Life insurance liabilities with policyholder behaviour and stochastic rates

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Preface

The present PhD thesis was written during the period January 2012 to October 2014, where I undertook the task of studying for the PhD degree at Department of Mathematical Sciences, University of Copenhagen. This was done as a part of Innovation Fund Denmark's industrial PhD programme, with PFA Pension as the industrial partner. The research was supervised by Prof. Mogens Steffensen from the University of Copenhagen, and Adj. Prof. Thomas Møller and Peter Holm Nielsen from PFA Pension. The PhD thesis consists of an introduction and five independent manuscripts. They appear as such and small notational discrepancies exist between the chapters. Two of the manuscripts are published in international peer reviewed journals at the time of writing.

It has been a pleasure for me to focus on the mathematics of reserving in life and pension insurance for the last three years, and I hope you will enjoy the result as well.

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From February to June 2013 Ann-Sophie and I had great pleasure in visiting the University of Lausanne, and I thank Hansjörg Albrecher and the Department of Actuarial Science for the kind hospitality we were shown.

Whether at the H. C. Ørsted Institute, or at the seminars and conferences I've attended the last three years, I've had numerous fruitful discussions with many academic colleagues. They helped shape my views on life insurance, probability theory, statistics and financial mathematics, for which I'm grateful. I thank all my colleagues throughout the period,

and it has been a joy coming to work each day, whether at Dorigny, at the H. C. Ørsted Institute or at Sundkrogsgade. I thank my family and friends who in one way or another supported me during the last three years.

At last a heartfelt thanks to Ann-Sophie for encouraging me to undertake the PhD studies more than three years ago. Without her confidence in me, and her continued support throughout the last three years, I would not have been able to complete the project.

Copenhagen, October 2014
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List of papers

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[9] A step forward with Kolmogorov.
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Summary

In any life and pension insurance company, it is a central task to calculate the value of the liabilities toward the policyholders. In the classic model for such valuations, a continuous time Markov chain in a finite state space describes the state of the insured, and the interest rate, mortality rate, disability rate, and other transition rates are assumed to be deterministic. Broadly speaking this PhD thesis consists of various extensions of this model to address the modern needs of life insurance companies. These extensions can be categorised into two types: the inclusion of policyholder behaviour in the model, and the modelling of the interest and transition rates as stochastic processes.

The forthcoming Solvency II rules from the EU have contributed to an increased focus on the modelling of policyholder behaviour, and in this thesis we focus on the so-called surrender and free policy options. A surrender is a cancellation of the policy in return for a payout of the technical value, and a free policy conversion is a cancellation of all future premiums only, whereafter the future benefits are reduced to account for the missing premiums. Since an exercise of such an option has an effect on the liabilities of the life insurance company, it is important to take them into account when calculating the liabilities in the first place. In this thesis they are modelled as random transitions of the Markov chain. Since the modelling of the free policy option introduces duration dependence, this increases the complexity of the model. The first main result is a modification of Kolmogorov's forward differential equation that allows us to include modelling of the free policy option without the extra duration dependence. This result is presented in both the Markov and the semi-Markov life insurance setup.

A central part of the forthcoming Solvency II rules is the requirement to quantify the risk inherent in the life insurance policies. Since a large part of this risk originates from changes in the underlying transition rates, for example the unpredictable decline of the mortality rate, it is necessary to model the underlying transition rates as stochastic processes. This significantly complicates matters, and one way to deal with this is to restrict oneself to the class of affine processes. The second main result is a generalisation of the formulae which make the affine processes mathematically convenient. This allows us to utilise affine processes for the calculation of the reserve for more complicated life

insurance products.

In the general setup with stochastic transition rates, where we are not restricted to affine processes, we present the third main result, namely Kolmogorov's forward partial integro-differential equation. In life insurance companies it is, for hedging purposes, important to know the interest rate sensitivity of the liabilities. For this sensitivity, the expected cash flows of the policies are often calculated since it is straightforward to calculate the interest rate sensitivity of a cash flow. Considering the general case of stochastic transition rates, it is complicated to calculate the cash flows, since the transition probabilities are required. To find these, one could, for each future time point, solve Kolmogorov's backward partial differential equation. Here Kolmogorov's forward partial integro-differential equation is advantageous, since it only needs to be solved once in order to obtain all the relevant transition probabilities for calculating the cash flow.

Resumé

I livsforsikrings- og pensionsselskaber er det en central opgave at opgøre værdien af forpligtelserne over for forsikringstagerne, også kaldet hensættelserne. I den klassiske model angiver en tidskontinuert Markovkæde på et endeligt tilstandsrum forsikringstagerens tilstand, og renten, dødeligheden, invaliditetshyppigheden og andre overgangshyppigheder er antaget at være deterministiske. Groft sagt består denne ph.d.-afhandling af forskellige udvidelser af den klassiske model for at adressere nogle af de moderne krav fra livsforsikringsselskaber. Disse udvidelser kan kategoriseres i to typer: modellering af policetageradfærd samt modellering af rente og overgangshyppigheder som stokastiske processer.

De kommende Solvens II regler fra EU har bidraget til et øget fokus på modellering af policetageradfærd, og i denne afhandling fokuseres specielt på de såkaldte genkøbs- og fripoliceoptioner. Et genkøb er en ophævelse af policen mod at få udbetalt den tekniske værdi af denne, og en fripolicekonvertering er en annullering af alle fremtidige præmier mod at de fremtidige ydelser bliver tilsvarende reduceret. Da et genkøb eller en fripolicekonvertering ændrer værdien af hensættelserne, er det væsentligt at modellere disse optioner i forbindelse med opgørelsen af hensættelserne. I denne afhandling modelleres genkøbs- og fripoliceoptionerne som tilfældige overgange i Markovkæden. Da modellering af fripoliceoptionen introducerer varighedsafhængighed kompliceres modellen. Det første hovedresultat er en modificeret version af Kolmogorov's forlæns differentialligning som tillader os at inkludere modellering af fripoliceoptionen uden at dette fører til den ekstra varighedsafhængighed. Dette resultat præsenteres både i Markov og semi-Markov livsforsikringssetuppet.

En central del af de kommende Solvens II regler er kravet om at kvantificere risikoen i livsforsikringspolicerne. Da en stor del af denne risiko stammer fra ændringer i de underliggende overgangshyppigheder, for eksempel det uforudsigelige fald i dødeligheden, er det nødvendigt at modellere overgangshyppighederne som stokastiske processer. Dette komplicerer modellen betydeligt, og en måde at håndtere dette er at begrænse sig til at modellere overgangshyppigheder med affine processer. Det andet andet hovedresultat er en generalisering af de formler der gør affine processer specielt fordelagtige. Dette tillader

os at udnytte den affine struktur til at beregne reserver for endnu mere komplicerede livsforsikringsprodukter.

I den generelle model med stokastiske overgangshyppigheder, hvor vi ikke begrænser os til affine processer, præsenterer vi det tredie hovedresultat, nemlig Kolmogorov's forlæns partielle integro-differentialligning. I livsforsikringsselskaber er det, for hedgingformål, væsentligt at kende hensættelsernes rentefølsomhed. For at bestemme denne følsomhed i praksis udregnes policernes cashflow ofte, da det er simpelt at opgøre rentefølsomheden af et cashflow. I den generelle model med stokastiske overgangshyppigheder er det kompliceret at bestemme cashflowet, idet alle overgangssandsynlighederne først skal findes. For at udregne disse kan man, for hvert fremtidigt tidspunkt, løse Kolmogorov's baglæns partielle differentialligning. I dette tilfælde er Kolmogorov's forlæns partielle integro-differentialligning imidlertid fordelagtig, da den blot skal løses en enkelt gang for at udregne alle de til cashflowet nødvendige overgangssandsynligheder.

Chapter 1

Introduction

This PhD thesis is broadly about techniques for valuation of life insurance liabilities. In this introduction we first outline some background for the research, and then we present the key mathematical ideas of the thesis and explain the main connections between the chapters. Throughout we refer to the different chapters, and for references to other literature, see the introductions in the relevant chapters where the research is presented and related to existing literature.

1.1 Background

1.1.1 The classic life insurance setup

Given an insurance policy, where certain future premiums and benefits are agreed upon, the insurance company must calculate the value of this policy in order to know what is required to honour the policy agreement. The starting point for this investigation is the classic Markov chain life insurance setup. Here, the stochastic process $\mathbf{Z} = (Z(t))_{t\geq 0}$ is a continuous time Markov chain in a finite state space $\mathcal{J} = \{0, 1, \dots, J\}$, where each state has an interpretation like *alive*, *disabled*, *dead* etc., such that the random insurance events can be interpreted as transitions between states. A classic example is found in Figure 1.1. Payments are then associated with sojourns in states, $b_j(t)$ at time t in state j, and transitions between states, $b_{ij}(t)$ for a transition $i \to j$ at time t. The payment at time t can then be written as

$$\mathrm{d}B(t) = \sum_j 1_{\{Z(t)=j\}} b_j(t)\,\mathrm{d}t + \sum_{i,j} b_{ij}(t)\,\mathrm{d}N_{ij}(t),$$

where $\mathbf{N}_{ij}$ is a counting process, counting the accumulated number of transitions $i \to j$.

The reserve A fundamental task of the actuary is to calculate the reserve for the balance sheet. This is given as the expected present value of the future payments.

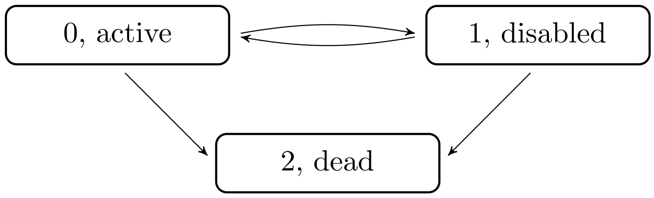

Figure 1.1: *The disability Markov model. The arrows represent the possible transitions.*

Conditional on being in state i at time t', this is usually denoted $V_i(t')$, and is given by

$$V_i(t') = \mathrm{E}\left[\int_{t'}^{\infty} e^{-\int_{t'}^{t} r(s)\,\mathrm{d}s}\, \mathrm{d}B(t) \middle| Z(t') = i \right],$$

where $r(t)$ is the interest rate, which in the classic setup is assumed to be deterministic. It is well known that $V_i(t)$ satisfies Thiele's differential equation.

1.1.2 Stochastic transition rates

Stochastic interest rate In Denmark and several other countries, it is mandatory to calculate the reserve in a market consistent manner. Thus, the interest rate assumptions used for reserving must be consistent with the interest rate assumptions used for trading interest rate derivatives on the financial market. In particular this means that the interest rate $r(t)$ is stochastic, and the results from the classic setup do not seem applicable. Luckily, if the stochastic interest rate is independent of the Markov chain $\mathbf{Z}$, which is a reasonable assumption, a simple result shows that one can simply replace the interest rate $r(t)$ with the forward interest rate at time t', $f_{t'}(t)$. This is defined through the relation

$$\mathrm{E}\left[e^{-\int_{t'}^{t} r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t') \right] = e^{-\int_{t'}^{t} f_{t'}(s)\,\mathrm{d}s},$$

where $(\mathcal{F}(t))_{t\geq 0}$ is the filtration generated by $(r(t))_{t\geq 0}$, and the expectation is taken using the risk neutral measure. The reserve $V_i(t')$ is found similar to in the classic setup, but we must condition on $\mathcal{F}(t')$ as well. Conditioning on the path of $\mathbf{Z}$ and using the independence, we find that

$$\begin{aligned} V_i(t') &= \mathrm{E}\left[\int_{t'}^{\infty} \mathrm{E}\left[e^{-\int_{t'}^{t} r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t'), \mathbf{Z} \right] \mathrm{d}B(t) \middle| \mathcal{F}(t'), Z(t') = i \right] \\ &= \mathrm{E}\left[\int_{t'}^{\infty} e^{-\int_{t'}^{t} f_{t'}(s)\,\mathrm{d}s}\, \mathrm{d}B(t) \middle| \mathcal{F}(t'), Z(t') = i \right]. \end{aligned}$$

Thus, using $f_u(t)$ as the interest rate in the classic setup and considering u as fixed, the usual results hold, e.g. Thiele's differential equation, and in this way it is simple to extend the classic setup to calculate the reserve with a market consistent interest rate model.

With a stochastic interest rate assumption, the reserve is dependent on the forward interest rate, which changes every day. In the recent years this has been decreasing, which has resulted in an increasing reserve. In order for the life insurance companies to handle this, it is vital to hedge the interest rate risk, for example by investing in bonds, which also increase in value when the interest rate decreases. A simple way to precisely manage this hedge in practice is to calculate the interest rate sensitivity of the liabilities and invest in assets such that the asset portfolio has a similar sensitivity with respect to the interest rate. It is thus central to calculate the interest rate sensitivity of a portfolio of life insurance contracts.

The cash flow A computationally efficient way of calculating the interest rate sensitivity of a large portfolio of life insurance policies is to calculate the expected cash flow (which we simply refer to as the cash flow) of each policy. This cash flow can be easily discounted with different interest rates in order to measure the sensitivity. The cash flow consists of the expected payments at future times $t > t'$, and conditional on being in state i at time t', we denote this by $a_i(t', t)$, satisfying

$$a_i(t', t)\,\mathrm{d}t = \mathrm{E}\left[\mathrm{d}B(t)\middle|\, Z(t') = i\right].$$

As can be seen from e.g. Chapter 2, $a_i(t', t)$ can be found by first calculating the transition probabilities for $\mathbf{Z}$, by e.g. Kolmogorov's forward differential equation. The reserve can then be calculated by a simple discounting,

$$V_i(t') = \int_{t'}^{\infty} e^{-\int_{t'}^{t} f_{t'}(s)\,\mathrm{d}s} a_i(t', t)\,\mathrm{d}t.$$

In all of the chapters below, one of the main objectives is to calculate the reserve in the respective model of the chapter. In Chapters 2, 3 and 6, the primary focus is on efficient calculation of the cash flow.

Stochastic mortality As the reserves in the balance sheet change due to a changing interest rate, they also change due to fluctuations in the underlying assumptions of the transition rates of $\mathbf{Z}$. In the classic setup the transition rates are deterministic. However, an example is the mortality, which has been declining the last centuries at an unpredictable pace. Thus, even though most agree that the mortality will continue to decline, it is difficult, if not impossible, to predict how much. What is important from the insurer's and regulator's perspective is to be aware of the risk originating from the unpredictable mortality, and a central way of modelling this mathematically is by letting the mortality transition rate be stochastic. In this situation, the classic setup again breaks down. In the simple survival model, a forward mortality rate can be defined, analogously to the forward interest rate, and the formulae of the classic setup then hold.

But in more advanced models, e.g. in the disability model from Figure 1.1, the forward mortality rate approach does not directly work if the mortality as active and as disabled differs.

It is worth noting that the rules on market consistent reserving in principle also apply for the mortality element. However, as there is yet to be a liquid market for trading mortality risk, there does not exist a unique risk neutral measure for the mortality element. In that case, expectation is taken using the physical measure, which is a best estimate, and some sort of safety margin may be added instead.

Systematic risk According to the forthcoming Solvency II rules from the EU, the insurance company must measure 1-year risk and have enough capital to withstand a loss corresponding to the 99.5% quantile. For large insurance companies, the main part of this loss originates from the so-called systematic risk, which can be thought of as the risk affecting all policies in the same way. As the transition rates of $\mathbf{Z}$ in practice change during the year, this becomes the main contribution to the systematic risk. The reserve in one year, $V(t'+1)$, is dependent on this development. Thus it is stochastic, and it is the 99.5% quantile in $V(t'+1)$ that constitutes the quantification of the systematic risk in a Solvency II context. For handling this problem mathematically, it is essential to model the transition rates as stochastic processes, and be able to calculate the reserves in such a model. In Chapter 4 we see how the class of affine processes can be utilised for modelling dependent stochastic interest and transition rates. This study is continued in Chapter 5, where we also consider the Solvency II capital requirement.

1.1.3 Policyholder behaviour

In the recent years, in part because it is required by the forthcoming Solvency II rules, there has been a focus on correctly modelling some of the policyholders' options inherent in life insurance policies. Two of these options, the surrender and free policy options, have attained a particular focus in Denmark. The surrender option is a right of the policyholder to cancel all future premiums, and in return have the (technical) value of the policy paid out immediately. This occurs often in Danish labour market pensions when a policyholder transfers his policy from one pension fund to another, typically in connection with a job change. The free policy option is a right of the policyholder to cancel all future premiums, where in return all future benefits are reduced to account for the missing premiums.

By considering the surrender option as a random transition of the Markov chain $\mathbf{Z}$, it is readily handled in the classic setup, since the payment upon surrender typically is a deterministic function of time. The free policy option can also be modelled as a random transition of $\mathbf{Z}$ if the classic setup is extended to include dependence on the duration

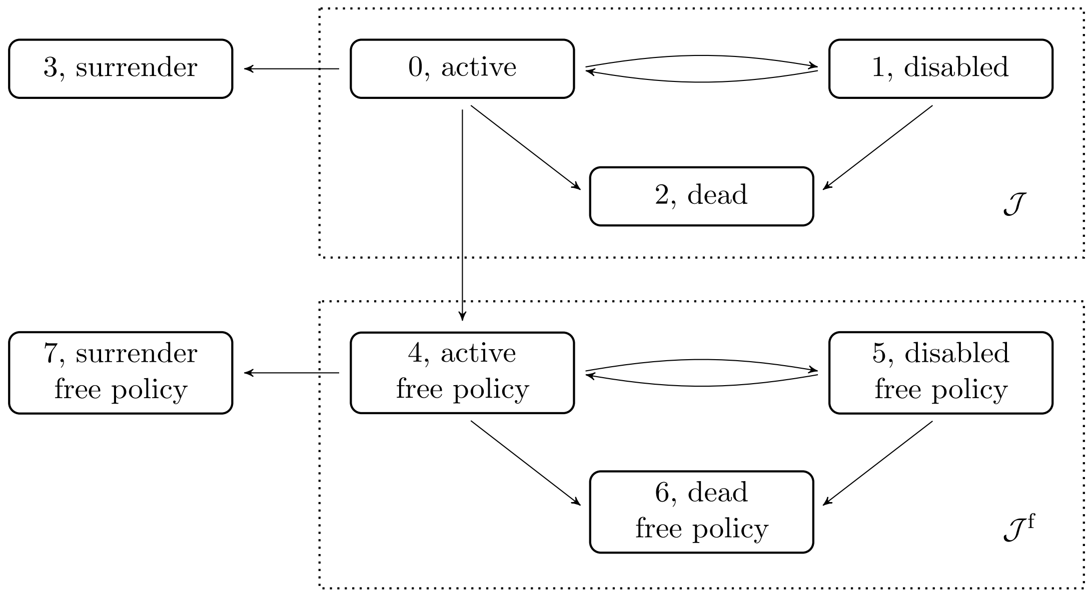

Figure 1.2: *The 8-state Markov model, with disability, surrender and free policy. Arrows illustrate the possible transitions, and each arrow can be associated with terms in Kolmogorov's forward differential equation. The free policy option can be handled by modifying a term of the transition* $0 \to 4$.

since the time of the free policy conversion, which brings us to the so-called semi-Markov setup. Modelling of the policyholder's options can then be included in the disability model from Figure 1.1 by an extension to the state space from Figure 1.2. The main challenge of this setup is that the payments are reduced upon the free policy conversion, i.e. the transition $0 \to 4$, and this reduction is dependent on the time of conversion. Therefore the payments in the free policy states are duration dependent, which is not part of the classic setup, but can be handled by the semi-Markov setup. The semi-Markov life insurance setup is an extension of the classic life insurance setup to include the duration in certain states, such that the transition rates and the payments are allowed to depend on this duration. However, the semi-Markov setup is more complicated than the classic Markov life insurance setup, and in particular the computational time required is higher. The main result of Chapters 2 and 3 is a modification of Kolmogorov's forward differential equation in order to account for this duration dependence and avoid dealing with the complications of the semi-Markov formulae. In Chapter 5 we also consider surrender modelling, but here we let the surrender rate be stochastic and dependent on the interest rate. In this model, we first see that the usual forward interest rate is not applicable due to the dependence. We introduce so-called dependent forward rates. Further, we examine the Solvency II capital requirement and present a numerical example.

1.2 Overview of the thesis and key ideas

The classic setup is well studied in the literature. In broad terms, this PhD thesis deals with different ways of generalising the classic setup, such that some of the challenges discussed above can be handled. The focus is primarily on calculation of the reserve and the cash flow in practice. Each following chapter of the thesis consists of a stand-alone paper. The papers are presented here not in chronological order of production[1], but in the order which the author thinks is best suited for presenting a combined story. Chapters 2 and 3 mainly deal with the question of including the modelling of policyholder behaviour, which is done in a setup with deterministic transition rates, and the semi-Markov setup is also included. Chapters 4 and 5 mainly deal with modelling the interest and transition rates as stochastic and dependent processes, in particular as affine processes, with the aim of finding efficient formulae for calculating the life insurance reserve. Chapter 6 somewhat unifies these two topics in the sense that Kolmogorov's forward differential equation is generalised, such that general (i.e. both affine and non-affine) diffusion processes can be used to model the transition rates. Both the semi-Markov setup and the setup with stochastic transition rates can be considered as special cases of this setup.

The main contributions of this thesis can at a stretch be formulated as three mathematical ideas. In the rest of the introduction, we try to explain these three ideas in an intuitive way. In addition, we highlight other relevant parts and connections in the thesis. Some mathematical precision is sacrificed, and for a precise treatment the reader is referred to the respective chapters.

The first mathematical idea is the duration elimination allowing us to efficiently model the free policy option. This is introduced in Section 1.2.1 below. In Chapter 2 this idea is made precise in the classic Markov setup, and in Chapter 3 the result is presented in the semi-Markov setup.

The second mathematical idea is the generalisation of certain expectations of transformations of affine processes that allows for a more general application of affine processes in life insurance. This is introduced in Section 1.2.2 below. In Chapter 4 we present the theory of affine processes and the results, and in Chapter 5 the results are applied more thoroughly in life insurance, including considerations on the Solvency II risk quantification and a numerical example.

The third mathematical idea is the generalisation of Kolmogorov's forward differential equation to handle e.g. the doubly stochastic Markov chain setup. This is introduced in Section 1.2.3 below and is made precise in Chapter 6.

[1]The chronological order of the chapters is: 4, 5, 3, 2 and 6.

1.2.1 Modelling of the free policy option

When the free policy option is modelled as a random transition of the Markov chain $\mathbf{Z}$, as in the state space shown in Figure 1.2, the payments in the free policy states (4, 5 and 6) become dependent on the time of the free policy conversion. This can be handled by the more complex semi-Markov setup. The first result is that we don't have to adopt the semi-Markov setup for modelling the free policy option, but can do something which in comparison is simple.

The idea behind the result is to consider and modify Kolmogorov's forward differential equation for the transition probabilities of $\mathbf{Z}$. Upon a free policy conversion at time t, all future benefits are reduced by a deterministic factor $\rho(t)$. Considering payments in the free policy states only, the cash flow can be written as

$$a_i(t',t) = \sum_{j\in\mathcal{J}^{\mathrm{f}}} \mathrm{E}\left[1_{\{Z(t)=j\}}\rho(t-W(t))\middle|\, Z(t')=i\right]\left(b_j(t)+\sum_{\ell;\ell\neq j}\mu_{j\ell}(t)b_{j\ell}(t)\right).$$

Here $W(t)$ is the time since the free policy conversion, thus $t-W(t)$ is the time of the transition $0\to 4$. Also, $\mu_{j\ell}(t)$ are the transition rates of $\mathbf{Z}$. If $\rho(t)=1$ for all t, the expectation is simply the transition probability for state $i\to j$ from time t' to t, and this can be calculated with Kolmogorov's forward differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t',t) = -p_{ij}(t',t)\sum_{\ell;\ell\neq j}\mu_{j\ell}(t)+\sum_{\ell;\ell\neq j}p_{i\ell}(t',t)\mu_{\ell j}(t).$$

The trick is to see that the expectation is essentially the transition probability, but multiplied with $\rho(t)$ at the time of the transition $0\to 4$. Interpreting Kolmogorov's differential equation, we identify the first of the two terms, which is negative, as probability mass leaving state j at time t to any state ℓ. The second term is identified as probability mass entering state j at time t from any state ℓ. Thus if we replace the term

$$p_{i0}(t',t)\mu_{04}(t),$$

with the term

$$p_{i0}(t',t)\mu_{04}(t)\rho(t),$$

we have effectively multiplied all the probability mass entering state 4 by the value of ρ at the time of transition. The resulting modified transition probabilities are then in all free policy states reduced by ρ at the time of the transition $0\to 4$, and this is essentially the result.

Chapter 2 consists of two parts and a numerical example. In the second part, we show the main result, which is the effective elimination of the duration. This is done in the classic 3-state disability model. In the first part of the chapter, we show a simple way of manipulating existing cash flows without inherent policyholder behaviour in order to include policyholder behaviour. This is done in a survival model, however applying the method for cash flows from a disability model can be thought of as an approximation which is easy to handle. In the end of the paper a numerical example is studied, where the approximation is compared with the correct formulae from the second part of the paper.

In Chapter 3 the outset is the semi-Markov setup. The purpose of this chapter is two-fold. First, we present Kolmogorov's forward integro-differential equation, which is the generalisation of Kolmogorov's forward differential equation to the semi-Markov setup, and we discuss how to solve it in practice. Second, we apply the same duration elimination trick as above to the semi-Markov model; modelling free policy behaviour in the semi-Markov setup still introduces an extra duration dependence, in practice giving us a double duration setup, and applying the result from above we can include the modelling of free policy behaviour without the extra duration dependence.

Some parts of Chapter 2 can be thought of as a pedagogical version of Chapter 3, where we can discuss the results without having to deal with the complications of the semi-Markov setup. In particular, the content of Section 2.4 constitutes a special case of Chapter 3.

1.2.2 Affine processes in life insurance

One of the disadvantages of the policyholder behaviour model from Chapters 2 and 3 is that the surrender and free policy options occur randomly and independent of the interest rate. A simple, reasonable conjecture is that if the market interest rate is high, then a low interest rate guarantee inherent in a life insurance policy is of little value, and surrender rates will be high. On the other hand, if the market interest rate becomes lower than the guaranteed interest rate, then the guarantee will be of high value, and the likelihood of surrender may be low. This effect can be modelled with stochastic and positively correlated interest and surrender rates, and this is a motivation for modelling the interest and transition rates as stochastic processes. Other reasons for modelling the interest and transition rates as stochastic processes include risk measurement of the systematic risk, e.g. in a Solvency II context, and also the study of hedging possibilities.

In Chapters 4 and 5 we study the application of affine processes for valuation of life insurance liabilities. Assume $\mathbf{X} = (X(t))_{t\geq 0}$ is an affine stochastic process, possibly multidimensional, and that the interest and transition rates are functions of $X(t)$: $r(t, X(t))$ and $\mu_{ij}(t, X(t))$. Conditional on the whole path of $\mathbf{X}$, we assume that $\mathbf{Z}$ is a Markov chain. Unconditionally $\mathbf{Z}$ is no longer a Markov chain, and the results from the classic life insurance setup do not hold. In particular, we can use neither the classic version of Thiele's differential equation nor the classic versions of Kolmogorov's forward and backward differential equations.

The idea is to consider hierarchical models (or decrement models) where you cannot return to a state after you have left it. In that case, both the reserve and the transition probabilities can be written as expectations of integral expressions. For example, for a term insurance with payout 1 and level premium π, one can show that the reserve, conditional on being alive, can be written as

$$V(t) = \mathrm{E}\left[\int_t^T e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u)))\,\mathrm{d}u} \left(\mu_{01}(s,X(s)) - \pi\right) \mathrm{d}s \,\middle|\, X(t)\right],$$

using an appropriate market-consistent measure. Interchanging expectation and integration we see that we must calculate the following expectations,

$$\mathrm{E}\left[e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u)))\,\mathrm{d}u} \,\middle|\, X(t)\right], \tag{1.2.1}$$

$$\mathrm{E}\left[e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u)))\,\mathrm{d}u} \mu_{01}(s,X(s)) \,\middle|\, X(t)\right]. \tag{1.2.2}$$

For more complicated models, e.g. a term insurance in a disability model, we also need to calculate expressions similar to

$$\begin{aligned} \mathrm{E}\Big[&e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u))+\mu_{02}(u,X(u)))\,\mathrm{d}u} \mu_{01}(s,X(s)) \\ &\times e^{-\int_s^v (r(u,X(u))+\mu_{12}(u,X(u)))\,\mathrm{d}u} \mu_{12}(v,X(v)) \Big| X(t)\Big], \end{aligned} \tag{1.2.3}$$

for $t < s < v$. It is well known that if $r(t,x)$ and $\mu_{ij}(t,x)$ are affine functions of x, then (1.2.1) can be written as an exponential affine function in $X(t)$,

$$\mathrm{E}\left[e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u)))\,\mathrm{d}u} \,\middle|\, X(t)\right] = e^{\phi(t,s)+\psi(t,s)^\top X(t)},$$

where ϕ and ψ are solutions to a set of Riccati differential equations. This relation is closely coupled with the definition of affine processes. Furthermore, a similar relationship for (1.2.2) exists,

$$\begin{aligned} &\mathrm{E}\left[e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u)))\,\mathrm{d}u} \mu_{01}(s,X(s)) \,\middle|\, X(t)\right] \\ &\qquad = e^{\phi(t,s)+\psi(t,s)^\top X(t)} \left(A^{01}(t,s,s) + B^{01}(t,s,s)^\top X(t)\right), \end{aligned} \tag{1.2.4}$$

where A and B are solutions to a set of linear differential equations.

The main result of Chapter 4 is twofold. First, we present a new proof of (1.2.4). Second, this proof is constructive, and the idea can be applied to obtain a similar relation for

(1.2.3),

$$
\begin{aligned}
&\mathrm{E}\Big[e^{-\int_t^s (r(u,X(u))+\mu_{01}(u,X(u))+\mu_{02}(u,X(u)))\,\mathrm{d}u}\mu_{01}(s,X(s)) \\
&\qquad \times e^{-\int_s^v (r(u,X(u))+\mu_{12}(u,X(u)))\,\mathrm{d}u}\mu_{12}(v,X(v))\Big|X(t)\Big] \\
&= e^{\phi(t,v)+\psi(t,v)^\top X(t)}\bigg(\Big(A^{01}(t,v,s)+B^{01}(t,v,s)^\top X(t)\Big) \\
&\qquad \times\Big(A^{12}(t,v,v)+B^{12}(t,v,v)^\top X(t)\Big)+C(t,v,s,v)+D(t,v,s,v)^\top X(t)\bigg),
\end{aligned}
$$

where the functions C and D are solutions to a set of differential equations. The idea can be reapplied to obtain relations for similar expectations, where $r(t,X(t))$ and $\mu_{ij}(t,X(t))$ appear inside an exponential function as terms and outside an exponential function as factors.

Using these three relations, we calculate transition probabilities and life insurance reserves using affine interest and transition rates, and these methods are explored in Chapters 4 and 5. In Chapter 4, the theory is presented with results and proofs. Further generalisations are discussed, and a numerical example of a doubly stochastic Markov chain is carried out to illustrate the numerical advantage compared with solving a partial differential equation or applying Monte Carlo methods.

In Chapter 5, we apply the relations in life insurance and discuss several aspects of the theory. First, from relation (1.2.4) it is natural to define so-called *dependent forward rates*, which generalise the concept of the forward interest rate and the forward mortality rate to the case of dependent rates. This is discussed and related to other proposals of forward rates in a setup with dependent interest and transition rates. Second, we study an example with surrender modelling. Letting the interest and surrender rate be dependent, the effect of the dependence on the reserve is calculated numerically. In the last part of Chapter 5, we study the systematic risk in a Solvency II context, and show how the affine processes can be exploited in measuring this risk. We also study numerically how the dependence of the interest and surrender rate affects the systematic risk, defined as the 99.5% quantile in the loss over one year.

1.2.3 Generalisation of Kolmogorov's forward differential equation

In the life insurance setup with affine stochastic transition rates in Chapters 4 and 5, there are two main restrictions. First, the rates must be an affine transformation of an affine process $\mathbf{X}$. However, we would like to consider the case where the transition rates are not necessarily affine. Second, we need a hierarchical model in order to apply the methods, so for example in the classic 3-state disability model as shown in Figure 1.1, we cannot include recovery to the original active state. In Chapter 6 we generalise

Kolmogorov's forward differential equation, such that it handles the general case where the transition rates are any diffusion process. Also, there is no restriction to hierarchical models. This result is of particular use for calculating cash flows, and it generalises Kolmogorov's forward differential equations from the classic Markov and the semi-Markov life insurance setup, wherein the transition rates are deterministic.

The key idea is to examine Kolmogorov's forward differential equations. To obtain a consistent notation in this introduction, the notation differs slightly from the notation used in some of the following chapters. In the classic setup where $\mathbf{Z}$ is a Markov chain with transition rates $\mu_{ij}(t)$, it reads

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t',t) = -p_{ij}(t',t)\sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq j}}\mu_{j\ell}(t) + \sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq j}}p_{i\ell}(t',t)\mu_{\ell j}(t), \tag{1.2.5}$$

where $p_{ij}(t',t) = \mathrm{P}(Z(t)=j|Z(t')=i)$. We interpret the right hand side as consisting of essentially two parts: what the location of the process is, and where it is headed. The probability density $p_{ij}(t',t)$ contains information about where $Z(t)$ is, and the transition rates $\mu_{Z(t-)j}(t)$ determine where $Z(t)$ is headed. The first term, which is negative, is transitions out of state j to any state ℓ, and the second term is transitions from any state ℓ to state j. The transition rates $\mu_{Z(t-)j}(t)$ depend on the location of $Z(t-)$, thus an idea is that this construction is possible because $\mathbf{Z}$ is a Markov chain: knowledge of the current location of $Z(t-)$ yields the transition rates that determine where $Z(t)$ is headed.

In the more complex semi-Markov case, the transition probabilities are solutions to Kolmogorov's forward integro-differential equation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}\left(t',u,t,D(t)\right) = &-\int_0^{D(t)} p_{ij}\left(t',u,t,\mathrm{d}z\right)\mu_{j\cdot}(t,z)\\ &+\sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq j}}\int_0^{u+t-t'} p_{i\ell}\left(t',u,t,\mathrm{d}z\right)\mu_{\ell j}(t,z),\end{aligned} \tag{1.2.6}$$

where

$$p_{ij}\left(t',u,t,D(t)\right) = \mathrm{P}(Z(t)=j, U(t)\leq D(t)|Z(t')=i, U(t')=u),$$

and $D(t) = d+t-t'$, see Theorem 3.3.1. This differential equation yields the transition probabilities for the process $(\mathbf{Z},\mathbf{U})$. We can again interpret the right hand side as consisting of the two parts: the location of the process, and where the process is headed. Again, the transition density $p_{ij}\left(t',u,t,\mathrm{d}z\right)$ determines the location in the state space. The main difference from above is that the transition rates $\mu_{Z(t-)j}(t,U(t-))$ determining where the process is headed are dependent on the current duration $U(t-)$, thus making it a requirement to know the location of $U(t-)$ in the state space. In particular, it does not seem possible to obtain a differential equation for $\mathbf{Z}$ alone. By itself, $\mathbf{Z}$ is not a Markov process, and we might again infer that the Markov assumption of the process is essential.

Consider now a diffusion process $\mathbf{X}$, satisfying the stochastic differential equation

$$\mathrm{d}X(t) = \beta(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t),$$

where $\beta(t,x)$ and $W(t)$ are d-dimensional and $\sigma(t,x)$ is $(d \times d)$-dimensional. This is also a Markov process, and we know that given certain regularity conditions, the probability density satisfies the Fokker-Planck partial differential equation,

$$\begin{aligned}\frac{\partial}{\partial t} p(t', x', t, x) = &-\sum_i \frac{\partial}{\partial x_i}\left(\beta_i(t,x) p(t', x', t, x)\right) \\ &+ \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_{ij}(t,x) p(t', x', t, x)\right),\end{aligned} \tag{1.2.7}$$

where $p(t', x', t, x)\,\mathrm{d}x = P(\,X(t) = \mathrm{d}x |\, X(t') = x')$ and $\rho(t,x) = \sigma(t,x)\sigma(t,x)^\top$. Again, the right hand side can be interpreted as consisting of the two parts, location and direction. The transition density appears on the right hand side, together with the drift $\beta(t,x)$ and the diffusion term $\rho(t,x)$, which, through the stochastic differential equation, determine where $X(t)$ is headed. We can repeat, that to determine the value of $\beta(t,x)$ and $\rho(t,x)$, we need to know where in the state space $X(t)$ is.

Common for the three examples above is that the Markov assumption seems necessary if we want to determine where the process is headed solely based on information about where the process is now. In the general case of a doubly stochastic Markov chain, the transition rates of $\mathbf{Z}$ are dependent on $\mathbf{X}$, and we write $\mu_{Z(t-)j}(t, X(t))$. In this setup, $(\mathbf{Z}, \mathbf{X})$ is a Markov process, and the idea is that we should be able to determine where $(Z(t), X(t))$ is headed since, if we know the location of $(Z(t-), X(t))$, we know the transition rates $\mu_{Z(t-)j}(t, X(t))$ of $\mathbf{Z}$, as well as the drift and diffusion term of $\mathbf{X}$. It turns out that this is indeed doable, and we refer to the result as Kolmogorov's forward partial differential equation,

$$\begin{aligned}\frac{\partial}{\partial t} p(t', k', x', t, k, x) = &-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\beta_k(t,x) p(t', k', x', t, k, x)\right) \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_{ij}(t,x) p(t', k', x', t, k, x)\right) \\ &+ \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq k}} \mu_{\ell k}(t,x) p(t', k', x', t, \ell, x) \\ &- \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq k}} \mu_{k\ell}(t,x) p(t', k', x', t, k, x),\end{aligned} \tag{1.2.8}$$

where $p(t', k', x', t, k, x)\,\mathrm{d}x = P(X(t) = \mathrm{d}x, Z(t) = k | X(t') = x', Z(t') = k')$. This is a key version of the main result of Chapter 6. We see that the right hand side is essentially (1.2.5) and (1.2.7) combined.

In Chapter 6 we first consider a general jump-diffusion, and assuming certain regularity conditions, we obtain a forward partial integro-differential equation for the transition probability. Similar results seem to exist in the literature, however the author has not been able to find such in the form presented and proven in Chapter 6. We then consider the doubly stochastic Markov chain with diffusion driven transition rates, and in particular obtain (1.2.8) as a special case of the general forward partial integro-differential equation. We relate the results to the life insurance setup and see that we can easily calculate the cash flow of a life insurance policy in the general doubly stochastic setup. This is the natural generalisation of the setup excluding policyholder behaviour from Chapters 2 and 3. However, the author believes that the duration elimination result from Chapters 2 and 3 can be generalised to the doubly stochastic setup; this is postponed for further research.

We remark that (1.2.8) does not seem directly connected to (1.2.6), which is because the semi-Markov setup is not a special case of the doubly stochastic Markov chain setup. However, the semi-Markov process is a special case of a general jump-diffusion, and (1.2.6) is indeed a special case of the general forward partial integro-differential equation that is presented in Section 6.2; this connection is studied in Section 6.5.

Forward rates as expectations In the last part of Chapter 6 we show that the forward mortality rate $f_{t'}(t)$, defined through the relation

$$\mathrm{E}\left[e^{-\int_{t'}^{t} \mu(s, X(s))\,\mathrm{d}s} \,\middle|\, X(t) \right] = e^{-\int_{t'}^{t} f_{t'}(s)\,\mathrm{d}s}, \tag{1.2.9}$$

can be represented as the expectation of the mortality rate $\mu(t, X(t))$, conditional on survival,

$$f_{t'}(t) = \mathrm{E}\left[\mu(t, X(t)) \middle| Z(t) = 0, X(t') \right]. \tag{1.2.10}$$

This representation follows by comparing (1.2.8) with the derivative of (1.2.9). Further, this is generalised to the dependent forward rates from Chapter 5, and we find that when $\mathbf{Z}$ is a doubly stochastic Markov chain in a survival model with multiple causes of death, the expectation in (1.2.10) equals the dependent forward rate.

Chapter 2

Life insurance cash flows with policyholder behaviour

This chapter is based on the paper [10], written jointly with T. Møller.

Abstract

The problem of valuation of life insurance payments with policyholder behaviour is studied. First a simple survival model is considered, and it is shown how cash flows without policyholder behaviour can be modified to include surrender and free policy behaviour by calculation of simple integrals. In the second part, a more general disability model with recovery is studied. Here, cash flows are determined by solving a modified Kolmogorov forward differential equation. This method has recently been suggested in a more general semi-Markov setup in Buchardt et al. [11]. We conclude the paper with numerical examples illustrating the methods proposed and the impact of policyholder behaviour.

2.1 Introduction

In a classic Markov chain multi-state life insurance setup, we show how policyholder behaviour can be included in cash flow projections. We study two approaches. First, we consider the survival model and show how simple integral expressions can solve the problem. Second, we consider the disability model, and present certain ordinary differential equations that solve the problem. We discuss how the integral expressions originating from the survival model can be used to approximate the more correct modelling in the disability model, for a very simple yet effective type of policyholder behaviour modelling.

In this paper, the policyholder behaviour consists of two policyholder options. First, the surrender option, where the policyholder may surrender the contract cancelling all future payments and instead receiving a single payment corresponding to the value of the contract on a technical basis. Second, the free policy option[1], where the policyholder may cancel the future premiums, and have the benefits reduced according to the technical basis. Policyholder modelling has a significant influence on future cash flows. If the technical basis differs considerably from the market basis, policyholder behaviour can also have a substantial impact on the market value of the contract.

The policyholder behaviour is modelled as random transitions in a Markov model as in [35] and [26], and rationality behind surrender and free policy modelling is thus disregarded. An empirical analysis of policyholder behaviour in the German market and further references on policyholder modelling can be found in [18]. In contrast, one can consider surrender and free policy exercises as rational, where they purely occur if it is beneficial for the policyholder with some objective measure, see e.g. [45]. For an introduction to policyholder modelling and a comparison of various approaches, see [38] and references therein. Attempts to couple the two approaches have been made for surrender behaviour, where surrender occurs randomly, but where the probability is somewhat controlled by rational factors, e.g. [23] and [8]. From a Solvency II point of view, the modelling of policyholder behaviour is required, see Section 3.5 in [12].

In the first part of the paper a simple survival model is considered. We calculate cash flows without policyholder behaviour as integral expressions. Then we extend the model by including first surrender behaviour and then both surrender and free policy behaviour. We see that these extensions can be obtained via simple modifications of the cash flows without policyholder behaviour. This can be viewed as a formula for extending current cash flows without policyholder behaviour. However, this modification of the cash flows is only correct for the survival model, and not for e.g. a disability model. If this method is applied to cash flows from a disability model, it could be viewed as an approximation to a more correct way of modelling policyholder behaviour. Also, we show that the cash flows with policyholder behaviour can be derived from cash flows with surrender behaviour. This method can be used in the case where one has access to cash flows with surrender behaviour but not free policy behaviour. In practice, many life insurance companies work with cash flows without policyholder behaviour, hence, the proposed method may be viewed as a simple alternative to full, combined modelling of policyholder behaviour and insurance risk. The quality of these formulae as an approximation is assessed numerically in the last part of the paper. This issue is also studied numerically in [26], where they examine ways to simplify the calculations when modelling policyholder behaviour.

In the second part, we consider the more correct way of modelling policyholder behaviour

[1] the free policy option is sometimes referred to as a “paid-up policy” in the literature.

in a multi-state life insurance setup. This model is presented in [11] for the general semi-Markov setup, and here we present the special case of a Markov process for the disability model with recovery. Within this setup, the transition probabilities are first calculated using Kolmogorov's forward differential equation, and then the cash flow can be determined. When including policyholder behaviour, duration dependence is introduced since the future payments are affected by the time of the free policy conversion. This complicates calculations significantly. We present the main result from [11] that allows us to effectively dismiss the duration dependence and calculate cash flows with policyholder behaviour by simply calculating a slightly modified version of Kolmogorov's forward differential equation. The complexity of the calculations is therefore not increased significantly by inclusion of policyholder behaviour.

In the third part of the paper a numerical example is studied, which illustrates, in part the importance of including policyholder modelling when valuating cash flows, and in part the quality of the approximating cash flows obtained by applying the integral expressions from the first part to cash flows without policyholder behaviour from a disability model. We see that the structure of the cash flows changes significantly in our example, and the dollar duration measuring interest rate risk is reduced by about 66%, when including policyholder behaviour. For hedging of interest rate risk, it is thus essential to consider policyholder behaviour. We compare the approximate method with the correct approach of solving the modified Kolmogorov differential equations, and find cash flows with policyholder behaviour in a disability model. We find that in our example the approximation is very precise.

Since the results obtained in the second part of the paper can be viewed as a special case of the ones presented in the more general semi-Markov framework in [11], we briefly describe the main differences between the two presentations. As mentioned above, in [11] the Kolmogorov forward integro-differential equation in the semi-Markov framework is studied and a modified version is presented that allows for the inclusion of policyholder behaviour in an efficient manner. The present paper contains three parts. In the first part, we discuss a simple approach to modelling policyholder behaviour, which is based on a modification of the underlying cash flows without policyholder behaviour. This construction provides simple pedagogical interpretations for the various new terms that arise in the cash flows when we introduce policyholder behaviour. Similar results are presented in [26], who compare with alternative modifications of the cash flows in more abstract models, such as the disability model. The second part presents the modified Kolmogorov equation in the classic Markov model. We believe that the presentation in this part could be accessible to a wider audience than [11], since we can avoid the more technical issues related to the semi-Markov framework with duration dependence. This leads to simpler results that are more easy to interpret, implement and more directly applicable than the semi-Markov framework. Moreover, the proofs are more direct and should be easy to follow for readers familiar with the classic Markov models as presented in e.g. [31].

2.2 Life insurance setup

The general setup is the classic multi-state setup in life insurance, consisting of a Markov process, $\mathbf{Z}$, in a finite state space $\mathcal{J} = \{0, 1, \dots, J\}$ indicating the state of the insured, see [27]. We associate payments with sojourns in states and transitions between states, and this specifies the life insurance contract. We go through the setup and basic results; for more details, see e.g. [39], [31] or [38].

Assume that $\mathbf{Z}$ is a Markov process in $\mathcal{J}$, and that $Z(0) = 0$. The transition probabilities are defined by

$$p_{ij}(s,t) = P\left(Z(t) = j \middle| Z(s) = i \right),$$

for $i, j \in \mathcal{J}$ and $s \leq t$. Define the transition rates, for $i \neq j$,

$$\begin{aligned} \mu_{ij}(t) &= \lim_{h \searrow 0} \frac{1}{h} p_{ij}(t, t+h), \\ \mu_{i\cdot}(t) &= \sum_{\substack{j \in \mathcal{J} \\ j \neq i}} \mu_{ij}(t). \end{aligned}$$

We assume that these quantities exist. Define also the counting processes $N_{ij}(t)$, for $i, j \in \mathcal{J}$, $i \neq j$, counting the transitions between state i and j. They are defined by

$$N_{ij}(t) = \# \left\{ s \in (0, t] \,|Z(s) = j, Z(s-) = i \right\},$$

where we have used the notation $f(t-) = \lim_{h \searrow 0} f(t-h)$.

The payments consist of continuous payment rates during sojourns in states, and single payments upon transitions between states. Denote by $b_i(t)$ the payment rate at time t if $Z(t) = i$, and let $b_{ij}(t)$ be the payment upon transition from state i to j at time t. Then, the accumulated payments at time t are denoted $B(t)$, and are given by

$$\mathrm{d}B(t) = \sum_{i \in \mathcal{J}} 1_{\{Z(t)=i\}} b_i(t) \mathrm{d}t + \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} b_{ij}(t) \mathrm{d}N_{ij}(t). \tag{2.2.1}$$

Positive values of the payment functions $b_i(t)$ and $b_{ij}(t)$ correspond to benefits, while negative values corresponds to premiums. It is also possible to include single payments during sojourns in states, but that is for notational simplicity omitted here.

We assume that the interest rate $r(t)$ is deterministic. Then, the present value at time t of all future payments is denoted $PV(t)$, and it is given by

$$PV(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}B(s).$$

The formula is interpreted as the sum over all future payments, $\mathrm{d}B(s)$, which are discounted by $e^{-\int_t^s r(\tau)\mathrm{d}\tau}$. For a current valuation, we take the expectation conditional on the current state, $\mathrm{E}\left[PV(t)\middle|\, Z(t) = i\right]$. This expected present value is called the prospective (state-wise) reserve.

Definition 2.2.1. *The prospective reserve at time t for state $i \in \mathcal{J}$ is denoted $V_i(t)$, and given as*

$$V_i(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}B(s) \middle|\, Z(t) = i\right].$$

The prospective reserve can be calculated using the following classic results.

Proposition 2.2.2. *The prospective reserve at time t given $Z(t) = i$, $i \in \mathcal{J}$, satisfies,*

$$V_i(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(b_j(s) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \mu_{jk}(s) b_{jk}(s) \right) \mathrm{d}s.$$

Proposition 2.2.3. *The prospective reserve at time t given $Z(t) = i$, $i \in \mathcal{J}$, satisfies* Thiele's differential equation*,*

$$\frac{\mathrm{d}}{\mathrm{d}t} V_i(t) = r(t)V_i(t) - b_i(t) - \sum_{j \in \mathcal{J}, j \neq i} \mu_{ij}(t) \left(b_{ij}(t) + V_j(t) - V_i(t) \right),$$

with boundary conditions $V_i(\infty) = 0$, for $i \in \mathcal{J}$.

Remark 2.2.4. If a time point $T \geq 0$ exists such that $b_i(t) = b_{ij}(t) = 0$ for $t > T$ and all $i, j \in \mathcal{J}$, then the boundary conditions $V_i(T) = 0$ for $i \in \mathcal{J}$ are used with Thiele's differential equation. $\diamond$

It can be convenient to calculate not only the expected present value (the prospective reserve), but also the expected cash flow. From here on, we simply refer to the expected cash flow as the cash flow, and it is a function giving the expected payments at any future time s. The cash flow is, in this setup, independent of the interest rate, and thus the cash flow can be useful for hedging and for an assessment of the interest rate risk associated with the life insurance liabilities.

Definition 2.2.5. *The cash flow at time t associated with the payment process $(B(t))_{t\geq 0}$, conditional on $Z(t) = i$, $i \in \mathcal{J}$, is the function $s \mapsto A_i(t,s)$, given by*

$$A_i(t,s) = \mathrm{E}\left[B(s) - B(t) \middle| Z(t) = i \right],$$

for $s \in [t, \infty)$.

A formal calculation yields an expression for the cash flow: From Definition 2.2.1, we note that

$$\begin{aligned}
V_i(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}(\mathrm{E}\left[B(s) \middle| Z(t) = i \right]) \\
&= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}(\mathrm{E}\left[B(s) - B(t) \middle| Z(t) = i \right]) \\
&= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}A_i(t,s),
\end{aligned}$$

where we have used that $B(t)$ is a constant and doesn't change the dynamics in s. We state the result in a proposition.

Proposition 2.2.6. *The cash flow $A_i(t,s)$ satisfies,*

$$V_i(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}A_i(t,s),$$

$$\mathrm{d}A_i(t,s) = \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(b_j(s) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \mu_{jk}(s) b_{jk}(s) \right) \mathrm{d}s.$$

The second result in Proposition 2.2.6 follows from the first result and from Proposition 2.2.2.

In order to actually calculate the cash flow, one must first calculate the transition probabilities $p_{ij}(s,t)$. In sufficiently simple models, so-called hierarchical models, where you can not return to a state after you left it, the transition probabilities can be calculated using only integrals and known functions. These kind of models are considered in Section 2.3. In general Markov models, closed form expressions for the transition probabilities typically do not exist. Instead, the transition probabilities can be found numerically by solving Kolmogorov's forward or backward differential equation.

Proposition 2.2.7. *The transition probabilities $p_{ij}(t,s)$, for $i, j \in \mathcal{J}$, are unique solutions to* Kolmogorov's backward differential equation*,*

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t,s) = \mu_{i\cdot}(t) p_{ij}(t,s) - \sum_{\substack{k \in \mathcal{J} \\ k \neq i}} \mu_{ik}(t) p_{kj}(t,s),$$

with boundary conditions $p_{ij}(s,s) = 1_{\{i=j\}}$, *and* Kolmogorov's forward differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s) = -p_{ij}(t,s)\mu_{j.}(s) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} p_{ik}(t,s)\mu_{kj}(s),$$

with boundary conditions $p_{ij}(t,t) = 1_{\{i=j\}}$.

Using Kolmogorov's differential equations, the transition probabilities needed in order to calculate the cash flow from Proposition 2.2.6 can be found. It is worth noting, that for calculating the cash flow, using the forward differential equation is the easiest way to obtain the desired transition probabilities.

2.2.1 Technical basis and market basis

In practice and in our examples, we distinguish between calculations on the so-called technical basis, used to settle premiums, and the market basis, used the calculate the market consistent value of the life insurance liabilities, referred to as the market value. A basis is a set of assumptions used for the calculations of life insurance liabilities, and it typically consists of an interest rate $r(t)$ and a set of transition rates $(\mu_{ij}(t))_{i,j\in\mathcal{J}}$. There can also be different administration costs associated with different bases, however administration costs are not considered in this paper. The Markov model can also be different in different bases, and the policyholder behaviour modelling of this paper is an example of this. Here, policyholder behaviour is not included in the technical basis, but is included in the market basis, so the Markov models differ by the surrender and free policy states.

Throughout the paper we let $\hat{r}(t)$ and $\hat{\mu}_{ij}(t)$ be the first order interest and transition rates, respectively, i.e. the interest and transition rates associated with the technical basis. We let $r(t)$ and $\mu_{ij}(t)$ be the interest and transition rates, respectively, for the market basis. In general, values marked with a ˆ are associated with the technical basis. Thus, $V(t)$ is the prospective reserve for the market basis, and $\hat{V}(t)$ is the prospective reserve for the technical basis.

2.2.2 The policyholder options

We study life insurance contracts with two options for the policyholder. She can surrender the contract at any time or she can stop the premium payments and convert the policy into a so-called free policy.

If the policyholder surrenders the contract at time t, all future payments are cancelled, and instead the policyholder receives a compensation for the premiums she has paid so far. Usually, the prospective reserve calculated on the technical basis, $\hat{V}_i(t)$, is paid out, but the formula allows it to be any deterministic value. In this paper, we allow for a deductible, and say that the payment upon surrender is $(1-\kappa)\hat{V}_i(t)$. Since any deterministic value can be chosen, in particular, we can choose κ to be time dependent.

If the policyholder stops the premium payments, i.e. exercises the free policy option, all future premiums are cancelled, and the size of the benefits are decreased to account for the missing future premium payments. If the free policy option is exercised at time t, all future benefits are decreased by a factor $\rho(t)$. In order to handle this, we split the payment process in positive and negative payments, corresponding to benefits and premiums, respectively. The benefit and premium cash flows are denoted by A^+ and A^-, respectively, and are given by

$$\mathrm{d}A_i^+(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s)^+ + \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \mu_{jk}(s)b_{jk}(s)^+\right)\mathrm{d}s,$$

$$\mathrm{d}A_i^-(t,s) = \sum_{j\in\mathcal{J}} p_{ij}(t,s)\left(b_j(s)^- + \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \mu_{jk}(s)b_{jk}(s)^-\right)\mathrm{d}s,$$

where the notation $f(x)^+ = \max(f(x),0)$ and $f(x)^- = \max(-f(x),0)$ for a function $f(x)$ is used. The prospective reserve can then be decomposed as well, and we have

$$V_i^+(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}A_i^+(t,s),$$

$$V_i^-(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}A_i^-(t,s),$$

and $V_i(t) = V_i^+(t) - V_i^-(t)$. The relations also hold on the technical basis, thus $\hat{V}_i(t) = \hat{V}_i^+(t) - \hat{V}_i^-(t)$, where $\hat{V}_i^+(t)$ and $\hat{V}_i^-(t)$ are the values of the future benefits and premiums, respectively, valuated on the technical basis.

If the free policy is exercised at time t, then at future time s, the payment rate while in state i is $\rho(t)b_i(s)^+$ and the payment if a transition from state i to j occurs, is $\rho(t)b_{ij}(s)^+$. Hence, the prospective reserve on the technical basis at time s in state i, given the free policy option is exercised at time $t \leq s$, is

$$\int_s^\infty e^{-\int_s^u \hat{r}(\tau)\mathrm{d}\tau}\rho(t)\mathrm{d}\hat{A}_i^+(s,u) = \rho(t)\hat{V}_i^+(s),$$

where $\mathrm{d}\hat{A}_i^+(s,u)$ is the cash flow calculated with the first order transition probabilities and rates, determined by $\hat{\mu}_{ij}(t)$.

The factor $\rho(t)$ should be deterministic and is usually chosen according to the equivalence principle on the technical basis: The prospective reserve for the technical basis should not change as a consequence of the exercise of the free policy option. We assume in this paper that the free policy conversion can only occur from state 0. Thus, if the free policy option is exercised at time t, the prospective reserve on the technical basis before the free policy option is exercised, $\hat{V}(t)$, should be equal to the prospective reserve after the exercise, $\rho(t)\hat{V}^{+}(t)$. Thus, we require $\hat{V}(t) = \rho(t)\hat{V}^{+}(t)$, yielding

$$\rho(t) = \frac{\hat{V}(t)}{\hat{V}^{+}(t)}.$$

Here, we omitted the subscript 0 from $\hat{V}_0(t)$, and we do that in general the rest of the paper when there is no ambiguity. We see that ρ is the value on the technical basis of benefits less premiums, divided by the value on the technical basis of the benefits only. We refer to $\rho(t)$ as the free policy factor.

2.3 The survival model

We consider the survival model and extend it gradually to include policyholder behaviour. First, we include the surrender option, and afterwards, we include the free policy option as well. The survival model consists of two states, 0 *(alive)* and 1 *(dead)*, corresponding to Figure 2.1.

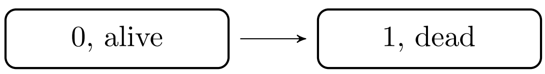

Figure 2.1: *Survival Markov model*

Assume the insured is x years old at time 0. The payments consist of a benefit rate $b(t)$ and a premium rate $\pi(t)$, and a payment $b_{\text{ad}}(t)$ upon death at time t. Referring to the general setup, we have

$$\begin{aligned} b_0(t) &= b(t) - \pi(t), \\ b_{01}(t) &= b_{\text{ad}}(t), \end{aligned}$$

and also, we denote the mortality intensity $\mu_{01}(t) = \mu_{\text{ad}}(t)$. The prospective reserve on the technical basis at time t in state 0 is given by Proposition 2.2.2, and we get

$$\hat{V}(t) = \int_t^\infty e^{-\int_t^s \hat{r}(u)\mathrm{d}u} {}_{s-t}\hat{p}_{x+t} \left(b(s) - \pi(s) + \hat{\mu}_{\text{ad}}(s) b_{\text{ad}}(s) \right) \mathrm{d}s,$$

We have used the actuarial notation for the survival probability, ${}_{s-t}\hat{p}_{x+t} = \hat{p}_{00}(t,s)$, and it is given by,

$${}_t\hat{p}_x = e^{-\int_0^t \hat{\mu}_{\text{ad}}(x+u)\mathrm{d}u}.$$

Thus, ${}_t\hat{p}_x$ is the survival probability of an x-year old reaching age $x+t$, calculated on the technical basis.

The market values of benefits and premiums, respectively, are then given by

$$\begin{aligned} V^+(t) &= \int_t^\infty e^{-\int_t^s r(u)\mathrm{d}u} {}_{s-t}p_{x+t} \left(b(s) + \mu_{\mathrm{ad}}(s) b_{\mathrm{ad}}(s)\right) \mathrm{d}s, \\ V^-(t) &= \int_t^\infty e^{-\int_t^s r(u)\mathrm{d}u} {}_{s-t}p_{x+t} \pi(s) \mathrm{d}s, \end{aligned}$$

and the associated cash flows, conditioning on being alive at time t, are

$$\begin{aligned} \mathrm{d}A^+(t,s) &= {}_{s-t}p_{x+t} \left(b(s) + \mu_{\mathrm{ad}}(s) b_{\mathrm{ad}}(s)\right) \mathrm{d}s, \\ \mathrm{d}A^-(t,s) &= {}_{s-t}p_{x+t} \pi(s) \mathrm{d}s, \end{aligned} \tag{2.3.1}$$

with $V(t) = V^+(t) - V^-(t)$ and $\mathrm{d}A(t,s) = \mathrm{d}A^+(t,s) - \mathrm{d}A^-(t,s)$ being the total prospective reserve and cash flow, respectively. Here, we have omitted the subscript 0 from the notation $\mathrm{d}\hat{A}_0(t,s)$.

The free policy factor is determined by

$$\rho(t) = \frac{\hat{V}(t)}{\hat{V}^+(t)},$$

where $\hat{V}^+(t)$ is the value on the technical basis of the benefits only. If the free policy option is exercised immediately, the market value is

$$\rho(t) V^+(t) = \frac{\hat{V}(t)}{\hat{V}^+(t)} V^+(t),$$

and in Denmark, this is often referred to as the market value of the guaranteed free policy benefits.

2.3.1 Survival model with surrender modelling

We continue the example from above and determine the market value including valuation of the surrender option. The Markov model is extended to include a surrender state, corresponding to Figure 2.2. The surrender modelling is only included in the market basis, and the valuation on the technical basis does not change.

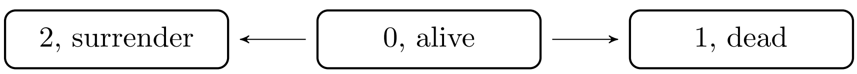

Figure 2.2: *Survival Markov model with surrender.*

On the market basis, we denote the surrender rate by $\mu_{\mathrm{as}}(t)$. We introduce a quantity ${}_tp^{\mathrm{s}}_x$ which is the probability that an x-year old does not die nor surrender before time $x+t$. It is thus the probability of staying in state 0, and is given by,

$$ {}_{s-t}p^{\mathrm{s}}_{x+t} := p_{00}(t,s) = e^{-\int_t^s(\mu_{\mathrm{ad}}(\tau)+\mu_{\mathrm{as}}(\tau))\mathrm{d}\tau} = {}_{s-t}p_{x+t}e^{-\int_t^s \mu_{\mathrm{as}}(\tau)\mathrm{d}\tau}. $$

Here, the transition rates μ_{ad} and μ_{as} are for an x-year old at time 0, which for simplicity is suppressed in the notation.

The payment upon surrender at time s is $(1-\kappa)\hat{V}(s)$, and the cash flow valuated at time t is, by Proposition 2.2.6,

$$ \mathrm{d}A^{\mathrm{s}}(t,s) = {}_{s-t}p^{\mathrm{s}}_{x+t}\Big(b(s)-\pi(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)+\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\Big)\,\mathrm{d}s. \tag{2.3.2} $$

We decompose the cash flow in all payments excluding the surrender payments,

$$ \begin{aligned} \mathrm{d}A^{\mathrm{s1}}(t,s) &= {}_{s-t}p^{\mathrm{s}}_{x+t}\left(b(s)-\pi(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)\right)\mathrm{d}s \\ &= e^{-\int_t^s \mu_{\mathrm{as}}(\tau)\mathrm{d}\tau}\mathrm{d}A(t,s), \end{aligned} $$

and the surrender payments,

$$ \mathrm{d}A^{\mathrm{s2}}(t,s) = {}_{s-t}p^{\mathrm{s}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\mathrm{d}s. $$

Here, $\mathrm{d}A(t,s)$ is the cash flow from the model in Figure 2.1, as defined by (2.3.1). The market value calculated on the market basis including surrender is denoted $V^{\mathrm{s}}(t)$, and is given by

$$ \begin{aligned} V^{\mathrm{s}}(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\left(\mathrm{d}A^{\mathrm{s1}}(t,s)+\mathrm{d}A^{\mathrm{s2}}(t,s)\right) \\ &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}e^{-\int_t^s \mu_{\mathrm{as}}(\tau)\mathrm{d}\tau}\mathrm{d}A(t,s) \\ &\quad+\int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}{}_{s-t}p^{\mathrm{s}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\mathrm{d}s. \end{aligned} \tag{2.3.3} $$

We see that the cash flow and market value including surrender modelling are found using the original cash flow without surrender modelling, $\mathrm{d}A(t,s)$, and multiplying the probability of no surrender $e^{-\int_t^s \mu_{\mathrm{as}}(\tau)\mathrm{d}\tau}$. Thus, finding the cash flow and the market value in the survival model with surrender is particularly simple when the existing cash flow is known.

2.3.2 Survival model with surrender and free policy modelling

We extend the model to include free policy modelling on the market basis, and the Markov model is extended in Figure 2.3 to include free policy states. The mortality and

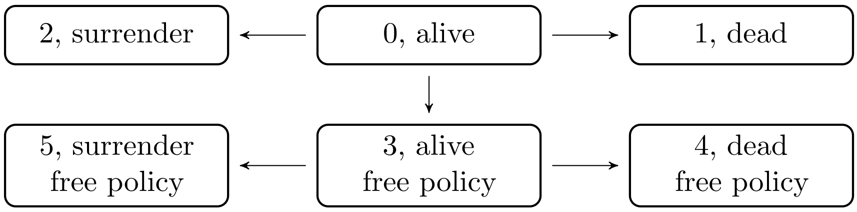

Figure 2.3: *Survival Markov model with surrender and free policy.*

surrender transition rates in the free policy states are identical to those in the premium paying states, μ_{ad} and μ_{as}.

We introduce a free policy rate $\mu_{\text{af}}(t)$, which is the transition rate of becoming a free policy at time t. We introduce the notation

$$\begin{aligned}{}_{s-t}p_{x+t}^{\text{fs}} &= e^{-\int_t^s (\mu_{\text{ad}}(\tau)+\mu_{\text{as}}(\tau)+\mu_{\text{af}}(\tau))\mathrm{d}\tau} \\ &= e^{-\int_t^s (\mu_{\text{as}}(\tau)+\mu_{\text{af}}(\tau))\mathrm{d}\tau}\, {}_{s-t}p_{x+t} \\ &= e^{-\int_t^s \mu_{\text{af}}(\tau)\mathrm{d}\tau}\, {}_{s-t}p_{x+t}^{\text{s}},\end{aligned}$$

which is the probability of staying in state 0, i.e. not becoming a free policy, surrendering nor dying.

If the free policy transition occurs at time t, the future benefits are reduced by a factor $\rho(t)$, and the future premiums are cancelled. Thus, in the free policy state at a later time s, the payment rate is $\rho(t)b(s)$, and the payment upon death is $\rho(t)b_{\text{ad}}(s)$. The surrender payment, if surrender occurs as a free policy, is $\rho(t)(1-\kappa)\hat{V}^{+}(s)$, where $\rho(t)\hat{V}^{+}(s)$ is the prospective reserve on the technical basis.

The payment process is dependent on the exact time of the free policy transition, i.e. the payments are dependent on the duration since the free policy transition. It can be shown that the cash flow valuated at time t is given by

$$\begin{aligned}\mathrm{d}A^{\text{f}}(t,s) = {}& {}_{s-t}p_{x+t}^{\text{fs}}\Big(b(s)-\pi(s)+\mu_{\text{ad}}(s)b_{\text{ad}}(s)+\mu_{\text{as}}(s)(1-\kappa)\hat{V}(s)\Big)\,\mathrm{d}s \\ &+\int_t^s {}_{\tau-t}p_{x+t}^{\text{fs}}\mu_{\text{af}}(\tau)\,{}_{s-\tau}p_{x+\tau}^{\text{s}} \\ &\times\Big(\rho(\tau)b(s)+\mu_{\text{ad}}(s)\rho(\tau)b_{\text{ad}}(s)+\mu_{\text{as}}(s)\rho(\tau)(1-\kappa)\hat{V}^{+}(s)\Big)\,\mathrm{d}\tau\mathrm{d}s.\end{aligned} \tag{2.3.4}$$

The result can be obtained as a special case of Proposition 2.4.1 below, where the disability rate is set to 0, but for completeness, a separate proof is given in Appendix 2.A.1. The first line is the payments in state 0 and the payments upon death and surrender. The second and third lines contain the payments as a free policy. This expression can be interpreted as the probability of staying in state 0 until time τ, then becoming a free policy at time τ, and then neither dying nor surrendering from time τ to time s. This is multiplied with the payments as a free policy at time s, given the free policy occurred at time τ. Finally, we integrate over all possible free policy transition times from s to t.

The cash flow is decomposed into four parts. First, the benefits and premiums, excluding surrender payments, while alive and not a free policy,

$$\begin{aligned} \mathrm{d}A^{\mathrm{f1}}(t,s) &= {}_{s-t}p^{\mathrm{fs}}_{x+t}\left(b(s)-\pi(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)\right)\mathrm{d}s \\ &= e^{-\int_t^s(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\mathrm{d}u}\left(\mathrm{d}A^{+}(t,s)-\mathrm{d}A^{-}(t,s)\right). \end{aligned} \tag{2.3.5}$$

Then, the surrender payments, if the free policy transition has not occurred,

$$\begin{aligned} \mathrm{d}A^{\mathrm{f2}}(t,s) &= {}_{s-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\mathrm{d}s \\ &= e^{-\int_t^s\mu_{\mathrm{af}}(u)\mathrm{d}u}{}_{s-t}p^{\mathrm{s}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\mathrm{d}s. \end{aligned} \tag{2.3.6}$$

Note that these cash flows correspond to the cash flows in the surrender model, but reduced with the probability of the free policy transition not happening.

The third cash flow is the benefits while a free policy

$$\begin{aligned} \mathrm{d}A^{\mathrm{f3}}(t,s) &= \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau){}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\mathrm{d}\tau\left(b(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)\right)\mathrm{d}s \\ &= \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\mathrm{d}u}\mathrm{d}A^{+}(\tau,s)\mathrm{d}\tau, \end{aligned} \tag{2.3.7}$$

and the fourth cash flow is the surrender payments while a free policy,

$$\mathrm{d}A^{\mathrm{f4}}(t,s) = \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau){}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\mathrm{d}\tau\cdot\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\mathrm{d}s. \tag{2.3.8}$$

The third cash flow (2.3.7) seems complicated, since the cash flows at time s evaluated at time τ, $\mathrm{d}A^{+}(\tau,s)$, is needed for any $\tau\in(t,s)$ and all $s\geq t$. However, a straightforward calculation yields,

$$\begin{aligned} &{}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\mathrm{d}u}\mathrm{d}A^{+}(\tau,s) \\ &= e^{-\int_t^\tau(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\mathrm{d}u}{}_{\tau-t}p_{x+t}\mathrm{d}A^{+}(\tau,s) \\ &= e^{-\int_t^\tau(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\mathrm{d}u}\mathrm{d}A^{+}(t,s), \end{aligned}$$

which simplifies things, and insertion of this into $\mathrm{d}A^{\mathrm{f3}}$ yields,

$$\mathrm{d}A^{\mathrm{f3}}(t,s) = \left(\int_t^s e^{-\int_t^\tau(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)e^{-\int_\tau^s\mu_{\mathrm{as}}(u)\mathrm{d}u}\mathrm{d}\tau\right)\mathrm{d}A^{+}(t,s).$$

Define the quantity

$$r^{\rho}(t,s) = \int_t^s e^{-\int_t^\tau\mu_{\mathrm{af}}(u)\mathrm{d}u}\mu_{\mathrm{af}}(\tau)\rho(\tau)\mathrm{d}\tau, \tag{2.3.9}$$

and note that

$$\begin{aligned}
\mathrm{d}A^{\mathrm{f3}}(t,s) &= r^{\rho}(t,s)e^{-\int_t^s \mu_{\mathrm{as}}(u)\mathrm{d}u}\mathrm{d}A^{+}(t,s),\\
\mathrm{d}A^{\mathrm{f4}}(t,s) &= r^{\rho}(t,s)\,{}_{s-t}p^{\mathrm{s}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\mathrm{d}s.
\end{aligned}$$

The market value including surrender and free policy modelling is denoted $V^{\mathrm{f}}(t)$, and it may finally be written as

$$\begin{aligned}
V^{\mathrm{f}}(t) &= \int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}\Big(\mathrm{d}A^{\mathrm{f1}}(t,s)+\mathrm{d}A^{\mathrm{f2}}(t,s)+\mathrm{d}A^{\mathrm{f3}}(t,s)+\mathrm{d}A^{\mathrm{f4}}(t,s)\Big)\\
&= \int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}e^{-\int_t^s(\mu_{\mathrm{as}}(u)+\mu_{\mathrm{af}}(u))\mathrm{d}u}\big(\mathrm{d}A^{+}(t,s)-\mathrm{d}A^{-}(t,s)\big)\\
&\quad+\int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}\,{}_{s-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\mathrm{d}s\\
&\quad+\int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}r^{\rho}(t,s)e^{-\int_t^s \mu_{\mathrm{as}}(u)\mathrm{d}u}\mathrm{d}A^{+}(t,s)\\
&\quad+\int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}r^{\rho}(t,s)\,{}_{s-t}p^{\mathrm{s}}_{x+t}\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\mathrm{d}s.
\end{aligned}\tag{2.3.10}$$

The last four lines in (2.3.10) have the following interpretation.

- The first line is the value of the original cash flow (2.3.1) without policyholder behaviour, reduced by the probability of not surrendering and not becoming a free policy.
- The second line is the value of the surrender payments, when not a free policy.
- The third line is the benefit payments as a free policy, i.e. the positive payments reduced with the free policy factor $\rho(\tau)$ at the time τ of the free policy transition.
- The fourth line is the surrender payments if surrender occurs after the free policy transition.

The formula gives the market value of future guaranteed payments, including valuation of the surrender and free policy options. In order to calculate the value, the following quantities are needed

- The original cash flows $\mathrm{d}A^{+}(t,s)$ and $\mathrm{d}A^{-}(t,s)$,
- The prospective reserve on the technical basis $\hat{V}^{+}(s)$ and $\hat{V}^{-}(s)$, for all future time points $s \geq t$, which allow us to determine the surrender payments and the free policy factor $\rho(s)$.
- The factor $r^{\rho}(t,s)$, which is a simple integral of the free policy transition rate.

2.3.3 Free policy modelling when surrender is already modelled

In the previous section, we found the market value including surrender and free policy modelling based on cash flows without any policyholder behaviour modelling. It is also possible to find this market value based on cash flows including surrender modelling. This could be relevant if the existing cash flows already include surrender modelling, and one wishes to modify these cash flows to include free policy modelling. Thus, we assume that the cash flow including surrender behaviour modelling, (2.3.2), are available, and that it is split in a cash flow associated with the benefits and a cash flow associated with premiums, i.e.

$$\begin{aligned} \mathrm{d}A^{\mathrm{s},+}(t,s) &= {}_{s-t}p^{\mathrm{s}}_{x+t}\left(b(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)+\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\right)\mathrm{d}s, \\ \mathrm{d}A^{\mathrm{s},-}(t,s) &= {}_{s-t}p^{\mathrm{s}}_{x+t}\left(\pi(s)+\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{-}(s)\right)\mathrm{d}s. \end{aligned} \tag{2.3.11}$$

Note that the payment upon surrender is split between the two cash flows, through the decomposition $\hat{V}(t)=\hat{V}^{+}(t)-\hat{V}^{-}(t)$, i.e. the value of the future benefits less the value of the future premiums.

The market value with surrender modelling, but not free policy modelling, $V^{\mathrm{s}}(t)$ from (2.3.3), is then given by

$$V^{\mathrm{s}}(t)=\int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}\left(\mathrm{d}A^{\mathrm{s},+}(t,s)-\mathrm{d}A^{\mathrm{s},-}(t,s)\right).$$

We find the cash flow including free policy modelling by modifying the existing cash flows into two cash flows: One, which is reduced by the probability of not becoming a free policy, and a special free policy cash flow. With a few calculations using (2.3.5), (2.3.6) and (2.3.11), we see that

$$\begin{aligned} \mathrm{d}A^{\mathrm{f1}}(t,s)+\mathrm{d}A^{\mathrm{f2}}(t,s) &= {}_{s-t}p^{\mathrm{fs}}_{x+t}\left(b(s)-\pi(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)+\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}(s)\right)\mathrm{d}s \\ &= e^{-\int_t^s \mu_{\mathrm{af}}(u)\mathrm{d}u}\left(\mathrm{d}A^{\mathrm{s},+}(t,s)-\mathrm{d}A^{\mathrm{s},-}(t,s)\right), \end{aligned}$$

and also, by (2.3.7), (2.3.8), (2.3.9) and (2.3.11),

$$\begin{aligned} &\mathrm{d}A^{\mathrm{f3}}(t,s)+\mathrm{d}A^{\mathrm{f4}}(t,s) \\ &= \int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\mathrm{d}\tau\left(b(s)+\mu_{\mathrm{ad}}(s)b_{\mathrm{ad}}(s)+\mu_{\mathrm{as}}(s)(1-\kappa)\hat{V}^{+}(s)\right)\mathrm{d}s \\ &= r^{\rho}(t,s)\mathrm{d}A^{\mathrm{s},+}(t,s). \end{aligned}$$

The total cash flow is then given as

$$\mathrm{d}A^{\mathrm{f}}(t,s)=e^{-\int_t^s \mu_{\mathrm{af}}(u)\mathrm{d}u}\left(\mathrm{d}A^{\mathrm{s},+}(t,s)-\mathrm{d}A^{\mathrm{s},-}(t,s)\right)+r^{\rho}(t,s)\mathrm{d}A^{\mathrm{s},+}(t,s).$$

This cash flow can be interpreted as a weighted average between the original cash flow, reduced with the probability of not becoming a free policy, and the payments as a free policy. The payments as a free policy are the positive payments multiplied with $r^\rho(t,s)$. The quantity $r^\rho(t,s)$ is interpreted as the probability of becoming a free policy multiplied with the free policy factor $\rho(\tau)$ at the time τ of the free policy transition.

The market value from before, $V^{\mathrm{f}}(t)$, can then be calculated as

$$\begin{aligned} V^{\mathrm{f}}(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \Big(\mathrm{d}A^{\mathrm{f1}}(t,s) + \mathrm{d}A^{\mathrm{f2}}(t,s) + \mathrm{d}A^{\mathrm{f3}}(t,s) + \mathrm{d}A^{\mathrm{f4}}(t,s)\Big) \\ &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} e^{-\int_t^s \mu_{\mathrm{af}}(u)\mathrm{d}u} \left(\mathrm{d}A^{\mathrm{s},+}(t,s) - \mathrm{d}A^{\mathrm{s},-}(t,s)\right) \\ &\quad + \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} r^\rho(t,s)\mathrm{d}A^{\mathrm{s},+}(t,s). \end{aligned}$$

If we only include surrender modelling, the needed extra quantities are simple integrals of the surrender rate $\mu_{\mathrm{as}}(t)$. If we in addition include free policy modelling, the free policy factor $\rho(t)$ must also be found, which requires access to future prospective reserves on the technical basis. When these are found, the market value is relatively simple to calculate.

2.3.4 Approximate method

An essential assumption for these calculations is that there are no payments after leaving the active state, i.e. that the prospective reserve is 0 in the dead and surrender states. That is, after the payment upon death or surrender, there are no future payments. If one adds a disability state, similar simple results can only be obtained if the prospective reserve is 0 in the disability state. This is typically not satisfied, and as such the methods of modifying the cash flows presented here are not applicable. However, assuming one has cash flows from a more general model without policyholder behaviour (e.g. a disability model), formula (2.3.10) can be applied to these cash flows in order to obtain an approximation to cash flows with policyholder behaviour. We refer to this method as the approximate method.

In the following section, we examine how to correctly model policyholder behaviour in a disability Markov model.

2.4 A general disability Markov model

In this section we consider the survival model extended with a disability state, from which it is possible to recover. We extend the model further by including states for surrender and free policy, and end up with an 8-state model, see Figure 2.4. By solving certain ordinary differential equations for the relevant transition probabilities and a special free

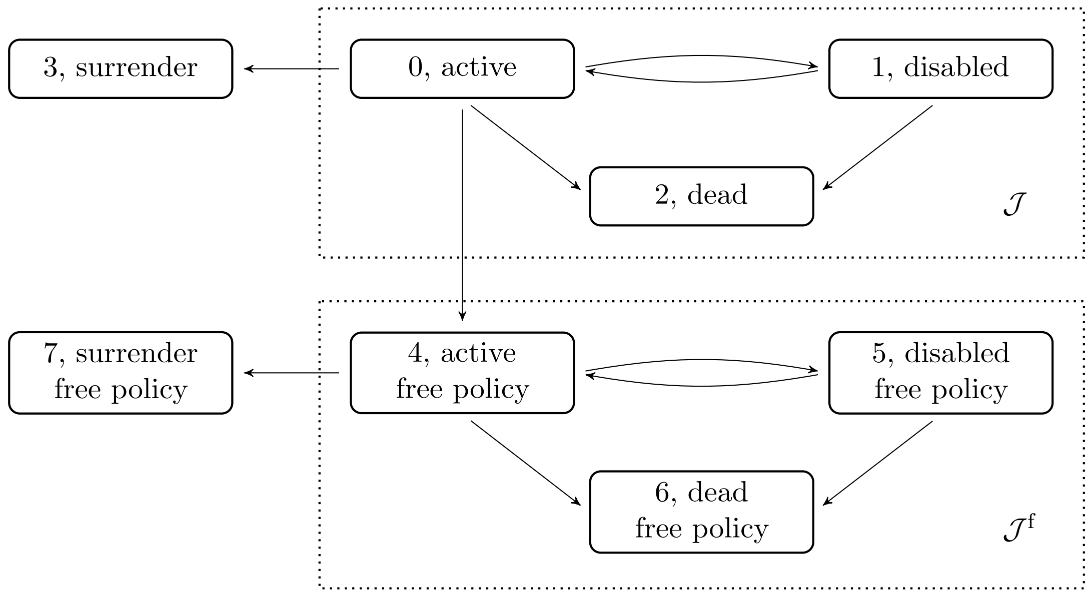

Figure 2.4: *The 8-state Markov model, with disability, surrender and free policy. The transition rates between states* 0*,* 1 *and* 2 *are identical to the transition rates between states* 4*,* 5 *and* 6*. The two surrender states can be considered one state, and then this model is known as the so-called "7-state model".*

policy quantity, similar to r^ρ from (2.3.9), the cash flow and prospective reserve can be found.

The results can easily be extended to more general Markov models than the disability model, as long as free policy conversion only occurs from the active state 0. A more general setup is studied in [11], which is here specialised to the case of the survival-disability model.

For valuation on the technical basis, the survival-disability Markov model, consisting of states 0, 1 and 2, are used. In this section, the payments are labelled by the state they correspond to instead of the labels used previously. Thus, the payment rate in state 0, *active*, is $b_0(t)$ and in state 1, *disabled*, it is $b_1(t)$. Upon disability there is a payment $b_{01}(t)$, upon death as active there is a payment $b_{02}(t)$, and upon death as disabled there is a payment $b_{12}(t)$. The payment function in state 0, $b_0(t)$, is decomposed in positive payments $b_0(t)^+$, which are benefits, and negative payments, $b_0(t)^-$, which are premiums. Thus,

$$b_0(t) = b_0(t)^+ - b_0(t)^-.$$

We assume that all other payments functions are positive. The notation corresponds to the notation used in (2.2.1) for the payment functions $b_0(t)$, $b_1(t)$, $b_{01}(t)$, $b_{02}(t)$ and

$b_{12}(t)$, and all other payment functions b_i and b_{ij} are zero. The transition rates are also labelled by numbers, e.g. the transition rate from state i to j is $\mu_{ij}(t)$.

Using Proposition 2.2.6, the cash flow for state 0 under the technical basis is,

$$\begin{aligned}\mathrm{d}\hat{A}(t,s) &= \hat{p}_{00}(t,s)\left(b_0(s) + \hat{\mu}_{02}(s)b_{02}(t) + \hat{\mu}_{01}(s)b_{01}(s)\right)\\ &\quad + \hat{p}_{01}(t,s)\left(b_1(s) + \hat{\mu}_{12}(s)b_{12}(s)\right),\end{aligned}$$

where the notation $\hat{p}$ and $\hat{\mu}$ refers to the transition probabilities and rates on the technical basis. The first line contains payments while in state 0, *active*, and payments during transitions out of state 0. The payments on the second line are payments in state 1, *disabled*, and payments during transitions out of state 1. We decompose the cash flow in positive and negative payments, and define,

$$\begin{aligned}\mathrm{d}\hat{A}^+(t,s) &= \hat{p}_{00}(t,s)\left(b_0(s)^+ + \hat{\mu}_{02}(s)b_{02}(s) + \hat{\mu}_{01}(s)b_{01}(s)\right)\mathrm{d}s\\ &\quad + \hat{p}_{01}(t,s)\left(b_1(s) + \hat{\mu}_{12}(s)b_{12}(s)\right)\mathrm{d}s,\\ \mathrm{d}\hat{A}^-(t,s) &= \hat{p}_{00}(t,s)b_0(s)^-\mathrm{d}s,\end{aligned}$$

such that $\mathrm{d}\hat{A}(t,s) = \mathrm{d}\hat{A}^+(t,s) - \mathrm{d}\hat{A}^-(t,s)$. The prospective reserve on the technical basis $\hat{V}(t)$ is also decomposed,

$$\begin{aligned}\hat{V}^+(t) &= \int_t^\infty e^{-\int_t^s \hat{r}(u)\mathrm{d}u}\mathrm{d}\hat{A}^+(t,s),\\ \hat{V}^-(t) &= \int_t^\infty e^{-\int_t^s \hat{r}(u)\mathrm{d}u}\mathrm{d}\hat{A}^-(t,s),\end{aligned}$$

and we have $\hat{V}(t) = \hat{V}^+(t) - \hat{V}^-(t)$. Here we again omit the notation 0 for the state in the reserves and cash flows.

For valuation on the market basis, we consider the extended Markov model in Figure 2.4. We define a duration, $U(t)$, which is the time since the free policy option was exercised (or since surrender),

$$U(t) = \inf\left\{s \geq 0 \,|\, Z(t-s) \in \{0,1,2\}\right\}.$$

If the free policy option is exercised, and the current time is t, the time of the free policy transition is then $t - U(t)$. Upon transition to a free policy, the benefits are reduced by the factor $\rho(t - U(t))$, and the premiums are cancelled. The payments in the free policy states are thus duration dependent, and at time t they are,

$$\begin{aligned}b_4(t,U(t)) &= \rho(t-U(t))b_0(t)^+,\\ b_5(t,U(t)) &= \rho(t-U(t))b_1(t),\\ b_{45}(t,U(t)) &= \rho(t-U(t))b_{01}(t),\\ b_{46}(t,U(t)) &= \rho(t-U(t))b_{02}(t),\\ b_{56}(t,U(t)) &= \rho(t-U(t))b_{12}(t).\end{aligned}$$

Upon surrender from state 0, an amount $(1-\kappa)\hat{V}(t)$ is paid out, where $\hat{V}(t)$ is the prospective reserve on the technical basis. If the free policy option is exercised and surrender occurs from state 4, the prospective reserve on the technical basis is the value of the future benefits, reduced by the free policy factor $\rho(t-U(t))$. Thus, the payment upon surrender as a free policy is $(1-\kappa)\rho(t-U(t))\hat{V}^{+}(t)$. The parameter κ is a surrender strain and is usually 0. We have,

$$\begin{aligned} b_{03}(t) &= (1-\kappa)\hat{V}(t), \\ b_{47}(t,U(t)) &= (1-\kappa)\rho(t-U(t))\hat{V}^{+}(t). \end{aligned}$$

The total payment process is then given by,

$$\begin{aligned} \mathrm{d}B(t) = {} & \big(1_{\{Z(t)=0\}}b_0(t) + 1_{\{Z(t)=1\}}b_1(t)\big)\,\mathrm{d}t \\ & + b_{01}(t)\mathrm{d}N_{01}(t) + b_{02}(t)\mathrm{d}N_{02}(t) + b_{12}(t)\mathrm{d}N_{12}(t) \\ & + (1-\kappa)\hat{V}(t)\mathrm{d}N_{03}(t) \\ & + \rho(t-U(t))\Big\{\big(1_{\{Z(t)=4\}}b_0(t)^{+} + 1_{\{Z(t)=5\}}b_1(t)\big)\,\mathrm{d}t \\ & \quad + b_{01}(t)\mathrm{d}N_{45}(t) + b_{02}(t)\mathrm{d}N_{46}(t) + b_{12}(t)\mathrm{d}N_{56}(t) \\ & \quad + (1-\kappa)\hat{V}^{+}(t)\mathrm{d}N_{47}(t)\Big\}. \end{aligned} \tag{2.4.1}$$

The first two lines contain the benefits and premiums in the states 0, *alive*, 1, *disabled* and 2, *dead*. Line three contains the payment upon surrender as a premium paying policy, and line six contains the payment upon surrender as a free policy. Lines four and five contain the payments as a free policy.

We find the cash flow, and to this end it is convenient to define the quantity

$$p_{ij}^{\rho}(t,s) = \mathrm{E}\left[1_{\{Z(s)=j\}}\rho(s-U(s))\,\middle|\, Z(t)=i\right],$$

for $i \in \{0,1,2\}$, $j \in \{4,5,6\}$, and $t \le s$. Then, it holds that

$$p_{ij}^{\rho}(t,s) = \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)p_{4j}(\tau,s)\rho(\tau)\mathrm{d}\tau. \tag{2.4.2}$$

For a proof of (2.4.2), see Appendix 2.A.2. For $\rho(t)=1$, this quantity is simply the transition probability from state i to j: It is the probability of going form state i to 0 at time τ, and then transitioning to state 4 at time τ, and finally going from state 4 to state j from time τ to s. Since a transition from a state $i \in \{0,1,2\}$ to a state $j \in \{4,5,6\}$ can only occur through a transition from state 0 to 4, this gives the transition probability. When $\rho(t) \neq 1$, the quantity corresponds to the transition probability multiplied by $\rho(t)$ at the time of transition to a free policy.

We now state the cash flow. The proof is found in Appendix 2.A.3.

Proposition 2.4.1. *The cash flow in state 0,* $\mathrm{d}A^{\mathrm{f}}(t,s)$*, for payments at time* s *valued at time* t*, is given by*

$$\begin{aligned}\mathrm{d}A^{\mathrm{f}}(t,s) = p_{00}(t,s)&\Big(b_0(s)+\mu_{01}(s)b_{01}(s)+\mu_{02}(s)b_{02}(s)+\mu_{03}(s)(1-\kappa)\hat{V}(s)\Big)\,\mathrm{d}s\\ &+p_{01}(t,s)\left(b_1(s)+\mu_{12}(s)b_{12}(s)\right)\mathrm{d}s\\ &+p^{\rho}_{04}(t,s)\Big(b_0(s)^{+}+\mu_{45}(s)b_{01}(s)+\mu_{46}(s)b_{02}(s)+\mu_{47}(s)(1-\kappa)\hat{V}^{+}(s)\Big)\,\mathrm{d}s\\ &+p^{\rho}_{05}(t,s)\left(b_1(s)+\mu_{56}(s)b_{12}(s)\right)\mathrm{d}s.\end{aligned}$$

Calculation of the cash flow requires $p^{\rho}_{ij}(t,s)$ to be calculated, and with (2.4.2), this requires the transition probabilities $p_{4j}(\tau,s)$ for all s and τ satisfying $t\le\tau\le s$. However, it turns out that this is not necessary, since there exists a differential equation for $p^{\rho}_{ij}(t,s)$ similar to Kolmogorov's forward differential equation. Using this, one can calculate all the usual transition probabilities and the $p^{\rho}_{ij}(t,s)$ quantities together. This eliminates the need to calculate $p_{4j}(\tau,s)$ for all τ and s satisfying $t\le\tau\le s$.

Proposition 2.4.2. *The quantities* $p^{\rho}_{ij}(t,s)$ *satisfy the forward differential equation, for* $i\in\{0,1,2\}$ *and* $j\in\{4,5,6\}$*,*

$$\frac{\mathrm{d}}{\mathrm{d}s}p^{\rho}_{ij}(t,s) = 1_{\{j=4\}}p_{i0}(t,s)\mu_{04}(s)\rho(s) - p^{\rho}_{ij}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\{4,5,6\}\\ \ell\neq j}} p^{\rho}_{i\ell}(t,s)\mu_{\ell j}(s),$$

with boundary conditions $p^{\rho}_{ij}(t,t)=0$*.*

A more general version of this result is presented in Theorem 4.2 in [11] for the general semi-Markov case, and can also be found for the general Markov case as equation (4.8) in [11]. For completeness, a straightforward proof is given in Appendix 2.A.4. For the proposition, we recall that $\mu_{j\cdot}(s)$ is the sum of all the transition rates out of state j. Note in particular, that if $j=4$, the last sum is simply the one term $p^{\rho}_{i5}(t,s)\mu_{54}(s)$, and if $j=5$, the last term is $p^{\rho}_{i4}(t,s)\mu_{45}(s)$.

The market value including surrender and free policy modelling is denoted $V^{\mathrm{f}}(t)$ and is given by,

$$V^{\mathrm{f}}(t) = \int_t^{\infty} e^{-\int_t^s r(\tau)\mathrm{d}\tau}\,\mathrm{d}A^{\mathrm{f}}(t,s)$$

$$
\begin{aligned}
&= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} p_{00}(t,s)\left(b_0(s) + \mu_{01}(s)b_{01}(s) + \mu_{02}(s)b_{02}(s)\right)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} p_{01}(t,s)\left(b_1(s) + \mu_{12}(s)b_{12}(s)\right)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} p_{00}(t,s)\mu_{03}(s)(1-\kappa)\hat{V}(s)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} p_{04}^{\rho}(t,s)\left(b_0(s)^+ + \mu_{45}(s)b_{01}(s) + \mu_{46}(s)b_{02}(s)\right)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} p_{05}^{\rho}(t,s)\left(b_1(s) + \mu_{56}(s)b_{12}(s)\right)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} p_{04}^{\rho}(t,s)\mu_{47}(s)(1-\kappa)\hat{V}^+(s)\mathrm{d}s.
\end{aligned}
$$

The first three lines are the payments when the free policy option is not exercised, and the last three lines are payments as a free policy. The first two lines are the payments without policyholder behaviour which is similar to the first line in (2.3.10). The third line is the surrender payments when the free policy option is not exercised and this is similar to line two in (2.3.10). The fourth and fifth line are the payments as a free policy, without the surrender payment, corresponding to the third line in (2.3.10). The last line is the surrender payments as a free policy which corresponds to the fourth line in (2.3.10). If the disability state is removed, the formula simplifies to (2.3.10).

2.5 Numerical example

We present a numerical example which illustrates the methods presented in the previous sections. First, we illustrate the impact of modelling policyholder behaviour, and in particular we see that the structure of the cash flows changes considerably. With the modelling of policyholder behaviour, the interest rate sensitivity (duration) of the cash flow is significantly reduced, which is of importance if one applies duration matching techniques in order to hedge the interest rate risk.

Second, we illustrate the error when using the simple model from Section 2.3.1 to modify existing cash flows from a disability model to include policyholder behaviour. In Section 2.3.1 it is shown how to manipulate cash flows from a life-death model to include policyholder behaviour according to the model shown in Figure 2.3. When using the methods on cash flows originating from the 3-state disability model shown in Figure 2.5, the formulae do not give the correct result, but they can serve as an approximation to the full model from Figure 2.4. As illustrated in this example, it can indeed be a very good approximation.

We consider a 40 year old male now at time 0, with retirement age 65. He enters a new policy with two products:

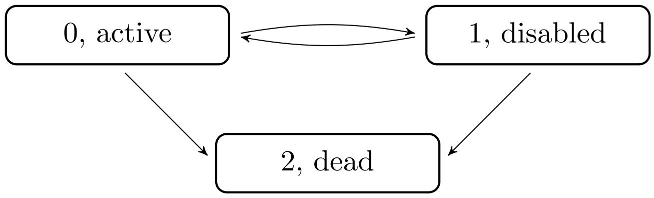

Figure 2.5: *Disability Markov model*

- A disability annuity consisting of an annual payment of 100,000 while disabled, until age 65.
- A life annuity consisting of an annual payment of 100,000 while alive, from age 65 until death.

He pays a yearly premium while active, determined by the principle of equivalence on the technical basis. The principle of equivalence states that the prospective reserve on the technical basis is 0 before any payments are made. The technical basis consists of

- 3-state disability Markov-model, as shown in Figure 2.5.
- Interest rate of 1%.
- Transition rates, where x is the age,

$$\begin{aligned}
\mu_{01}^{*}(x) &= \left(0.0004 + 10^{4.54+0.06x-10}\right) 1_{\{x\leq 65\}},\\
\mu_{10}^{*}(x) &= \left(2.0058 \cdot e^{-0.117x}\right) 1_{\{x\leq 65\}},\\
\mu_{02}^{*}(x) &= 0.0005 + 10^{5.88+0.038x-10},\\
\mu_{12}^{*}(x) &= \mu_{02}^{*}(x)\left(1 + 1_{\{x\leq 65\}}\right).
\end{aligned}$$

The transition rates are designed such that we do not distinguish between disabled and active after retirement, thus the mortality rate as disabled is simply set to the mortality rate as active after age 65, and this is simply interpreted as the average mortality rate from alive to dead.

With the technical basis and the equivalence principle, the premium paid while active is found to be of annual size 46,409 until age 65. Using the technical basis, we also find the technical reserve at future time points for the active and the free policy state. This determines the surrender payments and the free policy conversion factor, which are needed for the calculations below.

For the market basis, the interest rate provided by the Danish FSA of 2 September 2013 is used. The transition rates loosely resemble those used by a large Danish life and pension insurance company in the competitive market. With x being the age, they are given by,

- active $\to$ dead, $\mu_{02}(x)$: the mortality benchmark from 2012 from The Danish FSA,
- active $\to$ disabled: $\mu_{01}(x) = 1_{\{x \leq 65\}} \left(10^{5.662015+0.033462x-10}\right)$,
- disabled $\to$ active: $\mu_{10}(x) = 4.0116e^{-0.117x}$,
- disabled $\to$ dead: $\mu_{12}(x) = 0.010339 + 10^{5.070927+0.05049x-10}$.

The active to surrender (as), respectively to free policy (af), transition rates are for age $x \leq 65$ given as

$$\mu_{\mathrm{as}}(x) = 0.06 - 0.002 \cdot (x - 40)^+,$$
$$\mu_{\mathrm{af}}(x) = 0.05,$$

and they are set to 0 for $x > 65$.

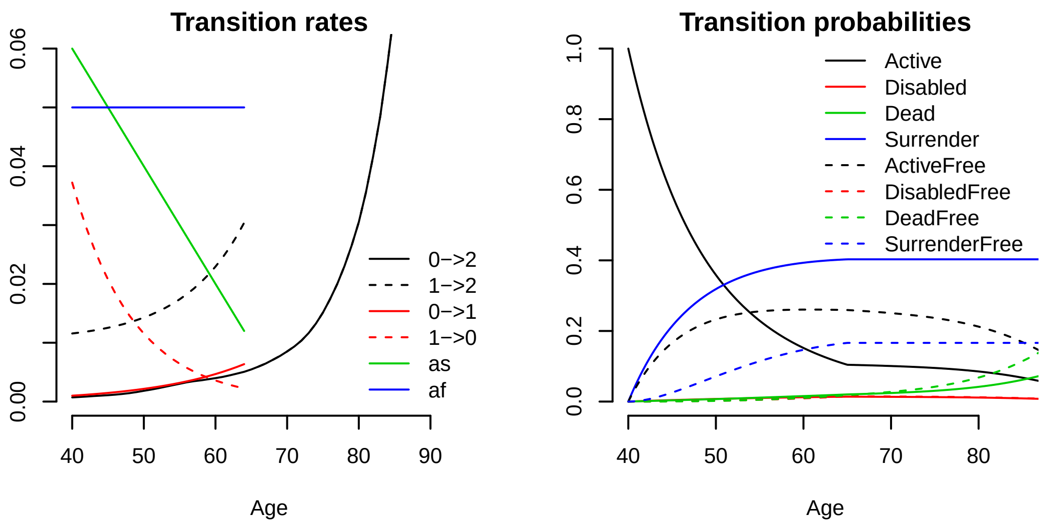

Figure 2.6: *Transition rates (left) and transition probabilities (right). In the left figure, the transition rates between the states in the disability model, (*0-active, 1-disabled *and* 2-dead*) are shown, as well as the surrender (*as*) and free policy (*af*) transition rates. In the right figure, the full lines are the non-free policy states and the dashed lines are corresponding free policy states. The active, free policy and surrender states are dominant.*

The transition rates are shown in Figure 2.6 together with the transition probabilities, which have been calculated using Kolmogorov's forward differential equation, Proposition 2.2.7. The probability of surrender and free policy conversion are significant, and already around age 47 the probability of having surrendered or made a free policy conversion is greater than the probability of still being active. We also note that the transition probabilities are smooth except at age 65 where the disability, recovery, surrender and free policy transition rates jump to 0.

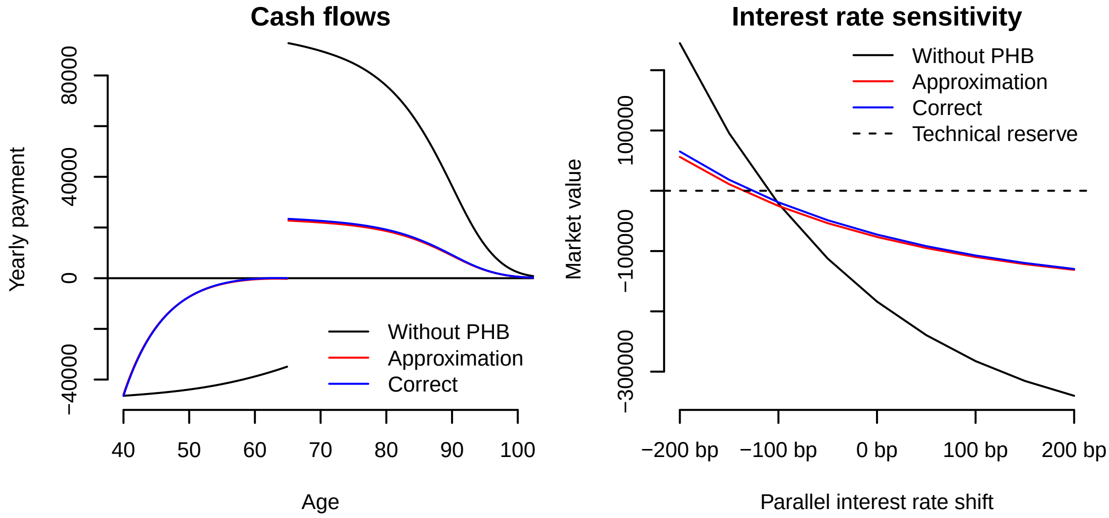

Figure 2.7: *Cash flows (left) and market value for different parallel shifts in the interest rate structure (right), measured in basis points. In the figures we see that the modelling of policyholder behaviour has a major impact. Also, it seems difficult to distinguish the approximate and the correct method using the eye-ball norm only. The approximation yields a slightly larger cash flow and market value than the correct modelling does.*

We perform three calculations. First, we consider the disability model without policyholder behaviour, as shown in Figure 2.5. In this model, the cash flow and the corresponding market value are calculated, using Propositions 2.2.6 and 2.2.7. In Figure 2.7, these results are shown as the black lines. In the beginning the premiums are paid, and the cash flow starts at the annual premium level −46,409. If the insured dies, the premium stops, and upon disability, the premium stops and the disability annuity is paid. Thus, we see a slightly increasing cash flow until age 65. After retirement, a life annuity is paid out instead. The value of the cash flow at age 65 is slightly less than 100,000, corresponding to a strictly positive probability of death before age 65. The cash flow decreases to 0 as the insured eventually dies.

In Table 2.1 the market value and the dollar duration are presented. The technical reserve is 0 due to the equivalence principle, corresponding to this being a new policy. The market value is negative, which expresses a surplus inherent in the contract, because the technical basis is on the safe side. We see the dollar duration is significant, and from Figure 2.7 it is seen that a decrease of a little more than 100 basis points in the interest rate leads to an increase in the market value, eliminating the surplus completely.

For the second calculation we consider the approximate method, as discussed in Section 2.3.4. The cash flow without any modelling of policyholder behaviour, $\mathrm{d}A_0(0,s) = \mathrm{d}A_0^+(0,s) - \mathrm{d}A_0^-(0,s)$ is modified by formula (2.3.10) in order to obtain approximate cash

	Without PHB	Approximation	Correct
Market value	-183,798	-76,599	-72,641
DV01	130,792	42,462	44,239

Table 2.1: *Market value of the cash flow and dollar durations (DV01), without policyholder behaviour, with the approximative method and the correct method. The duration is greatly reduced when policyholder behaviour is included, and the approximation is close to the result from the correct method.*

flows including policyholder behaviour. As mentioned above, this is only correct if the cash flows originate from a 2-state survival model as shown in Figure 2.1. However, we perform the modification anyway, and examine the quality of this as an approximation. One can show that this calculation corresponds to allowing surrender and free policy from the disability state, see [26]. In Figure 2.7, the results from this modification are shown as the red lines.

Third and last, we use the results from Section 2.4 and correctly calculate the cash flows, where surrender and free policy conversion are only possible from the active states. In Figure 2.7, this is shown as the blue lines.

We see that including modelling of policyholder behaviour has an effect on both the market value and the structure of the cash flow. The market value is increased, since the surplus inherent in the contract is reduced if the insured surrenders or converts to a free policy. In the right part of Figure 2.7, we however see that if the interest rate drops by 100 basis points, the market value with and without modelling of policyholder behaviour are identical, and we conclude that the effect on the market value of modelling policyholder behaviour is greatly influenced by the current market interest rate. The different structures of the cash flows also lead to significantly different interest rate sensitivities. The cash flow still begins at the premium level $-46{,}409$, but due to the large amount of surrenders and free policy conversions, it increases rapidly, in part because the premium stops, and in part because of the payment upon surrender. After retirement, the cash flow is significantly smaller than without policyholder modelling, in part due to surrender, in which case there are no life annuity, and in part due to free policy conversions reducing the size of the life annuity. In both the right part of Figure 2.7 and in Table 2.1, it is seen that the sensitivity of the market value with respect to the interest is reduced to about one third of the original sensitivity. Thus, for hedging the actual interest rate risk inherent in the cash flows, one sees that it is essential to model policyholder behaviour.

In this example, the results from the approximate and the correct method are almost identical. The cash flow and market value are slightly larger by using the approximation. We recall that the approximate method corresponds to allowing surrender and free policy conversion from the disabled state. The larger cash flow and market value are thus due to the fact that surrender and free policy conversion as disabled corresponds to the insured giving up the disability annuity in return for either a smaller technical reserve upon surrender, or a strictly smaller disability annuity upon free policy conversion. However, since the disability rate and probability are small, as seen in Figure 2.6, this error only has a minor effect on the cash flow.

Acknowledgements

We are grateful to Kristian Bjerre Schmidt for general comments and assistance with the numerical examples.

2.A Proofs

2.A.1 Cash flow for Section 2.3.2

We here prove the formula (2.3.4) for the model presented in Figure 2.3. Define first the duration $U(t)$ since entering the free policy state, that is

$$U(t) = \inf \{ s \geq 0 \,|\, Z(t-s) \in \{0,1,2\} \} .$$

Now, the payments in the setup lead to the payment process

$$\begin{aligned}
&\mathrm{d}B(t) \\
&= 1_{\{Z(t)=0\}}(b(t)-\pi(t))\mathrm{d}t + b_{\mathrm{ad}}(t)\mathrm{d}N_{\mathrm{ad}}(t) + (1-\kappa)\hat{V}(t)\mathrm{d}N_{\mathrm{as}}(t) \\
&\quad + \rho(t-U(t))\left(1_{\{Z(t)=3\}}b(t)\mathrm{d}t + b_{\mathrm{ad}}(t)\mathrm{d}N_{\mathrm{af,df}}(t) + (1-\kappa)\hat{V}^{+}(t)\mathrm{d}N_{\mathrm{af,sf}}(t)\right).
\end{aligned}$$

Here, N_{ad} is the counting process that counts the number of jumps from state *active* to state *dead*. Similarly, N_{as}, $N_{\mathrm{af,df}}$ and $N_{\mathrm{af,sf}}$ counts the number of jumps from state *active* to *surrender*, from state *active, free policy* to *dead, free policy*, and from state *active, free policy* to *surrender, free policy*, respectively.

The cash flow is then given as

$$\begin{aligned}
&\int_t^T \mathrm{d}A^{\mathrm{f}}(t,s) \\
&= \mathrm{E}\left[\int_t^T \mathrm{d}B(s) \,\middle|\, Z(t)=0\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left[\int_t^T 1_{\{Z(s)=0\}}(b(s)-\pi(s))\mathrm{d}s \,\middle|\, Z(t)=0\right] \\
&\quad + \mathrm{E}\left[\int_t^T b_{\mathrm{ad}}(s)\mathrm{d}N_{\mathrm{ad}}(s) + (1-\kappa)\hat{V}(s)\mathrm{d}N_{\mathrm{as}}(s) \,\middle|\, Z(t)=0\right] \\
&\quad + \mathrm{E}\left[\int_t^T \rho(s-U(s))1_{\{Z(s)=3\}}b(s)\mathrm{d}s \,\middle|\, Z(t)=0\right] \\
&\quad + \mathrm{E}\left[\int_t^T \rho(s-U(s))\left(b_{\mathrm{ad}}(s)\mathrm{d}N_{\mathrm{af,df}}(s) + (1-\kappa)\hat{V}^+(s)\mathrm{d}N_{\mathrm{af,sf}}(s)\right) \,\middle|\, Z(t)=0\right] \\
&= \int_t^T {}_{s-t}p_{x+t}^{\mathrm{fs}}(b(s)-\pi(s))\mathrm{d}s \\
&\quad + \int_t^T {}_{s-t}p_{x+t}^{\mathrm{fs}}\left(b_{\mathrm{ad}}(s)\mu_{\mathrm{ad}}(s) + (1-\kappa)\hat{V}(s)\mu_{\mathrm{as}}(s)\right)\mathrm{d}s \\
&\quad + \mathrm{E}\left[\int_t^T \rho(s-U(s))1_{\{Z(s)=3\}}b(s)\mathrm{d}s \,\middle|\, Z(t)=0\right] \\
&\quad + \mathrm{E}\left[\int_t^T \rho(s-U(s))\left(b_{\mathrm{ad}}(s)\mathrm{d}N_{\mathrm{af,df}}(s) + (1-\kappa)\hat{V}^+(s)\mathrm{d}N_{\mathrm{af,sf}}(s)\right) \,\middle|\, Z(t)=0\right].
\end{aligned}
$$

For the first expectation, we used that the expectation of an indicator function is a probability. For the second expectation, we recall that the counting process here can be replaced by the predictable compensator.

For the third expectation, we condition on the stochastic variable $s-U(s)$, which is the time of transition. Then, conditional on $Z(t)=0$, and with the indicator function $1_{\{Z(s)=3\}}$, we know that a transition has occured, thus $s-U(s)\in(t,s)$, which determine the integral limits. Also, the density of the time of the transition from state 0 to state 3 is $\tau\mapsto {}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)$. Using these observations, we calculate

$$
\begin{aligned}
&\int_t^T \mathrm{E}\left[\rho(s-U(s))1_{\{Z(s)=3\}}b(s) \,\middle|\, Z(t)=0\right]\mathrm{d}s \\
&= \int_t^T\int_t^s \mathrm{E}\left[\rho(\tau)1_{\{Z(s)=3\}}b(s) \,\middle|\, Z(t)=0, s-U(s)=\tau\right] \\
&\qquad \times \mathrm{d}P\left(s-U(s)\le\tau \,\middle|\, Z(t)=0\right)\mathrm{d}s \\
&= \int_t^T\int_t^s \rho(\tau)\,\mathrm{E}\left[1_{\{Z(s)=3\}} \,\middle|\, Z(\tau)=3\right] b(s)\,{}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\mathrm{d}\tau\mathrm{d}s \\
&= \int_t^T\int_t^s \rho(\tau)\,{}_{s-\tau}p_{x+\tau}^{\mathrm{s}}b(s)\,{}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\mathrm{d}\tau\mathrm{d}s \\
&= \int_t^T\int_t^s {}_{\tau-t}p_{x+t}^{\mathrm{fs}}\mu_{\mathrm{af}}(\tau)\rho(\tau)\,{}_{s-\tau}p_{x+\tau}^{\mathrm{s}}\mathrm{d}\tau\, b(s)\mathrm{d}s.
\end{aligned}
\tag{2.A.1}
$$

For the fourth expectation, we only consider the first part, since the second part is analogous. Since $U(s)$ is continuous whenever $N_{\mathrm{af,df}}(s)$ (and $N_{\mathrm{af,sf}}(s)$) increase in value, we can replace $U(s)$ by $U(s-)$. Using that $\rho(s-U(s-))$ is predictable, we can integrate with respect to the compensator of the counting process $N_{\mathrm{af,df}}(s)$ instead, so we get

$$\begin{aligned}
&\mathrm{E}\left[\int_t^T \rho(s-U(s))b_{\mathrm{ad}}(s)\mathrm{d}N_{\mathrm{af,df}}(s)\bigg|\, Z(t)=0\right] \\
&=\mathrm{E}\left[\int_t^T \rho(s-U(s-))b_{\mathrm{ad}}(s)\mathrm{d}N_{\mathrm{af,df}}(s)\bigg|\, Z(t)=0\right] \\
&=\mathrm{E}\left[\int_t^T \rho(s-U(s-))b_{\mathrm{ad}}(s)1_{\{Z(s-)=3\}}\mu_{\mathrm{ad}}(s)\mathrm{d}s\bigg|\, Z(t)=0\right] \\
&=\int_t^T \mathrm{E}\left[\rho(s-U(s-))1_{\{Z(s-)=3\}}\big|\, Z(t)=0\right] b_{\mathrm{ad}}(s)\mu_{\mathrm{ad}}(s)\mathrm{d}s \\
&=\int_t^T\int_t^s {}_{\tau-t}p^{\mathrm{fs}}_{x+t}\mu_{\mathrm{af}}(\tau)\rho(\tau)\,{}_{s-\tau}p^{\mathrm{s}}_{x+\tau}\mathrm{d}\tau\, b_{\mathrm{ad}}(s)\mu_{\mathrm{ad}}(s)\mathrm{d}s.
\end{aligned}$$

Since in the second last line, the expression is analogous to the third expectation, the last line was obtained using the same calculations as (2.A.1). Gathering the results, the cash flow $\mathrm{d}A^{\mathrm{f}}(t,s)$ is obtained.

2.A.2 Proof of equation (2.4.2)

Conditioning on the time of transition from state 0 to 4, $s-U(s)$, and using that the density for the transition time is $p_{i0}(t,\tau)\mu_{04}(\tau)$, we find

$$\begin{aligned}
&p^{\rho}_{ij}(t,s) \\
&=\mathrm{E}\left[1_{\{Z(s)=j\}}\rho(s-U(s))\big|\, Z(t)=i\right] \\
&=\int_t^s \mathrm{E}\left[1_{\{Z(s)=j\}}\rho(s-U(s))\big|\, Z(t)=i, s-U(s)=\tau\right]\mathrm{d}P\left(s-U(s)\le\tau\big|\, Z(t)=i\right) \\
&=\int_t^s \mathrm{E}\left[1_{\{Z(s)=j\}}\big|\, Z(t)=i, s-U(s)=\tau\right]\rho(\tau)p_{i0}(t,\tau)\mu_{04}(\tau)\mathrm{d}\tau \\
&=\int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)\,\mathrm{E}\left[1_{\{Z(s)=j\}}\big|\, Z(\tau)=4\right]\rho(\tau)\mathrm{d}\tau \\
&=\int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)p_{4j}(\tau,s)\rho(\tau)\mathrm{d}\tau.
\end{aligned}$$

At line five we used that if we know that $s-U(s)=\tau$, then in particular, we know that $Z(\tau)=4$. Since $\mathbf{Z}$ is Markov, we can then drop the condition that $Z(t)=i$ and $s-U(s)=\tau$. Note, that for the proof, it is essential that $i\in\{0,1,2\}$ and $j\in\{4,5,6\}$, since we at the conditioning on line three use that $s-U(s)\in(t,s)$, i.e. a transition to the free policy states occurs in the time interval (t,s). We can do that, since it must hold if $Z(t)=i\in\{0,1,2\}$ and $Z(s)=j\in\{4,5,6\}$.

2.A.3 Proof of Proposition 2.4.1

Proof. The cash flow is given as

$$\int_t^T \mathrm{d}A^{\mathrm{f}}(t,s) = \mathrm{E}\left[\int_t^T \mathrm{d}B(s)\middle| Z(t)=0\right],$$

where $B(t)$ is given in (2.4.1). Inserting $B(t)$ yields,

$$\begin{aligned}
&\int_t^T \mathrm{d}A^{\mathrm{f}}(t,s)\\
&=\int_t^T \mathrm{E}\left[\left(1_{\{Z(s)=0\}}b_0(s)+1_{\{Z(s)=1\}}b_1(s)\right)\middle| Z(t)=0\right]\mathrm{d}s &\text{(2.A.2)}\\
&\quad+\int_t^T \mathrm{E}\left[\rho(s-U(s))\left(1_{\{Z(s)=4\}}b_0(s)^{+}+1_{\{Z(s)=5\}}b_1(s)\right)\middle| Z(t)=0\right]\mathrm{d}s &\text{(2.A.3)}\\
&\quad+\mathrm{E}\left[\int_t^T \Big(b_{01}(s)\mathrm{d}N_{01}(s)+b_{02}(s)\mathrm{d}N_{02}(s)+b_{12}(s)\mathrm{d}N_{12}(s)\right.\\
&\qquad\left.+(1-\kappa)\hat{V}(s)\mathrm{d}N_{03}(s)\Big)\middle| Z(t)=0\right] &\text{(2.A.4)}\\
&\quad+\mathrm{E}\left[\int_t^T \rho(s-U(s))\Big(b_{01}(s)\mathrm{d}N_{45}(s)+b_{02}(s)\mathrm{d}N_{46}(s)+b_{12}(s)\mathrm{d}N_{56}(s)\right.\\
&\qquad\left.+(1-\kappa)\hat{V}^{+}(s)\mathrm{d}N_{47}(s)\Big)\middle| Z(t)=0\right] &\text{(2.A.5)}
\end{aligned}$$

The four expectations (2.A.2) – (2.A.5) are calculated separately. The first expectation (2.A.2) is the expectation of indicator functions, and we replace by the transition probabilities,

$$\int_t^T \left(p_{00}(t,s)b_0(t)+p_{01}(t,s)b_1(t)\right)\mathrm{d}s.$$

In the second expectation (2.A.3), the same calculations as in Section 2.A.2 can be performed to obtain,

$$\int_t^T \left(p^{\rho}_{04}(t,s)b_0(s)^{+}+p^{\rho}_{05}(t,s)b_1(s)\right)\mathrm{d}s.$$

In the third expectation (2.A.4), we integrate deterministic functions with respect to a counting process. Taking the expectation, we can instead integrate with respect to the predictable compensator, and we get,

$$\begin{aligned}&\int_t^T \mathrm{E}\Big[1_{\{Z(s-)=0\}}\left(b_{01}(s)\mu_{01}(s)+b_{02}(s)\mu_{02}(s)\right)+1_{\{Z(s-)=1\}}b_{12}(s)\mu_{12}(s)\\&\quad+1_{\{Z(s-)=0\}}(1-\kappa)\hat{V}(s)\mu_{03}(s)\mathrm{d}s\Big|Z(t)=0\Big]\mathrm{d}s\\&=\int_t^T\Big(p_{00}(t,s)\left(b_{01}(s)\mu_{01}(s)+b_{02}(s)\mu_{02}(s)\right)+p_{01}(t,s)b_{12}(s)\mu_{12}(s)\\&\qquad+p_{00}(t,s)(1-\kappa)\hat{V}(s)\mu_{03}(s)\Big)\mathrm{d}s.\end{aligned}$$

For the fourth expectation (2.A.5), we start by replacing $U(s)$ with $U(s-)$, since whenever any of $N_{45}(s)$, $N_{46}(s)$, $N_{56}(s)$ or $N_{47}(s)$ are increasing, then $U(s)$ is continuous. Thus, we integrate a predictable process with respect to a counting process, and we can integrate with respect to the predictable compensator instead,

$$\begin{aligned}&\mathrm{E}\Big[\int_t^T\rho(s-U(s-))\Big(b_{01}(s)\mathrm{d}N_{45}(s)+b_{02}(s)\mathrm{d}N_{46}(s)\\&\quad+b_{12}(s)\mathrm{d}N_{56}(s)+(1-\kappa)\hat{V}^{+}(s)\mathrm{d}N_{47}(s)\Big)\Big|Z(t)=0\Big]\\&=\int_t^T\mathrm{E}\Big[\rho(s-U(s-))\Big(1_{\{Z(s-)=4\}}\left(b_{01}(s)\mu_{45}(s)+b_{02}(s)\mu_{46}(s)\right)\\&\quad+1_{\{Z(s-)=5\}}b_{12}(s)\mu_{56}(s)+1_{\{Z(s-)=4\}}(1-\kappa)\hat{V}^{+}(s)\mu_{47}(s)\Big)\Big|Z(t)=0\Big]\mathrm{d}s\\&=\int_t^T\Big(p_{04}^{\rho}(t,s)\left(b_{01}(s)\mu_{45}(s)+b_{02}(s)\mu_{46}(s)\right)\\&\quad+p_{05}^{\rho}(t,s)b_{12}(s)\mu_{56}(s)+p_{04}^{\rho}(t,s)(1-\kappa)\hat{V}^{+}(s)\mu_{47}(s)\Big)\mathrm{d}s.\end{aligned}$$

For the last equality, we again used the calculations from Section 2.A.2. Gathering the four expectations, the result is obtained. □

2.A.4 Proof of Proposition 2.4.2

Proof. We differentiate $p_{ij}^{\rho}(t,s)$ for $i\in\{0,1,2\}$ and $j\in\{4,5,6\}$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}p_{ij}^{\rho}(t,s)\\&=\frac{\mathrm{d}}{\mathrm{d}s}\int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)p_{4j}(\tau,s)\rho(\tau)\mathrm{d}\tau\\&=p_{i0}(t,s)\mu_{04}(s)p_{4j}(s,s)\rho(s)+\int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)\frac{\mathrm{d}}{\mathrm{d}s}p_{4j}(\tau,s)\rho(\tau)\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&= 1_{\{j=4\}} p_{i0}(t,s)\mu_{04}(s)\rho(s) \\
&\quad + \int_t^s p_{i0}(t,\tau)\mu_{04}(\tau)\left(-p_{4j}(\tau,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\{4,5,6\}\\ \ell\neq j}} p_{4\ell}(\tau,s)\mu_{\ell j}(s)\right)\rho(\tau)\mathrm{d}\tau \\
&= 1_{\{j=4\}} p_{i0}(t,s)\mu_{04}(s)\rho(s) - p_{ij}^{\rho}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\{4,5,6\}\\ \ell\neq j}} p_{i\ell}^{\rho}(t,s)\mu_{\ell j}(s).
\end{aligned}
$$

For the third equality sign, we used that $p_{4\ell}(\tau,s)\mu_{\ell j}(s) = 0$ for $\ell \notin \{4,5,6\}$. $\square$

Chapter 3

Cash flows and policyholder behaviour in the semi-Markov life insurance setup

This chapter is based on the paper [11], written jointly with T. Møller and K. B. Schmidt.

Abstract

Within the setup of a semi-Markov process in a finite state space, we consider a life insurance contract. First, without the modelling of policyholder behaviour, we show how to calculate the expected cash flow associated with future payments, and to that end we present a version of Kolmogorov's forward integro-differential equation. The semi-Markov model is then extended to include modelling of surrender and free policy behaviour, and the main result is a modification of Kolmogorov's foward integro-differential equation, such that the cash flow can be calculated without significantly more complexity than the cash flow without policyholder modelling. The result is also demonstrated for the traditional Markov case where there is no duration dependence, and numerical examples are studied.

3.1 Introduction

In this paper we consider the problem of valuation of prospective reserves and expected cash flows for life insurance liabilities, including the modelling of policyholder behaviour. The setup consists of a semi-Markov process for the state of the insured in a multi-state model, and the special case of a Markov process is considered as well. When we include policyholder behaviour in the model, in particular the so-called free policy option[1], an extra duration dependence arises, such that we have a double duration setup. The main result of the paper is that we can effectively eliminate the extra duration. Thus, we simply have to solve a modified Kolmogorov forward integro-differential equation, which in complexity is equal to the differential equation without policyholder behaviour.

The setup of a semi-Markov model allows for dependence on the duration in the current state. Thus, the transition rates can be dependent on the duration, and the payment functions may also be dependent on the duration. An example of the presence of duration dependence in the transition rates is within a disability model, where a 50 year old disabled person might have a higher recovery transition rate if disability occured half a year ago than if disability occured 10 years ago, and hence the transition rate is not only age-dependent. For empirical evidence, see [44] and [24], wherein the recovery rate and the mortality rate for disabled are shown to be dependent on time since disability as well as age. An example of duration dependence in the payment functions is a disability annuity where one can model a 3 month waiting period after disability before the annuity starts. If only the payment functions depend on the duration, the semi-Markov process simplifies to a Markov process, however, in order to valuate the prospective reserve and/or the expected cash flow, one needs the duration dependent transition probabilities, and thus the duration dependent formulae are needed.

In terms of understanding the characteristics of the life insurance liabilities, in particular the future cash flow, it is of importance to take policyholder behaviour into account. This aspect is also given considerable attention from a Solvency 2 perspective, where life insurers are required to take into account policyholder behaviour when valuating the liabilities, see Section 3.5 in [12]. One aim of this paper is to show how the semi-Markov setup can be used to model policyholder behaviour, and how to efficiently calculate cash flows.

In the first part of the paper, Sections 2 and 3, we give an overview of current results for the multi-state semi-Markov life insurance setup, including a version of Kolmogorov's backward and forward differential equations, which does not seem to be well known in the semi-Markov setup. Semi-Markov models in life insurance were first introduced in [28], and later treatments include [36] and [13]. For a more theoretical treatment of the topic, see also [25], who first introduced the approach based on cumulative transition rates for the multi-state semi-Markov life insurance setup. We define the deterministic cash flow associated with the random future payments as the expectation of the future payments, and show how to efficiently valuate it using Kolmogorov's forward integro-differential equations. We specialise to the Markov case where there is no duration dependence, and

[1] "Free policy" is sometimes referred to as "paid-up policy" in the literature.

recover the classic results without duration dependence, which can be found in e.g. [39] or [31].

In the second part of the paper, Section 4, the semi-Markov model is extended by including policyholder behaviour in the form of a surrender and a free policy option. These are modelled by specifying transitions in an extended state space of the semi-Markov model, corresponding to exercises of the options. The surrender option is a right of the policyholder to cancel the contract, and receive an account value calculated on a technical basis. The free policy option is a right of the policyholder to cancel future payments and let the contract continue as a policy with no premiums and with reduced benefits, where the reduction is calculated on a technical basis. An exercise of any of these options thus changes future payments. While the modelling of these options has an effect on the prospective reserve, it is of even greater importance when considering the structure of future cash flows, and the interest rate sensitivity. In this paper, the exercise of the options occurs randomly, and a similar approach is taken in [26] where the insurance risk and policyholder behaviour is modelled by two separate Markov chains, possibly dependent. Earlier studies of random policyholder behaviour modelling include [34] and [35]. In contrast to this is the approach where the surrender occurs rationally, which is studied in [45]. For a comparison and overview of these approaches, see [38]. Attempts to couple rational and random behaviour are done in [23] and [8], where the common approach is that the level of random exercises is dependent on certain rational indicators.

In the extended semi-Markov model, the free policy option introduces a dependence in the payment functions on the duration since the free policy transition. Thus the transition probabilities needed for valuation are dependent on two durations, and we say that we have a setup with a double duration dependence. We show how the dependence on the duration since the free policy transition can be effectively eliminated, which is the main result of this paper. With this result, the evaluation of the prospective reserve and the cash flow can be carried out using a modified version of Kolmogorov's forward integro-differential equation, corresponding to the computational complexity of the original semi-Markov model with one duration. For a portfolio of life insurance contracts, this reduction, measured in time usage, is considerable.

We also present the results for the Markov case, which is the special case of the semi-Markov setup where there is no dependence on the duration. The extension to include policyholder behaviour is shown, and like in the semi-Markov setup, the free policy modelling introduces a dependence on the duration since the free policy transition. Thus, the Markov process becomes a semi-Markov process when the free policy option is included. As in the semi-Markov case, our result effectively eliminates this extra dependence on the duration, and a modification of Kolmogorov's forward differential equation makes it computationally simple to calculate cash flows, effectively as in a Markov model.

We conclude the paper by giving numerical results illustrating the cash flows for a simple life insurance policy. We show how the cash flows change due to policyholder modelling, and highlight that the dollar duration changes significantly, even though the prospective reserve might not change a lot. This is of particular interest if the cash flows are used for the hedging of interest rate risk, for example by duration-matching.

3.2 Setup

We consider the classic life insurance setup with a stochastic process $\mathbf{Z}$ in a finite state space $\mathcal{J}$, denoting the state of the insured, see e.g. [27] and [39], where the process $\mathbf{Z}$ is a Markov chain. In this paper, we let $\mathbf{Z}$ be a semi-Markov process as in [13] and [36]. To each state and each transition between states of $\mathbf{Z}$, we attach payments.

3.2.1 The semi-Markov model

We consider a semi-Markov process $\mathbf{Z} = (Z(t))_{t\geq 0}$ in a state space $\mathcal{J} = \{0, 1, \dots, J-1\}$ with J states, and let $Z(0) = 0$. Let $\mathbf{U} = (U(t))_{t\geq 0}$ be defined as the duration,

$$U(t) = \sup\{\, s \in [0,t] |\, Z(\tau) = Z(t), \tau \in [t-s, t]\},$$

which implies that $U(t)$ is the time $Z(t)$ has spent in its current state. We assume that $(\mathbf{Z}, \mathbf{U})$ is a Markov process, thus $\mathbf{Z}$ is a semi-Markov process. The processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration

$$\mathcal{F}(t) = \sigma(\, Z(s), U(s) |\, s \leq t) = \sigma(\, Z(s) |\, s \leq t),$$

with equality because $\mathbf{U}$ is constructed from $\mathbf{Z}$.

We define the transition probabilities for going from state i at time s with duration u to state j at time t with duration less than z,

$$p_{ij}\,(s,t,u,z) = P\,(\, Z(t) = j, U(t) \leq z |\, Z(s) = i, U(s) = u)\,, \tag{3.2.1}$$

and, for $F \subset \mathcal{J}$ and $G \in \mathcal{B}(\mathbb{R})$, we let

$$p_{iF}\,(s,t,u,G) = \sum_{j\in F} \int_G p_{ij}(s,t,u,\mathrm{d}z)$$

denote the probability measure of the distribution of $(Z(t), U(t))$ given $Z(s) = i$ and $U(s) = u$. This measure has finite support: With initial duration u at time s, the maximum duration at time t is $u+t-s$. Thus, the support of the measure is a subset of $\mathcal{J} \times [0, u+t-s]$, and we frequently use $(0, u+t-s]$ as integration bounds.

The transition rates are, for $i, j \in \mathcal{J}, i \neq j$ and $t, u \geq 0$, defined as

$$\begin{aligned} \mu_{ij}(t,u) &= \lim_{h \searrow 0} \frac{1}{h} p_{ij}(t, t+h, u, \infty), \\ \mu_{i\cdot}(t,u) &= \sum_{\substack{j \in \mathcal{J} \\ j \neq i}} \mu_{ij}(t,u), \end{aligned} \tag{3.2.2}$$

and throughout this paper we assume that they exist, and that they are continuous in t and u. We see later, in Proposition 3.2.4, that the transition rates determine the distribution of $\mathbf{Z}$.

Define the counting processes[2]

$$N_{ij}(t) = \#\left\{ s \in [0,t] \,|\, Z(s) = j, Z(s-) = i \right\}.$$

Then $N_{ij}(t)$ counts the number of jumps from state i to state j in the interval $[0,t]$. The counting process $N_{ij}(t)$ has intensity $1_{\{Z(t-)=i\}}\mu_{ij}(t, U(t-))$, and the process $\mathbf{M}_{ij}$ defined by

$$M_{ij}(t) = N_{ij}(t) - \int_0^t 1_{\{Z(s-)=i\}}\mu_{ij}(s, U(s-))\mathrm{d}s,$$

is a martingale. Using this, then for any predictable process $H(t)$, we obtain

$$\begin{aligned} \mathrm{E}\left[\int_0^t H(s)\mathrm{d}N_{ij}(s)\right] &= \mathrm{E}\left[\int_0^t H(s)1_{\{Z(s-)=i\}}\mu_{ij}(s, U(s-))\mathrm{d}s\right] \\ &= \int_0^t \int_0^s \mathrm{E}\left[H(s) | Z(s-) = i, U(s-) = z\right] p_{0i}(0, s, 0, \mathrm{d}z)\, \mu_{ij}(s,z)\mathrm{d}s. \end{aligned} \tag{3.2.3}$$

For more details on results of this type, see e.g. [5]. Throughout the paper we use the convention $\int_a^b = \int_{(a,b]}$ and $\int_a^\infty = \int_{(a,\infty)}$, such that the left endpoint is excluded from the integration range.

3.2.2 Life insurance payments

We model the payments associated with a life insurance contract by letting the semi-Markov process $\mathbf{Z}$ denote the state of the insured. To each state i, we associate a continuous payment rate $b_i(t,u)$ and single payments $\Delta B_i(t,u)$ at time t with duration u. Also, for each transition from i to j, we associate a single payment $b_{ij}(t,u)$ at time t

[2] The notation $f(x-) = \lim_{y \nearrow x} f(y)$ is used.

when the duration in the state i was u. We write the total payments at time t as $B(t)$, and the payment process $\mathbf{B}$ then satisfies

$$\begin{aligned} \mathrm{d}B(t) &= \sum_{i\in\mathcal{J}} 1_{\{Z(t)=i\}} \mathrm{d}B_i(t,U(t)) + \sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} b_{ij}(t,U(t-))\mathrm{d}N_{ij}(t), \\ \mathrm{d}B_i(t,U(t)) &= b_i(t,U(t))\mathrm{d}t + \Delta B_i(t,U(t)), \end{aligned} \tag{3.2.4}$$

where $\mathrm{d}B_i(t,u)$ are the payments in state i at time t if the duration is u. The functions b_i, ΔB_i and b_{ij} are assumed to be deterministic and piecewise continuous.

3.2.3 Interest rate

Let $r(t)$ denote the continuously compounded interest rate. We assume that $r(t)$ is deterministic, however, the results of this paper can easily be obtained with a stochastic short rate instead, if it is independent of $\mathbf{Z}$. In particular, the main results are about the semi-Markov chain $\mathbf{Z}$ and transition probabilities, and are independent of the interest rate.

3.2.4 Cash flows and valuation

For the balance sheet, one calculates the expected present value of future payments with appropriate interest rate assumptions. This is denoted the prospective reserve.

Definition 3.2.1. *The prospective reserve, $V_{i,u}(t)$, at time t of the future payments of $\mathbf{B}$, given that we are in state i with duration u, is*

$$V_{i,u}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}B(s) \,\middle|\, Z(t)=i, U(t)=u \right].$$

Using calculations similar to (3.2.3) and the linearity of conditional expectations yield the well known result

$$\begin{aligned} &V_{i,u}(t) \\ &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \sum_{j\in\mathcal{J}} \int_0^{u+s-t} p_{ij}\left(t,s,u,\mathrm{d}z\right) \left(\mathrm{d}B_j(s,z) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}(s,z) b_{jk}(s,z)\mathrm{d}s \right), \end{aligned} \tag{3.2.5}$$

for more details, see e.g. [28].

The payment at a future time s can formally be written as $\mathrm{d}B(s)$, and we say that $\mathrm{d}B(s)$ is the stochastic cash flow at time s. Since it is stochastic, it is not very convenient in practice, and one can be interested in a deterministic version. This can be obtained by taking the expectation of $\mathbf{B}$, thus considering the expected cash flow. For example, if one has a lot of identical and independent life insurance contracts, the total payments are, by the law of large numbers, close to the expected payments. We define the expected cash flow, and simply refer to it as the cash flow.

Definition 3.2.2. *The* cash flow *of a payment process* $\mathbf{B}$ *conditional on* $(Z(t) = i, U(t) = u)$ *is the payments of the deterministic payment process* $(A_{i,u}(t,s))_{s\geq t}$*, defined by*

$$A_{i,u}(t,s) = \mathrm{E}\left[B(s) - B(t) \middle| Z(t) = i, U(t) = u\right].$$

Using the cash flow process, the expected payments at a future time s given $(Z(t) = i, U(t) = u)$ can be written as $\mathrm{d}A_{i,u}(t,s) = A_{i,u}(t,\mathrm{d}s)$. The cash flows can be used to find the prospective reserve.

Proposition 3.2.3. *The cash flow* $(A_{i,u}(t,s))_{s\geq t}$ *satisfies*

$$V_{i,u}(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}A_{i,u}(t,s),$$

$$\mathrm{d}A_{i,u}(t,s) = \sum_{j\in\mathcal{J}}\int_0^{u+s-t} p_{ij}\left(t,s,u,\mathrm{d}z\right)\left(\mathrm{d}B_j(s,z) + \sum_{\substack{k\in\mathcal{J}\\k\neq j}}\mu_{jk}(s,z)b_{jk}(s,z)\mathrm{d}s\right).$$

Proof. Let $t \geq 0$. Define $\widetilde{B}_t(s) = B(s) - B(t)$ and note that Definition 3.2.1 can be written as

$$V_{i,u}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}\widetilde{B}_t(s)\middle| Z(t) = i, U(t) = u\right].$$

Let $F_t(s) = e^{-\int_t^s r(\tau)\mathrm{d}\tau}$, and use integration by parts to obtain,

$$\int_t^\infty F_t(s)\mathrm{d}\widetilde{B}_t(s) = F_t(\infty)\widetilde{B}_t(\infty) - F_t(t)\widetilde{B}_t(t) - \int_t^\infty \widetilde{B}_t(s)\mathrm{d}F_t(s).$$

Applying conditional expectation on both sides, conditioning on $(Z(t) = i, U(t) = u)$, yields

$$V_{i,u}(t) = F_t(\infty)A_{i,u}(t,\infty) - F_t(t)A_{i,u}(t,t) - \int_t^\infty A_{i,u}(t,s)\mathrm{d}F_t(s),$$

and the first result follows by integration by parts on the right hand side. The second result follows immediately from the first result and equation (3.2.5). □

3.2.5 Kolmogorov's backward differential equation

Kolmogorov's backward differential equation provides a way to calculate the transition probabilities given that the transition rates are specified. Here, we present a version for the semi-Markov chain, which can be found as a partial differential equation in [13] and in [25]. We present the differential equation as an ordinary differential equation. The differential equation can also be obtained from a similar integral equation version found in [28], by differentiation and some further calculations.

Proposition 3.2.4. (Kolmogorov's backward differential equation) *Let $0 \leq t_0 \leq t$ and $d, u \geq 0$ and $j \in \mathcal{J}$. Define $D(s) = d + s - t_0$. The transition probabilities $p_{ij}(t_0, t, d, u)$, for $i \in \mathcal{J}$, satisfy the system of differential equations given by,*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(s, t, D(s), u) &= \mu_{i\cdot}(s, D(s)) p_{ij}(s, t, D(s), u) \\ &\quad - \sum_{\substack{k \in \mathcal{J} \\ k \neq i}} \mu_{ik}(s, D(s)) p_{kj}(s, t, 0, u), \\ p_{ij}(t, t, D(t), u) &= 1_{\{i=j\}} 1_{\{D(t) \leq u\}}. \end{aligned} \tag{3.2.6}$$

From Kolmogorov's backward differential equations, we see in particular that the transition rates determine the distribution of $Z(t)$.

Solving Kolmogorov's backward differential equation, (3.2.6), is done for fixed $j \in \mathcal{J}$, $t_0 < t$ and $d, u \geq 0$ and yields the probabilities $(p_{ij}(t_0, t, d, u))_{i \in \mathcal{J}}$. We solve the system of differential equations (3.2.6), starting with boundary conditions $p_{ij}(t, t, d + t - t_0, u)$, $i \in \mathcal{J}$ and solving backwards to time t_0, corresponding to the blue line in Figure 3.1. This is not straightforward, since the quantities $(p_{ij}(s, t, 0, u))_{i \in \mathcal{J}}$ for $s \in (t_0, t)$ are needed, corresponding to the values on the red line in Figure 3.1. Thus, these values need to be calculated first. For fixed $j \in \mathcal{J}$, $t_0 \leq t$ and $d, u \geq 0$, we outline an algorithm for calculating the probabilities $(p_{ij}(t_0, t, d, u))_{i \in \mathcal{J}}$, with a numerical procedure using step size h, corresponding to a grid with step size h. We assume that t_0, d and t, u lie on the grid, in particular that there is $N \in \mathbb{N}$ such that $t = t_0 + Nh$.

1. With boundary condition $p_{ij}(t, t, h, u)$, calculate $p_{ij}(t - h, t, 0, u)$ for each $i \in \mathcal{J}$. Note, that the numerical solution is calculated in one step, so no values $(p_{ij}(s, t, 0, u))_{i \in \mathcal{J}}$ are needed for $s \in (t - h, t)$.

2. With boundary condition $p_{ij}(t, t, 2h, u)$, calculate $p_{ij}(t - 2h, t, 0, u)$. Here, the numerical algorithm requires two steps, where the values $(p_{ij}(t - h, t, 0, u))_{i \in \mathcal{J}}$ are used in the middle step.

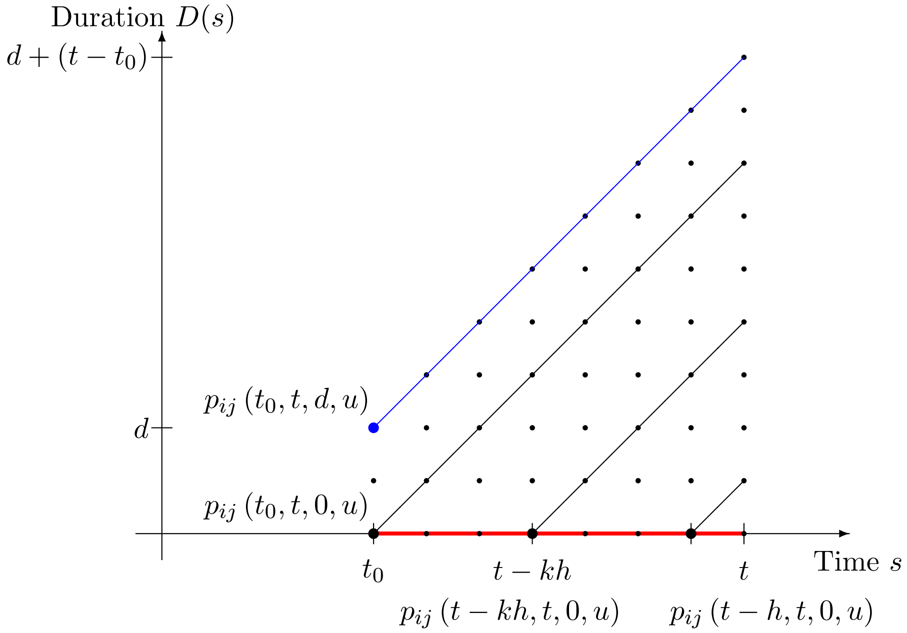

Figure 3.1: *Sketch of algorithm to calculate the transition probabilities* $p_{ij}(t_0, t, d, u)$ *for all* $i \in \mathcal{J}$ *using Kolmogorov's backward differential equation, using a numerical algorithm with step size* h*, matching the drawn grid. First, all values* $p_{ij}(s, t, 0, u)$*,* $s \in (t_0, t)$ *on the red line are calculated, and then the differential equation along the blue line can be calculated, yielding* $p_{ij}(t_0, t, d, u)$*.*

3. Continue calculating $(p_{ij}(t - nh, t, 0, u))_{i \in \mathcal{J}}$ for $n = 3, 4, \ldots$ as long as $t - nh \geq t_0$. In this way, we obtain the values $p_{ij}(s, t, 0, u)$ needed, that is, $i \in \mathcal{J}$ and $s \in [t_0, t)$, i.e. the red line on the first axis in Figure 3.1.

4. Finish with boundary condition $p_{ij}(t, t, d + t - t_0, u)$ and calculate $p_{ij}(t_0, t, d, u)$, where all the previously calculated values are used. Note, that if $d = 0$, this value was already calculated above with n such that $t - nh = t_0$.

Now, we have calculated $p_{ij}(t_0, t, d, u)$ for $i \in \mathcal{J}$ and fixed j, t and u. This algorithm must be repeated for each $j \in \mathcal{J}$ and $u \geq 0$, in order to get the distribution of $(Z(t), U(t))$ conditional on $Z(t_0) = i, U(t_0) = d$. Doing this for all $t \geq t_0$ as well, the cash flow in Proposition 3.2.3 can be calculated.

If we count the number of calculation steps required to calculate the transition probabilities needed for the cash flow, we find the time complexity to be of order $\mathcal{O}(N^4)$. Intuitively, N to the power of 4 is: one power for calculating one line, one power for all the lines, one power for all terminal durations, and one power for all terminal times. We see later that it can be of advantage to use Kolmogorov's forward integro-differential equation, which is presented below in Theorem 3.3.1 and is of order $\mathcal{O}(N^2)$. It should be noted, that it might be possible to optimise the backward algorithm such that it is $\mathcal{O}(N^3)$, by reusing results between the solutions for different terminal durations $u \geq 0$.

3.2.6 The Markov case

We consider the special case, where the transition rates do not depend on the duration, i.e. where $\mathbf{Z}$ is a Markov process. If also the payments do not depend on the duration, the duration dependent probabilities are not needed, and the formulae simplify. This corresponds to the classic setup, see e.g. [39] or [31]. For the transition probabilities, we write

$$p_{ij}(t,s) = P\left(Z(s) = j \middle| Z(t) = i \right).$$

Since nothing depends on the duration, we remove the dependency in the notation. Restating Proposition 3.2.3 in the Markov case, we get the cash flow and prospective reserve,

$$\begin{aligned} \mathrm{d}A_i(t,s) &= \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(\mathrm{d}B_j(s) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \mu_{jk}(s) b_{jk}(s) \mathrm{d}s \right), \qquad (3.2.7) \\ V_i(t) &= \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau} \mathrm{d}A_i(t,s). \end{aligned}$$

We now consider Proposition 3.2.4 in the Markov case, and obtain *Kolmogorov's backward differential equation for the Markov case*,

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(s,t) = \mu_{i\cdot}(s) p_{ij}(s,t) - \sum_{\substack{k \in \mathcal{J} \\ k \neq i}} \mu_{ik}(s) p_{kj}(s,t),$$

with boundary conditions $p_{ij}(t,t) = 1_{\{i=j\}}$.

When calculating the cash flow $(A_i(t_0,s))_{s \geq t_0}$ for fixed t_0, the transition probabilities $p_{ij}(t_0,s)$ need to be calculated for fixed t_0 and $s \geq t_0$. This can be done using Kolmogorov's backward differential equation. However, then one needs to solve the differential equation for each value of s, and one ends up with all the values $(p_{ij}(t,s))_{t \in (t_0,s)}$ for all $s \geq t_0$. It is then an advantage to use Kolmogorov's forward differential equation instead, which is a differential equation in s instead of t. This yields all the needed transition probabilities immediately.

Proposition 3.2.5. (Kolmogorov's forward differential equation for the Markov case) *Let $0 \leq t_0 \leq t$. If the transition rates do not depend on the duration, the transition* *probabilities* $p_{ij}(t_0, t)$ *satisfy the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t_0, s) = -p_{ij}(t_0, s)\mu_{j\cdot}(s) + \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} p_{i\ell}(t_0, s)\mu_{\ell j}(s),$$
$$p_{ij}(t_0, t_0) = 1_{\{i=j\}}.$$

By using Kolmogorov's forward differential equation, the problem is, formally speaking, reduced by 1 dimension. In the next section, we present a generalised version of Kolmogorov's forward differential equation for the semi-Markov case, which does not seem to be well-known in the insurance literature.

3.3 Kolmogorov's forward integro-differential equation

In this Section we present a version of Kolmogorov's forward differential equation for the semi-Markov case, which seems less known than the differential equations for the Markov case presented above and which can also be found in textbooks on life insurance mathematics, see for example [31]. In the semi-Markov case, Kolmogorov's forward differential equation is an ordinary integro-differential equation, and it is similar to the partial differential equation obtained by [25], Corollary 2.37. The result can also be obtained from a similar integral equation presented in [28], by differentiation and some further calculations. As discussed in the previous section in the Markov case, using Kolmogorov's forward differential equation is preferable when calculating cash flows, since this leads directly to the necessary transition probability measures.

Theorem 3.3.1. (Kolmogorov's forward integro-differential equation) *Let* $0 \leq t_0 \leq t$ *and* $u \geq 0$ *and* $i \in \mathcal{J}$. *The transition probabilities*

$$p_{ij}\left(t_0, t, u, d + t - t_0\right), \qquad \text{for } j \in \mathcal{J} \text{ and } d \in \mathbb{R} \text{ s.t. } d + t - t_0 \geq 0,$$

satisfy, with $D(s) = d + s - t_0$, *the system of integro-differential equations given by*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} p_{ij}\left(t_0, s, u, D(s)\right) = &- \int_0^{D(s)} p_{ij}\left(t_0, s, u, \mathrm{d}z\right) \mu_{j\cdot}(s, z) \\ &+ \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} \int_0^{u+s-t_0} p_{i\ell}\left(t_0, s, u, \mathrm{d}z\right) \mu_{\ell j}(s, z), \end{aligned} \tag{3.3.1}$$

with boundary conditions $p_{ij}\left(t_0, t_0, u, d\right) = 1_{\{i=j\}} 1_{\{u \leq d\}}$ *and, for* $s > t_0$, $p_{ij}\left(t_0, s, u, 0\right) = 0$.

For completeness we give a simple proof based on the Chapman-Kolmogorov equations and Kolmogorov's backward differential equations. The proof is presented in Appendix 3.A.

The terms in the differential equation have a straightforward interpretation. The term on the first line of (3.3.1) corresponds to the transitions out of state j at time s: It is the probability of being in state j at time s, and then leaving state j at time s. The transition rate out of state j is duration dependent, and thus the integral over the duration appears. We only integrate up to the current duration $D(s) = d + s - t_0$, as it is only probability mass with duration less than $d + s - t_0$ that is to be deducted. The term on the second line of (3.3.1) corresponds to new entries into state j at time s. First, it is the probability of being in state ℓ at time s, where $\ell \neq j$, and then at time s transitioning from state ℓ to state j. Again, this transition rate is duration dependent, and the integral appears. Here, we integrate up to the maximum duration $u + s - t_0$.

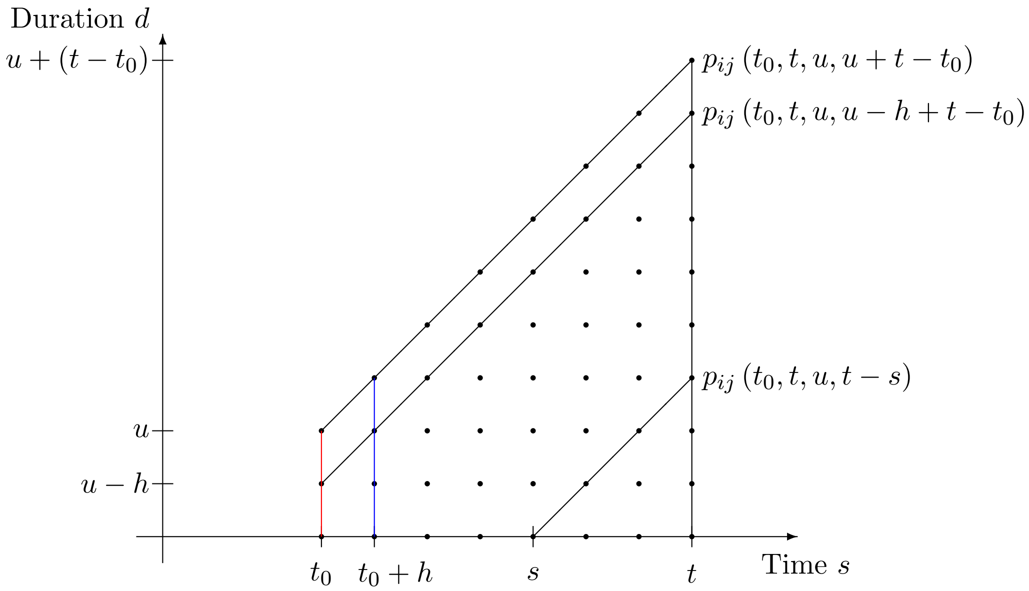

Figure 3.1: *Sketch of algorithm to calculate the transition probabilities* $\left(p_{ij}\left(t_0, s, u, d\right)\right)_{j\in\mathcal{J},\, s\in[t_0,t],\, d\geq 0}$ *using Kolmogorov's forward integro-differential equation, using a numerical algorithm with step size h, matching the drawn grid. The distribution along the red line is given by the boundary conditions, and then the integro-differential equation is used to shift the distribution to the right, thus the distribution along the blue line is reached in 1 step, and continuing yields the distribution along the vertical black line.*

We illustrate how to calculate $p_{ij}\left(t_0, s, u, d\right)$ for all $j \in \mathcal{J}$, $s \in [t_0, t]$ and $d \in [0, u + s - t_0]$ where $t_0 \leq t$ and $u \geq 0$ and $i \in \mathcal{J}$ are fixed. Assume a grid of size h according to Figure 3.1, and that t_0, t and u lie on the grid. We discretise the integral over the duration d using this grid as well.

1. At time t_0, the boundary conditions yield the values $p_{ij}(t_0, t_0, u, d)$ for all j and $d \in [0, u]$, corresponding to the distribution along the red line on Figure 3.1.

2. The differential equation (3.3.1) can now be used to calculate the probabilities $p_{ij}(t_0, t_0 + h, u, d + h)$, for all $j \in \mathcal{J}$ and $d \in (-h, u]$. Together with the new boundary condition at time $t_0 + h$ for duration 0, $p_{ij}(t_0, t_0 + h, u, 0)$, this yields the distribution along the blue line on Figure 3.1.

   In this step, note that the integral on the second line of (3.3.1) is independent of d, so the integral only needs to be solved once in this step. Also, the value of the integral on the first line can be reused for different durations: If the value is known for $d + s - t_0$, then for $d + h + s - t_0$, only the increment of the integral needs to be calculated.

3. Repeat this, calculating $p_{ij}(t_0, t_0 + nh, u, d + nh)$ for all $j \in \mathcal{J}$ and $d \in (-nh, u]$, for $n = 2, 3, \ldots$, until $t_0 + nh = t$ and the distribution along the vertical black line in Figure 3.1 is reached.

Note that the number of differential equations in the differential equation system is increased by one at each step, corresponding to the new diagonal line beginning at duration 0. The algorithm immediately yields all needed probabilities to calculate the cash flow in Proposition 3.2.3.

Compared to using the algorithm based on Kolmogorov's backward differential equation, as sketched in Figure 3.1, this is significantly simpler. The time complexity of solving Kolmogorov's forward integro-differential equations as described above, with the optimizations in step 2, is $\mathcal{O}(N^2)$. Intuitively, N to the power of two is: one power for the time, and one power for the calculations needed for each time point (the two integrals and the advancement of the differential equation for each duration). This is simpler than the time complexity of $\mathcal{O}(N^4)$ when using Kolmogorov's backward differential equations for calculating cash flows.

3.4 Modelling of policyholder behaviour

We consider a life insurance contract in the above setup without modelling of policyholder behaviour, and show how to extend it to include policyholder behaviour modelling. Our modelling of the options as random transitions in a semi-Markov model implies that they are not necessarily exercised rationally, but instead randomly. Rational surrender is studied in [45], and attempts to couple rational and random behaviour are carried out in [23] and [8]. In this paper, we carry on with the modelling of policyholder behaviour as occuring randomly, and this corresponds to a large extent to the behaviour observed in practice.

The prospective reserve and the cash flow are calculated on different bases. First, we consider calculations on the so-called *technical basis*, with separate (and typically conservative) interest and transition rates, in a state space $\mathcal{J}$, without any policyholder modelling. The technical basis is used to determine the premiums as well as the surrender benefits and the benefits after a free policy conversion. In particular, the policyholder modelling may be omitted on the technical basis since the surrender and free policy benefits are chosen such that the value on the technical basis is unaffected. Second, we consider market consistent calculations on a so-called *market basis*. This is done with a market consistent interest rate structure and a best estimate of future transition rates. These calculations are used for the balance sheet values. The calculations can be conducted in the same state space $\mathcal{J}$, which is then without policyholder modelling. In this paper, we extend the state space in the market basis in order to include policyholder modelling, in the form of a surrender and a free policy option. For a comprehensive treatment of the policyholder options, see [38].

3.4.1 The life insurance contract and the technical basis

We consider a life insurance contract where the state of the insured is modelled by the semi-Markov process $\mathbf{Z}$ on the state space $\mathcal{J}$, where $\mathcal{J}$ does not include any policyholder modelling. The payment process $B(t)$ is decomposed into the benefit payments (the positive payments), described by $B^+(t)$, and the premium payments (the negative payments), described by $B^-(t)$, such that the total payments are $B(t) = B^+(t) - B^-(t)$. That is, the benefit payments process $B^+(t)$ is defined as the payment process (3.2.4) associated with the positive parts of the payment functions $b_i(t)^+, \Delta B_i(t)^+$ and $b_{ij}(t)^+$. Similarly $B^-(t)$ is the payment process of the negative parts.[3] The premiums are settled according to the equivalence principle on the technical basis consisting of a technical interest rate $r^* = (r^*(t))_t$ (which is usually constant) and a technical set of transition rates $\mu^* = (\mu^*_{ij}(t,u))_{i,j,t,u}$. For calculations on the technical basis, policyholder behaviour is not modelled.

The *technical reserve* of the life insurance contract is the prospective reserve calculated on the technical basis. We denote the technical reserve by $V^*_{i,u}(t)$, and it is defined similar to the prospective reserve in Definition 3.2.1, using the technical basis (r^*, μ^*). We decompose the technical reserve into the value associated with the benefits and the

[3]The notation with + and − is a bit ambiguous: For a function $f(t)$, we have $f(t)^+ = \max\{0, f(t)\}$, and if we write $f^+(t)$, it is merely a label.

value associated with the premiums, defining, for $i \in \mathcal{J}$, $u \geq 0$,

$$\begin{aligned} V^{*,+}_{i,u}(t) &= \mathrm{E}^* \left[\int_t^\infty e^{-\int_t^s r^*(\tau)\mathrm{d}\tau} \mathrm{d}B^+(s) \middle| Z(t) = i, U(t) = u \right], \\ V^{*,-}_{i,u}(t) &= \mathrm{E}^* \left[\int_t^\infty e^{-\int_t^s r^*(\tau)\mathrm{d}\tau} \mathrm{d}B^-(s) \middle| Z(t) = i, U(t) = u \right]. \end{aligned} \tag{3.4.1}$$

Thus, the technical reserve is given by $V^*_{i,u}(t) = V^{*,+}_{i,u}(t) - V^{*,-}_{i,u}(t)$. Here, the notation E^* denotes expectation using the measure where $\mathbf{Z}$ has transition rates μ^*.

3.4.2 The exercise options

We consider the problem of market based valuation of the life insurance contract including two exercise options of the policyholder. The policyholder options are:

- The *surrender* option, where the policyholder stops payments and receives the technical reserve.
- The *free policy* (paid-up policy) option, where the policyholder stops premium payments, and the benefits are reduced proportionally according to the equivalence principle on the technical basis.

We assume that the policyholder may exercise the surrender option at any time while in state 0, and similarly, we assume that the free policy option can only be exercised in state 0. For notational and interpretational simplicity, we assume that on the technical basis, there is no duration dependence in state 0. That is, we assume that for $t, u \geq 0$,

$$V^*_{0,u}(t) = V^*_{0,0}(t). \tag{3.4.2}$$

If one has duration dependence in state 0 on the technical basis, certain interpretational complications arise: One has to consider whether or not the duration should be reset to 0 after a free policy conversion, and thereby also whether or not the duration on the technical basis and on the market basis are identical.

The free policy option

If the policyholder exercises the free policy option at time s, the benefits are recalculated on the technical basis, according to the fact that future premiums after time s are cancelled. We define $\rho(s)$ as the benefit scaling factor if the policyholder exercises the free policy option at time s.

The free policy benefit factor

On the technical basis, the payment process, conditioning on the exercise of the free policy option at time s, is, for $t \geq s$,

$$t \mapsto \rho(s)\mathrm{d}B^{+}(t).$$

That is, the payments are the positive payments $\mathrm{d}B^{+}(t)$ scaled by the factor $\rho(s)$ at the time of free policy conversion. Then, the technical reserve at time $t \geq s$ in state $i \in \mathcal{J}$ is, using (3.4.1),

$$\rho(s)V_{i,u}^{*,+}(t). \tag{3.4.3}$$

We stress that this is on the technical basis, where the state space $\mathcal{J}$ is without policyholder modelling.

The equivalence principle is used here to determine $\rho(s)$, which here states that there is no jump in the technical reserve due to a free policy conversion. Thus, we have the requirement $\rho(s)V_{0,0}^{*,+}(s) = V_{0,0}^{*}(s)$, which yields

$$\rho(s) = \frac{V_{0,0}^{*,+}(s) - V_{0,0}^{*,-}(s)}{V_{0,0}^{*,+}(s)}, \tag{3.4.4}$$

where we recall the assumption (3.4.2), which states that there is no duration dependence in state 0 for the technical reserve. In particular $\rho(s)$ is a deterministic function.

It should be noted, that the model of this paper can be used with other choices of $\rho(s)$, as long as it is a deterministic function. Also, we assumed that $V_{0,u}^{*}(t)$ is duration independent, and therefore $\rho(s)$ is duration independent. Proposition 3.4.1 and Theorem 3.4.2 below can readily be extended to allow for $\rho(s)$ being dependent on the duration immediately before the free policy conversion happened.

In practice there may be free policy options where the benefits do not scale proportionally as assumed here. For example, some benefits may be removed completely upon free policy conversion, and the rest may scale proportionally. In that case, for calculating the market value, the policy can be split into parts where ρ may be different in each part. Then, the results of this paper can be used separately for each part of the policy.

The surrender option

If the policyholder surrenders the contract at time t, all future payments are cancelled, and instead the policyholder receives the current technical reserve. If the policy is still with premium, the technical reserve is $V_{0,0}^{*}(t)$, since surrender can only happen from state 0, and since there is no duration dependence on the technical basis in state 0. If the policy became a free policy at time s, the technical reserve at time t is given by (3.4.3) with $i = u = 0$.

The surrender payment can be more general, and in particular it may depend on the duration at the time of surrender. Proposition 3.4.1 can readily be extended to allow for a duration dependent surrender payment, e.g. $V^*_{0,U(t-)}(t)$, if we had not assumed duration independence in state 0 on the technical basis.

3.4.3 Extending the Markov model

When valuing the life insurance contract on the market basis, we consider an extended state space, taking into account the policyholder exercise options. Thus, the difference between the technical basis and the market basis is not only the interest and transition rates, but also the state space and hence the payment process.

Consider the extended state space which includes policyholder behaviour (phb),

$$\mathcal{J}^{\mathrm{phb}} = \mathcal{J} \cup \mathcal{J}^{\mathrm{s}} \cup \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}},$$

see Figure 3.1. The original state space is $\mathcal{J} = \{0, 1, \ldots, J-1\}$, which is the one used for the technical basis. The new parts of the state space are;

- $\mathcal{J}^{\mathrm{s}} = \{J\}$, which contains the state for surrender when the free policy option is not exercised;
- $\mathcal{J}^{\mathrm{f}} = \mathcal{J} + (J+1) = \{J+1, J+2, \ldots, 2J\}$ which is a copy of $\mathcal{J}$, and is the states of the policy when the free policy option is exercised;
- $\mathcal{J}^{\mathrm{fs}} = \{2J+1\}$ which is the state for surrender as a free policy.

We assume that $\mathbf{Z}$ is a semi-Markov model on the extended state space $\mathcal{J}^{\mathrm{phb}}$, with the same assumptions as before, i.e. that the transition rates exist, defined by (3.2.2), and that $Z(0) = 0$. State 0 is the only state from which the free policy option can be exercised, and state 0 and $J+1$ are the only states from which the surrender option can be exercised. We say that the transition $0 \mapsto J+1$ is a free policy conversion, and that a transition $0 \mapsto J$ (or $J+1 \mapsto 2J+1$) is a surrender (as a free policy). The transition rates within $\mathcal{J}^{\mathrm{f}}$ are assumed to be identical to those of $\mathcal{J}$. The states in $\mathcal{J}^{\mathrm{s}}$ and $\mathcal{J}^{\mathrm{fs}}$ are absorbing.

3.4.4 The payment process

For the total payment process in the extended model, we introduce a second duration process, $\mathbf{W}$, defined by

$$W(t) = \inf\{s \geq 0 | Z(t-s) \in \mathcal{J}\},$$

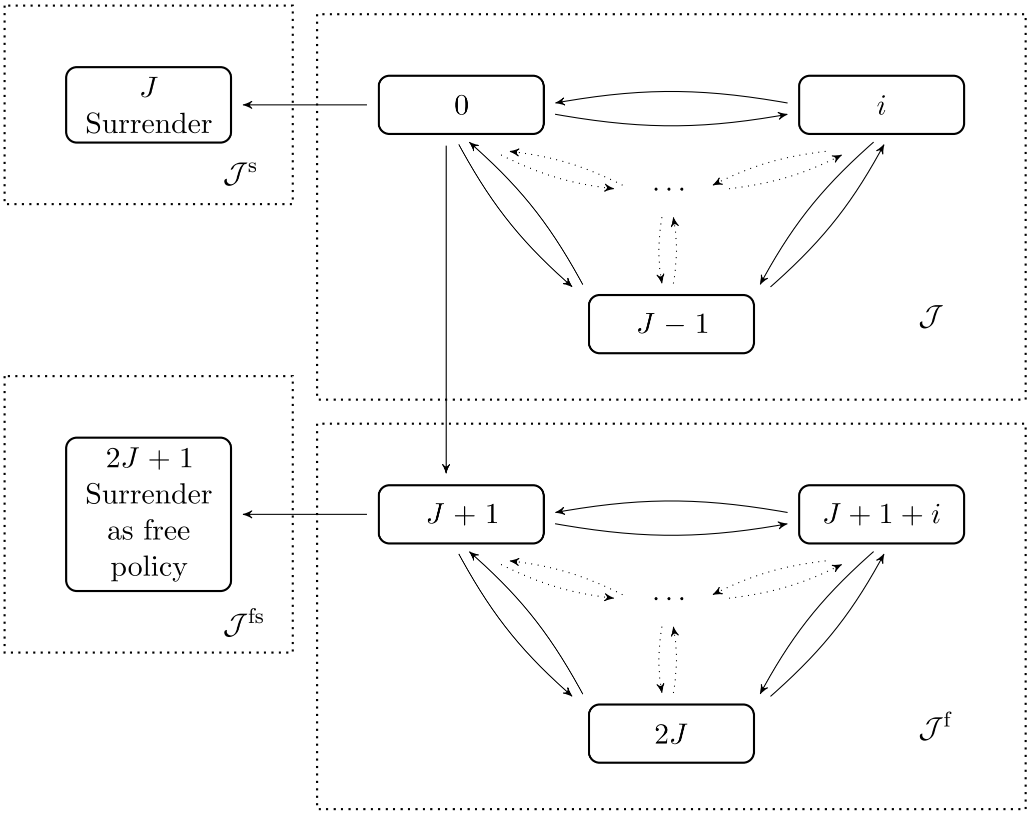

Figure 3.1: *A general finite state space $\mathcal{J}$ extended with free policy states $\mathcal{J}^{\mathrm{f}}$ and surrender states $\mathcal{J}^{\mathrm{s}}$ and $\mathcal{J}^{\mathrm{fs}}$. The state space $\mathcal{J}$ contains the state of the insured, e.g. active, disabled, dead, etc. The free policy states $\mathcal{J}^{\mathrm{f}}$ is a copy of $\mathcal{J}$, and a transition from $\mathcal{J}$ to $\mathcal{J}^{\mathrm{f}}$ corresponds to a free policy conversion. A transition to $J \in \mathcal{J}^{\mathrm{s}}$ or $2J+1 \in \mathcal{J}^{\mathrm{fs}}$ corresponds to a surrender of the policy.*

that is, $W(t)$ is the duration since $Z(t)$ left $\mathcal{J}$, or if $Z(t) \in \mathcal{J}$, then $W(t) = 0$. The duration $\mathbf{W}$ is required for the payments in the free policy states, because, if we are at time t and $Z(t) \in \mathcal{J}^{\mathrm{f}}$, the free policy option was exercised at time $t - W(t)$, and we recall that $\rho(t - W(t))$ is used to determine the size of the benefits.

The payment process for the extended model can now be stated. On the original part of the state space, $\mathcal{J}$, the payment process is the same as on the technical basis, $B(t)$, and is defined by (3.2.4). On the market basis, the payment process on the extended state space is denoted B^{phb}, and is given by,

$$\begin{aligned} \mathrm{d}B^{\mathrm{phb}}(t) &= \mathrm{d}B(t) + V_{0,0}^{*}(t)\mathrm{d}N_{0J}(t) \\ &\quad + \rho(t - W(t)) \left(\mathrm{d}B^{\mathrm{f},+}(t) + V_{0,0}^{*,+}(t)\mathrm{d}N_{J+1,2J+1}(t) \right). \end{aligned}$$

We briefly explain the payment process. For $Z(t) \in \mathcal{J}$, the payments are determined by the original payment process $B(t)$, that is, the original payment functions b_j, ΔB_j and b_{ij}, for $i,j \in \mathcal{J}$. By Section 3.4.2, the payment upon surrender as a premium paying policy is $V^*_{0,0}(t)$, and this is paid upon a transition from state 0 to state J. As a free policy, the payment upon surrender is $\rho(t-W(t))V^{*,+}_{0,0}(t)$, which is paid upon a transition from state $J+1$ to state $2J+1$. The payment process $B^{\mathrm{f},+}(t)$ is non-zero for $Z(t) \in \mathcal{J}^{\mathrm{f}}$ and is the payments as a free policy, excluding the scaling factor $\rho(t-W(t))$. Intuitively $B^{\mathrm{f},+}(t)$ is a copy on $\mathcal{J}^{\mathrm{f}}$ of the positive payments on $\mathcal{J}$, i.e. of $B^+(t)$. To be precise, define the payment functions b_i, ΔB_i and b_{ij} on $\mathcal{J}^{\mathrm{f}}$, by simply copying their value on $\mathcal{J}$, i.e. for $i,j \in \mathcal{J}^{\mathrm{f}}$ let,

$$b_i = b_{i-(J+1)}, \quad \Delta B_i = \Delta B_{i-(J+1)}, \quad b_{ij} = b_{i-(J+1),j-(J+1)}.$$

Then $B^{\mathrm{f},+}(t)$ satisfies

$$\begin{aligned}
\mathrm{d}B^{\mathrm{f},+}(t) &= \sum_{i\in\mathcal{J}^{\mathrm{f}}} 1_{\{Z(t)=i\}}\mathrm{d}B_i^+(t,U(t)) + \sum_{\substack{i,j\in\mathcal{J}^{\mathrm{f}}\\ i\neq j}} b_{ij}(t,U(t-))^+\mathrm{d}N_{ij}(t),\\
\mathrm{d}B_i^+(t,U(t)) &= b_i(t,U(t))^+\mathrm{d}t + \Delta B_i(t,U(t))^+.
\end{aligned}$$

3.4.5 Cash flows with policyholder behaviour

We now study the cash flow associated with the payment process $\mathbf{B}^{\mathrm{phb}}$ in the extended model. The cash flow at time t for the future payments is denoted $(A^{\mathrm{phb}}_{i,u}(t,s))_{s\geq t}$. By the use of transition probabilities dependent on the two durations, $\mathbf{U}$ and $\mathbf{W}$, we can find the cash flow. We note that $\mathbf{Z}$ is a semi-Markov process such that $(\mathbf{Z},\mathbf{U})$ is a Markov process, but the payments may depend on the additional duration $\mathbf{W}$.

Due to the simple structure for the extended model, the cash flow for state $i \in \mathcal{J}$ may be determined directly via a prospective argument. In this way, it can be shown that the cash flow satisfies, for $i \in \mathcal{J}$, $u \geq 0$ and $s \geq t$,

$$\begin{aligned}
\mathrm{d}A^{\mathrm{phb}}_{i,u}(t,s) = \mathrm{d}A_{i,u}(t,s) &+ \int_0^{u+s-t} p_{i0}\left(t,s,u,\mathrm{d}z\right)\mu_{0J}(s,z)V^*_{0,0}(s)\mathrm{d}s\\
&+ \sum_{j\in\mathcal{J}^{\mathrm{f}}}\int_0^\infty\int_0^\infty P(Z(s)=j,U(s)\leq \mathrm{d}z,W(s)\leq \mathrm{d}w|Z(t)=i,U(t)=u)\\
&\quad \times \rho(s-w)\left(\mathrm{d}B_j^+(s,z) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}}\mu_{jk}(s,z)b_{jk}(s,z)^+\mathrm{d}s\right)\\
&+ \int_0^\infty\int_0^\infty P(Z(s)=J+1,U(s)\leq \mathrm{d}z,W(s)\leq \mathrm{d}w|Z(t)=i,U(t)=u)\\
&\quad \times \rho(s-w)\mu_{J+1,2J+1}(s,z)V^{*,+}_{0,0}(s)\mathrm{d}s.
\end{aligned}$$

We interpret the right hand side. The first part of the first line is the insurance payments $\mathrm{d}A_{i,u}(t,s)$, which is the cash flow of the original payment process $B(s)$, i.e. the payments while in $\mathcal{J}$. Also, the second part of the first line are payments upon surrender from state 0. The second and third lines represent the payments of the payment process $B^{\mathrm{f},+}$, i.e. as a free policy. The last lines are surrenders as a free policy. In the last term, we have used that the first order reserve at time s in the free policy state $J+1$ is equal to $\rho(s-w)V_{0,0}^{*,+}(s)$ if the free policy option was exercised at time $s-w$.

The cash flow $\mathrm{d}A_{i,u}(t,s)$ is given by

$$\mathrm{d}A_{i,u}(t,s) = \sum_{j\in\mathcal{J}} \int_0^{u+s-t} p_{ij}\left(t,s,u,\mathrm{d}z\right) \left(\mathrm{d}B_j(s,z) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}(s,z) b_{jk}(s,z)\mathrm{d}s \right).$$

The transition probabilities can be calculated using Kolmogorov's forward integro-differential equation, (3.3.1), which in this case is on the state space $\mathcal{J}^{\mathrm{phb}}$. In particular, the cash flow $A_{i,u}(t,s)$ differs from the one obtained without policyholder modelling where the transition probabilities in the smaller semi-Markov model with state space $\mathcal{J}$ are used.

Because of the structure of the semi-Markov model, knowledge of $\mathbf{W}$ is the same as knowledge of the time of transition from state 0 to state $J+1$, and there can only be one such transition. Recall that by the design, a transition from state 0 to state $J+1$ is necessary to go from state $i\in\mathcal{J}$ to state $j\in\mathcal{J}^{\mathrm{f}}\cup\mathcal{J}^{\mathrm{fs}}$. With this, it can be shown that, for $i\in\mathcal{J}$ and $j\in\mathcal{J}^{\mathrm{f}}\cup\mathcal{J}^{\mathrm{fs}}$,

$$\begin{aligned} &\int_0^\infty P(Z(s)=j, U(s)\le z, W(s)\le \mathrm{d}w | Z(t)=i, U(t)=u)\rho(s-w) \\ &= \int_t^s \int_0^{u+\tau-t} p_{i0}\left(t,\tau,u,\mathrm{d}v\right)\mu_{0,J+1}(\tau,v)\rho(\tau)p_{J+1,j}\left(\tau,s,0,z\right)\mathrm{d}\tau \\ &=: p_{ij}^{\rho}\left(t,s,u,z\right). \end{aligned} \tag{3.4.5}$$

Here we defined p^{ρ}, and it is the probability, given we start in state $i\in\mathcal{J}$ at time t with duration u, of being in state $j\in\mathcal{J}^{\mathrm{f}}\cup\mathcal{J}^{\mathrm{fs}}$ with duration less than z at time s, multiplied by ρ at the time of transition from state 0 to $J+1$. Using the quantities p^{ρ}, we can rewrite the cash flow in the following way.

Proposition 3.4.1. *The cash flow $(A^{\mathrm{phb}}(t,s))_{s\ge t}$ at time t, with policyholder behaviour,* *is given by*

$$\begin{aligned}
\mathrm{d}A^{\mathrm{phb}}_{i,u}(t,s) &= \mathrm{d}A_{i,u}(t,s) + \int_0^{u+s-t} p_{i0}\left(t,s,u,\mathrm{d}z\right)\mu_{0J}(s,z)V^{*}_{0,0}(s)\mathrm{d}s \\
&\quad + \sum_{j\in\mathcal{J}^{\mathrm{f}}} \int_0^{u+s-t} p^{\rho}_{ij}\left(t,s,u,\mathrm{d}z\right)\left(\mathrm{d}B^{+}_{j}(s,z) + \sum_{\substack{k\in\mathcal{J}^{\mathrm{f}}\\ k\neq j}} \mu_{jk}(s,z)b_{jk}(s,z)^{+}\mathrm{d}s\right) \\
&\quad + \int_0^{u+s-t} p^{\rho}_{i,J+1}\left(t,s,u,\mathrm{d}z\right)\mu_{J+1,2J+1}(s,z)V^{*,+}_{0,0}(s)\mathrm{d}s,
\end{aligned}$$

for $i \in \mathcal{J}$, where $p^{\rho}_{ij}\left(t,s,u,z\right)$ is defined by (3.4.5).

We interpret the right hand side. The first part of the first line is the cash flow when none of the policyholder options has been exercised. The second part of the first line is the payment upon surrender when a free policy transition has not occured. The second line contains the payments after a free policy transition, and the last line is the payment associated with an exercise of the surrender option after a free policy conversion.

In order to calculate the cash flow, one needs to determine $p^{\rho}_{ij}\left(t,s,u,z\right)$. This is time-consuming compared to the model without policyholder behaviour, because the extra probabilities $p_{J+1,j}\left(\tau,s,0,z\right)$ in (3.4.5) need to be calculated for all $\tau \in (t,s)$ and $s \geq t$. That is, in excess of just solving Kolmogorov's forward integro-differential equation, we now need to solve for each $s \geq t$ as well. Formally, we can say that the dimension of the problem is doubled if we determine p^{ρ} directly via (3.4.5).

See, that if $\rho(t) = 1$ for all t, then, for $i \in \mathcal{J}$ and $j \in \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}}$, we have $p^{\rho}_{ij}\left(t,s,u,z\right) = p_{ij}\left(t,s,u,z\right)$. In that case, we can use Kolmogorov's forward integro-differential equation directly, which are significantly simpler than calculating (3.4.5). However, it turns out, that for $\rho(t) \neq 1$, we can find a similar forward integro-differential equation system for p^{ρ}, which is the main result of this paper.

Theorem 3.4.2. *Let $0 \leq t_0 \leq t$ and $u \geq 0$ and $i \in \mathcal{J}$. The quantities*

$$p^{\rho}_{ij}\left(t_0,t,u,d+t-t_0\right), \qquad \text{for } j \in \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}} \text{ and } d \in \mathbb{R} \text{ s.t. } d+t-t_0 \geq 0,$$

satisfy, with $D(s) = d + s - t_0$, the forward system of integro-differential equations,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}p^{\rho}_{ij}\left(t_0,s,u,D(s)\right) &= 1_{\{j=J+1\}}\int_0^{u+s-t_0} p_{i0}\left(t_0,s,u,\mathrm{d}z\right)\mu_{0,J+1}(s,z)\rho(s) \\
&\quad - \int_0^{D(s)} p^{\rho}_{ij}\left(t_0,s,u,\mathrm{d}z\right)\mu_{j\cdot}(s,z) \\
&\quad + \sum_{\substack{\ell\in\mathcal{J}^{\mathrm{f}}\\ \ell\neq j}} \int_0^{u+s-t_0} p^{\rho}_{i\ell}\left(t_0,s,u,\mathrm{d}z\right)\mu_{\ell j}(s,z),
\end{aligned} \tag{3.4.6}$$

with boundary conditions $p^{\rho}_{ij}(t_0, t_0, u, d) = 0$ *and, for* $s \geq t_0$, $p^{\rho}_{ij}(t_0, s, u, 0) = 0$.

The first line of the differential equation corresponds to new free policy conversions, i.e. transitions from state 0 to state $J+1$. We see that here the transition rate is multiplied with the free policy factor ρ. The two other lines correspond to the differential equation from Kolmogorov's forward integro-differential equations, Theorem 3.3.1. Line two is transitions away from state j at time s, where the duration is less than the current duration $D(s) = d + s - t$. The last line, line three, is transitions from other states in $\mathcal{J}^{\mathrm{f}}$ to state j, for any duration.

Proof of Theorem 3.4.2. Differentiate p^{ρ}, (3.4.5), and apply Kolmogorov's forward integro-differential equations (Theorem 3.3.1), to obtain the integro-differential equation,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} p^{\rho}_{ij}(t, s, u, D(s)) \\
&= \int_0^{u+s-t} p_{i0}(t, s, u, \mathrm{d}v)\, \mu_{0,J+1}(s, v)\rho(s) p_{J+1,j}(s, s, 0, D(s)) \\
&\quad + \int_t^s \int_0^{u+\tau-t} p_{i0}(t, \tau, u, \mathrm{d}v)\, \mu_{0,J+1}(\tau, v)\rho(\tau) \frac{\mathrm{d}}{\mathrm{d}s} p_{J+1,j}(\tau, s, 0, D(s))\, \mathrm{d}\tau \\
&= 1_{\{j=J+1\}} \int_0^{u+s-t} p_{i0}(t, s, u, \mathrm{d}v)\, \mu_{0,J+1}(s, v)\rho(s) \\
&\quad + \int_t^s \int_0^{u+\tau-t} p_{i0}(t, \tau, u, \mathrm{d}v)\, \mu_{0,J+1}(\tau, v)\rho(\tau) \Bigg(\\
&\qquad - \int_0^{D(s)} p_{J+1,j}(\tau, s, 0, \mathrm{d}z)\, \mu_{j\cdot}(s, z) \mathrm{d}\tau \\
&\qquad + \sum_{\substack{\ell \in \mathcal{J}^{\mathrm{f}} \\ \ell \neq j}} \int_0^{s-\tau} p_{J+1,\ell}(\tau, s, 0, \mathrm{d}z)\, \mu_{\ell j}(s, z) \mathrm{d}\tau \Bigg) \\
&= 1_{\{j=J+1\}} \int_0^{u+s-t} p_{i0}(t, s, u, \mathrm{d}v)\, \mu_{0,J+1}(s, v)\rho(s) \\
&\quad - \int_0^{D(s)} p^{\rho}_{ij}(t, s, u, \mathrm{d}z)\, \mu_{j\cdot}(s, z) \mathrm{d}\tau + \sum_{\substack{\ell \in \mathcal{J}^{\mathrm{f}} \\ \ell \neq j}} \int_0^{u+s-t} p^{\rho}_{i\ell}(t, s, u, \mathrm{d}z)\, \mu_{\ell j}(s, z) \mathrm{d}\tau.
\end{aligned}
$$

Here, the last equality is obtained by interchanging the order of integration. The boundary condition is obtained directly from (3.4.5). □

Considering the integro-differential equation (3.4.6) for p^{ρ}, we see that it is of similar structure as Kolmogorov's forward integro-differential equations, and that (3.4.6) depends on $p_{i0}(t, s, u, z)$. Thus, one can with advantage solve for p and p^{ρ} simultaneously, by solving the combined system of integro-differential equations, (3.3.1) and (3.4.6). Whereas, by letting the payments depend on ρ at the time of the free policy transition essentially creates a double-duration setup, Theorem 3.4.2 eliminates one duration. Thus, computationally, the calculation of cash flows in the model with the two policyholder options is almost as simple as without: one simply needs to solve a system of integro-differential equations, and no extra duration is effectively introduced.

3.4.6 Policyholder behaviour in the Markov case

We consider the policyholder behaviour model for the special case where there is no duration dependence, i.e. neither the transition rates nor the payment functions depend on $\mathbf{U}$. When the transition rates do not depend on the duration, $\mathbf{Z}$ is a Markov process, and the setup is similar to Section 3.2.6. We still model the free policy option, so the payment process is dependent on the free policy duration $\mathbf{W}$.

First, p^ρ simplifies a bit, since the integration of the duration disappears, and we have,

$$p^\rho_{ij}(t,s) = \int_t^s p_{i0}(t,\tau)\mu_{0,J+1}(\tau)\rho(\tau)p_{J+1,j}(\tau,s)\mathrm{d}\tau. \tag{3.4.7}$$

Then, the cash flow valuated at time t, given we are in state $i \in \mathcal{J}$, becomes

$$\begin{aligned}\mathrm{d}A^{\text{phb}}_i(t,s) = {} & \mathrm{d}A_i(t,s) + p_{i0}(t,s)\mu_{0J}(s)V^*_0(s)\mathrm{d}s \\ & + \sum_{j\in\mathcal{J}^{\text{f}}} p^\rho_{ij}(t,s)\left(\mathrm{d}B^+_j(s) + \sum_{\substack{k\in\mathcal{J}^{\text{f}}\\ k\neq j}} \mu_{jk}(s)b_{jk}(s)^+\mathrm{d}s\right) \\ & + p^\rho_{i,J+1}(t,s)\mu_{J+1,2J+1}(s)V^{*,+}_0(s)\mathrm{d}s,\end{aligned}$$

where $\mathrm{d}A_i(t,s)$ is given in (3.2.7), using transition probabilities from the state space $\mathcal{J}^{\text{phb}}$. The integro-differential equation for p^ρ reduces to a simple differential equation. For $i \in \mathcal{J}$ and $j \in \mathcal{J}^{\text{f}} \cup \mathcal{J}^{\text{fs}}$, p^ρ satisfies the forward differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}s}p^\rho_{ij}(t,s) = 1_{\{j=J+1\}}p_{i0}(t,s)\mu_{0,J+1}(s)\rho(s) - p^\rho_{ij}(t,s)\mu_{j\cdot}(s) + \sum_{\substack{\ell\in\mathcal{J}^{\text{f}}\\ \ell\neq j}} p^\rho_{i\ell}(t,s)\mu_{\ell j}(s), \tag{3.4.8}$$

with boundary condition $p^\rho_{ij}(t,t) = 0$.

From the differential equation for p^ρ, it is easy to see, that if $\rho(t) = 1$ for all t, then $p^\rho_{ij} = p_{ij}$ for $i \in \mathcal{J}$ and $j \in \mathcal{J}^{\text{f}} \cup \mathcal{J}^{\text{fs}}$. For $\rho(t) \neq 1$, we can interpret the differential equation intuitively. The second term of the right hand side corresponds to transitions out of state j, i.e. probability mass leaving state j. The third term corresponds to probability mass entering state j from one of the other free policy states $\ell \in \mathcal{J}^{\mathrm{f}}$. The first term then corresponds to new free policy conversions, i.e. transitions into $\mathcal{J}^{\mathrm{f}}$, and this probability mass is multiplied by ρ. Thus, the quantities p^{ρ} can be interpreted as manipulated transition probabilities in the way that probability mass that enters $\mathcal{J}^{\mathrm{f}}$ is manipulated through a modified transition rate.

Connection with retrospective reserves

The definition of $p^{\rho}_{ij}(t,s)$ in (3.4.7) can be interpreted as a form of retrospective reserve, where the interest rate is set to 0. To see this, consider the state space from Figure 3.1 and associate a single payment $(-\rho(t))$ upon transition from state 0 to state $J+1$. Then the retrospective reserve $W^{-}_{j}(t)$ in Section 5 E of [39] equals $p^{\rho}_{0j}(0,t)$ from (3.4.7), if the interest rate is set to 0. In particular, (3.4.8) equals the differential equation system (5.14) in [39] with $W^{-}_{j}(t) = 0$ for $j \in \mathcal{J} \cup \mathcal{J}^{\mathrm{s}}$.

To understand that a transitional payment of $(-\rho(t))$ is correct, note, that if an individual is in a free policy state now, e.g. state $J+1$, the value of the policy is the accumulated previous premiums, which is exactly $\rho(t - W(t))$. Then, a retrospective reserve for state $J+1$, with zero interest rate, is

$$p_{0,J+1}(0,t)\,\mathrm{E}\left[\rho(t-W(t))\right|Z(t)=J+1] = \mathrm{E}\left[\rho(t-W(t))1_{\{Z(t)=J+1\}}\right],$$

which is exactly $p^{\rho}_{0,J+1}(0,t)$, where $i \in \mathcal{J}$.

For a similar connection in the semi-Markov setup, one can compare with Chapter 5 in [25]: Using (3.4.5), one can see that for $a \in \mathcal{J}$ and $y \in \mathcal{J}^{\mathrm{f}} \cup \mathcal{J}^{\mathrm{fs}}$, the quantity $\frac{p^{\rho}_{ay}(0,t,0,v)}{p_{ay}(0,t,0,v)}$ is analogous to (5.2.1) in [25].

3.5 Numerical example

We consider a numerical example and study how the modelling of policyholder behaviour has an effect on the liabilities. Specifically we consider the cash flow and the interest rate sensitivity in form of the dollar duration, and the prospective reserve. We show that introducing policyholder behaviour in the model significantly changes the structure of the cash flows, and to an extent also the prospective reserve.

For simplicity, we consider a survival model, i.e. a 2-state Markov model with states 0, *alive*, and 1, *dead*. We consider an insured male of age 40 with pension age 65, and two products,

1. a life annuity, starting at age 65,
2. a 10 year annuity upon death, if death occurs before age 65.

We assume that the insured has already saved an amount of 100,000, and that he pays a yearly premium of 10,000 while alive, until age 65. We specify the payment functions from (3.2.4),

$$\begin{aligned} b_0(t,u) &= 37{,}404 \cdot 1_{\{t \geq 25\}}, \\ b_1(t,u) &= 18{,}702 \cdot 1_{\{t-u<25\}} 1_{\{u<10\}}. \end{aligned}$$

The life annuity is of size 37,404, and the annuity upon death corresponds to 50% of the life annuity. On the technical basis introduced below, it gives a technical reserve of 100,000, which is the already saved up amount.

The annuity upon death is dependent on the duration u, since it is only for the 10 first years after death that there is a payment. This implies that we need the transition probabilities with durations from Theorem 3.3.1, instead of the simpler ones from Proposition 3.2.5. In other words, we are in practice in the semi-Markov setup, even though the transition rates are not duration dependent and $\mathbf{Z}$ is a Markov process.

We consider valuation on a technical basis, as well as three different market bases. The technical basis is the one used for pricing, and it is the value on the technical basis, $V^*(t)$ that is paid out upon surrender. Quantities related to the technical basis are generally marked with $*$. A market basis is used for calculating market consistent values, i.e. the prospective reserve for the balance sheet. It consists of a market interest rate and a best estimate of the mortality rate. We consider three different market bases, with three different Markov models: One without policyholder behaviour modelling, one with surrender modelling only, and one with surrender and free policy modelling. The three different market bases are used to illustrate the effect of including policyholder modelling.

The technical basis consists of the following,

- a survival Markov model with 2 states, *alive* and *dead*,
- an interest rate, $r^*(t) = 0.015$,
- a mortality rate given by the Danish G82M mortality table, $\mu^*(x) = 0.0005 + 0.000075858 \cdot 1.09144^x$.

This technical basis has been used historically, and a more conservative technical basis would be used for new contracts. As stated above, the benefits on our policy are consistent with a technical reserve of 100,000, calculated using (3.2.5).

Common for the three market bases is the interest and mortality rates. The interest rate is the one published every day by the Danish FSA for discounting life insurance liabilities, and the one from 8 May 2013 is used; it is available at [20]. The mortality rate is the benchmark mortality rate for 2011, which is published by the Danish FSA; it is available at [21]. See [29] for a further treatment of this mortality benchmark.

The difference between the three market bases is the Markov model, where the first model is the survival model without any policyholder modelling, shown in Figure 3.1a. The second model is the 3-state model, where the survival model is extended with a surrender state, according to Figure 3.1b. The third model is the 6-state model including free policy modelling and surrender modelling, see Figure 3.1c. The surrender transition rate $\mu^{\mathrm{s}}(x)$ and the free policy transition rate $\mu^{\mathrm{f}}(x)$ are for $x \leq 65$ given by,

$$\mu^{\mathrm{s}}(x) = 0.06 - 0.002 \cdot (x - 40)^+,$$
$$\mu^{\mathrm{f}}(x) = 0.05,$$

where x is the age. We assume that surrender and free policy conversion cannot occur after age 65, and thus the rates are zero above age 65. The surrender and free policy parameters loosely resemble the ones used in practice by a large Danish life insurer in the competitive market. The transition rates are shown in Figure 3.2 together with the corresponding transition probabilities from the 6-state model in Figure 3.1c. The probability of surrender and free policy are significant and already at age 47 the probability of either surrender or free policy is higher than the probability of still being a premium paying policy.

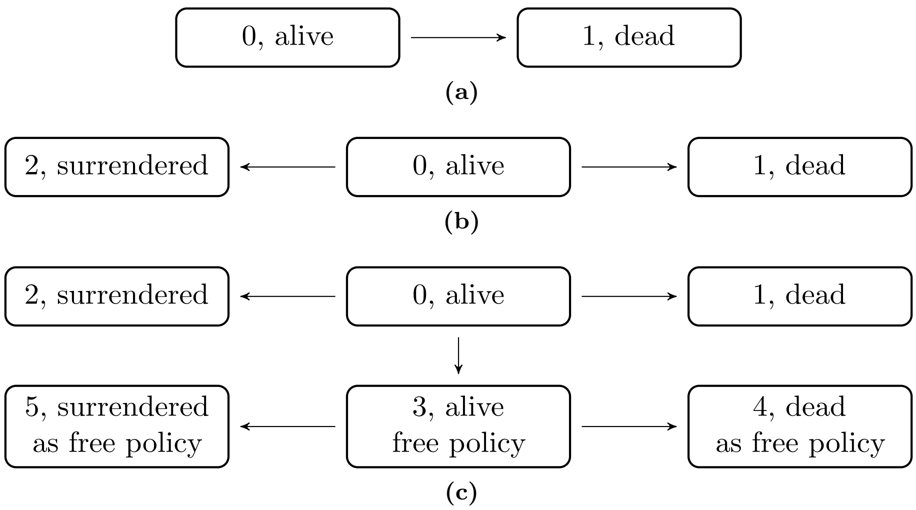

Figure 3.1: *The three state spaces for the market bases. In (a) there is no policyholder modelling. In (b) the surrender behaviour is modelled, and in (c) both the surrender and free policy behaviour is modelled.*

To find the free policy factor $\rho(t)$ from (3.4.4), the technical reserve is calculated for all future time points. This yields a first value $\rho(0) = 0.34$, and then it increases almost linearly, slightly concave, to $\rho(25) = 1$, at pension age 65.

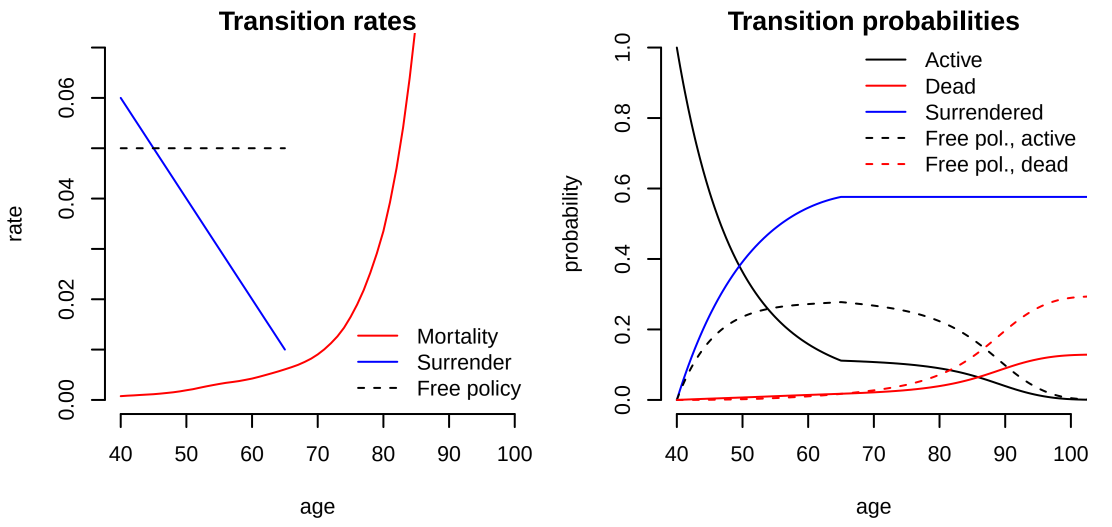

Figure 3.2: *Transition rates for the market basis (left) and transition probabilities in the 6-state surrender and free policy Markov model (right). The mortality rate is for a 40 year old male and shown including future improvements in mortality. The surrender rate is decreasing, and the free policy rate is constant. After age 65 it is not possible to surrender or convert to a free policy.*

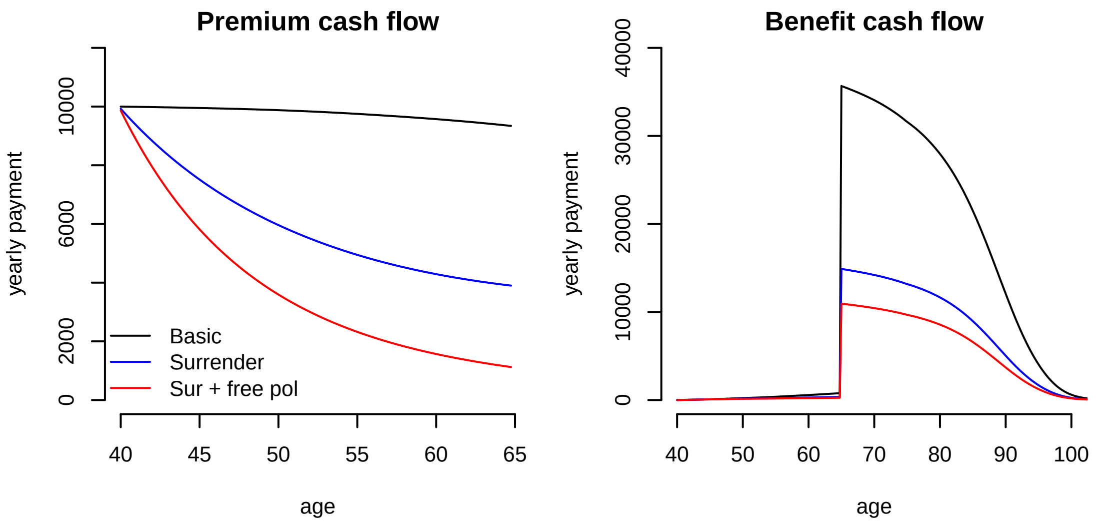

Figure 3.3: *Cash flow of premiums (left) and benefits (right). The premium cash flow in the basic model is slowly decreasing due to a small probability of death. With policyholder modelling, the premium cash flow is greatly reduced, since premiums are cancelled when either surrender or a free policy conversion occurs. The benefits are slightly increasing before age 65, due to a small probability of the annuity upon death being paid out. At age 65 the life annuity begins. Surrender and free policy modelling greatly reduces the benefits cash flow, since the benefits are reduced by a factor $\rho(t)$ upon transition to a free policy, and cancelled upon surrender.*

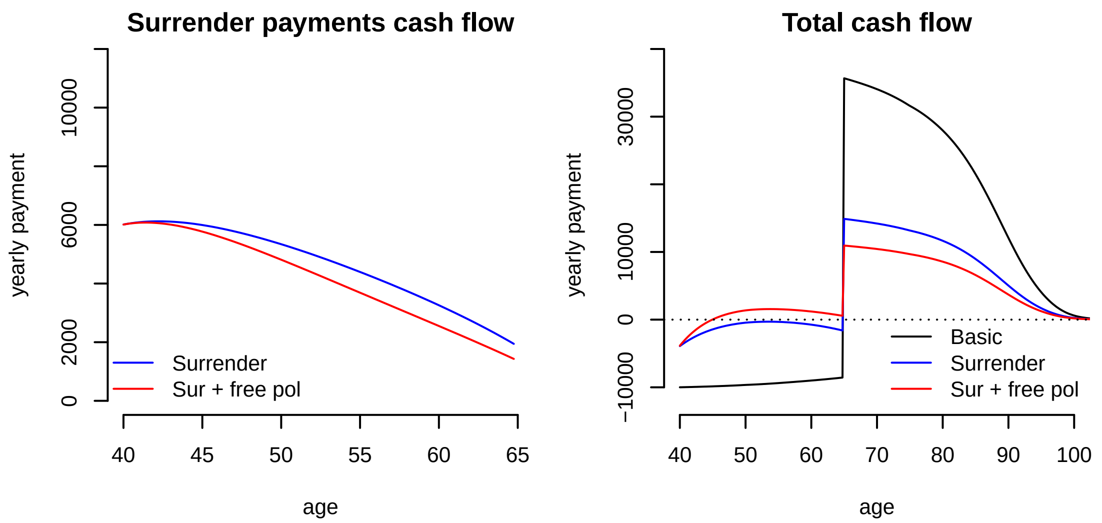

Figure 3.4: *Cash flow of surrender payments (left) and the total cash flow (right). There are no surrender payments in the basic model. With free policy modelling, the surrender payments are slightly smaller, since surrender as free policy yields a smaller payment. The total cash flows are the sum of the premium, benefit and surrender cash flows, and show that the payments, both positive and negative, are significantly reduced.*

On the three market bases we consider the cash flows. In Figure 3.3 the premiums and benefits cash flow are shown for the three models, and in Figure 3.4, the surrender payments and the total cash flow are shown. For the premiums, we see that introducing surrender greatly reduces the amount of premiums, and a free policy conversion reduces the premiums further. This is as expected, since if either a surrender or a free policy conversion occurs, future premiums are cancelled. For the benefits we see the same effect as for the premiums, where the introduction of both surrender and free policy modelling greatly reduces the benefits. Some of the reduction of the benefits due to the surrender modelling corresponds to value that is being paid out as surrender payments, shown in Figure 3.4, and some corresponds to the value of the future premiums that are cancelled upon surrender. The small benefits before age 65 are related to the annuity upon death.

The total cash flows in Figure 3.4 are the sum of the premiums, benefits and surrender payments and show the overall differences. Without policyholder modelling, there are first 25 years of negative payments, the premiums, and then after the pension age, the saved up value is paid out. If policyholder behaviour is included, the cash flows are in general smaller, and some of the benefits are also paid out earlier as surrender payments.

The prospective reserves can be calculated with a market interest rate, and here we use the one provided by the Danish FSA of 8 May 2013. The prospective reserves are shown in Table 3.1 and it is seen that the value does not change a lot when introducing

	Basic	Surrender	Sur. and free pol.
Prospective reserve	105,185	101,536	101,371
DV01 Total	83,407	41,838	34,547
DV01 Pos. payments	105,673	57,535	43,085
DV01 Premiums	22,266	15,697	8,538

Table 3.1: *Prospective reserves and dollar durations (DV01). The duration is greatly reduced when policyholder modelling is included, both for the total cash flows and also for the positive payments and premiums separately.*

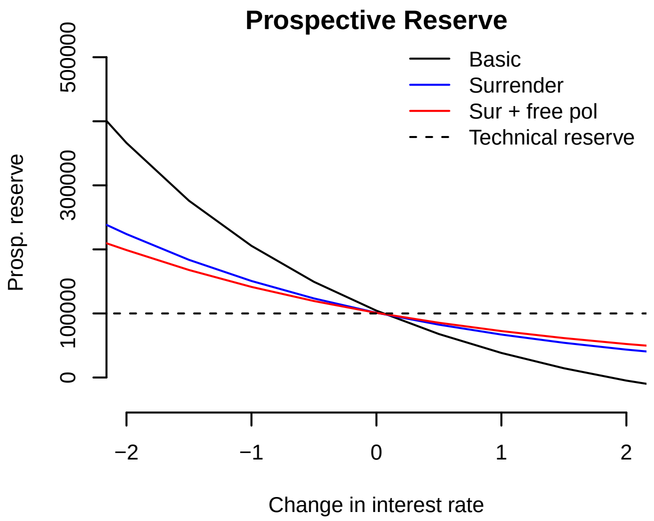

Figure 3.5: *Prospective reserves for the life insurance liabilities shown for different parallel shifts of the interest rate structure. The interest rate is capped at 0, so it is always non-negative. It is seen that modelling of policyholder behaviour reduces the interest rate sensitivity of the liabilities.*

policyholder modelling. However, this change is very interest rate dependent, and in Figure 3.5, the prospective reserves are shown for different shifts in the market interest rate. We stress that the lines do not cross in zero: The technical reserve is independent of the market interest and equal to 100,000, and as seen in Table 3.1, the market values in the three models are slightly different from 100,000. When the market interest rate changes it is seen from the figure that the value in the basic model without policyholder behaviour changes the most. With surrender modelling, the market value changes less, and even less with free policy modelling. Thus, we see that policyholder modelling greatly influences the interest rate sensitivity.

In Table 3.1 the interest rate sensitivity for the total cash flows is shown, in form of the dollar duration, DV01, which measures the absolute change in the value for a 100 basis point change in the interest rate. Surrender modelling reduces the dollar duration with about 50% and on top of that free policy modelling reduces the duration by another 17%.

Thus, if one applies duration matching for hedging the interest rate risk, it is essential to take into account both surrender and free policy modelling.

Acknowledgements

We are grateful to Ann-Sophie Buchardt for her assistance with the beautiful figures. Thorough comments and suggestions from an anonymous referee are gratefully acknowledged, and in particular we are grateful for the suggestion of a possible link to retrospective reserves.

3.A Proof of Kolmogorov's forward integro-differential equation

Before we can prove Theorem 3.3.1, we recall the Chapman-Kolmogorov equation.

Lemma 3.A.1. (The Chapman-Kolmogorov equation) *Let $t_0 \leq s \leq t$ and $u, z \geq 0$. The transition probabilities satisfy*

$$p_{ij}(t_0, t, u, z) = \sum_{\ell \in \mathcal{J}} \int_0^{u+s-t_0} p_{i\ell}(t_0, s, u, \mathrm{d}v)\, p_{\ell j}(s, t, v, z).$$

In the proof below, we use a slightly rewritten version, where, for $0 \leq h \leq t - t_0$, we write the Chapman-Kolmogorov equation as,

$$\begin{aligned} &p_{ij}(t_0, t, u, z) \\ &= \sum_{\ell \in \mathcal{J}} \int_0^{u+t-t_0} p_{i\ell}(t_0, t-h, u, \mathrm{d}v - h)\, p_{\ell j}(t-h, t, v-h, z), \end{aligned} \tag{3.A.1}$$

using the notation

$$p_{i\ell}(t_0, t-h, u, \mathrm{d}v - h) = P(Z(t-h) = \ell, U(t-h) + h \leq \mathrm{d}v | Z(t_0) = i, U(t_0) = u).$$

We also recall a result on weak convergence of measures, which is stated as Theorem 2.1 in [3].

Lemma 3.A.2. *Let $(F_h)_{h \in \mathbb{R}}$ and F be measures on $\mathbb{R}$, such that $F_h(\mathbb{R}) \leq 1$ and $F(\mathbb{R}) \leq 1$. Then the following are equivalent,*

1. $F_h \xrightarrow{\text{weak}} F$ *for* $h \to 0$,

2. *for all continuous bounded functions f,*

$$\int f(x)F_h(\mathrm{d}x) \to \int f(x)F(\mathrm{d}x) \quad \textit{for} \quad h \to 0,$$

3. *for all x where $F(x)$ is continuous, $F_h(x) \to F(x)$ for $h \to 0$.*

We are now ready to prove the result.

Proof of Theorem 3.3.1. First, see that the boundary condition is evident from inspection of (3.2.1). Second, we establish differentiability. By Kolmogorov's backwards differential equation (Proposition 3.2.4) and Theorem 9.2 in [1], we have that

$$t \to p_{ij}\left(s, t, u, d + t - s\right)$$

is continuously differentiable.

Now, let $h > 0$, and consider the Taylor expansion of $p_{ij}\left(s, t, u, z\right)$ around $(s, u) = (t, v)$,

$$\begin{aligned}
&p_{ij}\left(t - h, t, v - h, z\right) \\
&= p_{ij}\left(t, t, v, z\right) - h\,\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}\left(s, t, d + s, z\right)\Big|_{\substack{s=t \\ d+s=v}} + \mathrm{o}(h) \\
&= p_{ij}\left(t, t, v, z\right) - p_{ij}\left(t, t, v, z\right)\mu_{i\cdot}(t, v)h + \sum_{\substack{k\in\mathcal{J} \\ k\neq i}} p_{kj}\left(t, t, 0, z\right)\mu_{ik}(t, v)h + \mathrm{o}(h) \\
&= 1_{\{i=j\}}1_{\{v\le z\}} - 1_{\{i=j\}}1_{\{v\le z\}}\mu_{i\cdot}(t, v)h + 1_{\{i\neq j\}}\mu_{ij}(t, v)h + \mathrm{o}(h),
\end{aligned}$$

where, in the second equation, we applied Kolmogorov's backwards differential equation, Proposition 3.2.4.

Before we put it all together, let (t_0, u) and t be fixed, and define

$$F_{ij}(z, h) = p_{ij}\left(t_0, t + h, u, z + h\right),$$

and note that $F_{ij}(\mathrm{d}z, h)$ is a measure, and the support is a subset of $[0, u + t - t_0]$. Since the function $F_{ij}(z, h)$ is continuous in h, we have by Lemma 3.A.2 that $F_{ij}(\mathrm{d}z, h) \overset{\text{weak}}{\longrightarrow} F_{ij}(\mathrm{d}z, 0)$ for $h \to 0$.

Putting it all together, we insert into the Chapman-Kolmogorov equations (3.A.1),

$$\begin{aligned}
&F_{ij}(z, 0) \\
&= \sum_{\ell\in\mathcal{J}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, -h)p_{\ell j}\left(t - h, t, v - h, z\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell \in \mathcal{J}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, -h)\Big(1_{\{\ell=j\}} 1_{\{v \le z\}} \\
&\qquad - 1_{\{\ell=j\}} 1_{\{v \le z\}} \mu_{\ell \cdot}(t, v) h + 1_{\{\ell \neq j\}} \mu_{\ell j}(t, v) h + \mathrm{o}(h)\Big) \\
&= F_{ij}(z, -h) - h \int_0^z F_{ij}(\mathrm{d}v, -h) \mu_{j \cdot}(t, v) + h \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, -h) \mu_{\ell j}(t, v) + \mathrm{o}(h)
\end{aligned}
$$

and rearrange to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}h} F_{ij}(z, 0) \\
&= \lim_{h \searrow 0} \frac{F_{ij}(z, 0) - F_{ij}(z, -h)}{h} \\
&= \lim_{h \searrow 0} \left(- \int_0^z F_{ij}(\mathrm{d}v, -h) \mu_{j \cdot}(t, v) + \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, -h) \mu_{\ell j}(t, v) + \frac{1}{h} \mathrm{o}(h) \right) \\
&= - \int_0^z F_{ij}(\mathrm{d}v, 0) \mu_{j \cdot}(t, v) + \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} \int_0^{u+t-t_0} F_{i\ell}(\mathrm{d}v, 0) \mu_{\ell j}(t, v),
\end{aligned}
$$

where we used Lemma 3.A.2 for the last equality. □

Chapter 4

Continuous affine processes: transformations, Markov chains and life insurance

This chapter is based on the paper [7].

Abstract

Affine processes possess the property that expectations of exponential affine transformations are given by a set of Riccati differential equations, which is the main feature of this popular class of processes. In this paper we generalise these results for expectations of more general transformations. This is of interest in e.g. doubly stochastic Markov models, in particular in life insurance. When using affine processes for modelling the transition rates and interest rate, the results presented allow for easy calculation of transition probabilities and expected present values.

4.1 Introduction

The main results of this paper is a generalisation of a result from [17] which provides differential equations for calculating the expectation of a certain transformation of affine processes. We present a, to the author's knowledge, new proof of the result, which is constructive and allows for generalisation to expectations of more general transformations. These results are interesting as a part of the mathematical study of the class of affine processes. The results are also applicable, and the class of transformations are useful for the use of affine processes as transition rates in doubly stochastic Markov chains. Such Markov chains are used in e.g. life insurance mathematics and credit risk modelling, where it is also possible to model e.g. the interest rate jointly as an affine process. In life insurance, one of the main contributions from these results is the ability to handle dependent interest and transition rates while exploiting the structure of affine processes.

In traditional life insurance mathematics, a finite state Markov chain is often chosen to represent the state of the insured. Associating payments with sojourns in states and transitions between states, one can easily find the expected present value of life insurance liabilities when an interest rate is given. Traditionally these models have been studied with a deterministic interest rate and deterministic transition rates, however in recent life insurance mathematics, stochastic modelling of the interest and transition rates has gained attention. This is of particular interest either if one wants to study the risk associated with changes in the underlying interest and transition rates, and/or if one wants to hedge this risk in securities based on these same underlying rates. A basic reference on the life insurance setup with stochastic transition rates is [40], where the underlying rates are modelled by a finite state Markov chain. In particular, dependence between the rates is allowed. Basic treatment of stochastic interest rates applied in life insurance is given in [38], and for stochastic mortality rates see e.g. [14]. For combined models for stochastic interest and mortality rates see [15] and [2]. Common for several of the references mentioned is that interest and mortality rates are modelled as affine processes. This class of processes leads to mathematically tractable models, where one can solve a system of ordinary differential equations instead of partial differential equations in order to find expected present values. The results presented in this paper allow for the application of affine processes in more general life insurance models, thus making it easier to calculate expected present values.

Finite state Markov chains are also used when modelling credit risk, see e.g. [30]. A basic credit risk model is a two state Markov chain, where a jump from the initial state represents a default. A popular extension of this model is to let the default transition rate be modelled as a stochastic process itself such that it is possible for it to be dependent on the interest rate and other economic factors. This approach is studied in [33] where various Markov chain models are considered. A more general treatment of the Markov chain approach to credit risk modelling with stochastic transition rates is given in [32], and it is shown how prices generally satisfy a system of partial differential equations. In both papers, it is shown how one can benefit from affine stochastic processes as transition intensities and economic factors. If the model is particularly simple, the Riccati equations and a result from [17] can be used to reduce the problem of solving a system of partial differential equations to that of solving a system of ordinary differential equations. The results presented in this paper generalise these methods. This allows us to find prices in more general decrement Markov chain models solving only ordinary differential equations instead of partial differential equations.

The paper is structured as follows. In Section 4.2 we introduce the basics of affine processes that are multidimensional, continuous and time-inhomogeneous, allowing us to state the main results precisely. From this we present the main results: We give a new proof for a result from [17] on the expectation of a transformation of affine processes and use this to formulate and prove a more general result on affine processes. The results are discussed and further generalisations are considered. These results can be applied for doubly stochastic Markov chains, in particular in life insurance and credit risk modelling. In Section 4.3 we show how the results can be applied for calculation of transition probabilities in certain doubly stochastic Markov chains, and we present an example, which is studied numerically as well. In Section 4.4 we show how to apply the results for valuation of life insurance contracts.

4.2 Continuous affine processes

In this section we study continuous affine processes, inspired by Chapter 10 in [19]. In Section 4.2.1 we give a proper definition and then present Theorem 4.2.2, which is the usual theorem and basic property about affine processes, adapted to our setup. This gives e.g. bond prices and survival probabilities in the case where the short rate, respectively the mortality rate, is a stochastic affine process. This provides the basis for the main results which are presented in Section 4.2.2.

4.2.1 Definition and set-up

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in\mathbb{R}_+}, P)$ be a filtered probability space, satisfying the usual conditions, and denote by $\mathbf{W} = (W(t))_{t\in\mathbb{R}_+}$ an adapted d-dimensional Wiener process. Let $\mathbf{X} = (X(t))_{t\in\mathbb{R}_+}$ be a d-dimensional stochastic process satisfying the stochastic differential equation,

$$\mathrm{d}X(t) = \delta(t, X(t))\mathrm{d}t + \rho(t, X(t))\mathrm{d}W(t), \tag{4.2.1}$$

with $X(0) = x \in \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}^d$ is the state space. The functions $\delta : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}^d$ and $\rho : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}^{d\times d}$ are assumed to be measurable. We have not assumed any Lipschitz continuity of the parameter functions δ and ρ, and we allow for some discontinuities. In this paper we make the crucial assumption that $\mathbf{X}$ exists for all start values $x \in \mathcal{X}$, and start time points.

A continuous stochastic process $\mathbf{X}$ is affine on $[0, T]$ if the $\mathcal{F}(t)$-conditional characteristic function of $X(T)$ has an exponential affine form, and this is made precise in the following definition. We think of T as a fixed, long time horizon.

Definition 4.2.1. *The process* $\mathbf{X}$*, with initial value* $X(0) = x$*, is* affine *on* $[0, T]$ *if there exist functions* ϕ *and* ψ *such that for all* $x \in \mathcal{X}$*,* $0 \le t \le T$ *and* $z \in \mathbb{R}^d$*,*

$$\mathrm{E}\left[e^{\mathrm{i}z^\top X(T)} \,\middle|\, \mathcal{F}(t) \right] = e^{\phi(t,T,z)+\psi(t,T,z)^\top X(t)} \tag{4.2.2}$$

holds, where $\phi(t, T, z)$ *is* $\mathbb{C}$*-valued and* $\psi(t, T, z)$ *is* $\mathbb{C}^d$*-valued.*

It can be shown that for $\mathbf{X}$ to be affine, it is a necessary condition that the drift and diffusion parameter functions are affine of the form

$$\begin{aligned} \delta(t,x) &= b(t) + \sum_{i=1}^{d} \beta_i(t) x_i = b(t) + \mathcal{B}(t)x, \\ \rho(t,x)\rho(t,x)^\top &= a(t) + \sum_{i=1}^{d} \alpha_i(t) x_i, \end{aligned} \tag{4.2.3}$$

for some vector functions $b : \mathbb{R}_+ \to \mathbb{R}^d$ and $\beta_i : \mathbb{R}_+ \to \mathbb{R}^d$, $i = 1, \dots, d$, and matrix functions $a : \mathbb{R}_+ \to \mathbb{R}^{d \times d}$ and $\alpha_i : \mathbb{R}_+ \to \mathbb{R}^{d \times d}$, $i = 1, \dots, d$. We have $\mathcal{B}(t) = (\beta_1(t), \dots, \beta_d(t))$, i.e. column i equals $\beta_i(t)$. We use the notation $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

From now on we assume that (4.2.3) holds, and that the parameter functions b, β_i, a, α_i, $i = 1, \dots, i$ are bounded and piecewise continuous.

In order to allow for modelling flexibility, we consider affine transformations of $\mathbf{X}$. Let $p \geq 1$, and let c and Γ be a vector and matrix function respectively, $c : \mathbb{R}_+ \to \mathbb{R}^p$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p \times d}$. Define the p-dimensional process $\mathbf{Y}$ by

$$Y(t) = c(t) + \Gamma(t) X(t). \tag{4.2.4}$$

We assume that c_j and Γ_{ji}, $j = 1, \dots, p$, $i = 1, \dots, d$ are piecewise continuous and bounded, with limits everywhere. It is noted, that if $\Gamma(t)$ has a left inverse for all t, or, equivalently if it is injective for all t, then the affine transformation $\mathbf{Y}$ is an affine process on some state-space, as can be seen by the definition (4.2.2).

We use the column sums of Γ, so define the d-dimensional function $\gamma(t) = \mathbb{1}^\top \Gamma(t)$, where $\mathbb{1} = (1, \dots, 1)^\top$ is a column vector with 1 in all entries. Then $\gamma_i(t) = \mathbb{1}^\top \Gamma(t) e_i$ is the sum of column i in Γ, where e_i is the ith standard unit vector. Using this notation, we have $\mathbb{1}^\top Y(t) = \mathbb{1}^\top c(t) + \sum_{i=1}^{d} \gamma_i(t) X_i(t)$.

The following theorem states the essential feature of the affine processes, and the result is the reason for the great interest in affine processes.

Theorem 4.2.2. *Let* $\mathbf{X}$ *be an affine process. If* (ϕ, ψ) *exists as a solution to* (4.2.7) *and the stochastic process*

$$t \mapsto e^{-\int_0^t \mathbb{1}^\top Y(s) \mathrm{d}s + \phi(t,T) + \psi(t,T)^\top X(t)} \tag{4.2.5}$$

is a martingale, then

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s) \mathrm{d}s} \,\middle|\, \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\top X(t)}, \tag{4.2.6}$$

where the functions ϕ and ψ solve the Riccati differential equations,

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t,T) &= -\frac{1}{2}\psi(t,T)^\top a(t)\psi(t,T) - b(t)^\top\psi(t,T) + \mathbb{1}^\top c(t),\\
\phi(T,T) &= 0,\\
\frac{\partial}{\partial t}\psi_i(t,T) &= -\frac{1}{2}\psi(t,T)^\top \alpha_i(t)\psi(t,T) - \beta_i(t)^\top\psi(t,T) + \gamma_i(t), \quad i=1,\dots,d,\\
\psi(T,T) &= 0.
\end{aligned} \tag{4.2.7}$$

For the case of time-homogeneous parameters, the theorem is proven in [19], and this can be generalised to the case of time-inhomogeneous piece-wise continuous parameter functions presented here.

We briefly consider some important applications of Theorem 4.2.2. Consider the case where $\mathbf{X}$ is two-dimensional and jointly models the short rate and mortality rate, i.e. $X(t) = (X_1(t), X_2(t)) = (r(t), \mu(t))$. Choosing

$$\Gamma(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and $c(t) = 0$, we obtain by (4.2.4) that $Y^r(t) = r(t)$, i.e. $\mathbf{Y}^r$ is the short rate. If we work under a pricing measure, Theorem 4.2.2 can be applied to obtain bond prices,

$$P(t,T) = \mathrm{E}\left[e^{-\int_t^T Y^r(s)\mathrm{d}s}\,\middle|\, \mathcal{F}(t)\right] = e^{\Phi^r(t,T)+\Psi^r(t,T)^\top X(t)},$$

where $P(t,T)$ is the market value at time t of a payment of 1 at time T. Similarly, by choosing another Γ, we can let $\mathbf{Y}^\mu$ be the mortality rate, i.e. $Y^\mu(t) = \mu(t)$. Then, if we work under the physical measure, the survival probabilities are obtained,

$$S(t,T) = \mathrm{E}\left[e^{-\int_t^T Y^\mu(s)\mathrm{d}s}\,\middle|\, \mathcal{F}(t)\right] = e^{\Phi^\mu(t,T)+\Psi^\mu(t,T)^\top X(t)},$$

where $S(t,T)$ is the probability of surviving from time t till time T. Finally, choosing Γ as the identity matrix, $\mathbf{Y}$ is two-dimensional and equal to $\mathbf{X}$, hence jointly modelling the short rate and the mortality rate. In this case, if we work under a pricing measure, we obtain the price of a pure endowment,

$$V(t) = \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s}\,\middle|\, \mathcal{F}(t)\right] = e^{\Phi(t,T)+\Psi(t,T)^\top X(t)},$$

where $V(t)$ is the expected present value of a payment of 1 at time T, conditional on survival to time T given that the individual is alive at time t. Note that here the interest and mortality rate can be dependent. If we consider the special case of independence between the interest and mortality rate, we obtain $V(t) = P(t,T)S(t,T)$ and in particular that

$$\begin{aligned}
\Phi(t,T) &= \Phi^r(t,T) + \Phi^\mu(t,T),\\
\Psi(t,T) &= \Psi^r(t,T) + \Psi^\mu(t,T).
\end{aligned}$$

For an endowment insurance, where there is a payment upon death as well, Theorem 4.2.2 is not sufficient for pricing, and we have to use Theorem 4.2.3 below.

4.2.2 Main results

In this section we present the main contributions of the paper, which is basically a new proof of Theorem 4.2.3 below. The advantage of the proof is that it is constructive and the idea of the proof can be reapplied which allows us to state and prove Theorem 4.2.7, which, to the author's knowledge, is a new result. Together with Theorem 4.2.2 the two theorems presented in this section have applications for Markov chain modelling with stochastic transition rates, e.g. for life insurance valuation or credit risk modelling.

Consider the transformation

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) \middle| \mathcal{F}(t) \right],$$

for $k \in \{1, \dots, p\}$ and $u \in [t, T]$, and recall that $\mathbf{Y}$ is defined by (4.2.4). In [17] differential equations are derived for the expectation under slightly different conditions for time-homogeneous affine jump diffusions, though only for the case where $u = T$. The result presented here is for a general $u \in [t, T]$ and the case of continuous affine processes with time-inhomogeneous coefficients. The system of differential equations found in [17] is essentially the same as the one presented in Theorem 4.2.3.

Theorem 4.2.3. *Let $k \in \{1, \dots, p\}$ and $u \in [t, T]$. Then, if either Assumption 4.2.4 or Assumption 4.2.5 holds, and under the conditions of Theorem 4.2.2, it holds that*

$$\begin{aligned} &\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) \middle| \mathcal{F}(t) \right] \\ &= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \left(A^k(t,T,u) + B^k(t,T,u)^\top X(t) \right), \end{aligned} \tag{4.2.8}$$

where (ϕ, ψ) is a solution to (4.2.7) and (A^k, B^k) solves the linear differential equation system,

$$\begin{aligned} \frac{\partial}{\partial t} A^k(t,T,u) &= -\psi(t,T)^\top a(t) B^k(t,T,u) - b(t)^\top B^k(t,T,u), \\ A^k(u,T,u) &= e_k^\top c(u), \\ \frac{\partial}{\partial t} B_i^k(t,T,u) &= -\psi(t,T)^\top \alpha_i(t) B^k(t,T,u) - \beta_i(t)^\top B^k(t,T,u), \quad i = 1, \dots, d \\ B^k(u,T,u) &= e_k^\top \Gamma(u). \end{aligned} \tag{4.2.9}$$

The original proof of the result, from [17], is the classic one, which holds under the following assumption.

Assumption 4.2.4. *(Classic integrability condition) The process*

$$t \mapsto e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s+\phi(t,T)+\psi(t,T)^\top X(t)} \left(A^k(t,T,u)+B^k(t,T,u)^\top X(t)\right)$$

is a real martingale (i.e. not only a local martingale).

The assumption is that, when multiplied with $e^{-\int_0^t \mathbb{1}^\top Y(s)\mathrm{d}s}$, the right hand side of (4.2.8) is a martingale. Then, by an application of Itô's lemma to the martingale, and by setting the drift equal to zero, one can obtain the system of differential equations. Carrying out this proof, note that $A^k(t,T,u)$ and $B^k(t,T,u)$ are constant for $t > u$.

Instead of presenting the original proof, we give a new proof. Below is presented an outline of the proof, where certain details are omitted. The result holds under a slightly different integrability condition than in the original proof. It holds whenever we can interchange differentiation and expectation, and this is allowed if we can dominate an integrand. Then, the integrability condition is the following.

Assumption 4.2.5. *(New integrability condition) Let $t < u < T$ and let J be an open interval containing u. Then there exists a stochastic bound Z with finite expectation, such that*

$$\sup_{u' \in J}\left\{e^{-\int_t^T \left(\mathbb{1}-e_k+\left(1_{(s\le u')}+1_{(s>u)}\right)e_k\right)^\top \Gamma(s)X(s)\mathrm{d}s} e_k^\top \Gamma(u')X(u')\right\} \le Z. \tag{4.2.10}$$

Proof of Theorem 4.2.3. (Outline) The theorem clearly holds for $u = t$. We prove the case $c = 0$ and $u < T$. So, assume that $t < u \le r < T$. Define now the matrix function $\tilde{\Gamma}(t,u,r)$ by

$$\tilde{\Gamma}_{ji}(t,u,r) = \begin{cases} \Gamma_{ji}(t), & j \ne k, \\ (1_{(-\infty,u]}(t) + 1_{(r,\infty)}(t))\Gamma_{ji}(t), & j = k, \end{cases}$$

for $j = 1,\dots,p$ and $i = 1,\dots,d$. Notice that $\Gamma(t) = \tilde{\Gamma}(t,u,r)$ when $u = r$. We consider k fixed throughout the proof and suppress the functions' dependence on k. First, using the definition of $\tilde{\Gamma}$, and then applying Theorem 4.2.2,

$$\begin{aligned} &\mathrm{E}\left[e^{-\int_t^T \left(\mathbb{1}-e_k+\left(1_{(s\le u)}+1_{(s>r)}\right)e_k\right)\Gamma(s)X(s)\mathrm{d}s}\,\middle|\,\mathcal{F}(t)\right] \\ &= \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \tilde{\Gamma}(s,u,r)X(s)\mathrm{d}s}\,\middle|\,\mathcal{F}(t)\right] \\ &= e^{\tilde{\phi}(t,T,u)+\tilde{\psi}(t,T,u)^\top X(t)}. \end{aligned} \tag{4.2.11}$$

Here, $\tilde{\phi}$ and $\tilde{\psi}$ solve the differential equations (4.2.7) with $\tilde{\gamma} = \mathbb{1}^\top\tilde{\Gamma}$ in the place of γ, and we have added u as an argument to make clear the dependence of u in the solution. The solution also depends on r, but that is suppressed in the notation.

We apply $-\frac{\partial}{\partial u}$ on both sides of (4.2.11). On the left hand side we obtain,

$$\begin{aligned}
&-\frac{\partial}{\partial u}\mathrm{E}\left[e^{-\int_t^T\left(\mathbb{1}-e_k+\left(1_{(s\le u)}+1_{(s>r)}\right)e_k\right)^\top\Gamma(s)X(s)\mathrm{d}s}\middle|\mathcal{F}(t)\right]\\
&=-\frac{\partial}{\partial u}\mathrm{E}\left[e^{-\int_t^T\left(\mathbb{1}-e_k+1_{(s>r)}e_k\right)^\top\Gamma(s)X(s)\mathrm{d}s-\int_t^u e_k^\top\Gamma(s)X(s)\mathrm{d}s}\middle|\mathcal{F}(t)\right]\\
&=\mathrm{E}\left[e^{-\int_t^T\mathbb{1}^\top\tilde{\Gamma}(s,u,r)X(s)\mathrm{d}s}e_k^\top\Gamma(u)X(u)\middle|\mathcal{F}(t)\right],
\end{aligned}$$

where differentiation and expectation can be interchanged by the integrability condition (4.2.10). When $u=r$, this is the left hand side of (4.2.8) with $c=0$. For differentiation, we have implicitly assumed that $e_k^\top\Gamma(t)$ is continuous in a neighbourhood around u, however this is not a necessary assumption for the theorem.

Applying $-\frac{\partial}{\partial u}$ on the right hand side of (4.2.11), we obtain

$$\begin{aligned}
&-\frac{\partial}{\partial u}e^{\tilde{\phi}(t,T,u)+\tilde{\psi}(t,T,u)^\top X(t)}\\
&=e^{\tilde{\phi}(t,T,u)+\tilde{\psi}(t,T,u)^\top X(t)}\left(-\frac{\partial}{\partial u}\tilde{\phi}(t,T,u)+X(t)^\top\left(-\frac{\partial}{\partial u}\tilde{\psi}(t,T,u)\right)\right).
\end{aligned}$$

Let $\tilde{A}(t,T,u)=-\frac{\partial}{\partial u}\tilde{\phi}(t,T,u)$ and $\tilde{B}(t,T,u)=-\frac{\partial}{\partial u}\tilde{\psi}(t,T,u)$. We find, using (4.2.7),

$$\begin{aligned}
&\tilde{A}(t,T,u)\\
&=-\frac{\partial}{\partial u}\int_T^t\left(-\frac{1}{2}\tilde{\psi}(s,T,u)^\top a(s)\tilde{\psi}(s,T,u)-b(s)^\top\tilde{\psi}(s,T,u)\right)\mathrm{d}s\\
&=\int_T^t\left(-\tilde{\psi}(s,T,u)^\top a(s)\tilde{B}(s,T,u)-b(s)^\top\tilde{B}(s,T,u)\right)\mathrm{d}s.
\end{aligned}$$

Then, for $i=1,\dots,d$,

$$\begin{aligned}
&\tilde{B}_i(t,T,u)\\
&=-\frac{\partial}{\partial u}\int_T^t\left(-\frac{1}{2}\tilde{\psi}(s,T,u)^\top\alpha_i(s)\tilde{\psi}(s,T,u)-\beta_i(s)^\top\tilde{\psi}(s,T,u)+\tilde{\gamma}_i(s,u,r)\right)\mathrm{d}s\\
&=\int_T^t\left(-\tilde{\psi}(s,T,u)^\top\alpha_i(s)\tilde{B}(s,T,u)-\beta_i(s)^\top\tilde{B}(s,T,u)\right)\mathrm{d}s-\frac{\partial}{\partial u}\int_T^t\sum_{j=1}^p\tilde{\Gamma}_{ji}(s,u,r)\mathrm{d}s\\
&=\int_T^t\left(-\tilde{\psi}(s,T,u)^\top\alpha_i(s)\tilde{B}(s,T,u)-\beta_i(s)^\top\tilde{B}(s,T,u)\right)\mathrm{d}s-1_{(t<u)}\frac{\partial}{\partial u}\int_u^t\Gamma_{ki}(s)\mathrm{d}s\\
&=\int_T^t\left(-\tilde{\psi}(s,T,u)^\top\alpha_i(s)\tilde{B}(s,T,u)-\beta_i(s)^\top\tilde{B}(s,T,u)\right)\mathrm{d}s+1_{(t<u)}\Gamma_{ki}(u).
\end{aligned}$$

In particular, $\tilde{A}(T,T,u)=0$ and $\tilde{B}(T,T,u)=0$, and since the integrands are linear in $\tilde{B}$ for $t>u$, we have $\tilde{A}(t,T,u)=0$ and $\tilde{B}(t,T,u)=0$ for all $t>u$ as well.

Now let $r = u$. Then $\Gamma = \tilde{\Gamma}$ and thus $\psi = \tilde{\psi}$ and $\phi = \tilde{\phi}$, where ϕ and ψ are solutions to (4.2.7). Letting $A^k(t,T,u) = \tilde{A}(t,T,u)$ and $B^k(t,T,u) = \tilde{B}(t,T,u)$, the system of differential equations (4.2.9) is obtained, when $c = 0$.

For $u = T$, the result holds as well, which can be shown by taking limits as $u \nearrow T$. The left and right hand side of (4.2.8) are continuous in u when $u < T$, and it can be shown that the continuity also holds in the limit as $u \nearrow T$.

For a general integrable function c, we can rewrite in terms of the functions ϕ^0, ψ, $A^{k,0}$ and B^k corresponding to the case $c = 0$. (The functions ψ and B^k do not depend on c.) We get,

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top (c(s)+\Gamma(s)X(s))\mathrm{d}s} e_k^\top \left(c(u) + \Gamma(u)X(u)\right) \Big| \mathcal{F}(t)\right] \\
&= e^{-\int_t^T \mathbb{1}^\top c(s)\mathrm{d}s} \Big(\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \Gamma(s)X(s)\mathrm{d}s} \Big| \mathcal{F}(t)\right] e_k^\top c(u) \\
&\qquad + \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \Gamma(s)X(s)\mathrm{d}s} e_k^\top \Gamma(u)X(u) \Big| \mathcal{F}(t)\right] \Big) \\
&= e^{-\int_t^T \mathbb{1}^\top c(s)\mathrm{d}s + \phi^{c=0}(t,T) + \psi(t,T)^\top X(t)} \Big(e_k^\top c(u) + A^{k,c=0}(t,T,u) + B^k(t,T,u)^\top X(t) \Big) \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \Big(A^k(t,T,u) + B^k(t,T,u)^\top X(t) \Big),
\end{aligned}$$

where the last equality sign holds whenever ϕ and A^k solve the differential equations (4.2.7) and (4.2.9), respectively. This completes the proof. □

The interesting result is the identity (4.2.8), and this can immediately be extended, which is done in the following corollary. Since the expectation operator is linear, we have

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \left(Y_k(u) + Y_l(v)\right) \Big| \mathcal{F}(t)\right] \\
&= \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) \Big| \mathcal{F}(t)\right] + \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_l(v) \Big| \mathcal{F}(t)\right],
\end{aligned}$$

for $l \in \{1, \ldots, p\}$ and $v \in [t,T]$. By Theorem 4.2.3, the two expectations on the right hand side can be calculated, thus enabling us to find the left hand side. However, using that the system of differential equations for (A,B) is linear, we can actually calculate the left hand side directly. This is stated in a general way in the following corollary to Theorem 4.2.3. For the corollary, note that for a finite linear combination of elements of the type $Y_k(u)$, there exist $q \geq 1$, vectors $\kappa^1, \ldots, \kappa^q \in \mathbb{R}^p$ and time points $u_1, \ldots, u_q \in [t,T]$ such that the linear combination can be written as $\sum_{l=1}^q {\kappa^l}^\top Y(u_l)$.

Corollary 4.2.6. *For $q \geq 1$ let $\kappa^1, \ldots, \kappa^q \in \mathbb{R}^p$ be vectors and let $u_1, \ldots, u_q \in [t,T]$ be time points. If the conditions of Theorem 4.2.3 are satisfied for all combinations of time* *points $u_1, \ldots, u_q$ and dimensions $k = 1, \ldots, p$, then*

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \sum_{l=1}^{q} {\kappa^l}^\top Y(u_l) \middle| \mathcal{F}(t) \right] \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \left(A(t,T) + B(t,T)^\top X(t) \right),
\end{aligned}$$

where (ϕ, ψ) is a solution to (4.2.7) and (A, B) solves the linear system of differential equations with jumps,

$$\begin{aligned}
\frac{\partial}{\partial t} A(t,T) &= -\psi(t,T)^\top a(t) B(t,T) - b(t)^\top B(t,T), \\
A(u_l-,T) &= A(u_l,T) + {\kappa^l}^\top c(u_l), \quad l = 1, \ldots, q \\
A(T,T) &= 0, \\
\frac{\partial}{\partial t} B_i(t,u) &= -\psi(t,T)^\top \alpha_i(t) B(t,u) - \beta_i(t)^\top B(t,u), \quad i = 1, \ldots, d \\
B(u_l-,T) &= B(u_l,T) + {\kappa^l}^\top \Gamma(u_l), \quad l = 1, \ldots, q \\
B(T,T) &= 0.
\end{aligned} \tag{4.2.12}$$

Proof. By linearity of the expectation operator and Theorem 4.2.3

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \sum_{l=1}^{q} {\kappa^l}^\top Y(u_l) \middle| \mathcal{F}(t) \right] \\
&= \sum_{l=1}^{q} \sum_{k=1}^{p} \kappa_k^l \, \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) \middle| \mathcal{F}(t) \right] \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \sum_{l=1}^{q} \sum_{k=1}^{p} \kappa_k^l \left(A^k(t,T,u_l) + B^k(t,T,u_l)^\top X(t) \right),
\end{aligned}$$

where we used Theorem 4.2.3 for the last equality, such that A^k and B^k solve (4.2.9). Now, let

$$\begin{aligned}
A(t,T) &= \sum_{l=1}^{q} \sum_{k=1}^{p} \kappa_k^l A^k(t,T,u_l), \\
B(t,T) &= \sum_{l=1}^{q} \sum_{k=1}^{p} \kappa_k^l B^k(t,T,u_l),
\end{aligned}$$

and recall that $A^k(t,T,u) = 0$ and $B^k(t,T,u) = 0$ for $t > u$ and any $k = 1,\dots,d$. Then for $t \le T$,

$$\begin{aligned}
A(t,T) &= \sum_{l=1}^{q} 1_{(t\le u_l)} \sum_{k=1}^{p} \Big\{ \kappa_k^l e_k^\top c(u_l) \\
&\quad + \int_{u_l}^{t} \Big(-\psi(s,T)^\top a(s) \kappa_k^l B^k(s,T,u_l) - b(s)^\top \kappa_k^l B^k(s,T,u_l) \Big)\, \mathrm{d}s \Big\} \\
&= \sum_{l=1}^{q} 1_{(t\le u_l)} {\kappa^l}^\top c(u_l) + \int_{T}^{t} \Big(-\psi(s,T)^\top a(s) B(s,T) - b(s)^\top B(s,T) \Big)\, \mathrm{d}s.
\end{aligned}$$

The calculation for $B(t,T)$ is analogous, and the result is obtained. □

The functions A and B can be compared to A^k and B^k from Theorem 4.2.3. As is seen from the proof, they are a linear function of functions $A^k(t,T,u_l)$ and $B^k(t,T,u_l)$ that solves the linear differential equation system (4.2.9). Thus the linear differential equation system for A and B is also as in (4.2.9), except the different boundary conditions. In the system of differential equations for A and B extra jumps occur, which we can consider as gluing boundary conditions. These are exactly the boundary conditions of each of the system of differential equations for the functions A^k and B^k that add up to A and B.

The presentation in the corollary of the transformation yields another insight. We consider an affine transformation $\mathbf{Y}$ of $\mathbf{X}$, but the results also hold for the more general

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \kappa^\top X(u) \Big| \mathcal{F}(t) \right].$$

In this case, the boundary conditions in (4.2.12) are changed, such that $c(t) = 0$ and $\Gamma(u) = 1$. To give an intuition for the validity of this, consider for simplicity the case where $c(t) = 0$, and where $\Gamma(u)$ has a left inverse, $\Gamma^{-1}(u)$. Then apply the corollary for $q = 1$ and $\kappa = \tilde{\kappa}\Gamma^{-1}(u)$ for any $\tilde{\kappa}$. We get,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \kappa^\top Y(u) \Big| \mathcal{F}(t) \right] = \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \tilde{\kappa}^\top X(u) \Big| \mathcal{F}(t) \right].$$

In other words, the affine transformation of $\mathbf{X}$ in the exponentiated integral, does not need to be the same as the affine transformation outside the exponentiation.

We now present a result which, to the author's knowledge, is new. Theorem 4.2.2 is the essential result about affine processes in a multidimensional framework. This result was used to obtain Theorem 4.2.3, by differentiation in a specific way. This approach can be applied again, and the following theorem is obtained.

Theorem 4.2.7. *Let $k, l \in \{1, \dots, p\}$ and $u, v \in [t, T]$. Assuming sufficient integrability, then*

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) Y_l(v) \middle| \mathcal{F}(t)\right] \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \\
&\quad \times \Big\{ \left(A^k(t,T,u) + B^k(t,T,u)^\top X(t)\right) \left(A^l(t,T,v) + B^l(t,T,v)^\top X(t)\right) \\
&\qquad + C^{kl}(t,T,u,v) + D^{kl}(t,T,u,v)^\top X(t) \Big\},
\end{aligned} \tag{4.2.13}$$

where $\left(A^k(t,T,u), B^k(t,T,u)\right)$ and $\left(A^l(t,T,v), B^l(t,T,v)\right)$ are given by Theorem 4.2.3. The functions $C^{kl}(t,T,u,v)$ and $D^{kl}(t,T,u,v)$ are solutions to the following system of differential equations,

$$\begin{aligned}
\frac{\partial}{\partial t} C^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top a(t) B^l(t,T,v) - \psi(t,T)^\top a(t) D^{kl}(t,T,u,v) \\
&\quad - b(t)^\top D^{kl}(t,T,u,v), \\
C^{kl}(u \wedge v, T, u, v) &= 0, \\
\frac{\partial}{\partial t} D_i^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top \alpha_i(t) B^l(t,T,v) - \psi(t,T)^\top \alpha_i(t) D^{kl}(t,T,u,v) \\
&\quad - \beta_i(t)^\top D^{kl}(t,T,u,v), \quad i = 1, \dots, n \\
D^{kl}(u \wedge v, T, u, v) &= 0,
\end{aligned}$$

using the notation $x \wedge y = \min\{x, y\}$.

Proof. (Outline) The proof is analogous to the proof of Theorem 4.2.3. If either $u = t$ or $v = t$, the result follows from Theorem 4.2.3. We first prove the result for the case $c(t) = 0$ and $u, v < T$, so assume that $u, v \in (t, T)$ and $u \neq v$. As in the proof of Theorem 4.2.3, define now the matrix function $\tilde{\Gamma}^k(t, u, r)$ by

$$\tilde{\Gamma}_{ji}^k(t,u,r) = \begin{cases} \Gamma_{ji}(t) & j \neq k \\ \left(1_{(-\infty,u]}(t) + 1_{(r,\infty)}(t)\right) \Gamma_{ji}(t) & j = k \end{cases},$$

for $j = 1, \dots, p$ and $i = 1, \dots, d$. Notice that $\Gamma(t) = \tilde{\Gamma}^k(t, u, r)$ when $u = r$. An application of Theorem 4.2.3 yields

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \tilde{\Gamma}^k(s,u,r) X(s)\mathrm{d}s} e_l^\top \tilde{\Gamma}^k(v,u,r) X(v) \middle| \mathcal{F}(t)\right] \\
&= e^{\tilde{\phi}(t,T)+\tilde{\psi}(t,T)^\top X(t)} \left(\tilde{A}^l(t,T,v) + \tilde{B}^l(t,T,v)^\top X(t)\right).
\end{aligned}$$

Here, $\tilde{\phi}$ and $\tilde{\psi}$ solve the differential equations (4.2.7) with $\tilde{\gamma} = \mathbb{1}^\top \tilde{\Gamma}^k$ in place of γ. The functions $\tilde{A}^l$ and $\tilde{B}^l$ solve the differential equations (4.2.9) with boundary conditions

$\tilde{A}^l(v,T,v) = 0$ and $\tilde{B}^l(v,T,v) = e_l^\top \tilde{\Gamma}^k(v,u,r)$. Note that the functions $\tilde{\phi}$, $\tilde{\psi}$, $\tilde{A}^l$ and $\tilde{B}^l$ all depend on u. Applying $-\frac{\partial}{\partial u}$ on both sides, we obtain, after a few calculations similar to those in the proof of Theorem 4.2.3,

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \tilde{\Gamma}^k(s,u,r)X(s)\mathrm{d}s} e_k^\top \Gamma(u) X(u) e_l^\top \tilde{\Gamma}^k(v,u,r) X(v) \Big| \mathcal{F}(t) \right] \\
&= e^{\tilde{\phi}(t,T)+\tilde{\psi}(t,T)^\top X(t)} \\
&\quad \times \Bigg\{ \left(-\frac{\partial}{\partial u}\tilde{\phi}(t,T) - X(t)^\top \frac{\partial}{\partial u}\tilde{\psi}(t,T) \right) \left(\tilde{A}^l(t,T,v) + \tilde{B}^l(t,T,v)^\top X(t) \right) \\
&\qquad - \frac{\partial}{\partial u}\tilde{A}^l(t,T,v) - X(t)^\top \frac{\partial}{\partial u}\tilde{B}^l(t,T,v) \Bigg\}.
\end{aligned}$$

From the proof of Theorem 4.2.3 it follows that $\tilde{A}^k(t,T,u) = -\frac{\partial}{\partial u}\tilde{\phi}(t,T)$ and $\tilde{B}^k(t,T,u) = -\frac{\partial}{\partial u}\tilde{\psi}(t,T)$ are solutions to the differential equations (4.2.9), with boundary conditions $\tilde{A}^k(u,T,u) = 0$ and $\tilde{B}^k(u,T,u) = e_k^\top \Gamma(u)$, and $\tilde{\phi}$ and $\tilde{\psi}$ in the place of ϕ and ψ.

Now, let $\tilde{C}(t,T,u,v) = -\frac{\partial}{\partial u}\tilde{A}^l(t,T,v)$ and $\tilde{D}(t,T,u,v) = -\frac{\partial}{\partial u}\tilde{B}^l(t,T,v)$. By straightforward differentiation, we find

$$\begin{aligned}
\tilde{C}(t,T,u,v) &= -\frac{\partial}{\partial u}\int_v^t \left(-\tilde{\psi}(s,T)^\top a(s)\tilde{B}^l(s,T,v) - b(s)^\top \tilde{B}^l(s,T,v) \right) \mathrm{d}s \\
&= \int_v^t \Bigg\{ -\tilde{B}^k(s,T,u)^\top a(s)\tilde{B}^l(s,T,v) \\
&\qquad - \tilde{\psi}(s,T)^\top a(s)\tilde{D}(t,T,u,v) - b(s)^\top \tilde{D}(s,T,u,v) \Bigg\} \mathrm{d}s.
\end{aligned}$$

For $\tilde{D}$ we find $\tilde{D}_i$ for $i = 1,\ldots,d$,

$$\begin{aligned}
\tilde{D}_i(t,T,u,v) &= -\frac{\partial}{\partial u}\int_v^t \left(-\tilde{\psi}(s,T)^\top \alpha_i(s)\tilde{B}^l(s,T,v) - \beta_i(s)^\top \tilde{B}^l(s,T,v) \right) \mathrm{d}s \\
&= \int_v^t \Bigg\{ -\tilde{B}^k(s,T,u)^\top \alpha_i(s)\tilde{B}^l(s,T,v) \\
&\qquad - \tilde{\psi}(s,T)^\top \alpha_i(s)\tilde{D}(s,T,u,v) - \beta_i(s)^\top \tilde{D}(s,T,u,v) \Bigg\} \mathrm{d}s.
\end{aligned}$$

We have $\tilde{D}(t,T,u,v) = 0$ for $t > v$. Since $\tilde{B}^k(t,T,u) = 0$ for $t > u$, we also have $\tilde{D}(t,T,u,v) = 0$ for $t > u$. Similarly $\tilde{C}(t,T,u,v) = 0$ for $t > u \wedge v$.

Let $r = u$. Then $\Gamma = \tilde{\Gamma}^k$ and thus $\phi = \tilde{\phi}$ and $\psi = \tilde{\psi}$, where (ϕ,ψ) solves (4.2.7). In this case, $A^i(t,T,\eta) = \tilde{A}^i(t,T,\eta)$ and $B^i(t,T,\eta) = \tilde{B}^i(t,T,\eta)$ for $(i,\eta) \in \{(k,u),(l,v)\}$, where (A^i,B^i) solves (4.2.9). Also, in this case, let $C^{kl} = \tilde{C}$ and $D^{kl} = \tilde{D}$. The result is now obtained for $c = 0$ and $u, v < T$.

If $u = T$, $v = T$ or $u = v$, then take limits for $u \nearrow T$, $v \nearrow T$ or $u \to v$, respectively, which yields the result, since both the left hand and right hand side of (4.2.13) can be shown to be continuous in the arguments u and v on $[t,T]$.

The extension to $c \neq 0$ can be done analogously to the proof of Theorem 4.2.3. First use the linearity of the expectation operator, second apply Theorems 4.2.2 and 4.2.3, and last verify the differential equations. This completes the proof. $\square$

4.2.3 Generalisations of the transformations

Theorem 4.2.2 provides the relation (4.2.6), repeated here,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{\phi(t,T)+\psi(t,T)^\top X(t)},$$

where the functions ϕ and ψ can be found by solving a set of Riccati differential equations, (4.2.7). By a differentiation argument, as carried out in the proof, Theorem 4.2.3 gives us the relation (4.2.8), repeated here,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) \middle| \mathcal{F}(t) \right] = e^{\phi(t,T)+\psi(t,T)^\top X(t)} \left(A^k(t,T,u) + B^k(t,T,u)^\top X(t) \right).$$

The functions A^k and B^k are given by a set of differential equations, (4.2.9). Since they arise through a differentiation argument, we essentially think of them as ϕ and ψ differentiated, respectively. By an application of the exact same differentiation technique, but to relation (4.2.8) instead of (4.2.6), we then obtained Theorem 4.2.7, which is the relation

$$\begin{aligned} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) Y_l(v) \middle| \mathcal{F}(t) \right] &= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \\ \times \Big\{ \left(A^k(t,T,u) + B^k(t,T,u)^\top X(t) \right) &\left(A^l(t,T,v) + B^l(t,T,v)^\top X(t) \right) \\ + C^{kl}(t,T,u,v) + D^{kl}(t,T,u,v)^\top X(t) \Big\}, \end{aligned}$$

i.e. (4.2.13). The functions C^{kl} and D^{kl} are given by a set of differential equations. Again, they arise because of a differentiation, and we essentially think of them as A^k and B^k (or A^l and B^l) differentiated, respectively. This can also be seen from the proofs given.

There is no particular reason to stop here. We can apply the differentiation technique to (4.2.13), and obtain an expression for

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) Y_l(v) Y_r(w) \middle| \mathcal{F}(t) \right],$$

for some $r \in \{1, \dots, p\}$ and $w \in [t, T]$. To find the expression, one must apply the differentiation technique to the right hand side of (4.2.13). The result is obtainable, but the notation and number of differential equations grow with every differentiation, and thus becomes even more cumbersome. In principle, one can reapply the technique, and obtain expressions for any expectation of the form,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_{k_1}(u_1) \cdots Y_{k_q}(u_q) \middle| \mathcal{F}(t) \right], \tag{4.2.14}$$

for $k_1, \dots, k_q \in \{1, \dots, p\}$, $u_1, \dots, u_q \in [t, T]$ and $q \geq 0$. We can count the number of differential equations that need to be solved when using this approach. For the expression (4.2.6), corresponding to the case $q = 0$, the functions ϕ and ψ must be found, which is a system of differential equations of dimension $d + 1$. For the second expression, (4.2.8) (corresponding to $q = 1$), the functions A^k and B^k must also be found, which is $d + 1$ extra dimensions, in total $2(d + 1)$. For the third expression, (4.2.13) (corresponding to $q = 2$), A^l and B^l as well as C^{kl} and D^{kl} must be found, which is $2(d + 1)$ extra equations, in total $4(d + 1)$. It seems that the dimension of the system of differential equations that needs to be solved is increasing exponentially.

In the relation (4.2.13), one can choose $k = l$ and $u = v$, to obtain

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s) \mathrm{d}s} Y_k(u)^2 \middle| \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\top X(t)} \\
&\quad \times \left\{ \left(A^k(t, T, u) + B^k(t, T, u)^\top X(t) \right)^2 + C^{kk}(t, T, u, u) + D^{kk}(t, T, u, u)^\top X(t) \right\}.
\end{aligned}$$

Similarly, for particular choices of $k_1, \dots, k_q$ and $u_1, \dots, u_q$ in (4.2.14), one can obtain all moments and combinations of moments of different $Y_k(u)$. Using linearity of the expectation operator, one can use this to construct any polynomial in $Y_k(u)$.

A special case of transformations are moments of affine processes. For a process $\mathbf{Y}$ consider the modified process,

$$\hat{Y}(t) = 1_{\{u,v\}}(t) Y(t) = \hat{c}(t) + \hat{\Gamma}(t) X(t),$$

where $\hat{c}(t) = 1_{\{u,v\}}(t) c(t)$ and $\hat{\Gamma}(t) = 1_{\{u,v\}}(t) \Gamma(t)$. Then it holds that

$$\mathrm{E}\left[Y_k(u) \middle| \mathcal{F}(t) \right] = \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \hat{Y}(s) \mathrm{d}s} \hat{Y}_k(u) \middle| \mathcal{F}(t) \right] = A^k(t, T, u) + B^k(t, T, u)^\top X(t),$$

where $\phi(t, T) = \psi(t, T) = 0$, because $\hat{c}$ and $\hat{\Gamma}$ equal zero almost surely with respect to the Lebesgue measure. The system of differential equations for A^k and B^k simplifies to

$$\begin{aligned}
\frac{\partial}{\partial t} A^k(t, T, u) &= -b(t)^\top B^k(t, T, u), & A^k(u, T, u) &= e_k^\top c(u), \\
\frac{\partial}{\partial t} B^k(t, T, u) &= -\mathcal{B}(t)^\top B^k(t, T, u), & B^k(u, T, u) &= e_k^\top \Gamma(u).
\end{aligned}$$

Considering the second moment, we can obtain an interpretation of the functions C^{kl} and D^{kl} as a covariance. See that

$$\begin{aligned}
\mathrm{E}\left[Y_k(u) Y_l(v) \middle| \mathcal{F}(t) \right] &= \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top \hat{Y}(s) \mathrm{d}s} \hat{Y}_k(u) \hat{Y}_l(v) \middle| \mathcal{F}(t) \right] \\
&= \left(A^k(t, T, u) + B^k(t, T, u)^\top X(t) \right) \left(A^l(t, T, v) + B^l(t, T, v)^\top X(t) \right) \\
&\quad + C^{kl}(t, T, u, v) + D^{kl}(t, T, u, v)^\top X(t).
\end{aligned}$$

Combining with the result above, we conclude that

$$\operatorname{Cov}\left[Y_k(u), Y_l(v) \middle| \mathcal{F}(t)\right] = C^{kl}(t,T,u,v) + D^{kl}(t,T,u,v)^\top X(t).$$

We note here that the functions C^{kl} and D^{kl} of course depend on the functions $\hat{c}(t)$ and $\hat{\Gamma}(t)$ through $\psi(t,T)$, which is very special in this case because it is equal to zero. In general the functions C^{kl} and D^{kl} do not give the covariance of the stochastic variables $Y_k(u)$ and $Y_l(v)$.

4.3 Doubly stochastic decrement Markov chains

In this section, we consider so-called decrement Markov chains in finite state spaces with affine and dependent transition rates. The theorems presented above, and their generalisations, allow us to calculate transition probabilities for such doubly stochastic Markov chains. We use the notion decrement (or multiple decrement) Markov chain for the case where, for each state i, the Markov chain cannot return to state i after leaving it. This is also sometimes referred to in the literature as a hierarchical Markov chain model. The reason for the restriction to this class is, that in the case of deterministic rates, the transition probabilities can be written as integral expressions, which is necessary for the approach presented here.

Let a finite state space J be given. We associate a set of non-negative transition rates (μ_{ij}), $i,j \in \mathcal{J}$, with some of the transition rates identical to zero, such that it is a multiple decrement Markov chain. Let the set of non-zero transition rates be modelled as an affine transformation of a d-dimensional affine process $\mathbf{X}$. That is, assuming that there are p non-zero transition rates, let functions $c : \mathbb{R}_+ \to \mathbb{R}^p$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p\times d}$ be given, and define $Y(t) = c(t) + \Gamma(t)X(t)$. Then each of the stochastic transition rates is modelled as an element in $\mathbf{Y}$, i.e. for each non-zero transition rate μ_{ij}, there is a dimension in $\mathbf{Y}$, k say, such that $\mu_{ij}(t) = Y_k(t)$.

Define now the stochastic process $\mathbf{Z} = (Z(t))_{t\in\mathbb{R}_+}$ with $Z(0) = 0$, and let $\mathbf{Z}$ be a Markov chain in $\mathcal{J}$ with distribution given by the transition rates (μ_{ij}), conditional on $\mathbf{X}$. That is, we have defined $\mathbf{Z}$ through the conditional distribution, given the stochastic transition rates. With $(N_{ij}(t))_{t\in\mathbb{R}_+}$, $i,j \in \mathcal{J}$, being the process that counts the number of jumps for $\mathbf{Z}$ from state i to j, the compensated process

$$N_{ij}(t) - \int_0^t 1_{(Z(s-)=i)}\mu_{ij}(s)\mathrm{d}s \tag{4.3.1}$$

is a martingale, conditional on $\mathbf{X}$.

Let the filtrations $\mathbb{F}^{\mathbf{Z}} = (\mathcal{F}^{\mathbf{Z}}(t))_{t\in\mathbb{R}_+}$ and $\mathbb{F}^{\mathbf{X}} = (\mathcal{F}^{\mathbf{X}}(t))_{t\in\mathbb{R}_+}$ be the ones generated by the processes $\mathbf{Z}$ and $\mathbf{X}$, respectively, satisfying the usual hypotheses. Let the general filtration be given by $\mathbb{F} = (\mathcal{F}(t))_{t\in\mathbb{R}_+} = \left(\mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(t)\right)_{t\in\mathbb{R}_+}$.

The transition probability from state i at time s to state j at time t is now $\mathcal{F}^{\mathbf{X}}(s)$-measurable, and using the tower property and the Markov property of $\mathbf{X}$, it can be written as

$$\begin{aligned} p_{ij}(s,t) &:= P\left(Z(t)=j \,\middle|\, Z(s)=i, \mathcal{F}^{\mathbf{X}}(s)\right) \\ &= \mathrm{E}\left[P\left(Z(t)=j \,\middle|\, Z(s)=i, \mathcal{F}^{\mathbf{X}}(\infty)\right) \,\middle|\, \mathcal{F}^{\mathbf{X}}(s)\right] \\ &= \mathrm{E}\left[p_{ij}^{\mathbf{X}}(s,t) \,\middle|\, X(s)\right], \end{aligned}$$

where $p_{ij}^{\mathbf{X}}(s,t)$ is the transition probability in the conditional distribution of $\mathbf{Z}$ given $\mathbf{X}$. From this calculation it is seen, that if we know the conditional transition probabilities, corresponding to the case of known transition rates, we can find the unconditional ones by applying the expectation operator. For hierarchical models, this is exactly the expectations appearing in Theorems 4.2.2, 4.2.3 and 4.2.7, and the generalisations (4.2.14). We can, in principle, find transition probabilities in any doubly stochastic decrement Markov chain with affine transition rates. The conditional transition probabilities $p_{ij}^{\mathbf{X}}(s,t)$ are known explicitly and can be written in sums and integrals of expressions of the type (4.2.14). This can be verified using e.g. Kolmogorov's backward differential equation.

In order to illustrate the method in practice, we present an example in a simple state space and show how to find the transition probabilities, by applying Theorems 4.2.2, 4.2.3 and 4.2.7.

Example 4.3.1. Let $\mathcal{J}=\{0,1,2\}$, and assume that the non-zero transition rates are μ_{01}, μ_{02} and μ_{12}. This could be a life insurance disability model with state 0, 1 and 2 corresponding to *active*, *disabled* and *dead*, as shown in Figure 4.1. The transition rates are modelled by $(\mu_{01}(t), \mu_{02}(t), \mu_{12}(t))^\top = Y(t) = c(t) + \Gamma(t)X(t)$, for some affine process $\mathbf{X}$ and functions c and Γ.

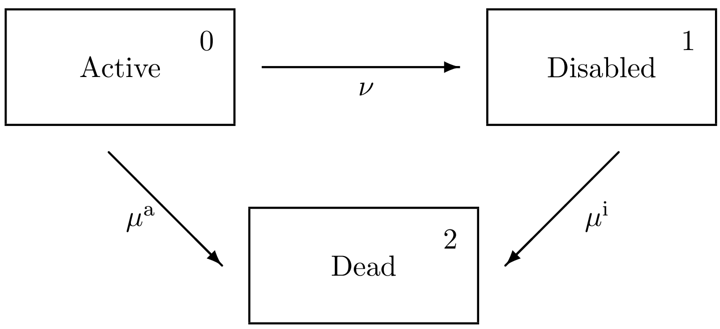

Figure 4.1: *State space for the disability model.*

Conditional on $\mathbf{X}$, the transition probabilities are known explicitly, and for state 0 we have

$$
\begin{aligned}
p_{00}^{\mathbf{X}}(t,s) &= e^{-\int_t^s(\mu_{01}(\tau)+\mu_{02}(\tau))\mathrm{d}\tau}, \\
p_{01}^{\mathbf{X}}(t,s) &= \int_t^s e^{-\int_t^u(\mu_{01}(\tau)+\mu_{02}(\tau))\mathrm{d}\tau}\mu_{01}(u)e^{-\int_u^s \mu_{12}(\tau)\mathrm{d}\tau}\mathrm{d}u, \\
p_{02}^{\mathbf{X}}(t,s) &= \int_t^s e^{-\int_t^u(\mu_{01}(\tau)+\mu_{02}(\tau))\mathrm{d}\tau}\left(\mu_{02}(u)+\mu_{01}(u)\int_u^s e^{-\int_u^v \mu_{12}(\tau)\mathrm{d}\tau}\mu_{12}(v)\mathrm{d}v\right)\mathrm{d}u \\
&= \int_t^s p_{00}^{\mathbf{X}}(t,u)\mu_{02}(u)\mathrm{d}u + \int_t^s p_{01}^{\mathbf{X}}(t,u)\mu_{12}(u)\mathrm{d}u,
\end{aligned}
\tag{4.3.2}
$$

and they can be verified by e.g. Kolmogorov's differential equations. Define

$$
I_{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_{(2,u)}(t) = \begin{bmatrix} 1_{(t\le u)} & 0 & 0 \\ 0 & 1_{(t\le u)} & 0 \\ 0 & 0 & 1_{(t>u)} \end{bmatrix},
$$

and let the processes $\mathbf{Y}_{(1)}$ and $\mathbf{Y}_{(2,u)}$ be given by

$$
\begin{aligned}
Y_{(1)}(t) &= I_{(1)}Y(t) = c_{(1)}(t) + \Gamma_{(1)}(t)X(t), \\
Y_{(2,u)}(t) &= I_{(2,u)}(t)Y(t) = c_{(2,u)}(t) + \Gamma_{(2,u)}(t)X(t),
\end{aligned}
$$

where $c_{(1)} = I_{(1)}c$, $\Gamma_{(1)} = I_{(1)}\Gamma$, $c_{(2,u)} = I_{(2,u)}c$ and $\Gamma_{(2,u)} = I_{(2,u)}\Gamma$. With these definitions, we have

$$
\begin{aligned}
\mathbb{1}^\top Y_{(1)}(t) &= \mu_{01}(t) + \mu_{02}(t), \\
\mathbb{1}^\top Y_{(2,u)}(t) &= 1_{(t\le u)}\left(\mu_{01}(t)+\mu_{02}(t)\right) + 1_{(t>u)}\mu_{12}(t)
\end{aligned}
$$

and the above conditional probabilities can be rewritten,

$$
\begin{aligned}
p_{00}^{\mathbf{X}}(t,s) &= e^{-\int_t^s \mathbb{1}^\top Y_{(1)}(\tau)\mathrm{d}\tau}, \\
p_{01}^{\mathbf{X}}(t,s) &= \int_t^s e^{-\int_t^s \mathbb{1}^\top Y_{(2,u)}(\tau)\mathrm{d}\tau} Y_{(2,u),1}(u)\mathrm{d}u, \\
p_{02}^{\mathbf{X}}(t,s) &= \int_t^s e^{-\int_t^u \mathbb{1}^\top Y_{(1)}(\tau)\mathrm{d}\tau} Y_{(1),2}(u)\mathrm{d}u \\
&\quad + \int_t^s \int_t^v e^{-\int_t^v \mathbb{1}^\top Y_{(2,u)}(\tau)\mathrm{d}\tau} Y_{(2,u),1}(u)Y_{(2,u),3}(v)\mathrm{d}u\mathrm{d}v
\end{aligned}
\tag{4.3.3}
$$

where we use the notation $Y_{(2,u),i}(t)$ for the ith entry in the vector $Y_{(2,u)}(t)$. The real transition probabilities can then be found as the conditional expectations,

$$
p_{ij}(t,s) = \mathrm{E}\left[p_{ij}^{\mathbf{X}}(t,s) \middle| X(t)\right]. \tag{4.3.4}
$$

Insertion of (4.3.3) into (4.3.4) and using linearity and interchanging expectation and integration, we find

$$p_{00}(t,s) = e^{\phi_{(1)}(t,s)+\psi_{(1)}(t,s)^\top X(t)}, \tag{4.3.5}$$

$$p_{01}(t,s) = \int_t^s e^{\phi_{(2,u)}(t,s)+\psi_{(2,u)}(t,s)^\top X(t)} \left(A^1_{(2,u)}(t,s,u) + B^1_{(2,u)}(t,s,u)^\top X(t)\right) \mathrm{d}u, \tag{4.3.6}$$

$$\begin{aligned} p_{02}(t,s) = &\int_t^s e^{\phi_{(1)}(t,s)+\psi_{(1)}(t,s)^\top X(t)} \left(A^2_{(1)}(t,u,u) + B^2_{(1)}(t,u,u)^\top X(t)\right) \mathrm{d}u \\ &+ \int_t^s \int_t^v e^{\phi_{(2,u)}(t,v)+\psi_{(2,u)}(t,v)^\top X(t)} \bigg(\left(A^1_{(2,u)}(t,v,u) + B^1_{(2,u)}(t,v,u)^\top X(t)\right) \\ &\quad \times \left(A^3_{(2,u)}(t,v,v) + B^3_{(2,u)}(t,v,v)^\top X(t)\right) \\ &\quad + C^{1,3}_{(2,u)}(t,v,u,v) + D^{1,3}_{(2,u)}(t,v,u,v)^\top X(t) \bigg) \mathrm{d}u \mathrm{d}v. \end{aligned} \tag{4.3.7}$$

We used Theorem 4.2.2 for (4.3.5), Theorem 4.2.3 for (4.3.6) and the first part of (4.3.7), and Theorem 4.2.7 for the second part of (4.3.7). These theorems provide us with differential equations for the functions ϕ, ψ, A, B, C and D. In order to calculate the integrals numerically, the integrand in (4.3.6) and the integrand in the first part of (4.3.7) must be calculated for each $u \in [t,s]$. Likewise, the integrand in the second part of (4.3.7) must be calculated for each $u, v \in [t,s]$ where $u \leq v$. In practice the expressions are found for u and v in a discretised grid, and these can then be used to carry out the numerical integration. ○

4.3.1 Numerical efficiency

A transition probability in a doubly stochastic Markov chain can be calculated with Monte Carlo methods, and it can also be characterised by a partial differential equation. In this section we extend Example 4.3.1 with an actual model, and calculate the transition probabilities with the method proposed. We discuss the advantages compared to Monte Carlo and PDE methods.

Example 4.3.2. (Example 4.3.1 continued.) Let the stochastic mortality be defined as

$$\begin{aligned} \mu_{02}(t) &= X_1(t) + X_2(t)c_1^{x+t} \\ \mathrm{d}X_i(t) &= -a_i X_i(t)\mathrm{d}t + \sigma_i \mathrm{d}W_i(t) \end{aligned}$$

for $i = 1, 2$, where x is the age at time $t = 0$. Here, the $\mathbf{X}_1$ and $\mathbf{X}_2$ are Ornstein-Uhlenbeck processes. This model is proposed in [43], in which other affine stochastic mortality models are studied as well. The mortality should be non-negative, which is not satisfied by this model. However, as is pointed out in [43], this drawback is not considered severe enough in practice to disregard the model, and we accept it. It is also pointed out that the probability of $\mathbf{X}_i$ becoming negative is negligible.

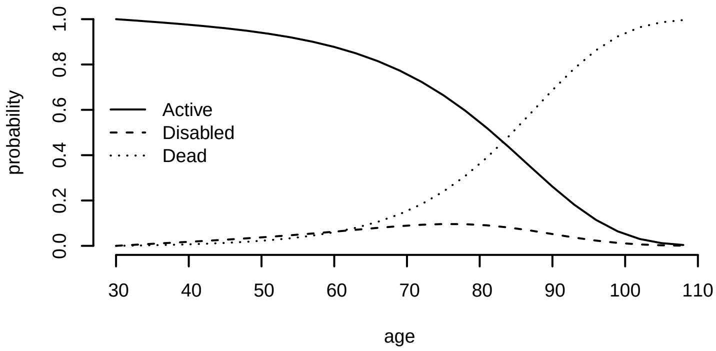

Figure 4.2: *Transition probabilities, starting in the active state at age 30.*

For simplicity, we choose to model $\mu_{02}(t)$ and $\mu_{12}(t)$ in a similar way, with correlation between the transition rates. Let the disability rate be given as

$$\begin{aligned}\mu_{01}(t) &= X_3(t) + X_4(t)c_2^{x+t} \\ \mathrm{d}X_i(t) &= -a_i X_i(t)\mathrm{d}t + \sigma_i \left(\sqrt{1-\rho_1^2}\mathrm{d}W_i(t) + \rho_1 \mathrm{d}W_{i-2}(t)\right),\end{aligned}$$

for $i = 3, 4$. Thus, by choice of ρ_1 it is correlated with the mortality rate as active. Also, let the mortality rate as disabled be given as

$$\begin{aligned}\mu_{12}(t) &= X_5(t) + X_6(t)c_3^{x+t} \\ \mathrm{d}X_i(t) &= -a_i X_i(t)\mathrm{d}t + \sigma_i \left(\sqrt{1-\rho_2^2}\mathrm{d}W_i(t) + \rho_2 \mathrm{d}W_{i-4}(t)\right),\end{aligned}$$

for $i = 5, 6$. Then, if $\rho_2 \neq 0$ it is correlated with the mortality rate. In particular, if $\rho_1 \neq 0$ and $\rho_2 \neq 0$, the mortality as disabled and the disability rate are correlated. Other dependency structures could also have been chosen.

By specifying $\Gamma(t)$ as

$$\Gamma(t) = \begin{bmatrix} 1 & c_1^{x+t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c_2^{x+t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & c_3^{x+t} \end{bmatrix},$$

we can write $(\mu_{02}(t), \mu_{01}(t), \mu_{12}(t))^\top = Y(t) = \Gamma(t)X(t)$ where $\mathbf{X}$ is the 6-dimensional affine process specified above. We choose parameters for illustrative purpose, partly taken from [43], and they are given in Table 4.1.

μ_{02}		μ_{01}		μ_{12}	
c_1	1.09144	c_2	1.09144	c_3	1.09144
a_1	0.028	a_3	0.028	a_5	0.028
a_2	0.0046	a_4	0.0046	a_6	0.0046
σ_1	$1.79 \cdot 10^{-5}$	σ_3	$1.5 \cdot 1.79 \cdot 10^{-5}$	σ_5	$10 \cdot 1.79 \cdot 10^{-5}$
σ_2	$3.83 \cdot 10^{-7}$	σ_4	$1.5 \cdot 3.83 \cdot 10^{-7}$	σ_6	$10 \cdot 3.83 \cdot 10^{-7}$
$X_1(0)$	$9.31 \cdot 10^{-5}$	$X_3(0)$	$1.5 \cdot 9.31 \cdot 10^{-5}$	$X_5(0)$	$10 \cdot 9.31 \cdot 10^{-5}$
$X_2(0)$	$2.19 \cdot 10^{-5}$	$X_4(0)$	$1.5 \cdot 2.19 \cdot 10^{-5}$	$X_6(0)$	$10 \cdot 2.19 \cdot 10^{-5}$
		ρ_1	0.5	ρ_2	0.5

Table 4.1: *Parameters for the stochastic transition rates. The disability rate μ_{01} is in distribution equal to $1.5\mu_{02}$ and the mortality as disabled is in distribution equal to $10\mu_{02}$. We have added a correlation. These intensities are stylised versions of what could be used in practice.*

In Figure 4.2 the transition probabilities are seen, starting as an active 30 year old. These are found calculating (4.3.5) – (4.3.7) numerically, by first finding ϕ, ψ, A, B, C and D by numerically solving their differential equations, and afterwards carrying out a numerical integration. In this example, some of the differential equations can be solved analytically, which is well known for Ornstein-Uhlenbeck processes, see e.g. [43]. We have solved them all numerically though, in order to present an example that generally applies to affine processes.

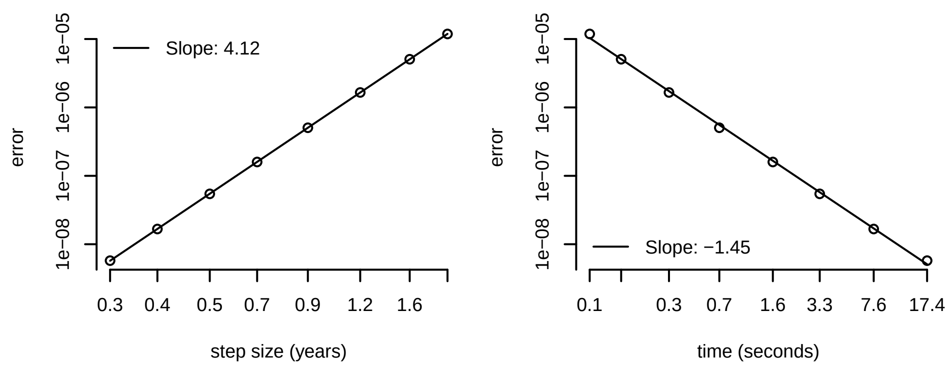

Figure 4.3: *Convergence of the transition probabilities of a 30 year old to age 80, $p_{00}(0, 50) + p_{01}(0, 50) + p_{01}(0, 50)$ towards 1, shown with logarithmic axes. To the left we see that the convergence is of order 4 in step size. To the right the same plot is shown, but with time on the first axis instead of step size. This illustrates that the convergence in time seems to be of order 1.5.*

The differential equations are solved using the 4th order Runge-Kutta method, and the integrals in (4.3.5) – (4.3.7) are solved using a 4th order Simpson integration method. The main work is done in the `C` programming language, and the rest in `R`. With the numerical methods used, the total convergence can be shown to be of order 4 in step size, and this is seen in the left part of Figure 4.3. The same step size is used for solving the differential equations and the integrals. From the right part of Figure 4.3, we see that the convergence order in time is close to 1.5. We also see that when using step size 0.9, we can calculate $p_{00}(0,50)$, $p_{01}(0,50)$ and $p_{02}(0,50)$ in 0.7 seconds, and that total error is smaller than 10^{-6}. The error and convergence order vary a bit for different ages. The calculation time increases with time-span, so it is significantly faster to calculate e.g. $p_{ij}(0,10)$ than $p_{ij}(0,50)$. ○

Monte Carlo and PDE methods

The stochastic process $\mathbf{X}$ can be simulated with Monte Carlo methods, and with each simulation, the transition probabilities can be calculated, e.g. by solving Kolmogorov's differential equations. Monte Carlo methods typically yields a convergence in step size of 0.5, and since one has to solve Kolmogorov's differential equations for each simulation, the convergence in time will be smaller. Thus, in our example above, exploiting the affine structure seems advantageous.

The transition probability can also be characterised by a partial differential equation: Since

$$\mathrm{E}\left[1_{(Z(t)=j)}\middle|\,\mathcal{F}(s)\right] = \sum_{i\in\mathcal{J}} 1_{(Z(s)=i)} p_{ij}(s,t)$$

is a martingale, Itô's formula can be applied in order to obtain a PDE that can be solved, typically by numerical methods. For more details about this approach see e.g. [14], where expectations of more general functions than the indicator function are considered. This partial differential equation has a time dimension and a dimension for each underlying process. Thus if $\mathbf{X}$ is d-dimensional, the PDE is $(d+1)$-dimensional. In Example 4.3.2 the process $\mathbf{X}$ is 6-dimensional, thus we obtain a 7-dimensional PDE. In practice one would use Monte Carlo methods instead of solving a 7-dimensional PDE with finite difference methods, since usually this is significantly faster. See e.g. Section 80.16 in [46], where it is stated that if there are 4 or more dimensions, a PDE problem is usually faster solved with Monte Carlo methods instead.

In Example 4.3.2, we considered a state space $\mathcal{J}$ with 3 states, and an affine process $\mathbf{X}$ with 6 dimensions. The PDE method suffers from the high dimension of $\mathbf{X}$, which is not a problem if the affine structure is exploited. On the other hand, if $\mathbf{X}$ is relatively simple, i.e. if it is 3-dimensional or less, the PDE method is applicable, and here a more complex state space $\mathcal{J}$ can be considered. If there are a lot of states in $\mathcal{J}$, one must solve high-dimensional integrals. Also, if the model is not hierarchical, e.g. if recovery from disability is possible, the affine methods do not work. The PDE method can handle a complex state space $\mathcal{J}$. To sum up, we can say that if the state space $\mathcal{J}$ is simple (and $\mathbf{X}$ is any affine process), the differential equations presented in this article are preferred, and on the other hand, if the stochastic process $\mathbf{X}$ is simple (and $\mathcal{J}$ is any state space), the PDE method is preferred.

4.4 An application in life insurance

We adopt the setup from Section 4.3 above, and consider a life insurance contract where $\mathbf{Z}$ describes the state of the insured. We present a brief example which shows that the methods from Example 4.3.1 also can be used for calculating the expected present value of a life insurance contract. This idea is further explored in [8] wherein affine processes are applied in life insurance, and in particular a surrender modelling example is studied.

Example 4.4.1. Let $\mathbf{Z}$ be as in Example 4.3.1, i.e. we consider the disability model shown in Figure 4.1. The interest rate process $(r(t))_{t\in\mathbb{R}_+}$ is allowed to be stochastic, and this is modelled jointly with the transition rates,

$$(r(t), \mu_{01}(t), \mu_{02}(t), \mu_{12}(t))^\top = c(t) + \Gamma(t)X(t).$$

We have thus specified a model where the interest and transition rates can have any dependent or independent affine structure. For applications, one could argue that the interest rate might not be dependent on the transition rates, however, because it does not increase the complexity of the mathematics in any significant way, the general case is considered here.

We specify a life insurance contract by the payments, given by the accumulated payment process $\mathbf{B}$, satisfying

$$\mathrm{d}B(t) = b_0(t)1_{(Z(t)=0)}\mathrm{d}t + b_1(t)1_{(Z(t)=1)}\mathrm{d}t + b_{02}(t)\mathrm{d}N_{02}(t) + b_{12}(t)\mathrm{d}N_{12}(t).$$

Here, $b_i(t)$ are continuous payments while in state i which, when negative, corresponds to premium payments. The functions $b_{ij}(t)$ are the payments upon a transition from state i to j at time t. The payment functions are deterministic, and we assume that there are no payments after time T.

Conditional on the transition rates and that the policyholder is active, we find the expected present value of the future payments as we do in the setup with deterministic interest and transition rates,

$$V^{\mathbf{X}}(t) = \int_t^T e^{-\int_t^s r(\tau)\mathrm{d}\tau}\bigg(p_{00}^{\mathbf{X}}(t,s)\left(b_0(s) + \mu_{02}(s)b_{02}(s)\right) \\ + p_{01}^{\mathbf{X}}(t,s)\left(b_1(s) + \mu_{12}(s)b_{12}(s)\right)\bigg)\mathrm{d}s,$$

see e.g. [8]. Here, the conditional transition probabilities $p_{0j}^{\mathbf{X}}(t,s)$ are from (4.3.3). The prospective reserve is now found by taking expectation, $V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t)\middle|\,\mathcal{F}(t)\right]$. By insertion of $p^{\mathbf{X}}_{0j}(t,s)$ and then interchanging expectation and integration, the reserve can be calculated analogously to the way we found the transition probabilities in Examples 4.3.1 and 4.3.2. ○

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Chapter 5

Dependent interest and transition rates in life insurance

This chapter is based on the paper [8].

Abstract

For market consistent life insurance liabilities modelled with a multi-state Markov chain, it is of importance to consider the interest and transition rates as stochastic processes, for example in order to consider hedging possibilities of the risks, and for risk measurement. In the literature, this is usually done with an assumption of independence between the interest and transition rates. In this paper, it is shown how to valuate life insurance liabilities using affine processes for modelling dependent interest and transition rates. This approach leads to the introduction of so-called dependent forward rates. We propose a specific model for surrender modelling, and within this model the dependent forward rates are calculated, and the market value and the Solvency II capital requirement are examined for a simple savings contract.

5.1 Introduction

Life insurance liabilities are traditionally modelled by a finite state Markov chain with deterministic interest and transition rates. In order to give a market consistent best estimate of the present value of future payments, it has become of increasing interest to let the interest and transition rates be modelled as stochastic processes. The stochastic modelling is important in order to consider hedging possibilities of the risks. With the Solvency II rules, stochastic modelling of the interest and transition rates is also important from a risk management perspective. Modelling the interest and transition rates as stochastic processes is traditionally done with an independence assumption.

In this paper, we relax the independence assumption, and consider basic valuation with dependence between the interest and one or more transition rates. This is done with continuous affine processes for the modelling of the dependent rates. The study of valuation of life insurance liabilities with dependent rates leads to the definition of so-called dependent forward rates. These are natural quantities that appear in case of dependence, replacing the usual forward rates, which are not directly applicable. Using the theory of dependent affine rates, we consider the case of surrender modelling, and propose a specific model for dependent interest and surrender rates. This is of particular interest from a Solvency II point of view. Within this model, a simple savings contract with a buy-back option is considered. We calculate the dependent forward rates, the market value and the Solvency II capital requirement. This is done in part without hedging, and in part with a simple static hedging strategy. We then examine the effect of correlation between the interest and surrender rate.

The study of valuation of life insurance liabilities with stochastic interest and transition rates has received considerable attention during the last decades. Primarily the interest and mortality rates have been modelled as stochastic, which is often done with affine processes. For basic applications of affine processes for valuation of life insurance liabilities, see [2]. Possibilities of hedging can be considered, which is important for market consistent valuation, and for the study of valuation and hedging of life insurance liabilities with stochastic interest and mortality rates, see [15] and [16]. Another approach to modelling stochastic interest and mortality is taken in [42], where the interest and mortality is modelled within a finite state Markov chain setup. In this paper we extend the study of affine interest and transition rates to the case of dependence. We consider how to valuate life insurance liabilities when the interest and one or more transition rates are modelled as dependent affine processes. This is possible in any decrement/hierarchical Markov chain setup, that is, in Markov chains where, when the process leaves a state, it cannot return. We adopt the theory presented in [7], which is reviewed in Section 5.2 of this paper. This provides the foundation for the study of multidimensional affine processes in life insurance mathematics. The theory presented in [7] is based partly on a result in [17], and partly on general theory for multidimensional affine processes presented in [19].

In the financial literature, the concept of a forward interest rate exists, which is convenient, e.g. for representing zero coupon bond prices. This quantity appears naturally in life insurance mathematics, when the interest rate is modelled as a stochastic process. If one also considers a stochastic mortality, independent of the interest rate, it becomes natural to define a forward mortality rate as well. With these forward rates, the expected present value of the life insurance liabilities has a particularly compelling representation. However, if one introduces dependence between the interest and mortality rates, the forward rates are no longer applicable. In this paper, we introduce so-called dependent forward rates that appear naturally and are applicable for representing the expected present value of the life insurance liabilities in a convenient form, in cases where the usual forward rates are not applicable. In [37], alternative forward mortality rates are defined in order to handle the case of dependence. In the present paper, we show that one of the forward mortality rates defined in [37] is in general not well defined. For a general discussion on forward rates, and their usefulness, see [41], wherein the case of dependence between the rates is discussed as well. One of the consistency problems with forward rates in the dependent setup that is raised in [41] is solved by the proposed dependent forward rates introduced in the present paper. Also, the dependent forward rates introduced here generalise the usual definitions of forward rates, in the sense that when there is independence between the rates, the dependent forward rates equal the usual forward rates.

Modelling policyholder behaviour has become of increasing importance with the proposed Solvency II rules, where one is required to consider any dependence between the economic environment and policyholder behaviour, see Section 3.5 in [12]. The study of surrender behaviour can either be made using a rational approach, where the outset is, that the policyholders surrender the contract if it is rational from some economic viewpoint, which is studied in [45]. This seems a bit extreme, given that this behaviour is not seen in practice. Another approach is the intensity approach, where the policyholders surrender randomly, regardless whether or not it is profitable in the current economic environment. This is not a perfect way of modelling either, since if the interest rates decrease a lot, a guarantee given in connection with the life insurance contract motivates the policyholders to keeping the contract. For an overview of some of the approaches, see [38]. In [23], an attempt is made on coupling the two approaches, using two different surrender rate models if it is rational or irrational, respectively, to surrender. In this paper, we propose another way of coupling the two approaches. We let the surrender rate be positively correlated with the interest rate, thus if the interest rate decreases a lot, the surrender rate also decreases, representing that the guarantee inherent in the life insurance contract is of value to the policyholder.

The Solvency II capital requirement is basically, that "the insurance company must have enough capital, such that the probability of default within the next year is less than 0.5%", representing that a default is a 200-year event. When the insurance company updates its mortality tables, or other transition rate tables, this represents a risk that must be taken into account when putting up the Solvency II capital requirement. Mathematically, this can be done using stochastic rates. For an examination of mortality modelling and the Solvency II capital requirement, see e.g. [4]. For a basic discussion of the mathematical formulation of the Solvency II capital requirement, see e.g. [6]. In this paper, we determine the Solvency II capital requirement for the simple savings contract where the interest and surrender rate risk is considered, both in the case of no hedging strategy, and also in the case of a simple strategy where interest rate risk is hedged.

The structure of the paper is as follows. In Section 5.2, we present basic results on multidimensional continuous affine processes, which provides the foundation for the application of dependent affine processes in life insurance mathematics. In Section 5.3, we present the general life insurance setup with stochastic interest and transition rates, and in Section 5.4, we propose the definition of dependent forward rates and compare to the usual forward rate definition. In Section 5.4.1, we discuss other definitions in the literature of forward rates in a dependent setup, and compare them to the dependent forward rates proposed here. In Section 5.5, we present a specific model for dependent interest and surrender rates. The model is introduced in Section 5.5.1. We first discuss how to find the Solvency II capital requirement, which is done in Section 5.5.3, and a simple hedging strategy for the interest rate risk is presented in Section 5.5.4. Numerical results are presented in Section 5.5.5, consisting of the dependent forward rates found, and the market value and Solvency II capital requirement, presented for different levels of correlation.

5.2 Continuous affine processes

The class of affine processes provides a method for modelling interest and transition rates, with the possibility of adding dependence. In this section, we consider general results about continuous affine processes, which we apply in this paper. For more details on the theory presented in this section, see [7].

Let $\mathbf{X}$ be a d-dimensional affine process, satisfying the stochastic differential equation

$$\mathrm{d}X(t) = (b(t) + \mathcal{B}(t)X(t))\mathrm{d}t + \rho(t, X(t))\mathrm{d}W(t),$$

where $\mathbf{W}$ is a d-dimensional Brownian motion. Here, $b : \mathbb{R}_+ \to \mathbb{R}^d$ is a vector function, and $\mathcal{B} : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is a matrix function, where we denote column i by $\beta_i(t)$, so that $\mathcal{B}(t) = (\beta_1(t), \dots, \beta_d(t))$. When squared, the volatility parameter function $\rho(t, x)$ must be affine in x, i.e.

$$\rho(t, x)\rho(t, x)^\top = a(t) + \sum_{i=1}^{d} \alpha_i(t)x_i,$$

for matrix functions $a : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ and $\alpha_i : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$. Consider now affine transformations of $\mathbf{X}$, by defining a vector function $c : \mathbb{R}_+ \to \mathbb{R}^p$ and a matrix function $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p\times d}$, thereby defining the p-dimensional process,

$$Y(t) = c(t) + \Gamma(t)X(t). \tag{5.2.1}$$

We think of $\mathbf{X}$ as socio-economic driving factors, and then $\mathbf{Y}$ is a collection of the stochastic interest rate and/or transition rates. In this section, we work in a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}(t))_{t \in \mathbb{R}_+}$ generated by the Brownian motion $\mathbf{W}$.

For applications of $\mathbf{Y}$ as interest and as transition rates in finite state Markov chain models, we present some essential relations. The results hold under certain regularity conditions, for details see [7]. Denote by $\mathbb{1}$ a vector with 1 in all entries, where the dimension is implicit. Also, denote by $\gamma_i(t)$ the sum of the ith column in $\Gamma(t)$, i.e. $\gamma_i(t) = \mathbb{1}^\top \Gamma(t) e_i$, where e_i is the ith unit vector, $i = 1, \ldots, d$.

The first relation, the basic pricing formula, is for $0 \leq t \leq T$ given by

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s) \mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\top X(t)}, \tag{5.2.2}$$

where $\phi(t,T)$ is a real function, and $\psi(t,T)$ is a d-dimensional function, given by the system of differential equations,

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,T) &= -\frac{1}{2}\psi(t,T)^\top a(t)\psi(t,T) - b(t)^\top \psi(t,T) + \mathbb{1}^\top c(t), \\ \frac{\partial}{\partial t}\psi_i(t,T) &= -\frac{1}{2}\psi(t,T)^\top \alpha_i(t)\psi(t,T) - \beta_i(t)^\top \psi(t,T) + \gamma_i(t), \quad i = 1, \ldots, d, \end{aligned} \tag{5.2.3}$$

with boundary conditions $\phi(T,T) = 0$ and $\psi(T,T) = 0$.

For the second relation, let a vector $\kappa \in \mathbb{R}^p$ be given, and let $u \in [t,T]$ be some time point. Then,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s) \mathrm{d}s} \kappa^\top Y(u) \middle| \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\top X(t)} \Big(A(t,T,u) + B(t,T,u)^\top X(t) \Big), \tag{5.2.4}$$

where (ϕ, ψ) is given by (5.2.3), A is a real function and B is a vector function, given by the system of differential equations,

$$\begin{aligned} \frac{\partial}{\partial t}A(t,T,u) &= -\psi(t,T)^\top a(t)B(t,T,u) - b(t)^\top B(t,T,u), \\ \frac{\partial}{\partial t}B_i(t,T,u) &= -\psi(t,T)^\top \alpha_i(t)B(t,T,u) - \beta_i(t)^\top B(t,T,u), \quad i = 1, \ldots, d, \end{aligned} \tag{5.2.5}$$

with boundary conditions $A(u,T,u) = \kappa^\top c(u)$ and $B(u,T,u) = \kappa^\top \Gamma(u)$. A particular example of importance is $\kappa = e_k$ for some $k = 1, \ldots, p$, and in this case, we write A^k and B^k to emphasize the dependence on k. This second relation (5.2.4) is proven in [17] for $u = T$, and the extension to the case $u < T$ is for example given in [7].

The third relation is, for another time point $v \in [t,T]$, and two integers $k, l = 1, \dots, p$, given by

$$\begin{aligned}
\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(u) Y_l(v) \middle| \mathcal{F}(t) \right] &= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \\
&\times \Big\{ \Big(A^k(t,T,u) + B^k(t,T,u)^\top X(t) \Big) \Big(A^l(t,T,v) + B^l(t,T,v)^\top X(t) \Big) \\
&+ C^{kl}(t,T,u,v) + D^{kl}(t,T,u,v)^\top X(t) \Big\},
\end{aligned} \tag{5.2.6}$$

where (ϕ, ψ) solves (5.2.3) and (A^k, B^k) and (A^l, B^l) both solve (5.2.5) with boundary conditions $A^k(u,T,u) = e_k^\top c(u)$, $B^k(u,T,u) = e_k^\top \Gamma(u)$ and $A^l(v,T,v) = e_l^\top c(v)$, $B^l(v,T,v) = e_l^\top \Gamma(v)$, respectively. The functions C^{kl} and D^{kl} are determined by the following system of differential equations,

$$\begin{aligned}
\frac{\partial}{\partial t} C^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top a(t) B^l(t,T,v) \\
&\quad - \psi(t,T)^\top a(t) D^{kl}(t,T,u,v) - b(t)^\top D^{kl}(t,T,u,v), \\
\frac{\partial}{\partial t} D_i^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top \alpha_i(t) B^l(t,T,v) \\
&\quad - \psi(t,T)^\top \alpha_i(t) D^{kl}(t,T,u,v) - \beta_i(t)^\top D^{kl}(t,T,u,v),
\end{aligned} \tag{5.2.7}$$

for $i = 1, \dots, d$, with boundary conditions[1] $C^{kl}(u \wedge v, T, u, v) = 0$ and $D^{kl}(u \wedge v, T, u, v) = 0$. This result is proven in [7].

5.3 The life insurance model

Consider the usual life insurance setup. Let $\mathbf{Z} = (Z(t))_{t \in \mathbb{R}_+}$ be a Markov process in the finite state space $\mathcal{J}$, indicating the state of the insured. The distribution of $\mathbf{Z}$ is defined via the transition rates $(\mu_{ij}(t))_{t \in \mathbb{R}_+}$, $i, j \in \mathcal{J}$. With $(N_{ij}(t))_{t \in \mathbb{R}_+}$, $i, j \in \mathcal{J}$ being the process that counts the number of jumps for $\mathbf{Z}$ from state i to j, the compensated process

$$N_{ij}(t) - \int_0^t 1_{(Z(s-)=i)} \mu_{ij}(s) \mathrm{d}s$$

is a martingale. We can allow the transition rates (μ_{ij}) to be stochastic. In this case, the distribution of $\mathbf{Z}$ is defined conditionally on the transition rates.

We model the transition rates as a time-dependent affine transformation of a d-dimensional continuous affine process $\mathbf{X}$. That is, for functions $c : \mathbb{R}_+ \to \mathbb{R}^p$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p \times d}$, let $\mathbf{Y}$ be defined as

$$Y(t) = c(t) + \Gamma(t) X(t).$$

[1] The notation $x \wedge y = \min\{x, y\}$ is used.

Hence, each of the stochastic transition rates are modelled as an element in $\mathbf{Y}$.

The interest rate process $(r(t))_{t\in\mathbb{R}_+}$ is also allowed to be stochastic. This is modelled in the same way, by specifying r as an element in $\mathbf{Y}$. By the design of Γ and $\mathbf{X}$, the interest and transition rates can be dependent, independent or deterministic.

Let the filtrations $\mathbb{F}^{\mathbf{Z}} = (\mathcal{F}^{\mathbf{Z}}(t))_{t\in\mathbb{R}_+}$ and $\mathbb{F}^{\mathbf{X}} = (\mathcal{F}^{\mathbf{X}}(t))_{t\in\mathbb{R}_+}$ be the ones generated by the processes $\mathbf{Z}$ and $\mathbf{X}$, respectively, satisfying the usual hypothesis. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in\mathbb{R}_+}$ is given by $\mathcal{F}(t) = \mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(t)$.

We consider a life insurance policy, with payment process $\mathbf{B} = (B(t))_{t\in\mathbb{R}_+}$, such that $B(t)$ is the accumulated payments until time t. Then we can think of $\mathrm{d}B(t)$ as the payment at time t, and we can specify $\mathbf{B}$ as

$$\mathrm{d}B(t) = \sum_{i\in\mathcal{J}} 1_{(Z(t)=i)} b_i(t)\mathrm{d}t + \sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} b_{ij}(t)\mathrm{d}N_{ij}(t),$$

for deterministic payment functions b_i and b_{ij}, $i, j \in \mathcal{J}$. Then $b_i(t)$ is the payment while in state i at time t, and $b_{ij}(t)$ is the payment if jumping from state i to j at time t.

The present value at time t of the future payments associated with the life insurance policy is given by

$$PV(t) = \int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}B(s).$$

For reserving and pricing, one considers the expected present value

$$V(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}B(s)\middle| \mathcal{F}(t)\right],$$

where the expectation is taken using a market, risk neutral or pricing measure. For actually calculating $V(t)$, the tower property is applied, that is, we condition on $\mathcal{F}^{\mathbf{X}}(\infty)$ to get

$$V^{\mathbf{X}}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}B(s)\middle| \mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(\infty)\right],$$

so that $V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t)\middle| \mathcal{F}(t)\right]$. Here, $V^{\mathbf{X}}(t)$ is the reserve conditional on the interest and transition rates, thus corresponding to the case of deterministic rates. When valuating $V^{\mathbf{X}}$ we need the conditional distribution of $\mathbf{Z}$, and thus $\mathbf{B}$, given the transition rates. By construction this is known, and well-established theory about life insurance reserves with deterministic interest and transition rates (see e.g. [40]) hold.

Example 5.3.1. Consider a surrender model with 3 states $\mathcal{J} = \{0, 1, 2\}$, corresponding to *alive*, *dead* and *surrendered* respectively. The Markov model is shown in Figure 5.1. Let the transition rate from state *alive* to state *dead*, i.e. the mortality rate, be deterministic. We model the interest rate r and the surrender rate η as dependent affine processes in the form,

$$(r(t), \eta(t))^\top = c(t) + \Gamma(t)X(t),$$

for a d-dimensional affine process $\mathbf{X}$. Hence, this specification is analog to (5.2.1). By the design of $\mathbf{X}$, the processes $\mathbf{X}_i$, $i = 1, \ldots, d$ can be dependent processes, such that the interest rate r and the surrender rate η can be dependent processes. Dependence can also arise through Γ.

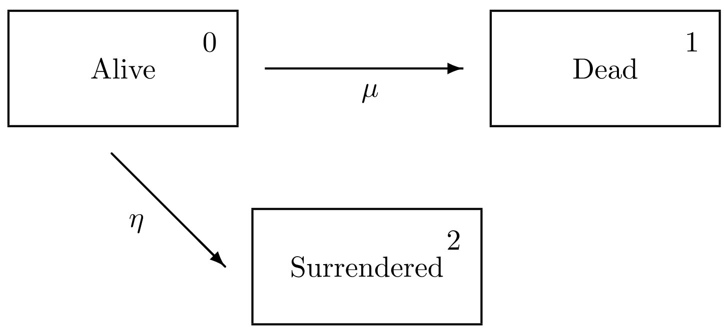

Figure 5.1: *Markov model for the survival-surrender model.*

Let the payments be defined by

$$\mathrm{d}B(t) = b(t)1_{(Z(t)=0)}\mathrm{d}t + b_d(t)\mathrm{d}N_{01}(t) + U(t)\mathrm{d}N_{02}(t),$$

where $b(t)$ is the continuous payment rate at time t while alive, $b_d(t)$ is the single payment if death occurs at time t, and $U(t)$ is the payment upon surrender at time t. The payment functions are deterministic.

Conditioning on the intensitites, the expected present value $V^{\mathbf{X}}(t)$ is the classic result,

$$\begin{aligned} V^{\mathbf{X}}(t) &= \mathrm{E}\left[PV(t) \mid \mathcal{F}^{\mathbf{X}}(\infty), Z(t) = 0\right] \\ &= \int_t^\infty e^{-\int_t^s (r(\tau)+\mu(\tau)+\eta(\tau))\mathrm{d}\tau} \left(b(s) + \mu(s)b_d(s) + \eta(s)U(s)\right) \mathrm{d}s, \end{aligned}$$

see e.g. [40]. Removing the condition, we find, using Equations (5.2.2) and (5.2.4),

$$V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t) \middle| \mathcal{F}(t)\right]$$

$$
\begin{aligned}
&= \int_t^\infty e^{-\int_t^s \mu(\tau)\mathrm{d}\tau} \Big\{ \mathrm{E}\left[e^{-\int_t^s (r(\tau)+\eta(\tau))\mathrm{d}\tau} \Big| \mathcal{F}(t) \right] (b(s)+\mu(s)b_d(s)) \\
&\quad + \mathrm{E}\left[e^{-\int_t^s (r(\tau)+\eta(\tau))\mathrm{d}\tau} \eta(s) \Big| \mathcal{F}(t) \right] U(s) \Big\} \mathrm{d}s \\
&= \int_t^\infty e^{-\int_t^s \mu(\tau)\mathrm{d}\tau + \phi(t,s)+\psi(t,s)^\top X(t)} \Big(b(s)+\mu(s)b_d(s) \\
&\quad + \Big(A^\eta(t,s,s) + B^\eta(t,s,s)^\top X(t) \Big) U(s) \Big) \mathrm{d}s.
\end{aligned}
$$

○

5.4 Dependent forward rates

The form of $V(t)$ in Example 5.3.1 motivates the definition of quantities similar to forward rates, that can be used to express the solution. In particular, this leads to a forward interest rate, but this is in general not equal the forward rate obtained using the usual definition. Hence, we apply the term *dependent forward rates.*

Let $\mathbf{X}$, $c(t)$ and $\Gamma(t)$ be given, and let $\mathbf{Y}$ be of the form (5.2.1). We consider some motivating calculations first, and then define the dependent forward rates. See that,

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \mathbb{1}^\top Y(T) \Big| \mathcal{F}(t) \right] \\
&= -\frac{\partial}{\partial T} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \Big| \mathcal{F}(t) \right] \\
&= -\frac{\partial}{\partial T} e^{\phi(t,T)+\psi(t,T)^\top X(t)} \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \left(-\frac{\partial}{\partial T}\phi(t,T) + X(t)^\top \left(-\frac{\partial}{\partial T}\psi(t,T) \right) \right),
\end{aligned}
$$

where we interchanged integration and differentiation, and applied (5.2.2). On the other hand, if we instead apply (5.2.4) with $\kappa = \mathbb{1}$, we find

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} \mathbb{1}^\top Y(T) \Big| \mathcal{F}(t) \right] \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \Big(A(t,T,T) + X(t)^\top B(t,T,T) \Big), \\
&= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \left(\sum_{k=1}^p A^k(t,T,T) + X(t)^\top \sum_{k=1}^p B^k(t,T,T) \right),
\end{aligned}
$$

where (A^k, B^k), $k = 1, \dots, p$ are solutions to (5.2.5) with boundary conditions given by $\kappa = e_k$, i.e. $A^k(T,T,T) = e_k^\top c(T)$ and $B^k(T,T,T) = e_k^\top \Gamma(T)$. The last equality sign is obtained using the relations $\sum_{k=1}^p A^k(t,T,T) = A(t,T,T)$ and $\sum_{k=1}^p B^k(t,T,T) = B(t,T,T)$, which hold since (A,B) also solves the linear system of differential equations

(5.2.5), with boundary conditions given by $\kappa = \mathbb{1}$. Gathering the two calculations above, we conclude that

$$-\frac{\partial}{\partial T}\phi(t,T) = \sum_{k=1}^{p} A^k(t,T,T), \qquad -\frac{\partial}{\partial T}\psi(t,T) = \sum_{k=1}^{p} B^k(t,T,T),$$

and, in particular, since $\phi(t,t) = 0$ and $\psi(t,t) = 0$, that

$$\phi(t,T) = -\int_t^T \sum_{k=1}^{p} A^k(t,s,s)\mathrm{d}s, \qquad \psi(t,T) = -\int_t^T \sum_{k=1}^{p} B^k(t,s,s)\mathrm{d}s. \tag{5.4.1}$$

Definition 5.4.1. *Let* $\mathbf{X}$ *be a* d*-dimensional continuous affine process, and let* c *and* Γ *be given, such that* $\mathbf{Y}$ *from* (5.2.1) *is defined. Let* $t \le s$ *and* $k \in \{1, \dots, p\}$*. The* dependent forward rate $f_t^k(s)$ *for the stochastic rate* $Y_k(s)$ *at time* t *is then given by*

$$f_t^k(s) = A^k(t,s,s) + X(t)^\top B^k(t,s,s), \tag{5.4.2}$$

where (A^k, B^k) *solves the system of differential equations* (5.2.5)*, with boundary conditions given by* $\kappa = e_k$*.*

Remark 5.4.2. Using the notation of the dependent forward rates, we can express the relation (5.2.2), and for $u = T$ also the relation (5.2.4), as

$$\begin{aligned}
\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s}\,\middle|\,\mathcal{F}(t)\right] &= e^{-\int_t^T \sum_{i=1}^p f_t^i(s)\mathrm{d}s}, \\
\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s} Y_k(T)\,\middle|\,\mathcal{F}(t)\right] &= e^{-\int_t^T \sum_{i=1}^p f_t^i(s)\mathrm{d}s} f_t^k(T).
\end{aligned} \tag{5.4.3}$$

$\diamond$

The dependent forward rates are in Definition 5.4.1 only for affine processes. However, if one wish, equation (5.4.3) can be used to extend the definition to any underlying process: The equations (5.4.3) uniquely determine the dependent forward rates, thus the dependent forward rates exist for any underlying process, and not only affine processes. This is not a constructive definition though, and in the present paper we only focus on the affine class.

Example 5.4.3. (Example 5.3.1 continued) Using the definition of the dependent forward rates, we can write the expected present value as,

$$V(t) = \int_t^\infty e^{-\int_t^s \left(f_t^r(\tau) + \mu(\tau) + f_t^\eta(\tau)\right)\mathrm{d}\tau} \Big(b(s) + \mu(s)b_d(s) + f_t^\eta(s)U(s)\Big)\mathrm{d}s. \tag{5.4.4}$$

We see that the expected present value is of the same form as the formula that appears in the case of deterministic rates, but with the interest and surrender rates exchanged by the corresponding dependent forward rates. Note that we used a slightly different notation, such that we write f^r instead of f^1 and f^η instead of f^2.

Often we want to consider both the quantity

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] = \mathrm{E}\left[e^{-\int_t^T (r(s)+\eta(s))\mathrm{d}s}\middle|\, \mathcal{F}(t)\right],$$

where $Y(t) = (r(t), \eta(t))$, as well as the quantities arising from the models $Y^1(t) = (r(t), 0)$ and $Y^2(t) = (0, \eta(t))$,

$$\begin{aligned}\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y^1(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] &= \mathrm{E}\left[e^{-\int_t^T r(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right],\\ \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y^2(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] &= \mathrm{E}\left[e^{-\int_t^T \eta(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right].\end{aligned}$$

In such cases, we add a more detailed superscript to the dependent forward rates f, and specify the model we think of after a colon. That is, we write

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\eta(s))\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] = e^{-\int_t^T (f_t^{r:(r+\eta)}(s)+f_t^{\eta:(r+\eta)}(s))\mathrm{d}s},$$

as well as

$$\begin{aligned}\mathrm{E}\left[e^{-\int_t^T r(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] &= e^{-\int_t^T f_t^{r:r}(s)\mathrm{d}s},\\ \mathrm{E}\left[e^{-\int_t^T \eta(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] &= e^{-\int_t^T f_t^{\eta:\eta}(s)\mathrm{d}s}.\end{aligned}$$

Note that $f_t^{r:r}(s)$ and $f_t^{\mu:\mu}(s)$ are the usual forward rates. ○

The representation (5.4.4) is convenient, since it allows us to use the classic formulae, and just plug in pre-calculated dependent forward rates. The result is only obtainable with the dependent forward rates defined here. If one used a spread-rate approach, as in [37], one would have had two different surrender rates and thus not obtained the formula (5.4.4). In [41], Section 5, the forward rate approach is criticised by the fact that the formula (5.4.4) is not available, and the dependent forward rates meet this critique. The difference between the dependent forward rates and the spread rate approach is examined with (5.4.11) in Section 5.4.1 below.

We briefly compare with the usual forward interest rate. Let the model $Y(t) = c(t) + \Gamma(t)X(t)$ be given, for $p > 1$, and let $r(t) = Y_1(t)$ be the interest rate. The *forward interest rate* is the function $g_t(s)$ that satisfies

$$\mathrm{E}\left[e^{-\int_t^T r(s)\mathrm{d}s}\middle|\, \mathcal{F}(t)\right] = e^{-\int_t^T g_t(s)\mathrm{d}s}.$$

This function also satisfies, as can be shown by differentiation,

$$\mathrm{E}\left[e^{-\int_t^T r(s)\mathrm{d}s} r(T)\middle|\, \mathcal{F}(t)\right] = e^{-\int_t^T g_t(s)\mathrm{d}s} g_t(T). \tag{5.4.5}$$

The dependent forward rate for the interest rate in our model $\mathbf{Y}$, as defined in Definition 5.4.1, is denoted $f_t^r(s)$. It satisfies,

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+Y_2(s)+\ldots+Y_p(s))\mathrm{d}s} r(T) \,\middle|\, \mathcal{F}(t)\right] = e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^p(s)\right)\mathrm{d}s} f_t^r(T), \quad (5.4.6)$$

where the other forward rates $f_t^i(s)$ satisfy analogue relations.

In the case that $\mathbf{r} = (r(t))_{t\in\mathbb{R}_+}$ is independent of $\mathbf{Y}_2, \ldots, \mathbf{Y}_p$, the dependent forward rate for the interest $\mathbf{r}$ simplifies to the usual forward interest rate. This can be seen by two simple calculations. First, see that

$$\begin{aligned}
&e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^p(s)\right)\mathrm{d}s} \\
&= \mathrm{E}\left[e^{-\int_t^T (r(s)+Y_2(s)+\ldots+Y_p(s))\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right] \\
&= \mathrm{E}\left[e^{-\int_t^T r(s)\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right] \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_p(s))\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right] \\
&= e^{-\int_t^T g_t(s)\mathrm{d}s}\, \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_p(s))\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right].
\end{aligned} \quad (5.4.7)$$

A similar calculation, using (5.4.6) and (5.4.5), yields

$$\begin{aligned}
&e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^p(s)\right)\mathrm{d}s} f_t^r(T) \\
&= e^{-\int_t^T g_t(s)\mathrm{d}s} g_t(T)\, \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_p(s))\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right].
\end{aligned}$$

Dividing with the identity (5.4.7) above, we conclude that

$$f_t^r(T) = g_t(T),$$

which holds for all T where $t < T$, and we conclude that the dependent forward interest rate equals the usual forward interest rate.

The calculations relied critically on the independence assumption, and in the general case the dependent forward rate for the interest is not equal to the forward interest rate. Intuitively, when the interest rate appears together with other dependent rates, the forward rates need to compensate for this dependence, and thus the difference of the dependent forward rates and the usual forward rates can be thought of as a "covariance" term.

5.4.1 Comparison with other dependent setups

For the case of dependent affine rates, there have been other proposals for the definition of forward rates. In [37], the model contains an interest rate and a mortality rate which are dependent. This corresponds to the case $p = 2$, where $r(t) = Y_1(t)$ is the interest rate and $\mu(t) = Y_2(t)$ is the mortality rate. Their approach is to keep the definition of the forward interest rate $g_t : [t, \infty) \to \mathbb{R}_+$ unchanged, and then find forward mortality rates that are compatible with this definition, thus interpreting the forward mortality rate as a spread rate. In order to make this idea work, they define two different mortality rates, one for pure endowments, and one for term insurances. We briefly review this approach and compare to the definition of the dependent forward rates in the previous section. This serves to highlight the advantage of the dependent forward rates, in particular that there is only one forward mortality rate when considering a pure endowment, a life annuity and a term insurance together.

The *forward mortality rate for pure endowments*, $h_t^{\mathrm{pe}} : [t, \infty) \to \mathbb{R}_+$, is defined as the function satisfying

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\mu(s))\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{-\int_t^T \left(g_t(s)+h_t^{\mathrm{pe}}(s)\right)\mathrm{d}s}.$$

In terms of the dependent forward rates, f_t^r and f_t^μ, we can use the first part of (5.4.3) and write the forward mortality rate for pure endowment as,

$$h_t^{\mathrm{pe}}(s) = f_t^r(s) + f_t^\mu(s) - g_t(s), \tag{5.4.8}$$

which in particular shows that it is well-defined. The forward mortality rate for pure endowment can be given an intuitive interpretation. Recall that the dependent forward rates are different from the usual definition of forward rates, because the mortality rate appears together with another dependent rate, thus the dependent forward rates contains a "covariance" part. The forward mortality rate for pure endowments corresponds to moving the "covariance" from the forward interest rate into the forward mortality rate, instead of having a part in each of the forward rates. In other words, $f_t^r + f_t^\mu$ contains the "covariance" terms, and subtracting g_t, which does not contain any "covariance" terms, the "covariance" terms are contained in h_t^{pe}. In this way, the original definition of the forward interest rate can be kept unaltered, but one can say that the forward mortality rate for pure endowment h_t^{pe} contains a "covariance" term belonging to the interest rate.

The *forward mortality rate for term insurances*, $h_t^{\mathrm{ti}} : [t, \infty) \to \mathbb{R}_+$, is defined as the function satisfying,

$$\mathrm{E}\left[\int_t^T e^{-\int_t^u (r(s)+\mu(s))\mathrm{d}s} \mu(u)\mathrm{d}u \middle| \mathcal{F}(t) \right] = \int_t^T e^{-\int_t^u \left(g_t(s)+h_t^{\mathrm{ti}}(s)\right)\mathrm{d}s} h_t^{\mathrm{ti}}(u)\mathrm{d}u. \tag{5.4.9}$$

To establish that h_t^{ti} is well-defined is not as easy as with the forward mortality rate for pure endowments. First, see that the definition depends on the choice of T. It is natural to make the assumption that the forward mortality rate for term insurances h_t^{ti} is independent of T. This assumption is implicity made in the notation used in [37], and the assumption is also made for the forward mortality rate for pure endowments. With this assumption of independence of T, we can differentiate with respect to T, and find the equivalent definition,

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\mu(s))\mathrm{d}s}\mu(T)\middle|\,\mathcal{F}(t)\right] = e^{-\int_t^T \left(g_t(s)+h_t^{\mathrm{ti}}(s)\right)\mathrm{d}s}h_t^{\mathrm{ti}}(T), \tag{5.4.10}$$

for $T \geq t$. We are now ready to answer the question of well-definedness, which is important for a fruitful definition of a forward rate. It turns out, that when using the definition (5.4.10), there are cases where the forward mortality rate for term insurances does not exist for all time points, and one should therefore be careful to use the definition in practice. This will in particular be the case for models with positive correlation. A proof is given in Appendix 5.A, where we also present a class of models where the forward mortality rate for term insurances does not exist. If one instead uses the definition (5.4.9) and allow $h_t^{\mathrm{ti}}(s)$ to depend on T as well, the extra parameter T probably makes it possible to show that it is well-defined.

One forward mortality/surrender rate

We are now ready to present the main difference between the spread rate approach and the dependent forward rate approach. We compare the forward mortality rates from [37] with the dependent forward rates, and for now assuming that the forward rate for term insurances exists, we consider a policy consisting of a life annuity with a payment rate b, and a term insurance with payment 1 upon death. The policy terminates at time T. The expected present value at time t is

$$\begin{aligned}
&\mathrm{E}\left[\int_t^T e^{-\int_t^s (r(s)+\mu(s))\mathrm{d}s}\left(b+\mu(s)\right)\mathrm{d}s\middle|\,\mathcal{F}(t)\right] \\
&= \int_t^T e^{-\int_t^s (f_t^{r:(r+\mu)}(s)+f_t^{\mu:(r+\mu)}(s))\mathrm{d}s}\left(b+f_t^{\mu:(r+\mu)}(s)\right)\mathrm{d}s \\
&= \int_t^T e^{-\int_t^s g_t(s)\mathrm{d}s}\left(e^{-\int_t^s h_t^{\mathrm{pe}}(s)\mathrm{d}s}b + e^{-\int_t^s h_t^{\mathrm{ti}}(s)\mathrm{d}s}h_t^{\mathrm{ti}}(s)\right)\mathrm{d}s,
\end{aligned} \tag{5.4.11}$$

where we first wrote it in terms of the dependent forward rates, and afterwards in terms of the forward mortality rates for pure endowments and term insurances, respectively. This illustrates the difference between the different types of forward rates. The dependent forward rate for mortality can be used for both the life annuity and the term insurance, whereas with the other forward mortality rate definitions, one need a different one for a different product. If the interest rate is independent of the mortality rate, the different forward mortality rates simplify and they all equal the usual forward mortality rate.

The fact that the dependent forward rates solve the problem of the two different forward rates for the different products in (5.4.11) is one of the main advantages. It is exactly this problem with existing forward rates that is criticised in Section 5 of [41]. With the dependent forward rates, this issue is resolved in that a unique forward mortality rate exists, that can be used for both the life annuity and the term insurance. In the article [41] a general discussion of the concept of forward rates, and generalisations to dependent models is carried out, including discussion of requirements for more generalised forward rates. Even though the critique from Section 5 of [41] is met with the dependent forward rates, they do not meet all the requirements set up in [41]. In particular, in life insurance models where one needs to use the relation (5.2.6), the dependent forward rates are not applicable.

The feature of a unique forward mortality rate in (5.4.11) does also apply to the surrender setup in Example 5.4.3. The dependent forward rates allow us to have one forward surrender rate, and if the spread approach was used, one would have had different forward surrender rates: One for reducing with the probability of not having surrendered, and another for calculating the probability of surrendering at an exact time. Thus, the formula (5.4.4) would have had two parts with two different forward surrender rates, similar to the last line in (5.4.11).

5.5 Modelling interest and surrender

In order to illustrate the methods proposed, we put up a specific model for dependent interest and surrender rate. The results are presented naturally using the dependent forward rates such that the formulae are in parallel with the classic life insurance results obtained with deterministic transition rates. This allows for convenient interpretation and comparison of the results. For example, using the replacement result, the effects of introducing stochastic rates can be measured in terms of forward rates, i.e. the difference between the original deterministic rates and the dependent forward rates.

We model the interest rate as a stochastic diffusion process r, and the surrender rate by the diffusion process η. The interest and surrender rates are then modelled as dependent processes, within the affine setup presented in Section 5.2. Within the Solvency II regime, one is required to model surrender behaviour, and also take into consideration any dependence of the interest rate (i.e. the economic environment), see Section 3.5 in [12]. This model is thus an example of how this can be done.

5.5.1 Correlated interest and surrender model

Let $\eta^0(t)$ be a deterministic surrender rate, corresponding to best estimate, i.e. the expectation of the future surrender rate. The interest rate $r(t)$ and surrender rate $\eta(t)$

are then modelled as an affine transformation of $\mathbf{X}$ of the form,

$$\begin{aligned} r(t) &= X_1(t), \\ \eta(t) &= \eta^0(t) X_2(t), \end{aligned}$$

where $\mathbf{X}$ is a 2-dimensional stochastic diffusion process. The process $\mathbf{X}$ satisfies the stochastic differential equation,

$$\begin{aligned} \mathrm{d}X_1(t) &= (b_1(t) - \beta_1 X_1(t))\,\mathrm{d}t + \sigma_1 \left(\sqrt{1-\rho^2}\mathrm{d}W_1(t) + \rho\sqrt{X_2(t)}\mathrm{d}W_2(t) \right), \\ \mathrm{d}X_2(t) &= (b_2 - \beta_2 X_2(t))\,\mathrm{d}t + \sigma_2\sqrt{X_2(t)}\mathrm{d}W_2(t), \end{aligned} \tag{5.5.1}$$

where $\mathbf{W}$ is a 2-dimensional standard Brownian motion. The parameters satisfy b_2, β_1, β_2, σ_1, $\sigma_2 \in \mathbb{R}_+$ and $\rho \in [-1, 1]$, and the function $b_1 : \mathbb{R}_+ \to \mathbb{R}$ is chosen such that an initial term structure is fitted.

The process $\mathbf{X}_2$ models relative deviations of the surrender rate from the best estimate, and it stays non-negative, hence the surrender rate $\eta(t)$ is non-negative. The interest rate process is a mix between a Hull-White Vašíček and a Heston model.

The model is affine, since $\mathbf{X}$ is affine and the surrender and interest rates are affine transformations of $\mathbf{X}$. By choosing no, or little, mean reversion, stress scenarios produced by the model are close to parallel shifts of the forward rates, which resemble the stress scenarios of the standard model of Solvency II.

Correlation

The correlation between the interest rate and the surrender rate is not in general equal to the dependency parameter ρ, which is due to the appearance of $\sqrt{X_2(t)}$ in the term $\rho\sqrt{X_2(t)}\mathrm{d}W_2(t)$ in the first line of (5.5.1). However, if we assume that $\mathrm{E}[X_2(t)] = 1$ for all t, we can calculate the correlation, using standard methods[2],

$$\mathrm{Corr}\left[r(t), \eta(t)\right] = \rho \frac{e^{(\beta_1+\beta_2)t} - 1}{\beta_1 + \beta_2} \sqrt{\frac{2\beta_1}{e^{2\beta_1 t} - 1}} \sqrt{\frac{2\beta_2}{e^{2\beta_2 t} - 1}}.$$

In the special case where $\beta_1 = \beta_2$, we get

$$\mathrm{Corr}\left[r(t), \eta(t)\right] = \rho.$$

When the parameters are chosen in Section 5.5.5 below, we see that indeed $\mathrm{E}[X_2(t)] = 1$ and $\beta_1 = \beta_2$ holds. We note that the correlation considered here is not the instantaneous correlation between the two stochastic processes $t \mapsto r(t)$ and $t \mapsto \eta(t)$, but the standard correlation between the two stochastic variables $r(t)$ and $\eta(t)$.

[2] The quantities $\mathrm{E}[r(t)]$ and $\mathrm{E}[\eta(t)]$ can be found taking expectation on the Itô representation, and solving a differential equation. The expectation $\mathrm{E}[r(t)\eta(t)]$ can be found analogously, by first finding a stochastic differential equation for the process $t \mapsto r(t)\eta(t)$.

5.5.2 The (life insurance) product

Consider a simple savings contract with a buy-back option. The savings contract consists of a guaranteed payment of 1 at retirement at time T. There is an account at the provider with a guaranteed interest rate $\hat{r}$ until time T. The value at time t of the account is then,

$$U(t) = e^{-\hat{r}(T-t)}. \tag{5.5.2}$$

The owner of the savings contract can then at any time before time T surrender the contract and receive the current account value $U(t)$.

The account value $U(t)$ is not necessarily identical to the reserve (market value) of the savings contract, thus the savings contract provider has a risk. In order to best estimate the value of the account, the surrender behaviour should be taken into account. There are different ways to valuate the surrender option, see [38] and references therein, and [23]. In this paper we adopt the intensity approach, and assume that the insured surrenders with rate $\eta(t)$ at time t, i.e. in a short time interval $[t, t + \mathrm{d}t]$, the insured surrenders with probability $\eta(t)\mathrm{d}t$, given that surrender has not occured before time t. We adopt the life insurance setup of Section 5.3, and consider the state of the insured in the state space $\mathcal{J}$ consisting of the two states *alive* and *surrendered*, corresponding to Figure 5.1.

This savings contract is a simplified version of the product considered in Example 5.3.1 and the Markov model shown in Figure 5.1, but in order to keep the notation simple and focus on the essential parts of the formulae, the mortality modelling is omitted. As long as the mortality rate is independent of the interest and surrender rate, e.g. if it is deterministic, it is straightforward to extend the formulae to include mortality. Including mortality intuitively corresponds to reducing all payments by the probability of death. For the Solvency II studies below, we can omit mortality because the independence assumption isolates it from our dependency considerations between the interest and surrender rate.

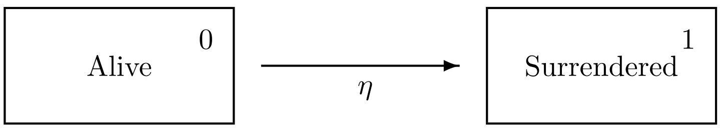

Figure 5.1: *Markov model for the surrender model.*

The payments of the contract consist of a single payment upon retirement at time T, and a payment upon surrender at time t of size $U(t)$. That is, the accumulated payments $B(t)$ at time t is given by

$$\mathrm{d}B(t) = U(t)\mathrm{d}N_{01}(t) + 1_{(Z(t)=0)}\mathrm{d}\varepsilon_T(t),$$

where ε_T is the Dirac measure at T. Analogously to the calculations in Example 5.3.1 and Example 5.4.3, we find the present value at time t of the contract as

$$\begin{aligned} PV^{\mathbf{L}}(t) &= \int_t^T e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}B(s) \\ &= \int_t^T e^{-\int_t^s r(\tau)\mathrm{d}\tau}U(s)\mathrm{d}N_{01}(s) + e^{-\int_t^T r(\tau)\mathrm{d}\tau}1_{(Z(T)=0)}, \end{aligned} \tag{5.5.3}$$

and the market value at time t is, given the savings contract has not been surrendered,

$$\begin{aligned} V(t) &= \mathrm{E}\left[PV^{\mathbf{L}}(t)\middle|\,\mathcal{F}(t), Z(t)=0\right] \\ &= \mathrm{E}\left[\int_t^T e^{-\int_t^s (r(\tau)+\eta(\tau))\mathrm{d}\tau}\eta(s)U(s)\mathrm{d}s + e^{-\int_t^T (r(\tau)+\eta(\tau))\mathrm{d}\tau}\middle|\,\mathcal{F}^{\mathbf{X}}(t)\right] \\ &= \int_t^T e^{-\int_t^s (f_t^{r:(r+\eta)}(\tau)+f_t^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_t^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &\quad + e^{-\int_t^T (f_t^{r:(r+\eta)}(\tau)+f_t^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau}. \end{aligned} \tag{5.5.4}$$

Here we used Remark 5.4.2. The notation used is introduced in Example 5.4.3 above.

5.5.3 Solvency II

For Solvency II purposes one wants to control the risk of default, such that it is less than 99.5% during the following year. In this section we specify how to interpret this in our setup, following the reasoning of Section 1.1 in [6].

We want to find the loss after one year, which is a stochastic variable, and find quantiles in the distribution of this stochastic variable. Let $PV(t)$ denote the present value at time t of future payments of the insurance company. At time 0, the Solvency II loss can be written as

$$\mathrm{E}\left[PV(0)\,|\mathcal{F}(1)\right] - \mathrm{E}\left[PV(0)\right],$$

where the expectation is taken using the market measure, or some reserving measure. For the rest of the paper, we refer to this measure as the market measure. The last term is the value of the future payments now, and the first term is the value conditional on the following year's information, which is uncertain. For simplicity, we ignore the so-called unsystematic risk during the first year, that is, we take average of the Markov chain $\mathbf{Z}$, conditionally on the underlying intensities $\mathbf{X}$. Formally, we define the *Solvency II loss after 1 year* as

$$L = \mathrm{E}\left[PV(0)\,|\mathcal{F}^{\mathbf{X}}(1)\right] - \mathrm{E}\left[PV(0)\right].$$

Both liabilities and assets must be taken into account, so the present value takes the form $PV(t) = PV^{\mathbf{L}}(t) - PV^{\mathbf{A}}(t)$, that is, the present value of the liabilities less the assets.

We consider our life insurance contract from Section 5.5.2. The simplest possible asset allocation is to deposit all capital in a savings account, earning the risk free interest rate. In that case, the present value of the assets is deterministic and equals the amount invested today. If the amount invested at time 0 is the value of the liabilities, (5.5.4), we have

$$PV^{\mathbf{A}}(0) = V(0).$$

Using this, we get the Solvency II loss,

$$\begin{aligned} L &= \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0) \middle| \mathcal{F}^{\mathbf{X}}(1)\right] - \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\right] \\ &= \mathrm{E}\left[PV^{\mathbf{L}}(0) \middle| \mathcal{F}^{\mathbf{X}}(1)\right] - V(0), \end{aligned}$$

and we see that the assets disappear from the formula, because they are essentially risk free. For our case, the first term is obtained from (5.5.3), and we get

$$\begin{aligned} &\mathrm{E}\left[PV^{\mathbf{L}}(0) \middle| \mathcal{F}^{\mathbf{X}}(1)\right] \\ &= \int_0^1 e^{-\int_0^s (r(\tau)+\eta(\tau))\mathrm{d}\tau} \eta(s)U(s)\mathrm{d}s \\ &\quad + e^{-\int_0^1 (r(s)+\eta(s))\mathrm{d}s} \int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &\quad + e^{-\int_0^1 (r(s)+\eta(s))\mathrm{d}s} e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau}. \end{aligned}$$

Subtracting $V(0)$, given by (5.5.4), yields the loss, and we rearrange the terms slightly,

$$\begin{aligned} L = &\int_0^1 e^{-\int_0^s (r(\tau)+\eta(\tau))\mathrm{d}\tau} \eta(s)U(s)\mathrm{d}s \\ &- \int_0^1 e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &+ e^{-\int_0^1 (r(s)+\eta(s))\mathrm{d}s} \int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &- \int_1^T e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &+ e^{-\int_0^1 (r(s)+\eta(s))\mathrm{d}s} e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} \\ &- e^{-\int_0^T (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau}. \end{aligned}$$

The first two lines correspond to the losses arising during the first year because of incorrect expectations of interest and surrender behaviour. The last four lines correspond to changed expectations of the future, arising because of information received during the first year. That is, the third and fourth line corresponds to changed expectations of the future about the surrender payments, and the last two lines correspond to changed expectations of the future about the payment occuring at retirement. Intuitively, the information received during the first year allows for an exact discounting during the first year, and a more precise valuation of the discounting and surrender behaviour occuring from year 1 and onwards.

The loss can be written in a simpler form. Using the notation that, for $s \leq t$, $f_t^{r:(r+\eta)}(s) = r(s)$ and $f_t^{\eta:(r+\eta)}(s) = \eta(s)$, we can write the Solvency II loss as

$$\begin{aligned} L = &\int_0^T e^{-\int_0^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &- \int_0^T e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\ &+ e^{-\int_0^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} - e^{-\int_0^T (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau}. \end{aligned} \tag{5.5.5}$$

This formula gives interpretation to the Solvency II loss, which arises due to the development of the forward rates. The loss is the difference in the expected present value with the forward rates evaluated in 1 year, and evaluated now. In practice the loss can thus be obtained by simulation of the dependent forward rates one year ahead. This is similar to how forward rates are used in finance, where the forward interest rate is simulated ahead to obtain future term structures. Here, we also simulate the dependent forward rates ahead in order to obtain the future valuation basis.

Recalling that the dependent forward rates $f_1^{r:(r+\eta)}$ and $f_1^{\eta:(r+\eta)}$ are $\mathcal{F}^{\mathbf{X}}(1)$ measurable, we can use that $\mathbf{X}$ is a Markov process and see that $f_1^{r:(r+\eta)}$ and $f_1^{\eta:(r+\eta)}$ are $r(1)$ and $\eta(1)$ measurable. Thus, the loss can be found by simulation of the underlying rates $r(s)$ and $\eta(s)$ for $0 \leq s \leq 1$. The simulation must be done under the real world probability measure. This is opposed to the market, or reserving, measure, that was used to find the loss. In this paper, we assume for simplicity that the two measures are identical, and do not adapt a change of measure approach, relieving us from discussions of preservation of the Markov property during measure changes.

5.5.4 Hedging strategy with a continuously paid coupon bond

In practice, an insurer tries to hedge the interest rate risk, thereby reducing the loss significantly. We consider a simple static hedging strategy, in a bond with continuous coupon payments of the form,

$$c(t) = e^{-\int_0^t f_0^{\eta:\eta}(\tau)\mathrm{d}\tau} f_0^{\eta:\eta}(t)U(t),$$

for $t \in (0, T)$, and a final payment at time T of

$$C(T) = e^{-\int_0^T f_0^{\eta:\eta}(\tau)\mathrm{d}\tau}.$$

For more details, see e.g. [38]. This corresponds to the expected payments of the life insurance contract, conditional on the interest rate. We can associate a payment process $\mathbf{A}^{\text{bond}}$ with the bond, given by $\mathrm{d}A^{\text{bond}}(t) = c(t)\mathrm{d}t + C(t)\mathrm{d}\varepsilon_T(t)$. The present value of future payments associated with the bond is then,

$$\begin{aligned} PV^{\text{bond}}(t) &= \int_t^T e^{-\int_t^s r(\tau)\mathrm{d}\tau}\mathrm{d}A^{\text{bond}}(s) \\ &= \int_t^T e^{-\int_t^s \left(r(\tau)+f_0^{\eta:\eta}(\tau)\right)\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\mathrm{d}s + e^{-\int_0^T \left(r(\tau)+f_0^{\eta:\eta}(\tau)\right)\mathrm{d}\tau}. \end{aligned}$$

This hedging strategy is the mean-variance optimal static hedging strategy when interest and surrender are independent. If there is a correlation between the interest and surrender rate, this strategy is not optimal. The mean-variance optimal static hedging strategy is in that case more complicated. These considerations are for simplicity omitted in this paper, and deferred for future studies.

In the case of dependence, the price of the hedging bond is smaller than the value of the liabilities, so the expected present value of the bond payments $\mathbf{A}^{\text{bond}}$ is less than of the payments from the savings contract $\mathbf{B}$. We choose to put this excess capital, which is given as

$$K = E\left[PV^{\mathbf{L}}(0) - PV^{\text{bond}}(0)\right],$$

in the bank account. For the assets, we thus have present value at time 0

$$PV^{\mathbf{A}}(0) = PV^{\text{bond}}(0) + K.$$

We note that the sign of the payments $\mathbf{A}^{\text{bond}}$ is opposite of $\mathbf{B}$, where the latter are payments to the insured and the former are payments to the insurer. Considering the life insurance contract and the hedging strategy together, we obtain a Solvency II loss,

$$\begin{aligned} L &= \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\middle|\, \mathcal{F}^{\mathbf{X}}(1)\right] - \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\right] \\ &= \mathrm{E}\left[\int_0^T e^{-\int_0^s r(\tau)\mathrm{d}\tau}(\mathrm{d}B(s) - \mathrm{d}A^{\text{bond}}(s))\middle|\, \mathcal{F}^{\mathbf{X}}(1)\right] - K \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 e^{-\int_0^s r(\tau)\mathrm{d}\tau} \left(e^{-\int_0^s \eta(\tau)\mathrm{d}\tau} \eta(s) - e^{-\int_0^s f_0^{\eta:\eta}(s)\mathrm{d}s} f_0^{\eta:\eta}(s) \right) U(s)\mathrm{d}s \\
&\quad + e^{-\int_0^1 (r(s)+\eta(s))\mathrm{d}s} \left(\int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \right. \\
&\quad \left. + e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} \right) \\
&\quad - e^{-\int_0^1 (r(s)+f_0^{\eta:\eta}(s))\mathrm{d}s} \left(\int_1^T e^{-\int_1^s (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\mathrm{d}s \right. \\
&\quad \left. + e^{-\int_1^T (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\mathrm{d}\tau} \right) - K \\
&= \int_0^T e^{-\int_0^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s \\
&\quad - \int_0^T e^{-\int_0^s (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\mathrm{d}s \qquad (5.5.6) \\
&\quad + e^{-\int_0^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\mathrm{d}\tau} - e^{-\int_0^T (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\mathrm{d}\tau} - K,
\end{aligned}
$$

Similar to (5.5.5), for $s \leq t$, the notation that $f_t^{r:(r+\eta)}(s) = f_t^{r:r}(s) = r(s)$ and $f_t^{\eta:(r+\eta)}(s) = \eta(s)$ is used for the last equality. When there is independence the bond value is the same as the value of the savings contract and $K = 0$. When there is dependence we have $K > 0$, which ensures that $\mathrm{E}[L] = 0$.

5.5.5 Numerical results

In this section we numerically show some consequenses of modelling interest and surrender as positively correlated processes. First, the model is specified by choosing a set of parameters, partly inspired by the stress levels in the Solvency II Standard Formula. With this model, we examine the consequenses for the balance sheet value of the liabilities, and the level of the Solvency II capital requirement, that is, the liabilities in 1 year's time.

For the Solvency II capital requirement, in practice in the industry, when there is no hedging, most of the risk is interest rate risk. Luckily, both in theory and practice, a lot of this can be hedged by e.g. buying bonds. For the numerical illustrations of the Solvency II capital requirement, we consider two different strategies for the assets, corresponding to the two strategies considered in Section 5.5.3 and Section 5.5.4, respectively. First, we consider the case where the interest rate risk is not hedged, and all assets are accumulated by the risk free interest rate. Second, we consider the case where the insurer tries to hedge the interest rate risk, and performs a static hedge.

Parameters

The numerical examples with the model (5.5.1) are carried out for different level of correlation, namely $\rho \in \{0, 0.3, 0.7\}$. Also, we consider two different guaranteed interest rates, namely $\hat{r} \in \{1\%, 4\%\}$. This corresponds to a low interest rate, which could be for a newly issued policy, and a high interest rate, which could be for a policy issued years ago, when the interest rate level was higher. We note that the base deterministic surrender rate η^0 corresponds to a person aged 40, thus with $T = 25$, the contract ends at age 65.

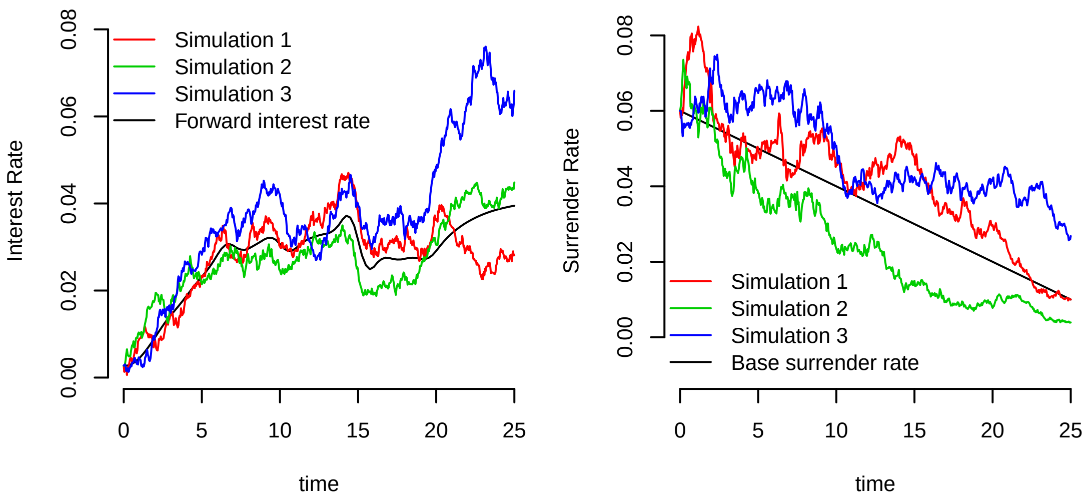

Figure 5.2: *Illustrative realisations of the interest rate (left) and the surrender rate (right), with* $\rho = 0.7$.

The parameters chosen for the interest and surrender rates are listed in Table 5.1, and in Figure 5.2 some realisations of the interest and surrender rates are shown. The initial value $X_1(0)$ and function $b_1(t)$ are chosen such that the term structure provided by the Danish FSA at August 17, 2012 is matched. Let $f^{\text{FSA}}(t)$ denote the forward rate provided by the Danish FSA. Then the parameters $X_1(0)$ and b_1 are fitted such that

$$\mathrm{E}\left[e^{-\int_0^t r(s)\mathrm{d}s}\right] = e^{-\int_0^t f^{\text{FSA}}(s)\mathrm{d}s},$$

for all $t \geq 0$. The parameters of the model correspond to the measure used for valuating the market value of the life insurance liabilities. Thus, with respect to the interest rate it is the risk neutral measure. For simplicity, we assume that this measure equals the real world probability measure.

Dependent forward rates

In Figure 5.3, the dependent forward rates are shown. They are calculated by solving the differential equations (5.2.3) and (5.2.5) numerically. For the interest rate, the forward

β_1	0.02	b_2	0.02	$\eta^0(t)$	$0.06 - 0.002 \cdot t$
σ_1	0.005	β_2	0.02	$X_2(0)$	1
		σ_2	0.15		

Table 5.1: *Parameters for correlated interest and surrender modelling. The initial value $X_1(0)$ and the function $b_1(t)$ are chosen such that the interest rate model matches the term structure provided by the Danish FSA for valuating life insurance liabilites, at August 17, 2012.*

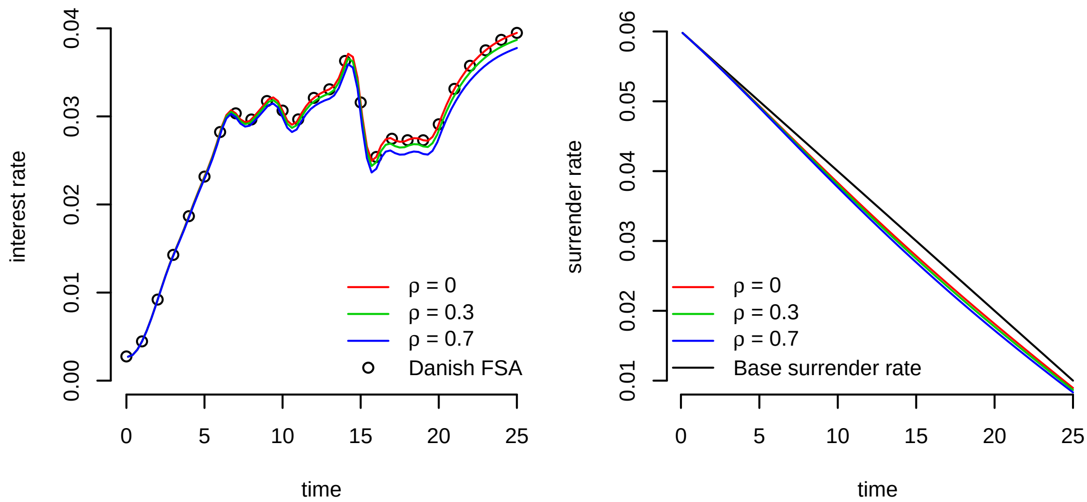

Figure 5.3: *Dependent forward rates. Left: for the interest rate, $f_0^{r:(r+\eta)}(t)$. Right: for the surrender rate, $f_0^{\eta:(r+\eta)}(t)$. The dependent forward rates are shown for different values of ρ. The forward interest rate extracted from the Danish FSA at August 17, 2012 is also shown, as well as the base deterministic surrender rate η^0. Higher values of ρ lead to lower values of the forward rates, corresponding to less discounting.*

interest rate supplied by the Danish FSA, f^{FSA}, is shown as well. We see that for the case $\rho = 0$ the dependent forward interest rate f_0^r is identical to the forward rate provided by the Danish FSA. This is as expected, since in the case $\rho = 0$ the interest rate and surrender rate are independent, and in this case the dependent forward rates are equal to the usual forward rates. For a positive correlation, the dependent forward rates are smaller. This is because the stochastic variable, $e^{-\int_0^t (r(s)+\eta(s))\mathrm{d}s}$, which is used to construct the dependent forward rates, has a heavier tail when the correlation is strictly positive, due to the exponential function. Intuitively, there is less diversification between the interest and surrender rate.

For the surrender rate, the basic deterministic surrender rate η^0 is shown as well as the dependent forward rates. Even though $\mathrm{E}[\eta(t)] = \eta^0(t)$, we see that the dependent forward rates are systematically lower than η^0. This is due to Jensens inequality, and to see this, consider the case $\rho = 0$, where we get,

$$\begin{aligned} e^{-\int_0^t (f_0^r(s)+f_0^\eta(s))\mathrm{d}s} &= \mathrm{E}\left[e^{-\int_0^t (r(s)+\eta(s))\mathrm{d}s}\right] \\ &= \mathrm{E}\left[e^{-\int_0^t r(s)\mathrm{d}s}\right] \mathrm{E}\left[e^{-\int_0^t \eta(s)\mathrm{d}s}\right] \\ &> \mathrm{E}\left[e^{-\int_0^t r(s)\mathrm{d}s}\right] e^{-\int_0^t \mathrm{E}[\eta(s)]\mathrm{d}s} \\ &= e^{-\int_0^t f_0^r(s)\mathrm{d}s} e^{-\int_0^t \eta^0(s)\mathrm{d}s}, \end{aligned}$$

for $t > 0$, using that the usual forward rate is identical to the dependent forward rate for $\rho = 0$. From this inequality, we obtain,

$$f_0^\eta(t) < \eta^0(t),$$

which is what was observed as the red and black lines in Figure 5.3. If there is a positive correlation, the dependent forward surrender rate, $f^{\eta:(r+\eta)}$, is even smaller, similar to the observation for the interest rates.

Market value

The market value at time 0, $V(0)$ from (5.5.4), can be calculated, solving the integral numerically. For this, first use (5.4.1) to get

$$V(t) = \int_t^T e^{\phi(t,s)+\psi(t,s)^\top X(t)} f_t^{\eta:(r+\eta)}(s)U(s)\mathrm{d}s + e^{\phi(t,T)+\psi(t,T)^\top X(t)},$$

which is easier to handle from a computational point of view, because the functions ϕ and ψ are obtained in the process of calculating the dependent forward rates $f^{r:(r+\eta)}$ and $f^{\eta:(r+\eta)}$ when solving (5.2.3) and (5.2.5). The market value $V(0)$, dependent upon the guaranteed interest rate $\hat{r}$ and the correlation ρ, is shown in Table 5.2. The market values can be compared to the value of the policyholders account which is paid out on surrender. This is given by (5.5.2), calculated using the guaranteed interest rate. The value at time 0 is presented in Table 5.3.

		$\hat{r}$	
		4%	1%
	0	0.4567	0.6167
ρ	0.3	0.4595	0.6191
	0.7	0.4631	0.6222

Table 5.2: *Market value at time 0, $V(0)$, of the life insurance contract. The value is shown using three different correlations, corresponding to three different sets of dependent forward rates, red, green and blue from Figure 5.3. Two different levels of guaranteed interest rate, $\hat{r}$, is used, which leads to different surrender payouts $U(t)$.*

$\hat{r}$	
4%	1%
0.3679	0.7788

Table 5.3: *Initial value of the policyholders account, $U(0)$. For the high guaranteed interest rate (4%), the value is lower than the market value from Table 5.2. For the low guaranteed interest rate (1%), the value is higher than the market value.*

The market value without surrender modelling, calculated setting the surrender rate equal to zero, is 0.5037. It is independent of the guaranteed interest rate. From Table 5.2 it is seen that when we include surrender modelling the market value is somewhere between the value of the policyholders account and the market value calculated without surrender modelling.

For both cases of guaranteed interest rates, the market value increases with correlation. When we discussed the dependent forward rates in Section 5.5.5, we saw that the dependent forward rates decrease with increasing correlation, which is basically due to the convexity of the exponential function and Jensen's inequality. A smaller dependent forward interest rate leads to an increasing market value. For the surrender rate, it is more complicated. For the case of a guaranteed interest rate of 4%, an increase in the dependent forward surrender rate leads to a decrease in the market value, because the market value come closer to the value paid out on surrender. For the case of a guaranteed interest rate of 1%, the same argument tells us that an increase in the dependent forward surrender rate instead leads to an increasing market value. We see that the effect of the decreasing dependent forward interest rate is largest, and in total, for both levels of guaranteed interest rate, the market value increases when the correlation increases.

Solvency II

We examine the effect on the Solvency II capital requirement with two different strategies for the assets. The first strategy is no hedging and the second strategy is a simple static hedging strategy. This corresponds to the two strategies discussed in Section 5.5.3 and Section 5.5.4, respectively. For the first strategy, where all assets are invested in the bank account, the Solvency II loss is given by (5.5.5). For the second strategy, where the interest rate risk is hedged statically in a bond with continuous payments, the Solvency II loss is given by (5.5.6).

In Table 5.4 the Solvency II loss for the different cases of hedging strategy, guaranteed interest rate risk and correlation is presented. It is immediately seen, that trying to hedge the interest rate risk by applying the simple hedging strategy significantly reduces the Solvency II loss.

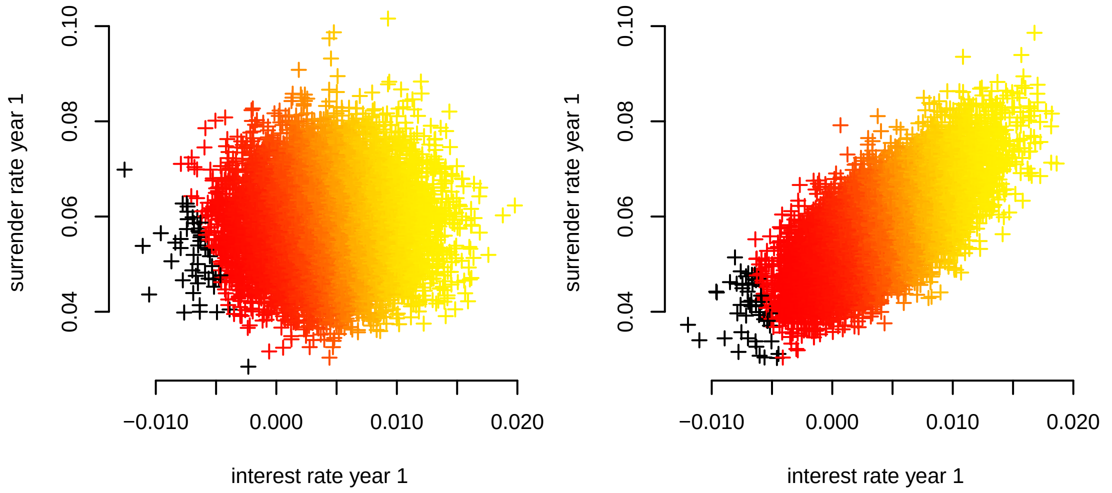

Figure 5.4: *Guaranteed interest rate 4%. Plot of the interest and surrender rate simulations after 1 year in the case without any hedging strategy and correlation $\rho = 0$ (left) and $\rho = 0.7$ (right). The color of a mark indicates the Solvency II loss* (5.5.5)*, where a darker color is a higher loss, and black colors are losses beyond the 99.5% quantile.*

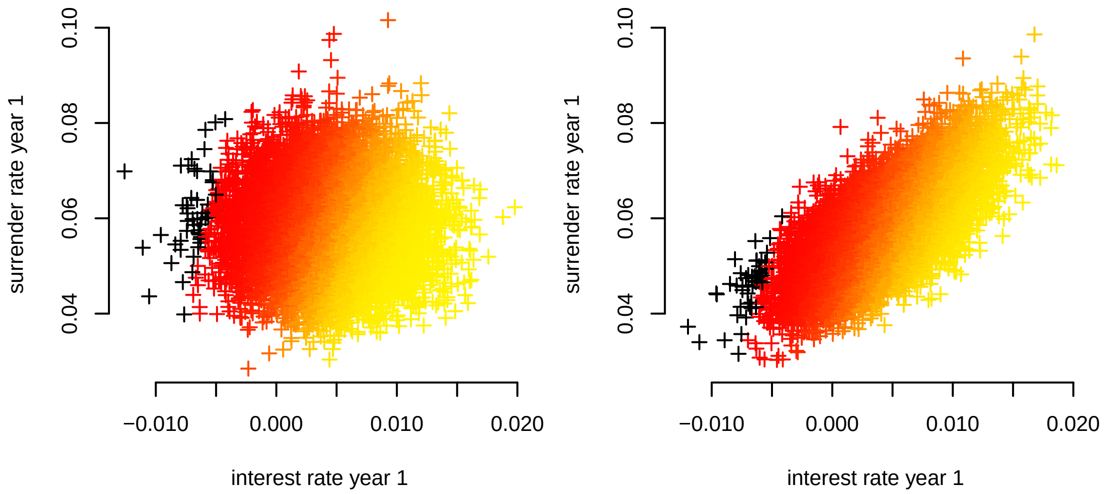

Figure 5.5: *Guaranteed interest rate 1%. Plot of the interest and surrender rate simulations after 1 year in the case without any hedging strategy and correlation $\rho = 0$ (left) and $\rho = 0.7$ (right). The color of a mark indicates the Solvency II loss* (5.5.5)*, where a darker color is a higher loss, and black colors are losses beyond the 99.5% quantile.*

		No Hedge		Hedge	
		$\hat{r}$		$\hat{r}$	
		4%	1%	4%	1%
	0	0.069	0.077	0.014	0.025
ρ	0.3	0.072	0.072	0.015	0.028
	0.7	0.078	0.060	0.017	0.029

Table 5.4: *Simulated Solvency II loss. Without hedging it is given by* (5.5.5) *and with the hedging strategy it is given by* (5.5.6). *Applying an interest hedging strategy significantly lowers the Solvency II loss. Also, modelling correlation between interest and surrender has a significant impact on the Solvency II loss.*

For the case of no hedging strategy, we see two different correlation effects. When the guaranteed interest rate is 4%, a higher correlation means a higher Solvency II loss, because a decrease in interest and surrender rate both increase the present value of the contract payments. This is depicted in Figure 5.4, where we see that the loss increases with both decreasing interest and decreasing surrender. A higher correlation means that the probability of simultaneous drops in the interest and surrender rate occurs simultaneously, which can be seen at the right graph in Figure 5.4. When the guaranteed interest rate is instead 1%, a decrease in the surrender rate now means that Solvency II loss decrease, which can be seen in Figure 5.5. Introducing a correlation leads to less observations with decreasing interest and increasing surrender, thus leading to more diversification and reducing the Solvency II loss. This can be seen in the right graph of Figure 5.5.

5.6 Conclusion

In this paper we review theory on continuous affine processes which sets the basis for the application in life insurance valuation and risk management. This allows us to introduce so-called dependent forward rates. They are compared to other forward rate definitions, and some desired properties about the dependent forward rates are highlighted. In particular, as is seen in Section 5.4.1, the dependent forward rates meet some of the critique of forward rates raised in [41]. However, a full answer is not reached, and it is open for further research whether the concept of forward rates in life insurance is fruitful beyond being a convenient representation for the quantities needed for calculation of certain life insurance liabilites under a stochastic intensity assumption.

In the second part of the paper, we apply the theory of the continuous affine processes and the dependent forward rates. A specific model for surrender modelling is proposed, where the interest and surrender rate is positively correlated. The surrender rate in this model is non-negative. We consider a simple life insurance like savings product with a buy-back option. The dependent forward rates are calculated for different correlations, and we see that they are decreasing with increasing correlation. This in part has the effect that the market value is increasing with correlation, since in part, this means we in practice use a smaller interest rate for discounting. We also consider the Solvency II capital requirement in the form of formulae for the value-at-risk in a one-year time horison. In particular we obtain the formula (5.5.5), where we see that the loss is given through the difference of the expected present value valuated with the dependent forward rates in 1 year and the current dependent forward rates. We calculate the actual Solvency II capital requirement in our example, with and without a simple static hedging strategy for the interest rate risk, and see how the introduction of correlation can both increase and decrease the Solvency II capital requirement in our example, depending on the guaranteed interest rate.

Acknowledgements

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5.A Forward mortality rate for term insurances not-so-well defined

Consider a interest and mortality rate model $(r(t), \mu(t))$. This give us a set of dependent forward rates, and a forward mortality rate for pure endowments. Assume that the following assumptions hold.

Assumption 5.A.1. *Let a model for the interest and mortality rates $r(t)$ and $\mu(t)$ be given. The assumptions are,*

1. $h_t^{\text{pe}}(s) > 0$ *for all* $s > t$.
2. h_t^{pe} *is bounded from below for some timepoint, i.e. there exist* $\varepsilon > 0$ *and* $t_0 > 0$ *such that* $h_t^{\text{pe}}(s) > \varepsilon$ *for all* $s > t_0$.
3. *The forward interest rate is greater than the dependent forward rate for the interest,* $g_t(s) > f_t^r(s)$*, for all* $s > t$.

It is indeed possible to construct models where these assumptions hold, and they will hold for most models when there is a positive correlation between the interest rate and mortality rate. The first two assumptions state that the forward mortality rate for pure endowments is positive and bounded below from some time, which is satisfied in reasonable models. The third assumption usually holds when there is a positive correlation between the interest and mortality rate.

The forward mortality rate for pure endowments, $h_t^{\mathrm{pe}}(s)$, present in the assumptions, is not the object of interest in this example. In view of (5.4.8), it can be thought of as a placeholder for $f_t^r + f_t^\mu - g_t$.

Proposition 5.A.2. *Under Assumption 5.A.1, there exists a $T > 0$ such that the forward mortality rate for term insurances $h_t^{\mathrm{ti}}(s)$ given by* (5.4.10) *does not exist for $s > T$.*

Proof. Combining (5.4.10) and (5.4.3), and then using (5.4.8) twice, we get that

$$\begin{aligned} e^{-\int_t^T h_t^{\mathrm{ti}}(s)\mathrm{d}s} h_t^{\mathrm{ti}}(T) &= e^{-\int_t^T \left(f_t^r(s)+f_t^\mu(s)-g_t(s)\right)\mathrm{d}s} f_t^\mu(T) \\ &= e^{-\int_t^T h_t^{\mathrm{pe}}(s)\mathrm{d}s} \left(h_t^{\mathrm{pe}}(T) + g_t(T) - f_t^r(T)\right), \end{aligned}$$

and by integration we find

$$e^{-\int_t^T h_t^{\mathrm{ti}}(s)\mathrm{d}s} = 1 - \int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau. \tag{5.A.1}$$

Since the left hand side must be positive for any T, we conclude that the condition

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau < 1 \tag{5.A.2}$$

is necessary for the forward mortality rate for term insurances to be well-defined.

Under the first assumption, the forward mortality rate for pure endowments, h_t^{pe}, defines a distribution in a two-state Markov chain, and we recognise the integral $\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\mathrm{d}s}$ $h_t^{\mathrm{pe}}(\tau)\mathrm{d}\tau$ as a probability: Let Z be a stochastic variable that denotes the lifetime in a survival model where death occurs with rate $h_t^{\mathrm{pe}}(s)$ at time s. Then

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\mathrm{d}s} h_t^{\mathrm{pe}}(\tau)\mathrm{d}\tau = P(Z \le T \mid Z > t).$$

Also, under the second assumption the probability converges to 1,

$$P(Z \le T \mid Z > t) \to 1 \text{ for } T \to \infty.$$

Consider now (5.A.2). Under the third assumption, $g_t(s) > f_t^r(s)$ for all $s > t$, there exists $\varepsilon > 0$ and $T^* \ge t$ such that

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\mathrm{d}s} \left(g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau > \varepsilon,$$

for all $T > T^*$. This allows us to conclude, for a $T > T^*$ large enough, such that $P(Z \leq T \mid Z > t) > 1 - \varepsilon$, that

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau > P(Z \leq T \mid Z > t) + \varepsilon > 1.$$

This contradicts (5.A.2), and the forward mortality rate for term insurances does not exist. □

We give an example of a model satisfying Assumption 5.A.1. Let the 2-dimensional process $\mathbf{X}$ satisfy

$$\begin{aligned}
\mathrm{d}X_1(t) &= (1 - X_1(t))\mathrm{d}t + \sigma \mathrm{d}W_1(t), \\
\mathrm{d}X_2(t) &= (1 - X_2(t))\mathrm{d}t + \sigma\lambda \mathrm{d}W_1(t) + \sigma\sqrt{1-\lambda^2}\mathrm{d}W_2(t),
\end{aligned}$$

with $X(0) = (1, 1)^\top$. Let the interest rate and mortality rate be given by

$$\begin{aligned}
r(t) &= r_0 X_1(t), \\
\mu(t) &= \mu^\circ(t) + X_2(t) - 1,
\end{aligned}$$

with parameters $\lambda = 0.8$, $\sigma = 0.07$ and base mortality

$$\mu^\circ(t) = 5 \cdot 10^{-4} + 7.5858 \cdot 10^{-5} \cdot 1.09144^{50+t}.$$

That this model satisfies Assumption 5.A.1 can be shown by solving relevant differential equations.

Chapter 6

A step forward with Kolmogorov

This chapter is, except for Section 6.7, based on the paper [9].

Abstract

In the general setup of a doubly stochastic Markov chain where the transition intensities are modelled as diffusion processes, we present a forward partial integro-differential equation for the transition probabilities. This is a generalisation of Kolmogorov's forward differential equation. These models are applicable in e.g. life insurance mathematics. The result presented follows from the general forward partial integro-differential equation for stochastic processes, of which the Fokker-Planck differential equation and Kolmogorov's forward differential equation are the two most known special cases. We furthermore consider the semi-Markov case, which relates to our results. We end the paper by using the forward partial integro-differential equation to find a novel representation of the forward mortality rate.

6.1 Introduction

We begin this paper by presenting a forward partial integro-differential equation for the transition probabilities of a general jump-diffusion, which is inspired from Chapter 3.4 in [22]. A forward partial differential equation exists for the case of a continuous diffusion, and this is known as the Fokker-Planck equation. For the case of a pure jump process, this is also known, and is called Kolmogorov's forward differential equation or the master equation. The more general result presented here we refer to as *Kolmogorov's forward partial integro-differential equation*, and this sets the basis for the rest of the results. We are interested in applying the result for a stochastic process $\mathbf{Z} = (Z(t))_{t\geq 0}$ taking values in a finite state space $\mathcal{J} = \{0, 1, \ldots, J\}$. A basic example is if $\mathbf{Z}$ itself is a Markov chain. Then there exists deterministic transition rates for each transition between states in $\mathcal{J}$.

This setup is widely used in e.g. life insurance to model the state of the policyholder, e.g. alive, disabled, dead etc. In this case, Kolmogorov's forward differential equation is well known, and can be used to find the transition probabilities, that is, given we are in a state i at time t, what is the probability of being in a state j at a future time point s. The transition probabilities are essential for e.g. calculating expected cash flows in life insurance. For details about this setup in life insurance, see e.g. [31] or [39], and for cash flows, see [11].

A more general case is if we allow the transition rates to depend on some underlying diffusion process $\mathbf{X} = (X(t))_{t\geq 0}$. Then $\mathbf{Z}$ is no longer Markov, but $(\mathbf{Z}, \mathbf{X})$ is a Markov chain. The main result of this paper is that we present Kolmogorov's forward differential equation for this setup, as a special case of the general equation. It is a partial integro-differential equation, and can be considered a generalisation of Kolmogorov's forward differential equation. Various examples of this setup have been studied in the life insurance literature, see [36], [14], [8], and also in credit risk, see e.g. [33]. It is well known that a backward partial differential equation exists, but for calculating life insurance cash flows, a forward differential equation is more efficient in practice.

Another example of a stochastic process $\mathbf{Z}$ on a finite state space is when $\mathbf{Z}$ is a semi-Markov process. Let $\mathbf{U} = (U(t))_{t\geq 0}$ be the duration in the current state, then if $(\mathbf{Z}, \mathbf{U})$ is a Markov process, $\mathbf{Z}$ is a semi-Markov process. In this case, the transition intensities may depend on the duration $\mathbf{U}$. It is already known that a version of Kolmogorov's forward differential equation exists for this case, see e.g. [11] or [25]. In this paper, we show how Kolmogorov's forward partial integro-differential equation specialises in the semi-Markov setup to Kolmogorov's forward integro-differential equation.

Common for these examples is that the obtained transition probabilities, or transition densities, describe the distribution of the multidimensional process $(\mathbf{Z}, \mathbf{X})$ or $(\mathbf{Z}, \mathbf{U})$. It seems impossible to obtain an ordinary differential equation for $\mathbf{Z}$ alone, since it is not a Markov process. However, in the simple survival model with the states alive and dead, where the transition rate depends on a diffusion process $\mathbf{X}$, such a forward differential equation is easy to find, and it is dependent on the so-called forward mortality rate. For literature on forward rates, see e.g. [41] or [8]. We show how Kolmogorov's forward partial integro-differential equation specialises in this case, and find that the forward mortality rate can in fact be represented as the expected mortality rate, conditional on survival. We also present an alternative proof for this result. To the author's knowledge, this relation has not been presented in the literature before.

The structure of the paper is as follows. In Section 6.2, we consider a jump-diffusion process and find a forward partial integro-differential equation for the transition probabilities, Kolmogorov's forward partial integro-differential equation. In Section 6.3, we consider the case where $\mathbf{Z}$ is dependent on a continuous diffusion process $\mathbf{X}$ and present

Kolmogorov's forward partial integro-differential equation for this case. In Section 6.4, we show how to apply this result in life insurance, and why it is an important result. In Section 6.5, we relate Kolmogorov's forward partial integro-differential equation to the semi-Markov setup, and see that we obtain the same integro-differential equation as in [11]. Finally, in Section 6.6 we examine the simple survival model and compare Kolmogorov's forward integro-differential equation to the simple forward differential equation, which we already know. Here we discover the interesting representation of the forward rate as the expectation of the mortality rate conditional on survival.

6.2 The forward partial integro-differential equation

Let $\mathbf{X}$ be a d-dimensional stochastic jump-diffusion in a state space $\mathcal{X} \subset \mathbb{R}^d$, defined on a probability space $(\Omega, \mathbb{F}, \mathrm{P})$. We assume that $\mathbf{X}$ is a solution to the stochastic differential equation,

$$\mathrm{d}X(t) = \beta(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t) + \mathrm{d}J(t), \tag{6.2.1}$$

for all $t \in [0, T]$. Throughout the article, T is some finite time-horizon. Let $\beta : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $\beta \in \mathcal{C}^{1,1}$ and $\sigma : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $\sigma \in \mathcal{C}^{1,2}$, where $\mathcal{C}^{n,m}$ is the set of functions that are n times continuously differentiable in the first argument, and m times continuously differentiable in the second argument. Let $\mathbf{W}$ be a d-dimensional standard Brownian motion, and $\mathbf{J}$ be a pure jump process. Define also $\rho(t, x) = \sigma(t, x)\sigma(t, x)^\top$.

The jump part of $\mathbf{X}$ is the pure jump process $\mathbf{J}$. The jump intensity measure is denoted $\mu : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^+$. We assume it exists, that $\mu(\mathbb{R}^d; t, x)$ is bounded for all t, x, and that the compensated process

$$t \mapsto J(t) - \int_0^t \int (y - X(s-))\, \mu(\mathrm{d}y; s, X(s-))\, \mathrm{d}s \tag{6.2.2}$$

is a martingale. Thus, $\mu(\mathrm{d}y; t, x)$ is the measure of the jump destinations of $\mathbf{X}$, and not jump sizes. Here and in general, when we omit integration bounds we integrate over the whole domain of the integrand. We use the notation $f(x-) = \lim_{y \nearrow x} f(y)$, and use τ_A as the counting measure on the set A.

The process $\mathbf{X}$ can be characterised by its infinitesimal generator $\mathcal{A}_t$, which is given as,

$$\begin{aligned}\mathcal{A}_t f(x) = &\sum_i \beta_i(t, x) \frac{\partial}{\partial x_i} f(x) + \frac{1}{2} \sum_{i,j} \rho_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) \\ &+ \int (f(y) - f(x))\, \mu(\mathrm{d}y; t, x),\end{aligned}$$

for suitable functions $f \in \mathcal{C}^2$. Last, we also define $\mathbf{N}$ as the process counting the number of jumps,

$$N(t) = \#\{s \in (0,t] \mid J(s-) \neq J(s)\}.$$

It follows that $\mathbf{N}$ has compensator $\int_0^t \mu(\mathbb{R}^d; s, X(s-))\mathrm{d}s$.

The conditional distribution of $\mathbf{X}$ is denoted P, and for $t \geq t'$ and a Borel set $A \subset \mathbb{R}^d$ we write

$$\mathrm{P}(X(t) \in A \mid X(t') = x') = P(t, A; t', x') = \int_A P(t, \mathrm{d}x; t', x').$$

The transition probability can be described by a forward partial integro-differential equation (PIDE). This result sets the basis for this article.

Theorem 6.2.1. (Kolmogorov's forward PIDE) *Assume that*

$$\int_0^t \beta_i(s, X(s)) f(X(s))\,\mathrm{d}W(s)$$

is a martingale for all $f \in \mathcal{C}^1$ with compact support. Assume there exists a set $\tilde{\mathcal{X}} \subset \mathcal{X}$ such that $\frac{\partial}{\partial t}P(t, \mathrm{d}x; t', x')$, $\frac{\partial}{\partial x_i}P(t, \mathrm{d}x; t', x')$ and $\frac{\partial^2}{\partial x_i \partial x_j}P(t, \mathrm{d}x; t', x')$ exist for all $i, j = 1, \ldots, d$, all $x \in \tilde{\mathcal{X}}$ and $t \in (t', T]$. Then the transition probability P of $\mathbf{X}$ satisfies, for $t \in (t', T]$, the PIDE,

$$\begin{aligned}
\frac{\partial}{\partial t}P(t, A; t', x') = &- \sum_i \int_A \frac{\partial}{\partial x_i}\left(\beta_i(t, x)P(t, \mathrm{d}x; t', x')\right) \\
&+ \frac{1}{2}\sum_{i,j}\int_A \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_{ij}(t, x)P(t, \mathrm{d}x; t', x')\right) \\
&+ \int_{A^\complement}\left(\int_A \mu(\mathrm{d}x; t, y)\right)P(t, \mathrm{d}y; t', x') \\
&- \int_A\left(\int_{A^\complement} \mu(\mathrm{d}y; t, x)\right)P(t, \mathrm{d}x; t', x'),
\end{aligned} \tag{6.2.3}$$

for any compact Borel set $A \subset \tilde{\mathcal{X}}$.

The notation $A^\complement$ is the complement of the set A.

Remark 6.2.2. The transition probability trivially satisfies the boundary condition

$$P(t', A; t', x) = 1_A(x).$$

◇

This differential equation does not seem to have any agreed name in the literature. If $\mathbf{X}$ is a continuous diffusion process, it is called the Fokker-Planck equation, and if $\mathbf{X}$ is a pure jump process it is often called Kolmogorov's forward differential equation, and also the master equation. Since we usually work with Kolmogorov's forward differential equation, and consider this a generalisation, we refer to it as *Kolmogorov's forward PIDE*. For more about Kolmogorov's differential equations and applications in life insurance, see e.g. [39]. For more on the Fokker-Planck and the master equation, see e.g. [22]. The following proof is inspired by the calculations in [22].

Proof. Let $f \in \mathcal{C}^2$, and assume that f has support on a compact set $A \subset \tilde{\mathcal{X}}$. An application of Itô's formula yields,

$$\begin{aligned} &f(X(t)) - f(X(t')) \\ &\quad= \int_{t'}^{t} \left\{ \sum_i \beta_i(s, X(s)) \frac{\partial f(X(s))}{\partial x_i} + \frac{1}{2} \sum_{i,j} \rho_{ij}(s, X(s)) \frac{\partial^2 f(X(s))}{\partial x_i \partial x_j} \right\} \mathrm{d}s \\ &\qquad + \int_{t'}^{t} \int \left(f(y) - f(X(s-)) \right) \mu(\mathrm{d}y; s, X(s-)) \mathrm{d}s \\ &\qquad + \int_{t'}^{t} \sum_i \beta_i(s, X(s)) \frac{\partial f(X(s))}{\partial x_i} \mathrm{d}W(s) + \tilde{J}(t) - \tilde{J}(t'), \end{aligned} \tag{6.2.4}$$

where $\tilde{J}(t) = \int_0^t \left(f(X(s)) - f(X(s-)) \right) \mathrm{d}N(s) - \int_0^t \left(f(y) - f(X(s-)) \right) \mu(\mathrm{d}y; s, X(s-)) \mathrm{d}s$. We know that $\tilde{J}(t)$ is a martingale. We also have,

$$\int_A f(x) P(t, \mathrm{d}x; t', x') - f(x') = \mathrm{E}\left[f(X(t)) - f(X(t')) \middle| X(t') = x' \right].$$

By insertion of (6.2.4), and using that the last line of (6.2.4) is a martingale, we obtain

$$\begin{aligned} &\int_A f(x) P(t, \mathrm{d}x; t', x') - f(x') \\ &\quad= \mathrm{E}\Bigg[\int_{t'}^{t} \left\{ \sum_i \beta_i(s, X(s)) \frac{\partial f(X(s))}{\partial x_i} + \frac{1}{2} \sum_{i,j} \rho_{ij}(s, X(s)) \frac{\partial^2 f(X(s))}{\partial x_i \partial x_j} \right\} \mathrm{d}s \\ &\qquad\quad + \int_{t'}^{t} \int \left(f(y) - f(X(s-)) \right) \mu(\mathrm{d}y; s, X(s-)) \mathrm{d}s \,\Bigg|\, X(t') = x' \Bigg] \\ &\quad= \int_{t'}^{t} \int_A \left\{ \sum_i \beta_i(s, x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j} \rho_{ij}(s, x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right\} P(s, \mathrm{d}x; t', x') \mathrm{d}s \\ &\qquad\quad + \int_{t'}^{t} \int \int \left(f(y) - f(x) \right) \mu(\mathrm{d}y; s, x) P(s, \mathrm{d}x; t', x') \mathrm{d}s. \end{aligned}$$

We differentiate with respect to t, and then apply partial integration to the two first terms,

$$\begin{aligned}
\frac{\partial}{\partial t}\int_A f(x)P(t,\mathrm{d}x;t',x') &= \sum_i \int_A \beta_i(t,x)\frac{\partial f(x)}{\partial x_i}P(t,\mathrm{d}x;t',x') \\
&\quad + \frac{1}{2}\sum_{i,j}\int_A \rho_{ij}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j}P(t,\mathrm{d}x;t',x') \\
&\quad + \int\int (f(y)-f(x))\,\mu(\mathrm{d}y;t,x)P(t,\mathrm{d}x;t',x') \\
&= -\sum_i \int_A f(x)\frac{\partial}{\partial x_i}\left(\beta_i(t,x)P(t,\mathrm{d}x;t',x')\right) \\
&\quad + \frac{1}{2}\sum_{i,j}\int_A f(x)\frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_{ij}(t,x)P(t,\mathrm{d}x;t',x')\right) \\
&\quad + \int\int (f(y)-f(x))\,\mu(\mathrm{d}y;t,x)P(t,\mathrm{d}x;t',x').
\end{aligned}$$

The boundary terms from the partial integration vanishes due to the fact that $f(x) = 0$ and $\frac{\partial}{\partial x_i}f(x) = 0$ on the boundary of A, since f is $\mathcal{C}^2$.

Let $f_n \in \mathcal{C}^2$ be a series of uniformly bounded functions with compact support in $\tilde{\mathcal{X}}$, such that $f_n \to 1_A$ for $n \to \infty$. This yields the result. $\square$

If $\mu(\mathbb{R}^d;t,x) = 0$ for all t,x, the process $\mathbf{X}$ is continuous. In that case, (6.2.3) reduces to an integral version of the well-known Fokker-Planck equation.

$$\begin{aligned}
\frac{\partial}{\partial t}P(t,A;t',x') &= -\sum_i \int_A \frac{\partial}{\partial x_i}\left(\beta_i(t,x)P(t,\mathrm{d}x;t',x')\right) \\
&\quad + \frac{1}{2}\sum_{i,j}\int_A \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_{ij}(t,x)P(t,\mathrm{d}x;t',x')\right).
\end{aligned}$$

In the opposite case, where $\beta(t,x) = 0$ and $\sigma(t,x) = 0$ for all t,x, the process $\mathbf{X}$ is a pure jump process. In that case, (6.2.3) reduces to an integral version of Kolmogorov's forward differential equation, also known as the master equation,

$$\begin{aligned}
\frac{\partial}{\partial t}P(t,A;t',x') &= \int_{A^\complement}\left(\int_A \mu(\mathrm{d}x;t,y)\right)P(t,\mathrm{d}y;t',x') \\
&\quad - \int_A\left(\int_{A^\complement}\mu(\mathrm{d}y;t,x)\right)P(t,\mathrm{d}x;t',x'),
\end{aligned}$$

Assuming that a density exists with respect to some measure, one can differentiate and obtain the Fokker-Planck respectively Kolmogorov's forward differential equation.

In particular the jump part of (6.2.3) is easily interpreted. The positive term leads to an increasing probability, and it is the probability of being somewhere in the complement of A and making a jump inside A. The negative term leas to a decreasing probability, and it is the probability of being somewhere in A, and making a jump to the complement of A. Jumps solely inside A or $A^{\complement}$ does not affect the probability.

6.3 The doubly stochastic Markov chain setup

In this section we consider the doubly stochastic Markov chain setup, which can be considered a special case of the stochastic process $\mathbf{X}$ from (6.2.1). We let a set of stochastic intensities be modelled as a continuous diffusion process. Conditional on these, we create a Markov chain with these intensities. For simplicity we restrict to the case of continuous intensities.

Let a finite state space $\mathcal{J} = \{0, \ldots, J\}$ be given. For each $k, \ell \in \mathcal{J}$, where $k \neq \ell$, we associate a transition rate $t \mapsto \mu_{k\ell}(t, X(t))$. Here, $\mathbf{X}$ is a d-dimensional diffusion process satisfying the stochastic differential equation

$$\mathrm{d}X(t) = \beta(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t),$$

where $\mathbf{W}$ is a d-dimensional diffusion process and β and σ are as in Section 6.2. Thus, the transition intensities $\mu_{k\ell}$ are stochastic.

Define now the stochastic process $\mathbf{Z}$ on $\mathcal{J}$, and assume it is cádlág. Conditional on $\mathbf{X}$, we let $\mathbf{Z}$ be a Markov chain, with transition rates $\mu_{k\ell}(t, X(t))$. Define the filtrations generated by $\mathbf{X}$ respectively $\mathbf{Z}$ as $\mathcal{F}^{\mathbf{X}}(t) = \sigma(X(s)|s \leq t)$ and $\mathcal{F}^{\mathbf{Z}}(t) = \sigma(Z(s)|s \leq t)$. Define also the larger filtration $\mathcal{F}(t) = \mathcal{F}^{\mathbf{X}}(t) \vee \mathcal{F}^{\mathbf{Z}}(t)$. Let $\mathbf{N}_{k\ell}$ be a counting process that counts the number of jumps from state k to state ℓ,

$$N_{k\ell}(t) = \#\{s \leq t | Z(s-) = k, Z(s) = \ell\}. \tag{6.3.1}$$

The assumption that $\mathbf{Z}$ is a Markov chain with stochastic transition rates $\mu_{k\ell}(t, X(t))$ means, that conditional on $\mathbf{X}$, the compensated process

$$N_{k\ell}(t) - \int_0^t 1_{(Z(s-)=k)} \mu_{k\ell}(s, X(s))\,\mathrm{d}s \tag{6.3.2}$$

is a martingale. That is, it is a martingale with respect to the filtration $\mathcal{F}^{\mathbf{X}}(T) \vee \mathcal{F}^{\mathbf{Z}}(t)$. In particular, it is straightforward to verify that (6.3.2) is also a martingale unconditionally on $\mathbf{X}$, that is, with respect to the filtration $\mathcal{F}(t)$.

We are interested in finding transition probabilities for the doubly stochastic Markov chain $Z(t)$,

$$p_{k\ell}(t; t', x) = P(Z(t) = \ell \mid Z(t') = k, X(t') = x). \tag{6.3.3}$$

These are dependent on $X(t')$; we think of time t' as *now*, and thus $X(t')$ is known. We find this transition probability by first finding the transition probability for the combined process $(Z(t), X(t))$. Thus, let $P(t, k, A; t', k', x')$ denote the transition probability of $(Z(t), X(t))$ conditional on $(Z(t'), X(t')) = (k', x')$. Considering the $(d+1)$-dimensional process $(Z(t), X(t))$ as a special case of (6.2.1), we obtain the following result.

Theorem 6.3.1. (Kolmogorov's forward PIDE for the doubly stochastic setup) *Assume that*

$$\int_0^t \beta_i(s, X(s)) f(X(s))\, \mathrm{d}W(s)$$

is a martingale for all $f \in \mathcal{C}^1$ with compact support, and that $\frac{\partial}{\partial t} P(t, k, \mathrm{d}x; t', k', x')$, $\frac{\partial}{\partial x_i} P(t, k, \mathrm{d}x; t', k', x')$ and $\frac{\partial^2}{\partial x_i \partial x_j} P(t, k, \mathrm{d}x; t', k', x')$ exist. Then the transition probability $P(t, k, A; t', k', x')$ for $k \in \mathcal{J}$ and a compact Borel-set $A \subset \mathbb{R}^d$ satisfy the forward PIDE

$$\begin{aligned}\frac{\partial}{\partial t} P(t, k, A; t', k', x') = &- \sum_i \int_A \frac{\partial}{\partial x_i} \left(\beta_i(t, x) P(t, k, \mathrm{d}x; t', k', x')\right) \\ &+ \frac{1}{2} \sum_{i,j} \int_A \frac{\partial^2}{\partial x_i \partial x_j} \left(\rho_{ij}(t, x) P(t, k, \mathrm{d}x; t', k', x')\right) \\ &+ \sum_{\ell:\ell \neq k} \int_A \mu_{\ell k}(t, x) P(t, \ell, \mathrm{d}x; t', k', x') \\ &- \int_A \sum_{\ell:\ell \neq k} \mu_{k\ell}(t, x) P(t, k, \mathrm{d}x; t', k', x'),\end{aligned} \tag{6.3.4}$$

subject to the boundary condition $P(t', k, A; t', k', x) = 1_{(k=k')} 1_A(x)$.

Proof. We have the dynamics

$$\mathrm{d}\begin{bmatrix} Z(t) \\ X(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \beta(t, X(t)) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 & 0 \\ 0 & \sigma(t, X(t)) \end{bmatrix} \mathrm{d}\widetilde{W}(t) + \begin{bmatrix} \mathrm{d}J(t) \\ 0 \end{bmatrix},$$

for $\widetilde{W}(t) = (\hat{W}(t), W(t))$, where $\hat{W}(t)$ is an adapted standard Brownian motion. In particular, the jump measure $\tilde{\mu}(\mathrm{d}(\ell, y); t, (k, x))$ of this process is,

$$\tilde{\mu}(\mathrm{d}(\ell, y); t, (k, x)) = \mu_{k\ell}(t, x) \cdot \mathrm{d}\left(\tau_{\mathcal{J}\setminus\{k\}}(\ell) \otimes \tau_{\{x\}}(y)\right).$$

Now the result follows from Theorem 6.2.1 applied to the process $(Z(t), X(t))$. □

Corollary 6.3.2. *Assume Theorem 6.3.1 holds and that a density with respect to the Lebesgue measure exists,*

$$P(t, k, \mathrm{d}x; t', k', x') = p(t, k, x; t', k', x')\, \mathrm{d}x. \tag{6.3.5}$$

Then the density satisfies, for $t > t'$, the PDE

$$\begin{aligned}\frac{\partial}{\partial t}p(t,k,x;t',k',x') = &-\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(\beta_k(t,x)p(t,k,x;t',k',x')\right)\\ &+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}\left(\rho_{ij}(t,x)p(t,k,x;t',k',x')\right)\\ &+\sum_{\ell\in\mathcal{J}}\mu_{\ell k}(t,x)p(t,\ell,x;t',k',x')\\ &-\sum_{\ell\in\mathcal{J}}\mu_{k\ell}(t,x)p(t,k,x;t',k',x'),\end{aligned} \tag{6.3.6}$$

subject to the boundary condition $p(t',k,x;t',k',x') = 1_{(k=k')}1_{(x=x')}$.

Assuming we have solved the PIDE (6.3.4) or the PDE (6.3.6), we have the transition probability for the process $(Z(t), X(t))$. If we are interested in the transition probabilities of $Z(t)$ only, (6.3.3), we can integrate over the underlying state $X(t)$,

$$p_{k'k}(t;t',x') = P(t,k,\mathbb{R}^d;t',k',x') = \int p(t,k,x;t',k',x')\,\mathrm{d}x. \tag{6.3.7}$$

Corollary 6.3.2 is a generalisation of Kolmogorov's forward differential equation. If the transition rates are deterministic, which without loss of generality can be characterised as $\mathbf{X}$ being constant, we have that $\beta(t,x) = 0$ and $\sigma(t,x) = 0$. In that case, (6.3.6) simplifies to Kolmogorov's forward differential equation,

$$\frac{\partial}{\partial t}p(t,k;t',k') = \sum_{\ell\in\mathcal{J}\setminus\{k\}}\left(\mu_{\ell k}(t)p(t,\ell;t',k') - \mu_{k\ell}(t)p(t,k;t',k')\right).$$

Here we removed x, y from the notation in $p(s,j,y;t,i,x)$ and $\mu_{k\ell}(t,x)$.

6.3.1 Backward partial differential equation

It is well known that a backward PDE exists for the transition probability; since

$$t \mapsto \mathrm{E}\left[1_{(Z(s)=k)}\middle|\,\mathcal{F}(t)\right] = \sum_{\ell}1_{(Z(t)=\ell)}p_{\ell k}(s;t,X(t)),$$

is a martingale, we can apply Itô's lemma and set the drift equal to zero. This yields a PDE in t, x, ℓ for $p_{\ell k}(s;t,x)$, together with the boundary conditions $p_{\ell k}(s;s,x) = 1_{(\ell=k)}$.

Proposition 6.3.3. (Kolmogorov's backward PDE) *For $\ell \in \mathcal{J}$, the transition probabilities $p_{\ell k}(s;t,x)$ satisfy the backward PDE*

$$\begin{aligned}\frac{\partial}{\partial t} p_{\ell k}(s;t,x) = &-\sum_{i=1}^{d} \beta_i(t,x) \frac{\partial}{\partial x_i} p_{\ell k}(s;t,x) - \frac{1}{2} \sum_{i,j=1}^{d} \rho_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} p_{\ell k}(s;t,x) \\ &- \sum_{m \in \mathcal{J} \setminus \{\ell\}} \mu_{\ell m}(t,x) \left(p_{mk}(s;t,x) - p_{\ell k}(s;t,x) \right),\end{aligned} \tag{6.3.8}$$

subject to the boundary condition $p_{\ell k}(s;s,x) = 1_{(\ell = k)}$.

If the transition intensities are deterministic, which can be modelled by setting $\beta(t,x) = 0$ and $\rho(t,x) = 0$, the first two terms of (6.3.8) disappear, and we are left with Kolmogorov's well known backward differential equation.

With the results presented, we have two ways of calculating $p_{\ell k}(s;t,x)$. Either, we solve the backward PDE in Proposition 6.3.3, or we solve the forward PDE from Corollary 6.3.2 and integrate over the intensities as in (6.3.7). In the next section about life insurance cash flows, we see that we need to find $p_{\ell k}(s;t,x)$ for fixed ℓ, t, x and varying k, s. In that case, the forward PDE seems preferable, as we only need to solve it once. If we use the backward PDE, we need to solve it for every fixed pair of k, s.

6.4 Life insurance cash flows

Let a finite state space $\mathcal{J}$ be given, where the states could be *alive*, *disabled*, *dead*, or similar. Then, let the doubly stochastic Markov chain $\mathbf{Z}$ from Section 6.3 describe the state of an insured in this state space. To each state k we associate a continuously paid payment rate $b_k(t)$, and to each transition we associate a payment, $b_{k\ell}(t)$. We assume both are continuous functions. The payments of the contract, accumulated until time t, is thus described by the payment process $B(t)$, satisfying

$$\mathrm{d}B(t) = \sum_{k \in \mathcal{J}} 1_{(Z(t)=k)} b_k(t) \mathrm{d}t + \sum_{\substack{k,\ell \in \mathcal{J} \\ k \neq \ell}} b_{k\ell}(t) \mathrm{d}N_{k\ell}(t).$$

We assume that all payments occur before the finite time horizon T, i.e. that $b_k(s) = b_{k\ell}(s) = 0$ for all k, ℓ and $s \geq T$. This setup constitutes our modelling of the life insurance contract. Here we let b_k and $b_{k\ell}$ be deterministic functions, but we later discuss the straightforward extension to dependence on the underlying stochastic process $\mathbf{X}$.

We define the cash flow associated with this contract, as the expected payments.

Definition 6.4.1. *The* accumulated cash flow *at time t' conditional on $Z(t') = k'$ and $X(t') = x'$ is the function,*

$$A_{k'}(t;t',x') = \mathrm{E}\left[B(t) - B(t') \middle| Z(t') = k', X(t') = x' \right].$$

Furthermore, if $A_{k'}(t;t',x')$ has a density with respect to the Lebesgue measure,

$$A_{k'}(\mathrm{d}t;t',x') = a_{k'}(t;t',x')\,\mathrm{d}t,$$

we refer to $a_{k'}(t;t',x')$ as the cash flow.

Remark 6.4.2. In the traditional life insurance setup, single payments can also happen at deterministic time points. In that case, one allows the cash flow to have a density with respect to a mixture between the Lebesgue measure and a counting measure. An extension to this more general case is straightforward, but complicates the notation and is thus omitted from the present article. ◇

One can now show the following result; a proof in the semi-Markov setup is given in [11], and the proof in the present setup is essentially identical.

Proposition 6.4.3. *The cash flow exists and is given by,*

$$a_{k'}(t;t',x') = \sum_{k\in\mathcal{J}} \int_{\mathcal{X}} P(t,k,\mathrm{d}x;t',k',x')\left(b_k(t) + \sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq k}} \mu_{k\ell}(t,x)b_{k\ell}(t)\right). \tag{6.4.1}$$

Given some continuously compounded interest rate $r(t)$, which we assume to be deterministic, we can calculate the expected present value at time t, conditional on $Z(t)=k$. This is defined as,

$$V_k(t,x) = \mathrm{E}\left[\int_t^T e^{-\int_t^s r(w)\mathrm{d}w}\mathrm{d}B(s)\,\middle|\, Z(t)=k, X(t)=x\right].$$

One can show that the expected present value is simply the sum of the discounted cash flow, which we state in the following proposition.

Proposition 6.4.4. *If r is deterministic, the expected present value at time t, conditional on $Z(t)=k$ and $X(t)=x$ is given as*

$$V_k(t,x) = \int_t^T e^{-\int_t^s r(w)\mathrm{d}w} a_k(s;t,x)\mathrm{d}s.$$

The main objective of the actuary is to calculate the expected present value, and this can be done e.g. by calculating the cash flow first, and then discounting it, as described above. Another possibility is to calculate it directly, since one can show that $V_k(t,x)$ is the solution of a backward PDE, similar in structure to the backward PDE for the transition probability, (6.3.8), see e.g. [15]. However, one is often interested in the cash flow, since it is convenient for an analysis of the interest rate risk in practice. For example, if one wants to calculate the expected present value with different interest rates, it is advantageous to first calculate the cash flows, which are independent of the interest rate. In particular, if a large portfolio of insurance contracts is considered, the cash flows can be accumulated to a single cash flow for the portfolio, which are easily discounted. For calculation of the cash flow, one needs the transition probabilities, which can be calculated with Corollary 6.3.2.

Remark 6.4.5. Mathematically it is straightforward to extend the current setup such that the payment functions b_k and $b_{k\ell}$ may depend on the value of the underlying process $\mathbf{X}$, and we can write $b_k(t,x)$ and $b_{k\ell}(t,x)$. In that case, the cash flow (6.4.1) would be

$$a_{k'}(t;t',x') = \sum_{k\in\mathcal{J}} \int_{\mathcal{X}} P(t,k,\mathrm{d}x;t',k',x') \left(b_k(t,x) + \sum_{\substack{\ell\in\mathcal{J}\\ \ell\neq k}} \mu_{k\ell}(t,x) b_{k\ell}(t,x) \right).$$

$\diamond$

Remark 6.4.6. In this section, we assumed a deterministic interest rate $r(t)$, however a stochastic interest rate is easily handled if it is independent of the underlying stochastic process $\mathbf{X}$ and $\mathbf{Z}$. For more on stochastic interest rates in life insurance, see e.g. [38] and references therein. $\diamond$

6.5 Semi-Markov models in life insurance

If we in the classic life insurance setup, where the transition intensities are deterministic, extend the setup and allow the transition intensities and the payment functions to depend on the time spent in the current state, we have introduced duration dependence in the setup. In this case, $\mathbf{Z}$ is a semi-Markov process, and this class of models are popular in e.g. life insurance. An example of such a model is a disability model with recovery, where the recovery rate and the mortality as disabled might decrease as a function of the time spent as disabled. For a treatment of this model, see [11] and references therein.

We construct the semi-Markov model and show that the semi-Markov process is a special case of the process in Section 6.2. This can be used to present Theorem 6.2.1 for the semi-Markov process, and in this case it becomes an integro-differential equation. This result also presented in [11], but the proof of the result is different.

Let $\mathbf{Z}$ be a stochastic process on a finite state space $\mathcal{J}$. Define $\mathbf{U}$ as the duration in the current state,

$$U(t) = \sup\left\{ s\in[0,t] \mid Z(w) = Z(t), w\in[t-s,t] \right\}.$$

We assume that $(\mathbf{Z}, \mathbf{U})$ is a Markov process. The process $(\mathbf{Z}, \mathbf{U})$ may jump, and the pure jump part can be written as

$$J(t) = \sum_{s \le t} \begin{bmatrix} \Delta Z(s) \\ \Delta U(s) \end{bmatrix}.$$

We denote the jump measure of $J(t)$, as defined in (6.2.2), by $\tilde{\mu}(\mathrm{d}(\ell, v); t, (k, u))$, and it has a density with respect to a counting measure,

$$\tilde{\mu}(\mathrm{d}(\ell, v); t, (k, u)) = \mu_{k\ell}(t, u) \cdot \mathrm{d}\left(\tau_{\mathcal{J}\setminus\{k\}}(\ell) \otimes \tau_{\{0\}}(v)\right).$$

Here, $J(t)$ is the jump sizes, and $\tilde{\mu}((\ell, v); t, (k, u))$ is intepreted as the instantaneous probability that if we are in state k with duration u, we make a jump to state ℓ and have duration v. By the definition of $\mathbf{U}$ we always jump to duration 0, so if $v \neq 0$ the density is zero. We interpret $\mu_{k\ell}(t, u)$ as the transition rate of $\mathbf{Z}$, and it can be shown to satisfy the relation

$$\mu_{k\ell}(t, u) = \lim_{\delta \searrow 0} \frac{1}{\delta} P\left(Z(t+\delta) = \ell | Z(t) = k, U(t) = u\right),$$

for $k \neq \ell$. We interpret this as the instantaneous probability of a jump of $\mathbf{Z}$ from state k to state ℓ, if we are at time t with duration u.

With $N_{k\ell}(t)$ given by (6.3.1), we can characterise $U(t)$ as

$$\mathrm{d}U(t) = 1\mathrm{d}t - U(t-) \sum_{\substack{k,\ell \in \mathcal{J} \\ k \neq \ell}} \mathrm{d}N_{k\ell}(t).$$

The first term on the right hand side is the constant increase of $U(t)$ with slope 1, and the second term states that whenever $\mathbf{Z}$ jumps, $\mathbf{U}$ jumps to zero. We remark that the state space of $U(s)$ is $[0, s]$, and, conditional on $U(t) = u$, the state space for $U(s), s \ge t$ is $[0, u + s - t]$.

We define the transition probability

$$P(t, k, u; t', k', u') = \mathrm{P}(Z(t) = k, U(t) \le u \mid Z(t') = k', U(t') = u').$$

We specialise Theorem 6.2.1 to this semi-Markov setup. Because of the structure of the process, in particular the fact that the duration process increases identically as time almost everywhere and possess no diffusion term, we can write it as an integro-differential equation (IDE). We note, that since $\frac{\partial}{\partial u} P(t, k, u; t', k', u')$ does not exist for all u, the result is not a direct special case of Theorem 6.2.1.

Theorem 6.5.1. *The transition probability $P(t,k,u;t',k',u')$, for $t \geq t'$, $k \in \mathcal{J}$ and $u \in [0, u'+t-t']$ satisfies the forward IDE*

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\right) P(t,k,u;t',k',u') = &\sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq k}} \int_0^{u'+t-t'} \mu_{\ell k}(t,v) P(t,\ell,\mathrm{d}v;t',k',u') \\ &- \int_0^u \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq k}} \mu_{k\ell}(t,v) P(t,k,\mathrm{d}v;t',k',u'),\end{aligned} \tag{6.5.1}$$

subject to the boundary conditions $P(t',k,u;t',k',u') = 1_{(k=k')} 1_{(u' \leq u)}$ and, for $t > t'$, $P(t,k,0;t',k',u') = 0$.

The differential equation can be interpreted both as a partial integro-differential equation, and as an ordinary integro-differential equation. The latter interpretation makes it easy to solve in practice, and for details on this see [11], where an algorithm is explained. The theorem is proven in [11], however, we give an outline of how to prove it with the tools presented in Section 6.2. Theorem 6.2.1 yields the IDE on the interior of the domain, so what is left is to make sure the IDE also holds on the boundary.

For the proof and subsequent results, we introduce the following notation,

$$\begin{aligned} p_{\overline{kk}}(t;t',u') &= \mathrm{P}(Z(t) = k, U(t) = u'+t-t' \mid Z(t') = k, U(t') = u') \\ &= \exp\left\{ -\int_0^{t-t'} \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq k}} \mu_{k\ell}(t'+w, u'+w) \mathrm{d}w \right\}. \end{aligned} \tag{6.5.2}$$

This quantity is the probability of staying in state k from time t' with duration u' until time t. It is well known that it has the above expression, see e.g. [28].

Proof. (Outline) The first boundary condition follows directly from the definition of the transition probability. Since, for any t, the probability that a jump occurs exactly at time t can be shown to equal 0, the probability that the duration is equal to zero is zero, for all t. This yields the second boundary condition.

We first claim that one can show, that on the open set

$$\{(t,u) \mid t \in (t',T), u \in (0, u'+t-t')\},$$

the partial derivatives $\frac{\partial}{\partial t} P(t,k,u;t',k',u')$ and $\frac{\partial}{\partial u} P(t,k,u;t',k',u')$ exist. This holds, since probability mass on that set originates from jumps. Thus, from Theorem 6.2.1, the result holds on this set.

Since $P(t,k,u;t',k',u')$ is right-continuous in u by definition, the result holds for $u = 0$.

For $u = u' + t - t'$, note that

$$\begin{aligned} P(t,k,u'+t-t';t',k',u') &= P(t,k,u'+t-t';t',k',u') - P(t,k,u'+t-t'-\delta;t',k',u') \\ &\quad + P(t,k,u'+t-t'-\delta;t',k',u'), \end{aligned}$$

and let $\delta \searrow 0$ to obtain

$$\begin{aligned} P(t,k,u'+t-t';t',k',u') &= \mathrm{P}(Z(t)=k, U(t)=u'+t-t' \mid Z(t')=k', U(t')=u') \\ &\quad + P(t,k,(u'+t-t')-;t',k',u'). \end{aligned} \tag{6.5.3}$$

The second term on the right hand side of (6.5.3) satisfies (6.5.1). The first term on the right hand side is, for $k = k'$, the probability of staying in state k' from time t' to t, and from (6.5.2) we know that this probability is of the form

$$\mathrm{P}(Z(t)=k, U(t)=u'+t-t' \mid Z(t')=k', U(t')=u') = 1_{(k=k')} p_{\overline{kk}}(t;t',u').$$

Applying $\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$ to both sides, we find that this functions satisfies a linear partial differential equation of the form (6.5.1).

Since both terms on the right hand side of (6.5.3) satisfy (6.5.1), which is linear, we use the fact that if two functions satisfy the same linear IDE, each with their own boundary condition, the sum of of these functions satisfy the same linear IDE, with the boundary condition being the sum of the two boundary conditions. From this we conclude that the whole of (6.5.3) satisfy (6.5.1). Thus, the result holds for all $u \in [0, u'+t-t']$ and $t \geq t'$. □

We can partly solve the forward IDE, and obtain the following integral equation, which is essentially a differentiated version of equation (4.7) in [28].

Proposition 6.5.2. *The transition density $p(t,k,u;t',k',u')$, for $k \in \mathcal{J}$, $u \in [0, u'+t-t']$ exists with respect to a mixture of the Lebesgue measure and a point measure. For fixed t,*

$$P(t,A,B;t',k',u') = \sum_{k\in A} \int_B p(t,k,u;t',k',u') \left(\mathrm{d}u + \mathrm{d}\tau_{\{u'+t-t'\}}(u)\right).$$

The density is given by

- *for $t - t' > u$, the integral equation*

$$\begin{aligned} p(t,k,u;t',k',u') = \sum_{\substack{\ell\in\mathcal{J} \\ \ell\neq k}} \int_0^{u'+t-u-t'} & p(t-u,\ell,v;t',k',u')\mu_{\ell k}(t-u,v) \\ & \times p_{\overline{kk}}(t;t-u,0) \left(\mathrm{d}v + \mathrm{d}\tau_{\{u'+t-u-t'\}}(v)\right), \end{aligned}$$

- *else,*

$$p(t, k, u; t', k', u') = 1_{(k=k')} 1_{(t-t'=u-u')} p_{\overline{kk}}(t, t', u').$$

Proof. (Outline) For $t - t' > u$, a jump must have occured between time t' and time t, and we argued in the proof of Theorem 6.5.1 that a density with respect to the Lebesgue measure exists. That the proposed density satisfies the IDE (6.5.1) can be seen by applying $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\right)$ to

$$P(t, k, u; t', k', u') = \int_0^u p(t, k, v; t', k', u') \left(\mathrm{d}v + \mathrm{d}\tau_{\{u'+t-t'\}}(v)\right),$$

and where the proposed density is inserted.

For the second part where $t - t' \leq u$, a jump cannot have occurred after time t', and we must have stayed in the initial state. Then the result is known from (6.5.2). □

We interpret the terms of the density. As was argued in the proof, the last part is the probability mass originating from the initial state, and is simply the probability of having made no jumps. The probability mass in the first part is all jumps into state k happening at time $t - u$. Inside the integral, we see the probability of being in a state ℓ at time $t - u$ with a duration v, multiplied with the probability of a jump to state k, which is again multiplied with the probability of then staying in state k from time $t - u$ to time t. Then we simply sum over all states ℓ and all durations v.

6.6 The survival model and the forward mortality rate

Common for the forward differential equations stated in this paper, is that they are for a Markov process; however, if we are mainly interested in $\mathbf{Z}$ alone, is it then possible to find a forward differential equation for the marginal transition probabilities for $\mathbf{Z}$? In this section, we adopt the doubly stochastic Markov chain setup from Section 6.3, and we consider a simple model. In this model, we already know that a forward differential equation for the transition probabilities of $\mathbf{Z}$ exists and it depends on the so-called forward mortality rate. We investigate how our forward differential equation relates to this.

Consider the 2-state survival model $\mathcal{J} = \{0, 1\}$ with the only non-zero transition rate being from state 0 to 1, which is denoted $\mu(t, X(t))$. State 0 corresponds to being alive, and state 1 corresponds to being dead, and we refer to $\mu(t, X(t))$ as the mortality rate. We assume that $\mathbf{X}$ is a 1-dimensional diffusion process, and that $\mathbf{W}$ is 1-dimensional. Further, we assume that the transition density for $(Z(t), X(t))$ exists on the whole state space, and is given by (6.3.5). Recall also the density for $\mathbf{Z}$ given by (6.3.3). We have the relation

$$p_{k'k}(t;t',x') = \int p(t,k,x;t',k',x')\,\mathrm{d}x.$$

The forward mortality rate $f_{t'}(t)$ is defined as the function satisfying the relation

$$p_{00}(t;t',X(t')) = \mathrm{E}\left[e^{-\int_{t'}^{t}\mu(s,X(s))\,\mathrm{d}s}\,\middle|\,X(t'),Z(t')=0\right] = e^{-\int_{t'}^{t} f_{t'}(s)\,\mathrm{d}s}. \tag{6.6.1}$$

One can furthermore show under Assumption 6.6.3 below, that

$$\begin{aligned} p_{01}(t;t',X(t')) &= \mathrm{E}\left[\int_{t'}^{t} e^{-\int_{t'}^{s}\mu(u,X(u))\,\mathrm{d}u}\mu(s,X(s))\,\mathrm{d}s\,\middle|\,X(t'),Z(t')=0\right] \\ &= \int_{t'}^{t} e^{-\int_{t'}^{s} f_{t'}(u)\,\mathrm{d}u} f_{t'}(s)\,\mathrm{d}s. \end{aligned} \tag{6.6.2}$$

It is simple to show by (6.6.1) and Jensen's inequality, that forward rate is not simply the expectation of $\mu(t,X(t))$. In fact, if $\mu(t,x)$ is increasing in x, and the distribution of $\mu(t,X(t))$ is not degenerate, we have

$$f_{t'}(t) < \mathrm{E}\left[\mu(t,X(t))\,|\,X(t'),Z(t')=0\right], \tag{6.6.3}$$

because of the convexity of $x \mapsto e^{-x}$. If we did not know this, we could be led to believe that the forward mortality rate could equal the expected mortality rate, which could make sense intuitively. This is not the case, but as the following theorem shows, we can actually write the forward mortality rate as the expected mortality rate, conditional on survival.

In the particularly simple case of this section, we can find differential equations for p_{0k} from (6.6.1) and (6.6.2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}p_{00}(t;t',X(t')) &= -f_{t'}(t)p_{00}(t;t',X(t')), \\ \frac{\mathrm{d}}{\mathrm{d}t}p_{01}(t;t',X(t')) &= f_{t'}(t)p_{00}(t;t',X(t')), \end{aligned} \tag{6.6.4}$$

with boundary condition $p_{0k}(t';t',x) = 1_{(k=0)}$. The PDE from Corollary 6.3.2 is also a forward differential equation, albeit that it includes the distribution of $\mathbf{X}$. If one integrates over x in (6.3.6), as we do in the first proof of Theorem 6.6.1 below, the following differential equations can be found,

$$\begin{aligned} \frac{\partial}{\partial t}p_{00}(t;t',x') &= -\,\mathrm{E}\left[\mu(t,X(t))\,|\,Z(t)=0,(Z,X)(t')=(0,x')\right]p_{00}(t;t',x'), \\ \frac{\partial}{\partial t}p_{01}(t;t',x') &= \mathrm{E}\left[\mu(t,X(t))\,|\,Z(t)=0,(Z,X)(t')=(0,x')\right]p_{00}(t;t',x'). \end{aligned}$$

Comparing the two systems of differential equations, we obtain the following theorem.

Theorem 6.6.1. *Under Assumption 6.6.2 or Assumption 6.6.3, the forward rate has representation*

$$f_{t'}(t) = \mathrm{E}\left[\mu(t, X(t))\right| X(t'), Z(t) = 0\right].$$

We note that we don't have to condition on $Z(t') = 0$ since this is necessarily true if $Z(t) = 0$. We present two different proofs, each with their own set of assumptions.

Assumption 6.6.2. *For all t, assume that $\mathrm{E}\left[|X(t)|\right| (Z,X)(t') = (0,x')] < \infty$, and that $\beta(t,x) \in \mathcal{O}(x)$, $\rho(t,x) \in \mathcal{O}(x^2)$, $\frac{\partial}{\partial x}\rho(t,x) \in \mathcal{O}(x)$, for $x \to \pm\infty$. Furthermore, assume there exists $K > 0$, such that for $x \leq -K$ and $x \geq K$, $p(t,k,x;t',k',x')$ is convex in x and*

$$x \mapsto \frac{\partial}{\partial x} p(t,k,x;t',k',x') \in \mathcal{O}\left(x^{-1}\right),$$

for $x \to \pm\infty$.

A brief word on Assumption 6.6.2: The three asymptotic assumptions on $\beta(t,x)$, $\rho(t,x)$ and $\frac{\partial}{\partial x}\rho(t,x)$ can with a few calculations be seen to follow from the usual Lipschitz condition on $\beta(t,x)$ and $\sigma(t,x) = \sqrt{\rho(t,x)}$ that ensures the existence of the stochastic process $\mathbf{X}$. We further assume that the density is convex for large $|x|$, which is satisfied for all the usual distributions. The last assumption, that $\frac{\partial}{\partial x}p(t,k,x;t',0,x') \in \mathcal{O}\left(x^{-1}\right)$, does not restrict us in practice; see for example from the proof below that

$$\int_K^\infty x^2 \frac{\partial}{\partial x} p(t,k,x;t',0,x')\mathrm{d}x < \infty.$$

Thus, if e.g. $\frac{\partial}{\partial x}p(t,k,x;t',0,x') = x^{-1}$, the integral would be far from finite, so intuitively, $\frac{\partial}{\partial x}p(t,k,x;t',0,x')$ is small compared to x^{-1}; the assumption simply safeguards us from erratic behaviour.

Assumption 6.6.3. *For each t, assume that there exists an open set J_t containing t, such that*

$$\hat{X}_t = \sup_{s \in J_t}\left(e^{-\int_{t'}^s \mu(u,X(u))\mathrm{d}u}\mu(s,X(s))\right)$$

has finite expectation.

For the first proof, we use Assumption 6.6.2, and then we compare the two forward differential equations, which is the one from Corollary 6.3.2 and (6.6.4).

Proof. (First proof of Theorem 6.6.1) We insert into the PDE (6.3.6) from Corollary 6.3.2, and integrate over x,

$$\begin{aligned}
\int \frac{\partial}{\partial t} p(t,k,x;t',0,x')\,\mathrm{d}x = &-\int \frac{\partial}{\partial x}\left(\beta(t,x)p(t,k,x;t',0,x')\right)\mathrm{d}x \\
&+\frac{1}{2}\int \frac{\partial^2}{\partial x^2}\left(\rho(t,x)p(t,k,x;t',0,x')\right)\mathrm{d}x \\
&+\int \left(1_{(k=1)}-1_{(k=0)}\right)\mu(t,x)p(t,0,x;t',0,x')\,\mathrm{d}x \\
= &-\left[\beta(t,x)p(t,k,x;t',0,x')\right]_{-\infty}^{\infty} && (6.6.5)\\
&+\frac{1}{2}\left[\rho(t,x)\frac{\partial}{\partial x}p(t,k,x;t',0,x')\right]_{-\infty}^{\infty} && (6.6.6)\\
&+\frac{1}{2}\left[p(t,k,x;t',0,x')\frac{\partial}{\partial x}\rho(t,x)\right]_{-\infty}^{\infty} && (6.6.7)\\
&+\left(1_{(k=1)}-1_{(k=0)}\right)\int \mu(t,x)p(t,0,x;t',0,x')\,\mathrm{d}x. && (6.6.8)
\end{aligned}$$

Assume first that (6.6.5), (6.6.6) and (6.6.7) equal zero. Then (6.6.8) is the only non-zero term. Using that $(k,x)\mapsto p(t,k,x;t',0,x')$ is the density of $(Z,X)(t)$ conditional on $(Z,X)(t')=(0,x')$, and that $k\mapsto p_{0k}(t;t',x')$ is the corresponding marginal density of $Z(t)$, we have that

$$x\mapsto \frac{p(t,0,x;t',0,x')}{p_{00}(t;t',x')}$$

is the conditional density of $X(t)$ given both $Z(t)=0$ and $(Z,X)(t')=(0,x')$. Thus, we can rewrite (6.6.8) to

$$\begin{aligned}
\frac{\partial}{\partial t}p_{00}(t;t',x') &= -\,\mathrm{E}\left[\mu(t,X(t))\middle|\, Z(t)=0,(Z,X)(t')=(0,x')\right]p_{00}(t;t',x'),\\
\frac{\partial}{\partial t}p_{01}(t;t',x') &= \mathrm{E}\left[\mu(t,X(t))\middle|\, Z(t)=0,(Z,X)(t')=(0,x')\right]p_{00}(t;t',x').
\end{aligned}$$

Comparing with (6.6.4), the result is obtained.

For the rest of the proof, which is to show that (6.6.5), (6.6.6) and (6.6.7) equal zero, see the appendix. □

We present an alternative proof, which avoids the differential equations. Here we use Assumption 6.6.3.

Proof. (Second proof of Theorem 6.6.1) We consider the expectation of the survival indicator multiplied with the transition rate. First,

$$\begin{aligned}
&\mathrm{E}\left[1_{(Z(t)=0)}\mu(t,X(t))\middle|\, X(t'),Z(t')=0\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[1_{(Z(t)=0)}\middle|\,\mathcal{F}^{\mathbf{X}}(T),Z(t')=0\right]\mu(t,X(t))\middle|\, X(t'),Z(t')=0\right] \\
&= \mathrm{E}\left[e^{-\int_{t'}^{t}\mu(s,X(s))\mathrm{d}s}\mu(t,X(t))\middle|\, X(t'),Z(t')=0\right] \\
&= -\frac{\partial}{\partial t}\mathrm{E}\left[e^{-\int_{t'}^{t}\mu(s,X(s))\mathrm{d}s}\middle|\, X(t'),Z(t')=0\right] \\
&= -\frac{\partial}{\partial t}e^{-\int_{t'}^{t}f_{t'}(s)\mathrm{d}s} \\
&= e^{-\int_{t'}^{t}f_{t'}(s)\mathrm{d}s}f_{t'}(t).
\end{aligned}$$

Here we could interchange differentiation and expectation by Assumption 6.6.3. Second, we begin with the same quantity, but perform another calculation,

$$\begin{aligned}
&\mathrm{E}\left[1_{(Z(t)=0)}\mu(t,X(t))\middle|\, X(t'),Z(t')=0\right] \\
&= \sum_{i=0}^{1}\mathrm{P}\left(Z(t)=i\mid X(t'),Z(t')=0\right)\mathrm{E}\left[1_{(Z(t)=0)}\mu(t,X(t))\middle|\, X(t'),Z(t')=0,Z(t)=i\right] \\
&= p_{00}(t;t',X(t'))\,\mathrm{E}\left[\mu(t,X(t))\mid X(t'),Z(t)=0\right],
\end{aligned}$$

which, with (6.6.1), yields the result. □

We can interpret the theorem in the following way: The forward rate is the expectation of the transition rate, given the fact that we have not yet made the transition. The inequality (6.6.3) also makes sense intuitively: If we condition on being alive, then we might infer that the mortality rate $\mu(t,X(t))$ is probably small rather than large, and hence the expectation conditional on being alive is smaller than if we do not know.

6.7 Dependent forward rates as expectations

This section does not appear in [9], but is added for this PhD thesis. It contains a last-minute addition, and may not be as polished as the rest of the thesis. In this section, we generalise Theorem 6.6.1, and find a relation with the dependent forward rates from [8] (Section 5.4 in this thesis).

We adopt the setup from Section 6.3, so let $\mathbf{X}$ be a d-dimensional diffusion process satisfying the stochastic differential equation

$$\mathrm{d}X(t) = \beta(t,X(t))\mathrm{d}t + \sigma(t,X(t))\mathrm{d}W(t).$$

We consider the state space $\mathcal{J} = \{0, 1, \dots, J\}$, where the only non-zero transition rates are $\mu_{0i}(t, X(t))$ for $i = 1, 2, \dots, J$. This can for example be interpreted as a survival model with multiple causes of death. The transition rates are dependent on the underlying diffusion process $\mathbf{X}$, and thus $\mathbf{Z}$ is a doubly stochastic Markov chain. In particular, the transition rates may be dependent.

In this setup, the transition probabilities can be written as

$$\begin{aligned} p_{00}(t; t', X(t')) &= \mathrm{E}\left[e^{-\int_{t'}^{t} \sum_{\ell=1}^{J} \mu_{0\ell}(s, X(s))\, \mathrm{d}s} \middle| X(t'), Z(t') = 0 \right] \\ &= e^{-\int_{t'}^{t} \sum_{\ell=1}^{J} f_{t'}^{\ell}(s)\, \mathrm{d}s}, \end{aligned} \tag{6.7.1}$$

$$\begin{aligned} p_{0k}(t; t', X(t')) &= \mathrm{E}\left[\int_{t'}^{t} e^{-\int_{t'}^{s} \sum_{\ell=1}^{J} \mu_{0\ell}(u, X(u))\, \mathrm{d}u} \mu_{0k}(s, X(s))\, \mathrm{d}s \middle| X(t'), Z(t') = 0 \right] \\ &= \int_{t'}^{t} e^{-\int_{t'}^{s} \sum_{\ell=1}^{J} f_{t'}^{\ell}(u)\, \mathrm{d}u} f_{t'}^{k}(s)\, \mathrm{d}s. \end{aligned} \tag{6.7.2}$$

for $k = 1, \dots, J$. Here, $f_{t'}^{k}(t)$ are the dependent forward rates. The definition of dependent forward rates, Definition 5.4.1, is only for affine transition rates, however, from the discussion following Remark 5.4.2, we see that the dependent forward rates could equally be defined as the functions that satisfy (5.4.3), which is identical to (6.7.1) and (6.7.2) in our setup. In that way, the dependent forward rates are defined for both affine and non-affine transition rates. It is noted that we do not allow for a stochastic interest rate that is dependent on the transition rates.

Following the reasoning in Section 6.6 leading up to Theorem 6.6.1, which is carried out in the first proof of that theorem, we obtain the following.

Theorem 6.7.1. *Under certain regularity conditions, the dependent forward rates have representation*

$$f_{t'}^{k}(t) = \mathrm{E}\left[\mu_{0k}(t, X(t)) \middle| X(t'), Z(t) = 0 \right],$$

for $k = 1, \dots, J$.

We present an outline of the proof and in particular we do not precisely state the regularity conditions. The proof is basically identical to the first proof of Theorem 6.6.1.

Proof. (Outline) By differentiation of (6.7.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{0k}(t; t', X(t')) = f_{t'}^{k}(t) p_{00}(t; t', X(t')), \tag{6.7.3}$$

for $k = 1, \dots, J$. From Corollary 6.3.2 we have (6.3.6), and by integration over x, this yields, for $k = 1, \dots, J$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p_{0k}(t; t', x') &= \frac{\partial}{\partial t} \int p(t, k, x; t', 0, x') \,\mathrm{d}x \\
&= -\sum_{i=1}^{d} \int \frac{\partial}{\partial x_i} \big(\beta_k(t, x) p(t, k, x; t', 0, x')\big) \,\mathrm{d}x && (6.7.4) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} \int \frac{\partial^2}{\partial x_i \partial x_j} \big(\rho_{ij}(t, x) p(t, k, x; t', 0, x')\big) \,\mathrm{d}x && (6.7.5) \\
&\quad + \int \mu_{0k}(t, x) p(t, 0, x; t', 0, x') \,\mathrm{d}x. && (6.7.6)
\end{aligned}$$

Similar to in the first proof of Theorem 6.6.1, we claim that the lines (6.7.4) and (6.7.5) can be shown to equal zero under certain regularity conditions. Further, using that

$$x \mapsto \frac{p(t, 0, x; t', 0, x')}{p_{00}(t; t', x')}$$

is the conditional probability density of $X(t)$, conditional on $X(t') = x'$ and $Z(t) = 0$, we can write (6.7.6) as

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{0k}(t; t', x') = \mathrm{E}\left[\mu_{0k}(t, X(t)) \,\middle|\, X(t') = x', Z(t) = 0\right] p_{00}(t; t', x').$$

Comparing this with (6.7.3), the result is obtained. □

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6.A Completion of the proof of Theorem 6.6.1

Here we complete the first proof of Theorem 6.6.1, and we recall that we use Assumption 6.6.2.

Proof. (Last part of first proof of Theorem 6.6.1) We show that (6.6.5), (6.6.6) and (6.6.7) equal zero. For (6.6.5), see that for x large there exists $b > 0$ such that,

$$0 \le |\beta(t, x)| p(t, k, x; t', 0, x') \le b|x| p(t, k, x; t', 0, x') \to 0,$$

for $x \to \infty$. For the limit, we used the assumption $\int x p(t, k, x; t', 0, x') \mathrm{d}x < \infty$ with Lemma 6.A.1 to conclude that $x p(t, k, x; t', 0, x') \to 0$ for $x \to \infty$. The same argument for $x \to -\infty$ allows us to conclude that (6.6.5) equals zero. Since $\frac{\partial}{\partial x}\rho(t,x)$ is bounded similarly to $\beta(t,x)$, we also conclude that (6.6.7) equals zero.

For (6.6.6), apply integration by parts to see that

$$\begin{aligned}\infty &> \int_K^\infty xp(t,k,x;t',0,x')\,\mathrm{d}x \\ &= \left[\frac{1}{2}x^2 p(t,k,x;t',0,x')\right]_K^\infty + \frac{1}{2}\int_K^\infty x^2\left(-\frac{\partial}{\partial x}p(t,k,x;t',0,x')\right)\mathrm{d}x.\end{aligned}$$

Since both terms are positive, we conclude that the latter is finite. Again, by Lemma 6.A.1, we conclude that

$$x^2\frac{\partial}{\partial x}p(t,k,x;t',0,x') \to 0,$$

for $x \to \infty$. Using this and that there for large x exists $b > 0$ such that $\rho(t,x) < bx^2$, we obtain

$$\begin{aligned}0 &\le \left|\rho(t,x)\frac{\partial}{\partial x}p(t,k,x;t',0,x')\right| \\ &\le bx^2\left|\frac{\partial}{\partial x}p(t,k,x;t',0,x')\right| \to 0,\end{aligned}$$

for $x \to \infty$. By the same argument for $x \to -\infty$, we conclude that (6.6.6) equals zero. $\square$

The following lemma was used for the proof.

Lemma 6.A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $K > 0$. Assume that for all $x \ge K$, $f'(x)$ exists, $f'(x) \le 0$ and $f(x) \ge 0$. Let $k \in \mathbb{N}$. If $\int_K^\infty x^k f(x)\mathrm{d}x < \infty$ and $f(x) \in \mathcal{O}\left(x^{-k+1}\right)$, then $x^k f(x) \to 0$ for $x \to \infty$.*

Proof. We prove the contrapositive statement, so assume that $x^k f(x) \not\to 0$ for $x \to \infty$. Then we can choose an $\varepsilon > 0$ such that

$$\forall N \ge 0\, \exists x > N : x^k f(x) \ge \varepsilon. \tag{6.A.1}$$

The idea of the proof is to use that $x^k f(x)$ infinitely many times becomes larger than ε to show that $\int_K^\infty x^k f(x)\mathrm{d}x = \infty$. Since $f(x) \in \mathcal{O}\left(x^{-k+1}\right)$, we can choose $L > K$ and $c > 0$ such that $f(x) \le cx^{-k+1}$ for $x \ge L$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k f(x)\right) = \left(kx^{k-1}f(x) + x^k f'(x)\right) \le kx^{k-1}f(x) \le kc.$$

Define now $\delta = \frac{\varepsilon}{kc}$, $x_0 = \inf\{x > L \mid x^k f(x) \ge \varepsilon\}$, and then recursively

$$x_n = \inf\{x > x_{n-1} + \delta \mid x^k f(x) \ge \varepsilon\},$$

for all $n \geq 1$. By (6.A.1), x_n is an infinite sequence. Now see, that for $n \geq 1$,

$$\begin{aligned}\int_{x_n-\delta}^{x_n} x^k f(x)\,\mathrm{d}x &= \int_{x_n-\delta}^{x_n} \left(\varepsilon - \int_x^{x_n} \left(ky^{k-1}f(y) + y^k f'(y)\right)\mathrm{d}y\right)\mathrm{d}x \\ &\geq \int_{x_n-\delta}^{x_n} \left(\varepsilon - \int_x^{x_n} kc\,\mathrm{d}y\right)\mathrm{d}x = \delta\varepsilon - \frac{1}{2}\delta^2 kc = \frac{1}{2}\varepsilon\delta.\end{aligned}$$

Since the intervals $(x_n - \delta, x_n]$ are disjoint, we obtain

$$\int_K^\infty x^k f(x)\,\mathrm{d}x \geq \sum_{n=1}^\infty \int_{x_n-\delta}^{x_n} x^k f(x)\,\mathrm{d}x \geq \sum_{n=1}^\infty \frac{1}{2}\varepsilon\delta = \infty.$$

$\square$

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