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ANALYTIC ASPECTS OF

THE THOMPSON GROUPS

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I always suppose anything people please, [...]

— Edmond Dantes, The Count of Monte Cristo

Abstract

In this thesis we study various analytic aspects of the Thompson groups, several of them related to amenability. In joint work with Uffe Haagerup, we prove that the Thompson groups T and V are not inner amenable, and give a criteria for non-amenability of the Thompson group F. More precisely, we prove that F is non-amenable if the reduced group C^*-algebra of T is simple. Whilst doing so, we investigate the C^*-algebras generated by the image of the Thompson groups in the Cuntz algebra $\mathcal{O}_2$ via a representation discovered by Nekrashevych. Based on this, we obtain new equivalent conditions to F being non-amenable.

Furthermore, we prove that the reduced group C^*-algebra of a non-inner amenable group possessing the rapid decay property of Jolissaint is simple with a unique tracial state. We then provide some applications of this criteria.

In the last part of the thesis, inspired by recent work of Garncarek, we construct one-parameter families of representations of the Thompson group F on the Hilbert space $\mathrm{L}^2([0,1], m)$, where m denotes the Lebesgue measure, and we investigate when these are irreducible and mutually inequivalent. In addition, we exhibit a particular family of such representations, depending on parameters $s \in \mathbb{R}$ and $p \in (0,1)$, and prove that these are irreducible for all values of s and p, and non-unitarily equivalent for different values of p. We furthermore show that these representations are strongly continuous in both parameters, and that they converge to the trivial representation, as p tends to zero or one.

Resumé

I denne afhandling studerer vi diverse analytiske aspekter af Thompson-grupperne, flere af hvilke er relateret til amenabilitet. I samarbejde med Uffe Haagerup viser vi, at Thompson-grupperne T og V ikke er indre amenable og giver et kriterium, for at Thompson-gruppen F ikke er amenabel. Helt konkret viser vi, at F ikke er amenabel, hvis den reducerede gruppe-C^*-algebra hørende til T er simpel. I processen undersøger vi de C^*-algebraer som bliver genereret af billederne af Thompson-grupperne via en repræsentation i Cuntz-algebraerne $\mathcal{O}_2$, der blev opdaget af Nekrashevych. Baseret på dette giver vi nye ækvivalente betingelser for, at F ikke er amenabel.

Derudover viser vi, at hvis en gruppe ikke er indre amenabel, men har rapid decay-egenskaben introduceret af Jolissaint, da er dens reducerede gruppe-C^*-algebra simpel og har en unik sportilstand. Efterfølgende præsenterer vi nogle anvendelser af dette resultat.

Den sidste del af afhandlingen er inspireret af resultater af Garncarek. I denne konstruerer vi ét-parameter-familier af repræsentationer af Thompson-gruppen F på Hilbertrummet $\mathrm{L}^2([0,1], m)$, hvor m betegner Lebesguemålet. Vi undersøger, hvornår disse repræsentationer er irreducible, og hvornår de ikke er parvist ækvivalente. Derudover producerer vi en konkret familie af sådanne repræsentationer, afhængig af to parametre $s \in \mathbb{R}$ og $p \in (0,1)$, samt viser at disse er irreducible for alle værdier af s og p, og ikke unitært ækvivalente for forskellige værdier af p. Ydermere viser vi, at repræsentationerne er stærkt kontinuerte i begge parametre og konvergerer mod den trivielle repræsentation, når p går med nul eller en.

Contents

Chapter 1

Introduction

In 1965 Richard Thompson introduced three groups now commonly referred to as the Thompson groups. These groups, denoted by F, T and V, have been the center of much study since their introduction, and a particularly famous open problem regarding these groups asks whether or not F is amenable. This question has been studied intensely and several unsuccessful attempts have been made to prove or disprove amenability of F. The three groups satisfy $F \subseteq T \subseteq V$, and it is well-known that the Thompson groups T and V are non-amenable, as they contain a copy of $\mathbb{F}_2$, the free group on two generators. However, F does not contain a copy of $\mathbb{F}_2$, as proved by Brin and Squire [8] in 1985. Thus F has been for a long time a candidate for a finitely generated counterexample to the von Neumann conjecture, stating that the only obstruction to amenability is containment of $\mathbb{F}_2$. The von Neumann conjecture was disproved in 1980 by Olshanskii [56], but his counterexample was not finitely generated. A finitely generated counterexample was found later by Olshanskii and Sapir [58], and, recently, several highly accessible counterexamples have been produced by Monod [53] and by Lodha and Moore [50]. In this thesis we approach the question of amenability of F by proving that the Thompson group F is non-amenable if the reduced group C^*-algebra of T is simple, thus giving an operator algebraic criterion for non-amenability of F. This result has been obtained in joint work with Uffe Haagerup, who was our main supervisor until his passing away in July, 2015. After announcing this result at various conferences, several partial converses have been obtained, first by Bleak and Juschenko [6] and later by Breuillard, Kalantar, Kennedy and Ozawa [7]. Recently, Le Boudec and Matte Bon [49] proved that the (full) converse statement holds, namely that F is non-amenable if and only if the reduced group C^*-algebra of T is simple. The study of groups whose reduced group C^*-algebra is simple, also known as C^*-simple groups, respectively, of groups whose reduced group C^*-algebra has a unique tracial state, also known as groups with the unique trace property, has been an important topic in operator algebra theory since 1975, when Powers [60] proved that $\mathbb{F}_2$ is C^*-simple and has the unique trace property. Over the years, many groups have been proven to be C^*-simple with the unique trace property, and a famous long time open question of de la Harpe asked whether or not C^*-simplicity and the unique trace property of a group are equivalent. Recently, a tremendous progress has been made regarding these two properties, and the question of de la Harpe has been completely settled. It was first proved by Poznansky [61] in 2009 that the two properties are equivalent for linear groups, however, it turns out that they are not equivalent in general. More precisely, in 2014 Kalantar and Kennedy [43] gave a characterization of C^*-simplicity in terms of certain boundary actions of the given group, and later the same year Breuillard, Kalantar, Kennedy and Ozawa [7] proved that C^*-simplicity implies the unique trace property. The year after, Le Boudec [48] gave an example of a C^*-simple group without the unique trace property, thus settling the question of de la Harpe. It is already known that the Thompson group T has the unique trace property, as shown by Dudko and Medynets [25] in 2012, but is remains an open question whether T is C^*-simple.

In order to prove that C^*-simplicity of the group T implies non-amenability of F, we use a representation of the Thompson groups in the unitary group of the Cuntz algebra $\mathcal{O}_2$, introduced by Nekrashevych [55] in 2004. In addition to this result, we also prove, in collaboration with Haagerup, that the C^*-algebras generated by the representations of F, T and V are distinct, and that the one generated by V is all of $\mathcal{O}_2$. We then give equivalent conditions to F being non-amenable in terms of whether certain ideals in the reduced group C^*-algebras of F and T are proper.

Another result we prove in collaboration with Haagerup is that the Thompson groups T and V are not inner amenable, thus settling a question that Chifan raised at a conference in Alba-Iulia in 2013. The notion of inner amenability was introduced by Effros [26] in 1975 in order to give a group theoretic characterization of property Γ of Murray and von Neumann for group von Neumann algebras of discrete ICC groups. Effros proved that a discrete ICC group is inner amenable if its group von Neumann algebra has Property Γ. He conjectured that the converse implication was also true, but it was only in 2012 that Vaes gave an example of an inner amenable discrete ICC group whose group von Neumann algebra does not possess Property Γ. Jolissaint [40] proved in 1997 that the Thompson group F is, in fact, inner amenable, and a year after, he strengthened his result by proving that the group von Neumann algebra of F is a McDuff factor, see [41]. A few years later a different, more elementary proof of inner amenability of F was given by Ceccherini-Silberstein and Scarabotti [12].

As a different approach to the question of C^*-simplicity of T, we prove that non-inner amenable groups with the rapid decay property are C^*-simple with the unique trace property. Rapid decay is a property introduced by Jolissaint [39] in 1990, inspired by a result of Haagerup [35] from 1979, that the free group on n generators has this property. With our criteria for C^*-simplicity we recover some of the known examples, and find further ones. However, we cannot settle the C^*-simplicity of T, since T does not have the rapid decay property, as kindly pointed out to us by Valette.

In 2012 Garncarek [30] proved that a certain one-parameter family of representations of F were all irreducible and, moreover, unitarily equivalent exactly when the parameters differ by an integer multiple of $\frac{2\pi}{\log 2}$. This one-parameter family is a natural analogue of the one-parameter family of representations of $\mathrm{SL}(2,\mathbb{R})$, known as the principal series. In this thesis, following a suggestion of Monod, we provide a method for obtaining one-parameter families of representations of F by means of other actions of F on the unit interval by homeomorphisms. For example, we expand upon Garncarek's family of representations by introducing another parameter. More precisely, for each $p \in (0,1)$ and $s \in \mathbb{R}$ we produce an irreducible representation $\pi_s^{\phi_p}$ of F on the Hilbert space $\mathrm{L}^2([0,1],m)$, where m is the Lebesgue measure, so that for $p = \frac{1}{2}$ this is the one-parameter family of Garncarek. We prove that these representations are strongly continuous in both parameters, and that $\pi_s^{\phi_p}$ converges to the trivial representation of F on $\mathrm{L}^2([0,1],m)$, as p tends to zero or one.

Let us end the introduction by supplying an overview of this thesis. Chapter 2 serves as an introduction to the Thompson groups. More precisely, the first two sections contain basic facts about these groups, including the notion of standard dyadic partitions, and how the elements of the Thompson groups can be represented as rearrangements of these. The third section introduces the Minkowski question mark function and a variant of it. These maps are then used in the fourth section to prove a result of Thurston realizing the Thompson group T as a group of piecewise projective linear maps. We would like to thank Sergiescu for several fruitful discussions and his help in sorting out the origin of the particular version of Thurston's result we are presenting.

The topic of Chapter 3 is inner amenability. The first section is devoted to amenable actions for discrete groups, and list some standard results about these. The second section contains a brief introduction to inner amenability and Property Γ, while the third section contains a proof of the result with Haagerup that the Thompson groups T and V are not inner amenable.

In Chapter 4 we look at the Thompson groups from an operator algebraic point of view. In the first section, after introducing the representation of Nekrashevych mentioned above, we prove the result concerning the C^*-algebras generated by the images of F, T and V via this representation announced above. The second section is devoted to C^*-simplicity and the unique trace property. We prove that the Thompson group F is non-amenable if T is C^*-simple, as well as a characterization of non-amenability of F in terms of whether certain ideals in the reduced group C^*-algebras of F and T are proper. These results are both joint work with Haagerup. In the third and fourth sections we provide a brief introduction to the rapid decay property for groups, and prove that non-inner amenable groups with the rapid decay property are C^*-simple with the unique trace property.

In the last chapter, motivated by work of Garncarek, we construct one-parameter families representations of the Thompson group F. In the first section we explain how to obtain a one-parameter family of representation of F from certain increasing homeomorphisms of the unit interval, as well as determine when these are irreducible and when they are unitarily equivalent for different homeomorphisms. In the second section we list some well-known results about measures and equivalence relations on the Cantor set, and in the third section we use these results to show that the representations $\pi_s^{\phi_p}$ mentioned above are irreducible and inequivalent for different values of p. Section Four is devoted to explaining a method for constructing the increasing homeomorphisms of the unit interval used in the first section to produce one-parameter families of representations of F. In section five we prove that all such homeomorphisms can be obtained via this construction. We end this chapter with a list of open questions and further projects related to the one-parameter families of representations of F which we have constructed.

1.1 Notation and terminology

Let us spend a few words setting some of the notation, terminology and conventions of this thesis. Most of the notation and terminology will be introduced along the way, and there is a subject index on page 125.

First of all, we denote the sets of complex numbers, real numbers and rational numbers by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$, respectively. The integers are denoted by $\mathbb{Z}$ and the set of natural numbers by $\mathbb{N}$. We do *not* include 0 in the set of natural numbers.

Groups will *always* be discrete, and will typically be denoted by the letters G and H, unless they are specific ones, such as the free non-abelian group $\mathbb{F}_n$ on n generators. We will use e to denote the neutral element of a generic group, and mainly g and h for general group elements. We use $\mathbb{C}G$ to denote the complex groups algebra of a group G and $C^*(G)$ the completion of $\mathbb{C}G$ into the *full group C^*-algebra*. Moreover, G is said be *ICC* if all of its non-trivial conjugacy classes are infinite, that is, if $\{ghg^{-1} : g \in G\}$ is an infinite set, for all $h \neq e$.

Ordinary measures and finitely additive measures will usually be denoted by μ and ν, except for the Lebesgue measure, which will always be denoted by m. Given a measure μ on a set $\mathfrak{X}_1$, and a measurable map $f \colon \mathfrak{X}_1 \to \mathfrak{X}_2$, we will let $f_*\mu$ denote the image measure on $\mathfrak{X}_2$, that is, the measure given by $f_*\mu(A) = \mu(f^{-1}(A))$.

We denote Hilbert spaces by $\mathcal{H}$, and sometimes by $\mathcal{K}$. The inner product and the norm on such spaces will be denoted by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\|_2$, respectively. Given a set $\mathfrak{X}$, we let $\ell^2(\mathfrak{X})$ denote the particular Hilbert space of complex valued square-summable functions on $\mathfrak{X}$, and, likewise, we will let $\ell^1(\mathfrak{X})$ and $\ell^\infty(\mathfrak{X})$ denote the spaces of summable and bounded complex valued function on $\mathfrak{X}$, respectively. We denote the norm on $\ell^1(\mathfrak{X})$ by $\|\cdot\|_1$ and the norm on $\ell^\infty(\mathfrak{X})$ by $\|\cdot\|_\infty$, respectively. The indicator function of a set $A \subseteq \mathfrak{X}$ will be denoted by $\mathbf{1}_A$. However, given a point $x \in \mathfrak{X}$, the indicator function of $\{x\}$ will be denoted by δ_x, and $\{\delta_x : x \in \mathfrak{X}\}$ will be referred to as the *standard orthonormal basis* of $\ell^2(\mathfrak{X})$. The space of bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted by $B(\mathcal{H})$, and the identity operator on $\mathcal{H}$ will be denoted by $\mathbf{1}$.

Given a group G, a representation of G on a Hilbert space $\mathcal{H}$ should *always* be understood as a unitary representation, that is, a group homomorphism from G to the group of unitary operators on $\mathcal{H}$. Representations will usually be denoted by π and σ, however, we will use λ and ρ to denote the *left regular representation* and the *right regular representation* of G, respectively, that is, the representations on $\ell^2(G)$ given by $(\lambda(g)f)(h) = f(g^{-1}h)$ and $(\rho(g)f)(h) = f(hg)$, for all $f \in \ell^2(G)$ and $g, h \in G$. Given a representation π of G on a Hilbert space $\mathcal{H}$, we will let $C^*_\pi(G)$ denote the C^*-algebra generated by $\pi(G)$ in $B(\mathcal{H})$. This is also the case for the reduced group C^*-algebra, that is, the C^*-algebra generated by $\lambda(G)$, which we will denote by $C^*_\lambda(G)$. The *group von Neumann algebra* associated to G will be denoted by $\mathrm{L}(G)$, and it is the strong operator closure of the reduced group C^*-algebra inside $B(\ell^2(G))$.

Chapter 2

The Thompson groups

The Thompson groups F, T and V were introduced by Thompson in 1965, and they are groups of piecewise linear maps of the unit interval. This chapter serves as an introduction to the Thompson group, and will provide an explanation of how they can be realized as fractional linear transformations.

No deep results about the Thompson groups are needed for the purpose of this thesis, and, indeed, besides the realization of the Thompson groups as piecewise fractional linear transformations, we only need a few elementary results about the dynamics of the action these groups exert on the unit interval. These latter results will all be included in Section 2.2. A more thorough introduction to the Thompson groups, as well as proofs of most statements made in the first two sections of this chapter, can be found in the comprehensive paper by Cannon, Floyd and Parry [10].

2.1 The definition

The Thompson groups F, T and V satisfy $F \subseteq T \subseteq V$, and, even though this might not be the most cost effective way, we will start by defining F, then T and lastly V, as this seems to be the more natural way to proceed.

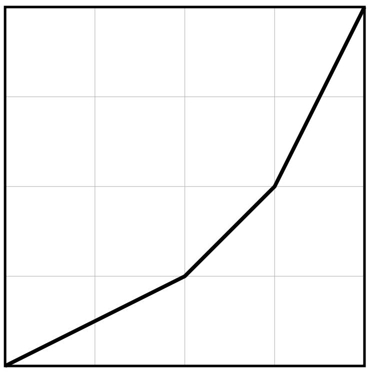
First, recall that the *dyadic rational numbers*, denoted by $\mathbb{Z}[\frac{1}{2}]$, is the set of rational numbers with a power of 2 in the denominator when in reduced form, that is, the set $\{\frac{m}{2^n} : n, m \in \mathbb{Z}\}$. To shorten notation, let us denote the set of dyadic rational numbers in $[0, 1]$ by $\mathcal{D}$. The *Thompson group F* is the set of all piecewise linear bijections of $[0, 1]$ which

(1) are homeomorphisms of $[0, 1]$;
(2) have finitely many points of non-differentiability;
(3) have all its points of non-differentiability in the set of dyadic rationals;
(4) have a derivative which is a power of 2 in each point of differentiability;
(5) map $\mathcal{D}$ bijectively onto itself.

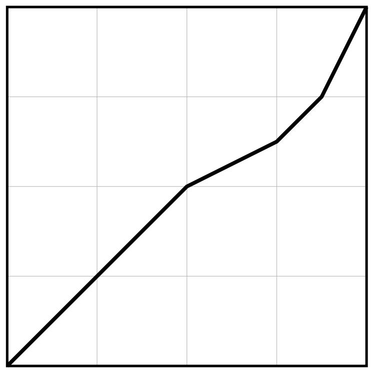
It is not difficult to see that F is a group with respect to composition of functions, and it also is worth noting that all the functions in F are increasing and fix the points 0 and 1. The following are examples of elements in F:

$$A(x) = \begin{cases} \frac{1}{2}x & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \text{for} \quad \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \text{for} \quad \frac{3}{4} \leq x \leq 1 \end{cases} \qquad B(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & \text{for} \quad \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \text{for} \quad \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \text{for} \quad \frac{7}{8} \leq x \leq 1 \end{cases}.$$

As one might notice, the element B is somehow "A squeezed into the upper right corner." The graphs of these elements look as follows:



A



B

None of these elements has finite order, which is easily realized by considering the slope of the graph near 1, as, for example, A^n will have slope 2^n just left of 1, for any $n \in \mathbb{Z}$. In fact, this method easily shows that F has no non-trivial elements of finite order.

Moving on, we define the *Thompson group* T as the set of piecewise linear bijections of $[0, 1)$ which

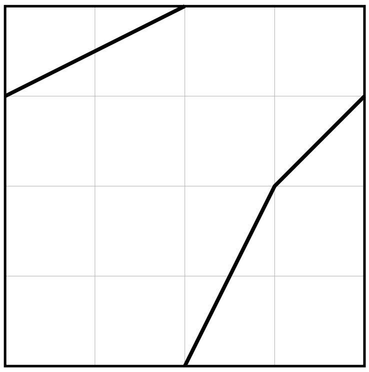
(1) are homeomorphisms of $[0, 1)$ when given the topology of a circle;
(2) have finitely many points of non-differentiability;
(3) have all its points of non-differentiability in the set of dyadic rationals;
(4) have a derivative which is a power of 2, in each point of differentiability;
(5) map $\mathcal{D} \setminus \{1\}$ bijectively onto itself.

Here "$[0, 1)$ with the topology of a circle" should be understood as the topology that $[0, 1)$ gets when identified with $\mathbb{R}/\mathbb{Z}$ in the natural way as representatives of equivalence classes. It is not difficult to see that in terms of the standard topology on $[0, 1)$, the first condition simply means that the bijection in question has at most one point of discontinuity. Examples of elements in T which are not elements of F

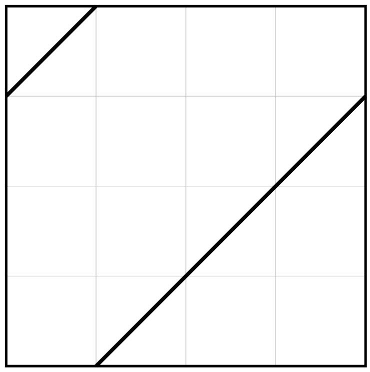
include the following:

$$C(x)=\begin{cases}\frac{1}{2}x+\frac{3}{4} & \text{for} \quad 0\le x\le \frac{1}{2}\\ 2x-1 & \text{for} \quad \frac{1}{2}\le x\le\frac{3}{4}\\ x-\frac{1}{4} & \text{for} \quad \frac{3}{4}\le x<1\end{cases} \qquad D(x)=\begin{cases}x+\frac{3}{4} & \text{for} \quad 0\le x\le\frac{1}{4}\\ x-\frac{1}{4} & \text{for} \quad \frac{1}{4}\le x<1\end{cases}.$$

A straightforward calculation shows that C and D have order 3 and 4, respectively, so that, even though F does not contains non-trivial elements of finite order, T does. The graphs of the elements C and D look as follows:



C

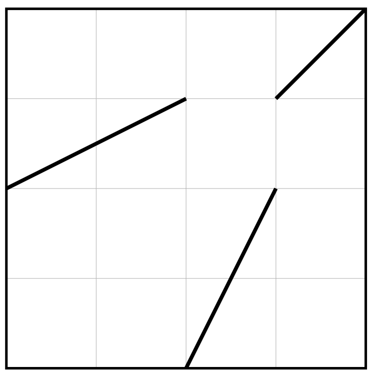


D

The Thompson group F is naturally a subgroup of T, as all the elements of F fix 1, so that there would have been no harm in defining F as maps of $[0, 1)$ instead of $[0, 1]$. In fact, depending on the context, we will think of the elements of F as maps of $[0, 1]$ or $[0, 1)$. It is easy to see that when considered as a subgroup of T, the elements of F are exactly those which fix 0.

Now, the *Thompson group* V is the set of right continuous piecewise linear bijections of $[0, 1)$ which satisfy the conditions (2)–(5) right above. An example of an element in V which is not in T is:

$$\pi_0(x)=\begin{cases}\frac{1}{2}x+\frac{1}{2} & \text{for} \quad 0\le x<\frac{1}{2}\\ 2x-1 & \text{for} \quad \frac{1}{2}\le x<\frac{3}{4}\\ x & \text{for} \quad \frac{3}{4}\le x<1\end{cases}$$



π_0

It is not difficult to realize that a bijection from $[0, 1)$ to $[0, 1)$ is a homeomorphism of $[0, 1)$ equipped with the topology of a circle if and only if it is right continuous and has at most one point of discontinuity when $[0,1)$ is considered with the regular topology. In particular, we see that T is a subgroup of V.

Let us end this section by mentioning a few known facts about the Thompson groups, most of which will not be used explicitly in this thesis. First of all, it turns out that the Thompson group T and V are simple groups, and so is the commutator subgroup of F. Besides this, all three Thompson groups are finitely generated; indeed, F is generated by $\{A, B\}$, T is generated by $\{A, B, C\}$ and V is generated by $\{A, B, C, \pi_0\}$. Moreover, all three groups have finite presentations in these generators. A fact we will need later, is that T is generated by the elements C and D, as $A = D^2C^2$ and $B = C^2DA$.

2.2 Rearrangement of dyadic partitions

In this section, we will explain how to think of the elements of the Thompson groups as rearrangements of certain partitions of the unit interval. This is closely related to the interpretation of the Thompson groups as tree diagrams, for which we refer the reader to the excellent treatment in Cannon, Floyd and Parry [10]. In this thesis, however, we will focus on the former interpretation, since it is more readily applicable in the analysis of the action on the unit interval.

To keep things simpler, we will first go through the explanation for the Thompson group F, and then discuss how this works for the Thompson groups T and V. First, we need the notions of standard dyadic intervals and standard dyadic partitions.

Definition 2.2.1. A *standard dyadic interval* is a closed interval in $[0,1]$ of the form $[\frac{m}{2^n}, \frac{m+1}{2^n}]$, for some non negative integers n and m with $m < 2^n$.

Definition 2.2.2. A sequence $(x_0, x_1, \ldots, x_n)$ of elements in $[0,1]$ is called a *standard dyadic partition* if $0 = x_0 < x_1 < \cdots < x_n = 1$ and each of the intervals $[x_{k-1}, x_k]$ is a standard dyadic interval, for $k = 1, 2, \ldots, n$.

Given a standard dyadic partition $(x_0, x_1, \ldots, x_n)$, we get a new one by adding the midpoint of two adjacent points. More precisely, if $k \in \{1, 2, \ldots, n\}$, then $(x_0, \ldots, x_{k-1}, \frac{x_{k-1}+x_k}{2}, x_k, \ldots, x_n)$ is a new standard dyadic partition. This is a refinement of the original partition, and it it not difficult to see that the only way to refine a standard dyadic partition is by repeating this procedure. Moreover, every standard dyadic partition is obtained in this way as a refinement of the partition $(0,1)$, that is, by starting out with the standard dyadic partition $(0,1)$ and then repeatedly adding the midpoints of choice.

The following two results are the key to the interpretation of the elements of F as rearrangements of dyadic partitions. The proofs are quite elementary, and can be found in [10, Section 2].

Lemma 2.2.3. *For each element g in F, there exists a standard dyadic partition $(x_0, x_1, \ldots, x_n)$ such that $(g(x_0), g(x_1), \ldots, g(x_n))$ is a standard dyadic partition as well, and g is linear on each of the intervals $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.*

Given an element $g \in F$, a standard dyadic partition satisfying the conditions in the above lemma is called a *standard dyadic partition associated to g* .

Lemma 2.2.4. *Given two standard dyadic partitions with the same number of elements $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$, there exists a unique element $g \in F$ for which $g(x_k) = y_k$, for each $k = 0, 1, \ldots, n$, and which is linear on each of the intervals $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.*

These results tell us that we can view the elements of the Thompson group F as pairs of dyadic partitions. However, given $g \in F$, there is more than one standard dyadic partition associated to g. Indeed, if we are given a standard dyadic partition associated to g, any refinement of it, in the sense described earlier, will also be a standard dyadic partition associated to g. This is, in fact, how every standard dyadic partition associated to g appears in the sense that there is a unique standard dyadic partition S_0 such that every other standard dyadic partition associated to g is a refinement of S_0. This partition is known as the *minimal standard dyadic partition associated to g*. To give some examples, the minimal standard dyadic partitions associated to the elements A and B are $(0, \frac{1}{2}, \frac{3}{4}, 1)$ and $(0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$, respectively.

Returning to the interpretation of elements of F as pairs of standard dyadic partitions, we have yet to explain why this is advantageous. In short, the reason is that it is easier to come up with standard dyadic partitions so that the corresponding element satisfies certain requirements, rather than to come up with the element itself. To illustrate this, let us sketch the proof of the following result.

Lemma 2.2.5. *Let $a, b \in [0, 1]$ with $a < b$. Given $c, d, c', d' \in (a, b)$ with $c < d$ and $c' < d'$, there exists some $g \in F$ with $g([c, d]) \subseteq [c', d']$ and $g(x) = x$, for all $x \notin (a, b)$.*

Sketch of proof. We will sketch the proof in the case where a, b, c, d, c' and d' are all dyadic rational numbers, and where $a > 0$ and $b < 1$. The rest of the cases can be obtained from this one by carefully squeezing dyadic rational numbers in-between the numbers a, b, c, d, c' and d'.

We start by choosing a standard dyadic partition S containing all the points a, b, c and d. This is easily done, as they must all be contained in the standard dyadic partition $\left(\frac{0}{2^m}, \frac{1}{2^m}, \ldots, \frac{2^m-1}{2^m}, \frac{2^m}{2^m}\right)$, for sufficiently large $m \in \mathbb{N}$. Now, the partition S has the form

$$S = \big(v_0, \ldots, v_i, a, \ldots, c, \ldots, d, \ldots, b, w_0, \ldots, w_j\big),$$

with possible gaps in-between a and c, c and d, and d and b. We then choose a standard dyadic partition R containing a, b, c' and d', but we can ensure that it has the form

$$R = \big(v_0, \dots, v_i, a, \dots, c', \dots, d', \dots, b, w_0, \dots, w_j\big),$$

again, with possible gaps in-between a and c', c' and d', and d' and b. Our goal is to refine these two partitions so that they end up defining an element g satisfying the requirements. To do this, we refine S by adding points in-between a and c, c and d, and d and b, and refine R by adding points in-between a and c', c' and d' d' and b. How we refine the two partitions is, to a certain degree, not important. The only important factor is that we add points so that the resulting refinement, S', of S has exactly as many points in-between a and c as the refinement, R', of R has in-between a and c', so that the number of element in-between c and d in S' is the same as the number of elements in-between c' and d' in R', and so that the number of points in-between d and b in S' is the same as the number of points in-between d' and b in R'. In this way we obtain two standard dyadic partition with the same number of points, and we may consider the element g in F corresponding to (S', R'). This g now satisfies the desired requirements. By construction it will satisfy $g(c) = c'$ and $g(d) = d'$ as these have the same positions in the two partitions, so, in particular, $g([c,d]) = [c', d']$. Moreover, as we have ensured that the two standard dyadic partitions have the same initial and final segments, we get that g is the identity outside (a, b). □

The proof of this lemma illustrates how we can choose the standard dyadic partitions cleverly to make the corresponding element suit our needs. A similar argument can be used to prove the following.

Lemma 2.2.6. *Let $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n$ be dyadic rational numbers in the interval $[0,1]$ with $0 = x_0 < x_1 < \dots < x_n = 1$ and $0 = y_0 < y_1 < \dots < y_n = 1$. Then there exists an element $g \in F$ with $g(x_k) = y_k$, for all $k = 0, 1, \dots, n$.*

So far we have only been dealing with the Thompson group F, but there is a similar view on the groups T and V. These just involve an additional permutation of the intervals of the standard dyadic partitions. For T and V the key lemmas in the interpretation of their elements as rearrangements of standard dyadic partitions are the following.

Lemma 2.2.7. *Given some element $g \in V$ there exist a permutation σ of $\{0, \dots, n\}$ and standard dyadic partitions $(x_0, x_1, \dots, x_n)$ and $(y_0, y_1, \dots, y_n)$ such that, with $I_k = [x_{k-1}, x_k)$ and $J_k = [y_{k-1}, y_k)$, the element g maps I_k linearly onto $J_{\sigma(k)}$, for each $k = 0, 1, \dots, n$. Moreover, if $g \in T$, then σ is cyclic, and, if $g \in F$, then σ is the identity.*

Lemma 2.2.8. *Given a permutation σ of $\{0, \dots, n\}$ and standard dyadic partitions $(x_0, x_1, \dots, x_n)$ and $(y_0, y_1, \dots, y_n)$, there exists an element $g \in V$ such that, with $I_k = [x_{k-1}, x_k)$ and $J_k = [y_{k-1}, y_k)$, the element g maps I_k linearly onto $J_{\sigma(k)}$, for each $k = 0, 1, \dots, n$. Moreover, if σ is cyclic, then $g \in T$, and if σ is the identity, then $g \in F$.*

We will not go more into detail with this interpretation of T and V, but just mention that there is a natural version of Lemma 2.2.6 for T and V, as well. This version simply allows one to permute the intervals of the standard dyadic partitions, as in the two lemmas above.

2.3 The Minkowski question mark function

In this section, we introduce the Minkowski question mark function, as well as a variant of this map. The Minkowski question mark function is a curious function, which was introduced by Minkowski in the beginning of the Twentieth Century in order to study quadratic irrational numbers. In this thesis, it is used for different purposes. Essentially, it allows us realize the Thompson group T as a different group of homeomorphisms; this will be explained in details in the next section. For more on the Minkowski question mark function, the reader is referred to [63] and [66].

The Minkowski question mark function, usually denoted by $?$, is a strictly increasing homeomorphism of the unit interval which maps the rational numbers onto the dyadic rational numbers. We will be particularly interested in a variant of this map, which we will denote by Q. This map is a strictly increasing homeomorphism from the extended real line to the unit interval, that is, from $[-\infty, \infty]$ to $[0, 1]$, which maps $\mathbb{Q} \cup \{\pm\infty\}$ onto the dyadic rational numbers in $[0, 1]$.

As the sets $\mathbb{Q} \cup \{\pm\infty\}$ and $\mathbb{Z}[\frac{1}{2}] \cap [0, 1]$ are dense in $[-\infty, \infty]$ and $[0, 1]$, respectively, it suffices to define Q on these sets, as long as it is order-preserving and bijective, because then there will be a unique extension to an order-preserving bijection from $[-\infty, \infty]$ to $[0, 1]$. Why this is the case will be explained later. Before beginning the construction, we will need to introduce a few notions, including the notion of reduced fractions, extended slightly to include $\pm\infty$.

Definition 2.3.1. A rational number $\frac{p}{q}$ is said to be a *reduced fraction* or to be in *reduced form* if p and q integers which are relatively prime, with q positive. We will use the convention that $\frac{-1}{0}$ and $\frac{1}{0}$ are the reduced form of $-\infty$ and ∞, respectively, and that $\frac{0}{1}$ is the reduced form of 0.

It will be important later on that we always require the denominator to be positive. In the definition above we have included reduced fraction forms of $\pm\infty$ as well, and, in fact, we will use the fractions $\frac{n}{0}$ and $\frac{-n}{0}$ to denote ∞ and $-\infty$, respectively, whenever $n \in \mathbb{N}$. However, the fractions are only said to be reduced if $n = 1$. Henceforth, when we let $\frac{p}{q}$ be a fraction in $\mathbb{Q} \cup \{\pm\infty\}$, it should be understood that either $q \in \mathbb{Z} \setminus \{0\}$, in which case $\frac{p}{q}$ is just an ordinary fraction representation of a rational number, or $q = 0$ and $p \in \mathbb{Z} \setminus \{0\}$, so that $\frac{p}{q}$ represents ∞ or $-\infty$.

Definition 2.3.2. Two fractions $\frac{p}{q}$ and $\frac{r}{s}$ in $\mathbb{Q} \cup \{\pm\infty\}$ are said to be *consecutive Farey fractions* if they are both non-negative or non-positive and the condition $|ps - rq| = 1$

is satisfied. Moreover, two elements of $\mathbb{Q} \cup \{\pm\infty\}$ are said to be *consecutive Farey numbers* if their reduced fractions are consecutive Farey fractions.

Let us make a few comments on the definition above. Firstly, the definition does not take the order of the fractions $\frac{p}{q}$ and $\frac{r}{s}$ into account, however, it is not difficult to see that $\frac{p}{q} < \frac{r}{s}$ if and only if $qr - ps = 1$. Secondly, it is not obvious that two consecutive Farey fraction are, in fact, two consecutive Farey numbers, as it might, a priori, happen that their reduced fractions where not consecutive Farey fractions. This, however, is never the case. It is easy to see that, if the denominators of two consecutive fractions are non-negative, then the fractions are in fact reduced, and, if this is not the case, $q < 0$ say, then $\frac{-p}{-q}$ is reduced, and $\frac{-p}{-q}$ and $\frac{r}{s}$ are again consecutive Farey fractions. Thirdly, the criteria that both fractions need to be non-negative or non-positive might seem a bit artificial. Indeed, we have only included it in the definition because we want to define the term for all of $\mathbb{Q} \cup \{\pm\infty\}$. The requirement would be superfluous if we only considered elements of $\mathbb{Q}$. More precisely, if we are given two reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$ in $\mathbb{Q}$, one of which is positive and the other one negative, then qr and $-ps$ are non-zero with the same sign, so that $|ps - rq| \geq 2$. To see why this is not automatically true if we allow the fractions $\frac{-1}{0}$ and $\frac{1}{0}$, we just have to note that $\frac{-1}{0}$ and $\frac{n}{1}$ would be consecutive Farey fractions without the requirement.

Definition 2.3.3. Let $\frac{p}{q}$ and $\frac{r}{s}$ be two distinct reduced fractions in $\mathbb{Q} \cup \{\pm\infty\}$ with $\{\frac{p}{q}, \frac{r}{s}\} \neq \{\frac{-1}{0}.\frac{1}{0}\}$. We define the *Farey median* of $\frac{p}{q}$ and $\frac{r}{s}$ to be the fraction $\frac{p+r}{q+s}$, which we denote by $\frac{p}{q} \oplus \frac{r}{s}$. Let x and y be two distinct elements in $\mathbb{Q} \cup \{\pm\infty\}$ with $\{x, y\} \neq \{-\infty, \infty\}$. Write x and y as reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$, respectively. We define the *Farey median* of x and y, denoted $x \oplus y$, to be the element of $\mathbb{Q} \cup \{\pm\infty\}$ whose reduced fraction is $\frac{p}{q} \oplus \frac{r}{s}$.

The reader might notice that we have, in the two last definitions, begun to distinguish between fractions and the numbers they represent. This fact will be important in the rest of this section and in the next section.

The following lemma justifies the use of the term "median" in the definition of Farey medians.

Lemma 2.3.4. *Let $x, y \in \mathbb{Q} \cup \{\pm\infty\}$ with $x < y$ and $\{x, y\} \neq \{-\infty, \infty\}$. Then*

$$x < x \oplus y < y.$$

Moreover, if $\frac{p}{q}$ and $\frac{r}{s}$ are reduced fractions and consecutive Farey fractions, then the same holds for $\frac{p}{q}$ and $\frac{p}{q} \oplus \frac{r}{s}$, as well as for $\frac{r}{s}$ and $\frac{p}{q} \oplus \frac{r}{s}$.

Proof. We start by proving the first claim. Let $\frac{p}{q}$ and $\frac{r}{s}$ be reduced fractions of x and y, respectively. If $x = -\infty$, then $y \in \mathbb{Q}$ and $x \oplus y = y - \frac{1}{s}$, so that $x < x \oplus y < y$. Similarly we see that the inequalities are satisfied if $y = \infty$, so we may assume that

$x, y \in \mathbb{Q}$. We will only prove that $x < x \oplus y$, as the other inequality is proved analogously. A straightforward calculation shows that

$$x \oplus y - x = \frac{s}{q+s}(y-x).$$

The right hand side is positive, as q and s are positive, so we conclude that $x < x \oplus y$.

Now, suppose that $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive Farey fractions. We may assume that $\frac{p}{q} < \frac{r}{s}$, which means that $qr - ps = 1$. Using this fact we see that

$$q(p+r) - p(q+s) = qr - ps = 1,$$

which shows that $\frac{p}{q}$ and $\frac{p}{q} \oplus \frac{r}{s}$ are consecutive Farey fractions if they are both non-negative or non-positive. This, however, is naturally the case, as $\frac{p}{q}$ and $\frac{r}{s}$ are both non-negative or non-positive. Moreover, as q and s are both non-negative, the same holds for $q+s$ so that $\frac{p+r}{q+s}$ is indeed a reduced fraction. That $\frac{p}{q} \oplus \frac{r}{s}$ and $\frac{p}{q}$ are also consecutive Farey fractions is proved similarly. $\square$

Let us define an increasing sequence of sets recursively, using the notion of Farey medians. We do this by defining $\mathcal{C}^0 = \{\frac{-1}{0}, \frac{1}{0}\}$ and $\mathcal{C}^1 = \{\frac{-1}{0}, \frac{0}{1}, \frac{1}{0}\}$, and, assuming that we have constructed $\mathcal{C}^n$, for some $n \in \mathbb{N}$, we obtain $\mathcal{C}^{n+1}$ from $\mathcal{C}^n$ by adding the Farey median of each pair of consecutive numbers in $\mathcal{C}^n$. More precisely, if $x_0, x_1, \ldots, x_m$ denote the elements of $\mathcal{C}^n$, with $x_0 < x_1 < \ldots < x_n$, then we let

$$\mathcal{C}^{n+1} = \mathcal{C}^n \cup \{x_{k-1} \oplus x_k : k = 1, 2, \ldots n\}.$$

To illustrate, we obtain $\mathcal{C}^2$ from $\mathcal{C}^1$ by adding $\frac{-1}{0} \oplus \frac{0}{1} = \frac{-1}{1}$ and $\frac{0}{1} \oplus \frac{1}{0} = \frac{1}{1}$, so that $\mathcal{C}^2 = \{\frac{-1}{0}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{1}{0}\}$. The two next sets look as follows:

$$\mathcal{C}^3 = \left\{\tfrac{-1}{0}, \tfrac{-2}{1}, \tfrac{-1}{1}, \tfrac{-1}{2}, \tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{0}\right\} \quad \text{and}$$

$$\mathcal{C}^4 = \left\{\tfrac{-1}{0}, \tfrac{-3}{1}, \tfrac{-2}{1}, \tfrac{-3}{2}, \tfrac{-1}{1}, \tfrac{-2}{3}, \tfrac{-1}{2}, \tfrac{-1}{3}, \tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}, \tfrac{3}{2}, \tfrac{2}{1}, \tfrac{3}{1}, \tfrac{1}{0}\right\}.$$

We know from Lemma 2.3.4 that the Farey median of two elements of $\mathcal{C}^n$ is strictly in-between these, so an easy induction argument shows that $|\mathcal{C}^n| = 2^n + 1$, for all $n \in \mathbb{N} \cup \{0\}$. It also follows from Lemma 2.3.4 and an induction argument that any two consecutive elements in $\mathcal{C}^n$ are, in fact, consecutive Farey fractions, for all $n \in \mathbb{N}$.

We want to investigate these sets further, but to do this, let us prove the following useful lemma about consecutive Farey fractions.

Lemma 2.3.5. *Let $\frac{p}{q}$ and $\frac{r}{s}$ be two consecutive Farey fractions in $\mathbb{Q} \cup \{\pm\infty\}$ in reduced form and with $\frac{p}{q} < \frac{r}{s}$. If $\frac{a}{b}$ is a third reduced fraction with $\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$, then $|a| \geq |p+r|$ and $b \geq q+s$.*

Proof. First of all, notice that since $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive Farey fractions, the numbers p, r, a are either all non-negative or all non-positive. It is not difficult to see that since $\frac{p}{q} < \frac{a}{b}$ and q and b are both non-negative, we must have that $aq - bp \geq 1$. Similarly, $br - as \geq 1$. Now, coupling this with the fact that $qr - ps = 1$, we see that

$$b = (qr - ps)b = (rb - as)q + (aq - pb)s \geq q + s,$$

since both q and s are non-negative. In the case that p and r are both non-negative, we see that

$$a = (qr - ps)a = (qa - bp)r + (br - as)p \geq r + p,$$

while, in the case where they are both non-positive, we get that

$$a = (qa - bp)r + (br - as)p \leq r + p.$$

In any case, we get that $|a| \geq |p + r|$. $\square$

Let $\mathcal{C}$ denote the union of the sets $\{\mathcal{C}^n : n \in \mathbb{N}\}$. As $\mathcal{C}^n$ is a subset of $\mathbb{Q} \cup \{\pm\infty\}$, for all $n \in \mathbb{N}$, their union $\mathcal{C}$ is also contained in $\mathbb{Q} \cup \{\pm\infty\}$. The following proposition tells us that the inclusion is actually an equality.

Proposition 2.3.6. *The set $\mathcal{C}$ contains all rational numbers.*

Proof. Let us proceed by contradiction. Assume that $\frac{a}{b}$ is a reduced fraction which is not in $\mathcal{C}$. We may assume that $\frac{a}{b}$ is non-zero, as $0 \in \mathcal{C}^1 \subseteq \mathcal{C}$. Also, we may assume that the $\frac{a}{b}$ is positive, as the negative case is handled analogously. Since $\frac{a}{b}$ is not in $\mathcal{C}^n$, for any $n \in \mathbb{N}$, we may chose reduced fractions $\frac{p_n}{q_n}$ and $\frac{r_n}{s_n}$ in $\mathcal{C}^n$ such that $\frac{p_n}{q_n}$ and $\frac{r_n}{s_n}$ are consecutive in $\mathcal{C}^n$ with $\frac{p_n}{q_n} < \frac{a}{b} < \frac{r_n}{s_n}$, for each $n \in \mathbb{N}$. Since the sets $\{\mathcal{C}^n : n \in \mathbb{N}\}$ are increasing, it is easy to see that $\frac{p_n}{q_n}$ is increasing in n and $\frac{r_n}{s_n}$ decreasing in n. From this, one easily sees that all the fractions are non-negative. It follows from Lemma 2.3.5 that

$$p_n + q_n + r_n + s_n \leq a + b,$$

for all $n \in \mathbb{N}$. Let us denote the left hand side by K_n. Our strategy is to prove that K_n tends to infinity, as this will clearly procure the desired contradiction.

At the step in the construction of the sets $\{\mathcal{C}^m : m \in \mathbb{N}\}$ where $\mathcal{C}^{n+1}$ is obtained from $\mathcal{C}^n$, the point $\frac{p_n}{q_n} \oplus \frac{r_n}{s_n}$ is added in-between $\frac{p_n}{q_n}$ and $\frac{r_n}{s_n}$. Since $\frac{a}{b} \neq \frac{p_n}{q_n} \oplus \frac{r_n}{s_n}$, we then either have

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} \qquad \text{and} \qquad \frac{r_{n+1}}{s_{n+1}} = \frac{p_n}{q_n} \oplus \frac{r_n}{s_n},$$

in the case when $\frac{a}{b} < \frac{p_n}{q_n} \oplus \frac{r_n}{s_n}$, or we have

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} \oplus \frac{r_n}{s_n} \qquad \text{and} \qquad \frac{r_{n+1}}{s_{n+1}} = \frac{r_n}{s_n},$$

in the case when $\frac{a}{b} > \frac{p_n}{q_n} \oplus \frac{r_n}{s_n}$. In the former case, we see that $K_{n+1} = 2(p_n + q_n) + r_n + s_n$, and in the latter case, $K_{n+1} = p_n + q_n + 2(r_n + s_n)$. In any case, we see that $K_{n+1} \geq K_n + 1$, which proves that K_n tends to infinity, contradicting the fact that $K_n \leq a + b$, for all $n \in \mathbb{N}$. This proves that $\mathcal{C}$ contains all of $\mathbb{Q}$. $\square$

So far we know that $\mathcal{C}$ contains all rational numbers and that, at any stage, two consecutive elements of $\mathcal{C}^n$ are, in fact, consecutive Farey fractions. The converse is also true, as the following proposition states.

Proposition 2.3.7. *Suppose we are given two fractions $\frac{p}{q}$ and $\frac{r}{s}$ in $\mathbb{Q} \cup \{\pm\infty\}$. Then these are consecutive Farey fractions if and only if they are consecutive in $\mathcal{C}^n$, for some $n \in \mathbb{N}$.*

Proof. As noted earlier, consecutive elements in $\mathcal{C}^n$ are consecutive Farey fractions, for all $n \in \mathbb{N}$. So assume that the fractions $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive Farey fractions, and let us prove that they are consecutive in some $\mathcal{C}^n$. We assume that $\frac{p}{q} < \frac{r}{s}$.

First of all, if $\frac{r}{s} = \frac{1}{0}$, then $\frac{p}{q} = \frac{p}{1}$ with $p \in \mathbb{N} \cup \{0\}$. It is easy to see that $\frac{1}{0}$ and $\frac{p}{1}$ are consecutive exactly in $\mathcal{C}^p$. Similarly, if $\frac{p}{q} = \frac{-1}{0}$, then $s = 1$, $r < 0$ and the fractions are consecutive exactly in $\mathcal{C}^{-r}$. Thus we may assume that both fractions are rational numbers.

Suppose towards a contradiction that the fractions are not consecutive in $\mathcal{C}^n$, for any $n \in \mathbb{N}$. Clearly this means that the fractions are not both in $\mathcal{C}^1$, as each pair of consecutive Farey fractions from this set is also consecutive in the set. Thus, if we choose the smallest $n \in \mathbb{N}$ so that both fractions $\frac{p}{q}$ and $\frac{r}{s}$ are contained in $\mathcal{C}^n$, then $n \geq 2$. Such n exists by Proposition 2.3.6. This means that one of the two fractions is not in $\mathcal{C}^{n-1}$. We assume that $\frac{p}{q}$ is not in $\mathcal{C}^{n-1}$. The case where $\frac{r}{s}$ is not in $\mathcal{C}^{n-1}$ is similar. As $\frac{p}{q}$ is in $\mathcal{C}^n$ but not in $\mathcal{C}^{n-1}$, we know from the construction of the sets that there exists consecutive Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$ in $\mathcal{C}^{n-1}$, with $\frac{a}{b} < \frac{c}{d}$ and $\frac{p}{q} = \frac{a}{b} \oplus \frac{c}{d}$. As $\frac{p}{q}$ and $\frac{r}{s}$ are both in $\mathcal{C}^n$ but are not consecutive, and as $\frac{p}{q}$ and $\frac{c}{d}$ are consecutive in $\mathcal{C}^n$, we know that $\frac{p}{q} < \frac{c}{d} < \frac{r}{s}$. By Lemma 2.3.5 this implies that $d \geq q + s$, but as $q = b + d$ by the choice of $\frac{a}{b}$ and $\frac{c}{d}$, we get that $d \geq q + s = c + d + s$, which is a contradiction, as c, d and s are all positive. $\square$

As it is, we have constructed an increasing sequence of sets containing all rational numbers. Let us introduce another increasing sequence of sets, namely, one that contains all the dyadic rational numbers in $[0, 1]$, We will denote these sets by $\mathcal{D}^n$, for $n \in \mathbb{N} \cup \{0\}$, and define $\mathcal{D}^n$ by

$$\mathcal{D}^n = \left\{ \tfrac{m}{2^n} : m = 0, 1, \ldots, 2^n \right\} \subseteq [0, 1] \cap \mathbb{Z}[\tfrac{1}{2}].$$

Clearly this forms an increasing sequence of sets, so that $\mathcal{D}^n$ has $2^n + 1$ elements. As it will be relevant for the construction of the map Q, let us describe how these sets can be constructed in an inductive fashion. Clearly the elements in the difference of the sets $\mathcal{D}^n$ and $\mathcal{D}^{n-1}$ are exactly those rational numbers whose reduced fraction form has 2^n in the denominator, that is, the set

$$\left\{\tfrac{2m-1}{2^n} : m \in \{1, 2, \ldots, 2^{n-1}\}\right\}.$$

As $\frac{m-1}{2^{n-1}}$ and $\frac{m}{2^{n-1}}$ are in $\mathcal{D}^{n-1}$, for all $m \in \{1, 2, \ldots, 2^{n-1}\}$, and $\frac{2m-1}{2^n}$ is the average of these two numbers, it follows that $\mathcal{D}^n$ is obtained from $\mathcal{D}^{n-1}$ by adding the average of two consecutive numbers in $\mathcal{D}^{n-1}$. We denote the union of all these sets by $\mathcal{D}$, that is $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}^n$.

At this point we are ready to construct the map Q. First we will define it on the set $\mathcal{C}$, and afterwards on the rest of $[-\infty, \infty]$. Now, given the way we have constructed the sets $\mathcal{C}^n$ and $\mathcal{D}^n$ inductively by repeatedly inserting numbers in-between each successive pair of elements, it is not difficult to see that the order-preserving bijections mapping $\mathcal{C}^n$ onto $\mathcal{D}^n$, for every $n \in \mathbb{N} \cup \{0\}$ are compatible in the sense that the one on $\mathcal{C}^{n+1}$ extends the one on $\mathcal{C}^n$. This is the short way to explain how the map Q is defined. Let us explain the construction in more detail, by defining the map recursively on the sets $\mathcal{C}^n$ one at a time. First, we define

$$Q(-\infty) = 0, \qquad Q(0) = \tfrac{1}{2} \qquad \text{and} \qquad Q(\infty) = 1,$$

so that, indeed, Q is a bijective order-preserving map from $\mathcal{C}^1$ to $\mathcal{D}^1$. Next, assuming that we have defined Q on $\mathcal{C}^n$, for some $n \in \mathbb{N}$, so that it is mapped bijectively onto $\mathcal{D}^n$ in an order-preserving way, we proceed to define it on $\mathcal{C}^{n+1}$, or, rather, we define it on $\mathcal{C}^{n+1} \setminus \mathcal{C}^n$. The elements in this difference of sets are precisely the elements of the form $x \oplus y$, for some consecutive numbers x and y in $\mathcal{C}^n$. Suppose that x and y are such elements, and define

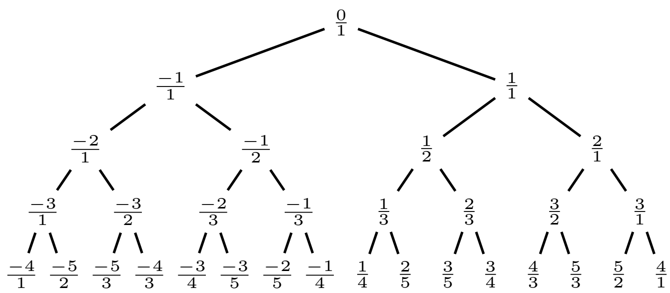
$$Q(x \oplus y) = \frac{Q(x) + Q(y)}{2}. \tag{2.1}$$

By assumption, Q is an order-preserving bijection between $\mathcal{C}^n$ and $\mathcal{D}^n$, so that $Q(x)$ and $Q(y)$ exactly are two consecutive elements of $\mathcal{D}^n$. As explained earlier, $\mathcal{D}^{n+1}$ is obtained from $\mathcal{D}^n$ by adding the average of each pair of consecutive elements in $\mathcal{D}^n$. This tells us that not only is the image of $\mathcal{C}^{n+1}$ contained in $\mathcal{D}^{n+1}$, but also that the map is bijective and order-preserving from $\mathcal{C}^{n+1}$ to $\mathcal{D}^{n+1}$. Now, as $\mathcal{C}$ is the union of all the sets $\{\mathcal{C}^n : n \in \mathbb{N}\}$, we have defined the map on all of $\mathcal{C}$. As the image of each $\mathcal{C}^n$ is $\mathcal{D}^n$, the image of $\mathcal{C}$ is exactly $\mathcal{D}$. Moreover, it is easy to see that the map is order-preserving. Thus we have defined the map Q on all of $\mathcal{C}$. Now, suppose that we are given some $x \in [-\infty, \infty]$ which is not in $\mathcal{C}$, then we define

$$Q(x) = \sup\{Q(y) : y \in \mathcal{C}, y \leq x\}.$$

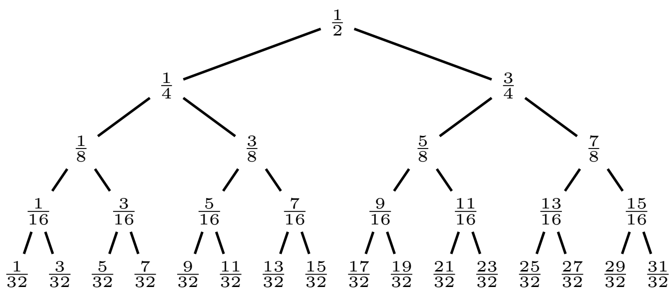
It is not difficult to see that this makes Q an order-preserving bijection from $[-\infty, \infty]$ to $[0, 1]$. Essentially, this is because the sets $\mathcal{C}$ and $\mathcal{D}$ are dense in $[-\infty, \infty]$ and $[0, 1]$, respectively, and as Q is an order-preserving bijection between these dense sets, the map Q ends up being an order-preserving bijection.

Now that we have constructed the function Q, let us relate it to the Minkowski question mark function. To do so, let us make a few comments on the construction of Q and try to give a more picturesque description of the construction. If we choose to label the infinite binary tree by labelling the 2^n nodes in the n'th level of the tree (the root being the zeroth level) from left to right by the element in $\mathcal{C}^{n+1} \setminus \mathcal{C}^n$, we obtain a labelled tree, the top of which looks as follows:



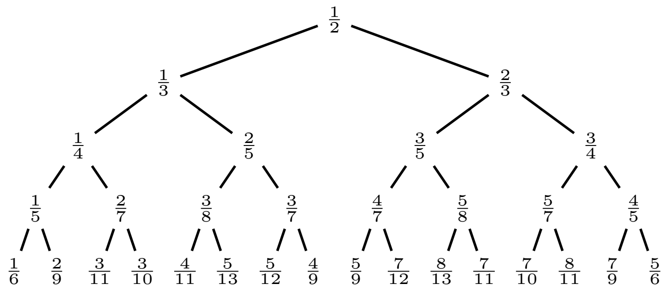
Infinite binary tree labelled with $\mathbb{Q}$

This tree is labelled with all the rational numbers, so that each number appears exactly once. We could label the tree with the dyadic rational numbers, as well, by using the sets $\mathcal{D}^{n+1} \setminus \mathcal{D}^n$ instead. In this way we obtain labelling of the tree with all dyadic rational numbers in $(0, 1)$, each one appearing exactly once. The top of this tree looks as follows:



Infinite binary tree labelled with $\mathcal{D}$

With this picture in mind, our map Q is simply obtained as the identification of the labels between the two trees, that is, mapping a $\mathbb{Q}$-label to the corresponding $\mathcal{D}$-label. Of course, afterwards we still extend it to all of $[-\infty, \infty]$, but the crucial thing is that these two different sets of labels produce the same order on the infinite binary tree. Now, if we wanted to construct a different function, we might have proceeded similarly, but with a different set of labels for the first tree. In fact, this is how the Minkowski question mark function is defined. If, instead of using the entire tree labelled with $\mathbb{Q}$, we only used the labelled subtree whose root is $\frac{1}{2}$, that is,



Infinite binary tree labelled with $\mathbb{Q} \cap (0,1)$

then we would get a map from the rational numbers in $(0,1)$ to the dyadic rational numbers in $(0,1)$. Extending this map to a map from $[0,1]$ to itself, we would get an order-preserving homeomorphism, denoted by $?$, which maps the rational numbers onto the dyadic rational numbers and satisfies

$$?\big(x \oplus y\big) = \frac{?(x)+?(y)}{2},$$

for consecutive Farey numbers x and y in $[0,1]$. Intuitively speaking, this is how the Minkowski question mark function is defined. More formally one would go through the construction by defining an increasing sequence of sets similar to $(\mathcal{C}^n)_{n\geq 0}$. Indeed, the construction is the same, but with the different starting conditions in the sense that one defines the first set as $\tilde{\mathcal{C}}^0 = \{\frac{0}{1}, \frac{1}{1}\}$, and then defines the set $\tilde{\mathcal{C}}^{n+1}$ as the union of the previous set $\tilde{\mathcal{C}}^n$ and of the set of Farey medians of successive elements of $\tilde{\mathcal{C}}^n$, as we did with the original sets. This yields a sequence of sets with similar properties as the original ones, but whose union, $\tilde{\mathcal{C}}$, is now only the rational numbers in $[0,1]$, as opposed to all of $\mathbb{Q} \cup \{\pm\infty\}$. Then one proceeds to construct a function from $\tilde{\mathcal{C}}$ to $\mathcal{D}$ in the same manner as when we constructed Q, and the map we end up with is then the original Minkowski question mark function $?$.

Clearly there is a great similarity between the two functions Q and $?$, as their constructions show, and it is not difficult to find explicit formulas relating the two. An example of such a formula is the following:

$$?(x) = 4Q(x) - 2,$$

for all $x \in [0,1]$. There are several ways to argue the validity of this formula, however, an easy way to do it is the following. If $f(x)$ denotes $4Q(x) - 2$, then $f(0) = 0$ and $f(1) = 1$ since $Q(0) = \frac{1}{2}$ and $Q(1) = \frac{3}{4}$. Moreover, the formula $f\big(x \oplus y\big) = \frac{1}{2}(f(x) + f(y))$ is satisfied for consecutive Farey numbers x and y in $[0,1]$ as Q satisfies (2.1). Now, an easy induction argument shows that then f agrees with $?$ on each of the sets $\tilde{\mathcal{C}}^n$, which, of course, means that f agrees with $?$ everywhere. This formula expresses the Minkowski question mark function in terms of the map Q. If, on the other hand, one would like to have the map Q expressed in terms of ?, then this is possible as well, for example, by the formula

$$Q(x) = \begin{cases} \frac{1}{2}?(\frac{x}{x+1}) + \frac{1}{2} & \text{for } x \geq 0 \\ \frac{1}{2}?(\frac{1}{1-x}) & \text{for } x \leq 0 \end{cases}.$$

This can be proved using the interpretation of T as a group of fractional linear transformations, which we discuss in the following section. We will not expand on this.

Let us end this section by mentioning that there are different descriptions of the Minkowski question mark function. Another description involves continued fractions, and was discovered by Salem [63] in 1943. Therein, the value at a certain point is given explicitly in terms of the continued fraction expansion of this point. The construction in this section is closely related to notions such as the Stern-Brocot tree and the Farey series. For a reference see, for example, [31].

2.4 Piecewise projective linear maps

The aim of this section is to prove a result due to Thurston, see [10], namely, that the map Q, constructed in the previous section, can be used to realize the Thompson group T as a certain group of piecewise projective linear homeomorphisms. We wish to thank Sergiescu for many fruitful discussions on this topic. We will start by giving a soft introduction to projective linear maps, which are a certain kind of homeomorphisms of $\mathbb{R} \cup \{\infty\}$, the one point compactification of $\mathbb{R}$. We will let $\hat{\mathbb{R}}$ denote the space $\mathbb{R} \cup \{\infty\}$.

A *fractional linear transformation*, *projective linear transformation* or *Möbius transformation* of the space $\hat{\mathbb{R}}$ is a transformation f of the form

$$f(x) = \frac{ax+b}{cx+d}, \qquad \text{for} \quad x \in \hat{\mathbb{R}}, \tag{2.2}$$

where a, b, c and d are fixed real numbers satisfying $ad - bc \neq 0$. As the expression above, strictly speaking, does not make sense in the ordinary fashion if $x = \infty$ or $x = -\frac{d}{c}$, let us explain how it should be understood. There are a few cases to consider. In the case when $c \neq 0$, the above expression evaluated at ∞ should be understood as the limit as $|x| \to \infty$, that is, $f(\infty) = \frac{a}{c}$. Moreover, we set $f(-\frac{d}{c}) = \infty$, which fits well with the fact that $|f(x)| \to \infty$, as $x \to -\frac{c}{d}$. In the case where $c = 0$, the map f is actually linear and we define $f(\infty) = \infty$. Strictly speaking, the map f is not linear, but affine, however, as it is customary to use the term linear, rather than affine, in connection with piecewise linear maps, we will adopt this terminology. It is easy to see from these choices that fractional linear maps are, in fact, homeomorphisms of $\hat{\mathbb{R}}$. Now, from the way we defined the maps, we see that the only fractional linear transformations which fix ∞ are the linear ones. This fact will be used frequently during this chapter.

Now, the requirement that $ad - bc \neq 0$, should get every mathematician thinking about invertible two-by-two matrices This is not a coincidence, as composition of fractional linear transformations corresponds to multiplication of the corresponding matrices of coefficients. Phrased differently, if f and g are fractional linear transformations with coefficients

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

respectively, meaning that

$$f(x) = \frac{a_1 x + b_1}{c_1 x + d_1} \qquad \text{and} \qquad g(x) = \frac{a_2 x + b_2}{c_2 x + d_2},$$

for all $x \in \hat{\mathbb{R}}$, then the composition fg is a fractional linear transformation whose coefficient matrix is exactly the product matrix

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

As it is easily seen that we get the identity on $\hat{\mathbb{R}}$ if we choose the identity matrix as coefficients, what we really have is an action on $\hat{\mathbb{R}}$ of the general linear group $\mathrm{GL}(2, \mathbb{R})$, of invertible two-by-two matrices with real entries. However, it is easy to see that many invertible matrices give rise to the same fractional linear transformations. Indeed, any non-zero scalar multiple of a matrix will produce the same transformation, and this is the only way this can happen. Indeed, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.3}$$

is an invertible matrix and f denotes the corresponding fractional linear transformation, then $f(x) = x$, for all $x \in \mathbb{R}$ if and only if

$$cx^2 + (d - a)x - b = 0,$$

for all $x \in \mathbb{R}$. Clearly this happens if and only if $c = b = 0$ and $a = d$, that is, when A is a scalar matrix. In other words, the kernel of the action is exactly the non-zero scalar matrices, as these are the only ones which act as the identity on $\hat{\mathbb{R}}$. Thus, what we really have is an action of $\mathrm{PGL}(2, \mathbb{R})$, the quotient of $\mathrm{GL}(2, \mathbb{R})$ by the normal subgroup $\mathbb{R}\mathbf{1}$. The action of $\mathrm{PGL}(2, \mathbb{R})$ on $\hat{\mathbb{R}}$ is therefore *faithful*, that is, no non-trivial element acts as the identity.

Before continuing the study of fractional linear transformations, let us mention a different view on these. First, recall that the real projective line, denoted by $\mathbb{RP}^1$, is the space of one dimensional subspaces of $\mathbb{R}^2$, or, more precisely, the quotient space of $\mathbb{R}^2 \backslash \{0\}$ by the relation of being proportional. As an invertible linear operator of $\mathbb{R}^2$ preserves the equivalence classes in $\mathbb{RP}^1$, and as a non-zero scalar multiple of such a map induces the same permutation of the equivalence classes, the group $\mathrm{PSL}(2, \mathbb{R})$

also acts on this space. Both of the spaces $\hat{\mathbb{R}}$ and $\mathbb{RP}^1$ are circles, topologically speaking, so naturally they are homeomorphic, but, in fact, the actions of $\mathrm{PGL}(2,\mathbb{R})$ on $\hat{\mathbb{R}}$ and $\mathbb{RP}^1$ are conjugate. More precisely, the homeomorphism from $\mathbb{RP}^1$ to $\hat{\mathbb{R}}$ which maps the equivalence class of $\mathbb{RP}^1$ containing (x, y) to $\frac{x}{y}$ if $y \neq 0$, and to ∞ if $y = 0$, is actually a homeomorphism which commutes with the action. This point of view makes some of the observations in the following very easy to realise, but we will, nonetheless, stick to the picture involving fractional linear transformations to get a feeling of these. However, in the following, we will not distinguish between the coefficient matrices and the fractional linear transformations, as this should not cause any confusion, and will make the formulations less cumbersome.

Theorem 2.4.1. *A fractional linear transformation is uniquely determined by its values on three points of* $\hat{\mathbb{R}}$.

Proof. It is clear that two fractional linear transformations f and g agree on three points if and only if $g^{-1}f$ fixes the same three points. Hence, it suffices to prove that a fractional linear transformation which fixes three points is actually trivial, meaning that it is the identity on $\hat{\mathbb{R}}$. So suppose that f is a fractional linear transformation fixing three points. At least two of these points must by in $\mathbb{R}$, so let y be one of these, and define a fractional linear transformation g by

$$g(x) = \frac{-1}{x - y}$$

Clearly $g(y) = \infty$, so gfg^{-1} fixes ∞ and two other points. Now, a fractional linear transformation fixes ∞ if and only if it is linear. Hence, gfg^{-1} is linear, and, since it fixes two points in $\mathbb{R}$, it must be the identity. This of course means that f must be the identity, as well. $\square$

An action is said to be *free* if no non-trivial element fixes a point, or, in other words, if every element is uniquely determined by its value in one point. Clearly the action of $\mathrm{PGL}(2,\mathbb{R})$ is not free, as the linear maps all fix ∞, however, the above theorem tells us that the action is, somehow, not too far from being free. The above result is the best one can hope for, as two point are not enough to uniquely determine the fractional linear transformation. Indeed, any linear maps going through zero fixes two points, namely, ∞ and 0.

Suppose that f is a fractional linear transformation, with coefficients as in (2.3). It is straightforward to check that the derivative of f is given by

$$\frac{\det(A)}{(cx + d)^2}$$

in each point x of differentiability. In particular, we see that the fractional linear transformation is increasing, or orientation preserving, if and only if the determinant is positive. Thus, $\mathrm{PSL}(2,\mathbb{R})$ as a subgroup of $\mathrm{PGL}(2,\mathbb{R})$ consists exactly of all the increasing fractional linear transformations. Here $\mathrm{PSL}(2,\mathbb{R})$ denotes the quotient of $\mathrm{SL}(2,\mathbb{R})$ by the subgroup $\{\pm\mathbf{1}\}$ as usual.

Recall that an action is said to be *transitive* if, given any two points, there is an element mapping one to the other. It is easy to see that the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\hat{\mathbb{R}}$ is transitive, for, given $x, y \in \mathbb{R}$, the fractional linear transformations

$$\begin{bmatrix} 1 & y-x \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} y & -1 \\ 1 & 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 0 & -1 \\ 1 & -x \end{bmatrix}$$

will map x to y, ∞ to y, and x to ∞, respectively. However, this is not all that can be said. The following proposition shows that the action of $\mathrm{PGL}(2,\mathbb{R})$ is, what is known as, *3-transitive*.

Proposition 2.4.2. *Suppose that x_1, x_2 and x_3 are three distinct points in $\hat{\mathbb{R}}$, and that y_1, y_2 and y_3 are three distinct points in $\hat{\mathbb{R}}$, as well. Then there is some fractional linear transformations A in* $\mathrm{PGL}(2,\mathbb{R})$ *so that $A(x_k) = y_k$, for $k = 1, 2, 3$.*

Proof. Suppose that none of the six points is ∞, and suppose that we are given a fractional linear transformation A, with coefficients (2.3) say. Rearranging the terms in the three equations $A(x_k) = y_k$, for $k = 1, 2, 3$, we see that they are equivalent to the following matrix equation:

$$\begin{bmatrix} x_1 & 1 & -x_1y_1 & -y_1 \\ x_2 & 1 & -x_2y_2 & -y_2 \\ x_3 & 1 & -x_3y_3 & -y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By dimension considerations, we see that this system of linear equations has a non-zero solution (a, b, c, d). This solution must necessarily satisfy $ad - bc \neq 0$. So choosing A with these coefficients, we get a fractional linear transformations which maps the points according to our wishes.

Now, if we drop the assumption that none of the points is ∞, then we just use a fractional linear transformations to move the points away from ∞ first. More precisely, choose some point $z \in \mathbb{R}$ different from our six points, and let B be the fractional linear transformation given by

$$\begin{bmatrix} 0 & -1 \\ 1 & -z \end{bmatrix}.$$

Then $B(z) = \infty$, which means that $B(x_k) \neq \infty$ and $B(y_k) \neq \infty$, for $k = 1, 2, 3$, so, by the first part, we may choose a fractional linear transformation A so that $A(B(x_k)) = B(y_k)$, for $k = 1, 2, 3$. Hence $B^{-1}AB$ is the desired fractional linear transformation mapping our original points as desired. $\square$

Let us make a comment on the proposition above. First of all, the statement is not true if we replace the group $\mathrm{PGL}(2,\mathbb{R})$ by $\mathrm{PSL}(2,\mathbb{R})$. Let us briefly explain why this is the case. If we are given three distinct points x_1, x_2 and x_3 in $\hat{\mathbb{R}}$, then these points give rise to an orientation of $\hat{\mathbb{R}}$ in a natural way, as there is a unique direction from x_1 to x_2 so that, travelling from x_1 to x_2 in this direction, one does not encounter x_3. As the elements of $\mathrm{PSL}(2,\mathbb{R})$ act in an orientation-preserving way, the above proposition fails if the two sets of points give rise to different orientations. In the case that both sets of points give rise to the same orientation, the element from $\mathrm{PGL}(2,\mathbb{R})$ will have positive determinant, and can therefore be rescaled to be in $\mathrm{PSL}(2,\mathbb{R})$.

Now, our real interest lies in the fractional linear transformations in $\mathrm{PSL}(2,\mathbb{Z})$. Here $\mathrm{PSL}(2,\mathbb{Z})$ denotes the quotient of the group $\mathrm{SL}(2,\mathbb{Z})$ of two-by-two integer matrices with determinant one by the subgroup $\{\pm\mathbf{1}\}$. The action of $\mathrm{PSL}(2,\mathbb{Z})$ is, of course, still faithful, but it is no longer transitive. This is obvious from the fact that $\mathrm{PSL}(2,\mathbb{Z})$ is countable, whereas $\hat{\mathbb{R}}$ is uncountable. However, $\hat{\mathbb{Q}}$ is an invariant subset on which $\mathrm{PSL}(2,\mathbb{Z})$ acts transitively. Indeed, the action on $\hat{\mathbb{Q}}$ actually satisfies something bordering on 2-transitivity. Let us make this more precise. Let us for the moment use $\frac{k}{0}$ to denote ∞, for every $k \in \mathbb{Z} \setminus \{0\}$. If we suppose that we are given two pairs, $\frac{p_1}{q_1}$ and $\frac{r_1}{s_1}$, and $\frac{p_2}{q_2}$ and $\frac{r_2}{s_2}$, of consecutive Farey fractions with $\frac{p_1}{q_1} < \frac{r_1}{s_1}$ and $\frac{p_2}{q_2} < \frac{r_2}{s_2}$, the fractional linear transformation

$$\begin{bmatrix} s_1p_2 - q_1r_2 & p_1r_2 - r_1p_2 \\ s_1q_2 - q_1s_2 & p_1s_2 - r_1q_2 \end{bmatrix} \tag{2.4}$$

is an element of $\mathrm{PSL}(2,\mathbb{Z})$ which maps $\frac{p_1}{q_1}$ to $\frac{p_2}{q_2}$ and $\frac{r_1}{s_1}$ to $\frac{r_2}{s_2}$. Using this we easily see that the action is transitive on $\hat{\mathbb{Q}}$, however, this can be done a bit easier. If $\frac{p}{q}$ is a rational number on reduced form, then p and q are relatively prime, and so we may choose integers r and s so that $pr + qs = 1$. Then

$$\begin{bmatrix} p & -r \\ q & s \end{bmatrix}$$

is a fractional linear transformation in $\mathrm{PSL}(2,\mathbb{Z})$ which maps ∞ to $\frac{p}{q}$. Thus, the orbit of ∞ is all of $\hat{\mathbb{Q}}$, which means that the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\hat{\mathbb{Q}}$ is transitive.

We proved earlier that elements of $\mathrm{PGL}(2,\mathbb{R})$ are uniquely determined by their values on three points. This is of course also true for the elements in $\mathrm{PSL}(2,\mathbb{Z})$, but for these fractional linear transformations two points in $\hat{\mathbb{Q}}$ are enough, as the following proposition shows.

Proposition 2.4.3. *An element in* $\mathrm{PSL}(2,\mathbb{Z})$ *is uniquely determined by its values on two distinct points in* $\hat{\mathbb{Q}}$.

Proof. Two fractional linear transformations A and B agree on two points if and only if $A^{-1}B$ fixes these two points, so what we need to prove is that, if a fractional linear transformations fixes two distinct points, then it is the identity. Suppose that A is a fractional linear transformation in $\mathrm{PSL}(2,\mathbb{Z})$, which fixes two points in $\hat{\mathbb{Q}}$. By replacing A with BAB^{-1}, for some $B \in \mathrm{PSL}(2,\mathbb{Z})$ with $B(x) = \infty$, where $x \in \hat{\mathbb{Q}}$ is a point fixed by A, we can make sure that one of the points fixed by A is ∞. Such a B exists since the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\hat{\mathbb{Q}}$ is transitive, as noted above. So we will assume that ∞ is one point fixed by A and $y \in \mathbb{Q}$ is another.

Now, as A fixes ∞, it must be linear, which means that A has the form

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

Since $ad = \det(A) = 1$, we see that a and d are either both equal to 1 or both equal to -1. In any case, A is linear with slope 1, more precisely $A(x) = x + \frac{b}{d}$, so if A fixes a point in $\mathbb{Q}$, then A is the identity. This concludes the proof. □

Scrutinizing the proof above, it will be apparent that only one of the two points needs to be in $\hat{\mathbb{Q}}$, that is, an element in $\mathrm{PSL}(2,\mathbb{Z})$ is uniquely determined by one point in $\hat{\mathbb{Q}}$ and one other point. Thus one might ask whether an element in $\mathrm{PSL}(2,\mathbb{Z})$ is actually determined uniquely by two arbitrary points. This is not the case though, as it is straightforward to check that

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

fixes $\frac{1}{2}(1 \pm \sqrt{5})$, but is obviously not the identity. The elements of $\mathrm{PSL}(2,\mathbb{Z})$ which fix two points are those known as hyperbolic elements.

Before explaining which piecewise fractional linear maps we are interested in, let us introduce some notation. As we will need to consider closed segments of the circle $\hat{\mathbb{R}}$, such as closed intervals, we will adopt the following notation: given some real number x, $[\infty, x]$ and $[x, \infty]$ will refer to the sets $[x, \infty) \cup \{\infty\}$ and $(-\infty, x] \cup \{\infty\}$, respectively. These are the closed segments of $\hat{\mathbb{R}}$ beginning or ending in ∞.

With this notation introduced, let us define the group of piecewise projective linear homeomorphisms of $\hat{\mathbb{R}}$ mentioned in the beginning of this section. We will denote this group by $\mathrm{PPSL}(2,\mathbb{Z})$, and we will define it to be the set of homeomorphisms of $\hat{\mathbb{R}}$ which are piecewise in $\mathrm{PSL}(2,\mathbb{Z})$ with breakpoints in $\hat{\mathbb{Q}}$. More precisely, a homeomorphism f of $\hat{\mathbb{R}}$ is in $\mathrm{PPSL}(2,\mathbb{Z})$ if and only if there exist rational numbers $x_1, x_2, \ldots, x_n \in \mathbb{Q}$ with

$$x_1 < x_2 < \cdots < x_n$$

and $A_0, A_1, \ldots, A_n \in \mathrm{PSL}(2,\mathbb{Z})$ such that, with $x_0 = x_{n+1} = \infty$, we have $f(y) = A_k(y)$, for all $y \in [x_k, x_{k+1}]$ and all $k = 0, 1, \ldots, n$.

It is easy to see that all of these piecewise fractional linear maps are orientation preserving, as the fractional linear transformations in $\mathrm{PSL}(2,\mathbb{Z})$ are so. Likewise, as the set $\hat{\mathbb{Q}}$ is invariant under fractional linear transformations from $\mathrm{PSL}(2,\mathbb{Z})$, this will also be the case for the elements in $\mathrm{PPSL}(2,\mathbb{Z})$.

In the definition above we used the word breakpoint, and, as this will be important later in Section 3.3, we will now make the notion precise.

Definition 2.4.4. Let $f \in \mathrm{PPSL}(2,\mathbb{Z})$ and $x \in \hat{\mathbb{R}}$. The point x is said to be a *breakpoint* for f if there does not exist a neighbourhood U around x and an element $A \in \mathrm{PSL}(2,\mathbb{Z})$ such that $f(y) = A(y)$, for all $y \in U$.

The reader should note that the points $x_0, \dots, x_n$ mentioned in the definition of the group $\mathrm{PPSL}(2,\mathbb{Z})$ are not necessarily all breakpoints. Indeed, some elements of $\mathrm{PPSL}(2,\mathbb{Z})$ do not have any breakpoints at all, namely, the elements of $\mathrm{PSL}(2,\mathbb{Z})$.

As mentioned earlier, the aim of this section is to prove that the Thompson group T is isomorphic to $\mathrm{PPSL}(2,\mathbb{Z})$, or, more precisely, that these groups are isomorphic via conjugation by a homeomorphism from $\hat{\mathbb{R}}$ to $[0,1)$, when the latter space is given the topology of a circle, as explained in the definition of T. This homeomorphism is the map from $\hat{\mathbb{R}}$ to $[0,1)$ induced by the map Q from Section 2.3. To be completely explicit, this is the map one gets by restricting Q to a map from $\mathbb{R}$ to $(0,1)$ and then extending it to a map from $\hat{\mathbb{R}}$ to $[0,1)$. We will use Q to denote this map as well, as it should not cause any confusion.

The fact that the Thompson group T can be realized as a group of piecewise projective linear transformations was originally discovered by Thurston, as explained in [10, §7]. Thurston realized T as piecewise projective linear transformations of the unit interval, and he did this by conjugating the elements of T with the Minkowski question mark function ?. The particular version of Thurston's result we are interested in was given by Imbert [38] (see Theorem 1.1 therein), and the difference consists in the fact that Imbert uses the map Q, rather than the map ?.

Theorem 2.4.5. *The map from* $\mathrm{PPSL}(2,\mathbb{Z})$ *to* T *given by* $f \mapsto Qf^{-1}Q$ *is an isomorphism.*

Sketch of proof. We will not give the complete proof as this involves to many details. Instead, we will try to explain why the theorem is true, and give sufficiently many of these details for the reader to be able to fill in the rest.

Let us denote this map from $\mathrm{PPSL}(2,\mathbb{Z})$ to T by Φ. It is clear that Φ is an injective group homomorphism, and the only things we need to check is that it is surjective and well-defined, that is, that $\Phi(f) \in T$ when $f \in \mathrm{PPSL}(2,\mathbb{Z})$. Fix some $f \in \mathrm{PPSL}(2,\mathbb{Z})$. Consulting the definition of T, we see that, for $\Phi(f)$ to be in T, it must satisfy the following conditions:

- it must be a homeomorphism of the circle;
- it must leave the set $\mathcal{D} \setminus \{1\}$ invariant;
- it must be piecewise linear with finitely many pieces;
- the slope on these pieces must be a power of 2;

- the points of non-differentiability are dyadic rational numbers.

It should be clear that $\Phi(f)$ satisfies the first condition as both Q and f are homeomorphisms. Also, we noted above that elements of $\mathrm{PPSL}(2, \mathbb{Z})$ leave the set $\hat{\mathbb{Q}}$ invariant, so, as Q restricts to a bijection between this set and $\mathcal{D} \setminus \{1\}$, it follows that $\Phi(f)$ leaves this latter set invariant. Therefore, we are left to prove that $\Phi(f)$ satisfies the last three conditions. It suffices to find rational numbers $x_1, x_2, \ldots, x_n$ with

$$x_1 < x_2 < \cdots < x_n$$

such that, with $y_0 = 0$, $y_{n+1} = 1$ and $y_k = Q(x_k)$, the map $\Phi(f)$ is linear on the interval $[y_k, y_{k+1})$ with a slope which is a power of 2, for $k = 0, 1, \ldots, n$.

A naive approach would be to try to prove that $\Phi(f)$ is linear with the correct slope on the pieces where f is fractional linear. However, this is not true, as, for example, it is only the identity of T which is linear on all of $[0, 1)$, whereas all the elements of $\mathrm{PSL}(2, \mathbb{Z})$ are fractional linear on all of $\hat{\mathbb{R}}$. Thus, we need be careful in choosing the points $x_1, \ldots, x_n$. The trick is to choose the points so that x_k and x_{k+1} are consecutive Farey numbers, but in way so that $\Phi(f)(x_k)$ and $\Phi(f)(x_{k+1})$ turn out to be consecutive Farey numbers, as well. For this, we choose some $j \in \mathbb{N}$ so that $\mathcal{C}^j \cap \mathbb{Q}$ contains all the breakpoints of f, as well as the points $f^{-1}(0)$ and $f^{-1}(\infty)$, possibly with the exclusion of ∞. This is possible by Lemma 2.3.6. Now, let $x_1, x_2, \ldots, x_n$ be the elements of $\mathcal{C}^j \cap \mathbb{Q}$ ordered increasingly. The claim is then that this choice of $x_1, \ldots, x_n$ works, that is, $\Phi(f)$ is linear on the interval $[y_k, y_{k+1})$ with slope a power of 2, for $k = 0, 1, \ldots, n$.

Fix some $k \in \{0, 1, \ldots, n\}$, choose a fractional linear transformation

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathrm{PSL}(2, \mathbb{Z})$ so that $f(x) = A(x)$, for $x \in [x_k, x_{k+1}]$, and let us show that $\Phi(f)$ is linear on the interval $[y_k, y_{k+1})$ with slope a power of 2. The reason is, in essence, that both A and Q behave well with respect to consecutive Farey fractions. Indeed, we would like to say that A maps consecutive Farey numbers to consecutive Farey numbers and satisfies

$$A(x \oplus y) = A(x) \oplus A(y),$$

but, unfortunately, this is a simplification of the story. The main problem is that being consecutive Farey fractions is a notion for the elements in $\mathbb{Q} \cup \{\pm\infty\}$, whereas A is a transformation of $\hat{\mathbb{R}}$ and not $[-\infty, \infty]$. Thus, for consecutive Farey fractions to make sense in $\hat{\mathbb{Q}}$, the point ∞ must play the role of both ∞ and $-\infty$, and this causes some technical difficulties. To avoid too many of these, we will assume that $A([x_k, x_{k+1}]) \subseteq \mathbb{R}$, so that, in particular, $A(x_k)$ and $A(x_{k+1})$ are rational numbers with $A(x_k) < A(x_{k+1})$. Notice that, since we have ensured that $A(x_l) = \infty$, for some $l \in \{0, 1, \ldots, n\}$, the condition $A([x_k, x_{k+1}]) \subseteq \mathbb{R}$ is satisfied for all but two values of k. We will explain briefly afterwards how to deal with these two special cases.

Let us, for the purpose of this argument, agree that, in $\hat{\mathbb{R}}$, the fraction $\frac{k}{0}$ represents ∞, for all $k \in \mathbb{Z}\setminus\{0\}$, so that we have two ways of representing ∞ as a reduced fraction, namely, $\frac{1}{0}$ and $\frac{-1}{0}$. With this convention, it is easy so see that

$$A(\tfrac{p}{q}) = \frac{ap+bq}{cp+dq}, \tag{2.5}$$

for every fraction $\frac{p}{q} \in \hat{\mathbb{Q}}$, even in the cases where $\frac{p}{q} = \infty$ or $A(\frac{p}{q}) = \infty$. As we are interested in consecutive Farey fractions, so that it matters which fraction we choose to represent the elements of $\hat{\mathbb{Q}}$, we will, again for the purpose of this argument, let $A(\frac{p}{q})$ denote the fraction on the right hand side of (2.5) rather than the point in $\hat{\mathbb{R}}$ it represents.

Now, write x_k and x_{k+1} as reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$, respectively, in a way so that, if $x_k = \infty$, then $p = -1$, and, if $x_{k+1} = \infty$, then $r = 1$. It is easy to see that the way we have chosen $x_1, \ldots, x_n$ ensures that $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive Farey fractions, by Proposition 2.3.7, as they will be consecutive in $\mathcal{C}^j$. Moreover, it is also easy to see that $qr - ps = 1$, as $\frac{p}{q} < \frac{r}{s}$, now as elements in $[-\infty, \infty]$. Recalling that A is a matrix with determinant one, we see that

$$(cp+dq)(ar+bs) - (ap+bq)(cr+ds) = (ad-bc)(qr-ps) = 1. \tag{2.6}$$

From this we deduce that, after possibly changing the sign of numerators and denominators, both the fractions

$$A(\tfrac{p}{q}) = \frac{ap+bq}{cp+dq} \qquad \text{and} \qquad A(\tfrac{r}{s}) = \frac{ar+bs}{cr+ds}$$

are, in fact, reduced fractions. Moreover, if both the denominators are already positive and if they both represent either non-negative or non-positive numbers, then they are consecutive Farey fractions. A straightforward computation shows that

$$(cp+dq)(cr+ds)\big(A(\tfrac{r}{s}) - A(\tfrac{p}{q})\big) = 1,$$

so we see that the denominators are either both positive or both negative, since $(A(\frac{r}{s}) - A(\frac{p}{q}))$ is positive. Thus, by possibly replacing the matrix A by $-A$, we can make sure that they are both positive. Moreover, as the identity (2.6) is clearly not affected by this change of sign, and, since the denominators of $A(\frac{p}{q})$ and $A(\frac{r}{s})$ are both positive, the numerators must either both be non-negative or non-positive. As mentioned earlier, this means that $A(\frac{p}{q})$ and $A(\frac{r}{s})$ are consecutive Farey fractions, and we then see that

$$A(\tfrac{p}{q}) \oplus A(\tfrac{r}{s}) = \frac{(ap+bq)+(ar+bs)}{(cp+dq)+(cr+ds)} = \frac{a(p+r)+b(q+s)}{c(p+r)+d(q+s)}.$$

The fraction on the right hand side is exactly $A(\frac{p}{q} \oplus \frac{r}{s})$, so, recalling that $x_k = \frac{p}{q}$ and $x_{k+1} = \frac{r}{s}$, we conclude that

$$A(x_k) \oplus A(x_{k+1}) = A(x_k \oplus x_{k+1}).$$

Let z_k and z_{k+1} denote $\Phi(f)(y_k)$ and $\Phi(f)(y_{k+1})$, respectively. Applying the equation above together with equation (2.1), we see that that

$$QAQ^{-1}\big(\tfrac{y_k+y_{k+1}}{2}\big) = QA(x_k \oplus x_{k+1}) = Q(A(x_k) \oplus A(x_{k+1})) = \tfrac{z_k+z_{k+1}}{2}.$$

Recalling that $\Phi(f)(x) = QAQ^{-1}(x)$, for all $x \in [y_k, y_{k+1})$, what we have shown is that $\Phi(f)(\frac{y_k+y_{k+1}}{2}) = \frac{z_k+z_{k+1}}{2}$. Now, this equation was indeed expected, as we are trying to show that $\Phi(f)$ is linear on the interval $[y_k, y_{k+1})$, and it is important to notice that, by letting $x' = x_k \oplus x_{k+1}$, both x_k and x', as well as x' and x_{k+1}, satisfy the same conditions as x_k and x_{k+1} in the sense that both pairs are consecutive Farey numbers and both $A([x_k, x'])$ and $A([x', x_{k+1}])$ are contained in $\mathbb{R}$. Thus, by repeating the argument with these two intervals, we see that, with $y' = Q(x')$ and $z' = \Phi(f)(x')$,

$$\Phi(f)(\tfrac{1}{4}y_k + \tfrac{3}{4}y_{k+1}) = \Phi(f)(\tfrac{1}{2}(y_k + y')) = \tfrac{1}{2}(z_k + z') = \tfrac{1}{4}z_k + \tfrac{3}{4}z_{k+1},$$

and similarly that $\Phi(f)(\frac{3}{4}y_k + \frac{1}{4}y_{k+1}) = \frac{3}{4}z_k + \frac{1}{4}z_{k+1}$. Continuing this procedure, we get that

$$\Phi(f)\big(\tfrac{m}{2^i}y_k + (1-\tfrac{m}{2^i})y_{k+1}\big) = \tfrac{m}{2^i}z_k + (1-\tfrac{m}{2^i})z_{k+1},$$

for every $i \in \mathbb{N}$ and every $m = 1, 2, \ldots, 2^m$. Evidently this means that $\Phi(f)$ maps $[y_k, y_{k+1})$ linearly onto $[z_k, z_{k+1})$, and we only need to know that the slope is a power of 2. Along the way we noticed that x_k and x_{k+1} are consecutive in $\mathcal{C}^j$, so from the construction of the map Q, it follows that y_k and y_{k+1} are consecutive in $\mathcal{D}^j$. In particular, the length of the interval $[y_k, y_{k+1})$ is 2^j. Likewise, we also noticed that $A(x_k)$ and $A(x_{k+1})$ are consecutive in some $\mathcal{C}^l$ so that the interval $[z_k, z_{k+1})$ has length 2^l. Hence, the slope of $\Phi(f)$ on $[y_k, y_{k+1})$ is 2^{l-j}, and we have proved that $\Phi(f)$ has the desired form on $[y_k, y_{k+1})$.

Now, let us make a few comments on how to deal with the two remaining cases, that is, to prove that $\Phi(f)$ is linear on $[A(x_{l-1}), \infty)$ and on $[\infty, A(x_{l+1}))$, when l is chosen so that $A(x_l) = \infty$. As we included $f^{-1}(0)$ in $x_1, \ldots, x_n$, we know that $A(x_{l-1}) \geq 0$ and $A(x_{l+1}) \leq 0$. Using this, one can do something similar to the case we tackled above to get that, in some sense, $A(x_{l-1})$ and $\frac{1}{0}$ must be consecutive Farey fractions with $A(x_{l-1} \oplus \frac{1}{0}) = A(x_{l-1}) \oplus \frac{1}{0}$, and similarly that, in some sense, $\frac{-1}{0}$ and $A(x_{l+1})$ are consecutive Farey fractions with $A(\frac{-1}{0} \oplus x_{l-1}) = \frac{-1}{0} \oplus A(x_{l-1})$.

This concludes the proof that Φ is well-defined. Before justifying that the map is surjective, let us make an important remark on the first part. Namely, that there was nowhere in the arguments have we used the fact that $x_1, \ldots, x_n$ were all of $\mathcal{C}^j$. We only used that they included $f^{-1}(\infty)$ and $f^{-1}(0)$, as well as all the breakpoints, and that each consecutive pair in the list

$$\tfrac{-1}{0}, x_1, x_2, \ldots, x_n, \tfrac{1}{0}$$

was consecutive in some $\mathcal{C}^j$. However, we did not use the fact that it was the same j in all cases. Indeed, we could possibly have chosen a smaller set of points, as long as consecutive elements in the list above were still consecutive Farey numbers. With this in mind, it is easy to exhibit elements $\tilde{C}$ and $\tilde{D}$ of $\mathrm{PPSL}(2,\mathbb{Z})$ so that $\Phi(\tilde{C}) = C$ and $\Phi(\tilde{D}) = D$. In particular, as C and D generate T, it follows that the map Φ is surjective. Define these elements by $\tilde{C}(x) = \frac{x-1}{x}$ and

$$\tilde{D}(x) = \begin{cases} \frac{x}{x+1}, & \text{for } x \in [\infty, -1] \\ \frac{-1}{x+1}, & \text{for } x \in [-1, 0] \\ x-1, & \text{for } x \in [0,1] \\ \frac{x-1}{x}, & \text{for } x \in [1,\infty] \end{cases}.$$

It follows from the remark above that, since $\tilde{C}$ is fractional linear on each of the intervals $[\infty, 0]$, $[0,1]$ and $[1,\infty]$, the map $\Phi(\tilde{C})$ is linear on each of the intervals $[0, \frac{1}{2})$, $[\frac{1}{2}, \frac{1}{4})$ and $[\frac{1}{4}, 1)$. Moreover, as the intervals $[\infty, 0]$, $[0,1]$ and $[1,\infty]$ are mapped onto the intervals $[1,\infty]$, $[\infty, 0]$ and $[0,1]$, respectively, $\Phi(\tilde{C})$ must map the interval $[0, \frac{1}{2})$, $[\frac{1}{2}, \frac{1}{4})$ and $[\frac{1}{4}, 1)$ onto the interval $[\frac{1}{4}, 1)$, $[0, \frac{1}{2})$ and $[\frac{1}{2}, \frac{1}{4})$, respectively. Hence $\Phi(\tilde{C}) = C$. Similar considerations using the standard dyadic partition $-1, 0, 1$ instead of $0, 1$ show that $\Phi(\tilde{D})$ must equal D. □

Remark 2.4.6. Let Λ denote the subgroup of T generated by C and D^2. As remarked by Fossas [29], the image of $\mathrm{PSL}(2,\mathbb{Z})$ under the isomorphism described in the section is Λ. More precisely, $\mathrm{PSL}(2,\mathbb{Z})$ is isomorphic to the free product $(\mathbb{Z}/3\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z})$ with $\frac{x-1}{x}$ and $\frac{-1}{x}$ being free generators of order 3 and order 2, respectively, and the former is mapped to C and the latter to D^2. From this, it follows that elements of Λ are uniquely determined by their values on two distinct dyadic rational points in $[0, 1)$. This follows from Proposition 2.4.3 and the fact that Q is a bijection between the rational and the dyadic rational numbers.

Note that we also obtain a new interpretation of V. Indeed, $Q^{-1}VQ$ is also a group of piecewise fractional linear maps, but right continuous bijections, rather than homeomorphisms. More precisely, if, in the definition of $\mathrm{PPSL}(2,\mathbb{Z})$, we had only required that the maps were right continuous bijections, then we would have gotten a group which is isomorphic to V via conjugation by Q.

Chapter 3

Inner amenability

The main theme of this chapter is, of course, inner amenability, but we will begin with a brief introduction to amenability for groups and actions, as these concepts are highly related to inner amenability. Afterwards, we will give an introduction to inner amenability and proceed to proving that the Thompson groups T and V are not inner amenable. For a thorough introduction to amenable groups, the reader may wish consult [11, Chapter 4].

All the results listed in Section 3.1 and 3.2 are standard results about amenability, inner amenability and amenable actions. The reader is refered to [9], [11], [3] and [32] for additional proofs.

3.1 Amenable actions

The notion of amenability for groups and actions is a very classical one, and goes back to von Neumann from around 1929, in connection with the Banach-Tarski Paradox. We will begin with a brief introduction to this, before discussing the more general notion of amenable actions.

Recall that, given a σ-algebra Σ on a set $\mathfrak{X}$, a function $\mu\colon \Sigma \to [0,\infty]$ satisfying $\mu(\emptyset) = 0$ is called a measure on $\mathfrak{X}$ if $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, for all families $\{A_n : n \in \mathbb{N}\} \subseteq \Sigma$ of pairwise disjoint sets. This condition is known as σ-additivity, however, in some cases, this is a very restrictive condition, and an important notion in the context of amenability is that of finitely additive measures. This notion makes perfectly good sense for general σ-algebras, but as we will only consider finitely additive measures on spaces equipped with the power set σ-algebra $\mathcal{P}(\mathfrak{X})$, we will restrict the definition to this case.

Definition 3.1.1. Given a set $\mathfrak{X}$, a function $\mu\colon \mathcal{P}(\mathfrak{X}) \to [0,\infty]$ satisfying $\mu(\emptyset) = 0$ is said to be a *finitely additive measure* on $\mathfrak{X}$ if it satisfies $\mu(\cup_{n=1}^{m} A_n) = \sum_{n=1}^{m} \mu(A_n)$, for all finite families $A_1, A_2, \dots, A_m$ of pairwise disjoint subsets of $\mathfrak{X}$.

Definition 3.1.2. A measure μ on a group G is called *left invariant*, respectively, *right invariant* if it satisfies $\mu(gA) = \mu(A)$ and $\mu(Ag) = \mu(A)$, respectively, for all $g \in G$ and $A \subseteq G$. A group G is said to be *amenable* if there exists a left invariant finitely additive probability measure on G.

The restriction to finitely additive measures is a very natural one, as it is easy to show that the only groups with a σ-additive, left invariant probability measure are the finite ones. This is seen easily if the group is countable as it will then be a countable union of singletons, which must all have the same measure by left invariance. This is of course not possible if the group is infinite. A similar argument works for uncountable groups, however, instead of singletons one should choose some countable infinite subgroup and a representative for each right coset. Then the group will be the disjoint union of countably many left translates of this set of representatives, which must then have the same measure.

Remark 3.1.3. It is not difficult to see that there exists a left invariant finitely additive probability measure on a group G if and only if there exists a right invariant one. Indeed, a finitely additive probability measure μ on G is left invariant if and only if the measure ν defined by $\nu(A) = \mu(A^{-1})$ is right invariant and vice versa. In fact, if G is amenable, then G has a finitely additive probability measure which is simultaneously left and right invariant. This is not as obvious, but still quite elementary. If μ_l and μ_r are left and right invariant finitely additive probability measures on G, then the measure μ, defined by

$$\mu(A) = \int_G \mu_l(Ag^{-1}) \, \mathrm{d}\mu_r(g),$$

will be a finitely additive probability measure which is both left and right invariant.

From the definition of amenability one sees straight away that the finite groups are all amenable, as the normalized counting measure will then be a probability measure which is both left and right invariant. Other examples of amenable groups include the abelian groups and the solvable groups, however, this is less obvious. As it is, amenable groups have rather nice stability properties, for example, the following groups are amenable:

(1) subgroups of amenable groups;

(2) quotients of amenable groups;

(3) extensions of amenable groups by amenable groups;

(4) inductive limits of amenable groups.

The above list is a selection of some of the more standard permanence properties, and, using these, one can easily come up with plenty more examples of amenable groups.

When it comes to non-amenable groups, the most natural example is $\mathbb{F}_2$, the free (non-abelian) group on two generators, and, indeed, often non-amenability is established by proving that the group in question contains $\mathbb{F}_2$.

Let us recall the following definition due to Day [21] from 1957.

Definition 3.1.4. Consider the smallest class of groups containing the finite groups and the abelian groups, and is closed under taking subgroups, quotients, extensions, as well as directed unions. Groups in this class are said to be *elementary amenable*.

To use the notation of Day, we will denote the class of elementary amenable groups by EG, the class of amenable groups by AG and the class of groups which do not contain a copy of the free group on two generators by NF. A question Day raised was whether $EG = AG$ or $AG = NF$. This question has been the subject of much attention, in particular whether or not $AG = NF$, which is commonly known as the *von Neumann problem* or the *von Neumann-Day problem*. The problem was solved for linear groups by Tits [64] in 1972, where he proved a result which is now known as the Tits Alternative. A consequence of this result is that linear groups are amenable if and only if they do not contain a copy of the free groups on two generators. In 1980 Chou showed [16] that $EG \neq NF$, so that either $EG \neq AG$ or $AG \neq NF$. It turns out that both $EG \neq AG$ and $AG \neq NF$. The latter was proved by Olshanskii [56] in 1980 and the former by Grigorchuk [33] in 1984. The counterexamples given by Olshanskii and Grigorchuk were not finitely presented, and for some time it was not known whether or not such finitely generated examples existed. However, in 1998 Grigorchuk [34] gave an example of a finitely presented amenable group which is not elementary amenable, and in 2003 Olshanskii and Sapir [58] gave an example of a finitely presented non-amenable group which does not contain a copy of $\mathbb{F}_2$. Recently, more accessible counterexamples for the von Neumann problem have been provided by Monod [53] in the form of groups of piecewise projective homeomorphisms resembling the group $\mathrm{PPSL}(2, \mathbb{Z})$ described in Section 2.4, or, rather, the subgroup of elements which fix ∞. Lodha and Moore [50] afterwards exhibited a finitely presented non-amenable subgroup of one of these groups.

Let us at this point make a comment on the Thompson groups. It is not difficult to show that the Thompson groups T and V are non-amenable as they contain a copy of $\mathbb{F}_2$, for example, because T contains a copy of $\mathrm{PSL}(2, \mathbb{Z})$, which is known to contain $\mathbb{F}_2$. However, it is a famous open problem whether the Thompson group F is amenable or not. Brin and Squire [8] proved in 1985 that F does not contain a copy of $\mathbb{F}_2$, and Cannon, Floyd and Parry [10] proved in 1996 that F is not elementary amenable. Thus F is either another counterexample to the von Neumann problem or another example of an amenable group which is not elementary amenable.

Let us move on to amenability for group actions. Recall that an action of a group G on a space $\mathfrak{X}$ is a group homomorphism α from G to the group of permutations of $\mathfrak{X}$.

Definition 3.1.5. An action α of a group G on a set $\mathfrak{X}$ is said to be *amenable* if there exists a finitely additive probability measure μ on $\mathfrak{X}$ which is invariant under the action, that is, $\mu(\alpha(g)(A)) = \mu(A)$, for all $g \in G$ and $A \subseteq G$.

Comparing the definitions, one sees right away that amenability of a group is nothing more than amenability of the action of the group on itself given by left translation. Moreover, it is also easy to see that actions of amenable groups are always amenable. Indeed, if μ is a left invariant finitely additive measure on a group G, and α is an actions of G on a set $\mathfrak{X}$, then, for any $x \in \mathfrak{X}$, the measure ν given by

$$\nu(A) = \mu\big(\{g \in G : \alpha(g)x \in A\}\big),$$

for all $A \subseteq \mathfrak{X}$, is a finitely additive measure on $\mathfrak{X}$ which is invariant under the action.

Given an action α of a group G on a set $\mathfrak{X}$, we get an induced action by linear maps on the space of complex valued functions on $\mathfrak{X}$. This action, which we will still denote by α, is given by $(\alpha(g)f)(x) = f(\alpha(g^{-1})x)$, for all functions $f\colon \mathfrak{X} \to \mathbb{C}$, all $g \in G$ and all $x \in \mathfrak{X}$. An important fact, which we will use implicitly throughout the thesis, is that the functions spaces $\ell^1(\mathfrak{X})$, $\ell^2(\mathfrak{X})$ and $\ell^\infty(\mathfrak{X})$ are invariant so that $\alpha(g)$ can be though of as a linear operator on each of these spaces. It is easy to see that it is isometric on each of these spaces. In particular, $\alpha(g)$ acts on $\ell^2(\mathfrak{X})$ as a unitary operator. Thus it is a representation of the group on $\ell^2(\mathfrak{X})$. In case of left translation action, this representation is, of course, the left regular representations λ.

There is an extraordinary number of characterizations of amenability, and many of these carry straight over to amenable actions. The following theorem lists some characterizations of amenable actions which, in the special case of left translations, are some of the more standard characterizations of amenability for groups. The proof is a straightforward adaptation of the proof in the case where the action is left translation action, however, a proof for general actions can be found in [42, Lemma 2.1].

Theorem 3.1.6. *Let α be an action of a group G on a set $\mathfrak{X}$. Then following are equivalent:*

(i) The action is amenable.

(ii) There exists a net $(\eta_i)_{i\in I}$ of unit vectors in $\ell^1(\mathfrak{X})$ such that, for all $g \in G$, $\lim_{i\in I} \|\alpha(g)\eta_i - \eta_i\|_1 = 0$.

(iii) The trivial representation is weakly contained in the representation on $\ell^2(\mathfrak{X})$ induced by α, that is, there exists a net $(\xi_i)_{i\in I}$ of unit vectors in $\ell^2(\mathfrak{X})$ such that $\lim_{i\in I} \|\alpha(g)\xi_i - \xi_i\|_2 = 0$, for all $g \in G$.

(iv) For every $\varepsilon > 0$ and every finite subset $K \subseteq G$, there exists a finite subset $F \subseteq \mathfrak{X}$ so that $|\alpha(g)F \Delta F| < \varepsilon|F|$, for every $g \in K$.

(v) There exists an invariant mean on $\ell^\infty(\mathfrak{X})$, that is, a state ϕ on $\ell^\infty(\mathfrak{X})$ such that $\phi(\alpha(g)f) = \phi(f)$, for all $f \in \ell^\infty(\mathfrak{X})$ and $g \in G$.

Condition (iv) above is called *Følner's condition*. Note that if G is an amenable group, then it follows easily from Følner's condition that there exists a net $(F_i)_{i\in I}$ of finite subsets of G such that $|gF_i\Delta F_i||F_i|^{-1}$ goes to zero, for all $g \in G$. Such a net is usually called a *Følner net*, and in the case the group is countable, the net can be replaced by a sequence, that is, a *Følner sequence*.

A few of the characterizations of amenability in the above theorem are operator algebraic in nature, and, indeed, there are many characterizations which have to do with operator algebra. The following theorem characterizes amenability in terms of certain operator norms. As a proof of this theorem is not as readily available in the literature, we have included a proof.

Theorem 3.1.7. *Suppose that G is a group acting on a set $\mathfrak{X}$, and let π denote the corresponding representation on $\ell^2(\mathfrak{X})$. Then the action of G on $\mathfrak{X}$ is non-amenable if and only if there exist elements $g_1, g_2, \dots, g_n$ of G, such that*

$$\Big\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\Big\| < 1.$$

Moreover, if G is finitely generated, then the action is non-amenable if and only if $\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\| < 1$, for any set of elements $g_1, g_2, \dots, g_n$ generating G.

Proof. We start by proving the first statement. Suppose that the action is amenable and let $g_1, \dots, g_n \in G$. By (iii) of Theorem 3.1.6, there exists a net $(\xi_i)_{i\in I}$ of unit vectors in $\ell^2(\mathfrak{X})$ so that $\lim_{i\in I}\|\pi(g)\xi_i - \xi_i\| = 0$. As

$$\Big\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\Big\| \geq \Big\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\xi_i\Big\|_2 \geq 1 - \Big\|\frac{1}{n}\sum_{k=1}^{n}(\pi(g_k)\xi_i - \xi_i)\Big\|_2$$

and $\lim_{i\in I}\|\frac{1}{n}\sum_{k=1}^{n}(\pi(g_k)\xi_i - \xi_i)\|_2 = 0$, we conclude that $\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\| = 1$.

Suppose instead that $\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\| = 1$, for all $g_1, \dots, g_n \in G$, and let us prove that the action is amenable by proving that (iii) of Theorem 3.1.6 holds. Let $h_1, \dots, h_n \in G$ and let $\varepsilon > 0$. Our goal is to find a unit vector $\xi \in \ell^2(\mathfrak{X})$ so that $\|\pi(h_k)\xi - \xi\|_2 < \varepsilon$, for all $k = 1, 2, \dots, n$. We may assume that $h_1 = e$. By assumption $\sum_{k=1}^{n}\pi(h_k)$ is a bounded operator of norm n, so we may choose a unit vector ξ with $\|\sum_{k=1}^{n}\pi(h_k)\xi\|^2 > n^2 - \frac{1}{2}\varepsilon^2$. Noting that $h_1^{-1}h_k = h_k$, we then see that

$$\sum_{k=1}^{n}\big\|\pi(h_k)\xi - \xi\big\|_2^2 \leq \sum_{i,j=1}^{n}\big\|\pi(h_i^{-1}h_j)\xi - \xi\big\|_2^2 = 2n^2 - 2\Big\|\sum_{k=1}^{n}\pi(h_k)\xi\Big\|_2^2 < \varepsilon^2.$$

In particular $\|\pi(h_k)\xi - \xi\|_2 < \varepsilon$, for all $k = 1, 2, \dots, n$, which means that we have found the desired ξ, and have thus proved amenability of the action.

For the last statement it suffices to prove that, if there exists a generating set $g_1, g_2, \dots, g_n$ in G with $\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)\| = 1$, then the action is amenable. We are going to do this in the same manner as above, that is, give some finite subset $F \subseteq G$ and $\varepsilon > 0$, we need to find a unit vector ξ such that $\|\pi(h)\xi - \xi\| < \varepsilon$ for all $h \in F$.

Choose some $N \in \mathbb{N}$ such that each $h \in F$ can be written as a product of at most N of the generators $g_1, g_2, \ldots, g_n$ and their inverses. By repeating the argument from the first part of the proof, we get that there exists a unit vector ξ such that $\|\pi(g_k)\xi - \xi\|_2 < \frac{1}{N}\varepsilon$, for all $k = 1, 2, \ldots, n$. Clearly we then also get that $\|\pi(g_k^{-1})\xi - \xi\|_2 < \frac{1}{N}\varepsilon$, for each $k = 1, 2, \ldots, n$. Now, given $h \in F$, we can write $h = h_1 h_2 \cdots h_m$ where $m \leq N$, and each h_k is one of the generators or its inverse. Using the triangle inequality and the fact that $\pi(h_k)$ is unitary, for each $k = 1, 2, \ldots, m$, we get that

$$\|\pi(h)\xi - \xi\| \leq \sum_{k=1}^{m} \|\pi(h_k)\xi - \xi\| < m\frac{1}{N}\varepsilon \leq \varepsilon.$$

This was exactly what we needed to prove, and we conclude that the action is amenable. □

Before continuing, recall the definition of weak containment of representations.

Definition 3.1.8. Let G be a group and let $\pi\colon G \to B(\mathcal{H})$ and $\sigma\colon G \to B(\mathcal{K})$ be representations of G on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then π is said to be *weakly contained* in σ if, for every $\xi \in \mathcal{H}$, $\varepsilon > 0$ and $F \subseteq G$ finite, there exist $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$\Big|\langle \pi(g)\xi \mid \xi\rangle - \sum_{k=1}^{n} \langle \sigma(g)\eta_k \mid \eta_k\rangle\Big| < \varepsilon, \tag{3.1}$$

for all $g \in F$ and $k = 1, \ldots, n$.

Recall that, if π is a representation of a group G on a Hilbert space $\mathcal{H}$, then a vector $\xi \in \mathcal{H}$ is called *cyclic* if $\{\pi(g)\xi : g \in G\}$ spans a dense subspace of $\mathcal{H}$. A straightforward approximation argument, which can be found in [24, Proposition 18.1.4] shows that, with the setup of the above definition, π is weakly contained in σ as long as the requirements are satisfied for a single ξ, as long as this ξ is a cyclic vector for π. More precisely, if ξ is cyclic for π, then π is weakly contained in σ if, for for all $\varepsilon > 0$ and $F \subseteq G$ finite, there exist $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that (3.1) is satisfied for this specific ξ.

The following theorem gives a characterization of weak containment in terms of maps between C^*-algebras. A proof of this result can be found in [24, Proposition 18.1.4].

Theorem 3.1.9. *Suppose that π and σ are representations of a group G. Then π is weakly contained in σ if and only if $\|\pi(x)\| \leq \|\sigma(x)\|$, for all $x \in C^*(G)$, that is, if and only if there exists a $*$-homomorphism $\kappa\colon C^*_\sigma(G) \to C^*_\pi(G)$ such that $\kappa\sigma = \pi$.*

Before we state the following result, which is a well-known result about weak containment, recall the following terminology about group actions. If α is an action of a group G on a set $\mathfrak{X}$, then, given $x \in \mathfrak{X}$, the subgroup $\{g \in G : \alpha(g)x = x\}$ is called the *stabilizer* of x. Also, the action α is said to be *transitive* if, for every $x, y \in \mathfrak{X}$, there exists $g \in G$ with $\alpha(g)x = y$.

Proposition 3.1.10. *Let G be a group acting on a set $\mathfrak{X}$ and let π denote the corresponding representation on $\ell^2(\mathfrak{X})$. If the stabilizers of the action are amenable, then π is weakly contained in the left regular representation of G.*

Proof. Suppose firrst that the actions is transitive, and let $x \in \mathfrak{X}$. As the action is transitive, the vector δ_x is cyclic for π. As noted above, this means that it suffices to prove that the function $g \mapsto \langle \pi(g)\delta_x \mid \delta_x\rangle$ is the limit of sums of functions of the form $g \mapsto \langle \lambda(g)\xi \mid \xi\rangle$ with $\xi \in \ell^2(G)$.

Denote the stabilizer of x by H, and note that the function $g \mapsto \langle \pi(g)\delta_x \mid \delta_x\rangle$ is actually the indicator function of H. As H is amenable, we know from Theorem 3.1.6 that we may choose a Følner net $(F_i)_{i\in I}$ for H. Now, letting $\xi_i = |F_i|^{-1/2}\mathbf{1}_{F_i}$ where $\mathbf{1}_{F_i}$ denotes the indicator function of the set F_i, it is straightforward to check that $\lim_{i\in I}\langle \lambda(g)\xi_i \mid \xi_i\rangle = 1$, for all $g \in H$, as $(F_i)_{i\in I}$ is a Følner net, and that $\lim_{i\in I}\langle \lambda(g)\xi_i \mid \xi_i\rangle = 0$, for all $g \in G \setminus H$. Thus π is weakly contained in λ.

Now, let us drop the assumption that the action is transitive. Let $\mathfrak{X} = \bigsqcup_{j\in J}\mathfrak{X}_j$ be a decomposition of $\mathfrak{X}$ into orbits. Then $\ell^2(\mathfrak{X}) = \bigoplus_{j\in J}\ell^2(\mathfrak{X}_j)$ is a decomposition of $\ell^2(\mathfrak{X})$ into π-invariant subspaces. Moreover, as G acts transtitively on each of the sets $\mathfrak{X}_j$, it is straightforwards to check, using the first part of the proof, that π is weakly contained in the left regular representation. $\square$

Now, let us mention the following theorem about amenable actions, due to Rosenblatt [62] from 1981. It gives a criterion for non-amenability of an action. We include a proof, different from Rosenblatt's original one.

Theorem 3.1.11. *Let G be a non-amenable discrete group acting on a set $\mathfrak{X}$. If the stabilizer of each point is amenable, then the action is itself non-amenable.*

Proof. Let π denote the representation of G on $\ell^2(\mathfrak{X})$ corresponding to the action. We aim to prove that the action does not satisfy Condition (iii) of Theorem 3.1.6. As G is non-amenable, Theorem 3.1.7 tells us that there exist elements $g_1, g_2, \ldots, g_n$ in G so that $\|\frac{1}{n}\sum_{k=1}^{n}\lambda(g_k)\| < 1$. Let us denote $\|\frac{1}{n}\sum_{k=1}^{n}\lambda(g_k)\|$ by c. Let $(\xi_i)_{i\in I}$ be a net of unit vectors in $\ell^2(\mathfrak{X})$. Our goal is to prove that

$$\frac{1}{n}\sum_{k=1}^{n}\|\pi(g_k)\xi_i - \xi_i\|_2 \geq 1 - c > 0,$$

for all $i \in I$. Clearly this implies that we cannot have $\lim_{i\in I}\|\pi(g_k)\xi_i - \xi_i\|_2 = 0$, for all $k \in \{1, 2, \ldots, n\}$, which means that Condition (iii) of Theorem 3.1.6 is not satisfied, and that the action is therefore not amenable.

From Proposition 3.1.10 we know that π is weakly contained in the left regular representation of G, so, by Theorem 3.1.9, we know that $\|\frac{1}{n}\sum_{k=1}^n \pi(g_k)\| \leq c$. As

$$\frac{1}{n}\sum_{k=1}^n \|\pi(g_k)\xi_i - \xi_i\|_2 \geq \frac{1}{n}\sum_{k=1}^n \big(\|\xi_i\|_2 - \|\pi(g_k)\xi_i\|_2\big) \geq 1 - \Big\|\frac{1}{n}\sum_{k=1}^n \pi(g_k)\xi_i\Big\|_2,$$

for all $i \in I$, we conclude that $\frac{1}{n}\sum_{k=1}^n \|\pi(g_k)\xi_i - \xi_i\|_2 \geq 1 - c$. We conclude that the action is not amenable. □

Let us end this section by discussing amenability of the natural action of the Thompson groups on the interval $[0,1)$. The first thing one might notice is that the actions of F on the interval is amenable, but for trivial reasons. All the elements of F fix the point 0, so the Dirac measure at the point will be a finitely additive probability measure on $[0,1)$ which is invariant under the action. The actions of the Thompson groups T and V on the interval, however, are non-amenable. This is not obvious, but it follows easily from Rosenblatt's result above, as well as Thurston's characterization of T as piecewise projective linear homeomorphisms. In order to do this, we start by proving the following proposition.

Proposition 3.1.12. *The subgroup of fractional linear transformations in* $\mathrm{PSL}(2,\mathbb{R})$ *which fix a given element* $x \in \hat{\mathbb{R}}$ *is amenable.*

Proof. As we noted in Section 2.4, the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\hat{\mathbb{R}}$ is transitive, so that all the stabilizer subgroups are conjugate, and, in particular, isomorphic. Thus, it suffice to prove that the stabilizer of the point ∞ is amenable. Let H denote the stabilizer subgroup of ∞. We claim that H is isomorphic to the semidirect product $\mathbb{R} \rtimes_\alpha \mathbb{R}_+$, where the action α of $\mathbb{R}_+$ on $\mathbb{R}$ is given by $\alpha(s)(t) = s^2 t$, for all $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$. As this semidirect product is the extension of an amenable group by another amenable group, it is amenable. Now, the elements of H are exactly the fractional linear transformations of the form

$$\begin{bmatrix} s & t \\ 0 & s^{-1} \end{bmatrix}$$

for some $s, t \in \mathbb{R}$ with $s \neq 0$, and it is straightforward to check that the map from $\mathbb{R} \rtimes_\alpha \mathbb{R}_+$ which sends (s,t) to the above fractional linear transformation above is an isomorphism from $\mathbb{R} \rtimes_\alpha \mathbb{R}_+$ to H. □

Proposition 3.1.13. *The actions of T and V on* $[0,1)$ *are non-amenable.*

Proof. As $T \subseteq V$, it suffices to prove that the action of T on $[0,1)$ is non-amenable, as a finitely additive V-invariant measure on $[0,1)$ would then also be T-invariant. By Theorem 2.4.5, it suffices to prove that the action of $\mathrm{PPSL}(2,\mathbb{Z})$ on $\hat{\mathbb{R}}$ is non-amenable, and as $\mathrm{PSL}(2,\mathbb{Z})$ is a subgroup of $\mathrm{PPSL}(2,\mathbb{Z})$, it suffices to prove that the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\hat{\mathbb{R}}$ is non-amenable. By Theorem 3.1.11, it suffices to prove that the stabilizers of the action are amenable as $\mathrm{PSL}(2,\mathbb{Z})$ itself is non-amenable. This follows directly from Proposition 3.1.12. $\square$

We mentioned earlier that the groups T and V are not amenable. As all actions of amenable groups are amenable, this is also a corollary of the above proposition.

Remark 3.1.14. Let V_0 denote the subgroup of V consisting of elements g satisfying $g(x) = x$, for all $x \in [0, \frac{1}{2})$. It is easy to see from the above proposition that the action of this group on the interval $[\frac{1}{2}, 1)$ is also non-amenable. Indeed, if we define a function $f\colon [\frac{1}{2}, 1) \to [0, 1)$ by $f(x) = 2x - 1$, then it follows directly from the fact that the map $g \mapsto fgf^{-1}$ from V_0 to V is a group isomorphism.

Similarly, we could consider the subgroup of V consisting of elements that fix every point in $[\frac{1}{2}, 1)$, instead of $[0, \frac{1}{2})$, and we then get the conclusion that the action of this subgroup on the interval $[0, \frac{1}{2})$ is non-amenable, this time using the function $x \mapsto 2x$.

3.2 Introduction to inner amenability and property Γ

Now that we have gone through the basics on amenable actions, the definition of inner amenability is straightforward. However, as the original motivations for introducing inner amenability was to study property Γ for II_1-factors, we will start by defining this property, or at least, define it for group von Neumann algebras.

Recall that a von Neumann algebra is a C^*-subalgebra of the bounded operators on some Hilbert space $\mathcal{H}$ which is closed in the topology of pointwise convergence and contains the identity operator $\mathbf{1}$ on that Hilbert space. As von Neumann algebras are unital C^*-algebras, they always have a non-empty center, that is, they contain non-zero operators which commute with all other operators, namely $\mathbb{C}\mathbf{1}$. Von Neumann algebras where $\mathbb{C}\mathbf{1}$ is all of the center are called *factors*. It is an easy exercise to prove that the group von Neumann algebra $\mathrm{L}(G)$ of a group G is a factor if and only if G is ICC. The group von Neumann algebra is always a finite von Neumann algebra as it has a faithful tracial state, which we denote by τ, and is given by $\tau(x) = \langle x\delta_e \mid \delta_e\rangle$. If the group is ICC, then the group von Neumann algebra is what is called a II_1-factor. We will denote the trace norm on $\mathrm{L}(G)$ by $\|\cdot\|_\tau$, that is, $\|x\|_\tau = \tau(x^*x)^{1/2}$.

Definition 3.2.1. The group von Neumann algebra $\mathrm{L}(G)$ of an ICC group G is said to have *property* Γ if there exists a net $(u_i)_{i\in I}$ of unitaries in $\mathrm{L}(G)$ with $\tau(u_i) = 0$ and $\lim_{i\in I} \|u_i x - x u_i\|_\tau = 0$, for all $x \in \mathrm{L}(G)$.

Property Γ was introduced by Murray and von Neumann [54] in 1943 to provide the first examples of non-isomorphic II_1-factors. They proved that all approximately finite II_1-factors are isomorphic and have property Γ, whereafter they proved that the group von Neumann algebra $\mathrm{L}(\mathbb{F}_2)$ of the free group on two generators does not have property Γ.

Definition 3.2.2. A group G is said to be *inner amenable* if the action of G on $G\setminus\{e\}$ by conjugation is an amenable action, in the sense of Definition 3.1.5.

Inner amenability was introduced by Effros [26] in 1975 in an attempt to characterize property Γ for group von Neumann algebras in terms of a purely group theoretic property. Effros proved that ICC groups are inner amenable if their group von Neumann algebra has property Γ, and he conjectured that the converse was also true. For many years if was an open problem, and it was only recently, in 2012, that Vaes [65] settled this in the negative. Vaes gave an explicit construction of an inner amenable ICC group whose group von Neumann algebra does not have property Γ. We have included a proof of Effros result, as it clearly illustrates the resemblance of inner amenability and property Γ. Note that the representation of G on $\ell^2(G)$ induced by the conjugation action is the product of the left and the right regular representations, that is, the representation $g \mapsto \lambda(g)\rho(g)$.

Theorem 3.2.3. *If G is an ICC group whose group von Neumann algebra has property Γ, then G is inner amenable.*

Proof. Suppose that $\mathrm{L}(G)$ has property Γ and let $(u_i)_{i\in I}$ be a net of unitaries in $\mathrm{L}(G)$ with $\tau(u_i) = 0$ and $\lim_{i\in I} \|u_i x - x u_i\|_\tau = 0$, for all $x \in \mathrm{L}(G)$. Put $\xi_i = u_i\delta_e$. As $\xi_i(e) = \tau(u_i) = 0$, we see that $(\xi_i)_{i\in I}$ is actually a net in $\ell^2(G\setminus\{e\})$, and so we want to prove that the action of G on $G \setminus \{e\}$ is amenable by proving that $(\xi_i)_{i\in I}$ satisfies the condition (iii) of Theorem 3.1.6. First of all, recalling that $\|x\|_\tau = \|x\delta_e\|_2$, for all $x \in \mathrm{L}(G)$, we notice that $\|\xi_i\|_2 = \|u_i\|_\tau = 1$, so that it is actually a net of unit vectors. Thus we only need to prove that $\lim_{i\in I} \|\lambda(g)\rho(g)\xi_i - \xi_i\|_2 = 0$. If we, furthermore, recall the fact that $\lambda(g)x\lambda(g)^*\delta_e = \lambda(g)\rho(g)x\delta_e$, for all $x \in \mathrm{L}(G)$ and all $g \in G$, we see that

$$\|\lambda(g)\rho(g)\xi_i - \xi_i\|_2 = \|\lambda(g)\rho(g)u_i\delta_e - u_i\delta_e\|_2 = \|\lambda(g)u_i\lambda(g)^*\delta_e - u_i\delta_e\|_2.$$

Using the fact that $\|vx\|_\tau = \|xv\|_\tau = \|x\|_\tau$, whenever $x, v \in \mathrm{L}(G)$ with v unitary, as well as the formula for the trace norm in terms of the norm on $\ell^2(G)$, we see that

$$\|\lambda(g)\rho(g)\xi_i - \xi_i\|_2 = \|\lambda(g)u_i\lambda(g)^* - u_i\|_\tau = \|\lambda(g)u_i - u_i\lambda(g)\|_\tau.$$

By the choice of $(u_i)_{i\in I}$ the right hand side tends to zero, and so we conclude that $\lim_{i\in I} \|\lambda(g)\rho(g)\xi_i - \xi_i\|_2 = 0$, which by Theorem 3.1.6 means that the action of G on $G \setminus \{e\}$ is amenable. Hence G is inner amenable. □

In the proof above it seems as though having the net of unitaries is almost the same as as having the net of unit vectors. Indeed, if G is an inner amenable group, then, because $\mathrm{L}(G)\delta_e$ is dense in $\ell^2(G)$, we could choose a net $(y_i)_{i\in I}$ of operators in $\mathrm{L}(G)$ with trace zero, such that $\lim_{i\in I} \|\lambda(g)\rho(g)y_i\delta_e - y_i\delta_e\|_2 = 0$. As in the proof above, this means that $\lim_{i\in I} \|\lambda(g)y_i - y_i\lambda(g)\|_\tau = 0$. Thus, it seems as though we are almost at a point where we can conclude property Γ. There are two problems, namely that the y_i's are not unitaries and that we need to replace $\lambda(g)$ by a general $x \in \mathrm{L}(G)$. These problems, however, cannot be overcome as we know from Vaes [65] that the converse of Effros theorem fail.

There are some obvious examples of inner amenable groups, namely, non-trivial non-ICC groups. Indeed, the normalized counting measure on a finite conjugacy class in G, different from $\{e\}$, will define a finitely additive probability measure on $G \setminus \{e\}$ which is conjugation invariant, so non-ICC groups are inner amenable. This also explains why it is the conjugacy action on $G \setminus \{e\}$, rather than on G, which is supposed to be amenable, as the Dirac measure at e is always conjugation invariant. Other obvious examples of inner amenable groups include the non-trivial amenable groups. We already know that finite groups are inner amenable as they are non-ICC, and, if G is an infinite amenable group, then by Remark 3.1.3 we know that G has a finitely additive probability measure which is both left and right invariant, so, in particular, it is conjugation invariant. As the group is infinite, all atoms have measure zero, so by restricting it to $G \setminus \{e\}$, we get a finitely additive probability measure which is conjugation invariant. Inner amenability has some atypical permanence properties. It is not passed onto subgroups and quotients and, however, directed unions of inner amenable groups are again inner amenable and every group containing an inner amenable, normal subgroup is again inner amenable.

There are no obvious examples of non-inner amenable groups, though there are some easy ones. Effros [26] gave an elementary proof that the free group on two generators $\mathbb{F}_2$ is not inner amenable. Combining this with the comments above, this means that the group $\mathbb{F}_2 \times \mathbb{Z}$ is inner amenable. Thus, inner amenability does not imply amenability. Another way to produce examples of non-inner amenable groups is the theorem of Rosenblatt, that is, Theorem 3.1.11. We have included a slightly modified version here for reference.

Proposition 3.2.4. *If G has a non-amenable subgroup H such that the centralizer subgroup $\{h \in H : hgh^{-1} = g\}$ is amenable, for each $g \in G \setminus \{e\}$, then G is not inner amenable.*

Proof. It follows directly from the assumptions and Theorem 3.1.11 that the conjugation action of H on $G \setminus \{e\}$ is non-amenable. In particular, the conjugation action of G on $G \setminus \{e\}$ is non-amenable as well, that is, G is not inner amenable. □

Let us also include a rephrasing of Theorem 3.1.7 for conjugacy actions, as we will need it later.

Proposition 3.2.5. *Let G be a non-trivial group. Then G is non-inner amenable if and only if there exist elements $g_1, g_2, \ldots, g_n$ of G and a positive real number $c < 1$, such that*

$$\Big\| \frac{1}{n} \sum_{k=1}^{n} \lambda(g_k) x \lambda(g_k)^* \Big\|_\tau \leq c \|x\|_\tau,$$

for all $x \in C^*_\lambda(G)$ with trace zero.

Proof. By Theorem 3.1.7 we know that G is non-inner amenable if and only if we can find elements $g_1, g_2, \dots, g_n$ of G and a positive real number $c < 1$ such that the restriction of $\frac{1}{n}\sum_{k=1}^{n}\lambda(g_k)\rho(g_k)$ to $\ell^2(G \setminus \{e\})$ has norm c. Phrased differently, $\|\frac{1}{n}\sum_{k=1}^{n}\lambda(g_k)\rho(g_k)\xi\|_2 \le c\|\xi\|$, for all $\xi \in \ell^2(G)$ with $\xi(e) = 0$. Now, that this is equivalent to the statement of the proposition follows from the facts that $\{x\delta_e : x \in \mathrm{L}(G), \tau(x) = 0\}$ is dense in $\ell^2(G \setminus \{e\})$, that $\rho(g)y\delta_e = y\lambda(g)^*\delta_e$, for all $y \in \mathrm{L}(G)$ and $g \in G$, and that $\|y\|_\tau = \|y\delta_e\|_2$, for all $y \in \mathrm{L}(G)$. $\square$

Let us end this section by recalling the definition of a group von Neumann algebra being a McDuff factor. For more on McDuff factors, see [52].

Definition 3.2.6. Let G be a countable ICC group. The group von Neumann algebra $\mathrm{L}(G)$ is said to be a *McDuff factor* if there exist sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of trace zero unitaries in $\mathrm{L}(G)$ satisfying, moreover, $\lim_{n\to\infty}\|u_nx - xu_n\|_\tau = \lim_{n\to\infty}\|v_nx - xv_n\|_\tau = 0$, for all $x \in \mathrm{L}(G)$, whereas $\|u_nv_n - v_nu_n\|_\tau$ does not tend to zero as $n \to \infty$.

It is clear that a McDuff factor has Property Γ. Hence, by Effros' result, a countable ICC group is inner amenable if its group von Neuman algebra is a McDuff factor.

3.3 The Thompson groups and inner amenability

The purpose of this section is to prove that the Thompson groups T and V are not inner amenable. The proof relies heavily on the interpretation of T as piecewise fractional linear transformations of $\hat{\mathbb{R}}$ from in Section 2.4. Recall that Λ denotes the subgroup of T generated by the elements C and D^2, and that it is the image of $\mathrm{PSL}(2,\mathbb{Z})$ via the isomorphism from $\mathrm{PPSL}(2,\mathbb{Z})$ to T, as explained in Remark 2.4.6.

That the Thompson group F is inner amenable was proved by Jolissaint [40] in 1997. A year after, he strengthened his result by proving that the group von Neumann algebra of F is a McDuff factor, see [41]. A few years later, Ceccherini-Silberstein and Scarabotti [12] gave a different, more elementary proof that F is inner amenable.

At a conference in 2013, Chifan raised the question whether the Thompson groups T and V are non-inner amenable. Together with Haagerup we were able to settle this.

Theorem 3.3.1. *The Thompson groups T and V are not inner amenable.*

Proof. Let us start by proving that T is not inner amenable. By Theorem 2.4.5 this is the same as proving that $\mathrm{PPSL}(2,\mathbb{Z})$ is not inner amenable. We will prove that the subgroup $\{g \in \mathrm{PSL}(2,\mathbb{Z}) : gf = fg\}$ is amenable, for all $f \in \mathrm{PPSL}(2,\mathbb{Z})\setminus\{e\}$. By Proposition 3.2.4, this shows that $\mathrm{PPSL}(2,\mathbb{Z})$ is non-inner amenable. Fix $f \neq e$, and let us prove that the associated above subgroup, which we denote by H, is amenable.

First suppose that $f \in \mathrm{PSL}(2,\mathbb{Z})$. Then H is the centralizer of f in $\mathrm{PSL}(2,\mathbb{Z})$, which is known to be cyclic; see for example Theorems 2.3.3 and 2.3.5 in [44]. In particular, H is amenable.

Suppose instead that $f \notin \mathrm{PSL}(2,\mathbb{Z})$. This means that f has a breakpoint, in the sense of Definition 2.4.4. However, it is not difficult to see that f must have at least two of them. Let $x_1,\ldots,x_n$ denote the breakpoints of f. Fix some $g \in \mathrm{PSL}(2,\mathbb{Z})$ with $gf = fg$. As g is a fractional linear transformation of $\hat{\mathbb{R}}$, it is easy to see that the breakpoints of the composition $g^{-1}f$ are still the points $x_1,\ldots,x_n$. For the same reason, the breakpoints of the composition fg^{-1} are $g(x_1),\ldots,g(x_n)$. As g commutes with f, we have $fg^{-1} = g^{-1}f$. In particular, the points $x_1,\ldots,x_n$ and the points $g(x_1),\ldots,g(x_n)$ are the same up to a permutation, that is, there is a permutation σ such that $g(x_k) = x_{\sigma(k)}$, for $k = 1,\ldots,n$. As we already noted that $n \geq 2$, we get from Proposition 2.4.3 that g is uniquely determined by the permutation of $x_1,\ldots,x_n$. Note that we here used that all the points $x_1,\ldots,x_n$ are in $\hat{\mathbb{Q}}$, by definition of $\mathrm{PPSL}(2,\mathbb{Z})$. As there are exactly $n!$ permutations of these points, we conclude that H is finite with at most $n!$ elements. In particular, H is amenable. This proves that H is amenable in all cases, and so T is not inner amenable.

Let us explain why V is not inner amenable, either. An easy way to see that would be to continue the argument above on the $\mathrm{PPSL}(2,\mathbb{Z})$ analogue of V, mentioned in Remark 2.4.6. We did not formally explain what the elements of this group look like, but we can actually, without problems, continue our strategy without this knowledge. Let us explain this more precisely. We know from Theorem 2.4.5 that if we conjugate the elements of T with Q, then we get the group $\mathrm{PPSL}(2,\mathbb{Z})$. As $T \subseteq V$, this means that if we conjugate the elements of V, that is, consider the group $Q^{-1}VQ$, then we obtain some group of bijections of $\hat{\mathbb{R}}$ which contains $\mathrm{PPSL}(2,\mathbb{Z})$. So if we would still use Proposition 3.2.4, then we need to argue that, if $f \in Q^{-1}VQ$ but $f \notin \mathrm{PPSL}(2,\mathbb{Z})$, then the group H is still amenable. As QfQ^{-1} is an element in V which is not in T, it must have a finite number of discontinuities, but at least two. However, as Q is a homeomorphism, this means that f must also have a finite number of points of discontinuity, and again at least two. As the elements of $\mathrm{PSL}(2,\mathbb{Z})$ are homeomorphisms, an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ with $fg = gf$ would permute these points of discontinuity, as in the first part. By definition of V, the discontinuities of $Q^{-1}fQ$ are all dyadic rational numbers, and, therefore, discontinuities of f are in $\hat{\mathbb{Q}}$. Thus, as in the first part, the element g is uniquely determined by how it permutes these finitely many points of discontinuity. Again we conclude that H is finite, and again, by Proposition 3.2.4, we conclude that V is not inner amenable. □

In combination with Theorem 3.2.3 we immediately get the following corollary.

Corollary 3.3.2. *Neither* $\mathrm{L}(T)$*, nor* $\mathrm{L}(V)$*, has property* Γ*. In particular, they are not McDuff factors, either.*

Chapter 4

Operator algebras and the Thompson groups

In this chapter, we begin by introducing a representation of the Thompson groups in the unitary group of the Cuntz algebra $\mathcal{O}_2$. Afterwards, we prove a statement connecting amenability of the Thompson group F with C^*-simplicity of the Thompson group T, as well as a statement about ideals in the reduced group C^*-algebras of F and T. Last, we introduce the rapid decay property of Jolissaint, and prove a criterion for C^*-simplicity of groups.

4.1 A representation in the Cuntz algebra

Let us start this section by introducing the Cuntz algebras, or, rather, one of them. The *Cuntz algebra* $\mathcal{O}_2$ is the universal C^*-algebra generated by two isometries s_0 and s_1 with orthogonal range projections summing up to the identity. In other words, it is the universal C^*-algebra generated by two elements, s_0 and s_1, subject to the relations

$$s_0^* s_0 = \mathbf{1}, \qquad s_1^* s_1 = \mathbf{1} \qquad \text{and} \qquad s_0 s_0^* + s_1 s_1^* = \mathbf{1}.$$

As it is a universal C^*-algebra given by generators and relations, it is, a priori, a rather abstract object. However, the Cuntz algebra $\mathcal{O}_2$ is known to be a simple C^*-algebra, as shown by Cuntz [19], which means that whenever two elements in a C^*-algebra satisfy the above relations, they will automatically generate a copy of $\mathcal{O}_2$. Thus we may easily find a concrete realization of the Cuntz algebra, and we do so below. For an introduction to universal C^*-algebras, the reader may consult [51, Chapter 3].

Let us list a few facts about the isometries s_0 and s_1. For this purpose, we will, given $\mu = (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$, denote the element $s_{i_1} s_{i_2} \cdots s_{i_n}$ by s_μ, and also let $s_\emptyset$ denote the identity operator $\mathbf{1}$, corresponding to the case $n = 0$.

(1) The elements $s_0s_0^*$ and $s_1s_1^*$ are both projections; indeed, they are the range projections of s_0 and s_1, respectively.

(2) the projections $s_0s_0^*$ and $s_1s_1^*$ are orthogonal, so that $s_0^*s_1 = 0$ and $s_1^*s_0 = 0$.

(3) Any product of s_0, s_1, s_0^* and s_1^* can be written in the form $s_\mu s_\nu^*$, for some $\mu \in \{0,1\}^n$ and $\nu \in \{0,1\}^m$. In particular, such elements span a dense subalgebra of $\mathcal{O}_2$.

As mentioned, our goal is to realize the Thompson groups as subgroups of the unitary group of $\mathcal{O}_2$ in a natural way. This fact was originally discovered by Nekrashevych [55] (see Section 9 therein), as kindly pointed out to us by Szymanski. Nekrashevych has a very explicit formula for expressing an element of the Thompson groups in terms of the isometries s_0 and s_1. We will not use these explicit formulas to show that the Thompson groups are subgroups of the unitary group of $\mathcal{O}_2$, but rather look at a natural representation of these on a concrete Hilbert space, and then show that they are, indeed, contained in a copy of the Cuntz algebra on this Hilbert space, in a natural way.

Let us for the rest of this section denote the set of dyadic rational numbers in $[0,1)$, $[0,\frac{1}{2})$ and $[\frac{1}{2},1)$ by $\mathfrak{X}$, $\mathfrak{X}_0$ and $\mathfrak{X}_1$, respectively, that is,

$$\mathfrak{X} = \mathbb{Z}[\tfrac{1}{2}] \cap [0,1), \qquad \mathfrak{X}_0 = \mathbb{Z}[\tfrac{1}{2}] \cap [0,\tfrac{1}{2}) \qquad \text{and} \qquad \mathfrak{X}_1 = \mathbb{Z}[\tfrac{1}{2}] \cap [\tfrac{1}{2},1).$$

Then $\mathfrak{X} = \mathfrak{X}_0 \sqcup \mathfrak{X}_1$ so that $\ell^2(\mathfrak{X}) = \ell^2(\mathfrak{X}_0) \oplus \ell^2(\mathfrak{X}_1)$. We will use $\{\delta_x : x \in \mathfrak{X}\}$ to denote the usual orthonormal basis of $\ell^2(\mathfrak{X})$, consisting of Dirac functions on the space $\mathfrak{X}$. Define the operators s_0 and s_1 on $\ell^2(\mathfrak{X})$ by

$$s_0\delta_x = \delta_{x/2} \qquad \text{and} \qquad s_1\delta_x = \delta_{(x+1)/2},$$

for all $x \in \mathfrak{X}$. It is straightforward to check that s_0 and s_1 are isometries satisfying $s_0s_0^* + s_1s_1^* = \mathbf{1}$, so, as mentioned earlier, they generate a copy of the Cuntz algebra $\mathcal{O}_2$. We emphasize that, from this point on, $\mathcal{O}_2$ will denote this particular copy of the Cuntz algebra. By construction, the range projections of s_0 and s_1 are the orthogonal projections onto $\ell^2(\mathfrak{X}_0)$ and $\ell^2(\mathfrak{X}_1)$, respectively, and it is easy to see that

$$s_0^*\delta_x = \delta_{2x} \qquad \text{and} \qquad s_1^*\delta_y = \delta_{2y-1},$$

for all $x \in \mathfrak{X}_0$ and all $y \in \mathfrak{X}_1$.

The groups F, T and V act by definition on the set $\mathfrak{X}$, so we get a corresponding representation on $\ell^2(\mathfrak{X})$, which we will denote by π. Just to recall, this means that π is the representation defined by $\pi(g)\delta_x = \delta_{g(x)}$, for all $x \in \mathfrak{X}$ and $g \in V$. We will use π to denote the representation of all the three groups. Using the functional expressions for the elements C, D and π_0 from Section 2.1, it is easy to check the following explicit identities:

$$\pi(C) = s_1s_1s_0^* + s_0s_0^*s_1^* + s_1s_0s_1^*s_1^*;$$
$$\pi(D) = s_1s_1s_0^*s_0^* + s_0s_0s_1^*s_0^* + s_0s_1s_0^*s_1^* + s_1s_0s_1^*s_1^*;$$
$$\pi(\pi_0) = s_1s_0s_0^* + s_0s_0^*s_1^* + s_1s_1s_1^*s_1^*.$$

As C, D and π_0 generate V, it follows that $\pi(V) \subseteq \mathcal{O}_2$. Thus, the C^*-algebra generated by $\pi(V)$ will also be a subset of $\mathcal{O}_2$. The above identities are exactly the same as one would get using the method of Nekrashevych [55], so that this is really the same representation.

Before studying the representation π, let us prove a small result about $\mathcal{O}_2$.

Lemma 4.1.1. *There exist unique states ϕ_0 and ϕ_1 on $\mathcal{O}_2$ such that $\phi_i(s_i^n(s_i^*)^m) = 1$, for all non-negative integers n and m, but $\phi_i(s_\mu s_\nu^*) = 0$ in all other cases, where $i = 0, 1$.*

Proof. We define ϕ_0 to be the vector state corresponding to the vector δ_0, that is, $\phi_0(x) = \langle x\delta_0 \mid \delta_0\rangle$. First of all, s_0 and s_0^* both fix δ_0, so it should be clear that $\phi_0(s_0^n(s_0^*)^m) = 1$, for all non-negative integers n and m. Let us prove that $\phi_0(s_\mu s_\nu^*) = 0$ in all the other cases, that is, when either the word μ or ν contains a 1. The state ϕ_0 is Hermitian, so that $\phi_0(s_\mu s_\nu^*) = \overline{\phi_0(s_\nu s_\mu^*)}$. Hence, $\phi_0(s_\mu s_\nu^*) = 0$ if and only if $\phi_0(s_\nu s_\mu^*) = 0$. Therefore we may assume that ν contains 1 so we may write $s_\nu = s_0^n s_1 s_{\nu'}$, for some non-negative integers n and $k \leq n-1$ and $\nu' \in \{0,1\}^{n-k-1}$, except when $k = n-1$, in which case $\nu' = \emptyset$. As the support projection of s_1^* is $s_1 s_1^*$, which is the projection onto $\ell^2(\mathfrak{X}_1)$, we know that $s_1^*\delta_0 = 0$. Hence

$$\phi_0(s_\mu s_\nu^*) = \langle s_\mu s_{\nu'}^* s_1^*(s_0^n)^*\delta_0 \mid \delta_0\rangle = \langle s_\mu s_{\nu'}^* s_1^*\delta_0 \mid \delta_0\rangle = 0$$

This shows that ϕ_0 has the desired properties.

By the universal property of $\mathcal{O}_2$, we know that there exists a $*$-isomorphism Θ of $\mathcal{O}_2$ satisfying $\Theta(s_0) = s_1$ and $\Theta(s_1) = s_0$. Clearly we may then define ϕ_1 by $\phi_1 = \phi_0 \circ \Theta$ to obtain a state with the desired properties.

That ϕ_0 and ϕ_1 are uniquely determined follows form the fact that elements of the form $s_\mu s_\nu^*$, for $\mu, \nu \in \{0,1\}^n$, span a dense subspace of $\mathcal{O}_2$. □

Let us explain what these states look like on the elements of $\pi(V)$. For this, let H_0 denote the subgroup of V consisting of elements g which map $[0, x)$ to $[0, y)$, for some $x, y \in \mathfrak{X}$. Likewise, let H_1 denote the subgroup of V consisting of elements g which map $[x, 1)$ to $[y, 1)$, for some $x, y \in \mathfrak{X}$. This is just a complicated way of defining H_0 to be the subgroup of elements which fix the point 0. Likewise, the group H_1 is just the elements which, intuitively speaking, fix the point 1. However, this does not really make sense, as 1 is not in $\mathfrak{X}$. Phrased more rigorously, H_1 is the subgroup of elements g with $\lim_{x\to 1} g(x) = 1$.

Lemma 4.1.2. *Let ϕ_0 and ϕ_1 denote the states on $\mathcal{O}_2$ from Lemma 4.1.1, and let H_0 and H_1 denote the groups introduced above. Then $\phi_j(\pi(g)) = 1$, for all $g \in H_j$, and $\phi_j(\pi(g)) = 0$, for all other $g \in V$, where $j = 0, 1$.*

Proof. Let us start by proving the statement for $j = 1$. Suppose that we are given some $\mu \in \{0,1\}^n$. Then the operator s_μ is an isometry as both s_0 and s_1 are isometries. In particular, their support is all of $\ell^2(\mathfrak{X})$. Moreover, if $\mu = (i_1, i_2, \ldots, i_n)$, then it is easy to prove by induction, that the range of s_μ is $\ell^2([\frac{m-1}{2^n}, \frac{m}{2^n}) \cap \mathfrak{X})$, where m is given by $m = 1 + \sum_{k=1}^{n} i_k 2^{n-k}$. From this we need to note two things. First of all, the range of s_μ, and therefore also the support of s_μ^*, is always $\ell^2([a, b) \cap \mathfrak{X})$, for some $a, b \in \mathfrak{X}$. Second, $b = 1$ if and only if $i_1 = \ldots = i_n = 1$. This latter fact follows from the expression of m in terms of $i_1, \ldots, i_n$.

Next we show that if $\pi(g)$ is a sum of operators of the form $s_\mu s_\nu^*$, then $\pi(g)$ include $s_1^n (s_1^*)^m$, for some $n, m \in \mathbb{N}$, if and only if $g \in H_1$, and there is only one such term. Let us write

$$\pi(g) = s_{\mu_1} s_{\nu_1}^* + \ldots + s_{\mu_n} s_{\nu_n}^*.$$

The first thing to notice is that the support of the different $s_{\mu_k} s_{\nu_k}^*$ are orthogonal and sum up to $\ell^2(\mathfrak{X})$. Given the form of the support that the different $s_{\mu_k} s_{\nu_k}^*$ have, this happens if each δ_x is in the support of exactly one $s_{\mu_k} s_{\nu_k}^*$. To see this, we need to note that evaluating the right hand side at the basis vector δ_x, we get a finite sum of these basis vectors, one for each k for which δ_x is in the support of $s_{\mu_k} s_{\nu_k}^*$. However, on the left hand side we get exactly one of these basis vectors, namely, $\delta_{g(x)}$, which means that the vector δ_x is in the support of exactly one of the operators $s_{\mu_k} s_{\nu_k}^*$. Repeating the argument with $\pi(g^{-1}) = \pi(g)^*$ we get that the ranges of the different $s_{\mu_k} s_{\nu_k}^*$ are orthogonal and sum up to $\ell^2(\mathfrak{X})$. Because of this, we can choose unique $k, l \in \{1, \ldots, n\}$ so that the support of $s_{\mu_k} s_{\nu_k}^*$ is $\ell^2([a, 1) \cap \mathfrak{X})$, for some $a \in \mathfrak{X}$, and the range of $s_{\mu_l} s_{\nu_l}^*$ is $\ell^2([b, 1) \cap \mathfrak{X})$, for some $b \in X$. In other words, we can choose unique $k, l \in \{1, \ldots, n\}$ so that $s_{\nu_k}^*$ is a power of s_1^* and s_{μ_l} is a power of s_1. Clearly, this shows that $g \in H_1$ if and only if $k = l$, and that $\phi_1(\pi(g)) = 1$ if $g \in H_1$, and $\phi_1(\pi(g)) = 0$ if $g \notin H_1$. This proves the statement in the case $j = 1$. The statement in the case $j = 0$ can be proved analogously. $\square$

Recall that we denote the C^*-algebras generated by $\pi(F)$, $\pi(T)$ and $\pi(V)$ inside $\mathcal{O}_2$ by $C^*_\pi(F)$, $C^*_\pi(T)$ and $C^*_\pi(V)$, respectively. The following proposition relates these three C^*-algebras.

Proposition 4.1.3. *With the notation above, we have*

$$C^*_\pi(F) \subsetneq C^*_\pi(T) \subsetneq C^*_\pi(V) = \mathcal{O}_2$$

Proof. We have already argued why $C^*_\pi(V) \subseteq \mathcal{O}_2$, while the inclusions $C^*_\pi(F) \subseteq C^*_\pi(T)$ and $C^*_\pi(T) \subseteq C^*_\pi(V)$ obviously follow directly from the inclusions $F \subseteq T$ and $T \subseteq V$. Thus, what we need to prove is that the first two inclusions are proper and that the last one is an equality. It is easy to see that $C^*_\pi(F) \neq C^*_\pi(T)$, since $\mathbb{C}\delta_0$ is a $C^*_\pi(F)$-invariant subspace which is not $C^*_\pi(T)$-invariant. To prove the rest, our strategy is to show that $C^*_\pi(V) = \mathcal{O}_2$ and afterwards that $C^*_\pi(T) \neq \mathcal{O}_2$.

We start by proving that $C^*_\pi(V)$ contains the projections $s_0 s_0^*$ and $s_1 s_1^*$. Let us denote these by p_0 and p_1, respectively. First, recall that V_0 denotes the subgroup of V consisting of elements g satisfying $g(x) = x$, for all $x \in [0, \frac{1}{2})$. As both the sets $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ are invariant for the elements of V_0, we see that $\ell^2(\mathfrak{X}_0)$ and $\ell^2(\mathfrak{X}_1)$ are invariant subspaces of $\pi(g)$, for all $g \in V_0$. Now, the representation $g \mapsto \pi(g)|_{\ell^2(\mathfrak{X}_1)}$ of V_0 is the one associated to the action of V_0 on $[\frac{1}{2}, 1)$, so, as this action is non-amenable by Remark 3.1.14, we know from Theorem 3.1.7 that there exist elements $g_1, g_2, \ldots, g_n$ in V_0, such that $\|\frac{1}{n}\sum_{k=1}^{n}\pi(g_k)|_{\ell^2(\mathfrak{X}_1)}\| < 1$. Set $x = \frac{1}{n}\sum_{k=1}^{n}\pi(g_k)$. Our claim is that $x^m \to p_0$, as $m \to \infty$, which proves that p_0 is in $C^*_\pi(V_0)$, and therefore, in particular, in $C^*_\pi(V)$. Let us explain why this is the case. First of all, as x commutes with the projections p_0 and p_1, we see that $x^m = (xp_0)^m + (xp_1)^m$. Since each element of V_0 is the identity on $[0, \frac{1}{2})$, the operator $\pi(g)$ is the identity operator on $\ell^2(\mathfrak{X}_0)$, for all $g \in V_0$. Thus, $\pi(g)p_0 = p_0$, for all $g \in V_0$, so that $xp_1 = p_0$. This means that $x^m = p_0 + (xp_1)^m$, but as $\|xp_1\| = \|x|_{\ell^2(\mathfrak{X}_1)}\| < 1$, we see that $(xp_1)^m$ converges to zero, as $m \to \infty$, which proves that x^m converges to p_0, as $m \to \infty$, so that $p_0 \in C^*_\pi(V)$. A similar argument shows that $p_1 \in C^*_\pi(V)$.

Once we know that p_0 and p_1 are in $C^*_\pi(V)$, it is not difficult to show that $C^*_\pi(V)$ also contains s_0 and s_1. The generator A of F, defined in Section 2.1, satisfies $A(x) = \frac{1}{2}x$ and $A^{-1}(y) = \frac{1}{2}y + \frac{1}{2}$, for $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1)$. Thus

$$\pi(A)p_0 = s_0 p_0 \qquad \text{and} \qquad \pi(A^{-1})p_1 = s_1 p_1.$$

Also, the element D^2 satisfies $D^2(x) = x + \frac{1}{2}$, for $x \in [0, \frac{1}{2})$, and $D^2(x) = x - \frac{1}{2}$, for $x \in [\frac{1}{2}, 1)$. Hence, $s_0 = \pi(D^2)s_1$. Putting these things together, we see that

$$s_0 = s_0 p_0 + s_0 p_1 = \pi(A)p_0 + \pi(D^2)s_1 p_1 = \pi(A)p_0 + \pi(D^2A^{-1})p_1,$$

which shows that s_0 is contained in $C^*_\pi(V)$. Similarly, s_1 is in $C^*_\pi(V)$ as $s_1 = \pi(D^2)s_0$, so we conclude that $C^*_\pi(V) = \mathcal{O}_2$.

We now prove that $C^*_\pi(T) \neq \mathcal{O}_2$. Let ϕ_0 and ϕ_1 be the states from Lemma 4.1.1. Clearly the two states are distinct, but our aim is to prove that they agree on $C^*_\pi(T)$. This will surely imply that $C^*_\pi(T) \neq \mathcal{O}_2$. As the linear span of $\pi(T)$ is dense in $C^*_\pi(T)$, it suffices to check that $\phi_0(g) = \phi_1(g)$, for all $g \in T$. This follows directly from Lemma 4.1.2, as it is easily seen that $T \cap H_0 = F$ and $T \cap H_1 = F$. $\square$

4.2 Simplicity and unique trace

In this section we will discus the notions of C^*-simplicity and unique trace property in connection with the Thompson groups. We start by proving a result relating amenability of the Thompson group F to C^*-simplicity of the Thompson group T. To do so, we begin by proving a result characterizing amenability of F in terms of weak containment of the representation π from the previous section.

Proposition 4.2.1. *With $\pi\colon T \to \mathcal{O}_2$ denoting the representation from Section 4.1, the following are equivalent:*

(1) *F is amenable;*

(2) *π is weakly contained in the left regular representation of T;*

(3) *$\pi|_F$ is weakly contained in the left regular representation of F;*

Proof. As in the last section, we let $\mathfrak{X}$ denote the set of dyadic rational points in the interval $[0,1)$ so that π is the representation of T on the Hilbert space $\ell^2(\mathfrak{X})$ induced by the action. Suppose now that F is amenable. As F is the stabilizer of the point $\{0\}$ of the action of T on $\mathfrak{X}$, we get from Proposition 3.1.10 that π is weakly contained in the left regular representation of T.

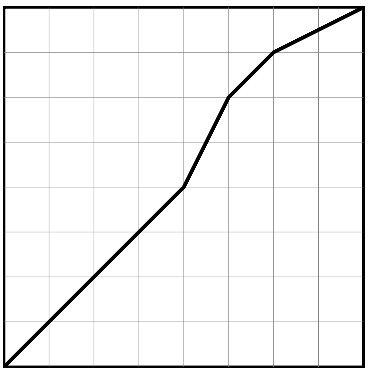
Assume that π is weakly contained in the left regular representation of T. Then $\pi|_F$ is weakly contained in the left regular representation of F, by Theorem 3.1.9.

Last, suppose that $\pi|_F$ is weakly contained in the left regular representation of F. Since each element of F fixes 0, we get that $\mathbb{C}\delta_0$ is a $\pi(F)$-invariant subspace of $\ell^2(\mathfrak{X})$. Let p denote the projection onto this subspace. Then $g \mapsto p\pi(g)p$ is the trivial representation of F. This representation is clearly weakly contained in $\pi|_F$, so by transitivity of weak containment, we get that the trivial representation of F is weakly contained in the left regular representation of F. By Condition (iii) of Theorem 3.1.6 this means that F is amenable. $\square$

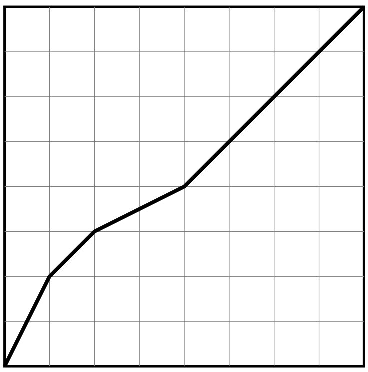
Theorem 4.2.2. *If $C^*_\lambda(T)$ is simple, then F is non-amenable.*

Proof. Suppose that F is amenable, and let us then prove that $C^*_\lambda(T)$ is not simple. By Proposition 4.2.1 we know that π is weakly contained in the left regular representation of T, so, by Theorem 3.1.9, there exists a $*$-homomorphism $\sigma\colon C^*_\lambda(T) \to \mathcal{O}_2$ so that $\sigma\lambda = \pi$. Our goal is to show that the kernel of σ is a non-trivial ideal in $C^*_\lambda(T)$. As the kernel of σ is clearly not all of $C^*_\lambda(T)$, it suffices to prove that the kernel contains some non-zero element. The left regular representation is clearly injective on the complex group algebra $\mathbb{C}T$, so, if we find an element $x \neq 0$ in $\mathbb{C}T$ such that $\pi(x) = 0$, then $\lambda(x)$ will be a non-zero element in the kernel of σ.

Consider the elements a and b of T given by $a = CDC$ and $b = D^2CDCD^2$, that is, the elements whose graphs look as follows:

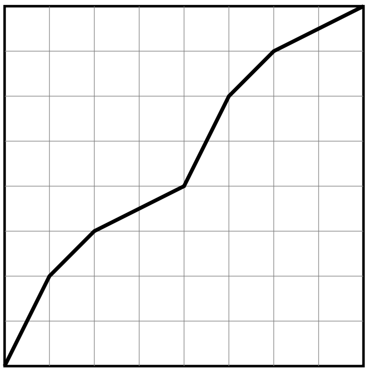


$a = CDC$



$b = D^2CDCD^2$

It is easy to see that the elements a and b commute, as the former fixes all points in the interval $[0, \frac{1}{2})$ and the latter fixes all points in the interval $[\frac{1}{2}, 1)$. Their product is the element with the following graph:

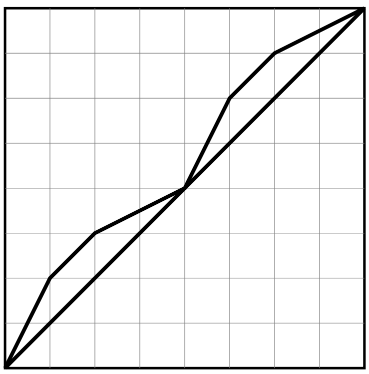


ab $(= ba)$

Let $x = a+b-ab-e$. We claim that $\pi(x) = 0$, so that $\lambda(x)$ is a non-trivial element of the kernel of σ. Through tedious calculations, this can be verified using the relations defining the Cuntz algebra $\mathcal{O}_2$, as well as the expressions of $\pi(a)$ and $\pi(b)$ in terms of s_0 and s_1. Namely,

$$\pi(a) = s_0 s_0^* + s_1 s_0 s_0^* s_0^* s_1^* + s_1 s_2 s_0 s_1^* s_0^* s_1^* + s_1 s_1 s_1 s_1^* s_1^*;$$
$$\pi(b) = s_1 s_1^* + s_0 s_0 s_0 s_0^* s_0^* + s_0 s_0 s_1 s_0^* s_1^* s_0^* + s_0 s_1 s_1^* s_1^* s_0^*.$$

However, there is a more intuitive explanation of why $\pi(x)$ is zero. Loosely speaking, the reason is that drawing the graphs of a and b on top of each other one gets the same picture as drawing the graphs of ab and e on top of each other, namely,



Formally speaking, the point is that $\{a(x), b(x)\} = \{ab(x), x\}$, because $a(x) = x$ and $b(x) = ab(x)$ on $[0, \frac{1}{2})$, and $a(x) = ab(x)$ and $b(x) = x$ on $[\frac{1}{2}, 1)$. From this it follows that

$$\pi(a+b)\delta_x = \delta_{a(x)} + \delta_{b(x)} = \delta_{ab(x)} + \delta_x = \pi(ab+e)\delta_x,$$

for all $x \in [0,1)$, so that $\pi(a+b-ab-e) = 0$. This proves that $\lambda(x)$ is in the kernel of σ, which is therefore a non-trivial ideal in $C^*(T)$. $\square$

Let us at this point properly introduce the notions of C^*-simplicity and unique trace property for groups.

Definition 4.2.3. A group G is said to be C^**-simple* if $C^*_\lambda(G)$ is simple, and it is said to have the *unique trace property* if $C^*_\lambda(G)$ has a unique tracial state.

The interest in these two properties arose in 1975 when Powers [60] proved that the free group on two generators has both of these properties. Since then many more groups have been shown to have these two properties, and a rather comprehensive list can be found in the survey by de la Harpe [23]. All of these groups are non-amenable, and, indeed, it is a well-known fact that non-trivial amenable groups are not C^*-simple and do not have the unique trace property. This follows from the fact that the full and the reduced group C^*-algebras of such groups coincide. More precisely, the full group C^*-algebra always has a one dimensional representation, so, if it agrees with the reduced group C^*-algebra, it also has a one dimensional representation. This representation is, in itself, a tracial state different from the usual trace on the reduced group C^*-algebra if the group is non-trivial, and its kernel is a non-trivial ideal. In fact, this is not only true for amenable groups, but for all groups containing an amenable subgroup. More precisely, Day [21] proved that every group possesses a largest normal amenable subgroup, called the *amenable radical*, and it is a well-known fact that groups with a non-trivial amenable radical are neither C^*-simple nor have the unique trace property.

A famous question of de la Harpe[22] from 1985 (see §2 question (2) therein) asks whether C^*-simplicity and the unique trace property are equivalent. This question received a lot of attention over the years, and many examples were found which seemed to suggest that these two properties were equivalent. As Dudko and Medynets [25] proved in 2012 that T and V have the unique trace property, a positive solution to this problem would prove that the Thompson group F was non-amenable, using Theorem 4.2.2. However, recently it has been proved that C^*-simplicity implies the unique trace property, whereas the unique trace property does not imply C^*-simplicity. More precisely, in 2014 Kalantar and Kennedy [43] gave a characterization of C^*-simplicity involving boundary actions. The same year Breuillard, Kalantar, Kennedy and Ozawa [7] proved a number of spectacular results, one of them being that C^*-simplicity implies the unique trace property. In fact, they proved that having the unique trace property is equivalent to the group having trivial amenable radical. The year after, Le Boudec [48] gave an example of a C^*-simple group without the unique trace property.

The results of Breuillard, Kalantar, Kennedy and Ozawa also provide a new proof of the fact that the Thompson groups T and V have the unique trace property. Indeed, both groups are simple and non-amenable, which clearly implies that their amenable radicals are trivial.

Remark 4.2.4. As we explained earlier, non-trivial amenable groups are neither C^*-simple, nor do they have the unique trace property, so, in particular, the Thompson group F is non-amenable if it is either C^*-simple or has the unique trace property. Using the characterization of the unique trace property above, it is easy to see that if F is non-amenable, then it has the unique trace property. To see this, one just has to recall that every non-trivial normal subgroup of F contains the commutator subgroup F' [10, Theorem 4.3], and, as this group contains a copy of F, then every non-trivial normal subgroup of F is non-amenable if F is.

We find it surprising that amenability of F is related to C^*-simplicity of T as shown in Theorem 4.2.2. Very recently, Le Boudec and Matte Bon [49] proved that the converse of this theorem also holds, so that F is non-amenable if and only if T is C^*-simple. Prior to this, in 2014, Bleak and Juschenko [6], as well as Breuillard, Kalantar, Kennedy and Ozawa [7], obtained partial converses of Theorem 4.2.2. The latter of these two results states that F is not C^*-simple if T is not C^*-simple. The converse of this result was also proved by Le Boudec and Matte Bon, so that either F and T are both C^*-simple, or none of them is. Let us mention that they, in addition, prove that the Thompson group V is, in fact, C^*-simple.

Let us now discuss the partial converse proved by Bleak and Juschenko. By scrutinizing the proof of Theorem 4.2.2, we see, with the notation therein, that the Thompson group F is non-amenable if the closed two-sided ideal generated by the element $\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)$ inside $C^*_\lambda(T)$ is the whole $C^*_\lambda(T)$. This element is not unique with this property, as the proof shows that this will be true for $\lambda(x)$, whenever x is an element of $\mathbb{C}T$ with $\pi(x) = 0$. Bleak and Juschenko proved that if the Thompson group F is non-amenable, then there exist disjoint finite subsets H_1 and H_2 of F so that $\sum_{g\in H_1} \pi(g) = \sum_{g\in H_2} \pi(g)$ and the ideal generated by$\sum_{g\in H_1} \lambda(g) - \sum_{g\in H_2} \lambda(g)$ is all of $C^*_\lambda(T)$. As it turns out, one may choose $H_1 = \{a, b\}$ and $H_2 = \{ab, e\}$. We will prove this in Proposition 4.2.7 below.

First we need a few results. The first one is well-known, and a proof, when the action is left translation of the group on itself, can be found in [57, Lemma 2.1(c)] or [36, Lemma 4.1].

Lemma 4.2.5. *Let G be group acting on a set $\mathfrak{X}$, and let σ denote the corresponding representation on $\ell^2(\mathfrak{X})$. Then $\|\sigma(x) + \sigma(y)\| \geq \|\sigma(x)\|$, for any $x, y \in \mathbb{R}_+G$.*

Haagerup [36] gave the following characterization of the unique trace property.

Theorem 4.2.6. *A group G has the unique trace property if and only if the closed convex hull of $\{\lambda(sts^{-1}) : s \in G\}$ contains zero, for all $t \in G \setminus \{e\}$.*

Using this characterization, we can prove the following proposition.

Proposition 4.2.7. *With a and b denoting the elements of F as above, the following are equivalent:*

(1) *The Thompson group F is non-amenable;*

(2) *The closed two-sided ideal generated by* $\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)$ *in* $C^*_\lambda(T)$ *is all of* $C^*_\lambda(T)$*;*

(3) *The closed convex hull of* $\{\lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in T\}$ *contains* 0*;*

(4) *The closed two-sided ideal generated by* $\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)$ *in* $C^*_\lambda(F)$ *is all of* $C^*_\lambda(F)$*;*

(5) *The closed convex hull of* $\{\lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in F\}$ *contains* 0.

Proof. We prove that (1) implies (4) and (5), that (4) implies (2), that (5) implies (3), that (2) implies (1), and that (3) implies (1). A few of these implications are straightforward. That (4) implies (2) follows from the fact that the inclusion $C^*_\lambda(F) \subseteq C^*_\lambda(T)$ is unital, that is, if (4) holds, then the closed two-sided ideal in $C^*_\lambda(T)$ generated by $\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)$ contains $C^*_\lambda(F)$, in particular, the unit. That (5) implies (3) follows directly from the fact that the inclusion $C^*_\lambda(F) \subseteq C^*_\lambda(T)$ is isometric.

That (2) implies (1) can be deduced from the proof of Theorem 4.2.2, as already mentioned. More precisely, since we know that $\pi(a + b - ab - e) = 0$, the ideal generated by $\lambda(a) + \lambda(b) - \lambda(ab) - \mathbf{1}$ must be contained in the kernel of any $*$-homomorphism from $C^*_\lambda(T)$ to $\mathcal{O}_2$. If this ideal is all of $C^*_\lambda(T)$, there are no non-zero $*$-homomorphisms from $C^*_\lambda(T)$ to $\mathcal{O}_2$. It follows that π is not contained in the left regular representation of T, by Theorem 3.1.9. Therefore F is non-amenable, by Proposition 4.2.1.

Let us now prove that (3) implies (1). This is done similarly to the previous implication. Again we know that $\pi(a) + \pi(b) = \mathbf{1} + \pi(ab)$, so it follows that

$$\overline{\mathrm{conv}}\{\pi(hah^{-1}) + \pi(hbh^{-1}) : h \in T\} = \mathbf{1} + \overline{\mathrm{conv}}\{\pi(habh^{-1}) : h \in T\}.$$

Now, if x is a finite convex combination of elements of the form $habh^{-1}$ with $h \in T$, then $x \in \mathbb{R}_+T$ and, by Lemma 4.2.5, we conclude that $\|\mathbf{1} + \pi(x)\| \geq \|\mathbf{1}\| = 1$. Thus we see that $\|\pi(x)\| \geq 1$, for all $x \in \mathrm{conv}\{hah^{-1} + hbh^{-1} : h \in T\} \subseteq \mathbb{C}T$. As $\overline{\mathrm{conv}}\{\lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in T\}$ contains 0 by assumption, we may choose a finite convex combination x of elements of the form $hah^{-1} + hbh^{-1}$ with $h \in T$ so that $\|\lambda(x)\| < 1$. However, by the first part we know that $\|\pi(x)\| \geq 1$, so we conclude that x is an element of $\mathbb{C}T$ with $\|\pi(x)\| > \|\lambda(x)\|$. It follows from Theorem 3.1.9 that π is not contained in the left regular representation of T, and, again, by Proposition 4.2.1, that F is non-amenable.

The only thing left is to prove that (1) implies (4) and (5), so assume that F is non-amenable. We will prove (4) and (5) in one step. To prove (4), it suffices to prove that the closed two-sided ideal in $C^*_\lambda(F)$ generated by $\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)$ contains $\mathbf{1}$. Clearly this ideal contains

$$\begin{aligned} &\overline{\mathrm{conv}}\,\big\{\lambda(h)\big(\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)\big)\lambda(h)^* : h \in F\big\} \\ &\qquad = \mathbf{1} + \overline{\mathrm{conv}}\,\big\{\lambda(habh^{-1}) - \lambda(hah^{-1}) - \lambda(hbh^{-1}) : h \in F\big\}, \end{aligned}$$

so it suffices to show that the closed convex hull on the right hand side contains 0. Let $\varepsilon > 0$ be given. As mentioned in Remark 4.2.4, we know that F has the unique trace property. Let F_1 and F_2 denote the subgroup of F consisting of elements $f \in F$ so that $f(x) = x$, for all $x \in [0, \frac{1}{2})$, and $f(x) = x$, for all $x \in [\frac{1}{2}, 1)$, respectively. Note that elements of these two subgroups commute, and that $b \in F_1$ and $a \in F_2$. Since F_1 and F_2 are isomorphic to F, they both have the unique trace property. By Theorem 4.2.6, this means that there exist positive real numbers $s_1, \ldots, s_n, t_1, \ldots, t_m$ with $\sum_{k=1}^n s_k = \sum_{k=1}^m t_k = 1$, as well as elements $g_1, \ldots, g_n \in F_1$ and $h_1, \ldots, h_m \in F_2$, so that

$$\Big\| \sum_{k=1}^{n} s_k \lambda(g_k b g_k^{-1}) \Big\| < \varepsilon \qquad \text{and} \qquad \Big\| \sum_{k=1}^{m} t_k \lambda(h_k a h_k^{-1}) \Big\| < \varepsilon.$$

For simplicity, let us denote $\sum_{k=1}^m t_k \lambda(h_k a h_k^{-1})$ and $\sum_{k=1}^n s_k \lambda(g_k b g_k^{-1})$ by $\tilde{a}$ and $\tilde{b}$, respectively. Using that the g_i's commute with the h_j's and b, as well as the fact that the h_j's commute with the g_i's and a, it is straightforward to check that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} s_i t_j \lambda(g_i h_j) \big(\lambda(a) + \lambda(b)\big) \lambda(g_i h_j)^* = \tilde{a} + \tilde{b}.$$

Since $\sum_{i=1}^n \sum_{j=1}^m s_i t_j = 1$, the left hand side above belongs to the convex hull of $\{\lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in F\}$, and, since $\|\tilde{a} + \tilde{b}\| < 2\varepsilon$, we conclude that the convex hull of $\{\lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in F\}$ contains elements of arbitrarily small norm. This proves (5). By similar calculations one can check that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} s_i t_j \lambda(g_i h_j) \big(\lambda(ab) - \lambda(a) - \lambda(b)\big) \lambda(g_i h_j)^* = \tilde{a}\tilde{b} - \tilde{a} - \tilde{b}.$$

The left hand side above is an element in the convex hull of

$$\{\lambda(habh^{-1}) - \lambda(hah^{-1}) - \lambda(hbh^{-1}) : h \in F\}.$$

Since $\|\tilde{a}\tilde{b} - \tilde{a} - \tilde{b}\| < \varepsilon^2 + 2\varepsilon$, we conclude that the closure of this convex hull contains 0, and as mentioned above, this means that the ideal generated by $\mathbf{1} + \lambda(ab) - \lambda(a) - \lambda(b)$ in $C^*_\lambda(F)$ is the whole $C^*_\lambda(F)$, that is, (4) holds. This proves that (1) implies (4) and (5), concluding the proof. $\square$

We point out that, since Le Boudec and Matte Bon proved the converse of Theorem 4.2.2 a slightly shorter proof of the proposition above can be given, using a characterization of C^*-simplicity discovered independently by Haagerup [36] and Kennedy [46].

4.3 The rapid decay property

In this section we will give a brief introduction to the rapid decay property and list some well-known results about it. The rapid decay property for groups was introduced by Jolissaint [39] in 1990, inspired by a result of Haagerup [35] from 1979 where he proves that the free group on n generators $\mathbb{F}_n$ has this property.

We will first recall the notion of a length function on a group.

Definition 4.3.1. A *length-function* on a group G is a function $L\colon G \to [0,\infty)$ satisfying

(1) $L(gh) \leq L(g) + L(h)$,

(2) $L(g) = L(g^{-1})$,

(3) $L(e) = 0$,

for all $g, h \in G$. If L_1 and L_2 are two length functions on G, we say that L_1 is *dominated* by L_2 if there exist $a, b > 0$ such that $L_1(g) \leq aL_2(g) + b$, for all $g \in G$. Two length functions are said to be *equivalent* if they dominate each other.

Given a length function L on a group G and a positive real number s, we define the norm $\|\cdot\|_{s,L}$ on $\mathbb{C}G$ by

$$\|f\|_{s,L} = \sqrt{\sum_{g\in G} |f(g)|^2(1 + L(g))^{2s}},$$

for all $f \in \mathbb{C}G$. Note that $\|f\|_{s,L} \leq \|f\|_{t,L}$, for all $f \in \mathbb{C}G$, when $s \leq t$.

Definition 4.3.2. Let L be a length function on G. Then G is said to have the *rapid decay property with respect to L* if there exist constants $C, s > 0$ such that

$$\|\lambda(f)\| \leq C\|f\|_{s,L},$$

for all $f \in \mathbb{C}G$. Furthermore, G is said to have the *rapid decay property* if it has the rapid decay property with respect to some length function.

Let us make some comments concerning to what degree the rapid decay property depends on the length function. It is not difficult to see that if G possesses the rapid decay property relative to some length function L_1 which is dominated by another length function L_2, then G has the rapid decay property with respect to L_2, as well. Indeed, we may choose $a, b > 0$ so that $L_1(g) \leq aL_2(g) + b$, for all $g \in G$, so with $d = \max\{a, b+1\}$, we have $L_1(g) + 1 \leq dL_2(g) + d$. In particular, we have

$$(1 + L_1(g))^{2s} \leq d^{2s}(1 + L_2(g))^{2s},$$

for all $g \in G$, and, therefore, also $\|f\|_{s,L_1} \leq d^s\|f\|_{s,L_2}$, for all $f \in \mathbb{C}G$. This clearly implies that G has the rapid decay property with respect to L_2. As a consequence, the rapid decay property with respect to a length function does not depend on the particular length function up to equivalence.

Remark 4.3.3. Suppose that we are given a finitely generated group G, with a generating set S. The *word length* on G with respect to S is the length function l defined as follows: we let $l(e) = 0$ and, for $g \in G \setminus \{e\}$, we let $l(g)$ be the minimal number k so that g can be written as a product of k elements from the set $S \cup S^{-1}$. It is straightforward to check that this defines a length function, and that l dominates every other length function L on G. Indeed, if we let $M = \max\{L(g) : g \in S\}$, it follows that

$$L(g_1 g_2 \cdots g_n) \leq L(g_1) + \ldots + L(g_n) \leq nM,$$

for all $g_1, \ldots, g_n \in S \cup S^{-1}$. This implies that $L(g) \leq l(g)M$, for all $g \in G$, so that L is dominated by l. In particular, it follows that all word length functions on G are equivalent, that is, up to equivalence, the length function l does not depend on the generating set S. Moreover, it also follows that a finitely generated group G has the rapid decay property if and only if it has the rapid decay property with respect to any word length function.

If L is a length function on G, denote $B_L(r) = \{g \in G : L(g) \leq r\}$ the *L-ball* of radius r centered at the identity element of G. The following result is due to Chatterji and Ruane [13]. We have included a proof for completeness.

Proposition 4.3.4. *Let G be a group with a length function L. Then G has the rapid decay property with respect to L if and only if there exists a polynomial P so that*

$$\|\lambda(f)\| \leq P(r)\|f\|_2,$$

for all $r > 0$ and $f \in \mathbb{C}$ supported on $B_L(r)$.

Proof. Suppose that G has the rapid decay property with respect to L. Choose some $C, s > 0$ so that $\|\lambda(f)\| \leq C\|f\|_{s,L}$, for all $f \in \mathbb{C}G$, and choose a natural number $n \geq s$. Fix $f \in \mathbb{C}G$ supported on $B_L(r)$, and let $P(x) = C(1+x)^n$. Then

$$\|f\|_{n,L}^2 = \sum_{g \in G} |f(g)|^2 (1 + L(g))^{2n} \leq \sum_{g \in G} |f(g)|^2 (1+r)^{2n} = \frac{1}{C} P(r)^2 \|f\|_2^2.$$

It follows that $\|\lambda(f)\| \leq C\|f\|_{s,L} \leq C\|f\|_{n,L} \leq P(r)\|f\|_2$, as wanted.

Now, suppose that there exists a polynomial P with the stated property, and let us prove that G has the rapid decay property with respect to L. Choose $n \in \mathbb{N}$ and $C > 0$ so that $P(x) \leq Cx^{n-1}$, for $x \geq 1$. Fix $f \in \mathbb{C}G$ and let us prove that $\|\lambda(f)\| \leq \frac{\pi C}{\sqrt{6}} \|f\|_{n,L}$. Write $f = \sum_{k=0}^{\infty} f_k$ with $f_k(g) = 0$, for all g with $L(g) \notin [k, k+1)$. Then $\|\lambda(f_k)\| \leq P(k+1)\|f_k\|_2$ by assumption, which means that $\|\lambda(f_k)\| \leq C(1+k)^{n-1}\|f\|_2$. Using the triangle inequality followed by the Cauchy-Schwarz inequality, we see that

$$\|\lambda(f)\| \leq C \sum_{k=0}^{\infty} (1+k)^{n-1} \|f_k\|_2 \leq C \Big(\sum_{k=0}^{\infty} (1+k)^{-2} \Big)^{1/2} \Big(\sum_{k=0}^{\infty} (1+k)^{2n} \|f_k\|_2^2 \Big)^{1/2}.$$

Now, as $k \leq L(g)$ for all g in the support of f_k, we see that $(1+k)^{2n}\|f_k\|_2^2 \leq \|f_k\|_{n,L}^2$. Moreover, it follows directly from the definition of the norm $\|\cdot\|_{n,L}$ that

$$\sum_{k=0}^{\infty} \|f_k\|_{n,L}^2 = \|f\|_{n,L}^2.$$

Since $\sum_{k=0}^{\infty}(1+k)^{-2} = \frac{\pi^2}{6}$, we conclude that $\|\lambda(f)\| \leq \frac{\pi C}{\sqrt{6}}\|f\|_{n,L}$, as desired. □

Given a length function L on a group G, we can restrict it to a subgroup H to obtain a length function L' on H. Clearly $\|f\|_{s,L'} = \|f\|_{s,L}$, for all $f \in \mathbb{C}H$, so it follows straight from the definition, that H has the rapid decay property with respect to L' if G has the rapid decay property with respect to L. This proves the following result from [39].

Proposition 4.3.5. *The rapid decay property passes to subgroups.*

Let us end this section by mentioning the following theorem due to Jolissaint [39], which relates rapid decay, amenability and growth of finitely generated group. Recall that a finitely generated group G with a word length L is said to have *exponential growth*, respectively, *polynomial growth* if the number of elements in the set $B_L(n)$ increases exponentially and polynomially in n, respectively. It is not difficult to see that this is independent of the choice of word length.

Theorem 4.3.6. *A finitely generated amenable group has the rapid decay property if and only if it is of polynomial growth.*

4.4 A criterion for C^*-simplicity

In this section we prove a criterion for a group to be C^*-simple involving the rapid decay property. Namely, we show that non-inner amenable groups with the rapid decay property are C^*-simple. Let us introduce the Dixmier property for C^*-algebras.

Definition 4.4.1. A unital C^*-algebra $\mathcal{A}$ is said to have the *Dixmier property* if the norm-closed convex hull of $\{uxu^* : u \in \mathcal{A} \text{ is unitary}\}$ intersects $\mathbb{C}\mathbf{1}$ non-trivially, for all $x \in \mathcal{A}$.

Remark 4.4.2. Suppose that $\mathcal{A}$ is a C^*-algebra with a tracial state τ. It is easy to see that if the closed convex hull of $\{uxu^* : u \in \mathcal{A} \text{ is unitary}\}$ intersects $\mathbb{C}\mathbf{1}$, then it must be in a single point, namely, $\tau(x)\mathbf{1}$. Indeed, as $\tau(uxu^*) = \tau(x)$, for all $x, u \in \mathcal{A}$ with u unitary, τ must be constant on the closed convex hull, and, therefore, also constant on the intersection with $\mathbb{C}\mathbf{1}$. From this it follows that $\mathcal{A}$ has the Dixmier property if and only if, for each $\varepsilon > 0$ and $x \in \mathcal{A}$, there exist unitaries $u_1, \ldots, u_n$ in $\mathcal{A}$ and positive real numbers $s_1, \ldots, s_n$ with $s_1 + \ldots + s_n = 1$ so that

$$\Big\| \sum_{k=1}^{n} s_k u_k x u_k^* - \tau(x)\mathbf{1} \Big\| < \varepsilon.$$

It easily follows from the observation above that a trace τ on a C^*-algebra $\mathcal{A}$ with the Dixmier property must necessarily be unique if it exists, since the intersection of the convex hull of $\{uxu^* : u \in \mathcal{A} \text{ is unitary}\}$ with $\mathbb{C}\mathbf{1}$ is $\tau(x)$, independent of the trace τ. In particular, a group whose reduced group C^*-algebra has the Dixmier property, also has the unique trace property. In fact, it is C^*-simple as well. This is an observation of Powers [60] and it was the method he used to prove that $\mathbb{F}_2$ is C^*-simple and has the unique trace property. We include the proof for convenience.

Proposition 4.4.3. *If the reduced group C^*-algebra of a group has the Dixmier property, then the group is C^*-simple and has the unique trace property.*

Proof. Suppose that G is a group such that $C^*_\lambda(G)$ has the Dixmier property. We have already explained above why G then has the unique trace property, so let us prove that it is also C^*-simple.

Let I be a non-zero ideal in $C^*_\lambda(G)$. Then we need to prove that I is all of $C^*_\lambda(G)$, that is, that I contains an invertible element. Choose some non-zero $x \in I$. As the canonical trace τ on $C^*_\lambda(G)$ is faithful, we know that $\tau(x^*x) > 0$. Now, letting $y = \tau(x^*x)^{-1}x^*x$, we see that y is an element of I with $\tau(y) = 1$. In particular, because of the Dixmier property, we can choose unitaries $u_1, \dots, u_n$ in $C^*_\lambda(G)$ and positive real numbers $s_1, \dots, s_n$ with $s_1 + \dots + s_n = 1$ so that $\| \sum_{k=1}^n s_k u_k y u_k^* - \mathbf{1}\| < 1$. Now, $z = \sum_{k=1}^n s_k u_k y u_k^*$ belongs to I, and it is invertible, as $\|z - \mathbf{1}\| < 1$. Thus $I = C^*_\lambda(G)$, and we conclude that G is C^*-simple. □

Remark 4.4.4. As it happens, the converse of the above proposition is also true. In fact, a group is C^*-simple if and only if its reduced group C^*-algebra has the Dixmier property. Let us explain why this is the case. In 1984 Haagerup and Zsidó [37] proved that a simple C^*-algebra with at most one tracial state has the Dixmier property. Combining this with the result by Breuillard, Kalantar, Kennedy and Ozawa that C^*-simplicity implies the unique trace property, we conclude that a group is C^*-simple if and only if its reduced group C^*-algebra has the Dixmier property.

Now we prove the criterion for C^*-simplicity and unique trace property announced in the beginning of the section.

Theorem 4.4.5. *Suppose that G is a non-inner amenable group with the rapid decay property. Then $C^*_\lambda(G)$ has the Dixmier property. In particular, G is C^*-simple and has the unique trace property.*

Proof. Suppose G is a non-inner amenable group with the rapid decay property. By Proposition 3.2.5, we may choose elements $g_1, g_2, \dots, g_n$ in G and a positive real number $c < 1$ so that

$$\Big\| \frac{1}{n} \sum_{k=1}^{n} \lambda(g_k) x \lambda(g_k)^* \Big\|_\tau \leq c \|x\|_\tau,$$

for all $x \in C^*_\lambda(G)$ with trace zero. Let $\sigma\colon C^*_\lambda(G) \to C^*_\lambda(G)$ be the map given by

$$\sigma(f) = \frac{1}{n}\sum_{k=1}^{n} \lambda(g_k) f \lambda(g_k)^*,$$

for all $f \in C^*_\lambda(G)$. Note that σ is a linear contraction and that $\sigma(f)$ is in the convex hull of $\{ufu^* : u \in C^*_\lambda(G) \text{ is unitary}\}$, for every $f \in C^*_\lambda(G)$. In particular, $\sigma^k(f)$ is in this convex hull, for every $k \in \mathbb{N}$, where σ^k denotes the k-fold composition of σ. Our strategy is to prove that $\sigma^k(f)$ converges to $\tau(f)\mathbf{1}$ in norm, as $k \to \infty$, for all $f \in C^*_\lambda(G)$. Since σ is a linear contraction with $\sigma(\mathbf{1}) = \mathbf{1}$, a standard approximation argument shows that it suffices to prove that $\sigma^k(\lambda(s))$ converges to 0, as $k \to \infty$, for all $s \in G \setminus \{e\}$, as the span of $\{\lambda(g) : g \in G\}$ is dense in $C^*_\lambda(G)$.

Fix $s \in G\setminus\{e\}$, and let H be the subgroup of G generated by s and $g_1, g_2, \dots, g_n$. By Proposition 4.3.5, H has the rapid decay property. By Remark 4.3.3 we conclude that H has the rapid decay property with respect to the word length function L corresponding to the generators $s, g_1, g_2, \dots, g_n$. Let P be the polynomial from Proposition 4.3.4 corresponding to this word length function. It is easy to see that if f an element in $\mathbb{C}G$ supported on $B_L(r)$, for some $r > 0$, then $\lambda(g_k)f\lambda(g_k)^*$ is supported on $B_L(r+2)$, for every $k \in \{1, 2, \dots, n\}$. In particular, $\sigma(f)$ is supported on $B_L(r+2)$. An induction argument shows that $\sigma^k(f)$ is supported on $B_L(r+2k)$, so it follows by the choice of P and $g_1, \dots, g_n$, that

$$\|\sigma^k(\lambda(s))\| \leq P(2+2k)\|\sigma^k(\lambda(s))\|_2 \leq P(2+2k)c^k\|\lambda(s)\|_2.$$

Since $c \in (0,1)$, we know that $P(2+2k)c^k$ goes to zero, as $k \to \infty$. This proves that $\sigma^k(\lambda(s))$ goes to zero in norm, as wanted. □

It is natural to ask at this point whether the theorem above can be used to prove that the Thompson group F is non-amenable. Recall that we have proved in Theorem 3.3.1 that T is not inner amenable and in Theorem 4.2.2 that F is non-amenable if T is C^*-simple. As kindly pointed out to us by Valette, T does not have the rapid decay property, so that the above theorem does not apply. This seems to be known, see [14], but we will briefly indicate the argument given to us by Valette. The key point is that T has an amenable subgroup with exponential growth, which by Proposition 4.3.5 and Theorem 4.3.6 will exclude the possibility that T has the rapid decay property. As explained in [17, Section 3], F contains a subgroup isomorphic to $\mathbb{Z}\wr\mathbb{Z}$. Recall that $\mathbb{Z}\wr\mathbb{Z}$ is the group $\bigoplus_{\mathbb{Z}}\mathbb{Z} \rtimes \mathbb{Z}$, that is, the semi-direct product of $\bigoplus_{\mathbb{Z}}\mathbb{Z}$ and $\mathbb{Z}$ where $\mathbb{Z}$ acts on the index set of $\bigoplus_{\mathbb{Z}}\mathbb{Z}$ by translation. This group is clearly amenable, and it is not difficult to see that it has exponential growth.

We end this section with some applications of the criterion for C^*-simplicity obtained above. First, let us recall the following theorem of Akemann and Walter [1]. For the definition of Kazhdan's property (T) the reader may consult [3].

Theorem 4.4.6. *ICC groups with Kazhdan's property (T) are non-inner amenable.*

Combining our criterion with the theorem above, we obtain C^*-simplicity and the unique trace property for the following groups.

Non-abelian free groups

That the non-abelian free groups are C^*-simple with unique trace was proved by Powers [60] in 1975. That these groups are not inner amenable was proved by Effros [26] the same year, and that they have the rapid decay property was proved by Haagerup [35] four years later.

Free products $G * H$ of rapid decay groups with $(|G|-1)(|H|-1) \geq 2$

Such free products were proved to be C^*-simple with the unique trace property by Paschke and Salinas [59] in 1979, even without the assumption of rapid decay. Furthermore, these groups were shown to be non-inner amenable by Chifan, Sinclair and Udrea [15] in 2016 (see Example 2.4(a), Proposition 4.4 and Theorem 4.5 therein). Moreover, Jolissaint [39] proved that the rapid decay property is preserved by free products (see Corollary 2.2.3 therein).

ICC amalgamated free products $G *_A H$ with A finite

These groups have the rapid decay property by work of Jolissaint [39], since A is finite (see Theorem 2.2.2 therein). Their non-inner amenability follows from Example 2.4(a), Theorem 4.5 and (the proof of) Proposition 4.4 in [15].

Co-compact ICC lattices in $\mathrm{SL}(3,\mathbb{R})$ *and* $\mathrm{SL}(3,\mathbb{C})$

These groups have the rapid decay property by [47]. They also have Kazhdan's property (T), see, for example, [3, Theorem 1.4.15] and [3, Theorem 1.7.1].

Let us end this section with a few relative versions of Theorem 4.4.5.

Proposition 4.4.7. *If G has the rapid decay property, and H is a subgroup such that the conjugation action of H on $G \setminus \{e\}$ is non-amenable, then*

$$\overline{\mathrm{conv}}\{uxu^* : u \in C^*_\lambda(H) \textit{ is unitary}\} \cap \mathbb{C}\mathbf{1} \neq \emptyset,$$

for all $x \in C^*_\lambda(G)$. In particular, both G and H are C^*-simple with the unique trace property.

Proof. The proof is exactly the same as the one of Theorem 4.4.5, with the change that the elements $g_1, g_2, \ldots, g_n$ are chosen in H. It is easy to see from the proof of Proposition 3.2.5 that we may choose the elements in H, as the action of H on $G \setminus \{e\}$ is assumed to be amenable. This means that $\sigma(x)$ is in $\{uxu^* : u \in C^*_\lambda(H) \text{ is unitary}\}$, for all $x \in C^*_\lambda(G)$, so that we get the desired conclusion. □

In order to state the next result, recall the following definition of Jolissaint [42].

Definition 4.4.8. A proper subgroup H of a group G is said to be *inner amenable relative to* G if the conjugation action of H on $G \setminus H$ is amenable.

Proposition 4.4.9. *Suppose that H is a C^*-simple subgroup of a group G, which is not inner amenable relative to G. Suppose further that there exist $C, r > 0$ and a length function on G such that $\|\lambda(f)\| \leq \|f\|_{r,L}$, for all $f \in \mathbb{C}G$ which are supported on $G \setminus H$. Then*

$$\overline{\mathrm{conv}}\{uxu^* : u \in C^*_\lambda(H) \textit{ is unitary}\} \cap \mathbb{C}\mathbf{1} \neq \emptyset,$$

for all $x \in C^*_\lambda(G)$. In particular, G is C^*-simple and has the unique trace property.

Proof. Let us start by making a few comments on the assumptions. First of all, it is easy so see from the proof of Proposition 3.2.5 that we can obtain a relative version of this. More precisely, we can choose elements $h_1, \ldots, h_m \in H$ and a positive real number $c < 1$, such that

$$\Big\| \frac{1}{m} \sum_{j=1}^{m} \lambda(g_j) x \lambda(g_j)^* \Big\|_\tau \leq c\|x\|_\tau,$$

this time not for all $x \in C^*_\lambda(G)$ with trace zero, but for all x with $\mathcal{E}_H(x) = 0$, where $\mathcal{E}_H$ is the conditional expectation from $C^*_\lambda(G)$ to $C^*_\lambda(H)$, that is, for all x in the closure of the subspace $\{\lambda(f) : f \in \mathbb{C}G$ supported on $G \setminus H\}$.

It is not difficult to see from the proof of Proposition 4.3.4 that, in the case of our relative version of the rapid decay property, we can choose a polynomial P such that

$$\|\lambda(f)\| \leq P(r)\|f\|_2,$$

for all $f \in \mathbb{C}G$ supported on $B_L(r)$.

By a standard approximation argument, it suffices to show that, given $\varepsilon > 0$ and $f \in \mathbb{C}G$ with trace zero, there exist unitaries $u_1, \ldots, u_n \in C^*_\lambda(H)$, such that

$$\Big\| \frac{1}{n} \sum_{j=1}^{n} u_j f u_j^* \Big\| < \varepsilon. \tag{4.1}$$

First of all, we may write $f = f_1 + f_2$ with f_1 supported on H and f_2 supported on $G \setminus H$. By arguments identical to those in the proof of Theorem 4.4.5, we get that $\|\sigma^k(f_2)\| \to 0$, as $k \to \infty$, where σ is defined by $\sigma(f) = \frac{1}{m}\sum_{j=1}^{m} \lambda(h_j) f \lambda(h_j)^*$ as in there . In particular, we may choose $k \in \mathbb{N}$ so that $\|\sigma^k(f_2)\| < \frac{\varepsilon}{2}$. Since $h_1, \ldots, h_m \in H$, we see that $\sigma^k(f_1)$ is an element of $\mathbb{C}G$ which is supported on H. Moreover, it is again an element of trace zero since $\{e\}$ is conjugation invariant. As H is C^*-simple by assumption, we get from Remark 4.4.4 that $C^*_\lambda(H)$

has the Dixmier property. Thus we may choose unitaries $v_1, \ldots, v_n$ in $C^*_\lambda(H)$ so that $\|\frac{1}{n}\sum_{j=1}^n v_j\sigma^l(f_1)v_j^*\| < \frac{\varepsilon}{2}$. By putting these things together, we see that

$$\Big\|\frac{1}{n}\sum_{j=1}^n v_j\sigma^k(f)v_j^*\Big\| \leq \Big\|\frac{1}{n}\sum_{j=1}^n v_j\sigma^k(f_1)v_j^*\Big\| + \frac{1}{n}\sum_{j=1}^n \|v_j\sigma^k(f_2)v_j^*\| < \varepsilon.$$

Since $\frac{1}{n}\sum_{j=1}^n v_j\sigma^k(f)v_j^*$ is an averaging of the form (4.1) with mnk unitaries, this concludes the proof. $\square$

Chapter 5

Families of irreducible representations

In this chapter we will construct one-parameter families of representations of the Thompson group F. We will investigate when these representations are irreducible and when they are unitarily equivalent. The work in this chapter is inspired by a result by Garncarek [30] from 2012. In this paper he constructs an analogue for the Thompson group F of a one-parameter family of representations of $\mathrm{PSL}(2, \mathbb{R})$, known as the principal series. Garncarek proves that these representations are all irreducible and mutually equivalent exactly when the parameters differ by an integer multiple of $\frac{2\pi}{\log 2}$. The representations we will construct in this chapter are similar to those of Garncarek, but with different realizations of F as homeomorphisms of the unit interval. We would like to thank Monod for suggesting this problem to us, during a longer stay in Lausanne.

Throughout this chapter we will let $\mathcal{H}$ denote the Hilbert space $\mathrm{L}^2([0,1], m)$. Moreover, given points $a, b \in [0,1]$ with $a < b$, we let $\mathcal{H}_{a,b}$ denote the Hilbert space $\mathrm{L}^2([a,b], m)$, and we let $F_{a,b}$ denote the subgroup of F consisting of elements f satisfying $f(x) = x$, for all $x \notin [a,b]$. For convenience we will return to thinking of the elements in F as functions of $[0,1]$ rather than $[0,1)$.

5.1 One map, lots of representations

In this section, we describe a way of constructing a one-parameter family of irreducible representations from certain homeomorphisms of $[0,1]$, as well as investigate when these representations are unitarily equivalent. First, however, we need to introduce a few notions from measure theory. Note that we only consider positive measures.

Recall that a set $\mathfrak{X}$ with a σ-algebra Σ is called a *measurable space*. The elements of Σ are called *measurable sets* and a map between measurable spaces is called a *measurable map* if pre-images of measurable sets are measurable. If μ is a measure on $\mathfrak{X}$, then $(\mathfrak{X}, \mu)$ is called a *measure space*. A measurable set is called a *null set* if it has measure zero, and the complement of a null set is called a *co-null set*. In case the space $\mathfrak{X}$ is a topological space, the σ-algebra generated by the topology is called the *Borel σ-algebra* and its elements are called *Borel sets*. Moreover, a measure on the Borel σ-algebra is called a *Borel measure*. As we will not be handling spaces with multiple σ-algebras at the same time, we will suppress the σ-algebra, and measurability of a set should be understood as relative to the given σ-algebra.

Definition 5.1.1. Let ν and μ be two measures on a measurable space $\mathfrak{X}$. The measure ν is said to be *absolutely continuous* with respect to the measure μ if every μ-null set is a ν-null set. The measures μ and ν are said to be *equivalent measures* if they are absolutely continuous with respect to each other.

Definition 5.1.2. A *measurable action* of a group G on a measurable space $\mathfrak{X}$ is an action α on the set $\mathfrak{X}$ such that the map $(g, x) \mapsto \alpha(g)x$ is measurable from $G \times \mathfrak{X}$ to $\mathfrak{X}$. If μ is a measure on $\mathfrak{X}$, then the action α is said to leave the measure *quasi-invariant* if the image measure $\alpha(g)_*\mu$ is equivalent to μ, for all $g \in G$.

It is easy to see that if G is a countable group and α is an action of G on a measurable space $\mathfrak{X}$, the action is measurable if and only if $\alpha(g)$ is a measurable map, for each $g \in G$. In this chapter all our action will be actions by homeomorphisms meaning that $\alpha(g)$ is a homeomorphism, for all $g \in G$. In particular, these actions will be measurable actions, and we will use this fact implicitly.

Recall that a measure space is said to be *σ-finite* if it is a countable union of sets of finite measure. The following is the Radon-Nikodym theorem, which characterizes when σ-finite measures are absolutely continuous with respect to each other. A proof can be found in [28, Proposition 3.8].

Theorem 5.1.3. *Let μ and ν be σ-finite measures on $\mathfrak{X}$. Then μ is absolutely continuous with respect to ν if and only if there exists a non-negative measurable function f, such that*

$$\mu(E) = \int_E f(x)\, \mathrm{d}\nu(x),$$

for all measurable sets $E \subset \mathfrak{X}$. Moreover, any two such functions are identical almost everywhere.

Definition 5.1.4. With the set-up of Theorem 5.1.3, the function f is called the *Radon-Nikodym derivative* of μ with respect to ν, and is denoted by $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

Remark 5.1.5. Suppose that μ and ν are two measures on a measurable space $\mathfrak{X}$ so that ν is absolutely continuous with respect to μ. If $E \subseteq \mathfrak{X}$, is measurable set such that μ and ν are equal when restricted to E, that is, $\mu(A) = \nu(A)$, for all $A \subseteq E$ measurable, then $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 1$ almost everywhere on E. Let us explain why this is the case. It is easy to see from the definition of absolute continuity that

$\nu|_E$, the restriction of ν to E, will be absolutely continuous with respect to $\mu|_E$, the restriction of μ to E. From the uniqueness of the Radon-Nikodym derivative, we see that $\frac{\mathrm{d}(\nu|_E)}{\mathrm{d}(\mu|_E)} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\big|_E$, and, since $\mu|_E = \nu|_E$, the right hand side must be equal to 1 almost everywhere on E.

We are now ready to define the representations of F we are interested in. Given an order preserving homeomorphism φ of $[0,1]$ and an element $g \in F$, we obtain a new order preserving homeomorphism, namely $\varphi^{-1} \circ g \circ \varphi$, which we denote by g^φ. To ease the notation, we will denote $(g^{-1})^\varphi$ by $g^{-\varphi}$ so that $g^{-\varphi}$ is the inverse of g^φ. Furthermore, we let F^φ denote the group $\{g^\varphi : g \in F\}$ of homeomorphism of $[0,1]$, which by construction is isomorphic to F.

We are particularly interested in certain homeomorphisms of $[0,1]$, and to simplify matters later on we will adopt the terminology that a function φ satisfies *condition ($\star$)* if

($\star$) φ is an order preserving homeomorphism of $[0,1]$, and the action of F^φ on $[0,1]$ leaves m quasi-invariant, that is, the image measure $g^\varphi_* m$ is equivalent to m, for all $g \in F$.

It is not difficult to find homeomorphisms φ which satisfy condition ($\star$), for example, any increasing homeomorphism which is continuously differentiable will do so. However, as it will be apparent later, we are more interested in less nice functions. In fact, an increasing homeomorphism will satisfy condition ($\star$) if and only if g^φ is an absolutely continuous function, for all $g \in F$. We will come back to this later.

Remark 5.1.6. Suppose that φ satisfies condition ($\star$). The fact that the action of F^φ leaves m quasi-invariant means that E is a null set if and only if $g^\varphi(E)$ is a null set, for all $g \in F$. It follows from the definition that if $\frac{\mathrm{d}g^\varphi_* m}{\mathrm{d}m}$ is zero on a measurable set E, then E is a $g^\varphi_* m$-null set. Thus we see that these Radon-Nikodym derivatives are strictly positive almost everywhere on $[0,1]$.

Now, given a measurable action of a group on a measure space such that the action leaves the measure quasi-invariant, there is a well-known way to construct a one-parameter family of representations. In our particular case, the representations are defined as follows. Given a real number s and a function φ satisfying condition ($\star$), we let π^φ_s denote the representation of F on $\mathcal{H}$ given by

$$\pi^\varphi_s(g)f(x) = \Big(\frac{\mathrm{d}g^\varphi_* m}{\mathrm{d}m}(x)\Big)^{\frac{1}{2}+is} f \circ g^{-\varphi}(x)$$

almost everywhere on $[0,1]$, for $f \in \mathcal{H}$ and $g \in F$. It is straightforward to check that this is actually a unitary representation of F. In the case $s = 0$, this representation is known as the Koopman representation corresponding to the action of F on the interval $[0,1]$.

Remark 5.1.7. If we choose φ to be the identity id on $[0,1]$, then it clearly satisfies condition ($\star$), and the one-parameter family of representations $(\pi^{\mathrm{id}}_s)_{s\in\mathbb{R}}$ is the one of Garncarek [30]. Among other things, he proved that these representations are all irreducible, and that π_s^{id} and π_t^{id} are unitarily equivalent if and only if $s = t + \frac{2\pi}{\log 2}k$, for some $k \in \mathbb{Z}$.

Our goal in this section is to investigate these representations, or, more precisely, to investigate when they are irreducible and when they are equivalent to each other. To begin with, we have an easy technical lemma about elements in the Thompson group.

Lemma 5.1.8. *Let φ be an increasing homeomorphism of $[0,1]$ and let $a, b \in [0,1]$ with $a < b$. Given $c, d, c', d' \in (a,b)$ with $c < d$ and $c' < d'$, there exists some $g \in F_{\varphi(a),\varphi(b)}$ satisfying $g^\varphi([c,d]) \subseteq [c', d']$.*

Proof. Given $g \in F$, it is straightforward to check that $g^\varphi([c,d]) \subseteq [c', d']$ if and only if $g([\varphi(c), \varphi(d)]) \subseteq [\varphi(c'), \varphi(d')]$, so the existence of $g \in F$ with the desired properties follows directly from Lemma 2.2.5. □

The following lemma is an easy adaptation of a lemma of Garncarek [30] and will be the key point of the investigation of the representations of the form π_s^φ.

Lemma 5.1.9. *Suppose that φ satisfies condition ($\star$). Fix some $0 \leq a < b \leq 1$ and let $f \in \mathcal{H}$. Then $f \in \mathcal{H}_{a,b}^\perp$ if and only if $\pi_s^\varphi(g)f = f$, for all $g \in F_{\varphi(a),\varphi(b)}$.*

Proof. Fix $g \in F_{\varphi(a),\varphi(b)}$ and $f \in \mathcal{H}_{a,b}^\perp$. It is easy to see that $g^\varphi \in F_{a,b}$ so that g^φ is constant on the intervals $[0,a]$ and $[b,1]$. In particular, the two measures $g_*^\varphi m$ and m agree on this interval, so it follows from Remark 5.1.5 that the radon Nikodym derivative in the definition of $\pi_s^\varphi(g)$ is constantly equal to 1 on the interval $[0,a]$. Obviously this implies that $\pi_s^\varphi(g)f = f$.

Suppose instead that $f \notin \mathcal{H}_{a,b}^\perp$, and let us find some $g \in F_{\varphi(a),\varphi(b)}$ such that $\pi_s^\varphi(g)f \neq f$. By assumption $\int_a^b |f(x)|^2 \,\mathrm{d}m(x) > 0$, so there exists $c, d \in (a,b)$ with $c < d$ such that $\int_c^d |f(x)|^2 \,\mathrm{d}m(x) > 0$. Choose also $c', d' \in (a,b)$ with $a < c' < d' < b$ such that

$$\int_{c'}^{d'} |f(x)|^2 \,\mathrm{d}m(x) < \int_c^d |f(x)|^2 \,\mathrm{d}m(x).$$

By Lemma 5.1.8, we choose $g \in F_{\varphi(a),\varphi(b)}$ satisfying $g^{-\varphi}([c,d]) \subseteq [c', d']$. Then

$$\begin{aligned}\int_c^d |\pi_s^\varphi(g)f(x)|^2 \,\mathrm{d}m(x) &= \int_c^d \left|f \circ g^{-\varphi}(x)\right|^2 \mathrm{d}(g^\varphi)_* m(x) \\ &= \int_{g^{-\varphi}(c)}^{g^{-\varphi}(d)} |f(x)|^2 \,\mathrm{d}m(x) \\ &\leq \int_{c'}^{d'} |f(x)|^2 \,\mathrm{d}m(x) < \int_c^d |f(x)|^2 \,\mathrm{d}m(x).\end{aligned}$$

This proves that $\pi_s^\varphi(g)f \neq f$, since we now have

$$\int_c^d |\pi_s^\varphi(g)f(x)|^2 \,\mathrm{d}m(x) < \int_c^d |f(x)|^2 \,\mathrm{d}m(x). \qquad \square$$

The above lemma describes certain subspaces of $\mathcal{H}$ in terms of the representations π_s^φ, which will enable us to say something about when these representations are unitary equivalent for different φ and s. However, first we will recall the notion of unitary equivalence of representations.

Definition 5.1.10. Let σ_1 and σ_2 be representations of G on Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. The two representations are said to be *unitarily equivalent* if there exists a unitary operator $U\colon \mathcal{K}_1 \to \mathcal{K}_2$ so that $\sigma_2(g) = U\sigma_1(g)U^*$, for all $g \in G$.

Now, let us prove the following lemma about unitary operators implementing a unitary equivalence between representations of the form π_s^φ.

Lemma 5.1.11. *Let $s, t \in \mathbb{R}$ and suppose that φ and ψ satisfy condition ($\star$) and the representations π_s^φ and π_t^ψ are unitarily equivalent. Let U be an implementing unitary, that is, a unitary operator on $\mathcal{H}$ with $U\pi_s^\varphi(g)U^* = \pi_t^\psi(g)$, for all $g \in F$. Then $f \in \mathcal{H}_{\varphi^{-1}(a),\varphi^{-1}(b)}$ if and only if $Uf \in \mathcal{H}_{\psi^{-1}(a),\psi^{-1}(b)}$, for every $f \in \mathcal{H}$ and a and b with $0 \le a < b \le 1$.*

Proof. Fix a and b with $0 \le a < b \le 1$ and let us for simplicity denote the Hilbert spaces $\mathcal{H}_{\varphi^{-1}(a),\varphi^{-1}(b)}$ and $\mathcal{H}_{\psi^{-1}(a),\psi^{-1}(b)}$ by $\mathcal{K}_\varphi$ and $\mathcal{K}_\psi$, respectively. Our goal is to prove that $f \in \mathcal{K}_\varphi$ if and only if $Uf \in \mathcal{K}_\psi$. Let us prove that the latter imply the former. The converse implication then follows by interchanging φ and ψ, as well as U and U^*.

Suppose that $f \notin \mathcal{K}_\varphi$. Write $f = f_1 + f_2$ with $f_1 \in \mathcal{K}_\varphi$ and $f_2 \in \mathcal{K}_\varphi^\perp$. Our assumption is that $f_2 \neq 0$, which, of course, implies that $Uf_2 \neq 0$. As U is unitary, Uf_1 and Uf_2 are orthogonal, so, if we can prove that $Uf_2 \in \mathcal{K}_\psi^\perp$, then this will mean that $Uf \notin \mathcal{K}_\psi$. Now, as $f_2 \in \mathcal{K}_\varphi^\perp$, we know from Lemma 5.1.9 that $\pi_s^\varphi(g)f = f$, for every $g \in F_{a,b}$. Note that the lemma was applied with the points $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$, rather than a and b. We now see that $\pi_t^\psi(g)Uf = U\pi_s^\varphi(g)f = Uf$, for every $g \in F_{a,b}$. By applying Lemma 5.1.9 again, we conclude that $Uf_2 \in \mathcal{K}_\psi^\perp$, as wanted. $\square$

Recall that $\mathrm{L}^\infty([0,1], m)$ acts as multiplication operators on $\mathcal{H}$, and it is a maximal abelian subalgebra of $B(\mathcal{H})$. Thus every operator which commutes with these is itself a multiplication operator. Moreover, as $\{\mathbf{1}_{[a,b]} : 0 \le a < b \le 1\}$ spans a strong operator dense subalgebra of $\mathrm{L}^\infty([0,1], m)$, every operator commuting with these is also a multiplication operator. With this in mind, let us prove the following lemma giving a characterization of unitary equivalence of π_s^φ for different s.

Lemma 5.1.12. *Suppose that φ satisfies condition ($\star$). If $s, t \in \mathbb{R}$ and U is a unitary operator on $\mathcal{H}$ with $U\pi_s^\varphi(g)U^* = \pi_t^\varphi(g)$, for all $g \in F$, then U is a multiplication operator, multiplication by u say, and satisfies*

$$u \circ g^{-\varphi}(x) = \Big(\frac{\mathrm{d}(g^\varphi)_* m}{\mathrm{d}m}(x)\Big)^{i(s-t)} u(x)$$

almost everywhere, for all $g \in F$. Moreover, if there exists some measurable function u from $[0,1]$ to the complex unit circle satisfying the above equality, then π_s^φ and π_t^φ are unitarily equivalent with multiplication by u as implementing unitary operator.

Proof. First assume that U is a unitary operator satisfying $U\pi_s^\varphi(g)U^* = \pi_t^\varphi(g)$, for all $g \in F$. By applying Lemma 5.1.11 in the case where $\psi = \varphi$, we see that the unitary U must satisfy $U(\mathcal{H}_{a,b}) = \mathcal{H}_{a,b}$, for all $a, b \in [0,1]$ with $a < b$, or, in other words, U commutes with $\mathbf{1}_{[a,b]}$, for all such a and b. As noted above, this means that U is a multiplication operator, in fact, multiplication by $u = U(\mathbf{1})$. The desired formula for u is obtained by writing out the equation $U\pi_s^\varphi(g)\mathbf{1} = \pi_t^\varphi(g)U\mathbf{1}$ and dividing by the Radon-Nikodym derivative from π_t^φ. This is possible, as the Radon-Nikodym derivative is non-zero almost everywhere by Remark 5.1.6.

It is straightforward to check that π_s^φ and π_t^φ are unitarily equivalent if a function u with the stated property exists. □

As we, among other things, are interested in determining when the representations π_s^φ are irreducible, let us first recall what this means.

Definition 5.1.13. A representation σ of a group G on a Hilbert space $\mathcal{K}$ is said to be *irreducible* if it has no invariant subspaces. That is, if $\mathcal{K}_0$ is a closed subspace of $\mathcal{K}$ so that $\sigma(g)\mathcal{K}_0 \subseteq \mathcal{K}_0$, for all $g \in G$, then either $\mathcal{K}_0 = \{0\}$ or $\mathcal{K}_0 = \mathcal{K}$.

The following theorem about irreducible representations is known as Schur's Lemma, and a proof can be found in [3, Theorem A.2.2].

Theorem 5.1.14. *Let σ be a representation of a group G on a Hilbert space $\mathcal{K}$. Then σ is irreducible if and only if $\sigma(G)' = \mathbb{C}\mathbf{1}$, that is, if and only if the only operators in $B(\mathcal{K})$ which commute with $\sigma(g)$, for all $g \in G$, are scalar multiples of the identity.*

Before continuing let us recall the definition of ergodic actions and essentially invariant functions, as well as the connection between these.

Definition 5.1.15. A measurable action α of a group G on a measure space $(\mathfrak{X}, \mu)$ is said to be *ergodic* if there are no non-trivial, invariant, measurable subsets of $\mathfrak{X}$. That is, if $A \subseteq \mathfrak{X}$ is measurable with $\alpha(g)A = A$, for all $g \in G$, then A is either a null set or a co-null set.

Definition 5.1.16. Let G be a group and α a measurable action of G on a measure space $(\mathfrak{X}, \mu)$. A measurable function $f \colon \mathfrak{X} \to \mathbb{C}$ is said to be *essentially invariant* if $f(\alpha(g)x) = f(x)$, for all $g \in G$ and μ-almost all $x \in \mathfrak{X}$.

The following is a characterization of ergodic actions in terms of essentially invariant functions, and a proof can be found in [4, Theorem 1.3].

Theorem 5.1.17. *An action of a group on a σ-finite measure space which leaves the measure quasi-invariant is ergodic if and only if every essentially invariant function is constant almost everywhere.*

We are now ready to give a characterization of when the representations of the form π_s^φ are irreducible.

Proposition 5.1.18. *Suppose that φ satisfies condition ($\star$) and let $s \in \mathbb{R}$. The representation π_s^φ is irreducible if and only if the action of F^φ on $([0,1], m)$ is ergodic.*

Proof. By Schur's Lemma, Theorem 5.1.14 above, the representation is irreducible if and only if the commutant of $\pi_s^\varphi(F)$ is trivial. As the commutant is a C^*-algebra, it is spanned by its unitaries, so we just need to prove that the action is ergodic if and only if every unitary operator commuting with $\pi_s^\varphi(F)$ is a multiple of the identity. Supposing that U is a unitary operator on $\mathcal{H}$ which commutes with $\pi_s^\varphi(g)$, for all $g \in F$, we get from Lemma 5.1.12, applied in the case where $t = s$, that U is a multiplication operator, namely, multiplication by $u = U\mathbf{1}$. Furthermore, the lemma tells us that $u \circ g^\varphi(x) = u(x)$ almost everywhere, for all $g \in F$. Now, if we assume that the action is ergodic, then u is constant almost everywhere, by Theorem 5.1.17, which means that U is a scalar multiple of the identity. Thus, as U was arbitrary, the commutant only consists of the scalars. If we instead assume that the action is not ergodic, then, by Theorem 5.1.17, there exists a measurable function $h\colon [0,1] \to \mathbb{C}$ which is essentially invariant and not constant almost everywhere. We may assume that the function is real valued, as we may just replace it with its real or imaginary part. Both of these will also be essentially invariant, and they cannot both be constant almost everywhere. Now, as h is real valued, the function u on $[0,1]$ defined by $u(x) = \exp(ih(x))$ will be a function whose values lie on the complex unit circle, which is not constant almost everywhere and is essentially invariant. By Lemma 5.1.12 the corresponding unitary operator lies in the commutant, and, as the function is not constant almost everywhere, this unitary is not a scalar multiple of the identity. $\square$

We now turn to the second problem, namely, to determine when the different representations of the form π_s^φ are unitarily equivalent. First we need a few more standard results from measure theory about measures on the unit interval. A proof of the following result can be found in [45, Theorem 17.10].

Theorem 5.1.19. *All finite Borel measures on $[0,1]$ are* regular*, that is, all finite Borel measures μ on $[0,1]$ satisfy*

$$\begin{aligned}\mu(A) &= \sup\{\mu(K) : K \subseteq [0,1] \text{ compact with } K \subseteq A\}\\ &= \inf\{\mu(U) : U \subseteq [0,1] \text{ open with } A \subseteq U\}\end{aligned}$$

for every measurable set $A \subseteq [0,1]$.

The above result holds in a much more general setting than the interval $[0,1]$, but we are only interested in this particular case. The following lemma is an easy application of the theorem above, Theorem 5.1.3 and the Lebesgue dominated convergence theorem.

Lemma 5.1.20. *Let μ and ν be finite Borel measures on $[0,1]$. If there exists a non-negative μ-integrable function h such that*

$$\nu(I) = \int_I h(x)\,\mathrm{d}\mu(x),$$

for all closed intervals $I \subseteq [0,1]$, then ν is absolutely continuous with respect to μ, and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = h$.

Recall the following straightforward result about absolute continuity and Radon-Nikodym derivatives.

Lemma 5.1.21. *Suppose that μ and ν are σ-finite measures on $[0,1]$ with ν absolutely continuous with respect to μ. If φ is a measurable bijection of $[0,1]$, then the image measure $\varphi_*\nu$ is absolutely continuous with respect to the image measure $\varphi_*\mu$ and*

$$\frac{\mathrm{d}\varphi^*\nu}{\mathrm{d}\varphi^*\mu} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \circ \varphi^{-1}.$$

Furthermore, if μ_1, μ_2 and μ_3 are finite measures on $[0,1]$ such that μ_3 is absolutely continuous with respect to μ_2 and μ_2 is absolutely continuous with respect to μ_1, then μ_3 is absolutely continuous with respect to μ_1 and

$$\frac{\mathrm{d}\mu_3}{\mathrm{d}\mu_2} \cdot \frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1} = \frac{\mathrm{d}\mu_3}{\mathrm{d}\mu_1}.$$

We are now ready to prove a result about when the representations of the form π_s^φ are unitarily equivalent for different φ and s.

Proposition 5.1.22. *Suppose that φ and ψ both satisfy condition ($\star$) and let $s, t \in \mathbb{R}$. If the representations π_s^φ and π_t^ψ are unitarily equivalent, then the measures $\varphi_* m$ and $\psi_* m$ are equivalent. Moreover, if the measures $\varphi_* m$ and $\psi_* m$ are equivalent and $s = t$, then the representations π_s^φ and π_t^ψ are unitarily equivalent.*

Proof. The last of the two statements is straightforward to check. Indeed, supposing that the measures are equivalent, if follows from Lemma 5.1.21 that both the measures $(\psi^{-1}\varphi)_* m$ and $(\varphi^{-1}\psi)_* m$ are absolutely continuous with respect to m, and a direct computation shows that the formula

$$Uf(x) = \Big(\frac{\mathrm{d}(\psi^{-1}\varphi)_* m}{\mathrm{d}m}(x)\Big)^{\frac{1}{2}+is} f\varphi^{-1}\psi(x)$$

defines a unitary operator U on $\mathcal{H}$, with inverse

$$U^* f(x) = \Big(\frac{\mathrm{d}(\varphi^{-1}\psi)_* m}{\mathrm{d}m}(x)\Big)^{\frac{1}{2}+is} f\psi^{-1}\varphi(x),$$

satisfying $U\pi_s^\varphi(g)U^* = \pi_s^\psi(g)$, for all $g \in F$. Hence the representations are unitarily equivalent if the measures are equivalent.

Assume that the two representation are unitarily equivalent. Choose some unitary operator U on $\mathcal{H}$ with $U\pi_s^\varphi(g)U^* = \pi_t^\psi(g)$, for all $g \in F$. This time U is not necessarily a multiplication operator, but it turns out that we can still say something about what U looks like. Let $h = U\mathbf{1}$. Then $h \in \mathcal{H}$, so $|h \circ \psi^{-1}|^2$ is $\psi_* m$-integrable. We will show that

$$\varphi_* m(I) = \int_I |h \circ \psi^{-1}(x)|^2 \,\mathrm{d}\psi_* m(x),$$

for all closed intervals $I \subseteq [0,1]$. In particular, this proves that $\varphi_* m$ is absolutely continuous with respect to $\psi_* m$ by Lemma 5.1.20.

Using Lemma 5.1.11 we see that, for all $a, b \in [0,1]$ with $a < b$, we have

$$U(f|_{(\varphi^{-1}(a),\varphi^{-1}(b))}) = U(f)|_{(\psi^{-1}(a),\psi^{-1}(b))}.$$

In particular, for all closed intervals $I \subseteq [0,1]$, we get that $U(\mathbf{1}_{\varphi^{-1}(I)}) = h\mathbf{1}_{\psi^{-1}(I)}$. This shows that

$$\begin{aligned}\varphi_* m(I) = \|\mathbf{1}_{\varphi^{-1}(I)}\|_2^2 = \|U(\mathbf{1}_{\varphi^{-1}(I)})\|_2^2 &= \int_{\psi^{-1}(I)} |h(x)|^2 \,\mathrm{d}m(x) \\ &= \int_I |h \circ \psi^{-1}(x)|^2 \,\mathrm{d}\psi_* m(x),\end{aligned}$$

for all closed intervals $I \subseteq [0,1]$. By Lemma 5.1.20, this means that $\varphi_* m$ is absolutely continuous with respect to $\psi_* m$. Repeating the argument with φ and ψ interchanged, we get that they are in fact equivalent. $\square$

As mentioned earlier, we are not particularly interested in too "nice" functions satisfying condition ($\star$), and the proposition above tells us exactly why. If the function is too nice, then it will preserve the Lebesgue measure class and we will not really get anything different than if we chose φ to be the identity on $[0,1]$. However, we do need g^φ to be nice enough to preserve the Lebesgue measure class, for every $g \in F$, so we are interested in a strange kind of functions. An example of such a function is the Minkowski question mark function, described in Section 2.3; see Remark 5.4.9.

So far we know exactly when the representations π_s^φ and π_t^ψ are unitarily equivalent for $s = t$, but for $s \neq t$ we only get a necessary criterion, namely, that the measures $\varphi_* m$ and $\psi_* m$ are equivalent. In case the two measures indeed are equivalent and $s \neq t$, we cannot, at the moment, say anything about whether or not the representations are equivalent. One would be tempted to conjecture that the representations π_s^φ and π_t^φ are always equivalent, or that they are never equivalent, except when $s = t$. Neither of these statements are true. In fact, Garncarek [30] proved that, if φ is the identity on $[0,1]$, then the representations are equivalent if and only if $s - t = \frac{2\pi k}{\log 2}$, for some $k \in \mathbb{Z}$. Thus, the question of equivalence for different values of the parameter is a bit more subtle than that, as evidenced also by the following proposition.

Proposition 5.1.23. *Suppose that φ satisfies condition $(\star)$ and let H denote the set of $t \in \mathbb{R}$ such that π_t^φ is equivalent to π_0^φ. Then H is a subgroup of $\mathbb{R}$, and, given $s, t \in \mathbb{R}$, the two representations π_s^φ and π_t^φ are unitarily equivalent if and only if $s + H = t + H$ in $\mathbb{R}/H$.*

Proof. It is easy to see from Lemma 5.1.12 that whether or not π_s^φ and π_t^φ are unitarily equivalent only depends on the difference $s - t$. Indeed, if u is the function from Lemma 5.1.12 whose multiplication operator implements the unitary equivalence between π_s^φ and π_t^φ for some $s, t \in \mathbb{R}$, then this same function implements the equivalence between $s + r$ and $t + r$, for all $r \in \mathbb{R}$. Thus, the latter part of the statement follows from the fact that H is a subgroup of $\mathbb{R}$. Now, using the above, we see that π_t^φ is equivalent to π_0^φ if and only if π_{-t}^φ is equivalent to π_0^φ, by setting $r = -t$. Also, clearly $0 \in H$. Thus, we only need to prove that $s+t \in H$ for $s, t \in H$. This is easily checked by hand. If u_s and u_t are functions from Lemma 5.1.12 satisfying the equations corresponding to s and t, respectively, then the function $x \mapsto u_s(x)u_t(x)$ will satisfy the equation corresponding to $s + t$. Thus, the corresponding multiplication operator will implement an equivalence between π_{s+t}^φ and π_0^φ. □

As mentioned in the beginning of this section, we use a well-known method of obtaining a one-parameter family of representation from an action on a measure space which leaves the measure quasi-invariant. It is well-known that such representations are strongly continuous in the parameter. This fact is easily proved directly using the Lebesgue dominated convergence theorem. We will give an argument for our particular representations by proving a well-known general result, which we will need later, and then deduce it from that.

Proposition 5.1.24. *Let $(\mathfrak{X}, \mu)$ be a measure space, Y be a subset of $\mathbb{R}^n$ and $y_0 \in Y$. Suppose that $f\colon \mathfrak{X} \times Y \to \mathbb{R}$ is measurable, and that h is a μ-integrable function with $|f(x,y)| \leq h(x)$, for all $x \in \mathfrak{X}$ and $y \in Y$. If the function $y \mapsto f(x,y)$ is continuous at the point y_0, for almost all $x \in \mathfrak{X}$, then the function from Y to $\mathbb{R}$ defined by*

$$y \mapsto \int f(x,y)\,\mathrm{d}\mu(x)$$

is continuous at the point y_0.

Proof. Clearly the function $x \mapsto f(x,y)$ is μ-integrable, for all $y \in Y$, as its absolute value is bounded above by h. Thus, the function $y \mapsto \int f(x,y)\,\mathrm{d}\mu(x)$ is well-defined from Y to $\mathbb{R}$, and we denote it by F. Suppose that $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y

converging to y_0, and let us prove that $F(y_n)$ converges to $F(y_0)$ as $n \to \infty$. Let $g_n \colon \mathfrak{X} \to \mathbb{R}$ be given by $g_n(x) = f(x, y_n)$, for all $x \in \mathfrak{X}$. By assumption there exists a null set $A \subseteq \mathfrak{X}$ so that $\lim_{n\to\infty} g_n(x) = f(x, y_0)$, for all $x \in \mathfrak{X} \setminus A$. From the Lebesgue dominated convergence theorem we conclude that

$$\lim_{n\to\infty} F(y_n) = \lim_{n\to\infty} \int_{\mathfrak{X}\setminus A} g_n(x)\, \mathrm{d}\mu(x) = \int_{\mathfrak{X}\setminus A} f(x, y_0)\, \mathrm{d}\mu(x) = F(y_0).$$

Note that the first and last equality follow from the fact that A is a null set. Thus, F is continuous at the point y_0, as wanted. □

Proposition 5.1.25. *Suppose that $(\mathfrak{X}, \mu)$ is a measure space. Let Y be a subset of $\mathbb{R}^n$ and F be a map from Y to $\mathrm{L}^2(\mathfrak{X}, \mu)$ so that there exists $\xi \in \mathrm{L}^2(\mathfrak{X}, \mu)$ with $|F(y)(x)| \leq \xi(x)$, for all $x \in \mathfrak{X}$. If, for each $y_0 \in Y$, the function $y \mapsto F(y)(x)$ is continuous at the point y_0, for almost all $x \in \mathfrak{X}$, then F is continuous.*

Proof. By assumption ξ^2 is μ-integrable. Fix $y_0 \in Y$, and let us prove that F is continuous at y_0. Let $f \colon \mathfrak{X} \times Y \to \mathbb{R}$ denote the function given by

$$f(x, y) = |F(y)(x) - F(y_0)(x)|^2.$$

Then $|f(x, y)| \leq 2\xi(x)^2$, for all $x \in \mathfrak{X}$. By assumption, the function $y \mapsto f(x, y)$ is continuous at y_0 for almost all $x \in \mathfrak{X}$. It follows by Proposition 5.1.24 that the map

$$y \mapsto \int f(x, y)\, \mathrm{d}\mu(x) = \|F(y) - F(y_0)\|_2^2.$$

is continuous at the point y_0. Clearly this means that F is continuous at y_0, and we conclude that F is continuous. □

We now easily get the continuity result mentioned earlier.

Proposition 5.1.26. *If φ is a map satisfying condition ($\star$), then the map $s \mapsto \pi_s^\varphi(g)$ from $\mathbb{R}$ to $B(\mathcal{H})$ is strongly continuous, for each $g \in F$.*

Proof. We need to prove that the map $s \mapsto \pi_s^\varphi(g)\xi$ is norm continuous, for all $g \in F$ and $\xi \in \mathcal{H}$. This follows directly from Proposition 5.1.25 applied with $F(y) = \pi_y^\varphi(g)\xi$ and $h = \pi_0^\varphi(g)$, as the map $y \mapsto (\pi_y^\varphi(g)\xi)(x)$ is clearly continuous, for all $x \in [0, 1]$, and $|\pi_y^\varphi(g)\xi| = |\pi_0^\varphi(g)\xi|$, for all $y \in \mathbb{R}$. □

Remark 5.1.27. Let us end this section with a remark about two different views on the representations we are concerned with. So far the focus has been the map φ, but there is also the possibility to think of the representations in terms of measures. Indeed, there is a natural one-to-one correspondence between atomless probability measures on $[0, 1]$ with full support and increasing homeomorphisms of $[0, 1]$. (By full support we mean that no open set has measure zero.) Let us elaborate on this correspondence.

If φ is an increasing homeomorphism of $[0,1]$, then the image measure $\varphi_* m$ is an atomless probability measure on $[0,1]$ with full support. Conversely, if μ is such a measure, then the function $x \mapsto \int_0^x \mathbf{1}\,\mathrm{d}\mu$ is an increasing homeomorphism of $[0,1]$. If we temporarily denote this latter function by F_μ, then the correspondence goes as follows:

$$\mu = (F_\mu^{-1})_* m \qquad \text{and} \qquad \varphi = F_{\varphi_* m}^{-1}.$$

Since we are interested in investigating when the action of F^φ on $([0,1], m)$ is ergodic, let us give a description of this in terms of the measure instead of the function. Clearly, a subset A of $[0,1]$ is invariant under the action of F if and only if the set $\varphi^{-1}(A)$ is invariant under the action of F^φ. From this it is easily seen that the action of F^φ on $([0,1], m)$ is ergodic if and only if the action of F on $([0,1], \varphi_* m)$ is ergodic. Thus finding maps φ that give rise to irreducible representations becomes a question of finding atomless probability measures on $[0,1]$ with full support which F leaves quasi-invariant and such that the action of F on $[0,1]$ is ergodic. In fact, it is not difficult to see that any measure on $[0,1]$ for which $\{0,1\}$ is a null set and which F leaves quasi-invariant must necessarily have full support.

5.2 Ergodicity and equivalence relations

In this section we explain how the Thompson group F acts on the Cantor set, as well as formulate ergodicity of the actions we are interested in in terms of ergodicity of the tail equivalence relation. We start by introducing the concept of a countable Borel equivalence relations and notions associated with these.

To do so, recall that a *Polish space* is a separable topological space which is completely metrizable, that is, there exists a complete metric on the space generating the topology. Examples of Polish spaces include the unit interval and the Cantor set with their usual topologies. A *standard Borel space* is a Polish space equipped with the Borel σ-algebra. An equivalence relation E on a standard Borel space $\mathfrak{X}$ is called a *Borel equivalence relation* if E is Borel as a subset of $\mathfrak{X} \times \mathfrak{X}$. Such an equivalence relation is said to be *countable* if each of its equivalence classes are countable.

A natural example of a countable Borel equivalence relation is the *orbit equivalence relation* induced by a measurable action α of a countable group G on a standard Borel space $\mathfrak{X}$, that is, the equivalence relation

$$\{(x, \alpha(g)x) : x \in \mathfrak{X}, g \in G\} \subseteq \mathfrak{X} \times \mathfrak{X}.$$

In other words, a point $x \in \mathfrak{X}$ is related to another point $y \in \mathfrak{X}$ if and only if there exists $g \in G$ such that $y = \alpha(g)x$. It is a non-trivial fact that all countable Borel equivalence relations arise in this fashion. This result was proved by Feldman and Moore [27] in 1977 and is known as the Feldman-Moore theorem, which we now state.

Theorem 5.2.1. *If E is a countable Borel equivalence relation on a standard Borel space $\mathfrak{X}$, then there exists a measurable action of a countable group on $\mathfrak{X}$ such that E is the orbit equivalence relation of this action.*

Let $\mathfrak{X}$ be a standard Borel space and E a countable Borel equivalence relation on $\mathfrak{X}$. Given a Borel subset A of $\mathfrak{X}$, the *restriction* of E to A is the equivalence relation $E \cap (A \times A)$. This will again be a countable Borel equivalence relation as A is in itself a standard Borel space; see for example [45, Corollary 13.4]. We will denote the restriction of E to A by $E|_A$. Now, the *saturation* or *E-saturation* of A, denoted by $[A]_E$, is the set of points in $\mathfrak{X}$ which are related to a point in A, that is,

$$[A]_E = \{x \in \mathfrak{X} : (x, y) \in E \text{ for some } y \in A\}.$$

It is not difficult to see from the Feldman-Moore theorem that the saturation of a Borel set is again a Borel set. A subset A is said to be *invariant* or *E-invariant* if it is equal to own saturation, that is, if $[A]_E = A$.

Definition 5.2.2. Let $\mathfrak{X}$ be a standard Borel space, let E be a Borel equivalence relation on $\mathfrak{X}$ and let μ be a Borel measure on $\mathfrak{X}$. If every E-invariant Borel set is either a null set or a co-null set, we say that the equivalence relation E is *ergodic* with respect to μ, or that the measure μ is *ergodic* with respect to the equivalence relation E.

Given a measurable action of a countable group on a standard Borel space $\mathfrak{X}$, it is easy to see that a Borel subset $A \subseteq \mathfrak{X}$ is invariant under the action if and only if it is invariant under the orbit equivalence relation. In particular, the action is ergodic if and only if the orbit equivalence relation is ergodic.

The following are well-known results about ergodicity for restricted measures and equivalence relations.

Proposition 5.2.3. *Let E be a Borel equivalence relation on a measure space $(\mathfrak{X}, \mu)$ and let $A \subseteq \mathfrak{X}$ be a measurable subset. If E is ergodic with respect to μ, then $E|_A$ is ergodic with respect to $\mu|_A$.*

Proof. If A is a null set, then the statement is trivial, since all equivalence relations on a space with the zero measure are ergodic. Thus, we may assume that $\mu(A)$ is not zero. Suppose that $B \subseteq A$ is an $E|_A$-invariant subset of positive measure. Then we want to prove that $A \setminus B$ has measure zero. The set $[B]_E$ is by definition E-invariant, so, since it has positive measure, it must be a co-null set in $\mathfrak{X}$. In other words, $\mathfrak{X} \setminus [B]_E$ has measure zero. It is easy to see that $(\mathfrak{X} \setminus [B]_E) \cap A = A \setminus B$, since B is $E|_A$-invariant. In particular, $A \setminus B$ is a null set. $\square$

Lemma 5.2.4. *Let E be a Borel equivalence relation on a measure space $(\mathfrak{X}, \mu)$ and let $A \subseteq \mathfrak{X}$ be a co-null set. Then E is ergodic with respect to μ if and only if $E|_A$ is ergodic with respect to $\mu|_A$.*

Proof. By Proposition 5.2.3 the restriction is ergodic if the original equivalence relation is ergodic, so let us assume that the restricted equivalence relation is ergodic. Let $B \subseteq \mathfrak{X}$ be E-invariant of positive measure. Since $\mathfrak{X} \setminus A$ is a set of measure zero, $B \setminus A$ is a set of measure zero, so $B \cap A$ must have positive measure. It is easy to check that $B \cap A$ is $E|_A$-invariant since B is E-invariant, so, by ergodicity, $B \cap A$ must be a co-null set in A. Since A is itself a co-null set in $\mathfrak{X}$, the set $B \cap A$, and therefore also B, must be a co-null set in $\mathfrak{X}$. $\square$

Remark 5.2.5. It is a well-known fact that the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\hat{\mathbb{R}}$ by fractional linear transformations is ergodic with respect to the Lebesgue measure (see [18, Example 3.5]). This means that the orbit equivalence relation of this action is ergodic, and, by Proposition 5.2.3, the restriction of this relation to $(0, 1)$ is ergodic with respect to the Lebesgue measure on $(0, 1)$. Clearly this relation is also the orbit equivalence relation of the action of $F^?$ on the interval $(0, 1)$, where ? is the Minkowski question mark function, as we know from Section 2.4 that $F^?$ is the representation of F as fractional linear transformations of $[0, 1]$. By Lemma 5.2.4 this means that the action of $F^?$ on $[0, 1]$ is also ergodic with respect to the Lebesgue measure. Therefore, we see that the representations $\pi_s^?$ are irreducible, for all $s \in \mathbb{R}$, by Proposition 5.1.18.

Let us now turn our attention to the mentioned realization of the Thompson groups as homeomorphisms of the Cantor set, by which we mean the space $\{0, 1\}^{\mathbb{N}}$ with the product topology. Loosely speaking, F can be thought of as the group of homeomorphisms of the Cantor set which are locally a change of prefix and which preserve the lexicographical order on the Cantor set. Let us make this more precise. There is a natural map from the Cantor set onto the unit interval, namely, the map that, to a sequence of zeroes and ones, associates the real number in $[0, 1]$ having the corresponding base-2 expansion, that is, the map κ given by

$$\kappa(x_1, x_2, x_3, \ldots) = \sum_{n=1}^{\infty} x_n 2^{-n}. \tag{5.1}$$

This map is continuous and it preserves the lexicographical order on $\{0, 1\}^{\mathbb{N}}$. It is not injective, though. However, since real numbers have a unique base-2 expansion unless they are dyadic rationals, in which case they have exactly two different base-2 expansions, it is close to being injective. For good measure, let us explain what we mean by the *lexicographic order* on $\{0, 1\}^{\mathbb{N}}$. It is the order described as follows: given $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\{0, 1\}$ we have $(x_n)_{n\in\mathbb{N}} < (y_n)_{n\in\mathbb{N}}$ if and only if $x_1 < y_1$ or there exists some $k \in \mathbb{N}$ such that $x_n = y_n$, for all $n \in \{1, 2, \ldots, k\}$, and $x_{k+1} < y_{k+1}$. We will explain in details how F is represented as homeomorphisms of the Cantor set. We first introduce some notation and terminology.

Definition 5.2.6. By a *cylinder set* we shall mean a subset of the Cantor space of the form $\{(x_1, x_2, \ldots, x_n)\} \times \{0, 1\}^{\mathbb{N}}$, for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \{0, 1\}$. We will denote such a set by $S_{x_1,\ldots,x_n}$.

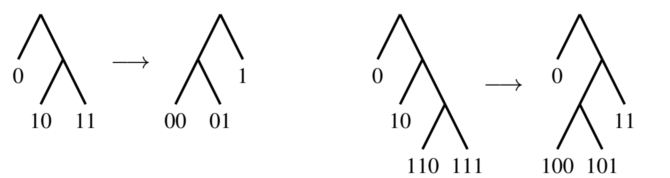
Now, back to the interpretation of F as homeomorphisms of the Cantor set. Consider the set of homeomorphisms f of $\{0,1\}^{\mathbb{N}}$ that preserve the lexicographical order and for which there exists a partition of $\{0,1\}^{\mathbb{N}}$ into cylinder sets such that each of these cylinder sets is mapped to a another cylinder set by changing the prefix. More precisely, if $S_{x_1,\dots,x_n}$ is one of the sets in the partition, then there exist $y_1,\dots,y_k \in \{0,1\}$ such that

$$f(x_1,\dots,x_n,z_1,z_2,\dots) = (y_1,\dots,y_k,z_1,z_2,\dots),$$

for all points $(x_1,\dots,x_n,z_1,z_2,\dots)$ in $S_{x_1,\dots,x_n}$. Note that the prefix $x_1,x_2,\dots,x_n$ need not have the same length as the prefix $y_1,y_2,\dots,y_k$. Since all these elements preserve the order, they fix the sequence which is constantly equal to one and that which is constantly equal to zero. Note that, since the cylinder sets are open and since the Cantor set is compact, the partition must necessarily be finite. It is easy to see that a homeomorphism f of this type defines a homeomorphism, $\tilde{f}$ say, of the unit interval as follows: given $x \in [0,1]$, let y be a pre-image of x under the map κ from (5.1) and define $\tilde{f}(x)$ to be $\kappa(f(y))$, which is easily seen to be well-defined. It is straightforward to check that the image of a cylinder set via κ is a standard dyadic interval, and that the map $\tilde{f}$ must be linear with slope being a power of 2 on each of the cylinder sets from the definition of f. In particular, $\tilde{f}$ is an element of the Thompson group F. Thus, the map defined by $f \mapsto \tilde{f}$ is an injective group homomorphism. It is easy to see that the elements A and B generating the Thompson group corresponds to the following prefix changes

$$A = \begin{cases} 0 & \mapsto & 00 \\ 10 & \mapsto & 01 \\ 11 & \mapsto & 1 \end{cases} \qquad B = \begin{cases} 0 & \mapsto & 0 \\ 10 & \mapsto & 100 \\ 110 & \mapsto & 101 \\ 111 & \mapsto & 11 \end{cases}$$

Similarly, one can realize the Thompson groups T and V as homeomorphisms of the Cantor set. To some people this will seem very familiar, and, indeed, what we have been describing so far is the representation of the Thompson group F as tree diagrams. In this interpretation, the elements A and B are represented by the following tree diagrams, on which we have added 0-1-labels in the natural way to indicate the connection:



Remark 5.2.7. Before continuing, let us make a remark concerning the already mentioned fact that κ from (5.1) maps cylinder sets to standard dyadic intervals. Let us fix some cylinder set $S_{x_1,\dots,x_n}$. The fact that $\kappa(S_{x_1,\dots,x_n})$ is an interval is easy to see, as κ is surjective and clearly order-preserving. The left endpoint of this interval is $\sum_{k=1}^{n} x_k 2^{-k}$ and the right endpoint is $\sum_{k=1}^{n} x_k 2^{-k} + \sum_{k=n+1}^{\infty} 2^{-k}$, or, in other words, $2^{-n} + \sum_{k=1}^{n} x_k 2^{-k}$. It is not difficult to see that a standard dyadic interval is uniquely determined by its midpoint, for if x is the midpoint of two standard dyadic intervals $[\frac{m}{2^j}, \frac{m+1}{2^j}]$ and $[\frac{m'}{2^{j'}}, \frac{m'+1}{2^{j'}}]$, then $\frac{2m+1}{2^{j+1}} = x = \frac{2m'+1}{2^{j'+1}}$, which clearly implies that $j = j'$, since $2m+1$ and $2m'+1$ are both odd. As this is the case, m and m' must also be equal, and the two standard dyadic intervals were then equal to begin with. Thus, the cylinder set $S_{x_1,\ldots,x_n}$ exactly corresponds to the standard dyadic interval whose midpoint is $\kappa(x_1, \ldots, x_n, 1, 0, 0, \ldots)$, that is, the point $2^{-(n+1)} + \sum_{k=1}^{n} x_k 2^{-k}$, and dyadic rational points in the interior of this interval are exactly those of the form

$$\sum_{k=1}^{n} x_k 2^{-k} + \sum_{k=1}^{j} z_k 2^{-(n+k)}$$

for some $j \in \mathbb{N}$ and $z_1, z_2, \ldots, z_j \in \{0,1\}$. Now, suppose that g maps this standard dyadic interval linearly onto the standard dyadic interval with midpoint $2^{-(m+1)} + \sum_{k=1}^{m} y_k 2^{-k}$, then

$$g\big(\sum_{k=1}^{n} x_k 2^{-k} + \sum_{k=1}^{j} z_k 2^{-(n+k)}\big) = \sum_{k=1}^{m} y_k 2^{-k} + \sum_{k=1}^{j} z_k 2^{-(m+k)}$$

for every $j \in \mathbb{N}$ and $z_{n+1}, z_{n+2}, \ldots, z_{n+j} \in \{0,1\}$.

Let us introduce a well-known equivalence relation on the Cantor set.

Definition 5.2.8. The *tail equivalence* relation on $\{0,1\}^{\mathbb{N}}$ is the equivalence relation defined as follows: $(x_k)_{k\in\mathbb{N}}$ is equivalent to $(y_k)_{k\in\mathbb{N}}$ if there exist natural numbers n and m such that $x_{n+k} = y_{m+k}$, for all $k \in \mathbb{N}$.

It is not difficult to see that the orbit equivalence relation generated by the action of either T or V on the Cantor set is exactly the tail equivalence relation. The orbit equivalence relation of F, though, is not the tail equivalence relations, since it fixes the points $(0,0,0,\ldots)$ and $(1,1,1,\ldots)$. In some sense, this is the only defect. More precisely, if $\mathfrak{X}$ denote the Cantor set minus the two constant sequences, then the restriction of the tail equivalence relation to $\mathfrak{X}$ is exactly the orbit equivalence relation of the action of F restricted to $\mathfrak{X}$. Using this, we easily obtain the following result.

Lemma 5.2.9. *Let μ be an atomless measure on the Cantor set which F leaves quasi-invariant. Then the action of F on $(\{0,1\}^{\mathbb{N}}, \mu)$ is ergodic if and only if μ is ergodic with respect to the tail equivalence relation.*

Proof. Let $\mathfrak{X}$ be defined as above. Since μ is atomless, we get from Lemma 5.2.4 that the tail equivalence relations is ergodic with respect to μ if and only if its restriction to $\mathfrak{X}$ is ergodic with respect to $\mu|_{\mathfrak{X}}$. Similarly, we get that the orbit equivalence relation of the action of F on the Cantor set is ergodic with respect to μ if and only if its restriction to $\mathfrak{X}$ is ergodic with respect to $\mu|_{\mathfrak{X}}$. However, we noted above that the tail equivalence relation and the orbit equivalence relation agree when restricted to $\mathfrak{X}$, so we conclude that they are ergodic with respect to μ exactly at the same time. That is, the action of F on the Cantor set is ergodic with respect to μ if and only if the tail equivalence is so. $\square$

Remark 5.2.10. Let us explain why we are interested in ergodicity of the action of F on the Cantor set. As explained in Remark 5.1.27, what we are really interested in are atomless measures on $[0, 1]$ which F leaves quasi-invariant and for which the action of F is ergodic. However, given an atomless measure μ on the Cantor set we get an atomless measure on the unit interval as the image measure of μ via the map (5.1). It is not difficult to see that this is a bijective correspondence between the atomless measures on the Cantor set and those on the unit interval. Moreover, since this map κ commutes with the action of F, it is easily seen that the measure μ is left quasi-invariant by the action of F on the Cantor set if and only if its image measure is left quasi-invariant by the action of F on the unit interval. For the same reason, we also get that the action of F on the Cantor set is ergodic with respect to μ if and only if the action of F on the unit interval is ergodic with respect to the image measure. Thus, in our search for measures producing different families of irreducible representations, we can look for measures on the Cantor set instead. Indeed, it is easy to see that two such measures on the Cantor set are equivalent if and only if the corresponding measures on the unit interval are so.

The simplest measures on the Cantor set are the product measures. In fact, the product measures which are atomless and which F leaves quasi-invariant are automatically measures for which the action of F is ergodic. Before we explain this in more detail, let us introduce the notion of product measures on the Cantor set. For this, we need a result known as the Hahn-Kolmogorov Theorem; see [45, Proposition 17.16]. Recall that a *Boolean algebra* on a set $\mathfrak{X}$ is a collection of subsets of $\mathfrak{X}$ which contains the empty set and is closed under finite union and complements.

Theorem 5.2.11. *Let $\mathcal{A}$ be a Boolean algebra on a set $\mathfrak{X}$, and let $\mu\colon \mathcal{A} \to [0, \infty]$ be a map satisfying $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, for any family $A_1, A_2, \dots$ of pairwise disjoint sets in $\mathcal{A}$ whose union is also in $\mathcal{A}$. Then μ has an extension to a measure on $\mathfrak{X}$ defined on the σ-algebra generated by $\mathcal{A}$. Furthermore, if μ is bounded, then the measure is finite, and, moreover, unique.*

Applying this theorem to the Boolean algebra on the Cantor set consisting of the empty set and all finite unions of cylinder sets, we arrive at a well-known description of the Borel probability measures on the Cantor set.

Theorem 5.2.12. *For each $n \in \mathbb{N}$, suppose that μ_n is a probability measure on the set $\prod_{k=1}^{n}\{0, 1\}$ with the power set σ-algebra. If the measures satisfy*

$$\mu_{n+1}(A \times \{0, 1\}) = \mu_n(A),$$

for all $n \in \mathbb{N}$ and $A \subseteq \prod_{k=1}^{n}\{0,1\}$, then there exists a unique Borel probability measure μ on the Cantor set $\{0,1\}^{\mathbb{N}}$ such that, given $A \subseteq \prod_{k=1}^{n}\{0,1\}$,

$$\mu(A \times \{0,1\}^{\mathbb{N}}) = \mu_n(A).$$

Moreover, all Borel probability measures on $\{0,1\}^{\mathbb{N}}$ are obtained in this fashion.

Proof. It is easy to see that the conditions put on the measures are exactly the conditions needed to define a measure on the Boolean algebra on $\{0,1\}^{\mathbb{N}}$ containing the empty set and all finite unions of cylinder sets. Hence the Hahn-Kolmogorov theorem gives us a unique extension to a measure on the σ-algebra generated by these sets. This σ-algebra is, of course, the Borel σ-algebra, as the cylinder sets are a basis for the topology on $\{0,1\}^{\mathbb{N}}$. Clearly this measure satisfies $\mu(A \times \{0,1\}^{\mathbb{N}}) = \mu_n(A)$, for all $n \in \mathbb{N}$ and $A \subseteq \prod_{k=1}^{n}\{0,1\}$. That all Borel probability measures are obtained in this way is obvious. $\square$

Remark 5.2.13. Given a sequence of probability measures $(\nu_n)_{n\in\mathbb{N}}$ on $\{0,1\}$, define μ_n to be the ordinary product measure $\nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$ on $\{0,1\}^n$, for each $n \in \mathbb{N}$, such that

$$\mu_n(x_1,\ldots,x_n) = \nu_1(x_1)\nu_2(x_2)\cdots\nu_n(x_n). \tag{5.2}$$

Clearly the sequence $(\mu_n)_{n\in\mathbb{N}}$ satisfies the conditions of Theorem 5.2.12, and, therefore, induces a unique Borel probability measure on the Cantor set. We will refer to these measures as *product measures*. With these measures, the measure of the cylinder sets are easily calculated. Indeed, for $x_1, x_2, \ldots, x_n \in \{0,1\}$, the measure of the cylinder set $S_{x_1,\ldots,x_n}$ is exactly given by (5.2). It is not difficult to see that a product measure has full support if and only if none of the ν_n's are concentrated in a single point. Since measures that F leaves quasi-invariant automatically have full support, as explained in Remarks 5.1.27 and 5.2.10, these are the only candidates for product measures that F leaves quasi-invariant. The measure of a single point $(x_1, x_2, \ldots)$ with respect to the product of $(\nu_n)_{n\in\mathbb{N}}$ is exactly $\prod_{n=1}^{\infty}\nu_n(x_n)$, so whether or not a product measure has atoms depends on whether one can choose a sequence so that the product does not converge to zero. One case where it is easy to see that there are no atoms is if there exists some $c \in (0,1)$ so that $\nu_n(x_k) < c$, for all $n \in \mathbb{N}$ and $k \in \{0,1\}$.

The following theorem is known as the Kolmogorov Zero-One Law; a proof can be found in [5, Theorem 4.5]. We will only state it in the context we are interested in.

Theorem 5.2.14. *Let μ be a product measure on $\{0,1\}^{\mathbb{N}}$ and let $A \subseteq \{0,1\}^{\mathbb{N}}$ be a Borel subset such that, if $(x_n)_{n\in\mathbb{N}} \in A$ and $(y_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ with $x_n = y_n$, for all but finitely many $n \in \mathbb{N}$, then $(y_n)_{n\in\mathbb{N}} \in A$. Then either $\mu(A) = 0$ or $\mu(A) = 1$.*

A subset A as in the above theorem is often referred to as a tail set, and it is easy to see that a subset of the Cantor set which is invariant under the tail equivalence relation is necessarily such a set, but not the other way around. Thus, we immediately get the following corollary.

Corollary 5.2.15. *The tail equivalence relation is ergodic with respect to every product measure on the Cantor set.*

In fact, one way to state the Kolmogorov Zero-One Law is that the equivalence relation usually denoted E_0 is ergodic with respect to all product measures on the Cantor set. This equivalence relation is given as follows: two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are equivalent if and only if there exists $k \in \mathbb{N}$ so that $x_n = y_n$, for all $n \geq k$. Note that this relation is distinct from the one we call the tail equivalence relation; see Definition 5.2.8.

Now, backtracking from a measure on the Cantor set to a measure on the unit interval and then to a map satisfying condition ($\star$), as explained in Remark 5.2.10 and Remark 5.1.27, we get the following result as an application of the above corollary.

Proposition 5.2.16. *If μ is an atomless product measure on the Cantor set $\{0,1\}^{\mathbb{N}}$ which F leaves quasi-invariant, then, with φ denoting the corresponding map satisfying condition ($\star$), the representation π_s^{φ} is irreducible, for all $s \in \mathbb{R}$.*

Proof. By Corollary 5.2.15 we know that the tail equivalence relation is ergodic with respect to the measure μ, so by Lemma 5.2.9 we know that the action of F on the Cantor set is ergodic with respect to μ. From Remark 5.2.10 we know that the measure μ corresponds to an atomless measure on $[0,1]$ which F leaves quasi-invariant and for which the action of F on $[0,1]$ is ergodic. By Remark 5.1.27, we know that this measure on $[0,1]$ corresponds to a map φ satisfying condition ($\star$) such that the action of F^{φ} on $([0,1],m)$ is ergodic. By Proposition 5.1.18, the representation π_s^{φ} is irreducible, for all $s \in \mathbb{R}$. $\square$

The above proposition gives us a large class of measures which produce irreducible representations of F when the construction from the previous section is applied. However, it is not obvious which of these measures will produce equivalent representations.

5.3 Bernoulli measures on the unit interval

In this section, we will give examples of different families of irreducible unitary representations of F arising in the fashion described in Section 5.1. As explained in Remark 5.2.10, we do not have to start the construction with a map φ satisfying condition ($\star$) to obtain a one-parameter family of representations. Instead, we may start with an atomless measure on the Cantor set which F leaves quasi-invariant, and this, indeed, is what we will do.

Given $p \in (0,1)$, we can consider the probability measure ν on $\{0,1\}$ given by $\nu(0) = p$ and $\nu(1) = 1-p$. Let μ_p denote the infinite product of this measure, which is a probability measure on the Cantor set $\{0,1\}^{\mathbb{N}}$. As mentioned in Remark 5.2.13 these product measures are atomless with full support. In particular, we get a corresponding atomless measure $\tilde{\mu}_p$ on $[0,1]$ with full support and a map ϕ_p satisfying condition ($\star$), as explained in Remark 5.2.10 and Remark 5.1.27. Recall that this map is an increasing homeomorphism of $[0,1]$, and its inverse is given by $\phi_p^{-1}(x) = \int_0^x \mathbf{1}\,\mathrm{d}\tilde{\mu}_p$.

Definition 5.3.1. With the notation described above, we will refer to the measures μ_p and $\tilde{\mu}_p$ as the *Bernoulli measures* for $p \in (0,1)$.

Let us briefly recall the notion of singularity of measures, as well as the Lebesgue decomposition theorem. A proof of the latter can be found in [28, Theorem 3.8].

Definition 5.3.2. Two measures μ and ν on a space $\mathfrak{X}$ are said to be *singular* with respect to each other if there exist two disjoint measurable subsets A and B of $\mathfrak{X}$ with $\mathfrak{X} = A \cup B$ so that A is a μ-null set and B is a ν-null set.

Theorem 5.3.3. *Let μ and ν be σ-finite measures on a space $\mathfrak{X}$. Then there exists σ-finite measures ν_a and ν_s on $\mathfrak{X}$ with $\nu = \nu_a + \nu_s$ so that ν_a is absolutely continuous with respect to μ, and ν_s and μ are singular with respect to each other.*

As mentioned, the Bernoulli measures are all atomless, and it is not difficult to realize that the action of F on the Cantor set leaves them quasi-invariant, but we postpone the argument. This means that the Bernoulli measures are of the kind we are interested in, and, as they are all product measures, Proposition 5.2.16 tells us that the representation $\pi_s^{\phi_p}$ is irreducible for each p in $(0,1)$ and $s \in \mathbb{R}$. In fact, we can say even more about these. Recall the following well-known fact about the Bernoulli measures. The proof is a simple application of the Strong Law of Large Numbers, which we include for completeness.

Proposition 5.3.4. *The Bernoulli measures are singular with respect to each other. In fact, the set*

$$\left\{(x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} \tfrac{1}{n}(x_1 + \cdots + x_n) = 1 - p\right\}$$

has measure 1 *with respect to* μ_p.

Proof. Clearly, the measures are singular with respect to each other if the indicated set has measure one, since these sets are disjoint for distinct values p. Note that the limit does not exists for all sequences, so it should be understood as the set of sequences for which the limit exists, and is as indicated.

For each $n \in \mathbb{N}$, let $X_n \colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be the n'th coordinate projection. Then $(X_n)_{n\in\mathbb{N}}$ is a sequence of real random variables. It is easy to see that these are independent, since μ_p is a product measure, and they are identically distributed as μ_p is a product of a single measure. By the Strong Law of Large Numbers (see for example [5, Theorem 6.1]) the averages $\frac{1}{n}(X_1 + X_2 + \ldots + X_n)$ converge almost surely to $E[X_1]$, the expected value of X_1. Hence there exists a set $A \subseteq \{0,1\}^{\mathbb{N}}$ of measure one such that $\lim_{n\to\infty} \frac{1}{n}(X_1(y)+X_2(y)+\ldots+X_n(y))$ exists, for all $y \in A$, and equals $E[X_1]$. Now, $E[X_1] = 0 \cdot p + 1 \cdot (1-p) = 1-p$, so A is contained in the set we are interested in, and it follows that it must have measure one, as well. □

We have not yet justified the fact that the action of F on the Cantor set leaves the Bernoulli measures quasi-invariant. This will follow easily when we know more about the elements of F^{ϕ_p}; see Remark 5.3.12. However, assuming that we know this, we can collect what we know about the representations $\pi_s^{\phi_p}$ so far in the following proposition.

Proposition 5.3.5. *Given $p, q \in (0,1)$ distinct and $s, t \in \mathbb{R}$, the representations $\pi_s^{\phi_p}$ and $\pi_t^{\phi_q}$ are irreducible, but not unitarily equivalent.*

Before continuing the investigation of these representations, let us describe the maps ϕ_p and the corresponding group of homeomorphisms F^{ϕ_p}. To do so, let us fix $p \in (0,1)$, and let us recursively define an increasing sequence $(\mathcal{A}_p^n)_{n\in\mathbb{N}}$ of finite subsets of $[0,1]$. We start by letting $\mathcal{A}_p^0 = \{0,1\}$. Supposing that we have defined the set $\mathcal{A}_p^n$, for some $n \geq 0$, we define

$$\mathcal{A}_p^{n+1} = \mathcal{A}_p^n \cup \big\{(1-p)x_{k-1} + px_k : k = 1, 2, \ldots, m\big\},$$

where $x_0, x_1, \cdots, x_m$ are the elements of $\mathcal{A}_p^n$ arranged increasingly. It is easy to see that the set $\mathcal{A}_p^n$ has exactly $2^n + 1$ elements, and, to illustrate, the first three sets are given by:

$$\mathcal{A}_p^0 = \big\{0,1\big\}, \quad \mathcal{A}_p^1 = \big\{0,p,1\big\} \quad \text{and} \quad \mathcal{A}_p^2 = \big\{0, p^2, p, (1-p)p + p, 1\big\}.$$

Now, let $\mathcal{A}_p$ denote the union $\bigcup_{k=1}^{\infty} \mathcal{A}_p^k$. It is easy to see that $\mathcal{A}_p$ is a dense subset of $[0,1]$ since the distance between two successive points of $\mathcal{A}_p^n$ is no greater than $\max\{p, 1-p\}^n$. In the case where $p = \frac{1}{2}$, the sets $\mathcal{A}_{1/2}^n$ and $\mathcal{A}_{1/2}$ are exactly the sets $\mathcal{D}^n$ and $\mathcal{D}$, respectively, which we introduced in Section 2.3, that is, $\mathcal{D}$ is the set of dyadic rational numbers in $[0,1]$, and $\mathcal{D}^n$ is the set $\{\frac{m}{2^n} : 0 \leq m \leq 2^n\}$.

Remark 5.3.6. Let $n \in \mathbb{N} \cup \{0\}$ and let x and y be two consecutive elements of $\mathcal{A}_p^n$ with $x < y$. It follows from the construction that the set $[x,y] \cap \mathcal{A}_p^{n+(k+1)}$ is obtained from the set $[x,y] \cap \mathcal{A}_p^{n+k}$ by adding $(1-p)x' + py'$, for x' and y' consecutive in $[x,y] \cap \mathcal{A}_p^{n+k}$ with $x' < y'$. Clearly, $[x,y] \cap \mathcal{A}_p^n = \{x,y\}$, as x and y are consecutive in $\mathcal{A}_p^n$. This identity can be rewritten as $[x,y] \cap \mathcal{A}_p^n = \{(1-s)x + sy : s \in \mathcal{A}_p^0\}$, and, in fact, an easy induction argument shows that

$$[x,y] \cap \mathcal{A}_p^{n+k} = \{(1-s)x + sy : s \in \mathcal{A}_p^k\},$$

for all $k \in \mathbb{N} \cup \{0\}$. In particular, $[x,y] \cap \mathcal{A}_p = \{(1-s)x + sy : s \in \mathcal{A}_p\}$.

We already know that ϕ_p satisfies $\phi_p^{-1}(x) = \int_0^x \mathbf{1}\,\mathrm{d}\tilde{\mu}_p$, so, in particular,

$$\tilde{\mu}_p\big([x,y]\big) = \phi_p^{-1}(y) - \phi_p^{-1}(x),$$

for all $x, y \in [0,1]$ with $x < y$. However, if the interval is a standard dyadic interval, then we can say something more as the following lemma shows.

Lemma 5.3.7. *Let $p \in (0,1)$ and suppose that $[x,y] \subseteq [0,1]$ is a standard dyadic interval. Then*

$$\phi_p^{-1}\big(\tfrac{x+y}{2}\big) = (1-p)\phi_p^{-1}(x) + p\phi_p^{-1}(y).$$

In particular, $\phi_p(\mathcal{A}_p^n) = \mathcal{D}^n$, for all $n \in \mathbb{N}$, and $\phi_p(\mathcal{A}_p) = \mathcal{D}$.

Proof. If $[x,y] = [0,1]$, then the statement is that $\phi_p^{-1}(\frac{1}{2}) = p$, which is easily seen from the formula $\phi_p^{-1}(x) = \int_0^x \mathbf{1}\,\mathrm{d}\tilde{\mu}_p$. Suppose instead that $[x,y]$ is a proper subset of $[0,1]$. As explained in Remark 5.2.7 the standard dyadic interval $[x,y]$ corresponds to some cylinder set $S_{x_1,\ldots,x_n}$, for some $x_1,\ldots,x_n \in \{0,1\}$. Moreover, the intervals $[x, \frac{x+y}{2}]$ and $[\frac{x+y}{2}, y]$ correspond to the cylinder sets $S_{x_1,\ldots,x_n,0}$ and $S_{x_1,\ldots,x_n,1}$, respectively. We see that

$$\tilde{\mu}_p\big([x, \tfrac{x+y}{2}]\big) = p\tilde{\mu}_p\big([x,y]\big) \quad \text{and} \quad \tilde{\mu}_p\big([\tfrac{x+y}{2}, y]\big) = (1-p)\tilde{\mu}_p\big([x,y]\big).$$

In particular, it follows that

$$\phi_p^{-1}\big(\tfrac{x+y}{2}\big) - \phi_p^{-1}(x) = \tilde{\mu}_p\big([x, \tfrac{x+y}{2}]\big) = p\tilde{\mu}_p\big([x,y]\big) = p\phi_p^{-1}(y) - p\phi_p^{-1}(x),$$

which, after rearranging the terms, is the desired identity.

The fact that $\phi_p(\mathcal{A}_p^n) = \mathcal{D}^n$, for all $n \in \mathbb{N}$, follows directly from this equation and the definition of the sets. Indeed, if $\phi_p(\mathcal{A}_p^n) = \mathcal{D}^n$, then $\phi_p(\mathcal{A}_p^{n+1}) = \mathcal{D}^{n+1}$, so, since $\phi_p(\mathcal{A}_p^0) = \mathcal{D}^0$, the statement follows by induction. Clearly this implies that $\phi_p(\mathcal{A}_p) = \mathcal{D}$. $\square$

Using the lemma above, we can express the value of $\phi_p^{-1}(x)$ explicitly in terms of the base-2 expansion of x, for all $x \in [0,1]$.

Lemma 5.3.8. *Let $p \in (0,1)$ and $(x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. Then, with $x_0 = 0$,*

$$\phi_p^{-1}\Big(\sum_{k=1}^{\infty} x_k 2^{-k}\Big) = \sum_{k=1}^{\infty} x_k p^k \prod_{j=0}^{k-1} \Big(\frac{1-p}{p}\Big)^{x_j}.$$

Moreover, every standard dyadic interval has measure $p^j(1-p)^i$ with respect to μ_p, for some $i, j \in \mathbb{Z}$ depending only on the standard dyadic interval.

Proof. First of all, it suffices to prove the statement for the dyadic rational numbers in $(0,1)$, as these are dense in $[0,1]$. The general statement will follow by continuity of ϕ_p^{-1}. Similarly, it will also follow by continuity that the expression is true for infinite base-2 expansions of dyadic rational numbers, if it is true for finite expansions. Thus we prove that the formula holds for finite base-2 expansions. We will do so by induction. More precisely, we will prove that the formula is true for finite base-2 expansions of elements of $\mathcal{D}^n \setminus \{1\}$ by induction on n. First of all, the case $n = 0$ is trivially satisfied and the case $n = 1$ amount to $\phi_p^{-1}(\frac{1}{2}) = p$, which we proved in Lemma 5.3.7. Thus we suppose that the formula holds for all finite base-2 expansions of elements in $\mathcal{D}^n \setminus \{1\}$, for some $n \in \mathbb{N}$, and want to prove that it holds for finite base-2 expansions of elements in $\mathcal{D}^{n+1} \setminus \{1\}$.

Clearly we just need to check the formula on $\mathcal{D}^{n+1} \setminus \mathcal{D}^n$. Note that the elements of $\mathcal{D}^n \setminus \{1\}$ are exactly the dyadic rational numbers which have a base-2 expansion of the form $\sum_{k=1}^n x_k 2^{-k}$, whereas the elements of $\mathcal{D}^{n+1} \setminus \mathcal{D}^n$ are exactly those which have a base-2 expansion of the form $\sum_{k=1}^{n+1} x_k 2^{-k}$ with $x_{n+1} = 1$. Therefore, suppose that we are given $x_1, \dots, x_n \in \{0,1\}$ and that $x_{n+1} = 1$, let $x = \sum_{k=1}^{n+1} x_k 2^{-k}$ and let us prove that the formula holds for x. Put $y_0 = \sum_{k=1}^n x_k 2^{-k}$ and $y_1 = y_0 + \frac{1}{2^n}$. Then $[y_0, y_1]$ is a standard dyadic interval and x is the midpoint of this interval. From Lemma 5.3.7, we know that

$$\phi_p(x) = \phi_p(y_0) + \tilde{\mu}_p([y_0, x]) = \phi_p(y_0) + p\tilde{\mu}_p([y_0, y_1]). \tag{5.3}$$

As $[y_0, y_1]$ is a standard dyadic interval with midpoint x and $x_{n+1} = 1$, we know from Remark 5.2.7 that $[y_0, y_1]$ corresponds to the cylinder set $S_{x_1,\dots,x_n}$ and therefore that

$$\tilde{\mu}_p\big([y_0, y_1]\big) = p^{n-(x_1+\dots+x_n)}(1-p)^{x_1+\dots+x_n} = p^n \prod_{k=1}^n \Big(\frac{1-p}{p}\Big)^{x_k}.$$

By the induction hypothesis, the formula holds for y_0, as $y_0 \in \mathcal{D}^n$. Thus, by inserting this expression for $\phi_p^{-1}(y_0)$ along with the one for $\tilde{\mu}_p([y_0, y_1])$ into the formula (5.3), we get the equation

$$\phi_p^{-1}\Big(\sum_{k=1}^{n+1} x_k 2^{-k}\Big) = \sum_{k=1}^n x_k p^k \prod_{j=0}^{k-1} \Big(\frac{1-p}{p}\Big)^{x_j} + p^{n+1} \prod_{k=1}^n \Big(\frac{1-p}{p}\Big)^{x_k}.$$

Recalling that $x_{n+1} = 1$, we see that this is exactly the formula we wanted to prove. Hence, by induction the formula holds for all finite base-2 expansions, and, by continuity, it holds for all base-2 expansions. The last claim about the measure of standard dyadic interval follows from the arguments we used to find the measure of the interval $[y_0, y_1]$. $\square$

Now that we have described the values of the map ϕ_p^{-1} at a point x in terms of the base-2 expansions of x, we recognize $(\phi_p^{-1})_{p\in(0,1)}$ as a family of maps introduced by

Salem [63] in 1943 in order to give concrete, explicit examples of strictly monotonic purely singular functions. We introduce the notion of singular functions later. For now, note that a strictly increasing homeomorphism of the unit interval is purely singular if and only if the corresponding measure, in the sense of Remark 5.1.27 is singular with respect to the Lebesgue measure. Thus, the combination of Lemma 5.3.8 and Proposition 5.3.4 is a different way of obtaining Salem's result.

Lemma 5.3.9. *Let $p \in (0,1)$, and let I and J be standard dyadic intervals. If $g\colon [0,1] \to [0,1]$ maps the interval I linearly and increasingly onto the interval J, then g^{ϕ_p} maps the interval $\phi_p^{-1}(I)$ linearly and increasingly onto the interval $\phi_p^{-1}(J)$ with slope $p^i(1-p)^j$, for some $i, j \in \mathbb{Z}$ depending only on the intervals I and J.*

Proof. First, write $I = [a, b]$ and $J = [c, d]$, for dyadic rational numbers a, b, c and d. For simplicity, we let a_p, b_p, c_p and d_p denote the points $\phi_p^{-1}(a)$, $\phi_p^{-1}(b)$, $\phi_p^{-1}(c)$ and $\phi_p^{-1}(d)$, respectively. Note that, as I and J are both standard dyadic intervals, g must have slope 2^k, for some $k \in \mathbb{Z}$.

As I is a standard dyadic interval, a and b are consecutive in $\mathcal{D}^n$, for some $n \in \mathbb{N} \cup \{0\}$. Choose such an n. Likewise, choose $m \in \mathbb{N} \cup \{0\}$ so that c and d are consecutive in $\mathcal{D}^m$. As g maps I linearly and increasingly onto J with slope being a power of 2, it maps the set $[a, b] \cap \mathcal{D}^{n+k}$ onto the set $[c, d] \cap \mathcal{D}^{m+k}$ in an order-preserving way, for all $k \geq 0$. We know from Lemma 5.3.7 that $\phi_p(\mathcal{A}_p^k) = \mathcal{D}^k$, for all $k \geq 0$, so we conclude that g^{ϕ_p} maps the set $[a_p, b_p] \cap \mathcal{A}_p^{n+k}$ onto the set $[c_p, c_p] \cap \mathcal{A}_p^{m+k}$ in an order-preserving way, for all $k \geq 0$. It now follows from Remark 5.3.6 that g^{ϕ_p} maps the set $\{(1-s)a_p + sb_p : s \in \mathcal{A}_p^k\}$ onto the set $\{(1-s)c_p + sd_p : s \in \mathcal{A}_p^k\}$ in an order-preserving way. Clearly this means that

$$g^{\phi_p}\big((1-s)a_p + sb_p\big) = (1-s)c_p + sd_p,$$

for all $s \in \mathcal{A}_p^k$ and $k \in \mathbb{Z}$. As the set $\mathcal{A}_p$ is dense in $[0,1]$, we conclude that the map g^{ϕ_p} is linear on $\phi_p^{-1}(I)$.

Now, the slope of the map g^{ϕ_p} on the interval $\phi_p^{-1}(I)$ is, of course, the length of the interval $\phi_p^{-1}(J)$ divided by the length of the interval $\phi_p^{-1}(I)$. The length of the intervals $\phi_p^{-1}(I)$ and $\phi_p^{-1}(J)$ are $\tilde{\mu}_p(I)$ and $\tilde{\mu}_p(J)$, respectively. As the measure of standard dyadic intervals with respect to $\tilde{\mu}_p$ is $p^i(1-p)^j$, for some $i, j \in \mathbb{Z}$, the statement about the slope of g^{ϕ_p} follows. □

Using the result above, we can say exactly what the elements of F^{ϕ_p} look like. Indeed, we immediately get the following result as a consequence.

Proposition 5.3.10. *Let $p \in (0,1)$. If $g \in F$, then g^{ϕ_p} is piecewise linear. In fact, if $x_0, \ldots, x_n$ is a standard dyadic partition associated to g, then there exist integers $i_1, \ldots, i_n, j_1, \ldots, j_n$, depending only on g, so that the map g^{ϕ_p} is linear on $[\phi_p^{-1}(x_{k-1}), \phi_p^{-1}(x_k)]$ with slope $p^{i_k}(1-p)^{j_k}$, for all $k = 1, 2, \ldots, n$.*

Let us, at this point, prove the following well-known result about transformations of the Lebesgue measure on an interval via a sufficiently nice map.

Proposition 5.3.11. *Let I and J be open intervals in $\mathbb{R}$, and let $\phi\colon I \to J$ be an order-preserving bijection. If ϕ^{-1} is continuously differentiable, the measure $\phi_* m$ on J is absolutely continuous with respect to the Lebesgue measure on J with Radon-Nikodym derivative*

$$\frac{\mathrm{d}\phi_* m}{\mathrm{d}m}(x) = (\phi^{-1})'(x),$$

for all $x \in J$.

Proof. It follows directly from the fundamental theorem of calculus that, for all $a, b \in J$ with $a < b$,

$$\int_a^b (\phi^{-1})'(x)\,\mathrm{d}m(x) = \phi^{-1}(b) - \phi^{-1}(a) = m\big([\phi^{-1}(a), \phi^{-1}(b)]\big) = \phi_* m\big([a,b]\big).$$

By Lemma 5.1.20, this means that $\phi_* m$ is absolutely continuous with respect to m with the indicated Radon-Nikodym derivative. □

Remark 5.3.12. At this point it is pretty clear that the action of F^{ϕ_p} on the unit interval leaves the Lebesgue measure quasi-invariant. Indeed, we can even say explicitly what the Radon-Nikodym derivative looks like. Suppose that $g \in F$ and that $x_0, \dots, x_n$ is a standard dyadic partition associated to g. We know from Proposition 5.3.10 that there exist $i_1, \dots, i_n, j_1, \dots, j_n$, depending only on g, so that, with

$$I_k = \big(\phi_p^{-1}(x_{k-1}), \phi_p^{-1}(x_k)\big) \qquad \text{and} \qquad J_k = g^{\phi_p}(I_k),$$

the map g^{ϕ_p} sends I_k linearly onto J_k with slope $p^{i_k}(1-p)^{j_k}$, for all $k = 1, 2, \dots, n$. Clearly, this means that $g^{-\phi_p}$ maps the interval J_k linearly onto the interval I_k with slope $p^{-i_k}(1-p)^{-j_k}$. Therefore, it easily follows from Proposition 5.3.11 that $(g^{\phi_p})_* m$ is absolutely continuous with respect to m, and that

$$\frac{\mathrm{d}(g^{\phi_p})_* m}{\mathrm{d}m}(x) = \sum_{k=1}^{n} (g^{-\phi_p})'(x)\mathbf{1}_{J_k}(x) = \sum_{k=1}^{n} \frac{\mathbf{1}_{(x_{k-1}, x_k)}(g^{-1}\phi_p(x))}{p^{i_k}(1-p)^{j_k}},$$

for all $x \in [0,1]$.

Our next goal is to prove that the representations $\pi_s^{\phi_p}$ are strongly continuous in the parameter p. To do so, we start with the following result.

Lemma 5.3.13. *Let $g \in F$. The three maps from $(0,1) \times [0,1]$ to $[0,1]$ given by*

$$(p,x) \mapsto \phi_p(x), \qquad (p,x) \mapsto \phi_p^{-1}(x) \qquad \textit{and} \qquad (p,x) \mapsto g^{\phi_p}(x)$$

are all continuous.

Proof. Denote these maps by Φ, Ψ and Θ, respectively, that is, $\Phi(p,x) = \phi_p(x)$, $\Psi(p,x) = \phi_p^{-1}(x)$ and $\Theta(p,x) = g^{\phi_p}(x)$. We start by showing that Φ is continuous. It suffices to prove that pre-images of sets of the form $[0,x)$, (x,y) and $(y,1]$ are open, for dyadic rational numbers $x, y \in [0,1]$ with $x < y$. As the three cases are very similar, we will prove that pre-images of set of the form (x,y) are open. So let x and y be dyadic rational numbers in $[0,1]$ with $x < y$. We know from Lemma 5.3.8 that, given a dyadic rational number z with base-2 expansion $\sum_{k=1}^n z_k 2^{-k}$,

$$\phi_p^{-1}(z) = \sum_{k=1}^{n} z_k p^k \prod_{j=0}^{k-1} \Big(\frac{1-p}{p}\Big)^{z_j},$$

where $z_0 = 0$. Obviously this expression varies continuously in p, so in particular, both $\phi_p^{-1}(x)$ and $\phi_p^{-1}(y)$ vary continuously in p. As ϕ_p^{-1} is an increasing homeomorphism of $[0,1]$, for fixed p, we see that

$$\Phi^{-1}\big((x,y)\big) = \big\{(p,z) \in (0,1) \times [0,1] : \phi_p^{-1}(x) < z < \phi_p^{-1}(y)\big\}.$$

Clearly this shows that the set $\Phi^{-1}\big((x,y)\big)$ is open, as we noted that both $\phi_p^{-1}(x)$ and $\phi_p^{-1}(y)$ vary continuously in p. Thus, Φ is continuous.

Now that we know that Φ is continuous, it is easy to see that Ψ is continuous. Given $a, b \in (0,1)$ with $a < b$, we see that

$$\Psi^{-1}\big((a,b)\big) = \big\{(p,x) \in (0,1) \times [0,1] : \Phi(p,a) < x < \Phi(p,b)\big\}.$$

Thus, by continuity of Φ, this set is open. Similarly one realizes that the pre-image of $[0,a)$ and $(b,1]$ are open, and we can, therefore, conclude that Ψ is continuous. Moreover, it is easily seen that $\Theta(p,x) = \Psi(p, g(\Phi(p,x)))$, so it follows that Θ is continuous, as well. □

At this point we are ready to prove that the representations $\pi_s^{\phi_p}$ are also continuous in the parameter p.

Proposition 5.3.14. *The representations $\pi_s^{\phi_p}$ are strongly continuous in both s and p, that is, for all $\xi \in \mathcal{H}$ and $g \in F$, the map $(p,s) \mapsto \pi_s^{\phi_p}(g)\xi$ is continuous from $(0,1) \times \mathbb{R}$ to $\mathcal{H}$.*

Proof. Fix $g \in F$. We want to show that the map $(p,s) \mapsto \pi_s^{\phi_p}(g)\xi$ is continuous, for all $\xi \in \mathcal{H}$. As $\{\mathbf{1}_{(a,b)} : 0 \le a < b \le 1\}$ spans a dense subset of $\mathcal{H}$, it suffices to prove the statement for ξ in this set, so suppose that $a, b \in [0,1]$ with $a < b$. Moreover, it suffices to prove that it is continuous on $[c,d] \times \mathbb{R}$, for all $c, d \in (0,1)$ with $c < d$, so fix such c and d.

Let $x_0, \dots, x_n$ be a standard dyadic partition associated to g, and choose integers $i_1, \dots, i_n, j_1, \dots, j_n$ as in Proposition 5.3.10. By Remark 5.3.12,

$$(\pi_s^{\phi_p}(g)1_{(a,b)})(x) = \sum_{k=1}^{n} \Big(\frac{\mathbf{1}_{(x_{k-1},x_k)}(g^{-1}\phi_p(x))}{p^{i_k}(1-p)^{j_k}}\Big)^{\frac{1}{2}+is} \mathbf{1}_{(a,b)}(g^{-\phi_p}(x)).$$

As $\pi_s^{\phi_p}(g)$ is linear, it suffices to prove continuity of each of the terms of the sum, that is, by letting

$$f_k(p,s) = \Big(\frac{\mathbf{1}_{(x_{k-1},x_k)}(g^{-1}\phi_p(x))}{p^{i_k}(1-p)^{j_k}}\Big)^{\frac{1}{2}+is}\mathbf{1}_{(a,b)}(g^{-\phi_p}(x)),$$

it suffices to prove that f_k is continuous from $[c,d]\times\mathbb{R}$ to $\mathcal{H}$, for all $k=1,\dots,n$. We will do so by using Proposition 5.1.25. Fix $k\in\{1,\dots,n\}$. As $[c,d]$ is compact, we may choose $M>0$ so that $p^{-i_k}(1-p)^{-j_k}\le M^2$, for all $p\in[c,d]$, which means that

$$|f_k(p,s)| \le (p^{-i_k}(1-p)^{-j_k})^{\frac{1}{2}} \le M,$$

for all $p\in[c,d]$ and $s\in\mathbb{R}$. Now, to get the desired conclusion from Proposition 5.1.25 we need to show that, for every $(p_0,s_0)\in[c,d]\times\mathbb{R}$, the map $(p,s)\mapsto f_k(p,s)(x)$ is continuous at the point (p_0,s_0), for almost all $x\in[0,1]$. It is easy to see from the definition of f_k that the map is continuous at (p_0,s_0), for all $x\in[0,1]$ possibly excluding

$$g^{\phi_{p_0}}(a), \qquad g^{\phi_{p_0}}(b), \qquad \phi_{p_0}^{-1}(g(x_{k-1})) \qquad \text{and} \qquad \phi_{p_0}^{-1}(g(x_k)),$$

as these are exactly the points where the transition between 0 and 1 happens for the indicator functions in the definition of f_k. Thus we can conclude from Proposition 5.1.25 that the map $(p,s)\mapsto f_k(p,s)$ is continuous on $[c,d]\times\mathbb{R}$, and, we conclude that the map $(p,s)\mapsto\pi_s^{\phi_p}(g)$ is strong operator continuous from $(0,1)\times\mathbb{R}$ to $\mathcal{H}$. $\square$

As we now know that our family of representations is continuous in both parameters, a natural question to ask is what happens when p approaches zero and one.

Lemma 5.3.15. *For every $x\in(0,1)$ and $g\in F$ we have*

$$\lim_{p\to 0}\phi_p^{-1}(x)=0, \quad \lim_{p\to 1}\phi_p^{-1}(x)=1 \quad \textit{and} \quad \lim_{p\to 0}g^{\phi_p}(x)=\lim_{p\to 1}g^{\phi_p}(x)=x.$$

Proof. We prove the statements for $p\to 1$. The case $p\to 0$ is analogous. Choose some $m\in\mathbb{N}$ so that $\frac{1}{2^m}<x$. Since ϕ_p^{-1} is increasing, for each $p\in(0,1)$, it suffices to prove that $\lim_{p\to 1}\phi_p^{-1}(\frac{1}{2^m})=1$. This, though, follows directly from Lemma 5.3.8 as it tells us that $\phi_p^{-1}(\frac{1}{2^m})=p^m$, which goes to one as p goes to one. Now, let us prove that $\lim_{p\to 1}g^{\phi_{p_n}}(x)=x$. By Lemma 5.3.10 we may choose dyadic rational numbers a and b such that g^{ϕ_p} maps $[0,\phi_p^{-1}(a)]$ linearly onto $[0,\phi_p^{-1}(b)]$, for all $p\in(0,1)$. This means that

$$g^{\phi_p}(y)=\frac{\phi_p^{-1}(b)}{\phi_p^{-1}(a)}y$$

for $y\in[0,\phi_p^{-1}(a)]$. As $\lim_{p\to 1}\phi_p^{-1}(a)=1$ by the first part of the proof, we know that $x\in[0,\phi_p^{-1}(a)]$, for p sufficiently close to 1, and, as we also know that

$\lim_{p\to 1} \phi_p^{-1}(b) = 1$, we see that

$$\lim_{p\to 1} g^{\phi_p}(x) = \lim_{p\to 1} \frac{\phi_p^{-1}(b)}{\phi_p^{-1}(a)} x = x,$$

as wanted $\square$

We can now prove that the representations $\pi_s^{\phi_p}$ converge strongly to the trivial representation on $\mathcal{H}$ for p going to zero or one.

Proposition 5.3.16. *Given $s \in \mathbb{R}$, the representation $\pi_s^{\phi_p}$ converges strongly to the trivial representation as p goes to 0 or 1, that is,*

$$\lim_{p\to 0} \|\pi_s^{\phi_p}(g)\xi - \xi\|_2 = 0 \qquad \textit{and} \qquad \lim_{p\to 1} \|\pi_s^{\phi_p}(g)\xi - \xi\|_2 = 0$$

for each $g \in F$ and $\xi \in \mathcal{H}$.

Proof. We will prove the statement for $p \to 0$. The case $p \to 1$ is analogue. Since the functions $\mathbf{1}_{(a,b)}$, for $a, b \in (0,1)$ with $a < b$, spans a dense subset of $\mathcal{H}$, it suffices to check the statement for ξ of this form. Fix such a and b, as well as $g \in F$, and choose a standard dyadic partition $x_0, x_1, \ldots, x_n$ associated to g. Let us for simplicity of notation let $c = x_{n-1}$ and $d = g(x_{n-1})$. We know from Lemma 5.3.15 that $\lim_{p\to 0} \phi_p^{-1}(d) = 0$ and $\lim_{p\to 0} g^{\phi_p}(a) = a$, so, as $a > 0$ by assumption, we know that $\phi_p^{-1}(d) < g^{\phi_p}(a)$ for sufficiently small p. Fix such a p. If $x \in [0,1]$ with $g^{-\phi_p}(x) \in (a,b)$, then $x \in (g^{\phi_p}(a), g^{\phi_p}(b))$. Now,

$$(g^{\phi_p}(a), g^{\phi_p}(b)) \subseteq (\phi_p^{-1}(d), 1) = (\phi_p^{-1}(g(c)), 1),$$

which means that $x \in (\phi_p^{-1}(g(c)), 1)$, so that $g^{-1}\phi_p(x) \in (c, 1)$. As the intervals (x_{k-1}, x_k) are disjoint for different values of k, we get that

$$\mathbf{1}_{(x_{k-1},x_k)}(g^{-1}\phi_p(x))\mathbf{1}_{(a,b)}(g^{-\phi_p}(x)) = 0,$$

for all $x \in [0,1]$ and all $k \in \{1, 2, \ldots, n-1\}$. Moreover, in the case $k = n$, we get

$$\mathbf{1}_{(x_{k-1},x_k)}(g^{-1}\phi_p(x))\mathbf{1}_{(a,b)}(g^{-\phi_p}(x)) = \mathbf{1}_{(a,b)}(g^{-\phi_p}(x)).$$

We now get from Remark 5.3.12 that

$$\pi_s^{\phi_p}(g)\mathbf{1}_{(a,b)}(x) = \Big(\frac{1 - \phi_p^{-1}(c)}{1 - \phi_p^{-1}(d)}\Big)^{\frac{1}{2}+is} \mathbf{1}_{(a,b)}(g^{-\phi_p}(x)) \tag{5.4}$$

for all $x \in [0,1]$, whenever p is sufficiently close to 0. From this point the conclusion easily follows by using the Lebesgue dominated convergence theorem. Indeed, if $(p_n)_{n\in\mathbb{N}}$ is any sequence in $(0,1)$ converging to 0, then it follows from expression (5.4) and Lemma 5.3.15 that

$$|\pi_s^{\phi_{p_n}}(g)\mathbf{1}_{(a,b)}(x) - \mathbf{1}_{(a,b)}(x)|^2$$

converges to zero as $n \to \infty$, for all $x \in [0,1]$ with the possible exception of a and b. By the Lebesgue dominated convergence theorem, this implies that

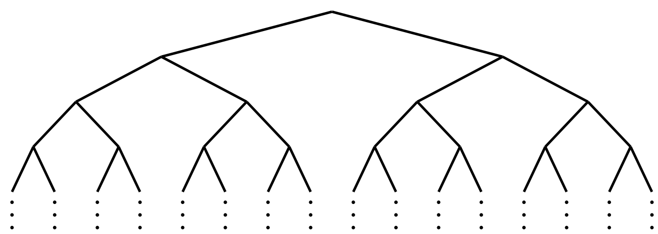
$$\|\pi_s^{\phi_{p_n}}(g)\mathbf{1}_{(a,b)} - \mathbf{1}_{(a,b)}\|_2^2 \to 0$$

as $n \to \infty$. We conclude that the representation converge strongly to the trivial representation on $\mathcal{H}$. $\square$

5.4 A construction

In this section we will describe a way of constructing maps satisfying condition $(\star)$ in order to generate families of irreducible representations of the Thompson group F. The strategy is to construct countable dense subsets of $[0,1]$ on which F acts by order-preserving bijections, define an order-preserving map φ from this set to the set of dyadic rational numbers in $[0,1]$ and then extend this map to a homeomorphism of the unit interval. By constructing the set in a certain way, we can say exactly how the elements g^φ will look like, for $g \in F$, and, therefore, decide whether or not the map satisfies condition $(\star)$.

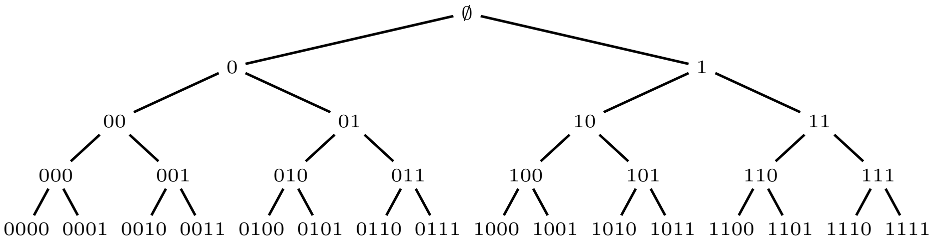
As explained in Section 5.2, the Thompson group F acts on the infinite binary rooted tree by certain permutations of finite subtrees, or, in other words, it acts on the Cantor set by certain prefix changes. We will construct the countable dense subsets of $[0,1]$ we are interested in as labellings for the infinite binary rooted tree, and then use the action of the Thompson group of this tree to obtain an action on these sets. Before we do this, let us discuss a natural order on the vertices of the this tree, as well as a natural set of labels. When we visualize the infinite binary tree rooted as follows.



We implicitly think of it as an ordered tree in a sense, as we have decided on some convention of left and right edges. This produces a natural ordering on the vertices of the tree, which, intuitively speaking, is given by saying that a vertex is less than another vertex if it is “to the left of” that other vertex. This ordering is exactly the same which is produced by labelling the vertices by dyadic rational numbers or rational numbers as shows in the end of Section 2.3. There is also a very natural way of labelling the tree with finite word in the letter $\{0, 1\}$, so let us explain this in details. Let us denote the set of finite words in the letters $\{0, 1\}$, including the empty word, by $\mathcal{S}$. The empty word will be denoted $\emptyset$. Let us define an order on $\mathcal{S}$ as follows. Given two distinct words α and β in $\mathcal{S}$, we can choose a largest common initial segment γ, meaning the largest (possibly empty) word γ such that $\alpha = \gamma\alpha'$ and $\beta = \gamma\beta'$, for some finite words α' and β' in $\mathcal{S}$. Then $\alpha \leq \beta$ if and only if one of the following statements are true:

- $\beta' = \emptyset$ and α' begins with a 0;
- $\alpha' = \emptyset$ and β' begins with a 1;
- α' begins with a 0 and β' begins with a 1.

Here "begins with" should be understood as the word not being empty, and the left most letter being the specified one. As an example, $000 \leq 00$, $00 \leq 001$ and $010 \leq 011$. It is easy to see that, if we label the root of the infinite binary rooted tree by $\emptyset$ and then label the left and right children of a vertex by adding a zero and a one to the right of the word, respectively, we get a labelling which is in accordance with the natural order on the tree. To illustrate, the first five levels of the tree looks as follows.



Now, we want to label the tree with a dense set in $(0, 1)$, and then obtain an action of F on $[0, 1]$ by increasing homeomorphisms. If we do have a labelling of the tree with a countable dense subset $\mathfrak{X}$, so that its order matches the order on the tree, then we get an order-preserving bijection φ from $\mathfrak{X}$ to the set of dyadic rational numbers in the interval $(0, 1)$, as we also have a labelling of the tree with the latter set. This map can then be extended to an increasing homeomorphism of $[0, 1]$ by first letting $\varphi(0) = 0$ and then

$$\varphi(x) = \sup\{\varphi(y) : y \in \mathfrak{X}, y \leq x\},$$

for $x \in (0, 1]$. As both $\mathfrak{X}$ and the set of dyadic rational numbers in $(0, 1)$ are dense in $[0, 1]$, we end up with an order-preserving bijection, in other words, an order-preserving homeomorphism. A natural question to ask is which dense subsets we can do this with, and, as it turns out, this is possible to do with all countable, dense subsets of $(0, 1)$. However, it will not be all of these which produce maps that satisfy condition $(\star)$, so we will need to do something particular to make sure this happens. Now, the reason that we can label the infinite binary tree with any countable dense subset of $(0, 1)$ in an way which matches the order of the tree is because all such sets are in fact order-isomorphic. This type of ordered set is known as a countable, dense, linear order without end-points, and it is a result due to Cantor that these are all isomorphic; see for example [20, Theorem 541].

Let us describe the construction of the countable, dense subsets of $(0,1)$. Fix some order preserving homeomorphism $f\colon [0,2] \to [0,1]$. This function will serve as our "parameter" on which the construction depends. From this map we obtain two injective, order preserving, continuous maps $f_0, f_1\colon [0,1] \to [0,1]$, with $f_0(1) = f_1(0)$, by defining

$$f_0(x) = f(x) \qquad \text{and} \qquad f_1(x) = f(x+1),$$

for all $x \in [0,1]$. Let us denote the point $f(1)$ by z_f, so that $f_0(1) = f_1(0) = z_f$. Note that $z_f \in (0,1)$ and that f_0 and f_1 are order-preserving homeomorphisms from $[0,1]$ to $[0,z_f]$ and $[z_f,1]$, respectively. Now, we want to use these functions to label the infinite binary tree, and we want to do in in a way so that f_0 and f_1 map the labels on the entire tree injectively onto the left and right half of the tree, respectively. This we will make more precise below.

Let us recursively define a sequence of subsets $(\mathcal{B}_f^n)_{n\in\mathbb{N}}$ of $[0,1]$. First we start by letting $\mathcal{B}_f^0 = \{0,1\}$, and then define

$$\mathcal{B}_f^n = f_0(\mathcal{B}_f^{n-1}) \cup f_1(\mathcal{B}_f^{n-1})$$

for all $n \in \mathbb{N}$. In this way we get a sequence of sets, and in the following lemma we will prove a few easy results about these sets. However, we will start by introducing some notation. Given a word $w \in \mathcal{S}$ different from the empty word, with $w = w_1 w_2 \cdots w_n$ say, for some $w_1, \dots, w_n \in \{0,1\}$, we will let f_w denote the composition $f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_n}$. Moreover, we will let $f_\emptyset$ denote the identity on the interval $[0,1]$. We clearly have $f_w \circ f_v = f_{wv}$, for every $w, v \in \mathcal{S}$.

Lemma 5.4.1. *With the notation above, the following statements hold:*

(1) *the sets* $\mathcal{B}_f^0, \mathcal{B}_f^1, \mathcal{B}_f^2, \dots$ *are increasing;*

(2) *for each* $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{B}_f^{n+1} \setminus \{0,1\} = \big\{ f_w(z_f) : w \in \mathcal{S} \text{ has length at most } n \big\},$$

so that, in particular

$$\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n = \big\{ f_w(z_f) : w \in \mathcal{S} \text{ has length } n \big\}$$

(3) *if* $w, v \in \mathcal{S}$ *are distinct, then* $f_w(z_f)$ *and* $f_v(z_f)$ *are distinct, so that, in particular, the sets* $\mathcal{B}_f^n$ *and* $\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1}$ *have* $2^n + 1$ *and* 2^{n-1} *elements, for every* $n \in \mathbb{N}$, *respectively;*

(4) *for each pair* $w, v \in \mathcal{S}$, *we have that* $f_w(z_f) < f_v(z_f)$ *if and only if* $w < v$ *with the usual order on* $\mathcal{S}$*;*

(5) *the sets* $f_0(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1})$ *and* $f_1(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1})$ *are disjoint, for every* $n \in \mathbb{N}$*, and*

$$f_0(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1}) \cup f_1(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1}) = \mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n.$$

Proof. Clearly (3) follows directly from (4) and (2), so let us start by proving (1). That $\mathcal{B}_f^0 \subseteq \mathcal{B}_f^1$ is obviously satisfied, as $\mathcal{B}_f^1 = \{0, z_f, 1\}$, and, supposing that $\mathcal{B}_f^{n-1} \subseteq \mathcal{B}_f^n$, we see that

$$\mathcal{B}_f^n = f_0(\mathcal{B}_f^{n-1}) \cup f_1(\mathcal{B}_f^{n-1}) \subseteq f_0(\mathcal{B}_f^n) \cup f_1(\mathcal{B}_f^n) = \mathcal{B}_f^{n+1}.$$

Thus, (1) follows by induction.

Next, let us argue that (5) holds. First of all, the images of f_0 and f_1 only has z_f in common, so the sets $f_0(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1})$ and $f_1(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1})$ are obviously disjoint if neither contains z_f. Since the maps f_0 and f_1 are injective and $0, 1 \notin \mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1}$, this follows from the fact that $z_f = f_0(1) = f_1(0)$. Now, along the same lines, we realize that

$$f_0(\mathcal{B}_f^n) \setminus (f_0(\mathcal{B}_f^{n-1}) \cup f_1(\mathcal{B}_f^{n-1})) = f_0(\mathcal{B}_f^n) \setminus f_0(\mathcal{B}_f^{n-1}),$$

as the images of f_0 and f_1 only have z_f in common. We get a similar statement with 0 and 1 interchanged. Using these two equations, we see that

$$\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n = \Big(f_0(\mathcal{B}_f^n) \setminus f_0(\mathcal{B}_f^{n-1})\Big) \cup \Big(f_1(\mathcal{B}_f^n) \setminus f_1(\mathcal{B}_f^{n-1})\Big).$$

Because both f_0 and f_1 are injective maps, the right hand side is exactly

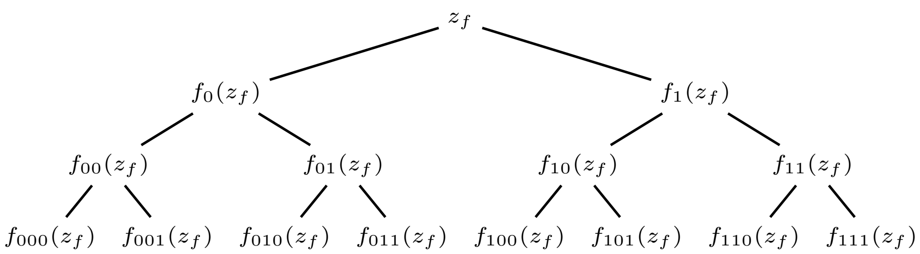
$$f_0(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1}) \cup f_1(\mathcal{B}_f^n \setminus \mathcal{B}_f^{n-1}),$$

which means that we have proved (5).

Let us proceed to prove (2). We start by proving the description $\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n$, and we do this by induction on n. Clearly it is true for $n = 0$, so suppose that the description holds for some $n \geq 0$, that is, $\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n$ is the set of elements of the form $f_w(z_f)$ with $w \in \mathcal{S}$ a word of length n. Clearly $f_0(\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n)$ and $f_1(\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n)$ are the sets of elements of the form $f_w(z_f)$ with $w \in \mathcal{S}$ a word of length $n+1$ beginning with 0 and 1, respectively. Thus their union is the set of elements of the form $f_w(z_f)$ with $w \in \mathcal{S}$ a word of length $n+1$. By (5) their union is exactly $\mathcal{B}_f^{n+2} \setminus \mathcal{B}_f^{n+1}$, so we have proved what we wanted to prove. Now, as $\mathcal{B}_f^{m+1} \setminus \{0,1\} = \bigcup_{k=0}^m \mathcal{B}_f^{k+1} \setminus \mathcal{B}_f^k$, the description of this set follows from what we have already proved.

Next, let us prove (4). Suppose that $w, v \in \mathcal{S}$, and let us prove the statement by induction over the maximum of the lengths of the two words. If both words have length equal to zero, then the statement is trivial. Suppose, therefore, that we have proved the statement for pairs of words for which one has length n and the other less than or equal to n, for some $n \geq 0$, and let us prove the statement for $n+1$. We may assume that w has length $n+1$ and v length less than or equal to $n+1$. The case where v has length $n+1$ is similar. Since w is not the empty word, we may write $w = w_1 w'$, for some $w_1 \in \{0,1\}$ and $w' \in \mathcal{S}$. Now, suppose that v is the empty word. Then $w < v$ if and only if $w_1 = 0$, by definition of the order on $\mathcal{S}$. Since v is the empty word and w is not, we know from (2) that $f_v(z_f) = z_f$ and $f_w(z_f) \neq z_f$. As $f_w(z_f) \in [0, z_f]$ if and only if $w_1 = 0$, we have proved the statement in the case where v was empty. Now, assume that v is not empty, so that we may write $v = v_1 v'$, for some $v_1 \in \{0,1\}$ and $v' \in \mathcal{S}$. We will consider a few cases. Suppose first that $w_1 = v_1$. In this case $f_w(z_f) < f_v(z_f)$ if and only if $f_{w'}(z_f) < f_{v'}(z_f)$ since f_{w_1} is injective and order-preserving. Moreover, $w < v$ if and only if $w' < v'$, so the conclusion follows from the induction hypothesis, as the maximum of the lengths of w' and v' is n. Suppose instead that $w_1 \neq v_1$. Then $w < v$ if and only if $w_1 = 0$ and $v_1 = 1$. Since both w and v are non-empty, both $f_w(z_f)$ and $f_v(z_f)$ are different from z_f, and, since $w_1 \neq v_1$, on of these lies on $[0, z_f]$ and one in $[z_f, 1]$. Hence, $f_w(z_f) < f_v(z_f)$ if and only if $w_1 = 0$ and $v_1 = 1$. This concludes the induction step, and so we have proved (4). $\square$

Letting $\mathcal{B}_f$ denote the union of the sets $\mathcal{B}_f^n$, the above lemma tells us that $\mathcal{B}_f = \{f_w(z_f) : w \in \mathcal{S}\} \cup \{0,1\}$. Moreover, it tells us that the order on this set, as a subset of $[0,1]$, agrees with the one on the set $\mathcal{S}$. Thus, we may label the infinite binary tree with the elements of $\{f_w(z_f) : w \in \mathcal{S}\}$ in the way we are interested in, namely, in a way that agrees with the order on the vertices we discussed. Indeed, if we consider the labelling by the elements of $\mathcal{S}$ described above, then the new labelling is replacing the label w with the label $f_w(z_f)$, for all $w \in \mathcal{S}$. The first four levels of the resulting labelling look as follows:



So far we have obtained a countable subset of $(0,1)$ and an action of F on this set, namely the one induced from the action on the tree. However, for it to be of interest to us, we need too know that it is dense and that the corresponding homeomorphism of $[0,1]$ satisfies condition ($\star$). We will start by focusing on the latter problem, namely, the problem of determining when the map satisfied condition ($\star$). If the set is dense, then we will denote the map we obtain by ϕ_f.

First, let us assume that $\mathcal{B}_f$ is dense and explain how the map ϕ_f is obtained from the set $\mathcal{B}_f$. We already explained above how the map was extended from $\mathcal{B}_f$ to all of $[0,1]$, so let us explain how the maps from $\mathcal{B}_f$ to $\mathcal{D}$ looks. This map is defined on $\mathcal{B}_f^n$ as the order-preserving bijection onto $\mathcal{D}^n$, for each $n \in \mathbb{N} \cup \{0\}$. However, we can say more explicitly how it looks on an element $f_w(z_f)$ in terms of w. Indeed, $\phi_f(z_f) = \frac{1}{2}$ and, given non-empty $w \in \mathcal{S}$, $w = w_1 \cdots w_n$ say, we have

$$\phi_f\big(f_w(z_f)\big) = \sum_{k=1}^{n} w_k 2^{-k} + 2^{-(n+1)}. \tag{5.5}$$

Moreover, we can actually say explicitly how the elements g^{ϕ_f} look in terms of the functions f_w, for $w \in \mathcal{S}$. To express this, we just need to note that, since both f_0 and f_1 are injective, the maps f_w, for $w \in \mathcal{S}$, are all injective, as well. Therefore, we can consider their inverses. Given some $w \in \mathcal{S}$, the image of $[0,1]$ by f_w is some closed interval in $[0,1]$, and on this interval the inverse f_w^{-1} is defined. It is easy to see that, given $w, v \in \mathcal{S}$, we have $f_w^{-1} \circ f_{wv} = f_v$.

Lemma 5.4.2. *Supposing that $\mathcal{B}_f$ is dense in $[0,1]$, and recall the definition of ϕ_f from above. Suppose that $g\colon [0,1] \to [0,1]$ maps a dyadic interval I, of length 2^{-n}, linearly onto another dyadic interval J, of length 2^{-m}, for some $n, m \in \mathbb{N}$. Let $\sum_{k=1}^{n+1} x_k 2^{-k}$ and $\sum_{k=1}^{m+1} y_k 2^{-k}$ be base 2 expansion of the midpoint of I and J, respectively. Then*

$$\phi_f^{-1} g \phi_f(z) = f_{y_1 y_2 \cdots y_m} f_{x_1 x_2 \cdots x_n}^{-1}(z)$$

for all z in the interval $\phi_f^{-1}(I)$.

Proof. First, let us note that $x_{n+1} = y_{m+1} = 1$, as the intervals I and J has length 2^{-n} and 2^{-m}, respectively. Thus, I and J are the images of the cylinder sets $S_{x_1,\ldots,x_n}$ and $S_{y_1,\ldots,y_m}$ via κ, respectively, as explained in Remark 5.2.7.

Now, let us rather prove that $\phi_f^{-1} g(z) = f_{y_1\cdots y_m} f_{x_1\cdots x_n}^{-1} \phi_f^{-1}(z)$, for all $z \in I$, as this will clearly imply the statement of the lemma. As the set of dyadic rational points in the interior of I is dense in I, it suffices to check the formula for these points. Let z be a dyadic rational point in the interior of I. From Remark 5.2.7 we know z has the form

$$z = \textstyle\sum_{k=1}^{n} x_k 2^{-k} + \sum_{k=1}^{j} z_k 2^{-(n+k)}$$

for some $j \in \mathbb{N}$ and $z_1, z_2, \ldots, z_j \in \{0,1\}$ with $z_j = 1$. From Equation (5.5) we know that $\phi_f^{-1}(z) = f_{x_1\ldots x_n z_1 \ldots z_{j-1}}(z_f)$, so that

$$\begin{aligned} f_{y_1\cdots y_m} f_{x_1\cdots x_n}^{-1} \phi_f^{-1}(z) &= f_{y_1\cdots y_m} f_{x_1\cdots x_n}^{-1}(f_{x_1\ldots x_n z_1\ldots z_{j-1}}(z_f)) \\ &= f_{y_1\cdots y_m}(f_{z_1\ldots z_{j-1}}(z_f)) \\ &= f_{y_1\cdots y_m z_1\ldots z_{j-1}}(z_f). \end{aligned}$$

As g maps I linearly onto J, we know from Remark 5.2.7 that

$$g(z) = \textstyle\sum_{k=1}^{m} y_k 2^{-k} + \sum_{k=1}^{j} z_k 2^{-(m+k)}.$$

Using Equation (5.5), we see that the right hand side is exactly $f_{y_1\cdots y_m z_1\ldots z_j}(z_f)$. This proves that $\phi_f^{-1}g(z) = f_{y_1\cdots y_m}f_{x_1\cdots x_n}^{-1}\phi_f^{-1}(z)$, which was what we needed to prove. $\square$

By applying the above lemma to standard dyadic partitions associated to the elements of F, we get a description the elements g^{ϕ_f}, namely, that they are piecewise of the form $f_{y_1y_2\cdots y_m}f_{x_1x_2\cdots x_n}^{-1}$. Using this, it is easy to determine when the action of F^{ϕ_f} leaves the Lebesgue measure quasi-invariant. To do this, let us introduce the notion of absolutely continuous functions and recall a few results about these.

Definition 5.4.3. Let $I \subseteq \mathbb{R}$ be a closed interval, and let h be a complex valued function on I. Then h is said to be *absolutely continuous* if, for every $\varepsilon > 0$, there exists some $\delta > 0$ so that, for any finite collection $(a_1, b_1), \ldots, (a_n, b_n)$ of disjoint subintervals of I with $\sum_{k=1}^n |b_k - a_k| < \delta$ we have $\sum_{k=1}^n |h(b_k) - h(a_k)| < \varepsilon$

It is easy to see from the definition that absolutely continuous functions are also uniformly continuous. The following theorem gives equivalent characterizations of absolutely continuous functions. A proof can be found in [28, Theorem 3.35]

Theorem 5.4.4. *Given some function $h\colon [a,b] \to \mathbb{C}$, the following conditions are equivalent:*

(1) *h is absolutely continuous on $[a,b]$;*

(2) *h is continuous and differentiable almost everywhere with h' integrable;*

(3) $h(x) = h(a) + \int_a^x g \,\mathrm{d}m$, *for some integrable function g;*

(4) *h is differentiable almost everywhere, the derivative h' is an integrable function and $h(x) = h(a) + \int_a^x h' \,\mathrm{d}m$.*

Moreover, if (3) *holds, then $g = h'$ almost everywhere.*

For our purposes, we will also need another characterization of absolute continuity, and to state this characterization, we need to introduce the notions of bounded variation and of Luzin's condition N.

Definition 5.4.5. A function $h\colon [a,b] \to \mathbb{C}$ is said to have *bounded variation* if there exists a constant $M > 0$ so that, for every partition

$$a = x_0 < x_1 < \cdots < x_n < x_n + 1 = b,$$

we have $\sum_{k=0}^n |h(x_k) - h(x_{k+1})| < M$

Definition 5.4.6. A function $f\colon [a,b] \to \mathbb{C}$ is said to satisfy the *Luzin condition* or satisfy *Luzin's condition N* if f maps null sets to null sets, that is, if $f(N)$ is a null set for every null set $N \subseteq [a,b]$.

The following characterization of absolutely continuous functions is known as the Vitali–Banach–Zaretskij Theorem. A proof can be found in [2, Theorem 3.9].

Theorem 5.4.7. *A function is absolutely continuous if and only if it is continuous, of bounded variation and satisfies Luzin's condition N.*

With the above theorem we immediately get the following easy result.

Proposition 5.4.8. *Let I and J be closed intervals in $\mathbb{R}$, and let $h\colon I \to J$ be an increasing homeomorphism. Then*

(1) *h is absolutely continuous if and only if h maps null sets to null sets.*

(2) *the image measure h_*m is absolutely continuous with respect to the Lebesgue measure if and only if h^{-1} is an absolutely continuous function.*

Moreover, compositions of strictly increasing, absolutely continuous functions are again absolutely continuous.

Proof. First of all, it is easy to see from the definition that h automatically has bounded variation, as h is increasing and bounded. Thus, it follows from Theorem 5.4.7 that it it absolutely continuous if and only if it satisfies Luzin's condition N, that is, if it maps null sets to null sets. This proves the first part.

Now, as $h_*m(A) = m(h^{-1}(A))$, for all measurable sets $A \subseteq J$, we see that h_*m is absolutely continuous if and only if $h^{-1}(A)$ is a null set whenever A is a null set. By the first part this is equivalent to the fact that h^{-1} is absolutely continuous, as h^{-1} is also an increasing homeomorphism.

By the first part we know that strictly increasing continuous maps are absolutely continuous if and only if they map null sets to null sets. Clearly compositions of increasing homeomorphisms are increasing homeomorphisms, so, since the composition of maps mapping null sets to null sets clearly also enjoys this property, it follows from the first part that the composition of increasing, absolutely continuous homeomorphisms are again absolutely continuous. $\square$

Remark 5.4.9. An increasing continuous function from $[0, 1]$ to $[0, 1]$ is called purely singular if it is differentiable almost everywhere with derivative 0. Such maps, obviously, cannot be absolutely continuous, as they would be constantly equal to zero by Theorem 5.4.4. As explained in [63], the Minkowski questionmark fucntion $?$ is a singular function. Thus by the above proposition and Proposition 5.1.22, the representation $\pi_s^?$ is not unitarily equivalent to the representaion π_t^{id}, for all $s, t \in \mathbb{R}$. This means that the one-parameter families $(\pi_s^?)_{s\in\mathbb{R}}$ and $(\pi_s^{\mathrm{id}})_{s\in\mathbb{R}}$ are really distinct.

The above proposition allows us to describe easily when the action of F^{ϕ_f} on the unit interval leaves the Lebesgue measure quasi-invariant in terms of the function f.

Proposition 5.4.10. *Suppose that we are in the situation where $\mathcal{B}_f$ is dense in $[0, 1]$. Then F^{ϕ_f} leaves the Lebesgue measure quasi-invariant if and only if both f and f^{-1} are absolutely continuous.*

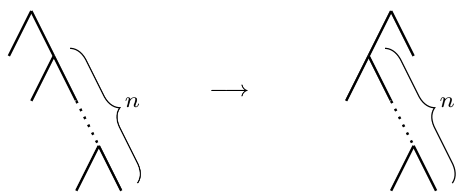
Proof. We know from Proposition 5.4.8, that the action of F^{ϕ_f} leaves the Lebesgue measure quasi-invariant if and only if g^{ϕ_f} is absolutely continuous, for all $g \in F$. Clearly, f and f^{-1} are absolutely continuous if and only if the maps f_0, f_1, f_0^{-1} and f_1^{-1} are all absolutely continuous, so what we need to prove is that g^{ϕ_f} is absolutely continuous, for all $g \in F$ if and only if f_0, f_1, f_0^{-1} and f_1^{-1} are all absolutely continuous.

One of the implications is easy. Suppose that f_0, f_1, f_0^{-1} and f_1^{-1} are absolutely continuous, and let $g \in F$. Choose a standard dyadic partition $x_0, x_1, \cdots, x_n$ associated to to g. By Lemma 5.4.2 we know that g^{ϕ_f} has the form $f_w \circ f_v^{-1}$, for some $w, v \in \mathcal{S}$, on each of the intervals $[\phi_f^{-1}(x_{k-1}), \phi_f^{-1}(x_k)]$, for $n = 1, 2, \ldots, n$. Since we know from Proposition 5.4.8 that the composition of strictly increasing, absolutely continuous maps are again absolutely continuous, it follows that functions of the form $f_w \circ f_v^{-1}$ are absolutely continuous, since they are compositions of the functions f_0, f_1, f_0^{-1} and f_1^{-1}. This shows that g^{ϕ_f} is absolutely continuous on each of these intervals. It is rather easy to see from the definition of absolute continuity that a function which is absolutely continuous on all the subintervals from a finite partition of the interval is itself absolutely continuous. Thus we conclude that g^{ϕ_f} is absolutely continuous.

Now, suppose instead that all the elements g^{ϕ_f} are absolutely continuous, for $g \in F$, and let us prove that the functions f_0, f_1, f_0^{-1} and f_1^{-1} are absolutely continuous. Since the proof in the four cases are very analogous, we will do the proof in the case of f_0 and then explain what you would need to change in the three other cases. For each $n \in \mathbb{N}$, let g_n be an element of F such that $g_n(x) = \frac{1}{2}x$, for $x \in [0, \frac{2^n-1}{2^n}]$. It is not difficult to see that such an element exists, and, indeed, by Lemma 2.2.4 we just need to specify two standard dyadic partitions in which will ensure this. For this one may choose the partitions

$$\{\tfrac{2^k-1}{2^k} : k = 0, \ldots, n+1\} \cup \{1\} \quad \text{and} \quad \{\tfrac{2^k-1}{2^{k+1}} : k = 0, \ldots, n\} \cup \{\tfrac{1}{2}, 1\}.$$

If the reader is familiar with the representation of the elements of F as tree diagrams, the element g_n is represented in the following way:



Now, put $x_n = \phi_f^{-1}(\frac{2^n-1}{2^n})$, put $\tilde{g}_n = g_n^{\phi_f}$ and let us prove that $\tilde{g}_n(x) = f_0(x)$, for all $x \in [0, x_n]$. Afterwards we will explain why this implies that f_0 is absolutely continuous. Suppose that we are given a word $w \in \mathcal{S}$ so that $f_w(z_f) \in [0, x_n]$. Then $\phi_f(f_w(z_f)) \in [0, \frac{2^n-1}{2^n}]$, which means that

$$g^{\phi_f}(f_w(z_f)) = \phi_f^{-1} g_n(\phi_f(f_w(z_f))) = \phi_f^{-1}(\tfrac{1}{2}\phi_f(f_w(z_f)))$$

Since $f_w(z_f)$ is a dyadic rational number and dividing a dyadic rational number by two amounts to shifting the coefficient of the dyadic expansion one place to the left and inserting a zero on the first place, it is easy to see from equation (5.5), that

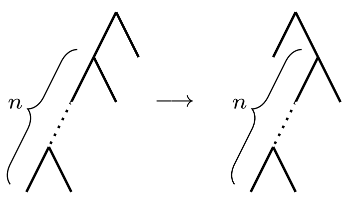
$$\phi_f^{-1}(\tfrac{1}{2}\phi_f(f_w(z_f))) = f_{0w}(z_f) = f_0(f_w(z_f)).$$

Hence, $\tilde{g}_n(x) = f_0(x)$, for all $x \in [0, x_n] \cap (\mathcal{B}_f \setminus \{0,1\})$, as we know from Lemma 5.4.1 that $\mathcal{B}_f \setminus \{0,1\} = \{f_v(z_f) : v \in \mathcal{S}\}$. Since we have assumed that $\mathcal{B}_f$ is dense in the unit interval, we conclude that $\tilde{g}_n(x) = f_0(x)$, for all $x \in [0, x_n]$. Let us now explain why this implies that f_0 is absolutely continuous. By Proposition 5.4.8 we just need to prove that f_0 maps null sets to null sets. Suppose that $N \subseteq [0,1]$ is a null set. We have assumed that g^{ϕ_f} is absolutely continuous, for all $g \in F$. In particular, f_0 is absolutely continuous on the interval $[0, x_n]$, for each $n \in \mathbb{N}$, as $\tilde{g}_n$ is absolutely continuous for all $n \in \mathbb{N}$. This means that $f_0(N \cap [0, x_n])$ is a null set, for each $n \in \mathbb{N}$. As $\lim_{n\to\infty} x_n = 1$, it follows that $f_0(N \setminus \{1\}) = \bigcup_{n=1}^{\infty} f(N \cap [0, x_n])$, which clearly means that $f_0(N)$ is a null set. Thus, we conclude that f_0 is absolutely continuous. Similarly we prove that f_0^{-1} is absolutely continuous by replacing g_n with g_n^{-1}, as f_0^{-1} is defined on $[0, z_f]$ and it follows from what we have already proved that $\tilde{g}_n^{-1}(x) = f_0^{-1}(x)$, for all $x \in [0, \phi_f^{-1}(\frac{2^n-1}{2^{n+1}})]$.

Now, to prove that f_1 and f_1^{-1} are absolutely continuous, one does the same, but replaces g_n with the element defined by the standard dyadic partitions:

$$\{0\} \cup \{\tfrac{1}{2^k} : k = 0, \dots, n+1\} \quad \text{and} \quad \{0, \tfrac{1}{2}\} \cup \{\tfrac{2^k+1}{2^{k+1}} : k = 0, \dots, n\}.$$

In term of tree diagrams, this is the element defined by:



This element has the form $x \mapsto \frac{1}{2}x + \frac{1}{2}$, for $x \in [\frac{1}{2^n}, 1]$. □

This proposition tells us exactly when the map ϕ_p satisfies condition ($\star$), assuming that $\mathcal{B}_f$ is dense in $[0,1]$. Let us consider the question of when $\mathcal{B}_f$ is actually dense. To do so, let start by introducing the notion of Lipschitz continuity. This makes sense in general metric space, but we only need if for functions on closed and bounded intervals of $\mathbb{R}$.

Definition 5.4.11. A function $h\colon [a,b] \to \mathbb{C}$ is called *Lipschitz* if there exists some $k \geq 0$ such that

$$|h(y) - h(x)| \leq k|y - x|,$$

for all $x, y \in [a,b]$. Furthermore, h is called a *contraction* if this happens for some $k < 1$. In general, the smallest possible k is called the *Lipschitz constant*.

It is easy to see that Lipschitz continuous maps are also absolutely continuous. The following propositions gives a characterization of Lipschitz continuity. As this is an easy consequence of the results we have already listed, we have included a proof of this.

Proposition 5.4.12. *A function $h\colon [a,b] \to \mathbb{C}$ is Lipschitz continuous if and only if it may be written as*

$$h(x) = h(a) + \int_a^x g \,\mathrm{d}m$$

for some essentially bounded function g. Moreover, the Lipschitz constant of h is exactly $\|g\|_\infty$.

Proof. Assume that the function h is Lipschitz, and let k be so that $|h(y) - h(x)| \leq k|y - x|$. As mentioned above, it is evident from the definitions that h is absolutely continuous, so, in particular, we know from Theorem 5.4.4 that h is differentiable almost everywhere. Clearly, $|h'(x)| \leq k$ in all points of differentiability, which means that h' is essentially bounded with $\|h'\|_\infty \leq k$. By Theorem 5.4.4 we know that

$$h(x) = h(a) + \int_a^x h' \,\mathrm{d}m,$$

so this takes care of the first implication.

Now, for the converse direction, assume that h has the desired form. Then h is absolutely continuous by Theorem 5.4.4, and, since

$$|h(y) - h(x)| \leq \int_x^y |g| \,\mathrm{d}m \leq (y - x)\|g\|_\infty,$$

for $x, y \in [a,b]$ with $x < y$, we see that h is Lipschitz with Lipschitz constant no greater than $\|g\|_\infty$. This proves that h is Lipschitz if and only if it has the desired form. Moreover, we have actually also shown that the Lipschitz constant must be equal to $\|g\|_\infty$. □

At this point, we are able to give a sufficient condition for when $\mathcal{B}_f$ is dense in $[0,1]$. Namely that f is a contraction.

Proposition 5.4.13. *If f is a contraction, then $\mathcal{B}_f$ is dense in $[0,1]$.*

Proof. By assumption we may choose some positive number $k < 1$ so that $|f(x) - f(y)| \leq k|x - y|$, for all $x, y \in [0,2]$. In particular, f_0 and f_1 are both contractions, as well, with the same constant k. Now, it is straightforward to check by induction that consecutive elements in $\mathcal{B}_f^n$ has distance at most k^n to each other. In particular, we get that, for each $x \in [0,1]$ and $n \in \mathbb{N}$, there exists an element $y \in \mathcal{B}_f^n$ with $|x - y| \leq \frac{k^n}{2}$. Evidently this means that $\mathcal{B}_f$ is dense in $[0,1]$. □

The proposition above is our only sufficient condition we have for $\mathcal{B}_f$ to be dense in $[0,1]$. We will see in Remark 5.5.2, that it is not necessary, though.

Proposition 5.4.14. *Suppose that $\mathcal{B}_f$ is dense in $[0,1]$. Then $f_0(x) \neq x$ and $f_1(x) \neq x$, for all $x \in (0,1)$, or, in other words, the graphs of f_0 and f_1 do not intersect the diagonal except in 0 and 1, respectively.*

Proof. Suppose that f_0 intersects the diagonal, that is, there is some $x \in (0,1)$ so that $f_0(x) = x$. The case where f_1 intersects the diagonal is similar. First note that $x < z_f$, since we otherwise would have $z_f = 1$, contradicting the fact that f is a homeomorphism. Let us prove that $\mathcal{B}_f \setminus \{0\}$ is contained in $[x,1]$. Since $x > 0$, this will prove that $(0,x) \cap \mathcal{B}_f$ is empty, so that, in particular, $\mathcal{B}_f$ is not dense in $[0,1]$. This is easily proved by induction. Clearly $y > x$, for all $y \in \mathcal{B}_f^0 \setminus \{0\} = \{1\}$, and supposing that $y > x$, for all $y \in \mathcal{B}_f^n \setminus \{0\}$, we see that, since f_0 is strictly increasing, $f_0(y) > f_0(x) = x$, for all $y \in \mathcal{B}_f^n \setminus \{0\}$. Moreover, we also see that $f_1(y) \geq z_f > x$, for all $y \in \mathcal{B}_f^n$. Since $\mathcal{B}_f^{n+1} \setminus \{0\} = f_0(B_f^n \setminus \{0\}) \cup f_1(B_f^n)$, we conclude that $y > x$, for all $y \in \mathcal{B}_f^{n+1}$. Thus, by induction, $\mathcal{B}_f$ is contained in $[x,1]$. □

This gives us a necessary criterion for when $\mathcal{B}_f$ is dense in $[0,1]$, but we can say a bit more about the set than this.

Proposition 5.4.15. *The limits $\lim_{n\to\infty} f_0^n(z_f)$ and $\lim_{n\to\infty} f_1^n(z_f)$ both exists. Let us denote them x_0 and x_1, respectively. Then x_0 is the largest point fixed by f_0, and x_1 is the smallest point fixed by f_1. Moreover, $\mathcal{B}_f \setminus \{0,1\}$ is contained in $[x_0, x_1]$.*

Proof. First note that the sets of fixed points of f_0 and f_1 are closed, and hence compact. Since $z_f < 1$ and f_0 strictly increasing, we get, by repeated application of f_0, that $f_0^n(z_f)$ is a strictly decreasing sequence. Hence it is convergent as it is bounded from below. Similarly the sequence $f_1^n(z_f)$ is increasing and also convergent. Let us prove that x_0 and x_1 are largest and smallest among the fixed points of f_0 and f_1, respectively, that is, $x_0' \leq x_0$ and $x_1' \geq x_1$, for all $x_0', x_1' \in [0,1]$ with $f_0(x_0') = x_0'$ and $f_1(x_1') = x_1'$. So suppose that we are given such x_0' and x_1'. It follows from the proof of Proposition 5.4.14 that $\mathcal{B}_f$ is contained $[x_0', x_1']$, but then clearly x_0 and x_1 are contained in this interval, since they are both in the closure of $\mathcal{B}_f$. This shows that $x_0' \leq x_0$ and $x_1' \geq x_1$, as desired. That $\mathcal{B}_f$ is contained in $[x_0, x_1]$ follows from the proof of Proposition 5.4.14, as well. □

Proposition 5.4.16. *Either $\mathcal{B}_f$ is dense in $[0,1]$ or nowhere dense in $[0,1]$.*

Proof. Suppose that $\mathcal{B}_f$ is not dense in $[0,1]$, and let us prove that it is nowhere dense in $[0,1]$. It suffices to prove that in-between each pair of element in $\mathcal{B}_f$ there is an open interval which does not intersect $\mathcal{B}_f$. For, if the closure of $\mathcal{B}_f$ contains an open interval, then this must contains infinitely many points from $\mathcal{B}_f$, and so will any interval in-between these points.

By assumption we may choose some open interval (a, b) in $[0, 1]$ which does not intersect $\mathcal{B}_f$. We are going to prove the statement by induction in the following sense. We will prove by induction that, for all $n \in \mathbb{N}$, given two consecutive elements in $\mathcal{B}_f^n$, there exist an open interval in-between these which does not intersect $\mathcal{B}_f$. This will clearly imply that there exists such an interval in-between each pair of points in $\mathcal{B}_f$. Clearly the statement is true for $n = 0$ since $\mathcal{B}_f^0 = \{0, 1\}$. So suppose that we have proved the statement for $\mathcal{B}_f^n$, and let us prove it for $\mathcal{B}_f^{n+1}$. Let $x, y \in \mathcal{B}_f^{n+1}$ be consecutive. Then x and y are either both in the image of f_0 or both in the image of f_1. We may assume that it is f_0 since the case of f_1 is similar. This means, by definition of $\mathcal{B}_f^{n+1}$, that $x = f_0(x')$ and $y = f_0(y')$, for some consecutive elements x' and y' in $\mathcal{B}_f^n$. Now, let I be an open interval in-between x' and y' which does not intersect $\mathcal{B}_f$. Then we see that

$$f_0(I) \cap \mathcal{B}_f = f_0(I) \cap f_0(\mathcal{B}_f) = f_0(I \cap \mathcal{B}_f) = \emptyset.$$

Here we used that $\mathcal{B}_f = f_0(\mathcal{B}_f) \cup f_1(\mathcal{B}_f)$, so that the intersection of $\mathcal{B}_f$ with the image of f_0 is exactly $f_0(\mathcal{B}_f)$. Since $f_0(I)$ is an interval in-between x and y since f_0 is strictly increasing, we have proved the desired statement. Thus it follows that $\mathcal{B}_f$ is nowhere dense in $[0, 1]$. □

5.5 A re-construction

In the last section we introduced a construction to obtain functions ϕ_f satisfying condition ($\star$), under the assumption that the set $\mathcal{B}_f$ is dense in $[0, 1]$. A natural question to ask is whether we can obtain interesting maps satisfying condition ($\star$) using this method. It turns out that this is indeed the case, as we will prove that all maps satisfying condition ($\star$) can be obtained in this fashion. Moreover, we already know of some maps satisfying condition ($\star$), and we will describe how to obtain these using this construction.

Proposition 5.5.1. *Suppose that φ is a map satisfying condition ($\star$). Then there exists a unique increasing homeomorphism $f\colon [0, 2] \to [0, 1]$ so that $\mathcal{B}_f$ is dense in $[0, 1]$ and $\varphi = \phi_f$. Indeed, this f is given by*

$$f(x) = \begin{cases} \varphi^{-1}(\frac{1}{2}\varphi(x)) & \textit{for } x \in [0, 1] \\ \varphi^{-1}(\frac{1}{2}\varphi(x-1) + \frac{1}{2}) & \textit{for } x \in [1, 2] \end{cases}.$$

In particular, the two functions f_0 and f_1 are given by $f_0(x) = \varphi^{-1}(\frac{1}{2}\varphi(x))$ and $f_1(x) = \varphi^{-1}(\frac{1}{2}\varphi(x) + \frac{1}{2})$, respectively.

Proof. Let f be defined as in the statement. In order to show that $\varphi = \phi_f$, it suffices to prove that $\varphi(\mathcal{B}_f^n) = \mathcal{D}^n$, for all $n \in \mathbb{N}$. Indeed, this implies that $\varphi^{-1}(\mathcal{D}) = \mathcal{B}_f$, and, as $\mathcal{D}$ is dense in $[0, 1]$ and φ^{-1} is a homeomorphism, we get that $\mathcal{B}_f$ is dense in $[0,1]$, as well. Moreover, because of this, ϕ_f is well-defined, and since $\phi_f^{-1}(\mathcal{B}_f^n) = \mathcal{D}^n$ by definition, it follows that φ and ϕ_f agree on the dense set $\mathcal{B}_f$. Hence, they agree on all of $[0,1]$ by continuity.

We prove that $\varphi(\mathcal{B}_f^n) = \mathcal{D}^n$ by induction on n. Clearly $\varphi(\mathcal{B}_f^0) = \mathcal{D}^0$, so assume that $\varphi(\mathcal{B}_f^n) = \mathcal{D}^n$, for some $n \geq 0$, and let us prove that $\varphi(\mathcal{B}_f^{n+1}) = \mathcal{D}^{n+1}$. Since φ is injective, we just have to prove that $\varphi(\mathcal{B}_f^{n+1})$ is contained in $\mathcal{D}^{n+1}$, as both sets are finite and has the same number of elements. In particular, by our induction hypothesis, we only need to prove that $\varphi(\mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n)$ is contained in $\mathcal{D}^{n+1}$. Let $x \in \mathcal{B}_f^{n+1} \setminus \mathcal{B}_f^n$. By definition, this means that there exits some $x' \in \mathcal{B}_f^n$ so that $x = f_0(x')$ or $x = f_1(x')$. Let us assume that we are in the former of the two cases. The latter case is analogue. Now,

$$\varphi(x) = \varphi(f_0(x')) = \varphi(\varphi^{-1}(\tfrac{1}{2}\varphi(x')) = \tfrac{1}{2}\varphi(x'),$$

so since $x' \in \mathcal{B}_f^n$, we get from our induction hypothesis that $\varphi(x') \in \mathcal{D}^n$. In particular, $\varphi(x) = \frac{1}{2}\varphi(x') \in \mathcal{D}^{n+1}$, as wanted. By induction we get that $\varphi(\mathcal{B}_f^n) = \mathcal{D}^n$, for all $n \in \mathbb{N}$, and as explained, this implies that $\varphi = \phi_f$.

It is easy to see that f is uniquely determined by the property $\varphi = \phi_f$. Indeed, the sets $\mathcal{B}_f^n$ is uniquely determined by φ, for all $n \in \mathbb{N} \cup \{0\}$, as $\mathcal{B}_f^n = \varphi^{-1}(\mathcal{D}^n)$. Thus, the function f_0 is uniquely determined by φ, as well, as $f_0(\mathcal{B}_f^n) = \mathcal{B}_f^{n+1} \cap [0, z_f]$, for all $n \in \mathbb{N} \cup \{0\}$, and $\mathcal{B}_f$ is dense in $[0,1]$. A similar argument shows that f_1 is uniquely determined. It follows that f is uniquely determined as well. $\square$

Let us describe how to re-construct the maps ϕ_p, for $p \in (0,1)$, from Section 5.3. Fix some $p \in (0,1)$ and let f^p denote the function from Proposition 5.5.1 satisfying $\phi_p = \phi_{f^p}$. Then it is straightforward to check, using Lemma 5.3.8, that $\phi_p^{-1}(\frac{1}{2}\phi_p(x)) = px$ and $\phi_p^{-1}(\frac{1}{2}\phi_p(x) + \frac{1}{2}) = (1-p)x + p$, so that the functions f_0^p and f_1^p are given by

$$f_0^p(x) = px \qquad \text{and} \qquad f_1^p(x) = (1-p)x + p.$$

It is not difficult to realize that, if we let f be the function from Proposition 5.5.1 corresponding to the Minkowski question mark function, then the functions f_0 and f_1 are given by

$$f_0(x) = \frac{x}{x+1} \qquad \text{and} \qquad f_1(x) = \frac{1}{-x+2}.$$

Remark 5.5.2. A thing to note, concerning Proposition 5.4.13, is that, although the function f^p is clearly a contraction, the function f, corresponding to the Minkowski question mark function, is not. This follows from the fact that the derivative of f_0 is $(1+x)^{-2}$, which is 1 at 0. It is still Lipschitz, though.

5.6 Further thoughts and projects

In this chapter we have left several questions unanswered. The aim of this section is to list some of these, and to discuss possible further related projects.

Probably the most interesting question we have not answered in this thesis, is the following about the representations $\pi_s^{\phi_p}$.

Question 5.6.1. Given $p \in (0,1) \setminus \{\frac{1}{2}\}$, when are the representations $\pi_s^{\phi_p}$ and $\pi_t^{\phi_p}$ unitarily equivalent, for s and t different real numbers?

As mentioned, Garncarek proved that, for $p = \frac{1}{2}$, the representations $\pi_s^{\phi_{1/2}}$ and $\pi_t^{\phi_{1/2}}$ are unitarily equivalent if and only if the parameters satisfy $s = t + \frac{2\pi}{\log 2}k$, for some $k \in \mathbb{Z}$. We have not been able to decide the question for $p \neq \frac{1}{2}$, but we suspect that, in this case, the representations $\pi_s^{\phi_p}$ and $\pi_t^{\phi_p}$ are never unitarily equivalent, when $s \neq t$.

Recalling that the set of $t \in \mathbb{R}$ such that π_t^{φ} is unitarily equivalent to π_0^{φ} forms a subgroup of $\mathbb{R}$, an interesting question is the following.

Question 5.6.2. Which subgroups $H \subseteq \mathbb{R}$ can be realized as the subgroup from Proposition 5.3.8 corresponding to a map φ satisfying condition $(\star)$?

Stating the comment above differently, Garncarek proved that, for $p = \frac{1}{2}$ the subgroup H corresponding to ϕ_p is equal to $\frac{2\pi}{\log 2}\mathbb{Z}$.

Another point which was not dealt with, was when the set $\mathcal{B}_f$ is dense. We proved in Proposition 5.4.13 that it suffices for f to be a contraction, however, as mentioned in Remark 5.5.2 it is not a necessary condition. In fact, it is not difficult to come up with an absolutely continuous map f so that $\mathcal{B}_f$ is dense, but the derivative of f is unbounded.

Question 5.6.3. Is the set $\mathcal{B}_f$ always dense in $[0,1]$? If not, can it be determined from the function f?

Now, for the construction of Section 5.4 to be really interesting, it would be nice to be able to determine when the representations ϕ_f are irreducible and equivalent for different f. This however, seems like a difficult question to answer.

Question 5.6.4. Can ergodicity of the action of F^{ϕ_f} on the unit interval be characterized as a property of f?

Question 5.6.5. Given two functions f and g, is there an easy way to determine whether or not the representations $\pi_s^{\phi_f}$ and $\pi_s^{\phi_g}$ are unitarily equivalent?

To determine this is equivalent to determining whether or not the function $\phi_f^{-1}\phi_g$ is absolutely continuous.

The last question we will mention is about whether or not the continuity result of Proposition 5.3.14 can be proved in a more general setting.

Question 5.6.6. Is it possible to prove some sensible sort of continuity of the representations $\pi_s^{\phi_f}$ in terms of the function f?

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