SPACES OF PIECEWISE LINEAR MANIFOLDS

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Abstract

In this thesis we introduce a Δ-set $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$ which we regard as the piecewise linear analogue of the space $\Psi_d(\mathbb{R}^N)$ of smooth d-dimensional submanifolds in $\mathbb{R}^N$ introduced by Galatius in [4]. Using $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$ we define a bi-Δ-set $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}$ whose geometric realization $B\mathcal{C}_d^{\mathrm{PL}}(\mathbb{R}^N) = \left\|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}\right\|$ should be interpreted as the PL version of the classifying space of the category of smooth d-dimensional cobordisms in $\mathbb{R}^N$, studied in [7], and the main result of this thesis describes the weak homotopy type of $B\mathcal{C}_d^{\mathrm{PL}}(\mathbb{R}^N)$ in terms of $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$, namely, we prove that there is a weak homotopy equivalence $B\mathcal{C}_d^{\mathrm{PL}}(\mathbb{R}^N) \simeq \Omega^{N-1}|\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet|$ when $N - d \geq 3$.

The proof of the main theorem relies on properties of $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$ which arise from the fact that this Δ-set can be obtained from a more general contravariant functor $\mathbf{PL}^{op} \to \mathbf{Sets}$ defined on the category of finite dimensional polyhedra and piecewise linear maps, and on a fiberwise transversality result for piecewise linear submersions whose fibers are contained in $\mathbb{R} \times (-1,1)^{N-1} \subseteq \mathbb{R}^N$. For the proof of this transversality result we use a theorem of Hudson on extensions of piecewise linear isotopies which is why we need to include the condition $N - d \geq 3$ in the statement of the main theorem.

Resumé

I denne afhandling introducerer vi en Δ-mængde $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$ som vi betragter som den stykkevis lineære analog til rummet $\Psi_d(\mathbb{R}^N)$ af glatte d-dimensionale delmangfoldigheder i $\mathbb{R}^N$ introduceret af Galatius i [4]. Ved at benytte $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$ definerer vi en bi-Δ-mængde $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}$ hvis geometriske realisation $B\mathcal{C}_d^{\mathrm{PL}}(\mathbb{R}^N) = \left\|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}\right\|$ bør fortolkes som PL versionen af det klassificerende rum for kategorien af glatte d-dimensionale kobordismer i $\mathbb{R}^N$, studeret i [7], og afhandlingens hovedresultatet beskriver den svage homotopitype af $B\mathcal{C}_d^{\mathrm{PL}}(\mathbb{R}^N)$ ved hjælp af $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$, nemlig, vi beviser at der findes en svag homotopiækvivalens $B\mathcal{C}_d^{\mathrm{PL}}(\mathbb{R}^N) \simeq \Omega^{N-1}|\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet|$ når $N - d \geq 3$.

Beviset for hovedsætningen bygger på egenskaper ved $\Psi_d^{\mathrm{PL}}(\mathbb{R}^N)_\bullet$, som stammer fra at denne Δ-mængde kan udledes fra en mere generel kontravariant funktor $\mathbf{PL}^{op} \to \mathbf{Sets}$ defineret på kategorien af endelig dimensioanle polyeder og stykkevis linære afbildninger, og på et fibervist transversalitetsresultat for stykkevis linære submersioner hvis fibre er indeholdt i $\mathbb{R} \times (-1,1)^{N-1} \subseteq \mathbb{R}^N$. I beviset af dette transversalitetsresultat benytter vi en sætning af Hudson om isotopiudvidelser, hvilket er grunden til at vi er nødt til at inkludere betingelsen $N - d \geq 3$ i formuleringen af hovedsætningen.

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Contents

1 Introduction

In the last two decadces there has been a great deal of success in applying homotopy theoretic methods to solve geometric problems, in particular, such methods have been applied extensively to understand the (co)homology and homotopy type of *diffeomorphism groups* $\mathrm{Diff}(M)$ and their classifying spaces $B\mathrm{Diff}(M)$. This trend was initiated with the work of Tillmann in [22] and later consolidated with the proof of the Mumford conjecture of Madsen and Weiss in [14]. This thesis can be seen as a starting point to work on similar problems in the piecewise linear category, i.e., use similar techniques to the ones applied to smooth manifolds in order to study the *(simplicial) groups of pl automorphisms* $PL(M)$ of a pl manifold M and their classifying spaces $BPL(M)$. Although the actual study of the algebraic topology of $PL(M)$ and $BPL(M)$ shall be saved for future projects the author believes that the necessary simplicial techniques and results on fiberwise piecewise linear transversality to start working on such problems are given in this thesis.

Background

a. Riemann surfaces and the Madsen-Weiss Theorem

Let Σ_g denote the oriented surface of genus g. The classifying spaces $B\mathrm{Diff}^+\Sigma_g$ of the groups of orientation preserving diffeomorphisms of the surfaces Σ_g initially attracted interest due to their connection with moduli spaces of Riemann surfaces $\mathcal{M}_g$. More precisely, for each $g \geq 0$ we have that the $B\mathrm{Diff}^+\Sigma_g$ is homotopy equivalent to $B\Gamma_g$, the classifying space of the mapping class group of Σ_g, which in turn has rational cohomology isomorphic to that of the moduli space of Riemann surfaces $\mathcal{M}_g$. A long standing conjecture, known as the *Mumford conjecture*, stated that in a range of degrees the rational cohomology ring $H^*(\mathcal{M}_g, \mathbb{Q})$ (or $H^*(B\mathrm{Diff}^+\Sigma_g, \mathbb{Q})$) is a polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ on certain classes κ_i known as the Mumford-Morita-Miller classes. This conjecture was answered in the affirmative by Madsen and Weiss in [14] by proving a much stronger result, now known as *the Madsen-Weiss Theorem*, than the one suggested by Mumford. More precisely, let $\Sigma_{g,1}$ be the oriented surface of genus g with one boundary component, let $\Gamma_{g,1} = \pi_0\mathrm{Diff}(\Sigma_{g,1}, \mathrm{rel}\partial)$ be the group of components of diffeomorphisms restricting to the identity on the boundary, let Γ_∞ be the stable mapping class group, i.e., the colimit of the diagram of maps

$$\cdots \to \Gamma_g \to \Gamma_{g+1} \to \Gamma_{g+2} \cdots,$$

where each map $\Gamma_g \to \Gamma_{g+1}$ is the one induced by gluing to $\Sigma_{g,1}$ the torus $\Sigma_{1,2}$ with two boundary components, and let $\mathbf{MT}\mathrm{SO}(2)$ be the Madsen-Tillmann spectrum, whose space at degree n is equal to the Thom space $\mathrm{Th}(\gamma^\perp_{d,N})$ of the normal bundle over the Grassmannian of oriented 2-planes in $\mathbb{R}^N$. In [14] Madsen and Weiss proved the following result.

Theorem (Madsen, Weiss). *There is a homology equivalence*

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbf{MT}\mathrm{SO}(2).$$

The Mumford conjecture is then obtained from this theorem by applying stability results of Harer [11] and by observing that the rational cohomology ring $H^*(\Omega^\infty_0 \mathbf{MT}\mathrm{SO}(2), \mathbb{Q})$ of the 0-th component of $\Omega^\infty \mathbf{MT}\mathrm{SO}(2)$ is a polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ generated by certain classes κ_i which pull back to the Mumford-Morita-Miller classes κ_i of $B\mathrm{Diff}^+\Sigma_g$ along a map $\alpha_g : B\mathrm{Diff}^+\Sigma_g \to \Omega^\infty_0 \mathbf{MT}SO(2)$ obtained by the fiberwise Pontrjagin-Thom construction.

b. The smooth cobordism category

The proof of the Madsen-Weiss Theorem given in [14] is extremely technical and several simplifications of it have been made over the last few years. The first of these was offered by Galatius, Madsen, Tillman and Weiss in [6] and the strategy they followed was to introduce a topological category $\mathcal{C}_d$ of *d-dimensional oriented cobordisms* such that in the case $d = 2$ the classifying space $B\mathcal{C}_2$ served as an intermediate step between $\mathbb{Z} \times B\Gamma_\infty$ and $\Omega^\infty \mathbf{MT}\mathrm{SO}(2)$.

Let us outline the definition of $\mathcal{C}_d$ as an untopologized category. This discussion will follow closely the one given in [6] and we shall restrict our attention only to manifolds without orientation. The set of objects of $\mathcal{C}_d$ is defined as follows: for a positive integer N let B_N denote the set of all $d-1$-dimensional closed submanifolds M in $\mathbb{R}^N$. The natural inclusion $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}$ induces a map $B_N \hookrightarrow B_{N+1}$ and the set of objects $\mathrm{Ob}\mathcal{C}_d$ is then defined to be the set of all tuples (M, a) where $a \in \mathbb{R}$ and M is an element of the colimit B_∞ of the following sequence of maps

$$\cdots \hookrightarrow B_N \hookrightarrow B_{N+1} \hookrightarrow \ldots.$$

Given any two objects (M_1, a_1) and (M_2, a_2) in $\mathrm{Ob}\mathcal{C}_d$ the set of non-identity morphisms $\mathcal{C}_d\big((M_1, a_1), (M_2, a_2)\big)$ is the set of all triples (W, a_0, a_1) where W is a d-dimensional compact submanifold

$$W \subseteq [a_0, a_1] \times \mathbb{R}^N$$

for some finite N for which there is an $\epsilon > 0$ such that

i) $W \cap ([a_0, a_0 + \epsilon] \times \mathbb{R}^N) = [a_0, a_0 + \epsilon] \times M_0$.

ii) $W \cap ((a_1 - \epsilon, a_1] \times \mathbb{R}^N) = (a_1 - \epsilon, a_1] \times M_1$.

iii) $\partial W = W \cap (\{a_0, a_1\} \times \mathbb{R}^N)$.

Two morphisms (W_1, a_0, a_1) and (W_2, a_1, a_2) in $\mathcal{C}_d$ are composable if the outgoing boundary of W_1 is equal to the incoming boundary of W_2 and their composition is equal to the triple

$$(W_1 \cup W_2, a_0, a_2).$$

In [6] a topology on both $\mathrm{ob}\mathcal{C}_d$ and $\mathrm{mor}\mathcal{C}_d$ is defined so that $\mathcal{C}_d$ becomes a topological category. In fact, with these topologies we have the following equivalences

$$ob\mathcal{C}_d \simeq \coprod_{[M]} B\mathrm{Diff}(M), \qquad \mathrm{mor}\mathcal{C}_d \simeq \coprod_{[W]} B\mathrm{Diff}(W, \partial W),$$

where the coproducts are indexed respectively by all diffeomorphism types of $d-1$-dimensional closed manifolds and diffeomorphism types of d-dimensional cobordisms.

The main result in [6] is the following theorem.

Theorem (Galatius, Madsen, Tillmann, Weiss). *There is a weak homotopy equivalence*

$$B\mathcal{C}_d \simeq \Omega^{\infty-1}\mathbf{MT}\mathrm{O}(d).$$

A more general version of this theorem is also proven in [6] for the case of manifolds with tangential structures. In particular, for the case of manifolds with orientations we obtain an equivalence

$$B\mathcal{C}_d^+ \simeq \Omega^{\infty-1}\mathbf{MT}\mathrm{SO}(d)$$

and the Madsen-Weiss Theorem is deduced by applying this result to the case $d = 2$ and then showing that there is a homology equivalence

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega B\mathcal{C}_2^+.$$

c. Scanning methods

Another significant simplification of the proof of the Madsen-Weiss Theorem came with the work of Galatius in the article [5] in which he used *scanning methods* and *spaces of graphs* $\Phi(\mathbb{R}^N)$ to prove a result about automorphism groups of free groups similar in spirit to the Madsen-Weiss theorem. Namely, in [5] Galatius shows that there is an isomorphism in homology

$$\mathbb{Z} \times \mathrm{Aut}_\infty \to QS^0 \tag{1}$$

where Aut_∞ is the colimit of the diagram

$$\ldots \to \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1}) \to \ldots$$

and QS^0 is the infinite loop space of the sphere spectrum S^0. An intermediate step in the proof of (1) is to introduce a topological category $\mathcal{C}^G$ of graph cobordisms and show a result similar to the main theorem of [6], namely, that there is a weak equivalence

$$\Omega B\mathcal{C}^G \to QS^0. \tag{2}$$

Galatius concludes [5] with an outline of how the scanning methods used to prove (1) still hold when $\Phi(\mathbb{R}^N)$ is replaced by a space $\Psi_d(\mathbb{R}^N)$ of *d-dimensional submanifolds of* $\mathbb{R}^N$ and how in this case the scanning methods yield the following unstable version of the main theorem of [6].

Theorem (Galatius, Randal-Williams). *There is a weak equivalence*

$$B\mathcal{C}_d(\mathbb{R}^N) \simeq \Omega^{N-1}\mathrm{Th}(\gamma_{d,N}^{\perp}).$$

In the limit $N \to \infty$ the main result of [6] is recovered from the theorem of Galatius and Randal-Williams. Details of this proof were later expanded in [7] in which the space $\Psi_d(\mathbb{R}^N)$ played a central role.

The underlying set of $\Psi_d(\mathbb{R}^N)$ is equal to the set of all d-dimensional submanifolds of $\mathbb{R}^N$ which are closed as subspaces. The empty set $\emptyset$ is also included in this set and it serves as a base boint for $\Psi_d(\mathbb{R}^N)$. Although the details of the definition of the topology on $\Psi_d(\mathbb{R}^N)$ are quite involved (see section § 2 of [7]) we can easily list the two crucial properties of $\Psi_d(\mathbb{R}^N)$ which suffice to study its weak homotopy type:

i) If M is an m-dimensional smooth manifold and $W \subseteq M \times \mathbb{R}^N$ is a smooth $(d+k)$-submanifold that is closed as a subspace and such that the natural projection $\pi : W \to M$ is a submersion of codimension d, then the function $f : M \to \Psi_d(\mathbb{R}^N)$ defined by $x \mapsto \pi^{-1}(x)$ is continuous. Such a map $f : M \to \Psi_d(\mathbb{R}^N)$ obtained from a submersion is called a *smooth map* in [7].

ii) Any continuous map $f : M \to \Psi_d(\mathbb{R}^N)$ is homotopic to a smooth map.

The weak equivalence $B\mathcal{C}_d(\mathbb{R}^N) \simeq \Omega^{N-1}\mathrm{Th}(\gamma_{d,N}^{\perp})$ is obtained by showing that there are two weak equivalences

$$B\mathcal{C}_d(\mathbb{R}^N) \simeq \Omega^{N-1}\Psi_d(\mathbb{R}^N), \tag{3}$$

and

$$\mathrm{Th}(\gamma_{d,N}^{\perp}) \xrightarrow{\simeq} \Psi_d(\mathbb{R}^N). \tag{4}$$

The proof of the first of these weak equivalences makes extensive use of the two properties of $\Psi_d(\mathbb{R}^N)$ listed above whereas the second one uses the fact that a manifold W in $\Psi_d(\mathbb{R}^N)$ which intersects the origin in $\mathbb{R}^N$ can be deformed by scanning to its germ at the origin, i.e., its tangent space T_0W at the point $0 \in W$.

Outline of this thesis and statement of results

The main result of this thesis is a piecewise linear version of the weak equivalence (3). More precisely, in this thesis I prove the following theorem.

Theorem (Gomez Lopez). *If $N - d \geq 3$ then there is a weak equivalence*

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|.$$

In this statement, $\Psi_d(\mathbb{R}^N)_\bullet$ is a Δ-*set* which should be viewed as a space of d-dimensional *piecewise linear* submanifolds of $\mathbb{R}^N$ and $BPL\mathcal{C}_d(\mathbb{R}^N)$ is the

space which should be regarded as the piecewise linear analogue of the classifying space of the category of the d-dimensional smooth cobordisms in $\mathbb{R}^N$ studied in [7].

The strategy of the proof of this theorem follows the one given in [7]. Namely, we introduce a filtration

$$\psi_d(N,1)_\bullet \hookrightarrow \psi_d(N,2)_\bullet \hookrightarrow \ldots \hookrightarrow \psi_d(N,N)_\bullet = \Psi_d(\mathbb{R}^N)_\bullet \tag{5}$$

and show that there are two weak equivalences

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq |\psi_d(N,1)_\bullet| \tag{6}$$

and

$$|\psi_d(N,1)_\bullet| \xrightarrow{\simeq} \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|. \tag{7}$$

This thesis is organized as follows: section §2 is an introduction to several concepts and results from piecewise linear topolology. In particular, we introduce the notions of piecewise linear submersion and regular values for piecewise linear maps which we need for several of the definitions involved in this thesis and which might be unfamiliar to most topologists. Furthermore, in §2 we introuduce Δ-sets (simplicial sets without degeneracies) and include a discussion about some of the geometric constructions that can be applied to these objects, like for example subdivision.

In section §3 we introduce the main definition of this thesis, namely, the Δ-set $\Psi_d(\mathbb{R}^N)_\bullet$. A p-simplex of $\Psi_d(\mathbb{R}^N)_\bullet$ should be viewed as a family of d-dimensional piecewise linear submanifolds of $\mathbb{R}^N$ parametrized piecewise linearly by Δ^p. We shall make this interpretation rigorous using the concept of piecewise linear submersion. We also introduce the sub-Δ-sets $\psi_d(N,k)_\bullet$ of the filtration (5). Furthermore, we show that these Δ-sets can be obtained from much more general contravariant functors $\psi_d(N,k) : \mathbf{PL}^{op} \to \mathbf{Sets}$ defined on the category of finite dimensional polyhedra and that for any compact polyhedron P the elements of the set $\psi_d(N,k)\big(P\big)$ are classified by $\psi_d(N,k)_\bullet$. This claim is stated a lot more rigorously in Theorem 3.5, and we conclude this section using Theorem 3.5 to show that each $\psi_d(N,k)_\bullet$ is a Kan Δ-set.

In section §4 we use the functor $\Psi_d(\mathbb{R}^N) : \mathbf{PL}^{op} \to \mathbf{Sets}$ to define a map $\rho : |\Psi_d(\mathbb{R}^N)_\bullet| \to |\Psi_d(\mathbb{R}^N)_\bullet|$, which we will call in this thesis *the subdivision map of* $\Psi_d(\mathbb{R}^N)_\bullet$, which has the following properties:

i) ρ is homotopic to the identity map on $|\Psi_d(\mathbb{R}^N)_\bullet|$.

ii) For any morphism of Δ-sets $f_\bullet : X_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ and any non-negative integer $r \geq 0$ there is a unique morphism $g_\bullet : \mathrm{sd}^r X_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$, defined on

the r-th barycentric subdivision of $X_\bullet$, which makes the following diagram commute

$$\begin{array}{ccc} |\mathrm{sd}^r X_\bullet| & \xrightarrow{|g_\bullet|} & |\Psi_d(\mathbb{R}^N)_\bullet| \\ \downarrow \cong & & \uparrow \rho^r \\ |X_\bullet| & \xrightarrow{|f_\bullet|} & |\Psi_d(\mathbb{R}^N)_\bullet| \end{array}$$

where the left vertical is the canonical homeomorphism $|\mathrm{sd}^r X_\bullet| \xrightarrow{\cong} |X_\bullet|$.

In section §5 we define the space $BPL\mathcal{C}_d(\mathbb{R}^N)$, which is the geometric of a *bi-Δ-set*, and we start the proof of the following theorem.

Theorem 1. *If $N - d \geq 3$ then there is a weak equivalence*

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq |\psi_d(N,1)_\bullet|.$$

The proof of the smooth version of this result given in [7] is very brief and it uses the following geometric fact:

Fact. *Let M^m be an m-dimensional smooth manifold, let W be a smooth $(d+m)$-submanifold of $M \times \mathbb{R}^k \times (0,1)^{N-1}$ such that the projection $\pi : W \to M$ is a smooth submersion of codimension d and let $\lambda \in M$. If $a_0 \in \mathbb{R}$ is a regular value for the projection $x_1 : \pi^{-1}(\lambda) \to \mathbb{R}$ onto the first component of $\mathbb{R} \times (0,1)^{N-1}$ then there is an open set U of λ such that a_0 is a regular value for*

$$x_1 : \pi^{-1}(\alpha) \to \mathbb{R}$$

for each $\alpha \in U$.

Unfortunately, this fact is definitely not true in the piecewise linear category. Counterexamples are really easy to produce and hence we cannot follow the proof of [7] too closely. However, one thing we can do is define a sub-Δ-set of $\psi_d(N,1)_\bullet$ for which this kind of *fiberwise regularity* does hold. Namely, in §5 we introduce a sub-Δ-set $\psi_d^R(N,1)_\bullet$ of $\psi_d(N,1)_\bullet$ where for each simplex W of this sub-Δ-set the projection $x_1 : W \to \mathbb{R}$ does have a *fiberwise regular value* (see defintion 5.2) and the rest of section §5 is devoted to proving that

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq |\psi_d^R(N,1)_\bullet|.$$

In section §5 we conclude the proof of Theorem 1 by proving the following theorem, which is one of the central results of this thesis.

Theorem 2. *The inclusion $\psi_d^R(N,1)_\bullet \hookrightarrow \psi_d(N,1)_\bullet$ is a weak homotopy equivalence when $N - d \geq 3$.*

The proof of this theorem uses a result of Hudson about extensions of piecewise linear isotopies (see [12]), which is why we need the condition $N - d \geq 3$.

In §7 we compare the spaces $|\psi_d(N,1)_\bullet|$ and $\Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|$. This is done by introducing, for each $k \geq 1$, a *scanning map*

$$\mathcal{S}_k : |\psi_d(N,k)_\bullet| \to \Omega|\psi_d(N,k+1)_\bullet|$$

and proving the following theorem.

Theorem 3. *If $N - d \geq 3$ and if $k \geq 1$ then the scanning map*

$$\mathcal{S}_k : |\psi_d(N,k)_\bullet| \to \Omega|\psi_d(N,k+1)_\bullet|$$

is a weak homotopy equivalence. Consequently, there is a weak homotopy equivalence

$$|\psi_d(N,1)_\bullet| \xrightarrow{\simeq} \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|$$

Finally, we obtain the main result of this thesis in Theorem 7.21 by combining Theorems 1 and 3.

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2 Preliminaries on piecewise linear topology

2.1 Basic definitions

We start this section with the three most basic definitions from pl topology.

Definition 2.1. A *simplex of dimension p* (a *p-simplex*) σ in $\mathbb{R}^n$ is the convex hull of a set of $p+1$ geometrically independent points $\{v_0, \ldots, v_p\}$ in $\mathbb{R}^n$. That is, each point x in σ can be expressed uniquely as $\Sigma t_i v_i$ where $0 \leq t_i \leq 1$ for $0 \leq i \leq p$ and $\Sigma t_i = 1$.

Convention 2.2. In this thesis the vertices of the standard basis of $\mathbb{R}^{p+1}$ shall be denoted by $e_0, e_1, \ldots, e_p$, i.e., we shall label the elements of this basis using the set $\{0, \ldots, p\}$ instead of $\{1, \ldots, p+1\}$.

The convex hull of the vectors $e_0, \ldots, e_p$ is called *the standard geometric p-simplex* and we will denoted by Δ^p.

Definition 2.3. A collection K of simplices in $\mathbb{R}^n$ is called a *simplicial complex* provided

i) If $\sigma \in K$ and $\tau < \sigma$ (τ is a *face* of σ), then $\tau \in K$.

ii) If $\sigma, \tau \in K$, then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$.

iii) K is *locally finite*, that is, given $x \in \sigma \in K$ then there is a neighborhood of x in $\mathbb{R}^n$ which meets finitely many simplices of K.

Definition 2.4. $P \subseteq \mathbb{R}^n$ is said to be an *Euclidean polyhedron* if it is equal to a finite union of simplices $\sigma_1, \ldots, \sigma_p$ in $\mathbb{R}^n$.

In particular, if K is a finite simplicial complex in $\mathbb{R}^n$ then the union of all the simplices of K, usually denoted by $|K|$, is an Euclidean polyhedron. $|K|$ is usually called *the underlying polyhedron of K*. The following proposition, whose proof can be found in [17], tell us that all Euclidean polyhedra are of the form $|K|$.

Proposition 2.5. *If P is an Euclidean polyhedron in $\mathbb{R}^n$ then there exists a finite simplicial complex K in $\mathbb{R}^n$ such that $P = |K|$.*

If $P = |K|$ then we say that K *triangulates* P. An Euclidean polyhedron P is said to be of *dimension p* if for any simplicial K such that $|K| = P$ we have that K has a simplex σ of dimension p but no simplices of dimension higher than p.

A compact subsapce Q of an Euclidean polyhedron P is said to be a *subpolyhedron of P* if Q is itself an Euclidean polyhedron. The notion of subpolyhedron allows us to formulate the following definition.

Definition 2.6. A continuous map $f : P \to Q$ between two Euclidean polyhedra P and Q is said to be *piecewise linear* if the graph $\Gamma(f) \subseteq P \times Q$ is a subpolyhedron of $P \times Q$.

The following proposition describes the image of a pl map.

Proposition 2.7. *The image of a pl map $f : P \to Q$ is a subpolyhedron of Q.*

The following is one of the most important defintions in the theory of simplicial complexes.

Definition 2.8. Let K be a simplicial complex in $\mathbb{R}^m$. A simplicial complex K_1 in $\mathbb{R}^m$ is a *subdivision of* K provided that $|K_1| = |K|$ and that each simplex τ of K_1 is contained in a simplex σ of K.

One particular important kind of subdivision is the *stellar subdivision*, which we will define in 2.11 below. In order to formulate this definition we need the following.

Definition 2.9. *i*) The *join* AB of two Euclidean polyhedra A and B is the set $AB = \{\lambda a + \mu b : \ a \in A, \ b \in B\}$ where $\lambda, \mu \in \mathbb{R}$, $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$.

ii) Two Euclidean polyhedra A and B are said to be *independent* if each point in AB may be written uniquely in the form $\lambda a + \mu b$ with $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$.

Example 2.10. The following are standard facts about simplicial complexes that we will need in order to formulate the definition of stellar subdivision:

1. The join AB of two independent polyhedra is again a polyhedron.
2. If A and B are independent simplices then AB is a simplex of dimension $\dim A + \dim B + 1$ spanned by the vertices of A and B.
3. If K and L are two finite simplicial complexes in $\mathbb{R}^m$ such that $P := |K|$ and $Q := |L|$ are independent then the polyhedron PQ is triangulated by the simplicial complex KL (the *simplicial join of K and L*) which consists of simplices of the form A, B and AB with $A \in K$ and $B \in L$.
4. For any p-simplex A in $\mathbb{R}^m$ and any $a \in \text{int}A$ we have that ∂A and a are independent.

The definition of stellar subdivision is done in the following two steps (see the definition given in page 3 of [1]):

Definition 2.11. *i*) If A is a p-simplex in $\mathbb{R}^m$, L a simplicial complex which triangulates ∂A and if $a_0 \in \text{int}A$ then

$$K := a_0 L = \{a_0\} \cup L \cup \{a_0 B : B \in L\}$$

is the simplicial complex which triangulates A obtained by *starring A at a_0 over L.*

ii) Let K be a simplicial complex in $\mathbb{R}^m$ and for each simplex A of K let c_A be a point in int$A$ (if A is a vertex we set $c_A = A$). *The stellar subdivision L of K obtained by starring at the points c_A* is the one obtained by the following inductive procedure: assume that the $(p-1)$-skeleton K^{p-1} of K is subdivided by a complex L_{p-1} and for each p-simplex A of K denote by $L_{p-1,\partial A}$ the subcomplex of L_{p-1} which triangulates ∂A. Then, we define L_p to be the simplicial complex

$$L_p := \bigcup_A c_A L_{p-1,\partial A}$$

where the union ranges over all p-simplices of K and where for each p-simplex A we have that $c_A L_{p-1,\partial A}$ is the simplicial complex obtained by starring A at c_A over $L_{p-1,\partial A}$. It is clear that L_p subdivides K^p.

A particular important kind of stellar subdivision is the following.

Definition 2.12. Let K be a simplicial complex in $\mathbb{R}^m$ and for each simplex A of K let bA denote its barycentric point. The *first barycentric subdivision of K*, denoted by sd$K$, is the stellar subdivision of K obtained by starring at the points bA.

Piecewise linear maps have the following alternate characterization in terms of simplicial complexes. The proof can be found in [17].

Proposition 2.13. *A continuous map $f : P \to Q$ is piecewise linear if and only if there exists simplicial complexes K and L such that $|K| = P$, $|L| = Q$ and such that $f : |K| \to |L|$ is simplicial.*

From this proposition we get the following corollary.

Corollary 2.14. *If $f : P \to Q$ is both a pl map and a bijection then the inverse f^{-1} is also a pl map.*

Proof. Observe that the inverse of f is continuous since P is compact and Q is Hausdorff. Let K and L be simplicial complexes such that f becomes a simplicial map if we traingulate P and Q with K and L respectively. Then, if σ is a simplex of L spanned by $f(v_0), \ldots, f(v_q)$ then for any point $x = \sum \lambda_j f(v_j)$ in σ we have that

$$f^{-1}(x) = \sum \lambda_j v_j,$$

that is, f^{-1} is also a simplicial map and by 2.13 we have that f^{-1} is also pl. □

2.2 Abstract pl spaces

In this section we introduce the notion of abstract pl spaces and state some of their basic properties but without giving many of the proofs, which can be found in for example [11]. We remark that in the literature the term *locally finite complex* is sometimes used instead of abstract pl space, see for example [23]. We also remark that this subsection follows closely the presentation given in §3 of [11].

Definition 2.15. Let X be a topological space.

i) A *coordinate map* is a tuple (f, P) with P an Euclidean polyhedron and $f : P \to X$ an embedding, i.e., a homeomorphism onto its imge.

ii) Two coordinate maps (f, P) and (g, Q) are said to be *compatible* if either $f(P) \cap g(Q) = \emptyset$ or there exists a coordinate map (h, R) such that $h(R) = f(P) \cap g(Q)$ and $f^{-1} \circ h$ and $g^{-1} \circ h$ are both piecewise linear in the sense of definition 2.6.

Definition 2.16. A *piecewise linear structure* $\mathcal{T}$ on a space X is a family of coordinate maps (f, P) such that

i) Any two elements of $\mathcal{T}$ are compatible.

ii) For every $x \in X$ there exists (f, P) such that $f(P)$ is a neighborhood of x in X.

iii) $\mathcal{T}$ is maximal, i.e., if (f, P) is compatible with every element of $\mathcal{T}$ then $(f, P) \in \mathcal{T}$.

Definition 2.17. An *abstract pl space* is a tuple $(X, \mathcal{T})$ with X a second countable Hausdorff space and $\mathcal{T}$ a piecewise linear structure on X. Furthermore, $(X, \mathcal{T})$ is said to be of dimension p if there exists $(f_0, P_0) \in \mathcal{T}$ with P_0 of dimension p and if for any other element $(f, P) \in \mathcal{T}$ the Euclidean polyhedron P is of dimension at most p.

A pl space is said to be *finite dimensional* if it is of dimension p for some non-negative integer p.

Definition 2.18. A family of coordinate maps $\mathcal{B}$ on X satisfying conditions *i*) and *ii*) of definition 2.16 is called a *base for a pl structure on* X.

Sometimes it is easier to define for a space X a base for a pl strucure instead of an actual pl strucure. However, the next proposition, which is proven in [11], tells us that both pieces of data contain the same amount of information.

Proposition 2.19. *Every base $\mathcal{B}$ for a piecewise linear structure on a space X is contained in a unique piecewise linear structure $\mathcal{T}$.*

Examples 2.20. 1. For any Euclidean polyhedron $P \subseteq \mathbb{R}^n$ let $i_P : P \hookrightarrow \mathbb{R}^n$ be the natural inclusion into $\mathbb{R}^n$. The collection of all tuples (P, i_P) with P an Euclidean polyhedron in $\mathbb{R}^n$ is a base for a pl structure on $\mathbb{R}^n$. This pl structure is usually called the *standard pl structure on* $\mathbb{R}^n$.

2. More generally, if U is open in $\mathbb{R}^n$ then the collection of tuples (P, i_P) with $P \subseteq U$ is also a base for a pl structure on U.

3. Any Euclidean polyhedron P can also be viewed as an abstract pl space. Indeed, if P is an Euclidean polyhedron then the singleton which contains the tuple (P, Id_P) is a base for a pl structure on P.

We conclude our discussion about abstract pl spaces with the following two useful lemmas about coordinate maps. Details for the proofs of both lemmas can be found in §3 of [11].

Lemma 2.21. *If $(X, \mathcal{T})$ is a pl space and if $C \subseteq X$ is a compact subspace then there exists $(h, R) \in \mathcal{T}$ such that $C \subseteq \mathrm{int}h(R)$.*

Lemma 2.22. *Let $(X, \mathcal{T})$ be a pl space. If (h, P) is a coordinate map whose image is covered by the images of a finite number of elements $(h_1, P_1), \ldots, (h_q, P_q)$ of $\mathcal{T}$ with which (h, P) is compatible then (h, P) is also an element of $\mathcal{T}$.*

Eventhough all the definitions and results in this subsection apply to any pl space we will from now on only work with pl spaces of finite dimension. Therefore, in the rest of this section and thesis the term *pl space* means *finite dimensional pl space.*

2.3 Abstract piecewise linear maps

The definition of pl map given in definition 2.6 can be used to formulate a definition of pl map between abstract pl spaces very much in the same way that smooth coordinate charts are used in order to formulate the defintion of smooth map between abstract smooth manifolds. Again, we are going to borrow the presentation given in §3 of [11].

Definition 2.23. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be abstract pl spaces. A continuous map $h : X \to Y$ is said to be a *piecewise linear map* if for all $(f, P) \in \mathcal{F}$ and all $(g, Q) \in \mathcal{G}$ we have that $f^{-1} \circ h^{-1} \circ g(Q)$ is either empty or $R := f^{-1} \circ h^{-1} \circ g(Q)$ is a subpolyhedron of P and, if the latter case holds,

$$g^{-1} \circ h \circ f : R \to Q$$

is a piecewise linear map in the sense of definition 2.6.

The following propostion implies that the pl spaces and pl maps are respectively the objects and morphisms of a category.

Proposition 2.24. *i) For any pl space $(X, \mathcal{T})$ the identity map $\mathrm{Id}_X : X \to X$ is a pl map.*

ii) The composition $g \circ f$ of two pl maps $f : X \to Y$ and $g : Y \to Z$ is again a pl map.

Defintion 2.25. The *piecewise linear category* **PL** is the category of (finite dimensional) pl spaces and pl maps in the sense of definition 2.23.

Proposition 2.14 can be used to prove the following.

Proposition 2.26. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be pl spaces. If $f : X \to Y$ is both a piecewsie linear map and a homeomorphism of topological spaces then the inverse $f^{-1} : Y \to X$ is also a piecewise linear map.*

Proof. Let $g : Q \to Y$ be any coordinate chart in $\mathcal{G}$. We wish to show that $f^{-1} \circ g$ is a chart compatible with any chart of $\mathcal{F}$. By the continuity of f^{-1} the image $f^{-1} \circ g(Q)$ is compact in X and by lemma 2.21 there is a chart $h : R \to X$ in $\mathcal{F}$ such that $f^{-1} \circ g(Q) \subseteq \text{int}h(R)$. We wish to show first that $h^{-1} \circ f^{-1} \circ g$ is a pl map in the sense of definition 2.6. Let then $g' : Q' \to Y$ be an element in $\mathcal{G}$ such that $f \circ h(R) \subseteq \text{int}g'(Q')$. By assumption, $g'^{-1} \circ f \circ h$ is pl. In fact, by 2.14 and 2.7, it is a pl homeomorphism onto its image R' in Q'. The inclusion $f^{-1} \circ g(Q) \subseteq \text{int}h(R)$ implies that $g'^{-1} \circ g(Q) \subseteq R'$. Also, the map $h^{-1} \circ f^{-1} \circ g$ can be expressed as the composition of two pl maps, namely, $g'^{-1} \circ g$ and the restriction of $h^{-1} \circ f^{-1} \circ g'$ on $g'^{-1} \circ g(Q)$. Since the composition of pl maps is again pl we have that $h^{-1} \circ f^{-1} \circ g$ is a pl map. But by corollary 2.14 we actually have that $f^{-1} \circ g : Q \to X$ is a coordinate chart compatible with $h : R \to X$. Since $h(R)$ covers the image of $f^{-1} \circ g$ it follows from lemma 2.22 that $f^{-1} \circ g$ is compatible with all the elements of $\mathcal{F}$. Since this argument holds for any chart in $\mathcal{G}$ we have that f^{-1} is also an abstract pl map.

□

2.4 Products of pl spaces

Let X and Y be two abstract pl spaces. Proposition 2.19 can be used to define a pl structure on the product $X \times Y$ such that, with this pl structure, the universal property of $X \times Y$ in the category of spaces also holds in the category **PL** (see proposition 2.29 below.)

Proposition 2.27. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be abstract pl spaces. The collection of coordinate charts on $X \times Y$ given by*

$$\mathcal{B} := \{(f \times g, P \times Q) : \ (f, P) \in \mathcal{T}, \ (g, Q) \in \mathcal{S}\}$$

is a base for a pl structure on $X \times Y$.

Proof. It is straightforward to verify that $\mathcal{B}$ satisfies condition *ii)* of 2.18. In order to verify that condition *i)* also holds one just needs to use the fact that the product $f_1 \times f_2$ of two pl maps $f_i : P_i \to Q_i$, $i = 1, 2$, between Euclidean polyhedra is also a pl map in the sense of definition 2.6. □

Definition 2.28. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be abstract pl spaces. The *product of X and Y* is the abstract pl space $(X \times Y, \mathcal{T} \times S)$ where $\mathcal{T} \times \mathcal{S}$ is the pl structure induced by the base defined in proposition 2.27.

The proof of the following proposition can be found in [23] (lemma 1.2.1, page 4).

Proposition 2.29. *Let X, Y_1, Y_2 be abstract polyhedra and let $P_i : Y_1 \times Y_2 \to Y_i$ be the projection onto the i-th factor. Then a map $f : X \to Y_1 \times Y_2$ is piecewise linear if and only if $P_1 f$ and $P_2 f$ are piecewise linear.*

2.5 Triangulations of polyhedra

Proposition 2.5 tells us that any Euclidean polyhedron P is just the underlying polyhedron of a simplicial complex K, i.e, $|K| = P$. In the case of abstract pl spaces we cannot make the exact same statement since these objects are not necesseraly contained in some Euclidean space. However, the following proposition says that any abstract pl space is pl homemomorphic to the underlying polyhedron of a simplicial complex K in some Euclidean space $\mathbb{R}^m$.

Proposition 2.30. *Let $(X, \mathcal{T})$ be an abstract pl space. There exists a simplicial complex K in some Euclidean space $\mathbb{R}^m$ and a homeomorphism $h : |K| \to X$ such that the restrctions of h on finite subcomplexes of K are elements in $\mathcal{T}$. Moreover, the set*

$$\mathcal{B} = \{h|_{|L|} : \quad L \subseteq K \text{ finite}\}$$

is a base for the pl structure $\mathcal{T}$.

Definition 2.31. Let $(X, \mathcal{T})$ be a pl space. A *triangulation of X* is a pair (K, h) with K a simplicial complex and $h : |K| \to X$ a homeomorphism such that for each finite subcomplex L of K the restriction $h|_{|L|}$ is an element in $\mathcal{T}$.

Remark 2.32. We remark that proposition 2.30 is not stated as a proposition in [11]. However, all the necessary arguments to prove this result are given in [11] and the reader is urged to consult page 82 of [11] for an outline of the proof.

Remark 2.33. We also remark that from proposition 2.30 it follows that any pl space is *first-countable*.

The following proposition, whose proof can be found in [11], describes a very useful fact about the interacton of pl maps and triangulations. Recall that a map $f : X \to Y$ is said to be *proper* if the pre-image of a compact subspace is again compact.

Proposition 2.34. *Let $f : X \to Y$ be a pl map and let $h : |K| \to X$ and $g : |L| \to Y$ be triangulations of X and Y respectively. If f is a proper map then there are subdivisions K' and L' of K and L such that the composition $g^{-1} \circ f \circ h : |K'| \to |L'|$ is a simplicial map.*

2.6 Piecewise linear subspaces

Definition 2.35. Let $(X, \mathcal{F})$ be a a piecewise linear space and let $(Y, \mathcal{F}')$ be another piecewise linear space with $Y \subseteq X$. Then $(Y, \mathcal{F}')$ is called a *piecewise linear subspace* of $(X, \mathcal{F})$ provided

1. Y has the subspace topology induced by X, and
2. $i : Y \to X$, $i(x) = x$, is a piecewise linear map.

If $(Y, \mathcal{F}')$ is a piecewise linear subspace of $(X, \mathcal{F})$ we shall usually denote the piecewise linear structure $\mathcal{F}'$ by $\mathcal{F}_Y$.

Remark 2.36. If $(Y, \mathcal{F}_Y)$ is a piecewise linear subspace of $(X, \mathcal{F})$ then it is easy to verify that we have the following equality

$$\mathcal{F}_Y = \{(f, P) \in \mathcal{F} : f(P) \subseteq Y\}.$$

Definition 2.37. A piecewise linear map $f : (X', \mathcal{F}') \to (X, \mathcal{F})$ is called a *piecewise linear embedding* provided that f is a topological embedding and that $\big(f(X'), \mathcal{G}\big)$ is a piecewise linear subspace of $(X, \mathcal{F})$, where $\mathcal{G} = \{f \circ g : g \in \mathcal{F}'\}$.

The definition of pl embedding given above allows us to formulate the following definitions.

Defintion 2.38. *i)* A pl space $(M, \mathcal{F})$ is a *piecewise linear manifold of dimension* m if for each point m_0 of M there is a piecewise linear embedding $h : \mathbb{R}^m \to M$ such that $h(\mathbb{R}^m)$ is an open neighborhood of m_0 in M.

ii) Let $\mathbb{R}^m_+$ denote the set of all points $(x_1, \ldots, x_m)$ in $\mathbb{R}^m$ such that $x_m \geq 0$. A pl space $(M, \mathcal{F})$ is a *piecewise linear manifold of dimension* m *with boundary* if for each point m_0 of M there is a piecewise linear embedding $h : V \to M$ defined on an open subspace V of $\mathbb{R}^m_+$ such that $h(V)$ is an open neighborhood of m_0 in M.

iii) Let $(M, \mathcal{F})$ be an m-dimensional piecewise linear manifold (with boundary) and let $0 \leq n \leq m$. A pl subspace $(N, \mathcal{F}')$ of $(M, \mathcal{F})$ is *an n-dimensional piecewise linear submanifold (with boundary) of M* if $(N, \mathcal{F}')$ is in itself an n-dimensional piecewise linear manifold (with boundary).

The following proposition tells us when the union of two sub pl spaces is again a pl subspace.

Proposition 2.39. *Let P_1 and P_2 be pl subspaces of a pl space $(X, \mathcal{F})$. Let $P := P_1 \cup P_2$ and suppose that P_1 and P_2 are open subspaces of P. Then, $\mathcal{F}_{P_1} \cup \mathcal{F}_{P_2}$ is a base for a piecewise linear structure $\mathcal{F}'$ on P. Furthermore, the inclusion $i : (P, \mathcal{F}') \hookrightarrow (X, \mathcal{F})$ is a piecewise linear map in the sense of defintion 2.23. In particular, $\mathcal{F}' = \mathcal{F}_P$.*

Proof. The union $\mathcal{F}_{P_1} \cup \mathcal{F}_{P_2}$ clearly satisfies condition i) of definition 2.18 since all the charts of this union are just charts that belong to $\mathcal{F}$. On the other hand, since every open subspace in either P_1 and P_2 is also an open subspace in P we have that $\mathcal{F}_{P_1} \cup \mathcal{F}_{P_2}$ also satisfies condition ii) of definition 2.18. Let then $\mathcal{F}'$ be the piecewise linear structure on P generated by this base. Since $\mathcal{F}_{P_1} \cup \mathcal{F}_{P_2} \subseteq \mathcal{F}$ and since $\mathcal{F}_{P_1} \cup \mathcal{F}_{P_2}$ is a base for $\mathcal{F}'$ we can use lemma 2.22 to prove that $i : (P, \mathcal{F}') \hookrightarrow (M, \mathcal{F})$ is a piecewise linear map. In particular we have that $\mathcal{F}' = \mathcal{F}_P$.

□

The following proposition says that we can triangulate *pairs* of pl spaces (X, Y) provided that Y is a closed subspace of X.

Proposition 2.40. *Let Y be a closed pl subspace of a pl space X. Then there exists a triangulation $h : |K| \to X$ of X such that K has a subcomplex K_0 which triangulates Y, i.e., the restriction $h|_{|K_0|}$ is a triangulation of Y.*

With this proposition we can prove the following result which tells us what happens when we intersect two closed pl subspaces of a pl space.

Proposition 2.41. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two closed pl subspaces of $(X, \mathcal{T})$. Then, if $X_1 \cap X_2 \neq \emptyset$ we have that $X_1 \cap X_2$ is the underlying space of a pl subspace of $(X, \mathcal{T})$.*

Proof. By proposition 2.40 we can find two triangulations $h_1 : (|K|, |K_1|) \to (X, X_1)$ and $h_2 : (|L|, |L_2|) \to (X, X_2)$ and by proposition 2.34 we can find subdivisions K' and L' of K and L respectively such that the composite $g := h_2^{-1} \circ h_1 : |K'| \to |L'|$ is a simplicial isomorphism. Then, K' contains a subcomplex K_1' which subdivides K_1 and the image of K_1' under g is going to be a subcomplex J of L'. There is also a subcomplex L_2' in L' which subivides L_2 and it is easy to see that the restriction of h_2 on $|J \cap L_2'|$ is a triangulation for $X_1 \cap X_2$ and thus we have that $X_1 \cap X_2$ is the underlying space of a pl subspace of X.

□

We will be working a lot with *closed pl subspaces* of Euclidean spaces $\mathbb{R}^N$ and hence it would be good to have a concrete way of describing such subspaces. This is done in the following two propositions.

Proposition 2.42. *Let $\mathcal{T}$ denote the standard pl strucure on $\mathbb{R}^N$ and let $(X, \mathcal{T}_X)$ be a closed pl subpace of $\mathbb{R}^N$. Then there exists a collection of simplices $\{\sigma_\lambda\}_{\lambda \in \Lambda}$ which is locally finite in $\mathbb{R}^N$ and such that $X = \bigcup_\lambda \sigma_\lambda$.*

Proof. That $\{\sigma_\lambda\}_{\lambda \in \Lambda}$ is locally finite in $\mathbb{R}^N$ means that each x in $\mathbb{R}^N$ has a neighborhood U which intersects only finitely many of the simplices of $\{\sigma_\lambda\}_{\lambda \in \Lambda}$. Pick any simplicial complex L in $\mathbb{R}^N$ such that $|L| = \mathbb{R}^N$, for example, we can pick a finite simplicial complex which triangulates the cube $[0, 1]^N$ and then we use translations of this simplicial complex to triangulate any cube of the form $[p, p+1]^N$ with $p \in \mathbb{Z}$. In particular, we have that the identity map $\mathrm{Id} : |L| \to \mathbb{R}^N$

is a triangulation in the sense of definition 2.30. By proposition 2.40 there is a pair of locally finite simplicial complexes (K, K_0) and a homeomorphism $h : (|K|, |K_0|) \to (\mathbb{R}^N, X)$ such that the restriction of h on each finite subcomplex of K is a chart that belongs to the standard pl strucure of $\mathbb{R}^N$ and by proposition 2.34 there are subdivisions K' and L' of K and L respectively such that the map

$$|K'| \xrightarrow{h} |L'| \tag{8}$$

becomes a simplicial map. In fact, it will be a *simplicial isomorphism.* K' contains a subdivision K_0' of K_0 and since (8) is a simplicial isomorphism we have that the image $h(K_0')$ of the subcomplex K_0' under h is a subcomplex of L', which we will denote by L'', which triangulates X. Since K' is a locally finite simplicial complex it follows that the set of all simplices in L'' is a collection of simplices which is locally finite in $\mathbb{R}^N$ and whose union equals X. □

Proposition 2.43. *Let $\mathcal{T}$ be the standard pl structure on $\mathbb{R}^N$, let $\{\sigma_\lambda\}_{\lambda\in\Lambda}$ be a collection of simplices in $\mathbb{R}^N$ which is locally finite in $\mathbb{R}^N$ and let $X = \bigcup_{\lambda\in\Lambda} \sigma_\lambda$. Then, taking the subspace topology on X, the collection*

$$\mathcal{B} = \{ \bigcup_{\lambda\in Z} \sigma_\lambda \hookrightarrow X : \quad Z \subseteq \Lambda \text{ finite}\}$$

is a base for a pl structure $\mathcal{T}'$ on X such that the inclusion $i : (X, \mathcal{T}') \hookrightarrow (\mathbb{R}^N, \mathcal{T})$ is a pl map, i.e., $(X, \mathcal{T}')$ is a pl subspace of $(\mathbb{R}^N, \mathcal{T})$. Furthermore, X is a closed subspace of $\mathbb{R}^N$

Proof. Lets show first that X is a closed subspace of $\mathbb{R}^N$. Pick a point $x \in \mathbb{R}^N$ which is not in X. By assumption, x has a neighborhood U which intersects only finitely many simplices $\sigma_1, \ldots, \sigma_q$ of $\{\sigma_\lambda\}_{\lambda\in\Lambda}$. But since each simplex $\sigma_1, \ldots, \sigma_q$ is compact we can find a value $\delta > 0$ small enough so that the ball $B(x, \delta)$ centered at x with radius δ is contained in U and does not intersect any of the simplices $\sigma_1, \ldots, \sigma_q$ and therefore none of the simplices in $\{\sigma_\lambda\}_{\lambda\in\Lambda}$. This shows that X is a closed subspace of $\mathbb{R}^N$.

Clearly we have that all the charts in $\mathcal{B}$ are compatible to each other since they are just charts in the standard pl structure of $\mathbb{R}^N$. Furthermore, each x in X has a neighborhood U in X which intersects only finitely many simplices $\sigma_1, \ldots, \sigma_q$ of $\{\sigma_\lambda\}_{\lambda\in\Lambda}$. Thus

$$x \in \operatorname{int}\big(\sigma_1 \cup \ldots \cup \sigma_q\big)$$

and it follows that $\mathcal{B}$ is indeed a base for a pl structure. Let then $\mathcal{T}'$ be the unique pl structure induced by $\mathcal{B}$. To see that the inclusion $i : (X, \mathcal{T}') \hookrightarrow (\mathbb{R}^N, \mathcal{T})$ is pl we just need to observe that the image of any chart (h, P) of $\mathcal{T}'$ can be covered by the images of finitely many charts in $\mathcal{B}$ and since these charts are in $\mathcal{T}$ we have by lemma 2.22 that (h, P) is compatible with all the charts in $\mathcal{T}$. Consequently, the inclusion map i is piecewise linear. □

The two previous propositions give us then the following corollary.

Corollary 2.44. *Let $\mathcal{T}$ be the standard pl structure on $\mathbb{R}^N$. Then, a subspace X of $\mathbb{R}^N$ is the underlying space of a pl subspace of $(\mathbb{R}^N, \mathcal{T})$ if and only if there is a collection of simplices $\{\sigma_\lambda\}_{\lambda\in\Lambda}$ in $\mathbb{R}^N$ which is locally finite in $\mathbb{R}^N$ and such that $X = \bigcup_{\lambda\in\Lambda} \sigma_\lambda$.*

Remark 2.45. At this point we introduce the following change in terminology: instead of using the term *pl space* we shall use from now on the term *polyhedron* and instead of using the term *sub pl space* we shall simply use the term *subpolyhedron*. Also, we will from now on denote a polyhedron $(X, \mathcal{T})$ simply by X whenever this simpler notation does not produce any confusion.

2.7 Regular values for pl maps

Definition 2.46. (See [23], page 4) Let $f : P \to Q$ be a piecewise linear map. A point $q \in Q$ is said to be a *regular value of f* if there is an open neighborhood U of q in Q and a pl homeomorphism $h : f^{-1}(q) \times U \to f^{-1}(U)$ such that $fh = p$, where p is the natural projection $f^{-1}(q) \times U \to U$.

One of the main results proven in [23] is the following theorem which plays a similar role in pl topology to the one played by Sard's theorem in the smooth category (see theorem 1.3.1 in [23]).

Theorem 2.47. *Let P be a polyhedron and let $f : P \to \Delta^p$ be a pl map. Suppose that there is a triangulation $h : |K| \to P$ of P such that the composition $f \circ h$ is a simplicial map. Then any point λ in $\mathrm{Int}\Delta^p$ is a regular value of f.*

The following result is a special case of a more general theorem about open neighborhoods in pl manifolds which have the structure of a *pl microbundle* (see section § 4.2 of [23] for the statement of this more general result).

Proposition 2.48. *If $M \times \mathbb{R}^p$ is a piecewise linear manifold of dimension $m+p$ then M is a piecewise linear manifold of dimension m.*

Remark 2.49. Proposition 2.48 is a special case of the lemma stated in § 4.2 of [23] in the sense that we are considering the case when the neighborhood is just a product of the form $M \times \mathbb{R}^p$. The way we are going to apply this proposition is as follows: suppose we have a pl map $f : M^m \to \Delta^p$ from an m-dimensional pl manifold M and suppose that $h : |K| \to M$ is a triangulation of M such that the composition $f \circ h$ is simplicial. Then, by theorem 2.47, for any point λ in $\mathrm{int}\Delta^p$ we have that $f^{-1}(\lambda)$ has a neighborhood which is pl homeomorphic to the product $f^{-1}(\lambda) \times \mathbb{R}^p$ and by proposition 2.48 we will have that $f^{-1}(\lambda)$ is a pl submanifold of M of dimension $m - p$.

2.8 Piecewise linear submersions

Definition 2.50. 1. A piecewise linear map $\pi : E \to P$ is called a *piecewise linear submersion* if for each $x \in E$ there is a piecewise linear space U, an

open neighborhood V of $\pi(x)$ in P and an open piecewise linear embedding $h : V \times U \to E$ onto an open neighborhood of x such that $\pi \circ f$ is equal to the projection $\mathrm{pr}_1 : V \times U \to V$.

2. $\pi : E \to P$ is said to be a *piecewise linear submersion of codimension* $d \in \mathbb{N}$ if in the previous definition we can take U to be equal to $\mathbb{R}^d$ and $h(\pi(x), 0) = x$ where 0 is the origin in $\mathbb{R}^d$. In particular, each fiber of π is a d-dimensional piecewise linear manifold.

If $\pi : E \to P$ is a piecewise linear submersion and if x is in E then any embedding $h : V \times U \to E$ onto an open neighborhood of x such that $\mathrm{pr}_1 = \pi \circ h$ will be called a *submersion chart around* x. Furthermore, if U is an open neighborhood of x in the fiber $\pi^{-1}(\pi(x))$ and if $h(\pi(x), y) = y$ for all $y \in U$ we say that h is a *normalized product chart around* x.

The proof of the following proposition is given in page 81 of [2].

Proposition 2.51. *Let P be a pl manifold, let $\pi : W \to P$ be a piecewise linear submersion of codimension $d \in \mathbb{N}$ and let $\lambda \in P$. For any y_0 in the fiber $\pi^{-1}(\lambda)$ and any compact subspace C of $\pi^{-1}(\lambda)$ such that $y_0 \in C$ we can find a normalized product chart $g : V \times U \to \pi^{-1}(V)$ of π around y_0 such that $C \subseteq U$.*

The following important proposition tells us that we can *pull back* the fibers of certain pl submersions along pl maps in order to obtain new pl submersions. This proposition is what we are going to use in order to define the structure maps of the Δ-set $\Psi_d(\mathbb{R}^N)_\bullet$ that we mentioned in the introduction.

Proposition 2.52. *Let $f : P \to Q$ be a piecewise linear map. If $W \subseteq Q \times \mathbb{R}^N$ is a closed subpolyhedron of $Q \times \mathbb{R}^N$ such that the projection $\pi : W \to Q$, $\pi(q, x) = q$, is a piecewise linear submersion of codimension d then*

$$f^*W = \{(p, x) \in P \times \mathbb{R}^N : (f(p), x) \in W\}$$

is a closed subpolyhedron of $P \times \mathbb{R}^N$ and the projection $\tilde{\pi} : f^*W \to P$, $\tilde{\pi}(p, x) = p$, is also a piecewise linear submersion of codimension d.

Proof. Observe first that the product $P \times W$ is a closed sub-polyhedron of $P \times Q \times \mathbb{R}^N$ and that the product $\mathrm{Id}_P \times \pi$ is a piecewise linear sumbersion of codimension d. Since the map f is piecewise linear we have that the graph $\Gamma(f)$ of f is a subpolyhedron of $P \times Q$. The pre-image $\left(\mathrm{Id}_P \times \pi\right)^{-1}(\Gamma(f))$ is a closed sub-polyhedron of $\Gamma(f) \times \mathbb{R}^N$ since it is equal to the intersection $(P \times W) \cap (\Gamma(f) \times \mathbb{R}^N)$ and the restriction of $\mathrm{Id}_P \times \pi$ on $\left(\mathrm{Id}_P \times \pi\right)^{-1}(\Gamma(f))$, which we shall denote by π', is also a piecewise linear submersion of codimension d since for any x in $\left(\mathrm{Id}_P \times \pi\right)^{-1}(\Gamma(f))$ and any submersion chart $h : V \times \mathbb{R}^d \to P \times W$ for $\mathrm{Id}_P \times \pi$ around x we have that the restriction of h on $(\Gamma(f) \cap V) \times \mathbb{R}^d$ is a submersion chart for π' around x. Let $g : \Gamma(f) \to P$ be the piecewise linear map which sends $(p, f(p))$ to p and let $G : \Gamma(f) \times \mathbb{R}^N \to P \times \mathbb{R}^N$ be equal to the product $g \times \mathrm{Id}_{\mathbb{R}^N}$. The map G is a piecewise linear homeomorphism and thus

the image of $\left(\mathrm{Id}_P \times \pi\right)^{-1}(\Gamma(f))$ under G is a closed sub-polyhedron of $P \times \mathbb{R}^N$. It is easy to verify that this image is equal to the subpace f^*W given in the statement of this propostion. Furthermore, the map $\tilde{\pi} : f^*W \to P$ given by $\tilde{\pi}(p, x) = p$ is equal to the composition $g \circ \pi' \circ G^{-1}|_{f^*W}$ and since both g and $G^{-1}|_{f^*W}$ are piecewise linear homeomorphisms we have that $\tilde{\pi}$ is a piecewise linear submersion of codimension d.

□

Definition 2.53. The closed sub-polyhedron f^*W of $P \times \mathbb{R}^N$ obtained in the previous proposition will be called *the pull back of W along f*.

Proposition 2.54. *Let M, N and P be polyhedra and let W be a closed sub-polyhedron of $M \times \mathbb{R}^N$ such that the projection $\pi_M : W \to M$ is a piecewise linear submersion of codimension d. If $f : N \to M$ and $g : P \to N$ are piecewise linear maps then $(f \circ g)^*W = g^* f^* W$.*

Proof. This lemma follows from the following equality

$$(f \circ g)^*W = \{(p, x) \in P \times \mathbb{R}^N : (f \circ g(p), x) \in W\} =$$

$$\{(p, x) \in P \times \mathbb{R}^N : (g(p), x) \in f^*W\} = g^* f^* W.$$

□

The argument used to prove 2.52 works as a template to prove also the following results.

Lemma 2.55. *Let P, M be polyhedra and let $h : P \times M \to P \times M$ be a pl homeomorphism which commutes with the projection onto P. If $f : Q \to P$ is any pl map then the map $g : Q \times M \to Q \times M$ defined by $g(q, m) = \left(q, h_{f(q)}(m)\right)$ is a pl homeomorphism which commutes with the projection onto Q.*

Proof. As we said before the method of this proof is somewhat similar to the one given in proposition 2.52. Consider first the product of maps $\mathrm{Id}_Q \times h : Q \times P \times M \to Q \times P \times M$. The graph $\Gamma(f)$ of the pl map f is a subpolyhedron of the product $Q \times P$ and hence we can restrict the map $\mathrm{Id}_Q \times h$ on the sub-polyhedron $\Gamma(f) \times M$. Let us denote this restriction by g'. Observe also that the map $r : Q \to \Gamma(f)$ which sends q to the tuple $(q, f(q))$ is a piecewise linear homeomorphism and that the map $g : Q \times M \to Q \times M$ given in the statement is equal to the following composite map

$$(r \times \mathrm{Id}_M)^{-1} \circ g' \circ (r \times \mathrm{Id}_M)$$

which clearly is a piecewise linear homeomorphism which commmutes with the projection onto Q. □

An argument similar to the one given in the previous proof can be used to prove the following.

Lemma 2.56. *Let P, M, N be polyhedra and let $h : P \times N \to P \times M$ be a pl embedding which commutes with the projection onto P. If $f : Q \to P$ is any pl map then the map $g : Q \times N \to Q \times M$ defined by $g(q, n) = \big(q, h_{f(q)}(n)\big)$ is a pl embedding which commutes with the projection onto Q.*

We can also adapt the methods of the proof of proposition 2.52 to prove the following result about *open pl embeddings*. A pl embedding is said to be *open* if its image is open in the target.

Proposition 2.57. *Let $F : P \times \mathbb{R}^N \to P \times \mathbb{R}^N$ be an open piecewise linear embedding which commutes with projection onto P and let $g : Q \to P$ be a piecewise linear map. Then, the map $G : Q \times \mathbb{R}^N \to Q \times \mathbb{R}^N$ which sends (q, x) to $(q, F_{f(q)}(x))$ is an open piecewise linear embedding which commutes with the projection onto Q.*

Proof. Consider first the piecewise linear map $\mathrm{Id}_Q \times F : Q \times P \times \mathbb{R}^N \to Q \times P \times \mathbb{R}^N$. This map is an open pl embedding which commutes with the projection onto $Q \times P$ and by pre-composing this map with the obvious inclusion $\Gamma(f) \times \mathbb{R}^N \hookrightarrow Q \times P \times \mathbb{R}^N$ we obtain an open piecewise linear embedding $H : \Gamma(f) \times \mathbb{R}^N \to \Gamma(f) \times \mathbb{R}^N$ which commutes with the projection onto $\Gamma(f)$. Since the map $G : Q \times \mathbb{R}^N \to Q \times \mathbb{R}^N$ given in the statement of this proposition is equal to the map obtained by pre-composing and post-composing H with the obvious pl homoemorphism $h : Q \times \mathbb{R}^N \to \Gamma(f) \times \mathbb{R}^N$ we have that G satisfies the desired properties.

□

2.9 Δ-sets and simplicial complexes

In this subsection we will introduce the notion of Δ-*set*. A good reference for this material is [18].

For each non-negative integer n let $[n]$ denote the set $\{0, \ldots, n\}$. The set $[n]$ has an obvious linear order.

Definition 2.58. The category Δ is the category whose set of objects is equal to $\{[n] : n \in \mathbb{N}\}$ and where the set of morphisms $\Delta\big([n], [m]\big)$ between $[n]$ and $[m]$ is empty if $n > m$ or it is equal to the set of all strictly increasing functions $[n] \to [m]$ if $n \leq m$.

Definition 2.59. A Δ-*set* X is a contravariant functor $X : \Delta^{op} \to \mathbf{Sets}$. We shall usually denote the set $X\big([p]\big)$ by X_p for any object $[p]$ in Δ. Also, for any morphism δ in Δ the function of sets $X(\delta)$ shall usually be denoted by δ^* and the functor X itself shall usually be denoted by $X_\bullet$.

The following more combinatorial definition of Δ-set is equivalent to definition 2.59 and it is sometimes more useful when one is trying to define Δ-sets by hand. See for example definition 2.69 below.

Definition 2.60. (See definition 2.6 in [3]) A Δ-*set* $X_\bullet$ consists of a sequence of sets $X_0, X_1, \ldots$ and, for each $n \geq 0$, maps $\partial_i : X_{n+1} \to X_n$ for each i, $0 \leq i \leq n+1$, such that

$$\partial_i \partial_j = \partial_{j-1} \partial_i$$

whenever $i < j$.

For each morphism $\alpha : [p] \to [q]$ in Δ let us denote by $\alpha_* : \Delta^p \to \Delta^q$ the simplicial map which sends e_j to $e_{\alpha(j)}$. The *geoemtric realization* of a Δ-set $X_\bullet$ is defined to be the quotient

$$|X| = \coprod_{n=0}^{\infty} X_n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation generated by the relations $\big(x, \alpha_*(\lambda)\big) \sim \big(\alpha^*(x), \lambda\big)$.

In the literature Δ-sets are sometimes described as a generalization of simplicial complexes. What people really mean by this is that from any simplicial complex K we can produce in a natural way, after ordering the vertices of K, a Δ-set $K_\bullet$ and that there is a canonical homeomorphism $|K_\bullet| \xrightarrow{\cong} |K|$. This shall be described in remark 2.63. But first we need to introduce the following definition.

Defintion 2.61. Let A be a set. An *ordered simplicial complex with vertices in* A is a tuple $(K, \leq)$ where K is a subset of $\mathcal{P}(A)$ (the power set of A) and $\leq$ is a relation on A which satisfy the following conditions:

1. $\emptyset \notin K$ and for every $a \in A$ the singleton $\{a\}$ is in K.
2. Every element σ in K is finite.
3. If $\sigma \in K$ then any non-empty subset of σ is also in K.
4. The restriction of $\leq$ on any $\sigma \in K$ is a linear order on σ, which we will denote by $\leq_\sigma$.

Definition 2.62. A tuple $(K, \leq)$ is called an *ordered Euclidean simplicial complex* if K is a simplicial complex in some Euclidean space and if $\leq$ is a relation on $\mathrm{Vert}(K)$ (the set of vertices of K) satisfying condition 4) of defintion 2.61.

Remark 2.63. Observe that from an ordered simplicial complex K we can obtain a Δ-set $K_\bullet$ by assigning to $[p]$ the subset of K consisting of all elements of cardinality $p+1$ and by assigning to a morphism $[n] \xrightarrow{s} [m]$ the function $s^* : K_m \to K_n$ which sends an element $a_0 \leq \ldots \leq a_m$ of cardinality $m+1$ to the set $a_{s(0)} \leq \ldots \leq a_{s(n)}$. In particular, if Δ^p is the standard geoemetric p-simplex we can use its canonical simplicial complex structure and the canonical

ordering $\leq$ on the set of vertices $\{e_0, \ldots, e_p\}$ to produce a Δ-set $\Delta^p_\bullet$ whose set of k-simplices consists of all strings

$$e_{i_0} \leq \ldots \leq e_{i_k}$$

of length k.

For any Δ-set $X_\bullet$ and any p-simplex σ of $X_\bullet$ the subset of X_0 consisting of all points x for which there is a morphism $s \in \Delta([0],[p])$ such that $s^*(\sigma) = x$ will be called *the set of vertices of* σ and we will denote it by $\mathrm{Vert}(\sigma)$. The following lemma offers a complete characterization of those Δ-sets which are isomorphic to Δ-sets obtained from ordered simplicial complexes as indicated in remark 2.63.

Lemma 2.64. *A Δ-set $X_\bullet$ is isomorphic to a Δ-set $K_\bullet$ obtained from an ordered simplicial complex K if and only if the following holds:*

1. *For any non-negative integer p and for any p-simplex σ the set* $\mathrm{Vert}(\sigma)$ *has cardinality $p+1$.*
2. *If σ_1 and σ_2 are simplices such that* $\mathrm{Vert}(\sigma_1) = \mathrm{Vert}(\sigma_2)$ *then $\sigma_1 = \sigma_2$.*

Proof. The first implication is obvious since conditions 1) and 2) hold for any Δ-set obtained from an ordered simplicial complex. Suppose then that $X_\bullet$ is a Δ-set which satisfies conditions 1) and 2). For $i = 0, \ldots, p$ let $\delta_{i,p} : [0] \to [p]$ be the morphism in Δ which sends 0 to i, let $\leq$ be the subset of $X_0 \times X_0$ which consists of all tuples (x, y) for which there is a p-simplex σ and morphisms $\delta_{i,p}, \delta_{j,p}$ with $i < j$ such that $\delta^*_{i,p}(\sigma) = x$ and $\delta^*_{j,p}(\sigma) = y$. Observe that this last condition is equivalent to the existence of a 1-simplex α such that $\delta^*_{0,1}(\alpha) = x$ and $\delta^*_{1,1}(\alpha) = y$ and by condition 2) such a 1-simplex must be unique. Let K now be the subset of $\mathcal{P}(X_0)$ which consists of all finite subsets β of X_0 such that $\beta = \mathrm{Vert}(\sigma)$ for some simplex σ of $X_\bullet$. We claim that for each element β in K the restriction of $\leq$ on β, which we denote by $\leq_\beta$, is a linear order. Indeed, let σ be the unique simplex of $X_\bullet$ such that $\mathrm{Vert}(\sigma) = \beta$ and let p be the dimension of σ. If x and y are two different elements in β then there are morphisms $\delta_{i,p}$ and $\delta_{j,p}$ in $\Delta\big([0],[p]\big)$ such that $i \neq j$ and such that $\delta^*_{i,p}(\sigma) = x$ and $\delta^*_{j,p}(\sigma) = y$. This implies that either $x \leq_\beta y$ or $y \leq_\beta x$ holds. However both relations cannot hold at the same time since if that were the case we would have two different 1-simplices with the same set of vertices which contradicts condition 2). Suppose now that x, y, z are elements in β such that $x \leq_\beta y$ and $y \leq_\beta z$. Observe that there are unique values j_0, j_1, j_2 such that $\delta^*_{j_0,p}(\sigma) = x$, $\delta^*_{j_1,p}(\sigma) = y$, $\delta^*_{j_2,p}(\sigma) = z$ and since $x \leq_\beta y$ and $y \leq_\beta z$ we must have that $j_0 < j_1 < j_2$ which implies that $x \leq_\beta z$. We will show now that if $\beta = \{v_0 \leq \ldots \leq v_p\}$ is in K then any subset β' of β is also in K and that the restriction of $\leq_\beta$ on β' is equal to $\leq_{\beta'}$. Let then $v_{i_0} \leq \ldots \leq v_{i_q}$ be the elements of β arranged in increasing order using the order relation $\leq_\beta$ ald let $s : [q] \to [p]$ be the morphism in Δ which sends j to i_j. Then we clearly have that $\delta^*_{j,q}\big(s^*(\sigma)\big) = v_{i_j}$ for $j = 0, \ldots, q$ and this implies both that $\beta' \in K$ and that the restriction of $\leq_\beta$ on β' is $\leq_{\beta'}$ and we conclude

that the tuple $(K, \leq)$ is an ordered simplicial complex with vertices in X_0. Let $K_\bullet$ be the ordered simplicial complex obtained from the tuple $(K, \leq)$ and for each non-negative integer p let $f_p : X_p \to K_p$ be the function which sends a p-simplex σ to the set $\mathrm{Vert}(\sigma)$. We claim that the functions f_p are the components of an isomorphism $f_\bullet : X_\bullet \to K_\bullet$ of Δ-sets. Let then σ be a p-simplex of $X_\bullet$, let $s : [q] \to [p]$ be a morphism in Δ and let $v_0 \leq \ldots \leq v_p$ be the vertices of σ arranged in increasing order. By the way we defined the structure maps of $K_\bullet$ we have that $s^*\big(\{v_0, \ldots, v_p\}\big) = \{v_{s(0)}, \ldots, v_{s(q)}\}$. On the other hand, we have that $\mathrm{Vert}\big(s^*(\sigma)\big)$ is equal to $\{\delta^*_{0,q}\big(s^*(\sigma)\big), \ldots, \delta^*_{q,q}\big(s^*(\sigma)\big)\} = \{v_{s(0)}, \ldots, v_{s(q)}\}$, which is exactly $f_q\big(s^*(\sigma)\big)$ and thus we conclude that the maps f_p are the components of a morphism $f_\bullet$ of Δ-sets. Furthermore, $f_\bullet$ is obviously surjective and by condition 2) we have that it is also injective.

□

We conclude this section with a very useful fact about *products of Δ-sets*. This result shall mostly be used in § 6. Recall that the product $X_\bullet \times Y_\bullet$ of two Δ-sets $X_\bullet$ and $Y_\bullet$ is the functor $\Delta^{op} \to \mathbf{Sets}$ which sends $[p]$ to $X_p \times Y_p$ and which sends a morphism $\delta : [q] \to [p]$ to the product $\delta^* \times \delta^* : X_p \times Y_p \to X_q \times Y_q$. For this proposition we shall omit the symbol $\bullet$ from the notation $X_\bullet$.

Proposition 2.65. *Let $\widetilde{X}_1$ and $\widetilde{X}_2$ be simplicial sets and let F : **Ssets** $\to$ Δ-**sets** be the forgetful functor from the category of simplicial sets to the category of Δ-sets. If X_1 and X_2 denote respectively the Δ-sets $F(\widetilde{X}_1)$ and $F(\widetilde{X}_2)$ and if $p_j : X_1 \times X_2 \to X_j$ is the projection onto X_j for $j = 1, 2$ then*

$$(|p_1|, |p_2|) : |X_1 \times X_2| \longrightarrow |X_1| \times |X_2|$$

is a weak homotopy equivalence where the topology on $|X_1| \times |X_2|$ is the product topology.

Proof. For $j = 1, 2$ let us denote by $|\widetilde{X}_j|_M$ the geometric realization of the simplicial set $\widetilde{X}_j$ (The subscript M stands for *Milnor*). We recall the following fact proven in [18]:

Fact: If F : **Ssets** $\to$ Δ-**sets** is the forgetful functor from **Ssets** to Δ-**sets** and if $\widetilde{X}$ is a simplicial set then there exists a natural map $|F(\widetilde{X})| \to |\widetilde{X}|_M$ which is a homotopy equivalence.

Consider the following diagram:

$$\begin{array}{ccc} |X_1 \times X_2| & \longrightarrow & |\widetilde{X}_1 \times \widetilde{X}_2|_M \\ \downarrow & & \simeq \downarrow \\ |X_1| \times |X_2| & \longrightarrow & |\widetilde{X}_1|_M \times |\widetilde{X}_2|_M \end{array}$$

where the topology on the spaces in the bottom row is the product topology. The horizontal maps in the square are the ones obtained by applying the fact

mentioned above to the simplicial sets $\widetilde{X}_1 \times \widetilde{X}_2$, $\widetilde{X}_1$ and $\widetilde{X}_2$. For $j = 1, 2$ the j-th component in either vertical map is just the geometric realization of the morphism p_j which projects the product onto the j-th component. The naturality of the horizontal maps implies that the square is commutative. The right verical map is a weak equivalence since it is equal to the composition

$$|\widetilde{X}_1 \times \widetilde{X}_2|_M \stackrel{\cong}{\to} \big(|\widetilde{X}_1|_M \times |\widetilde{X}_2|_M\big)_K \stackrel{\simeq}{\to} |\widetilde{X}_1|_M \times |\widetilde{X}_2|_M$$

where the middle term is endowed with the compactly generated topology. Since the two horizontal maps are also weak homotopy equivalences we conclude that the map

$$(|p_1|, |p_2|) : |X_1 \times X_2| \longrightarrow |X_1| \times |X_2|$$

is also a weak equivalence.

□

By induction we have the have the following corollary.

Proposition 2.66. *Proposition 2.65 continues to hold if we consider finitely many simplicial sets $\widetilde{X}_1, \ldots, \widetilde{X}_p$.*

2.10 Subdivisions of Δ-sets

Let $X_\bullet$ be a Δ-set. We can regard $X_\bullet$ as a category $\mathcal{X}$ with objects

$$\mathrm{ob}\mathcal{X} = \coprod_{p \geq 0} X_p$$

and with a morphism $\eta^* : \beta \to \sigma$ for each morphism η in Δ such that $\eta^*(\sigma) = \beta$. In particular, for each p-simplex σ , $p > 0$, and each j in $\{0, \ldots, p\}$ there is a morphism $\partial_j : \partial_j\sigma \to \sigma$ (see §1 of [16]).

From a *small* category $\mathcal{C}$ one can produce a Δ-set $N\mathcal{C}_\bullet$, called *the nerve of* $\mathcal{C}$, which is defined as follows (see [20]). Each object $[p]$ in the category Δ can be regarded as a category with objects $\{0, \ldots, p\}$ and with a unique morphism $i \to j$ whenever $i \leq j$. Furthermore, each morphism $\eta : [q] \to [p]$ induces a functor $[q] \to [p]$ in an obvious way, which by abuse of notation shall also be denoted by η. The Δ-set $N\mathcal{C}_\bullet$ is then defined to be the functor $\Delta^{op} \to \mathbf{Sets}$ which sends $[p]$ to $\mathbf{Func}\big([p], \mathcal{C}\big)$ and which sends a morphism $[q] \stackrel{\eta}{\to} [p]$ of Δ to the function

$$\eta^* : \mathbf{Func}\big([p], \mathcal{C}\big) \longrightarrow \mathbf{Func}\big([q], \mathcal{C}\big)$$

which sends a functor α to the composite $\alpha \circ \eta$. In particular, we have that the set of 0-simplices $N\mathcal{C}_0$ is just the set of objects ob$\mathcal{C}$ and for $p > 0$ the set of p-simplices $N\mathcal{C}_p$ is the the set of all strings of lenght p

$$x_0 \to \ldots \to x_p$$

of p composable morphisms.

Applying this construction to the category $\mathcal{X}$ obtained from a Δ-set $X_\bullet$ we can formulate the following definition.

Definition 2.67. (See definition 1.6 in [16]) The *barycentric subdivision of* $X_\bullet$ is the sub-Δ-set $\mathrm{sd}X_\bullet$ of $N\mathcal{X}_\bullet$ whose set of 0-simplices is equal to $N\mathcal{X}_0$ and whose set of p-simplices, $p > 0$, is equal to the set of all strings of lenght p

$$x_0 \to \ldots \to x_p$$

of *non-identity* composable morphisms.

Let $X_\bullet = \Delta^p_\bullet$ be the Δ-set obtained from the canonical ordered simplicial complex structure of the standard p-simplex Δ^p as indicated in remark 2.63. In order to spell out what $\mathrm{sd}X_\bullet$ looks like it is better if we identify each k-simplex $e_{i_0} \leq \ldots \leq e_{i_k}$ of $\Delta^p_\bullet$ with the face F of Δ^p spanned by the vertices $e_{i_0}, \ldots, e_{i_k}$. With these identifications we have that the category $\mathcal{X}$ is the category with objects the faces F of Δ^p and with a unique non-identity morphism $F_1 \to F_2$ whenever F_1 is a proper face of F_2. By definition 2.67 we have that $\mathrm{sd}\Delta^p_\bullet$ is the Δ-set whose of k-simplices consists of all *flags*

$$F_{i_0} \to \ldots \to F_{i_k}$$

of lenght k.

We can define the *m-barycentric subdivision* $\mathrm{sd}^m X_\bullet$ of $X_\bullet$ for $m > 1$ by iterating the previuos construction. In particular, for any Δ-set $X_\bullet$ we have the following result about $\mathrm{sd}^2 X_\bullet$.

Lemma 2.68. *For any Δ-set $X_\bullet$ the second barycentric subdivision $\mathrm{sd}^2 X_\bullet$ is isomorphic to a Δ-set $K_\bullet$ obtained from an ordered simplicial complex K.*

Proof. We are going to apply lemma 2.64 for the proof of this lemma. Let us start this proof by proving the following claim: if $X_\bullet$ is a Δ-set such that for each simplex σ the set $\mathrm{Vert}(\sigma)$ has $\dim\sigma + 1$ points then the first barycentric subdivision $\mathrm{sd}X_\bullet$ is isomorphic to a Δ-set $K_\bullet$ obtained from an ordered simplicial complex K. For a fixed simplex σ we will denote by $X^\sigma_\bullet$ the sub-Δ-set of $X_\bullet$ generated by σ. The fact that $\mathrm{Vert}(\sigma)$ has $\dim\sigma + 1$ points implies that the morphism $\Delta^{\dim\sigma}_\bullet \to X^\sigma_\bullet$ which sends $e_0 \leq \ldots \leq e_{\dim\sigma}$ to σ is an isomorphism of Δ-sets which implies that each Δ-set of the form $\mathrm{sd}X^\sigma_\bullet$ is isomorphic to a Δ-set obtained from an ordered simplicial complex. Since each simplex β of $\mathrm{sd}X_\bullet$ is contained in one of the sub-Δ-sets $\mathrm{sd}X^\sigma_\bullet$ we have that $\mathrm{sd}X_\bullet$ satisfies condition 1) of lemma 2.64. To see that it also satisfies condition 2) consider any two p-simplices β_1 and β_2 of $\mathrm{sd}X_\bullet$ such that $\mathrm{Vert}(\beta_1) = \mathrm{Vert}(\beta_2)$. We can find p-simplices σ_1 and σ_2 in $X_\bullet$ such that $\beta_i \in \mathrm{sd}X^{\sigma_i}_\bullet$ for $i = 1, 2$. Observe that we must have $\sigma_1 = \sigma_2$ since if this were not the case we would have that the barycentric points $s\sigma_1$ and $s\sigma_2$ are two different 0-simplices in $\mathrm{sd}X_0$ which

would contradict the assumption that $\mathrm{Vert}(\beta_1) = \mathrm{Vert}(\beta_2)$ since $s\sigma_1 \in \mathrm{Vert}(\beta_1)$ and $s\sigma_2 \in \mathrm{Vert}(\beta_2)$. Thus $\sigma_1 = \sigma_2$ and since $\mathrm{sd}X^{\sigma_1}$ is isomorphic to an ordered simplicial complex we have by condition 2) of 2.64 that $\beta_1 = \beta_2$ and thus we have that $\mathrm{sd}X_\bullet$ is isomorphic to a Δ-set obtained from an ordered simplicial complex.
Let $X_\bullet$ now be an arbitrary Δ-set. We claim that the first barycentric subdivision $\mathrm{sd}X_\bullet$ satisfies condition 1) of lemma 2.64. We remark that in order for a Δ-set to satisfy condition 1) of 2.64 it is sufficient that the condition holds for all of its 1-simplices. But $\mathrm{sd}X_\bullet$ satisfies this equivalent condition since for any 1-simplex α of $\mathrm{sd}X_\bullet$ we have that $\delta^*_{0,1}(\alpha) = s\sigma_0$ and $\delta^*_{1,1}(\alpha) = s\sigma_1$ where σ_0 is a proper face of σ_1 and thus $\delta^*_{0,1}(\alpha) \neq \delta^*_{1,1}(\alpha)$ and by the first part of this proof we conclude that $\mathrm{sd}^2X_\bullet$ is isomorphic to a Δ-set obtained from an ordered simplicial complex. □

2.11 Cone of a Δ-set

In this section we define the *cone of a Δ-set $X_\bullet$*. This is a construction that we are going to use to define the scanning map in § 6. This construction is similar to that of a cone of a simplicial set given in [8]. However the approach we are going to take is different in the sense that we are going to define the cone of a Δ-set $X_\bullet$ by specifying all the extra simplices we need to add to $X_\bullet$ and by indicating the images of these new simplices under face maps. More precisely, we have the following definition.

Definition 2.69. Let $X_\bullet$ be a Δ-set. The *cone $CX_\bullet$ of $X_\bullet$* is defined as follows: the set of 0-vertices CX_0 is equal to

$$X_0 \cup \{*\},$$

i.e., is the set of vertices X_0 plus a new vertex $*$. For $p \geq 0$ the set of $p+1$-simplices CX_{p+1} is equal to

$$X_{p+1} \cup \{\widetilde{\sigma} : \ \sigma \in X_p\},$$

i.e., for every p-simplex σ of $X_\bullet$ we add a new $p+1$-simplex $\widetilde{\sigma}$. Face maps are defined as follows:

i) On $X_{p+1} \subseteq CX_{p+1}$ the j-th face map $\partial_j : CX_{p+1} \to CX_p$ agrees with the j-th face map $\partial_j : X_{p+1} \to X_p$ of $X_\bullet$.

ii) For a $p+1$-simplex of the form $\widetilde{\sigma}$ we have the following identities: $\partial_{p+1}\widetilde{\sigma} = \sigma$ and $\partial_j\widetilde{\sigma} = \widetilde{\partial_j\sigma}$ for $j = 0, \ldots, p$.

iii) For a 1-simplex of the form $\widetilde{\sigma}$ we have that $\partial_1\widetilde{\sigma} = \sigma$ and $\partial_0\widetilde{\sigma} = *$.

The new vertex $*$ is meant to be the *tip* of the cone. It is straightforward, although slightly tedious, to verify that the functions ∂_j we just defined satisfy the Δ-set identity $\partial_i\partial_j = \partial_{j-1}\partial_i$ whenever $i < j$ and hence the sequence of sets

$\{CX_p\}_p$ and the functions ∂_j indeed determine a Δ-set. Details are left to the reader.

Furthermore, it is clear that there is a canonical homeomorphism

$$|CX_\bullet| \to C|X_\bullet|$$

where $C|X_\bullet|$ is the usual cone of $|X_\bullet|$

By the way we defined $CX_\bullet$ we have that there is a natural inclusion

$$j : X_\bullet \hookrightarrow CX_\bullet$$

which sends a p-simplex σ to its copy in CX_p. This inclusion shall be used in the following definition.

Definition 2.70. *The unreduced suspension* $SX_\bullet$ of $X_\bullet$ is the push out of the following diagram

$$\begin{array}{ccc} X_\bullet & \xrightarrow{j} & CX_\bullet \\ {\scriptstyle j}\downarrow & & \\ CX_\bullet. & & \end{array} \tag{9}$$

This is by no means a standard definition of suspension for either Δ-sets or simplicial sets. See for example section III.5 in [8]. However, the author found this construction useful in order to define the *scanning map* given in §7 and to show that it is a weak equivalence.

3 Spaces of PL manifolds

3.1 Definitions

In the following definition we introduce the piecewise linear analogue of the *space of smooth manifolds* $\Psi_d(\mathbb{R}^N)$ used in [7].

Definition 3.1. $\Psi_d(\mathbb{R}^N)_\bullet$ is the Δ-set which assigns to an object $[p]$ of Δ the set $\Psi_d(\mathbb{R}^N)_p$ of all closed sub-polyhedra W of $\Delta^p \times \mathbb{R}^N$ for which the natural projection $\pi : W \to \Delta^p$ is a piecewise linear submersion of codimension d. The i-th face map $\partial_i : \Psi_d(\mathbb{R}^N)_{p+1} \to \Psi_d(\mathbb{R}^N)_p$ is defined by $W \mapsto \delta_i^* W$, i.e., a p-simplex W is mapped to its pull back along the canonical inclusion $\delta_i : \Delta^p \to \Delta^{p+1}$ into the i-th face of Δ^{p+1} (See proposition 2.52).

We shall also need the following sub-Δ-sets of $\Psi_d(\mathbb{R}^N)_\bullet$.

Definition 3.2. $\psi_d(N,k)_\bullet$, $0 \le k \le N$, is the sub-Δ-set of $\Psi_d(\mathbb{R}^N)_\bullet$ given by

$$\psi_d(N,k)_p = \{W : \quad W \subseteq \Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k}\}.$$

In particular, we have that $\Psi_d(\mathbb{R}^N)_\bullet = \psi_d(N,N)_\bullet$.

In order to define the Δ-set $\Psi_d(\mathbb{R}^N)_\bullet$ we applied proposition 2.52 to very specific kinds of polyhedra and pl maps, namely, the standard geometric simplices Δ^p and simplicial injective maps $\Delta^q \hookrightarrow \Delta^p$ which preserve the canonical ordering on the set of vertices. However, proposition 2.52 applies to *all polyhedra* and all *pl maps* and therefore we can define a more general kind of functor.

Definition 3.3. For $0 \leq k \leq N$ let

$$\psi_d(N,k) : \mathbf{PL}^{op} \longrightarrow \mathbf{Sets} \tag{10}$$

be the contravariant from the category **PL** of polyhedra and piecewise linear maps to the category **Sets** which sends a polyhedron P to the set $\psi_d(N,k)\big(P\big)$ of all closed sub-polyhedra W of $P \times \mathbb{R}^k \times (-1,1)^{N-k}$ for which the natural projection map $\pi : W \to P$ is a piecewise linear submersion of codimension d and which sends a pl map $f : P \to Q$ to the *pull back along* f

$$\psi_d(N,k)\big(f\big) : \psi_d(N,k)\big(P\big) \stackrel{f^*}{\to} \psi_d(N,k)\big(Q\big),$$

i.e., the function which maps W to f^*W.

Observe that by proposition 2.54 we have for any pair of pl maps $f : P \to Q$ and $g : Q \to S$ that $(g \circ f)^* = f^* \circ g^*$ and hence we have that (10) is indeed a contravariant functor. Also, observe that the set of p-simplices $\psi_d(N,k)_p$ of the Δ-set defined in 3.2 is equal to

$$\psi_d(N,k)\big(\Delta^p\big)$$

and that the i-th face map $\Delta_i : \psi_d(N,k)_{p+1} \to \psi_d(N,k)_p$ is equal to

$$\psi_d(N,k)\big(\delta_i : \Delta^p \hookrightarrow \Delta^{p+1}\big)$$

and hence $\psi_d(N,k)_\bullet$ can be completely recovered from the functor (10).

We shall also make use of the following terminology.

Definition 3.4. Let P be a polyhedron. Two elements W_0 and W_1 of $\psi_d(N,k)\big(P\big)$ are said to be *concordant* if there is an element $\widetilde{W}$ of $\psi_d(N,k)\big([0,1] \times P\big)$ such that $i_0^*\widetilde{W} = W_0$ and $i_1^*\widetilde{W} = W_1$, where for $j = 0,1$ we have that $i_j : P \hookrightarrow [0,1] \times P$ is the inclusion defined by $\lambda \mapsto (j,\lambda)$. An element $\widetilde{W}$ in $\psi_d(N,k)\big([0,1] \times P\big)$ such that $i_j^*\widetilde{W} = W_j$ for $j = 0,1$ is said to be a *concordance between* W_0 *and* W_1.

3.2 Properties of $\Psi_d(\mathbb{R}^N)_\bullet$

The main result in this section is proposition 3.5 given below which describes a way of classifying all the elements of the set $\Psi_d(\mathbb{R}^N)\big(P\big)$ using the Δ-set $\Psi_d(\mathbb{R}^N)_\bullet$ when the polyhedron P is a compact Euclidean polyhedron and when P is triangulated by a simplicial complex K. Recall that an Euclidean simplicial

complex is said to be *ordered* if there is a relation $\leq$ on the set of vertices $\mathrm{Vert}(K)$ which restricts to a linear order on $\mathrm{Vert}(\sigma)$ for each simplex σ of K. Also, in the statement of propostion 3.5 we shall make use of the following convention: if A is a p simplex in some Euclidean space with vertices $v_0, \ldots, v_p$ and if $e_0, \ldots, e_p$ are the elements of the standard basis of $\mathbb{R}^{p+1}$ then $s_{v_0 \ldots v_p} : \Delta^p \to A$ is the simplicial map defined by $e_j \mapsto v_j$.

Proposition 3.5. *Let $(K, \leq)$ be a finite ordered simplicial complex in some Euclidean space $\mathbb{R}^m$, let P be the underlying polyhedron of K and let $K_\bullet$ be the Δ-set obtained from K as indicated in 2.63. Then the function of sets*

$$\mathcal{S}_K : \Psi_d(\mathbb{R}^N)(P) \to \Delta(K_\bullet, \Psi_d(\mathbb{R}^N)_\bullet)$$

which sends an element $W \subseteq P \times \mathbb{R}^N$ of $\Psi_d(\mathbb{R}^N)(P)$ to the morphism $f_W : K_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ of Δ-sets defined by $f_W(v_0 \leq \ldots \leq v_p) = s^*_{v_0 \ldots v_q} W$ is a bijection.

Definition 3.6. Let $(K, \leq)$ be a finite ordered simplicial complex in some Euclidean space $\mathbb{R}^m$. If W is an element in $\Psi_d(\mathbb{R}^N)\big(|K|\big)$ and if $h_\bullet := \mathcal{S}_K(W)$ we say that $h_\bullet$ *classifies the element* W.

Note 3.7. For the proof of 3.5 it will be useful to use the following convention: if $W \subseteq P \times \mathbb{R}^N$ is an element of $\Psi_d(\mathbb{R}^N)\big(P\big)$ and if S is a sub-polyhedron of P we denote by W_S the pre-image $\pi^{-1}(S)$, where π is the projection of W onto P, and we denote by π_S the restriction of the map π on W_S. In particular, $W_S \in \Psi_d(\mathbb{R}^N)\big(S\big)$.

Lemma 3.8. *Let*

$$H_+ := \{(x_1, \ldots, x_p) : \quad x_p \geq 0\}$$

be the closed half space in $\mathbb{R}^p$, let W be an element in $\Psi_d(\mathbb{R}^N)\big(H_+\big)$, let λ_0 and y_0 be points in ∂H_+ and $\pi^{-1}(\lambda_0)$ respectively, and let $f : V \times \mathbb{R}^d \to W$ and $g : U \times \mathbb{R}^d \to W_{\partial H_+}$ be submersion charts around y_0 for π and $\pi_{\partial H_+}$ respectively such that $\mathrm{Im}g \subseteq \mathrm{Im}f$ and such that $U \times [0, \delta) \subset V$ for a suitable value $0 < \delta < 1$. Then there exists a submersion chart $g' : U \times [0, \delta) \times \mathbb{R}^d \to W$ around y_0 for π such that $\mathrm{Im}g' \subseteq \mathrm{Im}f$ and $g'_{|U \times \mathbb{R}^d} = g$.

Proof. Let $h : U \times \mathbb{R}^d \to U \times \mathbb{R}^d$ be equal to the composite

$$U \times \mathbb{R}^d \xrightarrow{\ g\ } \mathrm{Im}g \xrightarrow{\ f^{-1}\ } U \times \mathbb{R}^d \ .$$

where the second map is just the restriction of f^{-1} on $\mathrm{Im}g$. Since both maps in this composition are piecewise linear embeddings then h is also a piecewise linear embedding. Furthermore, we also have that the image of h is open in $U \times \mathbb{R}^d$. Indeed, $\mathrm{Im}g$ is an open subset of $\mathrm{Im}f \cap W_{\partial H_+}$ and since the restriction of f^{-1} on $\mathrm{Im}f \cap W_{\partial H_+}$ is a pl homeomorphism between $\mathrm{Im}f \cap W_{\partial H_+}$ and $(V \cap \partial H_+) \times \mathbb{R}^d$ we have that $\mathrm{Im}h = f^{-1}(\mathrm{Im}g)$ is open in $(V \cap \partial H_+) \times \mathbb{R}^d$. However, since the image of h lies entirely in $U \times \mathbb{R}^d$ we have that $\mathrm{Im}h$ is open in $U \times \mathbb{R}^d$. This observation will be useful when we prove that the image of the

chart g' which extends g is open.

Let $\widetilde{h} : U \times [0,\delta) \times \mathbb{R}^d \to U \times [0,\delta) \times \mathbb{R}^d$ be the piecewise linear embedding which maps (u,t,x) to $\big(u,t,h_2(u,x)\big)$, where h_2 is the second component of the map $h : U \times \mathbb{R}^d \to U \times \mathbb{R}^d$, and let $g' : U \times [0,\delta) \times \mathbb{R}^d \to W$ be the following composite of piecewise linear embeddings

$$g' = f \circ \widetilde{h}.$$

It is easy to verify that $g'(\lambda_0, 0, 0) = y_0$ and that the following diagram commutes

$$\begin{array}{ccc} U \times [0,\delta) \times \mathbb{R}^d & \xrightarrow{g'} & W \\ \Big\downarrow{\scriptstyle pr_1} & & \Big\downarrow{\scriptstyle \pi} \\ U \times [0,\delta) & \hookrightarrow & H_+. \end{array}$$

where the bottom map is just the obvious inclusion $U \times [0,\delta) \hookrightarrow H_+$. Furthermore, since the image of the map h is open it follows that the image of $\widetilde{h}$ is also open which implies that the image $\mathrm{Im}g'$ is open in W and thus we have that g' is a submersion chart for π around y_0.

Finally, we also have that g' extends g since for any point $(u,0,x)$ in $U \times \{0\} \times \mathbb{R}^d$ we have the following:

$$\begin{aligned} g'(u,0,x) &= f \circ \widetilde{h}(u,0,x) \\ &= f(u,0,h_2(u,x)) \\ &= f(h(u,x)) \\ &= g(u,x). \end{aligned}$$

□

Lemma 3.9. *Let K be a finite simplicial complex in $\mathbb{R}^m$, let W be a closed sub-polyhedron of $|K| \times \mathbb{R}^N$ and let $\mathcal{K}$ be the poset of simplices of K. If $F : \mathcal{K} \to \mathbf{Top}$ is the functor which sends σ to $W_\sigma := W \cap \big(\sigma \times \mathbb{R}^N\big)$ and which sends the unique morphism $\beta \to \sigma$ to the inclusion $W_\beta \hookrightarrow W_\sigma$ whenever $\beta \leq \sigma$ then the map $f : \mathbf{colim}_{\mathcal{K}} F \to W$ obtained by the universal property of colimits is a homeomorphism.*

Proof. By the definition of colimit there is for each $\sigma \in K$ a unique map $h_\sigma : W_\sigma \to \mathbf{colim}_{\mathcal{K}} F$ such that the inclusion $W_\sigma \hookrightarrow W$ is equal to the composite $f \circ h_\sigma$. The result now follows from the fact that the map $g : W \to \mathbf{colim}_{\mathcal{K}} F$ defined by $g(x) = h_\sigma(x)$ if $x \in W_\sigma$ is a continuous inverse for f. □

Lemma 3.10. *Let K be a finite simplicial complex in $\mathbb{R}^m$ of the form $a_0 * L$, let W be a closed sub-polyhedron of $|K| \times \mathbb{R}^N$ and let $\pi : W \to |K|$ be the projection from W onto $|K|$. Suppose that for each simplex σ of K the projection $\pi_\sigma : W_\sigma \to \sigma$ is a piecewise linear submersion of codimension d. Then π is a pl submersion of codimension d on a neighborhood of $\pi^{-1}(a_0)$.*

Proof. Let q be the dimension of the simplicial complex K and let us fix a point y_0 in the fiber $\pi^{-1}(a_0)$. For each skeleton K^i of K let $\pi_{|K^i|} : W_{|K^i|} \to |K^i|$ be the restriction of π on $W_{|K^i|}$. The idea of the proof is to define, by induction on the dimension of the skeleta K^i, open pl embeddings $h_i : U_i \times \mathbb{R}^d \to W_{|K^i|}$ such that $h_i\big((a_0, 0)\big) = y_0$ and such that the following diagram commutes

$$\begin{array}{ccc} U_i \times \mathbb{R}^d & \xrightarrow{h_i} & W_{|K^i|} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi_{|K^i|}} \\ U_i & \hookrightarrow & |K^i|. \end{array}$$

Let then $\sigma_1, \ldots, \sigma_p$ be the 1-simplices of K which contain the vertex a_0 and let $f_i : \tilde{U}_i \times \mathbb{R}^d \to W_{\sigma_i}$ be a submersion chart around $y_0 \in \pi_{\sigma_i}^{-1}(a_0)$ for π_{σ_i}. We can assume that each $\tilde{U}_i$ doesn't intersect the link $\mathrm{lk}(a_0, K^1)$. Observe that $(\bigcap_{i=1}^p \mathrm{Im} f_i) \cap \pi^{-1}(a_0)$ is an open neighborhood of y_0 in $\pi^{-1}(a_0) = W_{a_0}$. Hence, since W_{a_0} is a d-dimensional piecewise linear manifold, we can find a piecewise linear embedding $g_0 : \mathbb{R}^d \to W_{a_0}$ with open image such that $g_0(0) = y_0$ and $\mathrm{Im} g_0 \subseteq \bigcap_{i=1}^p \mathrm{Im} f_i$. By lemma 3.8 there is for each $1 \leq i \leq p$ a submersion chart

$$g_i' : V_i' \times \mathbb{R}^d \to W_{\sigma_i}$$

for π_{σ_i} around y_0 such that $\mathrm{Im} g_i' \subseteq \mathrm{Im} f_i$ and such that g_i' extends g. Let V_i be a compact neighborhood of a_0 in σ_i contained in $\mathrm{int} V_i'$ and consider the restriction of g_i' on $V_i \times [-1, 1]^d$. By abuse of notation , we will also denote this restriction simply by g_i'. We now define a piecewise linear map

$$g_1 : \Big(\bigcup_{i=1}^p V_i \Big) \times [-1, 1]^d \to W_{|K^1|}$$

by setting $g_1(\lambda, x) = g_i'(\lambda, x)$ if (λ, x) is in $V_i \times [-1, 1]^d$. Observe that this map is well defined since for each g_i' we have $g_i'|_{\{a_0\} \times [-1,1]^d} = g_0|_{[-1,1]^d}$ after we identify $\{a_0\} \times [-1, 1]^d$ with $[-1, 1]^d$ and since each g_i' is a submersion chart we have both that $g_1(a, 0) = y_0$ and that the diagram

$$\begin{array}{ccc} \Big(\bigcup_{i=1}^p V_i \Big) \times [-1, 1]^d & \xrightarrow{g_1} & W_{|K^1|} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi_{|K^1|}} \\ \bigcup_{i=1}^p V_i & \longrightarrow & |K^1| \end{array}$$

commutes. Finally, let us denote the union $\bigcup_{i=1}^p \mathrm{int}_{\sigma_i} V_i$ by U_1 and let us now show that the image $g_1(U_1 \times (-1, 1)^d)$ is open in $W_{|K^1|}$. Observe that each $g_i'\big(\mathrm{int}_{\sigma_i} V_i \times (-1, 1)^d\big)$ is open in W_{σ_i} and since $g_1(U_1 \times (-1, 1)^d) \cap W_{\sigma_i} = g_i'\big(\mathrm{int}_{\sigma_i} V_i \times (-1, 1)^d\big)$ we have by lemma 3.9 that $\mathrm{Im} g_1$ is open in $W_{|K^1|}$. Thus, after identifying $(-1, 1)^d$ with $\mathbb{R}^d$, we obtain a submersion chart $g_1 : U_1 \times \mathbb{R}^d \to$

$W_{|K^1|}$ around y_0 for the projection $\pi_{|K^1|}$.

Suppose now that we have a submersion chart $g_n : U_n \times \mathbb{R}^d \to W_{|K^n|}$ for $\pi_{|K^n|}$ around $y_0 \in \pi^{-1}_{|K^n|}(a_0)$. We can assume that U_n doesn't intersect the link $\mathrm{lk}(a_0, K^n)$. Let $\sigma_1, \ldots, \sigma_p$ be the $n+1$-simplices of K which contain a_0. Observe that the restriction of g_n on $(U_n \cap \partial\sigma_i) \times \mathbb{R}^d$ is a submersion chart for $\pi_{\partial\sigma_i} : W_{\partial\sigma_i} \to \partial\sigma_i$ around y_0. Now, for each one of the $n+1$-simplices σ_i pick a submersion chart $f_i : \tilde{U}_i \times \mathbb{R}^d \to W_{\sigma_i}$ for π_{σ_i} around y_0. After possibly shrinking U_n and rescaling $\mathbb{R}^d$ we can assume that $g_n((U_n \cap \partial\sigma_i) \times \mathbb{R}^d) \subseteq \mathrm{Im} f_i$ for all $i = 1, \ldots, p$. Applying again lemma 3.8 we can find a submersion chart $g_i' : V_i' \times \mathbb{R}^d \to W_{\sigma_i}$ for π_{σ_i} around y_0 such that $\mathrm{Im} g_i' \subseteq \mathrm{Im} f_i$ and such that g_i' extends the restriction of g_i on $(U_n \cap \partial\sigma_i) \times \mathbb{R}^d$. As we did in the first step of the induction we can take smaller compact neighborhoods $V_i \subset \mathrm{int}_{\sigma_i} V_i'$ of a_0, restrict each g_i' on $\mathrm{int}_{\sigma_i} V_i \times (-1,1)^d$ and glue all of these restricitions together to obtain a piecewise linear embedding $g_{n+1} : U_{n+1} \times (-1,1)^d \to W_{|K^{n+1}|}$, where $U_{n+1} = \bigcup_{i=1}^p \mathrm{int}_{\sigma_i} V_i$, which commutes with the projection onto U_{n+1} and which has open image. After identifying $(-1,1)^d$ with $\mathbb{R}^d$ we have that g_{n+1} is a submersion chart for $\pi_{|K^{n+1}|}$ around y_0. Since the simplicial complex K is finite we conclude that there exists a submersion chart $h : U \times \mathbb{R}^d \to \pi^{-1}(U)$ for $\pi : W \to |K|$ around y_0.

□

We remark that in the previous proof we can assume that the open neighborhood U of a_0 in $|K|$ is contained in the open set $|K| - |L|$. This observation will be used in the following lemma.

Lemma 3.11. *Let K be a finite simplicial complex in $\mathbb{R}^m$ and let W be a closed sub-polyhedron of $|K| \times \mathbb{R}^N$ such that for each simplex σ of K the projection map $\pi_\sigma : W_\sigma \to \sigma$ is a piecewise linear submersion of codimension d. Then the projection map $\pi : W \to |K|$ is also a piecewise linear submersion of codimension d.*

Proof. Let y_0 be any point in the polyhedron W and let $a_0 = \pi(y_0)$. We can subdivide the simplicial complex K in order to obtain a simplicial complex K' which has a_0 as a vertex and such that $|K'| = |K|$. By proposition 2.52 we have that for each simplex σ of K' the projection $\pi_\sigma : W_\sigma \to \sigma$ is a piecewise linear submersion of codimension d . Let $\mathrm{lk}(a_0, K')$ be the link of a_0 in K' and let $\mathrm{st}(a_0, K')$ be the star of a_0 in K'. Observe that $\mathrm{st}(a_0, K') = \mathrm{lk}(a_0, K') * a_0$ and that $\mathrm{st}(a_0, K') - \mathrm{lk}(a_0, K')$ is an open neighborhood of a_0 in $|K'|$. Applying lemma 3.10 to the projection $\pi_{\mathrm{st}(a_0,K')} : W_{\mathrm{st}(a_0,K')} \to \mathrm{st}(a_0, K')$ we obtain a submersion chart $h : U \times \mathbb{R}^d \to W_{\mathrm{st}(a_0,K')}$ for $\pi_{\mathrm{st}(a_0,K')}$ around y_0. However, by the remark preceding this proof, the open subspace U of $\mathrm{st}(a_0, K')$ can be taken to be contained in $\mathrm{st}(a_0, K') - \mathrm{lk}(a_0, K')$ and thus we have that h is actually a submersion chart for $\pi : W \to |K|$ around y_0. Since we can do this argument for any $y_0 \in W$ we conclude that $\pi : W \to |K|$ is a piecewise linear submersion of codimension d. □

Recall that if σ is a p-simplex inside some eucildean space $\mathbb{R}^m$ spanned by a collection of points $v_0, \ldots, v_p$ in general position and if $\{e_0, \ldots, e_p\}$ is the standard basis of $\mathbb{R}^{p+1}$ then $s_{v_0 \ldots v_p} : \Delta^p \to \sigma$ is the simplicial map which sends e_j to v_j. Let us now denote the inverse of $s_{v_0 \ldots v_p}$ by $t_{v_0 \ldots v_p}$.

Lemma 3.12. *Let $(K, \leq)$ be a finite ordered simplicial complex of dimension k in $\mathbb{R}^m$, let $K_\bullet$ be the Δ-set obtained from K as indicated in 2.63 and let $g_\bullet : K_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ be a morphism of Δ-sets. For each simplex β of $K_\bullet$ let $W^\beta \subseteq \Delta^{\dim \beta} \times \mathbb{R}^N$ be the underlying polyhedron of the image of β under $g_\bullet$. Then the union*

$$W = \bigcup_{j=0}^{k} \Big(\bigcup_{v_0 \leq \ldots \leq v_j} t^*_{v_0 \ldots v_j} W^{v_0 \leq \cdots \leq v_j} \Big)$$

is a closed sub-polyhedron of $|K| \times \mathbb{R}^N$ and the projection $\pi : W \to |K|$ from W onto $|K|$ is a piecewise linear submersion of codimension d. In particular, $W \in \Psi_d(\mathbb{R}^N)\big(|K|\big)$.

Proof. Recall that we are denoting by W_S the pre-image $\pi^{-1}(S)$ of a sub-polyhedron S of $|K|$ under $\pi : W \to |K|$. For each simplex $v_0 \leq \ldots \leq v_p$ of $K_\bullet$ let $\langle v_0, \ldots, v_p \rangle$ denote the simplex of K spanned by the points $v_0, \ldots, v_p$. We begin this proof by observing that for each simplex $v_0 \leq \ldots \leq v_p$ of $K_\bullet$ we have

$$W_{\langle v_0, \ldots, v_p \rangle} = t^*_{v_0 \ldots v_p} W^{v_0 \leq \cdots \leq v_p}.$$

Indeed, if $w_0 \leq \ldots \leq w_p$ and $w'_0 \leq \ldots \leq w'_q$ are simplices of $K_\bullet$ such that $w_0 \leq \ldots \leq w_p$ is a face of $w'_0 \leq \ldots \leq w'_q$ and if $i : \langle w_0 \leq \ldots \leq w_p \rangle \hookrightarrow \langle w'_0 \leq \ldots \leq w'_q \rangle$ is the obvious inclusion from $\langle w_0 \leq \ldots \leq w_p \rangle$ into $\langle w'_0 \leq \ldots \leq w'_q \rangle$ then we have that $t^*_{w_0 \ldots w_p} W^{w_0 \leq \cdots \leq w_p} = i^* t^*_{w'_0 \ldots w'_q} W^{w'_0 \leq \cdots \leq w'_q}$ and this implies that $W_{\langle v_0, \ldots, v_p \rangle} = t^*_{v_0 \ldots v_p} W^{v_0 \leq \cdots \leq v_p}$ for any simplex $v_0 \leq \ldots \leq v_p$ of $K_\bullet$. Now, since for each simplex $\sigma = \langle v_0, \ldots, v_p \rangle$ the space $W_{\langle v_0, \ldots, v_p \rangle}$ is a closed sub-polyhedron of $|K| \times \mathbb{R}^N$ we have that the union $\bigcup_{\sigma \in K} W_\sigma$, which is equal to W, is a closed sub-polyhedron of $|K| \times \mathbb{R}^N$ since it is just the union of finitely many closed sub-polyhedrons. Finally, by lemma 3.10 we have that the projection $\pi : W \to |K|$ is a piecewise linear submersion of codimension d.

□

Note. The element W in $\Psi_d(\mathbb{R}^N)\big(|K|\big)$ obtained from the morphism $g_\bullet : K_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ shall be denoted by W^g. We shall use this notation in the following proof.

Proof of proposition 3.5: Let K be a finite ordered simpilcial complex in $\mathbb{R}^m$ and let $K_\bullet$ be the Δ-set obtained from K as indicated in definition 2.63. Recall that the function of sets

$$\mathcal{S}_K : \Psi_d(\mathbb{R}^N)(|K|) \to \Delta(K_\bullet, \Psi_d(\mathbb{R}^N)_\bullet)$$

sends an a element W of $\Psi_d(\mathbb{R}^N)(|K|)$ to the morphism of Δ-sets $f_{W,\bullet} : K_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ which maps a simplex $v_0 \leq \ldots \leq v_p$ to the simplex $s^*_{v_0 \ldots v_p} W_{\langle v_0 \leq \ldots \leq v_p \rangle}$.

To see that $f_{W,\bullet}$ is indeed a morphism of Δ-sets observe that if $w_0 \leq \ldots \leq w_q$ is a face of $v_0 \leq \ldots \leq v_p$ and if $i_1 : \langle w_0 \leq \ldots \leq w_q \rangle \hookrightarrow \langle v_0 \leq \ldots v_p \rangle$ and $i_2 : \Delta^q \hookrightarrow \Delta^p$ are the obvious inclusions then we have that

$$s^*_{w_0 \ldots w_q} i_1^* W_{\langle v_0 \leq \ldots \leq v_p \rangle} = i_2^* s^*_{v_0 \ldots v_p} W_{\langle v_0 \leq \ldots \leq v_p \rangle}$$

and this implies that $f_{W,\bullet}$ commutes with the structure maps of $K_\bullet$ and $\Psi_d(\mathbb{R}^N)_\bullet$. Let

$$\mathcal{T}_K : \Delta(K_\bullet, \Psi_d(\mathbb{R}^N)_\bullet) \to \Psi_d(\mathbb{R}^N)(|K|)$$

be the function of sets which sends a morphism $g_\bullet : K_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ of Δ-sets to the element W^g of $\Psi_d(\mathbb{R}^N)(|K|)$ defined in lemma 3.12. We will show that $\mathcal{S}_K$ and $\mathcal{T}_K$ are inverses of each other. For any morphism $g_\bullet \in \Delta(K_\bullet, \Psi_d(\mathbb{R}^N)_\bullet)$ let us denote by $W^{g(\sigma)}$ the underlying polyhedron of the simplex $g_{\dim \sigma}(\sigma)$. To see that the composition $\mathcal{S}_K \circ \mathcal{T}_K$ is the identity on $\Delta(K_\bullet, \Psi_d(\mathbb{R}^N)_\bullet)$ observe that for a morphism $g_\bullet \in \Delta(K_\bullet, \Psi_d(\mathbb{R}^N)_\bullet)$ the image $\mathcal{S}_K \circ \mathcal{T}_K(g_\bullet)$ is the morphism $f_{W^g,\bullet}$ which sends a simplex $\sigma = v_0 \leq \ldots \leq v_p$ to $s^*_{v_0 \ldots v_p} t^*_{v_0 \ldots v_p} W^{g(\sigma)}$. But since $s_{v_0 \ldots v_p}$ and $t_{v_0 \ldots v_p}$ are inverses of each other we have that $g_\bullet = f_{W^g,\bullet}$. A similar argument works to show that the composition $\mathcal{T}_K \circ \mathcal{S}_K$ is the identity on $\Psi_d(\mathbb{R}^N)(|K|)$ since for any W in $\Psi_d(\mathbb{R}^N)(|K|)$ we have that W and W^{f_W} can be expressed as

$$W = \bigcup_{\sigma \in K} W_\sigma$$

and

$$W^{f_W} = \bigcup_{\sigma \in K} W^{f_W}_\sigma$$

respectively, and as we observed in the proof of lemma 3.12 for each simplex $\sigma = \langle v_0 \leq \ldots \leq v_p \rangle$ of K the sub-polyhedron $W^{f_W}_\sigma$ is equal to $t^*_{v_0 \ldots v_p} W^{f_W(\sigma)}$ which in turn is equal to $t^*_{v_0 \ldots v_p} s^*_{v_0 \ldots v_p} W_\sigma$. Since $t_{v_0 \ldots v_p}$ and $s_{v_0 \ldots v_p}$ are inverses of each other we obtain that that $W = W^{f_W}$ and we can now conclude that $\mathcal{S}_K$ is a bijection.

□

Using this proposition and proposition 2.52 we can prove the following theorem.

Theorem 3.13. *$\Psi_d(\mathbb{R}^N)_\bullet$ is a Kan Δ-set.*

Proof. Let Λ^p_j be the j-th horn of Δ^p, let $\Lambda^p_{j\bullet}$ be the Δ-set obtained from Λ^p_j using the canonical order on $\mathrm{Vert}(\Lambda^p_j)$ and let $g_\bullet : \Lambda^p_{j\bullet} \to \Psi_d(\mathbb{R}^N)_\bullet$ be a morphism of Δ-sets. We need to show that $g_\bullet$ can be extended to $\Delta^p_\bullet$. The inclusion $i : \Lambda^p_j \hookrightarrow \Delta^p$ is actually an inclusion of ordered simplicial complexes

and using this fact it is easy to verify that the following diagram commutes

$$\begin{array}{ccc} \Psi_d(\mathbb{R}^N)\big(\Delta^p\big) & \xrightarrow{\mathcal{S}_{\Delta^p}} & \Delta(\Delta^p_\bullet, \Psi_d(\mathbb{R}^N)_\bullet) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle -\circ i_\bullet} \\ \Psi_d(\mathbb{R}^N)\big(\Lambda^p_j\big) & \xrightarrow{\mathcal{S}_{\Lambda^p_j}} & \Delta(\Lambda^p_{j\bullet}, \Psi_d(\mathbb{R}^N)_\bullet). \end{array}$$

Since $\mathcal{S}_{\Lambda^p_j}$ is surjective we can find W in $\Psi_d(\mathbb{R}^N)\big(\Lambda^p_j\big)$ such that $\mathcal{S}_{\Lambda^p_j}(W) = g_\bullet$ and by the commutativity of the above diagram it suffices to find a lift of W in $\Psi_d(\mathbb{R}^N)\big(\Delta^p\big)$ in order to find a lift of $g_\bullet$ in $\Delta(\Delta^p_\bullet, \Psi_d(\mathbb{R}^N)_\bullet)$. But if $r : \Delta^p \to \Lambda^p_j$ is any pl retraction onto Λ^p_j then the pull back r^*W is a lift for W and consequently $f_\bullet := \mathcal{S}_{\Delta^p}(r^*W)$ is a lift for $g_\bullet$. □

Kan Δ-sets are very convenient objects to work with in light of the following theorem which is proven in [18].

Theorem 3.14. *Suppose $Z_\bullet \subset Y_\bullet$ is a pair of Δ-sets and $X_\bullet$ is a Kan Δ-set. Suppose given a map $f : |Y_\bullet| \to |X_\bullet|$ such that $f|_{|Z_\bullet|}$ is the realization of a morphism of Δ-sets. Then f is homotopic relative to $|Z_\bullet|$ to the realization of a morphism of Δ-sets $f'_\bullet : Y_\bullet \to X_\bullet$.*

Note 3.15. We remark that the exact same argument used to prove theorem 3.5 also works to prove the exact same result for each sub-Δ-set $\psi_d(N,k)_\bullet$ of $\Psi_d(\mathbb{R}^N)_\bullet$ defined in 3.2. In particular, we have that each $\psi_d(N,k)_\bullet$ is a Kan-Δ-set.

3.3 The sheaf $\Psi_d(\mathbb{R}^N)$

In 3.1 we defined the *space of pl manifolds* $\Psi_d(\mathbb{R}^N)_\bullet$ using $\mathbb{R}^N$ as the background space. However, we could have easily defined a Δ-set $\Psi_d(U)_\bullet$ using *any* open subspace $U \subseteq \mathbb{R}^N$ as the background space and furthermore for any inclusion of open sets $V \hookrightarrow U$ we have that the function which sends a p-simplex W of $\Psi_d(U)_\bullet$ to the intersection $W \cap \big(\Delta^p \times V\big)$ is actually a morphism of Δ-sets. With these observations we can formulate the following definition.

Definition 3.16. Let

$$\Psi_d(-) : \mathcal{O}(\mathbb{R}^N)^{op} \longrightarrow \textbf{Δ-sets}$$

be the pre-sheaf of Δ-sets on $\mathbb{R}^N$ which sends an open set U to the Δ-set $\Psi_d(U)_\bullet$ whose set of p-simplices consists of all closed sub-polyhedrons W of $\Delta^p \times U$ such that the projection onto Δ^p is a piecewise linear submersion of codimension d. An inclusion of open sets $V \hookrightarrow U$ is mapped to the restriction morphism $\Psi_d(U)_\bullet \to \Psi_d(V)_\bullet$ which sends a p-simplex W to $W \cap \big(\Delta^p \times V\big)$.

The following proposition tells us that the functor $\Psi_d(-)$ is actually a sheaf.

Proposition 3.17. *Let U_1 and U_2 be open subsets of $\mathbb{R}^N$. If $W_1 \subseteq P \times U_1$ and $W_2 \subseteq P \times U_2$ are elements in $\Psi_d(U_1)(P)$ and $\Psi_d(U_2)(P)$ respectively such that $W_1 \cap \big(P \times U_2\big) = W_2 \cap \big(P \times U_1\big)$ then the union $W := W_1 \cup W_2$ is an element in $\Psi_d(U_1 \cup U_2)(P)$.*

Proof. Since W_1 and W_2 are both open subspaces of W and since W_1 and W_2 are both piecewise linear subspaces of $P \times U$ then by proposition 2.39 we have that W is a piecewise linear subspace of $P \times U$. To see that W is closed in $P \times U$ observe that the limit of any convergent sequence of points $(x_n)_n$ in W must be either in U_1 or U_2 and thus it must lie in either W_1 or W_2 since W_i is closed in U_i for $i = 1, 2$. Finally, the projection $\pi : W \to P$ must be a piecewise linear submersion of codimension d since the restrictions of π on W_1 and W_2, which are both open subspaces of W, are assumed to be piecewise linear submersions of codimension d. □

By applying this proposition to the case when the base space P is equal to one of the standard simplices Δ^p we obtain the following corollary.

Corollary 3.18. *The pre-sheaf*

$$\Psi_d(-) : \mathcal{O}(\mathbb{R}^N)^{op} \longrightarrow \Delta\textbf{-sets}$$

defined in 3.16 is a sheaf of Δ-sets.

We conclude this subsection with the following useful proposition which tells how to produce elements in $\Psi_d(\mathbb{R}^N)\big(P\big)$ using open pl embeddings $P \times \mathbb{R}^N \to P \times \mathbb{R}^N$ which commute with the projection onto P.

Proposition 3.19. *Let P be a compact polyhedron and let W be an element of $\Psi_d(\mathbb{R}^N)(P)$. Then for any open piecewise linear embedding $H : P \times \mathbb{R}^N \to P \times \mathbb{R}^N$ which commutes with the projection onto P we have that $H^{-1}(W)$ is an element of $\Psi_d(\mathbb{R}^N)(P)$.*

Proof. The intersection of W and $\mathrm{Im}H$ is a closed sub-polyhedron of $\mathrm{Im}H$ and since H is a piecewise linear homeomorphism from $P \times \mathbb{R}^N$ to $\mathrm{Im}H$ we also have that $H^{-1}(W)$ is a closed sub-polyhedron of $P \times \mathbb{R}^N$. Finally, the projection from $H^{-1}(W)$ to P is a piecewise linear submersion of codimension d since it is equal to the composition of $H|_{H^{-1}(W)}$, which is a piecewise linear homeomorphism from $H^{-1}(W)$ to W, and the projection from W onto P.

□

4 Functors $\mathcal{F} : \mathbf{Pl}^{op} \to \mathbf{Sets}$

In the previous section we introduced both the Δ-set $\Psi_d(\mathbb{R}^N)_\bullet$ and the more general contravariant functor $\Psi_d(\mathbb{R}^N) : \mathbf{PL}^{op} \to \mathbf{Sets}$ defined on the category **PL** of (finite dimensional) polyhedra and piecewise linear maps. It is the aim of this section to show how this more general functor defined on $\mathbf{PL}^{op}$ can be used to analyze the Δ-set $\Psi_d(\mathbb{R}^N)_\bullet$. In particular, we are going to show that there is a map $\rho : \big|\Psi_d(\mathbb{R}^N)_\bullet\big| \to \big|\Psi_d(\mathbb{R}^N)_\bullet\big|$ which satisfies the following properties:

1. ρ is homotopic to the identity map on $\left|\Psi_d(\mathbb{R}^N)_\bullet\right|$.
2. For any morphism of Δ-sets $f_\bullet : X_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ and for any non-negative integer r there is a unique morphism $g_\bullet : \mathrm{sd}^r X_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ which makes the following diagram commute

$$\begin{array}{ccc} |X_\bullet| & \xrightarrow{|f_\bullet|} & \left|\Psi_d(\mathbb{R}^N)_\bullet\right| \\ \cong \uparrow & & \downarrow \rho^r \\ |\mathrm{sd}^r X_\bullet| & \xrightarrow{|g_\bullet|} & \left|\Psi_d(\mathbb{R}^N)_\bullet\right| \end{array}$$

where the left vertical map is just the canonical homeomorphism from the geometric realization $|\mathrm{sd}^r X_\bullet|$ of the r-th barycentric subdivision of $X_\bullet$ to $|X_\bullet|$.

Because of property 2) the map ρ will be called the *subdivision map of* $\Psi_d(\mathbb{R}^N)_\bullet$.

4.1 Preliminary simplicial constructions

Before we can define the subdivision map $\rho : \left|\Psi_d(\mathbb{R}^N)_\bullet\right| \to \left|\Psi_d(\mathbb{R}^N)_\bullet\right|$ we need to introduce some basic constructions using simplicial complexes. We start by recalling the following definition given in 2.12 of § 1.

Definition 4.1. Let K be a simplicial complex in $\mathbb{R}^m$ and for each simplex F let bF be its barycentric point. The *(first) barycentric subdivision of* K, denoted by $\mathrm{sd}K$, is the stellar subdivision obtained by starring K at the barycentric points bF.

In order to define the subdivision map ρ we first need to introduce for each $p \in \mathbb{N}$ a simplicial complex $L(p)$ in $\mathbb{R}^{p+2}$ which triangulates the product $\Delta^p \times [0,1]$. We define these simplicial complexes by induction on p. In order to make our notation easier to follow we will throughout this section denote by $K(m)$ and $\partial K(m)$ the canonical simplicial complexes which triangulate Δ^m and $\partial\Delta^m$ respectively. Also, if K is any simplicial complex in $\mathbb{R}^M$, if a_0 is any point in $\mathbb{R}^P$ and if $T : \mathbb{R}^M \to \mathbb{R}^{M'}$ is any affine linear embedding we will denote by $K \times a_0$ and by $T(K)$ the simplicial complexes in $\mathbb{R}^{M+P}$ and in $\mathbb{R}^{M'}$ consisting of all simplices of the form $\sigma \times \{a_0\}$ and $T(\sigma)$ respectively with $\sigma \in K$.

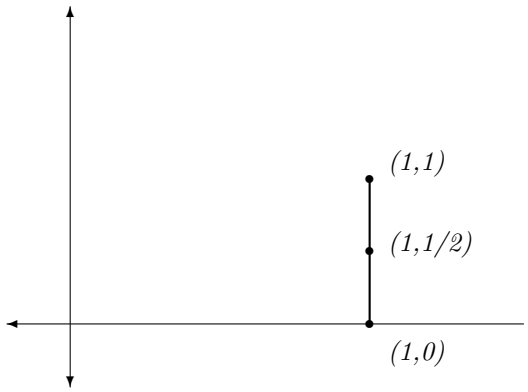
Recall that twe are denoting the canonical basis of $\mathbb{R}^{p+1}$ by $e_0, \ldots, e_p$. In order to distinguish the canonical bases of different Euclideant spaces we shall in this section denote the canonical basis of $\mathbb{R}^{p+1}$ by

$$e_{0,p+1}, \ldots, e_{j,p+1}, \ldots, e_{p,p+1}.$$

We begin now with the construction of the simplicial complex $L(p)$. For $p = 0$ let $L(0)$ be the simplicial complex in $\mathbb{R}^2$ consisting of the following simplices (see figure (1)):

$$(1,0),\ (1,1),\ (1,\frac{1}{2}),\ \ \{1\} \times [0,\frac{1}{2}],\ \ \{1\} \times [\frac{1}{2},1].$$

Figure (1)



Using $L(0)$ we define $L(1)$ in the following way: let $T_0 : \mathbb{R}^2 \to \mathbb{R}^3$ and $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ be the injective linear maps defined respectively by

$$T_0(e_{0,2}) = e_{0,3}, \quad T_0(e_{1,2}) = e_{2,3}$$

and

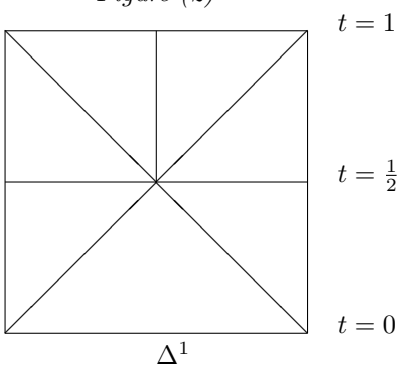
$$T_1(e_{0,2}) = e_{1,3}, \quad T_1(e_{1,2}) = e_{2,3}.$$

Also, let $L(1)'$ denote the following union of simplicial complexes in $\mathbb{R}^3$:

$$T_0\big(L(0)\big) \cup T_1\big(L(0)\big) \cup \big(\mathrm{sd}K(1) \times 1\big) \cup \big(K(1) \times 0\big).$$

$L(1)'$ is a simplicial complex which triangulates $\partial(\Delta^1 \times [0,1])$ and we define $L(1)$ to be the join $L(1)' * (b\Delta^1, \frac{1}{2})$, where $b\Delta^1$ is the barycentric point of Δ^1. Observe that by construction we have that $T_0\big(L(0)\big)$, $T_1\big(L(0)\big)$, $\mathrm{sd}K(1) \times \{1\}$ and $K(1) \times \{0\}$ are subcomplexes of $L(1)$ and, if we identify $\mathbb{R}^3$ with $\mathbb{R}^2 \times \mathbb{R}$, these subcomplexes triangulate respectively the polyhedra $\{e_{0,2}\} \times [0,1]$, $\{e_{1,2}\} \times [0,1]$, $\Delta^1 \times \{1\}$ and $\Delta^1 \times \{0\}$. Figure (2) below illustrates what $L(1)$ should look like.

Figure (2)



From now on we will denote the unit interval $[0,1]$ simply by I. Suppose now that for a positive integer m_0 we have defined simplicial complexes

$$L(0), \ldots, L(m_0)$$

which satisfy the following properties:

1. If $p \leq q \leq m_0$ and if $s : |K(p)| \hookrightarrow |K(q)|$ is any simplicial embedding which preserves the ordering of the vertices then the map $s \times \mathrm{Id}_I : \Delta^p \times I \hookrightarrow \Delta^q \times I$ becomes also a simplicial embedding if we triangulate $\Delta^p \times I$ and $\Delta^q \times I$ with $L(p)$ and $L(q)$ respectively.

2. For each $p \leq m_0$ the simplicial complex $L(p)$ contains $K(p) \times \{0\}$ and $\mathrm{sd}K(p) \times \{1\}$ as subcomplexes.

3. $L(p)$ contains a subcomplex $L(p)'$ which triangulates $\partial(\Delta^p \times I)$ and $L(p)$ is equal to the join $L(p)' * (b\Delta^p, \frac{1}{2})$, where $b\Delta^p$ denotes the barycentric point of Δ^p.

Let now $\delta_i : \Delta^{m_0} \hookrightarrow \Delta^{m_0+1}$ be the canonical inclusion of Δ^{m_0} into the ith face of Δ^{m_0+1} and let $L''(m_0+1)$ be the following union of simplicial complexes

$$L''(m_0+1) := \bigcup_{j=0}^{m_0+1} \delta_j \times \mathrm{Id}_I(L(m_0)).$$

This set of simplices covers $\partial\Delta^{m_0+1} \times I$ and by the first assumption given above it follows that it is actually a simplicial complex which triangulates $\partial\Delta^{m_0+1} \times I$. Furthermore, by the second assumption given above we have that $L''(m_0+1)$ contains both $\mathrm{sd}(\partial K(m_0+1)) \times \{1\}$ and $\partial K(m_0+1) \times \{0\}$ as subcomplexes and it follows that the union

$$L'(m_0+1) := L''(m_0+1) \cup \big(\mathrm{sd}K(m_0+1) \times \{1\}\big) \cup \big(K(m_0+1) \times \{0\}\big)$$

is a simplicial complex which triangulates $\partial(\Delta^{m_0+1} \times I)$. Finally, we define $L(m_0+1)$ to be the join $L'(m_0+1) * (b\Delta^{m_0+1}, \frac{1}{2})$ and by the way we constructed $L(m_0+1)$ we have that the sequence

$$L(0), \ldots, L(m_0), L(m_0+1)$$

satisfies the same three assumptions given above. This concludes the construction of the simplicial complexes $L(p)$.

Observe that the set of vertices of both $K(p)$ and $\mathrm{sd}K(p)$ have canonical partial orderings $\leq_{\Delta^p}$ and $\leq_{\mathrm{sd}\Delta^p}$. Indeed, for the vertices of K_p we have that $e_i \leq_{\Delta^p} e_j$ if and only if $i \leq j$ whereas for the vertices of $\mathrm{sd}K(p)$ we have that $sF_1 \leq_{\mathrm{sd}\Delta^p} sF_2$ if and only if F_1 is a proper face of F_2 (See definition 4.1). Both of these partial orderings yield Δ-sets $\Delta^p_\bullet$ and $s\Delta^p_\bullet$ for which there are canonical homeomorphisms $|\Delta^p_\bullet| \overset{\cong}{\to} \Delta^p$ and $|s\Delta^p_\bullet| \overset{\cong}{\to} \Delta^p$. Using the simplcial complexes

$L(p)$ we wish now to define for each p a Δ-set $L(p)_\bullet$ such that there is a canonical homeomorphism from the geometric realization $|L(p)_\bullet|$ to $\Delta^p \times I$. To this end, for each non-negative integer p and each simplex σ of $L(p)$ we are going to define a linear order $\leq_{p,\sigma}$ on the set of vertices $\mathrm{Vert}(\sigma)$ of σ so that whenever we have a pair of simplices σ_1 and σ_2 in $L(p)$ such that σ_2 is a face of σ_1 then the restriction of $\leq_{p,\sigma_1}$ on $\mathrm{Vert}(\sigma_2)$ is equal to $\leq_{p,\sigma_2}$. We will define all such orderings by induction on p. For $L(0)$ we declare the vertex $(1, \frac{1}{2})$ to be greater than the other two vertices of $L(0)$. Suppose now that for a non-negative integer m we have defined relations $\leq_0, \ldots, \leq_m$ on the sets $\mathrm{Vert}\big(L(0)\big), \ldots, \mathrm{Vert}\big(L(m)\big)$ such that for each non-negative integer $q \leq m$ the following holds:

1. For any simplex σ of $L(q)$ the restriction of $\leq_q$ on $\mathrm{Vert}(\sigma)$, which we will denote by $\leq_{q,\sigma}$, is a linear order.

2. If $i_{q,0} : \Delta^q \hookrightarrow \Delta^q \times [0,1]$ is the canonical inclusion of Δ^q into the bottom face of $\Delta^q \times [0,1]$ then $i_{q,0}(e_i) \leq_q i_{q,0}(e_j)$ if and only if $e_i \leq_{\Delta^q} e_j$.

3. Similarly, if $i_{q,1} : \Delta^q \hookrightarrow \Delta^q \times [0,1]$ is the inclusion into the top face then $i_{q,1}(bF_i) \leq_q i_{q,1}(bF_j)$ if and only if $bF_i \leq_{\mathrm{sd}\Delta^q} bF_j$.

4. If $q \leq q' \leq p$ and if $s : \Delta^q \hookrightarrow \Delta^{q'}$ is any simplicial embedding that preserves the ordering of the vertices then $s \times \mathrm{Id}_{[0,1]}(v_i) \leq_{q'} s \times \mathrm{Id}_{[0,1]}(v_j)$ if and only if $v_i \leq_q v_j$.

Let $\delta_i : \Delta^m \hookrightarrow \Delta^{m+1}$ be again the canonical inclusion from Δ^m into the i-th face of Δ^{m+1} and let $\tilde{\delta}_i$ denote the product $\delta_i \times \mathrm{Id}_I$. We define $\leq_{m+1}$ to be the relation on $\mathrm{Vert}\big(L(m+1)\big)$ such that $u \leq_{m+1} v$ if and only if one of the following holds:

- $v = (b\Delta^{m+1}, \frac{1}{2})$.
- $u, v \in \Delta^{m+1} \times \{0\}$ and $i_0^{-1}(u) \leq_{\Delta^{m+1}} i_0^{-1}(v)$.
- $u, v \in \Delta^{m+1} \times \{1\}$ and $i_1^{-1}(u) \leq_{\mathrm{sd}\Delta^{m+1}} i_1^{-1}(v)$.
- $u, v \in \tilde{\delta}_j(\Delta^m \times [0,1])$ for some $j \in \{0, \ldots, m+1\}$ and $\tilde{\delta}_j^{-1}(u) \leq_m \tilde{\delta}_j^{-1}(v)$.

It is straightforward to verify that the sequence of relations $\leq_0, \ldots, \leq_m, \leq_{m+1}$ satisfy conditions 1),2),3) and 4) given above for any non-negative integers $q', q \leq m+1$.

Let $L(p)_\bullet$ be the Δ-set induced by the relation $\leq_p$. For this Δ-set we have that $v_{i_0} \leq_p \ldots \leq_p v_{i_j}$ is a j-simplex of $L(p)_\bullet$ if and only if the vertices $v_{i_0}, \ldots, v_{i_j}$ span a j-simplex of the simplicial complex $L(p)$. Thus, the map

$$h'_p : \coprod_{j=0}^{p+1} L(p)_j \times \Delta^j \to |L(p)|$$

defined by

$$\big((v_{i_0},\ldots,v_{i_j}),\lambda_0+\ldots+\lambda_j\big)\mapsto\sum_{i=0}^{j}\lambda_i\cdot v_i$$

factors through $|L(p)_\bullet|$

$$\begin{array}{ccc}\coprod_{j=0}^{p+1}L(p)_j\times\Delta^j & & \\ \downarrow & \searrow^{h'_p} & \\ |L(p)_\bullet| & \xrightarrow{h_p} & |L(p)|\,.\end{array}$$

The vertical map in this diagram is just the obvious quotient map and it is straightforward to verify that h_p is bijective. Furthermore, since $|L(p)_\bullet|$ is compact and $|L(p)|$ is Haussdorf we have that h_p is a homeomorphism. In fact, h_p is linear on each simplex of $|L(p)_\bullet|$.

4.2 The subdivision map $\rho:\left|\Psi_d(\mathbb{R}^N)_\bullet\right|\longrightarrow\left|\Psi_d(\mathbb{R}^N)_\bullet\right|$

In this section, Δ is going to denote the category whose objects are all the standard simplices Δ^p and where the set of morphisms between any two simplices are just the simplicial maps which preserve the ordering on the set of vertices.

Instead of defining the subdivision map $\rho:\left|\Psi_d(\mathbb{R}^N)_\bullet\right|\longrightarrow\left|\Psi_d(\mathbb{R}^N)_\bullet\right|$ just for $\Psi_d(\mathbb{R}^N)_\bullet$ we are going to define it for any Δ-set $\mathcal{F}:\Delta^{\mathrm{op}}\to\mathbf{Sets}$ which comes from a functor defined on $\mathbf{PL}^{\mathrm{op}}$. We first introduce the following definition.

Definition 4.2. Let $\mathcal{F}:\mathbf{PL}^{\mathrm{op}}\to\mathbf{Sets}$ be a contravariant functor defined on $\mathbf{PL}$. The Δ-set $\mathcal{F}_\bullet$ obtained by restricting the functor $\mathcal{F}$ on the subcategory Δ is called *the underlying Δ-set of $\mathcal{F}$*.

Fix then a functor $\mathcal{F}:\mathbf{PL}^{\mathrm{op}}\to\mathbf{Sets}$ and let $\mathcal{F}_\bullet$ be its underlying Δ-set. In order to make our arguments easier to follow we will denote the functor $\mathcal{F}:\mathbf{PL}^{\mathrm{op}}\to\mathbf{Sets}$ by $\widetilde{\mathcal{F}}$.

$$\begin{array}{ccc}\mathbf{PL}^{\mathrm{op}} & & \\ \uparrow & \searrow^{\widetilde{\mathcal{F}}} & \\ \Delta^{\mathrm{op}} & \xrightarrow{\mathcal{F}} & \mathbf{Sets}\end{array}$$

We are going to break down the construction of the subdivision map ρ into several lemmas. Before stating the first of these we need to introduce a little bit of notation: for any q-simplex $F=F_{j_0}<\ldots<F_{j_q}$ of $\mathrm{sd}\Delta^p_\bullet$ we denote by ϕ_F the linear *pl* embedding $\Delta^q\hookrightarrow\Delta^p$ which maps e_i to the barycentric point bF_{j_i} of F_{j_i}.

Lemma 4.3. *Let $\mathcal{F}:\Delta^{\mathrm{op}}\to\mathbf{Sets}$ be the underlying Δ-set of the functor $\widetilde{\mathcal{F}}:\mathbf{PL}^{\mathrm{op}}\to\mathbf{Sets}$. Then, for any p-simplex β of $\mathcal{F}_\bullet$ the functions $\rho_0^\beta:\mathrm{sd}\Delta^p_0\to\mathcal{F}_0$,*

$\ldots$, $\rho_p^\beta : \mathrm{sd}\Delta_p^p \to \mathcal{F}_p$ *given by*

$$\rho_k^\beta(F = F_{j_0} < \ldots < F_{j_k}) = \widetilde{\mathcal{F}}(\phi_F : \Delta^k \hookrightarrow \Delta^p)(\beta)$$

are the components of a morphism of Δ*-sets* $\rho_\bullet^\beta : \mathrm{sd}\Delta_\bullet^p \to \mathcal{F}_\bullet$.

Proof. In the statement of this lemma $\widetilde{\mathcal{F}}(\phi_F : \Delta^k \hookrightarrow \Delta^p)(\beta)$ denotes the image of the p-simplex β of $\mathcal{F}_\bullet$ under the function of sets

$$\widetilde{\mathcal{F}}(\phi_F : \Delta^k \hookrightarrow \Delta^p) : \widetilde{\mathcal{F}}(\Delta^p) \to \widetilde{\mathcal{F}}(\Delta^k).$$

For any $k = 0, \ldots, p-1$ and any $i = 0, \ldots, k+1$ we have to show that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{sd}\Delta_{k+1}^p & \xrightarrow{\rho_{k+1}^\beta} & \mathcal{F}_{k+1} \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} \\ \mathrm{sd}\Delta_k^p & \xrightarrow{\rho_k^\beta} & \mathcal{F}_k. \end{array}$$

Let then $F = F_{j_0} < \ldots < F_{j_{i-1}} < F_{j_i} < F_{j_{i+1}} < \ldots < F_{j_{k+1}}$ be a $k+1$-simplex of $\mathrm{sd}\Delta_\bullet^p$. We have that the ith face of F is

$$\partial_i F = F_{j_0} < \ldots < F_{j_{i-1}} < F_{j_{i+1}} < \ldots < F_{j_{k+1}}$$

and that the diagram

$$\begin{array}{ccc} \Delta^k & \xrightarrow{\phi_{\partial_i F}} & \Delta^p \\ {\scriptstyle \delta_i}\downarrow & & \| \\ \Delta^{k+1} & \xrightarrow{\phi_F} & \Delta^p \end{array}$$

is commutative, where δ_i is the inclusion into the ith face of Δ^{k+1}. It follows then that

$$\widetilde{\mathcal{F}}(\phi_F \circ \delta_i)(\beta) = \widetilde{\mathcal{F}}(\phi_{\partial_i F})(\beta)$$
$$\widetilde{\mathcal{F}}(\delta_i) \circ \widetilde{\mathcal{F}}(\phi_F)(\beta) = \widetilde{\mathcal{F}}(\phi_{\partial_i F})(\beta)$$
$$\partial_i \widetilde{\mathcal{F}}(\phi_F)(\beta) = \widetilde{\mathcal{F}}(\phi_{\partial_i F})(\beta).$$

Since the left hand side of the last equality is equal to $\partial_i \rho_{k+1}^\beta(F)$ and the right hand side is equal to $\rho_k^\beta(\partial_i F)$ we conclude that $\rho_\bullet^\beta$ commutes with face maps and hence it is a morphism of Δ-sets. □

Let h_p be the inverse of the canonical homeomorphism $|\mathrm{sd}\Delta_\bullet^p| \xrightarrow{\cong} \Delta^p$. The subdivision map ρ will be obtained by applying the universal property of quotient spaces to a map $\widetilde{\rho} : \coprod_p \mathcal{F}_p \times \Delta^p \to |\mathcal{F}_\bullet|$ which is defined as follows: first take for each $p \geq 0$ the following composite g_p

$$g_p : \mathcal{F}_p \times \Delta^p \xrightarrow{I \times h_p} \mathcal{F}_p \times |\mathrm{sd}\Delta_\bullet^p| \longrightarrow |\mathcal{F}_\bullet|$$

where the second map sends a tuple (β, x) to $|\rho_\bullet^\beta|(x)$. The map $\widetilde{\rho} : \coprod_p \mathcal{F}_p \times \Delta^p \to |\mathcal{F}_\bullet|$ is then defined to be the coproduct of all the maps g_p

$$\widetilde{\rho} : \coprod_p \mathcal{F}_p \times \Delta^p \xrightarrow{\coprod g_p} |\mathcal{F}_\bullet|.$$

Proposition 4.4. *The map $\widetilde{\rho}$ factors through $|\mathcal{F}_\bullet|$, i.e. there is a unique map $\rho : |\mathcal{F}_\bullet| \to |\mathcal{F}_\bullet|$ making the following diagram commute*

$$\begin{array}{ccc} \coprod_p \mathcal{F}_p \times \Delta^p & & \\ \downarrow & \searrow^{\widetilde{\rho}} & \\ |\mathcal{F}_\bullet| & \xrightarrow{\rho} & |\mathcal{F}_\bullet| \end{array}$$

Proof. We just need to verify that any two points in $\coprod_p \mathcal{F}_p \times \Delta^p$ which are mapped to the same point in $|\mathcal{F}_\bullet|$ under the canonical quotient map $q : \coprod_p \mathcal{F}_p \times \Delta^p \to |\mathcal{F}_\bullet|$ have the same image under the map $\widetilde{\rho}$, i.e. for any map $\Delta^p \xhookrightarrow{s} \Delta^q$ in the category Δ we have to show that $\widetilde{\rho}(\sigma_1, \lambda_1) = \widetilde{\rho}(\sigma_2, \lambda_2)$ if $\mathcal{F}(s)(\sigma_2) = \sigma_1$ and $s(\lambda_1) = \lambda_2$. It is enough to consider the case when s is the inclusion δ_i of Δ^p into the i-th face of Δ^{p+1} for some $i = 0, \ldots, p+1$. Let us denote again by $K(p)$ the canonical simplicial complex in $\mathbb{R}^{p+1}$ which triangulates Δ^p. The inclusion $\delta_i : \Delta^p \hookrightarrow \Delta^{p+1}$ is also a simplicial map when we triangulate Δ^p and Δ^{p+1} with $\mathrm{sd}K(p)$ and $\mathrm{sd}K(p+1)$ and this simplicial map induces a morphism of Δ-sets $g_\bullet : \mathrm{sd}\Delta_\bullet^p \to \mathrm{sd}\Delta_\bullet^{p+1}$ which sends $F_0 < \ldots < F_k$ to $\delta_i(F_0) < \ldots < \delta_i(F_k)$. This morphism of Δ-sets makes the following diagram commute

$$\begin{array}{ccc} \mathrm{sd}\Delta_\bullet^p & & \\ \downarrow{\scriptstyle g_\bullet} & \searrow^{\rho_\bullet^{\sigma_1}} & \\ \mathrm{sd}\Delta_\bullet^{p+1} & \xrightarrow{\rho_\bullet^{\sigma_2}} & \mathcal{F}_\bullet \end{array}$$

and we obtain that the following diagram also commutes

$$\begin{array}{ccccc} \Delta^p & \xrightarrow[\cong]{h_p} & |s\Delta_\bullet^p| & & \\ \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle |g_\bullet|} & \searrow^{|\rho_\bullet^{\sigma_1}|} & \\ \Delta^{p+1} & \xrightarrow[\cong]{h_{p+1}} & |s\Delta_\bullet^{p+1}| & \xrightarrow{|\rho_\bullet^{\sigma_2}|} & |\mathcal{F}_\bullet| \end{array} .$$

It then follows that any two points which are identified in $|\mathcal{F}_\bullet|$ have the same image under $\widetilde{\rho}$. □

Definition 4.5. The map $\rho : |\mathcal{F}_\bullet| \longrightarrow |\mathcal{F}_\bullet|$ obtained in proposition 4.4 is called *the subdivision map of $\mathcal{F}_\bullet$*.

As we indicated at the beginning of this section the map ρ satisfies the following important property.

Theorem 4.6. *The subdivision map $\rho : |\mathcal{F}_\bullet| \to |\mathcal{F}_\bullet|$ is homotopic to the identity map on $|\mathcal{F}_\bullet|$.*

Proof. In this proof we are going to use the Δ-sets $L(p)_\bullet$ and the relations $\leq_p$ that we defined in § 3.1. For each p-simplex σ of $\mathcal{F}_\bullet$ denote by $\sigma \times I$ the image of σ in $\widetilde{\mathcal{F}}(\Delta^p \times I)$ under the function $\widetilde{\mathcal{F}}(pr_1 : \Delta^p \times I \to \Delta^p)$ where $pr_1 : \Delta^p \times I \to \Delta^p$ is just the projection onto the first component, and let $\mathcal{R}^\sigma_\bullet : L(p)_\bullet \to \mathcal{F}_\bullet$ be the morphism of Δ-sets which at the level of k-simplices is given by

$$\mathcal{R}^\sigma_k(V = v_0 \leq_p \ldots \leq_p v_k) = \widetilde{\mathcal{F}}(s_V : \Delta^k \to \Delta^p \times I)(\sigma \times I).$$

The map s_V that appears on the right side of this equality is just the simplicial map $|K(k)| \to |L(p)|$ which maps the vertex e_j to the vertex v_j of the simplex V. An argument completely analogous to the one given in the proof of 4.3 shows that $\mathcal{R}^\sigma_\bullet$ is indeed a morphism of Δ-sets. Let $i_{p,0} : |K(p)| \to |L(p)|$ and $i_{p,1} : |\mathrm{sd}K(p)| \to |L(p)|$ be the inclusions into the bottom and top face respectively of $\Delta^p \times [0,1]$. These two simplicial maps induce morphisms of Δ-sets $i_{p,0,\bullet} : \Delta^p_\bullet \to L(p)_\bullet$ and $i_{p,1,\bullet} : \mathrm{sd}\Delta^p_\bullet \to L(p)_\bullet$ which at the level of k-simplices are respectively given by

$$i_{p,0,k}(e_{i_0} \leq_{\Delta^p} \ldots \leq_{\Delta^p} e_{i_k}) = i_{p,0}(e_{i_0}) \leq_p \ldots \leq_p i_{p,0}(e_{i_k})$$

and

$$i_{p,1,k}(v_{i_0} \leq_{\mathrm{sd}\Delta^p} \ldots \leq_{\mathrm{sd}\Delta^p} v_{i_k}) = i_{p,1}(v_{i_0}) \leq_p \ldots \leq_p i_{p,1}(v_{i_k}),$$

and for any simplex σ of $\mathcal{F}_\bullet$ we have that

$$\rho^\sigma_\bullet = \mathcal{R}^\sigma_\bullet \circ i_{p,1,\bullet}$$

and that $\mathcal{R}^\sigma_\bullet \circ i_{p,0,\bullet}$ is the morphism $\phi_{\sigma,\bullet}$ which maps the unique p-simplex of $\Delta^p_\bullet$ to σ. For each non-negative integer p let f_p be the inverse of the homeomorphism $h_p : |L(p)_\bullet| \to |L(p)|$ given at the end of § 3.1 and for each p-simplex σ of $\mathcal{F}_\bullet$ let $\Gamma^\sigma : \Delta^p \times I \to |\mathcal{F}_\bullet|$ be equal to the following composite

$$\Gamma^\sigma := |\mathcal{R}^\sigma_\bullet| \circ f_p.$$

We want to show that the coproduct of maps

$$\coprod_p \coprod_{\sigma \in \mathcal{F}_p} \{\sigma\} \times \Delta^p \xrightarrow{\Gamma^\sigma} |\mathcal{F}_\bullet|$$

factors through $|\mathcal{F}_\bullet| \times I$. The desired homotopy between $\mathrm{Id}_{|\mathcal{F}_\bullet|}$ and ρ will be then the map $\Gamma : |\mathcal{F}_\bullet| \times I \to |\mathcal{F}_\bullet|$ that we obtain by the universal property of quotient spaces. In order to show that the coproduct of all the maps Γ^σ factors through $|\mathcal{F}_\bullet| \times [0,1]$ it is enough to show that the diagram

$$\begin{array}{ccc} \Delta^p \times I & \xrightarrow{s \times \mathrm{Id}_I} & \Delta^q \times I \\ & \searrow^{\Gamma^{\sigma_1}} & \downarrow{\scriptstyle \Gamma^{\sigma_2}} \\ & & |\mathcal{F}_\bullet| \end{array} \tag{11}$$

commutes for any injective simplicial map $s : \Delta^p \to \Delta^q$ which preserves the order of the vertices and for any simplices σ_1 and σ_2 of $\mathcal{F}_\bullet$ such that $\mathcal{F}(s)(\sigma_2) = \sigma_1$. Again, it is enough to consider the case when the map s is an inclusion δ_i from Δ^p into one of the faces of Δ^{p+1}. Fix then $p \in \mathbb{N}$ and fix an i in $\{0, \ldots, p+1\}$. The inclusion $g := \delta_i \times \mathrm{Id}_I : \Delta^p \times I \hookrightarrow \Delta^{p+1} \times I$ becomes a simplicial embedding when we triangulate $\Delta^p \times I$ and $\Delta^{p+1} \times I$ with the simplicial complexes $L(p)$ and $L(p+1)$ respectively. Furthermore, for any two vertices v_0 and v_1 of $L(p)$ such that $v_0 \leq_p v_1$ we have that $g(v_0) \leq_{p+1} g(v_1)$ and thus g induces a morphism of Δ-sets

$$g_\bullet : L(p)_\bullet \to L(p+1)_\bullet$$

which at the level of k-simplices is given by $g_k(V = v_0 \leq_p \ldots \leq_p v_k) = g(v_0) \leq_{p+1} \ldots \leq_{p+1} g(v_k)$. Observe that by the commutativity of the diagram

$$\begin{array}{ccc} \Delta^p \times I & \xrightarrow{g} & \Delta^{p+1} \times I \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \Delta^p & \xrightarrow{\delta_i} & \Delta^{p+1} \end{array}$$

we have that $\widetilde{\mathcal{F}}(g)(\sigma_2 \times I) = \sigma_1 \times I$ and for any k-simplex V of $L(p)$ which is spanned by vertices $v_0, \ldots, v_k$ such that $v_0 \leq_p \ldots \leq_p v_k$ we have that the diagram

$$\begin{array}{ccc} \Delta^k & = & \Delta^k \\ {\scriptstyle s_V}\downarrow & & \downarrow{\scriptstyle s_{g(V)}} \\ \Delta^p \times I & \xrightarrow{g} & \Delta^{p+1} \times I \end{array}$$

commutes, which implies that

$$\widetilde{\mathcal{F}}(s_V)(\sigma_1 \times I) = \widetilde{\mathcal{F}}(s_V)(\widetilde{\mathcal{F}}(g)(\sigma_2 \times I)) = \widetilde{\mathcal{F}}(g \circ s_V)(\sigma_2 \times I) = \widetilde{\mathcal{F}}(s_{g(V)})(\sigma_2 \times I).$$

By the way we defined the morphisms $\mathcal{R}^\sigma_\bullet$ we obtain from the equality $\widetilde{\mathcal{F}}(s_V)(\sigma_1 \times I) = \widetilde{\mathcal{F}}(s_{g(V)})(\sigma_2 \times I)$ that

$$\mathcal{R}^{\sigma_1}_k(v_0 \leq_p \ldots \leq_p v_k) = \mathcal{R}^{\sigma_2}_k(g(v_0) \leq_{p+1} \ldots \leq_{p+1} g(v_k)).$$

Since this holds for any simplex V of $L(p)_\bullet$ we conclude that $\mathcal{R}^{\sigma_1}_\bullet = \mathcal{R}^{\sigma_2}_\bullet \circ g_\bullet$ which implies that diagram (20) is commutative when $s = \delta_i$. It follows that all the maps Γ^σ can be glued together to produce a map

$$\Gamma : |\mathcal{F}_\bullet| \times I \to |\mathcal{F}_\bullet|\,. \tag{12}$$

Furthermore, since we had for each simplex σ of $\mathcal{F}_\bullet$ that $|\rho^\sigma_\bullet| = |\mathcal{R}^\sigma_\bullet \circ i_{1,\bullet}|$ and $|\phi_{\sigma,\bullet}| = |\mathcal{R}^\sigma_\bullet \circ i_{0,\bullet}|$ we have that $\Gamma_0 = \mathrm{Id}_{|\mathcal{F}_\bullet|}$ and $\Gamma_1 = \rho$ and we conclude that ρ is homotopic to $\mathrm{Id}_{|\mathcal{F}_\bullet|}$

□

Besides being homotopic to the identity map $\mathrm{Id}_{|\mathcal{F}_\bullet|}$ the subdivision map ρ also satisfies the following:

Proposition 4.7. *Let $\mathcal{F} : \Delta^{\mathrm{op}} \to \mathbf{Sets}$ be the underlying Δ-set of $\widetilde{\mathcal{F}} : \mathbf{PL}^{\mathrm{op}} \to \mathbf{Sets}$. If $f_\bullet : X_\bullet \to \mathcal{F}_\bullet$ is a morphism of Δ-sets and $h : |\mathrm{sd}X_\bullet| \to |X_\bullet|$ is the canonical homeomorphism between $|X_\bullet|$ and $|\mathrm{sd}X_\bullet|$ then there exists a unique morphism $g_\bullet : \mathrm{sd}X_\bullet \to \mathcal{F}_\bullet$ of Δ-sets which makes the following diagram commute*

$$\begin{array}{ccc} |\mathcal{F}_\bullet| & \xrightarrow{\rho} & |\mathcal{F}_\bullet| \\ {\scriptstyle |f_\bullet|}\uparrow & & \uparrow{\scriptstyle |g_\bullet|} \\ |X_\bullet| & \xleftarrow[h]{\cong} & |\mathrm{sd}X_\bullet| \end{array}$$

Proof. Consider first the case when $X_\bullet = \Delta^p_\bullet$ for some $p \geq 0$. Let then $f_\bullet : \Delta^p_\bullet \to \mathcal{F}_\bullet$ be a morphism of Δ-sets and let β be the image of $e_0 \leq_{\Delta^p} \ldots \leq_{\Delta^p} e_p$ under f_p. In this case we are going to show that the geometric realization of the morphism $\rho^\beta_\bullet : \mathrm{sd}\Delta^p_\bullet \to \mathcal{F}_\bullet$ defined in lemma 4.3 makes the following diagram commute

$$\begin{array}{ccc} |\mathcal{F}_\bullet| & \xrightarrow{\rho} & |\mathcal{F}_\bullet| \\ {\scriptstyle |f_\bullet|}\uparrow & & \uparrow{\scriptstyle |\rho^\beta_\bullet|} \\ |\Delta^p_\bullet| & \xleftarrow[h_{\Delta^p}]{\cong} & |\mathrm{sd}\Delta^p_\bullet| \,. \end{array}$$

The bottom map of this diagram is just the canonical homeomorphism from $|\mathrm{sd}\Delta^p_\bullet|$ to $|\Delta^p_\bullet|$, which can be expressed as the composition of the canonical homeomorphism $h_2 : |\mathrm{sd}\Delta^p_\bullet| \overset{\cong}{\to} \Delta^p$ and the inverse of the canonical homeomorphism $h_1 : |\Delta^p_\bullet| \overset{\cong}{\to} \Delta^p$. Instead of showing directly that the previous diagram commutes we will show that the outer rectangle of the following diagram commutes

$$\begin{array}{ccc} |\mathcal{F}_\bullet| & \xrightarrow{\rho} & |\mathcal{F}_\bullet| \\ {\scriptstyle |f_\bullet|}\uparrow & & \uparrow{\scriptstyle |\rho^\beta_\bullet|} \\ |\Delta^p_\bullet| & \xleftarrow[h_{\Delta^p}]{\cong} & |\mathrm{sd}\Delta^p_\bullet| \\ {\scriptstyle h_1^{-1}}\uparrow{\scriptstyle \cong} & & {\scriptstyle h_2^{-1}}\uparrow{\scriptstyle \cong} \\ \Delta^p & = & \Delta^p. \end{array} \tag{13}$$

The composition of the right vertical maps of this diagram is equal to the restriction of the map $\widetilde{\rho}$ defined before proposition 4.4 on $\{\beta\} \times \Delta^p$ and the composition of the left vertical maps is just the characteristic map of the simplex β and it then follows from proposition 4.4 that the outer rectangle of diagram (13) commutes. Since the lower square obviously commutes and since of all its maps are homeomorphisms we have that the top square also commutes.

Let now $f_\bullet : X_\bullet \to \mathcal{F}_\bullet$ be an arbitrary morphism of Δ-sets. For each p-simplex σ of $X_\bullet$ we have by the first part of this proof that

$$\rho \circ \left|\phi_{f(\sigma)_\bullet}\right| \circ h_{\Delta^p} = \left|\rho_\bullet^{f(\sigma)}\right| \tag{14}$$

where $\phi_{f(\sigma)_\bullet} : \Delta^p_\bullet \to \mathcal{F}_\bullet$ is the morphism which maps the unique p-simplex of $\Delta^p_\bullet$ to $f_p(\sigma)$. It is easy to verify that the diagram

$$\begin{array}{ccc} \mathrm{sd}\Delta^p_\bullet & \xrightarrow{\mathrm{sd}\theta_\bullet} & \mathrm{sd}\Delta^q_\bullet \\ {\scriptstyle \rho_\bullet^{f_p(\alpha)}}\downarrow & & \downarrow{\scriptstyle \rho_\bullet^{f_q(\beta)}} \\ \mathcal{F}_\bullet & = & \mathcal{F}_\bullet \end{array}$$

commutes (see note 4.8 below) for any morphism $\theta : \Delta^p \to \Delta^q$ in the category Δ and for any simplices β and α of $X_\bullet$ of dimension q and p respectively such that $\theta^*(\beta) = \alpha$, and since $\mathrm{sd}X_\bullet$ is equal to

$$\mathbf{colim}_{\Delta\downarrow X_\bullet} \mathrm{sd}\Delta^p_\bullet$$

we obtain a unique morphism $g_\bullet : \mathrm{sd}X_\bullet \to \mathcal{F}_\bullet$ such that for every simplex σ of $X_\bullet$ we have that

$$\rho_\bullet^{f(\sigma)} = g_\bullet \circ \mathrm{sd}\phi_{\sigma,\bullet}$$

where $\mathrm{sd}\phi_{\sigma,\bullet}$ is the map $\mathrm{sd}\Delta^{\dim\sigma}_\bullet \to \mathrm{sd}X_\bullet$ induced by the characteristic map $\phi_{\sigma,\bullet} : \Delta^{\dim\sigma}_\bullet \to X_\bullet$ of the simplex σ. Finally, since (14) is an equality which holds for any simplex σ of $X_\bullet$ it follows that the geometric realization $|g_\bullet|$ is equal to $\rho \circ |f_\bullet| \circ h$, where h is the canonical homeomorphism $|\mathrm{sd}X_\bullet| \overset{\cong}{\to} |X_\bullet|$. □

Note 4.8. In the previous proof we used the fact that for any morphism $\theta : \Delta^p \to \Delta^q$ in the category Δ and for any pair of simplices β and α of dimension q and p respectively such that $\theta^*(\beta) = \alpha$ we have that $\rho^\alpha_\bullet = \rho^\beta_\bullet \circ \mathrm{sd}\theta_\bullet$. To see this, let $F_0 < \ldots < F_m$ be an m-simplex of $\mathrm{sd}\Delta^p_\bullet$. By the definition of the morphisms $\rho^\beta_\bullet$ and $\rho^\alpha_\bullet$ we have that $\rho^\alpha_m(F_0 < \ldots < F_m) = \mathcal{F}(\Delta^m \overset{j_1}{\to} \Delta^p)(\alpha)$ and $\rho^\beta_m(\theta(F_0) < \ldots < \theta(F_m)) = \mathcal{F}(\Delta^m \overset{j_2}{\to} \Delta^q)(\beta)$, where $j_1 : \Delta^m \to \Delta^p$ is the linear map which maps e_i to the barycentric point of F_i and $j_2 : \Delta^m \to \Delta^q$ is the linear map which maps e_i to the barycentric point of $\theta(F_i)$. But since $j_2 = \theta \circ j_1$ and since $\theta^*(\beta) = \alpha$ we have that

$$\widetilde{\mathcal{F}}(\Delta^m \overset{j_1}{\to} \Delta^p)(\alpha) = \widetilde{\mathcal{F}}(\Delta^m \overset{j_1}{\to} \Delta^p) \circ \widetilde{\mathcal{F}}(\Delta^p \overset{\theta}{\to} \Delta^q)(\beta) = \widetilde{\mathcal{F}}(\Delta^m \overset{j_2}{\to} \Delta^q)(\beta)$$

and thus $\rho^\alpha_m(F_0 < \ldots < F_m) = \rho^\beta_m(\theta(F_0) < \ldots < \theta(F_m))$. Since $F_0 < \ldots < F_m$ was any simplex of $\mathrm{sd}\Delta^p_\bullet$ we conclude that $\rho^\alpha_\bullet = \rho^\beta_\bullet \circ \mathrm{sd}\theta_\bullet$.

From proposition 4.7 we obtain the following corollary.

Corollary 4.9. *For any morphism $f_\bullet : X_\bullet \to \mathcal{F}_\bullet$ and any $r > 0$ there is a unique morphism $g_\bullet : \mathrm{sd}^r X_\bullet \to \mathcal{F}_\bullet$ which makes the following diagram commute*

$$\begin{array}{ccc} |\mathcal{F}_\bullet| & \xrightarrow{\rho^r} & |\mathcal{F}_\bullet| \\ {\scriptstyle |f_\bullet|}\uparrow & & \uparrow{\scriptstyle |g_\bullet|} \\ |X_\bullet| & \xleftarrow[\cong]{} & |\mathrm{sd}^r X_\bullet| \end{array}$$

where the bottom map is the canonical homeomorphism $|\mathrm{sd}^r X_\bullet| \overset{\cong}{\to} |X_\bullet|$.

Corollary 4.9 can be used the prove the following useful result about maps from compact spaces into $|\mathcal{F}_\bullet|$.

Proposition 4.10. *Any map $f : P \to |\mathcal{F}_\bullet|$ from a compact space P is homotopic to a composition of the form*

$$P \to |K_\bullet| \overset{|g_\bullet|}{\to} |\mathcal{F}_\bullet|$$

where $K_\bullet$ is a finite Δ-set obtained from a finite ordered simplicial complex.

Proof. Since P is compact its image under f is going to intersect only finitely many simplices of $|\mathcal{F}_\bullet|$. If $Y_\bullet$ is the sub-Δ-set of $\mathcal{F}_\bullet$ generated by these simplices then the map f is equal to the composition $P \overset{f'}{\to} |Y_\bullet| \overset{|i_\bullet|}{\to} |\mathcal{F}_\bullet|$ where the first map is the map obtained from f by restricting the target to $|Y_\bullet|$ and the second map is just the geometric realization of the obvious inclusion of Δ-sets. By 4.9 we have that there exists a unique morphism $g_\bullet : \mathrm{sd}^2 Y_\bullet \to \mathcal{F}_\bullet$ of Δ-sets such that $\rho^2 \circ |i_\bullet| = |g_\bullet| \circ h$, where $h : |Y_\bullet| \overset{\cong}{\to} |\mathrm{sd}^2 Y_\bullet|$ is the inverse of the canonical homeomorphism from $|\mathrm{sd}^2 Y_\bullet|$ onto $|Y_\bullet|$ and since ρ is homotopic to the identity on $|\mathcal{F}_\bullet|$ we have that $|i_\bullet|$ is homotopic to $|g_\bullet| \circ h$ which implies that $f = |i_\bullet| \circ f'$ is homotopic to

$$P \overset{h \circ f'}{\to} |\mathrm{sd}^2 Y_\bullet| \overset{|g_\bullet|}{\to} |\mathcal{F}_\bullet|. \tag{15}$$

Furthermore, by lemma 2.68 we have that $\mathrm{sd}^2 Y_\bullet$ is isomorphic to a Δ-set obtained from a simplicial complex K and hence we can replace $\mathrm{sd}^2 Y_\bullet$ with $K_\bullet$ in (15). □

Note 4.11. Let $\rho : |\Psi_d(\mathbb{R}^N)_\bullet| \to |\Psi_d(\mathbb{R}^N)_\bullet|$ be the subdivision map of $\Psi_d(\mathbb{R}^N)_\bullet$. If $\rho_k : |\psi_d(N,k)_\bullet| \to |\psi_d(N,k)_\bullet|$ is the subdivision map of the sub-Δ-set $\psi_d(N,k)_\bullet$ then by the way we constructed the maps ρ and ρ_k we have that

$$\rho|_{|\psi_d(N,k)_\bullet|} = \rho_k.$$

Furthermore, if Γ is the homotopy between $\mathrm{Id}_{|\Psi_d(\mathbb{R}^N)_\bullet|}$ and ρ defined in the proof of theorem 4.6 then the restriction of Γ on $[0,1] \times |\psi_d(N,k)_\bullet|$ is a homotopy between $\mathrm{Id}_{|\psi_d(N,k)_\bullet|}$ and $\rho|_{|\psi_d(N,k)_\bullet|}$.

5 The piecewise linear cobordism category

In this section we are going to introduce the space $BPL\mathcal{C}_d(\mathbb{R}^N)$ which should be interpreted as the piecewise linear analogue of the classifying space of the smooth cobordism category defined in [7]. One of the central results of this thesis is that there is a weak equivalence

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq |\psi_d(N,1)_\bullet| \,. \tag{16}$$

if we assume that $N-d \geq 3$. This is done by showing that there is a sub-Δ-set $\psi_d^R(N,1)_\bullet$ of $\psi_d(N,1)_\bullet$ which fits into a chain of weak homotopy equivalences

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq \left|\psi_d^R(N,1)_\bullet\right| \overset{\simeq}{\hookrightarrow} |\psi_d(N,1)_\bullet| \,.$$

This section is organized as follows: we first introduce the notion of *fiberwise regular value* for the projection $x_1 : W \to \mathbb{R}$ from the underlying polyhedron of a simplex of $\psi_d(N,1)_\bullet$ onto the first coordinate of $\mathbb{R}^N$ and we introduce the Δ-set $\psi_d^R(N,1)_\bullet$. In the second subsection we discuss the notion of bi-Δ-sets, which is a notion that we need in order to define the space $BPL\mathcal{C}_d(\mathbb{R}^N)$, and in the third section we obtain the weak equivelence $BPL\mathcal{C}(\mathbb{R}^N)_d \simeq \left|\psi_d^R(N,1)_\bullet\right|$. In the next section we prove that the inclusion $\left|\psi_d^R(N,1)_\bullet\right| \hookrightarrow |\psi_d(N,1)_\bullet|$ is a weak homotopy equivalence when $N-d \geq 3$ and therefore conclude the proof of (16). We point out that we don't need to assume that $N-d \geq 3$ in order to show that $BPL\mathcal{C}_d(\mathbb{R}^N) \simeq \left|\psi_d^R(N,1)_\bullet\right|$.

5.1 The Δ-set $\psi_d^R(N,1)_\bullet$

For a p-simplex $W \subseteq \Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k}$ of $\psi_d(N,k)_\bullet$ let

$$x_k : W \to \mathbb{R}^k$$

be the projection from W onto the second factor of $\Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k}$. Observe that this map is piecewise linear since it is the restriction of the projection $\Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k} \to \mathbb{R}^k$, which is pl, on the subpolyhedron W. Furthermore, x_k is *proper* since $\Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k} \to \mathbb{R}^k$ is proper and W is a closed subspace of $\Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k}$.

Note 5.1. In the following definition $B(a,\delta)$ shall denote the closed ball centered at $a \in \mathbb{R}^k$ with radius $\delta > 0$ with respect to the norm $\|\cdot\|$ in $\mathbb{R}^k$ defined by

$$\|(x_1,\ldots,x_k)\| = \max\{|x_1|,\ldots,|x_k|\}.$$

Definition 5.2. Let $W \subseteq \Delta^p \times \mathbb{R}^k \times (-1,1)^{N-k}$ be a p-simplex of the Δ-set $\psi_d(N,k)_\bullet$ and let $\pi : W \to \Delta^p$ be the natural projection onto Δ^p. A value $a_0 \in \mathbb{R}^k$ is said to be a *fiberwise regular value* of the projection map $x_k : W \to \mathbb{R}^k$ if for every point w in the pre-image $x_k^{-1}(a_0)$ there is a $\delta > 0$, an open neighborhood V of $\lambda_0 = \pi(w)$ in Δ^p and a piecewise linear homeomorphism

$$h : (\pi, x_k)^{-1}\big((\lambda_0, a_0)\big) \times V \times B(a_0,\delta) \to (\pi, x_k)^{-1}\big(V \times B(a_0,\delta)\big)$$

such that $(\pi, x_k) \circ h$ is equal to the projection onto $V \times B(a_0,\delta)$.

Observe that if a_0 is a fiberwise regular value of $x_k : W \to \mathbb{R}^k$ then by proposition 2.48 we have that the pre-image $(\pi, x_k)^{-1}\big((\lambda_0, a_0)\big)$ is a $(d-k)$-dimensional piecewise linear manifold for any $\lambda_0 \in \Delta^p$.

In this section we shall only be concerned with the Δ-set $\psi_d(N,1)_\bullet$ and with fiberwise regular values of projections of the form $x_1 : W \to \mathbb{R}$ with W a simplex in the Δ-set $\psi_d(N,1)_\bullet$. As we said before, the Δ-set given in the following definition is going to serve as a bridge between $BPL\mathcal{C}_d$ and $|\psi_d(N,1)_\bullet|$.

Definition 5.3. $\psi_d^R(N,1)_\bullet$ is the sub-Δ-set of $\psi_d(N,1)_\bullet$ which consists of all the simplices W such that the projection $x_1 : W \to \mathbb{R}$ has a fiberwise regular value.

5.2 Bi-Δ-sets

As we indicated at the beginning of this section before we can define the space $BPL\mathcal{C}_d(\mathbb{R}^N)$ we need to introduce the notion of *bi-Δ-set.*

Definition 5.4. A *bi-Δ-set* $F_{\bullet,\bullet}$ is a functor $F : \Delta^{op} \times \Delta^{op} \to \mathbf{Sets}$.

Observe that from a bi-Δ-set $F_{\bullet,\bullet}$ we can produce two Δ-objects in the category of Δ-sets, namely, the functor $\Delta^{op} \to \mathbf{Fun}(\Delta^{op}, \mathbf{Sets})$ which sends $[m]$ to the Δ-set $F_{m,\bullet}$ and the functor which sends $[m]$ to $F_{\bullet,m}$. We can picture the sets $\{F_{m,n}\}_{m,n}$ as being arranged on a grid where the sets on the m-th column are the sets of simplices of the Δ-set $F_{m,\bullet}$ and the sets on the n-th row are the sets of simplices of $F_{\bullet,n}$. From now on the first parameter of a bi-Δ-set $F_{\bullet,\bullet}$ will be referred to as the *interior simplicial direction* of $F_{\bullet,\bullet}$ and the second parameter will be referred to as the *exterior simplicial direction.*

In view of all this a bi-Δ-set $F_{\bullet,\bullet}$ can equivalnetly be defined as a Δ-object in the category of Δ-sets, i.e, a functor $F : \Delta^{op} \to \mathbf{Fun}(\Delta^{op}, \mathbf{Sets})$. From this definition we recover definition 5.4 by setting $G([m],[n]) := F([m])_n$.

Recall that for any morphism $\alpha : [n] \to [m]$ in the category Δ the map $\alpha_* : \Delta^n \to \Delta^m$ is the simplicial map which sends the vertex e_j to the vertex $e_{\alpha(j)}$. We are going to use this notation in the following definition.

Definition 5.5. The geometric realization of a bi-Δ-set $F : \Delta^{op} \times \Delta^{op} \to \mathbf{Sets}$ is defined to be the quotient space

$$\|F_{\bullet,\bullet}\| := \coprod_{n,m} F_{n,m} \times \Delta^n \times \Delta^m / \sim$$

where $\sim$ is the equivalence relation generated by

$$\big(x, \alpha_*(\lambda_1), \beta_*(\lambda_2)\big) \sim \big((\alpha,\beta)^*(x), \lambda_1, \lambda_2\big)$$

for morphisms $\alpha : [p] \to [m]$ and $\beta : [q] \to [n]$ in Δ, $x \in X_{n,m}$ and $\lambda_1 \in \Delta^p$, $\lambda_2 \in \Delta^q$.

Let $F : \Delta^{op} \times \Delta^{op} \to \mathbf{Sets}$ be a bi-Δ-set and consider the functor $G : \Delta^{op} \to \mathbf{Fun}(\Delta^{op}, \mathbf{Sets})$ which sends $[m]$ to $F_{m,\bullet}$ and which sends a morphism $[m] \stackrel{s}{\to} [n]$ to the natural transformation $F_{n,\bullet} \Rightarrow F_{m,\bullet}$ whose q-th component is equal to $F(s, \mathrm{Id}_{[q]})$. We could have also defined the geometric realization of $F_{\bullet,\bullet}$ as the geometric realization of the Δ-space

$$\Delta^{op} \stackrel{G}{\to} \mathbf{Fun}(\Delta^{op}, \mathbf{Sets}) \stackrel{|-|}{\to} \mathbf{Spaces}$$

where $|-|$ is the usual geoemtric realization functor for Δ-sets. The next lemma shows that both geometric realizations are functorially homeomorphic.

Lemma 5.6. *Let $F : \Delta^{op} \times \Delta^{op} \to \mathbf{Sets}$ be a bi-Δ-set and let $G : \Delta^{op} \to \mathbf{Fun}(\Delta^{op}, \mathbf{Sets})$ be the functor which sends $[m]$ to $F_{m,\bullet}$ and which sends a morphism $s : [m] \to [n]$ to the Δ-set morphism $F_{n,\bullet} \Rightarrow F_{m,\bullet}$ whose q-th component is equal to $F(s, \mathrm{Id}_{[q]})$. For each m in $\mathbb{N}$ denote the geoemtric realization $|G([m])|$ by A_m. Then the map $H_F : \|F_{\bullet,\bullet}\| \longrightarrow \|A_\bullet\|$ given by*

$$\big[(x, \lambda, \beta)\big] \mapsto \big[\big([(x, \lambda)], \beta\big)\big]$$

is a natural homeomorphism.

Proof. Let $\tilde{H}_F : \coprod_{n,m} X_{n,m} \times \Delta^n \times \Delta^m \to \|A_\bullet\|$ be the composition

$$\coprod_{n,m} X_{n,m} \times \Delta^n \times \Delta^m \longrightarrow \coprod_m A_m \times \Delta^m \longrightarrow \|A_\bullet\|$$

where the first map is given by $(x, \lambda, \beta) \mapsto \big([x, \lambda], \beta\big)$ and the second map is just the usual quotient map onto $\|A_\bullet\|$. It is easy to verify that this map factors through $\|F_{\bullet,\bullet}\|$ and that the map $\|F_{\bullet,\bullet}\| \to \|A_{\bullet,\bullet}\|$ we obtain by the universal property of quotient spaces is equal H_F. This shows that H_F is continuous. To define an inverse for H_F we will define for each m in $\mathbb{N}$ a map $g_m : A_m \times \Delta^m \to \|F_{\bullet,\bullet}\|$ such that the coproduct of maps $\coprod_m g_m$ factors through $\|A_\bullet\|$. Both A_m and Δ^m have canonical CW-complex structures and since Δ^m is compact a CW-complex structure for $A_m \times \Delta^m$ is obtained by taking product of cells and characteristic maps of the CW-complex structures of A_m and Δ^m (see [10], page 524). If x is a p-simplex of $G([m])_\bullet$, i.e. an (m,p)-simplex of $F_{\bullet,\bullet}$, we define $G_{x,\Delta^m} : \Delta^p \times \Delta^m \to \|F_{\bullet,\bullet}\|$ to be the characteristic map corresponding to the (m,p)-simplex x. Furthermore, if x is a p-simplex of $G([m])_\bullet$ and σ is any face of Δ^m we define $G_{x,\sigma} : \Delta^p \times \sigma \to \|F_{\bullet,\bullet}\|$ to be the restriction of G_{x,Δ^m} on $\Delta^p \times \sigma$. It is straightforward to verify that the coproduct of maps

$$\coprod_{p \geq 0} \coprod_{x \in F_{m,p}} \coprod_{\sigma \leq \Delta^m} G_{x,\sigma}$$

factors through $A_m \times \Delta^m$ and that the map $g_m : A_m \times \Delta^m \to \|F_{\bullet,\bullet}\|$ obtained by the universal property of quotient spaces is given by $g_m([(x, \lambda)], \beta) = [(x, \lambda, \beta)]$. Furthermore, it is also an easy exercise to verify that the coproduct of maps

$$\coprod_m A_m \times \Delta^m \stackrel{\coprod g_m}{\longrightarrow} \|F_{\bullet,\bullet}\|$$

factors through $\|A_\bullet\|$ and that the map $g : \|A_\bullet\| \to \|F_{\bullet,\bullet}\|$ that we obtain is given by $[([(x,\lambda)],\beta)] \mapsto [(x,\lambda,\beta)]$ which is clearly a continuous inverse for H_F. $\square$

Remark 5.7. Observe that in the previous lemma we always took geometric realizations along the internal simplicial direction. A completely analogous version of this lemma holds when we take the external direction of $F_{\bullet,\bullet}$ instead of the internal one.

As we indicated earlier, by taking geometric realizations along the interior and external directions of a bi-Δ-set $F_{\bullet,\bullet}$ we obtain Δ-spaces. These are very convenient objects to work with for homotopy theoretic arguments as it is illustrated in the following lemma whose proof can be found in [15].

Lemma 5.8. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of Δ-spaces such that for each $n \in \mathbb{N}$ the map $f_n : X_n \to Y_n$ is a weak homotopy equivalence. Then the induced map $\|f_\bullet\| : \|X_\bullet\| \to \|Y_\bullet\|$ between geometric realizations is a weak homotopy equivalence.*

5.3 The space $BPL\mathcal{C}_d(\mathbb{R}^N)$

We now introduce the bi-Δ-set whose geometric realization should be interpreted as the piecewise linear analogue of the classifying space of the cobordism category given in [7]. In this subsection we are going to assume that the underlying polyhedron W of a p-simplex of $\psi_d(N,1)_\bullet$ is a sub-polyhedron of the product $\mathbb{R} \times \Delta^p \times (-1,1)^{N-1}$, i.e. we flip the first and second factor. Also, throughout this subsection we are going to denote by W_A the pre-image of a subset $A \subseteq \mathbb{R}$ under the projection $x_1 : W \to \mathbb{R}$.

Definition 5.9. Let $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}$ be the bi-Δ-set whose set of (p,q)-simplices is the subset of $\psi_d(N,1)_p \times \mathbb{R}^{q+1}$ which consists of tuples

$$(W \subseteq \mathbb{R} \times \Delta^p \times (-1,1)^{N-1}, a_0 < \ldots < a_q)$$

which satisfy the following conditions:

1. Each a_i is a fiberwise regular value of the projection $x_1 : W \to \mathbb{R}$.
2. There exists $\epsilon > 0$ such that

$$W \cap x_1^{-1}\big((a_i - \epsilon, a_i + \epsilon)\big) = (a_i - \epsilon, a_i + \epsilon) \times W_{a_i}$$

if $1 \leq i \leq q-1$, and such that

$$W \cap x_1^{-1}\big((-\infty, a_0 + \epsilon)\big) = (-\infty, a_0 + \epsilon) \times W_{a_0}$$

and

$$W \cap x_1^{-1}\big((a_q - \epsilon, \infty)\big) = (a_q - \epsilon, \infty) \times W_{a_q}.$$

Note that in the case when $q = 0$ the underlying polyhedron W of a $(p, 0)$-simplex (W, a_0) is equal to $\mathbb{R} \times W_{a_0}$.

The i-th-face map $\mathcal{C}_d(\mathbb{R}^N)_{k+1,q} \to \mathcal{C}_d(\mathbb{R}^N)_{k,q}$ is given by restricting the submersion over the i-th-face of Δ^{k+1}. In the other simplicial direction, if $i \neq 0, k+1$ then the i-th-face map $\mathcal{C}_d(\mathbb{R}^N)_{p,k+1} \to \mathcal{C}_d(\mathbb{R}^N)_{p,k}$ is just the map that forgets the term a_i in the second component. In the case when $i = 0$ in addition to forgetting the term a_0 we have to replace the underlying manifold W with the manifold

$$W_{(a_1,\infty)} \cup \big((-\infty, a_1 + \epsilon) \times W_{a_1}\big).$$

Similarly, in the case when $i = k+1$ we have to replace the underlying manifold W with the manifold

$$W_{(-\infty,a_k)} \cup \big((a_k - \epsilon, \infty) \times W_{a_k}\big).$$

We can finally introduce the other space that appears in the statement of the main theorem of this thesis.

Defintion 5.10. The space $BPL\mathcal{C}_d(\mathbb{R}^N)$ is defined to be the geometric realization of the bi-Δ-set $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}$.

We wish to show that $\big\|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}\big\|$ is weak homotopy equivalent to $\big|\psi^R_d(N,1)_\bullet\big|$. In order to do this we are going to introduce the following two poset models for the cobordism category.

Definition 5.11. Let $\mathcal{D}_d(\mathbb{R}^N)_{\bullet,\bullet}$ be the bi-Δ-set whose set of (p, q)-simplices is the subset of $\psi_d(N, 1)_p \times \mathbb{R}^{q+1}$ consisting of tuples

$$(W \subseteq \mathbb{R} \times \Delta^p \times (-1, 1)^{N-1}, a_0 < \ldots < a_q)$$

which satisfy condition 1) of defintion 5.9.

Definition 5.12. Let $\mathcal{D}^\perp_d(\mathbb{R}^N)_{\bullet,\bullet}$ be the bi-Δ-set whose set of (p, q)-simplices is the subset of $\psi_d(N, 1)_p \times \mathbb{R}^{q+1}$ consisting of tuples

$$(W \subseteq \mathbb{R} \times (-1, 1)^{N-1} \times \Delta^p, a_0 < \ldots < a_q)$$

which satisfy condition 1) of defintion 5.9 and for which there is an $\epsilon > 0$ such that

$$x_1^{-1}\big((a_i - \epsilon, a_i + \epsilon)\big) = \big((a_i - \epsilon, a_i + \epsilon)\big) \times W_{a_i}$$

for each i in $\{0, \ldots, q\}$.

The face maps in the p and q direction for both $\mathcal{D}_d(\mathbb{R}^N)_{\bullet,\bullet}$ and $\mathcal{D}^\perp_d(\mathbb{R}^N)_{\bullet,\bullet}$ are defined in the same way as for $\mathcal{C}(\mathbb{R}^N)_{\bullet,\bullet}$ with the exception that in the q-direction the face maps do not change the underlying manifold W.

We wish to show that there is a zig-zag

$$\big\|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}\big\| \xleftarrow{\simeq} \big\|\mathcal{D}^\perp_d(\mathbb{R}^N)_{\bullet,\bullet}\big\| \xrightarrow{\simeq} \big\|\mathcal{D}_d(\mathbb{R}^N)_{\bullet,\bullet}\big\| \xrightarrow{\simeq} \big|\psi^R_d(N,1)_\bullet\big|. \qquad (17)$$

of weak equivaleces. In order to do this we are going to need the following technical lemma which will allow us to produce homotopies $[0,1]\times\Delta^p \to \left|\Psi_d(\mathbb{R}^N)_\bullet\right|$ from concordances $\widetilde{W} \overset{\pi}{\to} [0,1]\times\Delta^p$. Before stating the lemma we need to introduce the following bit of terminology.

Note 5.13. Let P be a compact polyhedron of dimension p in $\mathbb{R}^m$ which is triangulated by a finite ordered simplicial complex $(K,\leq)$ and let $K_\bullet$ be the Δ-set obtained from $(K,\leq)$ as indicated in remark 2.63. For each q-simplex $V = v_0 \leq \ldots \leq v_q$ of $K_\bullet$ let $f_V : \Delta^q \to |K|$ be the simplicial map which for $0 \leq j \leq q$ maps the vertex e_j to v_j. The map

$$\coprod_{0\leq q\leq p} K_q \times \Delta^q \to |K| = P$$

which sends a tuple (V,λ) to $f_V(\lambda)$ factors through $|K_\bullet|$. Furthermore, it is easy to verify that the map $f : |K_\bullet| \to P$ we obtain by the universal property of quotient spaces is bijective and since $|K_\bullet|$ is compact and P is Hausdorff we have that f is a homeomorphism. This homeomorphism f shall be called the *canonical homeomorphism* from $|K_\bullet|$ to P.

Lemma 5.14. *For each $p \in \mathbb{N}$ there is an ordered simplicial complex $(K(p),\leq_p)$ in $\mathbb{R}^{p+2}$ which satisfies the following properties:*

i) *$K(p)$ triangulates the product $[0,1]\times\Delta^p$.*

ii) *The obvious inclusions $i_0, i_1 : \Delta^p \hookrightarrow |K(p)|$ into the bottom and top faces are embeddings of ordered simplicial complexes.*

iii) *If $\Delta^q \overset{\delta}{\hookrightarrow} \Delta^p$ is an embedding of ordered simplicial complexes then the product $\mathrm{Id}_{[0,1]} \times \delta : |K(q)| \to |K(p)|$ is also an embedding of ordered simplicial complexes.*

iv) *For any element W of $\Psi_d(\mathbb{R}^N)\big([0,1]\times\Delta^p\big)$, the map $F : [0,1]\times\Delta^p \to |\Psi_d(\mathbb{R}^N)_\bullet|$ obtained by pre-composing the geometric realization of the morphism $f_{W_\bullet} : K(p)_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ obtained from 3.5 with the inverse of the canonical homeomorphism $|K(p)_\bullet| \overset{\cong}{\to} [0,1]\times\Delta^p$ (see note 5.13) is a homotopy which at time $t = 0,1$ agrees with the characteristic map of the p-simplex i_t^*W obtained by pulling back W along the inclusion $i_t : \Delta^p \hookrightarrow [0,1]\times\Delta^p$ given by $\lambda \mapsto (t,\lambda)$.*

Proof. We are going to define the ordered simplicial complexes $(K(p),\leq_p)$ by induction on p. For $p = 0$ let $K(0)$ be the simplicial complex in $\mathbb{R}^2$ which consists of the points

$$(0,0),\ (\frac{1}{2},0),\ (1,0)$$

and the segments

$$[0,\frac{1}{2}]\times\{0\}, \quad [\frac{1}{2},1]\times\{0\}.$$

We define $\leq_0$ to be the following partial order relation:

$$(0,0) \leq_0 (\frac{1}{2},0), \quad (1,0) \leq_0 (\frac{1}{2},0).$$

Suppose that for every natural number $p \leq m$ we have already defined ordered simplicial complexes $(K(p), \leq_p)$ which satisfy properties *i*), *ii*) and *iii*) given in the statement of this lemma. In order to define $K(m+1)$ we are going to introduce the following conventions: the canonical simplicial complex structures of the bottom and top faces of $[0,1] \times \Delta^{m+1}$ shall be denoted simply by $\{0\} \times \Delta^{m+1}$ and $\{1\} \times \Delta^{m+1}$, and if $\delta_j : \Delta^m \to \Delta^{m+1}$ is the simplicial inclusion onto j-th face of Δ^{m+1} then we denote by $K^j(m)$ the image of $K(m)$ under the embedding $\mathrm{Id}_{[0,1]} \times \delta_j$. Let $K'(m+1)$ be the following collection of simplices:

$$\Big(\bigcup_{j=0}^{m+1} K^j(m) \Big) \cup \big(\{0\} \times \Delta^{m+1} \big) \cup \big(\{1\} \times \Delta^{m+1} \big).$$

Since all the $K(p)$ for $p \leq m$ satisfy properties *i*), *ii*) and *iii*) we have that $K'(m+1)$ is a simplicial complex which triangulates $\partial\big([0,1] \times \Delta^{m+1}\big)$ and we define $K(m+1)$ to be the join $K'(m+1) * \big(\frac{1}{2}, b(m+1)\big)$, where $b(m+1)$ denotes the barycentric point of Δ^{m+1}.

Let us now define the relation $\leq_{m+1}$ on the set of vertices of $K(m)$. On the set of vertices of $K'(m+1)$ we define a relation $\leq'_{m+1}$ by the following rule: $v_1 \leq'_{m+1} v_2$ if and only if there exists $j \in \{0 \ldots, m\}$ and vertices $w_1, w_2 \in K(m)$ such that $w_1 \leq_m w_2$ and $(\mathrm{Id}_{[0,1]} \times \delta_j)(w_i) = v_i$ for $i = 1, 2$. We leave it to the reader to verify that $\leq'_{m+1}$ is indeed a relation on the set of vertices of $K'(m+1)$ such that for each simplex σ of $K'(m+1)$ the restriction of $\leq'_{m+1}$ on $\mathrm{Vert}(\sigma)$ is a linear order. Finally, we define $\leq_{m+1}$ on the set of vertices of $K(m+1)$ in the following way:

$$v_1 \leq_{m+1} v_2 \text{ iff } \begin{cases} v_1, v_2 \in K'(m+1) \text{ and } v_1 \leq'_{m+1} v_2, \text{or} \\ v_1 \neq v_2 \text{ and } v_2 = (\frac{1}{2}, b(m+1)). \end{cases}$$

For each $p \in \mathbb{N}$ let $K(p)_\bullet$ be the Δ-set induced by $\leq_p$ (see the comment following definition 2.61). If W is any element in $\Psi_d(\mathbb{R}^N)\big([0,1] \times \Delta^p\big)$ and if $f_\bullet : K(p)_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ is the morphism which classifies the element W (see definition 3.6) then it is straightforward to verify that the composition

$$[0,1] \times \Delta^p \overset{\cong}{\to} |K(p)_\bullet| \xrightarrow{|f_\bullet|} |\Psi_d(\mathbb{R}^N)_\bullet|,$$

where the first map is the inverse of the canonical homeomorphism $|K(p)_\bullet| \overset{\cong}{\to} [0,1] \times \Delta^p$, is a homotopy which agrees with the characteristic maps of the p-simplices $i_0^* W$ and $i_1^* W$ at times $t = 0$ and $t = 1$ respectively. □

Although it is not going to be used in this section, the construction given in the previous lemma allows us also to obtain the following useful result.

Corollary 5.15. *Let L be a finite ordered simplicial complex in some Euclidean space $\mathbb{R}^m$. There is an ordered simplicial complex $\widetilde{L}$ in $[0,1]\times\mathbb{R}^m$ which satisfies the following properties*

i) *$\widetilde{L}$ triangulates the product $[0,1]\times|L|$.*

ii) *The inclusions $i_0, i_1 : |L| \hookrightarrow |\widetilde{L}|$ into the bottom and top face maps are embeddings of ordered simplicial complexes.*

iii) *If σ is a p-simplex of L and if $\Delta^p \xrightarrow{j_\sigma} \sigma$ is the simplicial isomorphism which preserves the order on the set of vertices then $\mathrm{Id}_{[0,1]}\times\delta_j : |K(p)| \to |\widetilde{L}|$ is an embedding of ordered simplicial complexes, where $K(p)$ is the ordered simplicial complex which triangulates $[0,1]\times\Delta^p$ defined in 5.14.*

iv) *For any element W of $\Psi_d(\mathbb{R}^N)\big([0,1]\times|L|\big)$ the map $F : [0,1]\times|L| \to |\Psi_d(\mathbb{R}^N)_\bullet|$ obtained by pre-composing the geometric realization of the morphism $f_{W\bullet} : \widetilde{L}_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ obtained from 3.5 with the inverse of the canonical homeomorphism $|\widetilde{L}_\bullet| \xrightarrow{\cong} [0,1]\times|L|$ (see note 5.13) is a homotopy which at time $t = 0, 1$ agrees with the geometric realization of the morphism $L_\bullet \to \Psi_d(\mathbb{R}^N)_\bullet$ which classifies the pull back i_t^*W of W along the inclusion $i_t : |L| \hookrightarrow [0,1]\times|L|$ given by $x \mapsto (t,x)$.*

Proof. For each p-simplex σ of L we triangulate the product $[0,1]\times\sigma$ using the image of $K(p)$ under the linear map $\mathrm{Id}_{[0,1]}\times j_\sigma$. Let us denote this simplicial complex by $\widetilde{L}_\sigma$. Let $\widetilde{L}$ denote the union of all the simplicial complexes $\widetilde{L}_\sigma$. We leave it to the reader to verify that $\widetilde{L}$ is indeed a simplicial complex which triangulates $[0,1]\times|L|$. We can also define a relation $\leq$ on $\mathrm{Vert}(\widetilde{L})$ which restricts to a linear order on each simplex of $\widetilde{L}$ by the follwoing rule: $v_1 \leq v_2$ iff there is a simplex σ, say of dimension p, in L such that $v_1, v_2 \in \widetilde{L}_\sigma$ and such that $v_1 \leq_p v_2$ once we identify $\widetilde{L}_\sigma$ with $K(p)$. We also leave it to the reader to check that $(\widetilde{L}, \leq)$ satisfies all the desired properties. □

We now define in the next proposition the first map in the zig-zag (17) and show that it is indeed a weak equivalence. We shall make use of the following notation: if W is a p-simplex of $\psi_d(N,1)_\bullet$ and if $(x_1,\pi) : W \to \mathbb{R}\times\Delta^p$ is the natural projection onto $\mathbb{R}\times\Delta^p$ then the pre-image of a product $A\times S \subseteq \mathbb{R}\times\Delta^p$ under (x_1,π) shall be denoted by $W_{A,S}$.

Proposition 5.16. *The morphism of bi-Δ-sets $\mathcal{D}_d^\perp(\mathbb{R}^N)_{\bullet,\bullet} \xrightarrow{f_{\bullet,\bullet}} \mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}$ defined by*

$$(W, a_0 < \ldots < a_q) \mapsto (W_{[a_0,a_q]} \cup W_{(-\infty,a_0]} \cup W_{[a_q,\infty)}, a_0 < \ldots < a_q),$$

induces a weak homotopy equivalence

$$\big\|\mathcal{D}_d^\perp(\mathbb{R}^N)_{\bullet,\bullet}\big\| \xrightarrow{\simeq} \big\|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,\bullet}\big\|$$

Proof. We wish to show that for each $q \geq 0$ the morphism

$$\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q} \xrightarrow{f_{\bullet,q}} \mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}$$

is a weak homotpy equivalence. If $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q} \xrightarrow{i_{q\bullet}} \mathcal{D}^{\perp}d(\mathbb{R}^N)_{\bullet,q}$ is the obvious inclusion of Δ-sets then we have that $f_{\bullet,q} \circ i_{q\bullet}$ is equal to $\mathrm{Id}_{\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}}$. In particular, the composition $f_{\bullet,q} \circ i_{q\bullet}$ is a weak homotopy equivalence. Thus, in order to show that $f_{\bullet,q}$ is a weak homotopy equivalence it suffices to show that $i_{q\bullet}$ is a weak homotopy equivalence. Consider then a map of pairs

$$g : (\Delta^k, \partial\Delta^k) \longrightarrow \big(\big|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q}\big| , \big|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}\big| \big).$$

We need to show that g is homotopic, as a map of pairs, to a map

$$g' : (\Delta^k, \partial\Delta^k) \longrightarrow \big(\big|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q}\big| , \big|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}\big| \big)$$

such that $g'(\Delta^k) \subseteq \big|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}\big|$. Let us identify the pair $(\Delta^k, \partial\Delta^k)$ with the realization of the pair of Δ-sets $(\Delta^k_\bullet, \partial\Delta^k_\bullet)$. Since both $\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q}$ and $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}$ are Kan Δ-sets we can assume that the map g is the geometric realization of a morphism of Δ-sets $h_\bullet$ which makes the following diagram commute

$$\begin{array}{ccc} |\partial\Delta^k_\bullet| & \xrightarrow{|h'_\bullet|} & \big|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}\big| \\ \big\downarrow & & \big\downarrow{\scriptstyle |i_{1\bullet}|} \\ |\Delta^k_\bullet| & \xrightarrow{|h_\bullet|} & \big|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q}\big| , \end{array}$$

where $h'_\bullet$ is the restriction of $h_\bullet$ on $\partial\Delta^k_\bullet$. The morphism $h_\bullet : \Delta^k_\bullet \to \mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q}$ classifies a k-simplex $(W \subseteq \mathbb{R} \times \Delta^k \times (-1,1)^{N-1}, a_0 < \ldots < a_q)$ of $\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q}$ for which there is an $\epsilon > 0$ such that

$$W_{(a_q-\epsilon,\infty),\partial\Delta^k} = (a_q - \epsilon, \infty) \times W_{a_q,\partial\Delta^k}$$

and

$$W_{(-\infty,a_0+\epsilon),\partial\Delta^k} = (-\infty, a_0 + \epsilon) \times W_{a_0,\partial\Delta^k}.$$

The strategy of the proof is to define an element $\widetilde{W}$ of $\psi_d(N,1)\big([0,1]\times\Delta^k\big)$ which is a concordance between W and an element W' which is the underlying polyhedron of a k-simplex of $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}$, and for which the values $a_0 < \ldots < a_q$ are fiberwise regular for the projection $x_1 : \widetilde{W} \to \mathbb{R}$. Once we have this concordance we can apply lemma 5.14 to obtain a homotopy $F : [0,1] \times |\Delta^k_\bullet| \to |\mathcal{D}_d^{\perp}(\mathbb{R}^{\mathbb{N}}{}_\bullet)|$ which at time $t = 0$ agrees with $|h_\bullet|$ and which at time $t = 1$ maps $|\Delta^k_\bullet|$ to $|\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}|$. Furthermore, the concordance $\widetilde{W}$ is going to be constant over $\partial\Delta^k$, i.e. $\widetilde{W}$ agrees with $[0,1] \times W$ over the product $[0,1] \times \partial\Delta^k$. It follows then that the homotopy F we obtain by applying lema 5.14 is actually a homotopy of maps of pairs $(|\Delta^k_\bullet|, |\partial\Delta^k_\bullet|) \to (|\mathcal{D}_d^{\perp}(\mathbb{R}^{\mathbb{N}}{}_\bullet)|, |\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}|)$.

In order to construct such a concordance we first pick a value ϵ' such that $0 < \epsilon' < \epsilon$ and an open pl embedding

$$f : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$$

which commutes with the projection onto $[0,1]$ and which satisfies the following properties:

- f_0 is the identity on $\mathbb{R}$.
- f fixes all points in $[0,1] \times [a_0 - \epsilon', a_q + \epsilon']$.
- f_1 maps $\mathbb{R}$ onto $(a_0 - \epsilon, a_q + \epsilon)$.

The product of maps $f \times \mathrm{Id}_{\Delta^k \times \mathbb{R}^{N-1}}$, which we shall denote by e, is an open piecewise linear embedding from $[0,1] \times \mathbb{R} \times \Delta^k \times \mathbb{R}^{N-1}$ from itself which commutes with the projection onto $[0,1] \times \Delta^k$ and thus, by lemma 3.19, the pre-image $\widetilde{W} := e^{-1}([0,1] \times W)$ is going to be a new element of $\psi_d(N,1)\big([0,1] \times \Delta^k\big)$. Since W agrees, over $\partial\Delta^k$, with the products $(-\infty, a_0) \times W_{a_0}$ and $(a_q, \infty) \times W_{a_q}$ at heights $x_1 < a_0$ and $x_1 > a_q$ respectively and since f fixes all points in $[0,1] \times [a_0 - \epsilon', a_q + \epsilon']$ we have that $\widetilde{W}$ agrees with $[0,1] \times W$ over $[0,1] \times \partial\Delta^k$. It also follows from the fact that f fixes all points in $[0,1] \times [a_0 - \epsilon', a_q + \epsilon']$ that $a_0 < \ldots < a_q$ are fiberwise regular values for the projection $x_1 : \widetilde{W} \to \mathbb{R}$. Finally, since f_1 maps $\mathbb{R}$ onto $(a_0 - \epsilon, a_q + \epsilon)$ we have that $\widetilde{W}$ is a concordance between W and an element W' which is the underlying polyhedron of a k-simplex of $\mathcal{C}_d(\mathbb{R}^N)_{\bullet,q}$. Applying 5.14 to $\widetilde{W}$ we obtain the desired relative homotopy F and we conclude that $i_{\bullet,q}$, and therefore $f_{\bullet,q}$, is a weak homotopy equivalence.

□

Proposition 5.17. *The inclusion map $i_{\bullet,\bullet} : \mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,\bullet} \to \mathcal{D}_d(\mathbb{R}^N)_{\bullet,\bullet}$ is a weak homotopy equivalence.*

Proof. As we did in the proof of proposition 5.16 we are going to show that $i_{\bullet,q} : \mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,q} \to \mathcal{D}_d(\mathbb{R}^N)_{\bullet,q}$ is a weak homotopy equivalence for all $q \in \mathbb{N}$. In order to make our arguments easier to follow we shall only do the proof in the case when $q = 1$. The remaining cases are handled similarly. Let then

$$g : (\Delta^p, \partial\Delta^p) \to \big(\left|\mathcal{D}_d(\mathbb{R}^N)_{\bullet,1}\right|, \left|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}\right|\big)$$

be a map of pairs. As we did in the proof of 5.16, we are going to show that g is homotopic, as a map of pairs, to a map $g' : (\Delta^p, \partial\Delta^p) \to \big(\left|\mathcal{D}_d(\mathbb{R}^N)_{\bullet,1}\right|, \left|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}\right|\big)$ such that $g'(\Delta^p) \subseteq \left|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}\right|$. Again, let us identify the pair $(\Delta^p, \partial\Delta^p)$ with the realization of the pair $(\Delta^p_\bullet, \partial\Delta^p_\bullet)$. Since both $\mathcal{D}_d(\mathbb{R}^N)_{\bullet,1}$ and $\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}$ are Kan Δ-sets we can assume that the map g is the geometric realization of a morphism of Δ-sets $h_\bullet$ which makes the following diagram commute

$$\begin{array}{ccc} |\partial\Delta^p_\bullet| & \xrightarrow{|h'_\bullet|} & \left|\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}\right| \\ \downarrow & & \downarrow{\scriptstyle |i_{1\bullet}|} \\ |\Delta^p_\bullet| & \xrightarrow{|h_\bullet|} & \left|\mathcal{D}_d(\mathbb{R}^N)_{\bullet,1}\right|, \end{array}$$

where $h'_\bullet$ is the restricition of $h_\bullet$ on $\partial\Delta^p_\bullet$. Let $(W, a_0 < a_1)$ be the p-simplex classified by $h_\bullet$. Since both a_0 and a_1 are fiberwise regular values of the projection $x_1 : W \to \mathbb{R}$ we have that there is an $\epsilon > 0$ smaller than $\frac{a_1 - a_0}{2}$ such that for $i = 0, 1$ the restriction of (x_1, π) on

$$W_{(a_i-\epsilon, a_i+\epsilon)}$$

is a piecewise linear submersion of codimension $d-1$. Also, we can assume that for this value ϵ it holds that

$$W_{(a_i-\epsilon,a_i+\epsilon),\partial\Delta^p} = (a_i - \epsilon, a_i + \epsilon) \times W_{a_i,\partial\Delta^p}$$

for $i = 0, 1$. Let $\delta > 0$ be a value less than $\frac{\epsilon}{2}$ and let $j : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a piecewise linear homotopy which satisfies the following properties:

- j_0 is the identity on $\mathbb{R}$.
- j fixes each point in $[0, 1] \times \big(\mathbb{R} - (a_0 - \frac{\epsilon}{2}, a_1 + \frac{\epsilon}{2})\big)$.
- j fixes each point in $[0, 1] \times [a_0 + \frac{\epsilon}{2}, a_1 - \frac{\epsilon}{2}]$
- For $i = 0, 1$ j maps the 2-simplex spanned by $(1, a_i - \delta), (1, a_i + \delta)$ and $(0, a_i)$ to a_i.

Let J be equal to the product $j \times \mathrm{Id}_{\Delta^p \times (-1,1)^{N-1}}$ and let $\widetilde{W}$ denote the preimage $J^{-1}([0, 1] \times W)$. Observe that $\widetilde{W}$ is indeed a closed sub-polyhedron of $[0, 1] \times \mathbb{R} \times \Delta^p \times (1, -1)^{N-1}$ since J is a pl map which fixes every point of $[0, 1] \times \mathbb{R} \times \Delta^p \times (1, -1)^{N-1}$ outside a compact subspace. In order to show that the natural projection $\widetilde{\pi} : \widetilde{W} \to [0, 1] \times \Delta^p$ is a piecewise linear submersion of codimension d we observe first that the restriction of $\widetilde{\pi}$ on

$$\widetilde{W}_{(-\infty,a_0-\frac{\epsilon}{2})} \cup \widetilde{W}_{(a_0+\frac{\epsilon}{2},a_1-\frac{\epsilon}{2})} \cup \widetilde{W}_{(a_1+\frac{\epsilon}{2},\infty)} \tag{18}$$

is a submersion of codimansion d since on this open set it agrees with the projection $[0, 1] \times W \to [0, 1] \times \Delta^p$. On the other hand, for $i = 0, 1$ the restriction of $\widetilde{\pi}$ on

$$\widetilde{W}_{(a_i-\epsilon,a_i+\epsilon)} \tag{19}$$

is equal to the composition

$$\widetilde{W}_{(a_i-\epsilon,a_i+\epsilon)} \xrightarrow{x_1 \times \widetilde{\pi}} (a_i - \epsilon, a_i + \epsilon) \times [0, 1] \times \Delta^p \to [0, 1] \times \Delta^p$$

where the second map is just the projection onto the last two coordinates, which is obviously a submersion of codimension 1. It follows from proposition 2.52 that the first map in this composition is a submersion of codimension $d-1$ since the map

$$(x_1, \pi) : W_{(a_i-\epsilon,a_i+\epsilon)} \to \mathbb{R} \times \Delta^p$$

is a submersion of codimension $d - 1$ and

$$\widetilde{W}_{(a_i-\epsilon,a_i+\epsilon)} = f^* W_{(a_i-\epsilon,a_i+\epsilon)}$$

where $f : [0,1] \times (a_i - \epsilon, a_i + \epsilon) \times \Delta^p \to (a_i - \epsilon, a_i + \epsilon) \times \Delta^p$ is the pl map defined by $(s,t,\lambda) \mapsto (j(s,t),\lambda)$. It follows then that the restriction of $\widetilde{\pi}$ on $\widetilde{W}_{(a_i \pm \epsilon)}$ for $i = 0,1$ is a submersion of codimension d. Since the open sets in (18) and (19) cover all of W we conclude that $\widetilde{\pi} : \widetilde{W} \to [0,1] \times \Delta^p$ is a submersion of codimension d. By construction, both a_0 and a_1 are fiberwise regular values for the projection $x_1 : \widetilde{W} \to \mathbb{R}$ and $\widetilde{W}$ is a concordance between W and an element W' of $\psi_d(N,1)\big(\Delta^p\big)$ which is the underlying polyhedron of a p-simplex of $\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}$. Furthermore, since for $i = 0,1$ we have that

$$W_{(a_i-\epsilon,a_i+\epsilon),\partial\Delta^p} = (a_i - \epsilon, a_i + \epsilon) \times W_{a_i,\partial\Delta^p}$$

then for each $t \in [0,1]$ we will have by construction that

$$(i_t^* \widetilde{W})_{(a_i-\epsilon,a_i+\epsilon),\partial\Delta^p} = (a_i - \epsilon, a_i + \epsilon) \times W_{a_i,\partial\Delta^p}$$

where $i_t : \Delta^p \hookrightarrow [0,1] \times \Delta^p$ is the inclusion defined by $\lambda \mapsto (t,\lambda)$. It follows that the homotopy $F : [0,1] \times |\Delta^p_\bullet| \to |\mathcal{D}_d(\mathbb{R}^N)_{\bullet,1}|$ we obtain by applying lemma 5.14 to $\widetilde{W}$ is a homotopy of maps of pairs $(|\Delta^p_\bullet|, |\partial\Delta^p_\bullet|) \to \big(|\mathcal{D}_d(\mathbb{R}^N)_{\bullet,1}|, |\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}|\big)$ such that $F_1(|\Delta^p_\bullet|) \subseteq |\mathcal{D}_d^{\perp}(\mathbb{R}^N)_{\bullet,1}|$ and we conclude that $i_{\bullet,1}$ is a weak homotopy equivalence. A completely analogous argument shows that $i_{\bullet,q}$ is a weak homotopy equivalence for all $q \geq 0$ and by lemma 5.8 we have that $i_{\bullet,\bullet}$ induces a weak homotopy equivalence between geometric realizations.

□

Finally, we compare the spaces $\big\|D(\mathbb{R}^N)_{\bullet,\bullet}\big\|$ and $\big|\psi_d^R(N,1)_\bullet\big|$.

Proposition 5.18. *There is a weak equivalence* $\big\|\mathcal{D}_d(\mathbb{R}^N)_{\bullet,\bullet}\big\| \stackrel{\simeq}{\to} \big|\psi_d^R(N,1)_\bullet\big|$.

Proof. For each p-simplex W of $\psi_d^R(N,1)_\bullet$ let $(\mathbb{R}_W, \leq)$ be the sub-poset of $(\mathbb{R}, \leq)$ consisting of those values $a \in \mathbb{R}$ which are fiberwise regular values for the projection $x_1 : W \to \mathbb{R}$. Observe that if we apply the geometric realization functor along the exterior simplicial direction of $\mathcal{D}_d(\mathbb{R}^N)_{\bullet,\bullet}$ we obtain a Δ-space $G_\bullet$ whose set of p-simplices is exactly equal to

$$G_p = \coprod_{W \in \psi_d^R(N,1)_p} \{W\} \times B(\mathbb{R}_W, \leq).$$

But since each $B(\mathbb{R}_W, \leq)$ is contractible we have that the map of Δ-spaces $g_\bullet : G_\bullet \to \psi_d^R(N,1)_\bullet$ defined by $(W,\lambda) \mapsto W$ induces by lemma 5.8 a weak homotopy equivalence

$$|g_\bullet| : \big\|\mathcal{D}_d(\mathbb{R}^N)_\bullet\big\| \stackrel{\simeq}{\to} \big|\psi_d^R(N,1)_\bullet\big|.$$

□

Combining propositions 5.16, 5.17 and 5.18 we obtain that

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq |\psi_d^R(N,1)_\bullet|.$$

In the following section we prove that the inclusion $\psi_d^R(N,1)_\bullet \hookrightarrow \psi_d(N,1)_\bullet$ is a weak equivalence when $N-d \geq 3$ and therefore finish the proof of the following claim.

Theorem 5.19. *If $N-d \geq 3$ then there is a weak homotopy equivalence*

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq |\psi_d(N,1)_\bullet|.$$

6 The inclusion $\psi_d^R(N,1)_\bullet \hookrightarrow \psi_d(N,1)_\bullet$

6.1 Preliminary lemmas and outline of the proof

The aim of this section is to prove the following result.

Theorem 6.1. *If $N-d \geq 3$ then the inclusion $\psi_d^R(N,1)_\bullet \hookrightarrow \psi_d(N,1)_\bullet$ is a weak homotopy equivalence.*

The following lemma is a first step towards understanding the relative homotopy groups $\pi_p\big(|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|\big)$.

Lemma 6.2. *Any map of pairs $f : (\Delta^p, \partial\Delta^p) \to \big(|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|\big)$ is homotopic, as a map of pairs, to a composite map of the form*

$$(\Delta^p, \partial\Delta^p) \xrightarrow{f'} \big(|K_\bullet|, |K'_\bullet|\big) \xrightarrow{|h_\bullet|} \big(|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|\big)$$

where $(K_\bullet, K'_\bullet)$ is a pair of finite Δ-sets obtained from a pair (K, K') of finite ordered Euclidean simplicial complexes.

Proof. Since Δ^p is compact the image of the map f intersects only finitely many of the simplices of $|\psi_d(N,k)_\bullet|$. Let $L_\bullet$ be the sub-Δ-set of $\psi_d(N,k)_\bullet$ generated by these simplices and let $L'_\bullet$ be the sub-Δ-set of $L_\bullet$ generated by thoses simplices which intersect the image $f(\partial\Delta^p)$. In particular, $L'_\bullet$ is a sub-Δ-set of $\psi_d^R(N,1)_\bullet$.

One of the properties of the subdivision map $\rho : |\psi_d(N,1)| \to |\psi_d(N,1)|$ proven in section §4 is that whenever we have a morphism $g_\bullet : S_\bullet \to \psi_d(N,1)_\bullet$ of Δ-sets and a non-negative integer $k \geq 0$ then there is a morphism

$$g'_\bullet : \mathrm{sd}^k(S)_\bullet \to \psi_d(N,1)_\bullet$$

from the k-th barycentric subdivision of $S_\bullet$ to $\psi_d(N,1)_\bullet$ whose geometric realization makes the following diagram commute

$$\begin{array}{ccc} |S_\bullet| & \xrightarrow{h_k} & \left|\mathrm{sd}^k(S)_\bullet\right| \\ {\scriptstyle |g|}\downarrow & & \downarrow{\scriptstyle |g'|} \\ |\psi_d(N,1)_\bullet| & \xrightarrow{\rho^k} & |\psi_d(N,1)_\bullet|\,, \end{array}$$

where h_k is the canonical homeomorphism between $|S_\bullet|$ and $\left|\mathrm{sd}^k(S)_\bullet\right|$. Then, since the map $f : (\Delta^p, \partial\Delta^p) \to (|\psi_d(N,1)|, |\psi_d^R(N,1)|)$ is equal to the composite map

$$(\Delta^p, \partial\Delta^p) \xrightarrow{f'} (|L_\bullet|, |L'_\bullet|) \hookrightarrow (|\psi_d(N,1)|, |\psi_d^R(N,1)|),$$

where f' is the map obtained by restricting the target of f to $|L_\bullet|$, we have by the property of the subdivision map ρ described above that f is homotopic to a composition of the form

$$(\Delta^p, \partial\Delta^p) \to (|\mathrm{sd}^2(L_\bullet)|, |\mathrm{sd}^2(L'_\bullet)|) \xrightarrow{|g_\bullet|} (|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|). \tag{20}$$

Furthermore, since $\Gamma\big([0,1] \times |\psi_d^R(N,1)_\bullet|\big) \subseteq |\psi_d^R(N,1)_\bullet|$, where Γ is the homotopy between ρ and $\mathrm{Id}_{|\psi_d(N,1)|}$ defined in 4.6, we have that the map (20) is homotopic to f as a map of pairs.

By lemma 2.68 we have that $(\mathrm{sd}^2(L_\bullet), \mathrm{sd}^2(L'_\bullet))$ is isomorphic to a pair of finite Δ-sets $(K_\bullet, K'_\bullet)$ obtained from a pair of finite ordered euclidean simplicial complexes (K, K') as indicated in remark 2.63 and thus in (20) we can replace $(\mathrm{sd}^2(L_\bullet), \mathrm{sd}^2(L'_\bullet))$ with $(K_\bullet, K'_\bullet)$ to obtain a composite map

$$(\Delta^p, \partial\Delta^p) \xrightarrow{f'} (K_\bullet, K'_\bullet) \xrightarrow{|g_\bullet|} \big(|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|\big)$$

which is homotopic to f. □

Remark 6.3. Recall that in section §4 we defined the subdvision map $\rho : |\psi_d(N,1)_\bullet| \to |\psi_d(N,1)_\bullet|$. Let now W be a p-simplex of $\psi_d^R(N,1)_\bullet$. If σ is any simplex, say of dimension q, of $\mathrm{sd}\Delta^p$ and if $\delta : \Delta^q \to \sigma$ is the simplicial isomorphism which preserves the order relation on the set of vertices then the pull back $\delta^* W$ is a q-simplex of $\psi_d^R(N,1)_\bullet$. It follows that ρ is a map of pairs from $\big(|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|\big)$ to itself. Similarly, if $\Gamma : [0,1] \times |\psi_d(N,1)_\bullet| \to |\psi_d(N,1)_\bullet|$ is the homotopy from $\mathrm{Id}_{|\psi_d(N,1)|}$ to ρ defined in the proof of theorem 4.6 then we also have that

$$\Gamma\big([0,1] \times |\psi_d^R(N,1)_\bullet|\big) \subseteq |\psi_d^R(N,1)_\bullet|.$$

These observations shall be used in the outline of the proof of theorem 6.1 that we are going to give at the end of this subsection.

Notation 6.4. Let P be an arbitrary polyhedron. For any product of the form $P \times \mathbb{R} \times (-1,1)^{N-1}$ we will denote by t_1 the projection $P \times \mathbb{R} \times (-1,1)^{N-1} \to \mathbb{R}$ onto $\mathbb{R}$ whereas for any element W in $\psi_d(N,1)\big(P\big)$ we will denote the restriction $t_1|W$ by x_1.

The following proposition, which we will prove in the next subsection, is what will allow us to show that any class in $\pi_p\big(|\psi_d(N,1)_\bullet|, |\psi_d^R(N,1)_\bullet|\big)$ is trivial.

Proposition 6.5. *Assume that $N-d \geq 3$. Let P be a compact polyhedron, let β be some fixed real constant and let W be an element in $\psi_d(N,1)\big(P\big)$. Then there is an element $\widetilde{W}$ in $\psi_d(N,1)\big([0,1] \times P\big)$ which satisfies the following properties:*

i) $\widetilde{W}$ *agrees with* $[0,1]\times W$ *in* $t_1^{-1}\big((-\infty,\beta]\big)$.

ii) $\widetilde{W}$ *is a concordance between* W *and an element* W' *in* $\psi_d(N,1)\big(P\big)$ *for which there is a finite open cover* $U_1,\ldots,U_q$ *of* P *such that for* $j=1,\ldots,q$ *the projection*

$$x_1 : W'_{U_j}\to\mathbb{R}$$

has a fiberwise regular value $a_j\in(\beta,\infty)$.

Proof of theorem 6.1. At this point it is convenient to give the proof of theorem 6.1 assuming proposition 6.5. We shall enumerate the steps of the proof and we remark that apart from step *iii*) none of the other steps need any further details:

i) Let $f : (\Delta^p,\partial\Delta^p)\to\big(\,|\psi_d(N,1)_\bullet|\,,\big|\psi_d^R(N,1)_\bullet\big|\,\big)$ be any map of pairs. By lemma 6.2 we can assume that f is homotopic as a map of pairs $(\Delta^p,\partial\Delta^p)\to\big(\,|\psi_d(N,1)_\bullet|\,,\big|\psi_d^R(N,1)_\bullet\big|\,\big)$ to a composite map of the form

$$(\Delta^p,\partial\Delta^p)\xrightarrow{f'}(|K_\bullet|,|K'_\bullet|)\xrightarrow{|g_\bullet|}\big(\,|\psi_d(N,1)_\bullet|\,,\big|\psi_d^R(N,1)_\bullet\big|\,\big),\qquad(21)$$

where $(K_\bullet,K'_\bullet)$ is a pair of finite Δ-sets obtained from a pair of finite ordered simplicial complexes (K,K') in some Euclidean space $\mathbb{R}^m$.

ii) By theorem 3.5 the morphism $g_\bullet$ classifies an element W of $\psi_d(N,1)\big(|K|\big)$ from which we can recover the morphism $g_\bullet$ as indicated in the statement of 3.5. Since $g_\bullet(K'_\bullet)\subseteq\psi_d^R(N,1)_\bullet$ then for each simplex σ of the simplicial complex K' there is a fiberwise regular value a_σ of the projection $x_1 : W_\sigma\to\mathbb{R}$. For each simplex $\sigma\in K'$ pick such a regular value a_σ and fix once and for all some real constant β larger than $\max\{a_\sigma :\ \sigma\in K'\}$.

iii) By proposition 6.5 there is an element $\widetilde{W}$ in $\psi_d(N,1)\big([0,1]\times|K|\big)$ which agrees with $[0,1]\times W$ in $t_1^{-1}\big((-\infty,\beta]\big)$, where β is the real constant we fixed in step *ii*), and which is a concordance between W and an element W' in $\psi_d(N,1)\big(|K|\big)$ for which there is a finite open cover $U_1,\ldots,U_q$ of $|K|$ such that for $j=1,\ldots,q$ the projection

$$x_1 : W'_{U_j}\to\mathbb{R}$$

has a fiberwise regular value $a_j\in(\beta,\infty)$.

iv) Using corollary 5.15 we can use $\widetilde{W}$ to produce a homotopy

$$F : [0,1]\times|K_\bullet|\to|\psi_d(N,1)_\bullet|$$

such that F_0 agrees with the map $|g_\bullet|$ of (21) and such that F_1 is equal to the geometric realization of the morphism

$$g'_\bullet : K_\bullet\to\psi_d(N,1)_\bullet$$

which classifies the element W' in $\psi_d(N,1)\big(|K|\big)$. Furthermore, since $\widetilde{W}$ agrees with $[0,1]\times W$ in $t_1^{-1}\big((-\infty,\beta]\big)$ we have that $F\big([0,1]\times|K'_\bullet|\big)\subseteq|\psi_d^R(N,1)_\bullet|$ and hence F is a homotopy of maps of pairs $(|K_\bullet|,|K'_\bullet|)\to\big(|\psi_d(N,1)_\bullet|,|\psi_d^R(N,1)_\bullet|\big)$.

v) By the Lebesgue number lemma we can find a large enough positive integer $k>0$ so that each simplex σ of $\mathrm{sd}^k K$ is contained in one of the open sets of the cover $U_1,\ldots,U_q$ given in step *iii)*. Thus for each simplex σ of $\mathrm{sd}^k K$ we have that the projection

$$x_1 : W'_\sigma \to \mathbb{R}$$

has a fiberwise regular value and hence the image of the composition $\rho^k\circ|h'_\bullet|$ lies entirely in $|\psi_d^R(N,1)_\bullet|$. Furthermore, by remark 6.3 we have that $|h'_\bullet|$ and $\rho^k\circ|h'_\bullet|$ are homotopic as maps of pairs $(|K_\bullet|,|K'_\bullet|)\to\big(|\psi_d(N,1)_\bullet|,|\psi_d^R(N,1)_\bullet|\big)$.

vi) Finally, by concatenating the following homotopies of maps of pairs $(\Delta^p,\partial\Delta^p)\to\big(|\psi_d(N,1)_\bullet|,|\psi_d^R(N,1)_\bullet|\big)$

$$f\ \sim\ |h_\bullet|\circ f'\ \sim\ |h'_\bullet|\circ f'\ \sim\ \rho^k\circ|h'_\bullet|\circ f'$$

we obtain that the map f that we started out with in step *i)* represents the trivial class in $\pi_p\big(|\psi_d(N,1)_\bullet|,|\psi_d^R(N,1)_\bullet|\big)$. Since f was an arbitrary map $(\Delta^p,\partial\Delta^p)\to\big(|\psi_d(N,1)_\bullet|,|\psi_d^R(N,1)_\bullet|\big)$ we conclude that the inclusion $\psi_d^R(N,1)_\bullet\hookrightarrow\psi_d(N,1)_\bullet$ is a weak homotopy equivalence.

□

In the next subsection we state some results about extensions of isotopies of piecewise linear embeddings that we need for the proof of 6.5. Finally, in §6.3 we give the proof of proposition 6.5 which completes the proof of theorem 6.1.

6.2 The isotopy extension theorem

Before we prove lemma 6.5 we need to state the Isotopy Extension Theorem, which was proven by Hudson in [12], and prove some additional results about isotopies of pl embeddings. We begin with the following definitions.

Definition 6.6. 1. Let M^m and Q^q be piecewise linear manifolds with M compact. A pl map $f : M\times I^n\to Q\times I^n$ is said to be an n-isotopy if it is a piecewise linear embedding and if it commutes with the projection onto I^n.

2. A piecewise linear n-isotopy $f : M\times I^n\to Q\times I^n$ is said to be *allowable* if for some piecewise linear $(m-1)$-submanifold N of ∂M we have that $i_t^{-1}(\partial Q)=N$ for all $t\in I^n$. N may be empty or it can be the whole of ∂M.

3. An *ambient n-isotopy* of Q is a piecewise linear homeomorphism $h : Q \times I^n \to Q \times I^n$ which commutes with the projection onto I^n and such that $h_0 = \mathrm{Id}_Q$

4. An n-isotopy f of M into Q is said to be *fixed* on $X \subseteq M$ if $f_t|X = f_0|X$ for all $t \in I^n$.

5. An ambient n-isotopy h of Q is said to *extend* the n-isotopy $f : M \times I^n \to Q \times I^n$ if $h_t \circ f_0 = f_t$ for all $t \in I^n$.

Proposition 6.7. *Let $f : M \times I^n \to Q \times I^n$ be an allowable pl isotopy of M in Q, fixed on $f_0^{-1}(\partial Q)$. If $q - m \geq 3$ then there is an ambient pl n-isotopy of Q, fixed on $\partial Q \times I^n$, which extends f. Furthermore, if B is a subpolyhedron of I^n such that B is a piecewise linear retract of I^n, $0 \in B$ and such that there is a pl homeomorphism $h' : Q \times B \to Q \times B$ which commutes with the projection onto B and which extends $f|B \times M$ then the ambient n-isotopy h can be chosen to agree with h' on $B \times Q$.*

Proof. The first claim of this proposition is theorem 2 of [12] and the proof can be found in that article. Assume now that there is a subpolyhedron B of I^n with $0 \in B$, a piecewise linear retraction $r : I^n \to B$ and a pl homeomorphism $h' : Q \times B \to Q \times B$ which commutes with the projection onto B, which extends $f|B \times M$, and which at time $0 \in I^n$ is equal to Id_Q. If $\tilde{h} : I^n \times Q \to I^n \times Q$ is any ambient n-isotopy which extends f then the map $h : I^n \times Q \to I^n \times Q$ which at time $t \in I^n$ is given by $h_t = \tilde{h}_t \circ \tilde{h}^{-1}_{r(t)} \circ h'_{r(t)}$ is also an ambient n-isotopy of Q by proposition 2.55. Clearly this map agrees with h' on $B \times Q$ and by pre-composing $\tilde{h}_t \circ \tilde{h}^{-1}_{r(t)} \circ h'_{r(t)}$ with f_0 we obtain

$$\tilde{h}_t \circ \tilde{h}^{-1}_{r(t)} \circ h'_{r(t)} \circ f_0 = \tilde{h}_t \circ \tilde{h}^{-1}_{r(t)} \circ f_{r(t)} = \tilde{h}_t \circ f_0 = f_t.$$

Thus we also have that h extends f. □

Using the second claim in proposition 6.7 and proposition 2.56 we can prove the following useful corollary. In this statement, b will denote the point $(0, \ldots, 0)$ in I^n.

Corollary 6.8. *Let Q^q be a pl manifold and let M^m be a compact submanifold of M such that $q - m \geq 3$. If $g : M \times I^n \to Q \times I^n$ is an allowable n-isotopy of embeddings of M into Q such that g_b is equal to the natural inclusion $M \hookrightarrow Q$ then there is an ambient isotopy*

$$G : [0, 1] \times I^n \times M \to [0, 1] \times I^n \times Q$$

parametrized by $[0, 1] \times I^n$ which satisfies the following:

i) $G_1 = \mathrm{Id}_{I^n \times Q}$.

ii) G_0 *extends the isotopy* g.

iii) $G_{(0,b)} = \mathrm{Id}_Q$.

Proof. Let $c : [0,1]\times I^n \to I^n$ be a pl map which satisfies the following properties

- If $i_0 : I^n \hookrightarrow [0,1]\times I^n$ is the map defined by $\alpha \mapsto (0,\alpha)$ then the composite map $I^n \overset{i_0}{\hookrightarrow} [0,1]\times I^n \overset{c}{\to} I^n$ is equal to the identity map Id_{I^n}.
- $c\big((1,\alpha)\big) = b$ for all points α in I^n.
- There is a subpolyhedron S of $[0,1]\times I^n$ such that

$$\{(0,b)\}\cup\{1\}\times I^n \subseteq S \subseteq c^{-1}(b)$$

and for which there is a pl retraction $r : [0,1]\times I^n \to S$.

Using proposition 2.56 we have that the map

$$\widetilde{g} : [0,1]\times I^n \times M \to [0,1]\times I^n \times Q$$

defined by $(t,\alpha,x) \mapsto \big(t,\alpha,g_{c(t,\alpha)}\big)$ is an isotopy of pl embeddings of M into Q parametrized by $[0,1]\times I^n$. Observe that by construction we have that the restriction $\widetilde{g}|_{S\times M}$ is equal to the natural inclusion $S\times M \hookrightarrow S\times Q$. By the isotopy extension theorem there is an ambient isotopy

$$G : [0,1]\times I^n \times Q \to [0,1]\times I^n \times Q$$

of Q such that

$$\widetilde{g} = G\circ(\mathrm{Id}_{[0,1]\times I^n}\times \widetilde{g}_b)$$

Furthermore, since the identity map covers $\widetilde{g}|_{S\times M}$ then by the second claim of proposition 6.7 we can assume that $G|_{S\times Q} = \mathrm{Id}_{S\times Q}$.

□

6.3 Proof of proposition 6.5

Proposition 6.5 is an immediate consequence of the following more technical lemma whose proof will occupy the rest of this section.

Lemma 6.9. *Let M^m be a pl manifold, possibly with boundary, let P be a compact subpolyhedron of M such that $P \subseteq M - \partial M$, let β be some fixed real constant and let W be an element in $\psi_d(N,1)\big(M\big)$. Then there is an element $\widetilde{W}$ in $\psi_d(N,1)\big([0,1]\times M\big)$ and finitely many open sets $V_1,\ldots,V_q$ in $M-\partial M$ which satisfy the following:*

i) $\widetilde{W}$ *agrees with* $[0,1]\times W$ *in* $t_1^{-1}\big((-\infty,\beta]\big)$.

ii) $V_1,\ldots,V_q$ *cover* P.

ii) $\widetilde{W}$ *is a concordance between* W *and an element* W' *in* $\psi_d(N,1)\big(M\big)$ *such that for each of the open sets* $V_1,\ldots,V_q$ *the projection*

$$x_1 : W'_{V_j} \to \mathbb{R}$$

has a fiberwise regular value $a_j \in (\beta,\infty)$.

Proof of proposition 6.5. Let us give the proof of proposition 6.5 assuming lemma 6.9. Let then P be a compact polyhedron, let β be some fixed real constant and let W be an element in $\psi_d(N,1)\big(P\big)$. Without loss of generality we can assume that P is embedded in some Euclidean space, say $\mathbb{R}^m$. Let M be a regular neighborhood of P in $\mathbb{R}^m$, let $r : M \to P$ be some pl retraction and let W' denote the pull back of W along r. M is an m-dimensional pl manifold with boundary such that $P \subseteq M - \partial M$. Applying lemma 6.9 to M, P, W' and β we obtain an element $\widetilde{W}$ in $\psi_d(N,1)\big([0,1]\times M\big)$ and finitely many open sets $V_1,\ldots,V_q$ in $M-\partial M$ which satisfy claims *i*), *ii*) and *iii*) of 6.9. Finally, it is clear that the concordance $\widetilde{W}_{[0,1]\times P}$ and the open sets

$$U_1 := V_1 \cap P, \ldots,\ U_q := V_q \cap P$$

satisfy claims *i*) and *ii*) of proposition 6.5. □

For the proof of lemma 6.9 we are going to need lemmas 6.10, 6.11 and 6.14 below. The reader is advised to skip the proofs of these three lemmas at a first reading and jump directly to the proof of 6.9, given after lemma 6.14, in order to grasp how these three lemmas are used to prove 6.9.

Lemmas 6.10 and 6.11 are the results we are going to use to produce fiberwise regular values. Also, is in 6.10 that we are going to make use of Hudson's isotopy extension theorem.

Note. In the proofs of 6.10 and 6.11 we are going to use the following notation: If W is an element in $\psi_d(N,1)\big(M\big)$ and if $\lambda \in M$ then the pre-image of a subspace $S \subseteq \mathbb{R}$ under the projection

$$x_1 : W_\lambda \to \mathbb{R}$$

shall be denoted by $W_{\lambda,S}$.

Lemma 6.10. *Let* M *be a compact* m*-dimensional piecewise linear manifold, possibly with boundary, let* W *be an element of* $\psi_d(N,1)\big(M\big)$ *and let* β *be a fixed real constant. Assume that* $N-d \geq 3$. *Then for each* $\lambda \in M-\partial M$ *there is a regular neighborhood* V_λ *of* λ *in* $M-\partial M$, *values* $a_{\lambda,0},\ldots,a_{\lambda,m}$ *in* $\mathbb{R}$ *and piecewise linear automorphisms* $F^{\lambda,0},\ldots,F^{\lambda,m}$ *of* $[0,1]\times V_\lambda \times \mathbb{R}\times(-1,1)^{N-1}$ *which satisfy the following:*

i) $a_{\lambda,j} \in (\beta+j+\frac{1}{4}, \beta+j+\frac{3}{4})$.

ii) $F^{\lambda,j}$ *commutes with the projection onto* $[0,1]\times V_\lambda$.

iii) $F_0^{\lambda,j}$ *is the identity map on* $V_\lambda \times \mathbb{R} \times (-1,1)^{N-1}$.

iv) $F^{\lambda,j}$ *is supported on* $t_1^{-1}\big([\beta + j + \frac{1}{4}, \beta + j + \frac{3}{4}]\big)$

v) $F^{\lambda,j}\big([0,1] \times W_{V_\lambda}\big)$ *is a concordance between* W_{V_λ} *and an element* W'_{V_λ} *of* $\psi_d(N,1)\big(V_\lambda\big)$ *such that* $a_{\lambda,j}$ *is a fiberwise regular value of the projection*

$$x_1 : W'_{V_\lambda} \to \mathbb{R}.$$

Proof. In order to make our notation less cumbersome we shall only prove this lemma in the case when $\beta = 0$. Fix a value j in $\{0, \dots, m\}$. By proposition 2.47 we can find a value

$$a \in (j + \frac{1}{4}, j + \frac{3}{4})$$

which is a regular value (in the sense of definition 2.46) of the projection

$$x_1 : W_\lambda \to \mathbb{R},$$

i.e. there is a pl homeomorphism

$$h : [a - \delta', a + \delta'] \times W_{\lambda,a} \xrightarrow{\cong} W_{\lambda,[a-\delta',a+\delta']} \tag{22}$$

such that $x_1 \circ h$ is equal to the projection onto $[a - \delta', a + \delta']$. We can assume without loss of generality that $[a - \delta', a + \delta'] \subseteq (j + \frac{1}{4}, j + \frac{3}{4})$. From now on we will denote the 'cylinder' $W_{\lambda,[a-\delta',a+\delta']}$ simply by $C_{\delta'}$. Also, the product $[j + \frac{1}{4}, j + \frac{3}{4}] \times (-1,1)^{N-1}$ shall be denoted by Q.

Let now V and U be open neighborhoods of λ and $C_{\delta'}$ in M and W_λ respectively for which there is a normalized product chart

$$g : V \times U \to W_V \tag{23}$$

around $C_{\delta'}$ for the submersion $\pi : W \to M$. The existence of such a chart g is guaranteed by corollary 2.51. Let us denote again by g the map obtained by composing (23) with the natural inclusion $W_V \hookrightarrow V \times \mathbb{R}^N$. After shrinking V we can further assume that

$$g_\alpha\big(C_{\delta'}\big) \subseteq Q$$

for each $\alpha \in V$ and that V is a regular neighborhood of λ in $M - \partial M$. The restriction of g on $V \times C_{\delta'}$ is then an m-isotopy of $C_{\delta'}$ in Q such that g_λ is just the natural inclusion $C_{\delta'} \hookrightarrow Q$. Applying corollary 6.8 we obtain an ambient isotopy

$$G : [0,1] \times V \times Q \to [0,1] \times V \times Q$$

parametrized by $[0,1] \times V$ which satisfies the following:

- $G_1 = \mathrm{Id}_{I^n \times Q}$.
- G_0 extends the isotopy g.

· $G_{(0,\lambda)} = \mathrm{Id}_Q$.

If we precompose G with $\mathrm{Id}_{[0,1]} \times G_0^{-1}$ we obtain a new ambient isotopy

$$F' : [0,1] \times V \times Q \to [0,1] \times V \times Q$$

which is the identity at time $t = 0$, which fixes all points in $[0,1] \times V \times \partial Q$ and which transforms the bundle $[0,1] \times g(V \times C_{\delta'})$, which is a bundle over $[0,1] \times V$ with fiber $C_{\delta'}$, into a new bundle such that over $t = 0$ it agrees with $g(V \times C_{\delta'})$ and at time $t = 1$ it is equal to

$$F'_1(V \times C_{\delta'}) = V \times C_{\delta'}. \tag{24}$$

Since F' fixes all points in $[0,1] \times V \times \partial Q$ we can extend F' to a pl automorphism

$$F : [0,1] \times V \times \mathbb{R} \times (-1,1)^{N-1} \to [0,1] \times V \times \mathbb{R} \times (-1,1)^{N-1}$$

just by setting $(t, \alpha, x) \mapsto (t, \alpha, x)$ for each point $x \in \mathbb{R} \times (-1,1)^{N-1}$ which is not in Q. By construction we have that F commutes with the projection onto $[0,1] \times V$ and that F_0 is just the identity and therefore $F\big([0,1] \times W_V\big)$ is a concordance between W_V and some new element $F_1(W_V)$ of $\psi_d(N,1)\big(V\big)$. Furthermore, by construction we also have that $F_{(1,\lambda)} = \mathrm{Id}_{\mathbb{R} \times (-1,1)^{N-1}}$, which implies that $F_1(W_V)_\lambda = W_\lambda$.

We now wish to show that, after possibly shrinking V, there is a small enough value $\delta > 0$ so that

$$F_1(W_V)_{[a-\delta, a+\delta]} = V \times W_{\lambda, [a-\delta, a+\delta]}. \tag{25}$$

If we manage to find such V and δ then we will have that a is a fiberwise regular value of $x_1 : F_1(W_V) \to \mathbb{R}$ since the product

$$\mathrm{Id}_V \times h : V \times [a-\delta, a+\delta] \times W_{\lambda,a} \to V \times W_{\lambda,[a-\delta,a+\delta]}, \tag{26}$$

where h is the map from (22), is a pl homeomorphism such that the composite $(\pi, x_1) \circ h$ is equal to the projection onto $V \times [a-\delta, a+\delta]$. In both (25) and (26) we are identifying the fiber $F_1(W_V)_\lambda$ with W_λ.

Fix first a value $\delta'' > 0$ smaller than $\delta' > 0$. Observe that V could have been chosen to be a small enough regular neighborhood of λ so that for the normalized product chart g given in (23) we have that

$$W_{\alpha,[a-\delta'',a+\delta'']} \subseteq g_\alpha\big(W_{\lambda,(a-\delta',a+\delta')}\big) \tag{27}$$

for each point α in V. In order to obtain (25) we shall use the following observation:

Observation. For any $\delta > 0$ smaller than $\delta'' > 0$ we can pick an m-dimensional pl ball $V' \subseteq V$ such that $\lambda \in \mathrm{int} V'$ and such that

$$F_{(1,\alpha)}\Big(\big([j + \frac{1}{4}, j + \frac{3}{4}] - (a - \delta'', a + \delta'')\big) \times (-1,1)^{N-1}\Big)$$

and

$$[a-\delta, a+\delta] \times (-1,1)^{N-1}$$

are disjoint for each $\alpha \in V'$.

This observation just follows from the fact that $F_{(1,\lambda)}$ is the identity map on $\mathbb{R} \times (-1,1)^{N-1}$.

Choose then any value $\delta > 0$ smaller than $\delta'' > 0$ and let V be a smaller regular neighborhood of λ for which the previous observation holds. We claim that equation (25) holds with this choice of δ and V. In order to show that the inclusion $V \times W_{\lambda,[a-\delta,a+\delta]} \subseteq F_1(W_V)_{[a-\delta,a+\delta]}$ holds we first observe that

$$V \times W_{\lambda,[a-\delta',a+\delta']} \subseteq F_1(W_V)_{[a-\delta',a+\delta']}$$

since $V \times W_{\lambda,[a-\delta',a+\delta']}$ is contained in the image of F_1. Intersecting both sides of this last inclusion with $t_1^{-1}\big([a-\delta, a+\delta]\big)$ gives us that

$$V \times W_{\lambda,[a-\delta,a+\delta]} \subseteq F_1(W_V)_{[a-\delta,a+\delta]}.$$

In order to verify the other inclusion in (25) we shall for each $\alpha \in V$ decompose the fiber W_α into the following three subspaces:

$$A_\alpha = x_1^{-1}\big([a\pm\delta'']\big), \quad B_\alpha = x_1^{-1}\big([j+\frac{1}{4}, j+\frac{3}{4}]-(a\pm\delta'')\big), \quad C_\alpha = x_1^{-1}\big(\mathbb{R}-[j+\frac{1}{4}, j+\frac{3}{4}]\big).$$

By the observation given above we have that

$$F_{(1,\alpha)}(B_\alpha) \cap [a \pm \delta] \times (-1,1)^{N-1} = \emptyset.$$

Also, since F_1 is supported on $V \times [j+\frac{1}{4}, j+\frac{3}{4}] \times (-1,1)^{N-1}$ we also have that

$$F_{1,\alpha}(C_\alpha) \cap [a \pm \delta] \times (-1,1)^{N-1} = \emptyset.$$

Finally, by (27) we have

$$A_\alpha \subseteq g_\alpha(W_{\lambda,(a\pm\delta')})$$

for each $\alpha \in V$ and since the right hand side of this inclusion is equal to $G_{(0,\alpha)}\big(W_{\lambda,(a\pm\delta')}\big)$ and since $F_1 = G_0^{-1}$ we obtain that

$$F_{(1,\alpha)}(A_\alpha) \subseteq W_{\lambda,(a\pm\delta')}$$

which in turn implies that

$$F_1(W_V)_{[a-\delta,a+\delta]} \subseteq V \times W_{\lambda,[a-\delta,a+\delta]}$$

since we already had that $F_{(1,\alpha)}(B_\alpha)$ and $F_{(1,\alpha)}(C_\alpha)$ do not intersect $[a-\delta, a+\delta] \times (-1,1)^{N-1}$ for each α in V.

The previous argument shows that for any $j = 0, \dots, m$ there is a regular neighborhood V_j of λ in $M - \partial M$, a value a_j in $(j + \frac{1}{4}, j + \frac{3}{4})$, and a pl automorphism F^j of $[0,1] \times V_j \times \mathbb{R} \times (-1,1)^{N-1}$ over $[0,1] \times V_j$ supported on $[j + \frac{1}{4}, j + \frac{3}{4}]$ such that $F^j\big([0,1] \times W\big)$ is a concordance between W_{V_j} and an element W' in $\psi_d(N,1)\big(V_j\big)$ such that a_j is a fiberwise regular value of $x_1 : W' \to \mathbb{R}$. Finally, if V is any regular neighborhood of λ contained in $\bigcap_{j=0}^{m} \mathrm{int} V_j$ we have that V, $a_0, \dots, a_m$ and $F_0, \dots, F_m$ satisfy all the claims listed in the statement of this lemma.

□

Lemma 6.11. *Let M^m be a compact piecewise linear manifold, possibly with boundary, let W be an element of $\psi_d(N,1)\big(M\big)$, let β be some fixed real constant and let $\lambda \in M - \partial M$. Let V_λ be the regular neighborhood of λ in $M - \partial M$ and $a_{\lambda,0}, \dots, a_{\lambda,m}$ the values in $\mathbb{R}$ obtained in lemma 6.10. Then for any $j \in \{0, \dots, m\}$ and any pair of m-dimensional piecewise linear balls (U', U) in $\mathrm{int} V_\lambda$ such that $U \subseteq \mathrm{int} U'$ there is a piecewise linear automorphism $\widetilde{F}$ of $[0,1] \times M \times \mathbb{R} \times (-1,1)^{N-1}$ which satistifies the following properties:*

i) $\widetilde{F}$ commutes with the projection onto $[0,1] \times M$.

ii) $\widetilde{F}_0$ is the identity map on $M \times \mathbb{R} \times (-1,1)^{N-1}$.

iii) $\widetilde{F}$ is supported on

$$[0,1] \times U' \times [\beta + j + \frac{1}{4}, \beta + j + \frac{3}{4}] \times (-1,1)^{N-1}.$$

iv) $\widetilde{F}\big([0,1] \times W\big)$ is a concordance between W and an element W' in $\psi_d(N,1)\big(M\big)$ such that $a_{\lambda,j}$ is a fiberwise regular value of the projection

$$x_1 : W'_U \to \mathbb{R}.$$

Proof. Again, without loss of generality we shall only work with the case when $\beta = 0$. Recall that in lemma 6.10 we obtained a piecewise linear automorphism $F^{\lambda,j}$ of $[0,1] \times V_\lambda \times \mathbb{R} \times (-1,1)^{N-1}$ which satisfies conditions $ii) - v)$ of lemma 6.10. Let us denote this pl homeomorphism simply by F. Let now $f : [0,1] \times V_\lambda \to [0,1] \times V_\lambda$ be a piecewise linear map which satisfies the following properties:

- $f|_{[0,1] \times U} = \mathrm{Id}_{[0,1] \times U}$.
- $f|_{\{0\} \times V_\lambda} = \mathrm{Id}_{\{0\} \times V_\lambda}$.
- $f(t,x) = (0,x)$ if $x \in \mathrm{cl}(V_\lambda - U')$.

See the remark following this proof for an explanation on how to define this map f. The map

$$\widetilde{F} : [0,1] \times V_\lambda \times \mathbb{R} \times (-1,1)^{N-1} \to [0,1] \times V_\lambda \times \mathbb{R} \times (-1,1)^{N-1}$$

defined by $\widetilde{F}(t,\alpha,x) = (t,\alpha,F_{f(t,\alpha)}(x))$ is a piecewise linear homeomorphism which commutes with the projection onto $[0,1]\times V_\lambda$. Furthermore, $\widetilde{F}$ can be extended to all of $[0,1]\times M\times\mathbb{R}\times(-1,1)^{N-1}$ just by setting $\widetilde{F}(t,\alpha,x)=(t,\alpha,x)$ if $\alpha\in M-U'$ and it is straightforward to verify that $\widetilde{F}$ satisfies claims $i)-iii)$ listed in the statement of this lemma. Finally, claim $iv)$ follows from the fact that $F_1(W_{V_\lambda})$ and $\widetilde{F}_1(W)$ give the same element in $\psi_d(N,1)\big(U\big)$. □

Remark 6.12. We are now going to indicate how to define the map $f : [0,1]\times V_\lambda \to [0,1]\times V_\lambda$ that we used at the beginning of the previous proof. For this construction we are going to need the following *piecewise linear product*: the real valued function $\cdot : [0,1]\times[0,1]\to[0,1]$ which maps a tuple (x,y) to its product $x\cdot y$ is not a pl function. However, *it is* piecewise linear when we restrict it on the boundary of $[0,1]\times[0,1]$. In fact, if L' is the obvious simplicial complex which triangulates $\partial\big([0,1]\times[0,1]\big)$ then $\cdot : [0,1]\times[0,1]\to[0,1]$ is linear on each simplex of L'. Let now L denote the join $\{(\frac{1}{2},\frac{1}{2})\}*L'$. L is a simplicial complex which triangulates $[0,1]\times[0,1]$ and we define $p : |L|\to[0,1]$ to be the unique map which is linear on each simplex of L and which sends each vertex (x,y) of L to the product $x\cdot y$. We shall refer to this function as *the piecewise linear product on* $[0,1]$.

In order to define the map $f : [0,1]\times V_\lambda\to[0,1]\times V_\lambda$ observe first that (V_λ,U',U) is a triad of regular neighborhoods of λ in $\mathrm{int}M$ such that $U\subseteq \mathrm{int}U'$ and $U'\subseteq \mathrm{int}V_\lambda$ and by the Combinatorial Annulus Theorem (see corollary 3.19 in [17]) we can then assume that

$$(V_\lambda,U',U)=([-3,3]^m,[-2,2]^m,[-1,1]^m).$$

Let $s:\mathbb{R}^m\to[0,\infty)$ be the piecewise linear map which sends a vector $(x_1,\dots,x_n)$ to $\max\{|x_1|,\dots,|x_n|\}$ and let $\phi:\mathbb{R}\to[0,1]$ be a piecewise linear map which is constantly equal to 1 on $[-1,1]$ and constantly equal to 0 on $\mathbb{R}-[-3,3]$. Finally, let

$$f : [0,1]\times[-3,3]^m\to[0,1]\times[-3,3]^m$$

be the map defined by

$$f(t,x)=\Big(p\big(t,\phi\circ s(x)\big),x\Big).$$

It follows from proposition 2.29 that f is indeed a piecewise linear map and we leave it to the reader to verify that f satisfies the three properties listed in the proof of lemma 6.11.

Lemma 6.14 below describes a convenient way of obtaining open covers for compact subpolyhedra in a manifold using simplicial complexes. Before stating the lemma we need to introduce the following definition.

Definition 6.13. Let K be a finite simplicial complex inside some Euclidean space. Let σ be a simplex of K with barycenric point $b(\sigma)$. A subset $\widetilde{\sigma}$ of σ is called a *concentric sub-simplex of* σ if:

1. $\widetilde{\sigma} = \sigma$ if $\dim\sigma = 0$

2. There is a value $0 < t_0 < 1$ such that

$$\widetilde{\sigma} = \{(1-t)\cdot b(\sigma) + t\cdot x : x \in \partial\sigma, 0 \leq t \leq t_0\}$$

if $\dim\sigma > 0$.

Lemma 6.14. *Let M^m be a compact piecewise linear manifold, possibly with boundary, embedded in some Euclidean space, let P be a compact subpolyhedron of M contained in $M - \partial M$ and let $\Lambda = \{V_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in M which cover P. Let K be a simplicial complex which traingulates P and which is subordinate to Λ. Then for each simplex σ of K there is a concentric sub-simplex $\widetilde{\sigma}$ and compact neighborhoods U_σ, U'_σ of $\widetilde{\sigma}$ which satisfy the following properties:*

i) $U_\sigma \subseteq \mathrm{int}U'_\sigma$ and U_σ, U'_σ are regular neighborhoods of $\widetilde{\sigma}$ in $M - \partial M$. In particular, both compact neighborhoods are m-dimensional piecewise linear balls.

ii) $U'_{\sigma_1} \cap U'_{\sigma_2} = \emptyset$ if $\sigma_1 \neq \sigma_2$ and $\dim\sigma_1 = \dim\sigma_2$.

iii) The collection $\{\mathrm{int}U_\sigma\}_{\sigma\in K}$ is an open cover for P.

Proof. Let p be the dimension of P. We are going to define the concentric simplices $\widetilde{\sigma}$ and the compact neighborhoods U_σ and U'_σ by induction on the dimension of the simplices of K. Let then $\alpha_1, \ldots, \alpha_q$ be the vertices of K and let V_i be an open set of the cover Λ which contains the vertex α_i. Since $|K|$ is Hausdorff we can find open sets $\tilde{U}_{\alpha_1}, \ldots, \tilde{U}_{\alpha_p}$ such that $\alpha_i \in \tilde{U}_{\alpha_i}$ and $\tilde{U}_{\alpha_i} \cap \tilde{U}_{\alpha_j} = \emptyset$ if $i \neq j$. Furthermore we can assume that each $\tilde{U}_{\alpha_i}$ is contained in V_i and in $M - \partial M$. Now for each α_i pick regular neighborhoods U_{α_i} and U'_{α_i} contained in $\tilde{U}_{\alpha_i}$ such that $U_{\alpha_i} \subseteq \mathrm{int}U'_{\alpha_i}$. This completes the first step of the induction argument. Suppose now that for every simplex σ of the q-skeleton K^q of K, $q < p$, there is a concentric simplex $\widetilde{\sigma}$ of σ and regular neighborhoods U_σ and U'_σ of $\widetilde{\sigma}$ in $M - \partial M$ such that $U_\sigma \subseteq \mathrm{int}U'_\sigma$, the collection of open sets $\{\mathrm{int}U_\sigma\}_{\sigma\in K^q}$ covers $|K^q|$, $U'_{\sigma_1} \cap U'_{\sigma_2} = \emptyset$ if σ_1 and σ_2 are two different simplices of the same dimension, and each U'_σ is contained in an element of Λ. For each $(q+1)$-simplex β of K pick a concentric sub-simplex $\widetilde{\beta}$ such that

$$\mathrm{cl}(\beta - \widetilde{\beta}) \subseteq \bigcup_{\sigma\in K^q} U_\sigma$$

and let V_β be an open set of the cover Λ which contains β. Observe that all the concentric simplices $\widetilde{\beta}$ are disjoint since the interiors of any pair of $(q+1)$-simplices are disjoint and since $|K|$ is a normal space we can find for each $\widetilde{\beta}$ an open neighborhood $\tilde{U}_\beta$ which is contained in $V_\beta \cap (M - \partial M)$ and such that $\tilde{U}_{\beta_1} \cap \tilde{U}_{\beta_2} = \emptyset$ whenever $\widetilde{\beta_1} \neq \widetilde{\beta_2}$. Finally, for each concentric simplex $\widetilde{\beta}$

pick regular neighborhoods U_β and U'_β in $\tilde{U}_\beta$ such that $U_\beta \subseteq \text{int}U'_\beta$. By the way we defined the regular neighborhoods U_β,U'_β we have that the collection $\{U'_\sigma, U_\sigma\}_{\sigma\in K^{q+1}}$ satisfies conditions $i)$ and $ii)$ given in the statement of this lemma and that the collection of open sets $\{\text{int}U_\sigma\}_{\sigma\in K^{q+1}}$ covers $|K^{q+1}|$.

□

Proof of lemma 6.9. By 6.10 we can find for each $\lambda \in P$ a regular neighborhood V_λ of λ in $M - \partial M$, values $a_{\lambda,0}, \ldots, a_{\lambda,m}$ in $\mathbb{R}$ and piecewise linear automorphisms $F^{\lambda,0}, \ldots, F^{\lambda,m}$ of $[0,1] \times V_\lambda \times \mathbb{R} \times (-1,1)^{N-1}$ which satisfy all the claims listed in lemma 6.10.

Pick finitely points $\lambda_1, \ldots, \lambda_q$ in P such that the interiors of the regular neighborhoods $V_{\lambda_1}, \ldots, V_{\lambda_q}$ cover P. Let K be a simplicial complex which triangulates P. By the Lebesgue number lemma we can assume that each simplex σ of K is contained in one of the open sets $\text{int}V_{\lambda_j}$. Now, by lemma 6.14 we can find for each simplex σ of K a concentric sub-simplex $\widetilde{\sigma}$ and regular neighborhoods U'_σ, U_σ of $\widetilde{\sigma}$ which satisfy the following conditions:

- $U_\sigma \subseteq \text{int}U'_\sigma$.
- There is an $i \in \{1, \ldots, q\}$ such that $U'_\sigma \subseteq \text{int}V_{\lambda_i}$.
- If $\sigma_1 \neq \sigma_2$ and $\dim\sigma_1 = \dim\sigma_2$ then $U'_{\sigma_1} \cap U'_{\sigma_2} = \emptyset$.
- $\{\text{int}U_\sigma\}_{\sigma\in K}$ is an open cover for P.

For each simplex σ in K we fix an open set V_{λ_i} such that $U'_\sigma \subseteq \text{int}V_{\lambda_i}$. Denote this set by V_σ and denote by $a_{\sigma,0}, \ldots, a_{\sigma,m}$ and $F^{\sigma,0}, \ldots, F^{\sigma,m}$ the corresponding values $a_{\lambda_i,0}, \ldots, a_{\lambda_i,m}$ and pl maps $F^{\lambda_i,0}, \ldots, F^{\lambda_i,m}$. Furthermore, for each simplex σ we shall denote the value $a_{\sigma,\dim\sigma}$ simply by a_σ.

By lemma 6.11 there is for each $\sigma \in K$ a piecewise linear automorphism $\widetilde{F}^\sigma$ of $[0,1] \times M \times \mathbb{R} \times (-1,1)^{N-1}$ which is supported on

$$[0,1] \times U'_\sigma \times [\beta + \dim\sigma + \frac{1}{4}, \beta + \dim\sigma + \frac{3}{4}] \times (-1,1)^{N-1},$$

which commutes with the projection onto $[0,1] \times M$, which is the identity at time $t = 0$ and such that the image

$$F^\sigma\big([0,1] \times W\big)$$

is a concordance between W and an element W' in $\psi_d(N,1)\big(M\big)$ such that a_σ is a fiberwise regular value of the projection

$$x_1 : W'_{U_\sigma} \to \mathbb{R}.$$

Observe that all the supports of the pl maps F^σ are disjoint. Indeed, if σ_1 and σ_2 are two simplices of different dimension then the the supports of F^{σ_1} and

F^{σ_2} are obviously disjoint, and if σ_1 and σ_2 are simplices of the same dimension we will have that $U'_{\sigma_1} \cap U'_{\sigma_2} = \emptyset$. We can then compose all the maps F^σ to obtain a pl automorphism

$$F : [0,1] \times M \times \mathbb{R} \times (-1,1)^{N-1} \to [0,1] \times M \times \mathbb{R} \times (-1,1)^{N-1}$$

such that the image $F\big([0,1] \times W\big)$, which we will denote by $\widetilde{W}$, is an element of $\psi_d(N,1)\big([0,1] \times M\big)$ which is a concordance between W and an element $\widehat{W}$ of $\psi_d(N,1)\big(M\big)$ such that for each simplex σ of K the value a_σ is a fiberwise regular value of the projection

$$x_1 : \widehat{W}_{U_\sigma} \to \mathbb{R}.$$

Thus, we have that $\widetilde{W}$ and $\{\text{int}U_\sigma\}_{\sigma \in K}$ satisfy claims *ii*) and *iii*) of lemma 6.9. Finally, since each of the pl maps F^σ fixes all points $t_1^{-1}\big((-\infty, \beta]\big)$ we also have that $\widetilde{W}$ satsifies point *i*) of lemma 6.9. □

7 The equivalence $|\psi_d(N,1)_\bullet| \xrightarrow{\simeq} \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|$

7.1 The scanning map

In order to show that there is a weak equivalence

$$|\psi_d(N,1)_\bullet| \xrightarrow{\simeq} \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet| \tag{28}$$

we shall try to follow the same stategy that was used in [7] to reach the same result in the smooth category. Let us review first how the map (28) is defined in [7]. For $k \geq 1$ consider the map

$$\mathbb{R} \times \psi_d(N,k) \to \psi_d(N,k+1)$$

defined by $(t, W) \mapsto W + t \cdot e_{k+1}$ where $W + t \cdot e_{k+1}$ denotes the image of W under the diffeomorphism $x \mapsto x + t \cdot e_{k+1}$. If we identify the unit circle S^1 with the one point compactification $\mathbb{R} \cup \{\infty\}$ then the previous map can be extended to $S^1 \times \psi_d(N,k)$ by setting $(\infty, W) \mapsto \emptyset$ for any W in $\psi_d(N,k)$, and by allowing $\emptyset$ to be the base point in both $\psi_d(N,k)$ and $\psi_d(N,k+1)$ we obtain a map $S^1 \wedge \psi_d(N,k) \to \psi_d(N,k+1)$ with adjoint map

$$\psi_d(N,k) \xrightarrow{S_k} \Omega\psi_d(N,k+1). \tag{29}$$

The strategy of the proof in [7] is then to show that (29) is a weak homotopy equivalence for $k \geq 1$ and the map (28) is defined to be the composite

$$\Omega^{N-2} S_{N-1} \circ \ldots \circ S_1$$

which is obviously also a weak equivalence.

In the rest of this subsection we will define a pl version

$$|\psi_d(N,k)_\bullet| \stackrel{S_k}{\to} \Omega|\psi_d(N,k+1)_\bullet| \tag{30}$$

of the map (29), which will be called the *scanning map*, and in the subsections that follow we will give all the details of the proof of the main theorem of this section, namely, that (30) is a weak homotopy equivalence.

Remark. In order to make our arguments easier to follow we will in this section assume that the underlying polyhedron W of a p-simplex of $\psi_d(N,k)_\bullet$ is contained in

$$\Delta^p \times \mathbb{R}^k \times (0,1)^{N-k}.$$

Also, the loops in $\Omega|\psi_d(N,k+1)_\bullet|$ will be parametrized by the interval $[-1,1]$ and not by $[0,1]$.

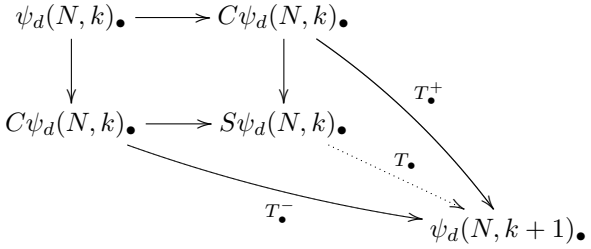
The scanning map (30) will be the adjoint of a map

$$T : [-1,1] \times |\psi_d(N,k)_\bullet| \to |\psi_d(N,k+1)_\bullet| \tag{31}$$

which is defined as follows: let $C\psi_d(N,k)_\bullet$ and $S\psi_d(N,k)_\bullet$ be respectively the cone and the unreduced suspension of $\psi_d(N,k)_\bullet$ (see definitions 2.69 and 2.70). The map (31) will be equal to a composite

$$[-1,1] \times |\psi_d(N,k)_\bullet| \stackrel{q}{\longrightarrow} S|\psi_d(N,k)_\bullet| \stackrel{\cong}{\longrightarrow} |S\psi_d(N,k)_\bullet| \stackrel{|T_\bullet|}{\longrightarrow} |\psi_d(N,k+1)_\bullet|$$

where the first map is the quotient map which collapses the bottom and top face of $[-1,1] \times |\psi_d(N,k)_\bullet|$, the second map is the natural homeomorphism $S|\psi_d(N,k)_\bullet| \stackrel{\cong}{\longrightarrow} |S\psi_d(N,k)_\bullet|$ and the third map is the geometric realization of a morphism of Δ-sets $T_\bullet : S\psi_d(N,k)_\bullet \to \psi_d(N,k+1)_\bullet$ obtained by applying the universal property of $S\psi_d(N,k)_\bullet$ to a diagram of the form



(32)

The upperscripts $+$ and $-$ in $T_\bullet^+$ and $T_\bullet^-$ mean that we are going to push elements of $\psi_d(N,k)_\bullet$ towards $+\infty$ and $-\infty$ respectively along the axis x_{k+1} of $\mathbb{R}^N$. Let us then define these two morphisms of Δ-sets. We will only describe how to construct $T_\bullet^+$ since the construction of $T_\bullet^-$ is done exactly the same way with some minor modifications.

For the construction of $T^+_\bullet$ we will denote by cW the extra $(p+1)$-simplex in $C\psi_d(N,k)_\bullet$ which corresponds to the p-simplex W of $\psi_d(N,k)_\bullet$ and we will denote by $*$ the extra 0-vertex of $C\psi_d(N,k)_\bullet$. Furthermore, we shall need the following:

i) Fix once and for all an increasing pl homeomorphism

$$f : [0,1) \to [0,\infty).$$

This is the map that we are going to use to push things to ∞.

ii) For each $p > 0$ let $\delta_p : \Delta^{p-1} \hookrightarrow \Delta^p$ be the inclusion into the p-th face of Δ^p, let e_p be the p-th vertex of Δ^p and let $\Delta^p_{\frac{1}{2}}$ be the convex compact subspace of Δ^p which consists of all points of the form

$$(1-t)\cdot\delta_p(\lambda) + t\cdot e_p$$

with $0 \leq t \leq \frac{1}{2}$.

iii) Let $\{h_p : \Delta^p \times [0,1] \to \Delta^{p+1}_{\frac{1}{2}}\}_p$ be a sequence of pl homeomorphisms with the following properties:

1. $h_p(\lambda, 0) = \delta_{p+1}(\lambda)$.
2. $h_p(\lambda, 1) = \frac{1}{2}\delta_{p+1}(\lambda) + \frac{1}{2}e_{p+1}$.
3. If $\Delta^q \overset{i}{\hookrightarrow} \Delta^p$ is an injective simplicial map which preserves the order relation on the set of vertices then the following diagram commutes

$$\begin{array}{ccc} \Delta^q \times [0,1] & \xrightarrow{h_q} & \Delta^{q+1}_{\frac{1}{2}} \\ \downarrow{\scriptstyle i\times \mathrm{Id}} & & \downarrow \\ \Delta^p \times [0,1] & \xrightarrow{h_p} & \Delta^{p+1}_{\frac{1}{2}} \end{array}$$

where the right vertical map is the restriction on $\Delta^{q+1}_{\frac{1}{2}}$ of the map defined by

$$(1-t)\cdot\lambda + t\cdot e_{q+1} \mapsto (1-t)\cdot i(\lambda) + t\cdot e_{p+1}.$$

Such a sequence of pl homeomorphisms can be defined by induction on p.

With these constructions we can now define for each p-simplex W of $\psi_d(N,k)_\bullet$ the $(p+1)$-simplex $\widetilde{W^+}$ of $\psi_d(N,k+1)_\bullet$ which will be equal to the image of cW under $T^+_\bullet$: first, from each p-simplex W of $\psi_d(N,k)_\bullet$ we can obtain an element

$$\widehat{W^+} \in \psi_d(N,k+1)\big([0,1]\times\Delta^p\big)$$

which is a concordance between W and $\emptyset$ by taking the image of the product $[0,1) \times W$ under the piecewise linear embedding

$$F : [0,1) \times \Delta^p \times \mathbb{R}^k \times (0,1)^{N-k} \to [0,1] \times \Delta^p \times \mathbb{R}^{k+1} \times (0,1)^{N-k-1}$$

defined by

$$(t, \lambda, x_1, \ldots, x_k, y_1, \ldots, y_{N-k}) \mapsto (t, \lambda, x_1, \ldots, x_k, y_1 + f(t), \ldots, y_{N-k}).$$

It is easy to verify that $\widehat{W}^+ := F\big([0,1) \times W\big)$ is a closed sub-polyhedron of $[0,1] \times \mathbb{R}^{k+1} \times (0,1)^{N-k-1}$ and that the projection $\widehat{W}^+ \to [0,1] \times \Delta^p$ is a piecewise linear submersion of codimension d. Finally, $\widetilde{W}^+$ is the $(p+1)$-simplex of $\psi_d(N, k+1)_\bullet$ obtained by first pulling back $\widehat{W}^+$ along the inverse $h_p^{-1} : \Delta^{p+1}_{\frac{1}{2}} \to [0,1] \times \Delta^p$ and by then taking the composite of the projection $\widetilde{W}^+ \to \Delta^{p+1}_{\frac{1}{2}}$ and the inclusion $\Delta^{p+1}_{\frac{1}{2}} \hookrightarrow \Delta^{p+1}$. Observe that this composite is indeed a piecewise linear submersion of codimension d since for any point α in $\Delta^{p+1}_{\frac{1}{2}}$ of the form $\alpha = \frac{1}{2}\delta_{p+1}(\lambda) + \frac{1}{2}e_{p+1}$ we have that the fiber $\widetilde{W}^+_\alpha$ is empty.

$T^+_\bullet : C\psi_d(N,k)_\bullet \to \psi_d(N,k+1)_\bullet$ is then the morphism of Δ-sets defined by

$$W \mapsto W, \quad cW \mapsto \widetilde{W}^+, \quad * \mapsto \emptyset.$$

Using propery 3) of the maps h_p defined in *iii)* above it is straightforward to verify that

$$T^+_p \partial_j = \partial_j T^+_p$$

for all $0 < p$ and all $0 \leq j \leq p$ and thus $T^+_\bullet$ is indeed a morphism of Δ-sets.

Furthermore, $T^-_\bullet : C\psi_d(N,k)_\bullet \to \psi_d(N,k+1)_\bullet$ is the morphism of Δ-sets defined by

$$W \mapsto W, \quad cW \mapsto \widetilde{W}^-, \quad * \mapsto \emptyset.$$

where the image $\widetilde{W}^-$ of $T^-_\bullet$ is defined in exactly the same way as $\widetilde{W}^+$ but using the pl homeomorphism

$$-f : [0,1) \to (-\infty, 0]$$

instead of $f : [0,1) \to [0,\infty)$. For both $T^+_\bullet$ and $T^-_\bullet$ we have that the composite map

$$\psi_d(N,k)_\bullet \hookrightarrow C\psi_d(N,k)_\bullet \xrightarrow{T^\pm_\bullet} \psi_d(N,k+1)_\bullet$$

is equal to the natural inclusion $\psi_d(N,k)_\bullet \hookrightarrow \psi_d(N,k+1)_\bullet$ and therefore $T^+_\bullet$ and $T^-_\bullet$ make the outer sub-diagram of (32) commute. By the universal property of $S\psi_d(N,k)_\bullet$ there exists a unique $T_\bullet$ making diagram (32) commute and finally we are now ready to give the following definition.

Definition 7.1. The *scanning map* is the adjoint map

$$\mathcal{S}_k : |\psi_d(N,k)_\bullet| \to \Omega|\psi_d(N,k+1)_\bullet|$$

of the composite

$$T : [-1,1] \times |\psi_d(N,k)_\bullet| \xrightarrow{q} S|\psi_d(N,k)_\bullet| \xrightarrow{\cong} |S\psi_d(N,k)_\bullet| \xrightarrow{|T_\bullet|} |\psi_d(N,k+1)_\bullet|.$$

The base point of the loops in $\Omega|\psi_d(N,k+1)_\bullet|$ is the vertex corresponding to the 0-simplex $\emptyset$. Also, observe that by construction we have for any x in $|\psi_d(N,k)_\bullet|$ that $T(\pm 1, x)$ is equal to the vertex of the 0-simplex $\emptyset$ and therefore the image of $\mathcal{S}_k$ lies entirely in $\Omega|\psi_d(N,k+1)_\bullet|$. From now on we will drop the subsript k from $\mathcal{S}_k$ and just denote the scanning map simply by $\mathcal{S}$. As we mentioned earlier the main result of this section is the following theorem.

Theorem 7.2. *The scanning map*

$$\mathcal{S} : |\psi_d(N,k)_\bullet| \to \Omega|\psi_d(N,k+1)_\bullet|$$

is a weak homotopy equivalence.

7.2 Decomposition of the scanning map

Instead of proving directly that the scanning map is a weak homotopy equivalence we are going to express it as the composite of three maps each of which is a weak homotopy equivalence.

Before we can give this decomposition we need to introduce the following sub-Δ-sets of $\psi_d(N,k)_\bullet$.

Definition 7.3.

1. $\psi_d^\emptyset(N,k)_\bullet$ will denote the path component of the vertex $\emptyset$ in $\psi_d(N,k)_\bullet$.
2. $\psi_d^0(N,k)_\bullet$ is the sub-Δ-set of $\psi_d(N,k)_\bullet$ whose set of p-simplices consists of those simplices $W \in \psi_d(N,k)_p$ for which there is a piecewise linear function $f : \Delta^p \to \mathbb{R}$ such that

$$W_\lambda \cap \left(\mathbb{R}^{k-1} \times \{f(\lambda)\} \times (0,1)^{N-k}\right) = \emptyset$$

for each $\lambda \in \Delta^p$.

Observe that $\psi_d^\emptyset(N,k)_\bullet$ is a Kan Δ-set. Also, it is easy to prove that each vertex of $\psi_d^0(N,k)_\bullet$ is concordant with the vertex $\emptyset$ and thus we have that $\psi_d^0(N,k)_\bullet$ is a sub-Δ-set of $\psi_d^\emptyset(N,k)_\bullet$.

We shall also need the following bi-Δ-set (see definition 5.4).

Definition 7.4. $N\psi_d(N,k)_{\bullet,\bullet}$ is the bi-Δ-set whose set of (p,q)-simplices consists of all tuples $(W, f_0, \ldots, f_q)$ where W is a p-simplex of $\psi_d(N,k+1)_\bullet$ and where the f_i's are piecewise linear maps $\Delta^p \to \mathbb{R}$ such that

$$f_0(\lambda) < \ldots < f_q(\lambda) \tag{33}$$

for all $\lambda \in \Delta^p$ and such that

$$W_\lambda \cap \big(\mathbb{R}^k \times \{f_j(\lambda)\} \times (0,1)^{N-k-1}\big) = \emptyset \tag{34}$$

for each j in $\{0, \ldots, q\}$ and for each $\lambda \in \Delta^p$.

The structure maps in the p-direction are the ones coming from the Δ-set structure of $\psi_d(N, k+1)_\bullet$. In the q-direction the j-th face map

$$\partial_j : N\psi_d(N,k)_{p,q+1} \to N\psi_d(N,k)_{p,q}$$

is given by deleting the term f_j.

Observe that for each non-negative integer q there is a functor $N_q\psi_d(N,k) : \mathbf{PL}^{op} \to \mathbf{Sets}$ which sends $[p]$ to the set of all tuples $(W, f_0 < \ldots < f_q)$ such that $W \in \psi_d(N, k+1)\big(P\big)$ and $f_0, \ldots, f_q$ satisfy (34) and which sends a pl map f to the pull-back function f^*. The Δ-set $N\psi_d(N,k)_{\bullet,q}$ is then just the restriction of this functor on the category Δ^{op} once we identify Δ with the image of its canonical embedding in $\mathbf{PL}$. Observe also that there is a forgetful map

$$\|N\psi_d(N,k)_{\bullet,\bullet}\| \to |\psi^0_d(N,k+1)_\bullet| \tag{35}$$

obtained by forgetting all the functions f_j.

We claim that the scanning map defined in 7.1 is homotopic to a composite map of the form

$$|\psi_d(N,k)_\bullet| \to \Omega\,\|N\psi_d(N,k)_{\bullet,\bullet}\| \to \Omega|\psi^0_d(N,k+1)_\bullet| \to \Omega|\psi^\emptyset_d(N,k+1)_\bullet| \tag{36}$$

where base point of the loops in $\Omega\,\|N\psi_d(N,k)_\bullet\|$ is the vertex which corresponds to the $(0,0)$-simplex $(\emptyset, 0)$. The last map in this composition is the one induced by the inclusion $\psi^0_d(N,k+1)_\bullet \hookrightarrow \psi^\emptyset_d(N,k+1)_\bullet$ and the second map is the one induced by the forgetful map (35). The first map will be defined in the following subsection. The aim now is to show that each map in the composite map (36) is a weak equivalence and that this composite is homotopic to the scanning map. We start with the following proposition.

Proposition 7.5. *The forgetful map $\|N\psi_d(N,k)_{\bullet,\bullet}\| \to |\psi^0_d(N,k+1)_\bullet|$ is a weak homotopy equivalence. Consequently, the second map in (36) is a weak homotopy equivalence.*

Proof. This proof is identical to the one given in proposition 5.18 and we refer the reader to that proof for more details. □

The rest of this section is organized as follows: In 7.3 we will introduce the first map in (36) and show that it is a weak homotopy equivalence using a *group-completion argument* which is also used in [7]. In 7.4 we prove that the inclusion $\psi^0_d(N,k)_\bullet \hookrightarrow \psi^\emptyset_d(N,k)_\bullet$ is a weak homotopy equivalence when $k > 1$. Finally, in 7.5 we show that the scanning map $|\psi_d(N,k)_\bullet| \to \Omega|\psi_d(N,k+1)_\bullet|$ is homotopic to the composite map (36) which concludes the proof of theorem 7.2.

7.3 The group-completion argument

We start this section by collecting in proposition 7.6 several properties of the bi-Δ-set $N\psi_d(N,k)_{\bullet,\bullet}$. Before we state this proposition we introduce the following notation: if W is a p-simplex of $\psi_d(N,k+1)_\bullet$ and $f:\Delta^p\to\mathbb{R}$ is a piecewise linear function we will denote by

$$W+f$$

the image of W under the piecewise linear automorphism of $\Delta^p\times\mathbb{R}^N$ defined by

$$(\lambda,x_1,\dots,x_{k+1},\dots,x_N)\mapsto(\lambda,x_1,\dots,x_{k+1}+f(\lambda),\dots,x_N).$$

It is clear that $W+f$ is again a p-simplex of $\psi_d(N,k+1)_\bullet$. Also, for any point a in $\mathbb{R}$ we will denote by c_a the constant map $\Delta^p\to\mathbb{R}$ which sends every point to a.

Proposition 7.6. *1. For $q>0$ the morphism of Δ-sets*

$$\eta^q_\bullet:\underbrace{\psi_d(N,k)_\bullet\times\ldots\times\psi_d(N,k)_\bullet}_{q}\to N\psi_d(N,k)_{\bullet,q}$$

which sends a q-tuple $(W_1,\dots,W_q)$ to

$$\big(\coprod_{j=1}^{q}(W_j+c_{j-1}),c_0,\dots,c_q\big),$$

is a weak homotopy equivalence.

2. *If $\emptyset_\bullet$ is the sub-Δ-set of $N\psi_d(N,k)_{\bullet,0}$ whose unique p-simplex is the tuple $(\emptyset,c_0)$ then the inclusion $\emptyset_\bullet\hookrightarrow N\psi_d(N,k)_{\bullet,0}$ is a weak homotopy equivalence. In particular $|N\psi_d(N,k)_{\bullet,0}|$ is contractible.*

3. *If $\beta_j:[1]\to[q]$ is the morphism in Δ defined by $\beta_j(0)=j-1$ and $\beta_j(1)=j$ then the morphism*

$$B^q:=(\beta_1^*,\dots,\beta_q^*):N\psi_d(N,k)_{\bullet,q}\to\underbrace{N\psi_d(N,k)_{\bullet,1}\times\ldots\times N\psi_d(N,k)_{\bullet,1}}_{q}$$

is a weak homotopy equivalence.

4. *The function obtained by composing the bijection*

$$\pi_0\big(N\psi_d(N,k)_{\bullet,1}\big)\times\pi_0\big(N\psi_d(N,k)_{\bullet,1}\big)\overset{\cong}{\to}\pi_0\big(\psi_d(N,k)_\bullet\big)\times\pi_0\big(\psi_d(N,k)_\bullet\big)$$

obtained from 1) and the function between path components induced by the composition

$$\big(\psi_d(N,k)_\bullet\times\psi_d(N,k)_\bullet\big)\overset{\eta^2_\bullet}{\to}N\psi_d(N,k)_{\bullet,2}\overset{\partial_{\bullet,1}}{\to}N\psi_d(N,k)_{\bullet,1}$$

is a product with respect to which $\pi_0\big(N\psi_d(N,k)_{\bullet,1}\big)$ is a group.

Proof. In order to make this proof easier to follow we are going to switch the roles of the coordinates x_1 and x_{k+1}. To prove 1) observe first that both $\psi_d(N,k)_\bullet \times \ldots \times \psi_d(N,k)_\bullet$ and $N\psi_d(N,k)_{\bullet,q}$ are Kan Δ-sets and hence in order to show that the morphism $\eta^q_\bullet$ given in the statement is a weak equivalence it suffices to show that for each commutative diagram of the form

$$\begin{array}{ccc} |\partial\Delta^p| & \xrightarrow{|g'_\bullet|} & |\psi_d(N,k)^q_\bullet| \\ \downarrow & & \downarrow{\scriptstyle |\eta^q_\bullet|} \\ |\Delta^p| & \xrightarrow{|g_\bullet|} & |N\psi_d(N,q)_{\bullet,q}| \end{array} \tag{37}$$

there exists homotopies $h_s : |\Delta^p| \to |N\psi_d(N,q)_{\bullet,q}|$ and $f_s : |\partial\Delta^p| \to |\psi_d(N,k)^q_\bullet|$, $s \in [0,1]$, with $|\eta^q_\bullet| \circ f_s = h_s||\partial\Delta^p|$ such that $f_0 = |g'_\bullet|$, $h_0 = |g_\bullet|$ and such that $f_1 : |\partial\Delta^p| \to |\psi_d(N,k)^q_\bullet|$ extends to a map $F : |\Delta^p| \to |\psi_d(N,k)^q_\bullet|$ such that $|\eta^q_\bullet| \circ F = h_1$ (see [5], exercise 1 in page 6). Consider then a diagram of the form (37) and let $(W, f_0, \ldots, f_q)$ be the p-simplex of $|N\psi_d(N,k)_{\bullet,q}|$ classified by $g_\bullet$. By assumption we have that the map f_j agrees with c_j on $\partial\Delta^p$. Using the simplicial approximation theorem (see [1]) we can find for each $0 \leq j \leq q$ a piecewise linear homotopy

$$H^j_t : \Delta^p \to \mathbb{R}$$

between f_j and c_j which agrees with the constant homotopy on $\partial\Delta^p$ and such that $H^i_t(\lambda) < H^j_t(\lambda)$ for all values $t \in [0,1]$ and $\lambda \in \Delta^p$ whenever $i < j$. Observe that the map

$$[0,1] \times \left(\Delta^p \times \{0,\ldots,q\}\right) \to [0,1] \times \Delta^p \times \mathbb{R} \tag{38}$$

defined by

$$(t,\lambda,j) \mapsto (t,\lambda,H^j_t(\lambda))$$

is actually a locally flat embedding since we can find a small enough ϵ such that (38) can be extended to a map

$$[0,1] \times \left(\Delta^p \times [-\epsilon,\epsilon] \times \{0,\ldots,q\}\right) \to [0,1] \times \Delta^p \times \mathbb{R} \tag{39}$$

by setting

$$(t,\lambda,s,j) \mapsto (t,\lambda,H^j_t(\lambda)+s).$$

The map (39) should be viewed as an isotopy of embeddings of q disjoint copies of $[-\epsilon,\epsilon]$ into $\mathbb{R}$ parametrized by $[0,1] \times \Delta^p$. By the isotopy extension theorem (see [12] and [13]) we can find a piecewise linear automorphism

$$F : [0,1] \times \Delta^p \times \mathbb{R} \to [0,1] \times \Delta^p \times \mathbb{R}$$

which commutes with the projection onto $[0,1] \times \Delta^p$, which is the identity on $\Delta^p \times \mathbb{R}$ at time $t = 0$ and such that $F(t,\lambda,f_j(\lambda)) = (t,\lambda,H^j_t(\lambda))$. Let $\tilde{F}$ denote the product $F \times \mathrm{Id}_{\mathbb{R}^{N-1}}$ and let $\widetilde{W}$ denote the image of $[0,1] \times W$ under $\tilde{F}$. $(\widetilde{W}, H^0, \ldots, H^q)$ is then a concordance between $(W, f_0, \ldots, f_q)$ and a p-simplex of $N\psi_d(N,k)_{\bullet,q}$ of the form $(W', c_0, \ldots, c_q)$. Using lemma 5.14

we can produce a homotopy $h : [0,1] \times \Delta^p \to |N\psi_d(N,k)_{\bullet,q}|$ between $|g_\bullet|$ and the geometric realization of the morphism $g'_\bullet$ which classifies the simplex $(W', c_0, \ldots, c_q)$. Furthermore by the way this homotopy is defined we also have by 5.14 that $h_t(\partial\Delta^p)$ is contained in $\mathrm{Im}|\eta^q_\bullet|$ for all t in $[0,1]$. Finally, since W' does not intersect $x_1^{-1}(0)$ nor $x_1^{-1}(q)$ we can push to infinity those parts of W' below $x_1^{-1}(0)$ and above $x_1^{-1}(q)$ to obtain a concordance between $(W', c_0, \ldots, c_q)$ and a p-simplex $(W'', c_0, \ldots, c_q)$ which lies in the image of $\eta_\bullet$. By applying again 5.14 we can produce a homotopy, relative to $\mathrm{Im}|\eta^q_\bullet|$, between h_1 and a map which maps Δ^p to $\mathrm{Im}|\eta^q_\bullet|$. This concludes the proof of 1). Also, observe that the arguments used in this proof can be used to show that any diagram

$$\begin{array}{ccc} |\partial\Delta^p| & \longrightarrow & |\emptyset_\bullet| \\ \downarrow & & \downarrow \\ |\Delta^p| & \longrightarrow & |N\psi_d(N,q)_{\bullet,0}| \end{array}$$

represents the trivial class in $\pi_p\big(|N\psi_d(N,k)_{\bullet,0}|, |\emptyset_\bullet|\big)$. This proves claim 2).

Point 3) is basically a consequence of 1), of the techniques used to prove 1) and of proposition 2.65. Indeed, let

$$N\psi_d(N,k)_{\bullet,1} \times \ldots \times N\psi_d(N,k)_{\bullet,1} \xrightarrow{p_j} N\psi_d(N,k)_{\bullet,1}$$

be the projection onto the jth factor and let

$$|N\psi_d(N,k)_{\bullet,1} \times \ldots \times N\psi_d(N,k)_{\bullet,1}| \xrightarrow{p} |N\psi_d(N,k)_{\bullet,1}| \times \ldots \times |N\psi_d(N,k)_{\bullet,1}|$$

be equal to the map $p = (|p_1|, \ldots, |p_q|)$. By 2.65 we have that p is a weak homotopy equivalence. Using the techniques that we used to prove 1) we can show that

$$p \circ |B^q| \circ |\eta^q|$$

is homotopic to

$$|\eta^1| \times \ldots \times |\eta^1|,$$

and since this map is a weak homotopy equivalence we have that $p \circ |B^q| \circ |\eta^q|$ is also a weak equivalence which implies finally that B^q is a weak homotopy equivalence.

In order to prove 4) we are first going to introduce a little bit of notation: for any interval (a,b) we are going to denote by $\psi_d^{(a,b)}(N,k+1)_\bullet$ the sub-Δ-set of $\psi_d(N,k+1)_\bullet$ whose set of p-simplices consists of all those simplices whose underlying polyhedron is strictly contained in $\Delta^p \times (a,b) \times \mathbb{R}^k \times (0,1)^{N-k-1}$. It is clear that any vertex of $N\psi_d(N,k)_{\bullet,1}$ is path connected to a vertex (W,a,b) with $W \in \psi_d^{(a,b)}(N,k)_0$. Let us denote by $\times$ the product defined in the statement of

4). If we take two vertices (W_1, a_1, b_1) and (W_2, a_2, b_2) with $W_i \in \psi_d^{(a_i,b_i)}(N, k+1)_0$ for $i = 1, 2$ and if we evaluate the product $\times$ on the tuple

$$\Big(\big[(W_1, a_1, b_1)\big], \big[(W_2, a_2, b_2)\big]\Big)$$

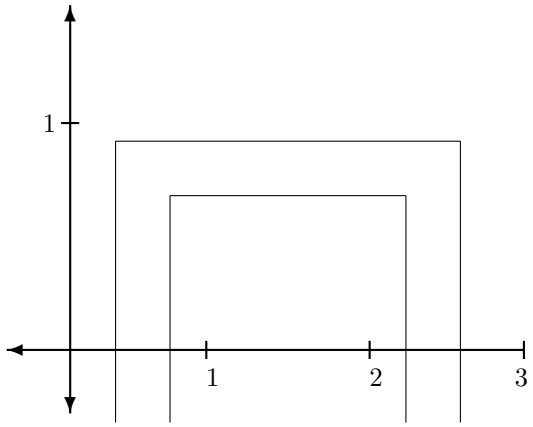
of the corresponding path components we obtain the path component

$$\big[(W_1 \cup (W_2 + b_1 - a_2), a_1, b_1 + (b_2 - a_2))\big].$$

From this we have immedeately that $\times$ is associative. Furthermore, it is clear that any vertex (W, a, b) with $W \in \psi_d^{(a,b)}(N, k+1)_\bullet$ is path connected to $(W, a - \lambda_1, b + \lambda_2)$ where λ_1, λ_2 are any non-negative constants and hence the path component which contains all the vertices of the form $(\emptyset, a, b)$ is the identity element with respect to $\times$.

It remains to show that every path component of $N\psi_d(N, k)_{\bullet,1}$ has an inverse with respect to $\times$. In order to this we again reeverse the roles of x_1 and x_{k+1}. Pick then a vertex (W, a, b). Assume without loss of generality that $a = 0$ and $b = 1$ and that $W \in \psi_d^{(0,1)}(N, k+1)_0$. The projection $p : W \to \mathbb{R}^k$ onto the first k euclidean factors is a proper pl map and by lemma 2.47 we can find a regular value $a \in \mathbb{R}^k$ of the projection $p : W \to \mathbb{R}^k$ in the sense of definiton 2.46. If N denotes the pre-image $p^{-1}(a)$, which we can view as a pl submanifold of $(0, 1)^{N-k}$, then by applying lemmas 7.15 and 3.19 we can show that $(W, 0, 1)$ is concordant to $(\mathbb{R}^k \times N, 0, 1)$ (see the last paragraph of this proof) and thus we can assume from now on that $(W, 0, 1)$ is of the form $(\mathbb{R}^k \times N, 0, 1)$.

Pick a pl embedding $e : \mathbb{R} \times (0, 1) \to \mathbb{R} \times (0, 3)$ whose image is equal to the one illustrated in the following figure:



Let $E : \mathbb{R}^N \to \mathbb{R}^N$ denote the product of maps $\mathrm{Id}_{\mathbb{R}^{k-1}} \times e \times \mathrm{Id}_{\mathbb{R}^{N-k-1}}$ and consider the image $E(W)$. This image is a vertex of $\psi_d^{(0,3)}(N, k+1)_\bullet$ which is concordant to $\emptyset$ since it is empty at all heights $x_{k+1} > 1$ and hence it can be pushed to $x_{k+1} = -\infty$. On the other hand, since W is of the form $\mathbb{R}^k \times N$ we obtain a concordance between $E(W)$ and a vertex of the form $W \coprod W'$ where $W' \in \psi_d^{(1,3)}(N, k+1)_0$ by pushing $E(W)$ to $x_{k+1} = \infty$. This implies that the product

$$\big[(W, 0, 1)\big] \times \big[(W', 1, 3)\big]$$

is equal to

$$\big[(\emptyset, 0, 3)\big]$$

and thus the element $\big[(W, 0, 1)\big]$ has an inverse with respect to $\times$.

It remains to justify why the vertex W is concordant to $\mathbb{R}^k \times N$. Recall that N is the pre-image of $a \in \mathbb{R}^k$ under the projection $p : W \to \mathbb{R}^k$. Assume without loss of generality that $a = 0$. By lemma 7.15, which we will state and proof in § 6.4, we have that W is concordant to another vertex W' which agrees with $\mathbb{R}^k \times N$ in $(-\delta, \delta)^k \times (0,1)^{N-k}$ for some $\delta > 0$. Let $e : [0,1] \times \mathbb{R}^k \to [0,1] \times \mathbb{R}^k$ be an open piecewise linear embedding which commutes with the projection onto $[0,1]$, which is the identity at time $t = 0$ and which maps $\mathbb{R}^k$ onto $(-\delta, \delta)^k$ at time $t = 1$. Then, by applying lemma 3.19 to the embedding $e \times \mathrm{Id}_{(0,1)^{N-k}}$ and to the constant concordance $[0,1] \times W'$ we obtain a concordance between W' and $\mathbb{R}^k \times N$.

□

The following proposition is the main tool that we are going to use in order to prove that the first map in (36) is a weak homotopy equivalence (compare with statement of lemma 3.14 in [7]).

Proposition 7.7. *Let $X_\bullet$ be a Δ-space such that the face maps induce a homotopy equivalence $X_k \simeq X_1 \times \ldots \times X_1$. (when $k = 0$ this means that X_0 is contractible). Then the natural map*

$$X_1 \to \Omega_{X_0} |X_\bullet|$$

is a homotopy equivalence if and only if $X_\bullet$ is group-like, i.e. $\pi_0(X_1)$ is a group with respect to the product induced by $d_1 : X_2 \to X_1$.

Remark 7.8. In this statement $\Omega_{X_0}|X_\bullet|$ denotes the space of paths $[-1,1] \to |X_\bullet|$ which map -1 and 1 to $X_0 \subseteq |X_\bullet|$. A version of this result for *simplicial spaces* is given in [19]. Although proposition 7.7 is about Δ-spaces, the proof of proposition 1.5 in [19] goes through when we work with Δ-spaces instead of simplicial spaces. The reader is referred to the proof given in [19] for more details.

If we apply the geometric realization functor to the external simplicial direction of $N\psi_d(N,k)_{\bullet,\bullet}$ we obtain a Δ-space, namely, the Δ-space whose space of k-simplices is equal to $|N\psi_d(N,k)_{\bullet,k}|$. In order to be able to apply proposition 7.7 we need to prove first the following lemma.

Lemma 7.9. *The Δ-space $[k] \mapsto |N\psi_d(N,k)_{\bullet,k}|$ satisfies the assumptions stated in proposition 7.7.*

Proof. In this proof we shall denote $N\psi_d(N,k)_{\bullet,k}$ simply by N_k and the Δ-space $[k] \mapsto |N\psi_d(N,k)_{\bullet,k}|$ shall simply be denoted by $|N_\bullet|$. The contractability of $|N_0|$ follows from point 2) of 7.6 and the fact that $|N_\bullet|$ is group-like follows from point 4) of 7.6.

For each $k \geq 1$ the map $|N_k| \stackrel{\simeq}{\to} |N_1| \times \ldots \times |N_1|$ induced by the face maps of $|N_\bullet|$ is equal to the diagonal map in the following diagram

$$\begin{array}{ccc} |N_k| & \xrightarrow{|B^k|} & |N_1 \times \ldots \times N_1| \\ & \searrow & \downarrow \\ & & |N_1| \times \ldots \times |N_1|, \end{array}$$

where the horizontal map B^k is the one defined in point 2) of 7.6 and the j-th component of the vertical map is just the geometric realization of the morphism p_j which projects the product onto the j-th component. Since B^k is a weak homotopy equivalence by proposition 7.6 and since the vertical map is a weak homotopy equivalence by 2.65 we have have that the diagonal map is also a weak homtopy equivalence.

□

The following composition

$$h_1' : |\psi_d(N,k)_\bullet| \stackrel{|\eta^1|}{\to} |N\psi_d(N,k)_{\bullet,1}| \to \Omega_{|N_0|} \|N\psi_d(N,k)_{\bullet,\bullet}\|, \qquad (40)$$

where the second map is the adjoint of the natural map $[-1,1]\times|N\psi_d(N,k)_{\bullet,1}| \to \|N\psi_d(N,k)_{\bullet,\bullet}\|$, is an auxiliary map that we are going to use to define the first map in the composition (36). We remind the reader that $|N_0|$ stands for the geometric realization of $N\psi_d(N,k)_{\bullet,0}$. Applyig 1) of 7.6 and proposition 7.7 we have that h_1' is a weak homotopy equivalence.

Using h_1' we will now define the first map in (36). Let T^- and T^+ denote the restrictions of the adjoint

$$T : [-1,1] \times |\psi_d(N,k)_\bullet| \to |\psi_d(N,k+1)_\bullet|$$

of the scanning map on $[-1,0]\times|\psi_d(N,k)_\bullet|$ and $[0,1]\times|\psi_d(N,k)_\bullet|$ respectively, and let $\psi_d^{0,0}(N,k+1)_\bullet$ and $\psi_d^{0,1}(N,k+1)_\bullet$ be the sub-Δ-sets of $\psi_d^0(N,k+1)_\bullet$ which consists of those simplices $W \in \psi_d(N,k+1)_p$ such that

$$W_\lambda \cap \mathbb{R}^k \times \{0\} \times (0,1)^{N-k-1} = \emptyset$$

and

$$W_\lambda \cap \mathbb{R}^k \times \{1\} \times (0,1)^{N-k-1} = \emptyset$$

rescpectively for each $\lambda \in \Delta^p$. Observe that both of these sub-Δ-sets embed into $N\psi_d(N, k+1)_{\bullet,0}$ by setting $W \mapsto (W, 0)$ and $W \mapsto (W, 1)$ respectively. By a slight abuse of notation, the image of a point x under the geometric realization of these embeddings will be denoted by $(x, 0)$ and $(x, 1)$ respectively. (This is a notation that we are going to use in equation (42) below). We remark that the image of T^- and T^+ land entirely in $|\psi_d^{0,1}(N, k+1)|$ and $|\psi_d^{0,0}(N, k+1)_\bullet|$ respectively.

It is easy to verify that the adjoint $g_1' : [-1, 1] \times |\psi_d(N, k)_\bullet| \to \|N\psi_d(N, k)_{\bullet,\bullet}\|$ of the map h_1' given in (40) is homotopic to the map

$$\tilde{g}_1 : [-1, 1] \times |\psi_d(N, k)_\bullet| \to \|N\psi_d(N, k)_{\bullet,\bullet}\|$$

defined by

$$\tilde{g}_1(t, x) = \begin{cases} \big(T^-(2t+1, x), 1\big) & \text{if } t \in [-1, \frac{-1}{2}] \\ g_1'(2t, x) & \text{if } t \in [-\frac{1}{2}, \frac{1}{2}], \\ \big(T^+(2t-1, x), 0\big) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{41}$$

Also, if

$$c : \Delta^1 \cong [0, 1] \to |N\psi_d(N, k)_{\bullet,0}|$$

is the characteristic map of the the 1-simplex $(\emptyset, i : [0, 1] \hookrightarrow \mathbb{R})$ with i the natural inclusion of $[0, 1]$ into $\mathbb{R}$ then it is easy to verify that $\tilde{g}_1$ is homotopic to the map

$$g_1 : [-1, 1] \times |\psi_d(N, k)_\bullet| \to \|N\psi_d(N, k)_{\bullet,\bullet}\|$$

defined by

$$g_1(t, x) = \begin{cases} c(1+t) & \text{if } t \in [-1, 0] \\ \tilde{g}_1'(-1+2t, x) & \text{if } t \in [0, 1] \end{cases} \tag{42}$$

Let

$$h_1 : |\psi_d(N, k)_\bullet| \to \Omega_{|N_0|} \|N\psi_d(N, k)_{\bullet,\bullet}\|$$

be the adjoint of the map g_1. Since g_1 and g_1' are homotopic it follows that h_1 and h_1' are also homotopic and thus h_1 is a weak equivalence as a map into $\Omega_{|N_0|} \|N\psi_d(N, k)_{\bullet,\bullet}\|$. But since the image of h_1 lies in $\Omega \|N\psi_d(N, k)_{\bullet,\bullet}\|$, where loops are based at the vertex corresponding to the tuple $(\emptyset, 0)$, we have that the diagram

$$\begin{array}{ccc} |\psi_d(N, k)_\bullet| & \xrightarrow{h_1} & \Omega_{|N_0|} \|N\psi_d(N, k)_{\bullet,\bullet}\| \\ & \searrow^{h_1} & \uparrow \\ & & \Omega \|N\psi_d(N, k)_{\bullet,\bullet}\| \end{array}$$

is commutative. Furthermore, by the contractability of $|N_0|$ we have that the inclusion $\Omega \|N\psi_d(N, k)_{\bullet,\bullet}\| \hookrightarrow \Omega_{|N_0|} \|N\psi_d(N, k)_{\bullet,\bullet}\|$ is a weak homotopy equivalence which implies finally that the map

$$h_1 : |\psi_d(N, k)_\bullet| \to \Omega \|N\psi_d(N, k)_{\bullet,\bullet}\| \tag{43}$$

i.e. h_1 as a map into $\Omega\,\|N\psi_d(N,k)_\bullet{}_\bullet\|$, is a weak homotopy equivalence. This is the first map in the decomposition (36).

7.4 The inclusion $\psi_d^0(N,k)_\bullet \hookrightarrow \psi_d^\emptyset(N,k)_\bullet$

It remains to show that the last map of the decompostion (36) is a weak homotopy equivalence. This will follow from the following proposition.

Proposition 7.10. *Assume that $N-d \geq 3$. Then the inclusion $\psi_d^0(N,k)_\bullet \hookrightarrow \psi_d^\emptyset(N,k)_\bullet$ is a weak homotopy equivalence when $k>1$.*

Note 7.11. 1. In this subsection we shall work with the norm $\|\cdot\|$ in $\mathbb{R}^k$ defined by

$$\|(x_1,\ldots,x_k)\| = \max\{|x_1|,\ldots,|x_k|\}.$$

Furthermore, $B(a,\delta)$ shall denote the closed ball centered at $a\in\mathbb{R}^k$ with radius $\delta>0$.

2. Also, in this subsection we will exchange again the rolles of the coordinates x_k and x_1. In particular, a p-simplex W of $\psi_d(N,k)_\bullet$ is in $\psi_d^0(N,k)_p$ if there is a piecewise linear function $f:\Delta^p\to\mathbb{R}$ such that for each λ in Δ^p we have that

$$W_\lambda \cap \big(\{f(\lambda)\}\times\mathbb{R}^{k-1}\times(0,1)^{N-k}\big).$$

Notation 7.12. Let W be an element in $\psi_d(N,k)\big(P\big)$. In particular, W is a closed subpolyhedron of $P\times\mathbb{R}^k\times(0,1)^{N-k}$. In this section we are going to use the following notation:

1. $x_k : W\to\mathbb{R}^k$ is the projection from W onto $\mathbb{R}^k$, i.e., the second factor of $P\times\mathbb{R}^k\times(0,1)^{N-k}$.
2. $t_k : P\times\mathbb{R}^k\times(0,1)^{N-k}\to\mathbb{R}^k$ is the projection onto $\mathbb{R}^k$.
3. $t_1 : P\times\mathbb{R}^k\times(0,1)^{N-k}\to\mathbb{R}$ is the projection onto the first component of $\mathbb{R}^k$.

The proof of proposition 7.10 is almost identical to that of theorem 6.1 once we have the following lemma at hand.

Lemma 7.13. *Assume that $N-d\geq 3$ and let $k>1$. Let P be a compact polyhedron, let β be some fixed real constant and let W be an element in $\psi_d^\emptyset(N,k)\big(P\big)$. Then there is an element $\widetilde{W}$ in $\psi_d^\emptyset(N,k)\big([0,1]\times P\big)$ which satisfies the following properties:*

i) $\widetilde{W}$ agrees with $[0,1]\times W$ in $t_1^{-1}\big((-\infty,\beta]\big)$.

ii) $\widetilde{W}$ is a concordance between W and an element W' in $\psi_d^\emptyset(N,k)\big(P\big)$ for which there is a finite open cover $U_1,\ldots,U_q$ of P and real values $a_1,\ldots,a_q>\beta$ such that for $j=1,\ldots,q$ we have

$$W'_\lambda \cap \big(\{a_j\} \times \mathbb{R}^{N-1}\big)$$

for each $\lambda \in U_j$.

Proof of proposition 7.10. Let us give a proof of 7.10 assuming lemma 7.13. Consider a map of pairs

$$f : (\Delta^p, \partial\Delta^p) \to \big(\big|\psi^\emptyset_d(N,k)_\bullet\big| , \big|\psi^0_d(N,k)_\bullet\big| \big). \tag{44}$$

By an argument completely analogous to the one given in the proof of lemma 6.2 we have that f is homotopic (as a map of pairs) to a composition of the form

$$(\Delta^p, \partial\Delta^p) \xrightarrow{f'} (|K_\bullet|, |K'_\bullet|) \xrightarrow{|h_\bullet|} \big(\big|\psi^\emptyset_d(N,k)_\bullet\big| , \big|\psi^0_d(N,k)_\bullet\big|\big)$$

where $(K_\bullet, K'_\bullet)$ is a pair of finite Δ-sets obtained from a pair of finite ordered simplicial complexes (K, K') in some Euclidean space $\mathbb{R}^m$.

Let W be the element of $\psi^\emptyset_d(N,k)\big(|K|\big)$ classified by the morhism $h_\bullet$ (see definition 3.6). Since $h_\bullet(K'_\bullet) \subseteq \psi^0_d(N,k)_\bullet$ then for each simplex σ of K' there is a piecewise linear function $f_\sigma : \sigma \to \mathbb{R}$ such that

$$W_\sigma \cap (\Gamma(f_\sigma) \times \mathbb{R}^{k-1} \times (0,1)^{N-k}) = \emptyset.$$

Fix some real constant β larger than all the maxima of the functions f_σ. By applying lemma 7.13 to W and β we can find an element $\widetilde{W}$ in $\psi^\emptyset_d(N,k)\big([0,1] \times |K|\big)$ which agrees with $[0,1] \times W$ in $t_1^{-1}\big((-\infty, \beta]\big)$ and which is a concordance between W and an element W' in $\psi^\emptyset_d(N,k)\big(|K|\big)$ for which there is a finite open cover $U_1, \ldots, U_q$ of $|K|$ such that for $j = 1, \ldots, q$ there is a real value $a_j > \beta$ such that

$$W'_\lambda \cap \big(\{a_j\} \times \mathbb{R}^{N-1}\big) = \emptyset$$

for each $\lambda \in U_j$. Finally, by arguments completely analogous to the ones given in steps *iv*), *v*) and *vi*) of the proof of theorem 6.1 we can conclude that the map f in (44) represents the trivial class in $\pi_p\big(\big|\psi^\emptyset_d(N,k)_\bullet\big| , \big|\psi^0_d(N,k)_\bullet\big| \big)$ and since f was arbitrary we conclude that $\psi^0_d(N,k)_\bullet \hookrightarrow \psi^\emptyset_d(N,k)_\bullet$ is a weak homotopy equivalence.

□

Lemma 7.13 is in turn an immediate corollary of the following lemma whose statement is similar to that of lemma 6.9.

Lemma 7.14. *Let M^m be a pl manifold, possibly with boundary, let P be a compact subpolyhedron of M such that $P \subseteq M - \partial M$, let β be some fixed real constant and let W be an element in $\psi^\emptyset_d(N,k)\big(M\big)$. Then there is an element $\widetilde{W}$ in $\psi^\emptyset_d(N,k)\big([0,1] \times M\big)$ and finitely many open sets $V_1, \ldots, V_q$ in $M - \partial M$ which satisfy the following:*

i) $\widetilde{W}$ agrees with $[0,1] \times W$ in $t_1^{-1}\big((-\infty, \beta]\big)$.

ii) $V_1, \ldots, V_q$ *cover* P.

ii) $\widetilde{W}$ *is a concordance between* W *and an element* W' *in* $\psi_d^{\emptyset}(N,k)\big(M\big)$ *such that for each of the open sets* $V_1, \ldots, V_q$ *there is a real value* $a_j > \beta$ *such that*

$$W'_\lambda \cap \big(\{a_j\} \times \mathbb{R}^{N-1}\big)$$

for each $\lambda \in V_j$.

Proof of lemma 7.13. Assuming lemma 7.14, the proof of lemma 7.13 is identical to the proof of lemma 6.5 and the reader is referred to that proof for more details. □

For the proof of 7.14 we shall need lemmas 7.15, 7.16, 7.17, 7.18 and 7.19 given below. The reader is advised to skip the proofs of 7.15, 7.16, 7.17 and 7.18 at a first reading and instead just read carefully their statements and then read the proof of lemma 7.19, which is the main tool we are going to use to prove lemma 7.14.

Lemma 7.15. *Let* W *be an element of* $\psi_d(N,k)\big([-1,1]^m\big)$ *and let* $\pi : W \to [-1,1]^m$ *be the projection onto* $[-1,1]^m$. *Let* $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ *and* $\delta' > 0$ *be such that the restriction*

$$(\pi, x_k)|_{x_k^{-1}\left(B(a,\delta')\right)} : x_k^{-1}\big(B(a,\delta')\big) \to [-1,1]^m \times B(a,\delta')$$

is a pl submersion of codimension $d-k$. *Finally, let* $M := (\pi, x_k)^{-1}\big((0,a)\big)$. *Then for any* $0 < \epsilon < 1$ *and any* $0 < \delta < \delta'$ *there is an element* $\widetilde{W}$ *in* $\psi_d(N,k)\big([0,1] \times [-1,1]^m\big)$ *which satisfies the following conditions:*

i) $i_0^*\widetilde{W} = W$ *where* i_0 *is the inclusion* $[-1,1]^m \hookrightarrow [0,1] \times [-1,1]^m$ *defined by* $\lambda \mapsto (0,\lambda)$

ii) $\widetilde{W}_{[0,1]\times V} = \big([0,1] \times W\big)_{[0,1]\times V}$ *for a neighborhood* V *of* $\partial[-1,1]^m$.

iii) *For any point* (t,λ) *in* $[0,1] \times [-1,1]^m$ *the fiber* $\widetilde{W}_{(t,\lambda)}$ *agrees with* W_λ *in* $t_1^{-1}\big(\mathbb{R} - [a_1 - \delta', a_1 + \delta']\big)$.

iv) *If* i_1 *is the inclusion* $[-1,1]^m \hookrightarrow [0,1] \times [-1,1]^m$ *defined by* $\lambda \mapsto (1,\lambda)$ *and if* $W' := i_1^*\widetilde{W}$ *then for each* $\lambda \in [-\epsilon, \epsilon]^m$ *we have*

$$W'_\lambda \cap t_k^{-1}\big(B(a,\delta)\big) = B(a,\delta) \times M.$$

Proof. Observe that $M := (\pi, x_k)^{-1}\big((0,a)\big)$ is a closed pl manifold of dimension $d-k$. Without loss of generality, we are only going to do this proof in the case when a is the origin in $\mathbb{R}^k$, $\delta' = 1$, (i.e., $B(a,\delta') = [-1,1]^k$) and $\delta = \epsilon$.

Consider first the element $[0,1] \times W$ in $\psi_d(N,k)\big([0,1] \times [-1,1]^m\big)$. Let $\pi' : [0,1] \times W \to [0,1] \times [-1,1]^m$ be the projection onto $[0,1] \times [-1,1]^m$ and let

$\widetilde{M}$ denote the pre-image of $[-1,1]^m \times (-1,1)^k$ under (π', x_k). By assumption we have that the restriction of (π', x_k) on $\widetilde{M}$ is a submersion of codimension $d-k$ and that each fiber of this restriction is pl homeomorphic to M. Pick now any value ϵ' such that $0 < \epsilon < \epsilon' < 1$ and let $f : [0,1] \times [-1,1]^{m+k} \to [0,1] \times [-1,1]^{m+k}$ be a piecewise linear map satisfying the following condtions:

- $f|_{\{0\}\times[-1,1]^{m+k}} = \mathrm{Id}_{\{0\}\times[-1,1]^{m+k}}$.
- f fixes each point in $[0,1] \times \big([-1,1]^{m+k} - (-\epsilon', \epsilon')^{m+k}\big)$.
- f maps $\{1\} \times [-\epsilon, \epsilon]^{m+k}$ to $(1,0)$.

Applying proposition 2.52 to $(\pi', x_k)|_{\widetilde{M}} : \widetilde{M} \to [0,1]\times[-1,1]^m \times (-1,1)^k$ and to the restriction of f on $[0,1] \times [-1,1]^m \times (-1,1)^k$ we obtain a subpolyhedron M' of $[0,1] \times [-1,1]^m \times (-1,1)^k \times (-1,1)^{N-k}$ such that the projection from M' onto $[0,1] \times [-1,1]^m \times (-1,1)^k$ is a piecewise linear submersion of codimension $d-k$ and such that

$$M' \cap \big(\{1\} \times [-\epsilon, \epsilon]^{m+k}\big) = \big(\{1\} \times [-\epsilon, \epsilon]^{m+k}\big) \times N.$$

Observe that the projection of M' onto $[0,1] \times [-1,1]^m$ is a piecewise linear submersion of codimension d since it can be expressed as the following composite of projections

$$M' \to [0,1] \times [-1,1]^m \times (-1,1)^k \to [0,1] \times [-1,1]^m.$$

Let F be the pl map from $[0,1] \times [-1,1]^m \times \mathbb{R}^k \times (-1,1)^{N-k}$ to itself which fixes all points in $[0,1] \times [-1,1]^m \times \big(\mathbb{R}^k - [-\epsilon', \epsilon']^k\big) \times (-1,1)^{N-k}$ and which agrees with $f \times \mathrm{Id}_{(-1,1)^{N-k}}$ on $[0,1] \times [-1,1]^m \times [-1,1]^k \times (-1,1)^{N-k}$, and let $\widetilde{W}$ denote the pre-image $F^{-1}\big([0,1] \times W\big)$. $\widetilde{W}$ is a closed subpolyhedron of $[0,1] \times [-1,1]^m \times \mathbb{R}^k \times (-1,1)^{N-k}$ since it is the pre-image of a sub-polyhedron under a pl map with compact support. Furthermore, the projection $\widetilde{\pi} : \widetilde{W} \to [0,1] \times [-1,1]^m$ is a pl submersion of codimension d since on $(\widetilde{\pi}, x_k)^{-1}\big([0,1] \times [-1,1]^m \times (\mathbb{R}^k - [-\epsilon', \epsilon']^k)\big)$ this map agrees with $\pi' : [0,1]\times W \to [0,1]\times[-1,1]^m$ and on the pre-image $(\widetilde{\pi}, x_k)^{-1}\big([0,1] \times [-1,1]^m \times (-1,1)^k\big)$, which is equal to M', it agees with projection $M' \to [0,1] \times [-1,1]^m$, which is also submsersion of codimension d. Furthermore, if $W' := i_1^* \widetilde{W}$ then it is clear that

$$W' \cap \big([-\epsilon, \epsilon]^m \times B(a,\epsilon) \times (-1,1)^{N-k}\big) = [-\epsilon, \epsilon]^m \times B(a,\epsilon) \times M.$$

□

Lemma 7.16. *Let $W \in \psi_d(N,k)\big([-1,1]^m\big)$, $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, $\delta' > 0$ and M^{d-k} be as in the statement of lemma 7.15. Then for any $0 < \epsilon < 1$ and any $0 < \delta < \delta'$ there exists a $\widetilde{W}$ in $\psi_d(N,k)\big([0,1] \times [-1,1]^m\big)$ which satisfies the following properties:*

i) $\widetilde{W}$ satisfies condtions i), ii) and iii) of lemma 7.15.

ii) *If* $W' := i_1^*\widetilde{W}$ *then for each* $\lambda \in [-\epsilon,\epsilon]^m$ *the fiber* W'_λ *agrees with* $\mathbb{R}^k \times M$ *in*

$$t_1^{-1}\big((a_1-\delta, a_1+\delta)\big).$$

Proof. Let ϵ' and δ'' be values such that $0<\epsilon<\epsilon'<1$ and $0<\delta<\delta''<\delta'$. By lemma 7.15 we can find an element $\widehat{W}$ in $\psi_d(N,k)\big([0,1]\times[-1,1]^m\big)$ which satisfies conditions $i), ii)$ and $iii)$ of 7.15 and which is a concordance between W and an element W' in $\psi_d(N,k)\big([-1,1]^m\big)$ such that

$$W'_\lambda \cap t_k^{-1}\big(B(a,\delta'')\big) = B(a,\delta'')\times M$$

for each λ in $[-\epsilon',\epsilon']^m$. The point now is to stretch the subpace $W'_\lambda \cap t_k^{-1}\big(B(a,\delta'')\big)$ in the direction x_k for each λ in $[-\epsilon,\epsilon]^m$ but withouth changing the fiber W'_λ in the region $t_1^{-1}\big(\mathbb{R}-(a_1-\delta', a_1+\delta')\big)$. This is done with the following *stretching map* E which is defined as follows: choose first an open piecewise linear embedding $e:[0,1]\times\mathbb{R}^k \to [0,1]\times\mathbb{R}^k$ over $[0,1]$ which satisfies the following properties:

- e is the identity on $\{0\}\times\mathbb{R}^k$ and on $[0,1]\times[a_1\pm\delta'']^c\times\mathbb{R}^{k-1}$.
- e maps $\{1\}\times(a_1\pm\delta)\times\mathbb{R}^{k-1}$ onto $\{1\}\times \mathrm{int}B(a,\delta)$.
- e commutes with the projection onto the first component of $\mathbb{R}^k$.

The construction of such a map e will be given at the end of this proof. Using e we can define an open piecewise linear embedding

$$E:[0,1]\times[-1,1]^m\times\mathbb{R}^k\times(0,1)^{N-k} \to [0,1]\times[-1,1]^m\times\mathbb{R}^k\times(0,1)^{N-k}$$

which satisfies the following properties:

- E commutes with the projection onto $[0,1]\times[-1,1]^m$.
- E agrees with the identity map on

  $$\{0\}\times[-1,1]^m\times\mathbb{R}^k\times(0,1)^{N-k},$$

  $$[0,1]\times\big([-1,1]^m-(-\epsilon',\epsilon')^m\big)\times\mathbb{R}^k\times(0,1)^{N-k}$$

  and

  $$[0,1]\times[-1,1]^m\times(a_1\pm\delta'')^c\times\mathbb{R}^{k-1}\times(0,1)^{N-k}.$$

- If

  $$q:[0,1]\times[-1,1]^m\times\mathbb{R}^k\times(0,1)^{N-k}\to[-1,1]^m\times[0,1]\times\mathbb{R}^k\times(0,1)^{N-k}$$

  is the map that flips the first two factors then E agrees with

  $$q^{-1}\circ\big(\mathrm{Id}_{[-1,1]^m}\times e_j\times\mathrm{Id}_{(0,1)^{N-k}}\big)\circ q$$

  on $[-\epsilon,\epsilon]^m\times[0,1]\times\mathbb{R}^k\times(0,1)^{N-k}$.

The details of the definition of the stretching map E will also be given at the end of this proof. For now let us conclude the proof of this lemma assuming the existence of such a map E. By proposition 3.19, the pre-image $\widetilde{W} := E^{-1}\big([0,1] \times W'\big)$ is an element in $\psi_d(N,k)\big([0,1] \times [-1,1]^m\big)$ which, by the properties of the stretching map E, satisfies conditions $i), ii)$ and $iii)$ of lemma 7.15 and which is a concordance between W' and a new element W'' in $\psi_d(N,k)\big([-1,1]^m\big)$ such that for each λ in $[-\epsilon,\epsilon]^m$ the fiber W''_λ agrees with $\mathbb{R}^k \times M$ in $t_1^{-1}\big((a_1-\delta, a_1+\delta)\big)$. By construction, the concatenation of the two concordances $\widehat{W}$ and $\widetilde{W}$ satisfies all the claims listed in the statement of this lemma.

Let us now explain how to define the map $e : [0,1] \times \mathbb{R}^k \to [0,1] \times \mathbb{R}^k$ used in the definition of the stretching map E. For simplicity a will be the origin in $\mathbb{R}^k$, $\delta' = 2$ and $\delta = 1$. Let A denote the product $\mathbb{R} \times [-2,2]^{k-1}$, and let

$$c : \mathbb{R}^k \to \mathrm{int} A$$

be a piecewise linear homeomorphism which is the identity when restricted on $\mathbb{R} \times [-1,1]^{k-1}$ and which commutes with the projection onto the first component $\mathbb{R}$. Let

$$D : [0,1] \times \mathbb{R} \times [-3,3]^{k-1} \to [0,1] \times \mathbb{R} \times [-3,3]^{k-1}$$

be a piecewise linear homeomorphism which commutes with the projection onto $[0,1] \times \mathbb{R}$, which is the identity when restricted on $\{0\} \times \mathbb{R} \times [-3,3]^{k-1}$ and on $[0,1] \times \big(\mathbb{R} - (-2,2)\big) \times [-3,3]^{k-1}$, and such that

$$(1, x_1, x_2, \ldots, x_k) \mapsto (1, x_1, \frac{x_2}{2}, \ldots, \frac{x_k}{2})$$

if $x_1 \in [-1,1]$ and if $(x_2, \ldots, x_k) \in [-2,2]^{k-1}$. The composition

$$e := (\mathrm{Id}_{[0,1]} \times c)^{-1} \circ D|_{[0,1] \times \mathrm{int} A} \circ (\mathrm{Id}_{[0,1]} \times c)$$

is then an open piecewise linear embedding $[0,1] \times \mathbb{R}^k \to [0,1] \times \mathbb{R}^k$ which commutes with the projection onto $[0,1]$ and with the projection onto the first factor of $\mathbb{R}^k$, which is the identity on $[0,1] \times \big(\mathbb{R} - (-2,2)\big) \times \mathbb{R}^{k-1}$ and which maps $\{1\} \times [-1,1] \times \mathbb{R}^{k-1}$ onto $\{1\} \times [-1,1] \times (-1,1)^{k-1}$.

Let us finally explain how to define the stretching map

$$E : [0,1] \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k} \to [0,1] \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k}.$$

Let E' be the map from $[0,1] \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k}$ to itself which is equal to the composite

$$q^{-1} \circ \big(\mathrm{Id}_{[-1,1]^m} \times e_j \times \mathrm{Id}_{(0,1)^{N-k}}\big) \circ q,$$

where q is the map defined on $[0,1] \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k}$ which just flips the first two components. Let $\phi : \mathbb{R} \to [0,1]$ be the piecewise linear function defined by

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [-\epsilon, \epsilon], \\ 0 & \text{if } t \in \mathbb{R} - [-\epsilon', \epsilon'], \\ \frac{t-\epsilon'}{\epsilon-\epsilon'} & \text{if } t \in [\epsilon, \epsilon'], \\ \frac{t+\epsilon'}{\epsilon'-\epsilon} & \text{if } t \in [-\epsilon', -\epsilon] \end{cases}$$

and let us denote by $|\cdot| : [-1,1]^m \to [0,1]$ the pl map which sends $(x_1, \ldots, x_m)$ to $\max\{|x_1|, \ldots, |x_m|\}$. Finally, let

$$c : [0,1] \times [-1,1]^m \to [0,1] \times [-1,1]^m$$

be the map defined by

$$(t, \lambda) \mapsto \big(p\big(t, \phi(|\lambda|)\big), \lambda\big),$$

where $p : [0,1] \times [0,1] \to [0,1]$ is the piecewise linear product introduced in the previous section. The map c is piecewise linear by proposition 2.29 and using proposition 2.57 we can pull back the map E', which is an open piecewise linear embedding which commutes with the projection onto $[0,1] \times [-1,1]^m$, along c to obatin a new open piecewise linear embedding

$$E : [0,1] \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k} \to [0,1] \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k}$$

which is defined by

$$E(t, \lambda, x) = \big(t, \lambda, E'_{c(t,\lambda)}(x)\big).$$

By construction the map E agrees with E' on $[0,1] \times [-\epsilon, \epsilon]^m \times \mathbb{R}^k \times (0,1)^{N-k}$ and agrees with the identity map on $\{0\} \times [-1,1]^m \times \mathbb{R}^k \times (0,1)^{N-k}$, $[0,1] \times \big([-1,1]^m - [-\epsilon', \epsilon']^m\big) \times \mathbb{R}^k \times (0,1)^{N-k}$ and $[0,1] \times [-1,1]^m \times \big(\mathbb{R} - (-2,2)\big) \times \mathbb{R}^{k-1} \times (0,1)^{N-k}$, and hence the map E satisfies all the claims listed in the proof of this lemma. □

Lemma 7.17. *Let $x_1 : [0,1] \times (0,1)^{N-1} \to [0,1]$ be the projection onto the first component and let $C \subseteq [0,1] \times (0,1)^{N-1}$ be a d-dimensional compact piecewise linear submanifold with boundary such that*

$$M := \partial C \subseteq x_1^{-1}(0), \ C \cap x_1^{-1}(1) = \emptyset$$

Then there is a 1-simplex W of $\psi_d(N,1)_\bullet$ such that

1. $W_0 = \mathbb{R} \times M$,
2. $W_{\frac{1}{2}} = \big((-\infty, 0] \times M\big) \cup C$, *and*
3. $W_1 = \emptyset$.

Proof. Let $\widetilde{M}$ denote the closed sub-polyhedron of $\mathbb{R} \times (0,1)^{N-1}$ obtained by taking the union $\big((-\infty, 0] \times M\big) \cup C$. Observe that the product

$$[0,1] \times \widetilde{M} \subseteq [0,1] \times \mathbb{R} \times (0,1)^{N-1}$$

is a 1-simplex of $\psi_d(N,1)_\bullet$. Let $e : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ be an open piecewise linear embedding which satisfies the following properties

. e commutes with the projection onto $[0,1]$.

. $e_{\frac{1}{2}}$ is the identity $\mathrm{Id}_{\mathbb{R}}$.

. $e_0(\mathbb{R}) = (-1,0)$.

. $e_1(\mathbb{R}) = (1,2)$.

If $E : [0,1] \times \mathbb{R} \times (0,1)^{N-1} \to [0,1] \times \mathbb{R} \times (0,1)^{N-1}$ is equal to the product $e \times \mathrm{Id}_{(0,1)^{N-1}}$ then by lemma 3.19 we have that the pre-image $W := E^{-1}([0,1] \times \widetilde{M})$ is also a 1-simplex of $\psi_d(N,1)_\bullet$ and by the way we chose the embedding e we have that W satisfies all the desired properties.

□

Lemma 7.18. *Let $W \subseteq \mathbb{R}^k \times (0,1)^{N-k}$ be a 0-vertex of $\psi_d^{\emptyset}(N,k)_\bullet$, let $a \in \mathbb{R}^k$ be a regular value of the projection $x_k : W \to \mathbb{R}^k$ in the sense of definition 2.46 and let $M := x_k^{-1}(a)$. Then there is a $(d-k+1)$-dimensional compact piecewise linear submanifold $C \subseteq [0,1] \times (0,1)^{N-k}$ with boundary such that*

$$\partial C \subseteq \{0\} \times (0,1)^{N-k},\ C \cap \big(\{1\} \times (0,1)^{N-k}\big) = \emptyset \tag{45}$$

and such that $M = \partial C$.

Proof. By assumption there is $\delta > 0$ such that the restricition of $x_k : W \to B(a,\delta)$ on $x_k^{-1}\big(B(a,\delta)\big)$

$$x_k|_{x_k^{-1}\left(B(a,\delta)\right)} : x_k^{-1}\big(B(a,\delta)\big) \to B(a,\delta) \tag{46}$$

is a proper piecewise linear submersion of codimension $d-1$. We begin this proof by observing that for any $b \in B(a,\delta)$ there is a $D \in \psi_{d-k}(N-k,0)\big([0,\frac{1}{2}]\big)$ such that $D_0 = M$ and $D_{\frac{1}{2}} = M'$ where M' is the pre-image of b under $x_k : W \to \mathbb{R}^k$. This is done by pulling back (46) along any pl map that identifies $[0,\frac{1}{2}]$ with the straightline connecting a and b. Then $D \subseteq [0,\frac{1}{2}] \times (0,1)^{N-k}$ is pl homeomorphic over $[0,\frac{1}{2}]$ to $[0,\frac{1}{2}] \times M$.

Now, since $\psi_d(N,k)_\bullet$ is a Kan Δ-set and since $W \in \psi_d^{\emptyset}(N,k)_0$ we can find a 1-simplex $\widetilde{W}'$ of $\psi_d(N,k)_\bullet$ such that $\widetilde{W}'_0 = W$ and $\widetilde{W}'_1 = \emptyset$. Furthermore, we can assume that there is $0 < \epsilon < 1$ such that

$$\widetilde{W}'_t = W, \quad \text{for all } t \in [0,\epsilon]. \tag{47}$$

Let $-\widetilde{W}' \subseteq [-1,0] \times \mathbb{R}^k \times (0,1)^{N-k}$ be the element in $\psi_d(N,k)\big([-1,0]\big)$ obtained by multiplying the first coordinate of each point in $\widetilde{W}'$ by -1 and let $\widetilde{W} \subseteq [-1,1] \times \mathbb{R}^k \times (0,1)^{N-k}$ be the element of $\psi_d(N,k)\big([-1,1]\big)$ obtained by gluing $\widetilde{W}'$ and $-\widetilde{W}'$ along $\widetilde{W}'_0$. Observe then that the projection $\pi : \widetilde{W} \to [-1,1]$ is a piecewise linear submersion of codimension d and that $\widetilde{W}$ is a piecewise linear manifold without boundary since $\pi^{-1}\big(\{-1,1\}\big) = \emptyset$. By applying theorem 2.47

we can find a regular value b in $\text{int}B(a,\delta)$ of the projection $x_k : \widetilde{W} \to \mathbb{R}^k$ and by proposition 2.48 the pre-image of b under $x_k : \widetilde{W} \to \mathbb{R}^k$ is going to be a $(d+1-k)$-dimensional closed (compact and without boundary) piecewise linear submanifold $\widetilde{C}$ of $\widetilde{W}$ and, after identifying $[-1,1]\times\{b\}\times(0,1)^{N-k}$ with $[-1,1]\times(0,1)^{N-k}$, we have that $\widetilde{C}$ is also a closed piecewise linear submanifold of $[-1,1]\times(0,1)^{N-k}$ which doesn't intersect the bottom and top face of this product. Furthermore, by (47) we have that

$$\widetilde{C}\cap\big((-\epsilon,\epsilon)\times(0,1)^{N-k}\big) = (-\epsilon,\epsilon)\times M',$$

where M' denotes the pre-image of b under $x_k : W \to \mathbb{R}^k$, and it follows that the intersection

$$C' := \widetilde{C}\cap\big([0,1]\times(0,1)^{N-k}\big)$$

is a piecewise linear manifold with boundary $\partial C'$ equal to M'. Without loss of generality we can assume that C' is contained in $[\frac{1}{2},1]\times(0,1)^{N-k}$. Finally, if $D\subseteq[0,\frac{1}{2}]\times(0,1)^{N-k}$ is the trivial cobordism from M to M' defined at the beginning of this proof then the union

$$C := D\cup C'$$

is a $(d-k+1)$-dimensional compact piecewise linear manifold in $[0,1]\times(0,1)^{N-k}$ which satisfies the desired properties.

□

As we said before, the following lemma is the key construction that we are going to use in the proof of lemma 7.14.

Lemma 7.19. *Let $W\in\psi_d(N,k)^{\emptyset}\big([-1,1]^m\big)$, $a=(a_1,\dots,a_k)\in\mathbb{R}^k$, $\delta'>0$ and M^{d-k} be as in the statement of lemma 7.15. Then for any $0<\epsilon<1$ there exists a $\widetilde{W}$ in $\psi_d(N,k)\big([0,1]\times[-1,1]^m\big)$ which satisfies the following properties:*

i) $\widetilde{W}$ satisfies conditions i), ii) and iii) of lemma 7.15.

ii) If $W' := i_1^*\widetilde{W}$ then

$$W'_\lambda\cap\big(\{a_1\}\times\mathbb{R}^{N-1}\big)=\emptyset.$$

for each λ in $[-\epsilon,\epsilon]^m$.

Proof. By lemma 7.16 we can assume that

$$W_\lambda\cap t_1^{-1}\big((a_1-\delta'',a_1+\delta'')\big) = (a_1-\delta'',a_1+\delta'')\times\mathbb{R}^k\times M$$

for every point λ in $[-\epsilon',\epsilon']^m$ where ϵ' and δ'' are values in the intervals $(\epsilon,1)$ and (δ,δ') respectively and by lemma 7.18 there is there is a $(d-k+1)$-dimensional compact piecewise linear submanifold $C\subseteq[0,1]\times(0,1)^{N-k}$ with boundary such that $M=\partial C$ and

$$\partial C\subseteq\{0\}\times(0,1)^{N-k},\ C\cap\big(\{1\}\times(0,1)^{N-k}\big)=\emptyset.$$

Let $\tilde{C}$ be the vertex of $\psi_{d-k+1}(N-k+1,1)_\bullet$ which agrees with the product $\mathbb{R}\times M$ in $t_1^{-1}\big((-\infty,0]\big)$ and which agrees with the cobordism C in $t_1^{-1}\big([0,\infty)\big)$. Using 7.17 we can find a 1-simplex $\tilde{D}$ of $\psi_{d-k+1}(N-k+1,1)_\bullet$ such that

$$\tilde{D}_0=\mathbb{R}\times M,\ \tilde{D}_{\frac{1}{2}}=\tilde{C},\ \tilde{D}_1=\emptyset.$$

Since $k\geq 2$ we have that each fiber W_λ of $W\to[-1,1]^m$ is allowed to be non-compact in at least to directions, one of them being x_1. The point of this proof is to use the concordance $\tilde{D}$ to push to $-\infty$ in the direction x_k the pre-image $x_1^{-1}(a_1)$ of a_1 under the projection $x_1:W_\lambda\to\mathbb{R}$ for each $\lambda\in[-\epsilon,\epsilon]^m$ but without changing any of the fibers W_λ in the region $t_1^{-1}\big(\mathbb{R}-[a_1-\delta',a_1+\delta']\big)$. This is done as follows: let D'' denote the product

$$\tilde{D}\times\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-2}$$

and let D' be the image of D'' under the map

$$[0,1]\times\mathbb{R}\times(0,1)^{N-k}\times\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-2}\to[0,1]\times\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-2}\times\mathbb{R}\times(0,1)^{N-k}$$

which just permutes the factors. D' is a closed sub-polyhedron of

$$[0,1]\times\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-1}\times(0,1)^{N-k},$$

once we have identified $\mathbb{R}^{k-2}\times\mathbb{R}$ with $\mathbb{R}^{k-1}$, and the natural projection $D'\to[0,1]\times[-\epsilon',\epsilon']^m\times\mathbb{R}$ is a piecewise linear submersion of codimension $d-1$, i.e., D' is an element of

$$\psi_{d-1}(N-1,k-1)\big([0,1]\times[-\epsilon',\epsilon']^m\times\mathbb{R}\big).$$

Studying closely the construction we just made we see that D' is a concordance between

$$\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-1}\times M$$

and $\emptyset$ such that at time $t=\frac{1}{2}$ we have

$$D'_{\frac{1}{2}}=\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-2}\times\tilde{C}.$$

Let $\tilde{\epsilon}>0$ and $\tilde{\delta}>0$ be such that $\epsilon<\tilde{\epsilon}<\epsilon'$ and $\delta<\tilde{\delta}<\delta''$. Using the technique for pulling back submersions given in proposition 2.52 we wish to obtain from the concordance D' a new concordance D in $\psi_{d-1}(N-1,k-1)\big([0,1]\times[-\epsilon',\epsilon']^m\times\mathbb{R}\big)$ which satisfies the following properties:

- At time $t=0$ the concordance is equal to $\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-1}\times M$.
- $D_\lambda=D'_\lambda$ for each λ in $[0,1]\times[-\epsilon,\epsilon]^m\times[a_1-\delta,a_1+\delta]$.
- D agrees with the constant concordance $[0,1]\times\big([-\epsilon',\epsilon']^m\times\mathbb{R}\big)\times\mathbb{R}^{k-1}\times M$ over
$$[0,1]\times[-\epsilon',\epsilon']^m\times\big(\mathbb{R}-[a_1-\tilde{\delta},a_1+\tilde{\delta}]\big)$$
and over
$$[0,1]\times\big([-\epsilon',\epsilon']^m-[-\tilde{\epsilon},\tilde{\epsilon}]^m\big)\times\mathbb{R}.$$

Let us indicate how to conclude this proof assuming the existence of such a concordance D. The composition of the submersion $D \to [0,1] \times [-\epsilon', \epsilon']^m \times \mathbb{R}$ and the natural projection $[0,1] \times [-\epsilon', \epsilon'] \times \mathbb{R} \to [0,1] \times [-\epsilon', \epsilon']^m$ is a submersion of codimension d and thus D can also be viewed as an element of

$$\psi_d(N,k)\big([0,1] \times [-\epsilon', \epsilon']^m\big)$$

and as element of this set it satisfies the following properties (here we are going to use the notation introduced in definition 3.16):

i) D agrees with $[0,1] \times W$ in

$$\Psi_d\Big(t_1^{-1}\big((a_1 - \delta', a_1 + \delta') - [a_1 - \tilde{\delta}, a_1 + \tilde{\delta}]\big)\Big)\big([0,1] \times [-\epsilon', \epsilon']^m\big)$$

ii) D agrees with $[0,1] \times W$ in

$$\Psi_d\Big(t_1^{-1}\big((a_1 - \delta', a_1 + \delta')\big)\Big)\big([0,1] \times ([-\epsilon', \epsilon']^m - [-\tilde{\epsilon}, \tilde{\epsilon}]^m)\big)$$

iii) For each point $(1, \lambda)$ in $\{1\} \times [-\epsilon, \epsilon]^m$ we have that

$$D_{(1,\lambda)} \cap \big(\{a_1\} \times \mathbb{R}^{N-1}\big) = \emptyset.$$

By *i)* and proposition 3.17 there is an element $\widehat{D}$ in

$$\psi_d(N,k)\big([0,1] \times [-\epsilon', \epsilon']^m\big)$$

which agrees with D in

$$t_1^{-1}\big((a_1 - \delta', a_1 + \delta')\big)$$

and with $[0,1] \times W_{[-\epsilon', \epsilon']^m}$ in

$$t_1^{-1}\big(\mathbb{R} - [a_1 - \tilde{\delta}, a_1 + \tilde{\delta}]\big)$$

Furthermore, by *ii)* we have that $\widehat{D}$ can be extended to an element $\widehat{W}$ in

$$\psi_d(N,k)\big([0,1] \times [-1,1]^m\big)$$

which agrees with $[0,1] \times W$ in $t_1^{-1}\big(\mathbb{R} - [a_1 - \tilde{\delta}, a_1 + \tilde{\delta}]\big)$ and over $[0,1] \times \big([-1,1]^m - [-\tilde{\epsilon}, \tilde{\epsilon}]^m\big)$. Finally, by condition *iii)* given above we have for each point $(1, \lambda)$ in $\{1\} \times [-\epsilon, \epsilon]^m$ that

$$\widehat{W}_{(1,\lambda)} \cap t_1^{-1}(a_1) = \emptyset.$$

This concludes the proof of this lemma assuming the existence of the concordance D.

It remains to explain how to obtain the concordance

$$D \in \psi_{d-1}(N-1, k-1)\big([0,1] \times [-\epsilon', \epsilon']^m \times \mathbb{R}\big)$$

from the concordance D'. Let then $c : [0,1] \times [-\epsilon', \epsilon']^m \to [0,1] \times [-\epsilon', \epsilon']^m$ be a piecewise linear map which fixes all points (t, λ) with $\lambda \in [-\epsilon, \epsilon]^m$, which maps (t, λ) to $(0, \lambda)$ if λ is in $[-\epsilon', \epsilon']^m - (-\tilde{\epsilon}, \tilde{\epsilon})^m$ and which commutes with the projection onto $[-\epsilon', \epsilon']^m$; and let $d : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ be a piecewise linear map which fixes all points (t, s) if $s \in [a_1 - \delta, a_1 + \delta]$, which maps (t, s) to $(0, s)$ if $s \in \mathbb{R} - (a_1 - \tilde{\delta}, a_1 + \tilde{\delta})$ and which commutes with the projection onto $\mathbb{R}$. Using d and c we define two pl maps $f, g : [0,1] \times [-\epsilon', \epsilon']^m \times \mathbb{R} \to [0,1] \times [-\epsilon', \epsilon']^m \times \mathbb{R}$ as follows

$$f(t, \lambda, s) = \big(d_1(t, s), \lambda, s\big), \quad g(t, \lambda, s) = \big(c_1(t, \lambda), \lambda, s\big).$$

Finally, applying proposition 2.52 to pull back the concordance D' along the map $f \circ g$ we produce a new element D in $\psi_{d-1}(N-1, k-1)\big([0,1] \times [-\epsilon', \epsilon']^m \times \mathbb{R}\big)$ which satisfies the desired properties.

□

Proof of lemma 7.14. By an argument completely analogous to the one used to prove lemma 6.9 we can show that there is an element

$$\widetilde{W} \in \psi_d^{\emptyset}(N, k)\big([0,1] \times M\big)$$

and a finite collection of pairs of pl m-balls $(V_1, U_1), \ldots, (V_q, U_q)$ in $\mathrm{int}M$ which satisfy the following conditions:

i) $\widetilde{W}$ agrees with $[0,1] \times W$ in $t_1^{-1}\big((-\infty, \beta)\big)$.

ii) $U_j \subseteq \mathrm{int}V_j$ for $j = 1, \ldots, q$ and the collection $\{\mathrm{int}U_j\}_{j=1}^q$ is a finite open cover for P.

iii) $\widetilde{W}$ is a concordance between W and an element W' of $\psi_d^{\emptyset}(N, k)(M)$ such that for each $j = 1, \ldots, q$ the projection

$$x_k : W'_{V_j} \to \mathbb{R}^k$$

has a fiberwise regular value $a^j = (a_1^j, \ldots, a_k^j)$ (see definition 5.2) with $a_1^j > \beta$.

Let $\pi : W' \to M$ be the projection onto M. From *iii)* we have that there is a $\delta > 0$ such that for each $j = 1, \ldots, q$ the restriction of (π, x_k) on

$$(\pi, x_k)^{-1}\big(V_j \times B(a^j, 2\delta)\big)$$

is a piecewise linear submersion of codimension $d-k$. After possibly perturbing the a^j's and shrinking $\delta > 0$ we can assume that all the intervals $[a_1^j - 2\delta, a_1^j + 2\delta]$ are mutually disjoint and that $a^j - 2\delta > \beta$ for $j = 1, \ldots, q$.

Finally, since all the intervals $[a_1^j - 2\delta, a_1^j + 2\delta]$ are mutually disjoint we can apply lemma 7.19 to define inductively an element

$$\widehat{W} \in \psi_d^{\emptyset}(N, k)\big([0,1] \times M\big)$$

which agrees with $[0,1]\times W'$ in $t_1^{-1}\big((-\infty,\beta)\big)$ and which is a concordance between W' and an element W'' in $\psi_d^{\emptyset}(N,k)\big(M\big)$ such that for $j=0,\dots,q$ we have that

$$W''_\lambda \cap \big(\{a_1^j\}\times\mathbb{R}^{N-1}\big)=\emptyset$$

for every $\lambda\in U_j$. The concordance obtained by concatenating $\widetilde{W}$ and $\widehat{W}$ and the open cover $\{\mathrm{int}U_j\}_{j=1}^p$ satisfy then the claims of lemma 7.14. □

7.5 Proof of the main theorem

As it was indicated in 7.2, the next step of our strategy to show that the scanning map $\mathcal{S}:|\psi_d(N,k)_\bullet|\to\Omega|\psi_d(N,k+1)_\bullet|$ is a weak homotopy equivalence is to show that $\mathcal{S}$ is homotopic to the composite map

$$|\psi_d(N,k)_\bullet|\overset{h_1}{\to}\Omega\,\|N\psi_d(N,k)_{\bullet,\bullet}\|\to\Omega|\psi_d^0(N,k+1)_\bullet|\to\Omega|\psi_d^{\emptyset}(N,k+1)_\bullet|,\ (48)$$

where h_1 is the map introduced in (43) and the other two maps are induced respectively by the forgetful map $f:\|N\psi_d(N,k)_{\bullet,\bullet}\|\to|\psi_d^0(N,k+1)_\bullet|$ and the inclusion $i:|\psi_d^0(N,k+1)_\bullet|\hookrightarrow|\psi_d^{\emptyset}(N,k+1)_\bullet|$. In order to do this, we are going to show instead that the adjoint $T:[-1,1]\times|\psi_d(N,k)_\bullet|\to|\psi_d(N,k+1)_\bullet|$ of the scanning map is homotopy equivalent to the adjoint of the map (48). The adjoint

$$g:[-1,1]\times|\psi_d(N,k)_\bullet|\to|\psi_d^{\emptyset}(N,k+1)_\bullet|$$

of the map (48) is equal to the composite map

$$[-1,1]\times|\psi_d(N,k)_\bullet|\overset{g_1}{\to}\|N\psi_d(N,k)_{\bullet,\bullet}\|\overset{f}{\to}|\psi_d^0(N,k+1)_\bullet|\overset{i}{\to}|\psi_d^{\emptyset}(N,k+1)_\bullet|,$$

where g_1 is the adjoint of h_1 (see equation (42)). Using the definition of the adjoint g_1 of h_1 given in (42) it is easy to verify that the map $g:[-1,1]\times|\psi_d(N,k)_\bullet|\to|\psi_d^{\emptyset}(N,k+1)_\bullet|$ is defined by the formula

$$g(t,x)=T\big(\phi(t),x\big)$$

where $\phi:[-1,1]\to[-1,1]$ is the continuous function defined by

$$\phi(t)=\begin{cases}-1 & \text{if } t\in[-1,0],\\ 4t-1 & \text{if } t\in[0,1/4],\\ 0 & \text{if } t\in[1/4,3/4],\\ 4t-3 & \text{if } t\in[3/4,1].\end{cases}$$

If $F:[0,1]\times[-1,1]\to[-1,1]$ is any homotopy between ϕ and $\mathrm{Id}_{[-1,1]}$ which is fixed on -1 and 1 then the map

$$H:[0,1]\times[-1,1]\times|\psi_d(N,k)_\bullet|\to|\psi_d(N,k+1)_\bullet|$$

defined by $(s,t,x)\mapsto T\big(F(s,t),x\big)$ is a homotopy between g and the adjoint T of the scanning map which is fixed on $\{-1,1\}\times|\psi_d(N,k)_\bullet|$. Consequently,

the scanning map $\mathcal{S}$ is homotopic to the composite map in (48), and since each map in (48) is a weak homotopy equivalence we have that the scanning map is a weak homotopy equivalence, thus concluding the proof of theorem 7.2.

Using theorem 7.2 we can prove by induction the following result.

Theorem 7.20. *If $N - d \geq 3$ then there is a weak homotopy equivalence*

$$|\psi_d(N,1)_\bullet| \xrightarrow{\simeq} \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|.$$

Finally, combining theorems 5.19 and 7.20 we obtain the main theorem of this thesis.

Theorem 7.21. *If $N - d \geq 3$ then there is a weak homotopy equivalence*

$$BPL\mathcal{C}_d(\mathbb{R}^N) \simeq \Omega^{N-1}|\Psi_d(\mathbb{R}^N)_\bullet|.$$

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