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Studies in the hyperbolic circle problem

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A Paolo, Anna, e Martina

Preface

This thesis collects the results of my research as a PhD student at the University of Copenhagen during the years 2013-2016, carried out under the supervision of Morten S. Risager. The thesis is organized in four chapters and investigates various aspects of a specific problem, namely the hyperbolic circle problem, but relates to different areas of mathematics such as number theory, harmonic analysis, fractional calculus, and the theory of almost periodic functions.

We state the main theorems in a short introduction, and we give an overview of background material in the first chapter. In chapter 2 we explain how to use the pretrace formula and classical techniques to obtain a mean value theorem for the error term in our problem. In chapter 3 we study almost periodic functions, giving sufficient conditions for the existence of asymptotic moments and limiting distribution, and we apply the results to the hyperbolic circle problem. Chapter 4 contains the results of the joint paper with Risager [16] and describes the results that can be obtained by using the methods of fractional integration.

I would like to express my profund gratitude to Morten Risager as he has been a wonderful supervisor, and I feel deeply indebted for all he has taught me. He has broadened my view on many subjects, and has repeatedly given me advises that improved my skills as writer, speaker, and overall mathematician. I also thank him for all the time and energy he spent with me writing the paper [16].

I thank Yiannis Petridis and Christian Berg for interesting discussions and for supporting me in the decision of continuing doing research. I consider Niko Laaksonen and Dimitrios Chatzakos as mathematical relatives, and I thank them for detailed conversations in several occasions. I thank Andreas Strömbergsson for sharing an unpublished note on the Bessel function. I thank Daniel Fiorilli and Jan-Christoph Schlage-Puchta for stimulating correspondence on the moments of certain error terms and almost periodic functions.

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I also thank all the other members of the algebra and number theory group, as well as the GAMP group and in particular my office mates, who contributed to make my life at the department a lively experience.

Dulcis in fundo, I thank my parents and my sister for all the support they provided me since the time when I chose to study mathematics until the present

day of conclusion of my PhD. Questa tesi è dedicata con affetto alla mia famiglia, che ha sostenuto e sostiene le mie scelte di perpetuare negli studi matematici, dovunque essi mi portino, e mi accoglie calorosamente ogni volta che riesco a tornare e a passare del tempo nel bel paese.

Copenhagen, April 29, 2016.

Abstract

In this thesis we study the remainder term $e(s)$ in the hyperbolic lattice point counting problem. Our main approach to this problem is that of the spectral theory of automorphic forms. We show that the function $e(s)$ exhibits properties similar to those of almost periodic functions, and we study different aspects of the theory of almost periodic functions, namely criteria for the existence of asymptotic moments and limiting distribution for such type of functions. This gives us the possibility to infer nontrivial bounds on higher moments of $e(s)$, and existence of asymptotic moments and limiting distribution for certain integral versions of it. Finally we describe what results can be obtained by application of fractional calculus, especially fractional integration to small order, to the problem.

Resumé

I denne afhandling studerer vi restleddet $e(s)$ i det hyperbolske gitterpunkt problem. Vores primære tilgang til problemet er via spektralteorien for automorfe funktioner. Vi viser at funktionen $e(s)$ har egenskaber der ligner egenskaberne ved næsten periodiske funktioner. Vi studerer forskellige aspekter af de næsten periodiske funktioners teori, og giver kriterier for, at asymptotiske momenter og fordeling eksisterer. Som en konsekvens kan vi bevise ikke trivielle estimater for højere ordens momenter af $e(s)$, og vi kan bevise at visse integralversioner af $e(s)$ har en asymptotisk fordeling. Til sidst beskriver vi hvilke resultater man kan vise hvis man introducerer integration af lille orden i problemet.

Contents

Introduction

Given the action of a group G on a metric space X, and a point $x \in X$, we can ask whether the orbit Gx intersects a given compact subset $K \subseteq X$, and if so, we can try to determine as accurately as possible the cardinality of $K \cap Gx$. The main problem studied in the present work is of this nature.

Description of the problem

The metric space we consider is the hyperbolic plane $\mathbb{H}$, and the action is that given by (a discrete subgroup of) the group $\mathrm{PSL}(2,\mathbb{R})$, acting on $\mathbb{H}$ by linear fractional transformations. More specifically, we deal with cofinite Fuchsian groups $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$. Given two points $z, w \in \mathbb{H}$, we study the cardinality

$$N(s,z,w) = |\Gamma w \cap B(z,s)| \tag{1}$$

where $B(z,s)$ is the closed hyperbolic ball of centre z and radius s. As s grows to infinity, the function $N(s,z,w)$ is asymptotic to the volume of the hyperbolic ball $B(z,s)$ divided by the volume of the quotient space $\Gamma\backslash\mathbb{H}$. Going further, we want to understand how accurate the approximation is, and to this end we ask if it is possible to obtain an expression for lower order terms in the asymptotic of $N(s,z,w)$.

The spectral theory of automorphic forms provides explicit expressions for a finite number of secondary terms, and we collect them all together to define a "completed" main term $M(s,z,w)$. This secondary terms involve the "small eigenvalues" $0 \leq \lambda_j \leq 1/4$ of the Laplace operator Δ acting on the space $L^2(\Gamma\backslash\mathbb{H})$, and the corresponding eigenfunctions (for the precise definition see section 2.6). We want to stress the fact that while the term "small eigenvalue" usually denotes eigenvalues strictly smaller than $1/4$, we also include in the main term $M(s,z,w)$ a contribution coming from the eigenvalue $\lambda = 1/4$. This is slightly different from what is done by Phillips and Rudnick [52] and Chamizo [15], who also studied the hyperbolic circle problem.

Fix $z, w \in \mathbb{H}$ and a cofinite Fuchsian group $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$. For $s \geq 0$ we define the remainder term $E(s,z,w)$ as

$$E(s,z,w) = N(s,z,w) - M(s,z,w). \tag{2}$$

We prove new results related to the size and the behaviour of the function $E(s,z,w)$. The tools that we use range from classical techniques of Cramér [18, 19] to the methods of fractional calculus, touching in passing the theory of almost periodic functions. Using these different approaches we prove various results that we state below in this short introduction.

The function $E(s,z,w)$ oscillates as s varies, taking both positive and negative values. Phillips and Rudnick have proven that these values can become, in absolute value, as big as $e^{s(1/2-\delta)}$ for any $\delta > 0$. They proved [52, Th. 1.2] that

$$E(s,z,w) = \Omega(e^{s(1/2-\delta)}) \ \ \forall\, \delta > 0 \tag{3}$$

and in the case of cocompact groups or subgroups of finite index of $\mathrm{PSL}(2,\mathbb{Z})$ they refine this to $\Omega(e^{s/2}(\log s)^{1/4-\delta})$ for every $\delta > 0$. The following conjecture suggests the maximal size of $E(s,z,w)$.

Conjecture 1. For Γ a cofinite Fuchsian group and $z, w \in \mathbb{H}$ we have, for every $\varepsilon > 0$, as s tends to infinity,

$$E(s,z,w) = O_\varepsilon(e^{s(1/2+\varepsilon)}). \tag{4}$$

This upper bound is, in view of (3), the best we can hope for. Some experts are skeptical about the bound predicted by the conjecture, at least in the greatest generality when no assumptions are made on the group or on the points z, w. The best known upper bound for $E(s,z,w)$ has been proven by Selberg in an unpublished note [63, pp. 4-5] and gives

$$E(s,z,w) = O(e^{2s/3}). \tag{5}$$

The implied constant depends in principle on all of Γ, z, w.

New results

Chamizo [15, Cor. 2.1.1] has proven a result that points towards the possible truth of Conjecture 1. He proved that, as T tends to infinity,

$$\frac{1}{T}\int_T^{2T} \Big|\frac{E(s,z,w)}{e^{s/2}}\Big|^2 ds \ll T^2. \tag{6}$$

The inequality would follow from a pointwise bound $E(s,z,w) \ll s e^{s/2}$, which is essentially (4), with the exponential factor $e^{\varepsilon s}$ replaced by a linear factor. Hence $E(s,z,w)$ is "on average" bounded by $O(se^{s/2})$, where the average is over $s \in [T, 2T]$. In other words, Conjecture 1 is true on average. We prove that the bound in (6) can be refined as follows.

Theorem 2. *Let Γ be a Fuchsian group, and let $z, w \in \mathbb{H}$. We have*

$$\frac{1}{T}\int_T^{2T} \Big|\frac{E(s,z,w)}{e^{s/2}}\Big|^2 ds \ll T \qquad \text{as } T \to \infty. \tag{7}$$

This proves that on average over $[T, 2T]$ we have $E(s,z,w) \ll \sqrt{s}e^{s/2}$. The method of proof consists in integrating the pretrace formula for a suitable automorphic kernel associated to the counting function $N(s,z,w)$, as suggested already in [13, p. 27], and proceeds in a similar way to what done by Cramér in [18] in the case of the remainder term in the prime number theorem (under the Riemann Hypothesis). However there are some technical obstructions that prevent us from proving a bound of the form $O(1)$, and possibly the existence of the limit as $T \to \infty$ of the integral in (7).

Almost periodic functions. Using the spectral expansion, we can prove that the function $e(s,z,w) = E(s,z,w)e^{-s/2}$ can be approximate by the sum

$$\sum_{0\le t_j\le X} r_j e^{ist_j}, \tag{8}$$

where $r_j \in \mathbb{C}$ are suitable coefficients, $X \gg 1$ a suitable parameter, and we have written the eigenvalues bigger than $1/4$ as $\lambda_j = 1/4 + t_j^2$, where we can assume that $t_j > 0$. This observation takes us to view the function $e(s,z,w)$ as a possible candidate for being in the class of almost periodic functions. Such functions are in the closure of the class of finite sums

$$\mathcal{H} = \Big\{ \sum r_n e^{i\lambda_n y} \Big\} \tag{9}$$

with $r_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$, with respect to a specified topology (here we follow the classical but probably unfortunate notation of indicating the frequencies of the exponential by λ_n, not to be confused with the eigenvalues of the Laplacian). Changing topology leads to different classes of almost periodic functions, and in what follows we have chosen to work with the class of Besicovitch B^p-almost periodic functions, defined for $p \ge 1$ as those functions ϕ for which for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in \mathcal{H}$ such that

$$\limsup_{Y\to\infty} \left(\frac{1}{Y} \int_0^Y |\phi(y) - f_\varepsilon(y)|^p dy \right)^{1/p} < \varepsilon. \tag{10}$$

We study the existence of the asymptotic moments and of the limiting distribution for a B^p-almost periodic function. We focus however only on functions that can be approximated by sums of type

$$\Re \left(\sum_{0<\lambda_n\le X} r_n e^{i\lambda_n y} \right) \tag{11}$$

with λ_n being an increasing sequence of real numbers. Such functions have been extensively studied already in the last century, when the theory of almost periodic function had its first developments, and in the context of number theory also more recently by for instance Heath-Brown [29], Rubinstein and Sarnak [56], and Akbary et al. [1]. We give a relatively simple sufficient criterion to prove our results, which is inspired by [1, Th. 1.2]. A quantity that will play an important role is the size of the coefficients r_n in a short window of λ_n of unit length. Assume that there exists $\beta \in \mathbb{R}$ such that we have, for $T \gg 1$,

$$\sum_{T\le\lambda_n\le T+1} |r_n| \ll \frac{1}{T^\beta}. \tag{12}$$

The main theorem on the existence of moments for almost periodic functions can be stated as follows.

Theorem 3. *Let $p \ge 1$ be an even integer and let ϕ be a B^p-almost periodic function that can be approximated by sums of type* (11)*, and such that* (12) *holds with $\beta > 1 - 1/p$. Then ϕ admits finite moments of every order $1 \le n \le p$.*

Remark. When p is an odd integer the theorem still holds if we assume that there exists $X = X(Y)$ such that the function

$$\mathcal{E}(y, X) = \phi(y) - \Re\left(\sum_{0<\lambda_n\leq X} r_n e^{i\lambda_n y}\right)$$

satisfies $\lim_{Y\to\infty} \sup_{y\in[0,Y]} |\mathcal{E}(y, X(Y))| = 0$.

We recall that a limiting distribution for a function $\phi : [0, \infty) \to \mathbb{R}$ is a probability measure μ on $\mathbb{R}$ such that the limit

$$\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y g(\phi(y))dy = \int_{\mathbb{R}} g d\mu \tag{13}$$

holds for every bounded continuous function g on $\mathbb{R}$. We prove the following theorem.

Theorem 4. *Let ϕ be a B^2-almost periodic function that can be approximated by sums of type* (11)*, and such that* (12) *holds with $1/2 < \beta < 1$. Then ϕ admits a limiting distribution μ with tails of size*

$$\mu\big((-\infty, -S] \cup [S, +\infty)\big) \ll S^{-(2\beta-1)/(2-2\beta)}. \tag{14}$$

For $\beta = 1$ we have exponential decay in S, and for $\beta > 1$ the measure μ is compactly supported.

Remark. The existence of the distribution is [1, Th. 1.2], and so the new result is the explicit estimate on the tails of the measure μ. The case of exponential decay is discussed in [56], but we couldn't find the intermediate result for $1/2 < \beta < 1$ in the literature.

Remark. The moments of a function ϕ and the moments of its limiting distribution μ need not to coincide a priori. We discuss when this is the case at the end of section 3.3.

Applications of almost periodic functions to the hyperbolic circle problem. We apply our study of almost periodic functions to the hyperbolic circle problem and we prove the following theorem.

Theorem 5. *Let Γ be a cocompact Fuchsian group, and let $e(s, z, w)$ be the normalized remainder term. For every even $N \in \mathbb{N}$ we have*

$$\frac{1}{Y}\int_Y^{2Y} |e(s, z, w)|^N ds \ll e^{2Y\left(\frac{N}{6} - \frac{2N}{3(3N-2)} + \varepsilon\right)}. \tag{15}$$

Remark. The theorem shows that some cancellation occurs in the higher moments of $e(s, z, w)$, and therefore a saving on what can be obtained by using (5). In the case $N = 2$ we recover, up to ε, Theorem 2.

A second application of the study of almost periodic function is the following theorem regarding certain integrated versions of $E(s, z, w)$. We define, for $s \geq 0$, the functions

$$\begin{aligned} G_1(s,z) &:= \int_{\Gamma\backslash\mathbb{H}} |e(s,z,w)|^2 d\mu(w), \\ G_2(s) &:= \iint_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} |e(s,z,w)|^2 d\mu(z)d\mu(w). \end{aligned} \tag{16}$$

The functions G_1, G_2 are thus defined by integrating away one or both of the spacial variables from $e(s,z,w)$. For Γ cocompact and $z \in \Gamma\backslash\mathbb{H}$, we define the following function.

$$G_3(s,z) := \frac{1}{e^{s/2}} \int_0^s E(x,z,z)dx. \tag{17}$$

Theorem 6. *Let Γ be cocompact, $z, w, \in \mathbb{H}/\Gamma$, and let $s \geq 0$. Then the functions G_1, G_2, G_3 are bounded in s, admit moments of every order, and limiting distributions μ_1, μ_2, μ_3 of compact support. The moments of G_i coincide with the moments of μ_i.*

The theorem shows that integration gives us functions that are better behaved than the original function $e(s,z,w)$ (or $E(s,z,w)$), in the sense that the integrated versions admit moments of every order and limiting distributions.

Fractional calculus. The final chapter of the thesis, which contains the material of the paper [16], joint work with M. Risager, investigates "how much integration" we need to prove that the second moment of $e(s,z,w)$ exists. This is done using the formalism of fractional integration. We consider a real number $0 < \alpha < 1$ and the function $e_\alpha(s) = e_\alpha(s,z,w)$ defined as the integration of order α of the function $e(s,z,w)$. For precise definitions we address the reader to chapter 4.

We prove the following theorems, that are analogous to the pointwise bound (5) and the first moment result known for the function $e(s,z,w)$, i.e. [52, Th. 1.1].

Theorem 7. *Let Γ be a cofinite group, $z, w \in \mathbb{H}$, and $0 < \alpha < 1$. Then as s tends to infinity we have*

$$e_\alpha(s) \ll \begin{cases} e^{s(1-2\alpha)/(6-4\alpha)} & 0 < \alpha < 1/2, \\ s & \alpha = 1/2, \\ 1 & 1/2 < \alpha < 1. \end{cases} \tag{18}$$

The implied constant depends on z, w, and the group Γ.

Theorem 8. *Let Γ be a cofinite group, $z, w \in \mathbb{H}$, and $0 < \alpha < 1$. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} e_\alpha(s)ds = 0. \tag{19}$$

To be able to prove finiteness of the variance for $e_\alpha(s)$ we need to make assumptions on the Eisenstein series. More precisely we need to assume, in the case where Γ is cofinite but not cocompact, that for $v = z$ and $v = w$ we have

$$\int_1^\infty \frac{|E_\mathfrak{a}(v, 1/2+it)|^{2p}}{t^{(3/2+\alpha)p}} dt < \infty \tag{20}$$

for some $1 < p < \min(2, \alpha^{-1})$, and all cusps $\mathfrak{a}$ of Γ.

Theorem 9. *Let $0 < \alpha < 1$ and assume* (20)*. Then we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} |e_\alpha(s)|^2 ds = 2\pi \sum_{\substack{t_j>0\\ distinct}} \frac{|\Gamma(it_j)|^2}{|t_j^\alpha \Gamma(3/2+it_j)|^2} \left| \sum_{t_{j'}=t_j} \phi_{j'}(z)\overline{\phi_{j'}(w)} \right|^2$$

and the sum on the right is convergent.

We conclude with the existence of a limiting distribution for $e_\alpha(s)$.

Theorem 10. *Let $0 < \alpha < 1$ and assume* (20)*. Then the function $e_\alpha(s)$ admits a limiting distribution μ_α. For $\alpha > 1/2$, μ_α is compactly supported.*

Remark. Condition (20) holds true for congruence groups: it is implied by the following stronger condition

$$|E_{\mathfrak{a}}(z, 1/2+it)| \ll_z |t|^{1/2+\varepsilon}, \qquad t \gg 1 \tag{21}$$

which holds for congruence groups. For cocompact groups (20) is vacuous, so Theorem 9 and Theorem 10 hold unconditionally in this case. Condition (20) holds also if $\alpha > 1/2$ and the Eisenstein series satisfy that they are bounded polynomially as $t \to \infty$.

Chapter 1

Background Material

In this chapter we introduce the fundamental concepts and results that will be needed in the rest of the thesis. We start by a short description of the hyperbolic plane and Fuchsian groups. Then we explain briefly how to do harmonic analysis on the hyperbolic plane, namely we introduce the notion of automorphic forms and state the pretrace formula.

1.1 Introduction to hyperbolic plane and Fuchsian groups

The hyperbolic 2-dimensional space can be represented by the model of the upper-half plane

$$\mathbb{H} = \{z = x + iy : \ x, y \in \mathbb{R}, \ y > 0\}.$$

This 2-dimensional manifold is equipped with the Riemannian metric given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

which is called *hyperbolic metric*. In particular, the length of a piecewise differentiable path $\gamma : [0,1] \to \mathbb{H}$ and the area of a region $Q = \{x+iy : \ x \in [a,b], \ y \in [c,d]\}$ are given respectively by

$$\ell(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{y(t)} dt, \qquad \mathcal{A}(Q) = \iint_Q \frac{dxdy}{y^2}. \tag{1.1}$$

The distance between two points $z, w \in \mathbb{H}$ is defined consequently as

$$d(z,w) = \inf\{\, \ell(\gamma)\}$$

where γ runs over piecewise differentiable paths $\gamma : [0,1] \to \mathbb{H}$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Given two points $z, w \in \mathbb{H}$ there is an explicit formula to compute the distance $d(z,w)$, which is given by

$$d(z,w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|},$$

but it is convenient to use a different relation to compute $d(z,w)$. We can indeed write

$$\cosh d(z,w) = 1 + 2u(z,w), \tag{1.2}$$

where

$$u(z,w) = \frac{|z-w|^2}{4\Im(z)\Im(w)}$$

which will be more practical for applications later. Consider now the linear fractional transformations

$$gz = \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$. These are represented by matrices in $\mathrm{PSL}(2,\mathbb{R})$ and are called Möbius transformations. A small computation shows that

$$\frac{|dgz|}{\Im(gz)} = \frac{|dz|}{\Im(z)}$$

and therefore the length of a path, as computed in (1.1), remains invariant under composition with $g \in \mathrm{PSL}(2,\mathbb{R})$. In particular, this shows that

$$d(gz,gw) = d(z,w),$$

i.e. the Möbius transformations act as isometries on $\mathbb{H}$. In fact more can be said: all the orientation-preserving isometries of $\mathbb{H}$ are given by Möbius transformations.

Proposition 1.1. *The group* $\mathrm{PSL}(2,\mathbb{R})$ *is a subgroup of* $\mathrm{Isom}(\mathbb{H})$ *of index* 2. *The group* $\mathrm{Isom}(\mathbb{H})$ *is generated by* $\mathrm{PSL}(2,\mathbb{R})$ *together with the orientation-reversing isometry* $z \mapsto -\bar{z}$.

Proof. See [41, Theorem 1.3.1]. □

Given two points $z,w \in \mathbb{H}$, the geodesic segment from z to w is the vertical line segment connecting them if $\Re(z) = \Re(w)$, and otherwise is the arc of the circle passing through z,w and meeting perpendicularly the real axis, of endpoints z and w. Geodesics are therefore vertical lines and semicircles perpendicular to the real axis.

If we fix a point $z \in \mathbb{H}$, and a positive number $r > 0$, the (open) hyperbolic ball is defined as the set

$$B(z,r) = \{w \in \mathbb{H} : \ d(z,w) < r\}.$$

A hyperbolic ball is an euclidean ball with a different center, but the area (or volume) of the ball is given by the formula

$$\mathrm{vol}(B(z,r)) = 4\pi \sinh^2\left(\frac{r}{2}\right),$$

and the length of the circumference is $\ell(\partial B(z,r)) = 2\pi \sinh r$. A difference compared to euclidean geometry clearly appears: the length of the boundary of the ball and the area of the ball are of the same order of magnitude for $r \gg 1$. The isoperimetric inequality gives a quantitative account of this phenomenon. In a space of constant Gaussian curvature K, given a domain with area A and boundary of length L, we have

$$4\pi A - KA^2 \leq L^2$$

and the inequality is sharp since equality is attained for disks. The Gaussian curvature measures the difference between the length of an euclidean circle of radius r, and the length $c(r)$ of a hyperbolic circle with the same radius r, for small r. More precisely, the following formula holds:

$$K = 3 \lim_{r \to 0} \frac{2\pi r - c(r)}{\pi r^3}.$$

In the euclidean space we obviously get $K = 0$, while in $\mathbb{H}$ we have $c(r) = 2\pi \sinh r$ and therefore we obtain $K = -1$. This means that the isoperimetric inequality reads

$$4\pi A \leq L^2$$

in the euclidean plane, hence allowing $L \asymp \sqrt{A}$, while

$$4\pi A + A^2 \leq L^2$$

in the hyperbolic plane, which means that in negative curvature the length L of the boundary is never smaller than the area A.

Dynamics of $\mathrm{PSL}(2, \mathbb{R})$ and Fuchsian groups

We describe now briefly how the elements of $\mathrm{PSL}(2, \mathbb{R})$ act on $\mathbb{H}$ and split into hyperbolic, elliptic, and parabolic classes. Consider an element $g \in \mathrm{PSL}(2, \mathbb{R})$, namely

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1,$$

and assume $g \neq \pm I$, so that g does not act trivially on $\mathbb{H}$. Then g is said to be *elliptic*, *parabolic*, or *hyperbolic*, according to whether it has respectively one fixed point in $\mathbb{H}$, one fixed point in $\partial\mathbb{H}$, or two fixed points in $\partial\mathbb{H}$. It is easily seen that these are the only possibilities: indeed, the fixed points of g are found by solving the quadratic equation

$$\frac{az + b}{cz + d} = z. \tag{1.3}$$

In particular, it is possible to give a characterization of the elements $g \in \mathrm{PSL}(2, \mathbb{R})$ in terms of their trace $\mathrm{Tr}(g) = a + d$, as solving (1.3) one finds that g is elliptic if and only if $|\,\mathrm{Tr}(g)| < 2$, g is parabolic if and only if $|\,\mathrm{Tr}(g)| = 2$, and g is hyperbolic if and only if $|\,\mathrm{Tr}(g)| > 2$. An elliptic element is conjugate to a matrix of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{for some } \theta \in \mathbb{R}.$$

This has fixed point i and it has finite or infinite order depending on whether θ is rational or irrational. A parabolic element is conjugate to a matrix of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{for some } x \in \mathbb{R}.$$

This has infinite order and the point at infinity as a fixed point (hence in $\partial\mathbb{H}$). Finally, a hyperbolic element is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{R},\ \lambda > 1.$$

This has two fixed points x^+ and x^- both belonging to the extended real line $\mathbb{R} \cup \{\infty\}$ and called respectively *attractive* point and *repulsive* point, the reason being that in a neighbourhood U of x^-, and every point $z \in U$, the (euclidean) distance between $g(z)$ and x^- is bigger than the distance between z and x^-, hence x^- repels points; on the contrary, x^- attracts points, since given any neighbourhood V of x^+ and any point $z \in \mathbb{H}$, there exists $\bar{n} \in \mathbb{N}$ such that $g^n(z) \in V$, for every $n \geq \bar{n}$. In other words, the orbit of any point under the action of g will eventually end up in a neighbourhood of x^+. We define the *displacement* (or *length*) $\ell(g)$ of an element $g \in \mathrm{PSL}(2,\mathbb{R})$ as

$$\ell(g) = \inf_{z \in \mathbb{H}} d(z, gz).$$

This quantity is well-defined and distinguishes further the elements, as expressed by the following proposition.

Proposition 1.2. *Let $g \in \mathrm{PSL}(2,\mathbb{R})$, $g \neq I$. Then*

1. *g is elliptic if and only if $\ell(g) = 0$, and the infimum is attained;*
2. *g is parabolic if and only if $\ell(g) = 0$, and the infimum is not attained;*
3. *g is hyperbolic if and only if $\ell(g) > 0$, and the infimum is attained.*

Proof. See [20, Proposition 2.8]. □

In view of the proposition one can say that the hyperbolic elements, at least regarding questions on elements length, are of greater interest than elliptic and parabolic. It results that considering subgroups of $\mathrm{PSL}(2,\mathbb{R})$ with only hyperbolic elements leads to slightly easier computations than when we also add parabolic and elliptic elements.

Definition 1.3. A subgroup $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ which is discrete with respect to the topology on $\mathrm{PSL}(2,\mathbb{R})$ induced by the euclidean distance on $\mathbb{R}^4$ is called a *Fuchsian* group.

Various basic notions related to Fuchsian groups are important. Here we mention that of primitive elements, and then we move to discuss fundamental domains and Fuchsian groups of finite covolume.

For a Fuchsian group Γ and a point $z \in \overline{\mathbb{H}}$, the stability group $\Gamma_z = \{g \in \Gamma : gz = z\}$ is cyclic (finite or infinite, [39, Proposition 2.2]). An element $g_0 \in \Gamma$ is called *primitive* if g_0 generates the stability group of its fixed points, and in the case when g_0 is elliptic, we require it to have the smallest angle of rotation. Every $g \in \Gamma$, $g \neq 1$, is a power of a primitive element, i.e. $g = g_0^n$, $n \in \mathbb{Z}$.

Fundamental Domains

We will be working with quotients of the hyperbolic plane by Fuchsian groups. For a Fuchsian group $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$, the quotient space is easily visualized by the so called fundamental domain of Γ.

Definition 1.4. Let Γ be a Fuchsian group. A closed region (non empty interior) $F \subseteq \mathbb{H}$ is a *fundamental domain* for the group Γ if:

$$(i) \quad \bigcup_{\gamma \in \Gamma} \gamma F = \mathbb{H}; \qquad (ii) \quad \mathring{F} \cap \gamma\mathring{F} = \emptyset.$$

The fundamental domain F_{z_0} of a Fuchsian group Γ, containing a given point z_0 not fixed by any element of $\Gamma - \mathrm{Id}$, can be characterized as the intersection

$$F_{z_0} = \bigcap_{\substack{\gamma\in\Gamma:\\ \gamma\neq \mathrm{Id}}} \mathbb{H}_{z_0}(\gamma),$$

where

$$\mathbb{H}_{z_0}(\gamma) = \{z \in \mathbb{H} \mid d(z, z_0) \leq d(z, \gamma z_0)\}.$$

With this characterization is not difficult to prove (see [20, Th. 2.11], [41, Th. 3.2.2]) that F_{z_0} is a connected convex fundamental domain for the group Γ. Moreover we have

$$\mu(\overset{\circ}{F}_{z_0}) = \mu(F_{z_0})$$

and all fundamental domains of a group Γ have the same measure (possibly infinite). Finally, the map

$$\theta : \Gamma\backslash F_{z_0} \longrightarrow \Gamma\backslash\mathbb{H}$$

defined by $\theta(\Gamma z \cap F_{z_0}) = \Gamma z$, is a homeomorphism [2, Theorem 9.2.4]. A Fuchsian group is said to be of the *first kind* if every point on the boundary $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{R})$ is a limit, in the $\widehat{\mathbb{C}}$-topology, of the orbit Γz for some $z \in \Gamma$. For such groups a fundamental domain can be chosen to be a convex polygon in $\mathbb{H}$ with a finite number of points in $\partial\mathbb{H}$. In fact the following is true (see [39, Proposition 2.3] and [64]).

Proposition 1.5. *Every Fuchsian group of the first kind has a finite number of generators and fundamental domain of finite volume.*

A group Γ with fundamental domain of finite volume is called *cofinite*. If moreover the fundamental domain is compact in $\mathbb{H}$, then we say that Γ is *cocompact*. The points of the fundamental polygon of a cofinite Fuchsian group that lie on $\partial\mathbb{H}$ are called *cusps*. Hence a group is cocompact if and only if it has no cusps. Two cusps are called equivalent if there is an element $g \in \Gamma$ that takes one of them to the other.

Before moving to show some explicit examples of Fuchsian groups of finite volume we conclude with a group-theoretical result (see [23]).

Theorem 1.6. *Any finite volume subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *is generated by primitive motions* $A_1, \dots, A_g$, $B_1, \dots B_g$, $E_1, \dots, E_\ell$, $P_1, \dots, P_h$ *satisfying the relations*

$$[A_1, B_1] \cdots [A_g, B_g] E_1 \cdots E_\ell P_1 \cdots P_h = 1, \quad E_j^{m_j} = 1,$$

where A_j, B_j *are hyperbolic motions,* $[A_j, B_j]$ *is the commutator of* A_j *and* B_j*,* g *is the genus of* $\Gamma\backslash\mathbb{H}$*,* E_j *are elliptic motions of order* $m_j \geq 2$*,* P_j *are parabolic motions, and* h *is the number of inequivalent cusps.*

The symbol $(g, m_1, \dots, m_\ell, h)$ is a group invariant which is called the *signature* of the group, and it satisfies the Gauss-Bonnet formula

$$2g - 2 + \sum_{j=1}^{\ell} \left(1 - \frac{1}{m_j}\right) + h = \frac{\mu(F)}{2\pi}.$$

REMARK 1.7. There exist some interesting groups with fundamental domain of infinite volume. This is especially true if we move to dimension three, and we study subgroups $\Gamma \leq \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$. The focus should be put on those groups Γ so called *geometrically finite*, that is, having convex fundamental domain for Γ with finitely many sides (the volume of the fundamental domain could still be infinite). Recent studies on the problem of Apollonian circle packings [50, 44, 24, 73, 57] exploit connections between the geometric formulation of the problem and certain geometrically finite subgroups $\Gamma \leq \mathrm{PSL}(2,\mathbb{C})$ of infinite covolume. For an overview see [50].

Examples of Fuchsian groups

There exists a wide variety of Fuchsian groups. Here we list some examples, ranging from trivial cyclic groups to arithmetic and Hecke triangle groups. In examples 1.9 and 1.10 we present the modular group and congruence subgroups, which are of fundamental importance in number theory, especially due to their relation to moduli spaces of elliptic curves.

Example 1.8. Consider the two elements $g_1, g_2 \in \mathrm{PSL}(2,\mathbb{R})$ given by

$$g_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where t is a fixed positive real number and λ is a fixed real number strictly bigger than one. Let Γ_1 and Γ_2 be the cyclic groups generated respectively by g_1 and g_2. Then Γ_1 and Γ_2 are geometrically finite Fuchsian groups of infinite volume. In particular they are not of the first kind. A fundamental domain for Γ_1 is the vertical strip

$$D_1 = \{z = x + iy \in \mathbb{H} \,:\, 0 \leq x \leq t, y > 0\},$$

while two possible fundamental domains for Γ_2 are the sets

$$D_2 = \{z \in \mathbb{H} \,:\, 1 \leq |z| \leq 2\log\lambda\},$$

$$\widetilde{D}_2 = \{z = x + iy \in \mathbb{H} \,:\, x \in \mathbb{R},\, 1 \leq y \leq 2\log\lambda\}.$$

The second set, which is a horizontal strip of width $2\log\lambda$, is sometimes more useful in computations since one can use rectangular coordinates x, y for the points z in this set. The number $2\log\lambda$ is the length $\ell(g_2)$ of the hyperbolic element g_2.

Example 1.9. The group $G = \mathrm{SL}(2,\mathbb{Z})$ is called the *modular group*. Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = G/\{\pm\,\mathrm{Id}\}$ be the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ consisting of matrices with integer coefficients[1]. Then Γ is a finite volume Fuchsian group. It is possible to calculate the volume of the quotient $\Gamma\backslash\mathbb{H}$ which equals $\mu(\Gamma\backslash\mathbb{H}) = \pi/3$. A fundamental domain for Γ is given by the set

$$D = \{z = x + iy \in \mathbb{H} \,:\, -1/2 \leq x \leq 1/2,\ |z| \geq 1\}.$$

In particular the fundamental domain is not compact, since the point at infinity is a cusp of Γ. Hence Γ is an example of cofinite non-cocompact Fuchsian group.

[1] There is some ambiguity on whether G or rather Γ is called modular group, see [38, §1.5], [39, (2.9)], [2, Example 2.3.1] versus [20, §II.3], [41, §5.5]).

For subgroups $\Gamma' \leq \Gamma$ of finite index $[\Gamma : \Gamma'] = d$, a fundamental domain is given by the union of d translates of D, so the number δ of inequivalent cusps of Γ' satisfies $1 \leq \delta \leq d$.

Example 1.10. Let $N \geq 1$ be a positive integer. The group $\Gamma(N) \leq G$ defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

is a finite index subgroup of the modular group G, and it is called the *principal* congruence subgroup of level N. The groups $\Gamma_1(N)$ and $\Gamma_0(N)$ defined by

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

are also finite index subgroups of G, and verify the chain of inclusions

$$1 \leq \Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N) \leq \Gamma(1) := G.$$

The group $\Gamma_0(N)$ is sometimes called the Hecke congruence group of level N ([38, (2.22)]). Formulas for the index of $\Gamma(N)$ and $\Gamma_1(N)$ in G are given in [38, §2.4]. Here we mention that $\Gamma_0(N)$ is a cofinite not cocompact Fuchsian group having a fundamental domain with

$$h = \sum_{ab=N} \varphi(a \wedge b)$$

inequivalent cusps, where φ denotes the Euler's function and $a \wedge b := (a, b)$ denotes the greatest common divisor of a and b ([38, Proposition 2.6]).

Example 1.11. For $q \geq 3$, the group $\Gamma_q \leq \mathrm{PSL}(2, \mathbb{R})$ generated by the two elements

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix}$$

is called Hecke triangle group; it is a cofinite not cocompact Fuchsian group having a fundamental domain D_q of volume $\mu(D_q) = \pi(1 - 2/q)$. There is only one cusp at infinity, and the genus is $g = 0$.

1.2 Harmonic analysis in Euclidean and Hyperbolic space

In this section we give the definition of automorphic functions and forms, and we state Selberg's pretrace formula. We start by recalling the Poisson summation formula in Euclidean space (we only state the formula in dimension one, [40, Theorem 4.4]), as the pretrace formula is built on similar ideas in the hyperbolic settings.

Theorem 1.12 (Poisson summation formula). *Let $f, \hat{f}$ be in $L^1(\mathbb{R})$ and have bounded variation. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)$$

where both series converge absolutely.

Proof. The function

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

is well defined because $f \in L^1(\mathbb{R})$ and has bounded variation, and it is periodic of period one. It admits therefore a Fourier series expansion

$$F(x) = \sum_{m\in\mathbb{Z}} \hat{f}(n)e(mx) \tag{1.4}$$

which is absolutely convergent, this time because of the summability of $\hat{f}$ and the fact that it has bounded variation too. Taking $F(0)$ we get the Poisson summation formula. □

A good point of view for interpreting the Poisson summation formula is by spectral theory. Consider the Laplace operator acting on twice differentiable functions of $\mathbb{R}^n$, namely

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

and consider the differential problem associated to the n-dimensional torus $\mathbb{R}^n/(2\pi\mathbb{Z})^n$, that is, to find a solution $f \in C^2(\mathbb{R}^n)$ to the system for $\lambda \in \mathbb{R}$

$$\begin{cases} (\Delta-\lambda)f = 0 & x \in (0,2\pi)^n \\ f(x+v) = f(x) & \forall\, v \in (2\pi\mathbb{Z})^n. \end{cases} \tag{1.5}$$

Consider only the one-dimensional case, i.e. $n = 1$. Then (1.5) reads

$$\begin{cases} f'' = \lambda f \\ f(x) = f(x+2\pi), \quad x \in \mathbb{R} \end{cases}$$

and all the solutions of this problem are found (up to multiplication by scalars) by the explicit formula

$$\phi_n(x) = e^{ixn} \tag{1.6}$$

for $\lambda = -n^2$, $n \in \mathbb{N}$ (for λ not of this form there are no solutions). Then $\{\phi_n\}$ is an orthonormal basis for $L^2(\mathbb{R}/(2\pi\mathbb{Z}))$, and (1.4) is the spectral expansion of the periodic function F with respect to this basis. Evaluating at zero one gets the Poisson summation formula.

Let us move now to the hyperbolic plane. A function $f : \mathbb{H} \to \mathbb{C}$ that is invariant under the action of a Fuchsian group Γ, i.e. satisfies $f(\gamma z) = f(z)$ for every $z \in \mathbb{H}$ and $\gamma \in \mathbb{H}$, is called an automorphic function, and is naturally defined on the quotient space $\Gamma\backslash\mathbb{H}$. The Laplace-Beltrami operator (or simply hyperbolic Laplacian) is defined by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and acts on the dense subset of $L^2(\Gamma\backslash\mathbb{H})$ consisting of twice differentiable automorphic functions. For Γ a cofinite Fuchsian group, the operator Δ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

which is either finite or satisfies $\lambda_n \to \infty$, and a continuous spectrum which covers $[1/4, \infty)$ with multiplicity equal to the number of cusps of Γ. The eigenvalues $\lambda_j \in (0, 1/4)$ are called *small eigenvalues*. Writing $\lambda_j = 1/4 + t_j^2$ with $\Im(t_j) \geq 0$ they correspond to t_j in the complex segment $t_j \in (0, i/2)$.

An automorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying $(\Delta - \lambda)f = 0$ is called an *automorphic form* (of Maass [49]). Automorphic forms associated to the discrete spectrum are called cusp forms, and play the role that the complex exponentials (1.6) had in the Euclidean case. The appearance of the continuous spectrum is interesting, since we do not have it in the Euclidean setting. For this we introduce, for each cusp $\mathfrak{a}$ of Γ, and $s \in \mathbb{C}$, $\Re(s) > 1$, the Eisenstein series

$$E_{\mathfrak{a}}(z, s) = \sum_{\Gamma_{\mathfrak{a}} \backslash \Gamma} (\Im \sigma_{\mathfrak{a}}^{-1} \gamma z)^s.$$

Here $\Gamma_{\mathfrak{a}}$ is the stabilizer of $\mathfrak{a}$ in Γ, while $\sigma_{\mathfrak{a}} \in \mathrm{PSL}(2, \mathbb{R})$ is such that $\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \Gamma_\infty$, where Γ_∞ is the subgroup of $\mathrm{PSL}(2, \mathbb{R})$ generated by the translation $z \mapsto z + 1$. The Eisenstein series is used to construct families of eigenforms associated to the continuous spectrum. For more details we address the reader to [39]. We want however to stress a few facts regarding cusp forms and Eisenstein series.

Remark 1.13. The Eisenstein series $E_{\mathfrak{a}}(z, s)$ admits meromorphic continuation to the whole complex plane. In particular it can be evaluated on the line $\Re(s) = 1/2$. This will be used repeatedly in the next chapters.

The cusp forms ϕ_j are square integrable over $\Gamma \backslash \mathbb{H}$, but the Eisenstein series are not. This is something that one must take care of when discussing cofinite Fuchsian groups that are not cocompact.

In contrast to the Euclidean situation, where the eigenvalues λ_n and the eigenfunctions ϕ_n are completely explicit by (1.6), the same is not true in the hyperbolic setting. No Fuchsian group is known for which all eigenvalues are explicit, and there is no explicit expression for cusp forms. In a recent work [11] many eigenvalues and eigenfunctions were computed to high accuracy for the modular group $\mathrm{SL}(2, \mathbb{Z})$.

A natural question to ask is if the discrete spectrum is finite or infinite, that is, if a Weyl law exists for a generic cofinite Fuchsian group. If the group is cocompact then there exists $c > 0$ such that

$$\sharp\{\lambda_j \leq T\} \sim cT \tag{1.7}$$

for $T \to \infty$. Hence there are infinitely many eigenvalues and cusp forms. In the non-cocompact case the modular group is special, and for it we can say that indeed the asymptotic (1.7) holds (see [32, Th. 2.28], [68, Th. 7.3]); the same is true for congruence subgroups of the modular group. However for the generic cofinite Fuchsian group (1.7) is not known, and while one is tempted to say that the rule should be the same, and a Weyl law should exist, the works of Phillips and Sarnak [54, 53], Colin de Verdière [17], and Sarnak [59] suggest that it should rather be true the opposite, namely that there are only finitely many cusp forms for the generic situation.

As a final remark on cusp forms we stress another difference with the Euclidean case. From (1.6) it is clear that ϕ_n is uniformly bounded in n. The same is not true in the hyperbolic setting, as we have [37]

$$\| \phi_j(z)\big|_K \|_\infty = \Omega(\sqrt{\log\log\lambda_j})$$

where ϕ_j are eigenforms for the modular group, $K \subseteq \mathbb{H}$ is a compact set, and j tends to infinity (see also [60] for a discussion on the growth of the unrestricted norm $\|\phi_j\|_\infty$ on the whole fundamental domain).

Automorphic forms are particularly useful to expand functions of the form

$$K(z,w) = \sum_{\gamma\in\Gamma} k(u(z,\gamma w))$$

where $k : \mathbb{H} \to \mathbb{C}$ is a function that depends only on the hyperbolic distance $d(z,w)$ (i.e. *point-pair* invariant). The function $K(z,w)$ is called an automorphic kernel (associated to k). For a point-pair invariant function k, its Selberg/Harish-Chandra transform is the integral transform $h(t)$ of k that is constructed in three steps as below:

$$q(v) = \int_v^{+\infty} \frac{k(u)}{(u-v)^{1/2}}du, \quad g(r) = 2q\left(\sinh^2\frac{r}{2}\right), \quad h(t) = \int_{-\infty}^{+\infty} e^{irt}g(r)dr.$$

The following theorem (which can be referred to as pretrace formula) states that we have a spectral expansion of automorphic kernel in terms of eigenfunctions of the Laplacian. The theorem requires certain properties on the Selberg/Harish-Chandra transform $h(t)$ of a point-pair invariant k to be fullfilled, more in details

$$\begin{cases} h(t) & \text{even} \\ h(t) & \text{holomorphic in the strip } |\Im t| \leq 1/2+\varepsilon \\ h(t) & \ll (1+|t|)^{-2-\varepsilon}, \text{ in the strip.} \end{cases} \tag{1.8}$$

Theorem 1.14 ([39, Theorem 7.4]). *Let $K(z,w)$ be an automorphic kernel given by a point-pair invariant k whose Selberg/Harish-Chandra transform $h(r)$ satisfies* (1.8)*. Then K has the spectral expansion*

$$\begin{aligned} K(z,w) &= \sum_j h(t_j)\phi_j(z)\overline{\phi_j(w)} \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{+\infty} h(r)E_{\mathfrak{a}}(z,1/2+ir)\overline{E_{\mathfrak{a}}(z,1/2+ir)}dr \end{aligned} \tag{1.9}$$

which converges absolutely and uniformly on compacta.

Chapter 2

About Circle Problems

In this chapter we introduce the hyperbolic circle problem and we work out in detail the computations that will be needed to prove the main results.

We start with a section in which we shortly describe the classical circle problem and its status, then we move to the hyperbolic counting. In sections 2.3–2.9 we carry out the computations to prove Theorem 2.9.

2.1 Euclidean

The circle problem is a classical problem in analytic number theory, that dates back to Gauss and has been studied by numerous authors. The problem refers to Hardy's conjecture from 1917, which can be explained as follows. Consider the function

$$N(x) = \sharp\{(a,b) \in \mathbb{Z}^2 : \ a^2 + b^2 \leq x\} = \sum_{0 \leq n \leq x} r_2(n), \tag{2.1}$$

where $r_2(n)$ is the number of ways of writing n as a sum of two squares. It is easy to see that $N(x) \sim \pi x$ for big x, as this is the area of an euclidean disk of radius $\sqrt{x}$. Denote now by $P(x)$ the function

$$P(x) = N(x) - \pi x.$$

In his paper [27], Hardy carefully conjectured: *"it is not unlikely that*

$$P(x) = O(x^{1/4+\varepsilon}) \tag{2.2}$$

for all positive values of ε, though this has never been proved".

Finding the right order of magnitude of the function $P(x)$ is a difficult task. A first estimate was given by Gauss, who proved with a geometric argument that $P(x) = O(x^{1/2})$. After that, Sierpinski first [65] and Landau later [45] showed that $P(x) = O(x^{1/3})$. Landau's proof was obtained by means of the Poisson summation formula, which has since then been the main tool for the analysis of the function $P(x)$.

The best known result is $P(x) = O(x^{131/416})$ and is due to Huxley (see [34]). It is the culminating point of a long series of works of several decades on the study of exponential sums. The estimate is remarkable, and the methods of proof apply to a large class of counting functions.

In a different direction it was proven by Hardy [26] that $P(x) = \Omega(x^{1/4}(\log x)^{1/4})$, which means that there exists a constant $C > 0$ such that

$$\limsup_{x\to\infty} \frac{|P(x)|}{x^{1/4}(\log x)^{1/4}} \geq C.$$

A number of improvements have been obtained. The best known result is a work of Soundararajan [66] in which he proves

$$P(x) = \Omega((x\log x)^{1/4}(\log\log x)^{b}(\log\log\log X)^{-5/8}), \tag{2.3}$$

with $b = 3(2^{1/3} - 1)/4$. A heuristic argument is also given that suggests that the constant b is the best possible.

It is easy to see that the estimate (2.2) is equivalent to the moments bound

$$\int_1^T (P(x))^n dx = O(T^{1+n/4+\varepsilon})$$

for all $\varepsilon > 0$, for every $n \geq 1$. It is thus of interest to understand the moments of the function $P(x)$. The question of whether these moments exist was initially studied by Hardy, who proved ([27, Theorem 2.1]) that

$$\int_1^T (P(x))^2 dx = O(T^{3/2+\varepsilon}).$$

This was refined by Cramér, who gave the asymptotic ([19, eq. (5)])

$$\int_1^T (P(x))^2 dx = C_2 T^{3/2} + O(T^{5/4+\varepsilon})$$

for an explicit constant $C_2 > 0$. Regarding the third and fourth moment, Tsang proved [67, Theorem 5] that

$$\int_2^T (P(x))^3 dx \sim C_3 T^{7/4} \quad \text{and} \quad \int_2^T (P(x))^4 dx \sim C_4 T^2$$

for some explicit constants C_3, C_4. It should be noticed that $C_3 < 0$, indicating that $P(x)^3$ is biased towards the negative side. The work of Ivić [35] (see also [36, Theorem 13.12]) gives the upper bound

$$\int_1^T |P(x)|^A dx \ll T^{1+A/4+\varepsilon} \quad 1 \leq A \leq 35/4$$

$$\int_1^T |P(x)|^A dx \ll T^{1+A/4+(8A-70)/108+\varepsilon} \quad A \geq 35/4.$$

This was used by Heath-Brown in [29, 30] to show that the limit

$$\lim_{T\to\infty} \frac{1}{T^{1+k/4}} \int_1^T (P(x))^k dx$$

exists for every $1 \leq k \leq 9$, which is at the moment the largest range for which we know finiteness of the moments. The explicit value of the limit was computed later by Zhai in [71, 72] and Lau [48], who both give an expression for higher moments as well, assuming that they exist.

A measure-theoretical approach to the problem came with the work of Heath-Brown [29] where he proved ([29, Theorem 1]) that the normalized remainder $p(R) = P(R^2)/R^{1/2}$ admits a limiting distribution. The statement of the theorem is as follows (see also [30]).

Theorem 2.1. *There exists a probability measure μ on $\mathbb{R}$ such that for every interval $A \subseteq \mathbb{R}$ we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T \mathbf{1}_A(p(R))dR = \int_{\mathbb{R}} \mathbf{1}_A d\mu \tag{2.4}$$

The measure μ is absolutely continuous with respect to the Lebesgue measure, and its density $\sigma(x)$ can be extended to an entire function. For real x, $\sigma(x)$ decays faster than $\exp(-|x|^{4-\varepsilon})$ for every $\varepsilon > 0$.

Remark 2.2. The fact that the density function of the limiting distribution decays faster than $\exp(-|x|^{4-\varepsilon})$ for every $\varepsilon > 0$ rules out some well-known distributions. In particular, the limiting distribution is not Gaussian. This is also hinted by the fact that the third moment of $P(x)$ is strictly negative, which is not the case for the third moment of the Gaussian distribution.

Remark 2.3. It follows from (2.4) that the tails of the distribution function associated to μ are exponentially small. More precisely we have for $S \gg 1$

$$\exp(-S^{4+\varepsilon}) \ll \mu((-\infty, -S] \cup [S, +\infty)) \ll \exp(-S^{4-\varepsilon})$$

for every $\varepsilon > 0$. This implies that all the moments of the measure μ exist, although it doesn't imply a priori that the moments of $p(R)$ exist too, since there exist for instance functions with infinite moments that admit limiting distribution with finite moments of every order. This is shown in Example 3.30.

The estimate on the tails of μ was refined by Lau in [47] in the form

$$\exp\left(-c_1\frac{S^4}{(\log S)^\theta}\right) \ll \mu((-\infty, -S] \cup [S, \infty)) \ll \exp\left(-c_2\frac{S^4}{(\log S)^\theta}\right)$$

with $c_1, c_2 > 0$ and $\theta = 3(2^{1/3} - 1)$. It is interesting to compare θ with the exponent b that appears in (2.3).

2.2 Hyperbolic

The hyperbolic circle problem, or hyperbolic lattice point counting problem, is the very analogue of the classical circle problem in the hyperbolic settings. For $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ a cofinite Fuchsian group and $z, w \in \mathbb{H}$, define the function

$$N(s, z, w) = \{\gamma \in \Gamma : \ d(z, \gamma w) \leq s\}. \tag{2.5}$$

This counts the number of translates of the point w by isometries of the group Γ that fall inside the hyperbolic ball $B(z, s)$ of centre z and radius s. Although the definition is very geometric in nature, for certain groups Γ and points $z, w \in \mathbb{H}$ the function $N(s, z, w)$ has a more arithmetic interpretation, in the same way as the counting function in the classical circle problem can be expressed in terms of the function $r_2(n)$ in (2.1). We explain this phenomenon in Lemma 2.4 below. For this purpose however, we use a different normalization for N: we define, for $X \geq 2$,

$$\begin{aligned} N^*(X, z, w) &= N(\cosh^{-1}(X/2), z, w) \\ &= \{\gamma \in \Gamma \mid d(z, \gamma w) \leq \cosh^{-1}(X/2)\} \\ &= \{\gamma \in \Gamma \mid 4u(z, \gamma w) + 2 \leq X\}, \end{aligned} \tag{2.6}$$

where in the last equality we have used (1.2) in order to pass from $d(z, w)$ to $u(z, w)$. Note that for large X we have $\cosh^{-1}(X/2) \approx \log X$, so we are roughly passing from the variable s to the variable $X = e^s$.

Lemma 2.4. *Let* $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ *and let* $z = w = i$. *We have*

$$N^*(X, i, i) = \frac{1}{2} \sum_{\substack{1 \leq N \leq X \\ N \equiv 2\ (4)}} r_2(N+2) r_2(N-2) + \frac{1}{4} \sum_{\substack{1 \leq N \leq X \\ N \equiv 3\ (4)}} r_2(N+2) r_2(N-2).$$

Proof. For $2 \leq N \leq X$ define the set

$$\mathcal{A}_N := \left\{ (a, b, c, d) \in \mathbb{Z}^4 \;\middle|\; \begin{matrix} a^2 + b^2 + c^2 + d^2 = N \\ ad - bc = 1 \end{matrix} \right\}. \tag{2.7}$$

If we write $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ as the matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then we have $4u(i, \gamma i) + 2 = a^2 + b^2 + c^2 + d^2$, and hence we deduce that

$$N^*(X, i, i) = \frac{1}{2} \sum_{2 \leq N \leq X} |\mathcal{A}_N|. \tag{2.8}$$

We claim that for $N \equiv 0, 1 \bmod 4$ we have $|\mathcal{A}_N| = 0$. Define the set $\mathcal{B}_N$ to be the set of quadruples $(k, \ell, m, n) \in \mathbb{Z}^4$ satisfying one of the equivalent conditions:

$$\begin{cases} k^2 + \ell^2 + m^2 + n^2 = 2N \\ k^2 - \ell^2 - m^2 + n^2 = 4 \end{cases} \iff \begin{cases} k^2 + n^2 = N + 2 \\ \ell^2 + m^2 = N - 2. \end{cases} \tag{2.9}$$

Consider the injective map $f : \mathcal{A}_N \to \mathcal{B}_N$

$$(a, b, c, d) \overset{f}{\longmapsto} (\, a{+}d,\ a{-}d,\ b{+}c,\ b{-}c\,).$$

For $N \equiv 0 \bmod 4$ we see from (2.7) and (2.9) that a point $(a, b, c, d) \in \mathcal{A}_N$ must satisfy

$$\begin{cases} a^2 + b^2 + c^2 + d^2 \equiv 0 & \pmod 4 \\ (a + d)^2 + (b - c)^2 \equiv 2 & \pmod 4 \\ (a - d)^2 + (b + c)^2 \equiv 2 & \pmod 4. \end{cases}$$

Since the only squares mod 4 are $0, 1$, we infer from the first line that it must be $a \equiv b \equiv c \equiv d \bmod 2$, while from the second and the third that $a \not\equiv d \bmod 2$ and $b \not\equiv c \bmod 2$, hence a contradiction. This implies that there are no points in $\mathcal{A}_N$, and therefore $|\mathcal{A}_N| = 0$.

For $N \equiv 1 \bmod 4$, using (2.9), we see that a point $(a,b,c,d) \in \mathcal{A}_N$ must satisfy

$$\begin{cases}(a+d)^2 + (b-c)^2 \equiv 3 \pmod 4\\(a-d)^2 + (b+c)^2 \equiv 3 \pmod 4.\end{cases}$$

This is not possible for any choice of $a,b,c,d \in \mathbb{Z}$. Hence again we deduce that $\mathcal{A}_N$ is empty and that we have $|\mathcal{A}_N| = 0$.

Consider now $N \equiv 2 \bmod 4$. We claim that in this case we have $|\mathcal{A}_N| = |\mathcal{B}_N| = r_2(N+2)r_2(N-2)$. It is convenient to decompose the set $\mathcal{A}_N$ as follows:

$$\mathcal{A}_N = A_{00} \cup A_{01} \cup A_{10} \cup A_{11},$$
$$A_{ij} = \{(a,b,c,d) \in \mathcal{A}_N : (a{+}d, b{+}c) \equiv (i,j) \pmod 2\}.$$

Similarly for $\mathcal{B}_N$ we write:

$$\mathcal{B}_N = \bigcup_{x \in \mathbb{F}_2^4} B_x, \quad B_x = \{y \in \mathcal{B}_N : y \equiv x \pmod 2\}.$$

If $N \equiv 2 \bmod 4$ we see from the definition of $\mathcal{B}_N$ that a point $(k,\ell,m,n) \in \mathcal{B}_N$ must satisfy $k \equiv \ell \equiv m \equiv n \equiv 0 \bmod 2$. In other words it is $\mathcal{B}_N = B_0$ in the notation above. Since moreover $f(A_{01} \cup A_{10} \cup A_{11}) \subseteq \mathcal{B}_N \setminus B_0 = \emptyset$, we deduce that $\mathcal{A}_N = A_{00}$. The map f is a bijection between A_{00} and B_0 since given a point $y = (k,\ell,m,n) \in B_0$, the point

$$\left(\frac{k+\ell}{2}, \frac{m+n}{2}, \frac{m-n}{2}, \frac{k-\ell}{2}\right) \tag{2.10}$$

is in the preimage $f^{-1}(y)$. Since f is injective, this shows that f is bijective between A_{00} and B_0. We deduce that we have the following equalities

$$|\mathcal{A}_N| = |A_{00}| = |B_0| = |\mathcal{B}_N|$$

and since $|\mathcal{B}_N| = r_2(N+2)r_2(N-2)$ (which follows directly from the definition of $\mathcal{B}_N$), this proves the claim for $N \equiv 2 \bmod 4$. Finally consider $N \equiv 3 \bmod 4$. We claim that in this case we have $|\mathcal{A}_N| = \frac{1}{2}r_2(N+2)r_2(N-2)$. From (2.9) we see that a point $(k,\ell,m,n) \in \mathcal{B}_N$ must satisfy $k^2 + n^2 \equiv \ell^2 + m^2 \equiv 1 \bmod 2$, hence we have

$$\mathcal{B}_N = B_{(1,1,0,0)} \cup B_{(1,0,1,0)} \cup B_{(0,1,0,1)} \cup B_{(0,0,1,1)}. \tag{2.11}$$

We also have, by the same condition and the definition of the map f,

$$\mathcal{A}_N = A_{10} \cup A_{01}.$$

The four sets in (2.11) are pairwise in bijection (by swapping two suitably chosen entries) and hence have the same cardinality, that equals $\frac{1}{4}r_2(N+2)r_2(N-2)$. Similarly, A_{01} and A_{10} are in bijection by the map $(a,b,c,d) \mapsto (b,-a,d,-c)$. We claim that A_{10} and $B_{(1,1,0,0)}$ are in bijection: from the definition of f we see that $f(A_{10}) \subseteq B_{(1,1,0,0)}$ and since f is injective it is sufficient to show that it is surjective from A_{10} to $B_{(1,1,0,0)}$. For a point $(k,\ell,m,n) \in B_{(1,1,0,0)}$, the point defined by (2.10) is a well-defined point in A_{10}. Hence f is a bijection between A_{10} and $B_{(1,1,0,0)}$ and we have the chain of equalities

$$|\mathcal{A}_N| = 2|A_{10}| = 2|B_{(1,1,0,0)}| = \frac{1}{2}r_2(N+2)r_2(N-2).$$

This proves the claim. In view of (2.8) we obtain

$$N^*(X,i,i) = \frac{1}{2}\sum_{\substack{1\le N\le X\\ N\equiv 2\ (4)}} r_2(N+2)r_2(N-2) + \frac{1}{4}\sum_{\substack{1\le N\le X\\ N\equiv 3\ (4)}} r_2(N+2)r_2(N-2)$$

and we conclude thus the proof of the lemma. □

As we have seen in the introduction, the main asymptotic of $N(s,z,w)$ equals the volume of the hyperbolic ball $B(z,s)$ for big s. However, in the general case of a cofinite Fuchsian group, explicit secondary terms will also appear. They are associated with the small eigenvalues of the Laplace operator.

The "completed" main term $M(s,z,w)$ is defined in section 2.6, and the remainder term $E(s,z,w)$ is defined as

$$E(s,z,w) := N(s,z,w) - M(s,z,w). \tag{2.12}$$

we will prove that for every cofinite Fuchsian group and every fixed $z,w,\in\mathbb{H}$ we have

$$\frac{1}{T}\int_0^T \Big|\frac{E(s,z,w)}{e^{s/2}}\Big|^2 ds \ll T. \tag{2.13}$$

In other words we show that the asymptotic variance of the normalized remainder in the hyperbolic circle problem, if not finite, diverges at most linearly with T.

Remark 2.5. Recall from the introduction that the best known upper bound on $E(s,z,w)$ is an unpublished result of Selberg [63, pp. 4-5] which gives

$$E(s,z,w) \ll e^{2s/3}. \tag{2.14}$$

The first moment of the normalized remainder $E(s,z,z)e^{-s/2}$ has been computed by Phillips and Rudnick in [52, Th. 1.1]. With the completed main term defined as in section 2.6 then their result reads

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{E(s,z,z)}{e^{s/2}}ds = 0. \tag{2.15}$$

In this chapter we will prove (2.13) in section 2.9. In fact we will prove (see Theorem 2.9) the estimate

$$\int_T^{T+1} \Big|\frac{E(s,z,w)}{e^{s/2}}\Big|^2 ds \ll T \tag{2.16}$$

from which (2.13) follows. Along the way we will give a proof of (2.14) and (2.15) (the latter we prove for any $z,w\in\mathbb{H}$ and with a rate of convergence that does not appear in [52]).

2.3 Convolution smoothing

The strategy of proof of (2.16) will consist in using the pretrace formula, i.e. Theorem 1.14, for the automorphic kernel

$$N(s,z,w)=\sum_{\gamma\in\Gamma}\mathbf{1}_{[0,s]}(d(z,\gamma w))=\sum_{\gamma\in\Gamma}\mathbf{1}_{[0,(\cosh s-1)/2]}(u(z,\gamma w)). \tag{2.17}$$

However, in view of Theorem 1.14 we need to ensure that the the Selberg/Harish-Chandra transform of the point-pair invariant satisfies conditions (1.8). As it will appear clear during the computations, the SHC of the point-pair invariant that appears in (2.17) would not satisfy those conditions. To get around this issue we start with a regularized version of the point-pair invariant, with the idea of getting rid of the regularization at the end, in order to recover the original function. We discuss in this section how to define the regularization.

Notation. When understood that the points $z,w\in\mathbb{H}$ are fixed, we will omit them and write $N(s)$ in place of $N(s,z,w)$. Similarly for $M(s,z,w)$ and $E(s,z,w)$.

Let $\delta\in(0,1)$ be a small positive parameter and consider the function

$$k_\delta(u):=\frac{1}{4\pi\sinh^2(\delta/2)}\mathbf{1}_{[0,(\cosh(\delta)-1)/2]}(u)$$

where $\mathbf{1}_{[0,A]}$ is the indicator function of the set $[0,A]$. This function satisfies

$$\int_{\mathbb{H}}k_\delta(u(z,w))d\mu(z)=1$$

which is a consequence of the fact that

$$\int_{\mathbb{H}}\mathbf{1}_{[0,(\cosh(\delta)-1)/2]}(u(z,w))d\mu(z)=\int_{\mathbb{H}}\mathbf{1}_{[0,\delta]}(d(z,w))d\mu(z)=\mathrm{vol}(B(w,\delta))$$

where $B(w,\delta)$ is the hyperbolic ball of radius δ centered at w, and its area is independent of w and equals

$$\mathrm{vol}(B(w,\delta))=4\pi\sinh^2(\delta/2).$$

Define $k^{\pm}(u)$ as the functions given by the convolution product

$$k^{\pm}(u):=\left(\mathbf{1}_{[0,(\cosh(s\pm\delta)-1)/2]}*k_\delta\right)(u), \tag{2.18}$$

where the hyperbolic convolution of two functions k_1,k_2 is defined [15, (2.11)] as

$$\int_{\mathbb{H}}k_1(u(z,v))k_2(u(v,w))d\mu(v).$$

Because of the triangle inequality $d(z,w)\le d(z,v)+d(v,w)$, for $Z\ge 0$ the convolution $\mathbf{1}_{[0,\cosh(Z)-1)/2]}*k_\delta$ satisfies

$$(\mathbf{1}_{[0,\cosh(Z)-1)/2]}*k_\delta)(u(z,w))=\begin{cases}1 & d(z,w)\le Z-\delta\\ 0 & d(z,w)\ge Z+\delta.\end{cases}$$

From this we deduce that

$$k^-(u) \leq \mathbf{1}_{[0,(\cosh(s)-1)/2]}(u) \leq k^+(u)$$

and summing over $\gamma \in \Gamma$:

$$K^-(s,\delta) = \sum_{\gamma\in\Gamma} k^-(u(z,\gamma w)) \leq N(s) \leq \sum_{\gamma\in\Gamma} k^+(u(z,\gamma w)) = K^+(s,\delta). \quad (2.19)$$

We will expand $K^{\pm}(s,\delta)$ using the pretrace formula (1.9). In order to do this we need to understand the properties of the Selberg-Harish-Chandra transform.

2.4 Computing the Selberg-Harish-Chandra transform

The Selberg-Harish-Chandra (SHC) transform turns convolutions into products (see [15, p. 323]), so if we denote by h_R the SHC transform of $\mathbf{1}_{[0,(\cosh(R)-1)/2]}$, and by $h^{\pm}$ the SHC transform of $k^{\pm}$, then we have

$$h^{\pm}(t) = \frac{1}{4\pi \sinh^2(\delta/2)} h_{s\pm\delta}(t) h_\delta(t).$$

Denote for simplicity

$$\tilde{h}_\delta(t) = \frac{1}{4\pi \sinh^2(\delta/2)} h_\delta(t).$$

The function $h_R(t)$ is explicitely computed in [15] and [52], and is given by

$$h_R(t) = 2^{3/2} \int_{-R}^{R} (\cosh R - \cosh u)^{1/2} e^{itu} du.$$

Observe that for every $t \in \mathbb{R}$ and $R > 0$ we have the estimate

$$|h_R(t)| \leq h_R(0) \leq Re^{R/2}. \quad (2.20)$$

This will be useful in later estimates for $h(t)$, especially for t close to 0.

In [15, Lemma 2.4], an expression is given for $h_R(t)$ in terms of special functions, which shows the asymptotic behaviour of $h_R(t)$ for big R. We have, for every $R > 0$ and every $t \in \mathbb{C}$ such that $it \notin \mathbb{Z}$,

$$h_R(t) = 2\sqrt{2\pi \sinh R}\, \Re \left(e^{its} \frac{\Gamma(it)}{\Gamma(3/2+it)} F\left(-\frac{1}{2}; \frac{3}{2}; 1-it; \frac{1}{(1-e^{2R})}\right)\right) \quad (2.21)$$

Looking at the series expansion of the hypergeometric function we can write, for $t \in \mathbb{R}$ and $R > \frac{1}{2}\log 2$,

$$F\left(-\frac{1}{2}; \frac{3}{2}; 1-it; \frac{1}{(1-e^{2R})}\right) = 1 + O\left(e^{-2R} \min\left\{1, \frac{1}{|t|}\right\}\right). \quad (2.22)$$

This expression is also valid for $t = 0$ (in this case the minimum is defined to be 1). We will be interested in purely imaginary values of t in the interval $[-i/2, i/2]$. In this case we know from [15, Lemma 2.4] and [52, Lemma 2.1] that for $R \geq 1$ and t purely imaginary we have

$$h_R(t) = \sqrt{2\pi \sinh R} e^{R|t|} \frac{\Gamma(|t|)}{\Gamma(3/2+|t|)} + O\left((1+|t|^{-1}) e^{R\left(\frac{1}{2}-|t|\right)}\right). \quad (2.23)$$

For $0 \le R \le 1$ and $t \in \mathbb{C}$ we can write (see [15])

$$h_R(t) = 2\pi R^2 \frac{J_1(Rt)}{Rt}\sqrt{\frac{\sinh R}{R}} + O\left(R^2 e^{R|\Im t|}\min\{R^2, |t|^{-2}\}\right) \tag{2.24}$$

where $J_1(z)$ is the J-Bessel function of order 1.

2.5 Contribution from small eigenvalues

In the pretrace formula we will split the spectral expansion into the contribution associated to the small eigenvalues and that associated to the rest.

In this section we compute the contribution from the small eigenvalues, among which we include also the case $\lambda = 1/4$, using the expressions given in the previous section.

We discuss first the contribution coming from the discrete spectrum and later the contribution coming from the continuous spectrum at $\lambda = 1/4$.

Eigenvalue $\lambda = 0$, i.e. $t = \frac{i}{2}$

For the case $t = i/2$ an explicit formula is given for $h_R(i/2)$, which reads as follows

$$h_R\left(\frac{i}{2}\right) = 2\pi(\cosh R - 1).$$

This means that we can compute $h^{\pm}(0)$ explicitely and obtain

$$h^{\pm}\left(\frac{i}{2}\right) = \frac{1}{4\pi\sinh^2(\delta/2)} h_{s\pm\delta}\left(\frac{i}{2}\right) h_\delta\left(\frac{i}{2}\right) = 2\pi(\cosh s - 1) + O(\delta e^s). \tag{2.25}$$

Eigenvalue $\lambda = 1/4$, i.e. $t = 0$ (discrete contribution)

An expression for $h_R(0)$ is given in [15] and [52] (explicit computations can be found in [52, Lemma 2.2]). We have

$$h_R(0) = 4\big(R + 2(\log 2 - 1)\big)e^{R/2} + O(e^{-R/2}). \tag{2.26}$$

We want to use now (2.24) to understand the value $h^{\pm}(0)$. First of all notice that since the function $J_1(z)$ verifies

$$\lim_{z\to 0}\frac{J_1(z)}{z} = \frac{1}{2} \tag{2.27}$$

and that we have, for $\delta \ll 1$,

$$\frac{\delta^2}{\sinh^2(\delta/2)} = 4(1 + O(\delta^2)), \qquad \sqrt{\frac{\sinh\delta}{\delta}} = 1 + O(\delta^2)$$

we obtain, in the limit as $t \to 0$ in (2.24), that for $\delta \ll 1$

$$\tilde{h}_\delta(0) = \frac{1}{4\pi\sinh^2(\delta/2)} h_\delta(0) = 1 + O(\delta^2).$$

This together with (2.26) gives

$$h^{\pm}(0) = h_{s\pm\delta}(0)\tilde{h}_\delta(0) = 4\big(s + 2(\log 2 - 1)\big)e^{s/2} + O(s\,\delta\, e^{s/2}) + O(e^{-s/2}). \tag{2.28}$$

Small eigenvalues $0 < \lambda < 1/4$, i.e. $it \in (0, \frac{i}{2})$

We analyze now the contribution coming from the small eigenvalues $0 < \lambda_j < 1/4$. These eigenvalues correspond to t_j chosen so that $\frac{t_j}{i} \in (0, 1/2)$. It is important to recall that there is only a finite number of such eigenvalues, which implies that there exists $0 < \varepsilon_\Gamma \leq 1/4$ such that $\frac{t_j}{i} \in (\varepsilon_\Gamma, 1/2 - \varepsilon_\Gamma)$. For our analysis we will make use of equations (2.23) and (2.24). We can write for $t = t_j$ purely imaginary corresponding to a small eigenvalue:

$$\begin{aligned} h^{\pm}(t) = &\left(\sqrt{2\pi \sinh(s \pm \delta)} e^{(s\pm\delta)|t|} \frac{\Gamma(|t|)}{\Gamma(3/2 + |t|)} + O\left((1+|t|^{-1}) e^{(1/2-|t|)(s\pm\delta)}\right)\right) \\ &\times \left(\frac{2\pi\delta^2}{4\pi \sinh^2(\delta/2)} \frac{J_1(\delta t)}{\delta t} \sqrt{\frac{\sinh \delta}{\delta}} + O\left(\frac{\delta^2 e^{\delta|t|} \min\{\delta^2, |t|^{-2}\}}{\sinh^2(\delta/2)}\right)\right). \end{aligned} \tag{2.29}$$

Now use the following approximations for $\delta \ll 1$ and $t = t_j$

$$\begin{aligned} \sqrt{2\pi \sinh(s \pm \delta)} &= \sqrt{\pi} e^{s/2} + O(\delta e^{s/2}) + O(e^{-3s/2}) \\ e^{(s\pm\delta)|t|} &= e^{s|t|} + O(\delta |t| e^{s|t|}) \\ 1 + |t|^{-1} &\leq 1 + \varepsilon_\Gamma^{-1} = O(1) \\ e^{(1/2-|t|)(s\pm\delta)} &= e^{(1/2-|t|)s} + O(\delta(1/2 - |t|) e^{(1/2-|t|)s}) \\ \frac{2\pi\delta^2}{4\pi \sinh^2(\delta/2)} &= 2(1 + O(\delta^2)) \\ \frac{J_1(\delta t)}{\delta t} &= \frac{1}{2} + O(\delta|t|) \\ \sqrt{\frac{\sinh \delta}{\delta}} &= 1 + O(\delta^2) \end{aligned}$$

to rewrite equation (2.29) in the more comfortable way

$$\begin{aligned} h^{\pm}(t) = &\left[\frac{\Gamma(|t|)}{\Gamma(3/2 + |t|)} \left(\sqrt{\pi} e^{s/2} + O(\delta e^{s/2}) + O(e^{-3s/2})\right) \left(e^{s|t|} + O(\delta e^{s|t|}\right)\right. \\ &\left. + O\left(e^{s(1/2-\varepsilon_\Gamma)}\right)\right] \left[1 + O(\delta) + O(\delta^2 e^{\delta|t|}\right]. \end{aligned}$$

Multiplying out all the terms we obtain that the contribution from a given small eigenvalue $0 < \lambda < 1/4$ is given by

$$h^{\pm}(t) = \sqrt{\pi} \frac{\Gamma(|t|)}{\Gamma(3/2 + |t|)} e^{s(1/2+|t|)} + O(\delta e^{s(1-\varepsilon_\Gamma)}) + O(e^{s(1/2-\varepsilon_\Gamma)}). \tag{2.30}$$

Eigenvalue $\lambda = 1/4$, i.e. $t = 0$ (continuous contribution)

We discuss now the contribution coming from the Eisenstein series at $\lambda = 1/4$. By this contribution we mean the expression

$$\frac{1}{4\pi} \sum_{\mathfrak{a}} E_{\mathfrak{a}}(z, 1/2) \overline{E_{\mathfrak{a}}(w, 1/2)} \int_{-\infty}^{+\infty} h^{\pm}(t)\, dt$$

where the sum runs over the cusps of $\Gamma\backslash\mathbb{H}$. Recalling (2.24) and (2.27) we get for $\delta \ll 1$

$$\tilde{h}_\delta(t) = \begin{cases} 1 + O(\delta|t| + \delta^2) & \delta|t| < 1 \\ O\left(\frac{1}{(\delta|t|)^{3/2}}\right) & \delta|t| \geq 1. \end{cases} \tag{2.31}$$

Consider now the integral

$$\begin{aligned} \int_{-\infty}^{+\infty} h^{\pm}(t)dt &= \int_{-\infty}^{+\infty} h_{s\pm\delta}(t)\, \tilde{h}_\delta(t)\, dt \\ &= \int_{-\infty}^{+\infty} h_{s\pm\delta}(t)dt + \int_{-\infty}^{+\infty} O(|h_{s\pm\delta}(t)(\tilde{h}_\delta(t) - 1)|)dt. \end{aligned} \tag{2.32}$$

Looking at (2.20), (2.21), and (2.22), we infer the bound for the function $h_{s\pm\delta}(t)$ as follows

$$h_{s\pm\delta}(t) = O\left(e^{s/2}\ \min\left\{s, \frac{1}{|t|}, \frac{1}{|t|^{3/2}}\right\}\right). \tag{2.33}$$

Inserting this in the second integral in (2.32) we obtain for $s \geq 1$

$$\begin{aligned} \int_{-\infty}^{+\infty} |h_{s\pm\delta}(t)(\tilde{h}_\delta(t) - 1)|dt &\ll \int_{|t|\leq 1} s\,\delta\, e^{s/2} + \int_{1<|t|\leq\frac{1}{\delta}} \frac{\delta e^{s/2}}{|t|^{1/2}}dt + \int_{|t|>\frac{1}{\delta}} \frac{e^{s/2}}{|t|^{3/2}}dt \\ &\ll s\,\delta^{1/2} e^{s/2}. \end{aligned}$$

Hence we can write

$$\int_{-\infty}^{+\infty} h^{\pm}(t)dt = \int_{-\infty}^{+\infty} h_{s\pm\delta}(t)dt + O(s\delta^{1/2}e^{s/2}).$$

Now the average of the function $h_R(t)$ can be computed via the Fourier inversion theorem, giving

$$\int_{\mathbb{R}} h_R(t)dt = 2\pi g_R(0)$$

where $g_R(u)$ is the Fourier inverse of $h_R(t)$ and is (see [52, eq. (2.9)])

$$g_R(u) = \begin{cases} 2^{3/2}(\cosh R - \cosh u)^{1/2} & |u| \leq R \\ 0 & \text{otherwise.} \end{cases}$$

For $R = s \pm \delta$ and $u = 0$ we obtain

$$\begin{aligned} 2\pi g_{s\pm\delta}(0) = 2^{5/2}\pi(\cosh(s\pm\delta) - 1)^{1/2} &= 8\pi \sinh\left(\frac{s\pm\delta}{2}\right) \\ &= 4\pi e^{s/2} + O(\delta e^{s/2}) + O(e^{-s/2}). \end{aligned}$$

This shows that we can write

$$\int_{\mathbb{R}} h^{\pm}(t)dt = 4\pi e^{s/2} + O(s\delta^{1/2}e^{s/2} + e^{-s/2}). \tag{2.34}$$

2.6 Main term: definition

We define the full main term associated to the counting problem as

$$\begin{aligned} M(s) = M_\Gamma(s,z,w) := &\frac{\pi e^s}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \\ &+ \sqrt{\pi} \sum_{t_j\in(0,\frac{i}{2})} \frac{\Gamma(|t_j|)}{\Gamma(3/2+|t_j|)} e^{s(1/2+|t_j|)}\phi_j(z)\overline{\phi_j(w)} \\ &+ 4\big(s+2(\log 2-1)\big)\, e^{s/2} \sum_{t_j=0} \phi_j(z)\overline{\phi_j(w)} \\ &+ e^{s/2}\sum_{\mathfrak{a}} E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}. \end{aligned} \tag{2.35}$$

If we denote by $M^{\pm}(s,\delta)$ the analogous main term associated to $K^{\pm}(s,\delta)$, defined as the contribution coming from the eigenvalues $\lambda_j \le 1/4$ for the kernels $K^{\pm}$, namely

$$\begin{aligned} M^{\pm}(s,\delta) = &\sum_{t_j\in[0,\frac{i}{2}]} h^{\pm}(t_j)\phi_j(z)\overline{\phi_j(w)} \\ &+ \frac{1}{4\pi}\sum_{\mathfrak{a}} E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)} \int_{\mathbb{R}} h^{\pm}(t)dt \end{aligned}$$

then we can summarize equations (2.25), (2.28), (2.30), and (2.34) by saying that

$$M^{\pm}(s,\delta) = M(s) + O\left(\delta e^s + s\delta^{1/2}e^{s/2} + e^{s(1/2-\varepsilon_\Gamma)}\right). \tag{2.36}$$

2.7 Recovering the pointwise bound

We have discussed so far the contribution coming from the eigenvalues $0 \le \lambda \le 1/4$. We want to analyze now the contribution coming from the rest of the spectrum. This corresponds to the quantity

$$\sum_{t_j>0} h^{\pm}(t_j)\phi_j(z)\overline{\phi_j(w)} \tag{2.37}$$

$$+\frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty} h^{\pm}(t)\big(E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w,1/2+it)} - E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}\big)dt.$$

If we bound every piece that appears in (2.37) in order to obtain some function depending on s and δ, and we then choose appropriately δ, we will deduce Selberg's bound.

First recall the inequality [39, eq. (13.8)], for short average of cusp forms and Eisenstein series. For z in a compact set $K \subset \mathbb{H}$, then

$$\sum_{T\le t_j\le T+1} |\phi_j(z)|^2 + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_T^{T+1} |E_{\mathfrak{a}}(z,1/2+it)|^2 dt \ll_K T. \tag{2.38}$$

Using the inequality $2|AB| \le A^2+B^2$ we can conclude that the same bound holds for $z \ne w \in K$, namely

$$\sum_{T\le t_j\le T+1} \phi_j(z)\overline{\phi_j(w)} + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_T^{T+1} E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w,1/2+it)}dt \ll_K T.$$

In view of this observation we will assume throughout this section that $z = w$ for simplicity.

Discrete spectrum. Using the bounds in (2.31) and (2.33) we arrive at the estimate

$$h^{\pm}(t) \ll e^{s/2} \min\left\{ s, \frac{1}{|t|}, \frac{1}{|t|^{3/2}}, \frac{1}{\delta^{3/2}\,|t|^3} \right\}. \tag{2.39}$$

Using this we can analyze the contribution coming from the discrete spectrum as follows.

$$\begin{aligned}\sum_{t_j>0} h^{\pm}(t_j)|\phi_j(z)|^2 &= \sum_{0<t_j\leq 1} h^{\pm}(t_j)|\phi_j(z)|^2 + \sum_{1<t_j<\frac{1}{\delta}} h^{\pm}(t_j)|\phi_j(z)|^2 \\ &\quad + \sum_{t_j\geq\frac{1}{\delta}} h^{\pm}(t_j)|\phi_j(z)|^2 \\ &\ll e^{s/2} \sum_{0<t_j\leq 1} \frac{1}{|t_j|}\,|\phi_j(z)|^2 \;+\; e^{s/2} \sum_{1<t_j<\frac{1}{\delta}} \frac{1}{|t_j|^{3/2}}\,|\phi_j(z)|^2 \\ &\quad + e^{s/2} \sum_{t_j\geq\frac{1}{\delta}} \frac{1}{\delta^{3/2}|t|^3}|\phi_j(z)|^2\end{aligned}$$

Recall that in a finite interval there is a finite number of eigenvalues, so the first sum is finite. Using (2.38) we obtain that the contribution coming from the discrete spectrum equals

$$\sum_{t_j>0} h^{\pm}(t_j)|\phi_j(z)|^2 = O(e^{s/2} + \delta^{-1/2}e^{s/2}).$$

Continuous spectrum. Let us move on to the continuous spectrum. We fix a cusp $\mathfrak{a}$ and we write E instead of $E_{\mathfrak{a}}$. Again we assume $z = w$. We need to bound the following integral

$$\frac{1}{4\pi}\int_{-\infty}^{+\infty} h^{\pm}(t)\big(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\big)dt. \tag{2.40}$$

Observe that by analytic continuation of the Eisenstein series, for $|t| < 1$ we have

$$|E(z,1/2+it)|^2 - |E(z,1/2)|^2 = O(|t|). \tag{2.41}$$

Now we split the integral in (2.40) over $|t| < 1$ and $|t| \geq 1$, and on the last part we apply the same argument used for the discrete spectrum (using (2.38) and (2.39)) to control the integral of the Eisenstein series. This part contributes as

$$\int_{|t|\geq 1} h^{\pm}(t)\big(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\big)dt = O(e^{s/2}) + O(\delta^{-1/2}e^{s/2}). \tag{2.42}$$

For the integral over the set $\{|t| < 1\}$ we use (2.39) and the above remark (2.41) to conclude

$$\int_{|t|<1} h^{\pm}(t)\big(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\big)dt = \int_{|t|<1} O(e^{s/2})dt = O(e^{s/2}). \tag{2.43}$$

Putting together equations (2.42) and (2.43) we obtain that the contribution coming from the continuous spectrum gives

$$\frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty} h^{\pm}(t)\big(E(z,1/2+it)\overline{E(w,1/2+it)} - E(z,1/2)\overline{E(w,1/2)}\big)dt$$
$$= O(e^{s/2} + \delta^{-1/2}e^{s/2}).$$

We have therefore shown that the following holds:

$$K^{\pm}(s,\delta) - M^{\pm}(s,\delta) = O(e^{s/2} + \delta^{-1/2}e^{s/2}).$$

Also, in view of equation (2.36) we obtain

$$K^{\pm}(s,\delta) - M(s) = O(\delta e^{s} + s\delta^{1/2}e^{s/2} + e^{s/2} + \delta^{-1/2}e^{s/2}). \tag{2.44}$$

Conclusion. We can thus end this section with a theorem that recovers Selberg's bound.

Theorem 2.6. *Let Γ be a cofinite Fuchsian group and $z, w \in K \subseteq \mathbb{H}$ be two points in a compact set K of the hyperbolic plane $\mathbb{H}$. Then* (2.14) *holds, that is*

$$E(s,z,w) \ll_K e^{2s/3}.$$

Proof. Take $K^{\pm}(s,\delta)$ as defined in (2.19). Then

$$K^{-}(s,\delta) \le N(s,z,w) \le K^{+}(s,\delta).$$

This implies that

$$|E(s,z,w)| \le \max\{|K^{-}(s,\delta) - M(s)|, |K^{+}(s,\delta) - M(s)|\}.$$

Using now equation (2.44) we infer the estimate

$$|E(s,z,w)| = O(\delta e^{s} + \delta^{1/2}e^{s/2} + e^{s/2} + \delta^{-1/2}e^{s/2}).$$

Balancing the first and last term we choose $\delta = e^{-s/3}$ and we obtain

$$|E(s,z,w)| = O(e^{2s/3}).$$

This proves Selberg's bound. □

2.8 Recovering Phillips and Rudnick's result

We give here a proof of the result obtained in [52, Theorem 1.1]. Define the mean normalized remainder to be the quantity

$$MNR = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{N(s) - M(s)}{e^{s/2}}ds.$$

The aim is to prove that

$$MNR = 0. \tag{2.45}$$

We will show that we if we start by the approximation $k^{\pm}$ defined in (2.18) of section 2.3, then choosing $\delta = \delta(T)$ in a suitable way, we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}}ds = 0. \tag{2.46}$$

In view of the inequality

$$K^-(s,\delta) \le N(s) \le K^+(s,\delta)$$

we deduce that

$$\int_0^T \frac{K^-(s,\delta)-M(s)}{e^{s/2}}ds \le \int_0^T \frac{N(s)-M(s)}{e^{s/2}}ds \le \int_0^T \frac{K^+(s,\delta)-M(s)}{e^{s/2}}ds \quad (2.47)$$

and so if (2.46) is true then (2.45) follows. We will prove the following.

Theorem 2.7. *Let Γ be a cofinite Fuchsian group and let $z, w \in K \subseteq \mathbb{H}$ be two points belonging to a compact set K of the hyperbolic plane $\mathbb{H}$. Then for every $0 < \varepsilon < 1$ we have, as $T \to \infty$,*

$$\int_0^T \frac{K^\pm(s,\delta) - M(s)}{e^{s/2}}ds = O\left(1 + \delta e^{T/2} + \delta^{1/2}T^2 + \varepsilon T + \log\frac{1}{\varepsilon}\right)$$

where the implied constant depends on none of δ, ε, T.

Proof. Observe first that (2.36) implies that

$$\int_0^T \frac{K^\pm(s,\delta) - M(s)}{e^{s/2}}ds = \int_0^T \frac{K^\pm(s,\delta) - M^\pm(s,\delta)}{e^{s/2}}ds + O(\delta e^{T/2} + \delta^{1/2}T^2 + 1). \quad (2.48)$$

Consider then the integral

$$\int_0^T \frac{K^\pm(s,\delta) - M^\pm(s,\delta)}{e^{s/2}}ds. \quad (2.49)$$

We remove a small neighbourhood of $s = 0$ from the domain of integration: since for $A > 0$ is $K^\pm(s,\delta) = O(1)$, and $M^\pm(s,\delta) = O(1)$ for every $s \in [0,A]$ and $\delta < 1$, we have

$$\int_0^A \frac{K^\pm(s,\delta) - M^\pm(s)}{e^{s/2}}ds = O(1). \quad (2.50)$$

Let A be fixed, chosen such that $A > 1$, so that in particular $A > \delta$. Then computing (2.49) is the same as computing the integral over $[A,T]$ up to a bounded quantity. Expand now the integrand via spectral expansion in the form

$$\frac{K^\pm(s,\delta) - M^\pm(s)}{e^{s/2}} = \sum_{t_j>0} e^{-s/2}h^\pm(t_j)\phi_j(z)\overline{\phi_j(w)} \quad (2.51)$$
$$+\frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty}\frac{h^\pm(t)}{e^{s/2}}\left(E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w,1/2+it)} - E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}\right)dt.$$

We will prove that the following estimates hold

$$\int_A^T \sum_{t_j>0} e^{-s/2}h^\pm(t_j)\phi_j(z)\overline{\phi_j(w)}ds = O(1) \quad (2.52)$$

$$\int_A^T \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty}\frac{h^{\pm}(t)}{e^{s/2}}\left(E(z,1/2+it)\overline{E(w,1/2+it)}-E(z,1/2)\overline{E(w,1/2)}\right)dtds$$
$$= O(1)+O(\varepsilon(T-A))+O\left(\log\frac{1}{\varepsilon}\right). \tag{2.53}$$

This together with equations (2.48) and (2.50) gives the theorem.

Discrete spectrum. The series and the integral in (2.51) are absolutely convergent in view of the discussion in section 2.7, so when integrating with respect to s we can interchange the order of integration (and summation). Consider the discrete spectrum first. We have to estimate the following quantity (as in the previous section we assume $z = w$)

$$\sum_{t_j>0}\int_A^T h^{\pm}(t_j)e^{-s/2}ds\,|\phi_j(z)|^2.$$

Now $h^{\pm}(t) = h_{s\pm\delta}(t)\tilde{h}_\delta(t)$ and $\tilde{h}_\delta(t)$ does not depend on s, so the previous sum reads

$$\sum_{t_j>0}|\phi_j(z)|^2\tilde{h}_\delta(t_j)\int_A^T h_{s\pm\delta}(t_j)e^{-s/2}ds. \tag{2.54}$$

Notice also that in view of (2.24), for every $t \in \mathbb{R}$ and $\delta \in [0,1]$, is

$$\tilde{h}_\delta(t) = O(1)$$

where the implied constant can be chosen to be absolute. Hence it is sufficient to understand the size of

$$\int_A^T h_{s\pm\delta}(t_j)e^{-s/2}ds$$

to bound the expression in (2.54). We have by (2.21) and (2.22)

$$\int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}}ds = \int_A^T \frac{2\sqrt{2\pi\sinh(s\pm\delta)}}{e^{s/2}}\left(\Re(e^{it(s\pm\delta)}G(t)) + O\left(\frac{|G(t)|}{1+|t|}e^{-2(s\pm\delta)}\right)\right)ds$$

where we have written for simplicity

$$G(t) := \frac{\Gamma(it)}{\Gamma(3/2+it)},$$

and we have used the fact that

$$\min\left\{1,\frac{1}{|t|}\right\} = O\left(\frac{1}{1+|t|}\right).$$

Notice also that $s\pm\delta > 0$ because $s \geq A > \delta$. Consider the error term. We have

$$\frac{|G(t)|}{1+|t|}\int_A^T \frac{\sqrt{\sinh(s\pm\delta)}}{e^{s/2}}e^{-2(s\pm\delta)}ds = O\left(\frac{|G(t)|}{1+|t|}\right).$$

Now we look at the integral

$$\begin{aligned}\sqrt{\pi}G(t)\int_A^T \frac{\sqrt{e^{s\pm\delta}-e^{-(s\pm\delta)}}}{e^{s/2}}e^{it(s\pm\delta)}ds \\ = \sqrt{\pi}G(t)\int_A^T e^{\pm\delta/2}\sqrt{1-e^{-2(s\pm\delta)}}e^{it(s\pm\delta)}ds.\end{aligned} \tag{2.55}$$

Similarly is discussed the complex conjugate term. We use the Taylor expansion of the square root in the form

$$\sqrt{1+x}=\sum_{n=0}^{\infty}a_nx^n = 1+\sum_{n=1}^{\infty}a_nx^n \tag{2.56}$$

which converges absolutely for every $|x|<1$. Since $s\pm\delta>0$, then is $e^{-2(s\pm\delta)}<1$ and hence we can write (2.55) as

$$\begin{aligned}&\sqrt{\pi}G(t)e^{\pm\delta(1/2+it)}\left(\int_A^T e^{its}ds+\sum_{n=1}^{\infty}(-1)^n a_n e^{\mp 2n\delta}\int_A^T e^{s(it-2n)}ds\right)\\ &= \sqrt{\pi}G(t)e^{\pm\delta(1/2+it)}\left(\frac{e^{itT}-e^{itA}}{it}+\sum_{n=1}^{\infty}(-1)^n a_n e^{\mp 2n\delta}\frac{e^{T(it-2n)}-e^{A(it-2n)}}{it-2n}\right)\\ &= \sqrt{\pi}G(t)e^{\pm\delta(1/2+it)}\frac{e^{itT}-e^{itA}}{it}+O\left(\frac{|G(t)|}{1+|t|}e^{\pm\delta/2}\sum_{n=1}^{\infty}|a_n|e^{-2n(A\pm\delta)}\right)\\ &= \sqrt{\pi}G(t)e^{\pm\delta(1/2+it)}\frac{e^{itT}-e^{itA}}{it}+O\left(\frac{|G(t)|}{1+|t|}\right).\end{aligned}$$

We are ready to calculate the contribution coming from the discrete spectrum. We have

$$\begin{aligned}&\sum_{t_j>0}\tilde{h}_\delta(t)\,|\phi_j(z)|^2\,\left(\sqrt{\pi}G(t)e^{\pm\delta(1/2+it)}\frac{e^{itT}-e^{itA}}{it}+O\left(\frac{|G(t)|}{1+|t|}\right)\right)\\ &\ll \sum_{t_j>0}|\phi_j(z)|^2\,\frac{1}{|t|^{5/2}} \ll \sum_{t_j\le 1}+\sum_{k=1}^{\infty}\sum_{t_j\in(k,k+1]} \ll 1+\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}=O(1)\end{aligned}$$

where we have used that there are only finitely many t_j's in $(0,1]$ and inequality (2.38) to estimate the series. The implied constant does not depend on δ. This proves (2.52).

Continuous spectrum. Let us analyze the continuous spectrum now. We fix a cusp $\mathfrak{a}$ and write E in place of $E_{\mathfrak{a}}$. Since, as in the case of the discrete spectrum, we have that the integral

$$\int_{-\infty}^{+\infty}\frac{h^{\pm}(t)}{e^{s/2}}\big(|E(z,1/2+it)|^2-|E(z,1/2)|^2\big)dt$$

is absolutely convergent, when integrating with respect to s we can interchange the order of integration. We can therefore write

$$\begin{aligned}&\int_A^T\int_{-\infty}^{+\infty}h^{\pm}(t)\big(|E(z,1/2+it)|^2-|E(z,1/2)|^2\big)dt\frac{ds}{e^{s/2}}\\ &= \int_{-\infty}^{+\infty}\big(|E(z,1/2+it)|^2-|E(z,1/2)|^2\big)\tilde{h}_\delta(t)\int_A^T\frac{h_{s\pm\delta}(t)}{e^{s/2}}ds\,dt.\end{aligned}$$

For the inner integral in s we conclude from the discussion for the discrete spectrum that

$$\int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}} ds = O\left(\frac{|G(t)|}{|t|}\right).$$

On the other hand, bounding trivially $h_{s\pm\delta}$ in the first place in (2.21) by

$$\frac{h_{s\pm\delta(t)}}{e^{s/2}} \ll |G(t)|$$

we can also write

$$\int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}} ds \ll T|G(t)|.$$

This will be useful to control the contribution of the continuous spectrum near $t = 0$. Let $0 < \varepsilon < 1$ be given. We have

$$\int_{-\infty}^{+\infty} \left(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\right) \tilde{h}_\delta(t) \int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}} ds\, dt$$
$$= \int_{|t|<\varepsilon} + \int_{\varepsilon\leq|t|<1} + \int_{|t|\geq 1} .$$

For $|t| \geq 1$, using (2.38), we obtain

$$\begin{aligned} &\int_{|t|\geq 1} \left(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\right) \tilde{h}_\delta(t) \int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}} ds\, dt \\ &\ll \int_{|t|\geq 1} \frac{|G(t)|}{|t|} \left(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\right) dt = O(1). \end{aligned} \tag{2.57}$$

For $|t| < \varepsilon$ we can write instead

$$\begin{aligned} &\int_{|t|<\varepsilon} \left(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\right) \tilde{h}_\delta(t) \int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}} ds\, dt \\ &\ll \int_{|t|<\varepsilon} \frac{|t|(T-A)}{|t|} dt = O(\varepsilon T). \end{aligned} \tag{2.58}$$

Finally, for $\varepsilon \leq |t| < 1$ we have

$$\begin{aligned} &\int_{\varepsilon\leq|t|<1} \left(|E(z,1/2+it)|^2 - |E(z,1/2)|^2\right) \tilde{h}_\delta(t) \int_A^T \frac{h_{s\pm\delta}(t)}{e^{s/2}} ds\, dt \\ &\ll \int_{\varepsilon\leq|t|<1} \frac{|t|}{|t|^2(1+|t|^{1/2})} dt = O\left(\log\frac{1}{\varepsilon}\right). \end{aligned} \tag{2.59}$$

In all the three cases the implied constants are independent of δ, ε, T. Summarizing (2.57), (2.58), and (2.59), we obtain that the contribution coming from the continuous spectrum equals

$$\frac{1}{4\pi}\sum_{\mathfrak{a}} \int_A^T \int_{-\infty}^{+\infty} \frac{h^\pm(t)}{e^{s/2}} \left(E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w,1/2+it)} - E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}\right) dt$$
$$= O\left(1 + \varepsilon T + \log\frac{1}{\varepsilon}\right).$$

This concludes the proof of (2.53), and hence the proof of Theorem 2.7. □

Conclusion. A direct consequence of Theorem 2.7 is the following, which is [52, Theorem 1.1]. The analysis gives a rate of convergence as follows.

Theorem 2.8. *Let Γ be a cofinite Fuchsian group and let $z, w \in K \subseteq \mathbb{H}$ be two points belonging to a compact set K of the hyperbolic plane $\mathbb{H}$. Then we have, as $T \to \infty$,*

$$\frac{1}{T}\int_0^T \frac{N(s) - M(s)}{e^{s/2}} ds = O\left(\frac{\log T}{T}\right).$$

Proof. Taking $\delta = e^{-T/2}$ and $\varepsilon = T^{-1}$ in Theorem 2.7 we get

$$\int_0^T \frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}} ds = O(\log T).$$

In view of equation (2.47) we obtain

$$\Big|\frac{1}{T}\int_0^T \frac{N(s) - M(s)}{e^{s/2}} ds\Big| \leq \Big|\frac{1}{T}\int_0^T \frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}} ds\Big| = O\left(\frac{\log T}{T}\right)$$

and the theorem is proven. □

2.9 Estimates on the second moment

The aim of this section is to prove Theorem 2.9 below, which gives an upper bound on the growth of the asymptotic variance of $E(s, z, w)e^{-s/2}$. The theorem is proven, in line with the proofs of Theorem 2.6 and Theorem 2.8, by means of the spectral expansion of the automorphic kernel $N(s, z, w)$ (or rather a regularized version of it). This approach is suggested already in [13], although the analysis below shows that by removing the eigenvalue $\lambda = 1/4$ (which means moving it to be part of the main term), it is possible to improve on [15, Cor. 2.1.1].

Theorem 2.9. *Let Γ be a cofinite Fuchsian group and $z, w \in K \subseteq \mathbb{H}$ be points in a compact set K of $\mathbb{H}$. Then*

$$\int_T^{T+1} \Big|\frac{E(s,z,w)}{e^{s/2}}\Big|^2 ds \ll_K T. \tag{2.60}$$

Proof. Start by the inequality

$$K^-(s,\delta) \leq N(s) \leq K^+(s,\delta).$$

This implies that

$$|E(s,z,w)| \leq \max\{\, |K^-(s,\delta) - M(s)|, |K^+(s,\delta) - M(s)|\, \}.$$

Squaring and integrating will preserve the inequality, so if we can prove that for $\delta = \delta(T)$ we have

$$\int_T^{T+1} \left|\frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}}\right|^2 ds \ll T \tag{2.61}$$

then we will deduce the bound in (2.60). We have seen that using (2.36) we have

$$K^{\pm}(s,\delta) - M(s) = K^{\pm}(s,\delta) - M^{\pm}(s) + O\left(\delta e^s + s\delta^{1/2}e^{s/2} + e^{s(1/2-\varepsilon_\Gamma)}\right). \tag{2.62}$$

Define the two quantities

$$H(T) := \int_T^{T+1} \left| \frac{K^{\pm}(s,\delta) - M^{\pm}(s)}{e^{s/2}} \right|^2 ds$$
$$F(T) := \int_T^{T+1} \left(\frac{\delta e^s + s\delta^{1/2} e^{s/2} + e^{s(1/2-\varepsilon_\Gamma)}}{e^{s/2}} \right)^2 ds.$$

If we square the terms in (2.62), expand the square, and use Cauhy-Schwarz inequality on the mixed product, we obtain

$$\int_T^{T+1} \left| \frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}} \right|^2 ds = H(T) + O\big(H(T)^{1/2} F(T)^{1/2} + F(T)\big). \quad (2.63)$$

Moreover, since it is

$$F(T) = \int_T^{T+1} \left(\delta e^{s/2} + s\delta^{1/2} + e^{-\varepsilon_\Gamma s} \right)^2 ds = O((\delta e^{T/2} + T\delta^{1/2} + e^{-\varepsilon_\Gamma T})^2)$$

we can write equation (2.63) as

$$\int_T^{T+1} \left| \frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}} \right|^2 ds = H(T) + O(H(T)^{1/2}(\delta e^{T/2} + T\delta^{1/2} + e^{-\varepsilon_\Gamma T})) + O((\delta e^{T/2} + T\delta^{1/2} + e^{-\varepsilon_\Gamma T})^2). \quad (2.64)$$

Assume one can choose δ depending on T in the form $\delta = \delta(T) = e^{-T/2}$ (we will show that such choice is allowed to get (2.60)). Then we will have

$$\int_T^{T+1} \left| \frac{K^{\pm}(s,\delta) - M(s)}{e^{s/2}} \right|^2 ds = H(T) + O(H(T)^{1/2}) + O(1).$$

The problem is reduced now to bounding the quantity

$$H(T) = \int_T^{T+1} \left| \frac{K^{\pm}(s,\delta) - M^{\pm}(s)}{e^{s/2}} \right|^2 ds \quad (2.65)$$

for which we can use the spectral expansion (2.37), similarly to what has been done for the proof of Theorem 2.6 and Theorem 2.8.

Recall that from Cauchy-Schwartz inequality is

$$|a_1 + \cdots + a_n|^2 = \left| (a_1, \ldots, a_n) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right|^2 \leq n(a_1^2 + \cdots + a_n^2).$$

This implies that in order to bound the square of the spectral expansion (2.37) it is sufficient to bound individually the square of the series and the square of the integral associated to each cusp.

Before proceeding to analyze the spectral expansion, we insert a weight function in (2.65), that will turn out useful later for multiple integration by parts. Let $\psi(s) \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \subseteq [-1/2, 3/2]$, and such that $\psi \geq 0$ and

$\psi(s) = 1$ for $s \in [0,1]$. Define, for $T \geq 0$, $\psi_T(s) = \psi(s-T)$. Then we have the inequality

$$H(T) \leq \int_{\mathbb{R}} \left| \frac{K^{\pm}(s,\delta) - M^{\pm}(s)}{e^{s/2}} \right|^2 \psi_T(s)\, ds,$$

and we will give bounds on this last integral.

We anticipate here the results we will obtain in the next two sections, and conclude the proof of the theorem. We will show that the bounds

$$\int_{\mathbb{R}} \Big| \sum_{t_j > 0} \phi_j(z)\overline{\phi_j(w)} e^{-s/2} h_{s\pm\delta}(t_j) \Big|^2 \psi_T(s) ds \ll \log(\delta^{-1}) + 1 \tag{2.66}$$

and

$$\int_{\mathbb{R}} \Big| \int_{-\infty}^{+\infty} \frac{h^{\pm}(t)}{e^{s/2}} E_{\mathfrak{a}}(z,w,t) dt \Big|^2 \psi_T(s) ds \ll \log(\delta^{-1}) + 1 \tag{2.67}$$

hold true, for every cusp $\mathfrak{a}$, where $E_{\mathfrak{a}}(z,w,t) = E_{\mathfrak{a}}(z, 1/2+it)\overline{E_{\mathfrak{a}}(w, 1/2+it)} - E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}$. Choosing $\delta = e^{-T/2}$ and looking at (2.61) and (2.64) we obtain the conclusion

$$\int_T^{T+1} \left| \frac{E(s,z,w)}{e^{s/2}} \right|^2 ds \ll T.$$

We will proceed now to prove the estimates (2.66) and (2.67) on the discrete and continuous spectrum.

Discrete spectrum. Consider the expansion (2.37) and analyze the square of the absolute value of the series. After multiplication by $\psi_T(s)$ and integration we have to analyze the expression

$$\sum_{t_j, t_\ell > 0} |\phi_j(z)|^2 |\phi_\ell(z)|^2 \int_{\mathbb{R}} \frac{h^{\pm}(t_j)\overline{h^{\pm}(t_\ell)}}{e^s} \psi_T(s) ds$$

Since $h^{\pm}(t) = \tilde{h}_\delta(t) h_{s\pm\delta}(t)$, it is important to understand the integral

$$\int_{\mathbb{R}} \frac{h_{s\pm\delta}(t_j)\overline{h_{s\pm\delta}(t_\ell)}}{e^s} \psi_T(s) ds.$$

Lemma 2.10. *Let $T \gg 1$ and $0 < \delta \ll 1$. Let $t_1, t_2 \in \mathbb{R}$, $t_1, t_2 \neq 0$. Then*

$$\int_{\mathbb{R}} \frac{h_{s\pm\delta}(t_1)\overline{h_{s\pm\delta}(t_2)}}{e^s} \psi_T(s) ds \ll \frac{1}{|t_1 t_2|^{3/2}} \min\left\{1, \frac{1}{|t_1 - t_2|^2}\right\} \tag{2.68}$$

where for $t_1 = t_2$ the minimum is understood to be 1.

Proof. Since the function $h_{s\pm\delta}(t)$ is even in t, we will assume that $t_1, t_2 > 0$. We will use in the proof an expression for $h_{s\pm\delta}$ given [52], which we find more suitable for integration with respect to s. In [52] the product of functions appearing in (2.21) is expressed as an explicit integral (this is proven in section 2.10). We have (see [52, Lemma 2.5])

$$\frac{h_{s\pm\delta(t)}}{e^{s/2}} = I(t, s\pm\delta) + \overline{I(t, s\pm\delta)} \tag{2.69}$$

where

$$I(t, s\pm\delta) = -2ie^{\pm\delta(1/2+it)}\int_0^\infty (1-e^{iv})^{1/2}(1-e^{-2(s\pm\delta)}e^{-iv})^{1/2}e^{-tv}\,dv\,e^{its}.$$

Multiplying equation (2.69) with its complex conjugate, we obtain

$$\begin{aligned}\frac{h_{s\pm\delta}(t_1)\overline{h_{s\pm\delta}(t_2)}}{e^s} &= I(t_1, s\pm\delta)\overline{I(t_2, s\pm\delta)} + I(t_1, s\pm\delta)I(t_2, s\pm\delta)\\ &+ \overline{I(t_1, s\pm\delta)}\,\overline{I(t_2, s\pm\delta)} + \overline{I(t_1, s\pm\delta)}I(t_2, s\pm\delta).\end{aligned} \tag{2.70}$$

The first and last term are conjugate, and so are the second and third term. Computations for each addend are similar, we carry out explicitely only the case of the first product. We have

$$\begin{aligned}I(t_1, s\pm\delta)\overline{I(t_2, s\pm\delta)} &= -4e^{\pm\delta(1+i(t_1-t_2))}\int_0^\infty\int_0^\infty (1-e^{iu})^{1/2}(1-e^{-iv})^{1/2}\\ \times\ & e^{-t_1u}e^{-t_2v}e^{is(t_1-t_2)}[1-e^{-2(s\pm\delta)}e^{-iu}]^{1/2}[1-e^{-2(s\pm\delta)}e^{iv}]^{1/2}\,dudv\,.\end{aligned} \tag{2.71}$$

Bounding trivially the double integral we obtain

$$|I(t_1, s\pm\delta)\overline{I(t_2, s\pm\delta)}| \ll \int_0^\infty\int_0^\infty |u|^{1/2}|v|^{1/2}e^{-t_1u}e^{-t_2v}dudv \ll \frac{1}{|t_1t_2|^{3/2}}$$

and integrating in s we arrive at the trivial bound

$$\int_{\mathbb{R}} I(t_1, \delta, s)\overline{I(t_2, \delta, s)}\psi_T(s)ds \ll \frac{1}{|t_1t_2|^{3/2}} \tag{2.72}$$

which gives one of the bounds appearing in the statement. Let us prove the other estimate. Notice that the implied constant depends on ψ but not on T. The double integral in (2.71) is absolutely convergent, so integration in s can be interchanged with integration in u and v. Consider then the integral $J(T, t_1, t_2)$ defined by

$$J(T, t_1, t_2) := \int_{\mathbb{R}} e^{is(t_1-t_2)}[1-e^{-2(s\pm\delta)}e^{-iu}]^{1/2}[1-e^{-2(s\pm\delta)}e^{iv}]^{1/2}\psi_T(s)ds.$$

(the integral depends actually on the parameters $A, T, \delta, t_1, t_2, u, v$). Using the Taylor expansion of $\sqrt{1+z}$ as in (2.56), which is absolutely convergent for $|z| < 1$, and integrating termwise, we obtain

$$\begin{aligned}J(T, t_1, t_2) = \sum_{m,n=0}^{\infty} &(-1)^{m+n}a_m a_n e^{-2m(\pm\delta-iu)-2n(\pm\delta+iv)}\\ &\times\int_{\mathbb{R}} e^{s(-2m-2n+i(t_1-t_2))}\psi_T(s)ds.\end{aligned}$$

Integrating by parts twice we get

$$\begin{aligned}J(T, t_1, t_2) = \sum_{m,n=0}^{\infty} &\frac{(-1)^{m+n}a_m a_n e^{-2m(\pm\delta-iu)-2n(\pm\delta+iv)}}{(-2m-2n+i(t_1-t_2))^2}\\ &\times\int_{\mathbb{R}} e^{s(-2m-2n+i(t_1-t_2))}\psi_T''(s)ds,\end{aligned}$$

from which we deduce

$$J(T, t_1, t_2) \ll \sum_{m,n=0}^{\infty} \frac{|a_m a_n| e^{-T(2m+2n)}}{|t_1 - t_2|^2} \ll \frac{1}{|t_1 - t_2|^2}.$$

Again the implicit constant depends on ψ but not on T. Looking at the integral in (2.71) we infer that

$$\begin{aligned} \int_{\mathbb{R}} I(t_1, s \pm \delta) \overline{I(t_2, s \pm \delta)} \psi_T(s) ds &\ll \int_0^{\infty} \int_0^{\infty} \frac{|u|^{1/2} |v|^{1/2} e^{-tu} e^{-tv}}{|t_1 - t_2|^2} dudv \\ &\ll \frac{1}{|t_1 - t_2|^2} \frac{1}{|t_1 t_2|^{3/2}} . \end{aligned} \tag{2.73}$$

The same bound holds for the last product in (2.70), whereas the same calculation for the second and third term leads to a similar estimate where $|t_1 - t_2|$ has to be replaced by $|t_1 + t_2|$. For $t_1, t_2 > 0$, then $|t_1 + t_2| \geq |t_1 - t_2|$ and hence we can use (2.73) to bound all terms in (2.70). Taking the minimum between this and (2.72) we conclude the proof of the lemma. □

We can now compute the contribution coming from the discrete spectrum.

Proposition 2.11. *With the same notation as before, we have*

$$\sum_{t_j, t_\ell > 0} |\phi_j(z)|^2 |\phi_\ell(z)|^2 \int_{\mathbb{R}} \frac{h^{\pm}(t_j) \overline{h^{\pm}(t_\ell)}}{e^s} \psi_T(s) ds \ll \log(\delta^{-1}) + 1. \tag{2.74}$$

Proof. By symmetry of the estimate (2.68) in Lemma 2.10 and positivity of the integral in (2.74) for $t_j = t_\ell$ it is sufficient to consider only the case when $t_\ell \geq t_j$. Recall that $h^{\pm}(t) = \tilde{h}_\delta(t) h_{s \pm \delta}(t)$ and that

$$\tilde{h}_\delta(t) \ll \min\{1, (\delta t)^{-3/2}\}.$$

We will use this bound together with (2.68) and (2.38). We have

$$\begin{aligned} &\sum_{\substack{t_j > 0 \\ t_j \leq t_\ell < t_j + 1}} |\phi_j(z)|^2 |\phi_\ell(z)|^2 \int_{\mathbb{R}} \frac{h^{\pm}(t_j) \overline{h^{\pm}(t_\ell)}}{e^s} \psi_T(s) ds \\ &\ll \sum_{\substack{t_j < \delta^{-1} \\ t_j \leq t_\ell < t_j + 1}} \frac{|\phi_j(z)|^2 |\phi_\ell(z)|^2}{(t_j t_\ell)^{3/2}} + \sum_{\substack{t_j \geq \delta^{-1} \\ t_j \leq t_\ell < t_j + 1}} \frac{|\phi_j(z)|^2 |\phi_\ell(z)|^2}{(\delta t_j t_\ell)^3} \\ &\ll \sum_{t_j < \delta^{-1}} \frac{|\phi_j(z)|^2}{t_j^2} + \frac{1}{\delta^3} \sum_{t_j \geq \delta^{-1}} \frac{|\phi_j(z)|^2}{t_j^5} \ll \log \delta^{-1} + 1. \end{aligned} \tag{2.75}$$

This shows that a neighbourhood of the diagonal $t_j = t_\ell$ (of width 1) gives a contribution of the order $O(\log \delta^{-1} + 1)$. Consider now the sum

$$\begin{aligned} &\sum_{\substack{t_j \geq \delta^{-1} \\ t_\ell \geq t_j + 1}} |\phi_j(z)|^2 |\phi_\ell(z)|^2 \int_{\mathbb{R}} \frac{h^{\pm}(t_j) \overline{h^{\pm}(t_\ell)}}{e^s} \psi_T(s) ds \\ &\qquad \ll \frac{1}{\delta^3} \sum_{t_j \geq \delta^{-1}} \frac{1}{t_j^3} |\phi_j(z)|^2 \sum_{t_\ell \geq t_j + 1} |\phi_\ell(z)|^2 \frac{1}{t_\ell^3} \frac{1}{|t_\ell - t_j|^2} \end{aligned}$$

$$\begin{aligned}
&\ll \frac{1}{\delta^3}\sum_{t_j\geq\delta^{-1}}\frac{1}{t_j^3}|\phi_j(z)|^2\sum_{n=1}^{\infty}\sum_{n\leq t_\ell-t_j\leq n+1}|\phi_\ell(z)|^2\frac{1}{t_\ell^3}\frac{1}{|t_\ell-t_j|^2}\\
&\ll \frac{1}{\delta^3}\sum_{t_j\geq\delta^{-1}}\frac{1}{t_j^3}|\phi_j(z)|^2\sum_{n=1}^{\infty}\frac{1}{n^2(t_j+n)^2}\\
&\ll \frac{1}{\delta^3}\sum_{t_j\geq\delta^{-1}}\frac{1}{t_j^5}|\phi_j(z)|^2\ll 1.
\end{aligned}\tag{2.76}$$

This shows that the "tail" of the double series gives a contribution of the order $O(1)$. Finally, consider the sum

$$\begin{aligned}
&\sum_{\substack{t_j<\delta^{-1}\\ t_\ell\geq t_j+1}}|\phi_j(z)|^2|\phi_\ell(z)|^2\int_{\mathbb{R}}\frac{h^{\pm}(t_j)\overline{h^{\pm}(t_\ell)}}{e^s}\psi_T(s)ds\\
&\qquad\ll \sum_{t_j<\delta^{-1}}\frac{1}{t_j^{3/2}}|\phi_j(z)|^2\sum_{t_\ell\geq t_j+1}|\phi_\ell(z)|^2\frac{1}{t_\ell^3/2}\frac{1}{|t_\ell-t_j|^2}\\
&\qquad\ll \sum_{t_j<\delta^{-1}}\frac{1}{t_j^{3/2}}|\phi_j(z)|^2\sum_{n=1}^{\infty}\frac{1}{n^2(t_j+n)^{1/2}}\\
&\qquad\ll \sum_{t_j<\delta^{-1}}|\phi_j(z)|^2\frac{1}{t_j^2}\ll\log\delta^{-1}.
\end{aligned}\tag{2.77}$$

this concludes the analysis of the contribution coming from the discrete spectrum. Summarizing (2.75), (2.76), and (2.77), we obtain that

$$\sum_{t_j,t_\ell>0}|\phi_j(z)|^2|\phi_\ell(z)|^2\int_{\mathbb{R}}\frac{h^{\pm}(t_j)\overline{h^{\pm}(t_\ell)}}{e^s}\psi_T(s)ds\ll\log\delta^{-1}+1.$$

as claimed. This proves the proposition. □

Continuous spectrum. We analyze here the contribution coming from the continuous spectrum. We assume again that $z=w$. We need to consider, for each cusp $\mathfrak{a}$ of Γ, the integral

$$\int_{\mathbb{R}}\Big|\int_{-\infty}^{+\infty}\frac{h^{\pm}(t)}{e^{s/2}}E_{\mathfrak{a}}(t)dt\Big|^2\psi_T(s)ds,$$

where for shortening notation we have set

$$E_{\mathfrak{a}}(t):=|E(z,1/2+it)|^2-|E(z,1/2)|^2.$$

Since both $h^{\pm}(t)$ and $E_{\mathfrak{a}}(t)$ are even in t, we can restrict the domain of integration (for the inner integral) to the positive real axis $[0,+\infty)$. So we have to discuss

$$4\int_{\mathbb{R}}\int_0^{+\infty}\int_0^{+\infty}E_{\mathfrak{a}}(t_1)E_{\mathfrak{a}}(t_2)\frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}dt_1dt_2\psi_T(s)ds.\tag{2.78}$$

Since the integrals in t_1 and t_2 are absolutely convergent, we can interchange order of integration. We need thus to deal with the quantity

$$\int_{\mathbb{R}}\frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}\psi_T(s)ds,$$

which we have already discussed in Lemma 2.10. A refinement of the argument, namely noticing that all the square roots appearing in the integral in (2.71) can be bounded by an absolute constant, gives the bound

$$\int_{\mathbb{R}} \frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}\psi_T(s)ds \ll \min\left\{1, \frac{1}{|t_1-t_2|^2}\right\} \frac{1}{|t_1|(1+\sqrt{t_1})} \frac{1}{|t_2|(1+\sqrt{t_2})},$$

which is better for $|t_1|, |t_2| \leq 1$. With this in hand we are going to bound the integral in (2.78). By symmetry we can analyze only the case when $t_2 \geq t_1$. We analyze first when t_1, t_2 are close to 0. We have

$$\begin{aligned} &\int_0^1 E_{\mathfrak{a}}(t_1) \int_{t_1}^{t_1+1} E_{\mathfrak{a}}(t_2) \int_{\mathbb{R}} \frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}\psi_T(s)ds\, dt_2\, dt_1 \\ &\quad \ll \int_0^1 \frac{t_1}{t_1(1+\sqrt{t_1})} \int_{t_1}^{t_1+1} \frac{t_2}{t_2(1+\sqrt{t_2})} dt_2\, dt_1 \ll 1. \end{aligned} \tag{2.79}$$

Similarly we have

$$\begin{aligned} &\int_1^{\delta^{-1}} E_{\mathfrak{a}}(t_1) \int_{t_1}^{t_1+1} E_{\mathfrak{a}}(t_2) \int_{\mathbb{R}} \frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}\psi_T(s)ds\, dt_2\, dt_1 \\ &\ll \int_1^{\delta^{-1}} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^{3/2}} \int_{t_1}^{t_1+1} \frac{|E_{\mathfrak{a}}(t_2)|}{t_2^{3/2}} dt_2 dt_1 \ll \int_1^{\delta^{-1}} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^2} dt_1 \ll \log\delta^{-1} + 1 \end{aligned} \tag{2.80}$$

and for the case $t_1 \geq \delta^{-1}$ we have

$$\begin{aligned} &\int_{\delta^{-1}}^{+\infty} E_{\mathfrak{a}}(t_1) \int_{t_1}^{t_1+1} E_{\mathfrak{a}}(t_2) \int_{\mathbb{R}} \frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}\psi_T(s)ds\, dt_2\, dt_1 \\ &\ll \frac{1}{\delta^3} \int_{\delta^{-1}}^{+\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^3} \int_{t_1}^{t_1+1} \frac{|E_{\mathfrak{a}}(t_2)|}{t_2^3} dt_2\, dt_1 \ll \frac{1}{\delta^3} \int_{\delta^{-1}}^{+\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^5} dt_1 \ll 1. \end{aligned} \tag{2.81}$$

Equations (2.79), (2.80), and (2.81) show that the contribution associated to the continuous spectrum, coming from a neighbourhood of the diagonal of width one, is bounded by $O(\log\delta^{-1}+1)$, as it was in the case of the discrete spectrum. Consider now the integral

$$\begin{aligned} &\int_{\delta^{-1}}^{+\infty} E_{\mathfrak{a}}(t_1) \int_{t_1+1}^{+\infty} E_{\mathfrak{a}}(t_2) \int_{\mathbb{R}} \frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s}\psi_T(s)ds\, dt_2\, dt_1 \\ &\quad \ll \frac{1}{\delta^3} \int_{\delta^{-1}}^{+\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^3} \int_{t_1+1}^{+\infty} \frac{|E_{\mathfrak{a}}(t_2)|}{t_2^3|t_2-t_1|^2} dt_2 dt_1 \\ &\quad \ll \frac{1}{\delta^3} \int_{\delta^{-1}}^{+\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^3} \Big(\sum_{n=1}^{\infty} \int_{t_1+n}^{t_1+n+1} \frac{|E_{\mathfrak{a}}(t_2)|}{t_2^3|t_2-t_1|^2} \Big) dt_2 dt_1 \\ &\quad \ll \frac{1}{\delta^3} \int_{\delta^{-1}}^{+\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^3} \Big(\sum_{n=1}^{\infty} \frac{1}{n^2(t_1+n)^2} \Big) dt_1 \\ &\quad \ll \frac{1}{\delta^3} \int_{\delta^{-1}}^{+\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^5} dt_1 \ll 1. \end{aligned}$$

This shows that also in the case of the continuous spectrum the tail gives a contribution of the order $O(1)$. Finally we estimate

$$
\begin{aligned}
&\int_1^{\delta^{-1}} E_{\mathfrak{a}}(t_1) \int_{t_1+1}^{+\infty} E_{\mathfrak{a}}(t_2) \int_{\mathbb{R}} \frac{h^{\pm}(t_1)\overline{h^{\pm}(t_2)}}{e^s} \psi_T(s) ds\, dt_2\, dt_1 \\
&\ll \int_1^{\delta^{-1}} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^{3/2}} \int_{t_1+1}^{+\infty} \frac{|E_{\mathfrak{a}}(t_2)|}{t_2^{3/2}|t_2-t_1|} dt_2\, dt_1 \\
&\ll \int_1^{\delta^{-1}} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^{3/2}} \Big(\sum_{n=1}^{t_1} \frac{1}{n^2 \ell(t_1+n)^{1/2}} \Big) dt_1 \\
&\ll \int_1^{\delta^{-1}} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^2}\, dt_1 \ll \log \delta^{-1}.
\end{aligned}
$$

This concludes the analysis of the contribution of the continuous spectrum, and so we end with the following proposition.

Proposition 2.12. *With the notation being as above, we have*

$$
\int_{\mathbb{R}} \Big| \int_{-\infty}^{+\infty} \frac{h^{\pm}(t)}{e^{s/2}} E_{\mathfrak{a}}(t) dt \Big|^2 \psi_T(s) ds \ll \log \delta^{-1} + 1.
$$

This concludes the proof of (2.67), and thus the proof of Theorem 2.9. □

Consider now the function $N^*(X,z,w)$ defined in (2.6), and define $M^*(X,z,w) := M(\log X, z, w)$. Then we have the following.

Corollary 2.13. *Let $T \gg 1$. We have*

$$
\frac{1}{T} \int_T^{2T} (N^*(X,z,w) - M^*(X,z,w))^2 dX \ll T \log T. \tag{2.82}
$$

Proof. Recalling the relation between the function $u(z,w)$ and the hyperbolic distance, we see that $N(s) = N^*(2\cosh s)$. Using the following estimates

$$
\begin{aligned}
(\cosh s)^{\alpha} &= e^s + O(e^{-s}) \quad \text{for every } \alpha \in [0,1], \quad \forall\, s \geq 0, \\
\log(\cosh s) &= \log e^s + \log(1 + e^{-2s}) = s + O(e^{-2s}) \quad \forall\, s \geq 0
\end{aligned}
$$

we obtain

$$
M^*(2\cosh s) = M(s) + O(e^{-s})
$$

for every $s \geq 0$. In particular we get

$$
\int_T^{T+1} \Big(\frac{N(s) - M^*(2\cosh s)}{e^{s/2}} \Big)^2 ds \ll T.
$$

Observe now that the following inequality holds

$$
\int_T^{T+1} \frac{(N(s) - M^*(2\cosh s))^2}{e^s} ds \geq \int_T^{T+1} \frac{(N(s) - M^*(2\cosh s))^2}{(2\cosh s)} \frac{2\sinh s}{2\cosh s} ds.
$$

Performing the change of variable $X = 2\cosh s$ and using that $2\cosh(T+1) \geq 4\cosh T$ we get

$$
\int_T^{2T} \frac{(N^*(X) - M^*(X))^2}{X^2} dX \ll \log T.
$$

This obviously implies (2.82). □

2.10 Integral representation of some special functions

We want to prove here that certain integrals can be expressed in terms of quotients of Gamma functions. Consider, for $t > 0$, the integrals $I(t)$ and $J(t)$ defined as follows.

$$J(t) = -\int_0^\infty (1 - e^{-iv})^{1/2} e^{-tv} dv.$$

$$I(t) = \int_0^\infty \int_0^\infty (1 - e^{-iv})^{1/2} (1 - e^{2i\tau} e^{iv})^{1/2} e^{-t\tau} e^{-tv} d\tau dv.$$

Proposition 2.14. *For every $t \in \mathbb{R}, t > 0$, we have*

$$I(t) = \frac{\pi}{8} \left| \frac{\Gamma(it/2)}{\Gamma(3/2 + it/2)} \right|^2 \qquad J(t) = -\frac{i\sqrt{\pi}}{2} \frac{\Gamma(it)}{\Gamma(3/2 + it)}.$$

The proof is an exercise in complex analysis which mainly uses the Cauchy first integral theorem, the functional equation of the gamma function and the relation between the gamma and the beta function. When applying Cauchy's theorem, special care is needed for the different pieces of the boundary of the domain of integration. A number of technical inequalities will allow us to control every piece. We state them in form of lemmata, of which we omit the proof, since they can be easily obtained using the Taylor expansion of exponential and trigonometric functions.

Lemma 2.15. *There exists a positive constant $L_1 > 0$ such that for every $\varphi \in [0, \pi]$ and $0 < \varepsilon < 1$ we have*

$$|1 - e^{i\varepsilon e^{i\varphi}}|^{-\frac{1}{2}} \leq L_1 \varepsilon^{-1/2}.$$

Lemma 2.16. *Let $\varphi_0 > 0$. There exists a positive constant $L_2 > 0$ such that for every $\varphi \in [\varphi_0, \pi/2]$ and for every $R \gg 1$, we have*

$$|1 - e^{iRe^{i\varphi}}|^{-\frac{1}{2}} \leq L_2.$$

Lemma 2.17. *Let $\varphi_0 > 0$, $0 < \varepsilon < 1$, and $R \gg 1$. Assume that*

$$d(R\cos\varphi_0, 2\pi\mathbb{Z}) = \inf_{n\in\mathbb{Z}} |R\cos\varphi_0 - 2\pi n| \geq \varepsilon.$$

Then there exists a positive constant $L_3 > 0$ such that, for every $x \in [0, R\sin\varphi_0]$ we have

$$|1 - e^{iR\cos\varphi_0 - x}|^{-1/2} \leq L_3\, \varepsilon^{-1/2} e^{R\sin\varphi_0/4}.$$

Lemma 2.18. *Let $\varphi_0 > 0$, $0 < \sigma < 1$, and $R \gg 1$. There exists a positive constant $L_4 > 0$ such that for $x \in [\sigma, R\sin\varphi_0]$ we have*

$$|1 - e^{iR\cos\varphi_0 - x}|^{-1/2} \leq L_4\, x^{-1/2}.$$

We prove instead the following lemma, which state how a certain integral representing the Beta function can be equally computed by integrating over the real or the imaginary axis.

Lemma 2.19. *The following integral can be computed equally by integration on the vertical semiaxis $I = [0, i\infty)$ or on the horizontal semiaxis $I' = [0, +\infty)$:*

$$B\left(1+it, \frac{1}{2}\right) = -i\int_0^{i\infty} e^{-(t-i)\theta}(1-e^{i\theta})^{-\frac{1}{2}}d\theta = -i\int_0^{+\infty} e^{-(t-i)\theta}(1-e^{i\theta})^{-\frac{1}{2}}d\theta.$$

Proof. The function $f(z) = e^{-(t-i)z}(1-e^{iz})^{\frac{1}{2}}$ is holomorphic in the open set $\Omega = \{z \in \mathbb{C} \mid z \neq 2\pi n, n \in \mathbb{Z}\}$. Consider the contour $\mathcal{C} \subseteq \Omega$ consisting of the vertical line $\ell = [i\varepsilon_0, iR]$, an arc of circle γ of the form $Re^{i\varphi}$ for $\varphi \in [\varphi_0, \pi/2]$, a series of semicircles C_{ε_n} of radius ε_n around the points $2\pi n$ (a quarter of circle in the case $n = 0$), the intervals I_n on the real line connecting the semicircles and γ, and the interval $J = [R\cos\varphi_0, R\cos\varphi_0 + iR\sin\varphi_0]$. By Cauchy's theorem [43, Theorem 2.3],

$$\int_{\mathcal{C}} f(z)dz = 0.$$

We want to show that the integral over the path γ, J, and all the C_{ε_n}, give a contribution that goes to zero when R goes to infinity. Consider the contribution coming from the circles C_{ε_n}. We have

$$\Big|\int_{C_{\varepsilon_n}} f(z)dz\Big| \leq \varepsilon_n\, e^{-tn}\int_0^{\pi} e^{-t\varepsilon_n\cos\varphi-\varepsilon_n\sin\varphi}|1-e^{i\varepsilon_n e^{i\varphi}}|^{-\frac{1}{2}}d\varphi$$

(for $n = 0$ the inequalities are the same but the integral is only over $[0, \pi/2]$), and using Lemma 2.15 we get

$$\Big|\int_{C_{\varepsilon_n}} f(z)dz\Big| \ll \varepsilon_n^{\frac{1}{2}}\, e^{-tn}\int_0^{\pi} e^{-t\varepsilon_n\cos\varphi-\varepsilon_n\sin\varphi}\,d\varphi \ll \varepsilon_n^{\frac{1}{2}} e^{-tn}$$

(the last estimate is true also for $n = 0$) Taking $\varepsilon_n = \varepsilon$ for every n, the total contribution coming from the circles C_{ε_n} is bounded by

$$\sum_{n=0}^{+\infty}\int_{C_{\varepsilon_n}} f(z)dz = O\left(\varepsilon^{\frac{1}{2}}\sum_{n=0}^{+\infty} e^{-tn}\right) = O(\varepsilon^{\frac{1}{2}}). \tag{2.83}$$

Consider now the integral over γ. Using Lemma 2.16 we get

$$\Big|\int_{\gamma} f(z)dz\Big| \ll R\int_{\varphi_0}^{\frac{\pi}{2}} e^{-tR\cos\varphi-R\sin\varphi}d\varphi \ll Re^{-R\sin\varphi_0}. \tag{2.84}$$

Consider now the integral over the vertical interval $J = [R\cos\varphi_0, R\cos\varphi_0 + iR\sin\varphi_0]$. Assume first that J does not intersect any of the circles C_{ε_n}. Using Lemma 2.17 we proceed as follows.

$$\begin{aligned}\int_J f(z)dz &= \int_0^{R\sin\varphi_0} e^{-tR\cos\varphi_0-itx+iR\cos\varphi_0-x}(1-e^{iR\cos\varphi_0-x})^{-1/2}\,i\,dx\\ &= O\left(\varepsilon^{-1/2}R\,e^{-R(t\cos\varphi_0-\sin\varphi_0/4)}\right).\end{aligned}$$

Taking φ_0 such that $\beta := t\cos\varphi_0 - \sin\varphi_0/4 > 0$, we can write

$$\int_J f(z)dz = O\left(\varepsilon^{-1/2}Re^{-\beta R}\right). \tag{2.85}$$

If we are in the special case where the line J hits one of the circles C_{ε_n}, then we integrate instead over the segment $J = [R\cos\varphi_0 + i\sigma, R\cos\varphi_0 + iR\sin\varphi_0]$, with $0 < \sigma < \varepsilon$.

Using Lemma 2.18 we can bound in this case the integral over J by

$$\Big|\int_J f(z)dz\Big| \ll e^{-tR\cos\varphi_0}\int_\sigma^{R\sin\varphi_0} x^{-1/2}dx \ll R^{1/2}e^{-tR\cos\varphi_0} \tag{2.86}$$

Summarizing (2.83), (2.84), (2.85) and (2.86), we conclude that the integral of $f(z)$ over the paths C_{ε_n}, γ and J is bounded by

$$\sum_{n=0}^{+\infty}\Big|\int_{C_{\varepsilon_n}} f(z)dz\Big| + \Big|\int_\gamma f(z)dz\Big| + \Big|\int_J f(z)dz\Big| \ll \varepsilon^{1/2} + Re^{-R\sin\varphi_0} + \varepsilon^{-1/2}R\,e^{-\beta R}.$$

Setting for instance $\varepsilon = R^{-1}$ and letting R go to infinity, we deduce that all the integrals must vanish. Calling I the vertical semiaxis $I = [0, i\infty)$ and I' the horizontal semiaxis $[0, +\infty)$, we get therefore that

$$\int_I f(z)dz = \int_{I'} f(z)dz.$$

This means that the beta function can be calculated integrating over the real semi-axis instead of the imaginary one, and the lemma is proven. □

Proof of Proposition 2.14. First we prove the formula for $J(t)$. Since the integration in τ can be resolved and gives a factor $1/t$, it is sufficient to prove that

$$\int_0^\infty (1-e^{-iv})^{1/2}e^{-tv}dv = \frac{i\sqrt{\pi}}{2}\frac{\Gamma(it)}{\Gamma(3/2+it)}.$$

Let us call $\alpha(t)$ the integral on the left hand side. Recalling the relation between the Gamma function and the Beta function, and using the functional equation of the Gamma function to shift the argument to the half-plane $\Re(z) > 0$, we can write

$$\frac{\Gamma(it)}{\Gamma(3/2+it)} = \frac{\Gamma(1+it)}{it\,\Gamma(3/2+it)} = \frac{1}{it\sqrt{\pi}}B\left(1+it, \frac{1}{2}\right),$$

and we need therefore to prove that $B(1+it, 1/2) = 2\,t\,\alpha(t)$. Starting by the definition of the Beta function

$$B\left(1+it, \frac{1}{2}\right) = \int_0^1 u^{it}(1-u)^{-\frac{1}{2}}du$$

and making the change of varibale $u = e^{i\theta}$ for $\theta \in [0, i\infty)$, we obtain

$$B\left(1+it, \frac{1}{2}\right) = -i\int_0^{i\infty} e^{-(t-i)\theta}(1-e^{i\theta})^{-\frac{1}{2}}d\theta.$$

The contour of integration can be moved to the real half-line $[0, \infty)$, as it is shown in Lemma 2.19. Once we have moved the path of integration to the real line, an integration by parts gives the result for $J(t)$. In order to prove the formula for $I(t)$ we observe that making the change of variable $w = 2\tau + v$ we can write the integral as

$$I(t) = \frac{1}{2}\left|\int_0^\infty (1-e^{-iv})^{1/2}e^{-tv/2}dv\right|^2 = \frac{1}{2}\left|\alpha\left(\frac{t}{2}\right)\right|^2 = \frac{\pi}{8}\left|\frac{\Gamma(it/2)}{\Gamma(3/2+it/2)}\right|^2$$

and so we conclude the proof. □

Chapter 3

Moments of almost periodic functions

In this chapter we study the moments of almost periodic functions, and we give a simple criterion to determine when an almost periodic function admits finite moments. We also study limiting distributions of almost periodic functions and their moments and behaviour.

The topic of almost periodic functions has been studied by many authors, and the theory dates back to the first half of the last century. Whereas the theory was developed in its origin to describe the class of functions f that, as the name naively suggests, barely fail from being periodic, and satisfy that for every $\varepsilon > 0$ there exists infinitely many $\ell \in \mathbb{R}$ (and "not too isolated", see [9, §44]) such that

$$|f(x+\ell) - f(x)| < \varepsilon$$

for every $x \in \mathbb{R}$, in a second time it evolved to study rather the class of functions that admit a Fourier-type expansion, as we can read in the following passage taken from a lecture by Harald Bohr in occasion of his 60th birthday [10]:

> " I was led to develop a general theory for a rather comprehensive class of functions – including functions of a real variable as well as functions of a complex variable – which I called almost periodic functions. The primary object was to characterize those functions which can be decomposed into a countably infinite number of so-called pure oscillations or, to speak in more geometrical terms, to characterize the class of movements in the plane which can be considered as arising from the superposition of simple circular movements with constant velocity."

Bohr was one of the first to develop the theory; he published in 1925-26 three rather long papers [6, 7, 8] that laid the basis of the study, and later a monograph [9] on the subject. The generalization that we use in this text is however that of Besicovitch [4, §II.2], which we describe later below.

To relate the theory of almost periodic functions to number theory, we shall mention the paper of Akbary, Ng, and Shahabi [1], in which they present a panorama of functions coming from number theory, they show that they are all almost periodic functions, and use the almost periodicity to prove that they all admit a limiting distribution.

The content of this chapter is to a big extent inspired by [1]. The methods of proofs are elementary, nevertheless they allow us not only to prove finiteness of the moments for certain almost periodic functions, but also to establish some upper bounds on the moments of functions that resemble but are not almost periodic ones. In particular, the bounds obtained show that there is some cancellation due to the fact that the terms in the Fourier-like expansion oscillate. We have in mind applications to the hyperbolic circle problem that are explained in section 3.4.

We start by recalling the definition of a Besicovitch almost periodic function ([1, §2]), and then we move to focus on a special subclass of such functions.

Definition 3.1. Let $p \in \mathbb{R}$, $p \geq 1$. A function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be a Besicovitch p-almost periodic function (B^p) if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ and a trigonometric polynomial

$$P_N(y) = \sum_{n=1}^{N} r_n e^{i\lambda_n y} \quad (y \in \mathbb{R}),$$

where $r_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$, such that

$$\limsup_{Y \to \infty} \left(\frac{1}{Y} \int_0^Y |\phi(y) - P_N(y)|^p dy \right)^{1/p} < \varepsilon.$$

Remark 3.2. Given a B^p-almost periodic function, the coefficients r_n can be recovered [1, §2] by the following formula

$$r_n = \lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \phi(y) e^{-i\lambda_n y} dy.$$

In what follows we are interested in a special type of (almost periodic) functions. Start by considering a function $\phi : [0, \infty) \to \mathbb{R}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n \xrightarrow{n \to \infty} \infty. \tag{3.1}$$

In other words $\{\lambda_n\}$ is a sequence of distinct positive increasing real numbers tending to infinity. Moreover, let $\{r_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Assume that there exists $y_0 \geq 0$ and $X_0 > 0$ such that for $y \geq y_0$ and $X \geq X_0$ we have

$$\phi(y) = c + 2\Re\Big(\sum_{0 < \lambda_n \leq X} r_n e^{i\lambda_n y} \Big) + \mathcal{E}(y, X), \tag{3.2}$$

where for some $k \geq 1$ we have

$$\int_0^{y_0} |\phi(y)|^k dy < \infty \quad \text{and} \quad \int_{y_0}^{Y} |\mathcal{E}(y, X(Y))|^k dy = o(Y), \tag{3.3}$$

and $X = X(Y)$ is some function of Y tending to infinity as Y tend to infinity, such that (3.3) holds. Finally, one more quantity is necessary in our discussion, namely the average of $|r_n|$ in a short windows. Assume hence that we have, for every $T > 0$,

$$\sum_{T \leq \lambda_n \leq T+1} |r_n| \ll \frac{1}{T^{\beta}}, \tag{3.4}$$

where β is some real number. With this setting we will essentially prove that the function ϕ admits finite moments of order $n \leq k$ whenever β is sufficiently large.

Remark 3.3. A function ϕ satisfying (3.2) and (3.3) is certainly a B^k-almost periodic function, since the limit superior

$$\limsup_{Y\to\infty}\left(\frac{1}{Y}\int_0^Y |\mathcal{E}(y,X(Y))|^k\right)^{1/k}$$

can be made smaller than ε for any $\varepsilon > 0$, by (3.3). In particular, the constant term appearing in (3.2) can be thought of as associated with the complex exponential with null frequency.

Remark 3.4. If (3.3) is satisfied for $k \geq 1$, then it is also satisfied for every real number $1 \leq k' \leq k$, by Hölder's inequality. For the same reason, B^p-almost periodic functions are also B^q-almost periodic functions if $1 \leq q \leq p$. Similarly, if (3.4) is satisfied for $\beta \in \mathbb{R}$, then it is also satisfied for every $\beta' \leq \beta$.

Remark 3.5. In the applications there might be repetitions of the λ_n. In this case, in order to recover a situation as described in (3.1)–(3.3), it will be convenient to group all terms associated to repeated λ_n, and replace the sequence $\{\lambda_n\}$ with the sequence $\{\varrho_n\}$ obtained by only listing once each distinct value of λ_n appearing. Once defined

$$s_n = \sum_{\lambda_m=\varrho_n} r_m,$$

we will request that (3.4) holds for s_n.

Remark 3.6. In some applications we found more convenient to have a control of $\mathcal{E}(y,X)$ on windows of the type $[Y,2Y]$ rather than $[y_0,Y]$. In this case we will require that there exists $X = X(Y)$ such that

$$\int_Y^{2Y} |\mathcal{E}(y,X(Y))|^2 dy = o(Y)$$

when $Y \to \infty$. The reason for we discuss the window $[0,Y]$ is mainly to follow the formalism in [1]. The only point where we will use intervals of type $[Y,2Y]$ will be in the proof of Theorem 3.33, and we will explain at that time how to perform the due changes.

3.1 First and Second moment

In this section we discuss the first and second moment of an almost periodic function ϕ for which (3.1)–(3.4) hold, with appropriate choices of k and β.

The existence of the first two moments has been also discussed in [1, (1.24)-(1.25)]. We present it here for the sake of completeness, and also to explain how the proof works. While the computation for the first moment is straightforward, the one for the second moment is slightly more convoluted; we therefore discuss it in detail, especially because the strategy of proof is similar to that used in section 3.3 to deal with higher moments.

Lemma 3.7. *Assume that* (3.3) *holds for* $k = 1$ *and* (3.4) *holds with* $\beta > 0$. *Then*

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y \phi(y)dy = c.$$

Proof. It follows from integrating (3.2) that we can write

$$\frac{1}{Y}\int_0^Y \phi(y)dy = c + O\bigg(\frac{1}{Y}\int_0^{y_0} (|\phi(y)|+|c|)dy \\ + \frac{1}{Y}\sum_{\lambda_n\le X(Y)} \frac{|r_n|}{\lambda_n} + \frac{1}{Y}\int_{y_0}^Y |\mathcal{E}(y,X(Y))|dy\bigg).$$

The first integral is finite and the last is $o(Y)$ by (3.3). Moreover the sum is finite since

$$\sum_{\lambda_n\le X(Y)} \frac{|r_n|}{\lambda_n} \le \sum_{k=0}^{\infty}\sum_{k<\lambda_n\le k+1} \frac{|r_n|}{\lambda_n} \ll 1+\sum_{k=1}^{\infty}\frac{1}{k^{1+\beta}} < \infty,$$

where we have used in the last inequality the fact that $\beta > 0$ by assumption. This gives

$$\frac{1}{Y}\int_0^Y \phi(y)dy = c + o(1)$$

and proves the lemma. □

Remark 3.8. The assumption of having $\beta > 0$ in (3.4) can be weakened to

$$\sum_{T\le\lambda_n\le T+1} |r_n| \ll f(T), \quad \text{with} \quad \int_1^\infty \frac{f(T)}{T}dT < \infty.$$

We now discuss the existence of the second moment of ϕ. We will from now on center the function ϕ so that it has mean zero, and so we will assume throughout the rest of the chapter that $c = 0$.

To set a notation that will turn out useful later on, we define the following function

$$S(y,X) := 2\Re\left(\sum_{0<\lambda_n\le X} r_n e^{i\lambda_n y}\right), \tag{3.5}$$

so that we have for $y \ge y_0$ and $X \ge X_0$ the relation

$$\phi(y) = S(y,X) + \mathcal{E}(y,X).$$

Lemma 3.9. *Assume that* (3.3) *holds with* $k = 2$ *and* (3.3) *holds with* $\beta > 1/2$. *Then*

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y |\phi(y)|^2 dy = 2\sum_{\lambda_n>0} |r_n|^2. \tag{3.6}$$

Proof. First remark that the infinte sum in (3.6) is convergent, since we can bound

$$\begin{aligned}\sum_{\lambda_n>0} |r_n|^2 &\ll 1+\sum_{\lambda_n>1}|r_n|^2 \ll 1+\sum_{k=1}^{\infty}\sum_{\lambda_n\in(k,k+1]}|r_n|^2 \\ &\ll 1+\sum_{k=1}^{\infty}\bigg(\sum_{\lambda_n\in(k,k+1]}|r_n|\bigg)^2 \ll 1+\sum_{k=1}^{\infty}\frac{1}{k^{2\beta}} < \infty\end{aligned} \tag{3.7}$$

where we use the assumption that (3.4) holds with $\beta > 1/2$ to infer finiteness in the last inequality. Consider now the function $S(y, X)$ defined by (3.5). Then computing the variance of ϕ amounts to compute the variance of $S(y, X)$. Choose indeed $X = X(Y)$ such that (3.3) holds with $k = 2$. If we can prove that

$$\frac{1}{Y}\int_{y_0}^{Y} S(y, X(Y))^2 dy = 2\sum_{\lambda_n>0} |r_n|^2 + o(1) \tag{3.8}$$

then we can write

$$\int_0^Y \phi^2(y)dy = \int_{y_0}^{Y} \big(S(y, X(Y)) + \mathcal{E}(y, X(Y))\big)^2 dy + \int_0^{y_0} \phi^2(y)dy,$$

and expanding the square and using Cauchy-Schwarz inequality on the mixed product we conclude that

$$\frac{1}{Y}\int_0^Y \phi^2(y)dy = \frac{1}{Y}\int_{y_0}^{Y} S(y, X(Y))^2 dy + o(1) = 2\sum_{\lambda_n>0} |r_n|^2 + o(1).$$

The problem is now reduced to prove (3.8). We can expand the square using the definition of $S(y, X)$, and writing $2\Re(z) = z + \bar{z}$ we obtain

$$\begin{aligned} &\frac{1}{Y}\int_{y_0}^{Y} S(y, X(Y))^2 dy \\ &= \frac{1}{Y}\int_{y_0}^{Y} \sum\sum_{0<\lambda_n,\lambda_m\le X(Y)} 2\Re(r_n r_m e^{iy(\lambda_n+\lambda_m)} + r_n\bar{r}_m e^{iy(\lambda_n-\lambda_m)})dy. \end{aligned} \tag{3.9}$$

Since the sum is finite we can interchange integration and summation and integrate termwise. The various exponentials are treated similarly, although there is a difference between the diagonal case, that is $n = m$, and the off-diagonal case $n \neq m$. For $n = m$ we have

$$\frac{1}{Y}\int_{y_0}^{Y} |r_n|^2(2 + 2\Re(e^{2i\lambda_n}))dy = 2|r_n|^2 + O\left(|r_n|^2\left(\frac{1}{Y} + \frac{1}{\lambda_n Y}\right)\right),$$

and using the fact that $\lambda_n > \lambda_1 > 0$ we can simply write

$$\frac{1}{Y}\int_{y_0}^{Y} |r_n|^2(2 + 2\Re(e^{2i\lambda_n}))dy = 2|r_n|^2 + O\left(\frac{|r_n|^2}{Y}\right).$$

For $n \neq m$ we can instead write

$$\left|\frac{1}{Y}\int_{y_0}^{Y} r_n r_m e^{iy(\lambda_n-\lambda_m)}dy\right| \le |r_n r_m| \min\left\{1, \frac{2}{Y|\lambda_n - \lambda_m|}\right\}. \tag{3.10}$$

The minimum is obtained by on one hand bounding the integrand in absolute value and then integrate, while on the other integrating first and then bounding. The same estimate (3.10) holds for the other terms in (3.9) (notice that $|\lambda_n + \lambda_m| \ge |\lambda_n - \lambda_m|$ since $\lambda_n, \lambda_m > 0$). Because of the inequality

$$\min(a^{-1}, b^{-1}) \le 2(a + b)^{-1} \le 2\min(a^{-1}, b^{-1})$$

which holds for every $a, b > 0$, we are allowed to conclude

$$\left| \frac{1}{Y} \int_{y_0}^{Y} 2\Re(r_n r_m e^{iy(\lambda_n+\lambda_m)} + r_n \bar{r}_m e^{iy(\lambda_n-\lambda_m)}) dy \right| \leq \frac{4|r_n r_m|}{1+Y|\lambda_n-\lambda_m|}.$$

Summarizing, we have

$$\begin{aligned} &\frac{1}{Y} \int_{y_0}^{Y} \sum_{0<\lambda_n,\lambda_m \leq X(Y)} 2\Re(r_n r_m e^{iy(\lambda_n+\lambda_m)} + r_n \bar{r}_m e^{iy(\lambda_n-\lambda_m)}) dy \\ &= \sum_{0<\lambda_n \leq X(Y)} |r_n|^2 \left(2 + O\left(\frac{1}{Y}\right)\right) + O\left(\sum_{\substack{0<\lambda_n,\lambda_m \leq X(Y) \\ \lambda_n \neq \lambda_m}} \sum \frac{|r_n r_m|}{1+Y|\lambda_n-\lambda_m|} \right) \end{aligned} \tag{3.11}$$

The first sum has already been analyzed in (3.7). In order to analyze the sum over the off-diagonal $n \neq m$ we apply a technique used by Cramér in [18]. Observe first that by symmetry we can write

$$T_Y := \sum_{\substack{0<\lambda_n,\lambda_m \leq X(Y) \\ \lambda_n \neq \lambda_m}} \sum \frac{|r_n r_m|}{1+Y|\lambda_n-\lambda_m|} = 2 \sum_{\substack{\lambda_n \leq X(Y) \\ \lambda_n < \lambda_m \leq X(Y)}} \sum \frac{|r_n r_m|}{1+Y|\lambda_n-\lambda_m|}. \tag{3.12}$$

If we can show that the sum is uniformly bounded as $Y \to \infty$, then by applying the Lebesgue dominated convergence theorem we can conclude that $T_Y \to 0$ as $Y \to \infty$. It is therefore sufficient to show that the sum

$$T := \sum_{\lambda_n > 0} \sum_{\lambda_m > \lambda_n} \frac{|r_n r_m|}{1+|\lambda_n-\lambda_m|}$$

is finite. Splitting the sum we can bound

$$\begin{aligned} T &\ll \sum_{\lambda_n>0} |r_n| \sum_{\lambda_m \in (\lambda_n, \lambda_n+1]} |r_m| \\ &\quad + \sum_{\lambda_n>0} |r_n| \sum_{\ell=1}^{\infty} \sum_{\lambda_m \in (\lambda_n+\ell, \lambda_n+\ell+1]} \frac{|r_m|}{|\lambda_n-\lambda_m|} \\ &\ll \sum_{\lambda_n>0} \frac{|r_n|}{\lambda_n^{\beta}} + \sum_{\lambda_n>0} |r_n| \sum_{\ell=1}^{\infty} \frac{1}{\ell(\lambda_n+\ell)^{\beta}} \end{aligned}$$

and splitting now the inner sum over ℓ at height λ_n we obtain

$$\begin{aligned} T &\ll \sum_{\lambda_n>0} \frac{|r_n|}{\lambda_n^{\beta}} + \sum_{\lambda_n>0} |r_n| \left(\sum_{1 \leq \ell \leq \lambda_n} \frac{1}{\ell\, \lambda_n^{\beta}} + \sum_{\ell > \lambda_n} \frac{1}{\ell^{1+\beta}} \right) \\ &\ll \sum_{\lambda_n>0} \frac{|r_n| \log(\lambda_n+2)}{\lambda_n^{\beta}} \\ &\ll 1 + \sum_{k=1}^{\infty} \sum_{\lambda_n \in (k,k+1]} \frac{|r_n| \log(\lambda_n+2)}{\lambda_n^{\beta}} \\ &\ll 1 + \sum_{k=1}^{\infty} \frac{\log(k+3)}{k^{2\beta}} < \infty. \end{aligned}$$

This proves that T_Y of (3.12) tends to zero as $Y \to \infty$. The right hand side in (3.11) becomes then

$$2 \sum_{0<\lambda_n \le X(Y)} |r_n|^2 + o(1) = 2 \sum_{\lambda_n > 0} |r_n|^2 + o(1)$$

where we have used that the tail of the series tends to zero as $Y \to \infty$ because the sum of the series is finite. This proves (3.8), and taking the limit as $Y \to \infty$ we obtain

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \phi(y)^2 dy = 2 \sum_{\lambda_n > 0} |r_n|^2$$

which is the claim. □

3.2 Limiting Distributions

In this section we discuss the existence of limiting distribution for an almost periodic function ϕ, and we prove upper bounds on the tails of the distribution in terms of the decay of the coefficients r_n associated with ϕ.

Definition 3.10. A function $\phi : [0, \infty) \to \mathbb{R}$ is said to admit a limiting distribution μ if there exists a probability measure μ on $\mathbb{R}$ such that the limit

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y g(\phi(y))dy = \int_{\mathbb{R}} g d\mu \tag{3.13}$$

holds for every bounded continuous function g on $\mathbb{R}$.

Remark 3.11. We can rephrase (3.13) in terms of weak convergence of measures: let λ be the Lebesgue measure, and consider the measure

$$\nu_Y = \frac{\mathbf{1}_{[0,Y]}}{Y} \lambda$$

so that for any measurable set A we have

$$\nu_Y(A) = \frac{1}{Y} \lambda(A \cap [0, Y]).$$

The image measure $\mu_Y = \phi_*(\nu_Y)$ is defined by

$$\mu_Y(B) = \frac{1}{Y} \lambda(\phi^{-1}(B) \cap [0, Y]) = \frac{1}{Y} \lambda(\{x \in [0, Y] : \phi(x) \in B\})$$

and condition (3.13) reads

$$\lim_{Y \to \infty} \int_{\mathbb{R}} g d\mu_Y = \lim_{Y \to \infty} \frac{1}{Y} \int_0^Y g(\phi(y))dy = \int_{\mathbb{R}} g d\mu$$

for every bounded continuous function g on $\mathbb{R}$. In other words, (3.13) is equivalent to requesting that the measures μ_Y converge weakly to μ as $Y \to \infty$.

Remark 3.12. Weak convergence of measures can be tested in certain cases on indicator functions of intervals instead that on bounded continuous functions g as in (3.13). By Portmanteau's theorem [42, Theorem 13.16], this is an equivalent problem if the measures satisfy

$$\lim_{Y \to \infty} \mu_Y(A) = \mu(A)$$

for every continuity set A of μ, that is a set such that $\mu(\partial A) = 0$. The convergence on intervals can be easier to visualize, and we give two examples below.

Example 3.13. Consider the function

$$E(x) = -\left(\sum_{1\le n\le x} 1 - x\right) = \{x\}.$$

Since E is bounded and attains values in $[0,1]$, for every subset $A \subseteq \mathbb{R}$ such that $A \cap [0,1] = \emptyset$ we will have

$$\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y \mathbf{1}_A(E(y))dy = 0.$$

On the other hand, for an interval $A = [a,b] \subseteq [0,1]$ we will have

$$\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y \mathbf{1}_A(E(y))dy = (b-a) = \lambda(A).$$

The limiting distribution of E is thus the Lebesgue measure λ on $[0,1]$ (and zero elsewhere). The convergence can also be checked for bounded continuous functions g as requested in (3.13).

Remark 3.14. Another method to show that the limiting distribution of E is the Lebesgue measure on $[0,1]$ is the method of moments: if the moments

$$s_k := \lim_{Y\to\infty} \frac{1}{Y}\int_0^Y E(y)^k dy$$

exist for every $k \in \mathbb{N}$, and if they determine uniquely a measure μ, then μ will be the limiting distribution of E. In this case the moments of E are given by

$$s_k = \lim_{Y\to\infty} \frac{1}{Y}\int_0^Y E(y)^k dy = \frac{1}{k+1}$$

and they determine uniquely (see e.g. [3, Theorem 2.1.7]) the measure λ on $[0,1]$.

Example 3.15. Consider now the function $\phi(x)$ defined by

$$\phi(x) = \begin{cases} n & x \in [n-1/3, n+1/3],\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that ϕ doesn't admit a limiting distribution. Let g be a bounded continuous function with compact support contained in $[a,b]$ and with $0 < a < b < \infty$. Then we have

$$\Big|\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y g(\phi(y))dy\Big| \le \lim_{Y\to\infty} \frac{\|g\|_\infty}{Y}\sum_{1\le n\le b}\frac{2}{3} = 0. \tag{3.14}$$

Let now $a > 0$ and consider the bounded continuous function

$$g_a(x) = \begin{cases} 0 & x \le a, \\ 2(x-a) & x \in [a, a+1/2], \\ 1 & x \ge a+1/2. \end{cases}$$

Then we have

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y g_a(\phi(y))dy = \lim_{Y\to\infty}\frac{1}{Y}\left(\sum_{a\le n\le Y}\frac{2}{3}+O(1)\right)=\frac{2}{3}. \tag{3.15}$$

Suppose that ϕ admits a limiting distribution μ. Then for every $n \ge 1$ let $f_n(x)$ be the bounded continuous function

$$f_n(x)=\begin{cases}0 & x\in(-\infty,n-1/2]\cup[n+3/2,+\infty),\\ 2-2|x-n-1/2| & x\in[n-1/2,n]\cup[n+1,n+3/2],\\ 1 & x\in[n,n+1).\end{cases}$$

Let A_n be the interval $[n, n+1)$. Then we have, using (3.14),

$$\mu(A_n)=\int_{\mathbb{R}}\mathbf{1}_{A_n}d\mu \le \int_{\mathbb{R}} f_n d\mu = \lim_{Y\to\infty}\frac{1}{Y}\int_0^Y f_n(\phi(y))dy = 0.$$

On the other hand, taking the set $B = [1,\infty)$ we get, using (3.15),

$$\mu(B)\ge\int_{\mathbb{R}} g_1(\phi(y))dy = \lim_{Y\to\infty}\frac{1}{Y}\int_0^Y g_1(\phi(y))dy=\frac{2}{3}.$$

Since the A_n are disjoint and their union is B, we get the contradiction

$$\frac{2}{3}\le\mu(B)=\mu\big(\bigcup_{n\ge1}A_n\big)=\sum_{n\ge1}\mu(A_n)=0.$$

This shows that ϕ doesn't admit a limiting distribution in the sense of (3.13). As a remark notice that all the moments of ϕ of order $k \ge 2$ are infinite, since

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y \phi(y)^k dy = \lim_{Y\to\infty}\frac{2}{3Y}\left(\sum_{1\le n\le Y} n^k + O(Y^k)\right) = +\infty.$$

Theorem 3.16. *Assume that the* (3.3) *and* (3.4) *are satisfied respectively with* $k = 2$ *and* $\beta > 1/2$. *Then* ϕ *admits a limiting distribution* μ *with tails of size*

$$\mu((-\infty,-S]\cup[S,+\infty)) \ll S^{-(2\beta-1)/(2-2\beta)}. \tag{3.16}$$

For $\beta = 1$ *we have exponential decay, and for* $\beta > 1$ *the measure* μ *is compactly supported.*

Remark 3.17. The proof of the existence of the measure μ can be found in several papers (see e.g. [29, 5, 56, 1]). The estimate on the tails for general β doesn't seem however to appear in these papers. We report here the proof following the argument of [56, p.178-181], which gives the opportunity of getting (3.16).

We start with two preparatory lemmata. Let $T \ge 2$ and consider the functions

$$\phi_T(y)=2\Re\Big(\sum_{0<\lambda_n\le T} r_n e^{i\lambda_n y}\Big),\quad \psi_T(y,X)=2\Re\Big(\sum_{T<\lambda_n\le X} r_n e^{i\lambda_n y}\Big)+\mathcal{E}(y,X).$$

Lemma 3.18. *Assume the hypothesis of Theorem 3.16. Then for $Y \gg 1$*

$$\frac{1}{Y}\int_{y_0}^{Y} |\psi_T(y, X(Y))|^2 dy \ll \frac{1}{T^{2\beta-1}}.$$

Proof. We have

$$\begin{aligned}
&\frac{1}{Y}\int_{y_0}^{Y} \big|2\Re\big(\sum_{T<\lambda_n\le X(Y)} r_n e^{i\lambda_n y}\big)\big|^2 dy \\
&\ll \sum_{T\le\lambda_n\le X(Y)} |r_n|^2\left(1+\frac{1}{\lambda_n Y}\right) + \sum_{\substack{T\le\lambda_n,\lambda_m\le X\\ \lambda_n\ne\lambda_m}} \frac{|r_n r_m|}{1+Y|\lambda_n-\lambda_m|} \\
&\ll \sum_{T\le\lambda_n\le X(Y)} \frac{|r_n|}{\lambda_n^{\beta}}\left(1+\frac{1}{TY}\right) + \sum_{T\le\lambda_n\le X(Y)} \frac{|r_n|\log(\lambda_n)}{Y\lambda_n^{2\beta}} \\
&\ll \frac{1}{T^{2\beta-1}} + \frac{1}{T^{2\beta}Y} + \frac{\log T}{YT^{2\beta-1}}.
\end{aligned}$$

Using then Cauchy-Schwarz to include the contribution from $\mathcal{E}(y, X(Y))$ we obtain

$$\frac{1}{Y}\int_{y_0}^{Y} |\psi_T(y, X(Y))|^2 dy \ll \frac{1}{T^{2\beta-1}}$$

which is the statement. □

Lemma 3.19. *For each $T \ge 2$ there exists a probability measure ν_T on $\mathbb{R}$ such that*

$$\nu_T(f) := \int_{\mathbb{R}} f(x)d\nu_T(x) = \lim_{Y\to\infty}\frac{1}{Y}\int_{y_0}^{Y} f(\phi_T(y))dy$$

for every bounded Lipschitz continuous function f on $\mathbb{R}$. In addition, there is a constant $c > 0$ such that the support of ν_T lies in the ball $B(0, cT^{1-\beta})$ for $\beta < 1$, and $B(0, c\log T)$ for $\beta = 1$. For $\beta > 1$ the support of ν_T is compact.

Proof. The existence of the measure is [56, Lemma 2.3]. The statement about the support of ν_T follows from the fact that

$$|\phi_T(y)| \ll T^{1-\beta}, \quad |\phi_T(y)| \ll \log T, \quad |\phi_T(y)| \ll 1$$

respectively for $\beta < 1$, $\beta = 1$, and $\beta > 1$. □

Proof (of Theorem 3.16). Consider a bounded Lipschitz continuous function f, with Lipschitz constant c_f, so that

$$|f(x) - f(y)| \le c_f|x-y|. \tag{3.17}$$

Then we have for $Y \gg 1$

$$\begin{aligned}
\frac{1}{Y}\int_{y_0}^{Y} \big(f(\phi(y)) - f(\phi_T(y))\big)dy &\ll c_f\big(\frac{1}{Y}\int_{y_0}^{Y} |\psi_T(y, X(Y))|^2 dy\big)^{1/2} \\
&\ll \frac{c_f}{T^{\beta-1/2}}
\end{aligned}$$

so that taking the limit as $Y \to \infty$ we obtain

$$\nu_T(f) - O\left(\frac{c_f}{T^{\beta-1/2}}\right) \le \liminf_{Y\to\infty} \frac{1}{Y}\int_{y_0}^{Y} f(\phi(y))dy$$
$$\le \limsup_{Y\to\infty} \frac{1}{Y}\int_{y_0}^{Y} f(\phi(y))dy \le \nu_T(f) + O\left(\frac{c_f}{T^{\beta-1/2}}\right).$$

Since T can be arbitrarily large, we conclude that the lim inf and lim sup coincide, i.e. that

$$\mu(f) := \lim_{Y\to\infty} \frac{1}{Y}\int_{y_0}^{Y} f(\phi(y))dy \tag{3.18}$$

exists. Thus there exists a Borel measure μ on $\mathbb{R}$ such that (3.18) holds for all f satisfying (3.17). Moreover, for such f's,

$$|\mu(f) - \nu_T(f)| \le \frac{c_f}{T^{\beta-1/2}}. \tag{3.19}$$

In view of Lemma 3.19 and (3.19) we also have for $\beta < 1$

$$\mu(B_\lambda^c)) = \nu_T(B_\lambda^c) + O\left(\frac{1}{T^{\beta-1/2}}\right) = O\left(\frac{1}{T^{\beta-1/2}}\right) \tag{3.20}$$

for $\lambda = cT^{1-\beta}$ (B_λ^c is the complement of the open ball of radius λ). This leads to

$$\mu(B_\lambda^c) = O(\lambda^{-(2\beta-1)/(2-2\beta)}).$$

In the case when $\beta = 1$ we get to substitute $\lambda = c\log T$ in (3.20), which gives $\mu(B_\lambda^c) = O(e^{-\lambda/2c})$. Finally, for $\beta > 1$ the compactness of the support of μ follows from the fact that ϕ_T is bounded. □

Remark 3.20. In the case of the prime number theorem and L-functions discussed in [56], since they have

$$\sum_{T\le\lambda_n\le T+1} |r_n| \ll \frac{\log T}{T} \tag{3.21}$$

they get exponential decay of type $O(\exp(-c\sqrt{\lambda}))$. Similarly, a bound in (3.21) of type $O(T^{-1}\log^m(T))$, $m \ge 0$, leads to an upper bound for the tails of μ of type $O(\exp(-c\lambda^{1/(m+1)}))$.

3.3 Higher moments

In view of the relation $\phi = S + \mathcal{E}$ as in (3.2), and the assumption (3.3) that $\mathcal{E}$ is "small", we expect that the moments of ϕ can be computed simply by computing the moments of its almost periodic expansion $S(y, X)$. This is the case for the first and second moment, as it has been shown in the section 3.1. In this section we show that the same occurs for higher moments. We start with two technical lemmata that will be used in the proof of the existence of higher moments of S.

Lemma 3.21. *Let λ_n be as in* (3.1)*, and assume that* (3.4) *holds with* $\beta \leq 1$*. Let* $0 < \alpha \leq 1$*,* $a \geq 0$*,* $B \gg 1$*, and* $\mu > 0$*. Then*

$$\sum_{0<\lambda_n<\mu} |r_n| \frac{\log^a(\mu - \lambda_n + 2B)}{(\mu - \lambda_n + B + 1)^\alpha} \ll \frac{\log^{a+1}(\mu + 2B)}{(\mu + B + 1)^{\alpha+\beta-1}}. \tag{3.22}$$

Lemma 3.22. *Let λ_n be as in* (3.1)*, and assume that* (3.4) *holds with* $\beta \leq 1$*. Let* $\alpha > 0$ *such that* $\alpha + \beta > 1$*,* $a \geq 0$*,* $B \gg 1$*, and* $\mu > 0$*. Then*

$$\sum_{\lambda_n>0} |r_n| \frac{\log^a(\mu + \lambda_n + 2B)}{(\mu + \lambda_n + B + 1)^\alpha} \ll \frac{\log^{a+1}(\mu + 2B)}{(\mu + B + 1)^{\alpha+\beta-1}}.$$

Remark 3.23. The terms $2B$ and $B+1$ appearing are rather flexible, and they are present only to avoid the situation where the argument of the logarithm or the denominator vanish.

Proof of Lemma 3.21. First observe that (3.4) implies, for fixed $B \gg 1$,

$$\sum_{T\leq\lambda_n\leq T+1} |r_n| \ll \frac{1}{(T+B)^\beta} \quad \forall\, T \geq 0.$$

Assume for now that both $\alpha, \beta < 1$. Write the sum in (3.22) in the form

$$\begin{aligned}
\sum_{0<\lambda_n<\mu} |r_n| \frac{\log^a(\mu - \lambda_n + 2B)}{(\mu - \lambda_n + B + 1)^\alpha} &= \sum_{k=0}^{\lfloor\mu\rfloor} \sum_{\substack{\lambda_n\in(0,\mu)\\ k<\lambda_n\leq k+1}} |r_n| \frac{\log^a(\mu - \lambda_n + 2B)}{(\mu - \lambda_n + B + 1)^\alpha} \\
&\ll \sum_{k=0}^{\lfloor\mu\rfloor} \frac{\log^a(\mu - k + 2B)}{(\mu - k + B)^\alpha (k+B)^\beta} \\
&\ll \log^a(\mu + 2B) \sum_{k=0}^{\lfloor\mu\rfloor} \frac{1}{(\mu - k + B)^\alpha (k+B)^\beta}
\end{aligned} \tag{3.23}$$

Write now $\lfloor\mu\rfloor = 2k_1 + \varepsilon$ for $\varepsilon \in \{0, 1\}$. Then we have

$$\frac{1}{2}(\mu - 2) \leq \frac{1}{2}(\lfloor\mu\rfloor - 1) \leq k_1 \leq \frac{1}{2}\lfloor\mu\rfloor \leq \frac{1}{2}\mu \leq \mu$$

and we split the last sum in (3.23) in two parts S_1 and S_2, respectively over $[0, k_1]$ and $[k_1 + 1, \lfloor\mu\rfloor\,]$. Let us analyze the first sum. We have

$$\begin{aligned}
S_1 &= \sum_{k=0}^{k_1} \frac{1}{(\mu + B - k)^\alpha} \frac{1}{(k+B)^\beta} \\
&\leq \frac{1}{(\mu + B - k_1)^\alpha} \left(\frac{(k_1 + B)^{1-\beta}}{(1-\beta)} + O(1) \right)
\end{aligned} \tag{3.24}$$

Since the power $1 - \beta$ is positive by the assumption that $\beta < 1$, we can simply estimate the big parenthesis by a constant times $(k_1 + B)^{1-\beta}$. We have therefore

$$S_1 \ll \frac{(k_1 + B)^{1-\beta}}{(\mu + B - k_1)^\alpha} \ll \frac{1}{(\mu + B + 1)^{\alpha+\beta-1}}.$$

Consider now the second sum S_2 over $[k_1+1, \lfloor \mu \rfloor\,]$. We have

$$S_2 = \sum_{k_1+1}^{\lfloor \mu \rfloor} \frac{1}{(\mu+B-k)^{\alpha}} \frac{1}{(k+B)^{\beta}} \le \frac{1}{(k_1+1+B)^{\beta}} \sum_{k_1+1}^{\lfloor \mu \rfloor} \frac{1}{(\mu+B-k)^{\alpha}}. \quad (3.25)$$

Make the change of variable $\ell = \lfloor \mu \rfloor - k$, so $\ell \in [0, \lfloor \mu \rfloor - k_1 - 1]$. We can write the sum as

$$\begin{aligned}\sum_{\ell=0}^{\lfloor \mu \rfloor - k_1 - 1} \frac{1}{(\mu+B-\lfloor \mu \rfloor + \ell)^{\alpha}} &\le \sum_{\ell=0}^{\lfloor \mu \rfloor - k_1 - 1} \frac{1}{(\ell+B)^{\alpha}} \\ &= \frac{(\lfloor \mu \rfloor - k_1 + B - 1)^{1-\alpha}}{(1-\alpha)} + O(1).\end{aligned}$$

Since the exponent $1-\alpha$ is positive by the assumption $\alpha < 1$, we can simply bound everything by a constant times $(\lfloor \mu \rfloor - k_1 + B - 1)^{1-\alpha}$. We obtain

$$S_2 \ll \frac{(\lfloor \mu \rfloor - k_1 + B - 1)^{1-\alpha}}{(k_1+1+B)^{\beta}} \ll \frac{1}{(\mu+B+1)^{\alpha+\beta-1}}$$

Adding together S_1 and S_2 and using (3.23), we obtain for $\alpha, \beta < 1$ the bound

$$\sum_{0<\lambda_n<\mu} |r_n| \frac{\log^a(\mu-\lambda_n+2B)}{(\mu-\lambda_n+B+1)^{\alpha}} \ll \frac{\log^a(\mu+2B)}{(\mu+B+1)^{\alpha+\beta-1}}$$

When $\alpha = 1$ or $\beta = 1$ minor modifications have to be done. In these cases we estimate S_1 and S_2 in (3.24) and (3.25) by

$$S_1 \ll \frac{\log(\mu+2B)}{(\mu+B+1)^{\beta}} \quad \text{and} \quad S_2 \ll \frac{\log(\mu+2B)}{(\mu+B+1)^{\beta}}$$

and thus we conclude that when $\alpha = 1$ or $\beta = 1$ we have

$$\sum_{0<\lambda_n<\mu} |r_n| \frac{\log^a(\mu-\lambda_n+2B)}{(\mu-\lambda_n+B+1)^{\alpha}} \ll \frac{\log^{a+1}(\mu+2B)}{(\mu+B+1)^{\beta}}.$$

This concludes the proof of Lemma 3.21. □

Proof of Lemma 3.22. Assume first that $\beta < 1$. Write the sum as

$$\sum_{k=0}^{\infty} \sum_{\lambda_n \in (k,k+1]} |r_n| \frac{\log^a(\mu+\lambda_n+2B)}{(\mu+\lambda_n+B+1)^{\alpha}} \le \sum_{k=0}^{\infty} \frac{\log^a(\mu+2B+k+1)}{(\mu+k+B+1)^{\alpha}(k+B)^{\beta}}. \quad (3.26)$$

Let $k_1 = \lfloor \mu \rfloor$ so that $\mu - 1 < k_1 \le \mu \le k_1 + 1$, and split the last sum in (3.26) at height k_1. This defines a finite sum S_1 over $0 \le k \le k_1$ and an infinite sum S_2 over $k \ge k_1 + 1$. Consider the sum S_1. We have

$$\begin{aligned} S_1 &= \sum_{k=0}^{k_1} \frac{\log^a(\mu+2B+k+1)}{(\mu+k+B+1)^{\alpha}(k+B)^{\beta}} \\ &\le \frac{\log^a(\mu+2B+k_1+1)}{(\mu+B+1)^{\alpha}} \sum_{k=0}^{k_1} \frac{1}{(k+B)^{\beta}} \\ &\ll \frac{\log^a(2\mu+2B+1)}{(\mu+B+1)^{\alpha}} \left(\frac{(k_1+B)^{1-\beta}}{(1-\beta)} + O(1) \right) \\ &\le \frac{\log^a(2\mu+2B+1)}{(\mu+B+1)^{\alpha+\beta-1}}. \end{aligned}$$

Consider now the sum S_2. We have

$$\begin{aligned} S_2 &= \sum_{k=k_1+1}^{\infty} \frac{\log^a(\mu+2B+k+1)}{(\mu+k+B+1)^\alpha (k+B)^\beta} \\ &\leq \sum_{k=k_1+1}^{\infty} \frac{\log^a(2k+2B+2)}{(k+B)^{\alpha+\beta}} \\ &\approx \int_{k_1+1}^{+\infty} \frac{\log^a(2u+2B+2)}{(u+B)^{\alpha+\beta}} du. \end{aligned}$$

For $\theta \geq 0$, $\varepsilon > 0$, $B \gg 1$, and $X \geq 1$, we have

$$\int_X^{+\infty} \frac{\log^\theta(2u+2B+2)}{(u+B)^{1+\varepsilon}} \ll \frac{\log^\theta(2X+2B+2)}{(X+B)^\varepsilon}$$

as $X \to \infty$. We conclude that

$$S_2 \ll \frac{\log^a(2k_1+2B+4)}{(k_1+B+1)^{\alpha+\beta-1}} \ll \frac{\log^a(\mu+2B)}{(\mu+B+1)^{\alpha+\beta-1}}$$

Putting together the two estimates for S_1 and S_2 we obtain for $\beta < 1$

$$\sum_{\lambda_n>0} |r_n| \frac{\log^a(\mu+\lambda_n+2B)}{(\mu+\lambda_n+B+1)^\alpha} \ll \frac{\log^a(\mu+2B)}{(\mu+B+1)^{\alpha+\beta-1}}.$$

In the case $\beta = 1$ we have the two estimates

$$S_1 \ll \frac{\log^{a+1}(\mu+2B)}{(\mu+B+1)^\alpha}, \quad S_2 \ll \frac{\log^a(\mu+2B)}{(\mu+B+1)^\alpha}$$

and so we conclude that

$$\sum_{\lambda_n>0} |r_n| \frac{\log^a(\mu+\lambda_n+2B)}{(\mu+\lambda_n+B+1)^\alpha} \ll \frac{\log^{a+1}(\mu+2B)}{(\mu+B+1)^\alpha}.$$

This concludes the proof of Lemma 3.22. □

We study now the moments of the function $S(y, X)$. We start by introducing the following notation. Let $G = \{+1, -1\}$ be the cyclic group with two elements. For $z \in \mathbb{C}$ and $g \in G$ we write

$$g(z) = \begin{cases} z & \text{if } g = 1 \\ \bar{z} & \text{if } g = -1. \end{cases}$$

In other words $g = 1$ acts on the complex plane as the identity, and $g = -1$ acts by complex conjugation. Consider the product

$$G^n = \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}.$$

An element $g \in G^n$ is a n-tuple $g = (g_1, \ldots, g_n)$ of entries equal to ± 1. Using this notation we can write

$$S(y, X)^n = \sum_{g \in G^n} \sum_{0<\lambda_{j_1},\ldots,\lambda_{j_n} \leq X} g_1(r_{j_1}) \cdots g_n(r_{j_n}) e^{y(g_1(i\lambda_{j_1})+\cdots+g_n(i\lambda_{j_n}))}. \quad (3.27)$$

Define for $g \in G^n$, $J = (j_1, \dots, j_n) \in \mathbb{N}^n$ a multiindex, and $\lambda_J = (\lambda_{j_1}, \dots, \lambda_{j_n})$, $r_J = (r_{j_1}, \dots, r_{j_n})$, the two quantities

$$A_g(r_J) := \prod_{s=1}^{n} g_s(r_{j_s}), \quad i\vartheta_g(\lambda_J) := \sum_{s=1}^{n} g_s(i\lambda_{j_s}). \tag{3.28}$$

Then (3.27) can be rewritten in the more compact form

$$S(y, X)^n = \sum_{g \in G^n} \sum_{\lambda_J \le X} A_g(r_J) e^{iy\vartheta_g(\lambda_J)},$$

where $\lambda_J \le X$ means that all the entries of λ_J are smaller than X. Setting $X = X(Y)$ and integrating in y we get

$$\frac{1}{Y} \int_{y_0}^{Y} S(y, X(Y))^n dy = \sum_{g \in G^n} \sum_{\lambda_J \le X(Y)} A_g(r_J) \frac{1}{Y} \int_{y_0}^{Y} e^{iy\vartheta_g(\lambda_J)} dy. \tag{3.29}$$

The integral can be bounded for every $g \in G^n$, by

$$\left| \frac{1}{Y} \int_{y_0}^{Y} e^{iy\vartheta_g(\lambda_J)} dy \right| \le \frac{2}{1 + Y|\vartheta_g(\lambda_J)|}. \tag{3.30}$$

For $g \in G^n$ define the sets

$$\mathscr{D}_g^Y := \{\lambda_J \le X(Y) \mid \vartheta_g(\lambda_J) = 0\}$$
$$\mathscr{O}_g^Y := \{\lambda_J \le X(Y) \mid \vartheta_g(\lambda_J) \neq 0\}.$$

For instance, for $g = e = (1, 1, \dots, 1)$, since $\lambda_{k_1}, \dots, \lambda_{k_n} > 0$, we have $\mathscr{D}_e^Y = \emptyset$ and $\mathscr{O}_e^Y = \{\lambda_{k_1}, \dots, \lambda_{k_n} \le X(Y)\}$. Similarly for $g = -e = (-1, \dots, -1)$ we have $\mathscr{D}_{-e}^Y = \emptyset$ and $\mathscr{O}_{-e}^Y = \{\lambda_{k_1}, \dots, \lambda_{k_n} \le X(Y)\}$.

We are almost ready to prove finiteness of higher moments of $S(y, X)$. We give a preparatory lemma that will help in the proof of Proposition 3.25 below. For $0 \le s \le n$ let $\mathcal{O}_s$ be the set

$$\mathcal{O}_s := \{\lambda_J \mid \lambda_{j_1} + \dots + \lambda_{j_s} - \lambda_{j_{s+1}} - \dots - \lambda_{j_n} \neq 0\}. \tag{3.31}$$

For every fixed $g \in G^n$, the exists a permutation σ on n elements such that the first s entries of $\sigma(g) \in G^n$ are $+1$ and the last $n - s$ entries are -1. The permutation σ gives then a bijection between the set $\mathscr{O}_g^\infty$ and the set $\mathscr{O}_{\sigma(g)}^\infty = \mathcal{O}_s$. This observation will be useful in the proof of Proposition 3.25. For now we prove the following lemma regarding the sets $\mathcal{O}_s$.

Lemma 3.24. *Let* λ_j *be as in* (3.1)*, and assume that* (3.4) *holds with* $\beta > 1 - 1/n$ *for some* $n \ge 2$*. Let* $\mathcal{O}_s$ *be the set defined in* (3.31)*, for* $0 \le s \le n$*. Then the sum*

$$\sum_{s=0}^{n} \sum_{\mathcal{O}_s} \frac{|r_{j_1} \cdots r_{j_n}|}{1 + Y|\lambda_{j_1} + \dots + \lambda_{j_s} - \lambda_{j_{s+1}} - \dots - \lambda_{j_n}|} \tag{3.32}$$

is uniformly bounded for $Y \ge 1$*.*

Proof. Observe that $\mathcal{O}_s = \mathcal{O}_s^+ \cup \mathcal{O}_s^-$, where the sets $\mathcal{O}_s^\pm$ are defined for $0 \le s \le n$ by

$$\mathcal{O}_s^+ = \{\lambda_J \mid \lambda_{j_1} + \cdots + \lambda_{j_s} - \lambda_{j_{s+1}} - \cdots - \lambda_{j_n} > 0\},$$
$$\mathcal{O}_s^- = \{\lambda_J \mid \lambda_{j_1} + \cdots + \lambda_{j_s} - \lambda_{j_{s+1}} - \cdots - \lambda_{j_n} < 0\}.$$

Moreover $\mathcal{O}_s^- = \mathcal{O}_{n-s}^+$, hence if we can deal with sums over sets of type $\mathcal{O}_s^+$ for every $s = 0, \ldots, n$, then we can deal with the sum in (3.32). Since $\mathcal{O}_0^+ = \emptyset$, we in fact only have to deal with $s = 1, \ldots, n$. It is then sufficient to prove that the sum

$$\mathcal{T} = \sum_{\mathcal{O}_s^+} \frac{|r_{j_1} \cdots r_{j_n}|}{1 + |\lambda_{j_1} + \cdots + \lambda_{j_s} - \lambda_{j_{s+1}} - \cdots - \lambda_{j_n}|} \tag{3.33}$$

is finite, for every $1 \le s \le n$. Define μ_{n-1} to be

$$\mu_{n-1} = \lambda_{j_1} + \cdots + \lambda_{j_s} - \lambda_{j_{s+1}} - \cdots - \lambda_{j_{n-1}}$$

so that the sum is shortly written as

$$\mathcal{T} = \sum_{\mu_{n-1} > \lambda_{j_n}} \frac{|r_{j_1} \cdots r_{j_n}|}{1 + |\mu_{n-1} - \lambda_{j_n}|}.$$

Observe now that for $\lambda_n < \mu_{n-1}$ we can write for any fixed $B \gg 1$

$$\frac{1}{1 + |\mu_{n-1} - \lambda_{j_n}|} \ll \frac{1}{\mu_{n-1} - \lambda_{j_n} + B + 1},$$

and therefore we can bound

$$\mathcal{T} \ll \sum_{\mu_{n-1} > 0} |r_{j_1} \cdots r_{j_{n-1}}| \sum_{0 < \lambda_n < \mu_{n-1}} \frac{|r_{j_n}|}{\mu - \lambda_n + B + 1}.$$

We can apply Lemma 3.21 (with $\alpha = 1$) to the inner sum and obtain

$$\mathcal{T} \ll \sum_{\mu_{n-1} > 0} |r_{j_1} \cdots r_{j_{n-1}}| \frac{\log(\mu_{n-1} + 2B)}{(\mu_{n-1} + B + 1)^{\beta}}. \tag{3.34}$$

The sum in (3.34) looks like that in (3.33), but we have reduced the problem to the $(n-1)$-tuples $(\lambda_{j_1}, \ldots, \lambda_{j_{n-1}})$ such that

$$\lambda_{j_1} + \cdots + \lambda_{j_r} - \lambda_{j_{r+1}} - \cdots - \lambda_{j_{n-1}} > 0.$$

We can therefore repeat the argument and reduce the problem to a sum over $(n$–$2)$-tuples, and so on until we arrive, after an application of Lemma 3.21 a total number of $n - s$ times, at the estimate

$$\mathcal{T} \ll \sum_{\lambda_{j_1} + \cdots + \lambda_{j_s} > 0} |r_{j_1} \cdots r_{j_s}| \frac{\log^{n-s}(\lambda_{j_1} + \cdots + \lambda_{j_s} + 2B)}{(\lambda_{j_1} + \cdots + \lambda_{j_s} + B + 1)^{(n-s)\beta - (n-s-1)}}.$$

Define now $\mu_{s-1} = \lambda_{j_1} + \cdots + \lambda_{j_{s-1}}$. Then $\mathcal{T}$ is bounded by

$$\mathcal{T} \ll \sum_{\mu_{r-1} > 0} |r_{j_1} \cdots r_{j_{s-1}}| \sum_{\lambda_{j_s} > 0} |r_s| \frac{\log^{n-s}(\mu_{s-1} + \lambda_{j_s} + 2B)}{(\mu_{s-1} + \lambda_{j_s} + B + 1)^{(n-s)\beta - (n-s-1)}}.$$

We can apply Lemma 3.22 and obtain the estimate

$$\mathcal{T} \ll \sum_{\mu_{s-1}>0} |r_{j_1} \cdots r_{j_{s-1}}| \frac{\log^{n-s+1}(\mu_{s-1} + 2B)}{(\mu_{s-1} + B + 1)^{(n-s+1)\beta-(n-s)}}.$$

Repeating the argument $s-2$ more times we arrive at

$$\mathcal{T} \ll \sum_{\lambda_{j_1}>0} |r_{j_1}| \frac{\log^{n-1}(\lambda_{j_1} + 2B)}{(\lambda_{j_1} + B + 1)^{(n-1)\beta-(n-2)}}$$

Since we have, in view of the assumption $\beta > 1 - 1/n$,

$$(n-1)\beta - (n-2) + \beta > 1$$

we can apply Lemma 3.22 one last time. This gives $\mathcal{T} < \infty$. Since this is true for every $1 \le s \le n$, we conclude. □

Proposition 3.25. *Let $S(y, X)$ be as in* (3.5)*, and assume that* (3.3) *holds for $k = n$, and* (3.4) *holds with $\beta > 1 - 1/n$. Then the limit*

$$L_n := \lim_{Y\to\infty} \frac{1}{Y} \int_{y_0}^{Y} S(y, X(Y))^n dy \tag{3.35}$$

exists, and is given by the absolutely convergent sum

$$L_n = \sum_{g\in G^n} \sum_{\lambda_J \in \mathscr{D}_g} A_g(r_J),$$

where $A_g(r_J)$ is defined in (3.28) *and $\mathscr{D}_g = \{\lambda_J \mid \vartheta_g(\lambda_J) = 0\}$.*

Proof. From (3.29) we can write

$$\begin{aligned} \frac{1}{Y} \int_{y_0}^{Y} S(y, X(Y))^n dy &= \sum_{g\in G^n} \sum_{\lambda_J \in \mathscr{D}_g^Y} A_g(r_J) \\ +O\Bigg(\frac{1}{Y}\Big| \sum_{g\in G^n} \sum_{\lambda_J \in \mathscr{D}_g^Y} A_g(r_J)\Big| &+ \sum_{g\in G^n} \sum_{\lambda_J \in \mathscr{O}_g^Y} \frac{|A_g(r_J)|}{1 + Y|\vartheta_g(\lambda_J)|}\Bigg). \end{aligned} \tag{3.36}$$

Using the observation before Lemma 3.24 that for every $g \in G^n$ the set $\mathscr{O}_g^\infty$ is in bijection with $\mathcal{O}_s$, for some $0 \le s \le n$, the last double sum is bounded by

$$\sum_{g\in G^n} \sum_{\lambda_J \in \mathscr{O}_g^\infty} \frac{|A_g(r_J)|}{1 + Y|\vartheta_g(\lambda_J)|} \ll \sum_{s=0}^{n} \sum_{\lambda_J \in \mathcal{O}_s} \frac{|A_g(r_J)|}{1 + Y|\vartheta_g(\lambda_J)|}$$

and it has been shown in Lemma 3.24 that this quantity is uniformly bounded as Y tends to infinity. By Lebesgue dominated convergence theorem, we conclude that as $Y \to \infty$ the sum vanishes.

If we can show that L_n is absolutely convergent, then we can bound the first term in the second line of (3.36) by $O(Y^{-1})$. It is sufficient to show that for every $0 \le s \le n$ we have

$$\mathcal{U} = \sum |r_{j_1} \cdots r_{j_n}| < \infty,$$

where the sum runs over the set $\lambda_{j_1} + \cdots + \lambda_{j_s} - \lambda_{j_{s+1}} - \cdots - \lambda_{j_n} = 0$. Define now $\mu_{n-2} = \lambda_{j_1} + \cdots + \lambda_{j_s} - \lambda_{j_{s+1}} - \cdots - \lambda_{j_{n-2}}$. Using (3.4) we can bound

$$\mathcal{U} \ll \sum_{\mu_{n-2}>0} |r_{j_1} \cdots r_{j_{n-2}}| \sum_{\lambda_{n-1}<\mu_{n-2}} \frac{|r_{j_{n-1}}|}{(\mu_{n-2} - \lambda_{j_{n-1}} + B + 1)^{\beta}}.$$

We are in a situation similar to that of Lemma 3.24. Applying Lemma 3.21 $n-s-1$ times and Lemma 3.22 $s-1$ times we arrive at the estimate

$$\mathcal{U} \ll \sum_{\lambda_{j_1}>0} |r_{j_1}| \frac{\log^{n-2}(\lambda_{j_1} + 2B))}{(\lambda_{j_1} + B + 1)^{(n-1)\beta-(n-2)}}$$

and since $(n-1)\beta - (n-2) + \beta > 1$ we can apply Lemma 3.22 one last time. This leads to $\mathcal{U} < \infty$, and concludes the proof of the proposition. □

Corollary 3.26. *Let $S(y, X)$ be as in (3.5), and assume that (3.3) holds for $k = N$, and (3.4) holds with $\beta > 1 - 1/N$. Then the limit*

$$L_n := \lim_{Y\to\infty} \frac{1}{Y} \int_{y_0}^{Y} S(y, X(Y))^n dy$$

exists for every $1 \le n \le N$, and is given by (3.35).

Proof. If (3.3) holds for N, then it holds for every $1 \le n \le N$, and if (3.4) holds with $\beta > 1 - 1/N$, then it holds for $\beta > 1 - 1/n$ for every $1 \le n \le N$. Applying Proposition 3.25 we infer that all the moments L_n exist, for $1 \le n \le N$. □

Lemma 3.27. *Assume that (3.3) holds with $k = n$, and (3.4) holds with $\beta > 1 - 1/n$. If n is odd assume the following*

$$\lim_{Y\to\infty} \sup_{y\in[y_0,Y]} |\mathcal{E}(y, X(Y))| = 0. \tag{3.37}$$

Then

$$\lim_{Y\to\infty} \frac{1}{Y} \int_{0}^{Y} \phi(y)^k dy = \lim_{Y\to\infty} \frac{1}{Y} \int_{y_0}^{Y} S(y, e^Y)^k dy \tag{3.38}$$

for every $1 \le k \le n$.

Proof. It suffices to prove the case $k = n$, since (3.3) is also satisfied for $1 \le k \le n$ and (3.4) is satisfied for $\beta > 1 - 1/k$ for every $1 \le k \le n$. Assume that n is even; from Proposition 3.25 we know that the n-th moment of S is finite. Hence we can write, using Hölder's inequality,

$$\frac{1}{Y} \int_{0}^{Y} \phi(y)^n dy = \frac{1}{Y} \int_{y_0}^{Y} S(y, X(Y))^n dy + O\left(\frac{1}{Y} \int_{0}^{y_0} |\phi(y)|^n dy\right)$$
$$+O\left(\sum_{k=0}^{n-1} \frac{1}{Y} \left(\int_{y_0}^{Y} |S(y, X(Y))|^n dy\right)^{\frac{k}{n}} \left(\int_{y_0}^{Y} |\mathcal{E}(y, X(Y))|^n dy\right)^{\frac{n-k}{n}}\right).$$

Because of (3.3) the parenthesis tend to zero as $Y \to \infty$, and we obtain (3.38). Assume now that n is odd. Then we can write

$$\frac{1}{Y}\int_0^Y \phi(y)^n dy = \frac{1}{Y}\int_{y_0}^Y S(y, X(Y))^n dy + O\left(\sum_{k=0}^{n-1} \frac{1}{Y}\int_{y_0}^Y |S(y, X(Y))|^k |\mathcal{E}(y, X(Y))|^{n-k} dy\right).$$

We use Proposition 3.25 to bound $Y^{-1}\int_{y_0}^Y |S(y, X(Y))|^k dy \ll 1$ for every $1 \leq k \leq n-1$. Hypothesis (3.37) shows that in the limit as $Y \to \infty$ the parenthesis tends to zero, and thus we obtain (3.38). This concludes the proof. □

Remark 3.28. In the case n odd in the above lemma, the condition (3.37) can be removed if we assume that β is big enough: if $\beta > 1 - 1/(n+1)$, then we can argue as in the case of even n, and bound the integral of $|S|^n$ by using Hölder's inequality and the fact that the $(n+1)$th moment of S (now $n+1$ is even) is bounded. This allows us to prove that (3.38) is true.

Summarizing the results of this section, we have proven the following theorem.

Theorem 3.29. *Let $\phi(y)$ satisfy* (3.3) *for $k = N$ and* (3.4) *with $\beta > 1 - 1/N$. If N is odd assume that* (3.37) *is satisfied. Then all the moments*

$$L_n = \lim_{Y\to\infty} \frac{1}{Y}\int_0^Y \phi(y)^n dy$$

exist for every $1 \leq n \leq N$, and they are explicitely given by (3.35).

A natural question to ask is whether the moments of ϕ agree with the moments of its limiting distribution. In general this needs not to be the case, in particular ϕ might have infinite moments while having limiting distribution with finite moments of every order. This is shown in the following example.

Example 3.30. Let $\phi(x) : [0, \infty) \to \mathbb{R}$ be the following function:

$$\phi(x) = \begin{cases} n & x \in [n - n^{-3}, n + n^{-3}], n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let g be a bounded continuous function on $\mathbb{R}$. Then we have

$$\frac{1}{Y}\int_0^Y g(\phi(y))dy = g(0) + O\left(\frac{\|g\|_\infty}{Y}\sum_{n=2}^\infty \frac{1}{n^3}\right)$$

and hence we get, as $Y \to \infty$,

$$\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y g(\phi(y))dy = g(0) = \int_{\mathbb{R}} g(x) d\delta_0(x)$$

for every bounded continuous function g on $\mathbb{R}$. Here δ_0 is the Dirac delta measure supported at 0. In other words, δ_0 is the limiting distribution of ϕ. Since δ_0 is compactly supported, it admits moments of every order. We have indeed

$$\int_{\mathbb{R}} x^n \delta_0 = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1. \end{cases}$$

Consider now the moments of ϕ. Let $k \geq 3$. Then we have

$$\frac{1}{Y}\int_0^Y \phi(y)^k dy \geq \frac{1}{Y}\sum_{2\leq n<Y} 2n^{k-3} \gg \frac{Y^{k-3}}{k-2}$$

which tends to $+\infty$ as $Y \to \infty$. Hence all the moments of ϕ of order $k \geq 3$ are infinite. This shows that the moments of ϕ and the moments of its limiting distribution δ_0 do not coincide.

If we know that the moments of ϕ exist, and if we assume that the limiting distribution decays sufficiently rapidly (i.e. its tails are sufficiently small), then we can show that the moments agree. This is proven in the following proposition.

Proposition 3.31. *Let ϕ satisfy* (3.3) *for $k = N \geq 2$ and* (3.4) *with $\beta > 1 - 1/(4+2N)$. Then the moments of ϕ of order $1 \leq n \leq N$ agree with the moments of its limiting distribution μ. In other words, for $1 \leq n \leq N$ we have*

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y \phi(y)^n dy = \int_{\mathbb{R}} x^n d\mu. \tag{3.39}$$

Proof. The proof follows the lines of [22, Lemma 2.5]. It suffices to show that (3.39) holds for $n = N$, as the case for $n < N$ is similar. First of all observe that by Lemma 3.27 and Remark 3.28 we have the bound as $Y \to \infty$

$$\frac{1}{Y}\int_0^Y |\phi(y)|^N dy \ll 1. \tag{3.40}$$

Consider for $S \gg 1$ the Lipschitz bounded continuous function

$$H_S(x) := \begin{cases} 0 & \text{if } |x| \leq S, \\ |x| - S & \text{if } S < |x| \leq S+1, \\ 1 & \text{if } |x| > S+1. \end{cases}$$

By Theorem 3.16 we have

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y H_S(\phi(y))dy = \int_{\mathbb{R}} H_S(x)d\mu \ll S^{-(2\beta-1)/(2-2\beta)}$$

It follows that

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_{\substack{0\leq y\leq Y\\ |\phi(y)|\geq S+1}} dR \leq \limsup_{Y\to\infty}\frac{1}{Y}\int_0^Y H_S(\phi(y))dy \ll S^{-(2\beta-1)/(2-2\beta)}.$$

In view of the bound (3.40) we can write

$$\begin{aligned}\limsup_{Y\to\infty}\frac{1}{Y}\int_{\substack{0\leq y\leq Y\\ |\phi(y)|\geq S}} |\phi(y)|^N dy &= \limsup_{Y\to\infty}\sum_{\ell=0}^{\infty}\frac{1}{Y}\int_{\substack{0\leq y\leq Y\\ |\phi(y)|\geq S+\ell}} |\phi(y)|^N dy \\ &\ll \sum_{\ell=0}^{\infty}(S+\ell+1)^N(S+\ell)^{-(2\beta-1)/(2-2\beta)} \\ &\ll S^{N+1-(2\beta-1)/(2-2\beta)}.\end{aligned}$$

Here we have used (3.40) to interchange the lim sup with the infinite series, and the fact that $\beta > 1 - 1/(4+2N)$ to estimate the sum of the series. Define the bounded Lipschitz continuous function

$$G_S(x) := \begin{cases} x^N & 0 \le x \le S, \\ S^N(S+1-x) & S < x \le S+1, \\ 0 & x > S+1 \end{cases}$$

for $x \ge 0$, and $G_S(-x) = (-1)^N G_S(x)$ for $x < 0$. We obtain

$$\begin{aligned} \limsup_{Y\to\infty} \frac{1}{Y}\int_0^Y \phi(y)^N dy &= \limsup_{Y\to\infty} \frac{1}{Y}\int_0^Y G_S(\phi(y))dy \\ &\qquad + O(S^{N+1-(2\beta-1)/(2-2\beta)}) \\ &= \int_{\mathbb{R}} G_S(x)d\mu + O(S^{N+1-(2\beta-1)/(2-2\beta)}) \\ &= \int_{\mathbb{R}} x^N d\mu + O(S^{N+1-(2\beta-1)/(2-2\beta)}). \end{aligned} \tag{3.41}$$

Notice that the equality sign in the first line of (3.41) is valid because the limsup on the right is a real limit and exists. Taking $S \to \infty$ we conclude that

$$\limsup_{Y\to\infty} \frac{1}{Y}\int_0^Y \phi(y)^N dy = \int_{\mathbb{R}} x^N \, d\mu.$$

A similar argument works for the liminf, and this proves the proposition. □

3.4 Applications to hyperbolic counting

The first application we have in mind for the results of the previous sections is to obtain some non-trivial bounds on the moments of the error term

$$e(s) = e(s,z,w) = \frac{N(s,z,w) - M(s,z,w)}{e^{s/2}}$$

(where N and M are defined in (2.5) and (2.35)) that we have studied in detail in chapter 2. As a consequence of Theorem 2.6, which gives the pointwise bound $e(s) = O(e^{s/6})$, we obtain

$$\int_Y^{2Y} |e(s)|^N ds \ll e^{NY/3}.$$

We will show that it is possible to obtain a saving in the exponential on the right. For this it is useful to prove a slightly modified version of Theorem 3.29.

We start by considering a function $\phi(y) : [0,\infty) \to \mathbb{R}$ such that for $y \ge y_0$ and $X \ge X_0$ we have

$$\phi(y) = \Re\Big(\sum_{\lambda_n \le X} r_n e^{i\lambda_n y}\Big)\psi(\delta,\lambda_n)\omega(y) + \mathcal{E}(y,X,\delta) \tag{3.42}$$

with λ_n as in (3.1), $r_n \in \mathbb{C}$, and we assume that they satisfy (3.4) for some $\beta \in \mathbb{R}$. We assume moreover an analogue of (3.37), namely that there exist $X = X(Y)$ and $\delta = \delta(Y)$ such that

$$\lim_{Y\to\infty} \sup_{y\in[Y,2Y]} |\mathcal{E}(y, X(Y), \delta(Y))| = 0. \tag{3.43}$$

Finally we assume that (3.3) holds with $k = N$. The functions ω and ψ are chosen as follows: $\omega \in C^1([y_0, \infty))$ such that

$$\|\omega\|_\infty + \|\omega'\|_1 < \infty, \tag{3.44}$$

while $\psi(\delta, \lambda)$ is a function such that for $\lambda > 0$ and $0 < \delta \ll 1$ we have the bound

$$|\psi(\delta, \lambda)| \ll \left(\frac{1}{1 + \delta\lambda}\right)^A, \tag{3.45}$$

where $A \in \mathbb{R}$ satisfies $A + \beta > 1$. The implied constant is allowed to depend on A but not on δ nor on λ.

Theorem 3.32. *Let ϕ satisfy (3.42)–(3.45) for some $\beta \in \mathbb{R}$, $k = N$ even, and $\delta(Y) \ll 1$ when $Y \gg 1$. Let $B \in \mathbb{R}$ be such that $B + \beta > 1 - 1/N$. Then*

$$\frac{1}{Y} \int_Y^{2Y} |\phi(y)|^N dy \ll \frac{1}{\delta(Y)^{BN}}. \tag{3.46}$$

Proof. From (3.43) and and an application of Hölder's inequality it suffices to show that the estimate (3.46) holds for the N-th moment of

$$S(y, X, \delta, \psi, \omega) := \Re\big(\sum_{\lambda_n \le X} r_n e^{i\lambda_n y}\big)\psi(\delta, \lambda_n)\omega(y)$$

for $X = X(Y)$ and $\delta = \delta(Y)$. The analysis of the N-th moment of $S(y, X, \delta, \psi, \omega)$ is similar to the one of the N-th moment of $S(y, X)$ in Proposition 3.25. The coefficients are replaced by $r_n\psi(\delta, \lambda_n)$ and satisfy (3.4) with exponent $B + \beta > 1 - 1/N$. More precisely, we have

$$\sum_{T \le \lambda_n \le T+1} |r_n \psi(\delta, \lambda_n)| \ll \frac{1}{\delta^B T^{B+\beta}}. \tag{3.47}$$

The inequality in (3.30) is replaced by

$$\begin{aligned}\frac{1}{Y} \int_Y^{2Y} e^{iy\vartheta} \omega^N(y) dy &\le \min\left\{ \|\omega\|_\infty^N, \frac{\|\omega\|_\infty^{N-1}(\|\omega\|_\infty + N\|\omega'\|_1)}{Y|\vartheta|} \right\} \\ &\ll_\omega \frac{1}{1 + Y|\vartheta|},\end{aligned}$$

and so the implied constant depends on the function ω. From here we can proceed to study multiple sums associated to the sets $\mathscr{O}_g^Y$, $\mathscr{D}_g^Y$ as in (3.36). For these we argue as in Lemma 3.24, where we showed that the sum over $\mathscr{O}_g^Y$ was uniformly bounded. At each application of Lemma 3.21 or Lemma 3.22, we will make use of (3.47), and so a factor $\delta(Y)^{-B}$ will appear. Since we need to apply (3.47) a total number of N times, we get $\delta(Y)^{-BN}$. A similar discussion applies to bounding the sum over $\mathscr{D}_g^Y$, and hence we conclude that

$$\frac{1}{Y} \int_Y^{2Y} S(y, X(Y), \delta(Y), \psi, \omega)^N dy \ll \frac{1}{\delta(Y)^{BN}}$$

which, together with the remark at the beginning of the proof, concludes. □

We are now ready to prove a theorem on the moments of the function $e(s)$.

Theorem 3.33. *Let* Γ *be a cocompact Fuchsian group, and let* $e(s,z,w)$ *be defined as in* (3.42). *For every* $N \in \mathbb{N}$ *even, we have*

$$\frac{1}{Y}\int_Y^{2Y} |e(s,z,z)|^N ds \ll e^{2Y\left(\frac{N}{6}-\frac{2N}{3(3N-2)}+\varepsilon\right)}. \tag{3.48}$$

Remark 3.34. For odd values of N one can obviously obtain a bound by using Hölder's inequality together with the estimate (3.48) for the even number $N+1$. However the bound one obtains in this way is sightly worse than what one gets by formally using (3.48) for N.

Remark 3.35. The ε in the statement of Theorem 3.33 can probably be improved to a power of Y with some effort. This is for instance what happens in the case $N = 2$, as Theorem 2.9 shows. We have however refrained from proving a similar estimate for higher moments.

Remark 3.36. As N becomes larger and larger, the saving in Theorem 3.33 becomes smaller and smaller. Indeed, in the limit as $N \to \infty$, the exponent in parenthesis in (3.48) tends (from below) to $N/6$, which corresponds to the pointwise bound of Selberg.

Proof. From (2.61) and (2.62) we know that

$$|e(s)| \le \max\{|e^+(s)|, |e^-(s)|\} + O(\delta e^{s/2} + s\delta^{1/2} + e^{-\varepsilon_\Gamma s}),$$

where we have set for shortening notation $e^{\pm}(s) := e^{-s/2}(K^{\pm}(s,\delta) - M^{\pm}(s))$. Since Γ is cocompact and we have chosen $z = w$, we have the spectral expansion

$$e^{\pm}(s) = \sum_{t_j>0} e^{-s/2} h^{\pm}(t_j)|\phi_j(z)|^2$$

which we have shown to be absolutely convergent in chapter 2. Moreover we have $h^{\pm}(t_j) = \tilde{h}_\delta(t_j) h_{s\pm\delta}(t_j)$, where from (2.69) we can write $h_{s\pm\delta}(t) = \Re\big(I(s\pm\delta,t)\big)$ with

$$I(s\pm\delta,t) = -2ie^{(s\pm\delta)(1/2+it)}\int_0^\infty (1-e^{iv})^{1/2}(1-e^{-2(s\pm\delta)-iv})^{1/2}e^{-tv}dv.$$

Using the Taylor expansion of $\sqrt{1+z}$ we can write

$$(1-e^{-2(s\pm\delta)-iv})^{1/2} = \sum_{k=0}^{K}(-1)^k a_k e^{-2k(s\pm\delta)-ikv} + O(e^{-2s(K+1)}).$$

Inserting this expression in the integral we get

$$\begin{aligned} I(s\pm\delta,t) = -2ie^{(s\pm\delta)(1/2+it)} &\sum_{k=0}^{K}(-1)^k a_k e^{-2k(s\pm\delta)} G(t,k) \\ &+ O\Big(e^{-2s(K+1)}\int_0^\infty |v|^{1/2}e^{-tv}dv\Big) \end{aligned}$$

where

$$G(t,k) = \int_0^\infty (1-e^{iv})^{1/2} e^{-ikv-tv} dv \tag{3.49}$$

and satisfies $G(t,k) \ll |t|^{-3/2}$ uniformly for $k \in [0,K]$. Hence we can write

$$\frac{I(s \pm \delta, t)}{e^{s/2}} = \sum_{k=0}^{K} A_k(t) B_k(t,\delta) \omega_k(s) e^{its} + O\big(|t|^{-3/2} e^{-2s(K+1)}\big)$$

with

$$A_l(t) = -2i(-1)^k a_k G(t,k)$$
$$B_k(t,\delta) = e^{\pm\delta(1/2+it)\mp 2k\delta}$$
$$\omega_k(s) = e^{-2ks}$$

for $k = 0, 1, \ldots, K$. Recalling that $\tilde{h}_\delta(t) \ll (1+\delta|t|)^{-3/2}$ for $\delta \ll 1$, we arrive at the expression, for $X > \delta^{-1}$,

$$e^{\pm}(s) = \sum_{k=0}^{K} f_k(s) + O\left(\frac{1}{X\delta^{3/2}} + \frac{e^{-2s(K+1)}}{\delta^{1/2}}\right),$$

where $f_k(s)$ is given by

$$f_k(s) = \Re\left(\sum_{0<t_j\le X} \big(A_k(t_j)|\phi_j(z)|^2\big)\big(\tilde{h}_\delta(t_j) B_k(t_j,\delta)\big)\omega_k(s) e^{it_j s}\right).$$

This means that $e^{\pm}(s)$ is a sum of K functions that satisfy (3.42) with $\mathcal{E}(s,\delta,X) = O(\delta^{-3/2}X^{-1} + \delta^{-1/2}e^{-2s(K+1)})$. Condition (3.4) is satisfied with $\beta = 1/2$, since by (2.38) and (3.49) we get

$$\sum_{T\le t_j\le T+1} \frac{|\phi_j(z)|^2}{t_j^{3/2}} \ll \frac{1}{T^{1/2}}.$$

Condition (3.43) is satisfied for instance by taking $\delta = e^{-\alpha Y}$ for some $0 < \alpha < 4K+4$ and $X = \delta^{-2}$. Hence we can apply Theorem 3.49 to obtain

$$\frac{1}{Y}\int_Y^{2Y} |e^{\pm}(s)|^N ds \ll \frac{1}{\delta(Y)^{BN}}$$

where B is any number such that $B > 1/2 - 1/N$. Since we have

$$\frac{1}{Y}\int_Y^{2Y} (\delta e^{s/2} + s\delta^{1/2} + e^{-\varepsilon_\Gamma s})^N ds \ll \frac{\delta^N e^{NY}}{Y} + Y^N \delta^{N/2} + \frac{e^{-\varepsilon_\Gamma NY}}{Y}$$

we obtain

$$\frac{1}{Y}\int_Y^{2Y} |e(s)|^N ds \ll \frac{1}{\delta^{BN}} + \frac{\delta^N e^{NY}}{Y} + Y^N \delta^{N/2} + \frac{e^{-\varepsilon_\Gamma NY}}{Y}.$$

Choosing $\delta = Y^{1/N(B+1)} e^{-Y/(B+1)}$ we conclude that

$$\frac{1}{Y}\int_Y^{2Y} |e(s)|^N ds \ll Y^{-B/(B+1)} e^{YBN/(B+1)}.$$

Since B can be any number bigger than $1/2 - 1/N$, we can write

$$\frac{1}{Y}\int_{Y}^{2Y} |e(s)|^N ds \ll e^{NY\left(\frac{1}{3}-\frac{4}{3(3N-2)}+\varepsilon\right)}$$

for every $\varepsilon > 0$. This proves the theorem. □

Another application of the previous sections is the proof of the existence of limiting distribution and moments for certain integrated versions of $e(s,z,w)$. We start by defining, for Γ cocompact, the following functions.

$$G_1(s,z) := \int_{\Gamma\backslash\mathbb{H}} |e(s,z,w)|^2 d\mu(w),$$

$$G_2(s) := \iint_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} |e(s,z,w)|^2 d\mu(z)d\mu(w).$$

The functions G_1, G_2 are thus defined by integrating away one or both of the spacial variables from $e(s,z,w)$. For Γ cocompact and $z \in \Gamma\backslash\mathbb{H}$, we define the following function.

$$G_3(s,z) := \frac{1}{e^{s/2}}\int_0^s \big(N(x,z,z) - M(x,z,z)\big)dx.$$

Theorem 3.37. *Let Γ be cocompact, $z, w, \in \mathbb{H}/\Gamma$, and let $s \geq 0$. Then the functions G_1, G_2, G_3 are bounded in s, admit moments of every order, and limiting distributions μ_1, μ_2, μ_3 of compact support. The moments of G_i coincide with the moments of μ_i.*

Proof. Since the group is cocompact the function $N(s,z,w)$ is uniformly bounded in z, w. We have indeed

$$N(s,z,w) = \sum_{\substack{\gamma\in\Gamma:\\ d(z,\gamma w)\leq s}} 1 = \frac{1}{\mathrm{vol}(F)}\int_{\mathbb{H}} \mathbf{1}_A(v)d\mu(v)$$

where F is a fundamental domain for Γ, and A is the set defined by

$$A = \bigcup_{\substack{\gamma\in\Gamma:\\ d(z,\gamma w)\leq s}} \gamma F.$$

Let D be the diameter of F, that is $D = \sup_{z,w\in F} d(z,w)$. Then for $v \in \mathbb{H}$, if $d(v,z) > s + D$, we have $\mathbf{1}_A(v) = 0$. Note that in the definition of $N(s,z,w)$ we can assume that z and w are in the same fundamental domain F for Γ. Assume by contradiction that there exists $\gamma \in \Gamma$ such that $d(z,\gamma w) \leq s$ and $v, \gamma w \in \gamma F$ (that is, assume that $v \in A$). Then we can write, by using the triangle inequality,

$$d(z,\gamma w) \geq d(z,v) - d(v,\gamma w) > s + D - D = s.$$

This means that $\mathbf{1}_A(v) = 0$ whenever $d(z,v) > s + D$. Denoting by $\mathrm{vol}(B(R))$ the area of a hyperbolic ball of radius R we deduce the inequality

$$N(s,z,w) \leq \frac{\mathrm{vol}(B(s+D)}{\mathrm{vol}(F)}$$

uniformly in $z, w \in F$. This proves that $N(s, z, w)$ is uniformly bounded in z, w (uniform boundedness, without an explicit estimate, is also a corollary of [61, Th. 6.1]). It follows that $N(s, z, w)$ is square-integrable in z, w (square summability is also discussed in [21, p. 278]), and by Parseval's theorem we get the expansion

$$G_1(s,z) = \sum_{t_j>0} \frac{h_s(t_j)^2}{e^s} |\phi_j(z)|^2 + \sum_{t_j \in [0,i/2]} f_s(t_j)^2 |\phi_j(z)|^2$$

where $f_s(t_j)$ is defined (recall (2.35)) for $t_j \in [0, i/2]$ by

$$\begin{aligned} f_s(i/2) &= \frac{1}{e^{s/2}} \left(h_s(i/2) - \frac{\pi e^s}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \right), \\ f_s(t_j) &= \frac{1}{e^{s/2}} \left(h_s(t_j) - \sqrt{\pi} \frac{\Gamma(|t_j|)}{\Gamma(3/2+|t_j|)} e^{s(1/2+|t_j|)} \right), \quad t_j \in (0, i/2), \\ f_s(0) &= \frac{1}{e^{s/2}} \left(h_s(0) - 4\big(s + 2(\log 2 - 1)\big) e^{s/2} \right), \end{aligned}$$

and it satisfies $f_s(t_j) = O(e^{-\varepsilon_\Gamma s})$ for every $t_j \in [0, i/2]$, for some $\varepsilon_\Gamma > 0$ (this follows from the discussion in section 2.5). We can therefore write

$$G_1(s,z) = C_1 + 2\Re \left(\sum_{0<t_j\le X} \frac{\pi\, \Gamma(it_j)^2 |\phi_j(z)|^2}{\Gamma(3/2+it_j)^2} e^{2it_j s} \right) + O\left(e^{-\varepsilon_\Gamma s} + \frac{1}{X} \right) \quad (3.50)$$

with

$$C_1 = 2\pi \sum_{t_j>0} \frac{|\Gamma(it_j)|^2}{|\Gamma(3/2+it_j)|^2} |\phi_j(z)|^2.$$

The coefficients in (3.50) satisfy (by (2.21) and (2.38))

$$\sum_{T \le 2t_j \le T+1} \frac{|\Gamma(it_j)\phi_j(z)|^2}{|\Gamma(3/2+it_j)|^2} \ll \frac{1}{T^{3/2}}.$$

In particular this means that $G_1(s, z)$ is bounded in s by

$$|G_1(s,z)| \ll C_1 + \sum_{t_j>0} \frac{|\phi_j(z)|^2}{t_j^3} + e^{-\varepsilon_\Gamma s} \ll_z 1.$$

The function $G_1(s, z)$ is of the form (3.2), satisfies (3.4) with $\beta = 3/2$, and choosing for instance $X(Y) = e^Y$ it satisfies (3.3) for every k. Hence we can apply Theorem 3.29 and infer the existence of all the moments, and Theorem 3.16 to infer the existence of the limiting distribution μ_1, which is of compact support. By Proposition 3.31 we also see that the moments of G_1 coincide with the moments of μ_1.

Consider now the function $G_2(s)$. In this case we have, again by Parseval, the expansion

$$G_2(s) = \sum_{t_j>0} \frac{h_s(t_j)^2}{e^s} + \sum_{t_j \in [0,i/2]} f_s(t_j)^2,$$

and therefore we can write

$$G_2(s) = C_2 + 2\Re\left(\sum_{0<t_j\le X}\frac{\pi\,\Gamma(it_j)^2}{\Gamma(3/2+it_j)^2}e^{2it_js}\right) + O\left(e^{-\varepsilon_\Gamma s}+\frac{1}{X}\right)$$

with

$$C_2 = 2\pi\sum_{t_j>0}\frac{|\Gamma(it_j)|^2}{|\Gamma(3/2+it_j)|^2}.$$

Using this time the estimate [68, Th. 7.3]

$$\sum_{T\le 2t_j\le T+1} 1 \ll T$$

we can write

$$\sum_{T\le 2t_j\le T+1}\frac{\Gamma(it_j)^2}{\Gamma(3/2+it_j)^2} \ll \frac{1}{T^2}$$

and choosing again $X = e^Y$ we see that $G_2(s)$ is of the form (3.2), satisfies (3.4) with $\beta = 2$, and (3.3) for every $k \ge 1$. Applying Theorem 3.29, 3.16, and Proposition 3.31, we conclude the proof for G_2.

Finally consider the function $G_3(s,z)$. The function $e^{-s/2}\int_0^s N(x,z,z)dx$ is an automorphic kernel associated to the function $k_s^*(u) = e^{-s/2}\int_0^s k_x(u)dx$, where $k_x(u) = \mathbf{1}_{[0,(\cosh x-1)/2)]}(u)$. The Selberg-Harish-Chandra h_s^* transform of k_s^* can be analysed with similar computations as done in section 2.5, and we claim that it is an admissible test function in the pretrace formula. First for the small eigenvalues we can write

$$h_s^*(i/2) = \frac{1}{e^{s/2}}\int_0^s(\pi e^x + O(1))dx = \pi e^{s/2} + O(se^{-s/2}).$$

Similarly we have

$$h_s^*(0) = 8(s + 2(\log 2 - 1)) + O(e^{-s/2})$$

and for $t_j \in (0, i/2)$

$$h_s^*(t_j) = \sqrt{\pi}\frac{\Gamma(|t_j|)}{(1/2+|t_j|)\Gamma(3/2+|t_j|)}e^{s|t_j|} + O(e^{-\varepsilon s})$$

for some $\varepsilon > 0$. For t_j real and positive we have

$$h_x(t_j) = 2\Re(I(x,t_j)e^{x(1/2+it_j)})$$

where

$$I(x,t_j) = -2i\int_0^\infty(1-e^{iv})^{1/2}(1-e^{-2x-iv})^{1/2}e^{-t_jv}dv.$$

Integrating in x we can write

$$\int_0^s h_x(t_j)dx = 2\Re\Big(\int_0^s I(x,t_j)e^{x(1/2++it_j)}dx\Big).$$

Moving the contour of integration to two vertical lines above 0 and s in the complex plane, we obtain

$$\begin{aligned}&\int_0^\infty I(x,t_j)e^{x(1/2+it_j)}dx\\&=2\int_0^\infty\int_0^\infty(1-e^{iv})^{1/2}(1-e^{-2i\lambda-iv})^{1/2}e^{-t_jv}e^{i\lambda/2-\lambda t_j}dvd\lambda\\&-2e^{s(1/2+it_j)}\int_0^\infty\int_0^\infty(1-e^{iv})^{1/2}(1-e^{-2s-2i\lambda-iv})^{1/2}e^{-t_jv}e^{i\lambda/2-\lambda t_j}dvd\lambda\end{aligned}$$

and isolating the oscillation e^{ist_j} and using the bound $|1-e^{iv}|\ll v$ to bound the other terms, we conclude that

$$h_s^*(t_j)=2\Re\big(A(t_j)e^{ist_j}\big)+O\left(\frac{1}{t_j^{5/2}}\right),$$

with

$$A(t_j)=-2\int_0^\infty\int_0^\infty(1-e^{iv})^{1/2}e^{-t_jv}e^{i\lambda/2-\lambda t_j}dvd\lambda=\frac{i\sqrt{\pi}\Gamma(it_j)}{(1/2-it)\Gamma(3/2+it_j)}.$$

Since $A(t_j)\ll t_j^{-5/2}$ we infer that h_s^* is an admissible test function in the pretrace formula. Observing moreover that the main terms that appear in the integration of the small eigenvalues correspond to the integration of the terms defining $M(s,z,z)$, we can write for any $X\gg 1$

$$G_3(s,z)=2\Re\Big(\sum_{0<t_j\le X}A(t_j)|\phi_j(z)|^2e^{ist_j}\Big)+O\left(e^{-\varepsilon s}+\frac{1}{X^{1/2}}\right).$$

This shows that $G_3(s,z)$ is of the form (3.2), its coefficients $A(t_j)|\phi_j(z)|^2$ satisfy (3.4) with $\beta=3/2$, and choosing for instance $X=e^Y$, $G_3(s,z)$ satisfy (3.3) for every $k\ge 1$. Applying Theorem 3.29, 3.16, and Proposition 3.31, we conclude the proof for G_3. □

Chapter 4

Fractional Calculus

The content of this chapter refers to the joint work with M. Risager [16] and describes the results we can obtain if we use the methods of fractional integration in the hyperbolic circle problem.

We have decided to include the paper (almost) verbatim in the thesis, so that this chapter can be considered essentially self-contained. The first section and the beginning of the second have some overlap with the rest of the thesis. However, we believe that they open smoothly the way to the discussion that follows in the rest of the chapter, and most important, they fix the notation used in the subsequent sections.

4.1 Introduction

Let $\mathbb{H}$ be the hyperbolic plane, and denote by $d(z,w)$ the hyperbolic distance between $z,w \in \mathbb{H}$. For Γ a cofinite Fuchsian group and $z,w \in \mathbb{H}$, consider the function

$$N(s,z,w) := \sharp\{\gamma \in \Gamma \ \mid \ d(z,\gamma w) \leq s\}, \tag{4.1}$$

which counts the number of translates γw of w, $\gamma \in \Gamma$ with hyperbolic distance from the point z not exceeding s. The hyperbolic lattice point problem asks for the behaviour of $N(s,z,w)$ for big values of s. It is known that

$$N(s,z,w) \sim \frac{\mathrm{vol}(B_z(s))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$$

as $s \to \infty$. Here $B_z(s)$ denotes the hyperbolic ball with center z and radius s. This can be proved in several ways, see e.g. [25, Section 1.3].

For our purposes it is convenient to appeal to the spectral theory of the Laplace-Beltrami operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

acting on a dense subset of $L^2(\Gamma\backslash\mathbb{H})$. The operator Δ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

which is either finite or satisfies $\lambda_n \to \infty$, and a continuous spectrum which covers $[1/4,\infty)$ with multiplicity equal to the number of cusps of Γ. The eigenvalues $\lambda_j \in (0,1/4)$ are called *small eigenvalues*. Writing $\lambda_j = 1/4 + t_j^2$ with

$\Im(t_j) \geq 0$ they correspond to t_j in the complex segment $t_j \in (0, i/2)$. One defines the following main term:

$$\begin{aligned} M(s,z,w) := \frac{\pi e^s}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} &+ \sqrt{\pi} \sum_{t_j \in (0, \frac{i}{2})} \frac{\Gamma(|t_j|)}{\Gamma(3/2+|t_j|)} e^{s(1/2+|t_j|)} \phi_j(z)\overline{\phi_j(w)} \\ &+ 4\big(s + 2(\log 2 - 1)\big)\, e^{s/2} \sum_{t_j=0} \phi_j(z)\overline{\phi_j(w)} \\ &+ e^{s/2} \sum_{\mathfrak{a}} E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)} \end{aligned} \tag{4.2}$$

where ϕ_j is the eigenfunction associated to λ_j, and $E_{\mathfrak{a}}(z,r)$ is the Eisenstein series associated to the cusp $\mathfrak{a}$. For the full modular group this expression simplifies to only the first term, but for general groups the small eigenvalues give rise to secondary terms in the expansion of the counting function $N(s,z,w)$. It is an unpublished result of Selberg (for a proof see e.g.[39, Thm. 12.1]) that

$$N(s,z,w) - M(s,z,w) \ll e^{\frac{2s}{3}}, \tag{4.3}$$

and it is conjectured that the true size of the difference should be not bigger than $e^{s(\frac{1}{2}+\varepsilon)}$ for any $\varepsilon > 0$. Define

$$e_\Gamma(s,z,w) = \frac{N(s,z,w) - M(s,z,w)}{e^{s/2}} \tag{4.4}$$

to be the normalized remainder in the hyperbolic problem.

Phillips and Rudnick have shown ([52, Theorem 1.1]) that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e_\Gamma(s,z,z)ds = 0.$$

We refer to the quantity on the left as the first (asymptotic) moment of $e_\Gamma(s,z,z)$.

It is an open problem whether the (asymptotic) *variance* of $e_\Gamma(s,z,w)$ exists and, if so, if it is finite. More precisely we are interested in knowing if

$$\mathrm{Var}(e_\Gamma) = \lim_{T\to\infty} \frac{1}{T} \int_T^{2T} |e_\Gamma(s,z,w)|^2\, ds$$

exists and is finite. Phillips and Rudnick remarked [52, Section 3.8] that they cannot show that the variance is finite but they prove non-zero *lower* bounds.

In order to simplify notation, from now on we will write $e(s)$ in place of $e_\Gamma(s,z,w)$, assuming that the group Γ and the points $z, w \in \mathbb{H}$ are fixed once and for all.

A first result on the size of the variance is due to Chamizo see [15, Corollary 2.1.1] who proves, using his large sieve in Riemann surfaces [14]

$$\frac{1}{T} \int_T^{2T} |e(s)|^2 ds \ll T^2 \tag{4.5}$$

(one gets from his statement to (4.5) by changing variable $X = 2\cosh(s)$). This doesn't prove finiteness of the variance of $e(s)$ but does shows that the integral in (4.5) grows at most polynomially in T. This is an improvement on what one gets by simply plugging Selberg's pointwise bound, and it is consistent with the conjecture $e(s) \ll e^{\varepsilon s}$. We remark that (4.5) can be improved to a bound $\ll T$ by using classical methods due to Cramér [18, 19].

REMARK 4.1. Cramér studied the analogous Euclidian problem [19], and in this case he was able to prove that the variance is finite and find an explicit expression for it. Like us, he also used a spectral expansion (coming in his case from Poisson summation), but contrary to our case the "eigenvalues" are explicitly known and the decay of the spectral coefficients is favorable. One difficulty in proving finiteness of the variance in our problem (using a spectral approach) relates to the following feature of the problem: the spectral coefficients do not decay sufficiently fast compared to the number of eigenvalues. The way we get around this problem is to slightly improve the decay of the coefficients using fractional integration. The formalism that we adopt follows the lines of [58].

Definition 4.2. Let $\varphi \in L^p([0,A])$ be a p-summable function on $[0,A]$ for $p \geq 1$, and let $\alpha > 0$ be a positive real number. The *fractional integral* of order α of φ is defined for $x \in [0,A]$ as the function

$$I_\alpha \varphi\,(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt.$$

The function $I_\alpha\varphi(x)$ will also be denoted by $\varphi_\alpha(x)$.

It is straightforward from the definition that the fractional integral of order $\alpha = 1$ coincides with the regular integral. It is interesting to consider integrals of small order $0 < \alpha < 1$ of a given function φ, because we have

$$\lim_{\alpha \to 0^+} \varphi_\alpha(x) = \varphi(x) \quad \text{for a.e. } x \in [0,A].$$

$$\lim_{\alpha \to 0^+} \|\varphi_\alpha - \varphi\|_p = 0.$$

The first condition is easy to check by integration by parts when φ is regular. If we integrate the function φ to a very small order, we expect thus the resulting function φ_α to be close to the original function. In addition to this, fractional integration enhances the properties of φ; Indeed, if $0 < \alpha < 1$ and $\varphi \in L^p$ with $1 < p < 1/\alpha$, then $\varphi_\alpha \in L^q$, for $q = p/(1-\alpha p) > p$, and therefore φ_α has better summability properties. Moreover, if $\varphi \in L^\infty$, then φ_α is Hölder of exponent α, and in general, if φ is Hölder of exponent $0 \leq \rho \leq 1$, then for $0 < \alpha < 1$ the function φ_α is Hölder of exponent $\rho + \alpha$[1]. Hence φ_α has better regularity properties than φ. For a reference on these and other results about fractional integration, see [58].

Definition 4.3. Let $0 < \alpha < 1$. We define, for $s > 0$, the α-integrated normalized remainder term in the hyperbolic lattice point counting problem as

$$e_\alpha(s,z,w) := I_\alpha e_\Gamma(s,z,w).$$

where the integration is with respect to the first s variable.

[1] The case $\rho + \alpha = 1$ is special, as in this situation φ_α is in a slightly bigger space than H^1 (see [58, Ch. 1, §3.3, Cor 1]).

The function $e_\alpha(s, z, w)$ is well-defined since for every $A > 0$ we have $e(s) \in L^1([0, A])$. When the group Γ and the points $z, w \in \mathbb{H}$ are fixed, we will simply write $e_\alpha(s)$.

We first prove a pointwise bound and an average result for $e_\alpha(s)$ that are analogous to the results for $e(s)$:

Theorem 4.4. *Let Γ be a cofinite group, $z, w \in \mathbb{H}$, and $0 < \alpha < 1$. Then*

$$e_\alpha(s) \ll \begin{cases} e^{s(1-2\alpha)/(6-4\alpha)} & 0 < \alpha < 1/2, \\ s & \alpha = 1/2, \\ 1 & 1/2 < \alpha < 1. \end{cases}$$

The implied constant depends on z, w, and the group Γ.

Remark 4.5. When $\alpha = 0$ this is Selberg's bound (4.3) (recall the normalization in (4.4)). When $\alpha > 0$ the exponent gets smaller approaching 0 as α increases to 1/2. For the threshold $\alpha = 1/2$ a polynomial factor appears, while for $\alpha > 1/2$ the function $e_\alpha(s)$ becomes bounded.

Theorem 4.6. *Let Γ be a cofinite group, $z, w \in \mathbb{H}$, and $0 < \alpha < 1$. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} e_\alpha(s) ds = 0.$$

Remark 4.7. When $\alpha = 0$ this corresponds to [52, Theorem 1.1].The case $\alpha = 1$ is delicate: if the group is cofinite but not cocompact we cannot show that the limit stays bounded, while if the group is cocompact then it is possible to show that the limit exists and is finite.

To be able to prove finite variance for $e_\alpha(s)$ we need to make assumptions on the Eisenstein series. More precisely we need to assume, in the case where Γ is cofinite but not cocompact, that for $v = z$ and $v = w$ we have

$$\int_1^\infty \frac{|E_{\mathfrak{a}}(v, 1/2 + it)|^{2p}}{t^{(3/2+\alpha)p}} dt < \infty, \qquad \text{for some } 1 < p < \min(2, \alpha^{-1}), \text{ and all } \mathfrak{a}. \tag{4.6}$$

Theorem 4.8. *Let $0 < \alpha < 1$ and assume (4.6). Then we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} |e_\alpha(s)|^2 ds = 2\pi \sum_{\substack{t_j>0 \\ distinct}} \frac{|\Gamma(it_j)|^2}{|t_j^\alpha \Gamma(3/2 + it_j)|^2} \left| \sum_{t_{j'}=t_j} \phi_{j'}(z)\overline{\phi_{j'}(w)} \right|^2$$

and the sum on the right is convergent.

Remark 4.9. Condition (4.6) holds true for congruence groups: it is implied by the following stronger condition

$$|E_{\mathfrak{a}}(z, 1/2 + it)| \ll_z |t|^{1/2+\varepsilon}, \qquad t \gg 1$$

which holds for congruence groups. For a proof see [70, Lemma 2.1] or combine [12, Eq. (2.4), ftnote2.] with a Maass-Selberg type argument as in the proof of [51, Lemma 6.1].

For cocompact groups (4.6) is vacuous, so Theorem 4.8 holds unconditional in this case.

Condition (4.6) holds also if $\alpha > 1/2$ and the Eisenstein series satisfy that they are bounded polynomially as $t \to \infty$. We note also, by using Theorem 4.4, that when $\alpha > 1/2$ the asymptotic variance is bounded.

REMARK 4.10. It is a straightforward exercise to show that if $f \in L^1_{loc}([0,\infty))$ then $T^{-1}\int_T^{2T} f(s)ds \to A$ as $T \to \infty$ if and only if $T^{-1}\int_0^T f(s)ds \to A$ as $T \to \infty$. It follows that theorems 4.6 and 4.8 are true also if we replace the integral from T to $2T$ by the integral from 0 to T. For various technical reasons it is convenient to consider the integral from T to $2T$.

REMARK 4.11. If we take $\alpha = 0$ we cannot prove that the infinite series appearing in Theorem 4.8 is convergent. However, for groups like $\mathrm{SL}_2(\mathbb{Z})$ this follows from standard (but probably very hard) conjectures (see Section 4.8). For groups Γ where

$$V = 2\pi \sum_{\substack{t_j>0 \\ \text{distinct}}} \frac{|\Gamma(it_j)|^2}{|\Gamma(3/2+it_j)|^2} \left| \sum_{t_{j'}=t_j} \phi_{j'}(z)\overline{\phi_{j'}(w)} \right|^2 < \infty, \tag{4.7}$$

and where the Eisenstein contribution is "small" it is tempting to speculate that V should be the variance of $e_\Gamma(s)$ i.e. that

$$\mathrm{Var}(e_\Gamma) = \lim_{\alpha \to 0^+} \mathrm{Var}(e_\alpha) = V. \tag{4.8}$$

In fact, by comparison of (4.7) with the explicit expression of the variance of error terms in other problems (see [18, 19, 1]), the quantity V seems the appropriate candidate for being the variance of e_Γ.

Finally, we conclude with a distributional result on $e_\alpha(s)$ which we prove as a by-product of bounds which emerge in the proof Theorem 4.8. Given a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ we say that g admits a *limiting distribution* if there exists a probability measure μ on $\mathbb{R}$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(g(s))ds = \int_{\mathbb{R}} f\, d\mu$$

holds for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$.

Theorem 4.12. *Let $0 < \alpha < 1$ and let Γ be as in Theorem 4.8. Then the function $e_\alpha(s)$ admits a limiting distribution μ_α. For $\alpha > 1/2$, μ_α is compactly supported.*

In view of Remark 4.9, the theorem applies to congruence groups and cocompact groups.

REMARK 4.13. The technique of regularizing functions that do not have sufficiently good properties is standard in analytic number theory. This can often be done for instance by convolution with some smooth functions f_ε that approximate a delta function as ε tends to zero, and it is in particular this type of smoothing that is used in [52] when proving lower bounds on $e(s)$. Using fractional integration corresponds to pushing the standard method to its limit. Indeed, the pre-trace formula for the integrated function $e_\alpha(s)$ for $\alpha \leq 1/2$ is not absolutely convergent, which is a characteristic of $e(s)$ but not of the smooth approximation. The small improvements given by the α integration allows to prove the above theorems.

4.2 Preliminaries

We recall here some basic facts on automorphic functions. Let $z, w \in \mathbb{H}$, and consider the standard point-pair invariant

$$u(z,w) = \frac{|z-w|^2}{4\Im(z)\Im(w)}.$$

We have

$$2u(z,w) + 1 = \cosh d(z,w) \tag{4.9}$$

Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a cofinite Fuchsian group. For $k : \mathbb{R} \to \mathbb{R}$ rapidly decreasing the function

$$K(z,w) = \sum_{\gamma\in\Gamma} k(u(z,\gamma w))$$

is an automorphic kernel for the group Γ. If we define $h(t)$ to be the Selberg–Harish-Chandra transform of $k(u)$, which is defined as an integral transform of k in three steps as follows

$$q(v) = \int_v^{+\infty} \frac{k(u)}{(u-v)^{1/2}} du, \quad g(r) = 2q\left(\sinh^2 \frac{r}{2}\right), \quad h(t) = \int_{-\infty}^{+\infty} e^{irt} g(r) dr,$$

then we have the following spectral expansion of $K(z,w)$, usually referred to as the pre-trace formula (see [39, Theorem 7.4]):

Proposition 4.14. *Let (k,h) be a pair such that $h(t)$ is even, holomorphic on a strip $|\Im(t)| \leq 1/2+\varepsilon$, and with $h(t) \ll (1+|t|)^{-2-\varepsilon}$ in the strip. We have the following expansion for $K(z,w)$:*

$$K(z,w) = \sum_{t_j} h(t_j)\phi_j(z)\overline{\phi_j(w)} + \frac{1}{4\pi}\sum_{\mathfrak{a}} \int_{\mathbb{R}} h(t) E_{\mathfrak{a}}(z, 1/2+it)\overline{E_{\mathfrak{a}}(w,1/2+it)}\, dt$$

and the right hand side converges absolutely and uniformly on compact sets.

The absolute convergence is a consequence of the local Weyl's law (See e.g. [52, Lemma 2.3])

$$\sum_{|t_j|<T} |\phi_j(z)|^2 + \frac{1}{4\pi}\sum_{\mathfrak{a}} \int_{-T}^{T} |E_{\mathfrak{a}}(z,1/2+it)|^2\, dt \sim cT^2 \tag{4.10}$$

as $T \to \infty$, for a positive constant $c > 0$, together with the assumption on the decay of h.

Using the pre-trace formula it is possible to give upper bounds on eigenfunctions averaged over short intervals. More precisely one can show (see [39, Eq. (13.8)])

$$\sum_{T \leq t_j \leq T+1} |\phi_j(z)|^2 \ll T. \tag{4.11}$$

In our proofs we also need to consider

$$b_j = \sum_{t_{j'}=t_j} \phi_{j'}(z)\overline{\phi_{j'}(w)} \ll t_j \tag{4.12}$$

where the bound follows immediately from (4.11). From (4.10) we find immediately that

$$\sum_{T\leq t_j\leq 2T}{}' |b_j| \ll T^2. \tag{4.13}$$

Here and in the rest of the paper a prime on a sum indexed over t_j means that in this sum the t_j are listed *without* multiplicity, i.e. $t_j \neq t_\ell$ for $j \neq \ell$.

Consider the counting function defined in (4.1). This can be written as an automorphic kernel as

$$N(s,z,w) = \sum_{\gamma\in\Gamma} k_s(u(z,\gamma w))$$

where $k_s(u) = \mathbf{1}_{[0,(\cosh s-1)/2]}$ is the indicator function of the set $[0,(\cosh s - 1)/2]$. This agrees with (4.1) by virtue of (4.9). In particular

$$k_s(u(z,w)) = \begin{cases} 1 & \text{if } d(z,w)\leq s \\ 0 & \text{if } d(z,w) > s. \end{cases}$$

To study the error term $e(s,z,w)$ we want to use the pre-trace formula. However the Selberg–Harish-Chandra transform of k_s only decays as fast as $O((1+|t|)^{-3/2})$ (see [52, Lemma 2.5 and 2.6]) and the pre-trace formula is therefore not absolutely convergent. The standard way to go around this is by regularizing the function k_s sufficiently to ensure that the associated Selberg–Harish-Chandra transform has better decay properties.

Since our purpose is to study the normalized remainder, we consider instead of $k_s(u)$ the function $k_s(u)e^{-s/2}$. This gives rise to the function $N(s,z,w)e^{-s/2}$, and subtracting from it the normalized main term $M(s,z,w)e^{-s/2}$ we obtain $e_\Gamma(s,z,w)$. The fractional integral of order α of $e_\Gamma(s,z,w)$ is defined by $e_\alpha(s) = I_\alpha e_\Gamma(s,z,w)$.
By linearity of the fractional integral we see that

$$e_\alpha(s) = I_\alpha e_\Gamma(s,z,w) = I_\alpha\left(\frac{N(s,z,w)}{e^{s/2}}\right) - I_\alpha\left(\frac{M(s,z,w)}{e^{s/2}}\right).$$

It is easy to compute directly what the second term is. We have indeed for $\beta > 0$

$$I_\alpha(e^{\beta s}) = \frac{e^{\beta s}}{\beta^\alpha} + O\left(\frac{1}{\beta\Gamma(\alpha)s^{1-\alpha}}\right) \tag{4.14}$$

and

$$I_\alpha(s) = \frac{s^{\alpha+1}}{\Gamma(\alpha+2)},\quad I_\alpha(1) = \frac{s^\alpha}{\Gamma(\alpha+1)}. \tag{4.15}$$

The implied constant in the first expression is absolute. It is now natural to define the α-integrated normalized main term to be

$$
\begin{aligned}
M_\alpha(s) := \frac{\pi e^{s/2}}{2^{-\alpha}\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sqrt{\pi} \sum_{t_j \in (0, \frac{i}{2})} \frac{\Gamma(|t_j|)}{|t_j|^\alpha \Gamma(3/2 + |t_j|)} e^{s|t_j|} \phi_j(z)\overline{\phi_j(w)} \\
+ 4\left(\frac{s^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2(\log 2 - 1)s^\alpha}{\Gamma(\alpha+1)} \right) \sum_{t_j=0} \phi_j(z)\overline{\phi_j(w)} \\
+ \frac{s^\alpha}{\Gamma(\alpha+1)} \sum_{\mathfrak{a}} E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}.
\end{aligned}
$$

We then have

$$
M_\alpha(s) = I_\alpha\Big(\frac{M(s,z,w)}{e^{s/2}}\Big) + O\left(\frac{1}{\Gamma(\alpha)s^{1-\alpha}}\right)
$$

where the implied constant depends on z, w, and the group Γ. We conclude that the α-integrated normalized remainder is expressed as

$$
e_\alpha(s) = N_\alpha(s) - M_\alpha(s) + O\left(\frac{1}{\Gamma(\alpha)s^{1-\alpha}}\right).
$$

The function $N_\alpha(s) = I_\alpha\big(N(s,z,w)e^{-s/2}\big)$can be expressed as an automorphic function associated to the kernel $k_\alpha(u) = I_\alpha(k_s(u)e^{-s/2})$ in the following way:

$$
N_\alpha(s) = I_\alpha\Big(\frac{N(s,z,w)}{e^{s/2}}\Big) = I_\alpha\left(\sum_{\gamma\in\Gamma} \frac{k_s(u(z,w))}{e^{s/2}}\right) = \sum_{\gamma\in\Gamma} k_\alpha(u(z,w)).
$$

Here we have used that the sum is finite, so we can interchange the order of integration and summation. In order to apply the pre-trace formula we need to understand the properties of the Selberg–Harish-Chandra transform $h'_\alpha(t)$ associated to $k_\alpha(u)$. We have the following expression (see [39, (1.62')]) for the Selberg-Harish-Chandra transform of a generic test function $k(u)$:

$$
h(t) = 4\pi \int_0^{+\infty} F_{1/2+it}(u)k(u)du,
$$

where $F_\nu(u)$ is the hypergeometric function. It follows, using that k_s is compactly supported in u, that

$$
\begin{aligned}
h'_\alpha(t) &= 4\pi \int_0^{+\infty} F_{1/2+it}(u)\, k_\alpha(u)\, du \\
&= 4\pi \int_0^{+\infty} F_{1/2+it}(u)\, I_\alpha(k_s(u)e^{-s/2})\, du = I_\alpha(h_s(t)e^{-s/2})
\end{aligned}
$$

where we have used that $I_\alpha k_s(u)$ is an integral in the variable s, and the double integral is absolutely convergent so that we can interchange the order of integration. The function $h'_\alpha(t)$ is the spectral function associated to the remainder $e_\alpha(s)$. We study in detail this function in section 4.3.

4.3 Analysis of the Selberg-Harish-Chandra transform

We prove some estimates on the function $I_\alpha(h_s(t)e^{-s/2})$ that will be useful to prove pointwise and average results for the remainder function $e_\alpha(s)$.

Integral representation

The Selberg-Harish-Chandra transform $h'_s(t)$ of the kernel

$$k'_s(u) = \mathbf{1}_{[0,(\cosh s-1)/2]}(u)e^{-s/2}$$

is given for $s \geq 0$ by (see [52, (2.10)] and [15, (2.6)])

$$h'_s(t) = \frac{2^{3/2}}{e^{s/2}} \int_{-s}^{s} (\cosh s - \cosh r)^{1/2} e^{irt}\, dr.$$

It is an important but non-obvious feature of $h'_s(t)$ that it decays as fast as $|t|^{-3/2}$ when $t \to \infty$. A flexible method to show such decay consists in shifting the contour of integration from the interval $[-s, s]$ to a pair of vertical half-lines in the complex plane, with base points $\pm s$. This is done in [52, Lemma 2.5], and this method can be used also to analyse the fractional integral of h'_s. Since the function $h'_s(t)$ is even, we will from now on only consider $t > 0$, using the reflection formula $h'_s(t) = h'_s(-t)$ to get negative values of t. We have (see [52, p. 89])

$$h'_s(t) = 2\Re(J_s(t)),$$

$$J_s(t) = -2i \int_0^{\infty} (1 - e^{iv})^{1/2}(1 - e^{-2s}e^{-iv})^{1/2} e^{-tv} dv e^{its}.$$

It is convenient also to set $J_s(t) = 0$ for $s < 0$. For technical reasons that will be clear later, it is convenient to consider a small shift of the function $h'_s(t)$. For $0 \leq \delta < 1$ and $s > 2$ consider the function $h'_{s\pm\delta}(t)$. We consider the fractional integral of order α of $h'_{s\pm\delta}(t)$, which we will denote by $h'_{\alpha,s\pm\delta}(t)$, i.e

$$\begin{aligned} h'_{\alpha,s\pm\delta}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^s \frac{2\Re(J_{x\pm\delta}(t))}{(s-x)^{1-\alpha}}\, dx \qquad (4.16) \\ &= 2\Re\left(\frac{-2i}{\Gamma(\alpha)} \int_0^{\infty} (1-e^{iv})^{1/2} e^{-tv} \int_{s_0}^{s} \frac{(1-e^{-2(x\pm\delta)-iv})^{1/2} e^{it(x\pm\delta)}}{(s-x)^{1-\alpha}} dx\, dv \right). \end{aligned}$$

Here s_0 denotes the quantity $s_0 = \max\{0, \mp\delta\}$. We will consider the innermost integral and move the contour of integration to two vertical half-lines in the upper half-plane of $\mathbb{C}$ with base points s_0, s. In order to move the contour we define for $\varepsilon > 0$ the set

$$\Omega_\varepsilon = \{z \in \mathbb{C} : z = x + iy,\ s_0 < x < s,\ y > 0,\ |z - s| > \varepsilon\}.$$

The integrand

$$f(z) = \frac{(1 - e^{-2(z\pm\delta)-iv})^{1/2} e^{it(z\pm\delta)}}{(s-z)^{1-\alpha}}$$

is holomorphic on Ω_ε and continuous on its boundary, so we can apply Cauchy's theorem and get

$$\int_{s_0}^{s-\varepsilon} f(z)dz = \int_{\ell_1} f(z)dz - \int_{\ell_{2,\varepsilon}} f(z)dz - \int_{\gamma_\varepsilon} f(z)dz \qquad (4.17)$$

where $\ell_1 = \{z = s_0 + iy,\ y \geq 0\}$, $\ell_{2,\varepsilon} = \{z \in \partial\Omega_\varepsilon,\ \Re(z) = s,\ |z - s| \geq \varepsilon\}$ and $\gamma_\varepsilon = \{z \in \partial\Omega_\varepsilon : |z - s| = \varepsilon\}$. Since we can bound

$$\left| \int_{\gamma_\varepsilon} f(z)dz \right| \leq \frac{\pi\varepsilon^\alpha}{\sqrt{2}}$$

we see, taking the limit as $\varepsilon \to 0$ in (4.17), that the integral over $[s_0, s]$ equals the integral over ℓ_1 minus the integral over $\ell_2 = \{s + iy,\ y \geq 0\}$. This gives

$$\begin{aligned}
&\Gamma(\alpha)h'_{\alpha,s\pm\delta}(t) \\
&= 2\Re\left(2\int_0^\infty (1-e^{iv})^{1/2}e^{-tv}\int_0^\infty \frac{(1-e^{-2(s_0\pm\delta+i\lambda/t)-iv})^{1/2}e^{-\lambda}}{((s-s_0)t-i\lambda)^{1-\alpha}}\frac{d\lambda dv}{t^\alpha}e^{it(s_0\pm\delta)}\right) \\
&- 2\Re\left(2\int_0^\infty (1-e^{iv})^{1/2}e^{-tv}\int_0^\infty \frac{(1-e^{-2(s\pm\delta+i\lambda/t)-iv})^{1/2}e^{-\lambda}}{(-i\lambda)^{1-\alpha}}\frac{d\lambda dv}{t^\alpha}\,e^{it(s\pm\delta)}\right).
\end{aligned}$$

In the rest of the section we will use this integral representation of $h'_{\alpha,s\pm\delta}(t)$ to obtain pointwise bounds for $h'_{\alpha,s\pm\delta}(t)$, bounds for the average $\frac{1}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t)ds$, and for products $\frac{1}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t_1)\overline{h'_{\alpha,s\pm\delta}(t_2)}ds$. These will be used in section 4.5 to get pointwise estimates on $e_\alpha(s)$, and in sections 4.6 and 4.7 to get estimates for the first and second moment of $e_\alpha(s)$.

In several of the proofs we will tacitly use the following elementary inequalities to interpolate between different bounds:

$$\begin{aligned}
\min(a^{-1}, b^{-1}) &\leq 2/(a+b) \leq 2\min(a^{-1}, b^{-1}) \\
\min(c,d) &\leq c^\sigma d^{1-\sigma}
\end{aligned}$$

valid for all $a, b, c, d > 0$ and $0 \leq \sigma \leq 1$.

Pointwise bounds

We now state and prove two lemmas in which we estimate pointwise the function $h'_{\alpha,s\pm\delta}(t)$. One is uniform in t but worse in s, while the second is sharper for $|t| \gg 1$ but it has a singularity when $t \to 0$.

Lemma 4.15. *Let $0 < \alpha < 1$ and let $t \in \mathbb{R}$. For $0 \leq \delta < 1$ and $s > 2$ we have*

$$h'_{\alpha,s\pm\delta}(t) \ll s^{\alpha+1}$$

where the implied constant is absolute.

Proof. Since the function $h'_s(t)$ satisfies the bound $|h'_s(t)| \leq |h'_s(0)| \ll s$ for every $t \in \mathbb{R}$ (see [52, Lemma 2.2]) and in view of the fact that fractional integration preserves inequalities, we get for every $t \in \mathbb{R}$

$$h'_{\alpha,s\pm\delta}(t) = I_\alpha\big(h'_{s\pm\delta}(t)\big) \ll I_\alpha(s) = \frac{s^{\alpha+1}}{\Gamma(\alpha+2)}$$

where the last equality follows from (4.15). □

Lemma 4.16. *Let $0 < \alpha < 1$ and $t \in \mathbb{R}$, $t \neq 0$. Then for $0 \leq \delta < 1$ and $s > 2$*

$$h'_{\alpha,s\pm\delta}(t) = 2\sqrt{\pi}\,\Re\left(\frac{\Gamma(it)e^{it(s\pm\delta)}}{(it)^\alpha\Gamma(3/2+it)}\right) + \ell(s,\delta,t)$$

where

$$\ell(s,\delta,t) = O\left(\frac{1}{|t|^{1+\alpha}(1+\sqrt{|t|})}\left(e^{-2s} + \frac{1}{(1+|st|^{1-\alpha}\Gamma(\alpha))}\right)\right)$$

and the implied constant is absolute.

Proof. We have from Section 4.3 that $h'_{\alpha,s\pm\delta}(t) = 2\Re(L_1) + 2\Re(L_2) + 2\Re(L_3)$, where

$$\begin{aligned}
L_1 =& \frac{-2}{\Gamma(\alpha)} \int_0^\infty (1-e^{iv})^{1/2} e^{-tv} dv \int_0^\infty \frac{e^{-\lambda}}{(-i\lambda)^{1-\alpha}} d\lambda \frac{e^{it(s\pm\delta)}}{t^\alpha}, \\
L_2 =& \frac{2}{\Gamma(\alpha)} \int_0^\infty (1-e^{iv})^{1/2} e^{-tv} \int_0^\infty \frac{(1-e^{-2(s_0\pm\delta+i\lambda/t)-iv})^{1/2}}{((s-s_0)t - i\lambda)^{1-\alpha}} e^{-\lambda} \frac{d\lambda dv}{t^\alpha} e^{it(s_0\pm\delta)}, \\
L_3 =& \frac{-2}{\Gamma(\alpha)} \int_0^\infty (1-e^{iv})^{1/2} e^{-tv} \qquad (4.18) \\
& \times \int_0^\infty \frac{[(1-e^{-2(s\pm\delta+i\lambda/t)-iv})^{1/2} - 1]}{(-i\lambda)^{1-\alpha}} e^{-\lambda} \frac{d\lambda dv}{t^\alpha} e^{it(s\pm\delta)}.
\end{aligned}$$

Integrating L_1 in v and λ, and using the relation

$$-2i \int_0^\infty (1-e^{iv})^{1/2} e^{-tv} dv = \sqrt{\pi} \frac{\Gamma(it)}{\Gamma(3/2+it)}$$

which can be proved using the functional equation of the Gamma function, its relation with the Beta function, and a a change of path in the integration, we obtain

$$L_1 = \sqrt{\pi} \frac{\Gamma(it)}{\Gamma(3/2+it)} \frac{e^{it(s\pm\delta)}}{(it)^\alpha}$$

and we recover the main term in the lemma. The error $\ell(s,\delta,t)$ is then given by the sum of $2\Re(L_2) + 2\Re(L_3)$. Bounding by absolute value and using $|(1-e^{iv})^{1/2}| \ll \min(1, v^{1/2})$we get

$$\begin{aligned}
\Re(L_2) &= O\left(\frac{1}{|t|^{1+\alpha}(1+\sqrt{|t|})} \frac{1}{(1+|st|^{1-\alpha}\Gamma(\alpha))} \right) \\
\Re(L_3) &= O\left(\frac{e^{-2s}}{|t|^{1+\alpha}(1+\sqrt{|t|})} \right)
\end{aligned}$$

with implied absolute constants, and the lemma is proven. □

Average bounds

We give now two lemmas to estimate the size of the average of $h'_{\alpha,s\pm\delta}(t)$. As in the pointwise bounds, the first estimate is uniform in t but worse in s.

Lemma 4.17. *Let $0 < \alpha < 1$ and $T > 2$, and let $t \in \mathbb{R}$. Then for $0 \le \delta < 1$*

$$\frac{1}{T} \int_T^{2T} h'_{\alpha,s\pm\delta}(t) ds \ll T^{\alpha+1}$$

where the implied constant is absolute.

Proof. This follows directly by integrating the bound in Lemma 4.15. □

Lemma 4.18. *Let $0 < \alpha < 1$, $T > 2$, and $t \in \mathbb{R}$, $t \neq 0$. For $0 \le \delta < 1$ we have*

$$\frac{1}{T} \int_T^{2T} h'_{\alpha,s\pm\delta}(t) ds \ll \frac{1}{|t|^{1+\alpha}(1+\sqrt{|t|})} \left(\frac{1}{1+T|t|} + \frac{e^{-2T}}{T} + \frac{1}{1+|Tt|^{1-\alpha}\Gamma(\alpha)} \right).$$

with implied absolute constant.

Proof. This follows directly by integrating the expression in Lemma 4.16. □

Products

Finally we give three lemmas on the size of the average of products of the form $h'_{\alpha,s\pm\delta}(t_1)\overline{h'_{\alpha,s\pm\delta}(t_2)}$. The first is a uniform estimate in t_1, t_2, the second deals with the diagonal $t_1 = t_2$, and the third gives a bound for the off $t_1 \neq t_2$.

Lemma 4.19. *Let $0 < \alpha < 1$, $T > 2$, and let $t_1, t_2 \in \mathbb{R}$. Then for $0 \leq \delta < 1$*

$$\frac{1}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t_1)\overline{h'_{\alpha,s\pm\delta}(t_2)}ds \ll T^{2+2\alpha}$$

where the implied constant is absolute.

Proof. This follows from using the bound of Lemma 4.15 for both factors and then integrating directly. □

Lemma 4.20. *Let $0 < \alpha < 1$, $T > 2$, and $t \in \mathbb{R}$, $t \neq 0$. Then for $0 \leq \delta < 1$*

$$\frac{1}{T}\int_T^{2T} |h'_{\alpha,s\pm\delta}(t)|^2 ds = 4\pi \left|\frac{\Gamma(it)}{t^\alpha \Gamma(3/2+it)}\right|^2 + O\left(\frac{1}{|t|^{2+2\alpha}(1+|t|)}\left(\frac{1}{1+|Tt|} + \frac{e^{-2T}}{T} + \frac{1}{1+|Tt|^{1-\alpha}\Gamma(\alpha)}\right)\right)$$

where the implied constant is absolute.

Proof. Recall from the proof of Lemma 4.16 that we can write $h'_{\alpha,s\pm\delta}(t) = 2\Re(L_1) + 2\Re(L_2) + 2\Re(L_3)$, where L_1, L_2, L_3 are defined in (4.18). In order to get an estimate on the integral of $|h'_{\alpha,s\pm\delta}(t)|^2$ it suffices to analyse the various products L_iL_j and $L_i\overline{L_j}$ for $i, j = 1, 2, 3$. The product $L_1\overline{L_1}$ gives

$$\frac{1}{T}\int_T^{2T} L_1\overline{L_1}ds = 2\pi \left|\frac{\Gamma(it)}{t^\alpha \Gamma(3/2+it)}\right|^2$$

which gives the first term in the statement. In order to get the error term we need an estimate on all the other products. We discuss one of the products and we state the bounds that we get on the others. Consider the product $L_1\overline{L_2}$. Then we have

$$\int_T^{2T} L_1\overline{L_2}ds = \frac{2\sqrt{\pi}\,\Gamma(it)}{i^\alpha t^{2\alpha}\,\Gamma(3/2+it)\Gamma(\alpha)}\int_0^\infty (1-e^{-iv})^{1/2}e^{-tv} \times \int_0^\infty (1-e^{-2s_0\mp 2\delta+2i\lambda/t+iv})^{1/2}e^{-\lambda}\int_T^{2T}\frac{e^{it(s-s_0)}}{((s-s_0)t+i\lambda)^{1-\alpha}}dsd\lambda dv$$

Bounding everything in absolute value and using that $0 < \alpha < 1$ in order to bound on one hand $|(s-s_0)t+i\lambda|^{\alpha-1} \leq \lambda^{\alpha-1}$ and on the other $|(s-s_0)t+i\lambda|^{\alpha-1} \leq 2^{1-\alpha}|st|^{\alpha-1}$ we obtain

$$\frac{1}{T}\int_T^{2T} L_1\overline{L_2}ds \ll \min\left\{\frac{1}{|t|^{2+2\alpha}(1+|t|)}, \frac{1}{|t|^{2+2\alpha}(1+|t|)}\frac{1}{\Gamma(\alpha)|Tt|^{1-\alpha}}\right\} \ll \frac{1}{|t|^{2+2\alpha}(1+|t|)}\frac{1}{\big(1+\Gamma(\alpha)|Tt|^{1-\alpha}\big)}$$

with implied absolute constant. Now we list the estimates one can get for the other products L_iL_j and $L_i\overline{L_j}$. For shortening notation write

$$A(i,j)=\frac{|t|^{2+2\alpha}(1+|t|)}{T}\int_T^{2T} L_iL_j ds, \quad B(i,j)=\frac{|t|^{2+2\alpha}(1+|t|)}{T}\int_T^{2T} L_i\overline{L_j} ds,$$

so for instance $B(1,1)$ gives the main term and $B(1,2)$ is the case that we just discussed explicitly. Then we have

$$\begin{gathered}A(1,1)\ll\frac{1}{1+|Tt|};\quad A(1,2),B(1,2)\ll\frac{1}{1+\Gamma(\alpha)|Tt|^{1-\alpha}};\\ A(1,3),B(1,3)\ll\frac{e^{-2T}}{T};\quad A(2,2),B(2,2)\ll\frac{1}{1+\Gamma(\alpha)^2|Tt|^{2-2\alpha}};\\ A(2,3),B(2,3)\ll\frac{e^{-2T}}{T}\frac{1}{\left(1+\Gamma(\alpha)|Tt|^{1-\alpha}\right)};\quad A(3,3),B(3,3)\ll\frac{e^{-4T}}{T}.\end{gathered}$$

All the implied constants are absolute. Summing up all the relevant bounds we conclude the proof of the lemma. □

Lemma 4.21. *Let $0<\alpha<1$, let $T>2$, and let $t_1,t_2\in\mathbb{R}$, $t_1,t_2\neq 0$, $t_1\neq t_2$. Then for $0\le\delta<1$ and $s>2$*

$$\begin{aligned}\frac{1}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t_1)\overline{h'_{\alpha,s\pm\delta}(t_2)}ds \ &\ll\ \frac{1}{|t_1t_2|^{1+\alpha}(1+\sqrt{|t_1|})(1+\sqrt{|t_2|})}\\ &\times\left(\frac{1}{1+T|t_1-t_2|}+\frac{1}{1+T|t_1+t_2|}+\frac{1}{1+\Gamma(\alpha)T^{2-2\alpha}|t_1t_2|^{1-\alpha}}\right)\end{aligned}$$

where the implied constant is absolute.

Proof. We argue similarly to the proof of the previous lemma. Since we only care about upper bounds it is convenient to consider the sum of the integrals L_1+L_3. Let us call then P the sum $P=L_1+L_3$. In analysing the product $h'_{\alpha,s\pm\delta}(t_1)\overline{h'_{\alpha,s\pm\delta}(t_2)}$ we need to analyse the products PP, $P\overline{P}$, L_2L_2, $L_2\overline{L_2}$, and the mixed products PL_2, $P\overline{L_2}$, where in writing the products we assume that one factor is evaluted at $t=t_1$ and the other at t_2. We discuss the case $P\overline{L_2}$ and we list the bounds that we obtain for the other products. We have

$$\begin{aligned}\int_T^{2T} P\,\overline{L_2}\,ds=&\frac{-4}{(t_1t_2)^\alpha\Gamma(\alpha)^2}\int_0^\infty(1-e^{iv})^{1/2}e^{-t_1v}\int_0^\infty(1-e^{-iu})^{1/2}e^{-t_2u}\\ &\times\int_0^\infty\frac{e^{-\lambda}}{(-i\lambda)^{1-\alpha}}\int_0^\infty(1-e^{-2(s_0\pm\delta-i\mu/t_2)+iu})^{1/2}e^{-\mu}\\ &\times\int_T^{2T}\frac{(1-e^{-2(s\pm\delta+i\lambda/t_1)-iv})^{1/2}e^{\pm i\delta(t_1-t_2)-it_2s_0}}{((s-s_0)t_2+i\mu)^{1-\alpha}}e^{ist_1}ds\,d\mu\,d\lambda\,du\,dv.\end{aligned}\tag{4.19}$$

Bounding everything in absolute value, and using $|(1-e^{iv})^{1/2}|\ll\min(1,v^{1/2})$ we get the estimate

$$\frac{1}{T}\int_T^{2T}P\overline{L_2}ds\ll\frac{1}{|t_1t_2|^{1+\alpha}(1+\sqrt{|t_1|})(1+\sqrt{|t_2|})}\min\left\{1,\frac{1}{\Gamma(\alpha)|Tt_2|^{1-\alpha}}\right\}.\tag{4.20}$$

If we instead integrate by parts in the inner integral we get from the exponential e^{ist_1} extra decay in t_1. If we then take absolute value we find the estimate

$$\frac{1}{T}\int_T^{2T} P\overline{L_2}ds \ll \frac{1}{|t_1t_2|^{1+\alpha}(1+\sqrt{|t_1|})(1+\sqrt{|t_2|})}\frac{1}{\Gamma(\alpha)|Tt_1|\,|Tt_2|^{1-\alpha}}.$$

Interpolating this with the second bound in (4.20), and combining the result with the first bound of (4.20), we arrive at the symmetric bound in t_1, t_2

$$\frac{1}{T}\int_T^{2T} P\overline{L_2}ds \ll \frac{1}{|t_1t_2|^{1+\alpha}(1+\sqrt{|t_1|})(1+\sqrt{|t_2|})}\frac{1}{1+\Gamma(\alpha)T^{2-2\alpha}|t_1t_2|^{1-\alpha}}.$$

The implied constant is absolute. Similarly is proven that, denoting by $g(t) = |t|^{-1-\alpha}(1+\sqrt{|t|})^{-1}$, then

$$\frac{1}{T}\int_T^{2T} PPds \ll \frac{g(t_1)g(t_2)}{1+T|t_1+t_2|}; \qquad \frac{1}{T}\int_T^{2T} P\overline{P}ds \ll \frac{g(t_1)g(t_2)}{1+T|t_1-t_2|};$$

$$\left|\frac{1}{T}\int_T^{2T} L_2L_2ds\right| + \left|\frac{1}{T}\int_T^{2T} L_2\overline{L_2}ds\right| + \left|\frac{1}{T}\int_T^{2T} PL_2ds\right| \ll \frac{g(t_1)g(t_2)}{1+\Gamma(\alpha)T^{2-2\alpha}|t_1t_2|^{1-\alpha}}.$$

All the implied constants are absolute. Summing up the relevant estimates finishes the proof. □

4.4 Additional smoothing

In order to have an absolutely convergent pretrace formula, and be able thus to manipulate the spectral series termwise, we need an automorphic kernel $K(z,w) = \sum k(u(z,w))$ such that the Selberg–Harish-Chandra transform $h(t)$ of $k(u)$ is decaying as fast as $|t|^{-2-\varepsilon}$ as $t \to \infty$. However we have seen in Lemma 4.16 that the function $h'_\alpha(t)$ only decays as fast as $|t|^{-3/2-\alpha}$, and therefore for $\alpha \le 1/2$ we don't get an absolutely convergent pretrace formula. In this case we need to use additional smoothing in order to approximate the remainder $e_\alpha(s)$. A standard procedure suffices for our purpose.

Convolution smoothing

Let $\delta > 0$ and consider the function

$$\tilde{k}_\delta(u) := \frac{1}{4\pi\sinh^2(\delta/2)}\mathbf{1}_{[0,(\cosh(\delta)-1)/2]}(u)$$

where $\mathbf{1}_{[0,A]}$ is the indicator function of the set $[0, A]$. It has unit mass, in the sense that

$$\int_{\mathbb{H}} \tilde{k}_\delta(u(z,w))d\mu(z) = 1.$$

Let $k_{s\pm\delta}(u) = \mathbf{1}_{[0,(\cosh(s\pm\delta)-1)/2]}(u)$ and define $k^\pm(u)$ as the functions given by

$$k^\pm(u) := \left(k_{s\pm\delta}(u) * \tilde{k}_\delta\right)(u) = \int_{\mathbb{H}} k_{s\pm\delta}(u(z,v)) * \tilde{k}_\delta(u(v,w))\,d\mu(v).$$

Using the triangle inequality $d(z,w) \leq d(z,v) + d(v,w)$, we observe that when $Z > 0$ the convolution $k_Z(u) * \tilde{k}_\delta$ satisfies

$$(k_Z * \tilde{k}_\delta)(u(z,w)) = \begin{cases} k_Z(u(z,w)) & d(z,w) \leq Z - \delta \\ 0 & d(z,w) \geq Z + \delta. \end{cases}$$

From this we deduce that for $z, w \in \mathbb{H}$

$$k^-(u(z,w)) \leq k_s(u(z,w)) \leq k^+(u(z,w))$$

and summing over $\gamma \in \Gamma$ we have

$$N^-(s,\delta) := \sum_{\gamma\in\Gamma} k^-(u(z,\gamma w)) \;\leq\; N(s,z,w) \;\leq\; \sum_{\gamma\in\Gamma} k^+(u(z,\gamma w)) =: N^+(s,\delta).$$

Defining $\tilde{e}^{\,\pm}(s) := \big(N^\pm(s,\delta) - M(s,z,w)\big)e^{-s/2}$ we obtain

$$\tilde{e}^{\,-}(s) \;\leq\; e(s) \;\leq\; \tilde{e}^{\,+}(s), \tag{4.21}$$

where we recall that $e(s) = (N(s,z,w) - M(s,z,w))\, e^{-s/2}$.The advantage of taking a convolution smoothing is that the Selberg–Harish-Chandra transform $h^\pm$ of the convolution kernel $k^\pm = k_{s\pm\delta} * \tilde{k}_\delta$ is the product $h^\pm(t) = h_{s\pm\delta}(t)\tilde{h}_\delta(t)$ of the two Selberg–Harish-Chandra transforms $h_{s\pm\delta}, \tilde{h}_\delta$ associated to the kernels $k_{s\pm\delta}, \tilde{k}_\delta$. In [15, Lemma 2.4], an expression is given for $h_R(t)$ in terms of special functions. We have, for every $R > 0$ and every $t \in \mathbb{C}$ such that $it \notin \mathbb{Z}$,

$$h_R(t) = 2\sqrt{2\pi \sinh R}\, \Re\left(e^{its}\frac{\Gamma(it)}{\Gamma(3/2+it)}F\left(-\frac{1}{2};\frac{3}{2};1-it;\frac{1}{(1-e^{2R})}\right)\right). \tag{4.22}$$

For t purely imaginary, $|t| < 1/2$, we get (see [52, Lemma 2.1] and [15, Lemma 2.4])

$$h_R(t) = \sqrt{2\pi \sinh R}\, e^{R|t|}\frac{\Gamma(|t|)}{\Gamma(3/2+|t|)} + O\left(\left(1+|t|^{-1}\right)e^{R\left(\frac{1}{2}-|t|\right)}\right). \tag{4.23}$$

For $R \leq 1$, there is a different expansion for $h_R(t)$ (see [15, Lemma 2.4]). Indeed, for $0 \leq R \leq 1$ and $t \in \mathbb{C}$ we can write

$$h_R(t) = 2\pi R^2 \frac{J_1(Rt)}{Rt}\sqrt{\frac{\sinh R}{R}} + O\left(R^2 e^{R|\Im t|}\min\{R^2, |t|^{-2}\}\right). \tag{4.24}$$

The expansion of $h_R(t)$ for small radius R implies that the function $\tilde{h}_\delta(t)$ satisfies

$$\tilde{h}_\delta(t) = \begin{cases} 1 + O(\delta|t| + \delta^2) & \delta|t| < 1 \\ O\left(\frac{1}{(\delta|t|)^{3/2}}\right) & \delta|t| \geq 1 \end{cases} \tag{4.25}$$

when $\Im(t)$ is bounded. Define

$$\begin{aligned} M^\pm(s,\delta) := &\sum_{t_j \in [0,\frac{i}{2}]} h^\pm(t_j)\phi_j(z)\overline{\phi_j(w)} \\ &\qquad + \frac{1}{4\pi}\sum_{\mathfrak{a}} E_{\mathfrak{a}}(z,1/2)\overline{E_{\mathfrak{a}}(w,1/2)}\int_{\mathbb{R}} h^\pm(t)dt, \end{aligned} \tag{4.26}$$

and set

$$e^\pm(s,\delta) := \frac{N^\pm(s,\delta) - M^\pm(s,\delta)}{e^{(s\pm\delta)/2}}. \tag{4.27}$$

Using the estimates above we can now prove the following lemma:

Lemma 4.22. *Let $s > 0$ and $0 < \delta < 1$. Then there exists functions $P^{\pm}(s,\delta)$ such that*

$$e^{-}(s,\delta) + P^{-}(s,\delta) \ \le\ e(s) \ \le\ e^{+}(s,\delta) + P^{+}(s,\delta)$$

Moreover there exist $0 < \varepsilon_\Gamma < 1/4$ such that

$$P^{\pm}(s,\delta) = O(\delta e^{s/2} + s\delta^{1/2} + e^{-\varepsilon_\Gamma s})$$

The implied constants depend on z, w, and the group Γ.

Proof. Using (4.21) we see that the inequality is satisfied if we set

$$\begin{aligned} P^{\pm}(s,\delta) &= \frac{N^{\pm}(s,\delta) - M(s,z,w)}{e^{s/2}} - \frac{N^{\pm}(s,\delta) - M^{\pm}(s,\delta)}{e^{(s\pm\delta)/2}} \\ &= \frac{M^{\pm}(s,\delta) - M(s,z,w)}{e^{s/2}} + \frac{|N^{\pm}(s,\delta) - M^{\pm}(s,\delta)|}{e^{s/2}} O(\delta). \end{aligned}$$

By discreteness there exist an $0 < \varepsilon_\Gamma < 1/4$ such that any imaginary $t_j \neq i/2$ satisfies $\varepsilon_\Gamma \le |t_j| \le 1/2 - \varepsilon_\Gamma$. Using the above expansions of $h_R(t)$ and $\tilde{h}_\delta$ together with various Taylor expansions one can show

$$\begin{aligned} h^{\pm}(i/2) &= 2\pi(\cosh s - 1) \ +\ O(\delta e^{s}) \\ h^{\pm}(0) &= 4\big(s + 2(\log 2 - 1)\big)e^{s/2} \ +\ O(s\,\delta\, e^{s/2} + e^{-s/2}) \\ h^{\pm}(t_j) &= \sqrt{\pi}\frac{\Gamma(|t_j|)}{\Gamma(3/2 + |t_j|)} e^{s(1/2+|t_j|)} \ +\ O(\delta e^{s(1-\varepsilon_\Gamma)} + e^{s(1/2-\varepsilon_\Gamma)}), \end{aligned}$$

for $t_j \in (0, i/2)$, and via Fourier inversion we see that

$$\int_{\mathbb{R}} h^{\pm}(t)dt = 4\pi e^{s/2} + O(\delta^{1/2} e^{s/2} + e^{-s/2}).$$

It follows that

$$M^{\pm}(s,\delta) = M(s,z,w) + O\left(1 + \delta e^{s} + s\delta\, e^{s/2} + \delta^{1/2} e^{s/2} + e^{s(1/2-\varepsilon_\Gamma)}\right). \tag{4.28}$$

From the pre-trace formula we find

$$N^{\pm}(s,\delta) - M^{\pm}(s,\delta) = \sum_{t_j > 0} h^{\pm}(t_j)\phi_j(z)\overline{\phi_j(w)} + \frac{1}{4\pi}\sum_{\mathfrak{a}} \int_{\mathbb{R}} h^{\pm}(t) E_{\mathfrak{a}}(t) dt$$

where

$$E_{\mathfrak{a}}(t) = E_{\mathfrak{a}}(z, 1/2 + it)\overline{E_{\mathfrak{a}}(w, 1/2 + it)} - E_{\mathfrak{a}}(z, 1/2)\overline{E_{\mathfrak{a}}(w, 1/2)} \tag{4.29}$$

We notice that by the previous expressions we find

$$h^{\pm}(t) \ll \frac{e^{s/2}}{|t|^{1}(1 + \sqrt{|t|})} \frac{1}{\big(1 + |\delta t|^{3/2}\big)} \tag{4.30}$$

so we may indeed apply the pre-trace formula. Using (4.30) and (4.10) we find $N^{\pm}(s,\delta) - M^{\pm}(s,\delta) = O(e^{s/2}\delta^{-1/2})$. Combining these estimates the bound on $P^{\pm}(s,\delta)$ follows easily. $\square$

Since we want to study the α-integrated problem, we will integrate the inequality in Lemma 4.22 to get an analogous inequality in the α-integrated case. We set

$$e_\alpha^\pm(s,\delta) = I_\alpha(e^\pm(s,\delta)). \tag{4.31}$$

Integration now gives the following corollary:

Corollary 4.23. *Let $0 < \alpha < 1$ and let $0 < \delta < 1 < s$. Then there exists functions $P_\alpha^\pm(s,\delta)$ such that*

$$e_\alpha^-(s,\delta) + P_\alpha^-(s,\delta) \;\leq\; e_\alpha(s) \;\leq\; e_\alpha^+(s,\delta) + P_\alpha^+(s,\delta),$$

where

$$P_\alpha^\pm(s,\delta) = O\left(\delta e^{s/2} + s^{1+\alpha}\delta^{1/2} + s^\alpha e^{-\frac{\varepsilon_\Gamma s}{2}} + \frac{1}{\Gamma(\alpha)s^{1-\alpha}}\right)$$

and the implied constant depends on z, w, and the group Γ.

Proof. From Lemma 4.4 and the fact that integration preserves inequalities we see that the inequality is satisfied for $P_\alpha^\pm(s,\delta) = I_\alpha(P^\pm(s,\delta))$.

Using now (4.14) and the bound

$$|I_\alpha(e^{-\beta s})| \leq \frac{(cs)^\alpha e^{-\beta(1-c)s}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)(cs)^{1-\alpha}\beta} \qquad \beta > 0, \forall\, 0 < c < 1 \tag{4.32}$$

we find, by integrating the inequality in Lemma 4.22 and choosing $c = 1/2$, the desired bound on $P_\alpha^\pm(s,\delta)$. ☐

4.5 Pointwise Estimates

In this section we prove Theorem 4.4. We start by considering the function $e_\alpha^\pm(s,\delta)$ constructed in Section 4.4. From Corollary 4.23 we conclude that

$$|e_\alpha(s)| \ll \max_\pm(|e_\alpha^\pm(s,\delta)| + |P_\alpha^\pm(s,\delta)|). \tag{4.33}$$

We will prove an upper bound on the right-hand side, which will then imply a bound on $e_\alpha(s)$. Consider the function

$$h_\alpha'^\pm(t) = I_\alpha\left(\frac{h_{s\pm\delta}(t)}{e^{(s\pm\delta)/2}}\right)\tilde{h}_\delta(t) = h'_{\alpha,s\pm\delta}(t)\tilde{h}_\delta(t)$$

where $h_{s\pm\delta}(t)$ and $\tilde{h}_\delta(t)$ are as in section 4.4. Using Lemma 4.16 and (4.25) we get the estimate

$$h_\alpha'^\pm(t) \ll \frac{1}{|t|^{1+\alpha}(1+\sqrt{|t|})}\frac{1}{(1+|\delta t|^{3/2})}.$$

We have therefore that $h_\alpha'^\pm(t) \ll |t|^{-2-\varepsilon}$ decays fast enough to ensure absolute convergence of the pretrace formula. Using the pre-trace formula and the definition of $e_\alpha^\pm(s,\delta)$ we find

$$e_\alpha^\pm(s,\delta) = \sum_{t_j>0} h_\alpha'^\pm(t_j)\phi_j(z)\overline{\phi_j(w)} + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{\mathbb{R}} h_\alpha'^\pm(t)E_{\mathfrak{a}}(t)dt \tag{4.34}$$

(recall (4.29)).

Consider first the discrete spectrum: Using the decay of $h'^{\pm}_{\alpha}(t)$ we can split the sum at $t_j = \delta^{-1}$ and using (4.10) and a standard dyadic decomposition we obtain the bound

$$\sum_{t_j>0} h'^{\pm}_{\alpha}(t_j)\phi_j(z)\overline{\phi_j(w)} \ll \sum_{0<t_j<\frac{1}{\delta}} \frac{1}{t_j^{3/2+\alpha}}\left(|\phi_j(z)|^2 + |\phi_j(w)|^2\right)$$
$$+ \frac{1}{\delta^{3/2}} \sum_{t_j \geq \frac{1}{\delta}} \frac{1}{t_j^{3+\alpha}}\left(|\phi_j(z)|^2 + |\phi_j(w)|^2\right)$$
$$\ll \left(\frac{1}{\delta}\right)^{2-\frac{3}{2}-\alpha} + \frac{1}{\delta^{3/2}}\left(\frac{1}{\delta}\right)^{-1-\alpha} + O(1) \ll \left(\frac{1}{\delta}\right)^{\frac{1}{2}-\alpha} + O(1).$$

Similarly, using the analyticity of the Eisenstein series we have that $E_{\mathfrak{a}}(t) = O(|t|)$ for $|t| < 1$, and so using (4.10) to bound the Eisenstein series we obtain

$$\int_{\mathbb{R}} h'^{\pm}_{\alpha}(t)\, E_{\mathfrak{a}}(t) dt \ll \int_{|t|<1} \frac{1}{|t|^{\alpha}} dt + \int_{1 \leq |t| < \frac{1}{\delta}} \frac{|E_{\mathfrak{a}}(t)|}{|t|^{3/2+\alpha}} dt + \frac{1}{\delta^{3/2}} \int_{|t| \geq \frac{1}{\delta}} \frac{|E_{\mathfrak{a}}(t)|}{|t|^{3+\alpha}} dt$$
$$\ll \left(\frac{1}{\delta}\right)^{1/2-\alpha} + O(1).$$

In the case when $\alpha = 1/2$ a logarithmic term $\log \delta^{-1}$ instead of a power of δ appears. Combining the result with (4.33) we obtain

$$e_{\alpha}(s) = O\left(\delta e^{s/2} + s^{1+\alpha}\delta^{1/2} + \delta^{-1/2+\alpha} + 1\right).$$

The theorem follows by choosing $\delta = e^{-s/(3-2\alpha)}$. For $\alpha = 1/2$ we get $e_{\alpha}(s) = O(\delta e^{s/2} + s^{3/2}\delta^{1/2} + \log \delta^{-1} + 1)$, and $\delta = e^{-s/2}$ gives the result.

Remark 4.24. In the case $\alpha = 1$ the analog result of Theorem 4.4 differs on whether the group Γ is cocompact or cofinite but not cocompact. In the first case the proof works fine and we obtain that $e_1(s) = O(1)$. If Γ is cofinite not cocompact, however, this type of proof doesn't provide $e_1(s) = O(1)$, due to the contribution of the Eisenstein series near the point $t = 0$. We can indeed in this case only bound as follows:

$$\int_{\mathbb{R}} h'^{\pm}_{1}(t) E_{\mathfrak{a}}(t) dt = \int_{|t|<\varepsilon} + \int_{\varepsilon \geq |t| < 1} + \int_{|t| \geq 1}$$
$$\ll \int_{|t|<\varepsilon} s^2 dt + \int_{\varepsilon \leq |t| < 1} \frac{1}{|t|} dt + \int_{|t| \geq 1} \frac{1}{|t|^{3/2}} dt$$
$$\ll \varepsilon s^2 + \log \frac{1}{\varepsilon} + 1.$$

Choosing $\varepsilon = s^{-2}$ (and $\delta = e^{-s}$ to bound the error coming from the approximation of the main term) we obtain that for Γ a cofinite not cocompact group and $\alpha = 1$ we have

$$e_1(s) \ll \log s$$

and hence we cannot show finiteness in this case.

4.6 First moment of integrated normalized remainder

In this section we prove Theorem 4.6. We will show that for $\delta = e^{-T}$ we have

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} e_\alpha^\pm(s)ds = \lim_{T\to\infty} \frac{1}{T}\int_T^{2T} P_\alpha^\pm(s)ds = 0 \tag{4.35}$$

which will allow us to conclude, using Corollary 4.23, that

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} e_\alpha(s)ds = 0.$$

The last part of (4.35) is easily proven by direct integration of the pointwise bounds on $P_\alpha^\pm(s)$ given in Lemma 4.22. Indeed we have

$$\begin{aligned}\frac{1}{T}\int_T^{2T} P_\alpha^\pm(s)ds &= O\left(\frac{1}{T}\int_T^{2T}\left(\delta e^{s/2} + s^{1+\alpha}\delta^{1/2} + s^\alpha e^{-\frac{\varepsilon_\Gamma s}{2}} + \frac{1}{\Gamma(\alpha)s^{1-\alpha}}\right)ds\right)\\ &= O\left(\frac{\delta e^T}{T} + T^{1+\alpha}\delta^{1/2} + T^\alpha e^{-\frac{\varepsilon_\Gamma T}{2}} + \frac{1}{\Gamma(\alpha)T^{1-\alpha}}\right).\end{aligned}$$

Plugging $\delta = e^{-T}$ we get

$$\frac{1}{T}\int_T^{2T} P_\alpha^\pm(s)ds \ll \frac{1}{T} + \frac{T^{1+\alpha}}{e^{T/2}} + \frac{T^\alpha}{e^{\frac{\varepsilon_\Gamma T}{2}}} + \frac{1}{\Gamma(\alpha)T^{1-\alpha}}$$

which tends to zero as $T \to \infty$.

In order to analyze the first integral in (4.35) use again the expansion (4.34). Since the series and the integral are absolutely convergent, we can integrate termwise and obtain

$$\int_T^{2T} e_\alpha^\pm(s,\delta)ds = \sum_{t_j>0}\int_T^{2T} h_\alpha'^\pm(t_j)ds\ \phi_j(z)\overline{\phi_j(w)} + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{\mathbb{R}} E_{\mathfrak{a}}(t)\int_T^{2T} h_\alpha'^\pm(t)ds\,dt,$$

Using now Lemma 4.18 and (4.25) we can bound

$$\begin{aligned}\frac{1}{T}\int_T^{2T} h_\alpha'^\pm(t)ds &= \frac{\tilde{h}_\delta(t)}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t)ds\\ &\ll \frac{1}{|t|^{1+\alpha}(1+\sqrt{|t|})(1+|\delta t|^{3/2})}\left(\frac{1}{1+T|t|} + \frac{e^{-2T}}{T} + \frac{1}{1+|Tt|^{1-\alpha}\Gamma(\alpha)}\right).\end{aligned}$$

Consider the contribution of the discrete spectrum. We get

$$\begin{aligned}&\sum_{t_j>0}\frac{1}{T}\int_T^{2T} h_\alpha'^\pm(t_j)ds\ \ \phi_j(z)\overline{\phi_j(w)}\\ &\ll \sum_{0<t_j\le\delta^{-1}}\left(\frac{1}{T|t_j|^{5/2+\alpha}} + \frac{e^{-2T}}{T|t_j|^{3/2+\alpha}} + \frac{1}{T^{1-\alpha}\Gamma(\alpha)|t_j|^{5/2}}\right)\left(|\phi_j(z)|^2 + |\phi_j(w)|^2\right)\\ &\quad + \frac{1}{\delta^{3/2}}\sum_{t_j>\delta^{-1}}\left(\frac{1}{T|t_j|^{4+\alpha}} + \frac{e^{-2T}}{T|t_j|^{3+\alpha}} + \frac{1}{T^{1-\alpha}\Gamma(\alpha)|t_j|^4}\right)\left(|\phi_j(z)|^2 + |\phi_j(w)|^2\right)\\ &\quad \ll \frac{1}{T} + \frac{e^{-2T}}{T\delta^{1/2-\alpha}} + \frac{1}{T^{1-\alpha}\Gamma(\alpha)},\end{aligned}$$

where we have used (4.10). For $\delta = e^{-T}$ this tends to zero as $T \to \infty$.

Consider next the continuous spectrum. Split the integral into three pieces, where we integrate respectively over $\{|t| \le 1\}$, $\{1 < |t| \le \delta^{-1}\}$, and $\{|t| > \delta^{-1}\}$. Let $\sigma = (1-\alpha)/2$ and $\lambda = 1/2$. We get

$$\begin{aligned}
&\frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{\mathbb{R}} E_{\mathfrak{a}}(t)\frac{1}{T}\int_T^{2T} h_\alpha'^{\pm}(t)ds\,dt\\
&\quad\ll \int_{|t|\le 1}\Big(\frac{1}{T^\sigma|t|^{\alpha+\sigma}} + \frac{1}{\Gamma(\alpha)^\lambda T^{(1-\alpha)\lambda}|t|^{\alpha+(1-\alpha)\lambda}} + \frac{e^{-2T}}{T|t|^\alpha}\Big)dt\\
&\quad + \int_{1<|t|\le\delta^{-1}}\Big(\frac{1}{T|t|^{5/2+\alpha}} + \frac{e^{-2T}}{T|t|^{3/2+\alpha}} + \frac{1}{\Gamma(\alpha)T^{1-\alpha}|t|^{5/2}}\Big)|E_{\mathfrak{a}}(t)|\,dt\\
&\quad + \frac{1}{\delta^{3/2}}\int_{|t|>\delta^{-1}}\Big(\frac{1}{T|t|^{4+\alpha}} + \frac{e^{-2T}}{T|t|^{3+\alpha}} + \frac{1}{\Gamma(\alpha)T^{1-\alpha}|t|^4}\Big)|E_{\mathfrak{a}}(t)|\,dt\\
&\quad \ll_\alpha \frac{1}{T^\sigma} + \frac{e^{-2T}}{T\delta^{1/2-\alpha}}.
\end{aligned}$$

Plugging $\delta = e^{-T}$ and taking the limit as $T\to\infty$ we get zero. Putting together the discrete and continuous contributions concludes the proof of Theorem 4.6.

4.7 Computing the variance

We have proved a pointwise bound and a mean value result for $e_\alpha(s)$. Now we look at the second moment of $e_\alpha(s)$. We start by using the pre-trace formula to write $e_\alpha^{\pm}(s)$ in the following way:

$$e_\alpha^{\pm}(s) = f_\alpha(s,\delta) + g_\alpha^{\pm}(s,\delta) + Q_\alpha^{\pm}(s,\delta) \tag{4.36}$$

where

$$f_\alpha(s,\delta) := \sum_{0<t_j<\delta^{-1}} \Re\big(r_\alpha(t_j)e^{it_js}\big)\phi_j(z)\overline{\phi_j(w)}, \tag{4.37}$$

with

$$r_\alpha(t) = \frac{2\sqrt{\pi}\,\Gamma(it)}{(it)^\alpha\,\Gamma(3/2+it)}, \tag{4.38}$$

and the functions $g_\alpha^{\pm}(s)$ are defined by

$$\begin{aligned}
g_\alpha^{\pm}(s,\delta) &= A^{\pm}(s,\delta) + B^{\pm}(s,\delta), \text{ where}\\
A^{\pm}(s,\delta) &= {\sum_{t_j\ge\delta^{-1}}}' \tilde{h}_\delta(t_j)h'_{\alpha,s\pm\delta}(t_j)b_j\\
B^{\pm}(s,\delta) &= {\sum_{0<t_j<\delta^{-1}}}' \Big(\tilde{h}_\delta(t_j)h'_{\alpha,s\pm\delta}(t_j) - \Re(r_\alpha(t_j)e^{it_js})\Big)b_j.
\end{aligned}$$

Here b_j is defined in (4.12), $h'_{\alpha,s\pm\delta}(t)$ in (4.16), and $\tilde{h}_\delta(t)$ is as in section 4.4. The functions $Q_\alpha^{\pm}(s,\delta)$ are the contributions coming from the Eisenstein series, given by

$$Q_\alpha^{\pm}(s,\delta) = \sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{\mathbb{R}} h_\alpha'^{\pm}(t)E_{\mathfrak{a}}(t)dt,$$

and $E_{\mathfrak{a}}(t)$ is as in (4.29).

In bounding the integral of the square of these terms we will often need the following simple estimate, which is extrapolated from [18, 19]:

Lemma 4.25. *For $a > 1$ and a given $t_j > 0$ we have*

$$\sum_{t_j < t_\ell}{}' \frac{|b_\ell|}{t_\ell^a(1 + T|t_\ell - t_j|)} \ll \frac{1}{t_j^{a-1}} \left(1 + \frac{1}{T(a-1)} + \frac{\log(t_j + 1)}{T}\right).$$

For $0 \le c \le 1$ and a given $t_j > 0$ we have

$$\sum_{t_j < t_\ell \le R}{}' \frac{|b_\ell|}{t_\ell^c(1 + T|t_\ell - t_j|)} \ll t_j^{1-c} \left(1 + \frac{\log(R+1)}{T}\right) + \frac{R^{1-c}}{T(1-c)} \quad 0 \le c < 1,$$

and the last term is to be replaced by $T^{-1} \log(R + 1)$ if $c = 1$. The implied constants are absolute.

Proof. Using (4.12) we find

$$\begin{aligned}\sum_{t_j < t_\ell}{}' \frac{|b_\ell|}{t_\ell^a(1 + T|t_\ell - t_j|)} &= \sum_{t_j < t_\ell \le t_j + 1}{}' + \sum_{n=1}^{\infty} \sum_{n < t_\ell - t_j \le n+1}{}' \ll \frac{1}{t_j^{a-1}} + \frac{1}{T} \sum_{n=1}^{\infty} \frac{t_j + n}{(t_j + n)^a n} \\ &\ll \frac{1}{t_j^{a-1}} + \frac{1}{T} \sum_{n \le t_j} \frac{1}{(t_j + n)^{a-1} n} + \frac{1}{T} \sum_{n \ge t_j} \frac{1}{(t_j + n)^{a-1} n} \\ &\ll \frac{1}{t_j^{a-1}} + \frac{1}{T t_j^{a-1}} \sum_{n \le t_j} \frac{1}{n} + \frac{1}{T} \sum_{n \ge t_j} \frac{1}{n^a} \\ &\ll \frac{1}{t_j^{a-1}} \left(1 + \frac{1}{T(a-1)} + \frac{\log(t_j + 1)}{T}\right).\end{aligned}$$

The second statement is proved analogously. □

We remark that by the above lemma, a symmetry argument, and partial summation and (4.10) we find from the above lemma that for $a > 3/2$ and $c > 0$ large we have

$$\sum_{\substack{t_j, t_\ell \ge c \\ t_j \ne t_\ell}}{}' \frac{|b_j b_\ell|}{(t_j t_\ell)^a} \frac{1}{(1 + T|t_j - t_\ell|)} \ll \frac{c^{3-2a}}{2a - 3} \left(1 + \frac{\log c}{T}\right) \tag{4.39}$$

The implied constant depends on z,w, and Γ.

We are now ready to show that the functions $g_\alpha^\pm(s, \delta)$ are small on average. More precisely we have the following lemma:

Lemma 4.26. *For $\delta = e^{-T}$ we have*

$$\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} |g_\alpha^\pm(s, \delta)|^2 ds = 0.$$

Proof. Using (4.25), Lemmata 4.20 and 4.21 together with (4.12), (4.10) and

(4.39) we find

$$\begin{aligned}
\frac{1}{T}\int_T^{2T}|A^{\pm}(s,\delta)|^2ds &= \sum_{t_j,t_\ell\geq\delta^{-1}}{}' \tilde{h}_\delta(t_j)\tilde{h}_\delta(t_\ell)\, b_j\overline{b_\ell}\,\frac{1}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t_j)\overline{h'_{\alpha,s\pm\delta}(t_\ell)}ds \\
&\ll \sum_{t_j\geq\delta^{-1}}{}' \frac{|b_j|^2}{1+|\delta t_j|^3}\frac{1}{|t_j|^{3+2\alpha}} \\
&\quad + \sum_{\substack{t_j\geq\delta^{-1}\\ t_j\neq t_\ell}}{}' \sum_{t_\ell\geq\delta^{-1}}{}' \frac{|b_jb_\ell|}{|t_jt_\ell|^{3/2+\alpha}(1+|\delta t_j|^{3/2})(1+|\delta t_\ell|^{3/2})} \qquad (4.40)\\
&\quad \times\left(\frac{1}{1+T|t_j-t_\ell|}+\frac{1}{1+\Gamma(\alpha)T^{2-2\alpha}|t_jt_\ell|^{1-\alpha}}\right)\\
&\ll \delta^{2\alpha}+\delta^{2\alpha}\left(1+\frac{\log\delta^{-1}}{T}\right)+\frac{\delta}{\Gamma(\alpha)\,T^{2-2\alpha}}.
\end{aligned}$$

The implied constant doesn't depend on α. Choosing $\delta=e^{-T}$ and taking the limit as $T\to\infty$ we get zero.

For the analysis of $B^{\pm}(s,\delta)$ a long and tedious computation like in the proof of Lemma 4.20 and Lemma 4.21 shows that for $0<\delta<1$ we have

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T}\Big(\tilde{h}_\delta(t_j)h'_{\alpha,s\pm\delta}(t_j)-\Re(r_\alpha(t_j)e^{it_js})\Big)\overline{\Big(\tilde{h}_\delta(t_\ell)h'_{\alpha,s\pm\delta}(t_\ell)-\Re(r_\alpha(t_\ell)e^{it_\ell s})\Big)}ds\\
&\ll \frac{1}{|t_jt_\ell|^{3/2+\alpha}}\Bigg(\frac{(\delta|t_j|+\delta^2)(\delta|t_\ell|+\delta^2)+e^{-2T}(\delta|t_\ell|+\delta^2)+(\delta|t_j|+\delta^2)e^{-2T}+e^{-4T}}{1+T|t_j-t_\ell|}\\
&\qquad\qquad + \frac{1}{1+\Gamma(\alpha)T^{2-2\alpha}|t_jt_\ell|^{1-\alpha}}\Bigg)
\end{aligned}$$

where we have used the estimate

$$|e^{\pm i\delta t_j}-\tilde{h}_\delta(t_j)| = O\big(\delta|t_j|+\delta^2\big).$$

With this we can estimate, with the same reasoning used in bounding $A^{\pm}(s,\delta)$, and choosing $\delta=e^{-T}$,

$$\frac{1}{T}\int_T^{2T}|B^{\pm}(s,\delta)|^2\,ds \ll \begin{cases}\frac{\delta^{2\alpha}}{2-2\alpha}\left(1+\frac{1}{T|\alpha-1/2|}\right)+\frac{1}{\Gamma(\alpha)T^{2-2\alpha}} & \alpha\neq 1/2\\ \delta+\frac{1}{\Gamma(\alpha)T^{2-2\alpha}} & \alpha=1/2.\end{cases} \qquad (4.41)$$

The implied constant doesn't depend on α. As $\delta=e^{-T}$, taking the limit as $T\to\infty$ this goes to zero, and this proves the lemma. □

Variance, cocompact groups

We are now ready to prove – in the co-compact case – that the variance of $e_\alpha(s)$ is finite. By Corollary 4.23 we find

$$|e_\alpha(s)-f_\alpha(s,\delta)|\leq \max_{\pm}\left\{\,|g_\alpha^{\pm}(s,\delta)+P^{\pm}(s,\delta)|\,\right\}.$$

Now we claim that for $\delta=\delta(T)=e^{-T}$ we have

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}|g_\alpha^{\pm}(s,\delta)|^2ds = \lim_{T\to\infty}\frac{1}{T}\int_T^{2T}|P_\alpha^{\pm}(s,\delta)|^2ds = 0.$$

The first limit is proven in Lemma 4.26, while the second limit can be proven by using the pointwise bound on $P_\alpha^{\pm}(s,\delta)$ from Corollary 4.23, since

$$\frac{1}{T}\int_T^{2T}|P_\alpha^{\pm}(s,\delta)|^2ds \ll \frac{1}{T}\int_T^{2T}\left(\delta e^{s/2}+s^{1+\alpha}\delta^{1/2}+s^{\alpha}e^{-\frac{\varepsilon_\Gamma s}{2}}+\frac{1}{\Gamma(\alpha)s^{1-\alpha}}\right)^2ds$$
$$\ll \frac{\delta^2e^{2T}}{T}+T^{2+2\alpha}\delta+\frac{T^{2\alpha}}{e^{\varepsilon_\Gamma T}}+\frac{1}{\Gamma(\alpha)^2T^{2-2\alpha}}, \tag{4.42}$$

so that choosing $\delta = e^{-T}$ and taking the limit as $T\to\infty$ we get zero. The implied constant is independent of α. If we can now compute the second moment of $f_\alpha(s,\delta)$ for $\delta = e^{-T}$ and show that it is asymptotically finite, then we may conclude

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}|e_\alpha(s)|^2ds = \lim_{T\to\infty}\frac{1}{T}\int_T^{2T}|f_\alpha(s,\delta)|^2ds.$$

The explicit expression for the right-hand side will give the sum appearing in the statement of the theorem, and this will conclude the proof. The problem therefore reduces to computing the second moment of $f_\alpha(s,\delta)$ for $\delta = e^{-T}$. For this we follow Cramér [18, p.149-150] and Landau [46, Proof of Satz 476]. We can write

$$f_\alpha(s,e^T) = {\sum_{0<t_j<e^T}}' \Re(r_\alpha(t_j)e^{it_js})b_j, \tag{4.43}$$

We obtain

$$\frac{1}{T}\int_T^{2T}|f_\alpha(s,\delta)|^2ds = {\sum_{0<t_j<e^T}}'{\sum_{0<t_\ell<e^T}}' b_j\overline{b_\ell}\frac{1}{T}\int_T^{2T}\Re(r_\alpha(t_j)e^{it_js})\Re(r_\alpha(t_\ell)e^{it_\ell s})\,ds$$
$$=\frac{1}{2}{\sum_{0<t_j<e^T}}'|b_jr_\alpha(t_j)|^2 \tag{4.44}$$
$$+O\left({\sum_{0<t_j<e^T}}'\frac{|b_jr_\alpha(t_j)|^2}{T|t_j|}\right)+O\left({\sum_{0<t_j<e^T}}'{\sum_{\substack{0<t_\ell<e^T\\ t_j\neq t_\ell}}}'\frac{|b_jb_\ell r_\alpha(t_j)r_\alpha(t_\ell)|}{1+T|t_j-t_\ell|}\right).$$

The middle sum is bounded (uniformly in α) by $O(T^{-1})$, while for $T>1$ the last sum is clearly bounded by

$${\sum_{0<t_j<e^T}}'{\sum_{\substack{0<t_\ell<e^T\\ t_j\neq t_\ell}}}'\frac{|b_jb_\ell r_\alpha(t_j)r_\alpha(t_\ell)|}{1+T|t_j-t_\ell|} = O\left({\sum_{0<t_j}}'{\sum_{\substack{0<t_\ell\\ t_j\neq t_\ell}}}'\frac{|b_jb_\ell r_\alpha(t_j)r_\alpha(t_\ell)|}{1+|t_j-t_\ell|}\right).$$

If the last sum is finite we may use the dominated convergence to conclude, since each term goes to zero as $T\to\infty$, that the left-hand side is $o(1)$. To prove finiteness, (4.39) and (4.11) allows us to estimate

$${\sum_{0<t_j}}'{\sum_{\substack{t_j<t_\ell\\ t_j\neq t_\ell}}}'\frac{|b_jb_\ell r_\alpha(t_j)r_\alpha(t_\ell)|}{1+|t_j-t_\ell|} \ll {\sum_{0<t_j}}'\frac{|b_jr_\alpha(t_j)|}{t_j^{1/2+\alpha}}\log t_j$$
$$\ll \sum_{0<t_j}\frac{\log t_j}{t_j^{2+2\alpha}}\left(|\phi_j(z)|^2+|\phi_j(w)|^2\right)\ll_\alpha 1.$$

Summarizing we have shown that as $T \to \infty$

$$\frac{1}{T}\int_T^{2T} |f_\alpha(s,\delta)|^2 ds = \frac{1}{2}\sideset{}{'}\sum_{0<t_j<e^T} |b_j r_\alpha(t_j)|^2 + o_\alpha(1).$$

Taking the limit as $T \to \infty$ and using the definition of $r_\alpha(t_j)$ and b_j proves the theorem. Observe that the series on the right is convergent as $T \to \infty$, again by (4.11).

Variance, cofinite groups

We now explain the changes needed in the cofinite case of Theorem 4.8: The proof given for cocompact groups extends to cofinite groups for the analysis of the discrete spectrum. It is in the control of the continuous spectrum that we need the assumption (4.6).

We have, from Corollary 4.23, that

$$|e_\alpha(s) - f_\alpha(s)| \ll \max_{\pm}\{\, |g_\alpha^\pm(s,\delta) + P^\pm(s,\delta) + Q_\alpha^\pm(s,\delta)|\,\}, \tag{4.45}$$

and we have shown in the proof of Theorem 4.8 in the cocompact case and Lemma 4.26 that for $\delta = e^{-T}$

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T} |g_\alpha^\pm(s,\delta)|^2 ds = \lim_{T\to\infty}\frac{1}{T}\int_T^{2T} |P_\alpha^\pm(s,\delta)|^2 ds = 0$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T} |f_\alpha(s,\delta)|^2 ds = 2\pi \sideset{}{'}\sum_{0<t_j} \frac{|\Gamma(it_j)|^2}{|t_j^\alpha \Gamma(3/2+it_j)|^2}\left|\sum_{t_{j'}=t_j}\phi_{j'}(z)\overline{\phi_{j'}(w)}\right|^2.$$

We will show that also the contribution coming from the Eisenstein series is negligible, namely that for $\delta = e^{-T}$ we have

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T} |Q_\alpha^\pm(s,\delta)|^2 ds = 0.$$

This, using (4.45) and Cauchy-Schwartz inequality, will give the result.

To this end we will show that

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}\left|\int_{\mathbb{R}} h_\alpha'^{\pm}(t)E_{\mathfrak{a}}(t)dt\right|^2 ds = 0. \tag{4.46}$$

We will also use the crude bound $|\tilde{h}_\delta(t)| \ll 1$ for every $0<\delta<1$ and $t\in\mathbb{R}$. For $T>2$ we find, using Lemma 4.21, that

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\left|\int_{\mathbb{R}} h_\alpha'^{\pm}(t)E_{\mathfrak{a}}(t)dt\right|^2 ds &= \int_{\mathbb{R}}\tilde{h}_\delta(t_1)E_{\mathfrak{a}}(t_1)\int_{\mathbb{R}}\tilde{h}_\delta(t_2)E_{\mathfrak{a}}(t_2)\\ &\quad\times\frac{1}{T}\int_T^{2T} h'_{\alpha,s\pm\delta}(t_1)\overline{h'_{\alpha,s\pm\delta}(t_2)}\,ds\,dt_1\,dt_2\end{aligned}$$

$$\ll \int_{\mathbb{R}} \frac{|E_{\mathfrak{a}}(t_1)|}{|t_1|^{1+\alpha}(1+\sqrt{|t_1|})} \int_{\mathbb{R}} \frac{|E_{\mathfrak{a}}(t_2)|}{|t_2|^{1+\alpha}(1+\sqrt{|t_2|})}$$
$$\times \left(\frac{1}{1+T|t_1-t_2|} + \frac{1}{1+T|t_1+t_2|} + \frac{1}{1+\Gamma(\alpha)T^{2-2\alpha}|t_1t_2|^{1-\alpha}} \right) dt_1\, dt_2.$$

Since $|E_{\mathfrak{a}}(t)| = |E_{\mathfrak{a}}(-t)|$ we can bound by the slightly simpler expression

$$\frac{1}{T}\int_T^{2T} \Big| \int_{\mathbb{R}} h'^{\pm}_{\alpha}(t) E_{\mathfrak{a}}(t) dt \Big|^2 ds \ll \int_0^{\infty} \frac{|E_{\mathfrak{a}}(t_1)|}{t_1^{1+\alpha}(1+\sqrt{t_1})} \int_0^{\infty} \frac{|E_{\mathfrak{a}}(t_2)|}{t_2^{1+\alpha}(1+\sqrt{t_2})}$$
$$\times \left(\frac{1}{1+T|t_1-t_2|} + \frac{1}{1+\Gamma(\alpha)T^{2-2\alpha}|t_1t_2|^{1-\alpha}} \right) dt_1\, dt_2. \tag{4.47}$$

For $x > 0$ we have $(1+x)^{-1} \le x^{-r}$ for all $0 \le r \le 1$, so choosing $r = 1/2$ we find – using that $|E_{\mathfrak{a}}(t)| = O(|t|)$ for $|t| < 1$ and the local Weyl law – that

$$\int_0^{\infty}\int_0^{\infty} \frac{|E_{\mathfrak{a}}(t_1)E_{\mathfrak{a}}(t_2)|}{\left(t_1^{1+\alpha}(1+\sqrt{t_1})\right)\left(t_2^{1+\alpha}(1+\sqrt{t_2})\right)\left(1+\Gamma(\alpha)T^{2-2\alpha}|t_1t_2|^{1-\alpha}\right)} dt_1\, dt_2$$
$$\ll \frac{1}{\Gamma(\alpha)^r T^{r(2-2\alpha)}} \left(\int_0^{\infty} \frac{|E_{\mathfrak{a}}(t)|}{t^{1+\alpha+r(1-\alpha)}(1+\sqrt{t})} dt \right)^2$$
$$\ll \frac{1}{\Gamma(\alpha)^{1/2}T^{1-\alpha}} \left(\int_0^1 \frac{1}{t^{(1+\alpha)/2}} dt + \int_1^{\infty} \frac{|E_{\mathfrak{a}}(t)|}{t^{2+\alpha/2}} dt \right)^2 \ll \frac{1}{\Gamma(\alpha)^{1/2}T^{1-\alpha}}.$$

In order to estimate the remaining part of (4.47) we will use the Hardy-Littlewood-Pólya inequality ([28, Theorem 382.]). This implies that given $0 < \sigma < 1$ and $p = 2/(2-\sigma)$, every non-negative function f satisfies

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)f(y)}{|x-y|^{\sigma}} dxdy \ll_{\sigma} \left(\int_0^{\infty} f(x)^p dx \right)^{2/p}. \tag{4.48}$$

Applying first $(1+x)^{-1} \le x^{-\sigma}$, and then (4.48) we find (4.6)

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty} \frac{|E_{\mathfrak{a}}(t_1)E_{\mathfrak{a}}(t_2)|}{(t_1t_2)^{1+\alpha}(1+\sqrt{t_1})(1+\sqrt{t_2})\left(1+T|t_1-t_2|\right)} dt_1\, dt_2 \\ &\qquad \ll \frac{1}{T^{\sigma}} \left(\int_0^{\infty} \frac{|E_{\mathfrak{a}}(t)|^p}{t^{(1+\alpha)p}(1+\sqrt{t})^p} dt \right)^{2/p} \end{aligned} \tag{4.49}$$

If we choose p as in (4.6) (and correspondingly $\sigma = 2 - 2/p$) the last integral is finite since we can bound

$$\int_0^{\infty} \frac{|E_{\mathfrak{a}}(t)|^p}{t^{(1+\alpha)p}(1+\sqrt{t})^p} dt \ll \int_0^1 \frac{dt}{t^{\alpha p}} + \int_1^{\infty} \frac{|E_{\mathfrak{a}}(t)|^p}{t^{(1+\alpha)p}(1+\sqrt{t})^p} dt \ll_{\alpha} 1$$

where for the first term we have used $E_{\mathfrak{a}}(t) = O(|t|)$ for $|t| \le 1$, and $p < \alpha^{-1}$, and in the second term we have used the bound in assumption (4.6). Summarizing, we have proven that

$$\frac{1}{T}\int_T^{2T} \Big| \int_{\mathbb{R}} h'^{\pm}_{\alpha}(t) E_{\mathfrak{a}}(t) dt \Big|^2 ds \ll \frac{1}{T^{1-\alpha}} + \frac{1}{T^{2-2/p}}.$$

Taking the limit as $T \to \infty$ we obtain (4.46), and this concludes the proof of theorem 4.8.

4.8 Hybrid limits

In the previous sections we have shown that for every $0 < \alpha < 1$ the variance of $e_\alpha(s)$ exists and is finite. Take for simplicity $z = w$. We would like to investigate the limit as $\alpha \to 0$ of $\mathrm{Var}(e_\alpha(s, z, z))$, and conclude that the variance of $e(s, z, z)$ should be given by

$$\mathrm{Var}(e(s, z, z)) = \sum_{0<t_j}{}' \frac{|\Gamma(it)|^2}{|\Gamma(3/2 + it)|^2} \left(\sum_{t_{j'}=t_j} |\phi_{j'}(z)|^2 \right)^2, \tag{4.50}$$

This involves an interchanging of limits that we do not know how to justify, and so we content ourselves with studying the sum appearing on the right hand side, and with giving a partial result in direction of (4.50) in Proposition 4.27 below. We cannot even prove that the sum is finite, unless we make assumptions on the eigenfunctions ϕ_j. It turns out that the sum barely fails to be convergent: if we assume

$$\sum_{t_j<T}{}' \left(\sum_{t_{j'}=t_j} |\phi_{j'}(z)|^2 \right)^2 \ll T^{3-\delta} \tag{4.51}$$

for some positive $\delta > 0$, then (4.50) becomes finite.

Observe that condition (4.51) with $\delta = 0$ is true, in view of (4.12) and (4.10).

For groups like $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ it is expected that we have strong bounds on the sup-norm and the multiplicity of eigenfunctions: It is expected that for any $0 < \delta_1, \delta_2 < 1/2$ we have

$$|\phi_j(z)| \ll_z t_j^{1/2-\delta_1} \tag{4.52}$$

and

$$m(t_j) = \sum_{t_{j'}=t_j} 1 \ll_z t_j^{1/2-\delta_2} \tag{4.53}$$

Iwaniec and Sarnak [37] has proved (4.52) with $\delta_1 = 1/12$, but we know no non-trivial bounds towards (4.53). If we knew (4.52) and (4.53) with $2\delta_1 + \delta_2 > 1/2$ the convergence of (4.51) would follow.

We conclude this section with the following proposition, which we state only for cocompact groups:

Proposition 4.27. *Let Γ be a cocompact Fuchsian group and let $z \in \mathbb{H}$. Assume that (4.51) holds for Γ. Let $\alpha = \alpha(T)$ such that*

$$\lim_{T\to\infty} \alpha(T) = 0, \quad \frac{1}{\alpha(T)e^{2T\alpha(T)}} \ll 1. \tag{4.54}$$

Then we have

$$\limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} |e_{\alpha(T)}(s)|^2 ds < \infty.$$

Proof. In (4.44) we can control the dependence on α in the last sum. We have indeed

$$\sideset{}{'}\sum_{0<t_j<e^T}\sideset{}{'}\sum_{\substack{0<t_\ell<e^T\\ t_j\neq t_\ell}}\frac{|b_j b_\ell r_\alpha(t_j)r_\alpha(t_\ell)|}{1+T|t_j-t_\ell|}\ll \sideset{}{'}\sum_{0<t_j<e^T}\sideset{}{'}\sum_{t_j<t_\ell<e^T}\frac{|b_j b_\ell r_\alpha(t_j)r_\alpha(t_\ell)|}{1+T|t_j-t_\ell|}$$
$$\ll \sideset{}{'}\sum_{0<t_j<e^T}\frac{|b_j|}{t_j^{3/2+\alpha}}\frac{1}{t_j^{1/2+\alpha}}\left(1+\frac{\log(t_j+1)}{T}\right)$$
$$\ll 1+\frac{1}{\alpha e^{2\alpha T}}$$

where we have used Lemma 4.25 and (4.10). The implied constant is now independent of α. Using this, and adding to (4.44) the estimates from (4.42), (4.40), and (4.41), we obtain, for $\delta = e^{-T}$,

$$\frac{1}{T}\int_T^{2T}|e_\alpha(s)|^2\,ds \ll 1+\frac{1}{T}+\frac{1}{\alpha e^{2\alpha T}}+\frac{1}{T^{2-2\alpha}}+\frac{T^{2+2\alpha}}{e^T}+\frac{T^{2\alpha}}{e^{\varepsilon_\Gamma T}}.$$

Take now $\alpha = \alpha(T)$ as in the statement. Condition (4.54) is sufficient for all the terms, in particular the third one, to be bounded as $T \to \infty$, and so we conclude

$$\limsup_{T\to\infty}\frac{1}{T}\int_T^{2T}|e_{\alpha(T)}(s)|^2 ds<\infty$$

which is the claim. □

4.9 Limiting Distribution

We are now ready to prove Theorem 4.12: In proving Theorem 4.8 (section 4.7) we have shown that if we write

$$e_\alpha(s) = \sideset{}{'}\sum_{0<t_j<X}\Re(r_\alpha(t_j)e^{it_js})b_j + \mathcal{E}(s,X)$$

for $r_\alpha(t_j), b_j \in \mathbb{C}$, defined as in (4.38) and (4.12), then we have

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}|\mathcal{E}(s,e^T)|^2 ds = 0. \tag{4.55}$$

We claim that also

$$\lim_{T\to\infty}\frac{1}{T}\int_0^{T}|\mathcal{E}(s,e^T)|^2 ds = 0. \tag{4.56}$$

To see this we note that

$$\frac{1}{T}\int_0^{T}|\mathcal{E}(s,e^T)|^2 ds = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{2^n}{T}\int_{\frac{T}{2^n}}^{\frac{2T}{2^n}}|\mathcal{E}(s,e^T)|^2 ds$$

We claim that for $T' \leq T$ we have

$$\frac{1}{T'}\int_{T'}^{2T'}|\mathcal{E}(s,e^T)|^2 ds \to 0 \text{ as } T'\to\infty \tag{4.57}$$

where the convergence is uniform in $T \geq T'$. By the dominated convergence theorem we may then conclude (4.56). To see (4.57) we note that

$$\frac{1}{T'}\int_{T'}^{2T'} |\mathcal{E}(s,e^T)|^2 ds \leq \frac{2}{T'}\int_{T'}^{2T'} |\mathcal{E}(s,e^{T'})|^2 ds + \frac{2}{T'}\int_{T'}^{2T'} |\sum_{e^{T'}<t_j\leq e^T}{}' \Re(r_\alpha(t_j)e^{it_j s})b_j|^2 ds.$$

The first term does not depend on T and tends to 0 as $T' \to \infty$ by (4.55). The second term can be analyzed as in Section 4.7 and we find that this term goes to zero uniformly in T. This proves (4.57) and proves therefore (4.56).

Eq. (4.56) implies that $e_\alpha(s)$ is in the closure of the set

$$\left\{\sum_{\text{finite}} r_n e^{is\lambda_n} : \lambda_n \in \mathbb{R}, r_n \in \mathbb{C}\right\}$$

with respect to the seminorm

$$\|f\| = \limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |f(s)|^2 ds\right)^{1/2}.$$

In other words, $e_\alpha(s)$ is an almost periodic function with respect to (4.9), i.e. a B^2-almost periodic function. We can then apply [1, Theorem 2.9] and conclude that $e_\alpha(s)$ admits a limiting distribution. The last part of the theorem is a direct consequence of the fact that $e_\alpha(s)$ is bounded for $1/2 < \alpha < 1$ (see Theorem 4.4).

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