, \partial Y)$

$$f : (Y, \partial Y) \rightarrow (X, \partial X),$$

such that $f|_{\partial Y} : \partial Y \rightarrow \partial X$ are diffeomorphisms, together with fiberwise isomorphisms of vector bundles $\hat{f} : \nu_Y \rightarrow \xi$, where ν_Y denotes the normal bundle of $Y \subset \mathbb{R}^N$ ($N \gg n$) and ξ is some vector bundle over X , such that $\xi|_{\partial X}$ is stably equivalent to $\nu_{\partial X}$. Two degree one normal maps

$$f_i : (Y_i, \partial Y_i) \rightarrow (X, \partial X), \quad i = 1, 2$$

are equivalent, if there exists a degree one normal cobordism between them. A degree one normal cobordism is a $(n + 1)$ -manifold $(W, \partial W)$, where $\partial W = Y_1 \cup U \cup Y_2$ and $\partial U = \partial Y_1 \cup \partial Y_2$, together with a degree one normal map

$$F : (W, \partial W) \rightarrow (X \times I; \partial(X \times I)),$$

such that $F|(Y_i, \partial Y_i) = f_i$ as normal maps for $i = 1, 2$ and $F|_U : U \rightarrow \partial X \times I$ is a diffeomorphism. The normal invariant map

$$\eta : \mathcal{S}^{G/O}(X, \partial X) \rightarrow \mathcal{N}^{G/O}(X, \partial X)$$

is defined by considering an element of the structure set as one of the normal invariant set, using that a diffeomorphism $F : (Y_1, \partial Y_1) \rightarrow (Y_2, \partial Y_2)$ gives rise to a degree one normal cobordism. The question now is, if a degree one normal map came from an element of the structure set. The failure is measured by the surgery obstruction

$$\sigma : \mathcal{N}^{G/O}(X, \partial X) \rightarrow L_n(\mathbb{Z}),$$

where

$$L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{otherwise.} \end{cases}$$

The result is a long exact sequence of groups of $k \geq 1$ and of sets for $k = 0$:

$$\dots \rightarrow \mathcal{S}^{G/O}(X \times D^k, \partial(X \times D^k)) \xrightarrow{\eta} \mathcal{N}^{G/O}(X \times D^k, \partial(X \times D^k)) \xrightarrow{\sigma} L_{n+k}(\mathbb{Z}) \rightarrow \dots$$

the so called Surgery exact sequence. There is a different description of the normal invariant set due to Sullivan. Let $O = \bigcup_n O(n)$, where $O(n)$ denotes the orthogonal group and $G = \bigcup_n G(n)$, where $G(n)$ is the topological monoid of homotopy self-equivalences of S^{n-1} . Denote by G/O the homotopy fiber of the map

$$BO \rightarrow BG,$$

where the map is induced by considering an element of the orthogonal group $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as a self-equivalence of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Sullivan showed that there is a natural isomorphism

$$\mathcal{N}^{G/O}(X, \partial X) \cong [X/\partial X, G/O].$$

Quinn shows in his thesis [Qui70] that there is a quasi-fibration

$$S(X, \partial X) \rightarrow \text{map}_*(X/\partial X, G/O) \rightarrow L(X)$$

and that its homotopy exact sequence is the surgery exact sequence. The structure space $S(X, \partial X)$ is related to automorphism spaces of the manifold X as we will now indicate. Denote by $\widetilde{\text{Aut}}_\partial(X)$ the Δ -monoid of block homotopy equivalences, with k -simplices face preserving homotopy equivalences

$$f : \Delta^k \times X \rightarrow \Delta^k \times X,$$

s.t. $f|_{\Delta^k \times \partial X}$ is the identity. The block diffeomorphisms $\widetilde{\text{Diff}}_\partial(X)$ are the sub- Δ -group with k -simplices, face preserving diffeomorphisms

$$f : \Delta^k \times X \rightarrow \Delta^k \times X.$$

We will not distinguish between Δ -objects and their realizations. Denote the inclusion $\widetilde{\text{Diff}}_{\partial}(X) \hookrightarrow \widetilde{\text{Aut}}_{\partial}(X)$ by \tilde{J} . By a result of Dold [Dol63] homotopy equivalences of fibrations over paracompact base spaces are homotopic to fiberwise homotopy equivalences. Hence the inclusion

$$\text{aut}_{\partial}(X) \hookrightarrow \widetilde{\text{Aut}}_{\partial}(X)$$

is a homotopy equivalences. The block diffeomorphisms $\widetilde{\text{Diff}}_{\partial}(X)$ and the diffeomorphisms $\text{Diff}_{\partial}(X)$ with the Whitney C^{∞} -topology on the other hand are not homotopy equivalent - the difference is related to algebraic K-theory (see [WW01]). The homogeneous space $\widetilde{\text{Aut}}_{\partial}(X)/\widetilde{\text{Diff}}_{\partial}(X)$ is by definition the homotopy fiber of the map $\tilde{J} : B\widetilde{\text{Diff}}_{\partial}(X) \rightarrow B\widetilde{\text{Aut}}_{\partial}(X)$. Denote by $\widetilde{\text{Aut}}_{\partial}(X)/\widetilde{\text{Diff}}_{\partial}(X)_{(1)}$ the component getting hit by the identity in $\widetilde{\text{Diff}}_{\partial}(X)$ and by $S(X, \partial X)_{(1)}$ the component of $S(X, \partial X)$ containing id_X .

Proposition 2.18 (see e.g. [BM13]). *There is a natural weak homotopy equivalence*

$$\widetilde{\text{Aut}}_{\partial}(X)/\widetilde{\text{Diff}}_{\partial}(X)_{(1)} \simeq_{w.e.} S(X, \partial X)_{(1)}.$$

We will conclude this section with a lemma that we will need for computations later. Consider three simply connected n -manifolds X_1 , X_2 and X_3 with boundary. Let $f : X_1 \rightarrow X_2$ and $g : X_2 \rightarrow X_3$ be homotopy equivalences such that $f|_{\partial X_1} : \partial X_1 \rightarrow \partial X_2$ and $g|_{\partial X_2} : \partial X_2 \rightarrow \partial X_3$ are diffeomorphisms.

Lemma 2.19 ([BM13, Lemma 3.3.]). *In $[X_3/\partial X_3, G/O]$ we have*

$$\eta(g \circ f) = (g^*)^{-1}(\eta(f)) + \eta(g).$$

3 On certain derivation Lie algebras

The goal of this chapter is to show that the Chevalley-Eilenberg chains of certain derivation Lie algebras can be described as the value of a Schur multifunctor. For this we establish graded version of [BM15, Section 5.2].

Let $n \in \mathbb{N}$. We define $\mathcal{S}_*^n(\mathbb{Q})$ to be the category whose objects are graded, finite dimensional, rational vector spaces V with a non-degenerate, graded symmetric, bilinear n -pairing. More precisely, a degree 0 homomorphism

$$\langle -, - \rangle_V : V \otimes V \rightarrow \mathbb{Q}[n],$$

such that $\langle x, y \rangle_V = (-1)^{|x||y|} \langle y, x \rangle_V$ for homogeneous elements $x, y \in V$ and such that the adjoint map

$$D_V : V \ni x \mapsto \langle x, - \rangle_V \in \text{Hom}_{\mathbb{Q}}(V, \mathbb{Q}[n])$$

is an isomorphism. The morphisms in $\mathcal{S}_*^n(\mathbb{Q})$ are linear maps $f : V \rightarrow W$ of degree 0 such that

$$\langle x, y \rangle_V = \langle f(x), f(y) \rangle_W, \text{ for all } x, y \in V.$$

Example Consider the category of closed, orientable n -manifold with non-empty boundary and degree 1 homotopy equivalences. The rational homology relative boundary together with the intersection pairing defines a functor to the category $\mathcal{S}_*^n(\mathbb{Q})$.

The adjoint $f^! : W \rightarrow V$ of $f : V \rightarrow W$ is the unique linear map such that

$$\langle f^!(x), y \rangle_V = \langle x, f(y) \rangle_W, \text{ for all } x \in W \text{ and } y \in V.$$

Morphisms in $\mathcal{S}_*^n(\mathbb{Q})$ are injective because $f^!f = \text{id}_V$. The adjoint induces in particular a splitting

$$W \cong V \oplus V^\perp, \quad x \mapsto (f^!(x), x - f f^!(x)),$$

where $V^\perp = \{x \in W \mid \langle x, f(y) \rangle_W = 0 \text{ for all } y \in V\}$.

Denote by $\mathcal{L}(V) = \mathbb{L}(s^{-1}V)$ the free graded Lie algebra generated by $s^{-1}V$, the desuspension of V (i.e. $(s^{-1}V)_* = V_{*+1}$). For $f : V \rightarrow W$ denote by $s^{-1}f : s^{-1}V \rightarrow s^{-1}W$ the desuspension of f and by $\mathcal{L}(f) : \mathcal{L}(V) \rightarrow \mathcal{L}(W)$ the morphism of graded Lie algebras induced by $s^{-1}f$. Denote by $\mathcal{L}^1(V)$ the elements of bracket length 1, i.e. the generating

vectorspace $s^{-1}V$. Recall that the derivations $\text{Der}\mathcal{L}(V)$ are a graded Lie algebra (Section 2.2.). We define

$$\text{Der}(\mathcal{L}(f)) : \text{Der}(\mathcal{L}(V)) \rightarrow \text{Der}(\mathcal{L}(W)),$$

to be the map that takes $\theta \in \text{Der}(\mathcal{L}(V))$ into the unique derivation of $\mathcal{L}(W)$ with

$$\text{Der}(\mathcal{L}(f))(\theta)(x) = \mathcal{L}(f) \circ \theta(s^{-1}f^!(x)) \text{ for all } x \in \mathcal{L}^1(W).$$

Lemma 3.1. *Let $f : V \rightarrow W$ be a morphism in $\mathcal{S}_*^n(\mathbb{Q})$. $\text{Der}(\mathcal{L}(f))$ is an injective morphism of graded Lie algebras. Moreover $\text{Der}(\mathcal{L}(-))$ defines a functor*

$$\text{Der}(\mathcal{L}(-)) : \mathcal{S}_*^n(\mathbb{Q}) \rightarrow g\text{Lie}.$$

Proof. Denote $\chi_f = \text{Der}(\mathcal{L}(f))$. To show that χ_f is a map of differential graded Lie algebras, it suffices to show that $\chi_f([\theta, \eta])(x) = [\chi_f(\theta), \chi_f(\eta)](x)$, for $\theta, \eta \in \text{Der}(\mathcal{L}(V))$ and $x \in \mathcal{L}^1(W)$, since both sides are derivations on a free graded Lie algebra. First we observe that $\chi_f(\theta) \circ \mathcal{L}(f) = \mathcal{L}(f) \circ \theta$. This follows because both sides are $\mathcal{L}(f)$ -derivations and on generators $x \in \mathcal{L}^1(W)$ we have

$$\chi_f(\theta) \circ \mathcal{L}(f)(x) = \mathcal{L}(f) \circ \theta(s^{-1}f^!(s^{-1}f(x))) = \mathcal{L}(f) \circ \theta(x).$$

Using this we can calculate

$$\begin{aligned} \chi_f([\theta, \eta])(x) &= \chi_f(\theta \circ \eta - (-1)^{|\theta||\eta|} \eta \circ \theta)(x) \\ &= \mathcal{L}(f) \circ \theta \circ \eta \circ s^{-1}f^!(x) - (-1)^{|\theta||\eta|} \mathcal{L}(f) \circ \eta \circ \theta \circ s^{-1}f^!(x) \\ &= \chi_f(\theta) \circ \mathcal{L}(f) \circ \eta \circ s^{-1}f^!(x) - (-1)^{|\theta||\eta|} \chi_f(\eta) \circ \mathcal{L}(f) \circ \theta \circ s^{-1}f^!(x) \\ &= \chi_f(\theta) \circ \chi_f(\eta)(x) - (-1)^{|\chi_f(\theta)||\chi_f(\eta)|} \chi_f(\eta) \circ \chi_f(\theta)(x) \\ &= [\chi_f(\theta), \chi_f(\eta)](x). \end{aligned}$$

To show injectivity we define a section (in general only of graded vector spaces)

$$\psi_f : \text{Der}(\mathcal{L}(W)) \rightarrow \text{Der}(\mathcal{L}(V)),$$

by $\psi_f(\theta)(x) = \mathcal{L}(f^!) \circ \theta(s^{-1}f(x))$ on generators $x \in s^{-1}W$ for $\theta \in \text{Der}(\mathcal{L}(W))$. To verify that ψ_f is a section, we calculate for $\theta \in \text{Der}(\mathcal{L}(W))$ and $x \in s^{-1}V$

$$\begin{aligned} \psi_f \circ \chi_f(\theta)(x) &= \mathcal{L}(f^!) \circ \chi_f(\theta)(s^{-1}f(x)) \\ &= \mathcal{L}(f^!) \circ \mathcal{L}(f) \circ \theta(s^{-1}f^! \circ s^{-1}f(x)) \\ &= \mathcal{L}(f^!) \circ \mathcal{L}(f) \circ \theta(x) = \theta(x). \end{aligned}$$

We leave it to the reader to show that $\chi_{gf} = \chi_g \chi_f$, for a $g : W \rightarrow Z$ in $\mathcal{S}_*^n(\mathbb{Q})$. \square

Denote by $\Lambda(V)$ the free graded commutative algebra over V . Let $\Lambda^2(V)_n$ be the elements of word length 2 and of degree n . Note that $\langle -, - \rangle_V$ gives us a homomorphism in $\text{Hom}_{\mathbb{Q}}(\Lambda^2(V)_n, \mathbb{Q}[n])$. Let $\mathcal{L}^2(V)_{n-2}$ be the elements of bracket length 2 and of degree $n - 2$. The suspension

$$s : s^{-1}V \ni x \mapsto sx \in V$$

induces an isomorphism of vector spaces $\mathcal{L}^2(V)_{n-2} \cong \Lambda^2(V)_n$ by

$$[x, y] \mapsto (-1)^{|x|} sx \wedge sy.$$

We define a non-degenerate pairing

$$\langle -, - \rangle : \mathcal{L}^2(V)_{n-2} \otimes \Lambda^2(V)_n \rightarrow \mathbb{Q}$$

by

$$\langle [x, y], a \wedge b \rangle = (-1)^{|x|} \langle sx, a \rangle_V \langle sy, b \rangle_V + (-1)^{|x|+|a||b|} \langle sy, a \rangle_V \langle sx, b \rangle_V.$$

It's adjoint

$$\mathcal{L}^2(V)_{n-2} \rightarrow \text{Hom}_{\mathbb{Q}}(\Lambda^2(V)_n, \mathbb{Q}[n])$$

is an isomorphism and hence there is a unique element $\omega_V \in \mathcal{L}^2(V)_{n-2}$ mapping to $\langle -, - \rangle_V$. We can characterize ω_V alternatively as the unique element, such that

$$\langle \omega_V, a \wedge b \rangle = \langle a, b \rangle_V \text{ for all } a, b \in V.$$

Given a basis e_1, \dots, e_m for the graded vector space V (i.e. with the e_i homogeneous), then we can form the dual basis $e_1^\#, \dots, e_m^\#$, which is uniquely determined by

$$\langle e_i, e_j^\# \rangle_V = \delta_{i,j}.$$

If we represent the bilinear form on V as the matrix $B = (\langle e_i, e_j \rangle_V)$, then $e_i^\# = B^{-1}e_i$ and $B^{-1} = (\langle e_i^\#, e_j^\# \rangle_V)$. Given this, one can check that

$$2\omega_V = \sum_{i,j} -(-1)^{|e_i|} \langle e_i^\#, e_j^\# \rangle_V [s^{-1}e_i, s^{-1}e_j].$$

For a general V we want to study the evaluation at ω_V :

$$ev_{\omega_V} : \text{Der}(\mathcal{L}(V)) \ni \theta \mapsto \theta(\omega_V) \in \mathcal{L}(V),$$

which is given explicitly in terms of a basis e_1, \dots, e_m for V by

$$ev_{\omega_V}(\theta) = \sum_{i,j} -(-1)^{|e_i|} \langle e_i^\#, e_j^\# \rangle_V [\theta(s^{-1}e_i), s^{-1}e_j].$$

Note that the kernel of the evaluation map is exactly the graded sub Lie algebra of derivations annihilating $\mathbb{L}[\omega_V]$, the Lie algebra generated by ω_V . We will denote it by

$$\text{Der}_{\omega_V}(\mathcal{L}(V)) = \text{Der}(\mathcal{L}(V), \text{rel. } \mathbb{L}[\omega_V]).$$

Lemma 3.2. For $f : V \rightarrow W$ in $\mathcal{S}_*^n(\mathbb{Q})$ the following diagram commutes

$$\begin{array}{ccc} \text{Der}(\mathcal{L}(V)) & \xrightarrow{ev_{\omega_V}} & \mathcal{L}(V) \\ \chi_f \downarrow & & \downarrow \mathcal{L}(f) \\ \text{Der}(\mathcal{L}(W)) & \xrightarrow{ev_{\omega_W}} & \mathcal{L}(W). \end{array}$$

Proof. We will use that $W \cong V \oplus V^\perp$ and since f respects the pairing, ω_W corresponds to $\omega_V \oplus \omega_{V^\perp}$, where $\omega_{V^\perp} \in \mathcal{L}(V^\perp)$. Assume without loss of generality that $W = V \oplus V^\perp$. For $\theta \in \text{Der}(\mathcal{L}(V))$ the image $\chi_f(\theta)$ is then given by the derivation on $\mathcal{L}(V \oplus V^\perp)$ that restricts to θ on $\mathcal{L}(V)$ and is constant 0 on $\mathcal{L}(V^\perp)$. The map f becomes just the inclusion of $\mathcal{L}(V)$ into $\mathcal{L}(V \oplus V^\perp)$. Hence we calculate

$$\chi_f(\theta)(\omega_W) = \chi_f(\theta)(\omega_V) + \chi_f(\theta)(\omega_{V^\perp}) = \chi_f(\theta)(\omega_V) = \theta(\omega_V).$$

□

Lemma 3.2 implies that the exact sequence:

$$0 \rightarrow \text{Der}_{\omega_V}(\mathcal{L}(V)) \rightarrow \text{Der}(\mathcal{L}(V)) \xrightarrow{ev_{\omega_V}} \mathcal{L}(V)$$

is natural with respect to maps in $\mathcal{S}_*^n(\mathbb{Q})$. This implies in particular that $\text{Der}_{\omega_{(-)}}(\mathcal{L}(-))$ extends to a functor $\mathcal{S}_*^n(\mathbb{Q}) \rightarrow gLie$. It sends a morphism $f : V \rightarrow W$ to the morphism defined by

$$\begin{aligned} \text{Der}_{\omega_V}(\mathcal{L}(V)) &\rightarrow \text{Der}_{\omega_V \oplus \omega_{V^\perp}}(\mathcal{L}(V \oplus V^\perp)) \cong \text{Der}_{\omega_W}(\mathcal{L}(W)) \\ \theta &\mapsto \theta \oplus 0_{V^\perp}. \end{aligned} \tag{3.1}$$

Consider $\mathcal{L}^1(V) \otimes \mathcal{L}(V)$ with grading given by $|x \otimes \xi| = |\xi| + |x| - n + 2$. For $x \in \mathcal{L}^1(V)$ and $\xi \in \mathcal{L}(V)$ we can define a derivation of $\mathcal{L}(V)$ by

$$\theta_{x,\xi}(y) = -(-1)^{|x|(|\xi|+1)+n} \langle sx, sy \rangle_V \xi, \quad y \in \mathcal{L}^1(V).$$

This extends to an isomorphism of graded vector spaces

$$\theta_{-,-} : \mathcal{L}^1(V) \otimes \mathcal{L}(V) \rightarrow \text{Der}(\mathcal{L}(V)).$$

Let e_1, \dots, e_n be a basis of V , then the inverse of $\theta_{-,-}$ is given by

$$\theta_{-,-}^{-1}(\theta) = \sum_i -(-1)^{|s^{-1}e_i^\#|(|\theta|+|s^{-1}e_i|+1)+n} s^{-1}e_i^\# \otimes \theta(s^{-1}e_i).$$

The map $\theta_{-, -}$ is natural with respect to morphisms in $\mathcal{S}_*^n(\mathbb{Q})$, i.e. $\theta_{-, -}$ fits into the following commutative diagram for any morphism $f : V \rightarrow W$ in $\mathcal{S}_*^n(\mathbb{Q})$

$$\begin{array}{ccc} \mathcal{L}^1(V) \otimes \mathcal{L}(V) & \xrightarrow{\theta_{-, -}} & \text{Der}(\mathcal{L}(V)) \\ \mathcal{L}^1(f) \otimes \mathcal{L}(f) \downarrow & & \downarrow \text{Der}(\mathcal{L}(f)) \\ \mathcal{L}^1(W) \otimes \mathcal{L}(W) & \xrightarrow{\theta_{-, -}} & \text{Der}(\mathcal{L}(W)). \end{array}$$

Let

$$b : \mathcal{L}^1(V) \otimes \mathcal{L}(V) \rightarrow \mathcal{L}(V)$$

be the map induced by

$$b(x, \xi) = [x, \xi] \text{ for } x \in \mathcal{L}^1(V) \text{ and } \xi \in \mathcal{L}(V).$$

Lemma 3.3. *The maps b and $\theta_{-, -}$ fit into the following commutative diagram*

$$\begin{array}{ccc} \mathcal{L}^1(V) \otimes \mathcal{L}(V) & \xrightarrow{b} & \mathcal{L}(V) \\ \theta_{-, -} \downarrow & & \parallel \\ \text{Der}(\mathcal{L}(V)) & \xrightarrow{ev_{\omega_V}} & \mathcal{L}(V). \end{array}$$

Proof. Let e_1, \dots, e_m be a basis for V and let B be the matrix defined by $(\langle e_i, e_j \rangle_V)$. It suffices to check for elements of the form $s^{-1}e_k \otimes \xi \in \mathcal{L}^1(V) \otimes \mathcal{L}(V)$. We calculate, remembering that we can assume $|e_i| + |e_j| = n$, since $\langle e_i^\#, e_j^\# \rangle_V = 0$, if $|e_i| + |e_j| \neq n$ and also $|e_i| + |e_k| = n$.

$$\begin{aligned} \theta_{s^{-1}e_k, \xi}(\omega_V) &= \sum_{i,j} (-1)^{|s^{-1}e_i|} \langle e_i^\#, e_j^\# \rangle_V [\theta_{s^{-1}e_k, \xi}(s^{-1}e_i), s^{-1}e_j] \\ &= \sum_{i,j} -(-1)^{|s^{-1}e_i| + |s^{-1}e_k|(|\xi|+1)+n} \langle e_i^\#, e_j^\# \rangle_V \langle e_k, e_i \rangle_V [\xi, s^{-1}e_j] \\ &= \sum_{i,j} (-1)^{|s^{-1}e_i| + |s^{-1}e_k|(|\xi|+1)+n + |s^{-1}e_j||\xi|} \langle e_i^\#, e_j^\# \rangle_V \langle e_k, e_i \rangle_V [s^{-1}e_j, \xi] \\ &= \sum_{i,j} B_{i,j}^{-1} B_{k,i} [s^{-1}e_j, \xi] = \sum_j \delta_{j,k} [s^{-1}e_j, \xi] \\ &= [s^{-1}e_k, \xi] = b(s^{-1}e_k, \xi). \end{aligned}$$

□

Remark 3.4. Let $n > 1$. We will later need that the evaluation map

$$\text{Der}(\mathcal{L}(V))_k \xrightarrow{ev_{\omega_V}} \mathcal{L}(V)_{k+n-2}$$

is surjective for $k \geq 1$ when V is concentrated in positive degrees and hence concentrated in degrees $1, \dots, n-1$. By the Lemma above it suffices to show that

$$b : (\mathcal{L}^1(V) \otimes \mathcal{L}(V))_k \rightarrow \mathcal{L}(V)_{k+n-2}$$

is surjective for $k \geq 1$. Now considering the standard grading on $\mathcal{L}^1(V) \otimes \mathcal{L}(V)$, i.e. $|x \otimes \xi| = |x| + |\xi|$, this is equivalent to asking that $b : (\mathcal{L}^1(V) \otimes \mathcal{L}(V))_{k+n-2} \rightarrow \mathcal{L}(V)_{k+n-2}$ is surjective, but this follows since all elements of degree $\geq n-1$ are of bracket length ≥ 2 when V is concentrated in positive degrees.

Denote by $\mathfrak{g}(V)$ the kernel of b . We can extend the commutative diagram of Lemma 3.3 to a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}(V) & \longrightarrow & \mathcal{L}^1(V) \otimes \mathcal{L}(V) & \xrightarrow{b} & \mathcal{L}(V) \\ & & \cong \downarrow \text{dotted} & & \theta_{-, -} \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Der}_{\omega_V}(\mathcal{L}(V)) & \longrightarrow & \text{Der}(\mathcal{L}(V)) & \xrightarrow{ev_{\omega_V}} & \mathcal{L}(V). \end{array} \quad (3.2)$$

As shown above all maps in (3.2) are natural with respect to maps in $\mathcal{S}_*^n(\mathbb{Q})$, hence we get a natural isomorphism $\mathfrak{g}(V) \cong \text{Der}_{\omega_V}(\mathcal{L}(V))$.

Note \mathfrak{g} factors through the forgetful functor $\mathcal{S}_*^n(\mathbb{Q}) \rightarrow \text{Vect}_*(\mathbb{Q})$, i.e. we can extend the definition of \mathfrak{g} to $\text{Vect}_*(\mathbb{Q})$.

We want to describe $\mathfrak{g} : \text{Vect}_*(\mathbb{Q}) \rightarrow \text{Vect}_*(\mathbb{Q})$ as a Schur functor.

Let $\mathcal{L}ie = \{\mathcal{L}ie(k)\}_{k \geq 0}$ be the Lie operad (see e.g. [LV12]). Picking a generator λ of $\mathcal{L}ie(2)$ defines a map

$$\mathcal{L}ie(k-1) \ni \gamma \mapsto \lambda \circ_2 \gamma \in \mathcal{L}ie(k).$$

This map is Σ_{k-1} -equivariant, where the Σ_{k-1} -action on $\mathcal{L}ie(k)$ is induced by the inclusion of $\Sigma_{k-1} \hookrightarrow \Sigma_k$ acting on the last $k-1$ elements. The adjoint map fits in a short exact sequence of Σ_k -representations

$$0 \rightarrow \mathcal{U}(k) \rightarrow \text{Ind}_{\Sigma_{k-1}}^{\Sigma_k} \mathcal{L}ie(k-1) \rightarrow \mathcal{L}ie(k) \rightarrow 0,$$

where $\mathcal{U}(k)$ denotes the kernel. Applying the functor $-\otimes_{\Sigma_k} W^{\otimes k}$ for any graded rational vector space W , we get a short exact sequence

$$0 \rightarrow \mathcal{U}(k) \otimes_{\Sigma_k} W^{\otimes k} \rightarrow \text{Ind}_{\Sigma_{k-1}}^{\Sigma_k} \mathcal{L}ie(k-1) \otimes_{\Sigma_k} W^{\otimes k} \rightarrow \mathcal{L}ie(k) \otimes_{\Sigma_k} W^{\otimes k} \rightarrow 0,$$

since $\text{Tor}_1^{\mathbb{Q}[\Sigma_k]}$ vanishes. We can naturally identify $\mathcal{L}ie(k) \otimes_{\Sigma_k} W^{\otimes k} \cong \mathbb{L}^k(W)$ and

$$\text{Ind}_{\Sigma_{k-1}}^{\Sigma_k} \mathcal{L}ie(k-1) \otimes_{\Sigma_k} W^{\otimes k} \cong W \otimes \mathbb{L}^{k-1}(W).$$

Under this identifications the map becomes the bracketing map. Note that the degree of elements in $\mathcal{U}(k)$ is constant $-n + 2$ under this identification. Hence we naturally identified

$$\mathfrak{g}^k(V) \cong \mathcal{U}(k) \otimes_{\Sigma_k} (s^{-1}V)^{\otimes k}.$$

We can describe the desuspension s^{-1} as a Schur functor \mathcal{D} with $\mathcal{D}(1)$ the trivial Σ_1 -representation \mathbb{Q} concentrated in degree -1 and trivial otherwise. Let \mathcal{U}' be the composite $\mathcal{U} \circ \mathcal{D}$. This implies the following

Proposition 3.5. *Let $V \in \mathcal{S}_*^n(\mathbb{Q})$. Then there is an isomorphism of graded rational vector spaces*

$$\mathrm{Der}_{\omega_V}(\mathcal{L}(V)) \cong \bigoplus_{k \geq 0} \mathcal{U}'(k) \otimes_{\Sigma_k} V^{\otimes k}$$

natural with respect to maps in $\mathcal{S}_*^n(\mathbb{Q})$ where elements of $\mathcal{U}'(k)$ are of constant degree $-n + 2 - k$.

Proof. By the discussion above we know that we get a natural isomorphism

$$\mathrm{Der}_{\omega_V}(\mathcal{L}(V)) \cong \mathcal{U}'(V) = \mathcal{U} \circ \mathcal{D}(V).$$

To show that elements of $\mathcal{U}'(k)$ are of constant degree $-n + 2 - k$, we use the description of a composition of Schur functors to show that

$$\mathcal{U}'(k) = \mathcal{U}(k) \otimes_{\Sigma_k} \mathrm{Ind}_{\Sigma_1 \times \dots \times \Sigma_1}^{\Sigma_k} \mathcal{D}(1) \otimes \dots \otimes \mathcal{D}(1),$$

but this is clearly of degree $-n + 2 - k$. □

We would like identify the positive degree derivations as a Schur functor in terms of $\mathcal{U}'(k)$, but this is not possible since we don't have enough control over the degrees of elements of $V^{\otimes k}$. Instead we consider the full subcategory $\mathcal{S}_+^n(\mathbb{Q})$ of $\mathcal{S}_*^n(\mathbb{Q})$ with objects concentrated in positive degrees (and hence concentrated in degrees $1, \dots, n-1$). Consider the functor

$$\mathcal{F} : \mathcal{S}_+^n(\mathbb{Q}) \rightarrow \mathrm{Vect}(\mathbb{Q})^{n-1},$$

defined on objects as $\mathcal{F}(V) = (V_i)_{i=1}^{n-1}$ and the functor

$$\mathcal{V} : \mathrm{Vect}(\mathbb{Q})^{n-1} \rightarrow \mathrm{Vect}_*(\mathbb{Q}),$$

defined on objects as $\mathcal{V}(V_i) = \bigoplus_{i=1}^{n-1} V_i[i]$. The forgetful functor $\mathcal{S}_+^n(\mathbb{Q}) \rightarrow \mathrm{Vect}_*(\mathbb{Q})$ factors as $\mathcal{V} \circ \mathcal{F}$. We can describe \mathcal{V} as a Schur multifunctor given by $\mathcal{V}(\epsilon^i)$ the trivial Σ_1 -representation \mathbb{Q} concentrated in degree i and trivial otherwise, where ϵ^i denotes the multi-index with a 1 at the i -th position and 0 otherwise.

Proposition 3.6. *Let $n > 1$, $V \in \mathcal{S}_*^n(\mathbb{Q})$ and $(V_i) \in \text{Vect}(\mathbb{Q})^{n-1}$ the image of the forgetful functor. Then there is an isomorphism of graded rational vector spaces*

$$\text{Der}_{\omega_V}^+(\mathcal{L}(V)) \cong \bigoplus_{\mu} \mathcal{W}(\mu) \otimes_{\Sigma_{\mu}} (V_i)^{\otimes \mu}$$

natural with respect to maps in $\mathcal{S}_+^n(\mathbb{Q})$, where $\mathcal{W}(\mu) = (\mathcal{U}' \circ \mathcal{V})(\mu)$, when

$$\sum_{i=1}^{n-1} m_i \frac{i-1}{n-1} \geq 1,$$

$\mu = (m_1, \dots, m_{n-1})$ and 0 otherwise. The elements of $\mathcal{W}(\mu)$ are of constant degree $\left(\sum_{i=1}^{n-1} m_i(i-1)\right) - n + 2$.

Proof. By the discussion above we see that we get a natural isomorphism

$$\text{Der}_{\omega_V}(\mathcal{L}(V)) \cong \bigoplus_{\mu} (\mathcal{U}' \circ \mathcal{V})(\mu) \otimes_{\Sigma_{\mu}} (V_i)^{\otimes \mu} \text{ for } V \in \mathcal{S}_+^n(\mathbb{Q}).$$

To understand the degree of $(\mathcal{U}' \circ \mathcal{V})(\mu)$ we use (2.2):

$$(\mathcal{U}' \circ \mathcal{V})(\mu) \cong \bigoplus_r \mathcal{U}'(r) \otimes_{\Sigma_r} \bigoplus \text{Ind}_{\Sigma_{\mu_1} \times \dots \times \Sigma_{\mu_r}}^{\Sigma_{\mu}} \mathcal{V}(\mu_1) \otimes \dots \otimes \mathcal{V}(\mu_r), \quad (3.3)$$

where the second sum runs over all r -tuples (μ_1, \dots, μ_r) , such that $\sum_{i=1}^r \mu_i = \mu$. But a summand corresponding to (μ_1, \dots, μ_r) is only non-trivial if $\mu_i = \epsilon^{j_i}$, for some j_i . Hence the only non-trivial summands of (3.3) occur for $r = |\mu|$ and they are of degree

$$\left(\sum_{i=1}^{n-1} m_i i\right) - n + 2 - |\mu| = \left(\sum_{i=1}^{n-1} m_i(i-1)\right) - n + 2.$$

This is ≥ 1 , when $\sum_{i=1}^{n-1} m_i \frac{i-1}{n-1} \geq 1$. □

We can now proceed to give a Schur multifunctor description of the Chevalley-Eilenberg chains of $\text{Der}_{\omega_V}(\mathcal{L}(V))$.

Proposition 3.7. *Let $n > 3$ and $V \in \mathcal{S}_+^n(\mathbb{Q})$. Then there exists a Schur multifunctor \mathcal{C} , such that there is an isomorphisms of graded vector spaces*

$$C_*^{CE}(\text{Der}_{\omega_V}^+ \mathcal{L}(V)) \cong \bigoplus_{\mu} \mathcal{C}(\mu) \otimes_{\Sigma_{\mu}} (V_i)^{\otimes \mu}$$

natural with respect to maps in $\mathcal{S}_+^n(\mathbb{Q})$, where the $\mathcal{C}(\mu)$ are concentrated in degrees

$$\geq \frac{2}{n-1} \left(\sum_{i=1}^{n-1} m_i(i-1)\right).$$

Proof. The underlying graded vector space of the Chevalley-Eilenberg chains of a graded Lie algebra can be described as the image of the Schur functor

$$C_*^{CE}(-) \cong \bigoplus_r \Lambda(r) \otimes_{\Sigma_r} (-)^{\otimes r},$$

where $\Lambda(r)$ is the trivial Σ_r -representation concentrated in degree r . The Chevalley-Eilenberg chains of $\text{Der}_{\omega_V}^+ \mathcal{L}(V)$ are naturally isomorphic as a graded vector space to $\mathcal{C}((V_i)) = \Lambda \circ \mathcal{W}((V_i))$ by the previous Proposition. We consider $\mathcal{C}(\mu)$ for fixed μ . It is given by

$$\mathcal{C}(\mu) = \bigoplus_r \Lambda(r) \otimes_{\Sigma_r} \bigoplus \text{Ind}_{\Sigma_{\mu_1} \times \dots \times \Sigma_{\mu_r}}^{\Sigma_\mu} \mathcal{W}(\mu_1) \otimes \dots \otimes \mathcal{W}(\mu_r),$$

where the sum is as before. For a fixed r we get by the previous Proposition that the summand corresponding to (μ_1, \dots, μ_r) , where $\mu_s = (m_{1,s}, \dots, m_{r,s})$, is only non-zero, if $\sum_{i=1}^{n-1} (m_{i,s} \frac{i-1}{n-1}) \geq 1$ for all $s = 1, \dots, r$. That implies that

$$\sum_{i=1}^{n-1} m_i \frac{i-1}{n-1} = \sum_{s=1}^r \left(\sum_{i=1}^{n-1} m_{i,s} \frac{i-1}{n-1} \right) \geq r. \quad (3.4)$$

If the summand is non-zero it is of degree

$$\begin{aligned} r + \sum_{s=1}^r \left(\left(\sum_{i=1}^{n-1} m_i^s (i-1) \right) - n + 2 \right) &= r - \left(\sum_{i=1}^{n-1} m_i (i-1) \right) - nr + 2r \\ &= \left(\sum_{i=1}^{n-1} m_i (i-1) \right) + r(3-n) \\ &\geq \left(\sum_{i=1}^{n-1} m_i (i-1) \right) + \left(\sum_{i=1}^{n-1} m_i \frac{i-1}{n-1} \right) (3-n) \\ &= \frac{2}{n-1} \left(\sum_{i=1}^{n-1} m_i (i-1) \right), \end{aligned}$$

where we used that $3-n$ is negative and (3.4). □

Denote by $\mathcal{C}_r : \text{Vect}(\mathbb{Q})^{n-1} \rightarrow \text{Vect}(\mathbb{Q})$ the functor defined by taking the degree r -part of \mathcal{C} .

Corollary 3.8. *Let $n > 3$ and $V \in \mathcal{S}_+^n(\mathbb{Q})$. Then there is an isomorphisms of graded vector spaces*

$$C_r^{CE}(\text{Der}_{\omega_V}^+ \mathcal{L}(V)) \cong \bigoplus_{\mu} \mathcal{C}_r(\mu) \otimes_{\Sigma_\mu} (V_i)^{\otimes \mu}$$

natural with respect to maps in $\mathcal{S}_+^n(\mathbb{Q})$, where the $\mathcal{C}_r(\mu)$ vanishes for $|\mu| > \frac{r}{2}$.

Remark 3.9. Note that this in particular implies that $\mathcal{C}_r : Vect(\mathbb{Q})^{n-1} \rightarrow Vect(\mathbb{Q})$ is a Polynomial functor of degree $\frac{r}{2}$.

Proof. By Proposition 3.7 $\mathcal{C}(\mu) \otimes_{\Sigma_\mu} (V_i)^{\otimes \mu}$, where $(V_i) \in Vect(\mathbb{Q})^{n-1}$, is concentrated in degrees $\geq \frac{2}{n-1} \left(\sum_{i=1}^{n-1} m_i(i-1) \right)$. This implies that $\mathcal{C}_r(\mu)$ has to vanish for

$$r < \frac{2}{n-1} \left(\sum_{i=1}^{n-1} m_i(i-1) \right) \leq 2 \sum_{i=1}^{n-1} m_i = 2|\mu|.$$

□

4 Mapping class groups

Let

$$N = N_I = (\#_{i \in I}(S^{p_i} \times S^{q_i})) \setminus \text{int}(D^n),$$

where $|I| < \infty$, $2 < p_i \leq q_i < 2p_i - 1$ and $p_i + q_i = n$ for $i \in I$. Note that this implies that necessarily $n \geq 6$. Denote by $in : \partial N \hookrightarrow N$ the inclusion of the boundary. We observe that $V_I = \bigvee_{i \in I}(S^{p_i} \vee S^{q_i}) \subset N$ is a deformation retract and denote by

$$\alpha_j : S^{p_j} \hookrightarrow \bigvee_{i \in I}(S^{p_i} \vee S^{q_i}) \text{ and } \beta_j : S^{q_j} \hookrightarrow \bigvee_{i \in I}(S^{p_i} \vee S^{q_i})$$

the canonical inclusions. We can consider in as an element of $\pi_{n-1}(\bigvee_{i \in I}(S^{p_i} \vee S^{q_i}))$ and we observe that it is given by $\sum_{i \in I}[\alpha_i, \beta_i]$. Denote by

$$\langle -, - \rangle : H_*(N) \otimes H_{n-*}(N) \rightarrow \mathbb{Z}$$

$$x \otimes y \mapsto (PD^{-1}(x) \cup PD^{-1}(y))([N, \partial N])$$

the intersection form, where $PD^{-1} : H_*(N) \rightarrow H^{n-*}(N, \partial N)$ denotes the Poincaré duality isomorphisms and we evaluate on the fundamental class $[N, \partial N]$. The $\{\alpha_i\}$ and $\{\beta_i\}$ define a basis for $\tilde{H}_*(N)$ via the Hurewicz homomorphism, which we will denote by $\{a_i\}$ respectively $\{b_i\}$. Note that we can assume that b_i dual to a_i . For $S^p \times S^q \setminus \text{int}(D^n)$ this is clear and it follows for N_I by restriction to the subspaces $S^{p_i} \times S^{q_i} \setminus \text{int}(D^n)$.

In the case $n = 2d$ is even we need to recall a further piece of structure from Wall's classification of highly connected even dimensional manifolds [Wal62]. The elements $x \in H_d(N)$ can be represented by embedded S^d . Denote by $\nu_x \in \pi_{d-1}(SO(d))$ the clutching function of the normal bundle of this embedding. It is independent of the choice of embedding, since homotopic embeddings are isotopic in this case. This defines a function

$$q : H_d(N) \rightarrow \pi_{d-1}(SO(d)), x \mapsto [\nu_x].$$

Denote by ι_d the class of the identity in $\pi_d(S^d)$ and by

$$\partial : \pi_d(S^d) \rightarrow \pi_{d-1}(SO(d))$$

the boundary map in the fibration $SO(d) \rightarrow SO(d+1) \rightarrow S^d$. The function q satisfies:

$$\langle x, x \rangle = HJq(x) \text{ and } q(x + y) = q(x) + q(y) + \langle x, y \rangle \partial \iota_d,$$

where $\pi_{d-1}(SO(d)) \xrightarrow{J} \pi_{2d-1}(S^d) \xrightarrow{H} \mathbb{Z}$ denote the J -homomorphism and the Hopf invariant. There is also a purely homotopy theoretic description of Jq in [KM63, Section 8]. Note that for $a_i, b_j \in H_d(N)$, we have $q(a_i) = q(b_j) = 0$. Hence $\text{Image}(q)$ is contained in the subgroup $\langle \partial\iota_d \rangle$ generated by $\partial\iota_d$. The J -homomorphism restricts to an isomorphism

$$J|\langle \partial\iota_d \rangle : \langle \partial\iota_d \rangle \rightarrow J(\langle \partial\iota_d \rangle) \cong \begin{cases} \langle [\iota_d, \iota_d] \rangle \xrightarrow[H]{\cong} 2\mathbb{Z} & \text{if } d \text{ even} \\ 0 & \text{if } d = 1, 3, 7 \\ \langle [\iota_d, \iota_d] \rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is odd and not } 1, 3 \text{ or } 7, \end{cases}$$

where the second isomorphism is induced by the Hopf invariant. Let

$$\text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, Jq) \text{ and } \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, q)$$

be the automorphisms of the reduced homology respecting the intersection form and the function Jq (q respectively). Note that

$$\langle x, y \rangle = \mu(x, y) + (-1)^{|x||y|} \mu(y, x),$$

where $\mu(-, -)$ is determined by

$$\mu(a_i, b_j) = \delta_{i,j} \text{ and } \mu(b_i, a_j) = \mu(a_i, a_j) = \mu(b_i, b_j) = 0.$$

Now let

$$\Lambda = \begin{cases} 0 & \text{if } n = 2d \text{ and } d \text{ is even} \\ \mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is } 3 \text{ or } 7 \\ 2\mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is odd and not } 3 \text{ or } 7. \end{cases}$$

Moreover $Jq = q_\mu$, where q_μ is the Λ -quadratic form associated to μ , where we identify $\langle [\iota_d, \iota_d] \rangle$ with \mathbb{Z} and $\mathbb{Z}/2\mathbb{Z}$ respectively. It suffices to check this for the elements $a_i + b_i$ and for these $Jq(a_i + b_i) = [\iota_d, \iota_d]$ and $q_\mu(a_i + b_i) = 1$.

By the discussion above we see that

$$\text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, q) \cong \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, Jq) \cong \Gamma_I = \text{Aut}(H_I) \text{ in } S_+^n(\mathbb{Z}, \Lambda).$$

For a representative f of $[f] \in \pi_0(\text{aut}_\partial(N))$ it is clear that $\tilde{H}_*(f) \in \Gamma_I$. We are now going to show that all elements of Γ_I can be realized by a homotopy self-equivalence, fixing the boundary pointwise.

Proposition 4.1. *The group homomorphism*

$$\pi_0(\text{aut}_\partial(N)) \xrightarrow{\tilde{H}_*} \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, Jq)$$

is surjective and has finite kernel.

Proof. The cofibration $in : \partial N \hookrightarrow N$ induces a fibration

$$\text{map}_{\partial}(N, N) \rightarrow \text{map}_*(N, N) \rightarrow \text{map}_*(\partial N, N),$$

where $\text{map}_{\partial}(N, N)$ is the fiber over in . Let $\text{map}_{\partial}(N, N)$ and $\text{map}_*(N, N)$ be based at the identity. Restricting the total space to invertible elements, we also get the following fibration:

$$\text{aut}_{\partial}(N) \rightarrow \text{aut}_*(N) \rightarrow \text{map}_*(\partial N, N).$$

We are going to analyze the long exact homotopy sequences

$$\begin{array}{ccccccc} \dots & \longrightarrow & \pi_1(\text{map}_*(\partial N, N), in) & \longrightarrow & \pi_0(\text{aut}_{\partial}(N)) & \longrightarrow & \pi_0(\text{aut}_*(N)) \longrightarrow [\partial N, N]_* \\ & & \parallel & & \downarrow & & \downarrow & \parallel \\ \dots & \longrightarrow & \pi_1(\text{map}_*(\partial N, N), in) & \longrightarrow & [N, N]_{\partial} & \longrightarrow & [N, N]_* \longrightarrow [\partial N, N]_* \end{array}$$

We want to consider the monoid homomorphism

$$\tilde{H}_* : [N, N]_* \rightarrow \text{End}(\tilde{H}_*(N))$$

and show that it is onto and with finite kernel. Using the relative Hurewicz isomorphism, it is easy to see that $V_I \hookrightarrow \prod_{i \in I} (S^{p_i} \times S^{q_i})$ is $(2 \min_{i \in I} \{p_i\} - 1)$ -connected and hence more than $\max_{i \in I} \{q_i\}$ -connected. Thus we get an isomorphism of sets

$$\begin{aligned} [N, N]_* &\cong [V_I, V_I]_* \cong [V_I, \prod_{i \in I} (S^{p_i} \times S^{q_i})]_* \\ &\cong \prod_{(i,j) \in I \times I} [S^{p_i}, S^{p_j}]_* \times \prod_{(i,j) \in I \times I} [S^{q_i}, S^{q_j}]_* \times \prod_{(i,j) \in I \times I} [S^{q_i}, S^{p_j}]_* \times \prod_{(i,j) \in I \times I} [S^{p_i}, S^{q_j}]_*. \end{aligned}$$

We write $I = \bigcup_l I_l$, where $I_l = \{i | p_i = l\}$. The only non-finite factors of the product above are

$$\prod_l \prod_{(i,j) \in I_l \times I_l} ([S^{p_i}, S^{p_j}]_* \times [S^{q_i}, S^{q_j}]_*). \quad (4.1)$$

We identify $\text{End}(\tilde{H}_*(N)) \cong \prod \text{Mat}_{r_l}(\mathbb{Z})$, where $r_l = \text{rank}(H_l(N))$, using the basis $\{a_i\} \cup \{b_i\}$. Note that for $l = n/2$ a b_i becomes a $(r_l/2 + i)$ -th basis element. Denote by $\alpha_1^l, \dots, \alpha_{r_l}^l$ and $\beta_1^l, \dots, \beta_{r_l}^l$ the inclusions $S^l \hookrightarrow V_I$ and $S^{n-l} \hookrightarrow V_I$ respectively. There is a multiplicative section of \tilde{H}_*

$$\prod \text{Mat}_{r_l}(\mathbb{Z}) \rightarrow [N, N]_*,$$

which is given by

$$(M^l) = (m_{i,j}^l) \mapsto f_{(M^l)} = \bigvee_{l=1}^{\lfloor n/2 \rfloor} f_{M^l} \quad (4.2)$$

where $f_{M^l} : \bigvee_{i \in I_l} S^{p_i} \vee S^{q_i} \rightarrow V_I$, is given by

$$f_{M^l} = \begin{cases} \bigvee_{i=1}^{r_l} (\sum_{j=1}^{r_l} m_{i,j}^l \alpha_j^l \vee \sum_{j=1}^{r_l} m_{i,j}^{n-l} \beta_j^l) & \text{if } l \neq n/2 \\ \bigvee_{i=1}^{r_l/2} (\sum_{j=1}^{r_l/2} m_{i,j}^l \alpha_j^l + \sum_{j=r_l/2+1}^{r_l} m_{i,j}^l \beta_{(j-r_l/2)}^l) \\ \vee \bigvee_{i=r_l/2+1}^{r_l} (\sum_{j=1}^{r_l/2} m_{i,j}^l \alpha_j^l + \sum_{j=r_l/2+1}^{r_l} m_{i,j}^l \beta_{(j-r_l/2)}^l) & \text{if } l = n/2. \end{cases}$$

We observe that the image of this section is precisely the sub-monoid of $[N, N]_*$ corresponding to the non-finite factors (4.1). Hence we get that

$$\tilde{H}_* : [N, N]_* \rightarrow \text{Aut}(\tilde{H}_*(N))$$

is surjective and has finite kernel. Restricting to the submonoids of invertible elements this implies upon using the section (4.2) that

$$\pi_0(\text{aut}_*(N)) \rightarrow \text{Aut}(\tilde{H}_*(N))$$

is surjective with finite kernel. The image of $\pi_0(\text{aut}_\partial(N)) \rightarrow \pi_0(\text{aut}_*(N))$ consists of the elements $[f] \in \pi_0(\text{aut}_*(N))$, such that $f \circ in \simeq in$ (we assume all homotopy equivalences in this proof to be pointed). Since (4.2) restricts to a section

$$\text{Aut}(\tilde{H}_*(N)) \rightarrow \pi_0(\text{aut}_*(N))$$

we get that the image of

$$\tilde{H}_* : \pi_0(\text{aut}_\partial(N)) \rightarrow \text{Aut}(\tilde{H}_*(N))$$

is given by the (M^l) such that $f_{(M^l)} \circ in \simeq in$. Using the Hilton-Milnor Theorem we identify

$$\pi_{n-1}(N) \cong \pi \oplus \bigoplus_l \pi_{n-1}(\bigvee_{i \in I_l} (S^{p_i} \vee S^{q_i})), \quad (4.3)$$

where π is some subgroup of $\pi_{n-1}(N)$. We observe that

$$f_{(M^l)} \circ in \simeq \sum_{i \in I} [f_{(M^l)} \circ \alpha_i, f_{(M^l)} \circ \beta_i] \simeq \sum_{l=1}^{\lfloor n/2 \rfloor} \sum_{i \in I_l} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l],$$

i.e. that the action of $f_{(M^l)}$ respects the summands of the identification (4.3). Thus it suffices to check that $f_{M^l} \circ \sum_{i \in I_l} [\alpha_i^l, \beta_i^l] \simeq \sum_{i \in I_l} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l]$ for all l . We will use that left homotopy composition is distributive for suspensions in the range we are in [Whi50], i.e. $(x + y) \circ \Sigma z \simeq x \circ \Sigma z + y \circ \Sigma z$. For $l \neq n/2$ we calculate

$$\begin{aligned} f_{M^l} \circ \sum_{i=1}^{r_l} [\alpha_i^l, \beta_i^l] &\simeq \sum_{i=1}^{r_l} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l] \simeq \sum_{i,j,k} [m_{i,j}^l \alpha_j^l, m_{i,k}^{n-l} \beta_k^l] \\ &\simeq \sum_{i,j,k} m_{i,j}^l m_{i,k}^{n-l} [\alpha_j^l, \beta_k^l] \simeq \sum_{j,k} ((M^l)^T M^{n-l})_{j,k} [\alpha_j^l, \beta_k^l]. \end{aligned}$$

This expression is homotopic to $\sum_{i=1}^{r_l} [\alpha_i^l, \beta_i^l]$ if

$$(M^l)^T M^{n-l} = \text{id}_{\text{Mat}_{r_l}(\mathbb{Z})}. \quad (4.4)$$

For $l = n/2$ we write

$$M^l = \begin{pmatrix} A^l & B^l \\ C^l & D^l \end{pmatrix}.$$

We calculate:

$$\begin{aligned} f_{M^l} \circ \sum_{i=1}^{r_l/2} [\alpha_i^l, \beta_i^l] &\simeq \sum_{i=1}^{r_l/2} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l] \\ &\simeq \sum_{i=1}^{r_l/2} \left[\sum_{j=1}^{r_l/2} (a_{i,j}^l \alpha_j^l + b_{i,j}^l \beta_j^l), \sum_{k=1}^{r_l/2} (c_{i,k}^l \alpha_k^l + d_{i,k}^l \beta_k^l) \right] \\ &\simeq \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} a_{i,j}^l d_{i,k}^l [\alpha_j^l, \beta_k^l] + \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} b_{i,j}^l c_{i,k}^l [\beta_j^l, \alpha_k^l] \\ &\quad + \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} a_{i,j}^l c_{i,k}^l [\alpha_j^l, \alpha_k^l] + \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} b_{i,j}^l d_{i,k}^l [\beta_j^l, \beta_k^l] \\ &\simeq \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} ((A^l)^T D^l + (-1)^{n/2} ((C^l)^T B^l))_{j,k} [\alpha_j^l, \beta_k^l] \\ &\quad + \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} ((A^l)^T C^l)_{j,k} [\alpha_j^l, \alpha_k^l] + \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} ((B^l)^T D^l)_{j,k} [\beta_j^l, \beta_k^l] \end{aligned}$$

This expression is homotopic to $\sum_{i=1}^{r_l/2} [\alpha_i^l, \beta_i^l]$, if

$$\begin{aligned} (A^l)^T D^l + (-1)^{n/2} (C^l)^T B^l &= 1 \\ (A^l)^T C^l + (-1)^{n/2} (C^l)^T A^l &= 0 \\ (B^l)^T D^l + (-1)^{n/2} (D^l)^T B^l &= 0 \\ (A^l)^T C^l \text{ and } (B^l)^T D^l &\text{ have diagonal entries in } \Lambda, \end{aligned}$$

where

$$\Lambda = \begin{cases} 2\mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is odd and not 3 or 7} \\ \mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is 3 or 7} \\ 0 & \text{if } n = 2d \text{ and } d \text{ is even.} \end{cases}$$

Where the diagonal entries of $(A^l)^T C^l$ and $(B^l)^T D^l$ have to be in Λ to kill the elements $[\alpha_i^l, \alpha_i^l]$ and $[\beta_i^l, \beta_i^l]$. These are exactly the conditions to be an automorphisms of $H(\mathbb{Z}^{g_{n/2}})$ in $Q^{(-1)^{n/2}}(\mathbb{Z}, \Lambda)$. Combining this with the condition in (4.4) we see that the image of \tilde{H}_* in $\text{Aut}(\tilde{H}_*(N))$ is given by

$$\Gamma_I \subset \prod_{k=1}^{\lfloor n/2 \rfloor} Gl_{r_k}(\mathbb{Z}).$$

Thus we proved that

$$\tilde{H}_* : \pi_0(\text{aut}_{\partial}(N)) \rightarrow \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, Jq)$$

is surjective. To show that the kernel is finite it suffices to check that $\pi_0(\text{aut}_{\partial}(N)) \rightarrow \pi_0(\text{aut}_*(N))$ has finite kernel. This follows from the fact that

$$\pi_1(in^*) : \pi_1(\text{aut}_*(N), \text{id}_N) \otimes \mathbb{Q} \rightarrow \pi_1(\text{map}_*(\partial N, N), in) \otimes \mathbb{Q}$$

is surjective as we will see in Lemma 5.1. □

Denote by

$$J_0 : \pi_0(\text{Diff}_{\partial}(N)) \rightarrow \pi_0(\text{aut}_{\partial}(N))$$

the map sending an isotopy class of diffeomorphisms to a homotopy class of homotopy automorphisms.

Proposition 4.2.

- (1) *The map $\tilde{H}_* : \pi_0(\text{Diff}_{\partial}(N)) \rightarrow \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, q)$ is surjective.*
- (2) *The image of $J_0 : \pi_0(\text{Diff}_{\partial}(N)) \rightarrow \pi_0(\text{aut}_{\partial}(N))$ has finite index.*

Proof. The first part follows from [Kre79] and [Wal63, Lemma 17]. Kreck shows that all elements of $\text{Aut}(\tilde{H}_{n/2}(N), \langle -, - \rangle, q)$ can be realized as self-diffeomorphisms of

$$(\#_{g_{n/2}}(S^{n/2} \times S^{n/2})) \setminus \text{int}(D^n)$$

fixing the boundary pointwise. Wall shows that for manifolds $\natural_g(D^{q+1} \times S^p)$, where $3 \leq p \leq q$ and \natural_g denotes the g -fold boundary connected sum, all automorphisms of the homology are realized by diffeomorphisms. Hence it follows for manifolds $\#_{g_i}(S^{p_i} \times S^{q_i})$. Since we can assume that a diffeomorphism fixes a disk up to isotopy, we get it in particular for $(\#_{g_i}(S^{p_i} \times S^{q_i})) \setminus \text{int}(D^n)$. Using the diffeomorphisms above and extending them by the identity on the complement of the manifolds above the claim follows. The second part follows from the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_0(\text{S Diff}_{\partial}(N)) & \longrightarrow & \pi_0(\text{Diff}_{\partial}(N)) & \xrightarrow{\tilde{H}_*} & \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, q) \longrightarrow 0 \\ & & \downarrow & & \downarrow J_0 & & \downarrow \cong \\ 0 & \longrightarrow & \pi_0(\text{S aut}_{\partial}(N)) & \longrightarrow & \pi_0(\text{aut}_{\partial}(N)) & \xrightarrow{\tilde{H}_*} & \text{Aut}(\tilde{H}_*(N), \langle -, - \rangle, Jq) \longrightarrow 0, \end{array}$$

where $\pi_0(\text{S Diff}_{\partial}(N))$ and $\pi_0(\text{S aut}_{\partial}(N))$ denote the kernels of the maps \tilde{H}_* and the fact that $\pi_0(\text{S aut}_{\partial}(N))$ is finite by Proposition 4.1. □

Remark 4.3. There is a lot of literature on the groups of components of mapping spaces of closed manifolds in different categories. Highly connected even dimensional manifolds are for example studied in [Kre79] and [Kah69]. Products of spheres are studied in [Lev69, Sat69, Tur69]. Homotopy self-equivalences of manifolds and in particular of connected sums of products of spheres are treated in [Bau96].

5 $\pi_1(-)$ -equivariant rational homotopy type of $B\text{aut}_\partial(-)\langle 1 \rangle$

Let N_I be as before. We introduce the following notation:

$$\begin{aligned} X_I &= \text{aut}_\partial(N_I) \\ \mathbb{L}_I &= (\mathbb{L}(s^{-1}\tilde{H}_*(N_I, \mathbb{Q})), 0) = (\mathbb{L}(s^{-1}H_I \otimes \mathbb{Q}), 0) \\ \omega &= \omega_I = \sum_i (-1)^{|\iota_i|} [\iota_i, \kappa_i], \text{ where } \iota_i = s^{-1}a_i \text{ and } \kappa_i = s^{-1}b_i \\ \text{Der}_\omega^+(\mathbb{L}_I) &= (\text{Der}^+(\mathbb{L}_I, \text{rel. } \mathbb{L}[\omega_I]), 0), \end{aligned}$$

the positive truncation of the derivation Lie algebra of \mathbb{L}_I annihilating $\mathbb{L}[\omega_I] \subset \mathbb{L}_I$, the Lie subalgebra generated by ω_I . We observe that \mathbb{L}_I with trivial differential is a minimal Quillen model for N_I .

Lemma 5.1. *The map*

$$\pi_k(\text{in}^*) : \pi_k(\text{aut}_*(N), \text{id}_{N_I}) \otimes \mathbb{Q} \rightarrow \pi_k(\text{map}_*(\partial N, N), \text{in}) \otimes \mathbb{Q}$$

is surjective for $k > 0$.

Proof. Let $\varphi : \mathbb{L}[\omega_I] \rightarrow \mathbb{L}_I$ be the obvious inclusion and we observe that it models the map in^* . Theorem 2.3 shows that

$$\text{Der}(\mathbb{L}_I)_k \cong \pi_k(\text{aut}_*(N), \text{id}_{N_I}) \otimes \mathbb{Q} \text{ and}$$

$$\text{Der}_\varphi(\mathbb{L}[\omega_I], \mathbb{L}_I)_k \cong \pi_k(\text{map}_*(\partial N, N), \text{in}) \otimes \mathbb{Q}, \text{ for } k > 0.$$

The map $\pi_k(\text{in}^*)$ is given by restricting a derivation $\theta \in \text{Der}(\mathbb{L}_I)$ to $\mathbb{L}[\omega_I]$. We can identify

$$\text{Der}_\varphi(\mathbb{L}[\omega_I], \mathbb{L}_I)_k \cong (\mathbb{L}_I)_{k+n-2}, \theta \mapsto \theta(\omega_I).$$

Under this identification the map $\pi_k(\text{in}^*)$ becomes the evaluation map, which we observed to be surjective in Remark 3.4. \square

Proposition 5.2. *The universal cover $B\text{aut}_\partial(N_I)\langle 1 \rangle$ is coformal.*

Proof. Compare [BM15, Proposition 4.8.]. In order to use Theorem 2.5 we need to replace $\varphi : \mathbb{L}[\omega_I] \rightarrow \mathbb{L}_I$ by a cofibration. For this we have to slightly enlarge the Quillen model of N_I . Denote by $\mathbb{L}_N = (\mathbb{L}_I * \mathbb{L}[\iota, \gamma], \partial(\gamma) = \omega_I - \iota)$, where ι is of degree $n - 1$ and γ is of degree n . The map

$$p : \mathbb{L}_N \rightarrow \mathbb{L}_I,$$

given as the identity on \mathbb{L}_I , $p(\iota) = \omega_I$ and $p(\gamma) = 0$ is a quasi-isomorphism and hence \mathbb{L}_N is a Quillen model for N_I . Let

$$q : \mathbb{L}[\iota] \rightarrow (\mathbb{L}_I * \mathbb{L}[\iota, \gamma], \partial(\gamma) = \omega_I - \iota)$$

denote the obvious inclusion. It is a cofibration (compare Remark 2.1) and hence we can apply Theorem 2.5. Thus $(\text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]), D)$ is a Quillen model for $B\text{aut}_{\partial}(N_I)\langle 1 \rangle$, where the differential is given by $D(-) = [\partial, -]$. Note that $pq : \mathbb{L}[\iota] \rightarrow \mathbb{L}_I$, maps the generator ι to ω_I and hence we can identify $\text{Der}_{\omega}^+(\mathbb{L}_I) \cong (\text{Der}^+(\mathbb{L}_I, \text{rel } \mathbb{L}[\iota]), 0)$. We are now going to show that there is a zig-zag of quasi-isomorphisms between the derivation Lie algebras $(\text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]), D)$ and $(\text{Der}_{\omega}^+(\mathbb{L}_I), 0)$, which implies the statement of the proposition.

The first step is to show that the map

$$p_* : \text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]) \rightarrow \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota])$$

is a surjective quasi-isomorphism. For this we consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]) & \longrightarrow & \text{Der}^+(\mathbb{L}_N) & \xrightarrow{q^*} & \text{Der}_q^+(\mathbb{L}[\iota], \mathbb{L}_N) & \longrightarrow & 0 \\ & & \downarrow p_* & & \downarrow p_* & & \downarrow p_* & & \\ 0 & \longrightarrow & \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota]) & \longrightarrow & \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I) & \xrightarrow{q^*} & \text{Der}_{pq}^+(\mathbb{L}[\iota], \mathbb{L}_I) & \longrightarrow & 0. \end{array}$$

Identify

$$\text{Der}^+(\mathbb{L}_N) \cong \text{Hom}^+(s^{-1}H_I \oplus \mathbb{Q}[\iota, \gamma], \mathbb{L}_N) \text{ and } \text{Der}_q^+(\mathbb{L}[\iota], \mathbb{L}_N) \cong \text{Hom}^+(s^{-1}\mathbb{Q}[\iota], \mathbb{L}_N).$$

Since q is induced by the inclusion $\mathbb{Q}[\iota] \subset s^{-1}H_I \oplus \mathbb{Q}[\iota, \gamma]$, we can describe the map q^* in terms of the right-hand sides as the restriction and we see that the upper q^* are surjective. The analogous argument shows that also the lower q^* is surjective. Since the left hand terms are by definition the kernels of the maps q^* , we can conclude that the rows are exact. The middle and right vertical maps p_* are quasi-isomorphisms, since p is a quasi-isomorphism (see [BM15, Lemma 3.5.]). Hence we can conclude that the left p_* is a quasi-isomorphism using the five Lemma. To see that the left p_* it is surjective, we identify $\text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]) \cong \text{Hom}^+(s^{-1}H_I \oplus \mathbb{Q}[\gamma], \mathbb{L}_N)$ and $\text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota]) \cong$

$\text{Hom}^+(s^{-1}H_I \oplus \mathbb{Q}[\gamma], \mathbb{L}_I)$. The map in terms of the right-hand sides is clearly surjective since \mathbb{L}_I is a sub differential graded Lie algebra of \mathbb{L}_N by definition.

The next step it to show that $p^* : \text{Der}^+(\mathbb{L}_I, \text{rel } \mathbb{L}[\iota]) \rightarrow \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota])$ is a quasi-isomorphism. For this we consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Der}^+(\mathbb{L}_I, \text{rel } \mathbb{L}[\iota]) & \longrightarrow & \text{Der}^+(\mathbb{L}_I) & \xrightarrow{(pq)^*} & \text{Der}_{pq}^+(\mathbb{L}[\iota], \mathbb{L}_I) \longrightarrow 0 \\ & & \downarrow p^* & & \downarrow p^* & & \parallel \\ 0 & \longrightarrow & \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota]) & \longrightarrow & \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I) & \xrightarrow{q^*} & \text{Der}_{pq}^+(\mathbb{L}[\iota], \mathbb{L}_I) \longrightarrow 0. \end{array}$$

We have already shown exactness of the lower row above. In Lemma 5.1 we have shown that the map $(pq)^*$ is surjective and the upper row is exact by the definition of relative derivations. The middle vertical p^* is a quasi-isomorphism since p is (see [BM15, Lemma 3.5.]). Hence we can conclude using the five Lemma that the left vertical p^* is a quasi-isomorphism.

To finish the proof we consider the following pullback diagram of chain complexes:

$$\begin{array}{ccc} P & \xrightarrow{pr_1} & \text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]) \\ \downarrow pr_2 & & \downarrow p_* \\ \text{Der}^+(\mathbb{L}_I, \text{rel } \mathbb{L}[\iota]) & \xrightarrow{p^*} & \text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota]), \end{array}$$

where

$$P = \text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]) \times_{\text{Der}_p^+(\mathbb{L}_N, \mathbb{L}_I, \text{rel } \mathbb{L}[\iota])} \text{Der}^+(\mathbb{L}_I, \text{rel } \mathbb{L}[\iota])$$

and pr_1, pr_2 denote the obvious projections. Defining the Lie-bracket and differential component wise turns P into a differential graded Lie algebra and the projections into morphisms of differential graded Lie algebras. We have shown that p_* is a surjective quasi-isomorphism and since surjective quasi-isomorphism are stable under pullbacks we get that pr_2 is a surjective quasi-isomorphism. Since p^* is also a quasi-isomorphism as shown above, we also see that pr_1 is a quasi-isomorphism. Thus we obtain a zig-zag of quasi-isomorphisms of differential graded Lie algebras:

$$(\text{Der}^+(\mathbb{L}_N, \text{rel } \mathbb{L}[\iota]), D) \xleftarrow[\simeq_{q.i.}]{pr_1} P \xrightarrow[\simeq_{q.i.}]{pr_2} (\text{Der}^+(\mathbb{L}_I, \text{rel } \mathbb{L}[\iota]), 0) \cong \text{Der}_{\omega}^+(\mathbb{L}_I),$$

which concludes the proof. □

Recall that the Samelson product makes $\pi_*^+(X_I) \otimes \mathbb{Q}$ into a graded Lie algebra. We can define a $\pi_0(X_I)$ -action on $\pi_k^+(X_I) \otimes \mathbb{Q}$ induced by pointwise conjugation. The homotopy

groups of the universal covering $B\text{aut}_\partial(N_I)\langle 1 \rangle$ are $\pi_1(B\text{aut}_\partial(N_I))$ -modules by the deck transformation action. Under the canonical isomorphism

$$\pi_{k+1}(B\text{aut}_\partial(X_I)) \cong \pi_k(\text{aut}_\partial(X_I))$$

this action corresponds to the conjugation action. We are now going to identify these actions in terms of $\text{Der}_\omega^+(\mathbb{L}_I)$.

Proposition 5.3. *There is a $\pi_0(X_I)$ -equivariant isomorphism of graded Lie algebras*

$$\pi_*^+(X_I) \otimes \mathbb{Q} \cong \text{Der}_\omega^+(\mathbb{L}_I),$$

where the action on the right hand side is induced by the natural action of Γ_I on $\tilde{H}_*(N_I; \mathbb{Q})$.

Proof. Compare [BM15, Proposition 4.7.]. Consider the fibration

$$\text{aut}_\partial(N_I) \rightarrow \text{aut}_*(N_I) \xrightarrow{in^*} \text{map}_*(\partial N_I, N_I),$$

where $\text{aut}_\partial(N_I)$ is the fiber over the inclusion of the boundary in . Using Lemma 5.1, we see that the long exact sequence of rational homotopy groups splits for $k \geq 1$ as

$$\begin{aligned} 0 \rightarrow \pi_k(\text{aut}_\partial(N_I), \text{id}_{N_I}) \otimes \mathbb{Q} &\rightarrow \pi_k(\text{aut}_*(N_I), \text{id}_{N_I}) \otimes \mathbb{Q} \\ &\xrightarrow{\pi_k(in^*)} \pi_k(\text{map}_*(\partial N_I, N_I), in) \otimes \mathbb{Q} \rightarrow 0. \end{aligned}$$

It can be identified using Theorem 2.3 with

$$0 \rightarrow \text{Der}_\omega(\mathbb{L}_I)_k \rightarrow \text{Der}(\mathbb{L}_I)_k \xrightarrow{\varphi_k^*} \text{Der}_\varphi(\mathbb{L}[\omega_I], \mathbb{L}_I)_k \rightarrow 0,$$

where we use that $\text{Der}_\omega(\mathbb{L}_I)_k$ is the kernel of φ_k^* . The resulting isomorphism

$$\pi_*^+(\text{aut}_\partial(N_I), \text{id}_{N_I}) \otimes \mathbb{Q} \cong \text{Der}_\omega^+(\mathbb{L}_I)$$

is in fact an isomorphism of graded Lie algebras. Indeed, since the inclusion $\text{aut}_\partial(N_I) \rightarrow \text{aut}_*(N_I)$ is a map of topological monoids, the induced maps on rational homotopy group respect the Samelson product and $\text{Der}^+(\mathbb{L}_I) \cong \pi_*^+(\text{aut}(N_I))$ is an isomorphism of Lie algebras. Hence we can calculate the Samelson product of $\pi_*^+(\text{aut}_\partial(N_I))$ in $\text{Der}^+(\mathbb{L}_I)$.

Now let $f, g \in \text{aut}_*(N_I)$. The action of $[f] \in \pi_0(\text{aut}_*(N_I))$ on $\pi_k(\text{aut}_*(N_I), \text{id}_{N_I})$ is induced by pointwise conjugation $g \mapsto f g f^{-1}$, where f^{-1} is some choice of homotopy inverse. Let ϕ_f be a Quillen model for f and $\theta \in \text{Der}(N_I)_k$. The action of $[f]$ on $\text{Der}(\mathbb{L}_I)_k$ is given by

$$\theta \mapsto \phi_f \circ \theta \circ \phi_f^{-1},$$

by the naturality of the identification $\pi_k(\text{aut}_*(N_I), \text{id}_{N_I}) \otimes \mathbb{Q} \cong \text{Der}(\mathbb{L}_I)_k$. For a homotopy self-equivalence f consider the induced map $f_* \in \text{Aut}(\tilde{H}_*(N_I))$. The map $\mathbb{L}(s^{-1}(f_* \otimes \mathbb{Q}))$ is in fact a Lie model for f , which shows that we can identify the conjugation action with the induced action of $\text{Aut}(\tilde{H}_*(N_I))$ on $\text{Der}(\mathbb{L}_I)_k$.

Using that $\text{Der}_\omega^+(\mathbb{L}_I)_k \rightarrow \text{Der}^+(\mathbb{L}_I)_k$ is injective, we can calculate the conjugation action of $\pi_0(\text{aut}_\partial(N_I))$ on $\pi_k(\text{aut}_\partial(N_I), \text{id}_{N_I})$ in terms of $\text{Der}_\omega(\mathbb{L}_I)_k$. Let f be an element of $\text{aut}_\partial(N_I)$, it is in particular also an element of $\text{aut}_*(N_I)$ and we know that its homotopy class $[f]$ in $\pi_0(\text{aut}_*(N_I))$ gives us an element in $\text{Aut}(\tilde{H}_*(N_I), \langle -, - \rangle, Jq)$. Considering $\theta \in \text{Der}_\omega(\mathbb{L}_I)_k$ as an element in $\text{Der}(\mathbb{L}_I)_k$ we see that $[f]$ acts by the action induced by $f_* \in \text{Aut}(\tilde{H}_*(N_I), \langle -, - \rangle, Jq)$ on $\tilde{H}_*(N_I)$. \square

6 Homological stability for the homotopy automorphisms

In this chapter we prove the first main result of this thesis, it follows the ideas in [BM15, Section 5.3 and 5.4]. Let N_I be as before. Denote by

$$V^{p,q} = S^p \times S^q \setminus \text{int}(D_1^n \sqcup D_2^n), \text{ where } 2 < p \leq q < 2p - 1 \text{ and } p + q = n.$$

We can define a new manifold

$$N' = N_I \cup_{\partial_1} V^{p,q},$$

by identifying one boundary component of $V^{p,q}$ with ∂N_I . Note that N' is canonically diffeomorphic to the manifold $N_{I'}$ with $I' = I \cup \{i'\}$ where $p_{i'} = p$ and $q_{i'} = q$. Using this diffeomorphisms we can define a map

$$\sigma : \text{aut}_{\partial}(N_I) \rightarrow \text{aut}_{\partial}(N_I \cup_{\partial_1} V^{p,q}) \xrightarrow{\cong} \text{aut}_{\partial}(N_{I'}),$$

by extending a self-map of N_I by the identity on $V^{p,q}$. We will refer to this map as stabilization map. In this section we are going to study the behavior of the induced map in the homology of classifying spaces

$$\sigma : H_*(\text{Baut}_{\partial}(N_I); \mathbb{Q}) \rightarrow H_*(\text{Baut}_{\partial}(N_{I'}); \mathbb{Q}).$$

For this we will first have to describe the induced map

$$\sigma_* : \pi_*(\text{Baut}_{\partial}(N_I)\langle 1 \rangle) \otimes \mathbb{Q} \rightarrow \pi_*(\text{Baut}_{\partial}(N_{I'})\langle 1 \rangle) \otimes \mathbb{Q}$$

in terms of the model in Proposition 5.3. Given an element of $\theta \in \text{Der}_{\omega}^+(\mathbb{L}_I)$ we can define an element $\theta' \in \text{Der}_{\omega}^+(\mathbb{L}_{I'})$, by letting $\theta' = \theta$ on generators ι_i, κ_i , where $i \in I$ and $\theta(\iota_{i'}) = \theta(\kappa_{i'}) = 0$. Using that $\mathbb{L}_{I'}$ is free we get a derivation θ' , which is indeed an element of $\text{Der}_{\omega}^+(\mathbb{L}_{I'})$, since $\omega_{I'} = \omega_I + (-1)^{|\iota_{i'}|}[\iota_{i'}, \kappa_{i'}]$. We will refer to this map again as stabilization map.

Lemma 6.1. *The isomorphisms $\pi_*^+(X_I) \otimes \mathbb{Q} \cong \text{Der}_{\omega}^+(\mathbb{L}_I)$ are compatible with the stabilization maps.*

Proof. Compare [BM15, Proposition 5.1.]. Denote by

$$i : N_I \hookrightarrow N_I \cup_{\partial_1} V^{p,q} \text{ and } j : V^{p,q} \hookrightarrow N_I \cup_{\partial_1} V^{p,q}$$

the inclusions. Let $f \in \text{aut}_{\partial}(N_I)$, we can characterize its image in $\text{aut}_{\partial}(N_I \cup_{\partial_1} V^{p,q})$ by the facts that it is f restricted to N_I and the identity restricted to $V^{p,q}$. This yields the following commutative diagram

$$\begin{array}{ccc}
 & \text{map}_*(N_I, N_I \cup_{\partial_1} V^{p,q}) & \\
 & \nearrow i_* & \uparrow i^* \\
 \text{aut}_{\partial}(N_I) & \xrightarrow{\sigma} & \text{aut}_{\partial}(N_I \cup_{\partial_1} V^{p,q}) \\
 \downarrow & & \downarrow j^* \\
 * & \xrightarrow[* \mapsto j]{} & \text{map}_*(V^{p,q}, N_I).
 \end{array} \tag{6.1}$$

The manifold $V^{p,q}$ is homotopy equivalent to $S^p \vee S^q \vee S^{p+q-1}$. Therefore the graded Lie algebra $\mathbb{L}[\iota, \kappa, \rho]$, where $|\iota| = p - 1$, $|\kappa| = q - 1$ and $|\rho| = p + q - 2$ together with the trivial differential, is a Quillen model for $V^{p,q}$. The maps i and j are modeled by

$$\varphi : \mathbb{L}_I \rightarrow \mathbb{L}_{I'} \text{ and } \psi : \mathbb{L}[\iota, \kappa, \rho] \rightarrow \mathbb{L}_{I'},$$

where φ is the obvious inclusion and $\psi(\iota) = \iota_{i'}$, $\psi(\kappa) = \kappa_{i'}$ and $\psi(\rho) = \omega_{I'}$. Lupton and Smith's natural isomorphism (Theorem 2.3) translates (6.1) into the following commutative diagram

$$\begin{array}{ccc}
 & \text{Der}_{\varphi}^+(\mathbb{L}_I, \mathbb{L}_{I'}) & \\
 & \nearrow \varphi_* & \uparrow \varphi^* \\
 \text{Der}_{\omega}^+(\mathbb{L}_I) & \xrightarrow{\sigma_*} & \text{Der}_{\omega}^+(\mathbb{L}_{I'}) \\
 \downarrow & & \downarrow \psi^* \\
 0 & \longrightarrow & \text{Der}_{\psi}^+(\mathbb{L}[\iota, \kappa, \rho], \mathbb{L}_{I'}).
 \end{array}$$

The commutativity of the upper triangle implies that for a derivation θ , $\sigma_*(\theta)$ equals θ on ι_i, κ_i for all $i \in I$ and the commutativity of the lower square implies that $\sigma_*(\theta)$ has to be zero on $\iota_{i'}$ and $\kappa_{i'}$, which proves the claim. \square

Denote by $\mathfrak{g}_I = \text{Der}_{\omega}^+(\mathbb{L}(s^{-1}\tilde{H}_*(N_I; \mathbb{Q}))) = \text{Der}_{\omega}^+(\mathbb{L}(s^{-1}\tilde{H}_I \otimes \mathbb{Q}))$. The Γ_I -action on \mathfrak{g}_I induces one on $C_r^{CE}(\mathfrak{g}_I)$. We define the $\Gamma_{I'}$ -modules $\mathfrak{g}_{I'}$ and $C_r^{CE}(\mathfrak{g}_{I'})$ analogously. Let

$$g_p = \begin{cases} \frac{1}{2} \text{rank}(H_p(N_I)) & \text{if } p = n/2 \\ \text{rank}(H_p(N_I)) & \text{otherwise.} \end{cases}$$

Proposition 6.2. *The map*

$$H_i(\Gamma_I; C_r^{CE}(\mathfrak{g}_I)) \rightarrow H_i(\Gamma_{I'}; C_r^{CE}(\mathfrak{g}_{I'}))$$

induced by the stabilization map is an isomorphism for $g_p > 2i + \frac{r}{2} + 2$ when $2p \neq n$ and $g_p > 2i + \frac{r}{2} + 4$ if $2p = n$ and an epimorphism for $g_p \geq 2i + \frac{r}{2} + 2$ respectively $g_p \geq 2i + \frac{r}{2} + 4$.

Remark 6.3. Note that we can assume that $g_p \geq 2$ (4 if $p = n/2$) because otherwise the statement is vacuous. We will assume this from now on.

Proof. The proposition will follow from Proposition 2.15 upon showing that there exist a commutative diagram

$$\begin{array}{ccc} C_r^{CE}(\mathfrak{g}_I) & \xrightarrow{\cong_{\Gamma_I}} & T(\lambda_I) \\ C_r^{CE}(\sigma) \downarrow & & \downarrow T(I_{p,q}) \\ C_r^{CE}(\mathfrak{g}_{I'}) & \xrightarrow{\cong_{\Gamma_{I'}}} & T(\lambda_{I'}) \end{array} \quad (6.2)$$

for some polynomial functor $T : \text{Mod}(\mathbb{Z})^{n-1} \rightarrow \text{Vect}(\mathbb{Q})$ of degree $\leq \frac{r}{2}$.

For this consider the functor $- \otimes \mathbb{Q} : \text{Mod}(\mathbb{Z})^{n-1} \rightarrow \text{Vect}(\mathbb{Q})^{n-1}$. We set $T(-) = \mathcal{C}_r(- \otimes \mathbb{Q})$, where \mathcal{C}_r is the Schur multifunctor in Corollary 3.8. Since $- \otimes \mathbb{Q}$ is additive, T is of degree $\leq \frac{r}{2}$. We observe that $- \otimes \mathbb{Q}$ defines a functor from $Q_+^n(\mathbb{Z}, \Lambda) \rightarrow \mathcal{S}_+^n(\mathbb{Q})$. This implies that the actions induced via $- \otimes \mathbb{Q}$ by standard actions of Γ_I on H_I and of $\Gamma_{I'}$ on $H_{I'}$ are through morphisms in $\mathcal{S}_+^n(\mathbb{Q})$. Moreover the image of the inclusion $I_{p,q} : H_I \rightarrow H_{I'}$ gives us a map in $\mathcal{S}_+^n(\mathbb{Q})$. The equation (3.1) implies that $\sigma : \mathfrak{g}_I \rightarrow \mathfrak{g}_{I'}$ and hence $C_r^{CE}(\sigma)$ is in fact induced by the inclusion $H_I \otimes \mathbb{Q} \hookrightarrow H_{I'} \otimes \mathbb{Q}$. By the naturality of the isomorphism in Proposition 3.8 we can conclude that (6.2) commutes. \square

Denote by F_I the finite kernel of the surjective map $\pi_1(BX_I) \cong \pi_0(X_I) \xrightarrow{H_*} \Gamma_I$ (Proposition 4.1). The Chevalley-Eilenberg chains become a $\pi_1(BX_I)$ -module via the action of Γ_I . Similarly for I' .

Corollary 6.4. *The map*

$$H_i(\pi_1(BX_I); C_r^{CE}(\mathfrak{g}_I)) \rightarrow H_i(\pi_1(BX_{I'}); C_r^{CE}(\mathfrak{g}_{I'}))$$

induced by the stabilization map induces isomorphisms and epimorphisms in the same range as above.

Proof. We observe that the E^2 -page of the Lyndon spectral sequence

$$E_{k,l}^2 = H_k(\Gamma_I; H_l(F_I; C_r^{CE}(\mathfrak{g}_I))) \Rightarrow H_{k+l}(\pi_1(BX_I); C_r^{CE}(\mathfrak{g}_I))$$

is given by $E_{k,0}^2 = H_k(\Gamma_I; C_r^{CE}(\mathfrak{g}_I))$ and 0 for $l \neq 0$ since F_I is finite and acts trivially on the rational vector space $C_r^{CE}(\mathfrak{g}_I)$. Similarly for I' . Using Proposition 6.2 the claim follows. \square

Hyperhomology will allow us to relate the homology with coefficients in the Chevalley-Eilenberg chain complex to homology with coefficients in Chevalley-Eilenberg homology.

Proposition 6.5. *The map*

$$\mathbb{H}_i(\pi_1(BX_I), C_*^{CE}(\mathfrak{g}_I)) \rightarrow \mathbb{H}_i(\pi_1(BX_{I'}), C_*^{CE}(\mathfrak{g}_{I'}))$$

induced by the stabilization map is an isomorphism for $g_p > 2i + 2$ when $2p \neq n$ and $g_p > 2i + 4$ if $2p = n$ and an epimorphism for $g_p \geq 2i + 2$ respectively $g_p \geq 2i + 4$.

Proof. Consider the first hyperhomology spectral sequence with E^1 -page:

$$E_{k,l}^1(I) = H_l((\pi_1(BX_g); C_k^{CE}(\mathfrak{g}_I))) \Rightarrow \mathbb{H}_{k+l}(\pi_1(BX_I); C_*^{CE}(\mathfrak{g}_I)).$$

By Corollary 6.4 $E_{k,l}^1(I) \rightarrow E_{k,l}^1(I')$ is an isomorphism for

$$g_p > \begin{cases} 2k + 2l + 4 \geq k + 2l + 4 & \text{if } p = n/2 \\ 2k + 2l + 2 \geq k + 2l + 2 & \text{otherwise} \end{cases}$$

and an epimorphism for " \geq ". Hence the claim follows by the spectral sequence comparison theorem. \square

Lemma 6.6. *The group $\pi_1(BX_I)$ is rationally perfect.*

Proof. By Lemma 2.16 the groups Γ_I are rationally perfect. Hence the finite extension $\pi_1(BX_I)$ is also rationally perfect. \square

Proposition 6.7. *The map*

$$H_k(\pi_1(BX_g), H_l^{CE}(\mathfrak{g}_g)) \rightarrow H_k(\pi_1(BX_{g+1}), H_l^{CE}(\mathfrak{g}_{g+1}))$$

induced by the stabilization map is an isomorphism for $g_p > 2k + 2l + 2$ when $2p \neq n$ and $g_p > 2k + 2l + 4$ if $2p = n$ and an epimorphism for $g_p \geq 2k + 2l + 2$ respectively $g_p \geq 2k + 2l + 4$.

Proof. By Lemma 6.6 and because $C_*^{CE}(\mathfrak{g}_I)$ is finite dimensional over \mathbb{Q} in each degree (by Corollary 3.8), we can use Proposition 2.17. Hence we get a chain homotopy equivalences $C_*^{CE}(\mathfrak{g}_I) \xrightarrow{\simeq} H_*^{CE}(\mathfrak{g}_I)$ such that

$$\begin{array}{ccc} C_*^{CE}(\mathfrak{g}_I) & \xrightarrow{\sigma_*} & C_*^{CE}(\mathfrak{g}_{I'}) \\ \simeq \downarrow & & \simeq \downarrow \\ H_*^{CE}(\mathfrak{g}_I) & \xrightarrow{\sigma_*} & H_*^{CE}(\mathfrak{g}_{I'}) \end{array}$$

commutes up to chain homotopy of $\mathbb{Q}[\Gamma_I]$ -chain complexes. Thus we get a commutative diagram

$$\begin{array}{ccc} \mathbb{H}_i(\pi_1(BX_I); C_*^{CE}(\mathfrak{g}_I)) & \xrightarrow{\sigma_i} & \mathbb{H}_i(\pi_1(BX_{I'}); C_*^{CE}(\mathfrak{g}_{I'})) \\ \cong \downarrow & & \cong \downarrow \\ \mathbb{H}_i(\pi_1(BX_I); H_*^{CE}(\mathfrak{g}_I)) & \xrightarrow{\sigma_i} & \mathbb{H}_i(\pi_1(BX_{I'}); H_*^{CE}(\mathfrak{g}_{I'})), \end{array}$$

where the vertical arrows are isomorphisms by the chain homotopy invariance of hyperhomology. By Proposition 6.5 the upper map is an isomorphism for $g_p > \begin{cases} 2i + 4 & \text{if } p = n/2 \\ 2i + 2 & \text{otherwise} \end{cases}$ and an epimorphism for " $>$ " \geq and hence also the lower maps. Ultimately we use the natural splitting for hyperhomology groups with coefficients in a chain complex with trivial differential

$$\begin{array}{ccc} \mathbb{H}_i(\pi_1(BX_I); H_*^{CE}(\mathfrak{g}_I)) & \xrightarrow{\sigma_i} & \mathbb{H}_i(\pi_1(BX_{I'}); H_*^{CE}(\mathfrak{g}_{I'})) \\ \cong \downarrow & & \cong \downarrow \\ \bigoplus_{k+l=i} H_k(\pi_1(BX_I); H_l^{CE}(\mathfrak{g}_I)) & \xrightarrow{\sigma_{k,l}} & \bigoplus_{k+l=i} H_k(\pi_1(BX_{I'}); H_l^{CE}(\mathfrak{g}_{I'})). \end{array}$$

Hence we see that the maps $\sigma_{k,l}$ are isomorphisms and epimorphisms in the range in the statement. □

Now we have everything together to prove the first main result of this thesis.

Theorem A. *The stabilization map*

$$H_i(\text{Baut}_\partial(N_I); \mathbb{Q}) \rightarrow H_i(\text{Baut}_\partial(N_{I'}); \mathbb{Q})$$

is an isomorphism for $g_p > 2i + 2$ when $2d \neq n$ and $g_p > 2i + 4$ if $2d = n$ and an epimorphism for $g_p \geq 2i + 2$ respectively $g_p \geq 2i + 4$.

Proof. We want to use the spectral sequence of the universal covering of BX_I with E_2 -terms:

$$H_k(\pi_1(BX_I); H_l(BX_I\langle 1 \rangle, \mathbb{Q})) \Rightarrow H_{k+l}(BX_I, \mathbb{Q}).$$

We claim there are isomorphisms of $\pi_1(BX_I)$ -modules

$$H_l(BX_I\langle 1 \rangle, \mathbb{Q}) \cong H_l^{CE}(\mathfrak{g}_I).$$

This can be seen by using the Quillen spectral sequence (Theorem 2.2). Since $BX_I\langle 1 \rangle$ is coformal (Proposition 5.2) it collapses and we get an isomorphisms of rational vector spaces

$$H_l(BX_I\langle 1 \rangle, \mathbb{Q}) \cong \bigoplus_{p+q=l} H_p^{CE}(\mathfrak{g}_I)^q \cong H_l^{CE}(\mathfrak{g}_I).$$

Since the Quillen spectral sequence is natural with respect to unbased maps it is in fact a spectral sequence of $\pi_1(BX_I)$ -modules. The $H_*^{CE}(g_I)$ are finite dimensional in each degree over \mathbb{Q} (since the $C_*^{CE}(g_I)$ are). Using that $\pi_1(BX_I)$ is rationally perfect (and hence all extensions of $\mathbb{Q}[\pi_1(BX_I)]$ -modules have to be trivial) it follows that the isomorphisms above are isomorphisms of $\pi_1(BX_I)$ -modules.

By Lemma 6.1 we get the commutativity of the following diagram

$$\begin{array}{ccc} H_k(\pi_1(BX_I); H_l(BX_I\langle 1 \rangle, \mathbb{Q})) & \xrightarrow{\sigma} & H_k(\pi_1(BX_{I'}); H_l(BX_{I'}\langle 1 \rangle, \mathbb{Q})) \\ \cong \downarrow & & \cong \downarrow \\ H_k(\pi_1(BX_I); H_l^{CE}(\mathfrak{g}_I)) & \xrightarrow{\sigma} & H_k(\pi_1(BX_{I'}); H_l^{CE}(\mathfrak{g}_{I'})). \end{array}$$

Proposition 6.7 and the spectral sequence comparison theorem now imply the statement of the theorem. The diagram above ensures that the isomorphism is induced by the stabilization map. \square

Consider another manifold

$$N_J = \#_{j \in J} (S^{p_j} \times S^{q_j}) \setminus \text{int}(D^n)$$

like above. Denote by $\tilde{I} = \{i \in I \mid p_i = p_j \text{ for some } j \in J\}$ and by

$$G_{I,J} = \min_{i \in \tilde{I}} \{\text{rank } H_{p_i}(N_I) \mid p_i \neq n/2\} \cup \{\text{rank } H_{p_i}(N_I) - 2 \mid p_i = n/2\}.$$

An immediate consequence of Theorem A is the following:

Corollary 6.8. *The map*

$$H_i(\text{Baut}_{\partial}(N_I); \mathbb{Q}) \rightarrow H_i(\text{Baut}_{\partial}(N_{I \cup J}); \mathbb{Q})$$

induced by iterated stabilization maps is an isomorphisms for $G_{I,J} > 2i + 2$ and an epimorphism for $G_{I,J} = 2i + 2$.

7 Homological stability for the block diffeomorphisms

In this chapter we prove the second main result of this thesis, it follows the ideas in [BM15, Section 5.5]. We will use the notations from the last section. Let $Y_I = \widetilde{\text{Diff}}_\partial(N_I)$. The stabilization map

$$H_i(B\widetilde{\text{Diff}}_\partial(N_I); \mathbb{Q}) \rightarrow H_i(B\widetilde{\text{Diff}}_\partial(N_{I'}); \mathbb{Q})$$

is defined in a similar fashion as before. Recall that $\widetilde{\text{Diff}}_\partial(N_I)$ is the geometric realization of a Δ -group whose k -simplices are diffeomorphisms

$$\varphi : \Delta^k \times N_I \rightarrow \Delta^k \times N_I,$$

such that for faces $\tau \subset \Delta^k$ we have $\varphi(\tau \times N_I) \subset \tau \times N_I$ and $\varphi|_{\Delta^k \times \partial N_I} = \text{id}_{\Delta^k \times \partial N_I}$. The stabilization map is induced by sending a k -simplex φ to the k -simplex in $\widetilde{\text{Diff}}_\partial(N_{I'})$

$$\Delta^k \times (N_I \cup_{\partial_1} V^{p,q}) \rightarrow \Delta^k \times (N_{I'} \cup_{\partial_1} V^{p,q}),$$

given by φ on $\Delta^k \times N_I$ and the identity on $\Delta^k \times V^{p,q}$, where we use that $N_{I'} \cong N_I \cup_{\partial_1} V^{p,q}$. Recall that

$$\tilde{J} : \widetilde{\text{Diff}}_\partial(N_I) \rightarrow \widetilde{\text{Aut}}_\partial(N_I)$$

just denoted the obvious inclusion. Proposition 4.2 implies (using Cerf's pseudo-isotopy theorem) that

$$\text{Image}(\pi_1(B\widetilde{\text{Diff}}_\partial(N_I)) \rightarrow \pi_1(B\widetilde{\text{Aut}}_\partial(N_I))) \quad (7.1)$$

has finite index in $\pi_1(B\widetilde{\text{Aut}}_\partial(N_I))$. Let $\bar{B}\widetilde{\text{Aut}}_\partial(N_I) \rightarrow B\widetilde{\text{Aut}}_\partial(N_I)$ denote the finite covering corresponding to the subgroup (7.1) above. By construction we get a lift

$$B\widetilde{\text{Diff}}_\partial(N_I) \rightarrow \bar{B}\widetilde{\text{Aut}}_\partial(N_I)$$

of \tilde{J} and call the homotopy fiber of this map \mathcal{F}_I . Note that

$$\mathcal{F}_I \simeq \widetilde{\text{Aut}}_\partial(N_I) / \widetilde{\text{Diff}}_\partial(N_I)_{(1)}$$

is the component of $\text{hofib}(\tilde{J})$ that can be studied using Surgery theory (see Proposition 2.18). We want to understand the rational homology of \mathcal{F}_I as $\pi_1(\bar{B}\widetilde{\text{Aut}}_\partial(N_I))$ -modules. By Proposition 2.18 we can use Surgery theory to understand its homotopy groups, i.e.

$$\pi_k(\mathcal{F}_I) \cong \pi_k(S(N_I, \partial N_I)) \cong \mathcal{S}^{G/O}(N_I \times D^k, \partial(N_I \times D^k)) \text{ for } k > 0.$$

We consider the rational homotopy exact sequence of the surgery fibration

$$\dots \rightarrow S(N_I, \partial N_I) \otimes \mathbb{Q} \xrightarrow{\eta} \pi_k(\text{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q} \xrightarrow{\sigma} \pi_k(L(N_I)) \otimes \mathbb{Q} \rightarrow \dots$$

Since the rational homotopy groups of G are trivial, we get that

$$G/O \simeq_{\mathbb{Q}} BO \simeq_{\mathbb{Q}} \prod_{i \geq 1} K(\mathbb{Q}, 4i).$$

Thus we can identify

$$\pi_k(\text{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q} \cong (H^*(N_I, \partial N_I; \mathbb{Q}) \otimes \pi_*(G/O))_k. \quad (7.2)$$

The rational homotopy groups of $\pi_k(L(N_I)) \otimes \mathbb{Q}$ given by the L -groups

$$L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{for } n+k \equiv 0 \pmod{4} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.1 ([BM13, Lemma 3.5.]). *The surgery obstruction map induces an isomorphism*

$$H^n(N_I, \partial N_I; \mathbb{Q}) \otimes \pi_{n+k}(G/O) \rightarrow L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q}$$

for $n+k \equiv 0 \pmod{4}$.

Proof. Consider the smooth and topological surgery exact sequences:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \pi_k(\text{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q} & \longrightarrow & L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q} & \longrightarrow & \dots \\ & & \downarrow & & \parallel & & \\ \dots & \longrightarrow & \pi_k(\text{map}_*(N_I/\partial N_I, G/Top)) \otimes \mathbb{Q} & \longrightarrow & L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q} & \longrightarrow & \dots \end{array}$$

The left hand vertical map is an isomorphism since $\pi_i(Top/O)$ is finite (see e.g. [KS77]). Milnor's Plumbing construction ensures that for $k+n$ even there is an element in $\mathcal{N}^{G/Top}(N_I \times D^k, \partial(N_I \times D^k))$ with non-trivial surgery obstruction. Since

$$\mathcal{N}^{G/Top}(N_I \times D^k, \partial(N_I \times D^k)) \cong \pi_k(\text{map}_*(N_I/\partial N_I, G/Top)) \text{ and}$$

$$\pi_k(\text{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q} \cong H^n(N_I, \partial N_I; \mathbb{Q}) \otimes \pi_{n+k}(G/O), \text{ for } n+k \equiv 0 \pmod{4},$$

this implies the claim because both sides are just one dimensional rational vectorspaces. \square

Using the surgery exact sequence, we see that for n odd $\pi_1(\mathcal{F}_I)$ is abelian, since it is a subgroup of the abelian group $[\Sigma(N_I/\partial N_I), G/O]_*$. For n even it is a finite extension of the abelian group $[\Sigma(N_I/\partial N_I), G/O]_*$ by a finite cyclic group (in case $L_{n+2}(\mathbb{Z}) \cong \mathbb{Z}$, the proof of Lemma 7.1 makes sure that the map to $L_{n+2}(\mathbb{Z})$ is non zero and hence the kernel of σ is a finite cyclic group). Denote by

$$\pi_k^{ab}(\mathcal{F}_I) = \begin{cases} \pi_1(\mathcal{F}_I) / \text{Image}(L_{n+2}(\mathbb{Z}) \rightarrow \pi_1(\mathcal{F}_I)) & \text{if } k = 1 \\ \pi_k(\mathcal{F}_I) & \text{if } k > 1. \end{cases}$$

Proposition 7.2. *There are isomorphism of $\pi_1(\widetilde{\text{BAut}}_\partial(N_I))$ -modules compatible with the stabilization maps*

- (1) $\pi_k^{ab}(\mathcal{F}_I) \otimes \mathbb{Q} \cong (\tilde{H}_*(N_I, \mathbb{Q}) \otimes \pi_*(G/O))_k$, where $|a \otimes \alpha| = |\alpha| - |a|$, $k \geq 1$
- (2) $H_*(\mathcal{F}_I, \mathbb{Q}) \cong \Lambda(\pi_*^{ab}(\mathcal{F}_I) \otimes \mathbb{Q})$,

where the actions on the right hand side are induced by the standard actions of Γ_I on $\tilde{H}_*(N_I)$.

Proof. Compare [BM13, p.26 and Theorem 3.6] and [BM15, p.32]. Observe that the rationalization $(\mathcal{F}_I)_{\mathbb{Q}}$ has rational homotopy groups $\pi_k^{ab}(\mathcal{F}_I) \otimes \mathbb{Q}$. Consider the splitting of the homotopy exact sequence of the surgery fibration as

$$0 \rightarrow L_{n+k+1}(\mathbb{Z}) / \text{Image}(\sigma) \rightarrow \pi_k(S(N_I, \partial N)) \rightarrow \text{Image}(\eta) \rightarrow 0.$$

By Lemma 7.1 we get

$$L_{n+k+1}(\mathbb{Z}) / \text{Image}(\sigma) \otimes \mathbb{Q} \cong 0 \text{ and } \text{Image}(\eta) \otimes \mathbb{Q} \cong (\tilde{H}^*(N_I, \mathbb{Q}) \otimes \pi_*(G/O))_k.$$

Using the isomorphism

$$\tilde{H}_*(N_I; \mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}}(\tilde{H}_*(N_I; \mathbb{Q}); \mathbb{Q}) \cong \tilde{H}^*(N_I; \mathbb{Q})$$

we get the isomorphism (1). We see that the action on the right hand side is induced by the standard action of Γ_I as follows: Use the identification

$$\pi_k(S(N_I, \partial N)) \cong \mathcal{S}^{G/O}(N_I \times D^k, \partial(N_I \times D^k)).$$

An element of $\mathcal{S}^{G/O}(N_I \times D^k, \partial(N_I \times D^k))$ is represented by a manifold $(X, \partial X)$ together with a homotopy equivalence $f : X \rightarrow N_I \times D^k$, such that $f|_{\partial X} : \partial X \rightarrow \partial(N_I \times D^k)$ is a diffeomorphism. Recall that J_0 denotes the map sending an isotopy class of self-diffeomorphisms to a homotopy class of homotopy automorphisms. The action of a

$$[\phi] \in \pi_1(\widetilde{\text{BAut}}_\partial(N_I)) \cong \text{Image}(\pi_1(\tilde{J})) \cong \text{Image}(J_0) \subset \pi_0(\text{aut}_\partial(N_I))$$

on f is given by the composition

$$X \xrightarrow{f} N_I \times D^k \xrightarrow{\phi \times \text{id}_{D^k}} N_I \times D^k,$$

where ϕ is a diffeomorphism representing $[\phi]$ considered as an element of $\text{Image}(J_0)$. Lemma 2.19 now implies that

$$\eta((\phi \times \text{id}_{D^k}) \circ f) = ((\phi \times \text{id}_{D^k})^*)^{-1}(\eta(f)),$$

using that the normal invariant of a diffeomorphism is trivial. This implies that $[\phi]$ acts on $\tilde{H}^*(N; \mathbb{Q}) \otimes \pi_*(G/O)_k$ via $(\phi^{-1})^* \otimes \text{id}_{\pi_*(G/O)}$. But this exactly corresponds to the standard action under the isomorphism

$$\tilde{H}^*(N_I; \mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}}(\tilde{H}_*(N_I; \mathbb{Q}); \mathbb{Q}) \cong \tilde{H}_*(N_I; \mathbb{Q}).$$

If ϕ lies in the kernel of the map

$$\pi_1(\widetilde{B\text{Aut}}_{\partial}(N_I)) \rightarrow \Gamma_I,$$

then it induces the identity on $\tilde{H}^*(N_I)$ and hence acts trivially on $\pi_k^{ab}(\mathcal{F}_I)$. The compatibility with the stabilization maps follows from the fact that the isomorphisms (7.2) is natural.

The statement (2) follows from (1) by using the fact that G/O and hence also the mapping-space $\text{map}_*(N_I/\partial N_I, G/O)$ are infinite loop spaces. Thus all rational k-invariants vanish for $\text{map}_*(N_I/\partial N_I, G/O)$. This is equivalent to:

For all $\alpha \in \pi_k(\text{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q}$ there exists a $c \in H^k(\text{map}_*(N_I/\partial N_I, G/O); \mathbb{Q})$, such that $c(h(\alpha)) \neq 0$, where h denotes the rational Hurewicz homomorphism. Since

$$\pi_k((\mathcal{F}_I)_{\mathbb{Q}}) \otimes \mathbb{Q} \rightarrow \pi_k(\text{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q}$$

is injective, it follows that all rational k-invariants also vanish for $(\mathcal{F}_I)_{\mathbb{Q}}$. This shows that $(\mathcal{F}_I)_{\mathbb{Q}}$ is a product of Eilenberg-MacLane spaces and hence its homology is given by the free graded commutative algebra on its homotopy groups. Moreover the $\pi_1(\widetilde{B\text{Aut}}_{\partial}(N_I))$ -action is induced by the standard action. \square

We can use the previous Proposition to give a Schur multifunctor description of $H_r(\mathcal{F}_I; \mathbb{Q})$. For a multi-index μ with $l(\mu) = n - 1$, consider the Σ_{μ} -modules $\Pi(\mu)$ given by

$$\Pi(0, \dots, 1, \dots, 0) = s^{-i} \pi_*(G/O) \otimes \mathbb{Q},$$

where the 1 sits in the i -th position and 0 otherwise. The corresponding Schur multifunctor

$$\Pi : \text{Mod}(\mathbb{Z})^{n-1} \rightarrow \text{Vect}_*(\mathbb{Q}),$$

has the property that there is an isomorphism of the induced Γ_I -modules

$$\Pi(H_I) \cong (\tilde{H}_*(N_I, \mathbb{Q}) \otimes \pi_*(G/O))^+.$$

It follows now that we get an isomorphisms of Γ_I -modules

$$\Lambda \circ \Pi(H_I) \cong \Lambda((\tilde{H}_*(N_I, \mathbb{Q}) \otimes \pi_*(G/O)^+)) \cong H_*(\mathcal{F}_I, \mathbb{Q}),$$

where the left-hand Λ denotes the free graded commutative algebra endofunctor of $Vect_*(\mathbb{Q})$. Recall that Λ is given as the Schur functor with $\Lambda(n) = \mathbb{Q}[n]$ and trivial Σ_n -action. Now setting $\mathcal{H}_r = \Lambda_r \circ \Pi$ and observing that Λ_r is of degree $\leq r$ and Π additive, we get the following:

Proposition 7.3. *There is an isomorphism of Γ_I -modules*

$$H_r(\mathcal{F}_I; \mathbb{Q}) \cong \bigoplus_{\mu} \mathcal{H}_r(\mu) \otimes_{\mu} H_I^{\otimes \mu},$$

compatible with the stabilization maps, where the $\mathcal{H}_r(\mu)$ are trivial for $|\mu| > r$.

Now we can proof the second main theorem of this thesis.

Theorem B. *The stabilization map*

$$H_i(\widetilde{BDiff}_{\partial}(N_I); \mathbb{Q}) \rightarrow H_i(\widetilde{BDiff}_{\partial}(N_{I'}); \mathbb{Q})$$

is an isomorphism for $g_p > 2i + 2$ when $2p \neq n$ and $g_p > 2i + 4$ id $2p = n$ and an epimorphism for $g_p \geq 2i + 2$ respectively $g_p \geq 2i + 4$.

Proof. Recall that we denote by $BY_I = \widetilde{BDiff}_{\partial}(N_I)$ and $\bar{B}X_I = \widetilde{BAut}_{\partial}(N_I)$, the finite cover of $BX_I = \widetilde{BAut}_{\partial}(N_I)$, corresponding to the subgroup $\text{Image}(\pi_1(B\tilde{J})) \subset \pi_1(\widetilde{BAut}_{\partial}(N_I))$. Consider the Serre spectral sequences of the homotopy fibration

$$\mathcal{F}_I \rightarrow BY_I \rightarrow \bar{B}X_I$$

and the analog for I' . The stabilization map induces maps on the E_2 -pages

$$\sigma_* : H_k(\bar{B}X_I; H_l(\mathcal{F}_I; \mathbb{Q})) \rightarrow H_k(\bar{B}X_{I'}; H_l(\mathcal{F}_{I'}; \mathbb{Q})).$$

The Theorem will follow upon showing that these are isomorphisms for $g_p > 2k + 2l + 2$ (+4 if $p = n/2$) and epimorphisms for $g_p \geq 2k + 2l + 2$ (+4 if $p = n/2$). For this we consider the universal coving spectral sequence

$$H_r(\pi_1(\bar{B}X_I); H_s(\bar{B}X_I(1); H_l(\mathcal{F}_I; \mathbb{Q}))) \Rightarrow H_{r+s}(\bar{B}X_I; H_l(\mathcal{F}_I, \mathbb{Q})).$$

The condition above would follow upon showing that maps induced by the stabilization map on the E^2 -page are isomorphisms for $g_p > 2r + 2s + 2l + 2$ (+4 if $p = n/2$) and epimorphisms for $g_p \geq 2r + 2s + 2l + 2$ (+4 if $p = n/2$). To show this we observe that there are isomorphism of Γ_I -modules compatible with the stabilization maps:

$$H_s(\bar{B}X_I\langle 1 \rangle; H_l(\mathcal{F}_I; \mathbb{Q})) \cong H_s(\bar{B}X_I\langle 1 \rangle) \otimes H_l(\mathcal{F}_I; \mathbb{Q}) \cong H_s^{CE}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q}),$$

where Γ_I acts on the 2-nd and 3-rd term diagonally. Note that $\bar{B}X_I$ and BX_I have the same universal cover, which is moreover naturally homotopy equivalent to $B\text{aut}_{\partial}(N_I)\langle 1 \rangle$. The stability for

$$H_r(\pi_1(\bar{B}X_I); H_s^{CE}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q}))$$

follows from stability for

$$H_r(\pi_1(\bar{B}X_I); C_s^{CE}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q})),$$

exactly like in the Propositions 6.5 and 6.7 upon using the two hyperhomology spectral sequences (using that $\pi_1(\bar{B}X_I)$ can be shown to be rationally perfect as in Lemma 6.6.) Hence we are left with showing that the stabilization maps

$$H_r(\pi_1(\bar{B}X_I); C_s^{CE}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q})) \rightarrow H_r(\pi_1(\bar{B}X_{I'}); C_s^{CE}(\mathfrak{g}_{I'}) \otimes H_l(\mathcal{F}_{I'}; \mathbb{Q}))$$

are isomorphisms for $g_p > 2r + 2s + 2l + 2$ (+4 if $p = n/2$) and epimorphisms for $g_p \geq 2r + 2s + 2l + 2$ (+4 if $p = n/2$). The Lyndon spectral sequence reduces this to the corresponding statement for

$$H_r(\Gamma_I; C_s^{CE}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q})) \rightarrow H_r(\Gamma_{I'}; C_s^{CE}(\mathfrak{g}_{I'}) \otimes H_l(\mathcal{F}_{I'}; \mathbb{Q})).$$

Proposition 7.3 and Corollary 3.8 give us isomorphisms of Γ_I -modules compatible with the stabilization map $C_s^{CE}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q}) \cong \mathcal{C}_s(H_I) \otimes \mathcal{H}_l(H_I)$. The functor \mathcal{C}_s is polynomial of degree $\leq s/2$ and the functor \mathcal{H}_l is polynomial of degree $\leq l$. The tensor product (in the sense of Schur multifunctors) $\mathcal{C}_s \otimes \mathcal{H}_l$ is of degree $\leq s/2 + l$. By Proposition 2.15 the stabilization maps

$$H_r(\Gamma_I; \mathcal{C}_s \otimes \mathcal{H}_l(H_I)) \rightarrow H_r(\Gamma_{I'}; \mathcal{C}_s \otimes \mathcal{H}_l(H_{I'}))$$

are isomorphisms for $g_p > 2r + s/2 + l + 2$ (+4 if $p = n/2$) and epimorphisms for $g_p \geq 2r + s/2 + l + 2$ (+4 if $p = n/2$), which finishes the proof. \square

Remark 7.4. The analog of Corollary 6.8 of course also holds in the block diffeomorphism case.

8 Perspectives for further research

8.1 Extension of the homological stability results

There are two possible directions to extend the homological stability results:

- (1) Lower the connectivity assumptions.
- (2) Enlarge the class of manifolds.

Lower the connectivity assumptions

We will begin with discussing the first direction. It seems reasonable that the condition $p_i \leq q_i < 2p_i - 1$ is actually unnecessary.

Conjecture 8.1. *Let $|I| < \infty$ and $1 < p_i \leq q_i$, $p_i + q_i = n$ for $i \in I$ and $1 < p \leq q$, $p + q = n$. Then the stabilization map*

$$\sigma : H_*(\text{Baut}_\partial(N_I), \mathbb{Q}) \rightarrow H_*(\text{Baut}_\partial(N_I \cup_{\partial_1} V^{p,q}); \mathbb{Q})$$

is an isomorphism in a range, where

$$N_I = (\#_{i \in I} (S^{p_i} \times S^{q_i})) \setminus \text{int}(D^n).$$

Remark 8.2. Of course also the analogous statement for the block diffeomorphisms seems likely, but the author doesn't know if an analog of Proposition 4.2 holds, i.e. if $\pi_0(\text{Diff}_\partial(N_I))$ has finite index in $\pi_0(\text{aut}_\partial(N_I))$. If this is not the case, then the group $\pi_0(S \text{Diff}_\partial(N_I))$ of isotopy classes of diffeomorphisms that induce the identity on homology and the map

$$\pi_0(S \text{Diff}_\partial(N_I)) \rightarrow \pi_0(S \text{aut}_\partial(N_I))$$

need to be understood. To the authors knowledge not even $\pi_0(S \text{Diff}_\partial(S^p \times S^q))$ has been considered for general $1 < p < q$.

We will describe a potential strategy to proof the Conjecture 8.1 for the special case

$$N_g = (\#_g (S^p \times S^q)) \setminus \text{int}(D^{p+q}).$$

The first step is to observe that there is an extension of groups:

$$0 \rightarrow K \rightarrow \pi_0(\text{aut}_\partial(N_g)) \rightarrow \text{Aut}(H_p(N_g)) \rightarrow 0,$$

where

$$K \otimes \mathbb{Q} \cong (\pi_q(N_g) \otimes \mathbb{Q})^g \cong (\mathbb{L}(s^{-1}\tilde{H}_*(N_g; \mathbb{Q}))_{q-1})^g.$$

This can be shown using the arguments in the proof of Proposition 4.1. The action on the kernel should be induced by the action of $\text{Aut}(H_p(N_g)) \cong \text{Aut}(\tilde{H}_*(N_g), \langle -, - \rangle)$ on $H_*(N_g; \mathbb{Q})$.

In Chapter 6 we now need to consider the $\pi_0(\text{aut}_\partial(N_g))$ -module

$$(\mathbb{L}(s^{-1}\tilde{H}_*(N_g; \mathbb{Q}))_{q-1})^g \otimes C_*^{CE}(\mathbb{L}(s^{-1}H_*(N_g; \mathbb{Q}))).$$

The only issue occurring is that the group $\pi_0(\text{aut}_\partial(N_g))$ is in general not rationally perfect. However, since the action of $\pi_0(\text{aut}_\partial(N_g))$ is most likely through $\text{Aut}(\tilde{H}_*(N_g), \langle -, - \rangle)$, the latter should not be a problem.

Enlarge the class of manifolds

One of the reasons why we choose to restrict to the manifolds N_I with $p_i \leq q_i < 2p_i - 1$ is that they are rationally homotopy equivalent to a large class of manifolds. Let X be a manifold and denote by $r_i = \text{rank } H_i(X)$.

Proposition 8.3. *Let X be a k -connected n -manifold with $k \geq \frac{n-1}{3}$ and $n > 5$. Then there is a rational homology equivalence if $n = 2d + 1$ is odd*

$$X \simeq_{\mathbb{Q}} (\#_{i=k+1}^d \#_{r_i}(S^i \times S^{n-i}))$$

and if $n = 2d$ is even

$$X \simeq_{\mathbb{Q}} (\#_{i=k+1}^{d-1} \#_{r_i}(S^i \times S^{n-i})) \# M_{[X]}.$$

The manifold $M_{[X]}$ is homotopy equivalent to a complex given by $\bigvee_{r_d} S^d \cup_{\varphi} e^n$, where

$$\varphi = \frac{1}{2} \sum_{i,j=1}^{r_d} \langle e_i, e_j \rangle [l_i^d, l_j^d] + \sum_{i=1}^{r_d} Jq(e_i) [l_i^d, l_i^d],$$

where e_i denotes some basis for the torsion free part of $H_d(X)$ and $l_i^d : S^d \hookrightarrow \bigvee_{r_d} S^d$ denote the canonical inclusions.

Proof. The manifold $X \setminus \text{int}(D^n)$ is homotopy equivalent to a CW-complex with cells only in dimensions $k + 1, \dots, 2k + 2$. This implies that the $(n - 1)$ -skeleton has to be (up to rational homotopy equivalence) a wedge of spheres

$$V = \bigvee_{i=k+1}^{2k+2} \left(\bigvee_{r_i} S^i \right).$$

This can be shown inductively, since the only possible non rationally trivial attaching maps give rise to contractible sub complexes.

We observe next that X splits rationally as a connected sum of compact manifolds with reduced rational homology only in degrees p and $n - p$. This follows since

$$\pi_{n-1}(X \setminus \text{int}(D^n)) \otimes \mathbb{Q} \cong \mathbb{L}(s^{-1}\tilde{H}_*(V))_{n-2} \cong \sum_p \mathbb{L}(s^{-1}(H_p(V) \oplus H_{n-p}(V)))_{n-2}$$

and the connected sum corresponds to the sum of the attaching maps of the top cell. A compact k -connected n -manifold $M_{p,n-p}$ with reduced rational homology only in degrees p and $n - p$ that are not the middle dimension is rationally equivalent to a connected sum of products of spheres. To see this we use that the $(n - 1)$ -skeleton of $M_{p,n-p}$ is rationally a wedge

$$V_{p,n-p} = \bigvee_{i=1}^{r_p} (S^p \vee S^{n-p})$$

and the attaching map of the top cell is given by $\sum_{i,j} \langle e_i^p, e_j^{n-p} \rangle [l_i^p, l_j^{n-p}]$, where l_i^p and l_j^{n-p} denote the inclusions of the S^p (S^{n-p} respectively) and the e_i^p and e_j^{n-p} denote elements in homology corresponding to them via the Hurewicz homomorphism. There is an automorphisms f of $V_{p,n-p}$, such that

$$f \left(\sum_{i,j} \langle e_i^p, e_j^{n-p} \rangle [l_i^p, l_j^{n-p}], \right) = \sum_i [l_i^p, l_i^{n-p}]$$

and it extends to a homotopy equivalence of the complexes obtained by attaching a cell - this proves the claim for manifolds that don't have homology in the middle dimension. To obtain the automorphisms f , we realize the automorphism of the homology given by the matrix

$$((\langle e_i^p, e_j^{n-p} \rangle)_{i,j})^{-1}$$

in degree p and the identity in degree $n - p$.

In the middle dimension the intersection form is more complicated and we can not in general simplify it. But in any case the attaching map is given by the formula for φ in the statement above.

□

This leads us to the following question:

Question 8.4. Assume $X \simeq_{\mathbb{Q}} Y$ are rationally homotopy equivalent spaces of finite \mathbb{Q} -type. Under which conditions are $H_*(\text{Baut}_*(X); \mathbb{Q})$ and $H_*(\text{Baut}_*(Y); \mathbb{Q})$ isomorphic?

One positive answer to Question 8.4 are rational homology spheres, i.e. compact manifolds with the rational homology of a sphere. We are going to give a quite complicated proof - but it illustrates the general problem.

Proposition 8.5. *Let X be a rational homology n -sphere, then $H_*(\text{Baut}_*(X); \mathbb{Q}) \cong H_*(\text{Baut}_*(S^n); \mathbb{Q})$.*

Proof. We know that the homology groups $H_1(X), \dots, H_{n-1}(X)$ are finite and that the automorphisms of $H_n(X)$ have to be $\{\pm 1\}$. Thus we get that the image of

$$\tilde{H}_* : \pi_0(\text{aut}_*(X)) \rightarrow \text{Aut}(\tilde{H}_*(X))$$

is finite. It is well known that for a n -CW-complexes $K^{n-1} \cup e^n$, the kernel of \tilde{H}_* is a quotient of $\pi_n(K^{n-1})$ (see e.g. [BB58]). But since $X \setminus \text{int}(D^n)$ is rationally homotopy equivalent to a rationally contractible $(n-1)$ -CW-complex, it follows that it is finite. Using Theorem 2.3, we now get that $\text{Der}^+(\mathbb{L}(\mathbb{Q}[n-1]))$ is a model for the rational homotopy groups of the universal covering of $\text{Baut}_*(X)$ and the action by deck transformations is via $\text{Aut}(H_n(X, \mathbb{Q})) \cong \mathbb{Z}/2\mathbb{Z}$ induced by the sign action on \mathbb{Q} . The statement now follows by considering the covering spectral sequence of the universal covering

$$\text{Baut}_*(X)\langle 1 \rangle \rightarrow \text{Baut}_*(X) \rightarrow B\pi_1(\text{Baut}_*(X))$$

with E^2 -terms given by

$$\begin{aligned} H_p(\pi_1(\text{Baut}_*(X)); H_q(\text{Baut}_*(X)\langle 1 \rangle; \mathbb{Q})) &\cong H_p(\mathbb{Z}/2\mathbb{Z}; H_q^{CE}(\text{Der}^+(\mathbb{L}(\mathbb{Q}[n-1]))) \\ &\cong H_p(\pi_1(\text{Baut}_*(S^n)); H_q(\text{Baut}_*(S^n)\langle 1 \rangle; \mathbb{Q})). \end{aligned}$$

□

This simple observation already allows us to extend Theorem A to manifolds $N_I \# X$, where X is a rational homology n -sphere.

In some sense this question is about the (co)homology of arithmetic groups: It is known for a finite CW-complex X that the group $\pi_0(\text{aut}_*(X))$ is commensurable with an arithmetic subgroup of $\pi_0(\text{aut}_*(X_{\mathbb{Q}}))$ (see e.g. [Wil76]). We know that the universal coverings of the classifying spaces of the homotopy automorphisms are rationally homotopy equivalent. The problem is that we in general not even have a maps between $\pi_0(\text{aut}_*(X))$ and $\pi_0(\text{aut}_*(Y))$. One way around this problem is to consider the homotopy automorphisms of the rationalizations of the spaces - this approach has been used in [Gul15] for formal and coformal spaces, but it doesn't help us to understand the homology of $\text{Baut}_*(X)$.

The next reasonable and interesting case to consider for Question 8.4 are wedges of spheres with one cell attached:

$$W_{\varphi_j} = \left(\bigvee_{i=1}^r S^{n_i} \right) \cup_{\varphi_j} e^n, \text{ where } 1 < n_i < n \text{ and } j = 1, 2.$$

The complexes W_{φ_1} and W_{φ_2} are rationally homotopy equivalent, if

$$f_*[\varphi_1] = [\varphi_2] \in \pi_{n-1}\left(\bigvee_{i=1}^r S^{n_i}\right) \otimes \mathbb{Q} \text{ for some } f \in \text{aut}_*\left(\bigvee_{i=1}^r S^{n_i}\right).$$

This case is interesting because of this related Question:

Question 8.6. Assume $W_{\varphi_1} \simeq_{\mathbb{Q}} W_{\varphi_2}$, is

$$H_*(\text{Baut}_{\partial}(W_{\varphi_1} \setminus \text{int}(D^n)); \mathbb{Q}) \cong H_*(\text{Baut}_{\partial}(W_{\varphi_2} \setminus \text{int}(D^n)); \mathbb{Q})?$$

A positive answer to Question 8.6 would allow us to extend Theorem A to all k -connected $(2n + 1)$ -manifolds with $k \geq \frac{2n}{3}$ and $n \geq 3$ using Proposition 8.3. However, we do not know if that is true. We want to discuss the problems occurring in the case that the $W_{\varphi_i} \setminus \text{int}(D^n)$ are rationally homotopy equivalent to N_I , relative boundary. Let W be a compact oriented manifold with boundary that is rationally homotopy equivalent to some (not necessarily highly connected) N_I , relative boundary. The image of

$$\tilde{H}_* : \pi_0(\text{aut}_{\partial}(W)) \rightarrow \text{Aut}(\tilde{H}_*(W))$$

fits into a group extension:

$$0 \rightarrow K_W \rightarrow \text{Image}(\tilde{H}_*) \rightarrow G_W \rightarrow 0,$$

where G_W is some subgroup of $\Gamma_I \subset \text{Aut}(H_I)$ and K_W is finite. The group K_W corresponds to automorphisms of the torsion part of the homology and we don't need to worry about it. The group G_W on the other hand is in general harder to understand. We know that the derivation Lie algebra \mathfrak{g}_I is a model for the rational homotopy groups of the universal cover of $\text{Baut}_{\partial}(W)$. It is reasonable to assume that the action of $\pi_0(\text{aut}_{\partial}(W))$ is through G_W and extends to the Γ_I -action that we have considered in this thesis. A first thing to try to use is the following consequence of Shapiro's Lemma:

$$\begin{aligned} H_r(G_W, \text{res}_{\Gamma_I}^{G_W} H_r^{CE}(\mathfrak{g}_I)) &\cong H_*(\Gamma_I; \text{ind}_{G_W}^{\Gamma_I} \text{res}_{\Gamma_I}^{G_W} H_r^{CE}(\mathfrak{g}_I)) \\ &\cong H_*(\Gamma_I; \mathbb{Q}[\Gamma_I/G_W] \otimes H_r^{CE}(\mathfrak{g}_I)), \end{aligned}$$

where the Γ_I acts diagonally on $\mathbb{Q}[\Gamma_I/G_W] \otimes H_r^{CE}(\mathfrak{g}_I)$. It is not obvious to the author what this means for the answer of Question 8.6 especially when Γ_I/G_W is not finite.

However, if $\mathbb{Q}[\Gamma_I/G_W]$ is a coefficient system satisfying homological stability for Γ_I , this would at least allow us to extend Theorem A to W .

The second approach to show homological stability for more manifolds is to find the

”right” notion of graded Λ -quadratic modules to describe the groups G_W (the definition we made in Section 2.5 is clearly not general enough) and that allows to show homological stability with twisted coefficients. The invariants of the homology that G_W respects can be found in Wall’s work on the classification of manifolds. This approach is likely to succeed, since the related problem, if $H_*(\pi_0(\text{Diff}_\partial(W\#_g(S^p \times S^q)))$ stabilizes follows from the methods in [Per14a].

8.2 The stable cohomology

It is desirable to understand the stable cohomologies of $B\text{aut}_\partial(N)$, $\widetilde{B\text{Diff}}_\partial(N)$ and $B\text{Diff}_\partial(N)$ for N odd-dimensional. We believe that this thesis lays the foundation to understand the first two and to at least give some information about the last.

Let

$$N_g^{p,q} = (\#_g(S^p \times S^q)) \setminus \text{int}(D^{p+q}), \text{ where } 3 \leq p \leq q.$$

In [BM15], Berglund and Madsen calculate the rational cohomology groups of

$$B\text{aut}_\partial(N_\infty^{d,d}) = \text{hocolim}_{g \rightarrow \infty} B\text{aut}_\partial(N_g^{d,d}).$$

A key step for this was to show that the Serre spectral sequence of the fibration

$$B\text{aut}_\partial(N_\infty^{d,d})\langle 1 \rangle \rightarrow B\text{aut}_\partial(N_\infty^{d,d}) \rightarrow B\pi_1(B\text{aut}_\partial(N_\infty^{d,d})) \tag{8.1}$$

collapses at the E_2 -page. Unfortunately the proof depends to some extent on the calculation of the rational cohomology of $B\text{Diff}(N_\infty^{d,d})$ by Galatius and Randal-Williams. They show that the map

$$H^*(B\pi_1(B\text{aut}_\partial(N_g^{d,d})); \mathbb{Q}) \rightarrow H^*(B\text{Diff}_\partial(N_g^{d,d+1}); \mathbb{Q})$$

is injective on indecomposables and hence also the map

$$H^*(B\pi_1(B\text{aut}_\partial(N_g^{d,d})); \mathbb{Q}) \rightarrow H^*(B\text{aut}_\partial(N_g^{d,d+1}); \mathbb{Q}).$$

This ensures that the Serre spectral sequence collapses, since $H^*(B\pi_1(B\text{aut}_\partial(N_g^{d,d})); \mathbb{Q})$ and $H^*(B\text{aut}_\partial(N_g^{d,d}); \mathbb{Q})$ are free graded commutative algebras.

We hope to find a way around this to calculate $H^*(B\text{aut}_\partial(N_\infty^{d,d+1}); \mathbb{Q})$ and possibly $H^*(\widetilde{B\text{Diff}}_\partial(N_\infty^{d,d+1}); \mathbb{Q})$, which then would give by Morlet’s Lemma of Disjunction some information about $H^*(B\text{Diff}_\partial(N_\infty^{d,d+1}); \mathbb{Q})$ in a range of degrees depending on the dimension.

The collapse of the Serre spectral sequence for the fibration (8.1) with $N_g^{d,d}$ replaced by $N_g^{d,d+1}$ would lead to a proof of:

Conjecture 8.7. *There is an isomorphism of graded rings*

$$H^*(\text{Baut}_{\partial}(N_{\infty}^{d,d+1}); \mathbb{Q}) \cong H^*(\text{Gl}(\mathbb{Z}), \mathbb{Q}) \otimes H_{CE}^*(\text{Der}_{\omega} \mathbb{L}_{\infty})^{\text{Gl}(\mathbb{Z})}.$$

The left-hand term is understood, due to Borel's calculation of $H^*(\text{Sl}(\mathbb{Z}), \mathbb{R})$. The right-hand term is more mysterious, but there is a theorem by Kontsevich about a similar object [CV03].

Homotopy automorphisms of wedges of spheres

A related but potentially easier to understand problem is the stable cohomology of

$$X_g^d = \text{Baut}_*(\bigvee_g S^d)$$

as g goes to infinity. Besides the stabilization map, which is induced by extending a homotopy self-equivalence as the identity on a new wedge summand, there is also a map induced by the suspension. The spaces X_g^d are related to Waldhausen's algebraic K-theory of spaces (see [Wal85]):

$$\text{colim}_{n,g \rightarrow \infty} (X_g^d) \simeq A(*).$$

The rational homotopy groups of the universal covering can be understood using Theorem 2.3. It is given by

$$\pi_{*-1}^+(X_g^d) \otimes \mathbb{Q} \cong \text{Der}^+(\mathbb{L}_{g,d}),$$

where $\mathbb{L}_{g,d}$ denotes the free graded Lie algebra on $\mathbb{Q}^g[d-1]$. The deck transformation action of $\pi_1(X_g^d) \cong \text{Gl}_g(\mathbb{Z})$ is just the action induced by the standard action of $\text{Gl}_g(\mathbb{Z})$ on \mathbb{Z}^g .

The derivations $\text{Der}^+(\mathbb{L}_{g,d})$ can be describes as the value of a functor from the category of finitely generated \mathbb{Z} -modules with injective maps and thus the $C_*^{CE} \text{Der}^+(\mathbb{L}_{g,d})$ form a coefficient system in the sense of [RWW15]. Hence we should be able to show rational homological stability for $\{X_g^d\}_{g \geq 1}$.

One might hope that the relation to A -theory might help to determine the stable value. But the maps

$$X_g^d \rightarrow X_g^{d+1}$$

are only $(d-2)$ -connected and that because the $X_g^d \langle 1 \rangle$ are $(d-2)$ -connected.

Botvinnik and Perlmutter identify the stable integral cohomology of

$$B \text{Diff}_{D^{2d}}(\natural_g(D^{d+1} \times S^d)), \quad d \geq 4,$$

the group of diffeomorphisms of the g -fold boundary connected sum that restrict to the identity near a disk, embedded in the boundary. It is given by

$$H^*(B \text{Diff}_{D^{2d}}(\natural_{\infty}(D^{d+1} \times S^d))) \cong H^*((Q(BO(2d+1)\langle d \rangle_+))_{(1)}),$$

where (1) indicates the component of the constant loop. The right-hand side is rationally easy to express: It is the free commutative algebra generated by the monomials in the Pontryagin classes $p_{\frac{d+1}{4}}, \dots, p_d$. Unfortunately the map

$$H^*(B\pi_1(X_{\infty}^d); \mathbb{Q}) \rightarrow H^*(X_{\infty}^d; \mathbb{Q}) \rightarrow H^*(B \text{Diff}_{D^{2d}}(\natural_{\infty}(D^{d+1} \times S^d)); \mathbb{Q}),$$

induced by the obvious map

$$B \text{Diff}_{D^{2d}}(\natural_g(D^{d+1} \times S^d)) \rightarrow B \text{aut}_{D^{2d}}(\natural_g(D^{d+1} \times S^d)) \xrightarrow{\cong} X_g^d$$

is trivial for degree reasons ($H^*(Gl_{\infty}(\mathbb{Z}); \mathbb{Q})$ is an exterior algebra on generators in degrees $4i+1$, $i > 0$). Instead one should consider $Y_{g,d} = \widetilde{B \text{Diff}}_{D^{2d}}(\natural_g(D^{d+1} \times S^d))$. The mapping class group was calculated by Wall ([Wal63, Lemma 17]). It is given by a split group extension

$$0 \rightarrow \pi_d(SO(d+1))^g \oplus \pi_{d+1}(S^{d+1})^{\binom{g}{2}} \rightarrow \pi_0(\text{Diff}_{D^{2d}}(\natural_g(D^{d+1} \times S^d))) \rightarrow Gl_g(\mathbb{Z}) \rightarrow 0.$$

Using the methods in this thesis we should be able show rational homological stability for the $Y_{g,d}$.

The space $\widetilde{B \text{Diff}}_{D^{2d}}(\natural_g(D^{d+1} \times S^d))$, sits in a commutative diagram

$$\begin{array}{ccc} B \text{Diff}_{D^{2d}}(\natural_g(D^{d+1} \times S^d)) & \longrightarrow & B \text{Diff}_{\partial}(N_g^{d,d}) \\ \downarrow ? & & \downarrow \\ \widetilde{B \text{Diff}}_{D^{2d}}(\natural_g(D^{d+1} \times S^d)) & \xrightarrow{??} & \widetilde{B \text{Diff}}_{\partial}(N_g^{d,d}) \end{array}$$

where the horizontal maps are induced by restriction to the boundary. The stable rational cohomology rings of the three other terms are understood, thus we hope to understand $H^*(Y_{\infty,d}; \mathbb{Q})$ upon studying the maps ? and ??. Moreover we hope that this will help us understanding $H^*(X_{\infty,d}; \mathbb{Q})$ and maybe even $H^*(B \text{aut}_{\partial}(N_{\infty}^{d,d+1}); \mathbb{Q})$.

Calculations for even dimensional manifolds

One of the motivations for the author to prove homological stability for the manifolds N_I is the fact that for N_I even dimensional the stable moduli space

$$B \operatorname{Diff}_{\partial}(N_I)$$

for $g_n/n \mapsto \infty$ is in principle understood in [GRW14]. For example for

$$N_{g_a, g_{a-1}} = ((\#_{g_a} S^d \times S^d) \# (\#_{g_{a-1}} S^{d-1} \times S^{d+1})) \setminus \operatorname{int}(D^{2d})$$

it is given by

$$(\Omega_{\bullet}^{\infty} MTSO(2d) \langle d \rangle \wedge K(\mathbb{Z}^{g_{a-1}}, d-1)_+)_{hGl_{g_{a-1}}(\mathbb{Z})},$$

where the action of $Gl_{g_{a-1}}(\mathbb{Z})$ is induced by the standard action on $K(\mathbb{Z}^{g_{a-1}}, d-1)$.

Upon understanding the homotopy orbits, it might be possible to extend Berglund and Madsen's calculations of

$$B\operatorname{aut}_{\partial}(N_{\infty}^{d,d}) \text{ and } B\widetilde{\operatorname{Diff}}_{\partial}(N_{\infty}^{d,d})$$

to even dimensional N_I -s. Understanding the role of the general linear groups in even dimensions might help to also understand it in odd dimensions.

Moreover Berglund and Madsen show that the map

$$H^*(B\widetilde{\operatorname{Diff}}_{\partial}(N_{\infty}^{d,d}); \mathbb{Q}) \rightarrow H^*(B \operatorname{Diff}_{\partial}(N_{\infty}^{d,d}); \mathbb{Q})$$

is an isomorphism for $* < 2d$, which is better than expected by concordance theory. It would be interesting to see if this is also true for even dimensional N_I -s.

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