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Property A and coarse embedding for locally compact groups

PhD thesis by

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Abstract

In the study of the Novikov conjecture, property A and coarse embedding of metric spaces were introduced by Yu and Gromov, respectively. The main topic of the thesis is property A and coarse embedding for locally compact second countable groups. We prove that many of the results that are known to hold in the discrete setting, hold also in the locally compact setting.

In a joint work with Deprez, we show that property A is equivalent to amenability at infinity and the strong Novikov conjecture is true for every locally compact group that embeds coarsely into a Hilbert space (see Article A).

In a joint work with Deprez, we show a number of permanence properties of property A and coarse embeddability into Hilbert spaces (see section 4).

In section 6 we give a completely bounded Schur multiplier characterization of locally compact groups with property A. In particular, weakly amenable groups have property A.

In a joint work with Knudby, we characterize the connected simple Lie groups with the discrete topology that have different approximation properties (see Article B). Moreover, we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (see Article C). Consequently, all locally compact groups whose weak Haagerup constant is 1 embed coarsely into Hilbert spaces.

In a joint work with Brodzki and Cave, we show that exactness of a locally compact second countable group is equivalent to amenability at infinity, which solves an open problem raised by Anantharaman-Delaroche (see section 8).

Resumé

I undersøgelsen af Novikovs formodning, blev egenskab A og grov indlejring af metriske rum indført af henholdsvis Yu og Gromov. Hovedemnet for denne afhandling er egenskab A og grov indlejring for lokalkompakte andentællelige grupper. Vi viser, at mange af de kendte resultater fra det diskrete tilfælde også holder i det lokalkompakte tilfælde.

I samarbejde med Deprez, vises, at egenskab A svarer til amenabilitet ved uendelig, og at den stærke Novikov-formodning er sand for enhver lokalkompakt gruppe, der indlejrer groft i et Hilbertrum (se artikel A).

I samarbejde med Deprez, vises, at egenskab A og grov indlejring i Hilbertrum bevares ved en række almindelige konstruktioner (se afsnit 4).

I afsnit 6 gives en karakterisering af lokalkompakte grupper med egenskab A ved brug af fuldstændigt begrænsede Schur-multiplikatorer. Dette viser, at svagt amenable grupper har egenskab A.

I samarbejde med Knudby, karakteriseres de sammenhængende simple Lie-grupper der, udstyret med den diskrete topologi, har forskellige approksimationsegenskaber (se artikel B). Desuden gives en karakterisering af lokalkompakte grupper groft indlejret i Hilbertrum ved brug af kontraherende Schur-multiplikatorer (se artikel C). Derfor indlejrer alle lokalkompakte grupper, hvis svage Haagerup konstant er 1, groft i Hilbertrum.

I samarbejde med Brodzki og Cave, vises, at eksakthed af en lokalkompakt andentællelig gruppe svarer til amenabilitet ved uendelig. Dette løser et åbent problem stillet af Anantharaman-Delaroche (se afsnit 8).

Contents

1. Introduction

Gromov introduced the notion of coarse embeddability of metric spaces and suggested that finitely generated discrete groups that are coarsely embeddable in a Hilbert space, when viewed as metric spaces with a word length metric, might satisfy the Novikov conjecture [33, 37]. Yu showed that this is indeed the case, provided that the classifying space is a finite CW-complex [89]. In the same paper Yu introduced a weak form of amenability on discrete metric spaces that he called property A, which guarantees the existence of a coarse embedding into Hilbert space. Higson and Roe proved in [50] that the metric space underlying a finitely generated discrete group has property A if and only if it admits a topologically amenable action on some compact Hausdorff space. Ozawa showed in [70] that a discrete group admits a topologically amenable action on a compact Hausdorff space if and only if the group is exact. In the case of property A groups, Higson strengthened Yu's result by removing the finiteness assumption on the classifying space [47]. Indeed, he proved that the Baum-Connes assembly map with coefficients, for any countable discrete group which has a topologically amenable action on a compact Hausdorff space, is split-injective. Baum, Connes and Higson showed that this implies the Novikov conjecture [7]. Using Higson's descent technique (see [47]), Skandalis, Tu and Yu [81] were able to generalize the split-injectivity result to arbitrary discrete groups which admit a coarse embedding into Hilbert space, and hence they answered Gromov's question.

All mathematical concepts mentioned above, except for the Novikov conjecture, not only make sense in the discrete setting but also in the locally compact setting. The main topic of my Ph.D. thesis is to study property A and coarse embedding on locally compact second countable groups. In the following, we show that many of the results that are known to hold in the discrete setting, hold also in the locally compact setting.

2. The Baum-Connes conjecture and the strong Novikov conjecture

The Baum-Connes conjecture was first introduced by Paul Baum and Alain Connes in 1982 [6][1] and its current formulation was given in [7]. The origins of the conjecture go back to Connes' foliation theory [21] and Baum's geometric description of K-homology theory [8].

Let us first state the Baum-Connes conjecture with coefficients and provide some scientific background for the conjecture. Consider a second countable locally compact group G and a separable G-C^*-algebra A. Let $\mathcal{E}(G)$ denote a locally compact universal proper G-space. Such a space always exists and unique up to G-homotopy equivalence [7, 57]. In many cases there is a natural model for $\mathcal{E}(G)$ with geometric interpretation ([7], [57]). The topological K-theory of G with coefficient A is defined as follows:

$$K_*^{\mathrm{top}}(G,A) = \lim_{X} KK_*^G(C_0(X),A),$$

where X runs through all G-invariant subspaces of $\mathcal{E}(G)$ such that X/G is compact, and $KK_*^G(C_0(X),A)$ denotes Kasparov's equivariant KK-theory [56]. Baum, Connes and Higson [7] produces a particular map between the topological K-theory of G with coefficient A, $K^{\mathrm{top}}(G,A)$ and the operator K-theory of the reduced crossed product C^*-algebra $A \rtimes_r G$, called the assembly map

$$\mu_A : K_*^{\mathrm{top}}(G,A) \to K_*(A \rtimes_r G).$$

[1] However, the paper was published 18 years later.

CONJECTURE 1 (The Baum-Connes Conjecture with Coefficients). Let G be a second countable locally compact group. The assembly map μ_A is an isomorphism of abelian groups for every separable G-C^*-algebra A.

The conjecture itself provides a way to compute the operator K-theory of $A \rtimes_r G$ from the equivariant K-homology theory. The Baum-Connes conjecture with coefficients has been proved for some large families of groups. In particular, Higson and Kasparov [49] proved the Baum-Connes conjecture with coefficients for groups having the Haagerup property, which contain all locally compact amenable groups. In 2012, Lafforgue [64] proved it for all word-hyperbolic groups. However, the Baum-Connes conjecture with coefficients is known to be false in general. The first counterexamples were obtained by Higson, Lafforgue and Skandalis in [48] for certain classes of Gromov's random groups [38], [4], [69].
The Baum-Connes conjecture with coefficients consists of two parts, injectivity of the assembly map and surjectivity of the assembly map. The injective part of the conjecture is called "the strong Novikov conjecture":

CONJECTURE 2 (The Strong Novikov Conjecture). Let G be a second countable locally compact group. The assembly map μ_A is injective for every separable G-C^*-algebra A.

The strong Novikov conjecture for countable discrete groups implies the Novikov conjecture on homotopy invariance of higher signatures [7]. This is the reason for the name of this conjecture. As far as I know, at present there are no known counterexamples to the strong Novikov conjecture. As mentioned in the introduction, the strong Novikov conjecture holds for countable groups which admit a coarse embedding into Hilbert space [81]. Later, Chabert, Echterhoff and Oyono-Oyono showed in [14] that the strong Novikov conjecture is still true for locally compact second countable groups that admit a topologically amenable action on some compact Hausdorff space.
The strong Novikov conjecture has been the main motivation for me to study property A and coarse embedding on locally compact second countable groups. In a joint work with Deprez [A], we are able to prove the strong Novikov conjecture for every locally compact second countable group that embeds coarsely into a Hilbert space. Hence we have answered Gromov's question in greater generality.

3. Property A and coarse embedding

In [37], Gromov introduced the notion of the coarse embedding of a metric space into another one. In the literature what we have called a coarse embedding is often referred to as an uniform embedding after M. Gromov. We use the term coarse embedding, because uniform embedding means something different in the Banach geometry (see [12]).

DEFINITION 3.1 ([37]). Let X and Y be *any* metric spaces. A map $f : X \to Y$ is called a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that
(i) $\rho_1(d(x, y)) \leq d(f(x), f(y)) \leq \rho_2(d(x, y))$ for all $x, y \in X$.
(ii) $\lim_{r\to\infty} \rho_i(r) = \infty$ for $i = 1, 2$.

Property A was first introduced by Yu on discrete metric spaces and the original motivation for introducing property A was that it is a sufficient condition to coarsely embed a discrete metric space into a Hilbert space.

DEFINITION 3.2 ([89]). A *discrete* metric space (X, d) is said to have *property A* if for any $R > 0$, $\varepsilon > 0$, there exist $S > 0$ and a family $(A_x)_{x\in X}$ of finite, non-empty subsets of $X \times \mathbb{N}$, such that
(i) $(y, n) \in A_x$ implies $d(x, y) \leq S$.
(ii) For all $x, y \in X$ with $d(x, y) \leq R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$.

The following result is inspired by the Bekka-Cherix-Valette therem which states that every amenable group admits a proper and isometric action on Hilbert space [10].

THEOREM 3.3 ([89]). *If a discrete metric space X has property A, then X admits a coarse embedding into Hilbert spaces.*

The coarse embeddability of locally compact (σ-compact) groups into Hilbert spaces has already studied by Anantharaman-Delaroche in [3]. Moreover, Roe in [78] generalized property A to proper metric spaces with bounded geometry in the sense of [76]. Deprez and I introduced in [A] our own notion of property A for locally compact second countable groups. This notion of property A is closely modelled on Yu's definition and unifies the coarse property on the underlying metric space and the topological property on the group in the following way. Every second countable locally compact group G has a proper left-invariant metric d that implements the topology on G, and such a metric is unique up to coarse equivalence (see [45] and [82]). Moreover, the proper metric space (G, d) has bounded geometry (see [45]). So Roe's property A makes sense for every locally compact second countable group and it agrees with our property A in this case. Furthermore, our property A is equivalent to amenability at infinity, which is a topological property on locally compact groups.

In the following we will recall the relevant definitions and state the main theorems in [A]. *From now on, G will always denote a locally compact, second countable, Hausdorff topological group.* Let μ' denote the product measure of a left Haar measure μ on G with the counting measure on $\mathbb{N}$. If $K \subseteq G$ is a subset, we denote $\mathrm{Tube}(K) := \{(s,t) \in G \times G : s^{-1}t \in K\}$.

DEFINITION 3.4. ([A]) A locally compact group G has *property A* if for any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a family $\{A_s\}_{s\in G}$ of Borel subsets of $G \times \mathbb{N}$ with $0 < \mu'(A_s) < \infty$ such that

- for all $(s,t) \in \mathrm{Tube}(K)$ we have $\frac{\mu'(A_s \Delta A_t)}{\mu'(A_s \cap A_t)} < \varepsilon$,
- $(t,n) \in A_s$ implies $(s,t) \in \mathrm{Tube}(L)$.

This definition can be regarded as a generalization of Yu's property A for finitely generated discrete groups with the counting measure and any word length metric.

DEFINITION 3.5 ([3]). A map u from a locally compact group G into a Hilbert space H is said to be a *coarse embedding* if u satisfies the following two conditions:

a) for every compact subset K of G there exists $R > 0$ such that

$$(s,t) \in \mathrm{Tube}(K) \Rightarrow ||u(s) - u(t)|| \leq R;$$

b) for every $R > 0$ there exists a compact subset K of G such that

$$||u(s) - u(t)|| \leq R \Rightarrow (s,t) \in \mathrm{Tube}(K).$$

It is not hard to see that the preceding definition of coarse embedding is equivalent to Gromov's notion of coarse embeddability of the metric space (G, d) into Hilbert spaces for the "unique" proper metric d as mentioned above.

Already observed by Anantharaman-Delaroche in [3] that a locally compact σ-compact group which is amenable at infinity, embeds coarsely into a Hilbert space. However, the converse implication is not true (see [69]). Recall that a locally compact group G is said to be *amenable at infinity* if G admits a topologically amenable action in the sense of [1] on some compact Hausdorff space X. It is well-known from [3] that a locally compact group G is amenable at infinity if and only if its action on $\beta^{lu}(G)$ by the left translation is topologically amenable, where $\beta^{lu}(G)$ is the universal compact Hausdorff left G-space equipped with a continuous G-equivariant inclusion of G as an open dense subspace, which has the following property: any (continuous) G-equivariant map from G into a compact Hausdorff left G-space K extends uniquely to a continuous G-equivariant map from $\beta^{lu}(G)$ into K. When G is discrete, the space $\beta^{lu}(G)$ is exactly the Stone-Čech compactification of the group G.

The implication (1) $\Rightarrow$ (2) in the next theorem is proved by Anantharaman-Delaroche in [3]. Deprez and I proved the converse implication and that both conditions in fact characterize property A for all locally compact second countable groups:

THEOREM 3.6. ([A]) *Let G be a locally compact second countable group. T.F.A.E.:*

(1) *G is amenable at infinity.*

(2) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a (continuous) positive type kernel $k : G \times G \to \mathbb{C}$ such that $\operatorname{supp} k \subseteq \operatorname{Tube}(L)$ and $\sup_{(s,t)\in \operatorname{Tube}(K)} |k(s,t) - 1| < \varepsilon$.*

(3) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a (continuous) map $\xi : G \to L^2(G)$ such that $||\xi_t||_2 = 1$, $\operatorname{supp} \xi_t \subseteq tL$ for every $t \in G$ and $\sup_{(s,t)\in \operatorname{Tube}(K)} ||\xi_s - \xi_t||_2 < \varepsilon$.*

(4) *G has property A.*

As a consequence of Theorem 3.6, all locally compact second countable groups with property A embed coarsely into Hilbert spaces. In fact, Deprez and I showed the following:

THEOREM 3.7. ([A]) *Let G be a locally compact second countable group. T.F.A.E.:*

(1) *G admits a coarse embedding into a Hilbert space.*

(2) *G admits an action with Haagerup property on some compact Hausdorff space.*

(3) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exists a (continuous) positive type kernel $k : G \times G \to \mathbb{C}$ such that $\sup_{(s,t)\in \operatorname{Tube}(K)} |k(s,t) - 1| < \varepsilon$ and for every $\delta > 0$, there exists a compact subset $L_\delta \subseteq G$ satisfying $|k(s,t)| > \delta \Rightarrow (s,t) \in \operatorname{Tube}(L_\delta)$.*

Recall that an action $G \curvearrowright X$ on a compact Hausdorff space X has the *Haagerup property* if the action admits a continuous proper conditionally negative type function. More precisely, a conditionally negative type function of the action $G \curvearrowright X$ is a function $\psi : X \times G \to \mathbb{R}$ such that
1) $\psi(x, e) = 0$ for all $x \in X$;
2) $\psi(x, g) = \psi(g^{-1}x, g^{-1})$ for all $(x, g) \in X \times G$;
3) $\sum_{i,j=1}^n t_i t_j \psi(g_i^{-1}x, g_i^{-1}g_j) \leq 0$ for all $\{t_i\}_{i=1}^n \subseteq \mathbb{R}$ satisfying $\sum_{i=1}^n t_i = 0$, $g_i \in G$ and $x \in X$.
Deprez and I applied Higson's descent technique in [**47**] and the going–down functor of Chabert, Echterhoff and Oyono-Oyono in [**14**], to obtain an analogue result of Skandalis, Tu and Yu (see [**81**]):

THEOREM 3.8. ([A]) *If G is a locally compact second countable group which admits a coarse embedding into a Hilbert space, then the Baum-Connes assembly map*

$$\mu_A : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_r G)$$

is split-injective for any separable G-C^*-algebra A.

We conclude this section by the following implications:

$$\begin{tikzcd}
G \text{ has property A} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & G \curvearrowright \beta^{lu}(G) \text{ is topologically amenable} \arrow[d, Rightarrow] \\
G \text{ is coarsely embedded into Hilbert space} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & G \curvearrowright \beta^{lu}(G) \text{ has Haagerup property} \\
\text{The Strong Novikov Conjecture} &
\end{tikzcd}$$

4. Property A pairs and proper cocycles

The material in this section is joint work with Steven Deprez and is based on the preprint "Permanence properties of property A and coarse embeddability for locally compact groups". Since the preprint contains some errors, I decide to rewrite a part of the preprint.

In this section we will discuss many important permanence properties of property A and coarse embeddability for locally compact groups. It is clear that property A and coarse embeddability into Hilbert spaces pass to closed subgroups. In the following we will be interested in the other direction: if G is a locally compact second countable group and a closed subgroup $H \subseteq G$ has property A or is coarsely embeddable into Hilbert spaces, under which conditions on the inclusion $H \subseteq G$ can we conclude that G has property A or is coarsely embeddable into Hilbert spaces?

THEOREM 4.1. *Let G and H be locally compact second countable groups. In each of the following situations, if H has property A or is coarsely embeddable into a Hilbert space, then G has property A or is coarsely embeddable into a Hilbert space.*

(1) *$H \subseteq G$ is a closed co-compact subgroup.*
(2) *$H \subseteq G$ is a closed co-amenable subgroup.*
(3) *H is a closed normal subgroup of G and the quotient group G/H has property A.*
(4) *$H = G/Q$ where $Q \subseteq G$ is a compact normal subgroup.*

Note that property A is closed under extensions. However, we have to require that H is coarsely embeddable into a Hilbert space and the quotient group G/H has property A in order to conclude that G is coarsely embeddable into a Hilbert space too. The reason is that the analogue condition of (3) in Theorem 3.6 can not be obtained in coarse embedding case (compare Theorem 3.7). Recall that a closed subgroup H of G is *co-compact* if the quotient space G/H is compact and is *co-amenable* if the homogeneous space G/H is amenable in Eymard's sense [**32**] that the quasi-regular representation $\lambda_{G/H}$ weakly contains the trivial representation 1_G. It is well-known that if the homogeneous space G/H is amenable, then H is amenable if and only if G is amenable (see e.g. [**9**, Corollary G.3.8]). In particular, the co-compactness does not imply the co-amenability (e.g. let $G = SL_2(\mathbb{R})$ and H be the closed subgroup of upper triangular matrices) unless the closed subgroup H is normal or discrete.

All the statements in Theorem 4.1 are special cases of a more general result, which is proved by Deprez and me. The crucial ingredients of this result are proper cocycles inspired by Jolissaint [**53, 54**] and property A pairs inspired by amenable pairs [**36, 32, 90, 52**].

Let $G \curvearrowright (X,\mu)$ be a measurable action of a locally compact second countable group G on a standard probability space (X,μ) (e.g. $X = [0,1]$ and μ the Lebesgue measure). We say the action is *non-singular* if the action preserves the measure class of μ (i.e. $\mu(A) = 0 \Leftrightarrow \mu(gA) = 0$ for every measurable set A and every $g \in G$). A family $\mathcal{A}$ of measurable sets in X is said to be *large* if it is closed under finite unions and under taking measurable subsets and moreover, for every $\varepsilon > 0$ there is a measurable set $A \in \mathcal{A}$ such that $\mu(X \backslash A) < \varepsilon$. It is clear that the σ-algebra on X is always large. If the standard probability space (X,μ) is also a σ-compact locally compact Hausdorff space, it is not hard to see that the family $\mathcal{A}_0$ of all precompact Borel sets in X is large. The reason to consider large families is explained by the following lemma:

LEMMA 4.2. *Let $\mathcal{A}$ be a large family on a standard probability space (X,μ) and $1 \leq p < \infty$. Then for every $\varepsilon > 0$ and $\xi \in L^p(X,\mu)$, there exist a measurable set $A \in \mathcal{A}$ and $\xi_0 \in L^p(X,\mu)$ such that $\operatorname{supp} \xi_0 \subseteq A$ and $||\xi - \xi_0||_p < \varepsilon$. In fact, we may choose ξ_0 to be $1_A\xi$.*

PROOF. Since the family $\mathcal{A}$ is large, we choose a sequence $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\mu(X\backslash B_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Since the family $\mathcal{A}$ is closed under finite unions, we define $A_n := \cup_{i=1}^n B_i \in \mathcal{A}$. Obviously, $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of elements in $\mathcal{A}$ satisfying $\mu(X\backslash \cup_{n=1}^\infty A_n) = 0$. It follows from Lebesgue's dominated convergence theorem that $||1_{A_n}\xi - \xi||_p \to 0$, for $n \to \infty$. The conclusion follows easily from this fact. □

DEFINITION 4.3. Let G and H be locally compact second countable groups and let $G \curvearrowright (X,\mu)$ be a non-singular measurable action on a standard probability space (X,μ).

- A measurable map $\omega : G \times X \to H$ is called a *cocycle* if the relation

$$\omega(gh,x) = \omega(g,hx)\omega(h,x)$$

holds for all $g, h \in G$ and for almost every $x \in X$.

- Let $\mathcal{A}$ be a large family on (X, μ). A cocycle $\omega : G \times X \to H$ is said to be *proper* with respect to $\mathcal{A}$ if
  (1) for every compact subset $K \subseteq G$ and every $A, B \in \mathcal{A}$, there exists a precompact subset $L(K, A, B)$ of H such that
  $$\omega(g, x) \in L(K, A, B)$$
  for all $g \in K$ and almost every $x \in A \cap g^{-1}B$.
  (2) for every compact subset $L \subseteq H$ and every $A, B \in \mathcal{A}$, the set
  $$K(L, A, B) := \{g \in G : \mu(\{x \in A \cap g^{-1}B : \omega(g, x) \in L\}) > 0\}$$
  is precompact in G.

EXAMPLE 4.4. Let G be a locally compact second countable group.

a) Let $H \subseteq G$ be a closed subgroup and consider the homogeneous G-space $X = G/H$ equipped with any quasi-invariant probability measure μ. Let $s : X \to G$ be a regular Borel section (see [**66**, Lemma 1.1]), i.e. $s(x)H = x$ for all $x \in X$ and $s(K)$ is precompact in G for every compact subset $K \subseteq X$. Then it is not difficult to check that the Borel cocycle $\omega : G \times X \to H$ defined by $\omega(g, x) = s(gx)^{-1}gs(x)$ is proper with respect to the large family $\mathcal{A}_0$ of all precompact Borel sets in X.
b) Let $Q \subseteq G$ be a closed normal subgroup and $H = G/Q$ be the quotient group. Moreover, let X be the one-point space with the counting measure and $\pi : G \times X \to H$ be the quotient map. Then the Borel cocycle π is proper with respect to the (trivial) σ-algebra on X if and only if Q is compact, which is also equivalent to the topological properness of the continuous map $\pi : G \to G/Q$.

In the following, we will recall the notion of amenable pairs and then define the notion of property A pairs. The properties and relations of these two notions will also be discussed. Let $G \curvearrowright (X, \mu)$ be a non-singular measurable action on a standard probability space (X, μ). *The Koopman representation* $\pi_X : G \to \mathcal{U}(L^2(X, \mu))$ of the action $G \curvearrowright (X, \mu)$ is given by

$$(\pi_X(g)\xi)(x) = \xi(g^{-1}x)\sqrt{\chi(g^{-1}, x)}, \text{ where } g \in G,\ x \in X \text{ and } \xi \in L^2(X, \mu).$$

Here $\chi : G \times X \to \mathbb{R}^*_+$ denotes the Radon-Nikodym derivative given by $\chi(g, x) := \frac{dg\mu}{d\mu}(x)$.

THEOREM 4.5 ([**90, 52**]). *Let $G \curvearrowright (X, \mu)$ be a non-singular measurable action on a standard probability space (X, μ). The following are equivalent:*

(1) *The Koopman representation π_X weakly contains the trivial representation 1_G: there is a sequence of unit vectors (ξ_n) in $L^2(X, \mu)$ satisfying $||\xi_n - \pi_X(g)\xi_n||_2 \to 0$ uniformly on compact subsets of G.*
(2) *There exists a G-invariant state on $L^\infty(X, \mu)$.*
(3) *The Koopman representation π_X is amenable in the sense of Bekka [**11**]: there exists a state M on $B(L^2(X, \mu))$ satisfying $M(\pi_X(g)T\pi_X(g^{-1})) = M(T)$ for $g \in G$ and $T \in B(L^2(X, \mu))$.*

If one of the conditions in Theorem 4.5 holds, we say that (G, X) is an *amenable pair*.

REMARK 4.6. The definition of amenable pairs is independent on the choices of the quasi-invariant measures within the same measure class. This is because two equivalent measures μ and ν give rise to unitarily equivalent Koopman representations and the intertwining unitary $U : L^2(X, \nu) \to L^2(X, \mu)$ is given by

$$U(f)(x) = \left(\frac{d\nu}{d\mu}(x)\right)^{\frac{1}{2}} f(x), \text{ for almost every } x \in X.$$

The basic example is that if μ is a G-invariant probability measure on X, then the pair (G, X) is amenable. It is well-known that a locally compact group G is amenable if and only if every pair (G, X) is amenable [**11**, Theorem 2.2]. If $X = G/H$ for H is a closed subgroup, then $\pi_{G/H}$ is equal to the quasi-regular representation $\lambda_{G/H}$. In particular, the pair $(G, G/H)$ is amenable if and only if the homogeneous space G/H is amenable.

Definition 4.7. Let $G \curvearrowright (X, \mu)$ be a non-singular measurable action on a standard probability space (X, μ) and let $\mathcal{A}$ be a large family on (X, μ). We say that the pair (G, X) has *property A with respect to* $\mathcal{A}$ if for any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a measurable set $A \in \mathcal{A}$ and a family $(\xi_t)_{t\in G}$ of unit vectors in $L^2(X, \mu)$ such that $\operatorname{supp} \xi_t \subseteq tA$ for every $t \in G$ and $\sup_{(s,t)\in \mathrm{Tube}(K)} ||\xi_s - \xi_t||_2 < \varepsilon$.

Since $L^2(X, \mu) \ni \xi \mapsto |\xi|^2 \in L^1(X, \mu)$ is uniformly continuous on the unit sphere, we can easily replace $L^2(X, \mu)$ by $L^1(X, \mu)$ in the above definition. Moreover, the definition of property A pairs with respect to the same large family $\mathcal{A}$ is independent on the choices of the quasi-invariant measures within the same measure class. This is because the intertwining unitary U in Remark 4.6 also preserves supports as $(d\nu/d\mu)(d\mu/d\nu) = 1$ almost everywhere.

Example 4.8. Let G be a locally compact second countable group.

a) If $H \subseteq G$ is a closed subgroup, then $(G, G/H)$ has property A with respect to the σ-algebra on G/H. Moreover, if H is co-compact, then the σ-algebra on G/H coincides with the large family $\mathcal{A}_0$ of all precompact Borel sets in G/H.
b) Let $H \subseteq G$ be a closed subgroup. If G has property A, then the pair $(G, G/H)$ has property A with respect to the large family $\mathcal{A}_0$ of all precompact Borel sets in G/H.
c) Let $H \subseteq G$ be a closed normal subgroup. If the quotient group G/H has property A, then the pair $(G, G/H)$ has property A with respect to the large family $\mathcal{A}_0$ of all precompact Borel sets in G/H. As a consequence of Example 4.8 b) the converse implication is not true in general. Otherwise, property A passes to quotients.
d) An amenable pair (G, X) has property A with respect to *any large* family $\mathcal{A}$ on X.

Proof. a): Let ξ_0 be a fixed unit vector in $L^2(G/H)$, then the constant family $(\xi_0)_{t\in G}$ implies that the pair $(G, G/H)$ has property A with respect to the σ-algebra on G/H.

b): We are going to use the extended formula of Mackey-Bruhat for quasi-invariant measures (see [**75**] Section 8.2) in the following.

Recall that a rho-function for the pair (G, H) is a continuous function $\rho : G \to \mathbb{R}_+^*$ satisfying

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x),$$

for all $x \in G$ and $h \in H$. Let dx and dh be a left Haar measure on G and H, respectively. If we denote $d_\rho(xH)$ the quasi-invariant regular Borel measure on G/H associated to the rho-function ρ, then the extended formula of Mackey-Bruhat is given by

$$\int_{G/H}\int_H \frac{f(xh)}{\rho(xh)}\, dh d_\rho(xH) = \int_G f(x)\, dx, \text{ for all } f \in L^1(G, dx).$$

Define a linear map $T_{H,\rho} : L^1(G, dx) \to L^1(G/H, d_\rho(xH))$ by

$$(T_{H,\rho}f)(xH) = \int_H \frac{f(xh)}{\rho(xh)}\, dh, \text{ for all } f \in L^1(G, dx).$$

It follows from the extended formula of Mackey-Bruhat that the linear map $T_{H,\rho}$ is contractive and isometric on $L^1(G)_+$.

Given a compact subset $K \subseteq G$ and $\varepsilon > 0$. Since the group G has property A, there exist a compact subset $L \subseteq G$ and a continuous map $\eta : G \to L^1(G)$ such that $||\eta_t||_1 = 1$, $\operatorname{supp} \eta_t \subseteq tL$ for every $t \in G$ and $\sup_{(s,t)\in \mathrm{Tube}(K)} ||\eta_s - \eta_t||_1 < \varepsilon$. We may assume that each η_t is non-negative, since $|||\eta_s| - |\eta_t|||_1 \leq ||\eta_s - \eta_t||_1$ for all $s, t \in G$. In particular, $||T_{H,\rho}(\eta_t)||_1 = ||\eta_t||_1$ for every $t \in G$.

Now, define a (continuous) map $\xi := T_{H,\rho} \circ \eta : G \to L^1(G/H)_1$. If $(s,t) \in \mathrm{Tube}(K)$, then

$$||\xi_s - \xi_t||_1 = ||T_{H,\rho}(\eta_s - \eta_t)||_1 \leq ||\eta_s - \eta_t||_1 < \varepsilon.$$

Let $\dot{L}$ be the image of L under the canonical quotient map $G \to G/H$. Obviously, $\dot{L}$ is a compact subset of G/H and belongs to $\mathcal{A}_0$. It is not hard to see that $\operatorname{supp} \xi_t \subseteq t\dot{L}$ for every $t \in G$: for every $t \in G$, if $\xi_t(xH) \neq 0$ then there exists $h \in H$ such that $\eta_t(xh) \neq 0$. In particular, $xh \in \operatorname{supp} \eta_t \subseteq tL$. We conclude that $xH = xhH \in t\dot{L}$.

c): Since property A pairs is independent on the choices of the quasi-invariant measures within the same measure class, we may choose a left Haar measure on the quotient group G/H.

Given a compact subset $K \subseteq G$ and $\varepsilon > 0$. Let $q : G \to G/H$ be the quotient homomorphism. Since the quotient group G/H has property A, there exist a compact subset $L \subseteq G/H$ and a continuous map $\xi : G/H \to L^2(G/H)$ such that $||\xi_x||_2 = 1$, $\operatorname{supp} \xi_x \subseteq xL$ for every $x \in G/H$ and $\sup_{(x,y)\in \mathrm{Tube}(q(K))} ||\xi_x - \xi_y||_2 < \varepsilon$.

Define a (continuous) map $\eta := \xi \circ q : G \to L^2(G/H)$. It is clear that $||\eta_t||_2 = ||\xi_{q(t)}||_2 = 1$ for every $t \in G$. If $(s,t) \in \mathrm{Tube}(K)$, then we see that $(q(s), q(t)) \in \mathrm{Tube}(q(K))$ and

$$||\eta_s - \eta_t||_2 = ||\xi_{q(s)} - \xi_{q(t)}||_2 < \varepsilon.$$

Finally, we notice that $L \in \mathcal{A}_0$ and for every $t \in G$,

$$\operatorname{supp} \eta_t = \operatorname{supp} \xi_{q(t)} \subseteq q(t)L = tL.$$

d): Given a compact subset $K \subseteq G$ and $\varepsilon > 0$. We choose a small $\varepsilon' > 0$ such that $\varepsilon'/(1-\varepsilon'/3) < \varepsilon$. Since (G,X) is an amenable pair, there exists a unit vector ξ in $L^2(X,\mu)$ such that $\sup_{g\in K} ||\pi_X(g)\xi - \xi||_2 < \varepsilon'/3$. By Lemma 4.2 we find a measurable set $A \in \mathcal{A}$ such that $\xi_0 = 1_A \xi$ and $||\xi_0 - \xi||_2 < \varepsilon'/3$. In particular, $||\xi_0||_2 > 1 - \varepsilon'/3$.

Now, define a (continuous) map $\eta : G \to L^2(X,\mu)_1$ by $\eta_g := \pi_X(g)(\xi_0/||\xi_0||_2)$. For each $g \in G$, $\{x \in X : \eta_g(x) \neq 0\} \subseteq gA$. In particular, $\operatorname{supp} \eta_g \subseteq gA$.

If $(g,h) \in \mathrm{Tube}(K)$, we see that

$$\begin{aligned} ||\eta_g - \eta_h||_2 &= \frac{1}{||\xi_0||_2} ||\xi_0 - \pi_X(g^{-1}h)\xi_0||_2 \\ &\leq \frac{1}{||\xi_0||_2} (||\xi_0 - \xi||_2 + ||\pi_X(g^{-1}h)(\xi - \xi_0)||_2 + ||\xi - \pi_X(g^{-1}h)\xi||_2) \\ &< \frac{\varepsilon'}{1 - \varepsilon'/3} \\ &< \varepsilon. \end{aligned}$$

□

In all examples above the map $G \ni t \mapsto \xi_t \in L^2(X,\mu)$ from Definition 4.7 is continuous. However, we don't need the continuity in the proofs. It is clear that Theorem 4.1 is a consequence of Example 4.4 and Example 4.8 as well as the following theorem.

THEOREM 4.9. *Suppose that both G and H are locally compact second countable groups. Let $G \curvearrowright (X,\mu)$ be a non-singular measurable action on a standard probability space (X,μ) and $\mathcal{A}$ be a large*[2] *family on (X,μ) such that*

- *the pair (G,X) has property A with respect to $\mathcal{A}$,*
- *there exists a measurable cocycle $\omega : G \times X \to H$ that is proper with respect to $\mathcal{A}$.*

If H has property A or is coarsely embeddable into a Hilbert space, then so does G.

PROOF. Suppose that H has property A or is coarsely embeddable into a Hilbert space. Given a compact subset $K \subseteq G$ and $\varepsilon > 0$, we want to find a positive type kernel $k : G \times G \to \mathbb{C}$ satisfying the following two conditions:

[2] The largeness will not be used in the proof.

- $\sup_{(s,t)\in \mathrm{Tube}(K)} |k(s,t)-1| < \varepsilon$ and
- for every $\delta \geq 0$, there exists a compact subset $L_\delta \subseteq G$ such that $(s,t) \notin \mathrm{Tube}(L_\delta) \Rightarrow |k(s,t)| \leq \delta$.

We note that the second condition for $\delta = 0$ is exactly the same as $\operatorname{supp} k \subseteq \mathrm{Tube}(L_0)$. So it follows from Theorem 3.6 and Theorem 3.7 that G has property A or is coarsely embeddable into a Hilbert space in that case.

Since the pair (G, X) has property A with respect to $\mathcal{A}$, we find a measurable set $A \in \mathcal{A}$ and a family $(\xi_t)_{t\in G}$ of unit vectors in $L^2(X,\mu)$ such that $\operatorname{supp} \xi_t \subseteq tA$ for every $t \in G$ and

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\xi_s - \xi_t||_2 < \varepsilon/2.$$

We apply the properness of the cocycle ω in order to find a compact subset $L := \overline{L(K,A,A)}$ of H such that $\omega(s,x) \in L$ for all $s \in K$ and almost every $x \in A \cap s^{-1}A$. By the assumption on H there exists a continuous positive type kernel $k_0 : H \times H \to \mathbb{C}$ satisfying

$$\sup_{(s,t)\in \mathrm{Tube}(L)} |k_0(s,t) - 1| < \frac{\varepsilon}{2}$$

and for every $\delta \geq 0$, there exists a compact subset $M_\delta \subseteq H$ such that

$$(s,t) \notin \mathrm{Tube}(M_\delta) \Rightarrow |k_0(s,t)| \leq \delta.$$

Since a positive type kernel is bounded if and only if it is bounded on the diagonal, we may assume that k_0 is bounded. Define a new kernel $k : G \times G \to \mathbb{C}$ by the formula

$$k(s,t) = \int_X \overline{\xi_s(x)}\xi_t(x)k_0(\omega(s,s^{-1}x), \omega(t,t^{-1}x))d\mu(x).$$

The new kernel k is well-defined, because for fixed $s,t \in G$, the function

$$X \ni x \mapsto \overline{\xi_s(x)}\xi_t(x)k_0(\omega(s,s^{-1}x), \omega(t,t^{-1}x))$$

belongs to $L^1(X,\mu)$. Moreover, It is routine to verify that the kernel k is of positive type, because k_0 is the case.

We note that if $(s,t) \in \mathrm{Tube}(K)$, then $\omega(s^{-1}t, t^{-1}x) \in L$ for almost every $x \in sA \cap tA$. The cocycle relation implies that

$$\omega(s,s^{-1}x)^{-1}\omega(t,t^{-1}x) = \omega(s^{-1}t, t^{-1}x) \tag{4.1}$$

for all $s,t \in G$ and almost every $x \in X$. In particular,

$$|k_0(\omega(s,s^{-1}x), \omega(t,t^{-1}x)) - 1| < \frac{\varepsilon}{2}$$

for all $(s,t) \in \mathrm{Tube}(K)$ and almost every $x \in sA \cap tA$.

Whenever $(s,t) \in \mathrm{Tube}(K)$, we see that

$$\begin{aligned}
|k(s,t)-1| &= \Big| \int_X \overline{\xi_s(x)}\xi_t(x)(k_0(\omega(s,s^{-1}x),\omega(t,t^{-1}x)) - 1)d\mu(x) + \int_X \overline{\xi_s(x)}\xi_t(x) - 1 \; d\mu(x) \Big| \\
&\leq \int_{sA\cap tA} |\xi_s(x)||\xi_t(x)||k_0(\omega(s,s^{-1}x),\omega(t,t^{-1}x)) - 1| \; d\mu(x) + |\langle \xi_s, \xi_t\rangle - 1| \\
&< \frac{\varepsilon}{2} \cdot ||\xi_s||_2||\xi_t||_2 + ||\xi_s - \xi_t||_2||\xi_t||_2 \\
&< \varepsilon.
\end{aligned}$$

Given $\delta \geq 0$, we would like to find a compact subset $L_\delta \subseteq G$ such that

$$(s,t) \notin \mathrm{Tube}(L_\delta) \Rightarrow |k(s,t)| \leq \delta.$$

Define $L_\delta := \overline{K(M_\delta, A, A)}$. It is a compact subset of G by the properness of ω. Let $(s,t) \notin \mathrm{Tube}(L_\delta)$. Then $s^{-1}t \notin K(M_\delta, A, A)$. It follows that $\{x \in A \cap (s^{-1}t)^{-1}A : \omega(s^{-1}t, x) \in M_\delta\}$ has measure zero. Since $t^{-1}\mu$ and μ are equivalent to each other, we conclude that

$$A_{s,t} := \{x \in tA \cap sA : \omega(s^{-1}t, t^{-1}x) \in M_\delta\}$$

also has measure zero. Since each ξ_t is supported in tA, the cocycle relation 4.1 implies that

$$\begin{aligned}|k(s,t)| &= |\int_{(tA\cap sA)\setminus A_{s,t}} \overline{\xi_s(x)}\xi_t(x)k_0(\omega(s,s^{-1}x),\omega(t,t^{-1}x))d\mu(x)| \\ &\leq \delta\cdot ||\xi_s||_2||\xi_t||_2 \\ &= \delta.\end{aligned}$$

Hence, we complete the proof. □

5. Uniform Roe algebras and ghost operators

In this section, we will introduce uniform Roe algebras and ghost operators and explain how these notions can be used to characterize Yu's property A. We will also discuss the connection between Yu's property A and property A on locally compact second countable groups G through metric lattices in G.

Definition 5.1. Let (X, d_X) and (Y, d_Y) be metric spaces. We say a map $f : X \to Y$ is *coarse* if the inverse image under f of any bounded subset in Y is bounded in X and if for any $R > 0$, there exists $S > 0$ such that $d_X(x, x') \leq R$ implies $d_Y(f(x), f(x')) \leq S$ for any $x, x' \in X$. Two coarse maps $f, g : X \to Y$ are *close* if $d_Y(f(x), g(x))$ is bounded on X.

Definition 5.2. We say that two metric spaces X and Y are *coarsely equivalent* if there exist two coarse maps $f : X \to Y$ and $g : Y \to X$ such that $g\circ f$ and $f\circ g$ are close to the identity maps on X and Y, respectively.

It is clear that a coarse embedding (see Definition 3.1) is a coarse map. In fact, a map $f : X \to Y$ is a coarse embedding if and only if it induces a coarse equivalence between X and $f(X)$ (see [40, Proposition A.2]). It is well-known that property A and coarse embeddability into Hilbert spaces both are coarsely invariant properties (See [87]).

Definition 5.3. A metric space (Z, d) is *uniformly locally finite* if $\sup_{z\in Z} |B(z, S)| < \infty$ for all $S > 0$, where $B(z, S)$ denotes the closed ball $\{x \in Z : d(z, x) \leq S\}$.

Since finite metric space is discrete and $Z = \cup_{n\in\mathbb{N}} B(z_0, n)$ for any $z_0 \in Z$, a uniformly locally finite metric space is necessarily discrete and countable. However, the uniformly local finiteness is not coarsely invariant even on uniformly discrete spaces [45, Example 3.4]:

Example 5.4. Consider the triple (D_n, d_n, x_n), where D_n is the discrete space with n points, d_n is the discrete metric on D_n and each x_n is a fixed element in D_n.
Let $Z = \sqcup_{n\in\mathbb{N}} D_n$ equipped with the following metric d:

$$d(z, y) = d_{j(z)}(z, x_{j(z)}) + |j(z) - j(y)| + d_{j(y)}(y, x_{j(y)}),$$

where $j(x) = n$ if and only if $x \in D_n$. So (Z, d) is a proper uniformly discrete space. It is not hard to see that Z is coarsely equivalent to $\mathbb{N}$, which is uniformly locally finite. In particular, (Z, d) has bounded geometry (see definition below). However, the metric space (Z, d) itself is not uniformly locally finite as $|B(x_n, 1)| \geq n$.

In order to make it coarsely invariant, we consider the following notion on metric spaces.

Definition 5.5. A metric space (X, d) has *bounded geometry* if it is coarsely equivalent to a uniformly locally finite (discrete) metric space (Z, d).

In fact, we can choose (Z, d) in the above definition to be a "lattice" of X:

Definition 5.6. Let (X, d) be a metric space. We say that a uniformly discrete subspace $Z \subseteq X$ is a *metric lattice*, if there is $R > 0$ such that $X = \cup_{z\in Z} B(z, R)$.

Note that the inclusion map $Z \subseteq X$ is a coarse equivalence and it follows easily from Zorn's lemma that every metric space always contains metric lattices.

Proposition 5.7 ([26], Proposition 3.D.15). *For a metric space (X, d), the following conditions are equivalent:*

(1) *The space (X, d) has bounded geometry.*
(2) *The space (X, d) contains a uniformly locally finite metric lattice (Z, d).*

Recall that every locally compact second countable group G admits a proper left-invariant compatible metric d and such a metric is unique up to coarse equivalence. Moreover, the proper metric group (G, d) has bounded geometry. There is a nice connection between Yu's property A and property A on locally compact groups through metric lattices.

Proposition 5.8 ([A], Theorem 2.3 and [78], Lemma 2.2). *Let G be a locally compact second countable group equipped with a proper left-invariant compatible metric d. Then the following conditions are equivalent:*

(1) *The group G has property A.*
(2) *Every uniformly locally finite metric lattice (Z, d) in G has Yu's property A.*
(3) *There exists a uniformly locally finite metric lattice (Z, d) in G satisfying Yu's property A.*

It is clear from Definition 3.5 that coarse embeddability into Hilbert spaces for locally compact groups is a coarsely invariant property. The proposition above implies the same conclusion for property A on locally compact groups.
There is a well-known operator algebraic characterization of Yu's property A on uniformly locally finite metric spaces in terms of uniform Roe algebras. Let us recall the definition of the uniform Roe algebra $C_u^*(Z)$ of a uniformly locally finite metric space (Z, d). Every $a \in B(l^2(Z))$ can be represented as a $Z \times Z$ matrix: $a = [a_{x,y}]_{x,y \in Z}$, where $a_{x,y} := \langle a\delta_y, \delta_x \rangle \in \mathbb{C}$. We define the *propagation* of $a = [a_{x,y}]_{x,y \in Z} \in B(l^2(Z))$ by

$$\sup\{d(x, y) : x, y \in Z, a_{x,y} \neq 0\}.$$

Let E_R be the set of all bounded operators on $l^2(Z)$ whose propagations are at most R. In fact, E_R is an operator system, i.e., a self-adjoint closed subspace of $B(l^2(Z))$ which contains the unit of $B(l^2(Z))$. Moreover, the union $\bigcup_{R>0} E_R$ is a $*$-subalgebra of $B(l^2(Z))$.

Definition 5.9. The C^*-algebra defined by the operator norm closure in $B(l^2(Z))$

$$C_u^*(Z) = \overline{\bigcup_{R>0} E_R}$$

is called the *uniform Roe algebra* of Z.

Theorem 5.10. [81, Theorem 5.3] *Let (Z, d) be a uniformly locally finite metric space. The following conditions are equivalent:*

(1) *The metric space (Z, d) has Yu's property A.*
(2) *The uniform Roe algebra $C_u^*(Z)$ is nuclear.*

Definition 5.11. (Yu) An operator $a \in C_u^*(Z)$ is called a *ghost* if $a_{x,y} \to 0$ as $x, y \to \infty$. We denote by $G^*(Z)$ the collection of all ghost operators, which forms a closed two sided ideal in $C_u^*(Z)$ and contains the compact operators on $l^2(Z)$.

A natural question is that are all ghost operators compact? It is easy to prove that for a uniformly locally finite space with Yu's property A, all ghost operators are compact ([77, Theorem 11.43]). Recently, the converse implication is proved by Roe and Willett:

Theorem 5.12. [79] *A uniformly locally finite metric space without Yu's property A always admits non-compact ghosts.*

6. Approximation properties for locally compact groups

The notion of amenability for groups was first introduced by von Neumann in order to study the Banach-Tarski paradox [86]. It is well-known that amenability has numerous characterizations and one of them, proved by Hulanicki around 1960, is the following: a locally compact Hausdorff group G is *amenable* if and only if there exists a net of continuous compactly supported, positive type functions on G tending to the constant function 1 uniformly on compact subsets of G [73].

In [41], Haagerup proved that the constant function 1 on free groups can be approximated pointwise by positive type functions vanishing at infinity. Since free groups are not amenable, it can not be approximated pointwise by compactly supported, positive type functions. In [41], Haagerup also showed that the Fourier algebra of the free groups admits an approximate unit which is bounded in multiplier norm. It is weaker than amenability, because the Fourier algebra of a locally compact group has an approximate unit bounded in norm if and only if the group is amenable [65]. In the light of Haagerup's ground-breaking work, two weak forms of amenability were introduced in the 1980s: the Haagerup property by Connes [22] and Choda [19] and the next one is weak amenability by Cowling and Haagerup [24]. They extend amenability in different directions as follows:

Definition 6.1 (Haagerup property [18]). A locally compact group G has the *Haagerup property* if there exists a net of positive type C_0-functions on G, converging uniformly to 1 on compact sets.

Definition 6.2 (Weak amenability [24]). A locally compact group G is *weakly amenable* if there exists a net $(\varphi_i)_{i\in I}$ of continuous, compactly supported Herz-Schur multipliers on G, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.

The *weak amenability constant* $\Lambda_{\mathrm{WA}}(G)$ is defined as the best (lowest) possible constant Λ such that $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$, where $(\varphi_i)_{i\in I}$ is as just described.

As we have already mentioned in Section 2, the Haagerup property implies the Baum-Connes conjecture with coefficients. It is natural to ask about the relation between the Haagerup property and weak amenability. In general, weak amenability does not imply the Haagerup property and vice versa. However, the following question was conjectured by Cowling [18]:

Conjecture 3 (The Cowling's Conjecture). Let G be a locally compact group. Is G weakly amenable with $\Lambda_{\mathrm{WA}}(G) = 1$ if and only if G has the Haagerup property?

In one direction of the conjecture, the group $\mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property [27], but is not weakly amenable, i.e., $\Lambda_{\mathrm{WA}}(G) = \infty$ [71]. In order to study the other direction, the weak Haagerup property was introduced in [61, 62], and the following questions were considered.

Definition 6.3 (The weak Haagerup property [62]). A locally compact group G has the *weak Haagerup property* if there exists a net $(\varphi_i)_{i\in I}$ of C_0 Herz-Schur multipliers on G, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.

The *weak Haagerup constant* $\Lambda_{\mathrm{WH}}(G)$ is defined as the best (lowest) possible constant Λ such that $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$, where $(\varphi_i)_{i\in I}$ is as just described.

Question 6.4. *For which locally compact groups G do we have $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$?*

Question 6.5. *Is $\Lambda_{\mathrm{WH}}(G) = 1$ if and only if G has the Haagerup property?*

Both Question 6.4 and Question 6.5 have positive answer for connected simple Lie groups by the work of many authors [23, 24, 25, 30, 42, 46, 43, 18]. Knudby and I consider the same class of groups, but made discrete:

Theorem 6.6. ([B]) *Let G be a connected simple Lie group, and let G_{d} denote the group G equipped with the discrete topology. The following are equivalent.*

(1) *G is locally isomorphic to* $\mathrm{SO}(3)$, $\mathrm{SL}(2,\mathbb{R})$ *or* $\mathrm{SL}(2,\mathbb{C})$.

(2) G_{d} *has the Haagerup property.*
(3) G_{d} *is weakly amenable with constant 1.*
(4) G_{d} *is weakly amenable.*
(5) G_{d} *has the weak Haagerup property with constant 1.*
(6) G_{d} *has the weak Haagerup property.*

In order to obtain the above theorem Knudby and I prove the following result, which follows ideas of Guentner, Higson and Weinberger [**39**].

THEOREM 6.7. ([B]) *Let K be any field. The discrete group* $\mathrm{GL}(2, K)$ *is weakly amenable with constant 1.*

The remaining of the section is devoted to compare the approximation properties mentioned so far with property A and coarse embeddability into Hilbert spaces. It follows clearly from Theorem 3.6 and Theorem 3.7 that locally compact amenable groups have property A and locally compact groups satisfying the Haagerup property are coarsely embeddable into Hilbert spaces. It is also well-known that *discrete* countable weakly amenable groups have property A [**42, 63, 70, 50**]. In the sequel, I will show that the same statement is even true for all locally compact second countable groups. The idea of the proof is to give a uniformly bounded Schur multiplier characterization of property A.
Let us start by recalling some basic definitions. Let X be a non-empty *set* and a kernel $k : X \times X \to \mathbb{C}$ is called a *Schur multiplier* on X if for every operator $a = [a_{x,y}]_{x,y \in X} \in B(l^2(X))$ the matrix $[k(x, y)a_{x,y}]_{x,y \in X}$ represents an operator in $B(l^2(X))$, denoted $m_k(a)$. If k is a Schur multiplier, it follows easily from the closed graph theorem that m_k defines a bounded operator on $B(l^2(X))$. We define the Schur norm $||k||_S$ to be the operator norm $||m_k||$ of m_k. For instance, any normalized (i.e., $k(x, x) = 1$ for every $x \in X$) positive type kernel is a Schur multiplier of norm 1. The following characterization of Schur multipliers is well-known and is essentially due to Grothendieck.

THEOREM 6.8 ([**74**], Theorem 5.1). *Let $k : X \times X \to \mathbb{C}$ be a function, and let $C \geq 0$ be given. The following conditions are equivalent:*

(1) *k is a Schur multiplier with $||k||_S \leq C$.*
(2) *There exist a Hilbert space H and two bounded maps $\xi, \eta : X \to H$ such that $k(x, y) = \langle \eta_y, \xi_x \rangle$ for all $x, y \in X$ and $\sup_{x \in X} ||\xi_x|| \cdot \sup_{y \in X} ||\eta_y|| \leq C$.*

Let G be a locally compact group. A continuous function $\varphi : G \to \mathbb{C}$ is a *Herz-Schur multiplier* if and only if the (continuous) function $\widehat{\varphi} : G \times G \to \mathbb{C}$ defined by

$$\widehat{\varphi}(s, t) = \varphi(s^{-1}t), \quad s, t \in G$$

is a Schur multiplier on G. We denote by $B_2(G)$ the Banach space of Herz-Schur multipliers on G equipped with the Herz-Schur norm $||\varphi||_{B_2} = ||\widehat{\varphi}||_S$.
The proof of the coming proposition is identical to the one used to prove [**77**, Theorem 11.43].

PROPOSITION 6.9. *Let (Z, d) be a uniformly locally finite metric space. If there is a constant $C > 0$ such that for any $R > 0$ and $\varepsilon > 0$, there exist $S > 0$ and a Schur multiplier $k : Z \times Z \to \mathbb{C}$ with $||k||_S \leq C$ such that*

- *If $d(x, y) > S$, then $k(x, y) = 0$.*
- *If $d(x, y) \leq R$, then $|k(x, y) - 1| < \varepsilon$.*

Then every ghost operator in $C^*_u(Z)$ is compact. In particular, it follows from Theorem 5.12 that the metric space (Z, d) has Yu's property A.

PROOF. For each $n \in \mathbb{N}$ there exist $S_n > 0$ and a Schur multiplier $k_n : Z \times Z \to \mathbb{C}$ with $||k_n||_S \leq C$ such that $|k_n(x, y) - 1| < \frac{1}{n}$ for $d(x, y) \leq n$ and $k_n(x, y) = 0$ for $d(x, y) > S_n$. From Theorem 6.8 we note that $\sup_{x,y \in Z} |k_n(x, y)| \leq C$ for all $n \in \mathbb{N}$.

Let $m_{k_n} : B(l^2(Z)) \to B(l^2(Z))$ be the bounded operator associated with k_n. For every $R > 0$ and any $a \in E_R$, it is not hard to see that

$$||m_{k_n}(a) - a|| \leq \sup_{z\in Z} |B(z,R)| \cdot ||a|| \cdot \sup_{\{(x,y)\in Z\times Z:\ d(x,y)\leq R\}} |k_n(x,y) - 1| \to 0, \text{ for } n \to \infty.$$

Since $||m_{k_n}|| \leq C$ for all $n \in \mathbb{N}$, we have that $||m_{k_n}(a) - a|| \to 0$ for all $a \in C_u^*(Z)$.
If $H \in C_u^*(Z)$ is a ghost operator, then each $m_{k_n}(H)$ is a compact operator (see [**16**, Theorem 3.1]). So H is compact, as $m_{k_n}(H) \to H$ in the operator norm. □

The following theorem extends [**29**, Theorem 6.1], which proved the same statement for discrete groups. However, the proof of [**29**, Theorem 6.1] relies heavily on the discreteness of groups. For instance, it used the fact from [**59**] that if a reduced discrete group C^*-algebra is exact, then the discrete group itself is exact.

THEOREM 6.10. *Let G be a locally compact second countable group. The following conditions are equivalent:*

(1) *The group G has property A.*
(2) *If there is a constant $C > 0$ such that for any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a (continuous) Schur multiplier $k : G \times G \to \mathbb{C}$ with $||k||_S \leq C$ such that $\operatorname{supp} k \subseteq \operatorname{Tube}(L)$ and $\sup_{(s,t)\in \operatorname{Tube}(K)} |k(s,t) - 1| < \varepsilon$.*

If G is weakly amenable, then in particular the group G has property A.

PROOF. (1) ⇒ (2): It follows easily from Theorem 3.6. In fact, we can take $C = 1$ if we assume that the positive type kernel in Theorem 3.6 is normalized.
(2) ⇒ (1): Let d be a proper left-invariant compatible metric on G such that the metric group (G, d) has bounded geometry. By the assumption, any uniformly locally finite metric lattice Z in (G, d) satisfies the assumption in Proposition 6.9, and hence (Z, d) has Yu's property A. We complete the proof by applying Proposition 5.8. □

In a joint work with Knudby, we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (see also [**29**, Theorem 5.3] for the discrete case) and this characterization can be regarded as an answer to the non-equivariant version of Question 6.5.

THEOREM 6.11. ([C]) *Let G be a locally compact second countable group. The following conditions are equivalent:*

(1) *The group G embeds coarsely into a Hilbert space.*
(2) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exists a (continuous) Schur multiplier $k : G \times G \to \mathbb{C}$ with $||k||_S \leq 1$ such that $\sup_{(s,t)\in \operatorname{Tube}(K)} |k(s,t) - 1| < \varepsilon$ and for every $\delta > 0$, there exists a compact subset $L_\delta \subseteq G$ satisfying $|k(s,t)| > \delta \Rightarrow (s,t) \in \operatorname{Tube}(L_\delta)$.*

If G has the weak Haagerup property with constant 1, then in particular the group G embeds coarsely into a Hilbert space.

7. Topologically amenable actions and crossed products of C^*-algebras

In this section we recall some definitions and state a few results on topologically amenable actions and crossed products of C^*-algebras. We refer to [**1, 3**] for topologically amenable actions and refer to [**72, 88**] for crossed products of C^*-algebras.
Let $\operatorname{Prob}(G)$ denote the space of Borel probability measures on a locally compact group G. It is the state space of the C^*-algebra $C_0(G)$ and it carries two natural topologies: the norm topology and the weak-$*$ topology. Recall that a locally compact group G acts on a locally compact Hausdorff space X if there exists a homomorphism $\alpha : G \to \operatorname{Homoe}(X)$ such that the map $G \times X \to X$ given by $(g, x) \mapsto \alpha(g)(x)$ is continuous.

DEFINITION 7.1. [1] We say that the action $G \curvearrowright X$ is *topologically amenable* if there exists a net $(m_i)_{i\in I}$ of weak-$*$ continuous maps $x \mapsto m_i^x$ from X into the space $\mathrm{Prob}(G)$ such that

$$\lim_i ||s.m_i^x - m_i^{s.x}|| = 0$$

uniformly on compact subsets of $X \times G$.

DEFINITION 7.2. [3] We say that a locally compact group G is *amenable at infinity* if it admits a topologically amenable action on a compact Hausdorff space X.

Since the following lemma is obvious from the above definition, we omit the proof.

LEMMA 7.3. *Let X and Y be compact Hausdorff G-spaces. Assume that there exists a continuous G-equivariant map $f : X \to Y$. If the action $G \curvearrowright Y$ is topologically amenable, so is the action $G \curvearrowright X$.*

If a locally compact group G is amenable at infinity, then there are some canonical choices of compact spaces on which the group G acts topologically amenable. For instance, let us denote by $C_b^{lu}(G)$ the C^*-algebra of bounded left-uniformly continuous functions on G. Let $\beta^{lu}(G)$ be the spectrum of $C_b^{lu}(G)$ and it is the universal compact Hausdorff left G-space equipped with a continuous G-equivariant inclusion of G as an open dense subspace. Let $\partial G := \beta^{lu}(G)\backslash G$ denote the boundary of the group G. It is also a compact Hausdorff space and the left translation action of G on $\beta^{lu}(G)$ restricts to an action on ∂G. The inclusion map from ∂G into $\beta^{lu}(G)$ is clearly equivariant. We obtain the following result from the lemma stated above.

PROPOSITION 7.4. [3, Proposition 3.4] *Let G be a locally compact group. The following conditions are equivalent:*

(1) *G is amenable at infinity.*
(2) *The left translation action of G on $\beta^{lu}(G)$ is topologically amenable.*
(3) *The left translation action of G on ∂G is topologically amenable.*

DEFINITION 7.5. Let G be a locally compact group and A be a G-C^*-algebra equipped with the action α. A *covariant representation* of the G-C^*-algebra A is a pair (π, U) where $\pi : A \to B(H)$ is a $*$-homomorphism and $U : G \to B(H)$ is a unitary representation of G such that $U_s\pi(a)U_s^* = \pi(\alpha_s(a))$ for all $s \in G$ and $a \in A$.

We denote by $C_c(G, A)$ the vector space of continuous A-valued functions on G with compact support. Define a convolution product and involution on $C_c(G, A)$ by

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))\ d\mu(t) \qquad \text{and} \qquad f^*(s) = \frac{\alpha_s(f(s^{-1})^*)}{\Delta(s)},$$

where Δ is the modular function and μ is a left Haar measure on G, respectively. In this way, $C_c(G, A)$ becomes a $*$-algebra.
Given a covariant representation (π, U) of a G-C^*-algebra A on a Hilbert space H. Then

$$\pi \rtimes U(f) = \int_G \pi(f(s))U_s\ d\mu(s)$$

defines a $*$-representation of $C_c(G, A)$ on the Hilbert space H.

DEFINITION 7.6. Let G be a locally compact group and A be a G-C^*-algebra equipped with the action α. The *full crossed product* $A \rtimes_\alpha G$ is the completion of $C_c(G, A)$ with respect to the universal C^*-norm $||\cdot||_u$ given by

$$||f||_u := \sup\{||\pi \rtimes U(f)|| : (\pi, U) \text{ is a covariant represention of } A\}.$$

It is clear from the definition of the full crossed product, $\pi \rtimes U$ extends to a $*$-representation of $A \rtimes_\alpha G$ for every covariant representation (π, U) of a G-C^*-algebra A.

To define the reduced crossed product, we begin with a *faithful* $*$-representation π of the G-C^*-algebra A on a Hilbert space H. Define a $*$-representation π_α^A of A on $L^2(G,H)$ by

$$(\pi_\alpha^A(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t),$$

for $a \in A$, $t \in G$ and $\xi \in L^2(G,H)$. Let λ denote the left regular representation of G on $L^2(G)$. Then $(\pi_\alpha^A, \lambda \otimes 1)$ is a covariant representation of A. The *regular representation* $\pi_\alpha^A \rtimes (\lambda \otimes 1)$ of $C_c(G,A)$ on $L^2(G,H)$ is easily seen to be faithful. In particular, the universal C^*-norm $||\cdot||_u$ really is a norm.

DEFINITION 7.7. Let G be a locally compact group and A be a G-C^*-algebra equipped with the action α. The *reduced crossed product* $A \rtimes_{\alpha,r} G$ is the completion of $C_c(G,A)$ with respect to the reduced C^*-norm $||\cdot||_r$ given by

$$||f||_r := ||\pi_\alpha^A \rtimes (\lambda \otimes 1)(f)||.$$

It is well-known that the reduced crossed product $A \rtimes_{\alpha,r} G$ does not depend on the choice of the faithful representation $\pi : A \to B(H)$. Moreover, we have a natural surjective $*$-homomorphism $A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$.

For a given locally compact group G, the full crossed product $(-)\rtimes G$ and the reduced crossed product $(-) \rtimes_r G$ form functors from the category of G-C^*-algebras to the category of C^*-algebras. Indeed, let A and B be G-C^*-algebras with actions α and β, respectively. If $\theta : A \to B$ is an equivariant $*$-homomorphism, then there are two canonical $*$-homomorphisms $\theta_u : A \rtimes_\alpha G \to B \rtimes_\beta G$ and $\theta_r : A \rtimes_{\alpha,r} G \to B \rtimes_{\beta,r} G$ whose restrictions in each case to A and G are θ and id_G, respectively.

EXAMPLE 7.8. Let $C_b(G)$ be the space of bounded continuous complex valued functions on G. Let $M : C_b(G) \to B(L^2(G))$ be the multiplication operator given by

$$(M(f)\xi)(x) = f(x)\xi(x),$$

where $f \in C_b(G)$, $\xi \in L^2(G)$ and $x \in G$. It is clear that M is a faithful $*$-representation.
Let L and R be the left and right translations on $C_b(G)$, respectively. More precisely,

$$(L_g f)(x) = f(g^{-1}x) \qquad \text{and} \qquad (R_g f)(x) = f(xg),$$

for $f \in C_b(G)$ and $x, g \in G$. We denote the space of bounded left (right) uniformly continuous functions on G by $C_b^{lu}(G)$ (respectively $C_b^{ru}(G)$). We have the left and right regular representations $\lambda, \rho\colon G \to U(L^2(G))$ given by

$$(\lambda_g\xi)(x) = \xi(g^{-1}x) \qquad \text{and} \qquad (\rho_g\xi)(x) = \xi(xg)\Delta(g)^{1/2},$$

where $\xi \in L^2(G)$ and $g, x \in G$. It is not hard to show that (M,λ) and (M,ρ) are covariant representations of the C*-dynamical systems $(C_b^{lu}(G), G, L)$ and $(C_b^{ru}(G), G, R)$, respectively. It is well-known that $M \rtimes \rho : C_b^{ru}(G) \rtimes_R G \to B(L^2(G))$ factors through a faithful $*$-representation $M \rtimes_r \rho$ of $C_b^{ru}(G) \rtimes_{r,R} G$. Moreover, $M \rtimes_r \rho$ induces a $*$-isomorphism between $C_0(G) \rtimes_{r,R} G$ and $K(L^2(G)$. We can also conclude the same facts for (M,λ) on $(C_b^{lu}(G), G, L)$.

We end this section with an important theorem, which will be used in the next section.

THEOREM 7.9. [**2, 3**] *Let G be a locally compact group and X be a locally compact Hausdorff G-space. Consider the following conditions:*

(1) *The action of G on X is topologically amenable.*
(2) *$(C_0(X) \otimes A) \rtimes G = (C_0(X) \otimes A) \rtimes_r G$ for every G-C^*-algebra A.*
(3) *$C_0(X) \rtimes_r G$ is nuclear.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *Moreover,* (3) $\Rightarrow$ (1) *if G is discrete.*

REMARK 7.10. The condition (3) does not imply the condition (1) in general. For example, the reduced group C^*-algebra $C_r^*(G)$ is always nuclear if the group G is connected (see [**20**]).

8. Exactness of locally compact groups

In this section we show that the exactness of a locally compact second countable group is equivalent to amenability at infinity, which solves an open problem raised by Anantharaman-Delaroche ([3], Problem 9.3). The material here is joint work with Jacek Brodzki and Chris Cave and is based on the unpublished manuscript "Exactness of locally compact groups".

Exact groups were introduced by Kirchberg and Wassermann in order to study the continuity of the reduced crossed product C^*-bundles. We recall the definition of exact groups:

Definition 8.1. [59, 60] We say that a locally compact group G is *exact*, if the reduced crossed product functor $A \to A \rtimes_{\alpha,r} G$ is exact for any G-C^*-algebra (A, α). To be more precise, for every G-equivariant short exact sequence of G-C^*-algebras $0 \to I \to A \to B \to 0$, the corresponding sequence $0 \to I \rtimes_r G \to A \rtimes_r G \to B \rtimes_r G \to 0$ of reduced crossed products is exact too.

Remark 8.2. The corresponding morphisms $\iota_r : I \rtimes_r G \to A \rtimes_r G$ and $q_r : A \rtimes_r G \to B \rtimes_r G$ are still injective and surjective, respectively. Moreover, Im $\iota_r \subseteq \ker q_r$. So the group G is exact if and only if this inclusion is always an equality. Note that Gromov's random groups and Osajda's groups are not exact [38, 4, 69]. However, the full crossed product functor is always exact by its universal property.

We would like to identify all elements in $A \rtimes_r G$ which are in $\ker q_r$. The next proposition provides a useful criterion in terms of slice maps. Recall that for any normal linear functional $\psi \in B(L^2(G))_*$, the slice map S_ψ corresponding to ψ is defined as follows

$$S_\psi : A \rtimes_{\alpha,r} G \xrightarrow{\pi_\alpha^A \rtimes (\lambda \otimes 1)} B(L^2(G, H)) \cong B(H)\overline{\otimes} B(L^2(G)) \xrightarrow{\mathrm{id}_{B(H)}\overline{\otimes}\psi} B(H),$$

where $\pi_\alpha^A \rtimes (\lambda \otimes 1)$ is the regular representation associated to the reduced crossed product $A \rtimes_{\alpha,r} G$. If $\psi = \omega_{\xi,\eta}$, where $\xi, \eta \in C_c(G)$ and $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$ for $x \in B(L^2(G))$. Then

$$S_\psi(f) = \int_G \int_G \xi(g^{-1}h)\overline{\eta(h)}\alpha_{h^{-1}}(f(g))\, d\mu(g) d\mu(h), \quad f \in C_c(G, A).$$

Proposition 8.3. [59, Proposition 2.2] *For $x \in A \rtimes_r G$, the following are equivalent:*

(1) $x \in \ker q_r$.
(2) $S_\psi(x) \in I$ *for all* $\psi \in B(L^2(G))_*$.
(3) $S_{\omega_{\xi,\eta}}(x) \in I$ *for all* $\xi, \eta \in C_c(G)$.

Before we prove the main theorem, we need the following proposition.

Proposition 8.4. *Let G be a locally compact second countable group equipped with a proper left-invariant compatible metric d. If Z is a (uniformly) locally finite metric lattice in the metric group (G, d), then there exist a faithful $*$-homomorphism $\Phi : C_u^*(Z) \to C_b^{ru}(G) \rtimes_{r,R} G$ and a c.c.p. map $E : C_b^{ru}(G) \rtimes_{r,R} G \to C_u^*(Z)$ satisfying the following properties:*

(1) $E \circ \Phi = \mathrm{Id}_{C_u^*(Z)}$.
(2) $T \in K(l^2(Z))$ *if and only if* $\Phi(T) \in C_0(G) \rtimes_{r,R} G$.
(3) Φ *maps* $G^*(Z)$ *into the kernel of* $C_b^{ru}(G) \rtimes_{r,R} G \to (C_b^{ru}(G)/C_0(G)) \rtimes_{r,R} G$.

Proof. Since Z is uniformly discrete, we fix $\delta > 0$ such that for all $z, w \in Z$, $d(z, w) > \delta$, whenever $z \neq w$. Let φ be a continuous compactly supported positive valued function on G such that $\operatorname{supp}\varphi \subseteq B(e, \delta/4)$ and $||\varphi||_2 = 1$. For $z \in Z$, set φ_z to be the function $g \mapsto \varphi(z^{-1}g)$ for $g \in G$. Clearly, each φ_z is supported on a $\delta/4$-neighbourhood around z. As Z is δ-uniformly discrete, each φ_z has disjoint support. In particular $\{\varphi_z : z \in Z\}$ forms an orthonormal set in $L^2(G)$.

Define an operator $W : \ell^2(Z) \to L^2(G)$, $\delta_z \mapsto \varphi_z$ and extend linearly. Hence for $\eta \in \ell^2(Z)$, $(W\eta)(x) = \sum_{z\in Z} \eta(z)\varphi_z(x)$ for all $x \in G$. For $\xi \in L^2(G)$, $W^*\xi(z) = \int \xi(y)\varphi_z(y)\, d\mu(y)$. It is clear that W is an isometry as it sends an orthonormal basis to an orthonormal set.

Let $T \in C_u^*(Z)$ be a finite propagation operator and denote $\langle T\delta_w, \delta_z\rangle$ by $T_{z,w}$. By left invariance of the Haar integral we have that for $x \in G$ and $\xi \in L^2(G)$ we have that

$$\begin{aligned}(WTW^*)(\xi)(x) &= \sum_{z\in Z}\varphi_z(x)\sum_{w\in Z}T_{z,w}\int_G \xi(y)\varphi_w(y)\,d\mu(y)\\ &= \int_G\sum_{z,w\in Z}\varphi_z(x)\varphi_w(xy)T_{z,w}\xi(xy)\,d\mu(y).\end{aligned}$$

As T has finite propagation and Z is (uniformly) locally finite, we are only performing finitely many sums. This means we are able to exchange the order of summation and integration. Now, we define a function $\widehat{T} : G\times G \to \mathbb{C}$ given by

$$\widehat{T}_y(x) = \sum_{z,w\in Z}\varphi_z(x)\varphi_w(xy)T_{z,w}\Delta(y)^{-1/2}. \tag{8.1}$$

The supports of φ_z are pairwise disjoint so for all $x, y \in G$, either $\widehat{T}_y(x) = 0$ or there exists at most one pair $(z, w) \in Z \times Z$ such that $\varphi_z(x)$ and $\varphi_w(xy)$ are non-zero. Observe in this case $z \in B(x, \delta/4)$ and $w \in B(xy, \delta/4)$ as the support of φ is contained in a ball of radius $\delta/4$. Moreover, if we choose $(x', y') \in G\times G$, which is very close to (x, y), then we can choose the same pair (z, w) for both (x, y) and (x', y') such that $\varphi_z(x), \varphi_z(x')$ and $\varphi_w(xy), \varphi_w(x'y')$ are non-zero. From these observations it is not hard to show that $\widehat{T}$ is continuous at y-variable and right uniformly continuous at x-variable. Moreover, the continuous map $y \mapsto \widehat{T}_y$ is compactly supported. This is because if there exists an $R > 0$ such that $T_{z,w} = 0$ whenever $d(z, w) > R$ then the function $y \mapsto \widehat{T}_y$ is supported on a ball of radius $R + \delta/2$. Finally, we notice that $\widehat{T}_y(x)$ is bounded at x-variable. Therefore the function $\widehat{T}$ belongs to $C_c(G, C_b^{ru}(G))$.

Let $M : C_b^{ru}(G) \to B(L^2(G))$ be the multiplication operator on $L^2(G)$ and $\rho : G \to B(L^2(G))$ be the right regular representation. Then $M \rtimes_r \rho : C_b^{ru}(G) \rtimes_{r,R} G \to B(L^2(G))$ induces a faithful $*$-representation of $C_b^{ru}(G) \rtimes_{r,R} G$. We claim that the operator WTW^* in $B(L^2(G))$ is the image of $\widehat{T}$ under the faithful $*$-representation $M \rtimes_r \rho$:

Indeed for all $\xi \in L^2(G)$ and $x \in G$

$$(M \rtimes_r \rho)(\widehat{T})(\xi)(x) = \int_G \widehat{T}_y(x)\xi(xy)\Delta(y)^{1/2}\,d\mu(y) = (WTW^*)(\xi)(x).$$

Since the image of $M \rtimes_r \rho$ is closed, we conclude that $WC_u^*(Z)W^*$ is contained in the image of $M \rtimes_r \rho$. Hence, there is a well-defined faithful $*$-homomorphism $\Phi : C_u^*(Z) \to C_b^{ru}(G) \rtimes_{r,R} G$ defined by $\Phi(T) = (M \rtimes_r \rho)^{-1}(WTW^*)$. If T has finite propagation, then $\Phi(T) = \widehat{T}$ given by the formula 8.1. Moreover, we define a c.c.p. map $E : C_b^{ru}(G) \rtimes_{r,R} G \to B(l^2(Z))$ by $E(a) = W^*M \rtimes_r \rho(a)W$, where $a \in C_b^{ru}(G) \rtimes_{r,R} G$.

We claim that the image of E is contained in $C_u^*(Z)$. Indeed, let $f \in C_c(G, C_b^{ru}(G))$ with support in $B(e, R)$ for some $R > 0$, then

$$\langle W^*M \rtimes_r \rho(f)W\delta_y, \delta_x\rangle = \langle M \rtimes_r \rho(f)(\varphi_y), \varphi_x\rangle = \int_{z\in G}\int_{s\in G} f_s(z)\varphi_y(zs)\Delta(s)^{1/2}\varphi_x(z)dsdz$$

It follows that

$$d(x, y) \le d(x, z) + d(z, y) \le d(x, z) + d(z, zs) + d(zs, y) \le \delta/4 + R + \delta/4.$$

That is to say that $E(f)$ has propagation at most $R + \delta/2$. Therefore, the claim follows.

The property (1) follows by the constructions of Φ and E. Since W is an isometry and $M \rtimes_r \rho$ induces a $*$-isomorphism between $C_0(G) \rtimes_{r,R} G$ and $K(L^2(G))$, the property (2) follows easily from the fact that the compact operators form a two-sided ideal.

We are ready to verify property (3). Let $H \in G^*(Z)$, we have to show $S_{\omega_{\xi,\eta}}(\Phi(H)) \in C_0(G)$ for all $\xi, \eta \in C_c(G)$ by Proposition 8.3.

Given $\varepsilon > 0$, we would like to find $C > 0$ such that

$$d(e,x) > C \Rightarrow |S_{\omega_{\xi,\eta}}(\Phi(H))(x)| < \varepsilon.$$

Consider the bounded continuous kernel $k : G \times G \to \mathbb{C}$ given by $k(g,h) = \xi(g^{-1}h)\overline{\eta(h)}$. Since both ξ and η are compactly supported, there exist two compact subsets K_1 and K_2 such that $\operatorname{supp} k \subseteq K_1 \times K_2$. Let D be a positive number such that

$$d(g,h) + d(e,h) + \Delta(g)^{-1/2} \leq D,$$

for all $(g,h) \in K_1 \times K_2$. Choose a small $\varepsilon' > 0$ such that

$$3\varepsilon' D||\xi||_\infty ||\eta||_\infty \mu(K_1)\mu(K_2)||\varphi||_\infty^2 \leq \varepsilon.$$

Since H is a ghost, we choose a $M > 0$ such that if $z, w \notin B(e,M)$ then $|H_{z,w}| < \varepsilon'$. As the slice map $S_{\omega_{\xi,\eta}}$ is continuous, we can choose an operator $T \in C_u^*(Z)$ of finite propagation such that $||T - H||_{B(l^2(Z))} + ||S_{\omega_{\xi,\eta}}(\Phi(T) - \Phi(H))||_\infty < \min\{\varepsilon/3, \varepsilon'\}$. In particular, $|T_{z,w} - H_{z,w}| < \varepsilon'$ for all $z, w \in Z$. We note that $\Phi(T) = \widehat{T}$ and

$$\begin{aligned}
S_{\omega_{\xi,\eta}}(\widehat{T})(x) &= \int_G \int_G \xi(g^{-1}h)\overline{\eta(h)} R_{h^{-1}}(\widehat{T}_g)(x)\ d\mu(g)d\mu(h) \\
&= \int_G \int_G k(g,h) \sum_{z,w\in Z} \varphi_z(xh^{-1})\varphi_w(xh^{-1}g) T_{z,w} \Delta(g)^{-1/2}\ d\mu(g)d\mu(h) \\
&= \int_{K_2} \int_{K_1} k(g,h)\varphi_{z_0}(xh^{-1})\varphi_{w_0}(xh^{-1}g) T_{z_0,w_0} \Delta(g)^{-1/2}\ d\mu(g)d\mu(h),
\end{aligned}$$

for some $(z_0, w_0) \in B(xh^{-1}, \delta/4) \times B(xh^{-1}g, \delta/4)$.
Set $C = M + D + \delta/4$. If $d(e,x) > C$ and $(g,h) \in K_1 \times K_2$, then

$$\begin{aligned}
d(e,z_0) &\geq d(x,e) - d(h,e) - d(z_0, xh^{-1}) > C - D - \delta/4 = M, \\
d(e,w_0) &\geq d(e,x) - d(xh^{-1}g, x) - d(w_0, xh^{-1}g) > C - D - \delta/4 = M.
\end{aligned}$$

In particular, $|T_{z_0,w_0}| \leq |T_{z_0,w_0} - H_{z_0,w_0}| + |H_{z_0,w_0}| < 2\varepsilon'$. It follows that

$$|S_{\omega_{\xi,\eta}}(\widehat{T})(x)| < 2\varepsilon' ||\xi||_\infty ||\eta||_\infty ||\varphi||_\infty^2 D\mu(K_1)\mu(K_2) \leq \frac{2\varepsilon}{3},$$

for all $x \notin B(e,C)$. We complete the proof by the following computation:

$$|S_{\omega_{\xi,\eta}}(\Phi(H))(x)| \leq |S_{\omega_{\xi,\eta}}(\Phi(H) - \Phi(T))(x)| + |S_{\omega_{\xi,\eta}}(\Phi(T))(x)| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

□

We are ready to prove the main theorem of this section. Recall that $C_b^{lu}(G) \cong C(\beta^{lu}(G))$ and $C_b^{lu}(G)/C_0(G) \cong C(\partial G)$.

Theorem 8.5. *Let G be a locally compact second countable group. Then the following conditions are equivalent.*

(1) *G is amenable at infinity.*
(2) *G is exact.*
(3) *The sequence*

$$0 \to C_0(G) \rtimes_{r,L} G \to C(\beta^{lu}(G)) \rtimes_{r,L} G \to C(\partial G) \rtimes_{r,L} G \to 0$$

is exact.
(4) *$C(\partial G) \rtimes_L G \cong C(\partial G) \rtimes_{r,L} G$ canonically.*
(5) *$C(\beta^{lu}(G)) \rtimes_{r,L} G$ is nuclear.*

Proof. We will show that (1) ⇒ (2) ⇒ (3) ⇒ (1), (1) ⇒ (4) ⇒ (3) and (1) ⇔ (5).
(1) ⇒ (2): This follows from [3, Theorem 7.2].
(2) ⇒ (3): This follows from the definition of the exactness of G.

(3) $\Rightarrow$ (1): We will show that if G is *not* amenable at infinity, then the sequence in condition (3) is *not* exact. It follows from Theorem 3.6 and Proposition 5.8 that there exists a uniformly locally finite metric lattice Z in G without Yu's property A. Theorem 5.12 and Proposition 8.4 imply that $C_u^*(Z)$ contains a non-compact ghost T and $\Phi(T)$ is an obstruction for the exactness of the following sequence:

$$0 \to C_0(G) \rtimes_{r,R} G \to C_b^{ru}(G) \rtimes_{r,R} G \to (C_b^{ru}(G)/C_0(G)) \rtimes_{r,R} G \to 0.$$

Finally, we claim that this sequence is exact if and only if the sequence in condition (3) is exact. Indeed, the inverse homeomorphism on G induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_0(G) \rtimes_{r,R} G \arrow[r] \arrow[d, "\cong"'] & C_b^{ru}(G)) \rtimes_{r,R} G \arrow[r] \arrow[d, "\cong"'] & (C_b^{ru}(G)/C_0(G)) \rtimes_{r,R} G \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & C_0(G) \rtimes_{r,L} G \arrow[r] & C_b^{lu}(G) \rtimes_{r,L} G \arrow[r] & (C_b^{lu}(G)/C_0(G)) \rtimes_{r,L} G \arrow[r] & 0.
\end{tikzcd}$$

The claim follows by an easy diagram chase.

(1) $\Rightarrow$ (4): This follows from Proposition 7.4 and Theorem 7.9.

(4) $\Rightarrow$ (3): Consider the following canonical diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_0(G) \rtimes_L G \arrow[r] \arrow[d] & C(\beta^{lu}(G)) \rtimes_L G \arrow[r] \arrow[d] & C(\partial G) \rtimes_L G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(G) \rtimes_{r,L} G \arrow[r] & C(\beta^{lu}(G)) \rtimes_{r,L} G \arrow[r] & C(\partial G) \rtimes_{r,L} G \arrow[r] & 0.
\end{tikzcd}$$

It is clear that the diagram commutes and the middle vertical arrow is surjective. Since the top sequence is exact and the right vertical arrow is injective, the bottom sequence is also exact by an easy diagram chase.

(1) $\Rightarrow$ (5): This follows from Proposition 7.4 and Theorem 7.9.

(5) $\Rightarrow$ (1): By assumption $C_b^{ru}(G) \rtimes_{r,R} G \cong C(\beta^{lu}(G)) \rtimes_{r,L} G$ is nuclear. Let Z be a uniformly locally finite metric lattice in G. Then it follows from Proposition 8.4 that the identity map on $C_u^*(Z)$ factors through the nuclear C^*-algebra $C_b^{ru}(G) \rtimes_{r,R} G$. Hence $C_u^*(Z)$ is also nuclear. The conclusion follows by Theorem 5.10, Proposition 5.8 and Theorem 3.6. $\square$

At the end of this section, we would like to mention a ground-breaking result of Hiroki Sako:

THEOREM 8.6 ([80]). *Let (Z, d) be a uniformly locally finite metric space. Then the following conditions are equivalent:*

(1) *The metric space (Z, d) has Yu's property A.*
(2) *The uniform Roe algebra $C_u^*(Z)$ is nuclear.*
(3) *The uniform Roe algebra $C_u^*(Z)$ is exact.*
(4) *The uniform Roe algebra $C_u^*(Z)$ is locally reflexive.*

For a general C^*-algebra, nuclearity implies exactness, and exactness implies local reflexivity. Moreover, a C^*-subalgebra of a locally reflexive C^*-algebra is also locally reflexive. We refer to [13] for more details. By Sako's result and Proposition 8.4 we obtain an analogous result on locally compact groups.

COROLLARY 8.7. *Let G be a locally compact second countable group. Then the following conditions are equivalent.*

(1) *The group G has property A.*
(2) *$C_b^{ru}(G) \rtimes_{r,R} G$ is nuclear.*
(3) *$C_b^{ru}(G) \rtimes_{r,R} G$ is exact.*
(4) *$C_b^{ru}(G) \rtimes_{r,R} G$ is locally reflexive.*

9. Future projects

In this section we list some questions related to my thesis for possible further research. Few of the questions have already been worked on, but most of them are still only ideas.

9.1. The Baum-Connes conjecture.

(1) One of the main ingredients in [15] to prove the Baum-Connes conjecture for linear algebraic groups over $\mathbb{Q}_p$ is Kirillov's orbit method for p-adic unipotent groups in [67, 51]. Is it possible to prove the Baum-Connes conjecture for linear algebraic groups over local fields of positive characteristic by a general Kirillov's orbit method developed in [31]? This is possible in many cases, e.g. the Jacobi group $\mathbb{H}^{2n+1}(k) \rtimes Sp(2n, k)$ is the semidirect product of the Heisenberg group $\mathbb{H}^{2n+1}(k)$ with the symplectic group $Sp(2n, k)$. The reason is that the unipotent radical $\mathbb{H}^{2n+1}(k)$ has smooth unitary dual. Note that the Jacobi group has Kazhdan's property (T), which is an obstacle to prove the Baum-Connes conjecture [55].

9.2. The strong Novikov conjecture.

(1) Recently, Kasparov and Yu [58] proved the strong Novikov conjecture for countable discrete groups coarsely embeddable into Banach spaces satisfying a geometric condition called property (H). For instance, uniformly convex Banach spaces with certain unconditional bases have property (H) [12]. Is it possible to extend the result of Kasparov and Yu to all locally compact second countable groups?
(2) Tu proved in [84, 83] that if a locally compact second countable group G admits a γ-element, then the strong Novikov conjecture is true for G. Later, he proved in [85] that every discrete group, which admits a coarse embedding into Hilbert spaces, has a γ-element. So we can ask whether it is true for locally compact (totally disconnected) groups coarsely embeddable into Hilbert spaces ([85], Final remark 2).

9.3. Measure equivalence of locally compact groups.

(1) The central notion of *measure equivalence* on discrete groups was suggested by Gromov in [37]. It is well-known that all approximation properties mentioned in section 6 are invariant under measure equivalence [68, 54]. Since the definition of measure equivalence also makes sense for locally compact second countable unimodular groups [34, 5, 35], it is natural to ask whether the same conclusion holds in the locally compact setting. It is also interesting to ask whether property A and coarse embedding into Hilbert spaces are invariant under measure equivalence.

9.4. Approximation properties versus property A and coarse embeddability.

(1) It follows from Theorem 6.10 that weak amenability implies property A for all locally compact second countable groups. In [44], Haagerup and Kraus introduced the approximation property (AP) for locally compact groups and (AP) is in general weaker than weak amenability. Does (AP) imply property A for all locally compact groups? It is true for all locally compact groups with group lattices [44, 63, 70, 28].
(2) Does the weak Haagerup property imply coarse embeddability into a Hilbert space for all locally compact groups? By Theorem 6.11 it is true for groups with weak Haagerup constant 1. I think the question is even open for discrete group case.

9.5. Characterization of weak amenability in coarse geometry.

(1) Property A can be regarded as coarse amenability. There are many reasons for that, but one of them is the following: let Γ be a finitely generated residually finite group. It is possible to construct a metric space $\mathrm{Box}(\Gamma)$, called the box space associated to the group Γ [77], such that Γ is amenable if and only if $\mathrm{Box}(\Gamma)$ has Yu's property A. Recently, the characterization of the Haagerup property is obtained in [17] by fibred

coarse embedding of the box space into Hilbert spaces. Is it possible to define a coarse property on metric spaces such that a finitely generated residually finite group is weakly amenable if and only if its box space has this property? Since Theorem 6.10 and Theorem 6.11 can be extended from locally compact second countable groups to proper metric spaces with bounded geometry, it might be useful to attack the Cowling's conjecture for finitely generated residually finite groups.

9.6. Exactness of reduced group C^*-algebras.

(1) A C^*-algebra A is said to be *exact* if the functor $A \otimes_{\min} (-)$ is exact on the category of C^*-algebras. From this definition it is rather easy to show that if a locally compact group G is exact, then its reduced group C^*-algebra $C^*_r(G)$ is exact as C^*-algebras. Kirchberg and Wassermann proved the converse assertion for all discrete groups (see [**59**]). It is natural to ask the following question: let G be a locally compact second countable group. Does exactness of the reduced group C^*-algebra $C^*_r(G)$ imply exactness of the group G?

Bibliography

[1] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain. 7, 18, 19

[2] Claire Anantharaman-Delaroche. Systèmes dynamiques non commutatifs et moyennabilité. *Math. Ann.*, 279(2):297–315, 1987. 20

[3] Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Trans. Amer. Math. Soc.*, 354(10):4153–4178 (electronic), 2002. 7, 18, 19, 20, 21, 23

[4] G.N. Arzhantseva and T. Delzant. Examples of random groups. Preprint, 2008, 2011 (revised). 6, 21

[5] Uri Bader, Alex Furman, and Roman Sauer. Integrable measure equivalence and rigidity of hyperbolic lattices. *Invent. Math.*, 194(2):313–379, 2013. 25

[6] Paul Baum and Alain Connes. Geometric K-theory for Lie groups and foliations. *Enseign. Math. (2)*, 46(1-2):3–42, 2000. 5

[7] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994. 5, 6

[8] Paul Baum and Ronald G. Douglas. K homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982. 5

[9] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. 9

[10] M. E. B. Bekka, P.-A. Cherix, and A. Valette. Proper affine isometric actions of amenable groups. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, volume 227 of *London Math. Soc. Lecture Note Ser.*, pages 1–4. Cambridge Univ. Press, Cambridge, 1995. 6

[11] Mohammed E. B. Bekka. Amenable unitary representations of locally compact groups. *Invent. Math.*, 100(2):383–401, 1990. 10, 11

[12] Yoav Benyamini and Joram Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1*, volume 48 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2000. 6, 25

[13] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. 24

[14] J. Chabert, S. Echterhoff, and H. Oyono-Oyono. Going-down functors, the Künneth formula, and the Baum-Connes conjecture. *Geom. Funct. Anal.*, 14(3):491–528, 2004. 6, 8

[15] Jérôme Chabert, Siegfried Echterhoff, and Ryszard Nest. The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups. *Publ. Math. Inst. Hautes Études Sci.*, (97):239–278, 2003. 25

[16] Xiaoman Chen and Qin Wang. Ghost ideals in uniform Roe algebras of coarse spaces. *Arch. Math. (Basel)*, 84(6):519–526, 2005. 18

[17] Xiaoman Chen, Qin Wang, and Xianjin Wang. Characterization of the Haagerup property by fibred coarse embedding into Hilbert space. *Bull. Lond. Math. Soc.*, 45(5):1091–1099, 2013. 25

[18] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability. 16

[19] Marie Choda. Group factors of the Haagerup type. *Proc. Japan Acad. Ser. A Math. Sci.*, 59(5):174–177, 1983. 16

[20] A. Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Ann. of Math. (2)*, 104(1):73–115, 1976. 20

[21] A. Connes. A survey of foliations and operator algebras. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 521–628. Amer. Math. Soc., Providence, R.I., 1982. 5

[22] Alain Connes. Property T, correspondences and factors, 1980. Lecture at Queen's University, Kingston. 16

[23] Michael Cowling. Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups). In *Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982)*, pages 81–123. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983. 16

[24] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989. 16

[25] Jean De Cannière and Uffe Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.*, 107(2):455–500, 1985. 16

[26] Y. de Cornulier and P. de la Harpe. *Metric geometry of locally compact groups*. arXiv:1403.3796v3, 2015. 15

[27] Yves de Cornulier, Yves Stalder, and Alain Valette. Proper actions of lamplighter groups associated with free groups. *C. R. Math. Acad. Sci. Paris*, 346(3-4):173–176, 2008. 16

[28] Steven Deprez and Kang Li. Permanence properties of property A and coarse embeddability for locally compact groups. Preprint, arXiv:1403.7111, 2014. 25

[29] Zhe Dong and Zhong-Jin Ruan. A Hilbert module approach to the Haagerup property. *Integral Equations Operator Theory*, 73(3):431–454, 2012. 18

[30] Brian Dorofaeff. Weak amenability and semidirect products in simple Lie groups. *Math. Ann.*, 306(4):737–742, 1996. 16

[31] Siegfried Echterhoff and Helma Klüver. A general Kirillov theory for locally compact nilpotent groups. *J. Lie Theory*, 22(3):601–645, 2012. 25

[32] Pierre Eymard. *Moyennes invariantes et représentations unitaires*. Lecture Notes in Mathematics, Vol. 300. Springer-Verlag, Berlin-New York, 1972. 9

[33] Steven C. Ferry, Andrew Ranicki, and Jonathan Rosenberg, editors. *Novikov conjectures, index theorems and rigidity. Vol. 1*, volume 226 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. Including papers from the conference held at the Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, September 6–10, 1993. 5

[34] Alex Furman. A survey of measured group theory. In *Geometry, rigidity, and group actions*, Chicago Lectures in Math., pages 296–374. Univ. Chicago Press, Chicago, IL, 2011. 25

[35] Maxime Gheysens and Nicolas Monod. Fixed points for bounded orbits in Hilbert spaces. Preprint arXiv:1508.00423, 2015. 25

[36] F. P. Greenleaf. Amenable actions of locally compact groups. *J. Functional Analysis*, 4:295–315, 1969. 9

[37] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993. 5, 6, 25

[38] M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003. 6, 21

[39] Erik Guentner, Nigel Higson, and Shmuel Weinberger. The Novikov conjecture for linear groups. *Publ. Math. Inst. Hautes Études Sci.*, (101):243–268, 2005. 17

[40] Erik Guentner and Jerome Kaminker. Exactness and uniform embeddability of discrete groups. *J. London Math. Soc. (2)*, 70(3):703–718, 2004. 14

[41] Uffe Haagerup. An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79. 16

[42] Uffe Haagerup. Group C^*-algebras without the completely bounded approximation property. Unpublished manuscript, 1986. 16, 17

[43] Uffe Haagerup and Søren Knudby. The weak Haagerup property II: Examples. Preprint, to appear in Int. Math. Res. Not. IMRN, 2014. 16

[44] Uffe Haagerup and Jon Kraus. Approximation properties for group C^*-algebras and group von Neumann algebras. *Trans. Amer. Math. Soc.*, 344(2):667–699, 1994. 25

[45] Uffe Haagerup and Agata Przybyszewska. Proper metrics on locally compact groups, and proper affine isometric actions on Banach spaces. Preprint, arXiv:math/0606794, 2006. 7, 14

[46] Mogens Lemvig Hansen. Weak amenability of the universal covering group of $\mathrm{SU}(1,n)$. *Math. Ann.*, 288(3):445–472, 1990. 16

[47] N. Higson. Bivariant K-theory and the Novikov conjecture. *Geom. Funct. Anal.*, 10(3):563–581, 2000. 5, 8

[48] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002. 6

[49] Nigel Higson and Gennadi Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001. 6

[50] Nigel Higson and John Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000. 5, 17

[51] Roger E. Howe. The Fourier transform for nilpotent locally compact groups. I. *Pacific J. Math.*, 73(2):307–327, 1977. 25

[52] Paul Jolissaint. Invariant states and a conditional fixed point property for affine actions. *Math. Ann.*, 304(3):561–579, 1996. 9, 10

[53] Paul Jolissaint. Borel cocycles, approximation properties and relative property T. *Ergodic Theory Dynam. Systems*, 20(2):483–499, 2000. 9
[54] Paul Jolissaint. Proper cocycles and weak forms of amenability. *Colloq. Math.*, 138(1):73–88, 2015. 9, 25
[55] Pierre Julg. Remarks on the Baum-Connes conjecture and Kazhdan's property T. In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, volume 13 of *Fields Inst. Commun.*, pages 145–153. Amer. Math. Soc., Providence, RI, 1997. 25
[56] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988. 5
[57] Gennadi Kasparov and Georges Skandalis. Groups acting properly on "bolic" spaces and the Novikov conjecture. *Ann. of Math. (2)*, 158(1):165–206, 2003. 5
[58] Gennadi Kasparov and Guoliang Yu. The Novikov conjecture and geometry of Banach spaces. *Geom. Topol.*, 16(3):1859–1880, 2012. 25
[59] Eberhard Kirchberg and Simon Wassermann. Exact groups and continuous bundles of C^*-algebras. *Math. Ann.*, 315(2):169–203, 1999. 18, 21, 26
[60] Eberhard Kirchberg and Simon Wassermann. Permanence properties of C^*-exact groups. *Doc. Math.*, 4:513–558 (electronic), 1999. 21
[61] Søren Knudby. Semigroups of Herz-Schur multipliers. *J. Funct. Anal.*, 266(3):1565–1610, 2014. 16
[62] Søren Knudby. The weak Haagerup property. Preprint, to appear in Trans. Amer. Math. Soc., 2014. 16
[63] Jon Kraus. The slice map problem and approximation properties. *J. Funct. Anal.*, 102(1):116–155, 1991. 17, 25
[64] Vincent Lafforgue. La conjecture de Baum-Connes à coefficients pour les groupes hyperboliques. *J. Noncommut. Geom.*, 6(1):1–197, 2012. 6
[65] Horst Leptin. Sur l'algèbre de Fourier d'un groupe localement compact. *C. R. Acad. Sci. Paris Sér. A-B*, 266:A1180–A1182, 1968. 16
[66] George W. Mackey. Induced representations of locally compact groups. I. *Ann. of Math. (2)*, 55:101–139, 1952. 10
[67] Calvin C. Moore. Decomposition of unitary representations defined by discrete subgroups of nilpotent groups. *Ann. of Math. (2)*, 82:146–182, 1965. 25
[68] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980. 25
[69] D. Osajda. Small cancellation labellings of some infinite graphs and applications. Preprint, arXiv:1406.5015v2, 2014. 6, 7, 21
[70] Narutaka Ozawa. Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8):691–695, 2000. 5, 17, 25
[71] Narutaka Ozawa. Examples of groups which are not weakly amenable. *Kyoto J. Math.*, 52(2):333–344, 2012. 16
[72] Gert K. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. 18
[73] Jean-Paul Pier. *Amenable locally compact groups*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1984. A Wiley-Interscience Publication. 16
[74] Gilles Pisier. *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 2001. Includes the solution to "The Halmos problem". 17
[75] Hans Reiter and Jan D. Stegeman. *Classical harmonic analysis and locally compact groups*, volume 22 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, second edition, 2000. 11
[76] John Roe. *Index theory, coarse geometry, and topology of manifolds*, volume 90 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. 7
[77] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003. 15, 17, 25
[78] John Roe. Warped cones and property A. *Geom. Topol.*, 9:163–178 (electronic), 2005. 7, 15
[79] John Roe and Rufus Willett. Ghostbusting and property A. *J. Funct. Anal.*, 266(3):1674–1684, 2014. 15
[80] Hiroki Sako. Translation c^*-algebras and property a for uniformly locally finite spaces. Preprint arXiv:1212.5900, 2013. 24
[81] G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002. 5, 6, 8, 15
[82] Raimond A. Struble. Metrics in locally compact groups. *Compositio Math.*, 28:217–222, 1974. 7
[83] Jean-Louis Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999. 25
[84] Jean Louis Tu. La conjecture de Novikov pour les feuilletages hyperboliques. *K-Theory*, 16(2):129–184, 1999. 25

[85] Jean-Louis Tu. The gamma element for groups which admit a uniform embedding into Hilbert space. In *Recent advances in operator theory, operator algebras, and their applications*, volume 153 of *Oper. Theory Adv. Appl.*, pages 271–286. Birkhäuser, Basel, 2005. 25

[86] John von Neumann. Zur allgemeinen Theorie des Maßes. *Fund. Math.*, 13:73–116, 1929. 16

[87] Rufus Willett. Some notes on property A. In *Limits of graphs in group theory and computer science*, pages 191–281. EPFL Press, Lausanne, 2009. 14

[88] Dana P. Williams. *Crossed products of C^*-algebras*, volume 134 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007. 18

[89] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000. 5, 6, 7

[90] Robert J. Zimmer. Amenable pairs of groups and ergodic actions and the associated von Neumann algebras. *Trans. Amer. Math. Soc.*, 243:271–286, 1978. 9, 10

Article A

Property A and uniform embedding for locally compact groups

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Journal of Noncommutative Geometry

Property A and uniform embedding for locally compact groups

Steven Deprez*,** and Kang Li**

Abstract. For locally compact groups, we define an analogue to Yu's property A that he defined for discrete metric spaces. We show that our property A for locally compact groups agrees with Roe's notion of property A for proper metric spaces, defined in [11]. We prove that many of the results that are known to hold in the discrete setting, hold also in the locally compact setting. In particular, we show that property A is equivalent to amenability at infinity (see [9] for the discrete case), and that a locally compact group with property A embeds uniformly into a Hilbert space (see [17] for the discrete case). We also prove that the Baum–Connes assembly map with coefficients is split-injective, for every locally compact group that embeds uniformly into a Hilbert space. This extends results by Skandalis, Tu and Yu [13], and by Chabert, Echterhoff and Oyono-Oyono [4].

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Keywords. Property A, Baum–Connes conjecture, uniform embeddability, locally compact groups.

1. Introduction

Gromov introduced the notion of uniform embeddability of metric spaces and suggested that finitely generated discrete groups that are uniformly embeddable in a Hilbert space, when viewed as metric spaces with a word length metric, might satisfy the Novikov conjecture [5, 6]. Yu showed that this is indeed the case, provided that the classifying space is a finite CW-complex [17]. In the same paper Yu introduced a weak form of amenability on discrete metric spaces that he called property A, which guarantees the existence of a uniform embedding into Hilbert space. Higson and Roe observed in [9] that the metric space underlying a finitely generated discrete group has property A if and only if it admits a topologically amenable action on some compact Hausdorff space. Ozawa showed in [10] that a discrete group admits a topologically amenable action on a compact Hausdorff space if and only if the

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group is exact. In the case of property A groups, Higson strengthened Yu's result by removing the finiteness assumption on the classifying space [8]. Indeed, he proved that the Baum–Connes assembly map with coefficients, for any countable discrete group which has a topologically amenable action on a compact Hausdorff space, is split-injective. Baum, Connes and Higson showed that this implies the Novikov conjecture [2]. Using Higson's descent technique (see [8]), Skandalis, Tu and Yu [13] were able to generalize the split-injectivity result to arbitrary discrete groups which admit a uniform embedding into Hilbert space, and hence they answered Gromov's question.

In [11], Roe generalized property A to proper metric spaces with bounded geometry (in the sense of [12]). All second countable locally compact groups have a proper left-invariant metric that implements the topology, and such a metric is unique up to coarse equivalence (see [7] and [14]). Moreover, locally compact groups with a proper left-invariant metric, have bounded geometry (see [7]). Roe already proved that his generalization of property A is equivalent to Ozawa's notion of exactness, see [11]. A locally compact group G satisfies Ozawa's notion of exactness if there is a net of positive type kernels $k_i : G \times G \to \mathbb{C}$ that tends to 1 uniformly on tubes, and such that each k_i is supported in a tube. We say that a subset $T \subseteq G \times G$ is a tube if it is contained in a set of the form $\mathrm{Tube}(K) = \{(s,t) : s^{-1}t \in K\} \subseteq G \times G$ for some compact subset $K \subseteq G$. Anantharaman-Delaroche has shown in [1] that whenever a locally compact group admits a topologically amenable action on a compact Hausdorff space, it also satisfies Ozawa's notion of exactness.

We give an alternative definition of property A, that resembles more closely Yu's definition, and we show that it is equivalent to Roe's definition. Moreover, we give a direct and elementary proof that it is equivalent to Ozawa's notion of exactness. We continue by showing that a locally compact group has property A if and only if it has a topologically amenable action on a compact Hausdorff space. This statement was proven for discrete groups by Higson and Roe [9].

Whenever a locally compact group admits a topologically amenable action on a compact Hausdorff space, it is uniformly embeddable into a Hilbert space (see [1]). By the above, this is also true for groups with property A. We also give an alternative characterisation of the locally compact groups that embed uniformly into a Hilbert space. We say that an action $G \curvearrowright X$ on a compact Hausdorff space X has the Haagerup property if its transformation groupoid $X \rtimes G$ admits a continuous proper conditionally negative type function. Then we show that a locally compact group embeds uniformly into a Hilbert space if and only if it admits a Haagerup action on a compact Hausdorff space. Finally, we apply Higson's descent technique and the going–down functor of Chabert, Echterhoff and Oyono-Oyono, to obtain an analogue of the result of Skandalis, Tu and Yu (see [13]): we show that the Baum–Connes assembly map with coefficients is split-injective for all locally compact groups that embed uniformly into Hilbert space.

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2. Property A for locally compact groups

In this section, we introduce our own notion of property A for locally compact second countable (l.c.s.c.) groups. Yu first introduced property A for discrete metric spaces in [17]. Our own notion of property A is closely modelled on Yu's definition. Roe has introduced a generalization of property A for proper metric spaces with bounded geometry (see [11]). Every second countable locally compact group G has a proper left-invariant metric d that implements the topology on G. This metric is unique up to coarse equivalence. Moreover, the proper metric space (G,d) has bounded geometry, see [7] and [14]. So Roe's property A makes sense for l.c.s.c. groups, and we show in Theorem 2.3 that it agrees with our property A. We combine Theorem 2.3 with Anantharaman-Delaroche's description of groups that are amenable at infinity [1, Proposition 3.4 and 3.5], i.e. groups that admit a topologically amenable action on a compact Hausdorff space. In this way we obtain Corollary 2.9: a l.c.s.c. group has property A if and only if it is amenable at infinity.

For the rest of the paper, G will always denote a l.c.s.c. group. We fix a left Haar measure μ on G. We also consider the measure μ' on $G \times \mathbb{N}$ which is the product measure of μ with the counting measure on $\mathbb{N}$.

Definition 2.1. Let G be a l.c.s.c. group and let $K \subseteq G$ be a compact subset. Then we write

$$\mathrm{Tube}(K) = \{(s,t) \in G \times G : s^{-1}t \in K\}.$$

We say that a subset $T \subseteq G \times G$ is a *tube* if $\{s^{-1}t : (s,t) \in T\}$ is precompact, or equivalently, if $T \subseteq \mathrm{Tube}(K)$ for some compact subset $K \subseteq G$.

Definition 2.2. A l.c.s.c group G has property A if for any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a family $\{A_s\}_{s \in G}$ of Borel subsets of $G \times \mathbb{N}$ with $0 < \mu'(A_s) < \infty$ such that

- for all $(s,t) \in \mathrm{Tube}(K)$ we have
$$\frac{\mu'(A_s \Delta A_t)}{\mu'(A_s \cap A_t)} < \varepsilon,$$
- $(t,n) \in A_s$ implies $(s,t) \in \mathrm{Tube}(L)$.

Theorem 2.3 below gives a number of equivalent characterizations of property A. It is an extention of [16] to the locally compact case. Condition (3) says that G has property A in the sense of Roe [11], as a proper metric space with bounded geometry. Condition (5) is Ozawa's notion of exactness for l.c.s.c. groups [10]. The equivalence between (3) and (5) has already been proven in [11], by reducing the problem to the discrete case. Nevertheless, we provide a simple direct proof of their equivalence, for the sake of completeness.

Theorem 2.3. *Let G be a l.c.s.c. group. The following are equivalent:*

1) *G has property A;*

2) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a continuous map $\eta : G \to L^1(G)$ such that $||\eta_t||_1 = 1$, $\operatorname{supp} \eta_t \subseteq tL$ for every $t \in G$ and*

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\eta_s - \eta_t||_1 < \varepsilon;$$

3) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a weak-$*$ continuous map $\nu : G \to C_0(G)^*_+$ such that $||\nu_t|| = 1$, $\operatorname{supp} \nu_t \subseteq tL$ for every $t \in G$ and*

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\nu_s - \nu_t|| < \varepsilon;$$

4) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a continuous map $\xi : G \to L^2(G)$ such that $||\xi_t||_2 = 1$, $\operatorname{supp} \xi_t \subseteq tL$ for every $t \in G$ and*

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\xi_s - \xi_t||_2 < \varepsilon;$$

5) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a continuous positive type kernel $k : G \times G \to \mathbb{C}$ such that $\operatorname{supp} k \subseteq \mathrm{Tube}(L)$ and*

$$\sup_{(s,t)\in \mathrm{Tube}(K)} |k(s,t) - 1| < \varepsilon.$$

In statements (2)–(5), we assume that the maps η, ν, ξ, k are continuous, because this is standard for locally compact groups. But in fact, each of the statements (2)–(5) is equivalent to the corresponding statement without the continuity assumption. In Lemma 2.5, we carefully state and prove this for statement (2). We omit the argument for statements (3)–(5), as it is entirely analogous. Part of the proof of Lemma 2.5 below consists of convolving η with a "nice" function. We use the same class of "nice" functions several times in the paper, so we give them a name.

Definition 2.4. A cut-off function for G is a function f in $C_c(G)$ such that

- $f \geq 0$;
- $f(t^{-1}) = f(t)$ for all $t \in G$;
- $\operatorname{supp} f$ is a compact neighborhood of the unit element e of G;
- $\int_G f(t)d\mu(t) = 1$.

Observe that every l.c.s.c. group has cut-off functions.

Lemma 2.5. *Let G be a l.c.s.c. group. Suppose that G satisfies the following property.*

2′) *For any compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a map $\eta : G \to L^1(G)$ such that $\|\eta_t\|_1 = 1$, $\operatorname{supp} \eta_t \subseteq tL$ for every $t \in G$ and*

$$\sup_{(s,t)\in \mathrm{Tube}(K)} \|\eta_s - \eta_t\|_1 < \varepsilon;$$

Then we can assume that the map $t \mapsto \eta_t$ is continuous, i.e., G satisfies property (2) from Theorem 2.3 above.

Proof. The proof proceeds in two steps. In step one, we show that we can assume that $t \mapsto \eta_t$ is a Borel map. In step two we use a convolution argument to make $t \mapsto \eta_t$ continuous.

Step 1: We can assume that $t \mapsto \eta_t$ is a piecewise constant Borel map.
Fix any compact subset $C \subseteq G$ with non-empty interior. Since G is second countable, we find a sequence (s_n) in G such that $G = \cup_n s_n C$. Define a sequence (C_n) of Borel subsets of G by induction as follows. We set $C_1 = s_1 C$ and for each $n > 1$, we set $C_n = s_n C \setminus (C_1 \cup \ldots \cup C_{n-1})$.

Let $\varepsilon > 0$ and let $K \subseteq G$ be a compact subset. Then we see that the product $CKC^{-1} \subseteq G$ is still a compact subset. Since G satisfies our condition (2′), we find a map $\eta : G \to L^1(G)$ and a compact subset $L \subseteq G$ such that $\|\eta_t\|_1 = 1$, $\operatorname{supp} \eta_t \subseteq tL$ for every $t \in G$ and

$$\sup_{(s,t)\in \mathrm{Tube}(CKC^{-1})} \|\eta_s - \eta_t\|_1 < \varepsilon;$$

Define a Borel map $\xi : G \to L^1(G)$ setting $\xi_t = \eta_{s_n}$ whenever $t \in C_n$. This is well-defined because G is the disjoint union of the Borel sets C_n. We see that $\|\xi_t\|_1 = 1$ for all $t \in G$. Observe that, if $t \in C_n$, then it follows that $s_n \in tC^{-1}$. Let $(s,t) \in \mathrm{Tube}(K)$ and take $n, m \in \mathbb{N}$ such that $s \in C_n$, $t \in C_m$. Then we see that $s_n^{-1} s_m \in CKC^{-1}$, so

$$\|\xi_s - \xi_t\|_1 = \|\eta_{s_n} - \eta_{s_m}\|_1 < \varepsilon.$$

Moreover, we compute that $\mathrm{supp}(\xi_t) = \mathrm{supp}(\eta_{s_m}) \subseteq s_m L \subseteq tC^{-1}L$. It is now clear that ξ is a Borel map that satisfies condition (2′).

Step 2: We can assume that $t \mapsto \eta_t$ is continuous.
Fix a cut-off function $f : G \to [0, \infty)$. Denote $C = \mathrm{supp}(f)$. Let $\varepsilon > 0$ and let $K \subseteq G$ be a compact subset. By step 1, we find a piecewise constant Borel map $\eta : G \to L^1(G)$ and a compact subset $L \subseteq G$ such that $\|\eta_t\|_1 = 1$, $\mathrm{supp}\, \eta_t \subseteq tL$ for every $t \in G$ and

$$\sup_{(s,t)\in \mathrm{Tube}(CKC^{-1})} \|\eta_s - \eta_t\|_1 < \varepsilon;$$

We define $\xi : G \to L^1(G)$ by the formula

$$\xi_t(v) = \int_G f(s^{-1}t)\, |\eta_s(v)|\, d\mu(s).$$

We check that ξ_t is indeed in $L^1(G)$ and has norm 1 and still satisfies condition (2′). Then we show that ξ is a continuous map, and hence staisfies condition (2) of Theorem 2.3.

We compute that

$$\begin{aligned}\|\xi_t\|_1 &= \int_G \int_G f(t^{-1}s)\, |\eta_s(v)|\, d\mu(s) d\mu(v) \\ &= \int_G f(t^{-1}s)\, \|\eta_s\|_1\, d\mu(s) = \int_G f(s) d\mu(s) = 1.\end{aligned}$$

It is clear that the support of ξ_t is contained in the compact subset tCL. Whenever we have $(s, t) \in \mathrm{Tube}(K)$, we see that

$$\begin{aligned}\|\xi_s - \xi_t\|_1 &= \int_G \left| \int_G f(r^{-1}s)\, |\eta_r(v)|\, d\mu(r) - \int_G f(r^{-1}t)\, |\eta_r(v)|\, d\mu(r) \right| d\mu(v) \\ &\leq \int_G \int_G f(r^{-1})\, |\eta_{sr}(v) - \eta_{tr}(v)|\, d\mu(r) d\mu(v) \\ &= \int_G f(r)\, \|\eta_{sr} - \eta_{tr}\|_1\, d\mu(r) \\ &< \varepsilon \int_G f(r) d\mu(r) = \varepsilon.\end{aligned}$$

where the last inequality follows because $(sr, tr) \in \mathrm{Tube}(CKC^{-1})$ whenever $r^{-1} \in C$.

Suppose that (t_n) is a sequence in G that tends to t. Without loss of generality, we can assume that t_n remains in the compact neighborhood tC. It follows that

$$\|\xi_{t_n} - \xi_t\|_1 \leq \int_G \int_G \left| f(s^{-1}t_n) - f(s^{-1}t) \right| |\eta_s(v)|\, d\mu(s) d\mu(v)$$

$$= \int_G \left| f(s^{-1}t_n) - f(s^{-1}t) \right| \|\eta_s\|_1 \, d\mu(s).$$

This last integral converges to 0 by the Lebesgue dominated convergence theorem.

We have shown that our continuous map $\xi : G \to L^1(G)$ satisfies the required conditions. □

Throughout this paper, we often need to "smoothen" a given kernel. For example, when we are given a measurable kernel k_0 on G, we can obtain a continuous kernel by convolving k_0 with a cut-off function. Lemma 2.6 below shows that this convolution procedure preserves a number of relevant properties of the kernel.

Lemma 2.6. *Let G be a l.c.s.c group and let $k_0 : G \times G \to \mathbb{C}$ be a measurable kernel that is bounded on every tube. Let $f : G \to [0,\infty)$ be a cut-off function for G. Define a new kernel $k : G \times G \to \mathbb{C}$ by the formula*

$$k(s,t) = \int_G \int_G f(v) f(w) k_0(sv, tw) d\mu(v) d\mu(w) \tag{2.1}$$

This kernel satisfies the following properties

1) *k is bounded on every tube.*
2) *k is continuous. In fact, k satisfies the following uniform continuity property: whenever $s_n \to s$ and $t_n \to t$ in G, then we have that*
$$\sup_{v \in G} |k(vs_n, vt_n) - k(vs, vt)| \to 0.$$
3) *if the support of k_0 is a tube, then also the support of k is a tube.*
4) *if k_0 is a positive type kernel, then so is k.*
5) *if $T \subseteq G \times G$ is a tube, then*
$$\sup_{(s,t) \in T} |k(s,t) - 1| \leq \sup_{(x,y) \in T \cdot (\operatorname{supp} f \times \operatorname{supp} f)} |k_0(x,y) - 1|.$$

Proof. Observe that the new kernel k is well-defined, because for fixed $s, t \in G$, the function

$$(v, w) \mapsto f(v) f(w) k_0(sv, tw)$$

is a bounded measurable function with compact support. We check that k satisfies properties (1) – (5).

Property (1). Let $T \subseteq G \times G$ be a tube. Observe that $T_0 = T(\operatorname{supp}(f) \times \operatorname{supp}(f))$ is still a tube. So k_0 is bounded on T_0, say by $C > 0$. For any $(s,t) \in T$ and $v, w \in \operatorname{supp}(f)$, we get that $(sv, tw) \in T_0$, hence

$$\begin{aligned} |k(s,t)| &\leq \int_G \int_G f(v) f(w) \, |k_0(sv, tw)| \, d\mu(v) d\mu(w) \\ &\leq C \int_G \int_G f(v) f(w) d\mu(v) d\mu(w) = C. \end{aligned}$$

So k is bounded by on T.

Property (2). Suppose that $s_n \to s$ and $t_n \to t$ in G. Let U be a compact neighborhood of identity. We can assume that $s_n \in sU$ and $t_n \in tU$ for all $n \in \mathbb{N}$. By assumption, k_0 is bounded on the tube $T = \mathrm{Tube}(\mathrm{supp}(f)^{-1}U^{-1}s^{-1}tU\,\mathrm{supp}(f))$, say by $C > 0$. Let $r \in G$ be arbitrary. We compute that

$$
\begin{aligned}
&|k(rs_n, rt_n) - k(rs, rt)| \\
&\leq \left| \int_G \int_G f(v) f(w) k_0(rs_n v, rt_n w) - f(v) f(w) k_0(rsv, rtw) d\mu(v) d\mu(w) \right| \\
&\leq \int_G \int_G \left| f(s_n^{-1}v) f(t_n^{-1}w) - f(s^{-1}v) f(t^{-1}w) \right| |k_0(rv, rw)|\, d\mu(v) d\mu(w) \\
&\leq C \int_G \int_G \left| f(s_n^{-1}v) f(t_n^{-1}w) - f(s^{-1}v) f(t^{-1}w) \right| d\mu(v) d\mu(w)
\end{aligned}
$$

The last inequality follows from the fact that $(rv, rw) \in T$ whenever either $s_n^{-1}v, t_n^{-1}w \in \mathrm{supp}(f)$ or $s^{-1}v, t^{-1}w \in \mathrm{supp}(f)$. The last line does not depend on r and converges to 0 by the Lebesgue dominated convergence theorem because f is continuous and has compact support.

Property (3). A simple direct computation shows that the support of k is contained in $\mathrm{supp}(k_0)(\mathrm{supp}(f) \times \mathrm{supp}(f))$.

Property (4). It is well-known that a kernel is of positive type if and only if there is a Hilbert space H and a map $\xi : G \to H$ such that $k(s,t) = \langle \xi(s), \xi(t) \rangle$ for all $s, t \in G$. The map ξ can be chosen to be weakly Borel if k is Borel.

Take such a Borel map $\xi^0 : G \to H$ for the kernel k_0. Observe that ξ^0 is bounded because k_0 is. So the formula

$$
\varphi_s(\eta) = \int_G f(v) \langle \xi^0(sv), \eta \rangle d\mu(v)
$$

defines a bounded anti-linear functional on H. By the Riesz representation theorem, there is a unique vector $\xi(s)$ such that $\varphi_s(\eta) = \langle \xi(s), \eta \rangle$ for all $\eta \in H$. It now suffices to observe that

$$
k(s,t) = \langle \xi(s), \xi(t) \rangle.
$$

Property (5). It is straightforward by the properties of the cut-off function f. □

We are now ready to prove Theorem 2.3

Proof of Theorem 2.3. **1) ⇒ 2):** Let $K \subseteq G$ be a compact subset and let $\varepsilon > 0$. By Lemma 2.5, we only have to find a map $\eta : G \to L^1(G)$ that satisfies the conditions in (2′), i.e, condition (2), but η is not necessarily continuous.

Since G has property A, we find a compact subset $L \subseteq G$ and $\{A_s\}_{s \in G}$ a family of Borel subsets in $G \times \mathbb{N}$ with $0 < \mu'(A_s) < \infty$ such that

- for all $(s,t) \in \mathrm{Tube}(K)$ we have
$$\frac{\mu'(A_s \Delta A_t)}{\mu'(A_s \cap A_t)} < \frac{\varepsilon}{2},$$
- $(t,n) \in A_s$ implies $(s,t) \in \mathrm{Tube}(L)$.

For $s, t \in G$, we denote $A_{t,s} = (\{s\} \times \mathbb{N}) \cap A_t$. It follows from Tonelli's theorem that
$$\int_G |A_{t,s}| d\mu(s) = \int_{G \times \mathbb{N}} \chi_{A_t}(s,n) d\mu'(s,n) = \mu'(A_t) < \infty.$$
For each $t \in G$, consider the almost everywhere defined measurable map $\eta_t : G \to \mathbb{C}$ defined by
$$\eta_t(s) = \frac{|A_{t,s}|}{\mu'(A_t)}.$$
It is clear that $0 \leq \eta_t \in L^1(G)$ and $\|\eta_t\|_1 = 1$ for all $t \in G$. Note that
$$\begin{aligned}
\|\eta_s \cdot \mu'(A_s) - \eta_t \cdot \mu'(A_t)\|_1 &= \int_G |\, |A_{s,x}| - |A_{t,x}|\, |\, d\mu(x) \\
&\leq \int_G |(\{x\} \times \mathbb{N}) \cap (A_s \Delta A_t)| d\mu(x) \\
&= \mu'(A_s \Delta A_t),
\end{aligned}$$
where the last equality follows from Tonelli's theorem. Hence we see that for all $(s,t) \in \mathrm{Tube}(K)$,
$$\begin{aligned}
||\eta_s - \eta_t||_1 &\leq \left\| \eta_s - \eta_t \cdot \frac{\mu'(A_t)}{\mu'(A_s)} \right\|_1 + \left\| \eta_t \cdot \frac{\mu'(A_t)}{\mu'(A_s)} - \eta_t \right\|_1 \\
&\leq \frac{\mu'(A_s \Delta A_t)}{\mu'(A_s)} + \left| \frac{\mu'(A_t)}{\mu'(A_s)} - 1 \right| \\
&\leq 2 \cdot \frac{\mu'(A_s \Delta A_t)}{\mu'(A_s)} \\
&\leq 2 \cdot \frac{\mu'(A_t \Delta A_s)}{\mu'(A_t \cap A_s)} < \varepsilon.
\end{aligned}$$
Note also that if $\eta_t(s) \neq 0$, then $(s,n) \in A_t$ for some n, whence $(t,s) \in \mathrm{Tube}(L)$. Hence $\operatorname{supp} \eta_t \subseteq tL$.

2) $\Rightarrow$ 1): Given a compact subset $K \subseteq G$ and $\varepsilon > 0$. We choose a small $0 < \varepsilon' < 1$ such that $\frac{6\varepsilon'}{2-5\varepsilon'} < \varepsilon$. By 2) there exist a compact subset $L \subseteq G$ and a map $\eta : G \to L^1(G)$ such that $||\eta_t||_1 = 1$, $\operatorname{supp} \eta_t \subseteq tL$ for every $t \in G$ and

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\eta_s - \eta_t||_1 < \varepsilon'.$$

We identify η_t with a representative function $\eta_t : G \to \mathbb{C}$. It is not hard to see that we can assume that $\{s \in G : \eta_t(s) \neq 0\} \subseteq tL$ and η_t may also be supposed to be non-negative, since $|| \, |\eta_s| - |\eta_t| \, ||_1 \leq ||\eta_s - \eta_t||_1$.

Note that $\mu(L) > 0$, for otherwise $||\eta_t||_1 = 0$ for all $t \in G$. Let $M := \mu(L)/\varepsilon' > 0$. For each $t \in G$, we set

$$A_t := \{(s,n) \in G \times \mathbb{N} : n \leq \eta_t(s) \cdot M\}.$$

It is clear that A_t is a Borel subset of $G \times \mathbb{N}$ for each $t \in G$. For every $t \in G$, we define a measurable map $\theta_t : G \to [0,\infty)$ by

$$\theta_t(s) = \frac{|A_{t,s}|}{M},$$

where $A_{t,s} := \{n \in \mathbb{N} : (s,n) \in A_t\}$. Then θ_t satisfies the following two relations

$$\mu'(A_t) = M \cdot ||\theta_t||_1$$
$$||\theta_t - \eta_t||_1 < \mu(L)/M = \varepsilon',$$

for all $t \in G$. It follows that $M(1-\varepsilon') < \mu'(A_t) < M(1+\varepsilon')$. In particular, $0 < \mu'(A_t) < \infty$ for each $t \in G$. Moreover,

$$\begin{aligned} \mu'(A_t \Delta A_s) &= \int_G |A_{t,x} \Delta A_{s,x}| d\mu(x) \\ &= \int_G ||A_{t,x}| - |A_{s,x}|| d\mu(x) = M \cdot ||\theta_t - \theta_s||_1. \end{aligned}$$

Hence,

$$\frac{\mu'(A_s \Delta A_t)}{\mu'(A_s \cap A_t)} = \frac{2\mu'(A_s \Delta A_t)}{\mu'(A_s) + \mu'(A_t) - \mu'(A_s \Delta A_t)} = \frac{2||\theta_s - \theta_t||_1}{||\theta_s||_1 + ||\theta_t||_1 - ||\theta_s - \theta_t||_1}.$$

Since $||\theta_t||_1 > 1-\varepsilon'$ for every $t \in G$ and $||\theta_s - \theta_t||_1 < 3\varepsilon'$ for all $(s,t) \in \mathrm{Tube}(K)$, we see that

$$\frac{\mu'(A_s \Delta A_t)}{\mu'(A_s \cap A_t)} < \frac{6\varepsilon'}{2(1-\varepsilon') - 3\varepsilon'} = \frac{6\varepsilon'}{2-5\varepsilon'} < \varepsilon$$

for all $(s,t) \in \mathrm{Tube}(K)$.

Finally, if $(s,n) \in A_t$, then $\eta_t(s) \neq 0$. It follows that $(t,s) \in \mathrm{Tube}(L)$.

2) $\Rightarrow$ **3)**: Recall that there is a linear isometric embedding $I : L^1(G) \hookrightarrow C_0(G)^*$ given by

$$I(f)(g) = \int_G g f d\mu.$$

Given a compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a continuous map $\eta : G \rightarrow L^1(G)$ such that $||\eta_t||_1 = 1$, $\operatorname{supp} \eta_t \subseteq tL$ for every $t \in G$ and

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\eta_s - \eta_t||_1 < \varepsilon.$$

η_t may be supposed to be non-negative (since $|||\eta_s| - |\eta_t|||_1 \leq ||\eta_s - \eta_t||_1$). Define the map $\nu = I \circ \eta : G \rightarrow C_0(G)^*_+$, which is obviously a weak-$*$ continuous map with $||\nu_t|| = 1$. Let $g \in C_0(G)$ be such that $g|_{tL} = 0$. Then

$$\nu_t(g) = I(\eta_t)(g) = \int_{tL} g \eta_t d\mu = 0,$$

i.e., $\operatorname{supp} \nu_t \subseteq tL$. Finally,

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\nu_s - \nu_t|| = \sup_{(s,t)\in \mathrm{Tube}(K)} ||I(\eta_s - \eta_t)|| = \sup_{(s,t)\in \mathrm{Tube}(K)} ||\eta_s - \eta_t||_1 < \varepsilon.$$

3) $\Rightarrow$ **2)**: Let f be a cut-off function for G. Using convolution we define a linear contraction $T_f : C_0(G)^* \rightarrow L^1(G)$ by $T_f(\nu)(s) = \nu(f_s)$ where f_s is defined by $f_s(t) = f(s^{-1}t)$ for $s, t \in G$. Indeed, this is clearly a linear map, and the following computation shows that T is a contraction. Moreover, T is isometric when restricted to the positive cone $C_0(G)^*_+$.

$$\begin{aligned} ||T_f(\nu)||_1 = \int_G |\nu(f_s)|\, d\mu(s) &\leq \int_G \int_G f(s^{-1}t) d\, |\nu|\, (t) d\mu(s) \\ &= \int_G \int_G f(t^{-1}s) d\mu(s) d\, |\nu|\, (t) = ||\nu||. \end{aligned}$$

Observe that the inequality above becomes an equality if ν is positive. Given a compact subset $K \subseteq G$ and $\varepsilon > 0$, there exist a compact subset $L \subseteq G$ and a weak-$*$ continuous map $\nu : G \rightarrow C_0(G)^*_+$ such that $||\nu_t|| = 1$, $\operatorname{supp} \nu_t \subseteq tL$ for every $t \in G$ and

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\nu_s - \nu_t|| < \varepsilon.$$

Define $\eta : G \rightarrow L^1(G)$ by the composition

$$G \xrightarrow{\nu} C_0(G)^*_+ \overset{T_f}{\hookrightarrow} L^1(G).$$

It is clear that $||\eta_t||_1 = 1$ and $\operatorname{supp} \eta_t \subseteq t(L \cdot \operatorname{supp} f)$ for every $t \in G$. Moreover, we see that

$$\sup_{(s,t)\in \mathrm{Tube}(K)} ||\eta_s - \eta_t||_1 = \sup_{(s,t)\in \mathrm{Tube}(K)} ||T_f(\nu_s - \nu_t)||_1 \leq \sup_{(s,t)\in \mathrm{Tube}(K)} ||\nu_s - \nu_t|| < \varepsilon.$$

Finally, we show that η is continuous. Let $t_n \to t$, we want to show that $||\nu_{t_n} * f - \nu_t * f||_1 \to 0$. We can assume that $\{t_n\}_{n\in\mathbb{N}} \subseteq t \cdot \operatorname{supp} f$. Since ν is weak-$*$ continuous, we get

$$(\nu_{t_n} * f)(s) = \nu_{t_n}(f_s) \to \nu_t(f_s) = (\nu_t * f)(s),$$

for all $s \in G$. Note that $\operatorname{supp}(\nu_{t_n} * f) \subseteq t_n(L \cdot \operatorname{supp} f) \subseteq t \cdot \operatorname{supp} f \cdot L \cdot \operatorname{supp} f$ and

$$(\nu_{t_n} * f)(s) = \nu_{t_n}(f_s) = \int_G f_s(x) d\nu_{t_n}(x) \leq ||f||_\infty ||\nu_{t_n}|| = ||f||_\infty.$$

It follows that

$$(\nu_{t_n} * f) \leq \chi_{t\cdot \operatorname{supp} f \cdot L \cdot \operatorname{supp} f} ||f||_\infty \in L^1(G)\ a.e.$$

We complete the proof by Lebesgue's dominated convergence theorem.

2) $\Rightarrow$ 4): Let $\eta : G \to L^1(G)$ be a map as in (2). For each $t \in G$, define $\xi_t = |\eta_t|^{1/2}$. Then

$$\begin{aligned} ||\xi_t - \xi_s||_2^2 &= \int_{x\in G} |\xi_t(x) - \xi_s(x)|^2 d\mu(x) \\ &\leq \int_{x\in G} |\xi_t(x)^2 - \xi_s(x)^2| d\mu(x) \\ &= \int_{x\in G} ||\eta_t(x)| - |\eta_s(x)|| d\mu(x) \\ &\leq ||\eta_t - \eta_s||_1. \end{aligned}$$

Now, the rest of the proof is obvious.

4) $\Rightarrow$ 2): Let $\xi : G \to L^2(G)$ be a map as in (4). For each $t \in G$, define $\eta_t = |\xi_t|^2$. Then by the Cauchy–Schwarz inequality, one has

$$\begin{aligned} ||\eta_t - \eta_s||_1 &= \int_{x\in G} ||\xi_t(x)|^2 - |\xi_s(x)|^2| d\mu(x) \\ &= \int_{x\in G} (|\xi_t(x)| + |\xi_s(x)|)||\xi_t(x)| - |\xi_s(x)|| d\mu(x) \\ &\leq |||\xi_t| + |\xi_s|||_2 |||\xi_t| - |\xi_s|||_2 \\ &\leq 2||\xi_t - \xi_s||_2. \end{aligned}$$

Now, it is not hard to complete the proof.

4) ⇒ 5): Given a compact subset $K \subseteq G$ and $\varepsilon > 0$, let $\xi : G \to L^2(G)$ be a continuous map as in (4). We identify $\xi_t \in L^2(G)$ with a representative $\xi_t : G \to \mathbb{C}$. We may assume that $\{s \in G : \xi_t(s) \neq 0\} \subseteq tL$. Then we define a continuous positive type kernel $k : G \times G \to \mathbb{C}$ by the formula

$$k(s,t) = \langle \xi_s, \xi_t \rangle.$$

It is clear that $\operatorname{supp} k \subseteq \operatorname{Tube}(L \cdot L^{-1})$ and we compute that

$$\sup_{(s,t)\in\operatorname{Tube}(K)} |k(s,t) - 1| = \sup_{(s,t)\in\operatorname{Tube}(K)} |\langle \xi_s - \xi_t, \xi_t \rangle| \leq \sup_{(s,t)\in\operatorname{Tube}(K)} ||\xi_s - \xi_t||_2 ||\xi_t||_2 < \varepsilon.$$

5) ⇒ 4): Let a compact subset $K \subseteq G$ and $0 < \varepsilon < 1/2$ be given. Let f be a cut-off function for G. Observe that $\operatorname{supp} f \cdot (K \cup \{e\}) \cdot \operatorname{supp} f$ is a compact subset of G. By (5), there exist a compact subset $L \subseteq G$ and a continuous positive type kernel k_0 on G such that $\operatorname{supp} k_0 \subseteq \operatorname{Tube}(L)$ and

$$\sup\{|k_0(s,t) - 1| : (s,t) \in \operatorname{Tube}(\operatorname{supp} f \cdot (K \cup \{e\}) \cdot \operatorname{supp}\ f)\} < \varepsilon.$$

Observe that k_0 is bounded because it is of positive type and $k_0(s,s) < 1 + \varepsilon$ for all $s \in G$. It follows from Lemma 2.6 that $k : G \times G \to \mathbb{C}$ given by

$$k(s,t) = \int_G \int_G f(v) k_0(sv, tw) f(w) d\mu(w) d\mu(v)$$

is a continuous bounded positive type kernel whose support is still a tube, say $\operatorname{supp}(k) \subseteq \operatorname{Tube}(L')$. If $(s,t) \in \operatorname{Tube}(K \cup \{e\})$, then

$$\begin{aligned}
|k(s,t) - 1| &= \Bigg| \int_G \int_G f(v) k_0(sv, tw) f(w) d\mu(w) d\mu(v) \\
&\qquad\qquad - \int_G f(v) d\mu(v) \int_G f(w) d\mu(w) \Bigg| \\
&\leq \int_{\operatorname{supp} f} \int_{\operatorname{supp} f} f(w) f(v) |k_0(sv, tw) - 1| d\mu(v) d\mu(w) \\
&\leq \sup\{|k_0(x,y) - 1| : (x,y) \in \operatorname{Tube}(\operatorname{supp} f \cdot (K \cup \{e\}) \cdot \operatorname{supp} f)\} \\
&< \varepsilon.
\end{aligned}$$

Let T_{k_0} be the integral operator on $L^2(G)$, which is induced by k_0, so we define T_{k_0} by $T_{k_0}(\xi)(s) = \int_G k_0(s,t)\xi(t) d\mu(t)$. Note that T_{k_0} is positive and bounded, and that

$$k(s,t) = \langle T_{k_0} f_t, f_s \rangle,$$

where $f_t(x) = f(t^{-1}x)$.

Let p be a polynomial such that $0 \leq p(t)$ and $|p(t)^2 - t| < \varepsilon/||f||_2^2$ for $t \in [0, ||T_{k_0}||]$. Define a continuous map $\eta : G \to L^2(G)$ by $\eta_t = p(T_{k_0}) f_t$. Note that

$$\begin{aligned} |\langle \eta_t, \eta_s \rangle - k(s,t)| &= |\langle p(T_{k_0}) f_t, p(T_{k_0}) f_s \rangle - k(s,t)| \\ &= |\langle p^2(T_{k_0}) f_t, f_s \rangle - \langle T_{k_0} f_t, f_s \rangle| \\ &\leq ||p^2(T_{k_0}) - T_{k_0}|| ||f_t||_2 ||f_s||_2 \\ &< \varepsilon, \end{aligned}$$

for all $s, t \in G$. It follows that $|\langle \eta_t, \eta_s \rangle - 1| \leq |\langle \eta_t, \eta_s \rangle - k(s,t)| + |k(s,t) - 1| < 2\varepsilon$ for all $(s,t) \in \mathrm{Tube}(K \cup \{e\})$, which implies that $1 - 2\varepsilon < \mathrm{Re}\langle \eta_t, \eta_s \rangle < 2\varepsilon + 1$ for all $(s,t) \in \mathrm{Tube}(K \cup \{e\})$.

Since $\varepsilon < 1/2$, we see that $\sqrt{1 + 2\varepsilon} > ||\eta_t||_2 > \sqrt{1 - 2\varepsilon} > 0$. We define a continuous map $\xi : G \to L^2(G)$ by $\xi_t = \eta_t / ||\eta_t||_2$. This map ξ satisfies

$$1 - \mathrm{Re}\langle \xi_t, \xi_s \rangle = 1 - \frac{\mathrm{Re}\langle \eta_t, \eta_s \rangle}{\langle \eta_t, \eta_t \rangle^{1/2} \langle \eta_s, \eta_s \rangle^{1/2}} \leq 1 - \frac{1 - 2\varepsilon}{1 + 2\varepsilon} = \frac{4\varepsilon}{1 + 2\varepsilon} < 4\varepsilon$$

for all $(s,t) \in \mathrm{Tube}(K \cup \{e\})$. Therefore we see that $||\xi_s - \xi_t||_2 = \sqrt{2 - 2\mathrm{Re}\langle \xi_t, \xi_s \rangle} < \sqrt{8\varepsilon}$ for all $(s,t) \in \mathrm{Tube}(K)$. Finally, if p is of degree d, then it is not hard to see that

$$\mathrm{supp}\, \xi_t \subseteq t \cdot \mathrm{supp}\, f \cdot ((L^d)^{-1} \cup \cdots \cup L^{-1} \cup \{e\}). \qquad \square$$

We end this section by showing that property A is equivalent to amenability at infinity. Recall that a locally compact group G is said to be amenable at infinity if there exists a topologically amenable action (in the sense of [1]) of G on some compact Hausdorff space X. In the discrete case, it is known that G is exact if and only if G is amenable at infinity if and only if the action of G on its Stone-Čech compactification is topologically amenable if and only if G has property A. In the locally compact case, we have to replace the Stone-Čech compactification by the space $\beta^u(G)$ that is defined in the following way. $\beta^u(G)$ is the universal compact Hausdorff left G-space equipped with a continuous G-equivariant inclusion of G as an open dense subspace, which has the following property: any (continuous) G-equivariant map from G into a compact Hausdorff left G-space K extends uniquely to a continuous G-equivariant map from $\beta^u(G)$ into K. We can identify $C(\beta^u(G))$ with the C*-algebra of bounded left-uniform continuous functions on G, i.e. the algebra of all bounded continuous functions f on G such that $f(t^{-1}s) - f(s)$ tends to 0 uniformly as t tends to the unit element of G. Anantharaman-Delaroche showed in [1, Proposition 3.4] that a l.c.s.c. group G is amenable at infinity if and only if its action on $\beta^u(G)$ by translation is topologically amenable.

As in [1], we denote by θ the homeomorphism of $G \times G$ that is given by $\theta(s,t) = (s^{-1}, s^{-1}t)$. Let $C_{b,\theta}(G \times G)$ be the algebra of bounded continuous functions f on $G \times G$ such that $f \circ \theta$ has a continuous extension to $\beta^u(G) \times G$. In fact, we have the following characterization of the continuous functions $f : G \times G \to \mathbb{C}$ such that $f \circ \theta$ has a continuous extension to $\beta^u(G) \times G$.

Observation 2.7. *Let $f : G \times G \to \mathbb{C}$ be a (continuous) function. Then $f \circ \theta$ extends to a continuous function on $\beta^u(G) \times G$ if and only if f satisfies the following two conditions:*

$$\sup_{v \in G} |f(v, vt)| < \infty \qquad \text{for all } t \in G$$

$$\sup_{v \in G} |f(vs_n, vt_n) - f(vs, vt)| \to 0 \qquad \text{for all } s_n \to s \text{ and } t_n \to t.$$

In [1], Anantharaman-Delaroche showed the following characterization of l.c.s.c. groups that are amenable at infinity.

Theorem 2.8 ([1, Proposition 3.4 and 3.5]). *Let G be a l.c.s.c. group. Then the following are equivalent.*

1) *G is amenable at infinity, i.e. there exists a topologically amenable action of G on a compact Hausdorff space.*
2) *the action of G on $\beta^u(G)$ is topologically amenable.*
3) *There exists a net (k_i) of positive type kernels in $C_{b,\theta}(G \times G)$ with support in tubes such that $\lim_i k_i = 1$, uniformly on tubes.*

The uniform continuity property in Observation 2.7 above is precisely the one we obtained in Lemma 2.6. So it follows from point 5) of Lemma 2.6 that we may assume that the kernel k in point 5) of Theorem 2.3 is contained in $C_{b,\theta}(G \times G)$. This proves the following result

Corollary 2.9. *A l.c.s.c. group G is amenable at infinity if and only if G has property A.*

One of our motivations to study groups with property A is that for such groups, the Baum–Connes assembly map with coefficients is split-injective. This was proven first by Higson [8, Theorem 1.1] in the discrete case. Later, Chabert, Echterhoff and Oyono-Oyono showed [4, Theorem 1.9] that this is still true for l.c.s.c. groups that are amenable at infinity. Since we have just shown that a l.c.s.c. group with property A is amenable at infinity, this is still true for groups with property A.

Corollary 2.10. *If G is a locally compact, second countable, Hausdorff group which has property A, then the Baum–Connes assembly map with coefficients for G is split-injective.*

3. Uniform embeddability into Hilbert space

In this section we study groups that admit a uniform embedding into Hilbert space, in the sense of Gromov [6], see definition 3.1. As a consequence of Corollary 2.9

in the previous section, all groups with property A embed uniformly into Hilbert space. In fact, we show that uniform embeddability into Hilbert space is equivalent to the existence of a Haagerup action on a compact Hausdorff space. We say that an action $G \curvearrowright X$ on a compact Hausdorff space has the Haagerup property if the associated transformation groupoid has a continuous proper conditionally negative type function. In the previous section, we mentioned that the Baum–Connes assembly map with coefficients is split-injective for groups with property A. In Theorem 3.5, we extend this result to groups that embed uniformly into Hilbert space.

In [6], Gromov introduced the notion of a uniform embedding of a metric space into another one. On any l.c.s.c. group G, there is a proper left-invariant metric d, and this metric is unique up to coarse equivalence, see [7] and [14]. This gives a well-defined notion of a uniform embedding of a l.c.s.c. group into Hilbert space. However, for the purpose of this paper, we use the following equivalent definition, that was first given by Anantharaman-Delaroche in [1].

Definition 3.1. Let G be a locally compact, second countable, Hausdorff topological group. A map u from G into a Hilbert space H is said to be a uniform embedding if u satisfies the following two conditions:

a) for every compact subset K of G there exists $R > 0$ such that

$$(s,t) \in \mathrm{Tube}(K) \Rightarrow ||u(s) - u(t)|| \leq R;$$

b) for every $R > 0$ there exists a compact subset K of G such that

$$||u(s) - u(t)|| \leq R \Rightarrow (s,t) \in \mathrm{Tube}(K).$$

We say that a l.c.s.c. group G embeds uniformly into Hilbert space (or admits a uniform embedding into Hilbert space) if there exists a Hilbert space H and a uniform embedding $u : G \to H$.

Anantharaman-Delaroche showed in [1] that l.c.s.c. groups that are amenable at infinity, embed uniformly into Hilbert space. As a consequence of Corollary 2.9, we obtain the following:

Proposition 3.2 ([1], Proposition 3.7). *If a l.c.s.c. group G has property A, then G admits a uniform embedding into Hilbert space.*

As with property A, whenever there is a uniform embedding of G into H, there also is a continuous uniform embedding of G into H.

Proposition 3.3. *Let G be a locally compact, second countable group. The following are equivalent:*

1) *G admits a uniform embedding into a Hilbert space;*
2) *G admits a Borel uniform embedding into a separable Hilbert space;*
3) *G admits a continuous uniform embedding into a separable Hilbert space.*

Proof. It is clear that 3) implies 1).

1) $\Rightarrow$ 2): Let $u : G \to H$ be a uniform embedding into a Hilbert space H. Let C be a compact neighborhood of identity in G. As in the proof of Lemma 2.5, we find group elements $(s_n)_n$ and Borel subsets $C_n \subseteq s_n C$ such that $G = \bigsqcup_n C_n$. Define $u' : G \to H$ by the property that $u'(s) = u(s_n)$ whenever $s \in C_n$. This way, u' is a Borel step function. Since u is a uniform embedding, we find an $R > 0$ such that $\|u(s) - u(t)\| \leq R$ whenever $s^{-1}t \in C$. Fix $n \in \mathbb{N}$ and observe that every $s \in C_n$ satisfies $s_n^{-1} s \in C$. As a consequence,

$$\left\|u'(s) - u(s)\right\| = \|u(s_n) - u(s)\| \leq R.$$

Since this is true for all $n \in \mathbb{N}$ and $s \in C_n$, we see that u' is at bounded distance from u, so u' is still a uniform embedding. Observe that u' takes values only in the separable closed subspace $H_0 \subseteq H$ that is spanned by $\{u(s_n) : n \in \mathbb{N}\}$. In other words, u' is a Borel uniform embedding into the separable Hilbert space H_0.

2) $\Rightarrow$ 3): Let $u : G \to H$ be a Borel uniform embedding into a separable Hilbert space H. Let f be a cut-off function for G. Since u is a uniform embedding, we find $R > 0$ such that $\|u(s) - u(t)\| \leq R$ whenever $s^{-1}t \in \operatorname{supp}(f)$. For a fixed $t \in G$, we define an anti-linear functional $\varphi_t : H \to \mathbb{C}$ by the formula $\varphi_t(v) = \int_G \langle f(s^{-1}t)u(s), v\rangle d\mu(s)$ for all $v \in H$. Observe that φ_t is bounded because

$$\begin{aligned}
|\varphi_t(v)| &\leq \int_G \left|\langle f(s^{-1}t)u(s), v\rangle\right| d\mu(s) \\
&\leq \int_G f(s^{-1}t)(\|u(t)\| + R)\, \|v\|\, d\mu(s) \\
&= (\|u(t)\| + R)\, \|v\|\,,
\end{aligned}$$

for every vector $v \in H$. By the Riesz–Fréchet theorem there exists a unique $u'(t) \in H$ such that $\varphi_t(v) = \langle u'(t), v\rangle$ for all $v \in H$. Observe that $u'(t)$ is at distance at most R from $u(t)$:

$$\begin{aligned}
\left|\langle u'(t) - u(t), v\rangle\right| &= \left|\int_G f(s^{-1}t)\langle u(s), v\rangle d\mu(s) - \langle u(t), v\rangle \int_G f(s^{-1}t) d\mu(s)\right| \\
&\leq \int_G f(s^{-1}t) R\, \|v\|\, d\mu(s) \\
&\leq R\, \|v\|\,.
\end{aligned}$$

In particular, we see that u' is still a uniform embedding of G into H.

We show that u' is continuous. This follows from the following computation: let $(t_n)_n$ be a sequence in G that converges to $t \in G$. The sequence $t^{-1}t_n$ remains in some compact neighborhood U of identity in G. Since u is a uniform embedding, we find $R_2 > 0$ such that $\|u(s) - u(t)\| \leq R_2$ whenever $t^{-1}s \in U \operatorname{supp}(f)$. Now we see that, for every $v \in H$ and $n \in \mathbb{N}$,

$$\begin{aligned} \left|\langle u'(t_n) - u'(t), v\rangle\right| &\leq \int_G \left|f(s^{-1}t_n) - f(s^{-1}t)\right| \|u(s)\| \, \|v\| \, d\mu(s) \\ &= \int_G \left|f(t_n^{-1}s) - f(t^{-1}s)\right| \|u(s)\| \, \|v\| \, d\mu(s) \\ &\leq \|t_n \cdot f - t \cdot f\|_\infty \, \mu(U \operatorname{supp}(f))(\|u(t)\| + R_2) \, \|v\| \, . \end{aligned}$$

Since f is continuous with compact support, we see that $\|t_n \cdot f - t \cdot f\|_\infty$ tends to 0. Therefore we also get that $\|u'(t_n) - u'(t)\|$ tends to 0. □

We give an alternative characterization of uniform embedding into Hilbert space in terms of transformation groupoids and conditionally negative type functions on it. For the convenience of our readers, we recall these concepts:

Let G be a locally compact group acting continuously on a locally compact Hausdorff space X. The transformation groupoid $X \rtimes G$ consists of all pairs (x, g) with $x \in X$, $g \in G$. Its base space is X, and the source and range maps are given by

$$s(x, g) = g^{-1}x, \quad r(x, g) = x.$$

The composition law is $(gx, g)(x, g') = (gx, gg')$ and the inversion is given by $(x, g)^{-1} = (g^{-1}x, g^{-1})$.

A conditionally negative type function on $X \rtimes G$ is a function $\psi : X \times G \to \mathbb{R}$ such that

1) $\psi(x, e) = 0$ for all $x \in X$;
2) $\psi(x, g) = \psi(g^{-1}x, g^{-1})$ for all $(x, g) \in X \times G$;
3) $\sum_{i,j=1}^n t_i t_j \psi(g_i^{-1}x, g_i^{-1}g_j) \leq 0$ for all $\{t_i\}_{i=1}^n \subseteq \mathbb{R}$ satisfying $\sum_{i=1}^n t_i = 0$, $g_i \in G$ and $x \in X$.

We say that an action $G \curvearrowright X$ of a group on a compact Hausdorff space has the Haagerup property if its transformation groupoid $X \rtimes G$ admits a continuous proper conditionally negative type function.

Theorem 3.4. *Let G be a l.c.s.c. group. The following are equivalent:*

1) *G admits a uniform embedding into a Hilbert space.*
2) *There exists a continuous conditionally negative type kernel k on $G \times G$ satisfying*
   - *k is bounded on every tube;*
   - *k is a proper kernel, i.e. $\{(s, t) \in G \times G : |k(s, t)| \leq R\}$ is a tube for all $R > 0$.*
3) *The action $G \curvearrowright \beta^u(G)$ has the Haagerup property, i.e, there exists a continuous proper conditionally negative type function on $\beta^u(G) \rtimes G$.*

4) *There exists a second countable compact Hausdorff left G-space Y which admits a continuous proper conditionally negative type function on $Y \rtimes G$.*

Proof. **2) $\Rightarrow$ 1):** Assume that k is a continuous conditionally negative type kernel on $G \times G$ satisfying the conditions in 2). It follows from the GNS construction (see Theorem C.2.3 [3]) that there exist a real Hilbert space H and a continuous map $u : G \to H$ such that

$$k(s,t) = ||u(s) - u(t)||^2.$$

By the conditions on k, it is easy to see that u is a uniform embedding.

1) $\Rightarrow$ 3): We may assume that G admits a continuous uniform embedding $u : G \to H$, where H is a (separable) Hilbert space. Define a continuous function $k_0 : G \times G \to \mathbb{R}$ by

$$k_0(s,t) = ||u(s) - u(t)||^2.$$

It is clear that k_0 is bounded on every tube. Let f be a cut-off function for G. It follows from Lemma 2.6 that the kernel $k : G \times G \to \mathbb{R}$ that is given by

$$k(s,t) = \int_G \int_G f(v) k_0(sv, tw) f(w) d\mu(w) d\mu(v)$$

is continuous and bounded on every tube. Moreover, $k \circ \theta$ has a continuous extension $\psi_0 : \beta^u(G) \times G \to \mathbb{R}$.

We have already seen in the proof of the previous proposition that there exists a unique continuous uniform embedding $u * f : G \to H$ such that

$$\langle u * f(t), \eta \rangle = \int_G \langle f(s^{-1}t) u(s), \eta \rangle d\mu(s) \quad \text{for } \eta \in H.$$

Now, we define a continuous conditionally negative type kernel $\varphi : G \times G \to \mathbb{R}$ by

$$\varphi(s,t) = ||u * f(s) - u * f(t)||^2.$$

By the definition of $u * f$, it is not hard to see that

$$\begin{aligned}
\varphi(s,t) &= \mathrm{Re} \int_G \int_G f(v) \langle u(sv) - u(tv), u(sw) - u(tw) \rangle f(w) d\mu(w) d\mu(v) \\
&= \int_G \int_G f(v) \mathrm{Re} \langle u(sv) - u(tv), u(sw) - u(tw) \rangle f(w) d\mu(w) d\mu(v) \\
&= \frac{1}{2} \int_G \int_G f(v) f(w) \\
&\qquad \begin{pmatrix} - \|u(sw)\|^2 + 2\mathrm{Re}\langle u(sv), u(sw) \rangle - \|u(sv)\|^2 \\ + \|u(sv)\|^2 - 2\mathrm{Re}\langle u(sv), u(tw) \rangle + \|u(tw)\|^2 \\ - \|u(tw)\|^2 + 2\mathrm{Re}\langle u(tv), u(tw) \rangle - \|u(tv)\|^2 \\ + \|u(tv)\|^2 - 2\mathrm{Re}\langle u(tv), u(sw) \rangle + \|u(sw)\|^2 \end{pmatrix} d\mu(w) d\mu(v)
\end{aligned}$$

$$= \frac{1}{2}\int_G\int_G f(v)f(w)(-k_0(sv,sw) + k_0(sv,tw) - k_0(tv,tw) + k_0(tv,sw))d\mu(w)d\mu(v)$$
$$= \frac{1}{2}(k(s,t) - k(s,s) - k(t,t) + k(t,s))$$
$$= \frac{1}{2}(2k(s,t) - k(s,s) - k(t,t)),$$

where the first equality follows from the fact that φ is real-valued.

Thus, the function $\psi : \beta^u(G) \times G \to \mathbb{R}$ that is given by $\psi(y,t) = \psi_0(y,t) - \frac{1}{2}(\psi_0(y,e) + \psi_0(t^{-1}y,e))$ extends $\varphi \circ \theta$ continuously. Note that φ is a conditionally negative type kernel on $G \times G$ if and only if $\varphi \circ \theta$ is a conditionally negative type function on $G \rtimes G$, which is also equivalent to ψ being a conditionally negative type function on $\beta^u(G) \rtimes G$. Moreover, ψ is proper because $\{(s,t) \in G \times G : |\varphi(s,t)| \leq R\}$ is a tube for all $R > 0$.

3) $\Rightarrow$ 4): Let $\varphi : \beta^u(G) \times G \to \mathbb{R}$ be a continuous proper conditionally negative type function on $\beta^u(G) \rtimes G$. If we identify $C(\beta^u(G) \times G)$ with $C(G, C(\beta^u(G)))$, then $G \ni t \mapsto \varphi(\cdot,t) \in C(\beta^u(G))$ is a continuous map. Let A be the C^*-algebra generated by the unit in $C(\beta^u(G))$ and the set $\{s.\varphi(\cdot,t) : s,t \in G\}$. It is clear that A is a unital, separable and G-invariant C^*-subalgebra of $C(\beta^u(G))$. Hence, there exists a compact Hausdorff, second countable left G-space Y such that $A \cong C(Y)$. It is not hard to see that there exist a continuous G-equivariant surjection $p : \beta^u(G) \to Y$ and a continuous function $\psi : Y \times G \to \mathbb{R}$ such that the following diagram

$$\begin{tikzcd}
\beta^u(G) \times G \arrow[d, "p\times\mathrm{id}"'] \arrow[dr, "\varphi"] & \\
Y \times G \arrow[r, "\psi"] & \mathbb{R}
\end{tikzcd}$$

commutes. The properness of ψ follows from the properness of φ and the surjectivity of p. Since p is also G-equivariant, ψ is a conditionally negative type function on $Y \rtimes G$, as desired.

4) $\Rightarrow$ 2): Let $\varphi : Y \times G \to \mathbb{R}$ be a conditionally negative type function on $Y \rtimes G$. Fix one point $y_0 \in Y$ and define a kernel $k : G \times G \to \mathbb{R}$ by the following formula:

$$k(s,t) = \varphi(s^{-1}y_0, s^{-1}t) \quad \text{for all } s,t \in G.$$

It is now clear that k is a continuous function. Because φ was a conditionally negative function, one easily computes that k is a conditionally negative type kernel on G. Moreover, because Y is compact and φ is continuous, it follows that k is bounded on tubes. Finally, the properness of φ translates to the properness of k, as a kernel on G. $\square$

Our final result shows that for every group G that embeds uniformly into a Hilbert space, the Baum–Connes assembly map with coefficients is split-injective.

The analogous result for discrete groups was first proven by Skandalis, Tu and Yu ([13] Theorem 6.1). The argument is almost identical to the one used to prove [4, Theorem 1.9].

Theorem 3.5. *If G is a l.c.s.c. group which admits a uniform embedding into Hilbert space, then the Baum–Connes assembly map*

$$\mu_A : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_r G)$$

is split-injective for any separable G-C^*-algebra A.

Proof. Suppose that ψ is a continuous proper conditionally negative type function on $Y \rtimes G$ as in Theorem 3.4 4). We show first that we can assume that Y is a compact convex space, on which G acts by affine transformations. Let X denote the space $\mathrm{Prob}(Y)$ of Borel probability measures on Y equipped with the weak-$*$ topology. Notice that X is a second countable compact Hausdorff left G-space (with the induced action from $G \curvearrowright Y$). We define $\varphi : X \times G \to \mathbb{R}$ by

$$\varphi(m,t) = \int_Y \psi(y,t)\, dm(y).$$

We claim that φ is a continuous proper conditionally negative type function on $X \rtimes G$. Indeed, if we identify X with the state space of $C(Y)$, then we see that $\varphi(m,t) = m(\psi(\cdot,t))$. Thus, the continuity of φ follows from the norm-boundedness of X and the continuity of the map $G \ni t \mapsto \psi(\cdot,t) \in C(Y,\mathbb{R})$. It is not hard to see that φ is a continuous proper conditionally negative type function since ψ is.

We now consider the following commutative diagram, which is called the Higson descent diagram (cf. the proof of Theorem 3.2 in [8]):

$$\begin{tikzcd}
K_*^{\mathrm{top}}(G; A) \arrow[r, "\mu_A"] \arrow[d, "i_*"'] & K_*(A \rtimes_r G) \arrow[d, "i_*"] \\
K_*^{\mathrm{top}}(G; A \otimes C(X)) \arrow[r, "\mu_{A\otimes C(X)}"'] & K_*((A \otimes C(X)) \rtimes_r G),
\end{tikzcd}$$

where the vertical arrows are induced by the inclusion $i : \mathbb{C} \to C(X)$ and the horizontal arrows are the Baum–Connes assembly maps. By Lemma 4.1 in [13], the Baum–Connes assembly map for the groupoid $X \rtimes G$ with coefficients in $A \otimes C(X)$ is the same as the one for the group G with coefficients in $A \otimes C(X)$. Because whenever the groupoid $X \rtimes G$ has a continuous proper conditionally negative type function, it also has a proper affine isometric action on a continuous field of Hilbert spaces over X [15]. Hence, by Theorem 9.3 in [15] the bottom horizontal arrow is an isomorphism. Since X is convex and the action of G on X is affine, the space X is

K-equivariantly contractible for any compact subgroup K of G. By Proposition 1.10 in [4], the left vertical arrow is an isomorphism. An easy diagram chase then shows the split-injectivity of the assembly map μ_A. □

References

[1] C. Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Trans. Amer. Math. Soc.* **354** (2002), 4153–4178 (electronic). Zbl 1035.46039 MR 1926869

[2] P. Baum, A. Connes and N. Higson. Classifying Space for Proper Actions and K-Theory of Group C^*-algebras. *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994. Zbl 0830.46061 MR 1292018

[3] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T)*. New Mathematical Monographs, 11, Cambridge University Press, 2008. Zbl 1146.22009 MR 2415834

[4] J. Chabert, S. Echterhoff and H. Oyono-Oyono. Going-down functors, the Künneth formula, and the Baum–Connes conjecture. *Geom. Funct. Anal.* **14** (2004), 491–528. Zbl 1063.46056 MR 2100669

[5] S. Ferry, A. Ranicki and J. Rosenberg (eds.). *Novikov conjectures, index theorems and rigidity*. Based on a conference of the Mathematisches Forschungsinstitut Oberwolfach held in September 1993. London Mathematical Society Lecture Notes, 226, Cambridge University Press, 1995. Zbl 0829.00027 MR 1388295

[6] M. Gromov. Asymptotic invariants of infinite groups. *Geometric Group Theory, Asymptotic invariants of infinite groups. Proceedings of the symposium held at Sussex University, Brighton, July 14–19, 1991*, 1–295. London Mathematical Society Lecture Note Series, 182, Cambridge University Press, 1993. Zbl 0841.20039 MR 1253544

[7] U. Haagerup and A. Przybyszewska. *Proper metrics on locally compact groups, and proper affine isometric actions on Banach spaces*, arXiv:math/0606794v1, 2006.

[8] N. Higson. Bivariant K-Theory and the Novikov Conjecture. *Geom. Funct. Anal.* **10** (2000), 563–581. Zbl 0962.46052 MR 1779613

[9] N. Higson and J. Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.* **519** (2000), 143–153. Zbl 0964.55015 MR 1739727

[10] N. Ozawa. Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Ser. I Math.* **330** (2000), 691–695. Zbl 0953.43001 MR 1763912

[11] J. Roe. Warped cones and property A. *Geom. Topol.* **9** (2005), 163–178. Zbl 1082.53036 MR 2115671

[12] J. Roe. *Index Theory, Coarse Geometry, and Topology of Manifolds*. CBMS Regional Conference Series in Mathematics, 90. American Mathematical Society, 1996. Zbl 0853.58003 MR 1399087

[13] G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum–Connes conjecture and groupoids. *Topology* **41** (2002), 807–834. Zbl 1033.19003 MR 1905840

[14] R. A. Struble. Metrics in locally compact groups. *Compositio Mathematica* **28** (1974), 217–222. Zbl 0288.22010 MR 348037

[15] J.-L. Tu. La conjecture de Baum–Connes pour les feuilletages moyennables. *K-Theory* **17** (1999), 215–264. Zbl 0939.19001 MR 1703305

[16] J. L. Tu. Remarks on Yu's "property A" for discrete metric spaces and groups. *Bull. Soc. Math. France* **129** (2001), 115–139. Zbl 1036.58021 MR 1871980

[17] G. Yu. The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139** (2000), 201–240. Zbl 0956.19004 MR 1728880

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Article B

Approximation properties of simple Lie groups made discrete

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Approximation Properties of Simple Lie Groups Made Discrete

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Abstract. In this paper we consider the class of connected simple Lie groups equipped with the discrete topology. We show that within this class of groups the following approximation properties are equivalent: (1) the Haagerup property; (2) weak amenability; (3) the weak Haagerup property (Theorem 1.10). In order to obtain the above result we prove that the discrete group $\mathrm{GL}(2,K)$ is weakly amenable with constant 1 for any field K (Theorem 1.11).
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1. Introduction

Amenability for groups was first introduced by von Neumann in order to study the Banach-Tarski paradox. It is remarkable that this notion has numerous characterizations and one of them, in terms of an approximation property by positive definite functions, is the following: a locally compact (Hausdorff) group G is amenable if there exists a net of continuous compactly supported, positive definite functions on G tending to the constant function 1 uniformly on compact subsets of G. Later, three weak forms of amenability were introduced: the Haagerup property, weak amenability and the weak Haagerup property. In this paper we will study these approximation properties of groups within the framework of Lie theory and coarse geometry.

Definition 1.1 (Haagerup property [10]). *A locally compact group G has the* Haagerup property *if there exists a net of positive definite C_0-functions on G, converging uniformly to 1 on compact sets.*

Definition 1.2 (Weak amenability [18]). *A locally compact group G is* weakly amenable *if there exists a net $(\varphi_i)_{i\in I}$ of continuous, compactly supported Herz-Schur multipliers on G, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.*

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The weak amenability constant $\Lambda_{\mathrm{WA}}(G)$ *is defined as the best (lowest) possible constant* Λ *such that* $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$*, where* $(\varphi_i)_{i\in I}$ *is as just described.*

Definition 1.3 (The weak Haagerup property [32]). *A locally compact group* G *has the* weak Haagerup property *if there exists a net* $(\varphi_i)_{i\in I}$ *of* C_0 *Herz-Schur multipliers on* G*, converging uniformly to* 1 *on compact sets, and such that* $\sup_i \|\varphi_i\|_{B_2} < \infty$*.*

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(G)$ *is defined as the best (lowest) possible constant* Λ *such that* $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$*, where* $(\varphi_i)_{i\in I}$ *is as just described.*

Clearly, amenable groups have the Haagerup property. It is also easy to see that amenable groups are weakly amenable with $\Lambda_{\mathrm{WA}}(G) = 1$ and that groups with the Haagerup property have the weak Haagerup property with $\Lambda_{\mathrm{WH}}(G) = 1$. Also, $1 \leq \Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WA}}(G)$ for any locally compact group G, so weakly amenable groups have the weak Haagerup property.

It is natural to ask about the relation between the Haagerup property and weak amenability. The two notions agree in many cases, like generalized Baumslag-Solitar groups (see [15, Theorem 1.6]) and connected simple Lie groups with the discrete topology (see Theorem 1.10). However, in the known cases where the Haagerup property coincides with weak amenability, this follows from classification results on the Haagerup property and weak amenability and not from a direct connection between the two concepts. In general, weak amenability does not imply the Haagerup property and vice versa. In one direction, the group $\mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property [14], but is not weakly amenable [39]. In the other direction, the simple Lie groups $\mathrm{Sp}(1,n)$, $n \geq 2$, are weakly amenable [18], but since these non-compact groups also have Property (T) [3, Section 3.3], they cannot have the Haagerup property. However, since the weak amenability constant of $\mathrm{Sp}(1,n)$ is $2n-1$, it is still reasonable to ask whether $\Lambda_{\mathrm{WA}}(G) = 1$ implies that G has the Haagerup property. In order to study this, the weak Haagerup property was introduced in [31, 32], and the following questions were considered.

Question 1.4. *For which locally compact groups* G *do we have* $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$*?*

Question 1.5. *Is* $\Lambda_{\mathrm{WH}}(G) = 1$ *if and only if* G *has the Haagerup property?*

It is clear that if the weak amenability constant of a group G is 1, then so is the weak Haagerup constant, and Question 1.4 has a positive answer. In general, the constants differ by the example $\mathbb{Z}/2 \wr \mathbb{F}_2$ mentioned before. There is an another class of groups for which the two constants are known to be the same.

Theorem 1.6 ([25]). *Let* G *be a connected simple Lie group. Then* G *is weakly amenable if and only if* G *has the weak Haagerup property. Moreover,* $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$*.*

By the work of many authors [16, 18, 9, 20, 24, 26], it is known that a connected simple Lie group G is weakly amenable if and only if the real rank of G is zero or one. Also, the weak amenability constants of these groups are known. Recently, a similar result was proved about the weak Haagerup property [25, Theorem B]. Combining the results on weak amenability and the weak Haagerup property with the classification of connected Lie groups with the Haagerup property [10, Theorem 4.0.1] one obtains the following theorem, which gives a partial answer to both Question 1.4 and Question 1.5.

Theorem 1.7. *Let G be a connected simple Lie group. The following are equivalent.*

1. *G is compact or locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ for some $n \geq 2$.*
2. *G has the Haagerup property.*
3. *G is weakly amenable with constant 1.*
4. *G has the weak Haagerup property with constant 1.*

The purpose of this paper is to consider the same class of groups as in theorem above, but made discrete. When G is a locally compact group, we let G_{d} denote the same group equipped with the discrete topology. The idea of considering Lie groups without their topology (or with the discrete topology, depending on the point of view) is not a new one. For instance, a conjecture of Friedlander and Milnor is concerned with computing the (co)homology of the classifying space of G_{d}, when G is a Lie group (see [34] and the survey [40]).

Other papers discussing the relation between G and G_{d} include [13], [2] and [4]. Since our focus is approximation properties, will we be concerned with the following question.

Question 1.8. *Does the Haagerup property/weak amenability/the weak Haagerup property of G_{d} imply the Haagerup property/weak amenability/the weak Haagerup property of G?*

It is not reasonable to expect an implication in the other direction. For instance, many compact groups such as $\mathrm{SO}(n)$, $n \geq 3$, are non-amenable as discrete groups. It follows from Theorem 1.10 below (see also Proposition 4.1) that when $n \geq 5$, then $\mathrm{SO}(n)$ as a discrete group does not even have the weak Haagerup property. It is easy to see that Question 1.8 has a positive answer for second countable, locally compact groups G that admit a lattice Γ. Indeed, G has the Haagerup property if and only if Γ has the Haagerup property. Moreover, $\Lambda_{\mathrm{WA}}(\Gamma) = \Lambda_{\mathrm{WA}}(G)$ and $\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(G)$.

Remark 1.9. *A similar question can of course be asked for amenability. This case is already settled: if G_{d} is amenable, then G is amenable [41, Proposition 4.21], and the converse is not true in general by the counterexamples mentioned above. A sufficient and necessary condition of the converse implication can be found in [2].*

Recall that $\mathrm{SL}(2,\mathbb{R})$ is locally isomorphic to $\mathrm{SO}(2,1)$ and that $\mathrm{SL}(2,\mathbb{C})$ is locally isomorphic to $\mathrm{SO}(3,1)$. Thus, Theorem 1.7 and the main theorem below together show in particular that Question 1.8 has a positive answer for connected simple Lie groups. This could however also be deduced (more easily) from the fact that connected simple Lie groups admit lattices [44, Theorem 14.1].

Theorem 1.10 (Main Theorem)**.** *Let G be a connected simple Lie group, and let G_{d} denote the group G equipped with the discrete topology. The following are equivalent.*

1. *G is locally isomorphic to $\mathrm{SO}(3)$, $\mathrm{SL}(2,\mathbb{R})$, or $\mathrm{SL}(2,\mathbb{C})$.*
2. *G_{d} has the Haagerup property.*
3. *G_{d} is weakly amenable with constant 1.*
4. *G_{d} is weakly amenable.*
5. *G_{d} has the weak Haagerup property with constant 1.*
6. *G_{d} has the weak Haagerup property.*

The equivalence of (1) and (2) in Theorem 1.10 was already done by de Cornulier [13, Theorem 1.14] and in greater generality. His methods are the inspiration for our proof of Theorem 1.10. That (1) implies (2) basically follows from a theorem of Guentner, Higson and Weinberger [21, Theorem 5.4], namely that the discrete group $\mathrm{GL}(2,K)$ has the Haagerup property for any field K. Here we prove a similar statement about weak amenability.

Theorem 1.11. *Let K be any field. The discrete group $\mathrm{GL}(2,K)$ is weakly amenable with constant 1.*

Theorem 1.11 is certainly known to experts. The result was already mentioned in [43, p. 7] and in [38] with a reference to [21], and indeed our proof of Theorem 1.11 is merely an adaption of the methods developed in [21]. However, since no published proof is available, we felt the need to include a proof.

To obtain Theorem 1.10 we use the classification of simple Lie groups and then combine Theorem 1.11 with the following results proved in Section 4: If G is one of the four groups $\mathrm{SO}(5)$, $\mathrm{SO}_0(1,4)$, $\mathrm{SU}(3)$ or $\mathrm{SU}(1,2)$, then G_{d} does not have the weak Haagerup property. Also, if G is the universal covering group of $\mathrm{SU}(1,n)$ where $n \geq 2$, then G_{d} does not have the weak Haagerup property.

2. Preliminaries

Throughout, G will denote a locally compact group. A kernel $\varphi\colon G\times G\to\mathbb{C}$ is a *Schur multiplier* if there exist bounded maps $\xi,\eta\colon G\to\mathcal{H}$ into a Hilbert space $\mathcal{H}$ such that $\varphi(g,h)=\langle\xi(g),\eta(h)\rangle$ for every $g,h\in G$. The Schur norm of φ is defined as

$$\|\varphi\|_S = \inf\{\|\xi\|_\infty\|\eta\|_\infty\}$$

where the infimum is taken over all $\xi, \eta\colon G \to \mathcal{H}$ as above. See [42, Theorem 5.1] for different characterizations of Schur multipliers. Clearly, $\|\varphi \cdot \psi\|_S \le \|\varphi\|_S \cdot \|\psi\|_S$ and $\|\check{\varphi}\|_S = \|\varphi\|_S$ when φ and ψ are Schur multipliers and $\check{\varphi}(x,y) = \varphi(y,x)$. Also, any positive definite kernel φ on G which is normalized, i.e., $\varphi(x,x) = 1$ for every $x \in G$, is a Schur multiplier of norm 1. Finally, notice that the unit ball of Schur multipliers is closed under pointwise limits.

A continuous function $\varphi\colon G \to \mathbb{C}$ is a *Herz-Schur multiplier* if the associated kernel $\widehat{\varphi}(g,h) = \varphi(g^{-1}h)$ is a Schur multiplier. The Herz-Schur norm of φ is defined as $\|\varphi\|_{B_2} = \|\widehat{\varphi}\|_S$. When φ is a Herz-Schur multiplier, the two bounded maps $\xi, \eta\colon G \to \mathcal{H}$ can be chosen to be continuous (see [6] and [29]). The set $B_2(G)$ of Herz-Schur multipliers on G is a unital Banach algebra under pointwise multiplication and $\|\cdot\|_\infty \le \|\cdot\|_{B_2}$. Any continuous, positive definite function φ on G is a Herz-Schur multiplier with $\|\varphi\|_{B_2} = \varphi(1)$.

Below we list a number of permanence results concerning weak amenability and the weak Haagerup property, which will be useful later on. General references containing almost all of the results are [1], [18], [24] and [32]. Additionally we refer to [17, Theorem III.9] and [8, Corollary 12.3.12].

Suppose Γ_1 is a co-amenable subgroup of a discrete group Γ_2, that is, there exists a left Γ_2-invariant mean on $l^\infty(\Gamma_2/\Gamma_1)$. Then

$$\Lambda_{\mathrm{WA}}(\Gamma_1) = \Lambda_{\mathrm{WA}}(\Gamma_2). \tag{2.1}$$

If $(G_i)_{i\in I}$ is a directed family of open subgroups in a locally compact group G whose union is G, then

$$\Lambda_{\mathrm{WA}}(G) = \sup \Lambda_{\mathrm{WA}}(G_i). \tag{2.2}$$

For any two locally compact groups G and H

$$\Lambda_{\mathrm{WA}}(G \times H) = \Lambda_{\mathrm{WA}}(G)\Lambda_{\mathrm{WA}}(H). \tag{2.3}$$

When H is a closed subgroup of G

$$\Lambda_{\mathrm{WA}}(H) \le \Lambda_{\mathrm{WA}}(G) \quad \text{and} \quad \Lambda_{\mathrm{WH}}(H) \le \Lambda_{\mathrm{WH}}(G). \tag{2.4}$$

When K is a compact normal subgroup of G then

$$\Lambda_{\mathrm{WA}}(G/K) = \Lambda_{\mathrm{WA}}(G) \quad \text{and} \quad \Lambda_{\mathrm{WH}}(G/K) = \Lambda_{\mathrm{WH}}(G). \tag{2.5}$$

When Z is a central subgroup of a discrete group G then

$$\Lambda_{\mathrm{WA}}(G) \le \Lambda_{\mathrm{WA}}(G/Z). \tag{2.6}$$

Recall that a *lattice* in a locally compact group G is a discrete subgroup Γ such that the quotient G/Γ admits a non-trivial finite G-invariant Radon measure. When Γ is a lattice in a second countable, locally compact G then

$$\Lambda_{\mathrm{WA}}(\Gamma) = \Lambda_{\mathrm{WA}}(G) \quad \text{and} \quad \Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(G). \tag{2.7}$$

When H is a finite index, closed subgroup in a group G then

$$\Lambda_{\mathrm{WH}}(H) = \Lambda_{\mathrm{WH}}(G). \tag{2.8}$$

3. Weak amenability of $\mathbf{GL}(2, K)$

This section is devoted to the proof of Theorem 1.11 (see Theorem 3.7 below). The general idea of our proof follows the idea of [21, Section 5], where it is shown that for any field K the discrete group $\mathrm{GL}(2, K)$ has the Haagerup property. Our proof of Theorem 1.11 also follows the same strategy as used in [22].

Recall that a *pseudo-length function* on a group G is a function $\ell\colon G \to [0, \infty)$ such that

- $\ell(e) = 0$,
- $\ell(g) = \ell(g^{-1})$,
- $\ell(g_1 g_2) \leq \ell(g_1) + \ell(g_2)$.

Moreover, ℓ is a length function on G if, in addition, $\ell(g) = 0 \implies g = e$.

Definition 3.1. *We say that the pseudo-length group (G, ℓ) is* weakly amenable *if there exist a sequence (φ_n) of Herz-Schur multipliers on G and a sequence (R_n) of positive numbers such that*

- $\sup_n \|\varphi_n\|_{B_2} < \infty$*;*
- $\operatorname{supp} \varphi_n \subseteq \{g \in G \mid \ell(g) \leq R_n\}$*;*
- $\varphi_n \to 1$ *uniformly on* $\{g \in G \mid \ell(g) \leq S\}$ *for every* $S > 0$*.*

The weak amenability constant $\Lambda_{\mathrm{WA}}(G, \ell)$ *is defined as the best possible constant* Λ *such that* $\sup_n \|\varphi_n\|_{B_2} \leq \Lambda$*, where* (φ_n) *is as just described.*

Notice that if the group G is discrete and the pseudo-length function l on G is proper (in particular, G is countable), then the weak amenability of (G, l) is equivalent to the weak amenability of G with same weak amenability constant. On other hand, every countable discrete group admits a proper length function, which is unique up to coarse equivalence ([46, Lemma 2.1]). If the group is finitely generated discrete, one can simply take the word-length function associated to any finite set of generators.

The next proposition is a variant of a well-known theorem, which follows from two classical results:

- The graph distance dist on a tree T is a conditionally negative definite kernel [23].
- The Schur multiplier associated with the characteristic function χ_n of the subset $\{(x, y) \in T^2 \mid \mathrm{dist}(x, y) = n\}$ has Schur norm at most $2n$ for every $n \in \mathbb{N}$ [7, Proposition 2.1].

The proof below is similar to the proof of [8, Corollary 12.3.5].

Proposition 3.2. *Suppose a group G acts isometrically on a tree T and that ℓ is a pseudo-length function on G. Suppose moreover* $\mathrm{dist}(g.v, v) \to \infty$ *if and only if $\ell(g) \to \infty$ for some (and hence every) vertex $v \in T$. Then $\Lambda_{\mathrm{WA}}(G, \ell) = 1$.*

Proof. Fix a vertex $v \in T$ as in the assumptions. For every $n \in \mathbb{N}$ we consider the functions $\psi_n(g) = \exp(-\frac{1}{n}\mathrm{dist}(g.v,v))$ and $\dot\chi_n(g) = \chi_n(g.v,v)$ defined for $g \in G$. Then

$$\dot\chi_m(g)\psi_n(g) = \exp(-m/n)\dot\chi_m(g)$$

holds for all $g \in G$ and every $n, m \in \mathbb{N}$. As G acts isometrically on T, each ψ_n is a unital positive definite function on G by Schoenberg's theorem and $\|\dot\chi_n\|_{B_2} \le 2n$ for every $n \in \mathbb{N}$. It follows that $\|\psi_n\|_{B_2} = 1$ and $\|\dot\chi_m\psi_n\|_{B_2} \le 2m \cdot \exp(-m/n)$ for every $n, m \in \mathbb{N}$. Therefore, for any $M \in \mathbb{N}$, we have

$$\left\| \sum_{m=0}^{M} \dot\chi_m\psi_n \right\|_{B_2} \le \|\psi_n\|_{B_2} + \left\| \sum_{m>M} \dot\chi_m\psi_n \right\|_{B_2} \le 1 + \sum_{m>M} 2m \cdot \exp(-m/n).$$

Hence, if we choose M_n suitably for all $n \in \mathbb{N}$, then the functions $\varphi_n = \sum_{m=0}^{M_n} \dot\chi_m\psi_n$ satisfy that

$$\|\varphi_n\|_{B_2} \le 1 + \tfrac{1}{n} \text{ and } \operatorname{supp}\varphi_n \subseteq \{g \in G \mid \mathrm{dist}(g.v,v) \le M_n\}.$$

The assumption

$$\mathrm{dist}(g.v,v) \to \infty \iff \ell(g) \to \infty$$

then insures that $\operatorname{supp}\varphi_n \subseteq \{g \in G \mid \ell(g) \le R_n\}$ for some suitable R_n and that $\varphi_n \to 1$ uniformly on $\{g \in G \mid \ell(g) \le S\}$ for every $S > 0$, as desired. ■

Remark 3.3. *The two classical results listed above have a generalization:*

- *The combinatorial distance* dist *on the 1-skeleton of a CAT(0) cube complex X is a conditionally negative definite kernel on the vertex set of X [37].*
- *The Schur multiplier associated with the characteristic function of the subset $\{(x,y) \in X^2 \mid \mathrm{dist}(x,y) = n\}$ has Schur norm at most $p(n)$ for every $n \in \mathbb{N}$, where p is a polynomial and X is (the vertex set of) a finite-dimensional CAT(0) cube complex [35, Theorem 2].*

To see that these results are in fact generalizations, we only have to notice that a tree is exactly a one-dimensional CAT(0) cube complex, and in this case the combinatorial distance is just the graph distance. Because of these generalizations and the fact that the exponential function increases faster than any polynomial, it follows with the same proof as the proof of Proposition 3.2 that the following generalization is true (see also [35, Theorem 3]): suppose a group G acts cellularly (and hence isometrically) on a finite-dimensional CAT(0) cube complex X and that ℓ is a pseudo-length function on G. Suppose moreover $\mathrm{dist}(g.v,v) \to \infty$ *if and only if $\ell(g) \to \infty$ for some (and hence every) vertex $v \in X$. Then $\Lambda_{\mathrm{WA}}(G,\ell) = 1$.*

In our context, a *norm* on a field K is a map $d\colon K \to [0,\infty)$ satisfying, for all $x, y \in K$

(i) $d(x) = 0$ implies $x = 0$,

(ii) $d(xy) = d(x)d(y)$,

(iii) $d(x+y) \le d(x) + d(y)$.

A norm obtained as the restriction of the usual absolute value on $\mathbb{C}$ via a field embedding $K \hookrightarrow \mathbb{C}$ is *archimedean*. A norm is *discrete* if the triangle inequality (iii) can be replaced by the stronger ultrametric inequality

(iii') $d(x+y) \le \max\{d(x), d(y)\}$

and the range of d on $K^\times$ is a discrete subgroup of the multiplicative group $(0,\infty)$.

Theorem 3.4 ([21, Theorem 2.1]). *Every finitely generated field K is* discretely embeddable: *For every finitely generated subring A of K there exists a sequence of norms d_n on K, each either archimedean or discrete, such that for every sequence $R_n > 0$, the subset*

$$\{a \in A \mid d_n(a) \le R_n \text{ for all } n \in \mathbb{N}\}$$

is finite.

Let d be a norm on a field K. Following Guentner, Higson and Weinberger [21] define a pseudo-length function ℓ_d on $\mathrm{GL}(n, K)$ as follows: if d is discrete

$$\ell_d(g) = \log \max_{i,j}\{d(g_{ij}), d(g^{ij})\},$$

where g_{ij} and g^{ij} are the matrix coefficients of g and g^{-1}, respectively; if d is archimedean, coming from an embedding of K into $\mathbb{C}$ then

$$\ell_d(g) = \log \max\{\|g\|, \|g^{-1}\|\},$$

where $\|\cdot\|$ is the operator norm of a matrix in $\mathrm{GL}(n,\mathbb{C})$.

Proposition 3.5. *Let d be an archimedean or a discrete norm on a field K. Then the pseudo-length group $(\mathrm{SL}(2,K), \ell_d)$ is weakly amenable with constant 1.*

Proof. The archimedean case: it is clear that the pseudo-length function on $\mathrm{SL}(2,K)$ is the restriction of that on $\mathrm{SL}(2,\mathbb{C})$, so clearly we only have to show $(\mathrm{SL}(2,\mathbb{C}), \ell_d)$ is weakly amenable with constant 1. Since ℓ_d is continuous and proper, this follows from the fact that $\mathrm{SL}(2,\mathbb{C})$ is weakly amenable with constant 1 as a locally compact group ([9, Remark 3.8]).

The discrete case: this is a direct application of [21, Lemma 5.9] and Proposition 3.2. Indeed, [21, Lemma 5.9] states that there exist a tree T and a vertex $v_0 \in T$ such that $\mathrm{SL}(2,K)$ acts isometrically on T and

$$\mathrm{dist}(g.v_0, v_0) = 2 \max_{i,j} -\frac{\log d(g_{ij})}{\log d(\pi)},$$

for all $g = [g_{ij}] \in \mathrm{SL}(2,K)$. Here dist is the graph distance on T and π, the uniformizer, is certain element of $\{x \in K \mid d(x) < 1\}$. Since the action is isometric, $\mathrm{dist}(g.v_0, v_0) \to \infty$ if and only if $\ell_d(g) \to \infty$. Hence, we are done by Proposition 3.2. ∎

Corollary 3.6. *Let K be a field and G a finitely generated subgroup of* $\mathrm{SL}(2,K)$*. Then there exists a sequence of pseudo-length functions ℓ_n on G such that $\Lambda_{\mathrm{WA}}(G,\ell_n) = 1$ for every n, and such that for any sequence $R_n > 0$, the set $\bigcap_n \{g \in G \mid \ell_n(g) \leq R_n\}$ is finite.*

Proof. As G is finitely generated, we may assume that K is finitely generated as well. Now, let A be the finitely generated subring of K generated by the matrix coefficients of a finite generating set for G. Clearly, $G \subseteq \mathrm{SL}(2,A) \subseteq \mathrm{SL}(2,K)$. Since K is discretely embeddable, we may choose a sequence of norms d_n on K according to Theorem 3.4. It follows from Proposition 3.5 that $\Lambda_{\mathrm{WA}}(G,\ell_{d_n}) = 1$. We complete the proof by observing that for any sequence $R_n > 0$,

$$\bigcap_n \{g \in G \mid \ell_{d_n}(g) \leq R_n\} \subseteq \mathrm{SL}(2,F),$$

where F is the finite set $\{a \in A \mid d_n(a) \leq \exp(R_n) \text{ for all } n \in \mathbb{N}\}$. ∎

Theorem 3.7. *Let K be a field. Every subgroup Γ of* $\mathrm{GL}(2,K)$ *is weakly amenable with constant 1 (as a discrete group).*

Proof. By the permanence results listed in Section 2 we can reduce our proof to the case where Γ is a finitely generated subgroup of $\mathrm{SL}(2,K)$. It then follows from the previous corollary that there exists a sequence ℓ_n of pseudo-length functions on Γ such that $\Lambda_{\mathrm{WA}}(\Gamma,\ell_n) = 1$ and for any sequence $R_n > 0$, the set $\bigcap_n \{g \in \Gamma \mid \ell_n(g) \leq R_n\}$ is finite.

For each fixed $n \in \mathbb{N}$ there is a sequence $(\varphi_{n,k})_k$ of Herz-Schur multipliers on Γ and a sequence of positive numbers $(R_{n,k})_k$ such that

1. $\|\varphi_{n,k}\|_{B_2} \leq 1$ for all $k \in \mathbb{N}$;
2. $\operatorname{supp}\varphi_{n,k} \subseteq \{g \in \Gamma \mid \ell_n(g) \leq R_{n,k}\}$;
3. $\varphi_{n,k} \to 1$ uniformly on $\{g \in \Gamma \mid \ell_n(g) \leq S\}$ for every $S > 0$ as $k \to \infty$.

Upon replacing $\varphi_{n,k}$ by $|\varphi_{n,k}|^2$ we may further assume that $0 \leq \varphi_{n,k} \leq 1$ for all $n,k \in \mathbb{N}$.

Given any $\varepsilon > 0$ and any finite subset $F \subseteq \Gamma$, we choose a sequence $0 < \varepsilon_n < 1$ such that $\prod_n (1-\varepsilon_n) > 1-\varepsilon$. It follows from (3) that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $1-\varepsilon_n < \varphi_{n,k_n}(g)$ for all $g \in F$. Consider the function $\varphi = \prod_n \varphi_{n,k_n}$. It is not hard to see that φ is well-defined, since $0 \leq \varphi_{n,k_n} \leq 1$. Additionally, since $\|\varphi_{n,k_n}\|_{B_2} \leq 1$ for all $n \in \mathbb{N}$ we also have $\|\varphi\|_{B_2} \leq 1$. Moreover, $\operatorname{supp}\varphi \subseteq \bigcap_n \{g \in \Gamma \mid \ell_n(g) \leq R_{n,k_n}\}$ and

$$\varphi(g) = \prod_n \varphi_{n,k_n}(g) > \prod_n (1-\varepsilon_n) > 1-\varepsilon$$

for all $g \in F$. This completes the proof. ∎

The remaining part of this section follows de Cornulier's idea from [12]. In [12] he proved the same results for Haagerup property, and the same argument actually works for weak amenability with constant 1.

Corollary 3.8. *Let R be a unital commutative ring without nilpotent elements. Then every subgroup Γ of $\mathrm{GL}(2,R)$ is weakly amenable with constant 1 (as a discrete group).*

Proof. Again by the permanence results in Section 2, we may assume that Γ is a finitely generated subgroup of $\mathrm{SL}(2,R)$, and hence that R is also finitely generated. It is well-known that every finitely generated ring is Noetherian and in such a ring there are only finitely many minimal prime ideals. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ be the minimal prime ideals in R. The intersection of all minimal prime ideals is the set of nilpotent elements in R, which is trivial by our assumption. So R embeds into the finite product $\prod_{i=1}^n R/\mathfrak{p}_i$. If K_i denotes the fraction field of the integral domain $R/\mathfrak{p}_i$, then Γ embeds into $\mathrm{SL}(2,\prod_{i=1}^n K_i) = \prod_{i=1}^n \mathrm{SL}(2,K_i)$. Now, the result is a direct consequence of Theorem 3.7, (2.3) and (2.4). ∎

Remark 3.9. *In the previous corollary and also in Theorem 3.7, the assumption about commutativity cannot be dropped. Indeed, the group $\mathrm{SL}(2,\mathbb{H})$ with the discrete topology is not weakly amenable, where $\mathbb{H}$ is the skew-field of quaternions. This can be seen from Theorem 1.10. Moreover, $\mathrm{SL}(2,\mathbb{H})_{\mathrm{d}}$ does not even have the weak Haagerup property by the same argument.*

Remark 3.10. *In the previous corollary, the assumption about the triviality of the nilradical cannot be dropped. Indeed, we show now that the group $\mathrm{SL}(2,\mathbb{Z}[x]/x^2)$ is not weakly amenable. The essential part of the argument is Dorofaeff's result that the locally compact group $\mathbb{R}^3 \rtimes \mathrm{SL}(2,\mathbb{R})$ is not weakly amenable [19]. Here the action $\mathrm{SL}(2,\mathbb{R}) \curvearrowright \mathbb{R}^3$ is the unique irreducible 3-dimensional representation of $\mathrm{SL}(2,\mathbb{R})$.*

Consider the ring $R = \mathbb{R}[x]/x^2$. We write elements of R as polynomials $ax+b$ where $a,b\in\mathbb{R}$ and $x^2=0$. Consider the unital ring homomorphism $\varphi\colon R\to\mathbb{R}$ given by setting $x=0$, that is, $\varphi(ax+b)=b$. Then φ induces a group homomorphism $\widetilde{\varphi}\colon \mathrm{SL}(2,R)\to\mathrm{SL}(2,\mathbb{R})$. Embedding $\mathbb{R}\subseteq R$ as constant polynomials, we obtain an embedding $\mathrm{SL}(2,\mathbb{R})\subseteq\mathrm{SL}(2,R)$ showing that $\widetilde{\varphi}$ splits. The kernel of $\widetilde{\varphi}$ is easily identified as

$$\ker\widetilde{\varphi} = \left\{ \begin{pmatrix} a_{11}x+1 & a_{12}x \\ a_{21}x & a_{22}x+1 \end{pmatrix} \middle| a_{ij}\in\mathbb{R},\ a_{11}+a_{22}=0 \right\} \simeq \mathfrak{sl}(2,\mathbb{R})$$

We deduce that $\mathrm{SL}(2,R)$ is the semidirect product $\mathfrak{sl}(2,\mathbb{R})\rtimes\mathrm{SL}(2,\mathbb{R})$. A simple computation shows that the action $\mathrm{SL}(2,\mathbb{R})\curvearrowright\mathfrak{sl}(2,\mathbb{R})$ is the adjoint action. Since $\mathfrak{sl}(2,\mathbb{R})$ is a simple Lie algebra, the adjoint action is irreducible. By uniqueness of the 3-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ (see [33, p. 107]) and from [19] we deduce that $\mathfrak{sl}(2,\mathbb{R})\rtimes\mathrm{SL}(2,\mathbb{R})\simeq\mathbb{R}^3\rtimes\mathrm{SL}(2,\mathbb{R})$ is not weakly amenable.

It is easy to see that $\mathrm{SL}(2,\mathbb{Z}[x]/x^2)$ is identified with $\mathfrak{sl}(2,\mathbb{Z})\rtimes\mathrm{SL}(2,\mathbb{Z})$ under the isomorphism $\mathrm{SL}(2,R)\simeq\mathfrak{sl}(2,\mathbb{R})\rtimes\mathrm{SL}(2,\mathbb{R})$. Since $\mathfrak{sl}(2,\mathbb{Z})\rtimes\mathrm{SL}(2,\mathbb{Z})$ is a lattice in $\mathfrak{sl}(2,\mathbb{R})\rtimes\mathrm{SL}(2,\mathbb{R})$, we conclude from (2.7) that $\mathfrak{sl}(2,\mathbb{Z})\rtimes\mathrm{SL}(2,\mathbb{Z})$ and hence $\mathrm{SL}(2,\mathbb{Z}[x]/x^2)$ is not weakly amenable.

Remark 3.11. *We do not know whether $\mathrm{SL}(2,\mathbb{Z}[x]/x^2)$ also fails to have the weak Haagerup property. As $\mathrm{SL}(2,\mathbb{Z}[x]/x^2)$ may be identified with a lattice in*

$\mathbb{R}^3 \rtimes \mathrm{SL}(2,\mathbb{R})$, by (2.7) the question is equivalent to the question [25, Remark 5.3] raised by Haagerup and the first author concerning the weak Haagerup property of the group $\mathbb{R}^3 \rtimes \mathrm{SL}(2,\mathbb{R})$.

Recall that a group Γ is residually free if for every $g \neq 1$ in Γ, there is a homomorphism f from Γ to a free group F such that $f(g) \neq 1$ in F. Equivalently, Γ embeds into a product of free groups of rank two. A group Γ is residually finite if for every $g \neq 1$ in Γ, there is a homomorphism f from Γ to a finite group F such that $f(g) \neq 1$ in F. Equivalently, Γ embeds into a product of finite groups. Since free groups are residually finite, it is clear that residually free groups are residually finite. On the other hand, residually finite groups need not be residually free as is easily seen by considering e.g. groups with torsion.

Corollary 3.12. *Any residually free group is weakly amenable with constant 1.*

Proof. Since the free group of rank two can be embedded in $\mathrm{SL}(2,\mathbb{Z})$, a residually free group embeds in $\prod_{i\in I} \mathrm{SL}(2,\mathbb{Z}) = \mathrm{SL}(2, \prod_{i\in I} \mathbb{Z})$ for a suitably large set I. We complete the proof by the previous corollary. ∎

4. Failure of the weak Haagerup property

In this section we will prove the following result.

Proposition 4.1. *If S is one of the four groups $\mathrm{SO}(5)$, $\mathrm{SO}_0(1,4)$, $\mathrm{SU}(3)$ or $\mathrm{SU}(1,2)$, then S_{d} does not have the weak Haagerup property.*

Also, if S is the universal covering group of $\mathrm{SU}(1,n)$ where $n \geq 2$, then S_{d} does not have the weak Haagerup property.

When $p, q \geq 0$ are integers, not both zero, and $n = p+q$, we let $I_{p,q}$ denote the diagonal $n \times n$ matrix with 1 in the first p diagonal entries and -1 in the last q diagonal entries. When g is a complex matrix, g^t denotes the transpose of g, and g^* denotes the adjoint (conjugate transpose) of g. We recall that

$$\begin{aligned} \mathrm{SO}(p,q) &= \{g \in \mathrm{SL}(p+q,\mathbb{R}) \mid g^t I_{p,q} g = I_{p,q}\} \\ \mathrm{SO}(p,q,\mathbb{C}) &= \{g \in \mathrm{SL}(p+q,\mathbb{C}) \mid g^t I_{p,q} g = I_{p,q}\} \\ \mathrm{SU}(p,q) &= \{g \in \mathrm{SL}(p+q,\mathbb{C}) \mid g^* I_{p,q} g = I_{p,q}\}. \end{aligned}$$

When $p, q > 0$, the group $\mathrm{SO}(p,q)$ has two connected components, and $\mathrm{SO}_0(p,q)$ denotes the identity component. In particular, by (2.8), the group $\mathrm{SO}(p,q)_{\mathrm{d}}$ has the weak Haagerup property if and only if the group $\mathrm{SO}_0(p,q)_{\mathrm{d}}$ has the weak Haagerup property.

Proof of Proposition 4.1. We follow a strategy that we have learned from de Cornulier [13], where the same techniques are applied in connection with the Haagerup property. The idea of the proof is the following.

If Z denotes the center of S, then we consider the group S/Z as a real algebraic group $G(\mathbb{R})$ with complexification $G(\mathbb{C})$. Let K be a number field of degree three over $\mathbb{Q}$, not totally real, and let $\mathcal{O}$ be its ring of integers. Then by the Borel Harish–Chandra Theorem (see [5, Theorem 12.3] or [36, Proposition 5.42]), $G(\mathcal{O})$ embeds diagonally as a lattice in $G(\mathbb{R}) \times G(\mathbb{C})$. If Γ is the inverse image in $S \times G(\mathbb{C})$ of $G(\mathcal{O})$, then Γ is a lattice in $S \times G(\mathbb{C})$.

The group $G(\mathbb{C})$ has real rank at least two, and we deduce that Γ does not have the weak Haagerup property by combining [25, Theorem B] with (2.7). The projection $S \times G(\mathbb{C}) \to S$ is injective on Γ, and hence (2.4) implies that S_{d} also does not have the weak Haagerup property. ∎

5. Proof of the Main Theorem

In this section we prove Theorem 1.10. The theorem is basically a consequence of Theorem 1.11 and Proposition 4.1 together with the permanence results listed in Section 2 and general structure theory of simple Lie groups.

When two Lie groups G and H are locally isomorphic we write $G \approx H$. An important fact about Lie groups and local isomorphims is the following [27, Theorem II.1.11]: Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

The following is extracted from [11, Chapter II] and [30, Section I.11] to which we refer for details. If G is a connected Lie group, there exists a connected, simply connected Lie group $\widetilde{G}$ and a covering homomorphism $\widetilde{G} \to G$. The kernel of the covering homomorphism is a discrete, central subgroup of $\widetilde{G}$, and it is isomorphic to the fundamental group of G. The group $\widetilde{G}$ is called the *universal covering group* of G. Clearly, $\widetilde{G}$ and G are locally isomorphic. Conversely, any connected Lie group locally isomorphic to G is the quotient of $\widetilde{G}$ by a discrete, central subgroup. If N is a discrete subgroup of the center $Z(\widetilde{G})$ of $\widetilde{G}$, then the center of $\widetilde{G}/N$ is $Z(\widetilde{G})/N$.

Let G_1 and G_2 be locally compact groups. We say that G_1 and G_2 are *strongly locally isomorphic*, if there exist a locally compact group G and finite normal subgroups N_1 and N_2 of G such that $G_1 \simeq G/N_1$ and $G_2 \simeq G/N_2$. In this case we write $G_1 \sim G_2$. It follows from (2.5) that if $G \sim H$, then $\Lambda_{\mathrm{WH}}(G_{\mathrm{d}}) = \Lambda_{\mathrm{WH}}(H_{\mathrm{d}})$.

A theorem due to Weyl states that a connected, simple, compact Lie group has a compact universal cover with finite center [28, Theorem 12.1.17], [27, Theorem II.6.9]. Thus, for connected, simple, compact Lie groups G and H, $G \approx H$ implies $G \sim H$.

Proof of Theorem 1.10. Let G be a connected simple Lie group. As mentioned, the equivalence (1) $\iff$ (2) was already done by de Cornulier [13, Theorem 1.14] in a much more general setting, so we leave out the proof of this part. We only prove the two implications (1) $\implies$ (3) and (6) $\implies$ (1), since the remaining implications then follow trivially.

Suppose (1) holds, that is, G is locally isomorphic to $\mathrm{SO}(3)$, $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$. If Z denotes the center of G, then by assumption G/Z is isomorphic to $\mathrm{SO}(3)$, $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$. It follows from Theorem 1.11 and (2.5) that the groups $\mathrm{SO}(3)$, $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$ equipped with the discrete topology are weakly amenable with constant 1 (recall that $\mathrm{SO}(3)$ is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$).

From (2.6) we deduce that G_{d} is weakly amenable with constant 1. This proves (3).

Suppose (1) does not hold. We prove that (6) fails, that is, G_{d} does not have the weak Haagerup property. We divide the proof into several cases depending on the real rank of G. We recall that with the Iwasawa decomposition $G = KAN$, the real rank of G is the dimension of the abelian group A.

If the real rank of G is at least two, then G does not have the weak Haagerup property [25, Theorem B]. By a theorem of Borel, G contains a lattice (see [44, Theorem 14.1]), and by (2.7) the lattice also does not have the weak Haagerup property. We conclude that G_{d} does not have the weak Haagerup property.

If the real rank of G equals one, then the Lie algebra of G is isomorphic to a Lie algebra in the list [30, (6.109)]. See also [27, Ch.X §6]. In other words, G is locally isomorphic to one of the classical groups $\mathrm{SO}_0(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ for some $n \geq 2$ or locally isomorphic to the exceptional group $\mathrm{F}_{4(-20)}$. Here $\mathrm{SO}_0(1,n)$ denotes the identity component of the group $\mathrm{SO}(1,n)$.

We claim that the universal covering groups of $\mathrm{SO}_0(1,n)$, $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$ have finite center except for the group $\mathrm{SO}_0(1,2)$. Indeed, $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$ are already simply connected with finite center. The K-group from the Iwasawa decomposition of $\mathrm{SO}_0(1,n)$ is $\mathrm{SO}(n)$ which has fundamental group of order two, except when $n = 2$, and hence $\mathrm{SO}_0(1,n)$ has fundamental group of order two as well. As the center of the universal cover is an extension of the center of $\mathrm{SO}_0(1,n)$ by the fundamental group of $\mathrm{SO}_0(1,n)$, the claim follows.

The universal covering group $\widetilde{\mathrm{SU}}(1,n)$ of $\mathrm{SU}(1,n)$ has infinite center isomorphic to the group of integers.

We have assumed that G is not locally isomorphic to $\mathrm{SL}(2,\mathbb{R}) \sim \mathrm{SO}_0(1,2)$ or $\mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SO}_0(1,3)$. If G has finite center, it follows that G is strongly locally isomorphic to one of the groups

$$\begin{array}{ll} \mathrm{SO}_0(1,n), & n \geq 4, \\ \mathrm{SU}(1,n), & n \geq 2, \\ \mathrm{Sp}(1,n), & n \geq 2, \\ \mathrm{F}_{4(-20)}, & \end{array}$$

and if G has infinite center, then G is isomorphic to $\widetilde{\mathrm{SU}}(1,n)$. Clearly, there are inclusions

$$\begin{aligned} \mathrm{SO}_0(1,4) &\subseteq \mathrm{SO}_0(1,n), \ n \geq 4, \\ \mathrm{SU}(1,2) &\subseteq \mathrm{SU}(1,n), \ n \geq 2, \\ \mathrm{SU}(1,2) &\subseteq \mathrm{Sp}(1,n), \ n \geq 2. \end{aligned}$$

The cases where G is strongly locally isomorphic to $\mathrm{SO}_0(1,n)$, $\mathrm{SU}(1,n)$ or $\mathrm{Sp}(1,n)$ are then covered by Proposition 4.1. Since $\mathrm{SO}(5) \subseteq \mathrm{SO}(9) \sim \mathrm{Spin}(9) \subseteq \mathrm{F}_{4(-20)}$ ([45, §.4.Proposition 1]), the case where $G \sim \mathrm{F}_{4(-20)}$ is also covered by Proposition 4.1. Finally, if $G \simeq \widetilde{\mathrm{SU}}(1,n)$, then Proposition 4.1 shows that G_{d} does not have weak Haagerup property.

If the real rank of G is zero, then it is a fairly easy consequence of [28, Theorem 12.1.17] that G is compact. Moreover, the universal covering group of G is compact and with finite center.

By the classification of compact simple Lie groups as in Table IV of [27, Ch.X §6] we know that G is strongly locally isomorphic to one of the groups $\mathrm{SU}(n+1)$ $(n \geq 1)$, $\mathrm{SO}(2n+1)$ $(n \geq 2)$, $\mathrm{Sp}(n)$ $(n \geq 3)$, $\mathrm{SO}(2n)$ $(n \geq 4)$ or one of the five exceptional groups

$$E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

By assumption G is not strongly locally isomorphic to $\mathrm{SU}(2) \sim \mathrm{SO}(3)$. Using (2.5) it then suffices to show that if G equals any other group in the list, then G_{d} does not have the weak Haagerup property. Clearly, there are inclusions

$$\begin{aligned} \mathrm{SO}(5) &\subseteq \mathrm{SO}(n),\ n \geq 5,\\ \mathrm{SU}(3) &\subseteq \mathrm{SU}(n),\ n \geq 3,\\ \mathrm{SU}(3) &\subseteq \mathrm{Sp}(n),\ n \geq 3. \end{aligned}$$

Since we also have the following inclusions among Lie algebras (Table V of [27, Ch.X §6])

$$\mathfrak{so}(5) \subseteq \mathfrak{so}(9) \subseteq \mathfrak{f}_4 \subseteq \mathfrak{e}_6 \subseteq \mathfrak{e}_7 \subseteq \mathfrak{e}_8$$

and the inclusion ([47])

$$\mathrm{SU}(3) \subseteq G_2,$$

it is enough to consider the cases where $G = \mathrm{SO}(5)$ or $G = \mathrm{SU}(3)$. These two cases are covered by Proposition 4.1. Hence we have argued that also in the real rank zero case G_{d} does not have the weak Haagerup property. ■

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References

[1] Anantharaman-Delaroche, C., *Amenable correspondences and approximation properties for von Neumann algebras*, Pacific J. Math. **171** (1995), 309–341.

[2] Bédos, E., *On the C^*-algebra generated by the left regular representation of a locally compact group*, Proc. Amer. Math. Soc. **120** (1994), 603–608.

[3] Bekka, B., P. de la Harpe, and A. Valette, "Kazhdan's property (T)," New Mathematical Monographs 11, Cambridge University Press, Cambridge 2008.

[4] Bekka, M. E. B., and A. Valette, *On duals of Lie groups made discrete*, J. Reine Angew. Math. **439** (1993), 1–10.

[5] Borel, A., and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535.

[6] Bożejko, M. and G. Fendler, *Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group*, Boll. Un. Mat. Ital. A (6) **3** (1984), 297–302.

[7] Bożejko, M., and M. A. Picardello, *Weakly amenable groups and amalgamated products*, Proc. Amer. Math. Soc. **117** (1993), 1039–1046.

[8] Brown, N. P., and N. Ozawa, "C^*-algebras and finite-dimensional approximations," Graduate Studies in Mathematics **88**, American Mathematical Society, Providence, RI 2008.

[9] de Cannière, J., and U. Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. **107** (1985), 455–500.

[10] Cherix, P.-A., M. Cowling, P. Jolissaint, P. Julg, and A. Valette, "Groups with the Haagerup property," Progress in Mathematics **197**, Birkhäuser Verlag, Basel 2001.

[11] Chevalley, C., "Theory of Lie Groups. I," Princeton Mathematical Series **8**, Princeton University Press, Princeton, N. J. 1946.

[12] de Cornulier, Y., *Haagerup property for subgroups of* SL_2 *and residually free groups*, Bull. Belg. Math. Soc. **13** (2006), 341–343.

[13] —, *Kazhdan and Haagerup properties in algebraic groups over local fields*, J. Lie Theory **16** (2006), 067–082.

[14] de Cornulier, Y., Y. Stalder, and A. Valette, *Proper actions of lamplighter groups associated with free groups*, C. R. Math. Acad. Sci. Paris **346** (2008), 173–176.

[15] de Cornulier, Y. and A. Valette, *On equivariant embeddings of generalized Baumslag-Solitar groups*, to appear in Geom. Dedicata, 2014.

[16] Cowling, M., *Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups)*, Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982), Ist. Naz. Alta Mat. Francesco Severi, Rome 1983, 81–123.

[17] —, *Rigidity for lattices in semisimple Lie groups: von Neumann algebras and ergodic actions*, Rend. Sem. Mat. Univ. Politec. Torino **47** (1989), 1–37 (1991).

[18] Cowling, M., and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. **96** (1989), 507–549.

[19] Dorofaeff, B., *The Fourier algebra of* $\mathrm{SL}(2,\mathbf{R})\rtimes\mathbf{R}^n$*,* $n\geq 2$*, has no multiplier bounded approximate unit*, Math. Ann. **297** (1993), 707–724.

[20] —, *Weak amenability and semidirect products in simple Lie groups*, Math. Ann. **306** (1996), 737–742.

[21] Guentner, E., N. Higson, and S. Weinberger, *The Novikov conjecture for linear groups*, Publ. Math. Inst. Hautes Études Sci. (2005), 243–268.

[22] Guentner, E., R. Tessera, and G. Yu, *Operator norm localization for linear groups and its application to K-theory*, Adv. Math. **226** (2011), 3495–3510.

[23] Haagerup, U., *An example of a nonnuclear C^*-algebra, which has the metric approximation property*, Invent. Math. **50** (1978/79), 279–293.

[24] —, *Group C^*-algebras without the completely bounded approximation property*, 1986, unpublished manuscript.

[25] Haagerup, U. and S. Knudby, *The weak Haagerup property II: Examples* (2014), preprint, to appear in Int. Math. Res. Not. IMRN.

[26] Hansen, M. L., *Weak amenability of the universal covering group of* $\mathrm{SU}(1,n)$, Math. Ann. **288** (1990), 445–472.

[27] Helgason, S., "Differential geometry, Lie groups, and symmetric spaces," Pure and Applied Mathematics **80**, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York 1978.

[28] Hilgert, J. and K.-H. Neeb, "Structure and geometry of Lie groups," Springer Monographs in Mathematics, Springer, New York 2012.

[29] Jolissaint, P., *A characterization of completely bounded multipliers of Fourier algebras*, Colloq. Math. **63** (1992), 311–313.

[30] Knapp, A. W., "Lie groups beyond an introduction," Progress in Mathematics **140**, Birkhäuser Boston Inc., Boston, MA 2002, second edition.

[31] Knudby, S., *Semigroups of Herz-Schur multipliers*, J. Funct. Anal. **266** (2014), 1565–1610.

[32] —, *The weak Haagerup property*, 2015, Preprint, to appear in Trans. Amer. Math. Soc.

[33] Lang, S., "$\mathrm{SL}_2(\mathbf{R})$," Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam 1975.

[34] Milnor, J., *On the homology of Lie groups made discrete*, Comment. Math. Helv. **58** (1983), 72–85.

[35] Mizuta, N., *A Bożejko-Picardello type inequality for finite-dimensional* $\mathrm{CAT}(0)$ *cube complexes*, J. Funct. Anal. **254** (2008), 760–772.

[36] Morris, D. W., "Introduction to arithmetic groups," Preliminary version, March 2014, http://people.uleth.ca/~dave.morris/books/IntroArithGroups.pdf.

[37] Niblo, G. and L. Reeves, *Groups acting on* $\mathrm{CAT}(0)$ *cube complexes*, Geom. Topol. **1** (1997), approx. 7 pp. (electronic).

[38] Ozawa, N., *Talk at the conference on von Neumann algebras and related topics, RIMS, Kyoto University, January 11, 2012.*

[39] —, *Examples of groups which are not weakly amenable*, Kyoto J. Math. **52** (2012), 333–344.

[40] Pascual Gainza, P., *Cohomology of Lie groups made discrete*, Publ. Mat. **34** (1990), 151–174.

[41] Pier, J.-P., "Amenable locally compact groups," Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York 1984.

[42] Pisier, G., "Similarity problems and completely bounded maps," Lecture Notes in Mathematics **1618**, Springer-Verlag, Berlin 2001, expanded edition, includes the solution to "The Halmos problem".

[43] Popa, S. and S. Vaes, *Unique Cartan decomposition for* II_1 *factors arising from arbitrary actions of hyperbolic groups*, 2012, Acta Math., to appear, arXiv:1201.2824.

[44] Raghunathan, M. S., "Discrete subgroups of Lie groups," Ergebnisse der Mathematik und ihrer Grenzgebiete **68**, Springer-Verlag, New York-Heidelberg 1972

[45] Takahashi, R., *Quelques résultats sur l'analyse harmonique dans l'espace symétrique non compact de rang* 1 *du type exceptionnel*, Analyse harmonique sur les groupes de Lie (Sém., Nancy-Strasbourg 1976–1978), II, Lecture Notes in Math. **739**, Springer, Berlin, 1979, 511–567.

[46] Tu, J.-L., *Remarks on Yu's "property A" for discrete metric spaces and groups*, Bull. Soc. Math. France **129** (2001), 115–139.

[47] Yokota, I., *Subgroups of type* A_2*-series in the exceptional Lie groups* G_2, F_4 *and* E_6, J. Fac. Sci. Shinshu Univ. **14** (1979), 87–94.

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Article C

A Schur multiplier characterization of coarse embeddability

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A Schur multiplier characterization of coarse embeddability

Søren Knudby Kang Li*

Abstract

We give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces. Consequently, all locally compact groups whose weak Haagerup constant is 1 embed coarsely into Hilbert spaces, and hence the Baum-Connes assembly map with coefficients is split-injective for such groups.

In this note we study coarse embeddability of locally compact groups into Hilbert spaces. An important application of this concept in [16], [13] and [5] is that the Baum-Connes assembly map with coefficients is split-injective for all locally compact groups that embed coarsely into a Hilbert space (see [2] and [15] for more information about the Baum-Connes assembly map). Here, we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (see also [6, Theorem 5.3] for the discrete case), and this characterization can be regarded as an answer to the non-equivariant version of [12, Question 1.5]. As a result, any locally compact group with weak Haagerup constant 1 embeds coarsely into a Hilbert space and hence the Baum-Connes assembly map with coefficients is split-injective for all these groups.

Let G be a σ-compact, locally compact group. A *(left) tube* in $G \times G$ is a subset of $G \times G$ contained in a set of the form

$$\mathrm{Tube}(K) = \{(x, y) \in G \times G \mid x^{-1}y \in K\}$$

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where K is any compact subset of G. Following [1, Definition 3.6], we say that a map u from G into a Hilbert space H is a *coarse embedding* if u satisfies the following two conditions:

- for every compact subset K of G there exists $R > 0$ such that

$$(s,t) \in \mathrm{Tube}(K) \implies \|u(s) - u(t)\| \leq R;$$

- for every $R > 0$ there exists a compact subset K of G such that

$$\|u(s) - u(t)\| \leq R \implies (s,t) \in \mathrm{Tube}(K).$$

We say that a group G *embeds coarsely into a Hilbert space* or *admits a coarse embedding into a Hilbert space* if there exist a Hilbert space H and a coarse embedding $u : G \to H$. Note that a coarse embedding need not be injective, and we also do not require it to be continuous.

Every second countable, locally compact group G admits a proper left-invariant metric d, which is unique up to coarse equivalence (see [14] and [9]). So the preceding definition is equivalent to Gromov's notion of coarse embeddability of the metric space (G, d) into Hilbert spaces. We refer to [5, Section 3] for more on coarse embeddability into Hilbert spaces for locally compact groups.

A kernel $\varphi\colon G \times G \to \mathbb{C}$ is a *Schur multiplier* if for every bounded operator $A = (a_{x,y})_{x,y\in G} \in B(\ell^2(G))$, the matrix $(\varphi(x,y)a_{x,y})_{x,y\in G}$ again defines a bounded operator, denoted $M_\varphi A$, on $\ell^2(G)$. In this case, it follows from the closed graph theorem that M_φ in fact defines a *bounded* operator $B(\ell^2(G)) \to B(\ell^2(G))$, and the *Schur norm* $\|\varphi\|_S$ of φ is defined to be the operator norm of M_φ.

A kernel $\varphi\colon G \times G \to \mathbb{C}$ *tends to zero off tubes*, if for any $\varepsilon > 0$ there is a tube $T \subseteq G \times G$ such that $|\varphi(x,y)| < \varepsilon$ whenever $(x,y) \notin T$. Note that if $\varphi\colon G \to \mathbb{C}$ is a function, then φ vanishes at infinity (written $\varphi \in C_0(G)$), if and only if the associated kernel $\widehat{\varphi}\colon G \times G \to \mathbb{C}$ defined by $\widehat{\varphi}(x,y) = \varphi(x^{-1}y)$ tends to zero off tubes.

Theorem 1. *Let G be a σ-compact, locally compact group. The following are equivalent.*

1. *G embeds coarsely into a Hilbert space.*
2. *There exists a sequence of Schur multipliers $\varphi_n\colon G \times G \to \mathbb{C}$ such that*
   - *$\|\varphi_n\|_S \leq 1$ for every natural number n;*
   - *each φ_n tends to zero off tubes;*
   - *$\varphi_n \to 1$ uniformly on tubes.*

If any of these conditions holds, one can moreover arrange that the coarse embedding is continuous and that each φ_n is continuous.

It is well-known that the notion of coarse embeddability into Hilbert spaces can be characterized by positive definite kernels (see [8, Theorem 2.3] for the discrete case and [4, Theorem 1.5] for the locally compact case).

Following [11], G has the *weak Haagerup property with constant 1*, if there is a sequence of continuous functions $\varphi_n \in C_0(G)$ converging uniformly to 1 on compact subsets of G and such that the associated kernels $\widehat{\varphi}_n : G \times G \to \mathbb{C}$ are Schur multipliers with $\|\widehat{\varphi}_n\|_S \leq 1$.

From Theorem 1 together with [5, Theorem 3.5] we immediately obtain the following.

Corollary 2. *If G is a σ-compact, locally compact group with the weak Haagerup property with constant 1, then G embeds coarsely into a Hilbert space. If G is moreover second countable, then in particular the Baum-Connes assembly map with coefficients is split-injective.*

We now turn to the proof of Theorem 1. It is not hard to see that the countability assumption in [10, Proposition 4.3] is superfluous. We thus record the following (slightly more general) version of [10, Proposition 4.3].

Lemma 3. *Let G be a group with a symmetric kernel $k : G \times G \to [0, \infty)$. The following are equivalent.*

1. *For every $t > 0$ one has $\|e^{-tk}\|_S \leq 1$.*
2. *There exist a real Hilbert space $\mathcal{H}$ and maps $R, S : G \to \mathcal{H}$ such that*

$$k(x, y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.$$

Recall that a kernel $k : G \times G \to \mathbb{R}$ is *conditionally negative definite* if k is symmetric ($k(x, y) = k(y, x)$), vanishes on the diagonal ($k(x, x) = 0$) and

$$\sum_{i,j=1}^{n} c_i c_j k(x_i, x_j) \leq 0$$

for any finite sequences $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} c_i = 0$. It is well-known that k is conditionally negative definite if and only if there is a function u from G to a real Hilbert space such that $k(x, y) = \|u(x) - u(y)\|^2$.

A kernel $k : G \times G \to \mathbb{C}$ is called *proper*, if $\{(x, y) \in G \times G \mid |k(x, y)| \leq R\}$ is a tube for every $R > 0$.

Theorem 1 is contained in Theorem 4 below, which extends both [6, Theorem 5.3] and [4, Theorem 1.5] in different directions. An important ingredient in the proof of Theorem 4 is the following result (which generalizes without change from the second countable case to the σ-compact case).

Theorem ([5, Theorem 3.4])**.** *Let G be a σ-compact, locally compact group. The following are equivalent.*

1. *The group G embeds coarsely into a Hilbert space.*
2. *There is a continuous conditionally negative definite kernel $h : G \times G \to \mathbb{R}$ which is proper and bounded on tubes.*

Theorem 4. *Let G be a σ-compact, locally compact group. The following are equivalent.*

1. *The group G embeds coarsely into a Hilbert space.*
2. *There exists a sequence of (not necessarily continuous) Schur multipliers $\varphi_n \colon G \times G \to \mathbb{C}$ such that*
   - *$\|\varphi_n\|_S \leq 1$ for every natural number n;*
   - *each φ_n tends to zero off tubes;*
   - *$\varphi_n \to 1$ uniformly on tubes.*
3. *There exists a (not necessarily continuous) symmetric kernel $k \colon G \times G \to [0, \infty)$ which is proper, bounded on tubes and satisfies $\|e^{-tk}\|_S \leq 1$ for all $t > 0$.*
4. *There exists a (not necessarily continuous) conditionally negative definite kernel $h \colon G \times G \to \mathbb{R}$ which is proper and bounded on tubes.*

Moreover, if any of these conditions holds, one can arrange that the coarse embedding in (1)*, each Schur multiplier φ_n in* (2)*, the symmetric kernel k in* (3) *and the conditionally negative definite kernel h in* (4) *are continuous.*

Proof. We show (1) $\iff$ (4) $\iff$ (3) $\iff$ (2).

That (1) implies (4) with h continuous follows directly from [5, Theorem 3.4].

Suppose (4) holds. By the GNS construction there are a real Hilbert space $\mathcal{H}$ and a map $u \colon G \to \mathcal{H}$ such that

$$h(x, y) = \|u(x) - u(y)\|^2.$$

It is easy to check that the assumptions on h imply that u is a coarse embedding. Thus (1) holds.

That (4) implies (3) follows with $k = h$ using Schoenberg's Theorem and the fact that normalized positive definite kernels are Schur multipliers of norm 1. Note also that conditionally negative definite kernels are symmetric and take only non-negative values.

Suppose (3) holds. We show that (4) holds. From Lemma 3 we see that there are a real Hilbert space $\mathcal{H}$ and maps $R, S \colon G \to \mathcal{H}$ such that

$$k(x, y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.$$

As k is bounded on tubes, the map S is bounded. If we let

$$h(x, y) = \|R(x) - R(y)\|^2,$$

then it is easily checked that h is proper and bounded on tubes, since k has these properties and S is bounded. It is also clear that h is conditionally negative definite. Thus (4) holds.

If (3) holds, we set $\varphi_n = e^{-k/n}$ when $n \in \mathbb{N}$. It is easy to check that the sequence φ_n has the desired properties so that (2) holds.

Finally, suppose (2) holds. We verify (3). Essentially, we use the same standard argument as in the proof of [11, Proposition 4.4] and [3, Theorem 2.1.1].

Since G is locally compact and σ-compact, it is the union of an increasing sequence $(U_n)_{n=1}^{\infty}$ of open sets such that the closure K_n of U_n is compact and contained in U_{n+1} (see [7, Proposition 4.39]). Fix an increasing, unbounded sequence (α_n) of positive real numbers and a decreasing sequence (ε_n) tending to zero such that $\sum_n \alpha_n \varepsilon_n$ converges. By assumption, for every n we can find a Schur multiplier φ_n tending to zero off tubes and such that $\|\varphi_n\|_S \leq 1$ and

$$\sup_{(x,y)\in \mathrm{Tube}(K_n)} |\varphi_n(x,y) - 1| \leq \varepsilon_n/2.$$

Upon replacing φ_n by $|\varphi_n|^2$ one can arrange that $0 \leq \varphi_n \leq 1$ and

$$\sup_{(x,y)\in \mathrm{Tube}(K_n)} |\varphi_n(x,y) - 1| \leq \varepsilon_n.$$

Define kernels $\psi_i : G \times G \to [0, \infty[$ and $\psi : G \times G \to [0, \infty[$ by

$$\psi_i(x,y) = \sum_{n=1}^{i} \alpha_n(1 - \varphi_n(x,y)), \qquad \psi(x,y) = \sum_{n=1}^{\infty} \alpha_n(1 - \varphi_n(x,y)).$$

It is easy to see that ψ is well-defined, bounded on tubes and $\psi_i \to \psi$ pointwise (even uniformly on tubes, but we do not need that).

To see that ψ is proper, let $R > 0$ be given. Choose n large enough such that $\alpha_n \geq 2R$. As φ_n tends to zero off tubes, there is a compact set $K \subseteq G$ such that $|\varphi_n(x,y)| < 1/2$ whenever $(x,y) \notin \mathrm{Tube}(K)$. Now if $\psi(x,y) \leq R$, then $\psi(x,y) \leq \alpha_n/2$, and in particular $\alpha_n(1 - \varphi_n(x,y)) \leq \alpha_n/2$, which implies that $1 - \varphi_n(x,y) \leq 1/2$. We have thus shown that

$$\{(x,y) \in G \times G \mid \psi(x,y) \leq R\} \subseteq \{(x,y) \in G \times G \mid 1 - \varphi_n(x,y) \leq 1/2\} \subseteq \mathrm{Tube}(K),$$

and ψ is proper.

We now show that $\|e^{-t\psi}\|_S \leq 1$ for every $t > 0$. Since ψ_i converges pointwise to ψ, it will suffice to prove that $\|e^{-t\psi_i}\|_S \leq 1$, because the set of Schur multipliers of norm at most 1 is closed under pointwise limits. Since

$$e^{-t\psi_i} = \prod_{n=1}^{i} e^{-t\alpha_n(1-\varphi_n)},$$

it is enough to show that $e^{-t\alpha_n(1-\varphi_n)}$ has Schur norm at most 1 for each n. And this is clear:

$$\|e^{-t\alpha_n(1-\varphi_n)}\|_S = e^{-t\alpha_n}\|e^{t\alpha_n\varphi_n}\|_S \leq e^{-t\alpha_n} e^{t\alpha_n\|\varphi_n\|_S} \leq 1.$$

The only thing missing is that ψ need not be symmetric. Put $k = \psi + \check{\psi}$ where $\check{\psi}(x,y) = \psi(y,x)$. Clearly, k is symmetric, bounded on tubes and proper. Finally, for every $t > 0$

$$\|e^{-tk}\|_S \leq \|e^{-t\psi}\|_S \|e^{-t\check{\psi}}\|_S \leq 1,$$

since $\|\check{\varphi}\|_S = \|\varphi\|_S$ for every Schur multiplier φ.

Finally, the statements about continuity follow from [5, Theorem 3.4] and the explicit constructions used in our proof of (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2). ∎

References

[1] Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Trans. Amer. Math. Soc.*, 354(10):4153–4178 (electronic), 2002.

[2] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[3] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability.

[4] Steven Deprez and Kang Li. Permanence properties of property A and coarse embeddability for locally compact groups. Preprint, arXiv:1403.7111, 2014.

[5] Steven Deprez and Kang Li. Property A and uniform embedding for locally compact groups. Preprint, to appear in J. Noncommut. Geom., 2015.

[6] Zhe Dong and Zhong-Jin Ruan. A Hilbert module approach to the Haagerup property. *Integral Equations Operator Theory*, 73(3):431–454, 2012.

[7] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[8] Erik Guentner and Jerome Kaminker. Exactness and the Novikov conjecture. *Topology*, 41(2):411–418, 2002.

[9] Uffe Haagerup and Agata Przybyszewska. Proper metrics on locally compact groups, and proper affine isometric actions on Banach spaces. Preprint, arXiv:math/0606794, 2006.

[10] Søren Knudby. Semigroups of Herz-Schur multipliers. *J. Funct. Anal.*, 266(3):1565–1610, 2014.

[11] Søren Knudby. The weak Haagerup property. Preprint, to appear in Trans. Amer. Math. Soc., 2015.

[12] Søren Knudby and Kang Li. Approximation properties of simple Lie groups made discrete. Preprint, to appear in J. Lie Theory, 2015.

[13] G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002.

[14] Raimond A. Struble. Metrics in locally compact groups. *Compositio Math.*, 28:217–222, 1974.

[15] Alain Valette. *Introduction to the Baum-Connes conjecture*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.

[16] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.

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